ADAPTIVE TIME STEPPING ANALYSIS FOR SOME DYNAMIC PROBLEMS

A DISSERTATION

submitted in partial fulfilment of the requirements for the award of the degree
of
MASTER OF ENGINEERING
in
EARTHQUAKE ENGINEERING
(With Specialization in Structural Dynamics)

By

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CANDIDATE'S DECLARATION

I hereby certify that the work which is being presented in this dissertation entitled "**ADAPTIVE TIME STEPPING ANALYSIS FOR SOME DYNAMIC PROBLEMS**" in partial fulfilment of the requirements for the award of the degree of MASTER OF ENGINEERING with specialization in Structural Dynamics, submitted in the **DEPARTMENT OF EARTHQUAKE ENGINEERING**, UNIVERSITY OF ROORKEE, Roorkee, is an authentic record of own work carried out for a period of about seven months from July 1996 to January 1997 under the supervision of **Dr. Pankaj & Dr. Ashok Kumar Mathur.**

The matter embodied in this dissertation has not been submitted by me for the award of any other degree.

Place: ROORKEE

Dated: 30th Jan, 1997.

(DALJEET SINGH SACHDEV)

CERTIFICATE

This is to certify that the above statement made by the candidate is correct to the best of my knowledge and belief.

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ABSTRACT

In dynamic analysis, direct time integration schemes are often used. Once a direct integration scheme is chosen, the accuracy of integration depends significantly on the time step size. As the step size decreases the accuracy of integration as well as computational cost increases. The adaptive time stepping procedures are aimed at seeking the largest possible step size to reduce the computational cost while maintaining a prescribed accuracy. For most direct integration schemes, a fixed time step size is usually used and is based frequently on intuition and experience. In practice, the dynamic process for a given problem can be, in some stages, very rapid and, in other stages, quite slow. It is therefore, unrealistic and unpractical to use a fixed step size in the whole process. To control the time discretization error, methods of estimating the error and then adjusting the time step accordingly for single step algorithm is introduced. To study the efficacy of the adaptive algorithm, the problems of two categories have been tested. One is with analytically defined forcing functions and second is for earthquake excitation (with acceleration time-history as input).

Direct integration of equations of motion may require a time step which is much smaller than the sampling interval at which the accelerogram has been provided. This necessitates the need for interpolating the digital accelerogram, which is conventionally done by linear interpolation between samples. However, as the original digital accelerogram is essentially a band limited signal, linear interpolation modifies the frequency content of the data and inserts spurious high frequency components at the cost of reducing power in the low frequency range.

High frequency insertion in input acceleration history, excites high frequency modes of the structure, thereby yielding a jittery response. So band limited interpolation is employed in this study in conjunction with adaptive time stepping schemes.

CONTENTS

CHAPTER 1

INTRODUCTION

1.1 GENERAL

The essential physical properties of any structural system subjected to an external source of excitation or dynamic loading are its mass, elastic properties (stiffness or flexibility), and the energy loss mechanism or damping. The equilibrium equation of multidegree of freedom system in motion are of the form:

$$M\ddot{X}(t)+C\dot{X}(t)+KX(t)=f(t) \tag{1.1}$$

where M, C and K are the mass, damping and stiffness matrices, $f(t)$ is the external load vector and $X(t)$, $\dot{X}(t)$ and $\ddot{X}(t)$ are the time dependent displacement, velocity and acceleration vectors of the nodes in the discretized element assemblage.

Eqs. (1.1) can alternatively be written as:

$$F_I(t)+F_D(t)+F_E(t)=f(t) \tag{1.2}$$

where $F_I(t)$ are the inertia forces, $F_D(t)$ the damping forces, and $F_E(t)$ the elastic forces

Mathematically Eqs. (1.1) represent a system of linear differential equations (for constant M, C and K matrices) of second order. However if any of the matrices M, C or K are functions of time (or X) the system becomes nonlinear. The solution procedures can be broadly divided into two categories, direct integration and mode superposition. While mode superposition can generally be employed only with linear problems, direct integration is suitable for both linear and nonlinear situations. In this study consideration is confined to direct integration methods.

1.2 DIRECT INTEGRATION METHODS

In direct integration methods, Eqs. (1.1) are integrated using a numerical step by step procedure. The term direct means that prior to numerical integration, no transformation of the equations is carried out [Bathe and Wilson, 1978]. In essence, direct numerical integration is based on two ideas. First, instead of trying to satisfy Eqs. (1.1) at any time t, it is aimed to satisfy Eqs. (1.1) only at discrete time intervals Δt apart. This means that a kind of static equilibrium, which includes the effect of inertia and damping forces is sought at discrete time points within the duration of analysis.

The second idea on which a direct integration method is based, is that a variation of displacements, velocities and accelerations within each time interval is assumed and on this assumption the accuracy, stability and cost of the solution procedure depends. Direct integration methods, assume that the displacement, velocity and acceleration vectors at time 0, denoted by $X_0, \dot{X}_0$ and $\ddot{X}_0$ respectively are known and the solution of Eqs. (1.1) is required from time 0 to time T. In the solution the time span under consideration T, is subdivided into n equal (or unequal) time intervals Δt and the integration scheme employed establishes an approximate solution at times 0, Δt, $2\Delta t$, $3\Delta t$ _ _ _ t, $t+\Delta t$, _ _ _T.

1.3 SINGLE STEP AND MULTISTEP ALGORITHMS

The methods in which the recurrence algorithms are valid within a single time step and relate the values of X_{n+1}, $\dot{X}_{n+1}$, $\ddot{X}_{n+1}$ etc. to X_n, $\dot{X}_n$, $\ddot{X}_n$ etc. are called Single Step Methods [Barthwal, 1992].

In multistep methods X_{n+1} are related to X_n, X_{n-1}, X_{n-2} etc. without introducing explicitly the derivatives and assuming that each set is separated by an equal interval Δt. Such algorithms are in general less convenient to use than the single step procedures as they do not permit an easy change of the time step size. Also these methods require a greater degree of bookkeeping as displacements of previous steps are required to be stored.

All algorithms that relate values at step n+1 to values at step n can be termed as single step algorithms. In this study adaptivity in time stepping analysis is applied to single step algorithms emanating from the generalization of the Newmark method [Zienkiewicz and Katona, 1985].

1.4 ADAPTIVE TIME STEPPING ANALYSIS

Once a direct integration scheme is chosen, the accuracy of the integration depends significantly on the time step size. The appropriate time step is determined by the stability and accuracy requirements. For integration schemes that are unconditionally stable accuracy requirement determines the step size.

For most direct integration schemes, a fixed time step size is usually used in the considered part of the time domain. In practice, the dynamic process for a given problem can be, in some stages, very rapid and, in other stages, quite slow depending on the response of the system to a given excitation. The time step is generally chosen based on the free vibration characteristics of the system as it can not obviously be selected a-priori on the basis of response to a given forcing function. It is therefore unrealistic and impractical to use a fixed time step for the whole process. If there is no automatic time stepping facility one has to suspend the analysis and run at certain time stations, assess the error and then change the step size before the solution process is resumed.

To control the time discretization error adaptive time stepping procedures are introduced that estimates the error and then adjust the step size accordingly. Adaptive time stepping procedures are aimed at seeking the largest possible step size while maintaining a prescribed accuracy.

1.5 BAND LIMITED INTERPOLATION

Seismic analysis of structure is often done using direct integration methods in the time domain, wherein the seismic input is provided in the form of an acceleration time history. This input accelerogram is a digital record of accelerations provided at a constant sampling interval (say 0.02 sec) to the analyst.

Direct integration of equations of motion may require a time step which is much smaller than the sampling interval at which the accelerogram has been provided. This necessitates the need for interpolating the digital accelerogram, which is conventionally done by linear interpolation between samples. However, as the original digital accelerogram is essentially a band limited signal, linear interpolation modifies the frequency content of the data and inserts spurious high frequency components at the cost of reducing power in the low frequency range [Basu et al., 1992].

High frequency insertion in input acceleration history, excites high frequency modes of the structure, thereby yielding a jittery response. So band limited interpolation technique is used in this study by virtue of which the band limited property of the signal is maintained.

CHAPTER 2

GENERALIZED NEWMARK SINGLE STEP DIRECT INTEGRATION PROCEDURES

2.1 GENERALIZED NEWMARK (GNpj) PROCEDURES [Zienkiewicz and Katona, 1985]

Single step algorithms for the solution of problems of dynamic problems have the inherent advantage over multistep algorithms that it is a simple matter to alter the time step as the requirement of the solution indicates.

In GNpj, p stands for the order of the polynomial of approximation of the function X(t), Eqs. (1.1) and j stands for the order of the differential Eqs. (1.1). Since this study is confined to dynamic analysis, the value of j equals 2.

This procedure applies Taylor series approach to derive a general form of single step algorithms that can be considered to be a generalization of Newmark method. It results in a scheme which is not self starting. The derivation considers the satisfaction of the governing Eqs. (1.1) only at the end points of the interval Δt and they can be written as [Zienkiewicz, 1977]

$$M\ddot{X}_{n+1} + C\dot{X}_{n+1} + KX_{n+1} = f_{n+1} \tag{2.1}$$

with appropriate approximations for the values of X_{n+1}, $\dot{X}_{n+1}$ and $\ddot{X}_{n+1}$. If the Taylor series expansion is considered these can be written as

$$
\left.
\begin{aligned}
X_{n+1} &= X_n + \Delta t\,\dot{X}_n + \dots\dots + \frac{\Delta t^p}{p!}\,\overset{p}{X}_n + \beta_p \frac{\Delta t^p}{p!}\left(\overset{p}{X}_{n+1} - \overset{p}{X}_n\right) \\
&= \tilde{X}_{n+1} + \beta_p \frac{\Delta t^p}{p!}\,\overset{p}{X}_{n+1} \\
\dot{X}_{n+1} &= \dot{X}_n + \Delta t\,\ddot{X}_n + \dots\dots + \frac{\Delta t^{p-1}}{(p-1)!}\,\overset{p}{X}_n + \beta_{p-1} \frac{\Delta t^{p-1}}{(p-1)!}\left(\overset{p}{X}_{n+1} - \overset{p}{X}_n\right) \\
&= \tilde{\dot{X}}_{n+1} + \beta_{p-1} \frac{\Delta t^{p-1}}{(p-1)!}\,\overset{p}{X}_{n+1} \\
&\dots\dots\dots\dots \qquad \dots\dots\dots\dots \\
&\dots\dots\dots\dots \qquad \dots\dots\dots\dots \\
\overset{p-1}{X}_{n+1} &= \overset{p-1}{X}_n + \Delta t\,\overset{p}{X}_n + \beta_p \Delta t\left(\overset{p}{X}_{n+1} - \overset{p}{X}_n\right) \\
&= \tilde{\overset{p-1}{X}}_{n+1} + \beta_1 \Delta t\,\overset{p}{X}_{n+1}
\end{aligned}
\right\} \qquad (2.2)
$$

where $\overset{p}{X}_n = \dfrac{d^p(X_n)}{dt^p}$ etc.

In Eq. (2.2) for a polynomial of degree p, a Taylor series remainder term has effectively been allowed in each of the expansion for the functions and its derivatives with parameter β_j, j=1,2,...... p which can be chosen to give good approximation properties to the algorithm.

Insertion of first three Eqs. of (2.2) into Eq. (2.1) gives a single equation from which $\overset{p}{X}_{n+1}$ can be found. When this is determined X_{n+1} to $\overset{p-1}{X}_{n+1}$ can be evaluated using Eqs. (2.2). The expression is

$$
\overset{p}{X}_{n+1} = \left[M\,\Delta t^{p-2}\,\frac{\beta_{p-2}}{(p-2)!} + C\,\Delta t^{p-1}\,\frac{\beta_{p-1}}{(p-1)!} + K\,\Delta t^{p}\,\frac{\beta_p}{p!}\right]^{-1} \qquad (2.3)
$$

$$
* \left[f_{n+1} - M\,\tilde{\ddot{X}}_{n+1} - C\,\tilde{\dot{X}}_{n+1} - K\,X_{n+1}\right]
$$

It can easily be shown that the commonly used Newmark method can be derived from the generalized procedure discussed above.

The above algorithm applies to both implicit and explicit schemes. In terms of generalized Newmark method an explicit scheme is simply defined by $\beta_p = 0$ for any order of p [Zeinkiewicz and Katona, 1985]. Conversely, an implicit scheme is defined by $\beta_p \neq 0$, irrespective of other integration parameters. To take full advantage of the computational efficiency of an explicit scheme, it is best to assume that the mass matrix M is diagonal (lumped) and the damping matrix is also diagonal. With these assumptions the system of Eqs. (1.1) written above are completely uncoupled because the stiffness matrix has a zero multiplier.

On the other hand, when $\beta_p \neq 0$ (implicit) stiffness matrix is populated (albeit banded) which requires triangularization at the onset or when the time step size is changed. It is thus problem dependent and it can not be said whether or not the accuracy/stability characteristics of an implicit scheme will provide a more efficient procedure than an explicit scheme whose accuracy/stability characteristics are generally inferior.

2.2 GENERALIZED NEWMARK (GN) ALGORITHM IN PREDICTOR CORRECTOR FORM FOR p=2, j=2

In many real physical situations nonlinearities may appear which do not allow the Eqs. (1.1) to be written in a simple form as Eqs. (1.1). The last term in the Eqs. (1.1) which corresponds to the internal (stiffness) forces may not obey simple linear elasticity laws and plastic or viscoplastic behaviour may intervene causing nonlinear dependence of this form on both the velocity ($\dot{X}$) and displacement (X). All such problems may be encompassed by a set of Eqs. (1.1) given below

$$M\ddot{X} + Q(X,\dot{X}) + P(X,\dot{X}) = f \tag{2.4}$$

in which Q and P are suitable vector valued functions of X and $\dot{X}$.

Clearly the linear case is now a specific one of the more general form in which

$$\begin{aligned} Q(X,\dot{X}) &= C\dot{X} \\ P(X,\dot{X}) &= KX \end{aligned} \tag{2.5}$$

The general nonlinear function may be defined in a variety of ways depending on the mechanics of the problem. The important thing at the moment is simply that for each problem such function can be defined either directly or in problems such as plasticity in an incremental form. The above discussed algorithm is presented in the predictor-corrector form for j=2. In this way this can be used for both linear as well as nonlinear problems.

Begin predictor phase

Set iteration counter i=0

1. $X_{n+1}^{i} = X_n + \Delta t\,\dot{X}_n + (1-\beta_2)\,\frac{\Delta t^2}{2}\,\ddot{X}_n$

$\dot{X}_{n+1}^{i} = \dot{X}_n + (1-\beta_1)\,\Delta t\,\ddot{X}_n$

$\tilde{X}_{n+1} = X_{n+1}^{i}$

$\dot{\tilde{X}}_{n+1} = \dot{X}_{n+1}^{i}$

$$\ddot{X}^{i}_{n+1} = \left[X^{i}_{n+1} - \tilde{X}_{n+1} \right] \frac{2}{\beta_2 \, \Delta t^2}$$

2. Form effective stiffness matrix K^* as

$$K^* = M \frac{2}{\beta_2 \, \Delta t^2} + C \frac{\beta_1}{\beta_2} \frac{2}{\Delta t} + K$$

3. Evaluate residual forces using

$$\psi_i = f_{n+1} - M \ddot{X}^{i}_{n+1} - C \dot{X}^{i}_{n+1} - K X^{i}_{n+1}$$

4. Perform factorisation, forward reduction and back substitution as required to solve

$$K^* \Delta X^i = \psi_i$$

5. Begin the corrector phase

$$X^{i+1}_{n+1} = X^{i}_{n+1} + \Delta X^i$$

$$\ddot{X}^{i+1}_{n+1} = \left[X^{i+1}_{n+1} - \tilde{X}_{n+1} \right] \frac{2}{\beta_2 \, \Delta t^2}$$

$$\dot{X}^{i+1}_{n+1} = \dot{\tilde{X}}_{n+1} + \beta_1 \, \Delta t \, \ddot{X}^{i+1}_{n+1}$$

6. If ΔX^i and/or ψ_i do not satisfy the convergence conditions then set i=i+1 and go to step (3), otherwise continue.

7. Now set

$$X_{n+1} = X_{n+1}^{i+1}$$

$$\dot{X}_{n+1} = \dot{X}_{n+1}^{i+1}$$

for use in the next time step. Also set n=n+1 and begin the next time step.

After convergence conditions are satisfied (step 6) a check for the appropriateness of the time step can also be introduced. This involves error estimates and a method of evaluating a new time step. This is considered in the following chapter.

CHAPTER 3

ERROR ESTIMATION AND STEP SIZE CONTROL

3.1 INTRODUCTION

In a single step method the error per step (often called the local error), is used to predict the step size [Zeng et al., 1992]. To estimate the local error, the traditional way is to compare the results when two different step sizes are used, or to compare the results given by two integration methods of different order. However to use such methods to estimate the local error can be very time consuming.

Zienkiewicz and Xie [1991] presented a simple local error estimate and adaptive time stepping procedure for integration schemes of Newmark type, in which the local error estimate is derived by using a Taylor series. Zeng et al. [1992] developed an *a posteriori* local error estimator based on the concept of a 'post-processing' technique.

In this study the error estimator developed by Zeng et al. [1992] is used for the *Newmark Single Step algorithm* discussed in the last chapter.

The step size is chosen in such a manner that the local error of each step is roughly equal to a prescribed tolerance.

3.2 *A-POSTERIORI* LOCAL ERROR ESTIMATION

Consider the original Newmark method (unconditionally stable), which can be derived from the generalized procedure discussed in Chapter 2 with p=2, $\beta_2 = 0.5$ and $\beta_1 = 0.5$. This becomes the constant-average-acceleration method known to yield an extremely small period elongation and no amplitude decay. This assumes the variation of the acceleration in each time step to be constant and equal to the average of the acceleration at the two ends of the time step as shown in Fig. 3.1.

