MATRIX ANALYSIS OF STATICALLY INDETERMINATE STRUCTURES

THESIS SUBMITTED

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C E R T I F I C A T E

CERTIFIED that the dissertation entitled "Matrix Analysis of Statically Indeterminate Structures" which is being submitted by Sri Subhash Chandra Goel in partial fulfilment for the award of the Degree of Master of Engineering in Structural Engineering including Concrete Technology, of University of Roorkee is a record of student's own work carried out by him under our supervision and guidance. The matter embodied in this dissertation has not been submitted for the award of any other Degree or Diploma.

This is further to certify that he has worked for a period of five months partly from July 1, 1960 to August 31,1960 and partly from May 1, 1961 to July31,1961, for preparing dissertation for Master of Engineering Degree at the University.

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C O N T E N T S

		Page
Acknowledgement	....	1
Summary	....	2
CHAPTER 1		
Introduction	....	4
CHAPTER 2		
Matrix Algebra	....	10
CHAPTER 3		
Matric Formulation of Structural Problem.	....	22
CHAPTER 4		
Computation	....	29
CHAPTER 5		
Flexibility Matrix Method	....	47
CHAPTER 6		
Stiffness Matrix Method	....	69
CHAPTER 7		
Automatic Digital Computers and Structural Analysis	......	136
References	....	157

A C K N O W L E D G E M E N T

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SUMMARY

This dissertation deals with the exact solution of linear algebraic equations governing the behaviour of statically indeterminate structures. Matrix Algebra has been employed as a tool for assisting the analysis, since it makes the discussion and formulation of complex structural problems a very convenient and systematic process which can easily be mechanised. This approach, being most general in its application, also reduces the chances of committing errors and results in a considerable saving in time and labour required for a particular problem. These matrix techniques are especially advantageous when we have to deal with highly complex and redundant problems, which would otherwise be impossible to solve by hand methods of computation.

The chief objective of this dissertation has been to represent the subject in a manner which is systematic and easily assimilable by a common civil engineer. In addition, a few easier and direct synthetic methods to assemble the matrix of a given structure have been developed. The obvious merits of the matrix methods over the existing conventional methods have been discussed while describing the techniques, and also the possibilities of making rapid design calculations on an electric desk

calculator, which is more easily available to a common structural engineer, have been fully discussed.

The systematic representation of the subject is contained in the following seven chapters.

In the first four chapters are given the matric formulation of the two complementary basic approaches to a structural problem and the explanation of various matrix operations and methods required for the analysis. Chapters 5 and 6 deal with a detailed discussion of the flexibility matrix method and the stiffness matrix method respectively. To illustrate the techniques described in these chapters, a good number of numerical examples are given which have been solved on a 'Marchant' electric desk calculator. The last chapter deals with Electronic Digital Computers - their brief functional description and as to how more complex problems are programmed for an automatic solution on such machines.

CHAPTER I

INTRODUCTION

1.1 Structural Analysis.

In the last 40 years, statically indeterminate structures have been used more and more extensively. This is no doubt due to their economy and increased rigidity under moving or movable loads. The details of reinforced-concrete and welded construction are such that structures of these types are usually wholly or partly continuous in their structural action and are therefore usually statically indeterminate. A knowledge of the analysis of indeterminate structures has thus become increasingly important as the use of these types of construction have become more extensive.

Statically indeterminate structures differ from statically determinate ones in two important respects, viz;

(1) Their stress analysis involves not only their geometry but also their elastic properties such as modulus of elasticity, cross-sectional area and moment of inertia. Thus the final design of an indeterminate structure involves assuming preliminary sizes for the members, making a stress analysis of this design, testing the members for these stresses, revising the design if necessary till the final design is arrived at.

(2) In general, stresses are developed in indeterminate structures, not only by loads, but also by temperature changes, support settlements, fabrication errors, etc.

