STUDY ON PIPING ON THE DOWNSTREAM OF A WEIR ON NON HOMOGENEOUS PERMEABLE SOIL

A DISSERTATION

Submitted in partial fulfillment of the requirements for the award of the degree of

MASTER OF TECHNOLOGY

in

WATER RESOURCES DEVELOPMENT

By

FREDDY VIJAYA

WATER RESOURCES DEVELOPMENT TRAINING CENTRE
INDIAN INSTITUTE OF TECHNOLOGY ROORKEE
ROORKEE-247 667 (INDIA)

JUNE, 2004

CANDIDATE'S DECLARATION

I herby declare that the dissertation titled **"Study on Piping on the Downstream of a Weir on Non Homogeneous Permeable Soil"** which is being submitted for partial fulfillment of the requirement for the award of master of technology in **Water Resources Development (Civil)** at Water Resources Development Training Centre (WRDTC), Indian Institute of Technology, Roorkee, is an authentic record of my own work carried out during the period of July 2003 to June 2004 under the supervision and guidance of Dr.G.C.Mishra, Professor, WRDTC, Indian Institute of Technology Roorkee, India.

I have never submitted the matter embodied in this dissertation for the award of any other degree.

Freddy V

Placed : Roorkee

Dated : 30.06.2004

(Freddy Vijaya)

47th WRD, (Civil)

This to certify that the above statement made by the candidate is correct to the best of my knowledge belief.

G. C. Mishra

(Dr. G.C. Mishra)

Professor,

WRDTC, IIT Roorkee.

ACKNOWLEDGEMENT

I express my profound gratitude to Dr.GC.Mishra, Professor, WRDTC, Indian Institute of Technology, Roorkee, for his guidance, advice and encouragement during the preparation of my dissertation.

I also express my gratitude to Head of WRDTC, all Professor and staff of WRDTC for their help and hospitality during the period of my study in WRDTC.

Thank to Indonesian Government and Indian Government for providing me a chance and opportunity to study in WRDTC, Indian Institute of Technology Roorkee.

(Freddy Vijaya)

Placed : Roorkee

Dated : 30.06.2004

ABSTRACT

The exit gradient theory up to 19th century that used for designing various irrigartion structures was empirical method based on experience and intuition. Some of the structures failed because of subsurface flow. The subsurface flow may cause the failure of the impervious floor either by piping or by uplift pressure.

Bligh went a step forward and gave a creep theory. According to this theory, the percolating water creeps along the contact surface of the base structure with subsoil. As the water creeps from the upstream end to the downstream end, the head loss occurs. The head loss is proportional to the creep distance traveled.

Lane brought out the deficiencies in Bligh's creep theory. The theory gives the vertical creep three times more weightage as compared to the horizontal creep.

Koshla and his associates determined the flow pattern below the impervious base of hydraulic structures on permeable foundation. They started with potential flow theory and found the solution of Laplace's equation for different configuration of floors. From the flow pattern, the distribution of uplift pressure on the base of the hydraulic structures and exit gradient were found. Piping starts from the downstream side, when the hydraulic gradient at the exit end is greater than the critical gradient of the soil. To ensure that the piping does not occur, there must be a downstream pile and the exit gradient should be safe.

A general method of determining the functional relationship for confined flow problem was first introduced by Pavlovsky. If the all boundaries of the flow domain are completely defined, such flow is said to be confined. All the flow characteristics could be obtained once the function $w = f(z)$ was known. By Schwarz-Cristoffel transformation the flow region in each of

these planes can be mapped conformally onto the same half on an auxiliary t plane, yielding the function $z = f_1(t)$ and $w = f_2(t)$.

In the present study, using methods of fragments and conformal mapping, confined flow under a weir with a downstream cut off founded on a porous medium of finite depth with a highly porous slit in the foundation soil, has been analysed and the distribution of exit gradient, which is the prime cause of the piping, has been studied. The presence of a slit changes the distribution of exit gradient. It shifts the place vulnerable to piping from the sheet pile to its own location.

CONTENS

LIST OF SYMBOLS

e	=	soil void ratio
F	=	seepage force
$F(\theta,m)$	=	incomplete elliptical integral of the first kind without amplitudo θ
$F(\frac{\pi}{2},m)$	=	complete elliptical integral of the first kind without amplitudo θ
G_s	=	specific gravity of soil particles
h	=	total difference head
i	=	imaginary unit ($\sqrt{-1}$)
I_E	=	exit gradient
k	=	coefficient of permeability
L	=	Length of impervious floor
L_1	=	distance between downstream end of weir and top end of non homogeneous permeable soil
l	=	distance between downstream end of weir and bottom end of non homogeneous permeable soil
m	=	depth of non homogeneous permeable soil
$M_{1,}...,M_4$	=	complex constant
$N_{1,}\ N_2$	=	complex constant
n	=	soil porosity
q	=	quantity of seepage
s	=	depth of sheetpile
T	=	finite depth of homogeneous permeable soil
t	=	transformation plane
u	=	velocity in x direction
v	=	velocity in z direction

w = complex variable in w plane

W_s = weight of soil

z = complex variable in z plane

ϕ = velocity potential function

ψ = stream function

α = slope of non homogeneous permeable soil

δ = coefficient of seepage quantity

γ_w = unit weight of water

CHAPTER I

INTRODUCTION

1.1. General.

Hydraulic structure such as weir or barrage may either be founded on an impervious solid rock foundation or on a permeable foundation. Whenever a weir is constructed on permeable foundation, it is subjected to seepage of water beneath the structure. The water seeping below the body of a weir may cause failure of the structure due to piping. When the seepage water retains sufficient residual force at the downstream end, it may lift up the soil particles and increase the the flow channel by progressive removal of soil from the downstream end toward the upstream end of weir.

For the soil to remain stable, the seepage force should be less than the submerged weight of soil or the gradient of water pressure which is called the exit gradient should be less than the safe exit gradient. Therefore, it is necessary to provide downstream sheetpile in order to reduce the exit gradient.

The permeable soil foundation of a weir is not always homogeneous. Sometime it consists of another soil of different permeability. If there is another permeable soil at the downstream of a weir which is more porous or its permeability is higher than the permeability of soil foundation, it is possible the piping will occur through that layer.

1.2. Objectives of the study.

The objectives of the study is to investigate the piping on the downstream of weir, where there is an inclined soil layer of high permeability or more porous than the permeable soil foundation, as shown in fig.I-1.

The weir is assumed to rest on permeable soil foundation of finite depth. An approximate analytical method of solution for any confined flow system of finite depth, directly applicable to design, was furnished by Pavlovsky. The fundamental assumtion of this method, called method of fragments, is that equipotential lines at various critical parts of flow region can be approximated by straight vertical lines that divide the region into sections or fragments.

In the thesis the effect of the presence of a thin highly porous layer, which would act as a conduit, on exit gradient distribution has been studied using method of fragments.

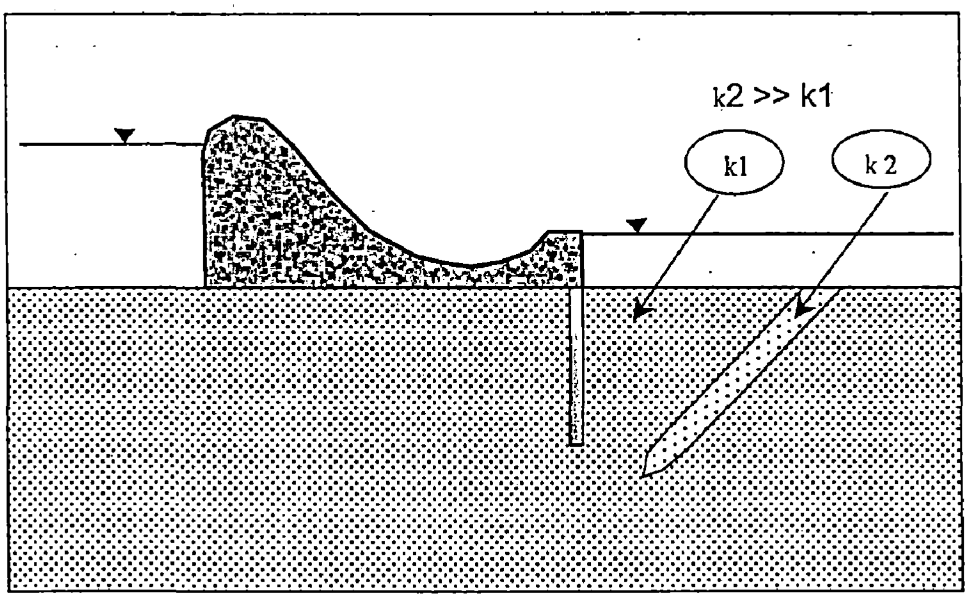


Fig. I.1

CHAPTER II

LITERATURE REVIEW

2.1. Two Dimensional Flow

Physically, all flow systems extend in three dimensions. How ever, in many problems the features of groundwater motion are essentially planar, with the motion being substantially the same in parallel planes. For this problem, the flow system can be simplified as two-dimensional flow.

In general, Darcy's law may be written as:

$$v = -k\frac{dh}{ds} \tag{2.1.1}$$

and velocity components in x-y plane can be derived as :

$$u = -k\frac{\partial h}{\partial x} = \frac{\partial \phi}{\partial x} \tag{2.1.2}$$

$$v = -k\frac{\partial h}{\partial y} = \frac{\partial \phi}{\partial y} \tag{2.1.3}$$

where :

k = coefficient of permeability of isotropic soil

h = total head = $\left(\frac{p}{\gamma_w} + y\right)$

ϕ = velocity potential = -kh + c

If the coefficient of permeability is independent of direction of velocity, the soil is said to be an isotropic. Moreover, if the soil has the same coefficient of permeability at all points within the region of flow, the soil is said to be homogeneous and isotropic.

In non homogeneous and isotropic soil the coefficient of permeability is independent on the direction of velocity but dependent on the space coordinate.

2.2. Steady Flow

For steady flow there is no change of velocity with respect to time and the equation of continuity becomes :

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \qquad \text{or} \tag{2.2.1}$$

$$\frac{\partial^2 \phi}{\partial x^2}+\frac{\partial^2 \phi}{\partial y^2}=0 \tag{2.2.2}$$

and this expression is known as the Laplace's equation for two-dimensional flow.

The velocity potential curves $\phi(x,y)$=constant are orthogonal trajectories of the stream function curves $\psi(x,y)$=constant, and the flow velocity components can be defined as :

$$u=\frac{\partial \phi}{\partial x}=\frac{\partial \psi}{\partial y} \tag{2.2.3}$$

$$v=\frac{\partial \phi}{\partial y}=-\frac{\partial \psi}{\partial x} \tag{2.2.4}$$

and the equation of continuity become :

$$\frac{\partial^2 \psi}{\partial x \partial y}-\frac{\partial^2 \psi}{\partial y \partial x}=0 \tag{2.2.5}$$

The relation of ϕ and ψ can be found from the Cauchy-Riemann equations as follows :

$$\phi=\int\left(\frac{\partial \psi}{\partial y}dx-\frac{\partial \psi}{\partial x}dy\right) \tag{2.2.6}$$

$$\psi=\int\left(\frac{\partial \phi}{\partial x}dy-\frac{\partial \phi}{\partial y}dx\right) \tag{2.2.7}$$

A combination of the function ϕ and ψ is called *complex potential* and defined by :

$$w=\phi+i\psi \tag{2.2.8}$$

2.3. Schwarz Cristoffel Transformation

The shape of the flow net depends on the configuration of impervious floor and the homogeneity of soil permeability in the flow region. In practical case where the sheet pile is provided below the weir or if there is non homogeneous soil in the flow region, the flow net will be distorted. The streamline and equipotential line do not consist of confocal ellipse and hyperbolas respectively, as in the case of a horizontal floor on homogeneous permeable foundation.

The distorted flow can be mapped conformally onto the upper half of *t-plane* by using Schwarz-Cristoffel transformation. The Schwarz-Christoffel Transformation is the method of mapping from one or more planes onto the upper half of another plane as shown in fig 2.1.

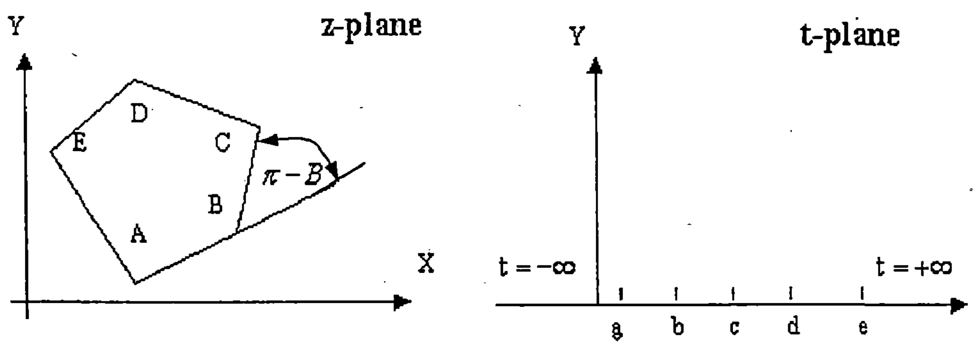


Fig 2.1

If a polygon is located in z plane, then the transformation that maps it conformally onto the upper half of the t plane (t=r+is) is :

$$z = M\int \frac{dt}{(t-a)^{1-\frac{A}{\pi}}(t-b)^{1-\frac{B}{\pi}}(t-c)^{1-\frac{C}{\pi}}+\dots} + N \qquad (2.3.1)$$

The equation above is called *Schwarz Cristoffel Transformation,* where:

- M and N are complex constants.
- A,B,C,...are the interior angle (rad) of the polygon in the z-plane.