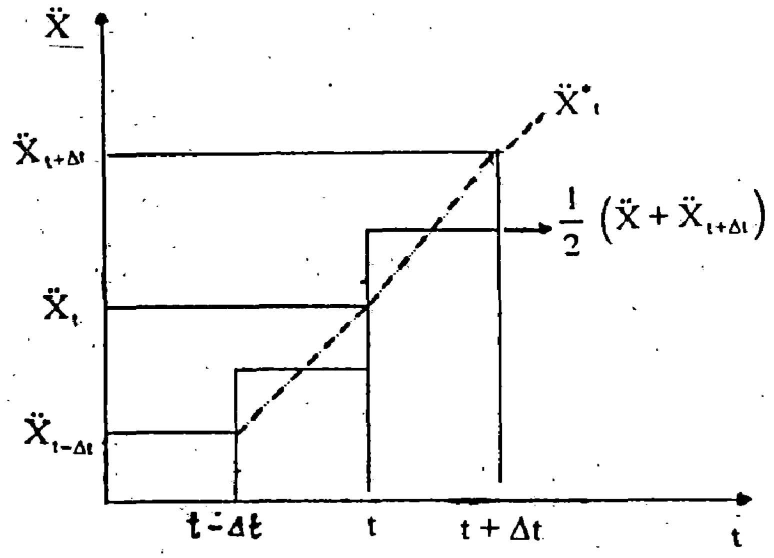

Fig. 3.1 Accelerations assumed in the Newmark time integration (———) and a post-processed continuous acceleration (--------) for $\beta_2 = 0.5$.

The average of $\ddot{X}_n$ and $\ddot{X}_{n+1}$ is used as the acceleration in the time interval (n, n+1). This approximation yields a discontinuous distribution for the acceleration in the time domain, which is similar to the discontinuous stress distribution obtained in a finite element analysis when displacements are approximated by C^0 continuous linear functions [Zeng et al., 1992]. In the finite element analysis, a smoothed stress field given by a proper post-processing technique can be much more accurate than the discontinuous one. Error estimation for spatial discretization, obtained by comparing the discontinuous stress field given by the finite element analysis and the post-processed stress field, has been found to be efficient and practical [Zienkiewicz and Zhu, 1987]. Zeng et al. [1992] extended this idea to time integration and derived an *a-posteriori* local error estimate.

The assumptions used in the Newmark scheme for dynamic analysis can be derived from the procedure discussed in section 2.1 are

$$\begin{aligned} X_{n+1} &= X_n + \dot{X}_n \Delta t + \left[(1-\beta_2)\ddot{X}_n + \beta_2 \ddot{X}_{n+1}\right] \frac{\Delta t^2}{2} \\ \dot{X}_{n+1} &= \dot{X}_n + \left[(1-\beta_1)\ddot{X}_n + \beta_1 \ddot{X}_{n+1}\right] \Delta t \end{aligned} \tag{3.1}$$

If a continuous linear vector-valued function of time, $\ddot{X}^*$, cf. the dashed line in Fig.3.1 is used for the approximation for the acceleration a more accurate result can be obtained. Errors in the displacements, velocities and accelerations are denoted by vector-valued functions of time, e, ė, and ë, respectively. Considering a time interval [n, n+1] and assume that $\tau \in [n,\ n+1]$. The linear approximation of the acceleration is given by

$$\ddot{X}^* = \frac{\ddot{X}_{n+1} - \ddot{X}_n}{\Delta t} (\tau - t) + \ddot{X}_n \tag{3.2}$$

The error of acceleration at time τ is then estimated through

$$\begin{aligned} \ddot{e}(\tau) &\approx \frac{1}{2}(\ddot{X}_{n+1} + \ddot{X}_n) - \ddot{X}^* \\ &= -\frac{\ddot{X}_{n+1} - \ddot{X}_n}{\Delta t} (\tau - t) + \frac{1}{2} (\ddot{X}_{n+1} - \ddot{X}_n) \end{aligned} \tag{3.3}$$

Suppose that the solutions at time station n are exact. Then, the error of velocity solution at time τ can be estimated by

$$\dot{e}(\tau) = \int_t^{\tau} \ddot{e}(\tau')\,d\tau' \approx -\frac{\ddot{X}_{n+1} - \ddot{X}_n}{2\Delta t}(\tau - t)^2 + \frac{1}{2}(\ddot{X}_{n+1} - \ddot{X}_n)(\tau - t) \tag{3.4}$$

Hence the error of the displacement at time station n+1 can be estimated by

$$e(t+\Delta t) = \int_t^{t+\Delta t} \dot{e}(\tau)\,d\tau \approx \frac{1}{12}\Delta t^2 (\ddot{X}_{n+1} - \ddot{X}_n) \tag{3.5}$$

Now $e(t+\Delta t)$ is a vector. Taking a certain norm, *a-posteriori* local error estimate is obtained as

$$\|e\| \approx \frac{1}{12}\Delta t^2 \|(\ddot{X}_{n+1} - \ddot{X}_n)\| \tag{3.6}$$

It reads in L_2 norm as

$$\|e\|_{L_2} = \sqrt{e^T e} \approx \frac{1}{12}\Delta t^2 \left[(\ddot{X}_{n+1} - \ddot{X}_n)^T (\ddot{X}_{n+1} - \ddot{X}_n)\right]^{\frac{1}{2}} \tag{3.7}$$

and in the strain energy norm as

$$\|e\|_E = \sqrt{e^T K e} \approx \frac{1}{12}\Delta t^2 \left[(\ddot{X}_{n+1} - \ddot{X}_n)^T K (\ddot{X}_{n+1} - \ddot{X}_n)\right]^{\frac{1}{2}} \tag{3.8}$$

To be able to obtain a general *a-posteriori* local error estimate for the whole family of Newmark schemes, the following procedure is adopted.

From Eq. (3.1) it can be deduced that the whole family of Newmark schemes uses the acceleration expressed by

$$\ddot{X}(\tau) = (1-\beta_2)\,\ddot{X}_n + \beta_2\,\ddot{X}_{n+1} \ . \tag{3.9}$$

which is a constant for a given β_2. Thus, the linear approximation Eq. (3.2) is a higher order approximation than Eq. (3.9). Hence, error can be estimated in the acceleration given by the Newmark schemes by

$$\begin{aligned} \ddot{e}(\tau) &\approx \left(\ddot{X}(\tau) - \ddot{X}^{*}\right) \\ &= -\frac{\ddot{X}_{n+1} - \ddot{X}_n}{\Delta t}\,(\tau - t) + \beta_2\left(\ddot{X}_{n+1} - \ddot{X}_n\right) \end{aligned} \tag{3.10}$$

Repeating the integration that has been done for Eq. (3.5), the local error estimate for the displacement solution at station n+1 as

$$e(t+\Delta t) \approx \left(-\frac{1}{6} + \beta_2\right)\Delta t^2\left(\ddot{X}_{n+1} - \ddot{X}_n\right) \tag{3.11}$$

which can be used for the whole family of Newmark schemes except for the case when $\beta_2 = \frac{1}{6}$. The reason is that $\beta_2 = \frac{1}{6}$ implies a linear acceleration method. Clearly to estimate the errors this case a higher order approximation needs to be employed. Work in this direction using the Taylor's series expansion has been done by Zienkiewicz and Xie [1991].

3.3 ADAPTIVE TIME-STEPPING

An adaptive time stepping algorithm should, ideally, find a discretization at the least cost, such that the local error is uniformly distributed and the global error is within a given tolerance. For time-dependent problems, however the global error estimation is generally difficult to obtain. Adaptive algorithms for dynamic analysis are thereafter usually designed based on the control of the local error. The aim becomes here to adjust/select the step size in an efficient and economic way so that for each step, the local error is roughly equal to a prescribed error tolerance ε. [Zienkiewicz and Xie, 1991; Zeng et al., 1992].

The following condition has been given in adaptive literature for step size control [Zienkiewicz and Xie, 1991; Zeng et al., 1992].

$$\gamma_1\varepsilon \le \|e\| \le \gamma_2\varepsilon \tag{3.12}$$

where, $0 \le \gamma_1 \le 1$ and $\gamma_2 \ge 1$ are two parameters and ε is the Prescribed error tolerance.

When the condition (3.12) is satisfied, the solution is accepted and the time integration proceeds to the next time step without change of the step size. The right hand side of the inequality (3.12), i.e. $\gamma_2\varepsilon$, is an 'upper error limit'. If the estimated error is larger than this upper limit, the solution is not accepted; and the step size is reduced. The left hand side of the inequality (3.12), i.e. $\gamma_1\varepsilon$, is a threshold, rather than a lower error limit. When the estimated error is less than this threshold, the solution is accepted; however, the step size should be enlarged before stepping to the next time step. It is, from the efficiency point of view, necessary not to set a lower error limit, but instead, to set this threshold.

When a step size should be updated, the prediction of the new step size has to be made such that the prescribed accuracy can be achieved with the least cost. It is known that, for Newmark integration, the rate of convergence of the global error can be $O(\Delta t^2)$. Correspondingly, the rate of convergence of the local error should achieve $O(\Delta t^3)$.

Suppose that the current step size is Δt, then

$$\|e\| \approx C_1 \Delta t^3 \tag{3.13}$$

where C_1 is a constant depending on the exact solution. With the new step size $\Delta t'$, the aim is to obtain the local error equal to the given tolerance, i.e.

$$\|e\|' \approx C_2 (\Delta t')^3 = \varepsilon \tag{3.14}$$

where C_2 is an another constant. Comparing equation (3.13) with equation (3.14) and assuming that $C_1 = C_2$, the new step size can be obtained as

$$\Delta t' = \left(\frac{\varepsilon}{\|e\|}\right)^{\frac{1}{3}} \Delta t \tag{3.15}$$

In practice, an implicit algorithm can not be efficient if the step size varies too frequently, as the effective stiffness matrix has to be factorized when a new step size is used. Therefore the step size should not be enlarged until

$$\|e\| < \gamma_1 \varepsilon \tag{3.16}$$

has been registered consecutively for K_0 time steps.

3.4 FLOWCHART FOR ADAPTIVE GENERALIZED NEWMARK (GNpj) PROCEDURE

The flowchart which is based on the algorithm discussed in last chapter is shown in Fig.3.2. It will be observed that a box for estimating error and adjusting the time step has been included. The procedure followed for this part is as follows

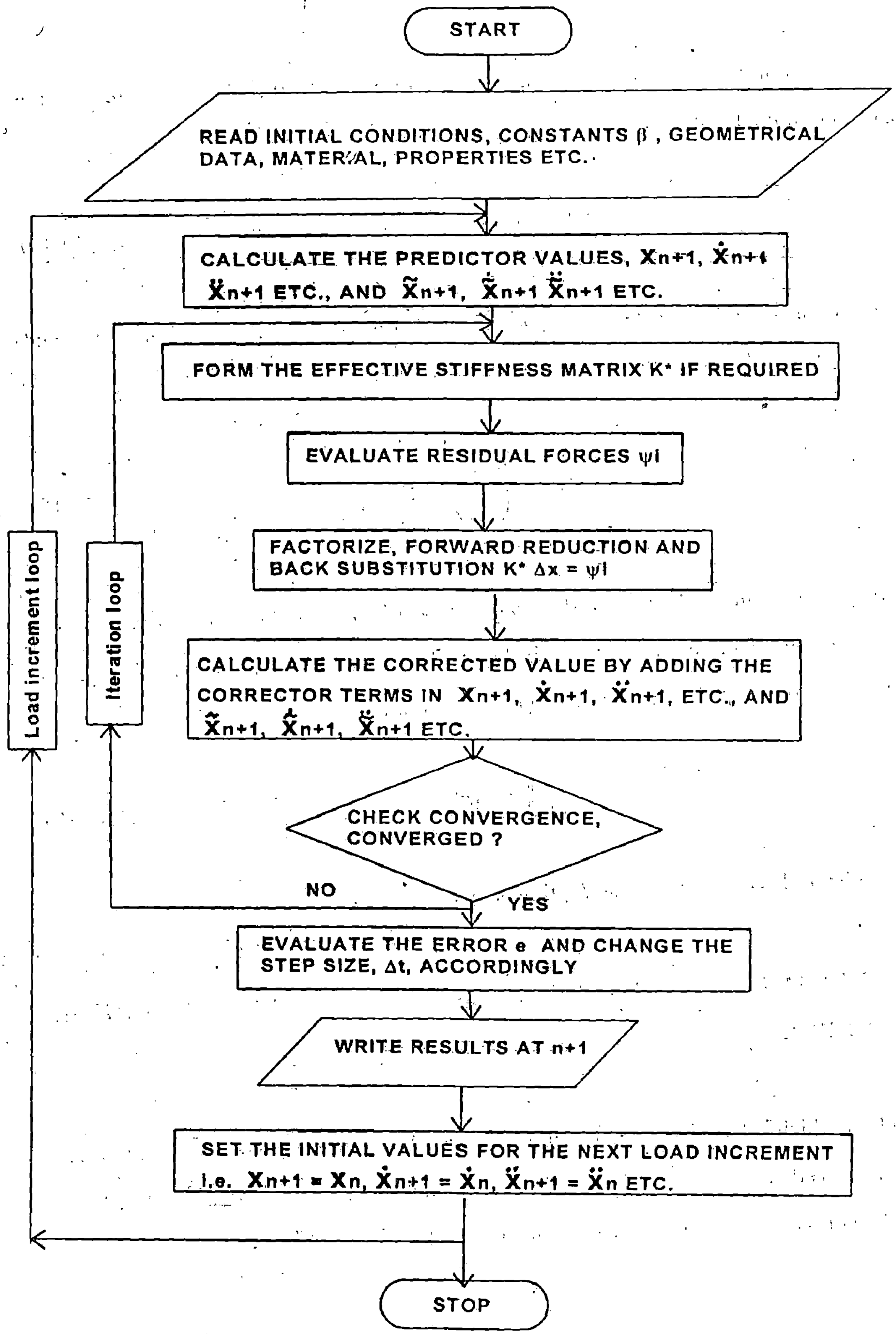

Fig. 3 .:2 FLOW CHART OF ADAPTIVE GENERALISED NEWMARK (GNpj) PROCEDURE

(a) Evaluate the error

$$\|e\| = \left(-\frac{1}{6}+\beta_2\right)\frac{\Delta t^2}{2}\left\|\ddot{X}_{n+1}-\ddot{X}_n\right\|$$

(b) Change the step size Δt if required using

$$\Delta t_{new} = \left(\frac{\varepsilon}{\|e\|}\right)^{\frac{1}{3}} \Delta t \text{ , where } \varepsilon = \text{prescribed error tolerance}$$

and set $\Delta t = \Delta t_{new}$.

CHAPTER 4

BAND LIMITED INTERPOLATION FOR ADAPTIVE TIME STEPPING ANALYSIS

4.1 INTRODUCTION

In direct integration analysis of structure the appropriate time step (or sampling rate) is determined by the stability and accuracy requirements.

Integration schemes that are unconditionally stable require a small time step Δt from the point of view of accuracy. It has been suggested that results are reasonably accurate when the time step is limited as $\Delta t / T \leq 0.01$ [Bathe and Wilson, 1978; Owen and Hinton, 1980], where T is the fundamental period of the structure.

For the Newmark method (with $\beta_2 = 0.5$ and $\beta_1 = 0.5$) it has been shown that the period elongation for an undamped single degree of freedom system is less than 3% for $\Delta t / T \leq 0.01$ and the method does not decay response amplitudes.

For conditionally stable schemes, stability considerations may require a time step $\Delta t \leq \frac{T_{min}}{\Pi}$, where T_{min} is the smallest natural period of the structure [Owen and Hinton, 1980].

For nonlinear problems it becomes necessary to iterate within a time step to obtain a converged solution. It has been felt that it is better to reduce the time step rather than pushing iteration of nonlinear quantities within a time increment [Zienkiewicz et al., 1984]. In other words the time step to be used is determined by the problem and the numerical scheme being employed.

Also, to control the time discretization error adaptive time stepping procedures are introduced that estimates the error and then adjust the step size accordingly. This also necessitates the interpolation of the seismic input.

The point that emerges from the above discussion is that the accelerogram provided for the analysis (generally the group deciding upon the seismic loading history is different from the group doing structural analysis) may not be at the required sampling rate. Therefore the seismic record is required to be interpolated.

4.2 BAND LIMITED SIGNAL

Seismic analysis using direct integration schemes employ a ground acceleration history that is either recorded or is synthetically generated. In either case these are the digital values at equally spaced discrete time intervals. Since the majority of the accelerograms are analog in nature the history obtained from these requires digitization.

The instrument and the digitization process introduces noise that has to be removed by band pass filtering [Kumar, 1993]. The upper cut-off frequency of this band is generally 25Hz. to 27Hz. Moreover, most corrected recorded accelerations are available at a sampling interval $\Delta t=0.02$ sec. As per the Nyquist theorem in digital signal processing the highest frequency that a signal sampled at Δt, is capable of representing is $(1/2\Delta t)$. Thus a sampling interval of 0.02 sec implies a Nyquist frequency of 25Hz. In other words the highest frequency content of the accelerogram also gets dicided by the sampling interval and, therefore any digital signal is band limited signal.

4.3 BAND LIMITED INTERPOLATION

The sampling rate at which the input accelerogram is available is required to be interpolated for adaptive direct integration. It has been considered reasonable to assume that ground acceleration varies linearly in the time interval, while recognising simultaneously that this may result in loss of accuracy [Zienkiewicz et al., 1984] as linear interpolation introduces a high frequency content which is absent in the original record provided.

In this study the band limited interpolation technique [Basu et al., 1992] is used for adaptive time stepping analysis by virtue of which band limited property of the signal is maintained.

Band limited interpolation is done by zero packing the data to an extent required for analysis. This zero packed accelerogram is low-passed to recover the bare band signal of interest and eliminate the unwanted image of components generated by sampling rate expander. Thus this technique maintains the band limited property in the interpolated data.

4.4 INTERPOLATION USED IN THIS STUDY

In this study the Uttarkashi Earthquake (Oct 20, 1991; N 75 E; Transverse) corrected accelerogram available at a sampling interval of 0.02 sec was used for some seismic studies (to be discussed in Chapter 5). The first 15 sec record of this accelerogram is shown in Fig. 4.1. Clearly the digital points at 0.02 sec intervals have been joined using straight lines and a linear interpolation between sampling interval would also yield an accelerogram as shown in above Fig. 4.1.

In order to preserve the frequency content which has a maximum frequency of 25Hz. (Nyquist frequency corresponding to sampling interval of 0.02 sec) the accelerogram was interpolated using Band Limited Interpolation at sampling intervals of 0.01, 0.005, 0.0025 sec. (Fig. 4.2 to 4.4).