Structural analysis involves computing not only the external reactions and the internal forces(and stresses) of a structure, but also strains and deflections throughout. In so called statically determinate structures, almost invariably the external reactions and internal forces (and stresses) are computed first; then, from the stresses, the strains can be determined, and subsequently deflections can be computed. Same order of computations is followed in many methods of analysing statically indeterminate structures; that is, first the redundants and corresponding statically determinate primary structure (also called the released structure) are selected; then these redundant forces (and/or couples) are computed by solving an equal number of simultaneous equations, each of which expresses a known deflection condition for the released structure in terms of the redundants; and then once the redundants are known, the stresses, strains, and deflections for the entire structure may be computed as in the case of a statically determinate structure. Such a method of structural analysis is referred to as a "force method of structural analysis", since the first step in the computations involves determining the unknown external and internal forces (and/or couples) in the structure.

In other cases of statically indeterminate structures, the above order of computation is completely inverted. Such a method of analysis is called a "displacement method of structural analysis". In this approach, first the internal forces (and couples), are expressed in terms of the key displacement components of the structure; such expressions are substituted into the key equilibrium equations of the structure, thereby obtaining a system of linear simultaneous equations involving the key displacements as the unknowns; then the values of the displacements obtained from the solution of these equations are substituted into the original expressions for the internal forces (and couples) to obtain the values of the latter; and finally, once all the internal forces (and couples) are known, it is easy to compute the reactions of the structure.

According to the above classification, the superposition equation method, the application of Castigliano's second theorem, and the use of the three moment equation are all forces methods of analysis. The slope-deflection method is a displacement method. The moment distribution method is a successive approximation procedure based on the same philosophy as the displacement methods.

1.2 Structural Analysis and Matrix Algebra.

Whatever method of analysis we employ, the solution of multiply redundant structures require solving simultaneous equations. For structures having only a few degrees of redundancy, say upto five, any of the conventional methods can be used. With the recent advance in modern construction, complex structures, like multi-storeyed building frames are becoming more and more common. These structures contain a very large number of redundants and if solved by either of the direct methods or the numerical iterative methods, the calculation becomes a superhuman task, requiring many months of human labor-labor subject to the inherent shortcomings of human beings which produce errors, omissions, and the like. The problem becomes much more difficult if the structure is to be analysed for a number of loading conditions.

For such complex and highly redundant problems of structural analysis, the conventional methods have to be abandoned and some newer approach has to be resorted to, which will enable the structural engineer to discuss his complex problems in a more compact and convenient form. This should also make the method of computations most systematic and easy to be mechanised. Matrix notation provides just this requirement.

During the last few years a number of methods of analysing statically indeterminate structures have been suggested which use matrix notation. Most of these involve rather more numerical work than is required in the traditional methods, but the computing is entirely systematic and can easily be mechanised. They appear somewhat cumbersome when applied to simple structures, but form a very powerful tool for dealing with complex highly-redundant systems. They have already been used extensively in the design and analysis of aircraft structures.

Matrix notation is simply a useful shorthand invented by mathematicians for discussing problems of linear algebra. Almost any method of analysis which treats a structure as a linear elastic system (i.e., the changes in the geometry of the structure under load are sufficiently small to have a negligible effect upon loads and their corresponding stress distribution, and the structural materials obey Hooke's law) can be written in matrix form, but the notation appears to its best advantage when it is used to set up the load-dis-

-placement equations in explicit form. Such an approach leads, of course, to the computational problem of solving sets of linear simultaneous equations and for this reason it has in the past been restricted to simple structures with only a few degrees of freedom. However, the development of the desk calculating machine and more recently the automatic digital computer has made it easy to solve large sets of equations, so that the main objection to direct methods of this type has disappeared.

Matrix algebra may be regarded as a 'shorthand' technique for representing a system of linear equations by a simgle equation and then solving that single matrix equation. The rules of matrix algebra provide a computational procedure which is more rapid on a mechanised basis, than the numerical process in common usage. Since all indeterminate structures are governed by linear equations, the possibility of useful application of matrix methods by the structural engineer is suggested in the following contents of this work.