- a,b,c,...(a<b<c...) are points on the real axis of the t plane corresponding to the respective vertices A,B,C,...

The complex constant N correspondents to the point on the perimeter of the polygon that has its image at t=0.

2.4. Critical Gradient

The seepage water exerts a force on soil particles. This force (F) acts in the direction of flow or tangential to the streamline if the soil is isotropic. The force is known as *seepage force.* The seepage force per unit volume is proportional to the hydraulic gradient at that point, denoted by :

$$F = \gamma_w \left(\frac{\partial h}{\partial s} \right) \tag{2.4.1}$$

The seepage force has an upward component when the flow line turn upward. At the downstream end the flow line emerge vertically, because it has to be orthogonal to the equipotential at the downstream bed. Therefore, at the exit end the seepage forces acts vertically upward, as shown in Fig.2.1.

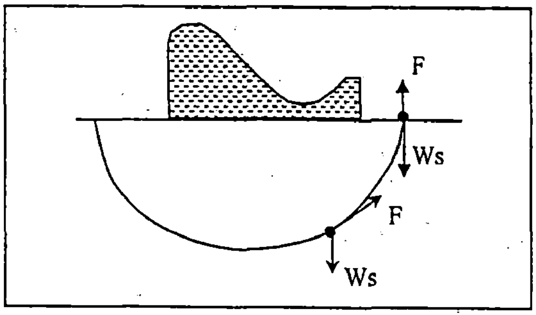


Fig. 2.1. Seepage Force

The soil remains stable and there will be no piping if the downward force due to submerged weight of the soil (Ws') is equal to or greater than the seepage force. If the seepage force exceeds the downstream force, the piping will occur. The submerged unit weight of soil is given by :

$$Ws' = \left(\frac{Gs-1}{1+e}\right)\gamma_w = \gamma_w(1-n)(Gs-1) \qquad (2.4.2)$$

where : Gs = the specific gravity of soil particles

n = the porosity

e = the void ratio

γ_w = the specific weight of water

In the critical condition, the upward force will be just balanced by the submerged unit weight of soil. Thus from the equation (2.4.1) and (2.4.2) the critical gradient I_{cr} will be found as follows :

$$F = Ws'$$

$$\gamma_w\left(\frac{\partial h}{\partial s}\right) = \gamma_w(1-n)(Gs-1)$$

$$\left(\frac{\partial h}{\partial s}\right) = (1-n)(Gs-1)$$

$$I_{cr} = (1-n)(Gs-1) \qquad (2.4.3)$$

CHAPTER III

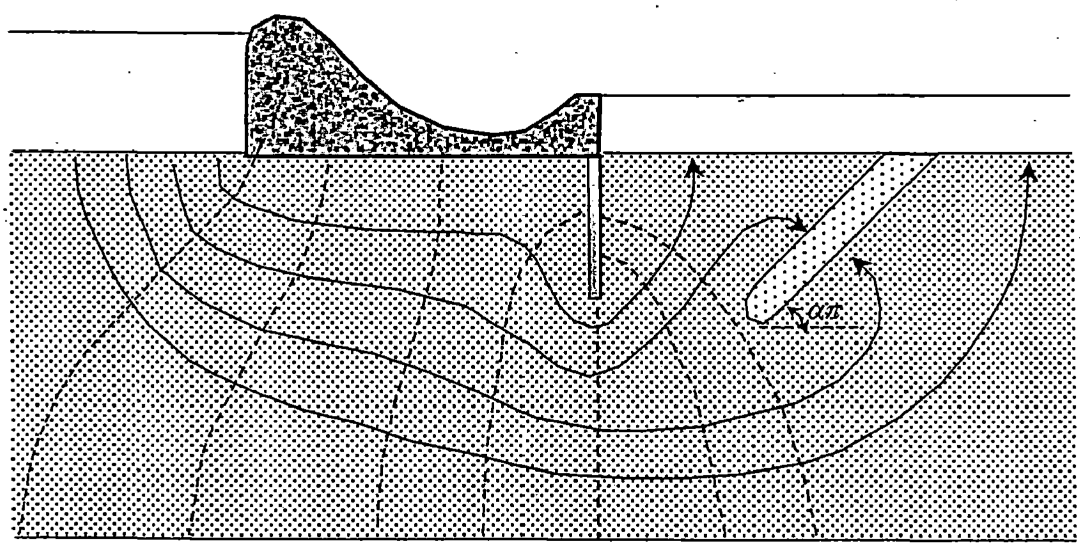
ANALYSIS

3.1. Statement of problem.

A weir with a downstream sheet pile rests on permeable soil foundation of finite depth. At the downstream side of the weir there is a soil of high permeability and inclined at an angle $\alpha\pi$.

The water flows from the upstream side to the downstream side. Some quantity of the flow come out to the downstream surface and others come out through the inclined layer of high permeability. The inclined highly porous layer acts as a constant head boundary. The flow lines are perpendicular to the inclined equipotential line as shown in Fig.3.1.

For computation the confined flow of finite depth, the flow domain is devided into two fragment as shown in Fig.3.2.



Flow domain with a highly draining layer

Fig. 3.1 Flow Net

3.2. Seepage Flow Analysis.

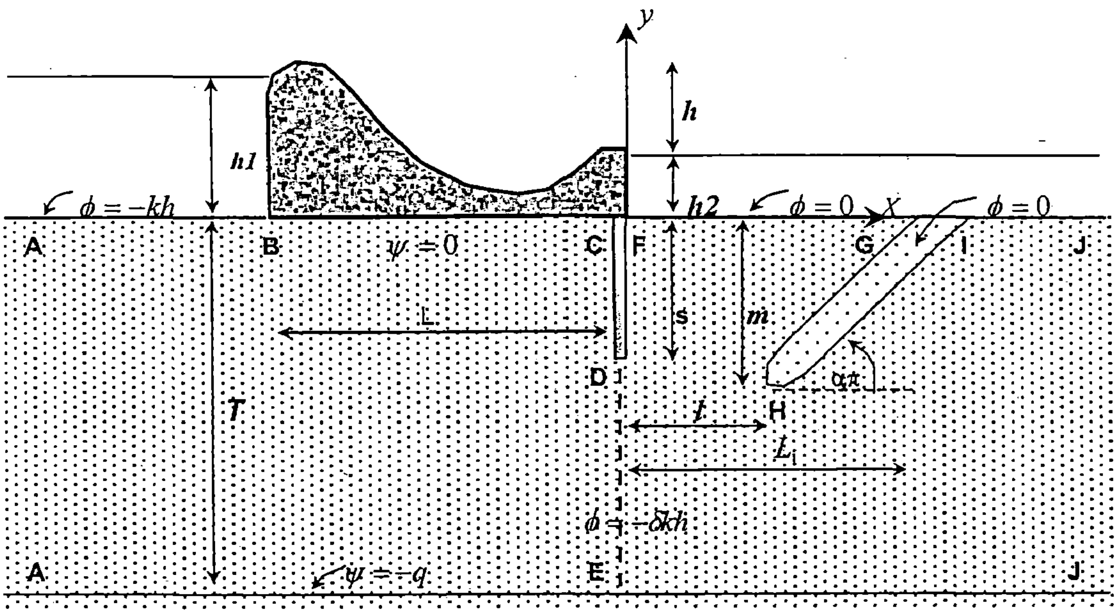
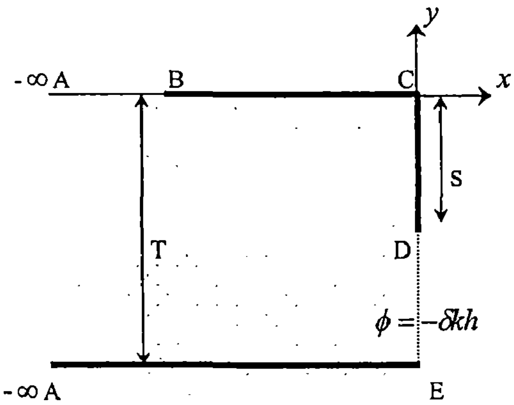


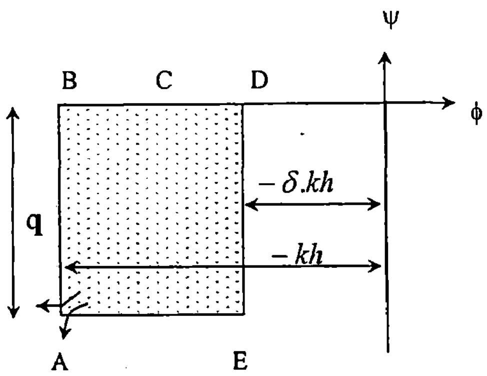
Fig. 3.2

3.2.1. Fragment I.



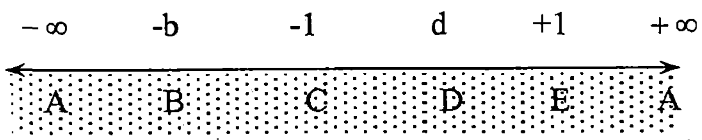
z-plane

Fig. 3.3



w-plane

Fig. 3.4



t-plane

Fig.3.5

Applying Schwarz–Cristoffel transformation, the conformal mapping of **fragment I** in ***z-plane*** onto the auxialiary ***t-plane*** *is given by :*

$$z = M_1 \int \frac{dt}{(1+t)^{1/2}(1-t)^{1/2}} + N_1$$

After integrating :

$$z = M_1 \sin^{-1} t + N_1 \qquad (3.1.1)$$

At point C : t = -1 and $z = 0$

$$0 = M_1 \sin^{-1}(-1) + N_1$$

$$N_1 = M_1\left(\frac{\pi}{2}\right) \qquad (3.1.2)$$

At point E : $t = 1$ and $z = -iT$

$$-iT = M_1 \sin^{-1}(1) + M_1(\frac{\pi}{2})$$

$$M_1 = -\frac{iT}{\pi} \qquad (3.1.3)$$

Subtituting M from (3.1.3) in (3,1.2) the constant N_1 is found to be :

$$N_1 = -\frac{iT}{2} \qquad (3.1.4)$$

Hence :

$$z = -\frac{iT}{\pi}\sin^{-1}(t) - \frac{iT}{2} \qquad (3.1.5)$$

At Point B : $t = -b$ and $z = -L$

$$-L = -\frac{iT}{\pi}\sin^{-1}(-b) - \frac{iT}{2}$$

$$-L = \frac{iT}{\pi}\sin^{-1}(b) - \frac{iT}{2} \qquad (3.1.6)$$

Let $\sin^{-1}(b)$ be equal to θ and b > 1. $Sin^{-1}(b)$ is derived as follows :

$$e^{i\theta} = i(b \pm \sqrt{b^2 - 1}) = e^{\frac{i\pi}{2}}(b \pm \sqrt{b^2 - 1})$$

Taking logarithm on either side

$$i\theta = \frac{i\pi}{2} + \ln(b \pm \sqrt{b^2 - 1})$$

$$\theta = \frac{\pi}{2} + \frac{1}{i}\ln(b \pm \sqrt{b^2 - 1}) \qquad (3.1.7)$$

Subtituting θ from (3.1.7) in (3.1.6) :

$$-L = \frac{iT}{\pi}\left\{\frac{\pi}{2} + \frac{1}{i}\ln(b \pm \sqrt{b^2 - 1})\right\} - \frac{iT}{2}$$

$$-L = \frac{T}{\pi}\ln(b \pm \sqrt{b^2 - 1})$$

$$\frac{L}{T} = -\frac{1}{\pi}\ln(b - \sqrt{b^2 - 1}) \qquad (3.1.8)$$

From (3.1.8) the value of parameter b can be found by iteration.

