STATISTICAL DESIGN
OF
SAMPLED DATA CONTROL SYSTEMS

A Dissertation
submitted in partial fulfilment
of the requirements for the Degree
of
MASTER OF ENGINEERING
in
ADVANCED ELECTRONICS

By

S.K. AGARWAL

DEPTT. OF ELECTRONICS & COMMUNICATION ENGINEERING
UNIVERSITY OF ROORKEE
ROORKEE
September, 1966

ACKNOWLEDGEMENT

Author wishes to express his deep sense of gratitude to Dr. M. Lal, Reader in Electronics and Communication Engineering , University of Roorkee, Roorkee , for suggesting the problem, continuous encouragement and valuable guidance throughout the preparation of this work, without which it would not have been possible to complete the present dissertation.

CERTIFICATE

CERTIFIED that the dissertation entitled "STATISTICAL DESIGN OF SAMPLED DATA CONTROL SYSTEMS" which is being submitted by Sri S.K. Agarwal in partial fulfilment for the award of degree of MASTER OF ENGINEERING in ADVANCED ELECTRONICS of the University of Roorkee is a record of student's own work carried out by him under my supervision and guidance. The matter embodied in this dissertation has not been submitted for the award of any other degree or diploma.

This is to further certify that he has worked for a period of more than seven months from 1st January 1966 to 13th August, 1966 for preparing this dissertation at the University.

Dated September 8, 1966.

(M. LAL)
READER IN ELECTRONICS &
COMMUNICATION ENGINEERING
UNIVERSITY OF ROORKEE
ROORKEE

ABSTRACT

Analytical technique for the design of control systems is adopted to overcome the drawbacks of classical trial and error approach which is based on the system response to a selected input in absence of noise. Statistical properties of noise and input signals are used to make the system design more realistic.

In this dissertation, analytical design of sampled data systems, using statistical properties of signals, is carried out, extending the technique for design of continuous data systems. Optimization of sampled-data systems is carried out in time-domain, minimizing the mean-square value of error sequence. Z- transform, optimum-system pulse-transfer function is obtained in terms of pulse-spectral density of input signals and cross-spectral density of input, ideal output signals. Optimization is done for free and semi-free configurations.

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CHAPTER I

1.1 INTRODUCTION

Sampled data systems have been in vogue recently. Their usé is increasing rapidly with advances in other fields of science and technology . Sampled data systems are the systems where the signal is sampled at one or more points.

The sampling of the signal may be inherent. The input to a radar tracking system is in the form of a pulse train. In time multiplexed control or communication systems, data from several channels is sampled and multiplexed for transmission over the same channel.

Some times the sampling of the continuous signal is introduced intentionally to improve the performance of the system. Continuous system with transportation lag can be stabilized by introducing sampling systems with sampled data can, in general, facilitate the realization of adaptive principles. Pulsed data systems are also used for improved sensitivity, by sampling the low power signal, the sensing device can be made extremely sensitive in terms of power gain. The sampling is also introduced, because of the ease and accuracy with which the digital signals can be stored, transmitted and processed.

Some of the applications of sampled data systems are :

1 Semi- Automatic Ground Environment (SAGE) system used by U.S. Department of Defence, a large complex system consisting of telemetry link for weapon guidance.

2 Pulse control of low power motor.

3 Digital control in controlling machine tools, for precision components.

4 Communication and control systems for outerspace communication use sampled data signals.

1.2. CLASSICAL DESIGN TECHNIQUES FOR CONTINUOUS-DATA CONTROL SYSTEMS.

Control system transfer function can be represented either in time domain or in the frequency domain. The methods of design for control systems can also be classified along these lines.

In time domain, transient and steady state behavior of the system are of interest. System specifications, e.g., maximum overshoot, time of first overshoot, maximum settling time etc., can be expressed in terms of the damping ratio and undamped natural frequency of the system. Transient behaviour of a system is determined by the roots of the characteristic equation,

$$1 + G(s)\,H(s) = 0$$

where $G(s)\,H(s)$ is open-loop transfer function of the system.

Stability roots are key to the dynamic performance of a system. Root solution of the characteristic equation by direct analytical method is laborious and impractical for design purposes. A graphical approach makes plotting of root locus practical for complex systems. This makes available a complete picture of stability changes due to the effect of individual elements. Original root locus is modified by insertion of compensation element that places the roots of characteristic equation at a more favourable point.

Root locus approach is essentially an analytical approach, where the characteristic equation of the system must be known.

Other method of control system design is that in frequency domain. The system can be represented by its response to a sinusoidal signal of constant amplitude. This is essentially a graphical method of system design. Frequency response transfer function of a system can be represented by Polar plot, Bode plots or magnitude versus phase shift plot. The frequency response design methods are preferable, because the experimental data is in the form of frequency response and final design can also be checked by frequency analysis.

Various frequency domain specifications are the system bandwidth, resonance peak, resonance frequency, cutoff rate , gain margin phase margin etc.

Bode plot method of control system design and compensation is preferred, because the effect of compensation is easily obtained by adding magnitude and phase shift curves of individual elements.

1.3. EXTENSION OF CLASSICAL DESIGN TECHNIQUES TO SAMPLED- DATA SYSTEMS

Compensation and design techniques for the continuous data control systems also extended to the design of sampled-data systems (7) . But the sampling operation makes the design of feedback system compensation more difficult. Compensation of sampled- data systems may be effected by two general methods.

1. Compensation by continuous devices, making use of continuous data compensation networks in series with other components of the system.

2. Compensation by pulsed data devices whose output is sampled in synchronism with its input at a constant rate.

When the transfer function of a system is in factorised form, it is preferable to work with Bode- diagram, because of the ease and simplicity with which the asymptotic Bode-plot of a transfer function can be plotted and reshaped.

Characteristic equation of a sampled data control system is a transcedental equation and open loop transfer function of the system is a transcedental function in s. Difficulties encountered in plotting of transcedental functions, discourage the application of Bode-plot techniques for the design of sampled data control systems. This difficulty is overcome by converting transcedental function in s into a rational function in z, by the transformation

$$z = e^{sT}$$

This process maps the primary and complementary strips of left half of s - plane into unit circle in z plane. Bilinear transformation,

$$z = \frac{1 + w}{1 - w}$$

maps, unit circle in Z - plane onto imaginary axis of another plane w, and interior of the unit circle into entire left- half of w - plane.

In the extension of root-locus techniques to the design of sampled- data control systems, numerous difficulties are encountered in construction of root- loci from a starred transfer function, because of infinite number of poles and zeroes. Complicated nature of root-locus plot in s plane makes it difficult to study the effect of added compensation.

These difficulties are overcome by the use of Z - transform technique. The overall system characteristic equation is transformed to

$$1 + A(z) = 0$$

where, $A(z)$ is open-loop pulse transfer function, a rational function of Z containing a finite number of poles and zeroes. Stable operation of the system requires that theroot locus of sampled data system be confined to the unit circle in Z plane.

1.4. DRAWBACKS OF CLASSICAL DESIGN TECHNIQUES

Classical design, techniques suffer from several drawbacks, that hamper the systematic design of control systems and compensating networks;

1 Compensating networks and control systems are designed, assuming, that a known input like transi-ent like step, or sinusoidal, is available. Such signals do not occur in general, and the design becomes unrealistic.

2 System is designed only for processing the signal. No consideration is taken of inevitability of noise in the system due to physical nature of the system, e.g., shot noise in the tubes, manufacturing tolerances or other disturbances.

3. Classical design techniques are based on trial and error approach. The designer makes some changes in the system, studies the changes in response, again makes changes in system parameter until system performance is within the satisfactory limits. The system designer, except experience does not have any means to recognize an inconsistant set of specifications. Trial and error cycles may be repeated one after the other, without achieving the desired results.

1.5. ANALYTICAL DESIGN METHODS

To overcome the drawbacks of repeated trials and noise considerations, analytical design approach is adopted. In analytical procedure the design of control system begins, with the specifications of system input and desired output. By analytical approach, an inconsistent set of specifications can be recognised, and either a new set of specifications is prescribed or the design is given up as not being feasible.

Considerable work has been done on the analytical design of control systems with continuous signals, based on the mean square error criterion and integral square error criterion (9,10,14,15). Analytical design procedure adopted by Newton, Gould and Kaiser (14), requires one more specification that is not explicitly used for trial and error procedure. This concerns the degree of freedom allowed in compensation. The system to be designed or fixed configuration, according to , if there are no constraints of the system configuration or fixed elements, constraints of fixed elements only, and both the systems configuration and the fixed elements being specified. Attempts have also been made for the analytical design of systems with constraints like saturation(12) or system bandwidth (13).

1.6. ANALYTICAL DESIGN METHODS EXTENDED TO SAMPLED- DATA CONTROL SYSTEMS

Some work has been done on the analytical design of sampled-data systems. In a paper concerning the statistical treatment of sampled data control systems for random signals(11), Mori , deals with the correlation function of time series, and pulsed-spectral densities for sampled data control systems. This paper also deals with modified Z- transforms, when the signals are considered at sampling instants and during intervals between sampling instants.

Work has been done on the statistical design of sampled data control systems, utilizing statistical properties of input and output signals (2,3,16). The design of control system is carried in Z- domain, minimizing the mean-square error between desired and actual outputs of the system. Tou has also considered compensation of sampled-data control system in Z- domain (17) . Tou has also considered design procedure for discrete-data control system subject to power limitations(17). Bergen (1) has considered statistical design of sampled-data systems with randomly varying sampling.

The statistical design procedure has its own limitations, For example, design of systems using statistical procedure is unrealistic, since the final result requires poles and zeroes of the initial system to be completely cancelled out and poles and zeroes placed at new and more suitable locations by the equilizer.

Statistical method of analytical design is highly restrictive as far as inputs are concerned. The inputs are assumed to be stationery and ergodic. In general the conditions of ergodicity are nebulous, it may be rather difficult to deduce from the actual data whether the assumption of ergodicity is true or not.

1.7. STATEMENT OF THE PROBLEM

The problem under investigation is to analytically design the sampled-data systems, utilizing the statistical properties of the desired signals, control signal component, and the noise.

A control system can be calssified as a free, semi-free or fixed configuration system, depending whether, system performance specifications alone, or system performance specifications alongwith some elements of the system, or system performance specifications alongwith the system configuration, are specified.

In the present work, the design of sampled-data control system, has been attempted in the time domain.

In the second chapter of this dissertation, statistical nature of input signals, correlation techniques utilizing the statistical properties of the signals, appropriateness of mean-square error as performance index are considered for the statistical design procedure.

In the Third Chapter, time-domain equation is developed for the optimum sampled - data system weighting sequences that gives minimum mean-square error, this equation is similar to Wiener - Hopf equation for the continuous data systems:

$$\int_{-\infty}^{\infty} dt_2 \, w_m(t_2) \, \phi_{rr}(t_1 - t_2) - \phi_{ri}(t_1) = 0, \text{for } t_1 \geq 0$$

where

$w_m(t)$ - weighting function of optimum linear control system.

$\phi_{rr}(t_1 - t_2)$ - auto correlation function of input signal.

$\phi_{ri}(t_1)$ - Cross-correlation of function of input, ideal output signals.

An explicit solution is found for the pulse-transfer-function of the optimum system, with mean-square value of error sequence.

Time- domain solution , is extended to the optimum systems, in mean-square error sense, with fixed elements , or semi-free configuration systems.

A-nalytical design of sampled-data systems with determinist 'ic signals is also attempted.

CHAPTER II

PRELIMINARY CONSIDERATIONS

2.1 INTRODUCTION

Inputs to the control systems are of random nature. These random signals can be represented statistically, in terms of probability density functions, average value and moments of various orders. Mean-square value of the system error may be taken as a convenient criterion of system performance. Integral-square error is a convenient performance index for the control system with deterministic inputs.

Wiener-Hopf equation gives the optimum-system impulse response, of the continuous data systems, in the mean-square sense. By modification of the Wiener-Hopf equation transfer function of the optimum system is obtained in terms of spectral densities of signals in the system. Tou's approach to the design of optimum sampled-data systems is also considered in the second chapter.

2.2 STATISTICAL CHARACTERISTICS OF CONTROL SIGNALS

An automatic control system is seldom designed to perform a task, which may be completely specified beforehand. A control system is designed to perform a task selected at random from a range of possible tasks, but within certain limits. Many performance inaccuracies are essentially random functions of time and can be prescribed in statistical sense only . Because of the manufacturing tolerances in a system the sensitivity and other fixed parameters of operating components are subject to random fluctuations. When examined in detail all physical processes are discontinuous and indeterminate. The voltage output of a vacuum tube oscillator is considered as a continuous smooth function. But , on microscopic examination, the wave is found to be relatively rough because of shot noise.