CHAPTER 2

MATRIX ALGEBRA

2.1 Principle.

When using an indeterminate structure it is found that the evaluation of the unknown displacement components or the force components has to be made through the solution of a system of linear simultaneous equations of the type,

$$k_{11}\,x_1 + k_{12}\,x_2 + k_{13}\,x_3 = u_1$$
$$k_{21}\,x_1 + k_{22}\,x_2 + k_{23}\,x_3 = u_2$$
$$k_{31}\,x_1 + k_{32}\,x_2 + k_{33}\,x_3 = u_3$$

Here there are three equations with k and x mixed but in an ordered pattern. It would seem advantageous if these equations could be reduced to "parcels" of the form

$$\boxed{kx} = \boxed{u}$$

which it might be possible to separate into further parcels so that

$$\boxed{k}\,\boxed{x} = \boxed{u}$$

The process; perhaps, could be carried further by writing

$$\boxed{x} = \frac{\boxed{u}}{\boxed{k}}$$

so that the left hand side of the equation could be unparcelled to give the required values of the unknowns. We shall find that matrices provide convenient form of parcel and as such they may be regarded as a tool for assisting the analysis. Since we are to use matrices as a tool we shall not need to know much of the pure mathematical properties of matrices but only some of the simple operations in which they can be used.

2.2 Definition.

A matrix is defined as a rectangular array of coefficients (numbers or linear operators) which obey certain laws of combination, to be specified. We shall adopt the following notations for a matrix and call on array of m rows and n columns an "$m \times n$ matrix".

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

The element of the matrix in the ith row and the jth column is donated by a_{ij} with the subscripts in this order. It is to be noted that the matrix is enclosed in square brackets and is to be distinguished from the determinant $|A|$. A determinant must be square and can be evaluated by the rules of algebra. A matrix need not be square and can never be evaluated.

In the special case of a column matrix (vector) i.e., a matrix with one column only, we shall use a lower case letter. Thus

$$U = \begin{bmatrix} u_1 \\ u_2 \\ \cdot \\ \cdot \\ u_m \end{bmatrix}$$

Similarly we may have row matrices which are also written with a lower case letter:

$$V = [\, v_1 \;\; v_2 \;\; \cdots \;\; v_n \,]$$

2.3 Addition, Substraction and the Null Matrix.

The sum of two $m \times n$ matrices $[a_{ij}]$ and $[b_{ij}]$ is defined to be $m \times n$ matrix $[a_{ij} + b_{ij}]$. This is to say that we add the corresponding elements of the two matrices to form the elements of the summation matrix, so that two matrices can only be added when they have an equal number of rows and columns. If this is not the case, the sum has no meaning.

The law of addition shows that it is

(a) commutative , i.e.,

$$A + B = B + A$$

and (b) associative i.e.,

$$A + (B + C) = (A + B) + C$$

The addition law also shows that multiplication of a matrix by a single quantity results in a matrix each of whose elements is multiplied by that quantity i.e.,

$$\text{if } A = [a_{ij}]$$

$$\text{then } kA = [k a_{ij}]$$

This shows that the distributive property holds, namely that

$$k\ (A + B) = k\ A + kB$$

The above laws of addition include substraction. Two matrices are said to be equal

$$A = B$$

if each of the elements of A equals the corresponding element of B. Two matrices can only be equal if they each have the same number of rows and columns.

If two matrices are equal and we substract one from the other, then the resulting matrix has zero, as each of its elements and is called the null matrix:

$$A - B = A - A = 0$$

2.4 Multiplication.

The product AB of two matrices A and B is defined as a matrix C whose element in the i^{th} row and the j^{th} column is the inner product of the i^{th} row of A and the j^{th} column of B.