At Point D : $t = \mathrm{d}$, $z = -\mathrm{is}$

$$-is = -\frac{iT}{\pi}\sin^{-1}(d) - \frac{iT}{2}$$

$$d = \sin\left(\frac{s\pi}{T} - \frac{\pi}{2}\right)$$

$$= -\cos\left(\frac{s\pi}{T}\right) \qquad (3.1.9)$$

Applying Schwarz-Christoffel transformation, the conformal mapping of ***w-plane*** onto ***t- plan*** is given by :

For $(-\infty < t' \le -b)$:

$$w = M_2 \int_{-\infty}^{t'} \frac{dt}{\sqrt{(-b-t)(d-t)(1-t)}} - kh - iq$$

$$= M_2 \frac{2}{\sqrt{1+b}} \int_0^{\sqrt{\frac{1+b}{1-t'}}} \frac{d\theta}{\sqrt{1-m_1{}^2 \sin^2\theta}} - kh - iq$$

$$= M_2 \frac{2}{\sqrt{1+b}} F(\theta_1, m_1) - kh - iq \qquad (3.2.1)$$

in which : $\theta_1 = \sin^{-1} = \sqrt{\frac{1+b}{1-t'}}$ and $m_1 = \sqrt{\frac{1-d}{1+b}}$

At point B : $t' = -b$ and $w = -kh$

$$-kh = M_2 \frac{2}{\sqrt{1+b}} F(\pi/2, m_1) - kh - iq$$

$$q = M_2 \frac{2F(\pi/2, m_1)}{i\sqrt{1+b}} \qquad (3.2.2)$$

For $(-b < t' \le d)$:

$$w = M_2 \frac{1}{i} \int_{-\infty}^{t'} \frac{dt}{\sqrt{(b+t)(d-t)(1-t)}} - kh$$

$$= M_2 \frac{2}{i\sqrt{1+b}} \int_0^{\sqrt{\frac{t'+b}{d+b}}} \frac{d\theta}{\sqrt{1-m_2{}^2 \sin^2\theta}} - kh$$

$$= M_2 \frac{2}{i\sqrt{1+b}} F(\theta_2, m_2) - kh \qquad (3.2.3)$$

in which : $\theta_2 = \sin^{-1}\sqrt{\frac{t'+b}{d+b}}$ and $m_2 = \sqrt{\frac{d+b}{1+b}}$

At point D : $t' = d$ and $w = -\delta kh$

$$-\delta kh = M_2 \frac{2}{i\sqrt{1+b}} F(\pi/2, m_2) - kh$$

$$(1-\delta)kh = M_2 \frac{2}{i\sqrt{1+b}} F(\pi/2, m_2)$$

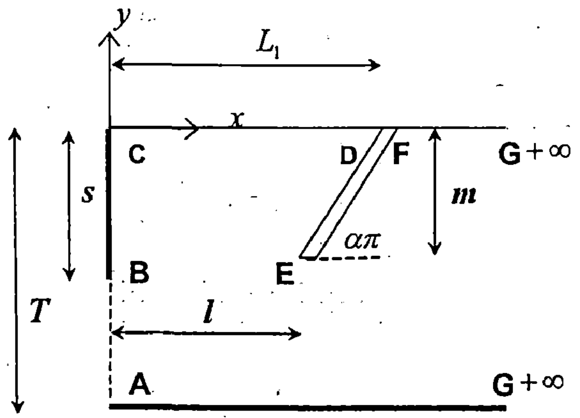
$$M_2 = (1-\delta)kh \frac{i\sqrt{1+b}}{2F(\pi/2, m_2)} \qquad (3.2.4)$$

Subtituting M_2 from (3.2.2) in (3.2.4)

$$q = \left[(1-\delta)kh \frac{i\sqrt{1+b}}{2F(\pi/2, m_2)} \right] \frac{2F(\pi/2, m_1)}{i\sqrt{1+b}}$$

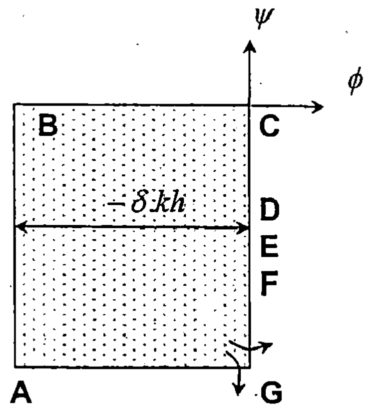
$$q = (1-\delta)kh \frac{F(\pi/2, m_1)}{F(\pi/2, m_2)} \qquad (3.2.5)$$

3.2.2. Fragment II.



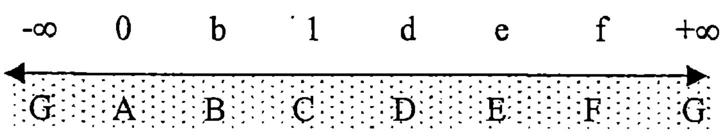
z-plane

Fig. 3.6



w-plane

Fig. 3.7



t-plane

Fig. 3.8

Applying Schwarz–Cristoffel transformation, the conformal mapping of **fragment II** in ***z-plane*** onto the auxiliary ***t-plane*** is given by :

$$z = M_3 \int_0^{t'} \frac{(t-e)dt}{t^{1/2}(1-t)^{1/2}(d-t)^{1-\alpha}(f-t)^{\alpha}} + N_3$$

$$= M_3 \left[\int_0^{t'} \frac{t^{1/2}\,dt}{(1-t)^{1/2}(d-t)^{1-\alpha}(f-t)^{\alpha}} - e\int_0^{t'} \frac{dt}{t^{1/2}(1-t)^{1/2}(d-t)^{1-\alpha}(f-t)^{\alpha}} \right] + N_3 \qquad (3.3.1)$$

At point A : $t' = 0$ and $z = -iT$, hence $N_3 = -iT$ (3.3.2)

At point C : $t = 1$ and $z = 0$

$$0 = M_3\left[\int_0^1 \frac{t^{1/2}\,dt}{(1-t)^{1/2}(d-t)^{1-\alpha}(f-t)^{\alpha}} - e\int_0^1 \frac{dt}{t^{1/2}(1-t)^{1/2}(d-t)^{1-\alpha}(f-t)^{\alpha}}\right] - iT$$

$$iT = M_3\left[\int_0^1 \frac{t^{1/2}\,dt}{(1-t)^{1/2}(d-t)^{1-\alpha}(f-t)^{\alpha}} - e\int_0^1 \frac{dt}{t^{1/2}(1-t)^{1/2}(d-t)^{1-\alpha}(f-t)^{\alpha}}\right]$$

$$iT = M_3(I_1 - e\,I_2)$$

$$M_3 = \frac{iT}{(I_1 - e\,I_2)} \qquad (3.3.3)$$

$$I_1 = \int_0^1 \frac{t^{1/2}\,dt}{(1-t)^{1/2}(d-t)^{1-\alpha}(f-t)^{\alpha}}$$

Let assume :

$1 - t = v^2$ $\qquad t = 1 - v^2$ $\qquad dt = -\,2v\,dv$

at : $t = 0$ $\qquad v = 1$

$t = 1$ $\qquad v = 0$

$$I_1 = \int_0^1 \frac{(1-v^2)^{1/2}(2v\,dv)}{\left[1-(1-v^2)\right]^{1/2}\left[d-(1-v^2)\right]^{1-\alpha}\left[f-(1-v^2)\right]^{\alpha}}$$

$$= \int_0^1 \frac{2(1-v^2)^{1/2}\,dv}{(d-1+v^2)^{1-\alpha}(f-1+v^2)^{\alpha}}$$

Further assume :

$v = \frac{1}{2}(1+u)$ $\qquad dv = \frac{1}{2}du$

$$I_1 = \int_{-1}^{1} \frac{\left[1-\frac{1}{4}(1+u)^2\right]^{1/2} du}{\left[d-1+\frac{1}{4}(1+u)^2\right]^{1-\alpha}\left[f-1+\frac{1}{4}(1+u)^2\right]^{\alpha}} \qquad (3.3.4)$$

$$I_2 = \int_0^1 \frac{dt}{t^{1/2}(1-t)^{1/2}(d-t)^{1-\alpha}(f-t)^{\alpha}}$$

$$= \int_0^{1/2} \frac{dt}{t^{1/2}(1-t)^{1/2}(d-t)^{1-\alpha}(f-t)^{\alpha}} + \int_{1/2}^1 \frac{dt}{t^{1/2}(1-t)^{1/2}(d-t)^{1-\alpha}(f-t)^{\alpha}}$$

$$I_2 = I_{21} + I_{22} \qquad (3.3.5)$$

$$I_{21} = \int_0^{1/2} \frac{dt}{t^{1/2}(1-t)^{1/2}(d-t)^{1-\alpha}(f-t)^{\alpha}}$$

Let us assume :

$t = v^2$ $\qquad dt = 2vdv$

at : $\quad t = 0 \qquad v = 0$

$\qquad t = 1/2 \qquad v = \sqrt{1/2}$

$$I_{21} = \int_0^{\sqrt{1/2}} \frac{2vdv}{\left(v^2\right)^{1/2}\left(1-v^2\right)^{1/2}\left(d-v^2\right)^{1-\alpha}\left(f-v^2\right)^{\alpha}}$$

$$= \int_0^{\sqrt{1/2}} \frac{2dv}{\left(1-v^2\right)^{1/2}\left(d-v^2\right)^{1-\alpha}\left(f-v^2\right)^{\alpha}}$$

Further assume :

$v = \frac{1}{2}\left(\frac{1}{2}\right)^{1/2}(1+u)$ $\qquad dv = \frac{1}{2}\left(\frac{1}{2}\right)^{1/2} du$

at : $\quad v = 0 \qquad u = -1$

$\qquad v = \left(\frac{1}{2}\right)^{1/2} \qquad u = 1$

$$I_{21} = \int_{-1}^{1} \frac{\left(\frac{1}{2}\right)^{1/2} du}{\left(1-\frac{1}{8}(1+u)^2\right)^{1/2}\left(d-\frac{1}{8}(1+u)^2\right)^{1-\alpha}\left(f-\frac{1}{8}(1+u)^2\right)^{\alpha}} \qquad (3.3.5a)$$

$$I_{22} = \int_{1/2}^{1} \frac{dt}{t^{1/2}(1-t)^{1/2}(d-t)^{1-\alpha}(f-t)^{\alpha}}$$

Let us assume :

$1-t = v^2$ $\qquad t = 1-v^2$ $\qquad dt = -2vdv$

at : $t = 1/2$ $\qquad v = \sqrt{1/2}$

$t = 1$ $\qquad v = 0$

$$I_{22} = \int_{0}^{\sqrt{1/2}} \frac{2vdv}{\left(1-v^2\right)^{1/2}\left[1-\left(1-v^2\right)\right]^{1/2}\left[d-\left(1-v^2\right)\right]^{1-\alpha}\left[f-\left(1-v^2\right)\right]^{\alpha}}$$

$$= \int_{0}^{\sqrt{1/2}} \frac{2dv}{\left(1-v^2\right)^{1/2}\left(d-1+v^2\right)^{1-\alpha}\left(f-1+v^2\right)^{\alpha}}$$

Further assume :

$v = \frac{1}{2}\left(\frac{1}{2}\right)^{1/2}(1+u)$ $\qquad dv = \frac{1}{2}\left(\frac{1}{2}\right)^{1/2} du$

at : $v = 0$ $\qquad u = -1$

$v = \left(\frac{1}{2}\right)^{1/2}$ $\qquad u = 1$

$$I_{22} = \int_{-1}^{1} \frac{\left(\frac{1}{2}\right)^{1/2} du}{\left[1-\frac{1}{8}(1+u)^2\right]^{1/2}\left[d-1+\frac{1}{8}(1+u)^2\right]^{1-\alpha}\left[f-1+\frac{1}{8}(1+u)^2\right]^{\alpha}} \qquad (3.3.5b)$$

At point D : $t = d$ and $z = L_1$

$$L_1 = M_3\left[\int_1^d \frac{t^{1/2}\,dt}{(1-t)^{1/2}(d-t)^{1-\alpha}(f-t)^{\alpha}} - e\int_1^d \frac{dt}{t^{1/2}(1-t)^{1/2}(d-t)^{1-\alpha}(f-t)^{\alpha}}\right]$$

$$= \frac{M_3}{i}\left[\int_1^d \frac{t^{1/2}\,dt}{(t-1)^{1/2}(d-t)^{1-\alpha}(f-t)^{\alpha}} - e\int_1^d \frac{dt}{t^{1/2}(t-1)^{1/2}(d-t)^{1-\alpha}(f-t)^{\alpha}}\right]$$

$$= \frac{M_3}{i}\left(I_3 - e\,I_4\right) \qquad (3.3.6)$$

Subtituting M_3 from equation (3.3.3) in (3.3.6)

$$L_1 = \frac{1}{i}\frac{iT}{(I_1 - e\,I_2)}\left(I_3 - e\,I_4\right)$$

$$\frac{L_1}{T} = \frac{(I_3 - e\,I_4)}{(I_1 - e\,I_2)}$$

$$F_1 = \frac{L_1}{T} - \frac{(I_3 - e\,I_4)}{(I_1 - e\,I_2)} \qquad (3.3.7)$$

$$I_3 = \int_1^d \frac{t^{1/2}\,dt}{(t-1)^{1/2}(d-t)^{1-\alpha}(f-t)^{\alpha}}$$

$$= \int_1^{\frac{1+d}{2}} \frac{t^{1/2}\,dt}{(t-1)^{1/2}(d-t)^{1-\alpha}(f-t)^{\alpha}} + \int_{\frac{1+d}{2}}^{d} \frac{t^{1/2}\,dt}{(t-1)^{1/2}(d-t)^{1-\alpha}(f-t)^{\alpha}}$$

$$I_3 = I_{31} + I_{32} \qquad (3.3.8)$$

$$I_{31} = \int_{1}^{\frac{1+d}{2}} \frac{t^{1/2}\, dt}{(t-1)^{1/2}(d-t)^{1-\alpha}(f-t)^{\alpha}}$$

Let us assume :

$t-1=v^2$ $\qquad$ $t=1+v^2$ $\qquad$ $dt = 2vdv$

at : $\quad t=1 \qquad v=0$

$\quad t=\frac{1+d}{2} \qquad v=\sqrt{\frac{d-1}{2}}$

$$I_{31} = \int_{0}^{\sqrt{\frac{d-1}{2}}} \frac{\left(1+v^2\right)^{1/2}(2vdv)}{\left[\left(1+v^2\right)-1\right]^{1/2}\left[d-\left(1+v^2\right)\right]^{1-\alpha}\left[f-\left(1+v^2\right)\right]^{\alpha}}$$

$$= \int_{0}^{\sqrt{\frac{d-1}{2}}} \frac{2\left(1+v^2\right)^{1/2} dv}{\left(d-1-v^2\right)^{1-\alpha}\left(f-1-v^2\right)^{\alpha}}$$

Further assume :

$v=\frac{1}{2}\left(\frac{d-1}{2}\right)^{1/2}(1+u) \qquad dv=\frac{1}{2}\left(\frac{d-1}{2}\right)^{1/2} du$

at : $\quad v=0 \qquad u=-1$

$\quad v=\left(\frac{d-1}{2}\right)^{1/2} \qquad u=1$

$$I_{31} = \int_{-1}^{1} \frac{\left(1+\frac{1}{8}(d-1)(1+u)^2\right)^{1/2}\left(\frac{d-1}{2}\right)^{1/2} du}{\left[d-1-\frac{1}{8}(d-1)(1+u)^2\right]^{1-\alpha}\left[f-1-\frac{1}{8}(d-1)(1+u)^2\right]^{\alpha}} \qquad (3.3.8a)$$

$$I_{32} = \int_{\frac{1+d}{2}}^{d} \frac{t^{1/2}\, dt}{(t-1)^{1/2}(d-t)^{1-\alpha}(f-t)^{\alpha}}$$