Auto-and- cross correlation functions, and spectral densities for random signals are given by

autocorrelation function of signal r(t)

$$\phi_{rr}(\tau) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} r(t)\, r(t+\tau)\, dt \qquad ..(1)$$

Cross correlation function of signals r(t) and i(t)

$$\phi_{ri}(\tau) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} r(t)\, i(t+\tau)\, dt \qquad ..(2)$$

Spectral density of signal r(t)

$$\Phi_{rr}(w) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \phi_{rr}(\tau)\, e^{-jw\tau}\, d\tau \qquad ..(3)$$

$$\Phi_{ri}(w) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \phi_{ri}(\tau)\, e^{-jw\tau}\, d\tau \qquad ..(4)$$

2.3 CORRELATION SEQUENCES AND SEPECTRAL DENSITIES FOR SAMPLED SIGNALS

Correlation sequences for sampled, random signals (11) are defined as follows :

Auto correlation sequence,

$$\phi_{rr}(kT) = \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=-N}^{N} r(nT)\, r(n+k)T \qquad ..(1)$$

Cross correlation sequence

$$\phi_{ri}(kT) = \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=-N}^{N} r(nT)\, i(n+k)T \qquad ..(2)$$

Pulse spectral density is the Fourier -transform of the correlation function.

$$\Phi_{rr}(w) = T \sum_{k=-\infty}^{\infty} \phi_{rr}(kT)\, e^{-jwkT} \qquad ..(3)$$

Cross pulse spectral density,

$$\Phi_{ri}(w) = T \sum_{k=-\infty}^{\infty} \phi_{ri}(kT)\, e^{-jwkT} \qquad ..(4)$$

Correlation sequences and pulsed-spectral density of pulsed data signals are characterized by

$$\phi_{ii}(kT) = \phi_{ii}(-kT) \qquad \ldots. \qquad ..(5)$$

$$\phi_{ic}(kT) = \phi_{ci}(-kT) \qquad \ldots. \qquad ..(6)$$

$$\bar{\phi}_{ii}(z^{-1}) = \bar{\phi}_{ii}(z) \qquad \ldots \qquad ..(7)$$

$$\bar{\phi}_{ic}(z^{-1}) = \bar{\phi}_{ci}(z) \qquad \ldots. \qquad ..(8)$$

If the response of the pulsed data system $G(z)$ to an input $r^*(t)$ is $c^*(t)$, then the response of this system to an input $\phi_{rr}(kT)$ is $\phi_{rc}(kT)$ and the response of this system to an input $\phi_{cr}(kT)$ is $\phi_{cc}(kT)$

$$\sum_{n=-\infty}^{\infty} g(nT)\ \phi_{rr}(kT - nT) = \phi_{rc}(kT) \qquad \ldots \qquad ..(9)$$

$$\sum_{n=-\infty}^{\infty} g(nT)\ \phi_{cr}(kT - nT) = \phi_{cc}(kT) \qquad \ldots \qquad \ldots(10)$$

$$\phi_{rc}(z) = G(z)\ \phi_{rr}(z) \qquad \ldots \qquad \ldots(11)$$

$$\phi_{cc}(z) = G(z)\ \phi_{cr}(z) \qquad \ldots \qquad \ldots(12)$$

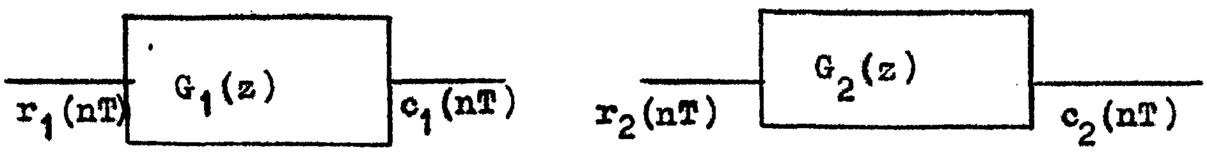

Fig. 1

With reference to Fig. 1 cross correlation sequence and pulse cross spectral density for the output sequences of sampled data systems $G_1(z)$ and $G_2(z)$ possess following characteristics :

$$\phi_{r_2c_1}(kT) = \sum_{n=-\infty}^{\infty} g_1(nT)\, \phi_{r_2r_1}(kT - nT) \qquad ..(13)$$

$$\phi_{c_1c_2}(kT) = \sum_{n=-\infty}^{\infty} g_2(nT)\, \phi_{c_1r_2}(kT - nT) \qquad ...(14)$$

$$\phi_{r_1c_2}(kT) = \sum_{n=-\infty}^{\infty} g_2(nT)\, \phi_{r_1r_2}(kT - nT) \qquad ...(15)$$

$$\phi_{c_2c_1}(kT) = \sum_{n=-\infty}^{\infty} g_1(nT)\, \phi_{c_2r_1}(kT - nT) \qquad ...(16)$$

$$\Phi_{r_2c_1}(z) = G_1(z)\, \Phi_{r_2r_1}(z) \qquad ...(17)$$

$$\Phi_{c_1c_2}(z) = G_2(z)\, \Phi_{c_1r_2}(z)$$

$$\Phi_{c_1c_2}(z) = G_1(z^{-1})\, G_2(z)\, \Phi_{r_1r_2}(z) \qquad ...(18)$$

$$\Phi_{r_1c_2}(z) = G_2(z)\, \Phi_{r_1r_2}(z) \qquad ...(19)$$

$$\Phi_{c_2c_1}(z) = G_1(z)\, \Phi_{c_2r_1}(z)$$

$$\Phi_{c_2c_1}(z) = G_1(z)\, G_2(z^{-1})\, \Phi_{r_2r_1}(z) \qquad ...(20)$$

2.4 PERFORMANCE INDEX

Performance index is defined as some mathematical function of measured response, the function being chosen to give emphasis to system specifications of interest(5,6) Principles for selection of a performance index are:

1. Reliability : Performance index should express the quality of performance as closely as possible.
2. Selectivity : Optimum value of system parameters should be clearly discernible from some characteristics, such as a minimum value of the performance index versus system parameters.

3. Performance index should be easily calculable from existing techniques

4. Performance index should be unaffected by unlikely short-lived deviations from the mean, or shifts in time-axis; instead it should be a measure of the average behaviour of control system.

The mean-square error criterion is widely used, it is defined as follows :

$$\overline{e^2} = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} e^2(t)\, dt$$

$$= \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} \left[c(t) - i(t)\right]^2 dt$$

Where

$c(t)$ = Actual output of the system.

$i(t)$ = Ideal output of the system

$e(t)$ = Error in the output

One of the main reason for wide usage of mean-square error criterion stems from its mathematical convenience. A different error criterion may be preferable except for attendant mathematical difficulties. Mean square error criterion is adequate wherever the undesirability of an error grows with the magnitude.

Mean square value of a random process is one of the easier parameters to evaluate experimentally. Mean-square error together with the mean value of a random process yields information about the process when it is Gaussian. The Central limit theorem is often invoked to assume Gaussian Process.

Integral square error is a measure of transient response of the system. It was first applied by A.C. Hall. ISE is defined as :

$$ISE = \int_{-\infty}^{\infty} \left[e(t)\right]^2 \, dt$$

Both integral-square error and mean-square error can be represented in terms of the system impulse response and correlation functions amongst system input and desired output. Both ISE and MSE criteria can be extended to the sampled data systems also the systems being characterized as having sampled inputs and outputs.

Mean - square error,

$$1 \quad \overline{e^2(nT)} = \overline{\left[i(nT) - c(nT)\right]^2}$$

$$= \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=-N}^{N} e^2(nT)$$

Total square err-or

$$= \sum_{n=-\infty}^{\infty} \left[i(nT) - c(nT)\right]^2$$

$$= \sum_{n=-\infty}^{\infty} e^2(nT)$$

2.5 WIENER- HOPF EQUATION FOR OPTIMUM LINEAR SYSTEMS

In Newton, Gould and Kaiser approach (14) to the design of optimum linear systems, a system is classified as free, semi-free or fixed configuration system, depending on the restrictions placed on configuration.

In the following section, the design of an optimum free configuration, continuous - data system with minimum mean square error criterion is given (14).

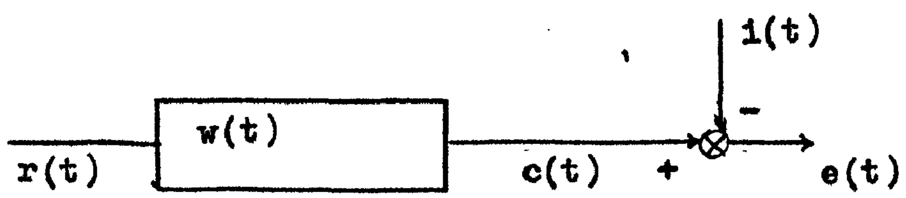

Fig. 2

The error signal at the output of the system, with reference to Fig. 2,. is defined as

$$e(t) = c(t) - i(t) \qquad ...(1)$$

Where $r(t)$ = actual random input to the system

$i(t)$ = ideal output of the system

$c(t)$ = actual output of the system

$w_m(t)$ = impulse response of optimum linear system

The mean square error expressed in terms of correlation functions of system input and ideal output is given by

$$\overline{e^2(t)} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} w_m(\tau_1)\, w_m(\tau_2)\, \phi_{rr}(\tau_1 - \tau_2)\, d\tau_1\, d\tau_2$$

$$- 2\int_{-\infty}^{\infty} w_m(\tau_2)\, \phi_{ri}(\tau_1)\, d\tau_1 + \phi_{ii}(0) \qquad ...(2)$$

The bar indicates time average of the function.

By calculus of variation, equation for optimum system minimizing the mean square error is obtained. The equation, known as Wiener - Hoff equation is as follows :

$$\int_{-\infty}^{\infty} w_m(\tau_2)\, \phi_{rr}(\tau_1 - \tau_2)\, d\tau_2 - \phi_{ri}(\tau_1) = 0 \quad \text{for } \tau_1 \geq 0 \qquad ....(3)$$

Transfer function fo the optimum system, in terms of spectral - and cross-spectral density of system input and ideal output is given by the equation

$$W_m(s) = \frac{\left[\dfrac{\phi_{ri}(s)}{\phi_{rr}^{-}(s)}\right]_+}{\phi_{rr}^{+}(s)} \qquad ..(4)$$

Where

$W_m(s)$ = Transfer function of the optimum system

$\phi_{rr}^{+}(s)$ = Factor of $\phi_{rr}(s)$ which includes all the poles and zeroes of $\phi_{rr}(s)$ in the left half plane.

$\phi_{rr}^{-}(s)$ = Factor of $\phi_{rr}(s)$ which includes all the poles and zeroes of $\phi_{rr}(s)$ in the right half plane.

$\left[\dfrac{\phi_{ri}(s)}{\phi_{rr}^{-}(s)}\right]_+$ = Component of $\dfrac{\phi_{ri}(s)}{\phi_{rr}^{-}(s)}$, which has all its poles in the left half plane.

$\left[\dfrac{\phi_{ri}(s)}{\phi_{rr}^{-}(s)}\right]_-$ = Component of $\dfrac{\phi_{ri}(s)}{\phi_{rr}^{-}(s)}$, which has all its poles in the right half plane.

2.6 TOU'S APPROACH TO DESIGN OF OPTIMUM FREE CONFIGURATION SAMPLED DATA CONTROL SYSTEMS

According to J.T. Tou's approach(16), the optimum design of a sampled- data system is obtained for sampled stochastic control signal $r_s(nT)$ and sampled noise $r_n(nT)$. Both the signal and noise are

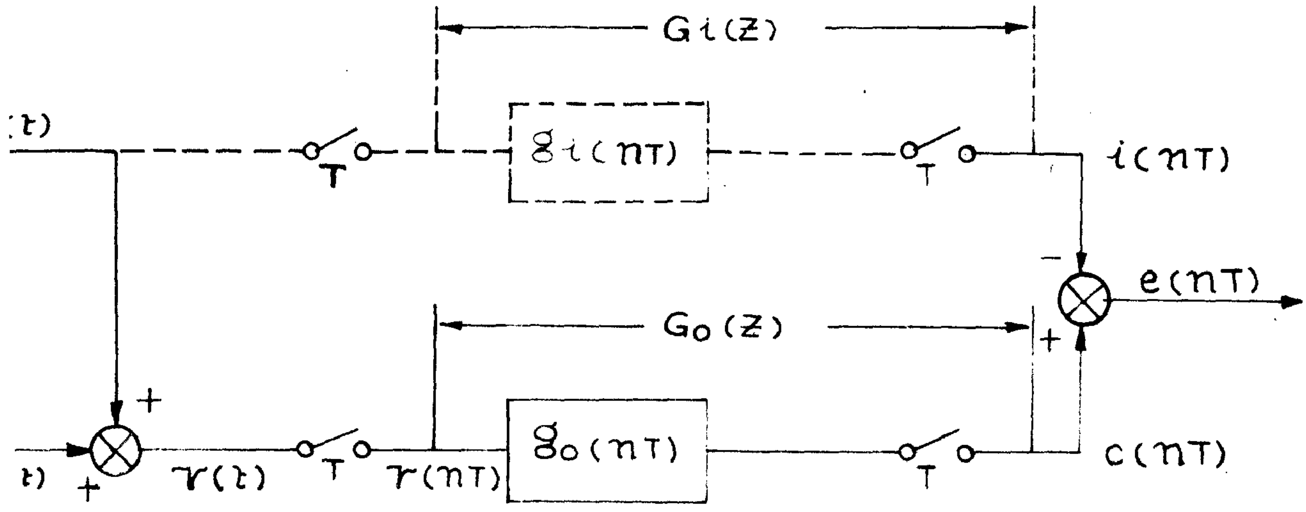

Figure 3.

assumed to be stationary random functions.

For the system shown in Fig. 3, $G_o(z)$ is the pulse- transfer function of optimum control system, $G_i(z)$ is pulse transfer function of the ideal system when, there is no noise present.

Error sequence of the system is,

$$e(nT) = c(nT) - i(nT) \qquad ..(1)$$

Where

T = Sampling period.