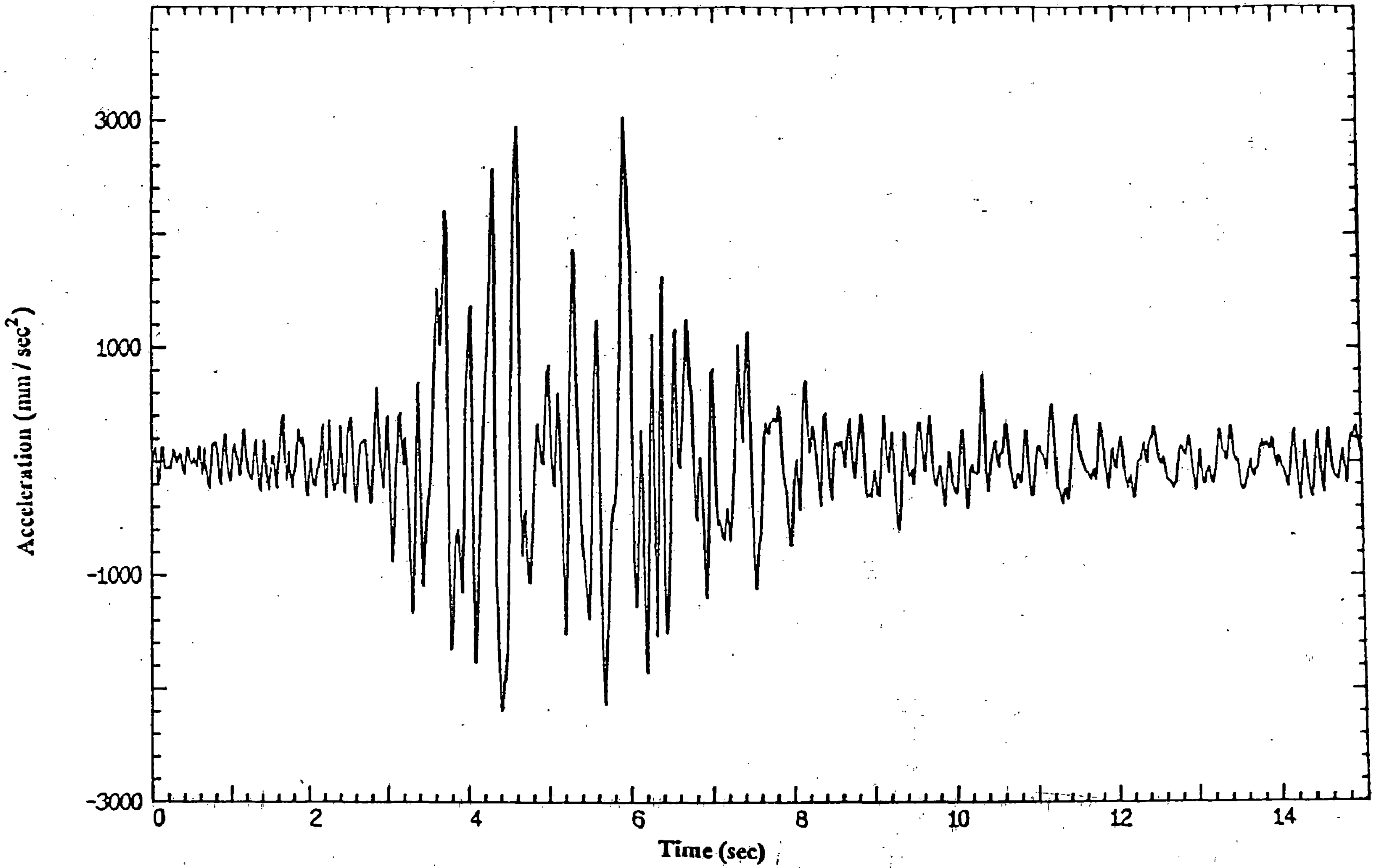

Fig. 4.1 Accelerogram of Uttarkashi Earthquake at sampling interval of 0.02 sec

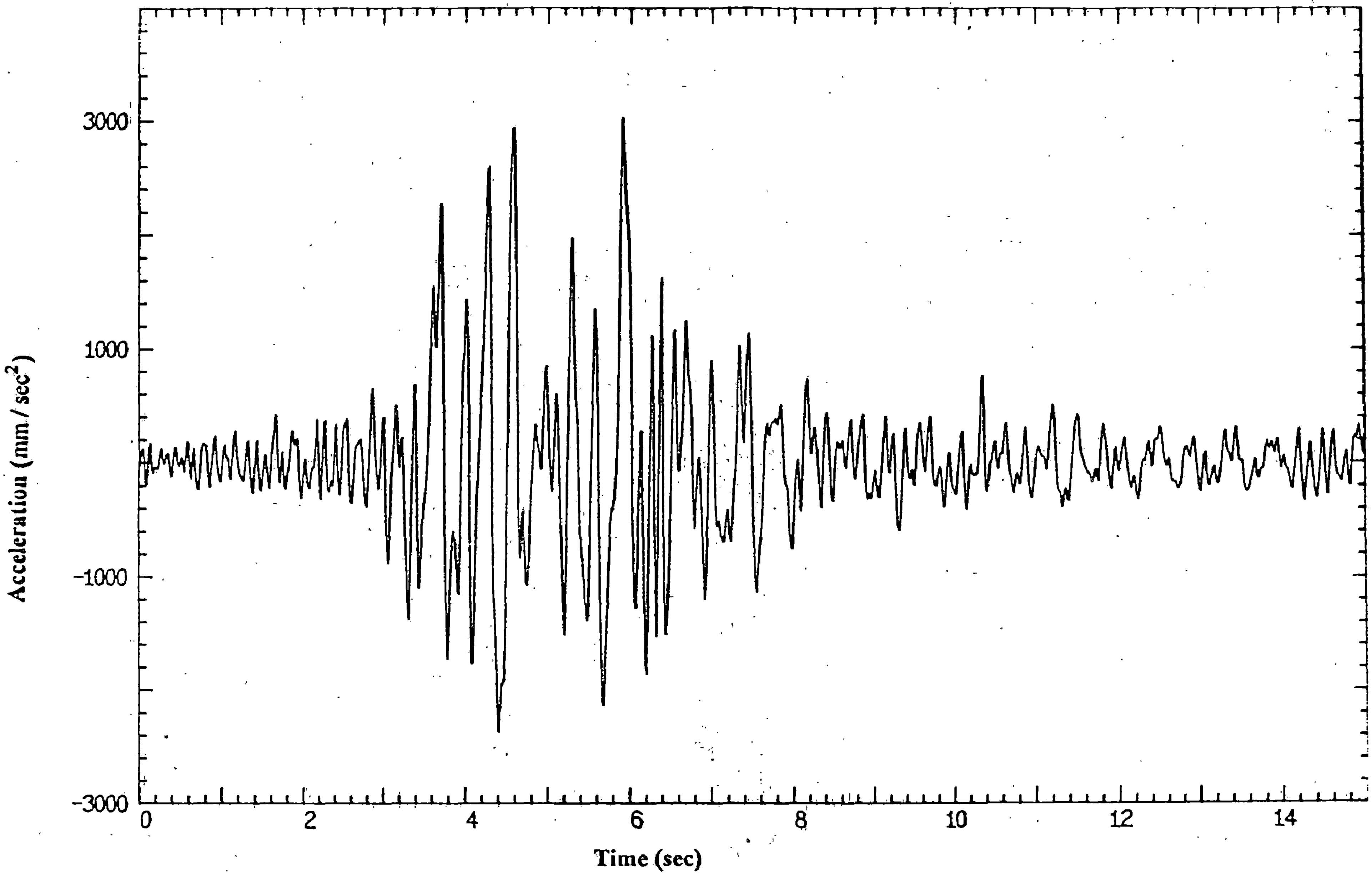

Fig 4.2 Band interpolated accelerogram with sampling interval of 0.01 sec.

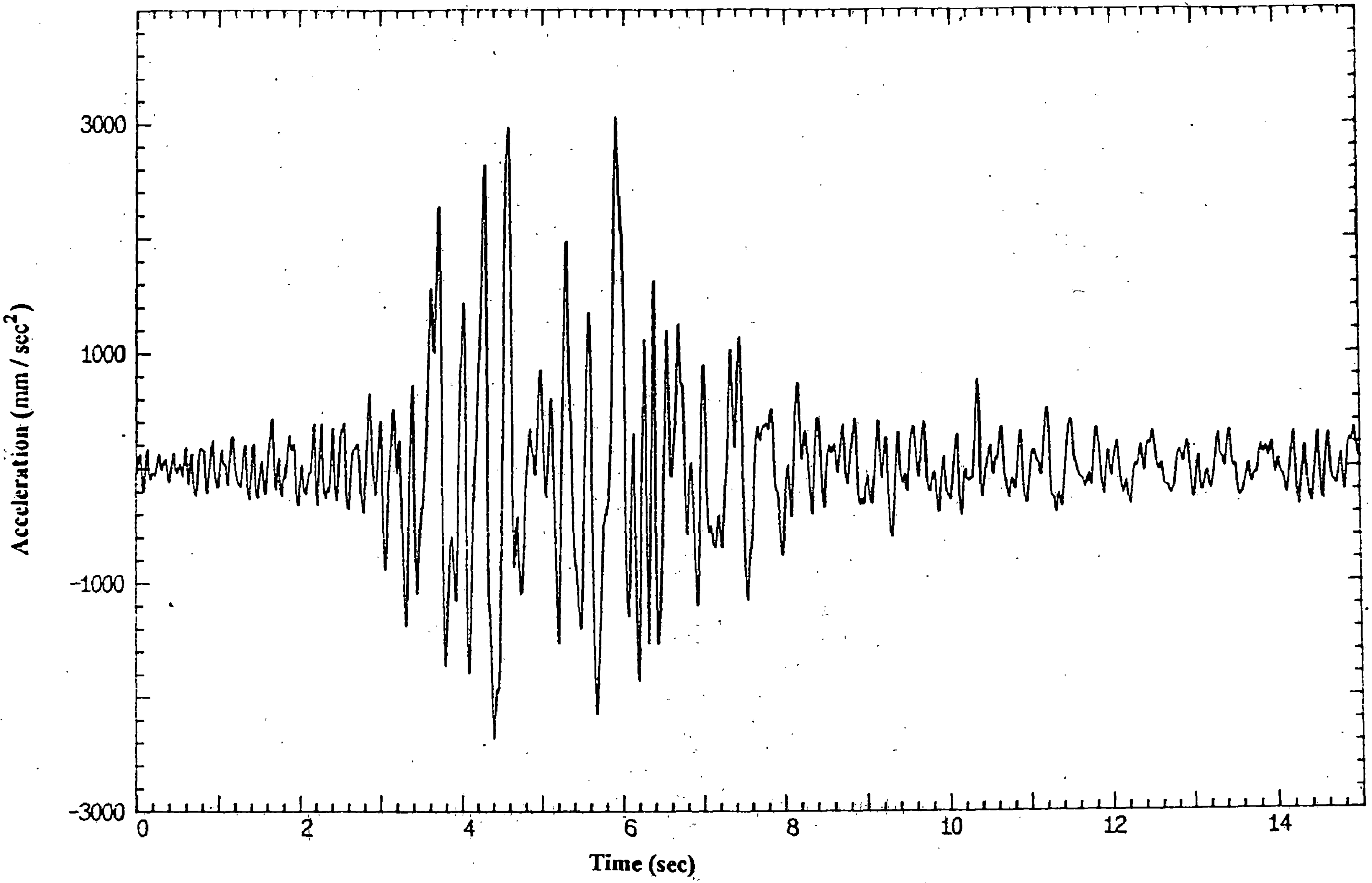

Fig. 4.3 Band interpolated accelerogram with sampling interval of 0.005 sec.

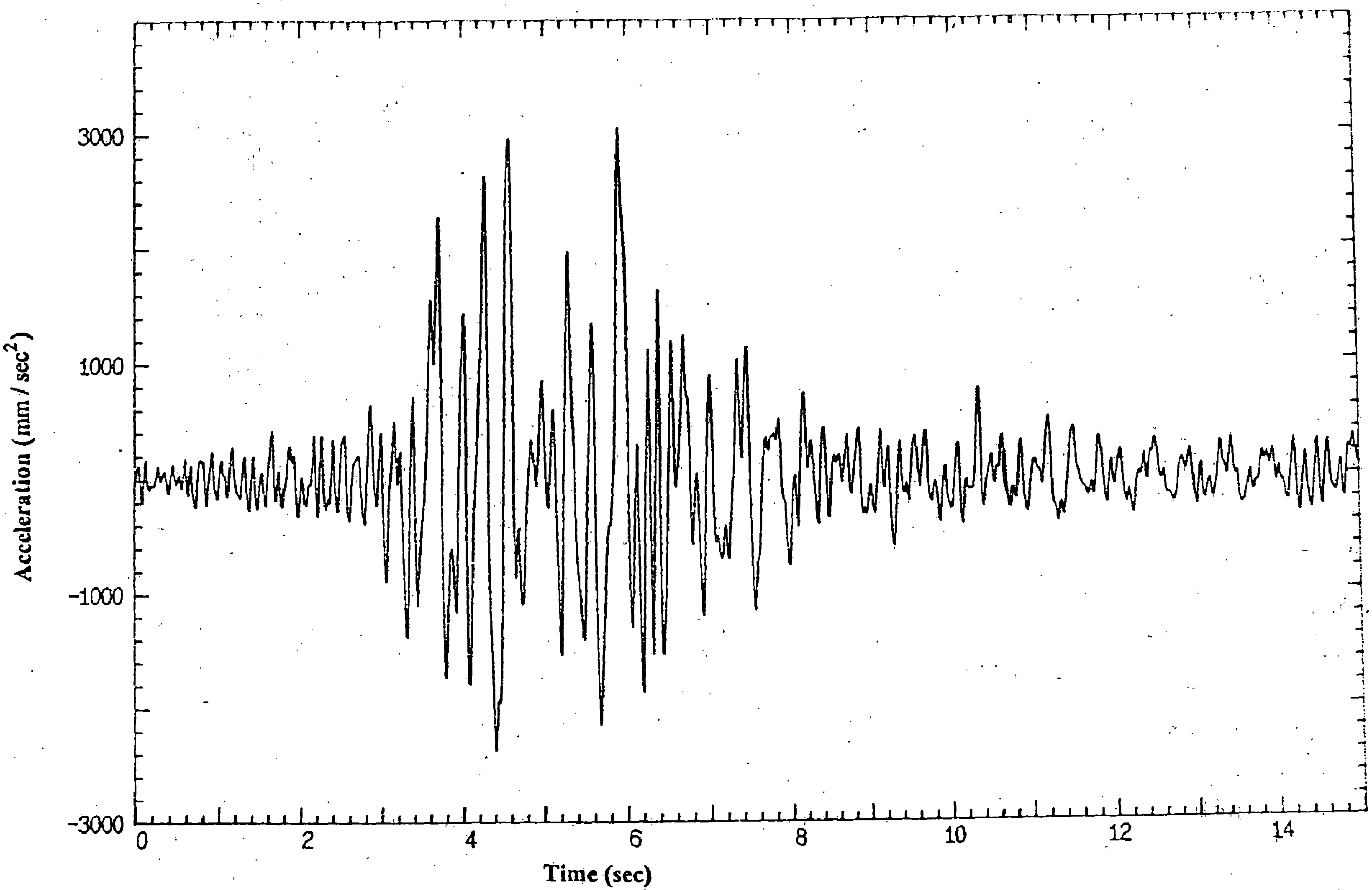

Fig. 4.4 Band interpolated accelerogram with sampling interval of 0.0025 sec.

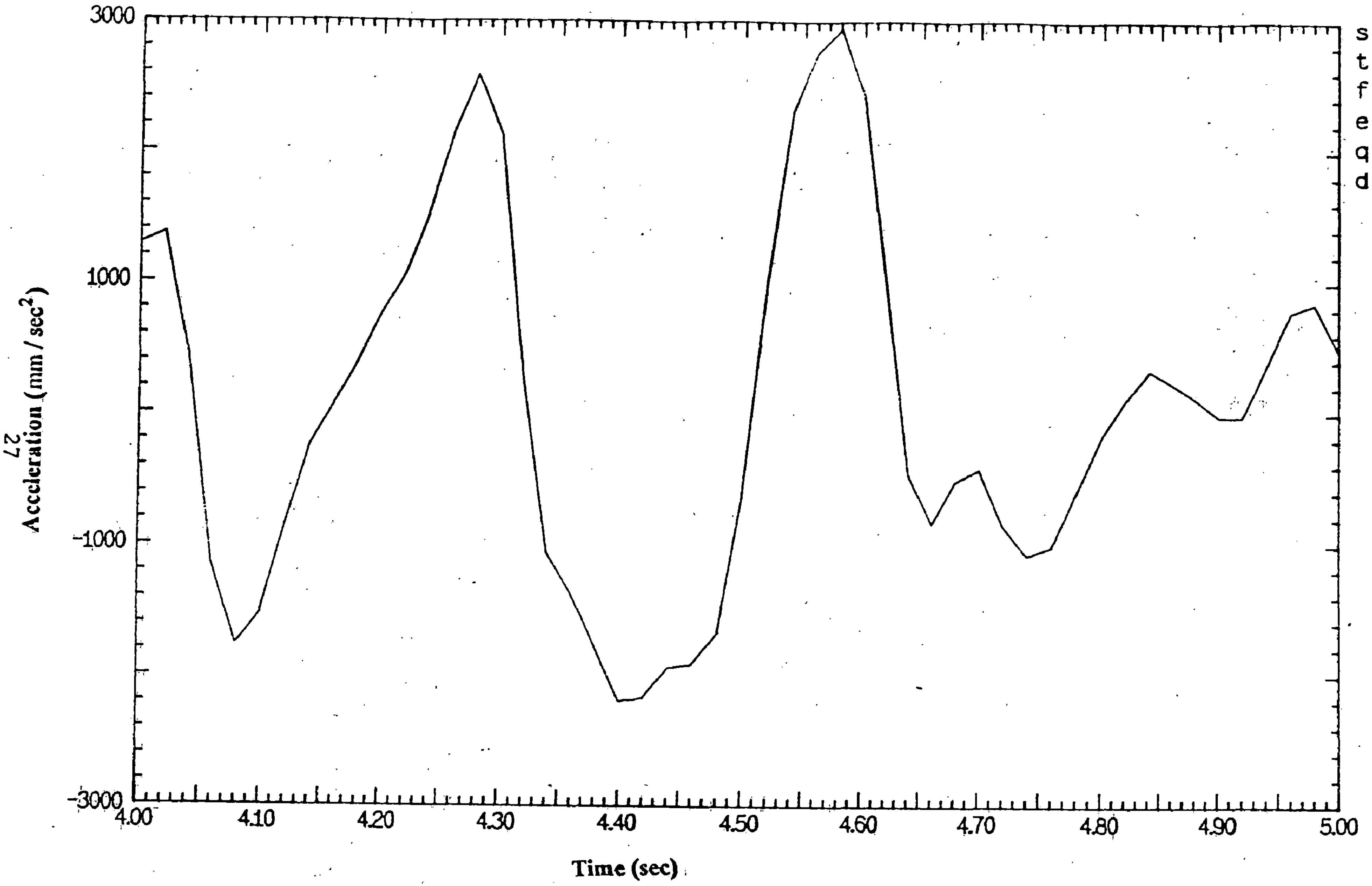

Fig. 4.5 Accelerogram with sampling interval of 0.02 sec.
(zoomed between 4 to 5 sec)