Let us assume :

$$d - t = v^{p} \qquad t = d - v^{p} \qquad dt = -p v^{p-1} dv$$

at : $t = \dfrac{1+d}{2}$ $\qquad v = \left(\dfrac{d-1}{2}\right)^{1/p}$

$t = d$ $\qquad v = 0$

$$I_{32} = \int_{0}^{\left(\frac{d-1}{2}\right)^{1/p}} \frac{p\left(d - v^{p}\right)^{1/2} v^{p-1}\, dv}{\left[\left(d - v^{p}\right)-1\right]^{1/2}\left[d-\left(d-v^{p}\right)\right]^{1-\alpha}\left[f-\left(d-v^{p}\right)\right]^{\alpha}}$$

$$= \int_{0}^{\left(\frac{d-1}{2}\right)^{1/p}} \frac{p\left(d - v^{p}\right)^{1/2} v^{\alpha p-1}\, dv}{\left(d - v^{p} - 1\right)^{1/2}\left(f - d + v^{p}\right)^{\alpha}}$$

LET : $\alpha p - 1 = 0$ $\qquad p = \dfrac{1}{\alpha}$ $\qquad 0 < \alpha < 1$

For $0 < \alpha < 1$

$$I_{32} = \int_{0}^{\left(\frac{d-1}{2}\right)^{\alpha}} \frac{\frac{1}{\alpha}\left(d - v^{\frac{1}{\alpha}}\right)^{1/2} dv}{\left(d - v^{\frac{1}{\alpha}} - 1\right)^{1/2}\left(f - d + v^{\frac{1}{\alpha}}\right)^{\alpha}}$$

Further assume :

$$v = \frac{1}{2}\left(\frac{d-1}{2}\right)^{\alpha}(1+u) \qquad dv = \frac{1}{2}\left(\frac{d-1}{2}\right)^{\alpha} du$$

$$I_{32} = \int_{-1}^{1} \frac{\frac{1}{2\alpha}\left(d - \left[\frac{1}{2}\left(\frac{d-1}{2}\right)^{\alpha}(1+u)\right]^{\frac{1}{\alpha}}\right)^{1/2}\left(\frac{d-1}{2}\right)^{\alpha} du}{\left(d - \left[\frac{1}{2}\left(\frac{d-1}{2}\right)^{\alpha}(1+u)\right]^{\frac{1}{\alpha}} - 1\right)^{1/2}\left(f - d + \left[\frac{1}{2}\left(\frac{d-1}{2}\right)^{\alpha}(1+u)\right]^{\frac{1}{\alpha}}\right)^{\alpha}} \qquad (3.3.8b)$$

$$I_4 = \int_1^d \frac{dt}{t^{1/2}(t-1)^{1/2}(d-t)^{1-\alpha}(f-t)^\alpha}$$

$$= \int_1^{\frac{1+d}{2}} \frac{dt}{t^{1/2}(t-1)^{1/2}(d-t)^{1-\alpha}(f-t)^\alpha} + \int_{\frac{1+d}{2}}^{d} \frac{dt}{t^{1/2}(t-1)^{1/2}(d-t)^{1-\alpha}(f-t)^\alpha}$$

$$I_4 = I_{41} + I_{42} \tag{3.3.9}$$

$$I_{41} = \int_1^{\frac{1+d}{2}} \frac{dt}{t^{1/2}(t-1)^{1/2}(d-t)^{1-\alpha}(f-t)^\alpha}$$

Let us assume :

$t-1=v^2$ $\quad$ $t=1+v^2$ $\quad$ $dt = 2vdv$

at : $t = 1$ $\quad$ $v = 0$

$t = \dfrac{1+d}{2}$ $\quad$ $v = \sqrt{\dfrac{d-1}{2}}$

$$I_{41} = \int_0^{\sqrt{\frac{d-1}{2}}} \frac{2vdv}{\left(1+v^2\right)^{1/2}\left[\left(1+v^2\right)-1\right]^{1/2}\left[d-\left(1+v^2\right)\right]^{1-\alpha}\left[f-\left(1+v^2\right)\right]^\alpha}$$

$$= \int_0^{\sqrt{\frac{d-1}{2}}} \frac{2dv}{\left(1+v^2\right)^{1/2}\left(d-1-v^2\right)^{1-\alpha}\left(f-1-v^2\right)^\alpha}$$

Further assume :

$v = \dfrac{1}{2}\left(\dfrac{d-1}{2}\right)^{1/2}(1+u)$ $\quad$ $dv = \dfrac{1}{2}\left(\dfrac{d-1}{2}\right)^{1/2} du$

at : $v = 0$ $\quad$ $u = -1$

$v = \left(\dfrac{d-1}{2}\right)^{1/2}$ $\quad$ $u = 1$

$$I_{41} = \int_{-1}^{1} \frac{\left(\frac{d-1}{2}\right)^{1/2} du}{\left[1+\frac{1}{8}(d-1)(1+u)^2\right]^{1/2}\left[d-1-\frac{1}{8}(d-1)(1+u)^2\right]^{1-\alpha}\left[f-1-\frac{1}{8}(d-1)(1+u)^2\right]^\alpha}$$

(3.3.9a)

$$I_{42} = \int_{\frac{1+d}{2}}^{d} \frac{dt}{t^{1/2}(t-1)^{1/2}(d-t)^{1-\alpha}(f-t)^{\alpha}}$$

Let us assume :

$d - t = v^p \qquad t = d - v^p \qquad dt = -pv^{p-1}dv$

at : $t = \dfrac{1+d}{2} \qquad v = \left(\dfrac{d-1}{2}\right)^{1/p}$

$t = d \qquad v = 0$

$$I_{42} = \int_0^{\left(\frac{d-1}{2}\right)^{1/p}} \frac{p\, v^{p-1}\, dv}{(d-v^p)^{1/2}\left[(d-v^p)-1\right]^{1/2}\left[d-(d-v^p)\right]^{1-\alpha}\left[f-(d-v^p)\right]^{\alpha}}$$

$$= \int_0^{\left(\frac{d-1}{2}\right)^{1/p}} \frac{p\, v^{\alpha p-1}\, dv}{(d-v^p)^{1/2}(d-v^p-1)^{1/2}(f-d+v^p)^{\alpha}}$$

LET : $\alpha p - 1 = 0 \qquad p = \dfrac{1}{\alpha} \qquad 0 < \alpha < 1$

$$I_{42} = \int_0^{\left(\frac{d-1}{2}\right)^{\alpha}} \frac{\frac{1}{\alpha}\, dv}{\left(d - v^{\frac{1}{\alpha}}\right)^{1/2}\left(d - v^{\frac{1}{\alpha}} - 1\right)^{1/2}\left(f - d + v^{\frac{1}{\alpha}}\right)^{\alpha}}$$

Further assume :

$$v = \frac{1}{2}\left(\frac{d-1}{2}\right)^{\alpha}(1+u) \qquad dv = \frac{1}{2}\left(\frac{d-1}{2}\right)^{\alpha} du$$

$$I_{42} = \int_{-1}^{1} \frac{\frac{1}{2\alpha}\left(\frac{d-1}{2}\right)^{\frac{1}{\alpha}} du}{\left(d - \left[\frac{1}{2}\left(\frac{d-1}{2}\right)^{\alpha}(1+u)\right]^{\frac{1}{\alpha}}\right)^{1/2}\left(d - \left[\frac{1}{2}\left(\frac{d-1}{2}\right)^{\alpha}(1+u)\right]^{\frac{1}{\alpha}} - 1\right)^{1/2}\left(f - d + \left[\frac{1}{2}\left(\frac{d-1}{2}\right)^{\alpha}(1+u)\right]^{\frac{1}{\alpha}}\right)^{\alpha}}$$

(3.3.9b)

$$\lim_{R\to\infty} M_3 \int_{\pi}^{2\pi} \frac{\left(\mathrm{Re}^{i\theta}-e\right)\ \mathrm{Re}^{i\theta}\, id\theta}{\left(\mathrm{Re}^{i\theta}\right)^{1/2}\left(1-\mathrm{Re}^{i\theta}\right)^{1/2}\left(d-\mathrm{Re}^{i\theta}\right)^{1-\alpha}\left(f-\mathrm{Re}^{i\theta}\right)^{\alpha}} = iT$$

$$M_3 \int_{\pi}^{2\pi} \frac{e^{2i\theta}\, id\theta}{e^{\frac{i\theta}{2}}\left(-e^{i\theta}\right)^{1/2}\left(-e^{i\theta}\right)^{1-\alpha}\left(-e^{i\theta}\right)^{\alpha}} = iT$$

$$M_3 \int_{\pi}^{2\pi} \frac{e^{2i\theta-\frac{i\theta}{2}-\frac{i\theta}{2}-i\theta(1-\alpha)-i\theta\alpha}\, id\theta}{(-1)^{1/2}(-1)^{1-\alpha}(-1)^{\alpha}} = iT$$

$$M_3 \int_{\pi}^{2\pi} \frac{id\theta}{(-1)^{3/2}} = iT$$

$$M_3\,\pi = (-1)^{3/2}\,T$$

$$M_3 = \pm\frac{iT}{\pi}$$

Let : $M_3 = -\dfrac{iT}{\pi}$ (3.3.10)

Subtituting M_3 from (3.3.3) in (3.3.10)

$$\frac{iT}{(I_1 - e\,I_2)} = -\frac{iT}{\pi}$$

$$e = \frac{I_1 + \pi}{I_2} \tag{3.3.11}$$

At point E : $t = e$ and $z = l - im$

$$l - im = M_3\left[\int_d^e \frac{t^{1/2}\,dt}{(1-t)^{1/2}(d-t)^{1-\alpha}(f-t)^{\alpha}} - e\int_d^e \frac{dt}{t^{1/2}(1-t)^{1/2}(d-t)^{1-\alpha}(f-t)^{\alpha}}\right] + L_1$$

$$= \frac{M_3}{i(-1)^{1-\alpha}}\left[\int_d^e \frac{t^{1/2}\,dt}{(t-1)^{1/2}(t-d)^{1-\alpha}(f-t)^{\alpha}} - e\int_d^e \frac{dt}{t^{1/2}(t-1)^{1/2}(t-d)^{1-\alpha}(f-t)^{\alpha}}\right] + L_1$$

$$= \frac{1}{(-1)^{\frac{3}{2}-\alpha}}\left(\frac{-iT}{\pi}\right)(I_5 - eI_6) + L_1$$

$$= (-1)^{\alpha-\frac{3}{2}}(-i)\left(\frac{T}{\pi}\right)(I_5 - eI_6) + L_1$$

$$= (-1)^{\alpha-\frac{3}{2}}\ (i)\left(\frac{T}{\pi}\right)(eI_6 - I_5) + L_1$$

$$= (-\sin\alpha\pi \pm i\cos\alpha\pi)(i)\left(\frac{T}{\pi}\right)(eI_6 - I_5) + L_1$$

$$= (-i\sin\alpha\pi \pm \cos\alpha\pi)\left(\frac{T}{\pi}\right)(eI_6 - I_5) + L_1$$

$$l = (-\cos\alpha\pi)\left(\frac{T}{\pi}\right)(eI_6 - I_5) + L_1$$

$$F_2 = L_1 - l - \cos\alpha\pi)\left(\frac{T}{\pi}\right)(eI_6 - I_5) \qquad (3.3.12)$$

$$I_5 = M_3 \int_d^e \frac{t^{1/2}\, dt}{(t-1)^{1/2}(t-d)^{1-\alpha}(f-t)^{\alpha}}$$

Let us assume :

$t - d = v^p \qquad t = d + v^p \qquad dt = p\, v^{p-1} dv$

at : $\quad t = d \qquad v = 0$

$\qquad t = e \qquad v = (e-d)^{1/p}$

$$I_5 = \int_0^{(e-d)^{1/p}} \frac{(d+v^p)^{1/2}\; p\,v^{p-1} dv}{\left[(d+v^p)-1\right]^{1/2}\left[(d+v^p)-d\right]^{1-\alpha}\left[f-(d+v^p)\right]^{\alpha}}$$

$$= \int_0^{(e-d)^{1/p}} \frac{p(d+v^p)^{1/2} v^{\alpha p-1} dv}{(d+v^p-1)^{1/2}(f-d-v^p)^{\alpha}}$$

LET : $\qquad \alpha\, p - 1 = 0 \qquad p = \frac{1}{\alpha} \qquad 0 < \alpha < 1$

$$I_5 = \int_0^{(e-d)^{\alpha}} \frac{\frac{1}{\alpha}\left(d + v^{\frac{1}{\alpha}}\right)^{1/2} dv}{\left(d + v^{\frac{1}{\alpha}} - 1\right)^{1/2}\left(f - d - v^{\frac{1}{\alpha}}\right)^{\alpha}}$$

Further assume :