Error sequence of the system may be written

as $$e(nT) = c_s(nT) + c_n(nT) - i(nT) \qquad ..(2)$$

Where,

$c_s(nT)$ = System response to control signal $r_s(nT)$

$c_n(nT)$ = System response to control signal $r_n(nT)$

The mean - square value of error sequence of the system is defined as

$$\overline{e^2(nT)} = \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=-N}^{N} e^2(nT) \quad ..(3)$$

$$= \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} \Big[c_s^2(nT) + c_n^2(nT) + i_n^2(nT) + c_s(nT)c_n(nT) + c_n(nT)c_s(nT) - c_n(nT)c_i(nT) - c_i(nT)c_n(nT) - c_s(nT)c_i(nT) - c_i(nT)c_s(nT) \Big] \quad ..(4)$$

$$= \overline{c_s^2(nT)} + \overline{c_n^2(nT)} + \overline{i_n^2(nT)} + \overline{c_s(nT)\ c_n(nT)} + \overline{c_n(nT)c_s(nT)} - \overline{c_n(nT)c_i(nT)} - \overline{c_i(nT)\ c_s(nT)} - \overline{c_s(nT)c_i(nT)} - \overline{c_i(nT)\ c_s(nT)} \quad ..(5)$$

From the equqtions 2.3-1 to 2.3-4 the mean square value of error sequence may be written in terms of correlation sequences, and pulse - spectral densities.

$$\overline{e^2(nT)} = \phi_{c_s c_s}(0) + \phi_{c_n c_n}(0) + \phi_{c_i c_i}(0) + \phi_{c_s c_n}(0) + \phi_{c_n c_s}(0) - \phi_{c_n c_i}(0) - \phi_{c_i c_n}(0) - \phi_{c_s c_i}(0) - \phi_{c_i c_s}(0) \quad ...(6)$$

$$e^2(nT) = \frac{1}{2\pi j} \oint_{\Gamma} \Big[\Phi_{c_s c_s}(z) + \Phi_{c_n c_n}(z) + \Phi_{c_i c_i}(z)$$

$$+ \Phi_{c_s c_n}(z) + \Phi_{c_n c_s}(z) - \Phi_{c_n c_i}(z) - \Phi_{c_i c_n}(z)$$

$$- \Phi_{c_s c_i}(z) - \Phi_{c_i c_s}(z) \Big] \cdot z^{-1} d_z \qquad ..(7)$$

where , contour of integration Γ is the unit circle in the z plane.

These pulse spectral densities may be expressed in terms of pulse spectral densities of input signals and pulse transfer functions with the help of equations (2.3-13) to (2.3-20).

$$\Phi_{c_s c_s}(z) = \Phi_{r_s r_s}(z)\, G_0(z)\, G_0(z^{-1}) \qquad ..(8A)$$

$$\Phi_{c_n c_n}(z) = \Phi_{r_n r_n}(z)\, G_0(z)\, G_0(z^{-1}) \qquad ..(8B)$$

$$\Phi_{c_i c_i}(z) = \Phi_{r_s r_s}(z)\, G_i(z)\, G_i(z^{-1}) \qquad ..(8C)$$

$$\Phi_{c_s c_n}(z) = \Phi_{r_s r_n}(z)\, G_0(z)\, G_0(z^{-1}) \qquad ..(8D)$$

$$\Phi_{c_n c_s}(z) = \Phi_{r_n r_s}(z)\, G_0(z)\, G_0(z^{-1}) \qquad ..(8E)$$

$$\Phi_{c_n c_i}(z) = \Phi_{r_n r_s}(z)\, G_i(z)\, G_0(z^{-1}) \qquad ..(8F)$$

$$\Phi_{c_i c_n}(z) = \Phi_{r_s r_n}(z)\, G_0(z)\, G_i(z^{-1}) \qquad ..(8G)$$

$$\Phi_{c_s c_i}(z) \qquad = \Phi_{r_s r_s}(z)\, G_i(z)\, G_o(z^{-1}) \qquad ..(8H)$$

$$\Phi_{c_i c_s'}(z) \qquad = \Phi_{r_s r_s}(z)\, G_o(z)\, G_i(z^{-1}) \qquad ..(8I)$$

The mean square error is reduced to

$$\overline{e^2(nT)} = \frac{1}{2\pi j} \oint \Phi_{ee}(z)\, Z^{-1}\, dz \qquad ..(9)$$

Where

$$\begin{aligned}\Phi_{ee}(z) = & \left[G_o(z) - G_i(z)\right] \left[G_o(z^{-1}) - G_i(z^{-1})\right] \Phi_{r_s r_s}(z) \\ & + G_o(z) \left[G_o(z^{-1}) - G_i(z^{-1})\right] \Phi_{r_s r_n}(z) \\ & + G_o(z^{-1}) \left[G_o(z) - G_i(z)\right] \Phi_{r_n r_s}(z) \\ & + G_o(z)\, G\,(z^{-1})\, \Phi\ \Phi_{r_n r_n}(z) \end{aligned} \qquad ..(10)$$

Condition for the minimum mean square sampled error is obtained by applying calculus of variations to the integral of equation (2.6-9). The first order variation of mean square error $\delta\ \overline{e^2(nT)}$ is obtained from equation (9) by substituting $G_o\ (z) + \eta\ (z)$ for $G_o\ (z)$ and $G_o(z^{-1}) + \eta(z^{-1})$ for $G_o(z^{-1})$

$$\therefore\ e^2(nT) = \frac{1}{2\pi j} \oint_{\Gamma} \eta(z) \left\{ G_o\ (z^{-1}) \Phi\ (z) - G_d(z^{-1}) \left[\Phi_{r_s r_s}(z) + \Phi_{r_s r_n}(z)\right]\right\} z^{-1} dz + \frac{1}{2\pi j} \oint_{\Gamma} \eta(z^{-1}) \left\{ G_o(z) \Phi(z) - G_d(z)\, \Phi_{r_s r_s}(z) + \Phi_{r_n r_s}(z) \right\} z^{-1} dz \quad ...(11)$$

$\lceil$ G $\rfloor$

Where

$$\Phi(z) = \Phi_{r_s r_s}(z) + \Phi_{r_n r_n}(z) + \Phi_{r_s r_n}(z) + \Phi_{r_n r_s}(z) \qquad ..(12)$$

For reasons of stability all the poles and zeroes of $G_o(z)$ and $\eta(z)$ lie inside the unit circle in z plane, and those of $G_o(z^{-1})$ and $\eta(z^{-1})$ lie outside the unit circle. From equations 2.3-5 to 2.3-8

$$\Phi(z) = \Phi(z^{-1}) \qquad ..(13)$$

and $\Phi(z)$ may be written as

$$\Phi(z) = \Phi^{+}(z)\ \Phi^{-}(z) \qquad ..(14)$$

Where $\Phi^{+}(z)$ is a function of z with poles and zeroes lying inside the unit circle in z plane, and $\Phi^{-}(z)$ is a function with poles and zeroes lying outside the unit circle.

$$\therefore\ \delta\overline{e^2(nT)} = \frac{1}{2\pi j}\oint_{\Gamma} \eta(z)\left[G_o(z^{-1})\,\Phi^{+}(z)\Phi^{-}(z) - G_i(z^{-1})\left[\Phi_{r_s r_s}(z) + \Phi_{r_s r_n}(z)\right]\right] z^{-1}dz + \frac{1}{2\pi j}\oint_{\Gamma}\eta(z^{-1})\left[G_o(z)\,\Phi^{+}(z)\,\Phi^{-}(z) - G_i(z)\left[\Phi_{r_s r_s}(z) + \Phi_{r_n r_s}(z)\right]\right] z^{-1}dz \qquad ..(15$$

$$= \frac{1}{2\pi j}\oint_{\Gamma}\eta(z)\,\Phi^{+}(z)\left[G_o(z^{-1})\cdot\Phi^{-}(z) - \frac{G_i(z^{-1})\ \Phi_{r_s r_s}(z) + \Phi_{r_s}}{\Phi^{+}(z)}\right] z^{-1}\,dz + \frac{1}{2\pi j}\oint_{\Gamma}\eta(z^{-1})\,\Phi^{-}(z)\left[G_o(z)\,\Phi^{+}(z) - \frac{G_i(z)\ \Phi_{r_s r_s}(z) + \Phi_{r_n r_s}(z)}{\Phi^{-}(z)}\right] z^{-1}\,dz \qquad ..(16)$$

Second term in braces of equation (2.6-16) contains poles inside the unit circle in Z- plane as well as outside the unit circle, this may be written as

$$\frac{\left[G_1(z)\left\{\bar{\Phi}_{r_s r_s}(z)+\bar{\Phi}_{r_n r_s}(z)\right\}\right]}{\bar{\Phi}^{-}(z)} = \left[\frac{G_1(z)\left[\bar{\Phi}_{r_s r_s}(z)+\bar{\Phi}_{r_n r_s}(z)\right]}{\bar{\Phi}^{-}(z)}\right]_+ + \left[\frac{G_1(z)\left[\bar{\Phi}_{r_s r_s}(z)+\bar{\Phi}_{r_n r_s}(z)\right]}{\bar{\Phi}^{-}(z)}\right]_-$$

Symbol $[\;]_-$ implies operation of picking part of a function of z with poles outside unit circle in Z- plane and $[\;]_+$ implies the operation of picking part of a function of z with poles inside unit circle in Z- plane. The contour integral vanishes, if the integrand has its poles either all inside the unit circle or all outside the unit circle.

Hence

$$\frac{1}{2\pi j}\oint_{\Gamma} \eta(z)\,\bar{\Phi}^{+}(z)\left[G_1(z^{-1})\frac{\left[\bar{\Phi}_{r_s r_s}(z)+\bar{\Phi}_{r_s r_n}(z)\right]}{\bar{\Phi}^{+}(z)}\right]_+ z^{-1}dz=0 \quad ..(18A)$$

$$\frac{1}{2\pi j}\oint_{\Gamma} \eta(z^{-1})\bar{\Phi}^{-}(z)\left[G_1(z\;\;)\frac{\left[\bar{\Phi}_{r_s r_s}(z)+\bar{\Phi}_{r_n r_s}(z)\right]}{\bar{\Phi}^{-}(z)}\right]_- z^{-1}dz = 0 \quad ..(18B)$$

The equation (2.7-15) is reduced to

$$\delta\overline{e^2(nT)} = \frac{1}{2\pi j}\oint_{\Gamma} \eta(z)\bar{\alpha}^{+}(z)\Bigg\{\Bigg[G_o(z^{-1})\bar{\alpha}^{-}(z) - \left[\frac{G_i(z^{-1})\left[\bar{\alpha}^{-}_{r_s r_s}(z)+\bar{\alpha}_{r_s r_n}(z)\right]}{\bar{\alpha}^{+}(z)}\right]_{-}\Bigg]\Bigg\} z^{-1}dz$$

$$+\frac{1}{2\pi j}\oint_{\Gamma} \eta(z^{-1})\bar{\alpha}^{-}(z)\Bigg\{\Bigg[G_o(z)\,\bar{\alpha}^{+}(z) - \left[\frac{G_i(z)\left[\bar{\alpha}_{r_s r_s}(z)+\bar{\alpha}_{r_n r_s}(z)\right]}{\bar{\alpha}^{-}(z)}\right]_{+}\Bigg]\Bigg\} z^{-1}\,dz \quad ...(19)$$

If $G_o(z)$ is overall pulse transfer function of the optimum system which minimizes the mean square error of sampled data system, the variation $\delta\,\overline{e^2\,(nT)}$ should vanish for arbitrary $\eta(z)$, hence the quantities in the brackets should be zero.

$$G_o(z)\bar{\alpha}^{+}(z) - \left[\frac{\left[G_i(z)\left[\bar{\alpha}_{r_s r_s}(z)+\bar{\alpha}_{r_n r_s}(z)\right]\right]}{\bar{\alpha}^{-}(z)}\right]_{+} = 0 \quad ...(20)$$

The optimum Pulse-transfer function is given by

$$G_o(z) = \left[\frac{G_i(z)\left[\bar{\alpha}_{r_s r_s}(z)+\bar{\alpha}_{r_n r_s}(z)\right]}{\bar{\alpha}^{-}(z)}\right]_{+} \times \frac{1}{\bar{\alpha}^{+}(z)} \quad ..(21)$$

2.7 TOU'S APPROACH TO THE DESIGN OF OPTIMUM SEMI FREE CONFIGURATION SAMPLED DATA CONTROL SYSTEM

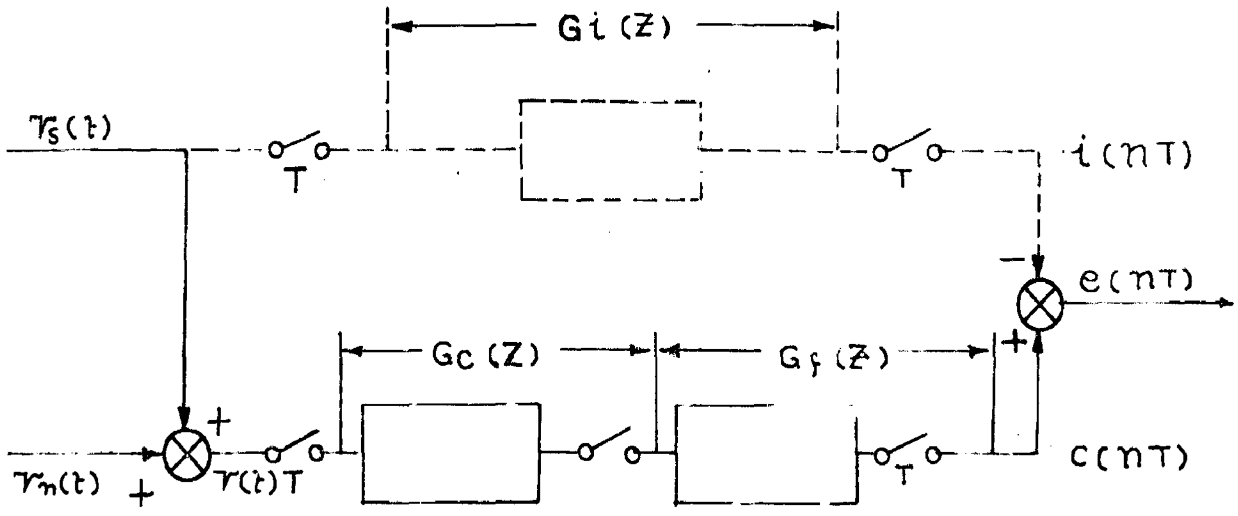

Figure 4

The optimum-design of compensator pulsetransfer function in the mean square sense, when the control system is partially specified can be achieved in a way similar to that adopted in previous section (17). Sampled -data control system is subjected to a sampled stochastic control signal and a sampled random noise, both being stationary functions.