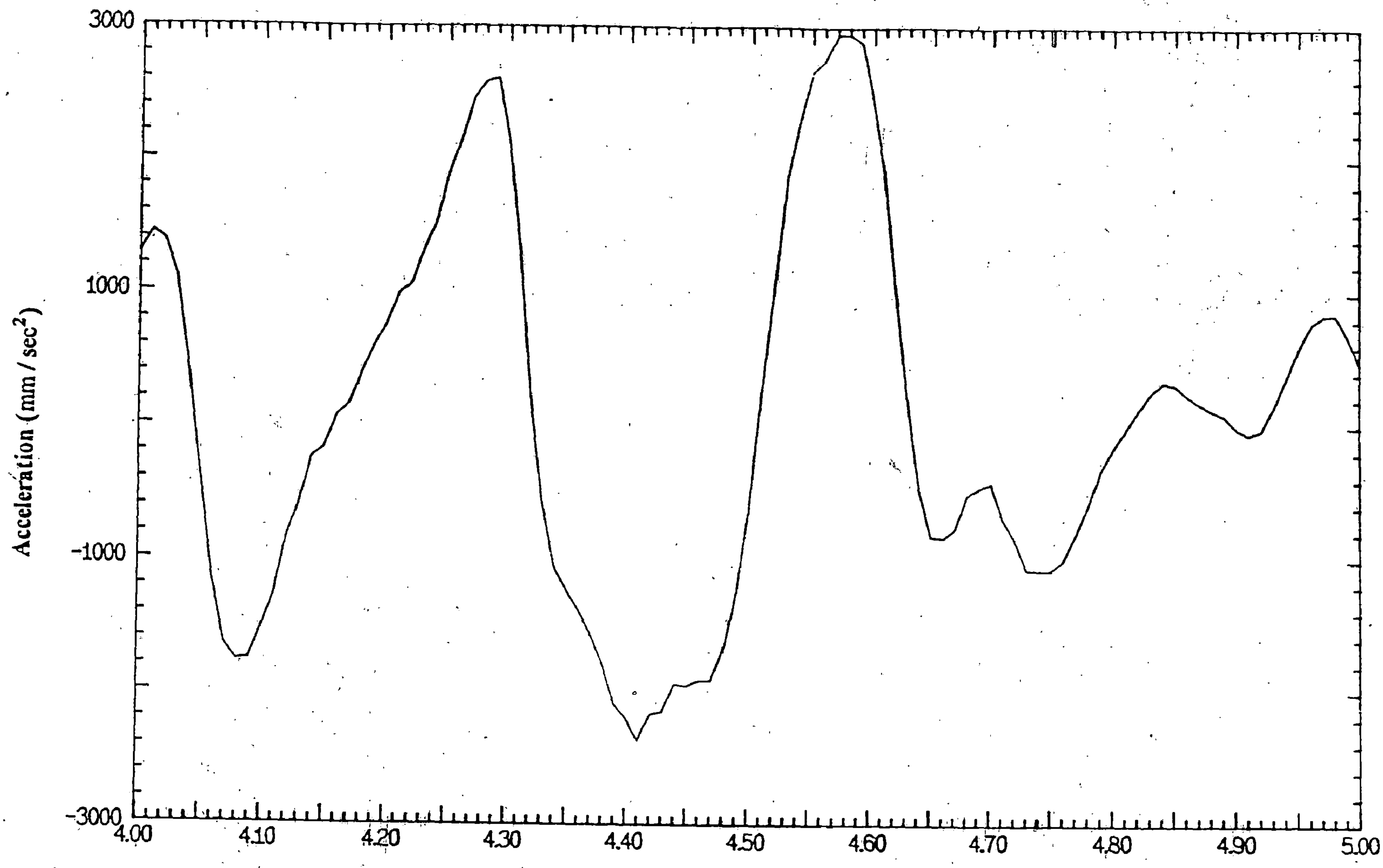

Fig. 4.6 Band interpolated accelerogram with sampling interval of 0.01 sec.
(zoomed between 4 to 5 sec)

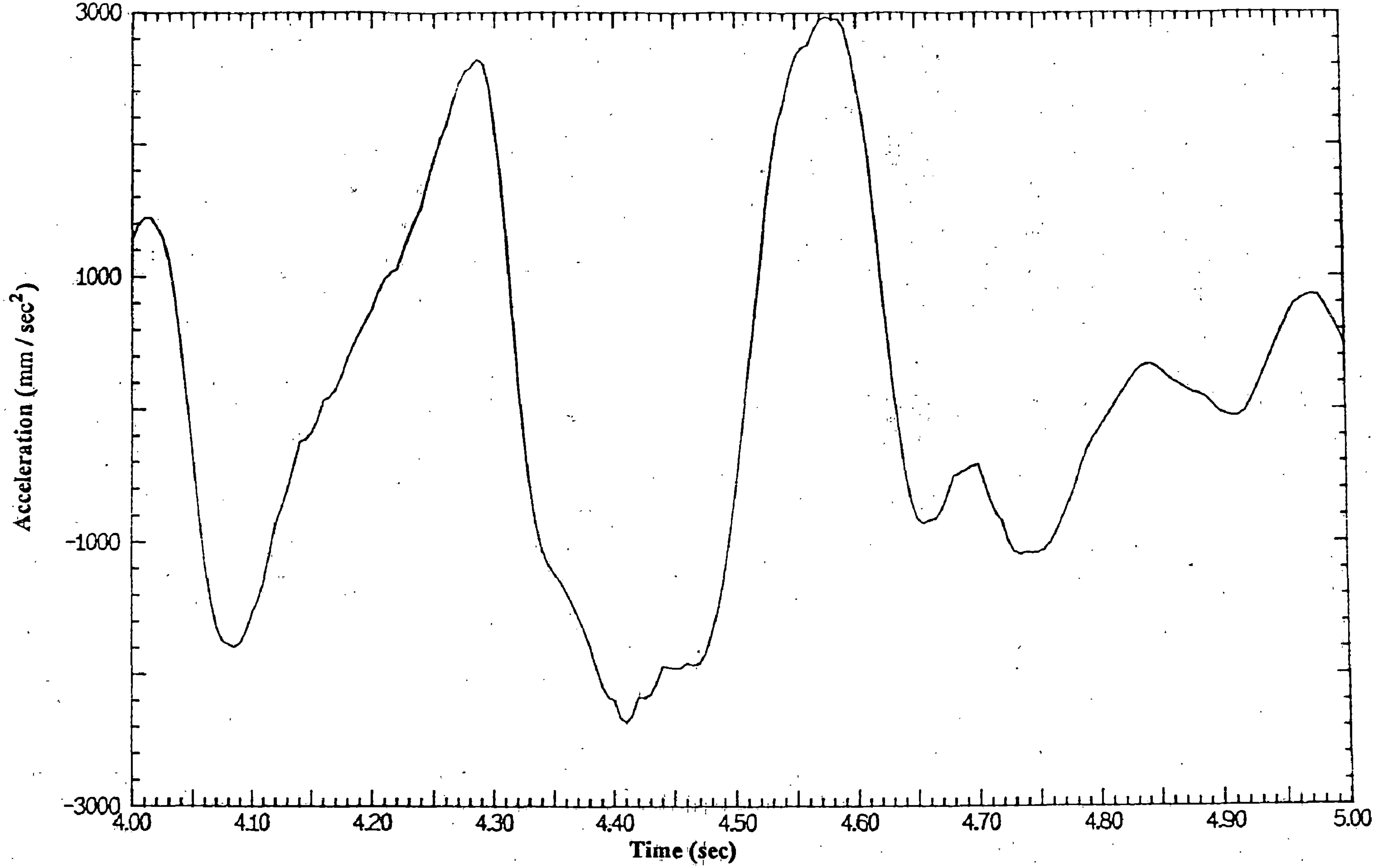

Fig. 4.7 Band interpolated accelerogram with sampling interval of 0.005 sec. (zoomed between 4 to 5 sec)

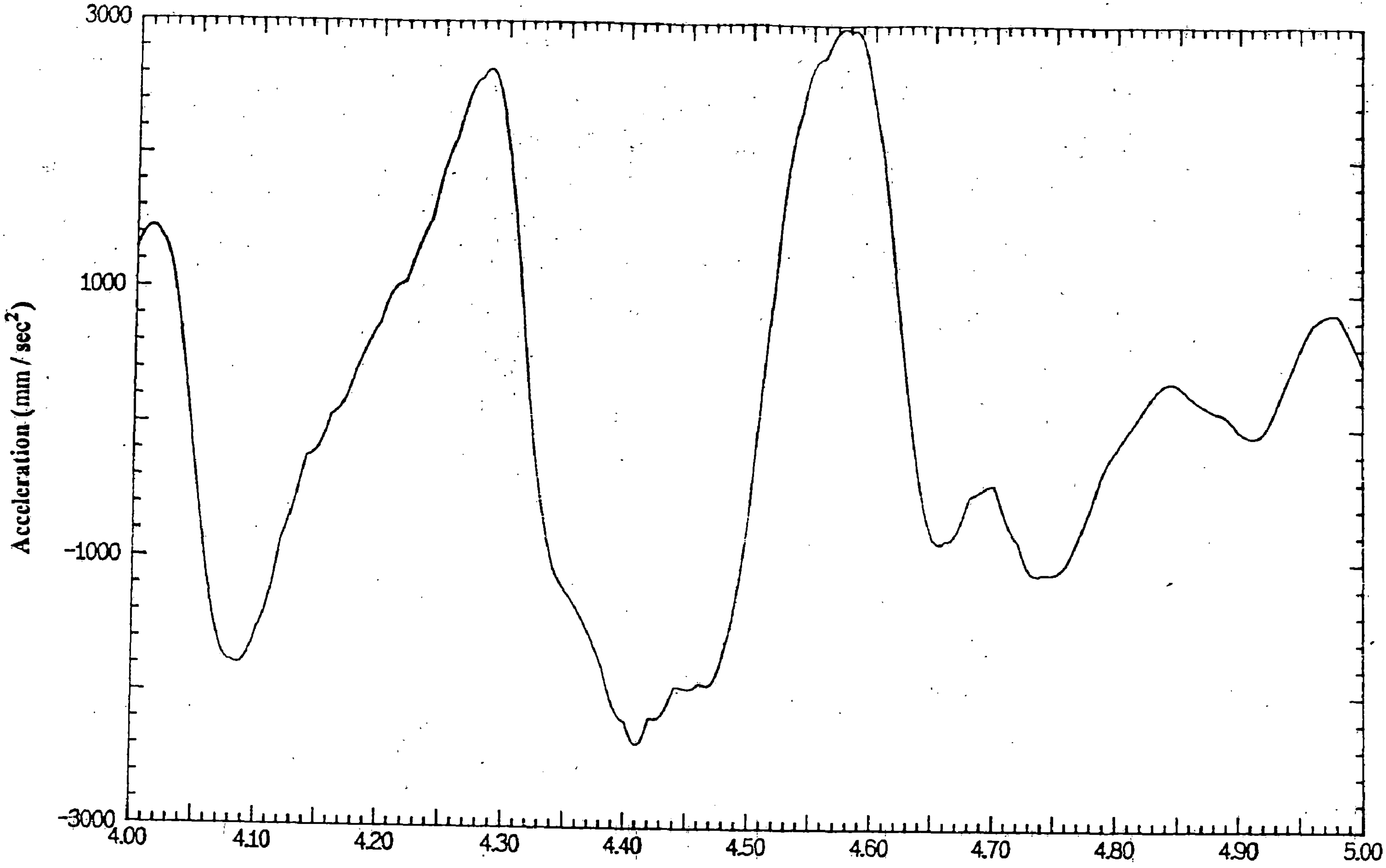

Fig. 4.8 Band interpolated accelerogram with sampling interval of 0.0025 sec. (zoomed between 4 to 5 sec)

One can observe from visual comparison of these accelerograms that they look almost similar. However a zoomed picture shows clear differences. For the purpose of illustration the duration between 4 to 5 sec was selected and is shown in Fig. 4.5 to 4.8. It will be observed that for smaller sampling intervals the sharp corners are replaced by smoother curves. These accelerograms at different sampling intervals will subsequently be employed for adaptive direct integration analysis of structures.

PROBLEMS, RESULTS AND DISCUSSION

5.1 INTRODUCTION

To study the efficacy of the adaptive time stepping procedures with unconditionally stable Newmark algorithm both base excitation and direct mass excitation problems were studied. For problems with an analytical function as excitation it is possible to use any arbitrary time step. However for most problems (such as earthquake excitation) it is not possible to vary the time step arbitrarily. In such cases several files with varying sampling intervals can be used as discussed in the following section.

5.2 TIME STEP-SIZE CONTROL PROCEDURE ADOPTED

In adaptive literature [Zienkiewicz and Xie, 1991; Zeng et al., 1992], the following condition have been given for the control of step-size, as discussed in section 3.3

$$\gamma_1\varepsilon \leq \|e\| \leq \gamma_2\varepsilon \tag{5.1}$$

where, $0 \leq \gamma_1 \leq 1$ and $\gamma_2 \geq 1$ are two parameters. The R.H.S. of the above inequality is an 'upper error limit' and the L.H.S. is the threshold rather than a 'lower error limit'.

In this study, the adaptive scheme for step-size control has been modified by making the R.H.S. of the inequality (5.2) as threshold rather than a 'upper error limit'. If the estimated error is larger then this threshold, the solution is accepted; however the step-size is reduced for the next time step. Clearly a more appropriate procedure would be to return to the previous step and recalculate with a smaller time step. This involves extensive bookkeeping and was, therefore, not used. Similarly the L.H.S. of the inequality is applied as threshold. When the estimated error is less than this threshold, the solution is accepted; however the step-size is being enlarged before stepping to the next time step.

In this study instead of arbitrarily accepting the new time step as calculated from the equation

$$\Delta t' = \left(\frac{\varepsilon}{\|e\|}\right)^{\frac{1}{3}} \Delta t \tag{5.2}$$

as discussed in section 3.3, a number of excitation histories with varying time step sizes were created *a-priori*. The procedure was then made to adopt the step size (enlarged / reduced) which was closest to the step size computed by equation 5.2. This modification was necessitated by the complexity posed by interpolation for the cases when the forcing function is not well defined. For the problems where forcing function is not analytically defined, the time step size can be chosen arbitrarily as required.

In order to select the appropriate pre-interpolated value the excitation records at different time steps were stored in different files. The algorithm kept a record of the elapsed time t. When the computation suggested a step size Δt_i from the i^{th} interpolation file, the value $\frac{t}{\Delta t_i}$ indicated the record to be used. Appropriate procedure to avoid truncation errors in this critical division was adopted.

It should be mentioned at this stage that all earlier studies with adaptive time stepping algorithms [Zienkiewicz and Xie, 1991; Zeng et al., 1992] have either (a) used simple analytic functions as excitation where it was straight forward to select any arbitrary step or (b) have initiated adaptive procedures only after the non-analytic has ceased.

For elastic problems the additional computation that becomes essential in adaptive schemes is the computation and triangularization of the effective stiffness matrix K^* whenever time step is changed. Clearly it is possible to compute K_i^* a-priori for all the time steps i being employed and use the appropriate matrix when required. The later procedure was, however, not used.

5.3 STEP FUNCTION LOAD WITH SUDDEN LOAD REVERSAL

A single degree of freedom (SDOF) system as shown in Fig. 5.1 with m=100 kg, k=4100 N/m and damping $\xi = 20\%$ was excited by a step function load (f_0=2000 N) which was suddenly reversed at 5 sec as shown in Fig. 5.2. For the system used the natural period T=0.98 sec.