$v = \frac{1}{2}(e-d)^{\alpha}(1+u) \qquad dv = \frac{1}{2}(e-d)^{\alpha} du$

$$I_5 = \int_{-1}^{1} \frac{\frac{1}{2\alpha}\left\{d + \left[\frac{1}{2}(e-d)^{\alpha}(1+u)\right]^{\frac{1}{\alpha}}\right\}^{1/2} (e-d)^{\alpha} du}{\left\{d + \left[\frac{1}{2}(e-d)^{\alpha}(1+u)\right]^{\frac{1}{\alpha}} - 1\right\}^{1/2}\left\{f - d - \left[\frac{1}{2}(e-d)^{\alpha}(1+u)\right]^{\frac{1}{\alpha}}\right\}^{\alpha}} \qquad (3.3.13)$$

$$I_6 = \int_d^e \frac{dt}{t^{1/2}(t-1)^{1/2}(t-d)^{1-\alpha}(f-t)^{\alpha}}$$

Let us assume :

$t-d = v^p$ $\qquad t = d + v^p$ $\qquad dt = p v^{p-1} dv$

at : $\quad t = d \qquad v = 0$

$\qquad t = e \qquad v = (e-d)^{1/p}$

$$I_6 = \int_0^{(e-d)^{1/p}} \frac{p v^{p-1} dv}{(d+v^p)^{1/2}\left[(d+v^p)-1\right]^{1/2}\left[(d+v^p)-d\right]^{1-\alpha}\left[f-(d+v^p)\right]^{\alpha}}$$

$$= \int_0^{(e-d)^{1/p}} \frac{p v^{\alpha p-1} dv}{(d+v^p)^{1/2}(d+v^p-1)^{1/2}(f-d-v^p)^{\alpha}}$$

LET : $\qquad \alpha p - 1 = 0 \qquad p = \frac{1}{\alpha} \qquad 0 < \alpha < 1$

$$I_6 = \int_0^{(e-d)^{\alpha}} \frac{\frac{1}{\alpha} dv}{\left(d+v^{\frac{1}{\alpha}}\right)^{1/2}\left(d+v^{\frac{1}{\alpha}}-1\right)^{1/2}\left(f-d-v^{\frac{1}{\alpha}}\right)^{\alpha}}$$

Further assume :

$v = \frac{1}{2}(e-d)^{\alpha}(1+u) \qquad dv = \frac{1}{2}(e-d)^{\alpha} du$

$$I_6 = \int_{-1}^{1} \frac{\frac{1}{2\alpha}(e-d)^{\alpha} du}{\left\{d+\left[\frac{1}{2}(e-d)^{\alpha}(1+u)\right]^{\frac{1}{\alpha}}\right\}^{1/2}\left\{d+\left[\frac{1}{2}(e-d)^{\alpha}(1+u)\right]^{\frac{1}{\alpha}}-1\right\}^{1/2}\left\{f-d-\left[\frac{1}{2}(e-d)^{\alpha}(1+u)\right]^{\frac{1}{\alpha}}\right\}^{\alpha}}$$

(3.3.14)

The equation (3.3.7) and (3.3.12) contains the unknown parameter d, f and those equations are non-linier. Newton-Raphson technique has been used to solve the equations as explained in Appendix-I.
The solution is given by the Jacobian matrix.

$$\begin{bmatrix} \frac{\partial F_1}{\partial d} & \frac{\partial F_1}{\partial f} \\ \frac{\partial F_2}{\partial d} & \frac{\partial F_2}{\partial f} \end{bmatrix} \begin{bmatrix} \Delta d \\ \Delta f \end{bmatrix} = \begin{bmatrix} F_1(d,f) \\ F_2(d,f) \end{bmatrix}$$

At point B : $t = b$ and $z = -is$

$$-is = \left(\frac{-iT}{\pi}\right)\int_0^b \frac{(t-e)\,dt}{t^{1/2}(1-t)^{1/2}(d-t)^{1-\alpha}(f-t)^{\alpha}} - iT$$

$$s = \left(\frac{T}{\pi}\right)\int_0^b \frac{(t-e)\,dt}{t^{1/2}(1-t)^{1/2}(d-t)^{1-\alpha}(f-t)^{\alpha}} + T$$

$$(s-T)\left(\frac{\pi}{T}\right) = \int_0^b \frac{(t-e)\,dt}{t^{1/2}(1-t)^{1/2}(d-t)^{1-\alpha}(f-t)^{\alpha}}$$

$$\left(\frac{s}{T}-1\right)\pi = \int_0^b \frac{(t-e)\,dt}{t^{1/2}(1-t)^{1/2}(d-t)^{1-\alpha}(f-t)^{\alpha}}$$

$$\left(1-\frac{s}{T}\right)\pi = \int_0^b \frac{(e-t)\,dt}{t^{1/2}(1-t)^{1/2}(d-t)^{1-\alpha}(f-t)^{\alpha}} \qquad (3.3.15)$$

The value of parameter b can be found by iteration.

Applying Schwarz-Christoffel transformation, the conformal mapping of **segment II** in ***w- plane*** onto ***t plan*** is given by :

For $(-\infty < t' \leq 0)$:

$$w = M_4 \int_{-\infty}^{t'} \frac{dt}{(-t)^{1/2}(b-t)^{1/2}(1-t)^{1/2}} - iq$$

$$= M_4 2 \int_0^{\sqrt{\frac{1}{1-t'}}} \frac{d\theta}{\sqrt{1-m_3{}^2 \sin^2\theta}} - iq$$

$$= 2M_4 F(\theta_3, m_3) - iq \qquad (3.4.1)$$

in which : $\theta_3 = \sin^{-1} = \sqrt{\frac{1}{1-t}}$ and $m_3 = \sqrt{1-b}$

At point A : $t' = 0$ and $w = -\delta kh - iq$

$-\delta kh - iq = 2M_4 F(\pi/2, m_3) - iq$

$$M_4 = \frac{-\delta kh}{2F(\pi/2, m_3)} = \frac{-\delta kh}{2F(\pi/2, \sqrt{1-b})} \qquad (3.4.2)$$

For $(0 < t' \leq b)$:

$$w = M_4 \int_0^{t'} \frac{dt}{(-1)^{1/2}(t)^{1/2}(b-t)^{1/2}(1-t)^{1/2}} - iq - \delta kh$$

$$= M_4 \frac{2}{i} \int_0^{\sqrt{\frac{t'}{b}}} \frac{d\theta}{\sqrt{1-m_4{}^2 \sin^2\theta}} - iq - \delta kh$$

$$= \frac{2}{i} M_4 F(\theta_4, m_4) - iq - \delta kh$$

in which : $\theta_4 = \sin^{-1} = \sqrt{\frac{t'}{b}}$ and $m_4 = \sqrt{b}$

At point B : $t' = b$ and $w = -\delta kh$

$$-\delta kh = M_4 \frac{2}{i} F(\pi/2, m_4) - iq - \delta kh$$

$$q = -2M_4 F(\pi/2, m_4) \qquad (3.4.3)$$

Subtituting M_4 from (3.4.2) in (3.4.3)

$$q = -2\left[\frac{-\delta\ kh}{2F(\pi/2, m_3)}\right] F(\pi/2, m_4)$$

$$q = \delta\ kh \frac{F(\pi/2, m_4)}{F(\pi/2, m_3)} \qquad (3.4.4)$$

3.3. Computation of Seepage discharge.

The discharge of seepage flow at Fragment I should be the same with discharge of seepage flow at fragment II.

From (3.3.5) discharge at Fragment I :

$$(1-\delta)\ kh \frac{F(\pi/2, m_1)}{F(\pi/2 m_2)}$$

From (3.4.4) discharge at Fragment II :

$$q = \delta\ kh \frac{F(\pi/2, m_4)}{F(\pi/2, m_3)}$$

Equating (3.2.5) and (3.4.4)

$$(1-\delta)\ kh \frac{F(\pi/2, m_1)}{F(\pi/2 m_2)} = \delta\ kh \frac{F(\pi/2, m_4)}{F(\pi/2, m_3)}$$

$$\frac{1-\delta}{\delta} = \frac{F(\pi/2, m_4)}{F(\pi/2, m_3)} \frac{F(\pi/2, m_2)}{F(\pi/2, m_1)}$$

If : $X = \dfrac{F(\pi/2, m_4)}{F(\pi/2, m_3)} \dfrac{F(\pi/2, m_2)}{F(\pi/2, m_1)}$

$$\frac{1-\delta}{\delta} = X$$

$$\delta = \frac{1}{X+1} \qquad (3.5.1)$$

The values of seepage discharge of various depth of sheet pile, various distance and slope of porous layer are shown in Fig.3.1.

Fig. 3.1. Seepage discharge

L/T	s/T	l/T	α	q/kh
1.25	0.5	0.5	0.1	0.4035
1.25	0.5	0.5	0.5	0.4035
1.25	0.5	0.5	0.25	0.4035
1.25	0.5	1	0.25	0.4035
1.25	0.5	2	0.25	0.4035
1.25	0.5	3	0.25	0.4035
1.25	0.1	0.5	0.25	0.4239
1.25	0.2	0.5	0.25	0.4116
1.25	0.3	0.5	0.25	0.4056
1.25	0.4	0.5	0.25	0.4046
1.25	0.5	0.5	0.25	0.4035

3.4. Computation of Exit Gradient

Let the complex potential $w = \phi + i\psi$ be the analytic function of the complex variable z, as $w = f(z)$

$$\frac{dw}{dz} = \frac{\partial \phi}{\partial x} + i\frac{\partial \psi}{\partial x} \tag{3.5.2}$$

which, substituting the velocity components, yields the complex velocity

$$\frac{dw}{dz} = u - iv \tag{3.5.3}$$

Along the downstream horizontal boundary $u = 0$, hence

$$\frac{dw}{dz} = -\,iv \tag{3.5.4}$$

From Darcy's Law

$$v = -\,I\,k \tag{3.5.5}$$

Subtituting (3.3.5) in (3.5.4)

$$\frac{dw}{dz} = -\,i\,I\,k \tag{3.5.6}$$

$$i\,I\,k = \frac{dw}{dt}\frac{dt}{dz}$$

$$I = \frac{1}{i\,k}\frac{dw}{dt}\frac{dt}{dz} \tag{3.5.7}$$

The exit gradient can be computed from Fragment II.

$$I_E = \frac{1}{i\,k}\left[\frac{-\delta kh}{2F(\pi/2,\sqrt{1-b})}\left(\frac{1}{(-t)^{1/2}(b-t)^{1/2}(1-t)^{1/2}}\right)\right]\left[-\frac{\pi}{iT}\left(\frac{t^{1/2}(1-t)^{1/2}(d-t)^{1-\alpha}(f-t)^{\alpha}}{(t-e)}\right)\right]$$

$$= -\frac{1}{k}\left[\frac{\delta kh}{2F(\pi/2,\sqrt{1-b})}\left(\frac{1}{(t)^{1/2}(t-b)^{1/2}(1-t)^{1/2}}\right)\right]\left[\frac{\pi}{T}\left(\frac{t^{1/2}(1-t)^{1/2}(d-t)^{1-\alpha}(f-t)^{\alpha}}{(t-e)}\right)\right]$$

$$\frac{I_E\,T}{h} = \pi\left(\frac{\delta}{2F(\pi/2,\sqrt{1-b})}\right)\left(\frac{(d-t)^{1-\alpha}(f-t)^{\alpha}}{(t-b)^{1/2}(e-t)}\right) \tag{3.5.7}$$

3.5. Results and Discussions

The variations of exit gradient in the region between the hydraulic structure and porous layer with distance from the downstream sheet pile are shown in Fig. 3.9 through 3.18 for various inclination of the porous layer, distance from the sheet pile and depth of sheet pile.

Due to presence of the highly porous layer, the exit gradient decrease from a maximum value at the sheet pile to zero at the highly porous layer. Thus the presence of highly porous layer, which acts as relief well, decreases the exit gradient on the downstream side. Hence piping will not commence in the vicinity of the sheet pile. However, the exit gradient at the entry of porous slit (i.e. at point E) would be infinite. Therefore the slit will act as conduit and the soil particles will escape to the downstream side trough the slit which would act as a pipe.