For the sampled - data system, Fig. 4 , the symbols used are as follows :

$r_s(nT)$ = Sampled stochastic control signal

$r_n(nT)$ = Sampled random noise

$G_i(z)$ = Pulse transfer function of ideal system

$G_f(z)$ = Pulse transfer function of part of the system partially specified.

$G_c(z)$ Pulse transfer function of compensation system.

Error sequence of the system is,

$$e(nT) = c(nT) - i(nT) \qquad \ldots(1)$$

Mean - sequare value of error sequence can also be written in terms of correlation sequences

$$\overline{e^2(nT)} = \phi_{cc}(0) - \phi_{ci}(o) - \phi_{ic}(o) + \phi_{ii}(o) \qquad \ldots(2)$$

Writing $e^2(nT)$ in terms of spectral densities

$$\overline{e^2(nT)} = \frac{1}{2\pi j} \oint_{\Gamma} \Phi_{ee}(z) Z^{-1} dz \qquad \ldots(3)$$

Where $\Phi_{ee}(z) = \Phi_{ec}(z) - \Phi_{ci}(z) - \Phi_{ic}(z) + \Phi_{ii}(z) \qquad \ldots(4)$

With the help of equations (2.3-13 to 2.3-20)and

$$G_o(z) = G_c(z)\, G_f(z)$$

$$\begin{aligned}\overline{e^2(nT)} &= \Phi_{r_s r_s}(z) \left[G_c(z) G_f(z) - G_i(z)\right] \left[G_c(z^{-1}) G_f(z^{-1}) - G_i(z^{-1})\right] \\ &+ \Phi_{r_s r_n}(z) \left[G_c(z^{-1}) G_f(z^{-1}) - G_i(z^{-1})\right] G_c(z) G_f(z) \\ &+ \Phi_{r_n r_s}(z) \left[G_c(z)\, G_f(z) - G_i(z)\right] G_c(z^{-1}) G_f(z^{-1}) \\ &+ \Phi_{r_n r_n}(z) \left[G_c(z) G_f(z)\, G_c(z^{-1}) G_f(z^{-1})\right] \qquad \ldots(6)\end{aligned}$$

By calculus of variations, putting $\left[G_c(z) + \eta(z)\right]$ instead of $G_c(z)$ and $\left[G_c(z^{-1}) + \eta(z^{-1})\right]$ instead of $G_c(z^{-1})$, $\delta\, \overline{e^2(nT)}$ first order variation of the mean square value of error sequence is obtained.

$$\overline{e^2(nT)} = \frac{1}{2\pi j}\oint_\Gamma \eta(z) \Big[G_f(z) G_f(z^{-1}) G_c(z^{-1}) \bar{\Phi}(z) - G_i(z^{-1}) G_f(z) \left[\bar{\Phi}_{r_s r_s}(z) + \bar{\Phi}_{r_s r_n}(z)\right]\Big] z^{-1}\, dz + \frac{1}{2\pi j}\oint_\Gamma \eta(z^{-1})$$

$$\Big[G_f(z^{-1}) G_f(z) G_c(z) \bar{\Phi}(z) - G_i(z) G_f(z^{-1}) \left[\bar{\Phi}_{r_s r_s}(z) + \bar{\Phi}_{r_n r_s}(z)\right]\Big] z^{-1} dz \qquad ..(7)$$

Where $\bar{\Phi}(z) = \bar{\Phi}_{r_s r_s}(z) + \bar{\Phi}_{r_s r_n}(z) + \bar{\Phi}_{r_n r_s}(z) + \bar{\Phi}_{r_n r_n}(z)$ ..(8)

Defining $\bar{\Phi}(z) = \bar{\Phi}^{+}(z)\, \bar{\Phi}^{-}(z)$ ......(9)

$$\left[G_f(z) G_f(z^{-1})\right] = \left[G_f(z) G_f(z^{-1})\right]^{+} \left[G_f(z)\, G_f(z^{-1})\right]^{-} \qquad ..(10)$$

$\left[\bar{\Phi}(z)\right]^{+}$ indicates selecting portion of function $\bar{\Phi}(z)$ that lie within the unit circle and $\left[\bar{\Phi}(z)\right]^{-}$ indicates selecting the portion of function $\bar{\Phi}(z)$ that lie outside the unit circle in Z- plane .

$$\overline{e^2(nT)} = \frac{1}{2\pi j}\oint_\Gamma \left[\eta(z) \bar{\Phi}^{+}(z) \left[G_f(z) G_f(z^{-1})\right]^{+} \left\{ \left[G_f(z) G_f(z^{-1})\right]^{-} G_c(z^{-1})\, \bar{\Phi}^{-}(z) - \frac{G_i(z^{-1}) G_f(z) \left[\bar{\Phi}_{r_s r_s}(z) + \bar{\Phi}_{r_s r_n}(z)\right]}{\bar{\Phi}^{+}(z) \left[G_f(z)\, G_f(z^{-1})\right]^{+}} \right\} \right] z^{-1} dz$$

$$+ \frac{1}{2\pi j}\oint_{\Gamma}\left\{\left[\eta(z^{-1})\bar{a}^{-1}(z)\left[G_f(z)G_f(z^{-1})\right]^{-}\left[\left[G_f(z)G_f(z^{-1})\right]^{+}\bar{a}^{+}(z)\,G_c(z) - \frac{G_i(z)\,G_f(z^{-1})\left[\bar{a}_{r_s r_s}(z) + \bar{a}\,r_n r_s(z)\right]}{\bar{a}^{-}(z)\left[G_f(z)\,G_f(z^{-1})\right]^{-}}\right]\right]\right\} z^{-1}.\,dZ \quad \ldots(11)$$

The second term in the braces contains poles inside the unit circle in the z- plane, as well as outside the unit circle. The contour integral vanishes if the integral has its poles either all inside the unit circle or all outside the unit circle. Thus,

$$\frac{1}{2\pi j}\oint_{\Gamma}\eta(z^{-1})\,\bar{a}^{-}(z)\left[G_f(z)\,G_f(z^{-1})\right]^{-}\left[\frac{G_i(z)G_f(z^{-1})\;\bar{a}_{r_s r_s}(z)+\bar{a}_{r_n r_s}(z)}{\bar{a}^{-}(z)\left[G_f(z)\,G_f(z^{-1})\right]^{-}}\right]_{-} = 0 \quad \ldots(12)$$

$$\frac{1}{2\pi j}\oint_{\Gamma}\eta(z)\bar{a}^{+}(z)\left[G_f(z)G_f(z^{-1})\right]^{+}\left[\frac{G_i(z)G_f(z^{-1})\left[\bar{a}_{r_s r_s}(z)+\bar{a}_{r_s r_n}(z)\right]}{\bar{a}^{+}(z)\left[G_f(z)G_f(z^{-1})\right]^{+}}\right]_{+} z^{-1}dz = 0 \quad \ldots(13)$$

where, symbol $\left[\;\right]_{+}$ implies the operation of picking the part of a function of z with poles inside the unit circle in Z- plane, and the symbol $\left[\;\right]_{-}$ implies the operation of picking the part of a function of z with poles outside the unit circle in z plane.

Hence equation (2.7-11) reduces to :

$$\overline{e^2(nT)} = \frac{1}{2\pi j}\oint_{\Gamma} \eta(z)\, \bar{\Phi}^{+}(z) \left[G_f(z)G_f(z^{-1})\right]^{+} \left\{ \left[G_f(z)\, G_f(z^{-1})\right]^{-} G_c(z^{-1})\bar{\Phi}^{-}(z) - \left[\frac{G_i(z^{-1})G_f(z)\left[\bar{\Phi}_{r_s r_s}(z) + \bar{\Phi}_{r_s r_n}(z)\right]}{\bar{\Phi}^{+}(z)\left[G_f(z)\, G_f(z^{-1})\right]^{+}}\right]_{-} \right\} z^{-1}dz$$

$$+ \frac{1}{2\pi j}\oint_{\Gamma} \eta(z^{-1})\bar{\Phi}^{-}(z)\left[G_f(z)G_f(z^{-1})\right]^{-} \left\{ \left[G_f(z)G_f(z^{-1})\right]^{+} G_c(z)\, \bar{\Phi}^{+}(z) - \left[\frac{G_i(z)\, G_f(z^{-1})\, \left[\bar{\Phi}_{r_s r_s}(z) + \bar{\Phi}_{r_n r_s}(z)\right]}{\bar{\Phi}^{-}(z)\, \left[G_f(z)\, G_f(z^{-1})\right]^{-}}\right]_{+} \right\} z^{-1}\, dz \qquad ..(14)$$

For the compensation to be optimum, giving minimum mean square error, $\delta\overline{e^2(nT)}$ should vanish for arbitrary $\eta(z)$. Optimum compensation is given by

$$G_c(z)\left[G_f(z)G_f(z^{-1})\right]^{+} \bar{\Phi}^{+}(z) - \left[\frac{G_f(z^{-1})G_i(z)\left[\bar{\Phi}_{r_s r_s}(z) + \bar{\Phi}_{r_n r_s}(z)\right]}{\bar{\Phi}^{-}(z)\left[G_f(z)\, G_f(z^{-1})\right]^{-}}\right]_{+} = 0 \qquad ...(15)$$

$$G_c(z) = \left[\frac{G_f(z^{-1})\, G_i(z)\, \left[\bar{\Phi}_{r_s r_s}(z) + \bar{\Phi}_{r_n r_s}(z)\right]}{\bar{\Phi}^{-}(z)\, \left[G_f(z)G_f(z^{-1})\right]^{-}}\right]_{+} \times \frac{1}{\left[G_f(z)G_f(z^{-1})\right]^{+}\bar{\Phi}^{+}(z)} \qquad ...(16)$$

Equation (2.7-16), gives the pulse-transfer function of the compensation that will optimize the system in the minimum mean square error sense. This can be obtained in terms of pulse-transfer function of fixed components and pulse spectral density of input signals.

CHAPTER III

OPTIMUM SAMPLED DATA SYSTEMS

3.1 INTRODUCTION

In the present Chapter, optimum design for the systems, whose input and output are sampled in synchronism, is carried out in time-domain. The optimum system equation for sampled-data systems with random input signals, is obtained by minimization of the mean-square value of error sequence. The system equation obtained is as follows :

$$\sum_{m=-\infty}^{\infty} g(mT)\ \phi_{rr}(k-m)T - \phi_{ri}(kT) = 0 \qquad \text{for } k \geqslant 0 \ \ldots \quad 0$$

Where

$g(mT)$ - weighting sequence of the optimum sampled data systems.

$\phi_{rr}\ (k-m)T$ - Autocorrelation sequence of the input signal

$\phi_{ri}\ (kT)$ - Input-ideal output cross- correlation sequence.

which is quite similar to Wiener-Hopf equation for optimum continuous data systems. A solution of the equation is suggested and is carried out in z- domain. The equation is modified as

$$\sum_{m=-\infty}^{\infty} g(mT)\ \phi^{+}_{rr}\ (q-m)T = \beta_{+}(qT)$$

where $\beta_{+}\ (qT)$ is defined by equations (3.3-4) and (3.3-9B)

Pulse transfer function of the optimum system is obtained by taking z- transforms of both sides of the modified system equation. Pulse transfer function of the optimum sampled-data system in terms of spectral densities of systems signals, is given by

$$G_m(z) = \frac{1}{\Phi_{rr}^{+}(z)} \times \left[\frac{\Phi_{ri}(z)}{\Phi_{rr}^{-}(z)}\right]_{+}$$

Where ,

$G_m(z)$ Optimum system pulse-transfer function

$\phi_{rr}(z)$ Pulse-spectral density of input signals

$\Phi_{ri}(z)$ Cross-spectral density of input-ideal output signals.

Optimization of the sampled-data with deterministic input signals, minimizing total square error, has also been discussed and it is shown that an equation for sampled data system with random input signals satisfying minimum mean square error criterion can be obtained.

It is further shown that when the control signal component and the noise enter the system at different points, minimization of mean square error yields the optimum sampled-data system equation.

$$\sum_{p=-\infty}^{\infty}\left[g_s(pT)\,\phi_{ss}(k-p)T + g_n(pT)\,\phi_{ns}(p-k)T\right] - \phi_{si}(kT) = 0 \quad \text{for } k \geqslant 0$$

Where,

$g_s(nT)$ Weighting sequence for stochastic control signal component.

$g_n(nT)$ weighting sequence for random noise component

$r_s(nT)$ input control signal component sequence

$r_n(nT)$ input noise sequence

Optimization of sampled - data system has been extended to semi-free configuration systems . For random input signals, it is shown that the minimization of mean square error yields, the optimum system equation,

$$\sum_{p_1=-\infty}^{\infty} g_f(p_1T)\,\phi_{ri}\,(k_1+p_1)T - \sum_{p_1=\infty}^{\infty}\ \sum_{p_2=-\infty}^{\infty}\ \sum_{k_1=-\infty}^{\infty} g_f(p_1T)$$

$$g_f(p_2T)\,g_c(k_2T)\,\phi_{rr}\,(k_2+p_2-k_1-p_1)T \quad = 0 \text{ for } k_1 \geqslant 0$$

Where

$g_f(pT)$ - Weight sequence of fixed elements

$g_c(pT)$ - Weighting sequence of conpensation elements.