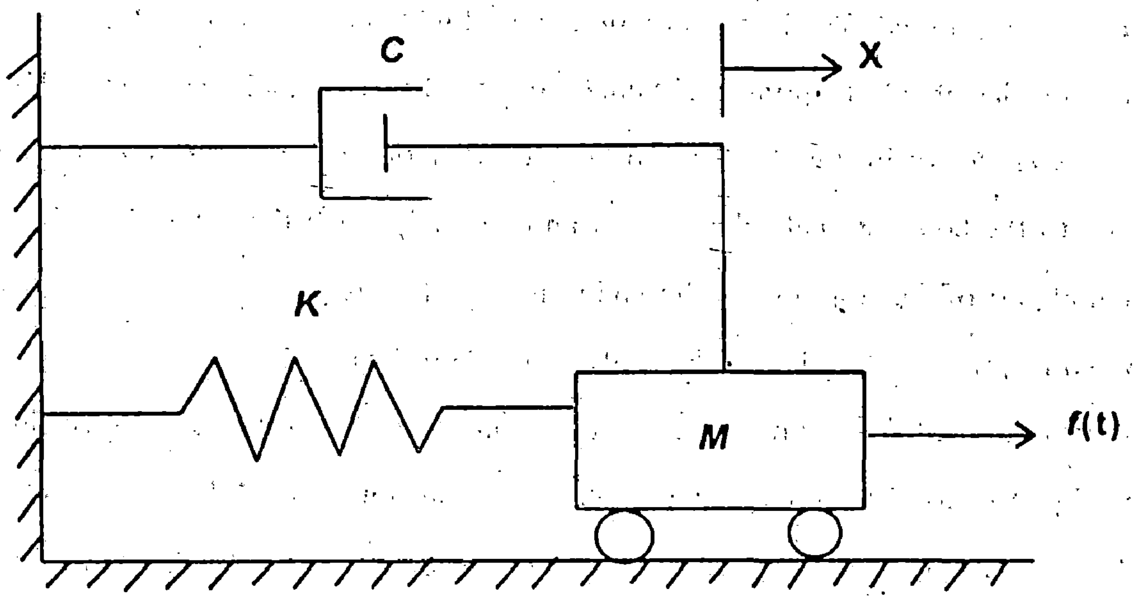

FIG. 5.1 : SINGLE DEGREE OF FREEDOM SYSTEM ANALYSED

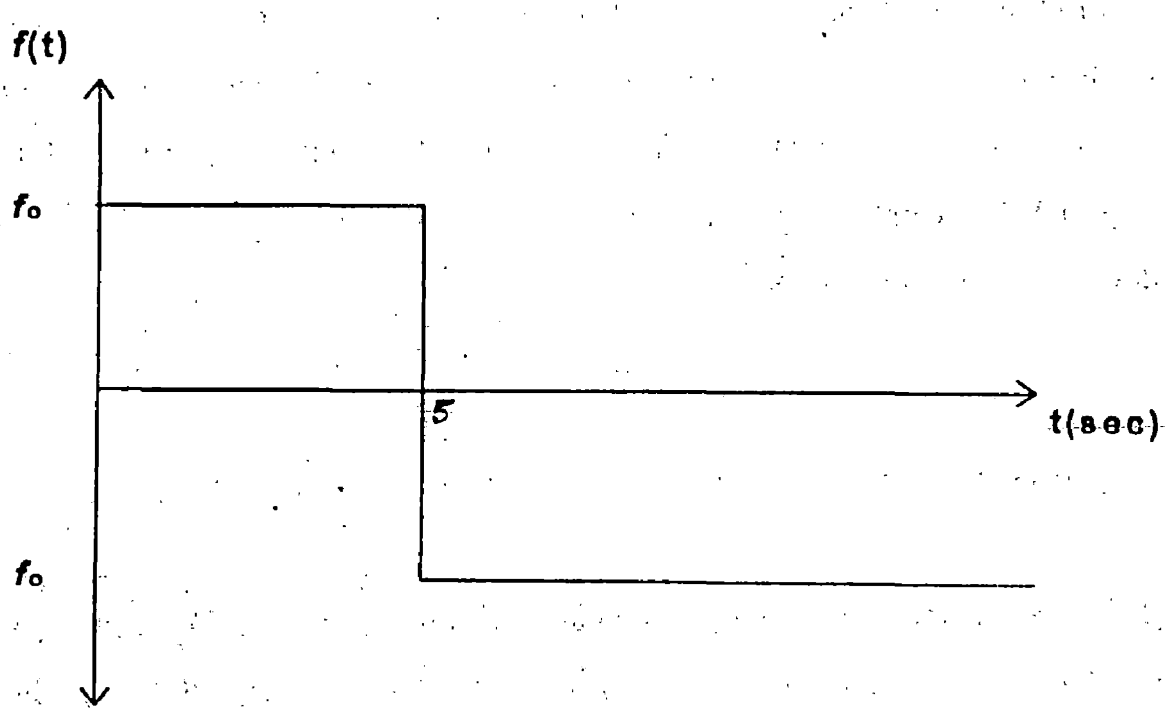

FIG.5.2 : SUDDENLY REVERSING FORCE

Clearly a step size Δt=T/5 should be appropriate. The analysis was conducted using a conventional (non-adaptive) scheme with Δt=0.2 sec and Δt=0.025 sec. The analysis was also conducted using an adaptive scheme wherein Δt_i=0.2, 0.1. 0.05, 0.025 sec were used as discussed. The integration was started with a time step of 0.1 sec.

The displacement response obtained using the conventional and adaptive methodologies is shown in Fig. 5.3. It can be seen that the adaptive scheme is a much better match to the conventional scheme with an extremely small time step. This is more so in the initial part of the response as shown in Fig. 5.4. This is achieved inspite of using a large step in a majority of duration of the analysis as shown in Fig. 5.5.

The variation of a absolute error using the three schemes as a function of time is shown in Fig. 5.6 to 5.8. For Δt=0.2 sec with conventional analysis the error is considerably higher (Fig. 5.6) for most of the duration as compared to the conventional analysis with Δt=0.025 sec (Fig. 5.7). For the adaptive scheme the error remains small for most of the duration except at the time of load reversal. Apparently this is the reason why

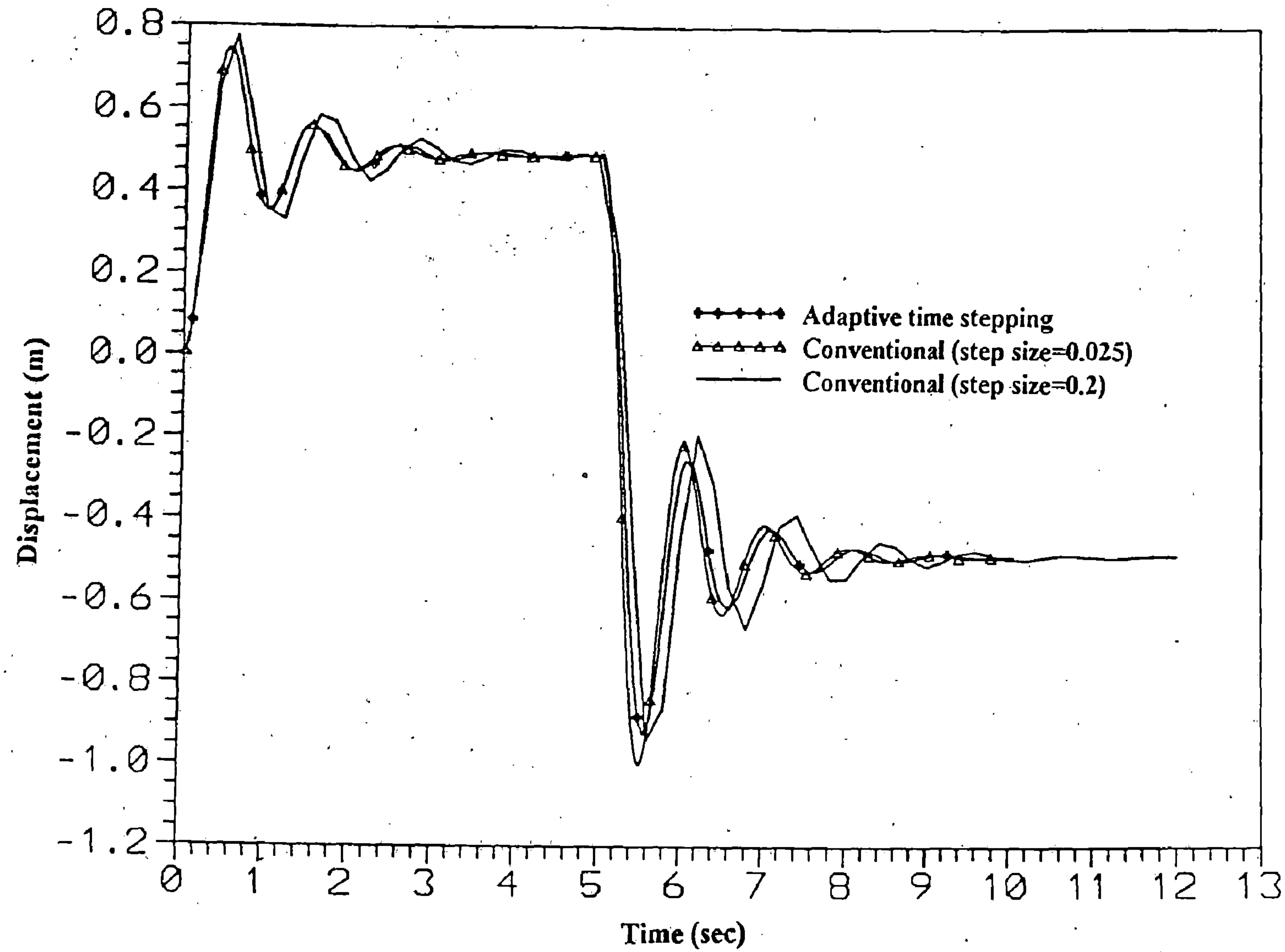

Fig.5.3 Sudden Load Reversal, comparison of solutions by Adaptive and Conventional time stepping analysis.

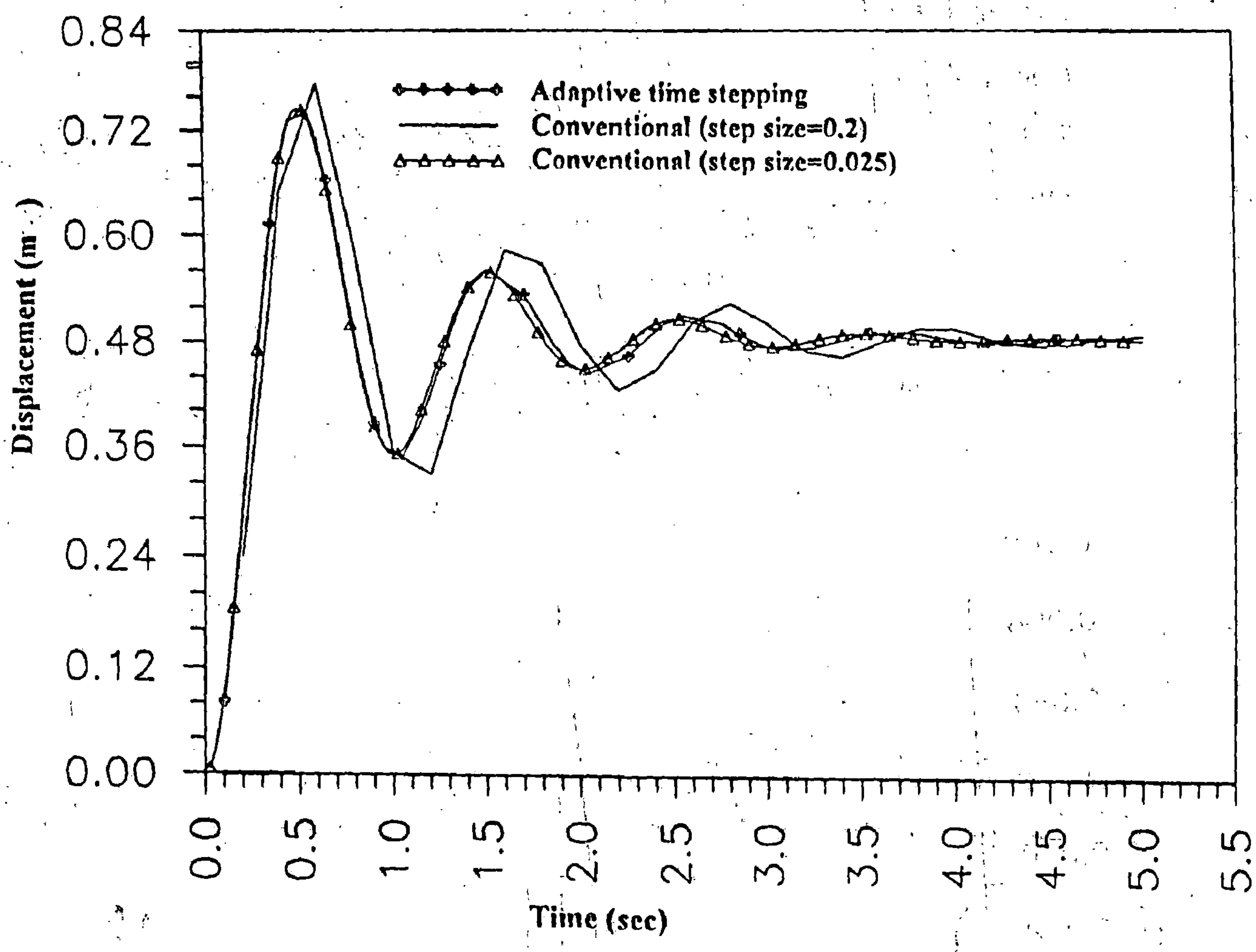

Fig5 . 4 Sudden Load Reversal, comparison of solutions by Adaptive and Conventional time stepping analysis (zoomed, upto 5 sec).

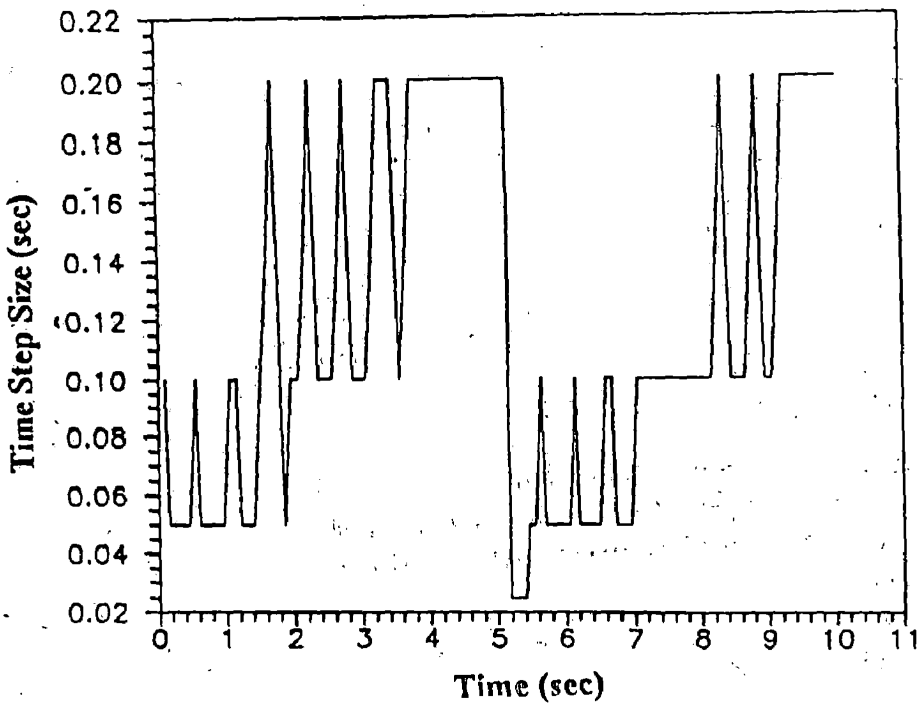

Fig5.5 Sudden Load Reversal, Time Vs Time Step Size variation of solution by Adaptive time stepping analysis.

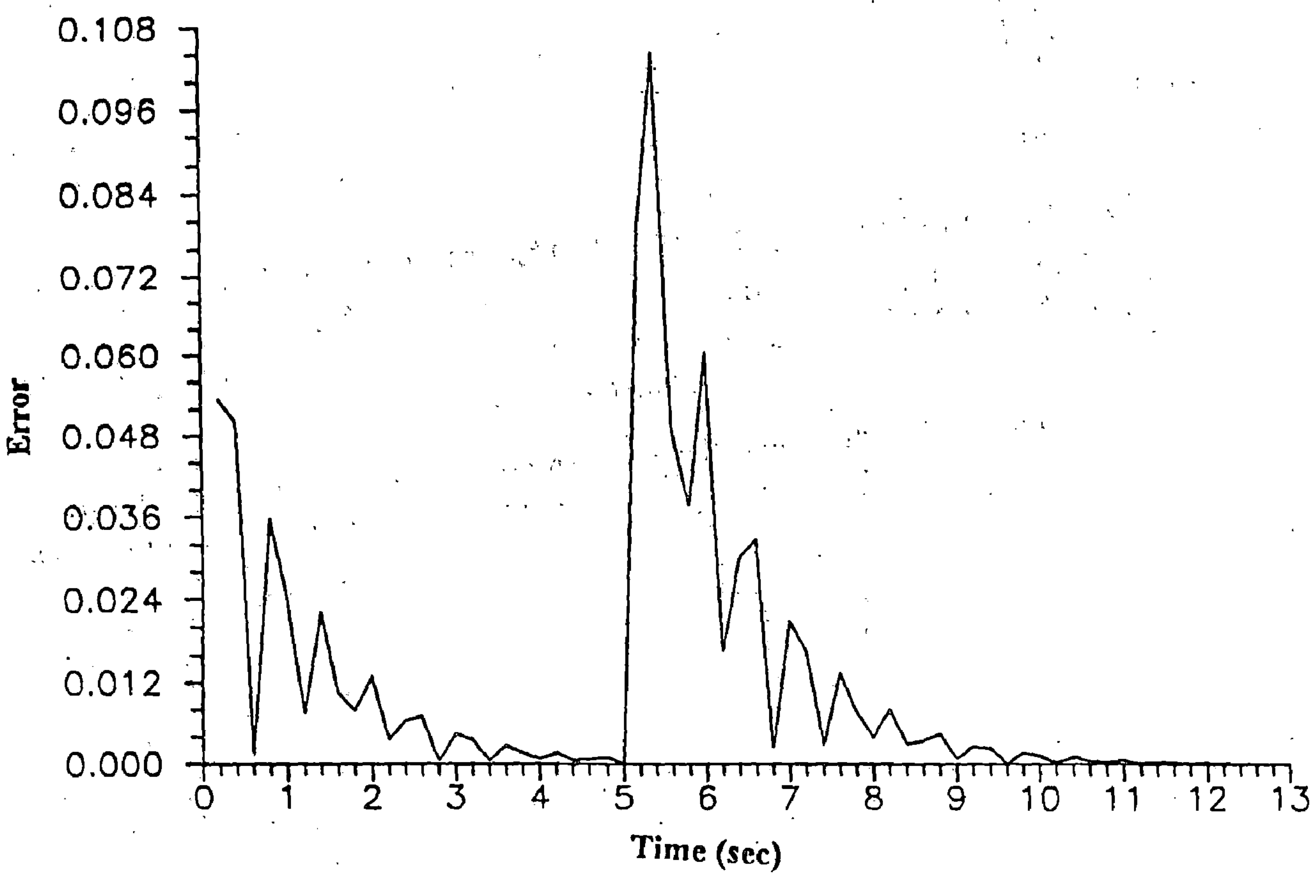

Fig5.6 Sudden Load Reversal, Time Vs Error variation of solution by Conventional time stepping analysis (step size=0.2 sec).