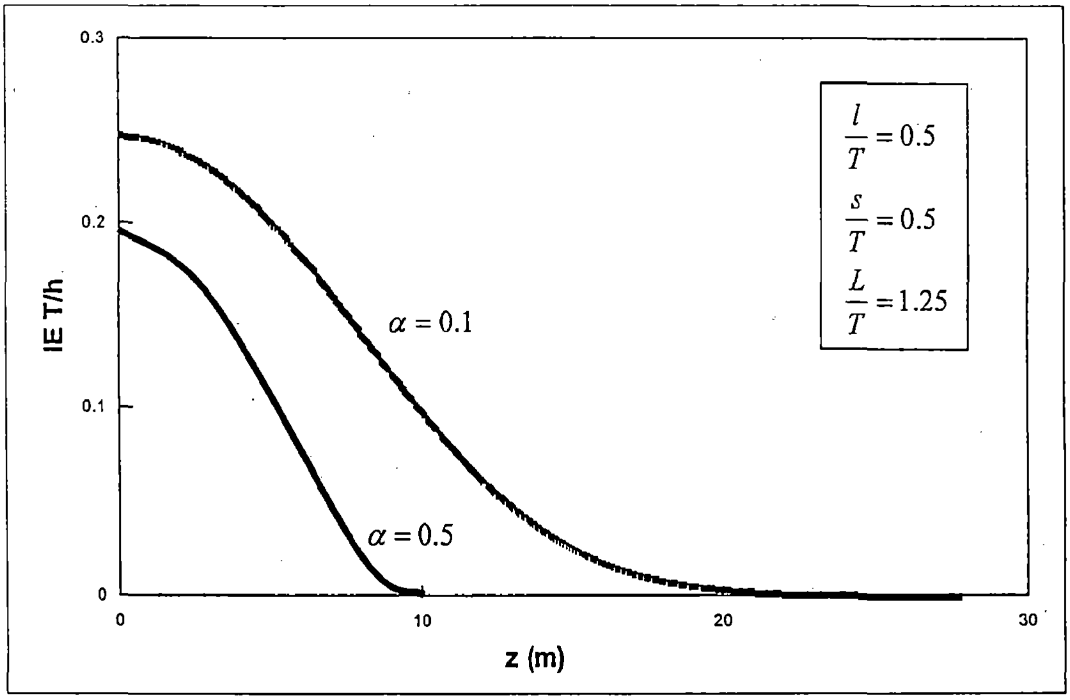


Fig. 3.9 **Exit gradient distribution along the downstream bed**

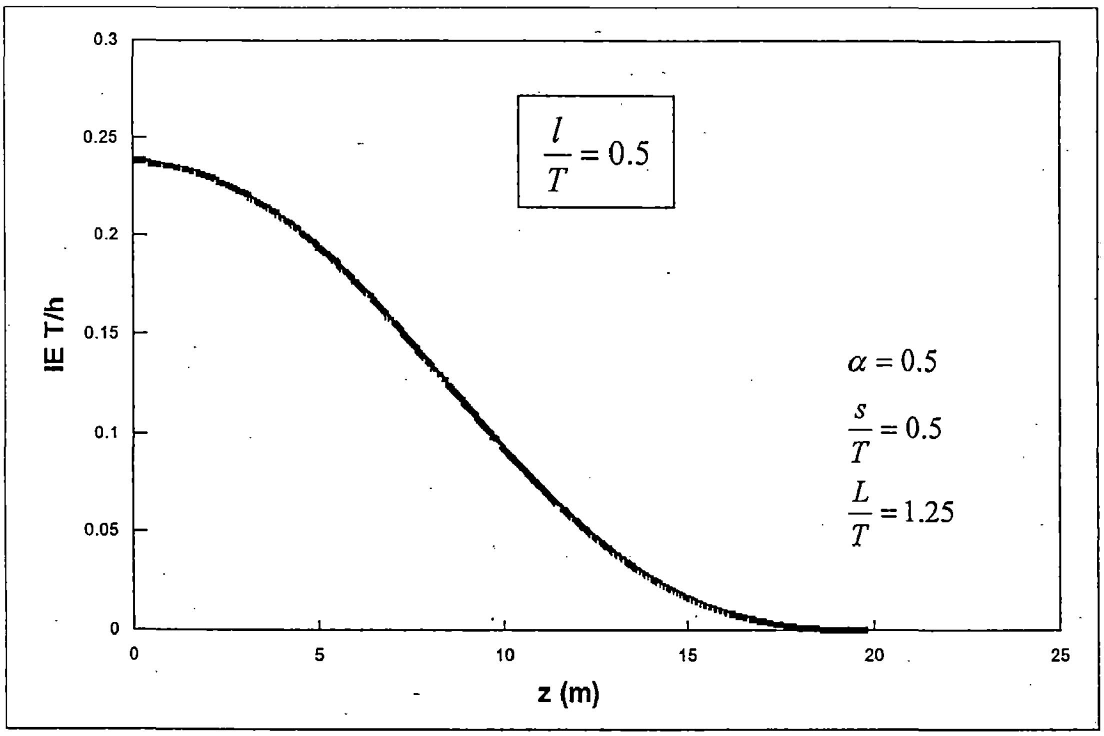


Fig. 3.10 **Exit gradient distribution along the downstream bed**

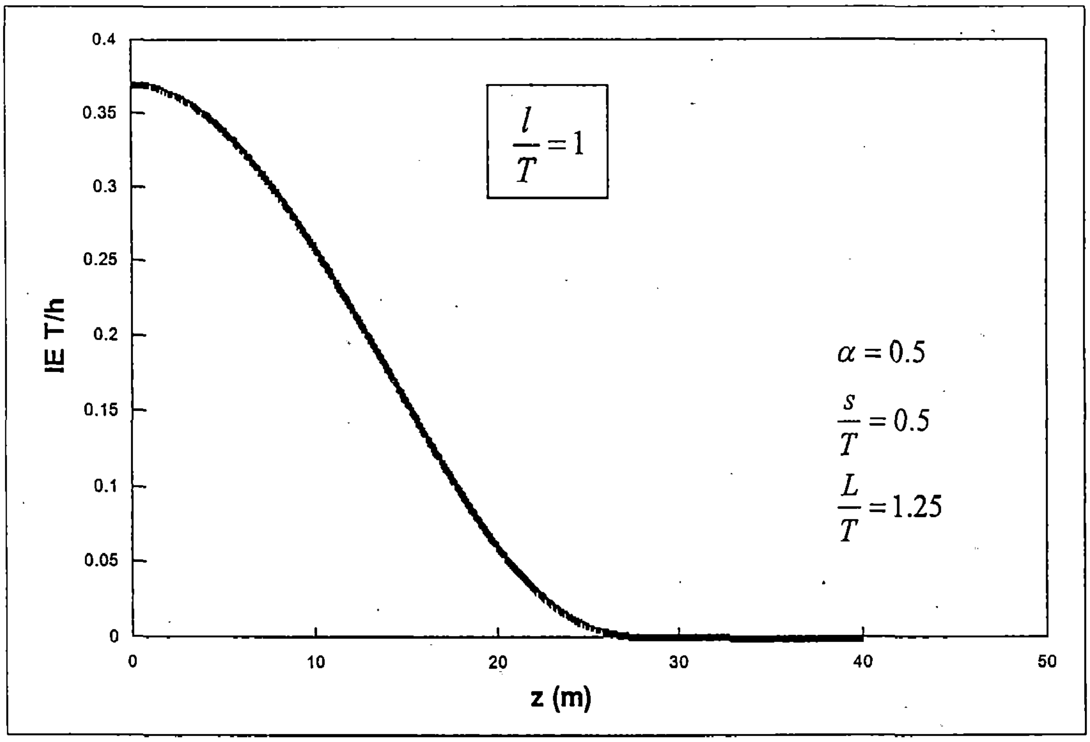


Fig. 3.11 Exit gradient distribution along the downstream bed

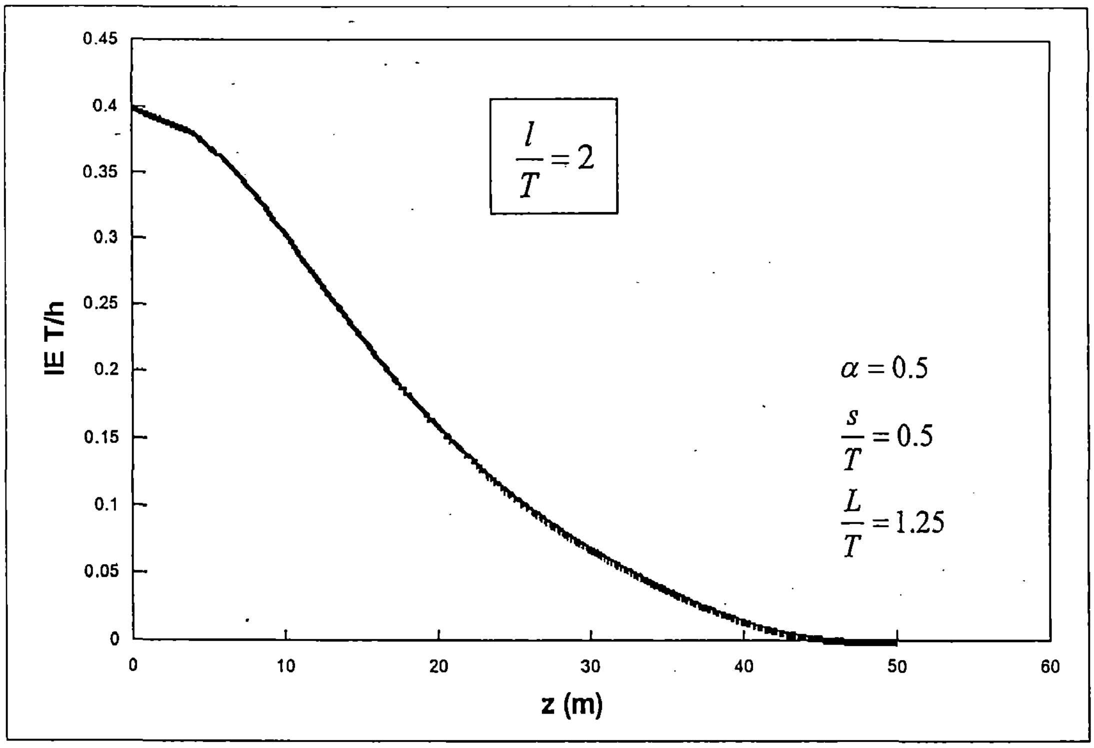


Fig. 3.12 **Exit gradient distribution along the downstream bed**

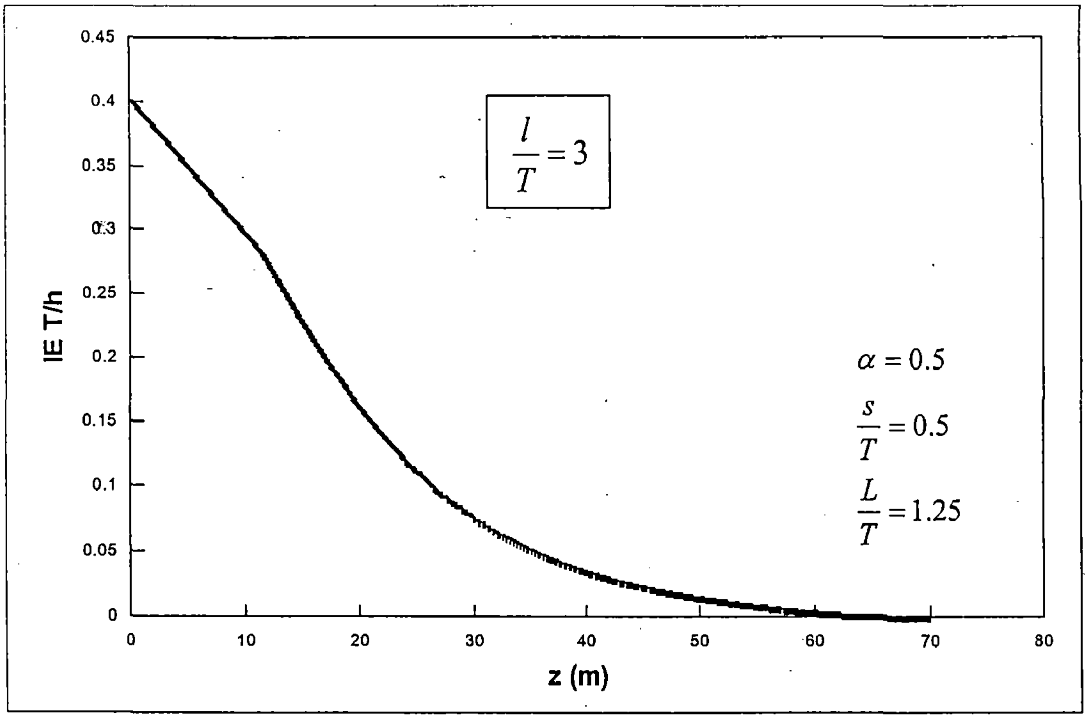


Fig. 3.13 **Exit gradient distribution along the downstream bed**

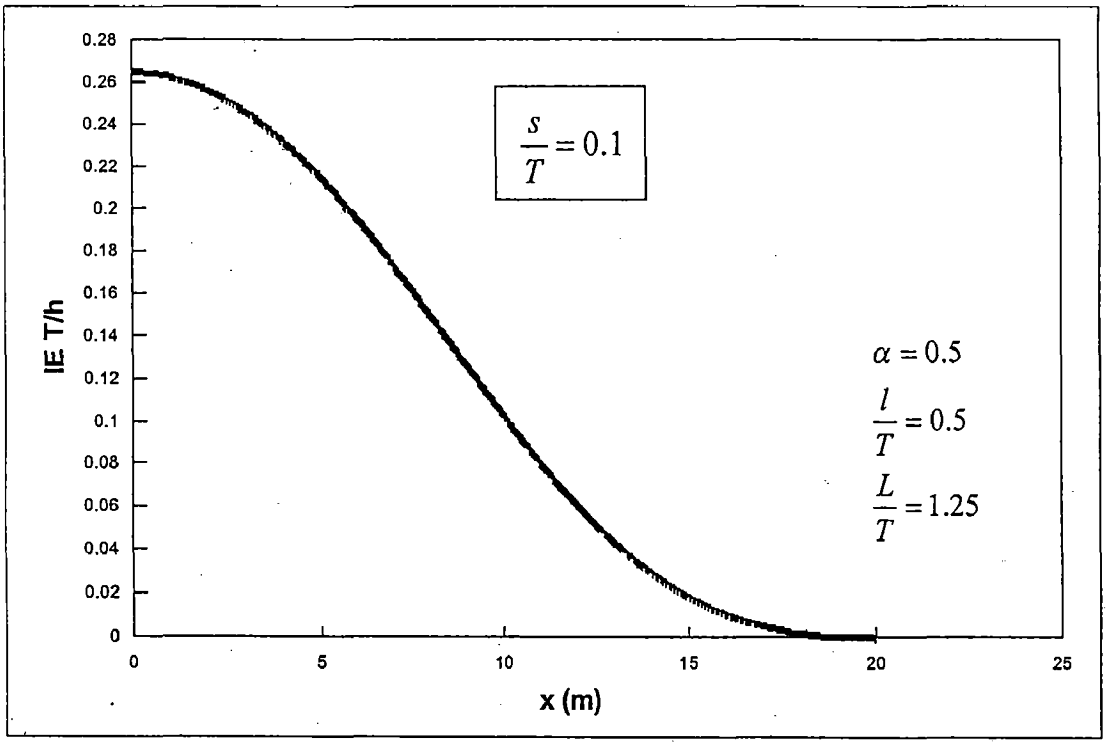


Fig. 3.14 **Exit gradient distribution along the downstream bed**

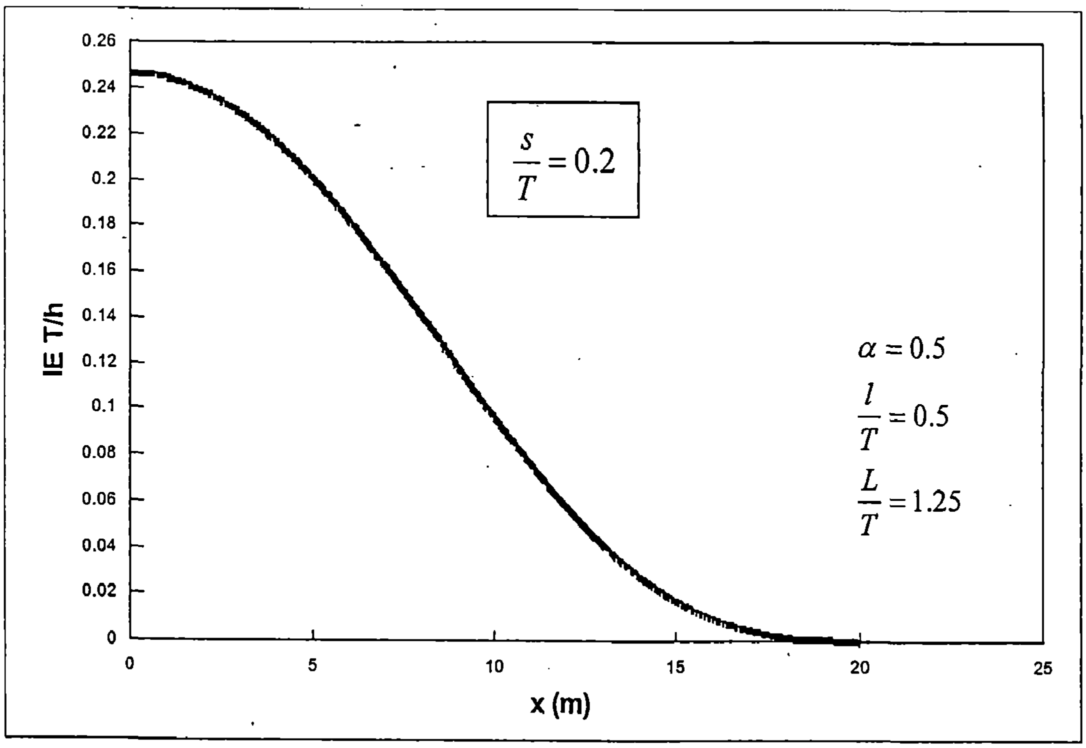


Fig. 3.15 Exit gradient distribution along the downstream bed

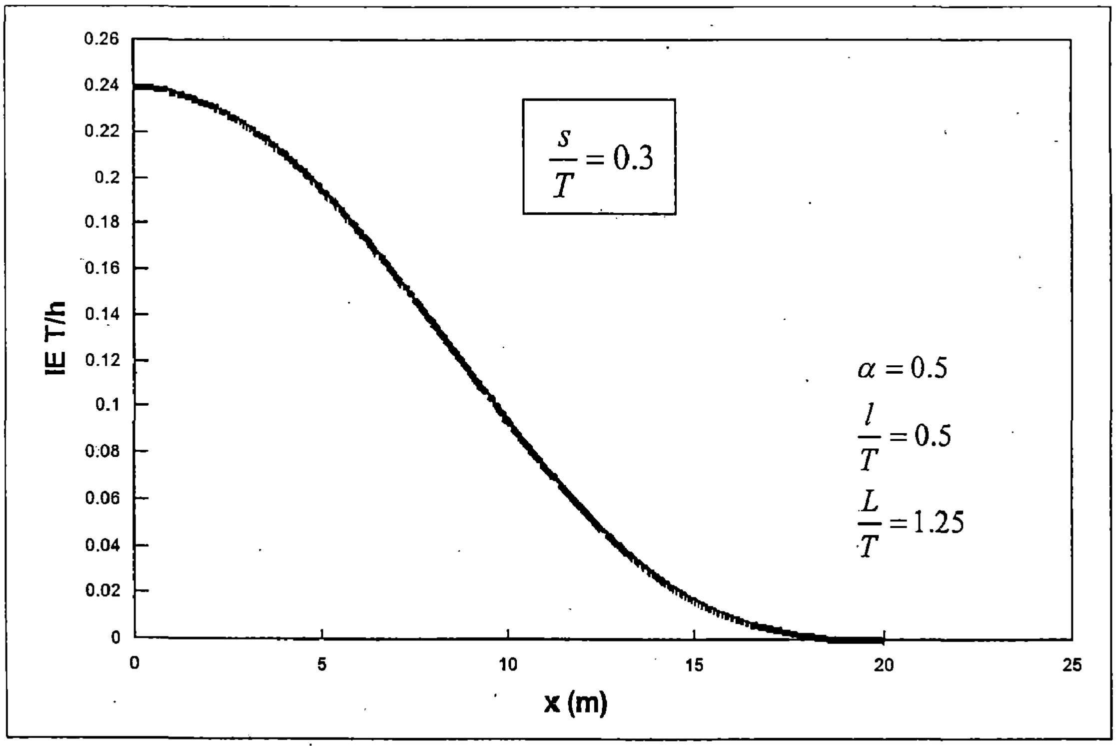


Fig. 3.16 **Exit gradient distribution along the downstream bed**

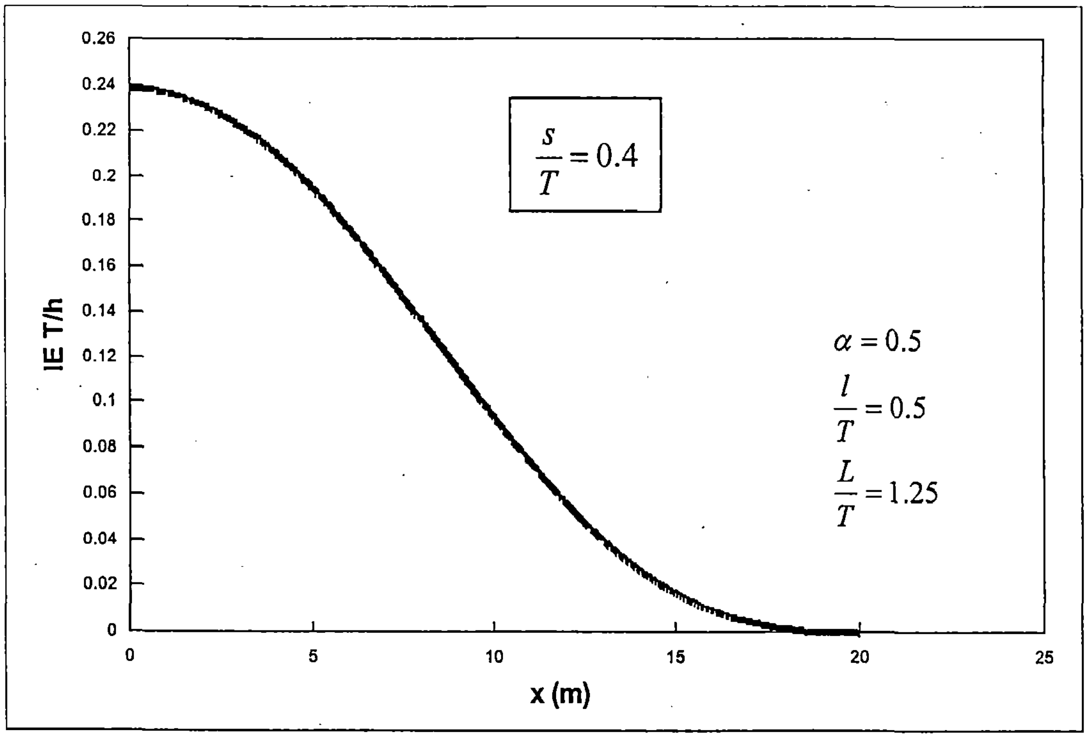


Fig. 3.17 Exit gradient distribution along the downstream bed

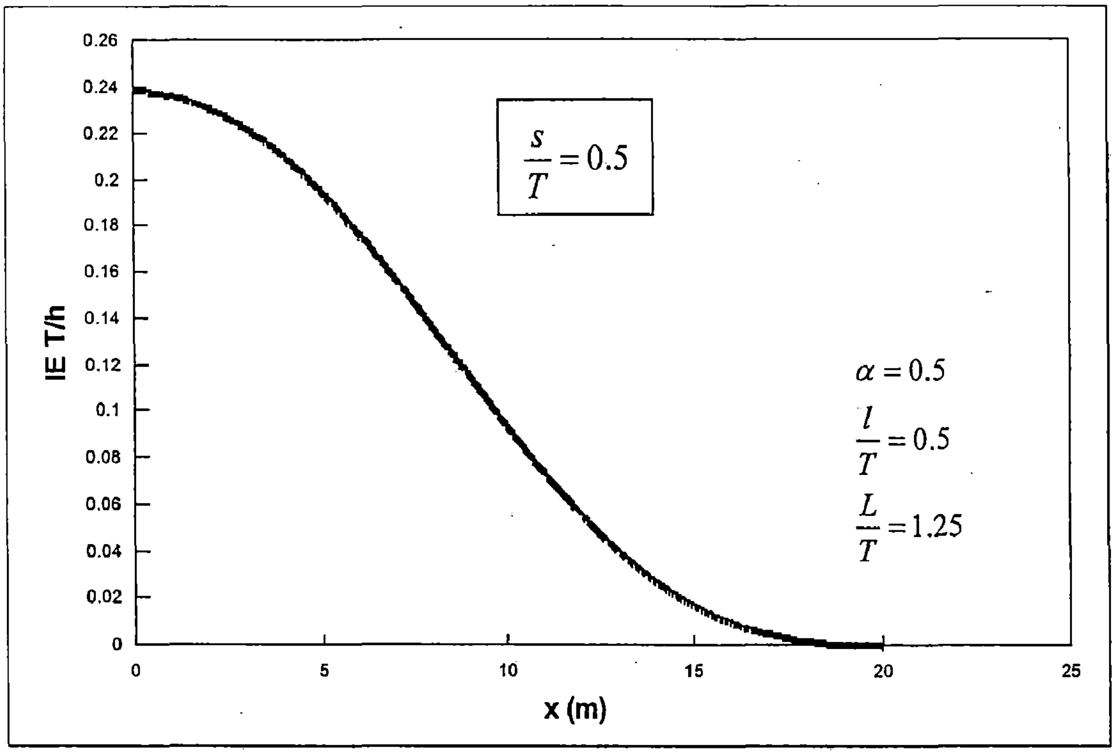


Fig. 3.18 Exit gradient distribution along the downstream bed

CHAPTER IV

CONCLUSIONS

Vertical slit reduces the exit gradient more than by horizontal slit. The zone nearer the sheet pile is vulnerable to piping in the absence of a highly porous slit. The presence of slit changes the location of vulnerable zone to piping as presence of a slit reduces the exit gradient. However the exit gradient is infinite at the entry of the slit. Piping channel will advance to the upstream side from the slit, which will act as a conduit for escape of particles to the downstream side.