Pulse transfer function of the compensation elements in terms of pulse spectral densities of input and ideal output-signals and the pulse- tranfer function of fixed elements is obtained and the expression obtained is as follows:

$$G_c(z) = \left[\frac{G_f\,(z^{-1})\,\Phi_{ri}(z)}{\left[G_f(z)\,G_f(z^{-1})\right]^{-}\,\Phi_{rr}^{-}(z)}\right]_{+} \times \left[\frac{1}{\left[G_f(z)G_f(z^{-1})\right]^{+}\,\Phi_{rr}^{+}(z)}\right]$$

Next it is shown that when fixed element pulse-transfer function has all its poles and zeroes within the unit circle in z- plane, the pulse transfer function of compensation system is the ratio of pulse-transfer function of a free configuration system., with same performance specifications, and the pulse transfer function of fixed elements.

The expression for minimum - mean square error of the sampled-data system, has been obtained in terms of the optimum system pulse transfer function and spectral densities of the system signals.

3.2 OPTIMAL SYSTEM EQUATION FOR FREE CONFIGURATION SAMPLED DATA SYSTEM

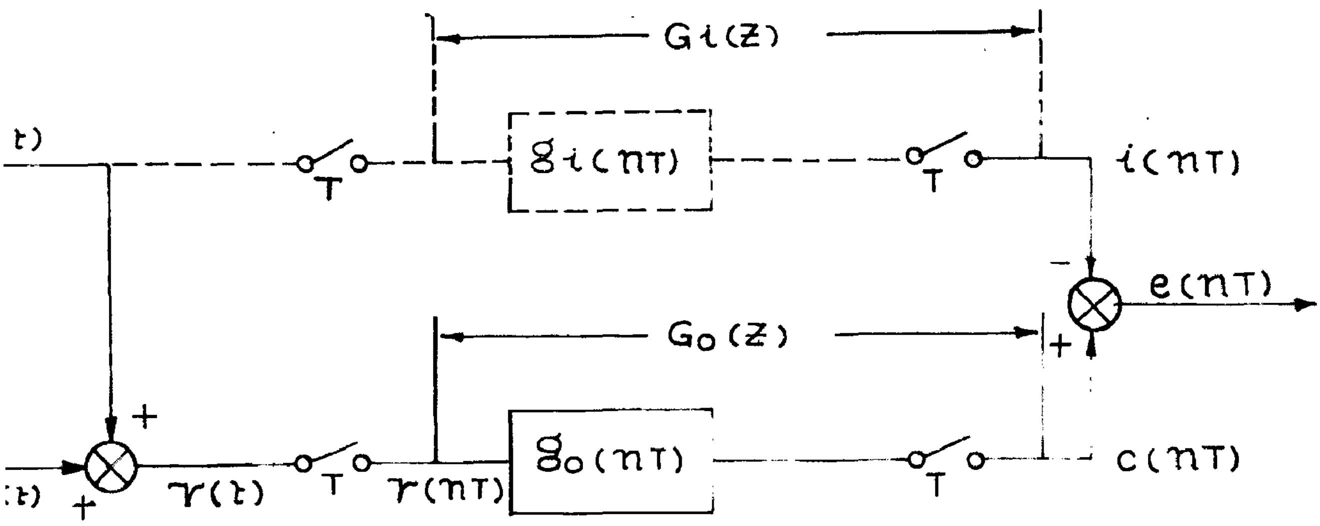

Fig. 5

In free-configuration case, there are no constraints, either on the form of the system ~~desi~~ configuration or of the fixed elements and the designer has to choose both the form and elements. In the present section, time-domain equation of the optimum , sampled data system weighting sequence, in the mean square sense has been derived for the free configuration case.

The output of a system, Fig. 5 , input and output of which are sampled in synchronism, will be given by

$$c(nT) = \sum_{k=-n^*}^{n} g(kT)\, r(nT-kT) \quad ..(1)$$

Where,

c(nT) - Sampled output of the system at nth sampling instant

r(nT) - Sampled input of the system at nth sampling instant

g(kT) - Weighting sequence of sampled-data system.

T - Sampling period

Difference between optimum system output gc(nT) and ideal output sequence , i.e, system error sequence Equation $e_g(nT)$ will be given by

$$e_g(nT) = c(nT) - i(nT) \quad ..(2)$$

$$\therefore \quad e_g^2(nT) = c^2(nT) - 2c(nT)\, i(nT) + i^2(nT) \quad ..(3)$$

e_g^2 Taking mean square value of error sequence

$$\overline{e^2_g(nT)} = \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=-N}^{N} e_g^2(nT)$$

---

* Weighting sequence g(kT) is zero for negative values of k.

$$\overline{e_g^2(nT)} = \overline{c^2(nT)} - \overline{2c(nT)\,i(nT)} + \overline{i^2(nT)} \qquad ..(4)$$

$$\overline{c^2(nT)} = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} c(nT)\,c(nT)$$

$$= \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} \sum_{k=-n}^{n} g(kT)r(n-k)T \sum_{m=-n}^{n} g(mT)r(n-m)T$$

Since k, m & n are dummy variables interchanging them

$$\overline{c^2(nT)} = \sum_{k=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} g(kT)g(mT) \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} r(n-k)Tr(n-m)T$$

From equation (2.3-1)

$$\overline{c^2(nT)} = \sum_{k=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} g(kT)\,g(mT)\,\phi_{rr}(k-m)T \qquad ...(5)$$

Similarly from equation (2.3-2)

$$\overline{c(nT)i(nT)} = \sum_{k=-\infty}^{\infty} g(kT)\,\phi_{ri}(kT) \qquad .... \qquad ..(6)$$

and

$$\overline{i^2(nT)} = \phi_{ii}(o) \qquad ... \qquad ... \quad ..(7)$$

Hence, from equqtions (3.2-4),(3.2-5), (3.2-6),(3.2-7)

$$e_g^2(nT) = \phi_{ii}(o) - 2\sum_{k=-\infty}^{\infty} g(kT)\phi_{ri}(kT) + \sum_{k=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} g(kT)g(mT)\phi_{rr}(k-m)T \qquad ..(8)$$

If the optimum system g(kT) is replaced by another system $[g(kT) + \epsilon\, h(kT)]$, where h(kT) is any arbitrarily realizable weighting sequence, ϵ is a parameter, that is varied to test the optimality of g(kT). Error with

$\left[g(kT)+\epsilon\, h(kT)\right]$ is greater than that for optimal system and will be given by

$$\overline{e_{g+\epsilon h}^{2}(nT)} = \phi_{ii}(o) - 2\sum_{k=-\infty}^{\infty}\left[g(kT)+\epsilon h(kT)\right]\phi_{ri}(kT)$$

$$+\sum_{k=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\left[g(kT)+h(kT)\right]\left[g(mT)+h(mT)\right]\phi_{rr}(k-m)T \quad ..(9)$$

$$\frac{\partial}{\partial\epsilon}\left[\overline{e^{2}_{g+h}(nT)}\right] = -2\sum_{k=-\infty}^{\infty}h(kT)\phi_{ri}(kT) + 2\epsilon\sum_{k=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}h(kT)h(mT)\phi_{rr}(k-m)T$$

$$+\sum_{k=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\left[h(kT)g(mT)+h(mT)g(kT)\right]\phi_{rr}(k-m)T \quad ..(10)$$

$$\frac{\partial^{2}}{\partial\epsilon^{2}}\left[\overline{e^{2}_{g+\epsilon h}(nT)}\right] = 2\sum_{k=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}h(kT)h(mT)\phi_{rr}(k-m)T \quad ...(11)$$

$\frac{\partial^{2}}{}\left[\overline{e^{2}_{g+\epsilon h}(nT)}\right]$ is a positive quantity.

If $g(kT)$ is the weighting sequence of optimum system giving minimum mean square error, then

$$\frac{\partial}{\partial\epsilon}\left[\overline{e^{2}_{g+\epsilon h}(nT)}\right]_{\epsilon=0} = 0 \quad ...(12)$$

From equation 3.2-9 the condition of optimum weighting sequence is given by

$$\sum_{k=-\infty}^{\infty}h(kT)\left[\sum_{m=-\infty}^{\infty}g(mT)\,\phi_{rr}(k-m)T - \phi_{ri}(kT)\right] = 0 \quad ...(13)$$

Since $h(kT)$ is a realizable weighting sequence,

$$h(kT) = 0 \quad \text{for} \quad kT < 0 \quad \text{or} \quad k < 0 \qquad \ldots(14)$$

Equation (3.2-13) reduces to

$$\sum_{k=0}^{\infty} h(kT)\left[\sum_{m=-\infty}^{\infty} g(mT)\,\phi_{rr}(k-m)T - \phi_{ri}(kT)\right] = 0 \quad \text{for } k \geqslant 0 \qquad ..(15)$$

Only way, the above equation can be satisfied for $k \geqslant 0$, is for the expression within the brackets to be equal to zero for $k \geqslant 0$

$$\sum_{m=-\infty}^{\infty} g(mT)\,\phi_{rr}(k-m)T - \phi_{ri}(kT) = 0 \quad \text{for} \quad k \geqslant 0 \qquad \ldots(16)$$

Optimum system, weighting sequence $g(mT)$ in the minimum mean square sense is given by the above equation.

3.3. SOLUTION OF TIME DOMAIN EQUATION FOR OPTIMUM SAMPLED DATA SYSTEM

Weighting sequence $g(mT)$, of the optimum system that will minimize mean square error, is given by :

$$\sum_{m=-\infty}^{\infty} g(mT)\,\phi_{rr}(k-m)T - \phi_{ri}(kT) = 0 \quad \text{for} \quad k \geqslant 0 \qquad (1)$$

The above equation holds good for $k \geqslant 0$, but need not necessarily hold good for $k < 0$ also, since $\phi_{ri}(kT)$ and $\phi_{rr}(k-m)T$ in general will not be zero for $k < 0$.

Writing $\phi_{rr}(kT)$ as

$$\phi_{rr}(kT) = \sum_{p=-\infty}^{\infty} \phi_{rr}^{+}(k-p)T \; \phi_{rr}^{-}(pT) \qquad \ldots \quad (2A)$$

Where $\phi_{rr}^{+}(pT) = 0$ for $p < 0$ ... (2B)

$\phi_{rr}^{-}(pT) = 0$ for $p > 0$ ... (2C)

Multiplying both sides of equation (3.3-2A by z^{-k} and summing for all values of k from $-\infty$ to $+\infty$.

$$\sum_{k=-\infty}^{\infty} \phi_{rr}(kT)\, z^{-k} = \sum_{k=-\infty}^{\infty} z^{-k}\left[\sum_{p=-\infty}^{\infty} \phi_{rr}^{+}(k-p)T\; \phi_{rr}^{-}(pT)\right]$$

Changing order of summation,

$$\Phi_{rr}(z) = \sum_{p=-\infty}^{\infty} \phi_{rr}^{-}(pT)\, z^{-pT} \sum_{k=-\infty}^{\infty} \phi_{rr}^{+}(k-p)T\; z^{-(k-p)T}$$

$$\Phi_{rr}(z) = \Phi_{rr}^{-}(z)\, \Phi_{rn}^{+}(z) \qquad \ldots (3)$$

$\Phi_{rr}^{+}(z)$ will have poles and zeroes inside the unit circle in z - plane only, since $\phi_{rr}^{+}(kT)$ is zero for negative values of k . Similarly, $\Phi_{rr}^{-}(z)$ will have poles and zeroes outside the unit circle in z-plane only, since $\phi_{rr}^{-}(kT)$ is zero for positive values of k .

By equation (3.3-3) , spectral density $\Phi_{rr}(z)$ can be expressed as product of two parts, one containing poles and zeroes inside the unit circle in z- plane, while the other has poles and zeroes outside the unit circle only.