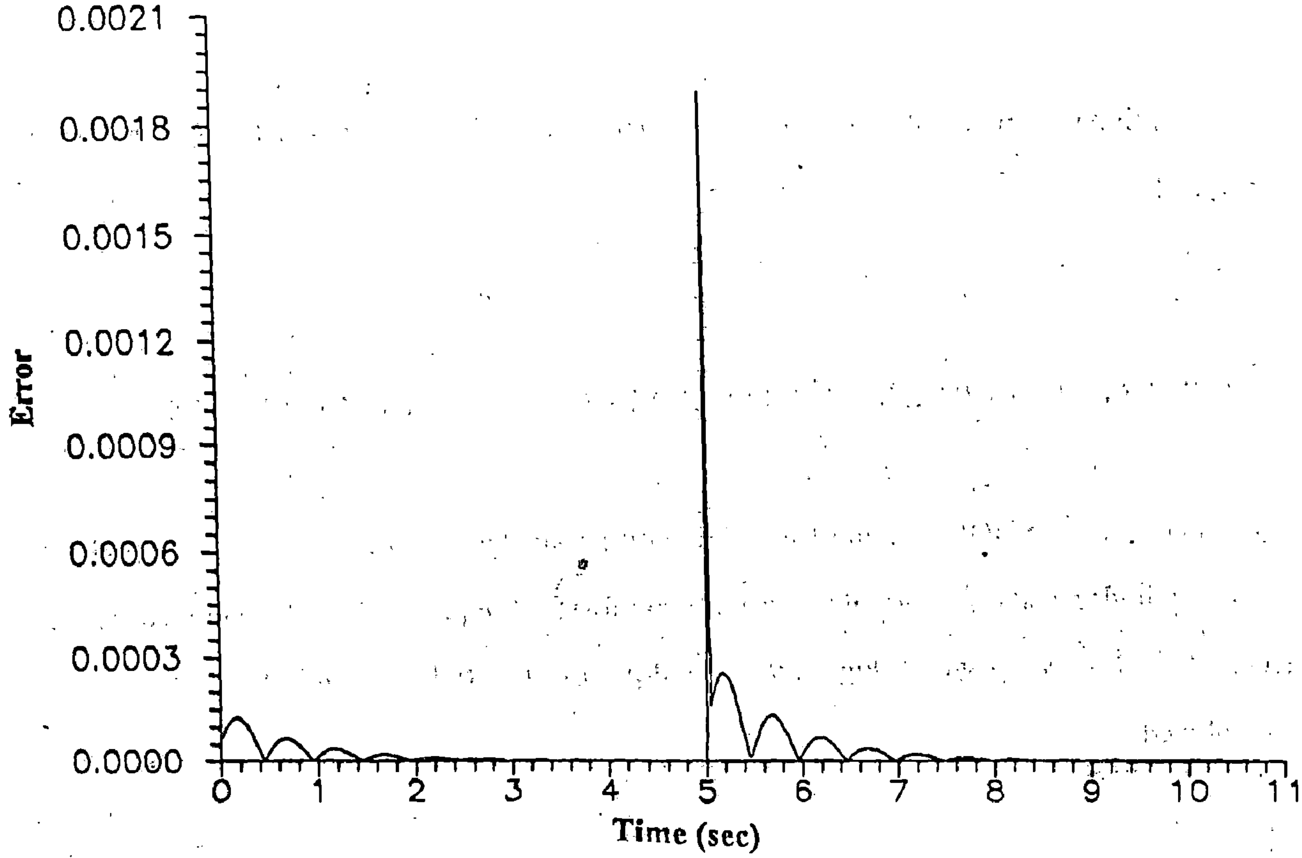

Fig.5 . 7 Sudden Load Reversal, Time Vs Error variation of solution by Conventional time stepping analysis (step size=0.025 sec).

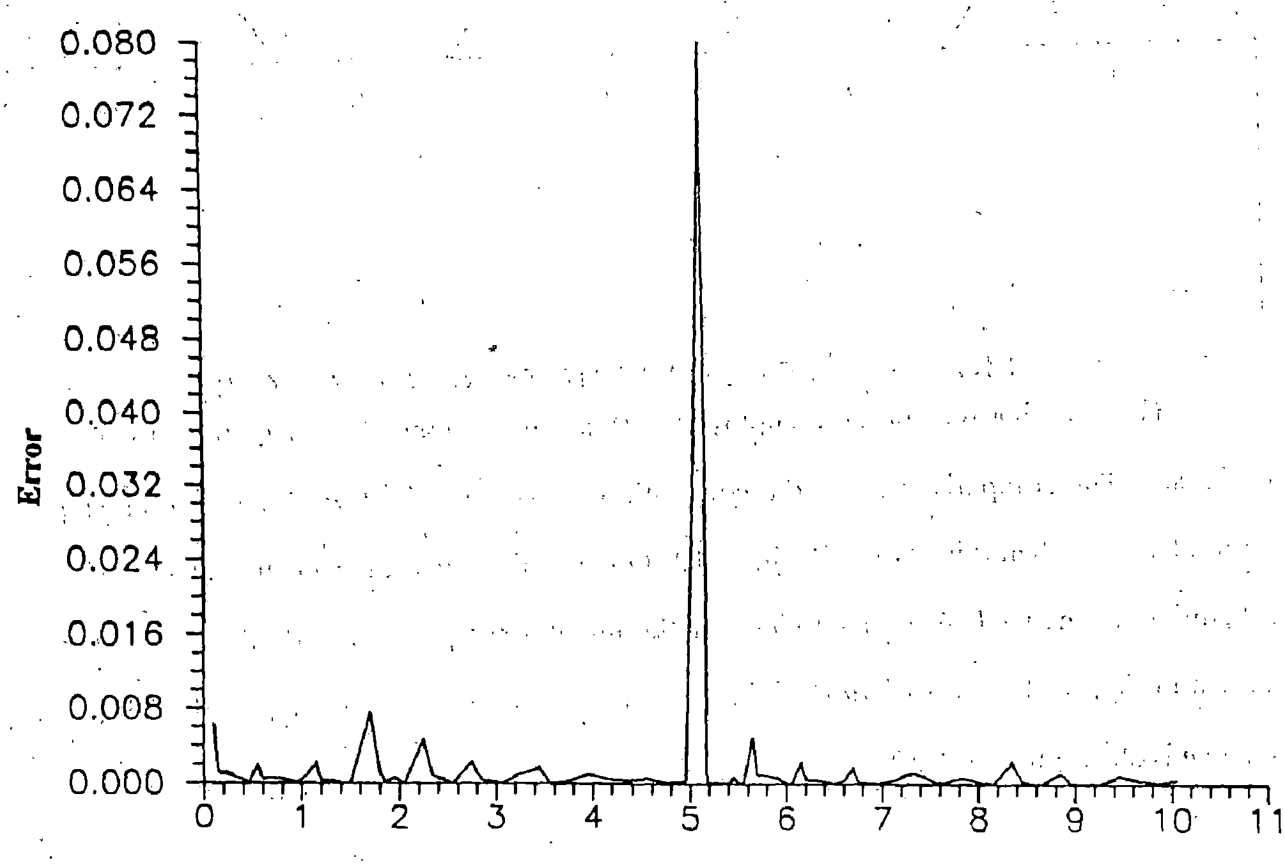

Fig5 . 8 Sudden Load Reversal, Time Vs Error variation of solution by Adaptive time stepping analysis.

the adaptive scheme results deviate slightly from those obtained using Δt=0.025 sec after reversal.

5.4 EXCITATION DUE TO EQUALLY SPACED TRIANGULAR SPIKES

The mass of the SDOF system of Fig. 5.1 with parameters used for the previous problem was excited by a time varying force in the form of equally spaced triangular spikes (f_0=2000 N) as shown if Fig. 5.9. The duration $t_1 = 0.4$ sec and $t_2 = 4.4$ sec was employed.

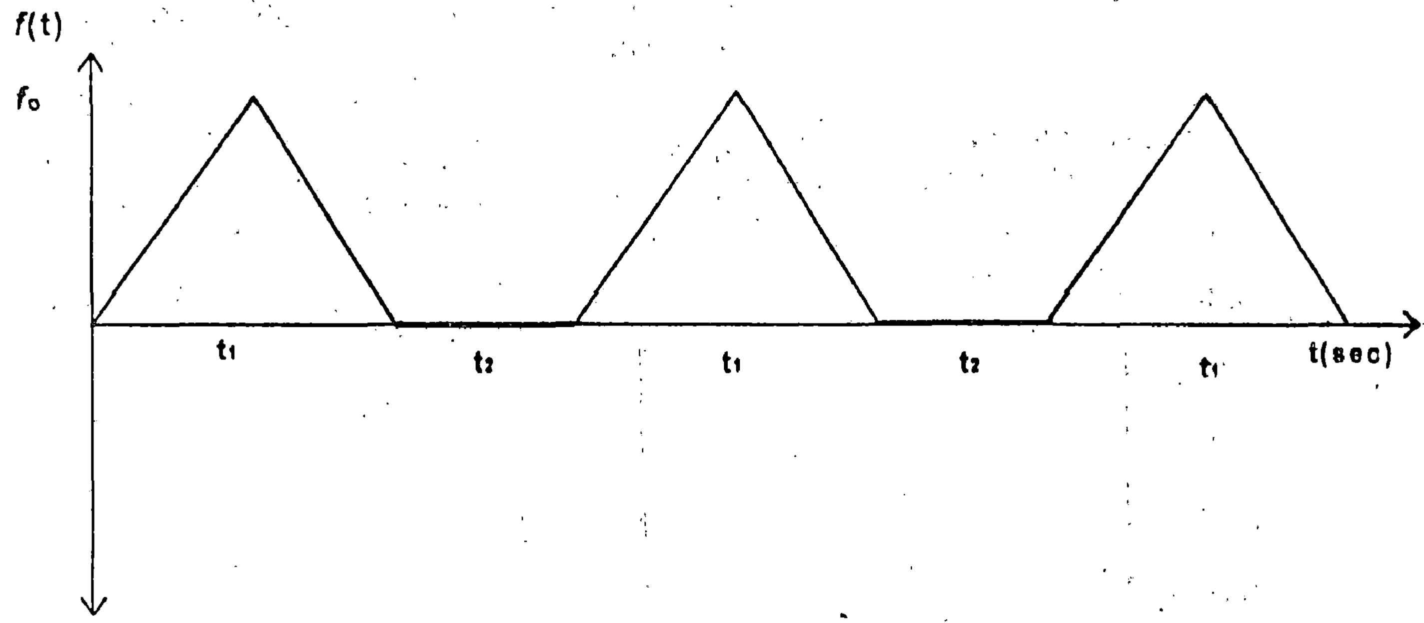

FIG. 5.9 : FORCE IN THE FORM OF SPIKES.

The adaptive scheme was employed with sampling intervals of Δt_i=0.2, 0.1, 0.05, 0.025 sec. The comparison using the conventional schemes with two different time steps with adaptive scheme is shown if Fig. 5.10. Once again it can be seen that the adaptive scheme performs well. A zoomed view for the initial 4 sec period clearly illustrate this as shown in Fig. 5.11. The variation of time step as a function of time used by the adaptive scheme is shown in Fig. 5.12.

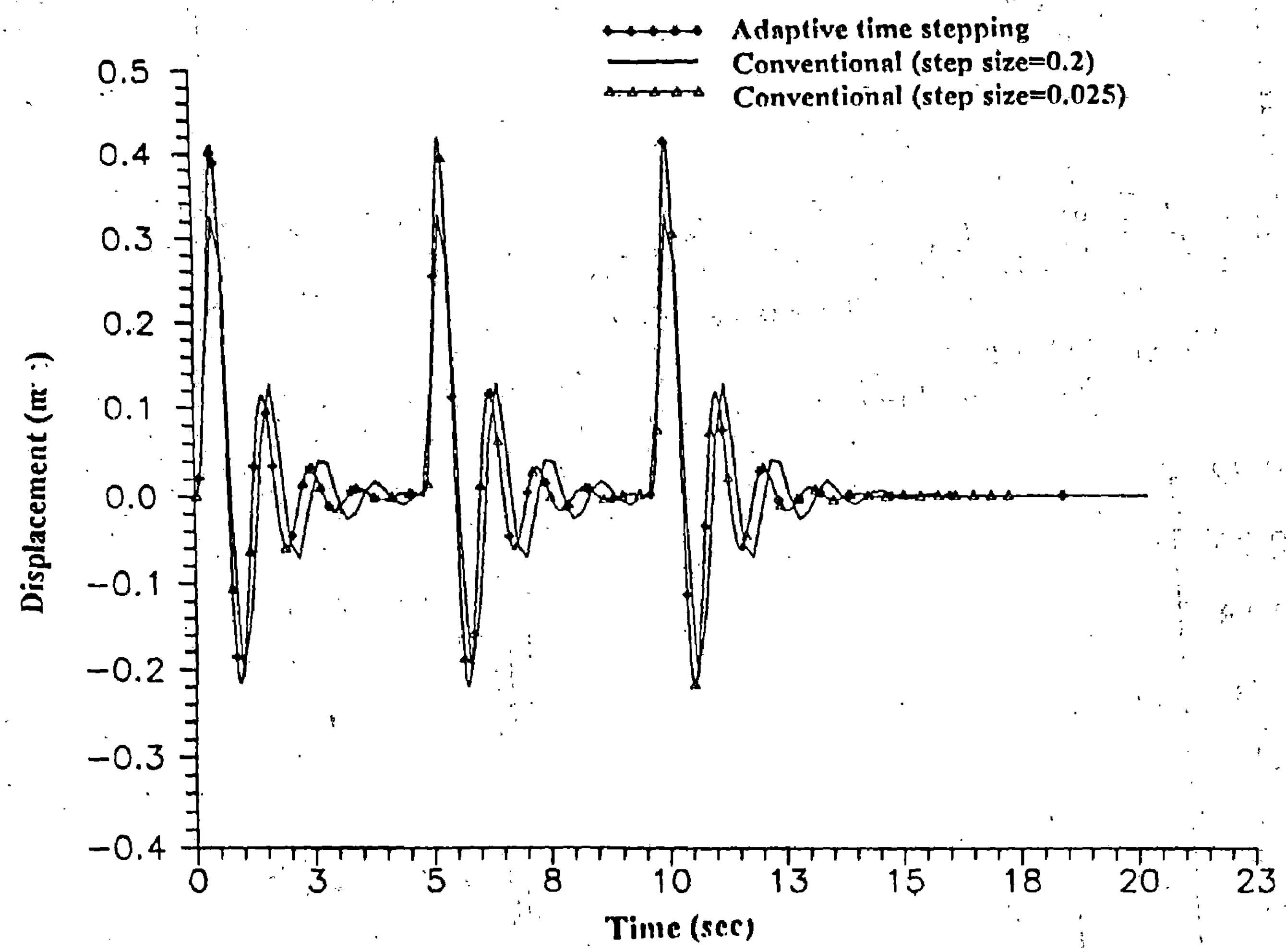

Fig.5.10Spikes Problem, comparison of solutions by Adaptive and Conventional time stepping analysis.

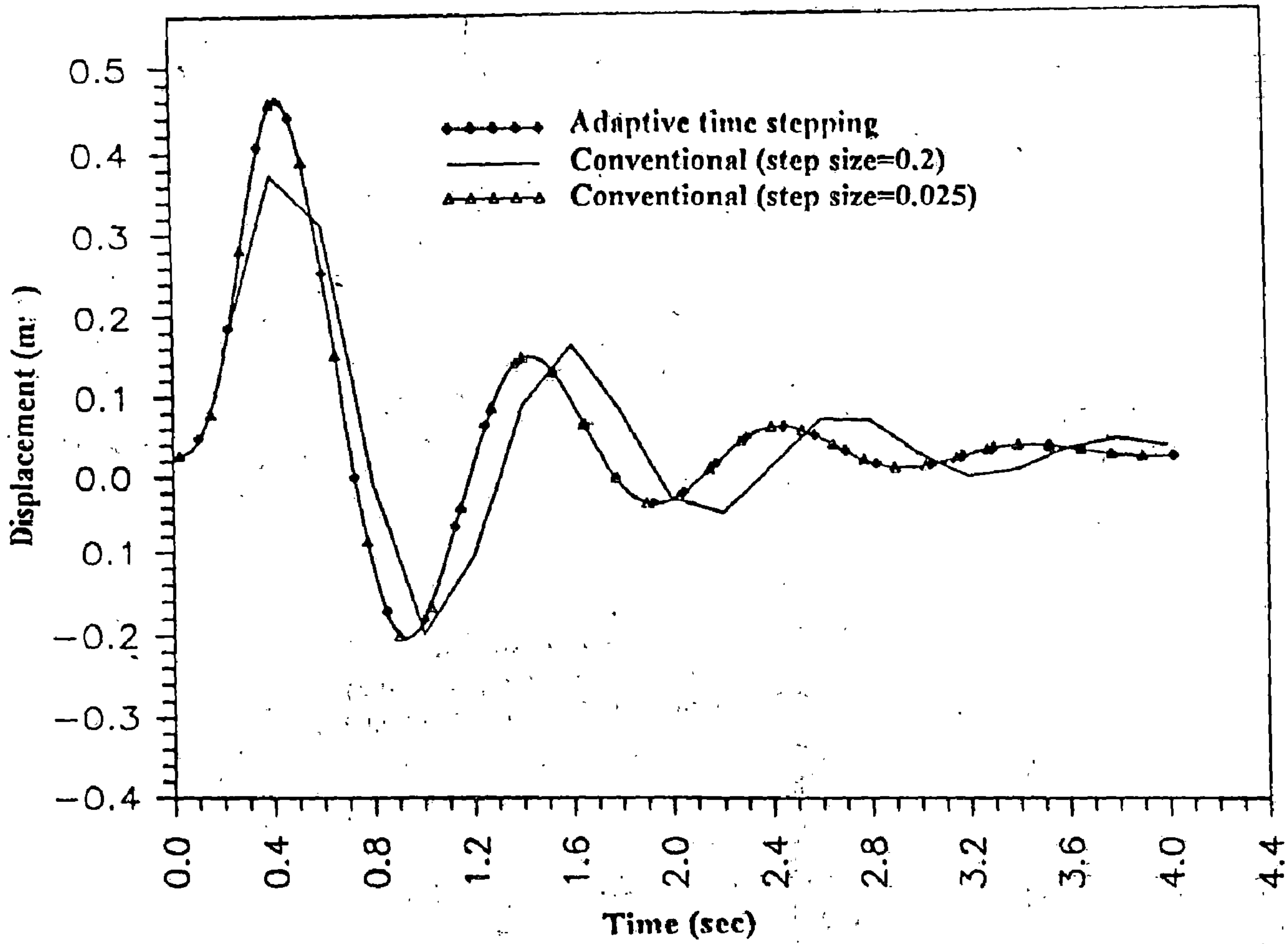

Fig.5.11 Spikes Problem, comparison of solutions by Adaptive and Conventional time stepping analysis (zoomed, upto 4 sec).