The inner product of a row and a column is the sum of the products of the elements in the following order:

$$(a_{i1} \quad a_{i2} \; \cdots \; a_{in}) \begin{pmatrix} b_{1j} \\ b_{2j} \\ \bullet \\ \bullet \\ b_{nj} \end{pmatrix} = a_{i1} b_{1j} + a_{i2} b_{2j} + \cdots a_{in} b_{nj} = C_{ij}$$

It follows from the above definition that two matrices are only conformable for multiplication in the order AB when the number of columns of A is equal to the number of rows of B. Any two matrices, $m \times n$ and $n \times r$, when multiplied produce a matrix of order $m \times r$.

For example

$$\begin{bmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & k_{23} \\ k_{31} & k_{32} & k_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} k_{11} x_1 + k_{12} x_2 + k_{13} x_3 \\ k_{21} x_1 + k_{22} x_2 + k_{23} x_3 \\ k_{31} x_1 + k_{32} x_2 + k_{33} x_3 \end{bmatrix}$$

(3 x 3) (3x1) (3 x 1)

From the law of multiplication stated above, it follows immediately that the product AB is not necessarily equal to BA which may in fact not exist at all. In case of the product AB we say either that B is premultiplied by A or that A is postmultiplied by B.

The product of two matrices leads to the form of result we wanted when parcelling $\boxed{kx}$, i.e.,

$$\boxed{kx} = \boxed{k}\boxed{x} = K\,X$$

We see that the product matrix KX is column matrix $(n\times 1)$, as that it can be equal to the column matrix U which is also $(n\times 1)$ i.e.,

$$KX = U$$

The law of multiplication shows that the associative and distributive properties apply, provided the order of the matrices is kept unaltered, viz.,

$$A(B\ C) = (A\ B)\ C = A\ B\ C$$

and $A(B+C) = A\ B + A\ C$

and $(B+C)\ A = B\ A + C\ A$

As a result of the form of a matrix product it is possible that the product matrix may be a null matrix

$$A\ B = 0$$

with neither A nor B being a null matrix. This is exemplified by the following numerical case:

$$A\ B = \begin{bmatrix} 3 & 4 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 4 & 8 \\ -3 & -6 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\text{but } B\ A = \begin{bmatrix} 4 & 8 \\ -3 & -6 \end{bmatrix} \begin{bmatrix} 3 & 4 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 12 & 16 \\ -9 & -12 \end{bmatrix} \neq 0$$

2.5 Transposition.

If a matrix B is made from a matrix A by writing the ith row of B with the same elements in the same relative position as the i^{th} column of A , then B is said to be the transpose of A and is written

$$B = A^*$$

For example,

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$$

$$A^* = \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \\ a_{13} & a_{23} \end{bmatrix}$$

If the transpose of a square matrix is equal to the original matrix, then the matrix is said to be symmetrical

$$A = \begin{bmatrix} 4 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 4 \end{bmatrix}$$

i.e., $A^* = A$

It can be shown by the law of multiplication that the transpose of a product of two matrices is the reversed product of the transposed matrices viz.,

$$(A\,B)^* = B^*A^*$$

2.6 Submatrices.

It is sometimes convenient to partition a matrix into submatrices. This is to say that the elements of the matrix are themselves matrices. For example,

$$A = \left[\begin{array}{c:cc} a_{11} & a_{12} & a_{13} \\ \hdashline a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{array}\right] = \begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix}.$$

Where

$$\alpha_{11} = a_{11} \qquad \alpha_{12} = [a_{12} \quad a_{13}]$$

$$\alpha_{21} = \begin{bmatrix} a_{21} \\ a_{31} \end{bmatrix} \qquad \alpha_{22} = \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix}$$

If two matrices are of the same order in rows and columns and are similarly partitioned, then the rules of matrix addition are still applicable and yield the same result as addition of the unpartitioned matrices. Further, it can also be proved by rules of matrix multiplication that if two matrices are conformable for multiplication and they are each partitioned so that they still remain conformable for multiplication, then the resulting product matrix from either operation will be the same.