Any complex flow problem relating to weir on homogeneous porous medium can be solved using the method of fragments and Newton Raphson Technique.

REFFERENCES

1. Harr,ME, "Ground Water and Seepage", McGraw-Hill Book Company (1962)
2. Khosla.R.B.A.N, Bose.N.K,Taylor.E.McK, "Design of Weirs on Permeable Foundations", CBIP, India, publication No.12 (1962).
3. Garg.S.K, "Irrigation Engineering and Hydraulic Structures", Khanna Publishers(1976).
4. Arora.K.R, "Irrigation Water Power and Water Resources Engineering", Standard Publishers Distributors.
5. Polubarinova-Kochina.P.Ya, "Theory of Ground Water Movement", Princenton University Press (1962).
6. Byrd.Paul.F, Friedman, Morris.D, "Hand Book of Elliptical Integrals for Engineers and Scientists", Spriger-Verlag, New York (1971).
7. Gerald Curtis.F, Wheatley, Patrick.O, "Applied Numerical Analysis", Pearson Education (1999).

APPENDIX A

NEWTON RAPHSON METHOD

The result of conformal mapping often in non-linear equations, which require a technique to compute the unknown parameters. A suitable method to solve the non-linear equations is known as Newton-Raphson method.

Chapter-III reveals that the problem consists of highly non-linier equations involving multi variables, which makes it difficult to be solved by analytically. The process of numerical application is explained below.

Let :

$$F_1(x_1, x_2, \dots x_n) = 0$$

$$\vdots$$

$$F_n(x_1, x_2, \dots x_n) = 0$$

and x denoted the enire vector of values x_i.