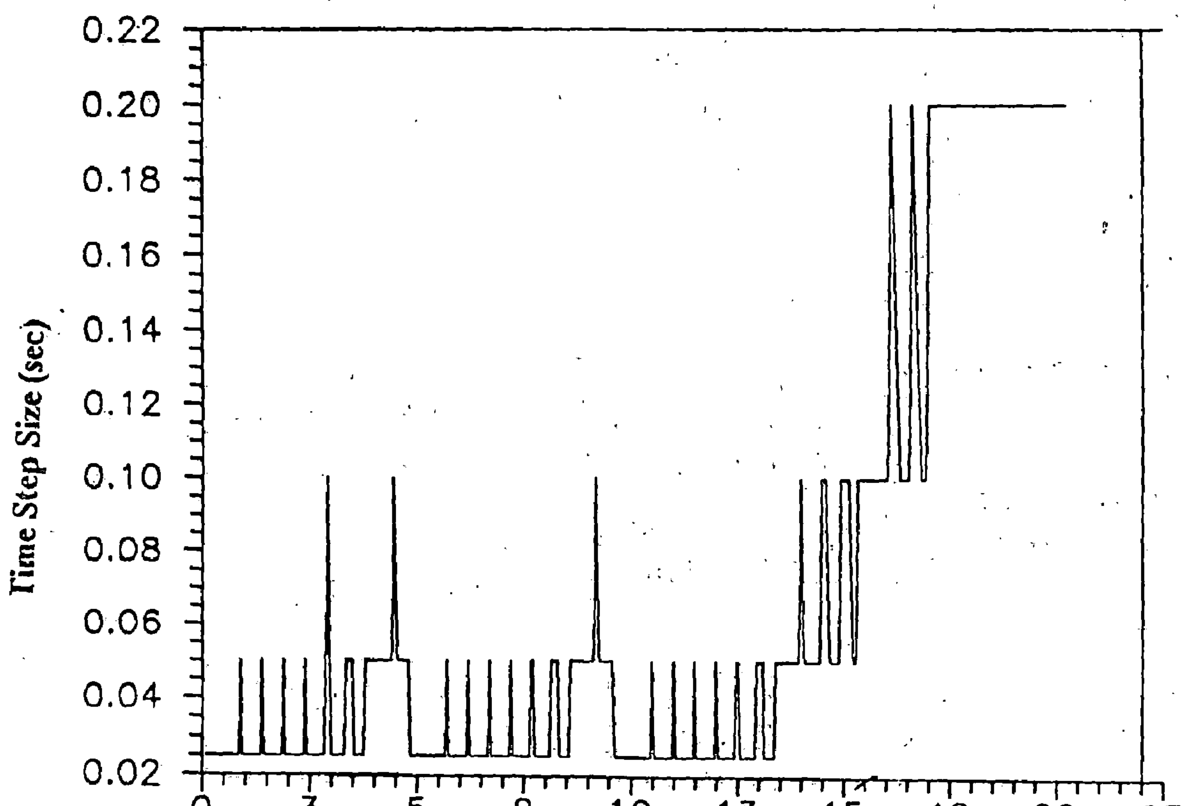

Fig.5.12 Spikes Problem, Time Vs Time Step Size variation of solution by Adaptive time stepping analysis.

The variation of computed error as a function of time for conventional and adaptive schemes is shown Fig. 5.13 to 5.15. The conclusions drawn from these figures are the same as drawn from the earlier problem.

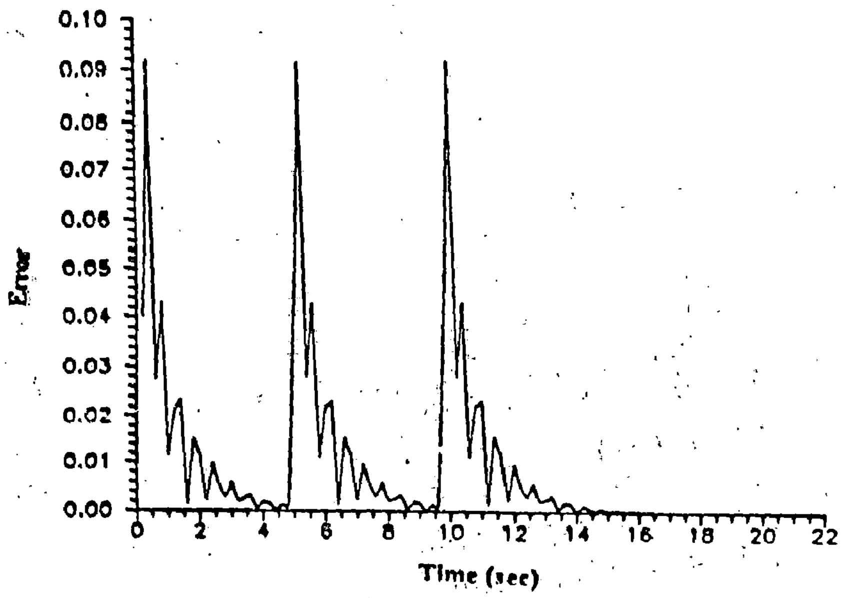

Fig.5.13 Spikes Problem, Time Vs Error variation of solution by Conventional time stepping analysis (step size=0.2 sec).

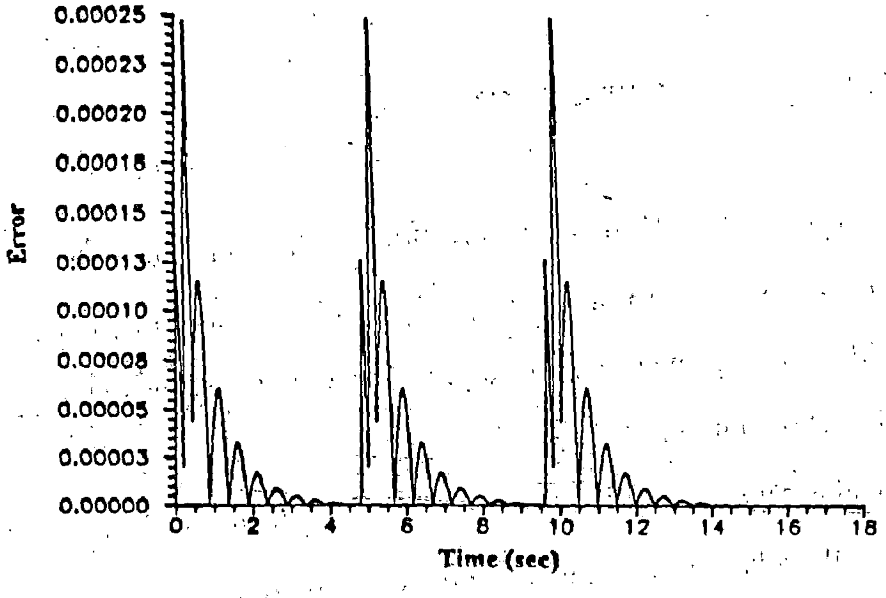

Fig.5.14 Spikes Problem, Time Vs Error variation of solution by Conventional time stepping analysis (step size=0.025 sec).

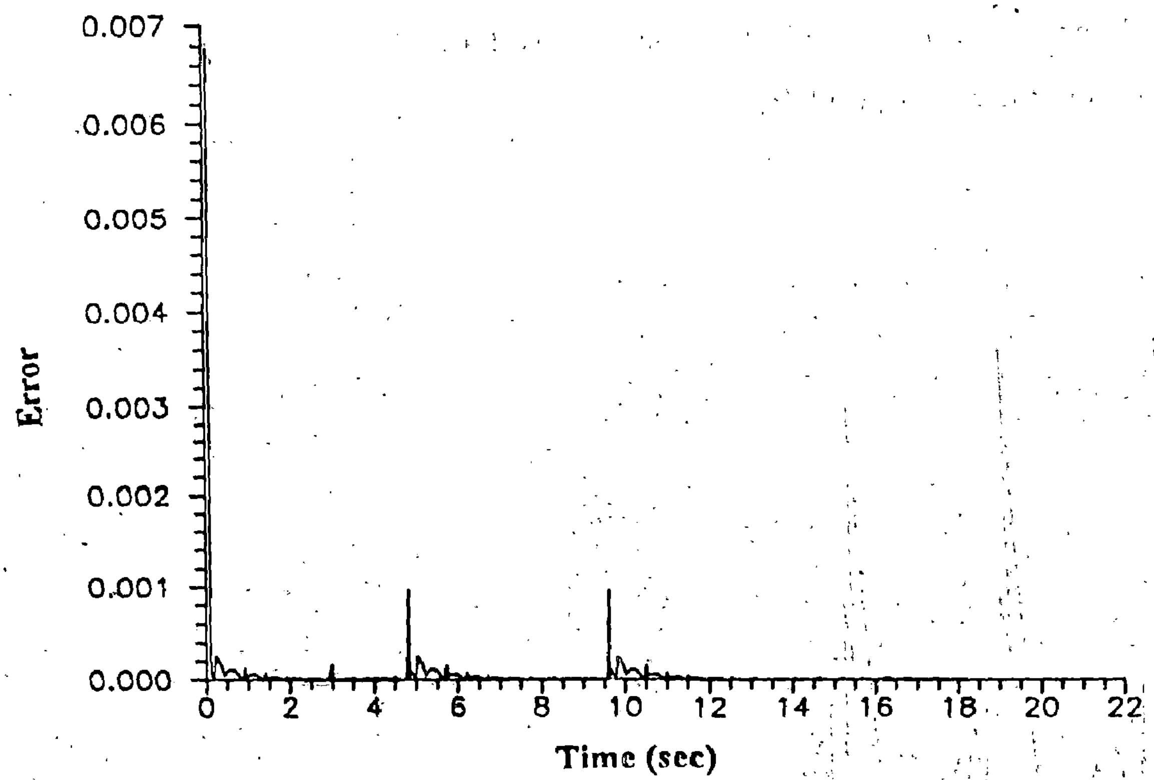

Fig. 5.15 Spikes Problem, Time Vs Error variation of solution by Adaptive time stepping analysis.

5.5 ELASTOPLASTIC SYSTEM WITH STEP FUNCTION LOAD

In order to study the possibility of using the adaptive scheme with nonlinear problems the SDOF system shown in Fig. 5.1 was assumed to be elasto-perfectly plastic with an yield load of 2460 N. Once again the displacement response using the conventional and adaptive schemes was compared as shown in Fig. 5.16.

Once again the problem illustrates that the adaptive scheme performs well. No difficulty was encountered in using the adaptive scheme in a nonlinear situation.

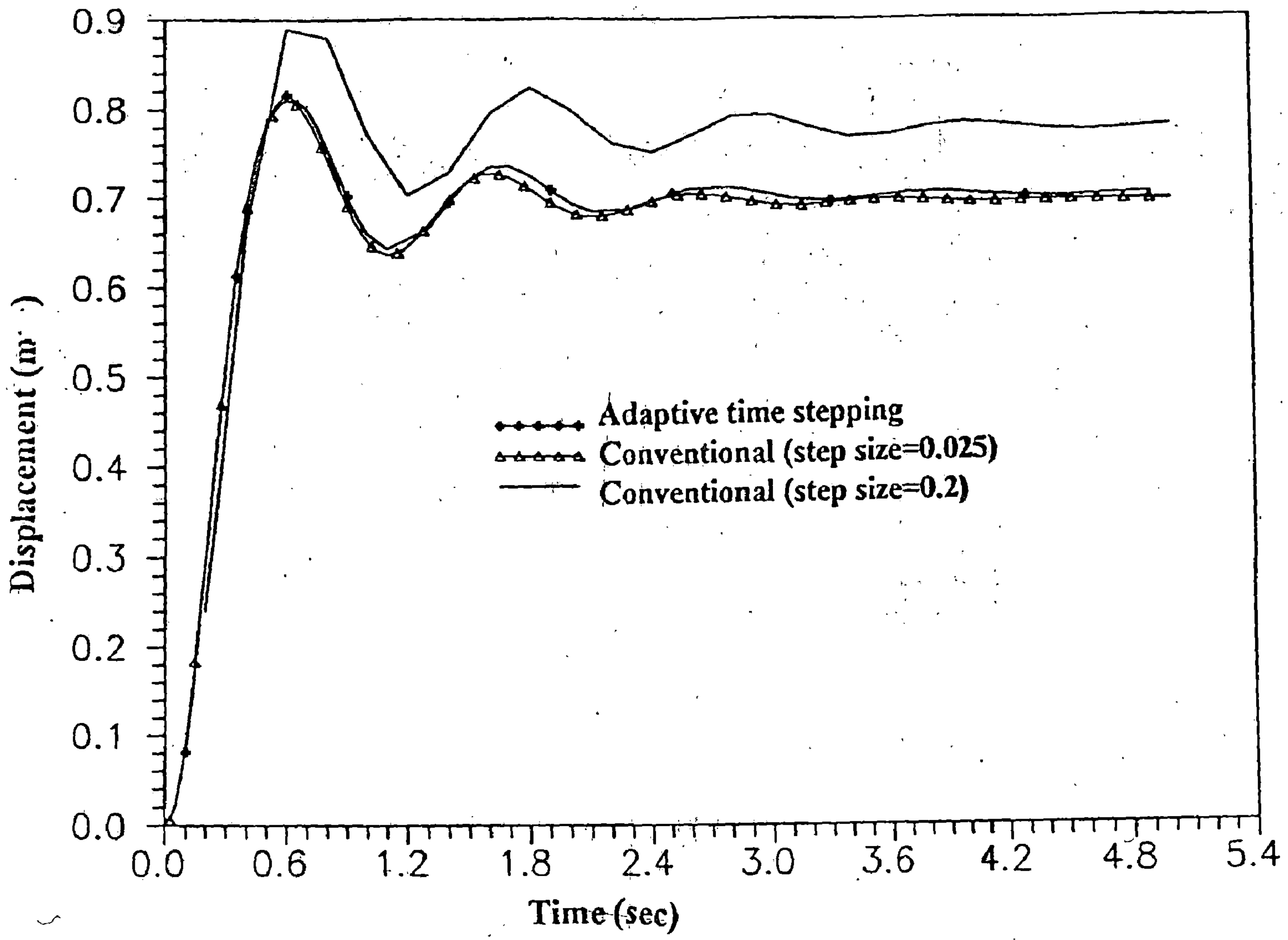

Fig5.16 NONLINEAR, STEP FUNCTION LOADING; comparison of solutions using Adaptive and Conventional analysis.

5.6 SDOF SYSTEM WITH EARTHQUAKE EXCITATION

The SDOF system shown in Fig. 5.1 was excited by the Uttarkashi earthquake interpolated as discussed in Chapter 4.

The conventional schemes with Δt=0.02 sec, Δt=0.0025 sec and the adaptive scheme did not show any difference in results. The reason clearly was that the time step of 0.02 sec was sufficiently small for the natural period of the system (T=0.98 sec).

In order to perform a more stringent test the natural frequency of the SDOF system was increased to 40 Hz by enhancing its stiffness. The damping was reduced to 5 %. A comparison of the displacement response is shown in Fig. 5.17. A zoomed view of the same response is shown in Fig. 5.18.

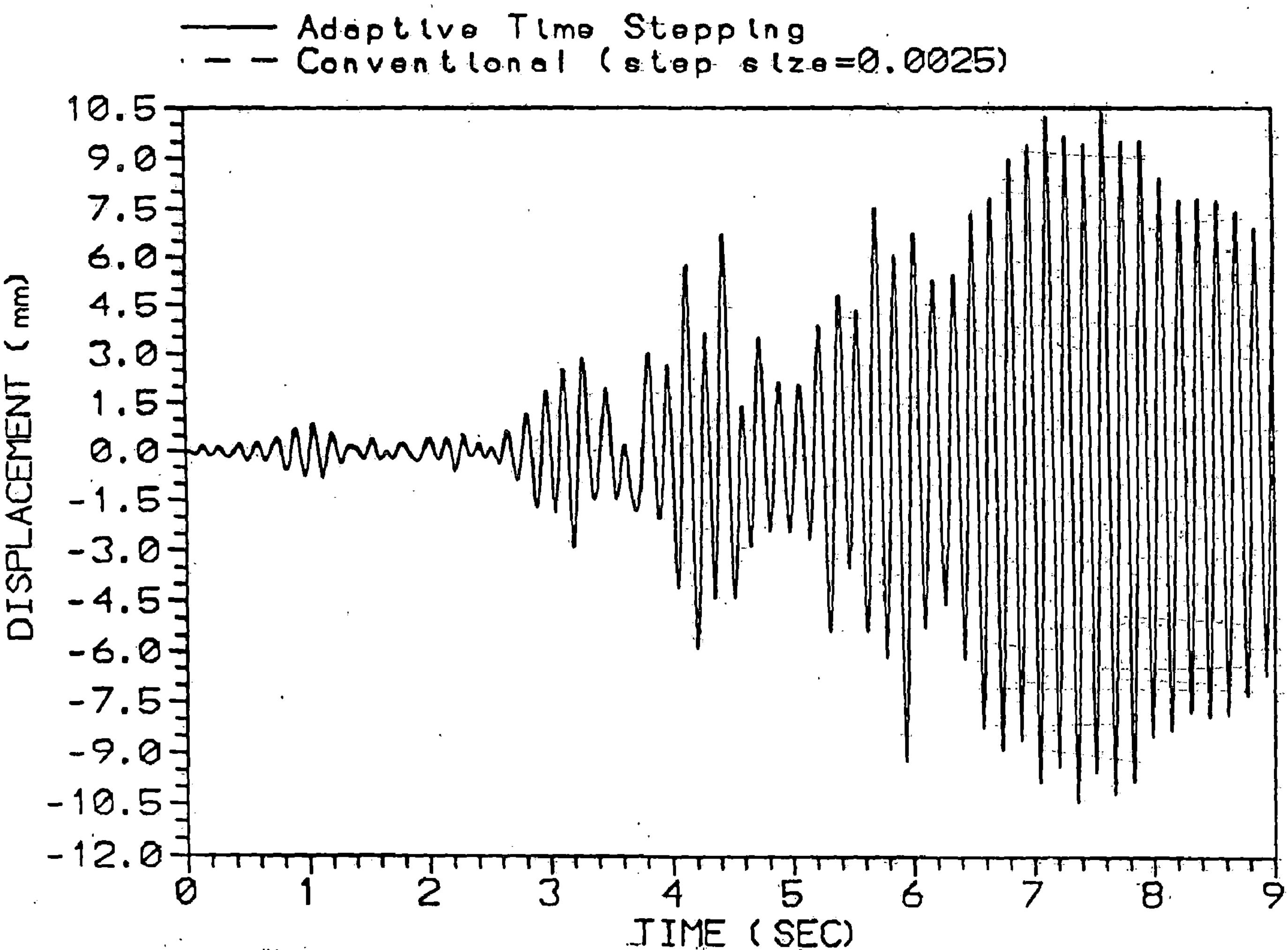

Fig.5.17SDOF system with earthquake excitation, comparison of solutions by Adaptive and Conventional time stepping analysis