2.7 The Unit matrix and the Inverse matrix.

A square matrix with its leading diagonal elements a_{ii} equal to unity and all other elements zero is called a a unit matrix (sometimes called Identity Matrix also) and is represented by [I].

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The rules of matrix multiplication will show that

$$A\ I = I\ A = A$$

where A is an arbitrary matrix and I the unit matrix of the same order.

If we now find a square matrix B which is related to the square matrix A in the following way

$$A\ B = B\ A = I$$

then the matrix B which is unique, is called the "inverse" of A and is written

$$B = A^{-1}$$

so that $A\ A^{-1} = A^{-1}\ A = I$

We must note that the matrix A must be non-singular; a non-singular matrix being defined as one in which the determinant with the same coefficients, in the same positions, is not equal to zero, i.e.,

$$|A| \neq 0$$

Each row and each column of a non-singular matrix must contain at least one non-zero element. A singular matrix has

$$|A| = 0$$

It is to be noted that when the product A B equals zero then although, as has been shown, neither of the individual matrices A nor B is necessarily zero, one or both of the matrices must be singular.

By the law of multiplication of matrices the inverse of a product of matrices can be written as the reversed product of the inverse matrices.

$$(A B)^{-1} = B^{-1} A^{-1}$$

also, that $(A^*)^{-1} = (A^{-1})^*$

If the original matrix has only elements on the leading diagonal, e.g.,

$$A = \begin{bmatrix} a & o & o \\ o & b & o \\ o & o & c \end{bmatrix}$$

then the rules of matrix multiplication show that since $A A^{-1} = I$ thus

$$A^{-1} = \begin{bmatrix} 1/a & 0 & 0 \\ 0 & 1/b & 0 \\ 0 & 0 & 1/c \end{bmatrix}$$

That is, the inverse of a diagonal matrix is also a diagonal matrix whose elements are the reciprocals of the elements of the original matrix.

The inverse has special value in our structural analysis problem for we have seen that the relevant equations can be written in matrix notation as

$$K\,X = U$$

which premultiplied by k^{-1} gives

$$K^{-1}\,K\,X = K^{-1}\,U$$

or

$$I\,X = K^{-1}\,U$$

$$X = K^{-1}\,U$$

Apart from its conciseness, matrix notation is useful in that it clearly separates the constants appearing in a set of simultaneous equations from the particular sets of variables which happen to be related. In any physical system whose behaviour is governed by linear algebraic equation, the matrix K is always an invariant function and can indeed be regarded as forming a complete mathematical statement of the properties of the system. The vectors X and U on the other hand are merely related to one particular set of conditions.

CHAPTER 3

MATRIC FORMULATION OF STRUCTURAL PROBLEM

It has already been pointed out that there are possible two complementary formulations of a structural problem.

(a) The stiffness method in which geometrically compatible states are combined to give equilibrium and

(b) The flexibility method in which equilibrium states are combined to give geometrical compatibility.

It is proposed to discuss, here, the general energy theory and matrix formulation of the two complementary approaches.

(a) Stiffness Matrix Method.

Let U represent the strain energy stored within a structure which is loaded by the forces P_1 , P_2 , ..., P_i ,... P_n . The temperature of the material remains constant and the supports are rigid. Applying Castigliano's Theorem I of Structure Equilibrium one obtains;

$$P_i = \frac{\partial U}{\partial \Delta_i} \qquad (3.1)$$

where Δ_i is the deflection of the point of application of the load P_i in the direction of P_i.