Cross correlation sequence $\phi_{ri}(kT)$ can be written as

$$\phi_{ri}(kT) = \sum_{p=-\infty}^{\infty} \beta(kT-pT)\,\phi^{-}_{rr}(pT) \qquad ..(4)$$

Substituting (3.3-2A), (3.3-4) in (3.3-1), one gets

$$\sum_{m=-\infty}^{\infty} g(mT) \sum_{p=-\infty}^{\infty} \phi^{+}_{rr}(k-p-m)T\,\phi^{-}_{rr}(pT) - \sum_{p=-\infty}^{\infty} \beta(k-p)T\,\phi^{-}_{rr}(pT) = 0$$

$$\text{for } k \geqslant o \qquad ..(5)$$

Rearranging terms,

$$\sum_{p=-\infty}^{\infty} \phi^{-}_{rr}(pT) \left[\sum_{m=-\infty}^{\infty} g(mT)\,\phi^{+}_{rr}(k-p-m)T - \beta(k-p)T \right] = 0$$

$$\text{for } k \geqslant o \qquad ..(6)$$

From equation(3.3 -2C)

$$\phi^{-}_{rr}(pT) = 0 \qquad \text{for} \quad p > o$$

Hence, the equation will hold good for $p < o$, if the expression within the brackets is zero.

or $$\sum_{m=-\infty}^{\infty} g(mT)\,\phi^{+}_{rr}(k-p-m)T - \beta(k-p)T = 0 \quad \text{for} \quad k \geqslant o \ \& \ p < o \qquad ..(7)$$

The above equation holds good for $k \geqslant 0$, hence, it will hold for $k-p \geqslant o$ also, where p is a negative number. The equation (3.3-7) may be written as, with a change of variable

$$q = k - p$$

$$\sum_{m=-\infty}^{\infty} g(mT)\,\phi^{+}_{rr}(q-m)T - \beta(qT) = 0, \text{for} \quad q \geqslant 0 \qquad ..(8)$$

for a realizable weighting sequence,

$$g(mT) = 0 \qquad \text{for } m < o$$

also $$\phi^{+}_{rr}(mT) = 0 \qquad \text{for } m < o$$

First term of expression (3.3-8) is zero for $q < o$ and $m < o$ but the second term may not be zerd for $q < o$. Resolving second term $\beta(qT)$ in two parts

$$\beta(qT) = \beta_+(q\,T) + \beta_-(qT) \qquad ..(9A)$$

such that

$$\beta_+(qT) = 0 \qquad \text{for } q < 0 \qquad ..(9B)$$

$$\beta_-(qT) = ? \qquad \text{for } q > 0 \qquad ..(9C)$$

Equation (3.3-8) reduces to

$$\sum_{m=-\infty}^{\infty} g(mT)\,\phi_{rr}^{+}(q-m)T = \beta_+(qT) \quad \text{for all values of } q \qquad ..(10)$$

Equation (3.3-10) holds for all values of q i.e., it holds good for the whole time range. This equation differs from (3.3-8) that holds good only for positive values of q, by the term $^{-}(q\,T)$.

Multiplying both sides of equation (3.3-10) by z^{-q} and summing it for all values of q from $-\infty$ to $+\infty$, one gets

$$\sum_{q=-\infty}^{\infty} z^{-q} \sum_{m=-\infty}^{\infty} g(mT)\,\phi_{rr}^{+}(q-m)T = \sum_{q=-\infty}^{\infty} \beta_+(q\,T)\; z^{-q}$$

$$\text{or } \sum_{m=-\infty}^{\infty} g(mT) z^{-m} \sum_{q=-\infty}^{\infty} \phi_{rr}^{+}(q-m)T\; z^{-(q-m)} = \sum_{q=0}^{\infty} \beta_+(qT)\; z^{-q}$$

$$G(z)\,\Phi_{rr}^{+}(z) = \left[\beta(z)\right]_+ \qquad ...(11)$$

The symbol $[\;]_+$ indicates, only that portion of the function has been taken which has poles inside the unit circle in z- plane.

$$\therefore \quad G(z) = \frac{[\beta(z)]_+}{\phi_{rr}^{+}(z)} \qquad ..(12)$$

But taking transform of both sides of equation (3.3-4)

$$\Phi_{ri}(z) = \beta(z)\ \Phi_{rr}^{-}(z) \qquad ..(13)$$

From (3.3-12) and (3.3-13) optimum system pulse-transfer function G(z) is given by ,

$$G(z) = \frac{1}{\Phi_{rr}^{+}(z)} \left[\frac{\Phi_{ri}(z)}{\Phi_{rr}^{-}(z)} \right]_{+} \qquad ..(14)$$

If we consider the ideal output due to stochastic control signal component $r_s(nT)$ only, while the actual input to the c ntrol system consists of sampled stochastic control component $r_s(nT)$ and sampled random noise component $r_n(nT)$ pulse-spectral densities $\Phi_{ri}(z)$ and $\Phi_{rr}(z)$ can be writt-en in terms of pulse-spectral densities of input signals and pulse- transfer functions of ideal and optimum systems.

$$\Phi_{ri}(z) = G_i(z)\left[\Phi_{r_s r_s}(z) + \Phi_{r_n r_s}(z)\right] \qquad ..(15)$$

$$\Phi_{rr}(z) = \Phi_{r_s r_s}(z) + \Phi_{r_n r_s}(z) + \Phi_{r_s r_n}(z) + \Phi_{r_n r_n}(z) \qquad ..(16)$$

$$G(z) = \frac{1}{\Phi_{rr}^{+}(z)} \left[\frac{G_i(z)\left[\Phi_{r_s r_s}(z) + \Phi_{r_n r_s}(z)\right]}{\Phi_{rr}^{-}(z)} \right]_{+} \qquad ..(17)$$

A relatively shorter proof can also be given as follows :

The optimum sampled data system equation 3.2-16 is

$$\sum_{m=-\infty}^{\infty} g(mT)\, \phi_{rr}(k-m)T - \phi_{ri}(kT) = 0 \qquad \text{for } k \geqslant o$$

$$\text{Let } \sum_{m=-\infty}^{\infty} g(mT)\, \phi_{rr}(k-m)T - \phi_{ri}(kT) = q(kT) \qquad \text{for } k < o \qquad \ldots(18)$$

$q(kT)$ is in general nonzero for $k < o$, but vanishes for $k \geqslant o$. The optimum sampled data system equation can be written as

$$\sum_{m=-\infty}^{\infty} g(mT)\, \phi_{rr}(k-m)T - \phi_{ri}(kT) = q(kT) \qquad \ldots(19)$$

Taking z-transform of $q(kT)$

$$Q(z) = \sum_{k=-\infty}^{\infty} q(kT)\, z^{-k}$$

$$Q(z) = \sum_{k=-\infty}^{o} q(kT)\, z^{-k} \qquad \ldots(20)$$

$Q(z)$ has poles outside the unit circle in z plane only,

Taking double sided z transform of equation 3.3-19,

$$G_m(z)\, \Phi_{rr}(z) - \Phi_{ri}(z) = Q(z) \qquad \ldots(21)$$

$\Phi_{rr}(z)$ is an even function and can be expressed by equation (3.3-3) as the product of $\Phi_{rr}^{+}(z)$ and $\Phi_{rr}^{-}(z)$. Equation (3.3-21) can be written as

$$G_m(z)\, \Phi_{rr}^{+}(z)\, \Phi_{rr}^{-}(z) - \Phi_{ri}(z) = Q(z)$$

$$G_m(z)\, \Phi_{rr}^{+}(z) - \frac{\Phi_{ri}(z)}{\Phi_{rr}^{-}(z)} = \frac{Q(z)}{\Phi_{rr}^{-}(z)} \qquad \ldots(22)$$

$G_m(z)$ has all its poles inside the unit circle in z- plane for reasons of stability, $\Phi_{rr}^{+}(z)$ also has poles and zeroes inside the unit circle. First term of the left hand side expression of equation 3.3-22 , has poles inside unit cicle only.

$\Phi_{ri}(z)$ may have poles outside the unit circle, $\Phi_{rr}^{-}(z)$ has zeroes outside the unit circle. the function $\left[\Phi_{ri}(z)/\Phi_{rr}^{-}(z)\right]$ can have poles inside, as well as outside the unit circle. It can be resolved in two parts, one having poles inside the unit circle only and the other outside the unit circle.

$$\left[\frac{\Phi_{ri}(z)}{\Phi_{rr}^{-}(z)}\right] = \left[\frac{\Phi_{ri}(z)}{\Phi_{rr}^{-}(z)}\right]_{+} + \left[\frac{\Phi_{ri}(z)}{\Phi_{rr}^{-}(z)}\right]_{-} \qquad ..(23)$$

$\left[Q(z) / \Phi_{rr}^{-}(z)\right]$ has poles outside the unit circle only.

Since $Q(z)$ has poles outside the unit circle and $\Phi_{rr}^{-}(z)$ has zeroes outside the unit cirlle only. Thus equation 3.3-22 can be written as

$$G_m(z)\,\Phi_{rr}^{+}(z) - \left[\frac{\Phi_{ri}(z)}{\Phi_{rr}^{-}(z)}\right]_{+} - \left[\frac{\Phi_{ri}(z)}{\Phi_{rr}^{-}(z)}\right]_{-} = \frac{Q(z)}{\Phi_{rr}^{-}(z)} \qquad ...(24)$$

Considering the equation 3.3-24 inside the unit circle in z- plane only,

$$G_m(z)\,\Phi_{rr}^{+}(z) - \left[\frac{\Phi_{ri}(z)}{\Phi_{rr}^{-}(z)}\right]_{+} = 0 \qquad ..(25)$$

Thus the optimum sampled data system pulse transfer function is given by

$$G_m(z) = \frac{1}{\Phi_{rr}^{+}(z)} \cdot \left[\frac{\Phi_{ri}(z)}{\Phi_{rr}^{-}(z)}\right]_{+} \qquad ..(26)$$

When $\left[\Phi_{ri}(z)/\ \Phi_{rr}^{-}(z)\right]$ is not a rational function of z

$$G_m(z) = \frac{1}{\Phi_{rr}^{+}(z)} \left[\sum_{k=0}^{\infty} z^{-k} \frac{1}{2\pi j}\oint_{\Gamma} \frac{\Phi_{ri}(z)}{\Phi_{rr}^{-}(z)}\, z^{-1}\, dz\right] \qquad ..(27)$$

3.4 OPTIMUM DESIGN WITH DETERMINISTIC SIGNALS

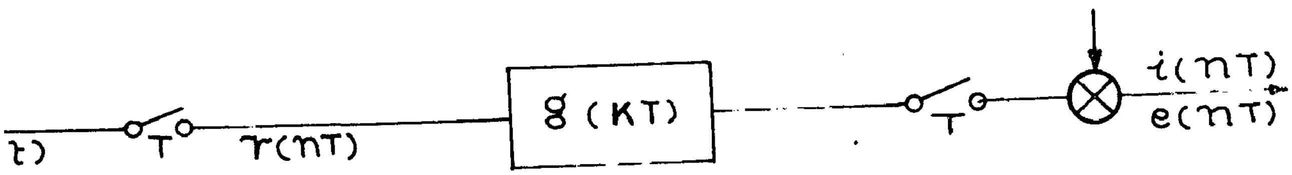

Fig. 6

Optimum design of a sampled data system for deterministic signals can be achieved on the basis of "total square error" criterion in a way analogus to the design of continuous data control systems with integral square error criterion.

Error Sequence for the Sampled data system shown in Fig. 6 is given by

$$e(nT) = c(nT) - i(nT) \qquad ..(1)$$

Where,

$$c(nT) = \sum_{k=-n}^{n} g(kT)\, r(n-k)T \qquad ..(2)$$

= Output sequence

$r(nT)$ = Input sequence

$i(nT)$ = Desired output sequence

$g(kT)$ = Weighting sequence of optimum sampled data system.

Total square error, T may be defined as

$$T = \sum_{n=-\infty}^{\infty} e^2(nT)$$

$$T = \sum_{n=-\infty}^{\infty} \left[c^2(nT) - 2c(nT)i(nT) + i^2(nT)\right] \qquad ..(3)$$

Correlation sequences for deterministic signals may be defined as

$$\phi_{ii}(kT) = \sum_{n=-\infty}^{\infty} i(nT)\, i(nT+kT) \qquad ..(4)$$

$$\phi_{ic}(kT) = \sum_{n=-\infty}^{\infty} i(nT)\, c(nT+kT) \qquad ..(5)$$

Total square error T may be expressed in terms of correlation sequences and weighting sequence by equation (3.4-2) , (3.4-4), (3.4-5) , as

$$T = \sum_{k=-\infty}^{\infty} \sum_{p=-\infty}^{\infty} g(kT)g(pT)\phi_{rr}(k-p)T - 2\sum_{k=-\infty}^{\infty} g(kT)\phi_{ri}(kT) + \phi_{ii}(o) \qquad ..(6)$$

If the optimum system $g(kT)$ is replaced by another one $g(kT)+\epsilon\, h(kT)$ where $h(kT)$ is arealisable weighting sequence and ϵ is a parameter to test optimality of $g(kT)$ the error will be increased. Increase in error is given by

$$\delta T = T_{g(kT)+\ h(kT)} - T_{g(kT)}$$

$$\delta T = \sum_{k=-\infty}^{\infty} \sum_{p=-\infty}^{\infty} \left[2\epsilon\, g(pT)h(kT) + \epsilon^2 h(kT)h(pT)\right] \phi_{rr}(k-p)T$$
$$- 2\epsilon \sum_{k=-\infty}^{\infty} h(kT)\ \phi_{ri}(kT) \qquad ..(7)$$

$$\therefore \frac{\partial}{\partial\epsilon}(\delta T) = 2 \sum_{k=-\infty}^{\infty} h(kT) \left[\sum_{p=-\infty}^{\infty} g(pT)\phi_{rr}(k-p)T - \phi_{ri}(kT)\right]$$
$$+ 2\epsilon \sum_{k=-\infty}^{\infty} \sum_{p=-\infty}^{\infty} h(kT)\, h(pT)\ \phi_{rr}(k-p)T \qquad ..(8)$$

$$\frac{\partial^2}{\partial\epsilon^2}(\delta T) = 2 \sum_{k=-\infty}^{\infty} \sum_{p=-\infty}^{\infty} h(kT)\, h(pT)\ \phi_{rr}(k-p)T \qquad ..(9)$$

For an optimum system

$$\frac{\partial}{\partial\epsilon}(\delta T)\Big|_{\epsilon=0} = 0$$

or
$$\sum_{k=-\infty}^{\infty} h(kT) \left[\sum_{p=-\infty}^{\infty} g(pT)\ \phi_{rr}(k-p)T - \phi_{ri}(kT)\right] = 0 \qquad ..(10)$$

Since $h(kT)$ is a realizable sequence,

$$h(kT) = 0 \qquad \text{for } k < 0$$

Equation 3.4-10 will hold good for $k \geq 0$, also, if, the expression within brackets is equal to zero for $k \geq 0$

$$\sum_{p=-\infty}^{\infty} g(pT)\ \phi_{rr}(k-p)T - \phi_{ri}(kT) = 0 \qquad \text{for } k \geq 0 \qquad ...(11)$$

Equation 3.4-11 gives the optimum system, with minimum total-square error criterion, since as seen from (3.4-9) $\frac{\partial^2}{\partial \epsilon^2}(\delta_T)$ is a positive quantity.