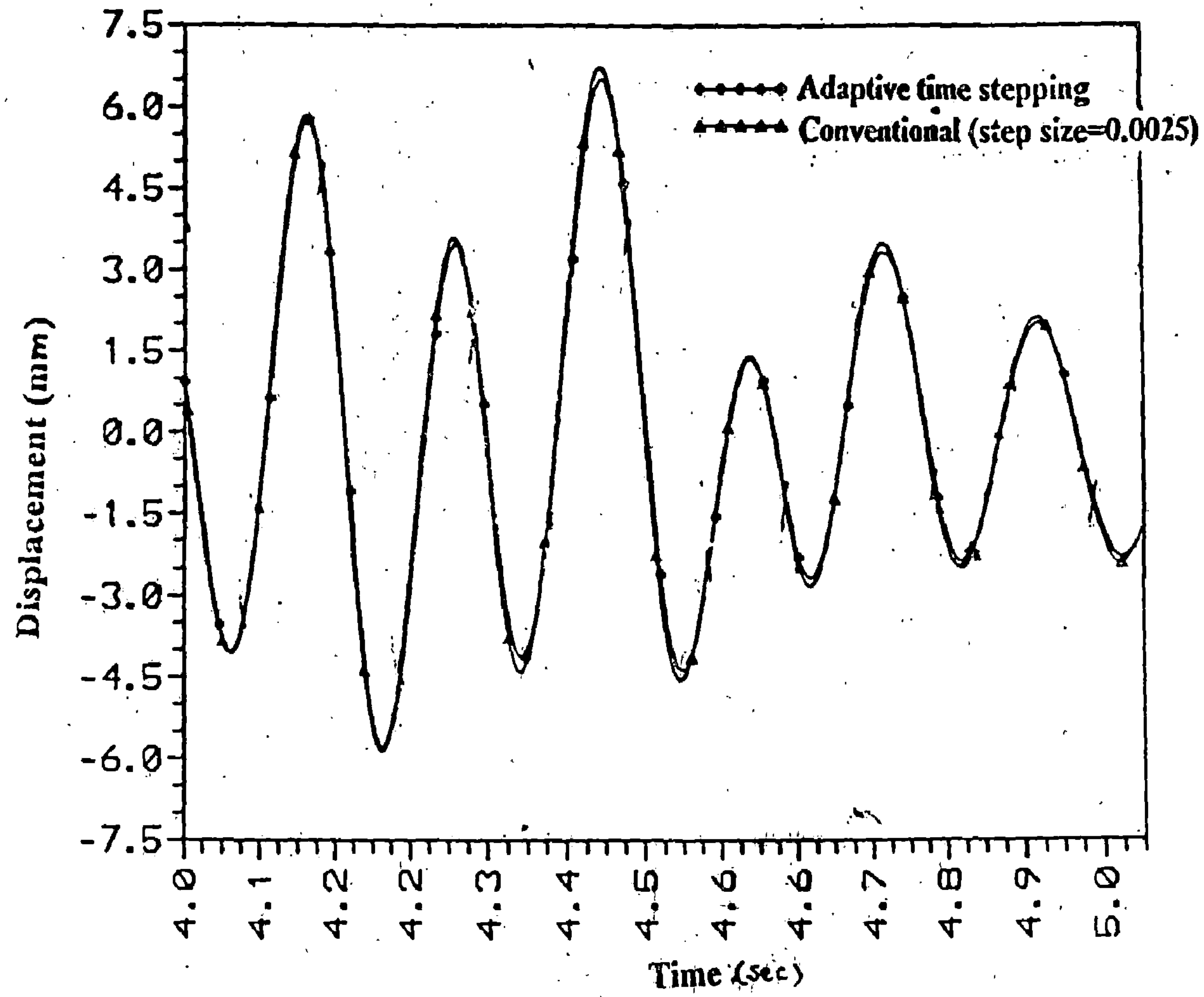

Fig.5.18 SDOF system with earthquake excitation, comparison of solutions by Adaptive and Conventional time stepping analysis (b/w 4-5 sec).

The comparison of displacement response using the adaptive and the conventional schemes in a typical duration is shown in Fig. 5.21 and 5.22 for floors one and five respectively.

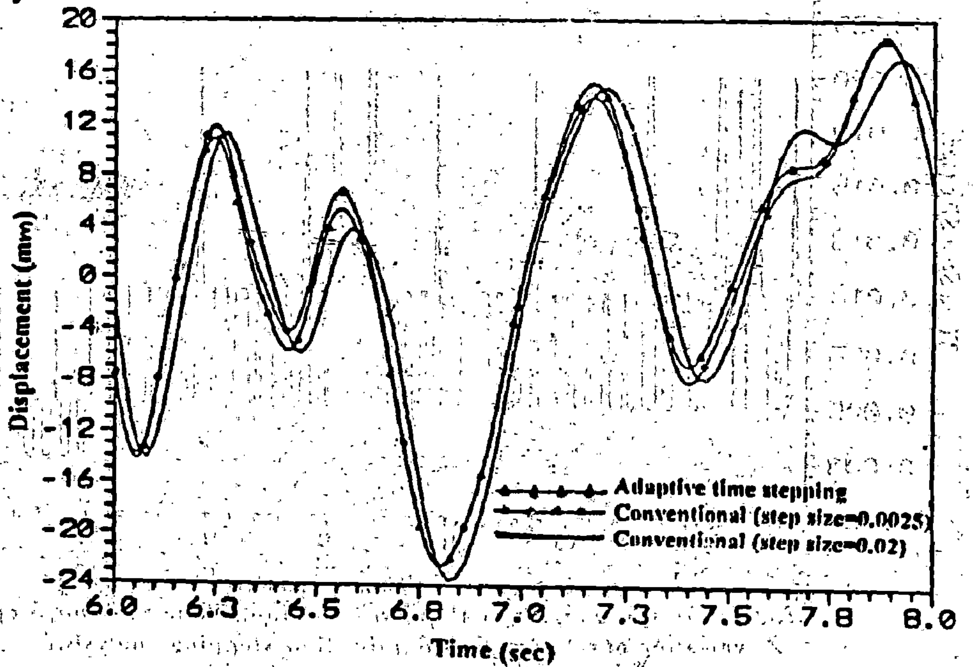

Fig.5.21Multistorey building with earthquake excitation, comparison of solutions for first floor by Adaptive and Conventional time stepping analysis.

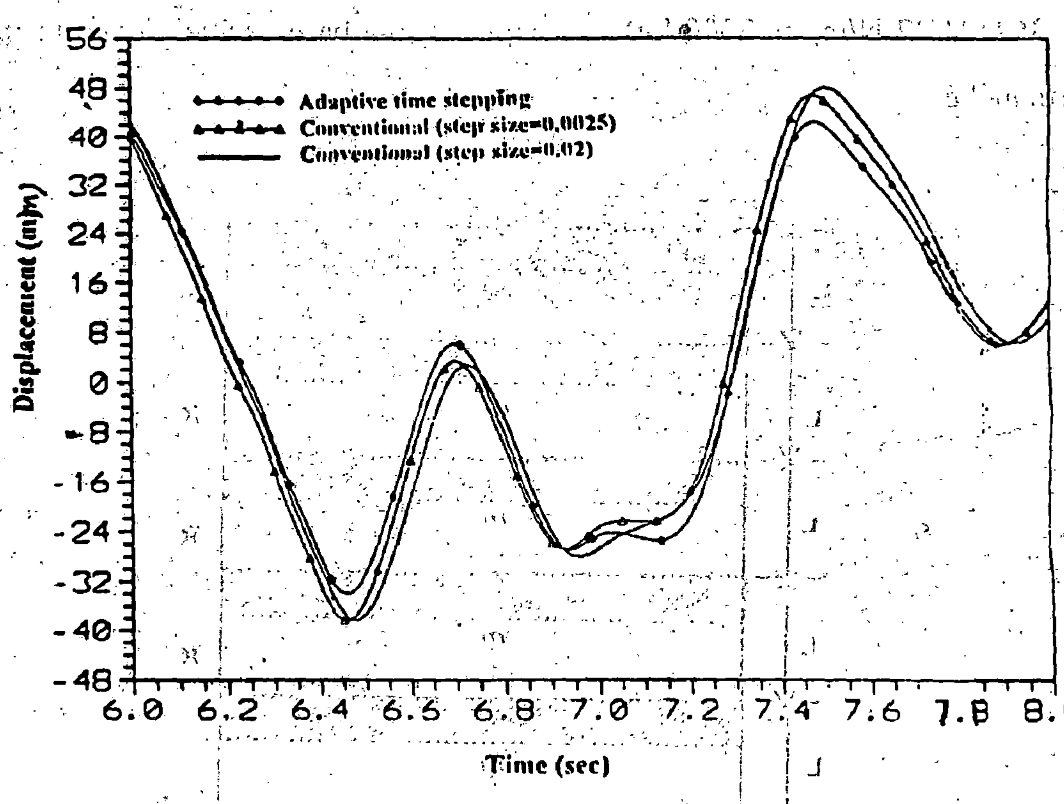

Fig.5.22Multistorey building with earthquake excitation, comparison of solutions for 5^{th} floor by Adaptive and Conventional time stepping analysis

The variation of the time step with time for the period of 3.5 sec is shown in Fig 5.19 which shows that the scheme advocates time step changes for earthquake problems.

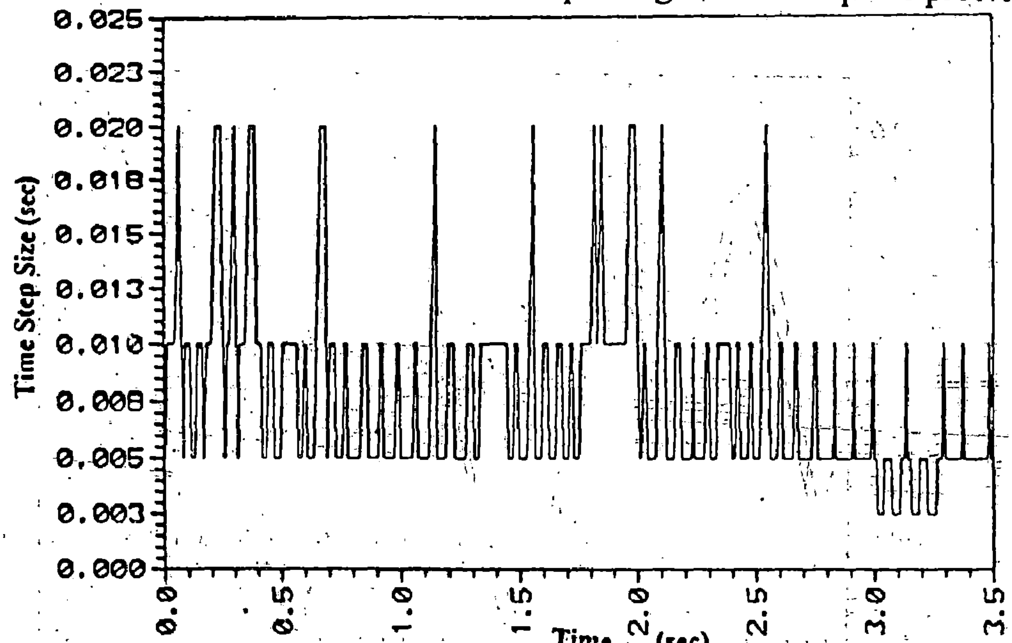

Fig.5.19 SDOF system with earthquake excitation, Time Vs Time Step Size variation of solution by Adaptive time stepping analysis.

5.7 MULTISTOREY BUILDING WITH EARTHQUAKE EXCITATION

A five storey building idealized as a knotted cantilever as shown in Fig. 5.20 (K=364141.32 N/m, m=3000 kg) was exposed to a base excitation due to the Uttarkashi Earthquake.

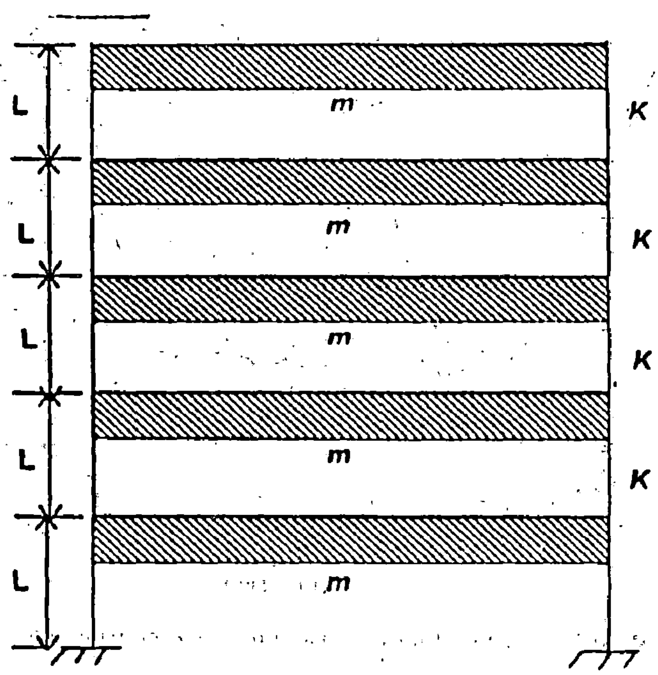

FIG5.20 5 Storey Building under consideration, where displacement response of first and fifth floor level is measured by adaptive and conventional time stepping analysis

The variation of time step in the initial stages of excitation is illustrated in Fig.5.23 Once again it can be seen that the adaptive scheme performs well with appropriate time step changes.

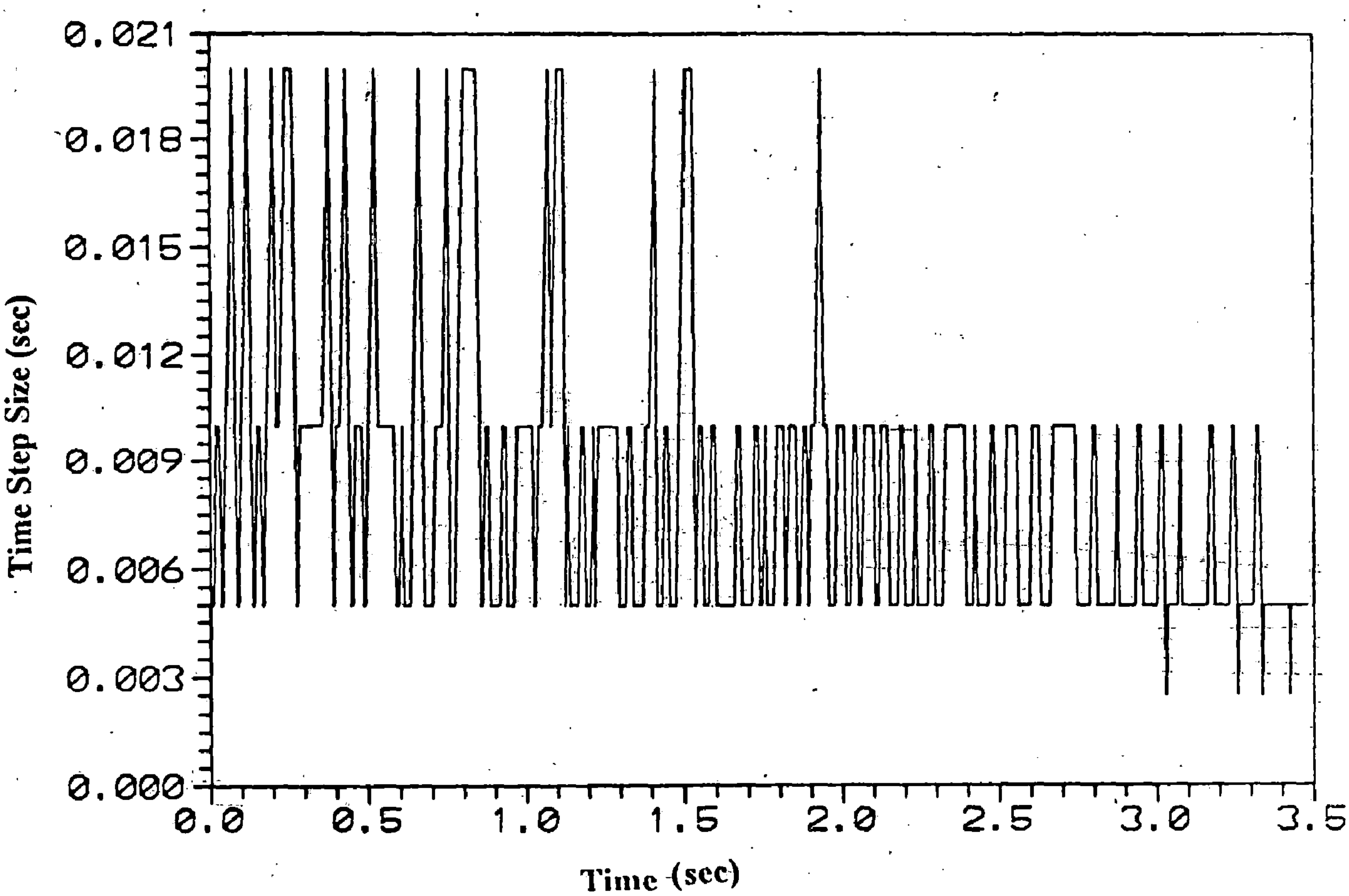

Fig.5.23 Multistorey building with earthquake excitation, Time Vs Time Step Size variation of solution by Adaptive time stepping analysis.

EPITOME

The Single Step direct integration procedures have an advantage of being adaptable to time step changes. A simple algorithm can easily be incorporated in direct integration packages for evaluation of errors and effecting time step changes. Thus the time step can be enlarged when the variation of response with time is small. The previous studies with these procedures were with simple loading functions like suddenly applied loads wherein the damped dynamic system would adopt a static response pattern after passage of some time. Some studies with seismic response where interpolation with arbitrary time steps is difficult used a constant time step during the excitation period and resorted to adaptive schemes only after the excitation had ceased. In this study a novel approach wherein specific interpolation excitation values at various time step sizes are provided *a-priori* and used as necessary was proposed. The advantage of the scheme is that it can easily be used for non analytic functions such as those due to earthquakes. This scheme also has the advantage in seismic analysis that it easily offers itself to excitation histories that are band limited and preserves the frequency content of the original digital signal.

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