If the strain energy is evaluated in terms of the loads P_i acting upon the structure we may expand Equation (3.1) as follows:

$$P_i = \frac{\partial U}{\partial \Delta_i} = \sum_j \left(\frac{\partial U}{\partial P_j}\right)\left(\frac{\partial P_j}{\partial \Delta_i}\right) \qquad (3.2)$$

If the structure is assumed to be elastic then Castigliano's Theorem II for Linear Structures may be applied.

$$\Delta_j = \frac{\partial U}{\partial P_j} \qquad (3.3)$$

Substituting Equation (3.3) into Equation (3.2),

$$P_i = \sum_j \Delta_j \left(\frac{\partial P_j}{\partial \Delta_i}\right) \qquad (3.4)$$

The partial derivative $\frac{\partial P_j}{\partial \Delta_i}$ represents the force developed at point j due to a unit deflection of point i, all other points assumed to be fixed. This force is represented by the symbol k_{ji}. The subscript j represents the point at which the force acts and the subscript i the point at which the unit deflection is imposed. With this substitution Eq. (3.4) becomes:

$$P_i = \sum_j \Delta_j k_{ji} \qquad (3.5)$$

From the generalized Maxwell's Law of Reciprocal Deflections we obtain the relation

$$k_{ji} = k_{ij} \qquad (3.6)$$

and hence

$$P_i = \sum_j \Delta_j k_{ij} \qquad (3.7)$$

Writing Eq. (3.7) in its expanded form

$$P_i = k_{i1}\Delta_1 + k_{i2}\Delta_2 + \dots + k_{ij}\Delta_j + \dots k_{in}\Delta_n \qquad (3.8)$$

It is evident from the expanded form that Eq. (3.7) is a superposition equation expressing the total load at joint i as the sum of the loads developed by each deflection component Δ_j acting by itself. Each portion of Eq.(3.7) describes an independent component of the structural behaviour. The components may represent translation or rotation. The total number of components is the number of degrees of freedom which the idealized structure possesses.

Using matrix algebra notation, Eq. (3.7) may be rewritten as

$$P = K\Delta \tag{3.9}$$

Where P is a vector or column matrix made up of the load components P_1, P_2, ... P_i, ... P_n. Δ is a vector made up of the deflection components $\Delta_1, \Delta_2,$ ... $\Delta_i,$... Δ_n.

K is a square matrix consisting of an ordered array of the stiffness influence coefficients k_{ij} of Eq.(3.7). Matrix K is called the stiffness matrix of the structure. In the expanded form Eq. (3.9) appears as follows:

$$\begin{bmatrix} P_1 \\ P_2 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ P_n \end{bmatrix} = \begin{bmatrix} k_{11} & k_{12} \cdots\cdots & k_{1n} \\ k_{21} & k_{22} \cdots\cdots & k_{2n} \\ \cdot & & \\ \cdot & & \\ \cdot & & \\ \cdot & & \\ k_{n1} & k_{n2} \cdots\cdots & k_{nn} \end{bmatrix} \begin{bmatrix} \Delta_1 \\ \Delta_2 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \Delta_n \end{bmatrix} \tag{3.10}$$

(b) Flexibility Matrix Method.

Following the principle of superposition of deflections we can write in matrix form the force-deflection equations for a general structure as

$$-\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_i \\ \vdots \\ u_n \end{bmatrix} = \begin{bmatrix} f_{11} & f_{12} \cdots & f_{1j} \cdots & f_{1n} \\ f_{21} & f_{22} \cdots & f_{2j} \cdots & f_{2n} \\ \vdots & & & \\ f_{i1} & f_{i2} \cdots & f_{ij} \cdots & f_{in} \\ \vdots & & & \\ f_{n1} & f_{n2} \cdots & f_{nj} \cdots & f_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_j \\ \vdots \\ x_n \end{bmatrix} \qquad (3.11)$$

or

$$-U = F\,X \qquad (3.12)$$

In this matrix equation X is a column matrix composed of the unknown redundant forces x_1, x_2, ... x_n, which are to be removed in order to make the structure statically determinate. The elements of the vector U are the deflections of the released structure at the position and in the direction of the respective redundants due to applied loads.