F denoted the entire vector of values F_i.

$$F_i(x+\delta x) = F_i(x) + \sum_{j=1}^{n} \frac{\partial F_i}{\partial x_j} \Delta x_j + \delta\ x^2$$

in matrix notation, the above equation can be written as :

$$F_i(x+\delta x) = F_i(x) + J.\Delta x_j + \delta\ x^2$$

Now neglecting the terms of the order $\delta\ x^2$ and set $F_i(x+\delta x) = 0$,

hence

$$F_i(x) + J.\Delta x_j = 0$$

$$J.\Delta x_j = -F_i(x)$$

$$\begin{bmatrix} \frac{\partial F_1}{\partial x_1}, \dots\dots\dots\dots \frac{\partial F_1}{\partial x_n} \\ \vdots \\ \frac{\partial F_n}{\partial x_1} \dots\dots\dots\dots \frac{\partial F_2}{\partial x_n} \end{bmatrix} \begin{bmatrix} \Delta x_1 \\ \vdots \\ \vdots \\ \Delta x_n \end{bmatrix} = - \begin{bmatrix} F_1(x_1, \dots . x_n) \\ \vdots \\ \vdots \\ F_n(x_1, \dots . x_n) \end{bmatrix}$$

and $x_i = x_j + \Delta x$

APPENDIX B

FORTRAN PROGRAM

```
PROGRAM PIPING

INTEGER::IOS
INTEGER, PARAMETER ::dsk=1


! STRUCTURE MODEL DIMENSION
REAL:: L0=25.,L1, l=10, m = 10, T=20.0, al=0.25, S=10
REAL:: pi=3.141592654

!** FRAGMENT I**

REAL:: b1=1.01, dx=0.001,  d1

INTEGER,DIMENSION(2)::Xr=(/0,1/)
DIMENSION FX(2)
DIMENSION b(2)

! ** FRAGMENT II**

! INTEGRAL COMPONENT
REAL:: I1,I2, I21,I22,I3,I31,I32,I4,I41,I42,I5,I6
REAL:: d0=1.01, dd=0.001
REAL:: f0=1.02, df=0.001
REAL:: e,v,dv,v2,b2
INTEGER::h

! MATRIX
REAL::a11,a12,a21,a22
REAL::b11,b12,b21,b22
REAL::j11,j12,j21,j22


! DISCHARGE FLOW
REAL::IB0,IBA,IBB,DELV

REAL::m1,m2,m3,m4
REAL::Fm1,Fm2,Fm3,Fm4
REAL::q1,q2,XF,DELTA

!ITERATION

REAL::TP,I10,I1a,I1b,I20,I2a,I2b,IE,z0=0,c0=1


REAL,DIMENSION(3)::dr=(/0,1,0/)
REAL,DIMENSION(3)::fr=(/0,0,1/)
```

```
DIMENSION W(96),X(96)
DIMENSION F1(3),F2(3)
DIMENSION d(3),f(3)

OPEN (1,FILE='GAUSS.dat',STATUS='OLD')
READ (1,*) (W(i), i=1,96)
READ (1,*) (X(i), i=1,96)


OPEN(unit=dsk,FILE="OUTPUT.txt",STATUS="OLD",IOSTAT=ios)
IF (IOS/=0) THEN
PRINT *,"FILE CAN NOT BE OPENED"
STOP
END IF
REWIND DSK

L1=l+m/tan(al*pi)

WRITE (unit=dsk, FMT=14)
14 FORMAT (T15,"(l)",T25,"(m)",T35,"(L1)",T45,"(alfa)")

WRITE (unit=dsk, FMT=15) l,m,L1,al
15 FORMAT (10x, F10.4, F10.4, F10.4, F10.4)

WRITE (unit=dsk, FMT=16)
16 FORMAT (" ")



!** CALCULATION **

! ITERATION FOR PARAMETER (b) AT FRAGMENT I.
write (*,*) 'FRAGMENT I :'
write (*,*) ' '

DO
      DO ii=1,2
      b(ii) =b1+Xr(ii)*dx
      FX(ii)=L0/T+1/pi*log(b(ii)-sqrt(b(ii)**2-1))
      END DO
      b1=b(1)-FX(1)*dx/(FX(2)-FX(1))
      IF ( abs(b1-b(1))<=0.0001) THEN
      EXIT
      END IF
END DO
write (*,*) 'b = ',b1

pause

! CALCULATION FOR PARAMETER (d) AT FRAGMENT I.

d1= -cos(S*pi/T)
WRITE (*,*) 'd = ',d1

PAUSE


! ITERATION FOR PARAMETER (b) AT FRAGMENT I.
```

```
DO h = 1,50

WRITE (*,*) 'ITERATION NUMBER :',h,'     FOR PARAMETER d,e,f at FRAGMENT
II'

WRITE (*,*) 'dd, df   =', dd, df


  DO i=1,3

      I1=0; I21=0; I22=0; I2=0; I31=0; I32=0; I41=0; I42=0; I5=0; I6=0

      d(i) = d0 + dr(i)*dd
      f(i) = f0 + fr(i)*df

      DO J=1,96

            ! INTEGRATION I1

            v =0.5*(1+X(j))
            v2=v**2
            dv=0.5

            I1=I1+W(j)*2*(1-v2)**0.5*dv/(d(i)-1+v2)**(1-al)/(f(i)-
1+v2)**al

            ! INTEGRATION I2
            v =0.5*(0.5)**0.5*(1+X(j));                    v2=v**2
            dv=0.5*(0.5)**0.5

            I21=I21+ W(j)*2*dv/(1-v2)**0.5/(d(i)-v2)**(1-al)/(f(i)-v2)**al
            I22=I22+ W(j)*2*dv/(1-v2)**0.5/(d(i)-1+v2)**(1-al)/(f(i)-
1+v2)**al
            I2 =I21+I22

      END DO

      e  =(I1+pi)/I2

      DO k=1,96


            ! INTEGRATION I3

            v  =0.5*(0.5*d(i)-0.5)**0.5*(1+X(k));
            dv =0.5*(0.5*d(i)-0.5)**0.5
            v2 =v**2;
            I31=I31+W(k)*2*(1+v2)**0.5*dv/(d(i)-1-v2)**(1-al)/(f(i)-1-
v2)**al

            v  =0.5*(0.5*d(i)-0.5)**al*(1+X(k))
            dv =0.5*(0.5*d(i)-0.5)**al
            v2 =v**(1/al)
            I32=I32+ W(k)*1/al*(d(i)-v2)**0.5*dv/(d(i)-v2-1)**0.5/(f(i)-
d(i)+v2)**al
```

```
          I3 =I31+I32

          ! INTEGRATION I4

          v  =0.5*(0.5*d(i)-0.5)**0.5*(1+X(k))
          dv =0.5*(0.5*d(i)-0.5)**0.5
          v2 =v**2;
          I41=I41+W(k)*2*dv/(1+v2)**0.5/(d(i)-1-v2)**(1-al)/(f(i)-1-
v2)**al

          v  =0.5*(0.5*d(i)-0.5)**al*(1+X(k))
          dv =0.5*(0.5*d(i)-0.5)**al
          v2 =v**(1/al)
          I42=I42+ W(k)*1/al*dv/(d(i)-v2)**0.5/(d(i)-v2-1)**0.5/(f(i)-
d(i)+v2)**al

          I4 =I41+I42

          ! INTEGRATION I5

          v  =0.5*(e-d(i))**al*(1+X(k))
          dv =0.5*(e-d(i))**al
          v2 =v**(1/al)
          I5 =I5+W(k)*1/al*(d(i)+v2)**0.5*dv/(d(i)+v2-1)**0.5/(f(i)-
d(i)-v2)**al


          ! INTEGRATION I6

          v  =0.5*(e-d(i))**al*(1+X(k))
          dv =0.5*(e-d(i))**al
          v2 =v**(1/al)
          I6 =I6+W(k)*1/al*dv/(d(i)+v2)**0.5/(d(i)+v2-1)**0.5/(f(i)-
d(i)-v2)**al

     END DO


     F1(i)=L1/T-(I3-e*I4)/(I1-e*I2)
     F2(i)=L1-1-T/pi*(e*I6-I5)*cos(al*pi)

     WRITE (*,*) 'd, f, e  =',d(i),f(i),e
     WRITE (*,*) 'I1,I2,I3 =',I1,I2,I3

     WRITE (*,*) 'I4,I5,I6 =',I4,I5,I6

     WRITE (*,*) 'F1,F2    =',F1(i),F2(i)
     WRITE (*,*) ''


  END DO


a11=(F1(2)-F1(1))/dd;       a12=(F1(3)-F1(1))/df;
a21=(F2(2)-F2(1))/dd;       a22=(F2(3)-F2(1))/df;

DET=a11*a22-a21*a12
```

```
! INVERS MATRIX A
b11=(a22)/DET;                    b12=(-a12)/DET;
b21=(-a21)/DET;                   b22=(a11)/DET;

dd=b11*(-F1(1))+b12*(-F2(1));
df=b21*(-F1(1))+b22*(-F2(1));

! (MATRIX A)  * (INVERS MATRIX A)
J11=b11*a11+b12*a21;              J12=b11*a12+b12*a22;
J21=b21*a11+b22*a21;              J22=b21*a12+b22*a22;

d0=d0+dd
f0=f0+df

WRITE (*,*) '          ** MATRIX A **                    ** INVERS MATRIX A
**'
WRITE (*,*) a11,a12,b11,b12
WRITE (*,*) a21,a22,b21,b22
WRITE (*,*) '                                                  dd, df          d ,
f'
WRITE (*,*) J11,J12,dd,d0
WRITE (*,*) J12,J22,df,f0


IF (F2(1)<0.0001) THEN
     IF (F1(1)<0.0001) THEN
     EXIT
     END IF
     ELSE
     CONTINUE
END IF
END DO
PAUSE


WRITE (unit=dsk, FMT=12)
12 FORMAT (T15,"(d)",T25,"(e)",T35,"f")

WRITE (unit=dsk, FMT=26) d(1),e,f(1)
26 FORMAT (10x, F10.6, F10.6, F10.6)


! ITERATION FOR PARAMETER (b) at FRAGMENT II

TP=0

DO
     IB0=0
     v=0
     IBA=2*(e-v**2)/(1-v**2)**0.5/(d(1)-v**2)**(1-al)/(f(1)-v**2)**al
     TP=TP+0.001
     DELV=(TP**0.5)/100

     DO BB=1,100

     v=v+DELV
     IBB=2*(e-v**2)/(1-v**2)**0.5/(d(1)-v**2)**(1-al)/(f(1)-v**2)**al
     IB0=IB0+(IBA+IBB)/2*DELV
```

```
      IBA=IBB

      END DO

      IF (((1-S/T*pi)-IB0)<=0.0001) THEN
      EXIT
      END IF

END DO

WRITE (*,*) 'b = ',tp


! DISCHARGE
b2=tp
m1=sqrt((1-d1)/(1+b1))
m2=sqrt((d1+b1)/(1+b1))
m3=sqrt(1-b2)
m4=sqrt(b2)

Fm1=0.5*pi*(1+0.25*(m1**2)+9/64*(m1**4)+25/256*(m1**6))
Fm2=0.5*pi*(1+0.25*(m2**2)+9/64*(m2**4)+25/256*(m2**6))
Fm3=0.5*pi*(1+0.25*(m3**2)+9/64*(m3**4)+25/256*(m3**6))
Fm4=0.5*pi*(1+0.25*(m4**2)+9/64*(m4**4)+25/256*(m4**6))

XF     = Fm4*Fm2/(Fm3*Fm1)
DELTA = 1/(1+XF)
q1     = (1-DELTA)*Fm1/Fm2
q2     = DELTA*Fm4/Fm3

WRITE (*,*) 'Delta = ',DELTA
WRITE (*,*) 'q1/kh = ',q1
WRITE (*,*) 'q2/kh = ',q2

WRITE (unit=dsk, FMT=17)
17 FORMAT (T15,"(q1/kh)",T25,"(q2/kh)",T35,"DELTA")

WRITE (unit=dsk, FMT=18) q1,q2,DELTA
18 FORMAT (10x, F10.4, F10.4,F10.4)

WRITE (unit=dsk, FMT=19)
19 FORMAT (" ")


PAUSE


! ITERATION FOR  t vs ( Z and Exit Gradient )

write (*,*) '  t                        Z                        IE'

WRITE (unit=dsk, FMT=27)
27 FORMAT (T15,"(t)",T25,"(z)",T35,"IE")
WRITE (unit=dsk, FMT=28)
28 FORMAT (" ")

IE=pi*DELTA/(2*Fm3)*(d(1)-1)**(1-a1)*(f(1)-1)**a1/(1-b2)**0.5/(e-1)
WRITE (unit=dsk, FMT=29) c0,z0,IE
29 FORMAT (10x, F10.4, F10.4, E12.5)
```

```
TP=1

DO

TP=TP+0.0001
v=0
I1a=2*sqrt(1+v**2)/(d(1)-1-v**2)**(1-al)/(f(1)-1-v**2)**al
I2a=2/sqrt(1+v**2)/(d(1)-1-v**2)**(1-al)/(f(1)-1-v**2)**al
DELV=sqrt(TP-1)/1000
I10=0
I20=0


DO ii=1,1000

     v=v+DELV
     I1b=2*sqrt(1+v**2)/(d(1)-1-v**2)**(1-al)/(f(1)-1-v**2)**al
     I2b=2/sqrt(1+v**2)/(d(1)-1-v**2)**(1-al)/(f(1)-1-v**2)**al
     I10=I10+(I1a+I1b)/2*DELV
     I20=I20+(I2a+I2b)/2*DELV
     I1a=I1b
     I2a=I2b

END DO

Z=T/pi*(e*I20-I10)
IE=pi*DELTA/(2*Fm3)*(d(1)-TP)**(1-al)*(f(1)-TP)**al/(TP-b2)**0.5/(e-TP)

WRITE (unit=dsk, FMT=30) TP,Z,IE
30 FORMAT (10x, F10.4, F10.4, E12.5)

write (*,*) TP,Z,IE

IF ((L1-Z)<=0.0001) THEN
     EXIT
END IF
END DO

END FILE dsk
REWIND dsk
CLOSE (dsk)

PAUSE

END PROGRAM PIPING
```