3.5 OPTIMUM SYSTEM WITH SIGNAL AND NOISE AT DIFFERENT POINTS

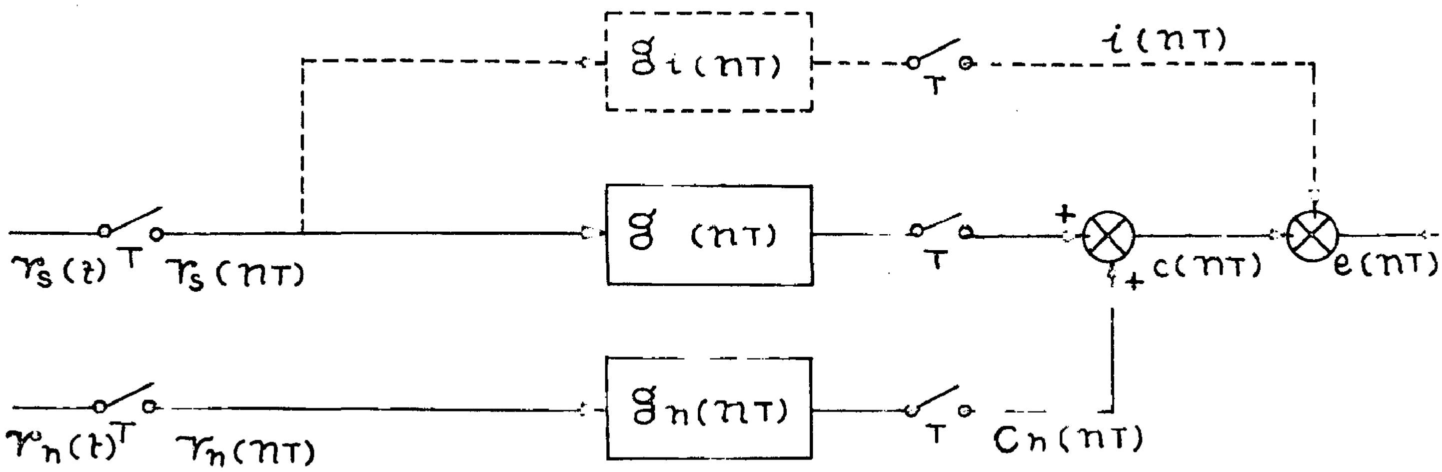

Fig. 7

In the present section, optimization of the sampled data system, Fig. 7, is considered, when noise and signal components enter the system at different points. In the block diagram shown in Fig. 7 $g_s(nT)$ is the weighting sequence for the stochastic control signal components and $g_n(nT)$ the weighting sequence for random noise samples. Both the signal and noise components are assumed to be stationary random functions.

Error sequence of the system

$$e(nT) = c_s(nT) + c_n(nT) - i(nT) \qquad ..(1)$$

Where

$c_s(nT)$ = output sequence due to signal component

$c_n(nT)$ - output sequence due to noise

$c(nT)$ - output sequence of the system

$i(nT)$ - ideal output sequence.

$c_s(nT)$ and $c_n(nT)$ can be written in terms of system input and weighting sequence,

$$c_s(nT) = \sum_{k=-n}^{n} g_s(kT)\, r_s(nT-kT) \qquad ..(2)$$

$$c_n(nT) = \sum_{k=-n}^{n} g_n(kT)\, r_n(n-k)T \qquad ..(3)$$

Mean square value of error seq uence, from the equations 2.3-1, 2.3-2 , 3.5-1 , 3.5-2, 3.5-3 will be given by

$$\overline{e^2(nT)} = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} e^2(nT)$$

$$\overline{e^2(nT)} = \phi_{ii}(o) - 2 \sum_{k=-\infty}^{\infty}\Big[g_s(kT)\, \phi_{si}(kT) + g_n(kT)\, \phi_{ri}(kT)\Big]$$

$$+ \sum_{k=-\infty}^{\infty} \sum_{p=-\infty}^{\infty} \Big[g_s(kT) g_s(pT) \phi_{ss}(k-p)T + g_n(kT) g_n(pT) \phi_{nn}(k-p)T$$

$$+ 2g_n(kT) g_s(pT)\, \phi_{ns}(k-p)T\Big] \qquad ..(4)$$

Following procedure, analogus to that followed by free - configuration case, when control signals and noise components enter the system at the same point , the optimum system equation obtained is,

$$\sum_{p=-\infty}^{\infty}\left[g_s(pT)\ \phi_{ss}(k-p)T + g_n(pT)\phi_{ns}(p-k)T\right] - \phi_{si}(kT) = 0$$
$$\text{for } k \geqslant 0 \qquad ..(5)$$

When the noise and control signal component are statistically independent, cross sorrelation sequence is zero and optimum system equation reduces to

$$\sum_{p=-\infty}^{\infty} g_s(pT)\ \phi_{ss}(k-p)T - \phi_{si}(kT) = 0 \qquad \text{for } k \geqslant 0 \qquad ..(6)$$

3.6 OPTIMUM COMPENSATION FOR SEMI FREE SAMPLED DATA CONTROL SYSTEM

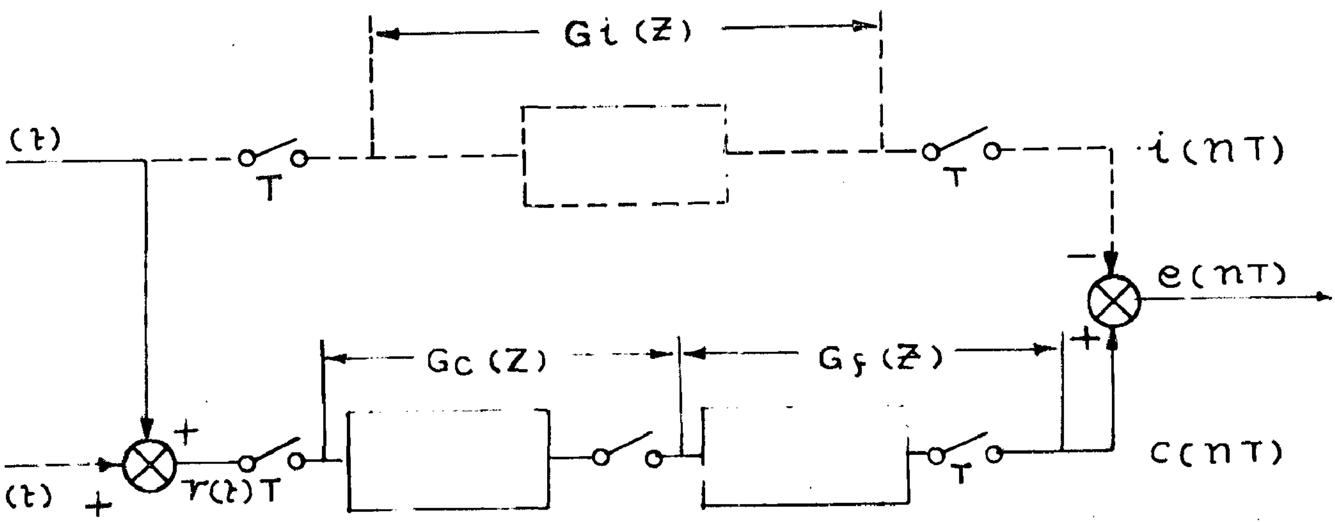

Fig. 8

In this section, the compensator, that gives minimum mean-sequare error, is designed. Input to the system is assumed to be stationary and ergodic sampled signals. For the sampled data system block diagram of figure 8, $G_f(z)$ is the pulse transfer function of the fixed elements of the system, and $G_c(z)$ compensator pulse transfer function has to be obtained.

Output sequence of the system is given by

$$c(nT) = \sum_{p=-n}^{n} g_f(pT)\, r_c(n-p)T \qquad \ldots(1)$$

and

$$c(nT) = \sum_{p=-n}^{n} \sum_{k=-n}^{n} g_f(pT)\, g_c(kT)\, r\,(n-k-p)T \qquad \ldots(2)$$

Where

$r(nT)$ = input sequence to the system

$g_f(nT)$ - weighting sequence of fixed elements

$g_c(nT)$ - weighting sequence of compensating elements

Error sequence of the system is given by

$$e(nT) = c(nT) - i(nT) \qquad \ldots(3)$$

Mean square value of err or - sequence is given by

$$\overline{e^2(nT)} = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} e^2(nT)$$

$$= \overline{c^2(nT)} - \overline{2c(nT)i(nT)} + \overline{i^2(nT)} \qquad \ldots(4)$$

$\overline{e_g^2(nT)}$, mean- square error, for the optimum system compensator weighting sequence $g_c(nT)$, can be expressed, making use of equations 2.3-1 , 2.3-2 as

$$e^2_{g_c}(nT) = \phi_{ii}(o) - 2\sum_{p_1=-\infty}^{\infty} \sum_{k_1=-\infty}^{\infty} \left[g_f(p_1T) g_c(k_1T)\, \phi_{ri}(k_1+p_1)T\right]$$

$$\sum_{p_1=-\infty}^{\infty} \sum_{p_2=-\infty}^{\infty} \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} \left[g_f(p_1T) g_f(p_2T) g_c(k_1T) g_c(k_2T)\, \phi_{rr}(k_2-k_1+p_2-p_1)T\right] \qquad ..(5)$$

Following the procedure, analogus to that adopted for optimum system weighting sequence for free configuration system, optimum compensation weighting sequence is given by

$$\sum_{p_1=-\infty}^{\infty}\ \sum_{k_1=-\infty}^{\infty}\ \sum_{k_2=-\infty}^{\infty} g_f(p_1T)\, g_f(p_2T)\ g_c(k_2T)\,\phi_{rr}(k_2-k_1+p_2-p_1)T$$

$$-\sum_{p_1=-\infty}^{\infty} g_f(p_1T)\ \phi_{ri}(k_1+p_1)T = 0 \qquad \text{for } k_1 \geqslant 0 \qquad ..(6)$$

By explicit solution (section 3.3), the pulse - transfer function of the compensating system will be given by

$$G_c(z) = \frac{1}{\left[G_f(z)G_f(z^{-1})\right]^{+}\Phi_{rr}^{\ +}(z)} \left[\frac{G_f(z^{-1})\ \Phi_{ri}(z)}{\left[G_f(z)\ G_f(z^{-1})\right]^{-}\Phi_{rr}^{\ -}(z)}\right]_{+} \qquad ..(7)$$

When input to the actual system consists of stochastic signal $r_s(nT)$ and random noise $r_n(nT)$, both being sampled, stationary and ergodic functions. The ideal output corresponds to the sampled control $r_s(nT)$. From equations (2.3-17 to 23 -2

$$\Phi_{rr}(z) = \Phi_{r_sr_s}(z) + \Phi_{r_sr_n}(z) + \Phi_{r_nr_s}(z) + \Phi_{r_nr_n}(z) \qquad ..(8)$$

$$\Phi_{ri}(z) = G(z)\ \left[\Phi_{r_sr_s}(z) + \Phi_{r_nr_s}(z)\right] \qquad ..(9)$$

Pulse transfer function of compensation system is given by

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$$G_c(z) = \frac{\left[\dfrac{G_f(z^{-1})\, G_i(z) \left[\Phi_{r_s r_s}(z) + \Phi_{r_n r_s}(z)\right]}{\left[G_f(z) G_f(z^{-1})\right]^{-} \left[\Phi_{r_s r_s}(z) + \Phi_{r_n r_s}(z) + \Phi_{r_s r_n}(z) + \Phi_{r_n r_n}(z)\right]^{-}} \right]_{+}}{\left[G_f(z) G_f(z^{-1})\right]^{+} \left[\Phi_{r_s r_s}(z) + \Phi_{r_n r_s}(z) + \Phi_{r_s r_n}(z) + \Phi_{r_n r_n}(z)\right]^{+}} \qquad \ldots(10)$$

If $G_f(z)$, the pulse - transfer- function of the fixed elements has no poles or zeroes outside the unit circle in z- plane , then

$$\left[G_f(z)\; G_f(z^{-1})\right]^{+} = G_f(z) \qquad ..(11A)$$

$$\left[G_f(z)\, G_f(z^{-1})\right]^{-} = G_f(z^{-1}) \qquad ..(11B)$$

Pulse transfer function of optimum compensation is given by

$$G_c(z) = \frac{1}{\Phi_{rr}^{+}(z)} \times \left[\frac{G_i(z)\left[\Phi_{r_s r_s}(z) + \Phi_{r_n r_s}(z)\right]}{\Phi_{r r}^{-}(z)} \right]_{+} \qquad ..(12)$$

$$= \frac{\text{Pulse transfer function of optimum system when no fixed elements are present.}}{\text{Pulse- transfer function of fixed elements}}$$

3.7 MEAN SQUARE ERROR

Mean square error of sampled data system is given by

$$\overline{e^2(nT)} = \left[c(nT) - i(nT)\right]^2 \qquad ..(1)$$

$$\overline{e^2(nT)} = \phi_{cc}(o) + \phi_{ii}(o) - \phi_{ci}(o) - \phi_{ic}(o) \qquad ..(2)$$

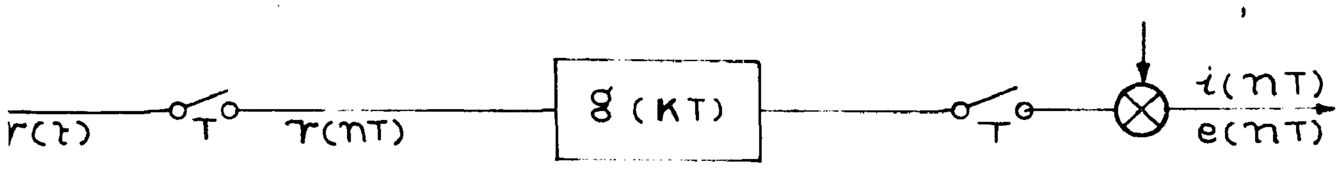

Fig. 9

Mean square error also can be given in terms of pulse spectral densities

$$\overline{e^2(nT)} = \frac{1}{2\pi j}\oint_{\Gamma}\left[\bar{\Phi}_{cc}(z)+\bar{\Phi}_{ii}(z)-\bar{\Phi}_{ci}(z)-\bar{\Phi}_{ic}(z)\right] z^{-1}\,dz$$

Mean square error of an optimum sampled data system may be given by

$$\overline{e^2_{opt}(nT)} = \overline{\left[\sum_{k=-\infty}^{\infty} g_m(kT)\, r(n-k)T - i(nT)\right]^2} \qquad ..(3)$$

Where $g_m(kT)$ - weighting sequence of the optimum system.

$$\overline{e^2_{opt}(nT)} = \sum_{k=-\infty}^{\infty}\ \sum_{m=-\infty}^{\infty} g(kT)\, g(mT)\, \phi_{rr}(k-m)T - 2\sum_{k=-\infty}^{\infty} g(kT)\phi_{ri}(kT) + \phi_{ii}(o) \qquad ...(4)$$

From the optimum system equqtion, (3.2-16) ,

$$\sum_{m=-\infty}^{\infty} g(mT)\, \phi_{rr}(k-m)T - \phi_{ri}(kT) = 0 \qquad \text{for } k \geqslant o$$

$$\overline{e^2_{opt}(nT)} = \phi_{ii}(o) - \sum_{k=-\infty}^{\infty} g_m(kT)\, \phi_{ri}(kT) \qquad ...(5)$$

Pulse transfer function of the optimum sampled data system is given by

$$G_m(z) = \frac{1}{\Phi_{rr}^{+}(z)} \left[\frac{\Phi_{ri}(z)}{\Phi_{rr}^{-}(z)}\right]_{+} \qquad ..(6)$$

and can also be written as,

$$G_m(z) = \frac{1}{\Phi_{rr}^{+}(z)} \sum_{k=0}^{\infty} \Psi(kT)\, z^{-k}$$

Where

$$\Psi(kT) = \frac{1}{2\pi j} \oint_{\Gamma} \frac{\Phi_{ri}(z)}{\Phi_{rr}^{-}(z)}\, z^{-1}\, dz \qquad ...(7)$$

$$\therefore \overline{e^2_{opt.}(nT)} = \phi_{ii}(o) - \sum_{k=-\infty}^{\infty} \phi_{ri}(kT) \frac{1}{2\pi j} \oint_{\Gamma} G_m(z)\,.z^{-1} dz \qquad .(8)$$

$$\overline{e^2_{opt}(nT)} = \phi_{ii}(o) - \sum_{k=-\infty}^{\infty} \phi_{ri}(kT) \frac{1}{2\pi j} \oint_{\Gamma} dz\, z^{-1} \left[\frac{1}{\Phi_{rr}^{+}(z)} \sum_{k=0}^{\infty} \Psi(kT) z^{-k}\right] \qquad ...(9)$$

Interchanging the summation limits,

$$\overline{e^2_{opt}(nT)} = \phi_{ii}(o) - \sum_{k=0}^{\infty} \Psi(kT) \frac{1}{2\pi j} \oint_{\Gamma} dz \frac{z^{-1}}{\Phi_{rr}^{+}(z)} \sum_{k=-\infty}^{\infty} \Phi_{ri}(kT) z^{-k} \qquad ..(10)$$

$$= \phi_{ii}(o) - \sum_{k=0}^{\infty} \Psi(kT) \frac{1}{2\pi j} \oint_{\Gamma} dz\, z^{-1} \cdot \frac{\Phi_{ri}(z)}{\Phi_{rr}^{+}(z)} \qquad ...(11)$$

$$\therefore \overline{e^2_{opt}(nT)} = \phi_{ii}(o) - \sum_{k=0}^{\infty} \Psi^2(kT) \qquad ...(12)$$

The equation (3.7-12), gives the minimum value of mean square error for the optimum sampled data system.

3.8 EXAMPLE

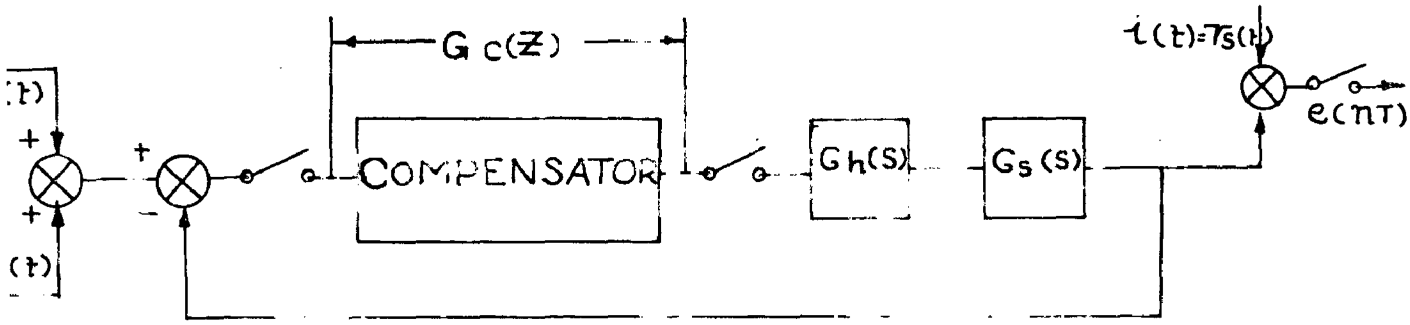

Fig. 10

A sampled data system has the configuration, as shown in Fig. 10. The transfer functions and input signals are described by

$$G_s(s) = \frac{1+0.005s}{s\left[1+\left(\frac{s}{10}\right)+\left(\frac{s}{10}\right)^2\right]}, \quad G_h(s) = \frac{1-e^{-Ts}}{s}$$

$$\phi_{r_s r_s}(s) = \frac{4.5}{(0.25-s^2)}, \quad \phi_{r_n r_n}(s) = \frac{0.1}{\pi}$$

$$\phi_{r_s r_n}(s) = 0$$

Sampling period is taken as 0.1 second. The design of sampled data compensator is to be carried out, that will minimize the mean square sampled error $\overline{e^2(nT)}$.

The block diagram of sampled data system Fig. 10, can be reshaped as that of Fig. 4. Here $G_i(z)$ and $G_o(z)$ are the ideal and optimum systems respectively.

From the tables of z transforms, following results are obtained.

$$\Phi_{r_s r_s}(z) = \frac{-1.436\,z}{(z-0.9512)\,(z-1.0513)}\ , \quad \Phi_{r_n r_n}(z) = 0.0318$$

$$\Phi_{r_s r_n}(z) = \Phi_{r_n r_s}(z) = 0$$

$$G_h G_s(z) = \frac{0.0096\,(z^2-8.87z+1.845)}{(z-1)(z^2-0.786z+0.368)}$$

$$\therefore \quad \Phi_{rr}(z) = \Phi_{r_s r_s}(z) + \Phi_{r_n r_s}(z) + \Phi_{r_n r_s}(z) + \Phi_{r_n r_n}(z)$$

$$= \frac{0.0318\,(z-47.08)\,(z-0.0212)}{(z-0.9512)\,(z-1.0513)}$$

Factorising $\Phi_{rr}(z)$ into $\Phi_{rr}^{+}(z)$ and $\Phi_{rr}^{-}(z)$, following results are obtained.

$$\therefore \Phi_{rr}^{+}(z) = \frac{0.178\,(z-0.0212)}{(z-0.9512)}$$

$$\Phi_{rr}^{-}(z) = \frac{0.178\,(z-47.08)}{(z-1.0513)}$$

For the system under consideration, $G_1(z) = 1.$

Here,

$$\frac{G_1(z)\left[\Phi_{r_s r_s}(z) + \Phi_{r_n r_s}(z)\right]}{\Phi_{rr}^{-}(z)} = \frac{-8.06\,z}{(z-0.9512)(z-47.08)}$$

$$= \frac{0.162}{z-0.9512} + \frac{8.2}{z-47.08}$$

Picking part of $\left[G_1(z)\left[\Phi_{r_s r_s}(z) + \Phi_{r_n r_s}(z)\right]\right] / \Phi_{rr}^{-}(z)$ that has poles inside the unit circle in z - plane.

$$\left[\frac{\left[\Phi_{r_s r_s}(z) + \Phi_{r_n r_s}(z)\right] \; G_i(z)}{\Phi_{rr}^{-}(z)}\right]_{+} = \frac{0.162}{z-0.9512}$$

Pulse transfer function of optimum system is given by

$$G_o(z) = \frac{1}{\Phi_{rr}^{+}(z)} \times \left[\frac{\left[\Phi_{r_s r_s}(z) + \Phi_{r_s r_n}(z)\right] G_i(z)}{\Phi_{rr}^{-}(z)}\right]_{+}$$

$$= \frac{0.162}{z-0.9512} \quad X \quad \frac{z-0.9512}{0.178(z-0.0212)}$$

$$= \frac{0.807}{z-0.0212}$$

Pulse transfer function of optimum system compensator is given by

$$G_c(z) = \frac{1}{G_h G_s(z)} \left[\frac{G_o(z)}{1-G_o(z)}\right]$$

$$= \frac{(z-1)(z^2-0.786z+0.368)}{0.0096\,(z^2-8.87z+1.845)} \times \left[\frac{0.907}{z-0.9282}\right]$$

$$= \frac{94.5\,(z-1)(z^2-0.786z+0.368)}{(z-0.9282)(z^2-8.87z+1.845)}$$

CHAPTER IV

SUMMARY AND CONCLUSIONS

4.1 SUMMARY

In the present dissertation, statistical design of sampled-data control systems in time domain has been considered.

In the first Chapter, limitations of classical approach to the design of control systems have been pointed out, namely, this is a trial and error approach for design with idealized signals only, disregarding random nature and the presence of noise. The drawbacks of classical design techniques are overcome by analytical design approach. The analytical design proceeds directly from the system specifications. It considers both control system component and noise in the system, as well as the random nature of input signals to an actual system.

The control signal component and noise component can be represented only statistically. For random signal input to a control system, mean-square value of error output is the most convenient performance index, because of its mathematical amenability, besides being selective. For deterministic signal inputs, integral square error is a convenient performance index.

In the second Chapter, the Wiener-Hopf equation, that gives optimum system impulse response, in terms of correlation functions of system input and ideal output was considered.

Optimization of sampled - data systems following Tou's approach has also been considered . In Tou's approach the random system input consists of control-signal component and noise component. The system is optimized by minimizing the mean-square value of error sequence in Z- plane, and the pulse-transfer function obtained.

In the third chapter of the present dissertation, the statistical design of a sampled data control system is carried out in t ime domain. An equation, similar to Wiener Hopf equation for continuous data systems, has been obtained for the sampled data systems with random signals, by minimizing the mean square value of error sequence. The optimum system equation is modified and taking z transforms, the pulse transfer function of the optimum system is obtained in terms of pulse spectral density of input signals, cross-spectral density of the input and desired output. Optimum system pulse transfer function is also obtained, when input consists of noise and control system component. The ideal output is due to control signal component only. The expression for optimum system pulse transfe r function is identical to the one obtained by Tou's approach.

Optimization equation for compensator has also been obtained for the semi-free configuration. Pulse transfer function of the compensator has been obtained in terms of the pulse spectral densities of input signal pulse-transfer functions of ideal system and fixed elements . If the fixed-element pulse transfer function has no poles or zeroes outside the unit circle in z- plane pulse-transfer function of the compensator, is the ratio of pulse-transfer function of the optimum system without fixed elements, and the fixed element pulse transfer function.

Optimum system equation has also been obtained for the sampled data systems, where the control signal and noise enter the system at different points.

4.2 FURTHER PROBLEMS SUGGESTED BY THIS INVESTIGATION

The study of statistical design is restricted to the sampled-data control systems with infinitesimal sampling duration. But all the physical sampled data system have finite sampling duration. The design will be more realistic especially when the time constant of the system is not large enough compared to the sampling duration, if the finite width of sampled pulses is also considered.

Optimum sampled data system design is restricted to the cases, where the input and output signals are sampled in synchronism. It should be possible to extend the statistical design approach to the systems with input and output being sampled at different rates. The technique may also be extended to the systems where sampling itself is random.

Statistical design procedure for sampled data systems is restricted to the systems, where control signal component and noise enter the system at the same point. It should be possible to extend the concept to the systems, where control signal component and the noise enter the system at different points, as well as to the multiple input systems.

With the advancements in mathematical techniques, it should be possible to extend the statistical design approach to the nonlinear sampled-data control systems, and also the time varying systems.

In conclusion it can be said that the problem of statistical design of sampled data control systems in time domain, has been discussed briefly in this dissertation. It is hoped that further investigations in this field will lead to more worthwhile results.

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