F is called the flexibility matrix of the structure. It is composed of the elements like f_{ij}, called the flexibility influence coefficients and defined as follows.

f_{ij} = the deflection of the released structure at the position and in the direction of x_i for a unit value of the redundant x_j acting alone.

It is evident from the very definition of the coefficients k_{ij} and f_{ij} that

$$k_{ij} = k_{ji} \quad \text{and} \quad f_{ij} = f_{ji}$$

Hence, for any structure, both the stiffness matrix K and the flexibility matrix F will be symmetrical.

In the typical problem Eq. (3.10) must be constructed and solved for the deflections in terms of the applied loads. The direct determination of the coefficients f_{ij} of Eq.(3.11) is difficult and impractical for a large indeterminate structure. However, the coefficients k_{ij} of Eqn. (3.9) can be readily calculated. The usual procedure is thus to assemble the stiffness matrix directly from the known properties of the individual members of the structure. Then the matrix equation (3.10) is solved for the unknown deflection components of the vector Δ, which, when substituted into the load-deflection equations for individual members, give the internal stresses everywhere inside the structure.

However, in some problems where the number of unknown joint deflections is large as compared to the number of redundant forces, as in the case of indeterminate pin-jointed trusses, the number of equations in the displacement method will be much larger as compared to that in force method where the redundant forces are taken as unknowns. Hence, the solution of such problems will involve less labour if solved by the flexibility method. The details of these methods will be discussed later as we proceed.

4.1 Introduction.

We have seen, by now, that matrix analysis of indeterminate structures consists in expressing the load-displacement equations for the structure in the form of one single matrix equation and then solving that matrix equation for the unknowns. For instance, in the Stiffness Matrix method, the matrix equation obtained is of the form

$$P = K \triangle$$

To determine the elements of the unknown displacement vector $\triangle$ we premultiply both sides of the above equation by K^{-1}, the inverse of the stiffness matrix K . Thus,

$$\triangle = K^{-1}P$$

The computational work, thus, lies in obtaining the reciprocal matrix K^{-1} and then computing the deflections by determining the matrix product of the applied load vector and the matrix K . But the task of inverting the matrix K , if attempted by hand calculation on a slide rule or a desk calculator, is a very cumbersome process. This job is best suited to Modern High Speed Automatic Digital Computers which will invert a matrix of ordinary size in a few minutes and the matrix as large as having 100 rows and 100 columns in a few hours - the job which is impossible to attempt by hand methods. Also the chances to commit an error by these computers are very remote, rather the results may be taken as accurate as the data fed into the machine.

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CHAPTER 4

COMPUTATION

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$$\triangle = K^{-1}P$$

The computational work, thus, lies in obtaining the reciprocal matrix K^{-1} and then computing the deflections by determining the matrix product of the applied load vector and the matrix K . But the task of inverting the matrix K , if attempted by hand calculation on a slide rule or a desk calculator, is a very cumbersome process. This job is best suited to Modern High Speed Automatic Digital Computers which will invert a matrix of ordinary size in a few minutes and the matrix as large as having 100 rows and 100 columns in a few hours - the job which is impossible to attempt by hand methods. Also the chances to commit an error by these computers are very remote, rather the results may be taken as accurate as the data fed into the machine.

Due to the non-availability of such big digital computers for routine design and research work in our country, same method has to be looked for, which will make the best use of more easily available electrically operated desk calculating machines. Keeping in view the constantly increasing complexity of structural problems it is recommended that the traditional slide rule be replaced by a desk calculator made available to every structural engineer in the design office.

Prescott D. Crout has evolved an "auxiliary matrix method" for solving a matrix equation - as obtained in a structural problem. His method is best suited to an electric desk calculator which can store the products of numbers. It is proposed, for our work, to solve the typical problems of structural analysis by this method on a similar machine available. For this reason the method shall be discussed in details, and also the Doolittle technique of matrix inversion, which is most commonly adopted for solution of large size matrix equations on automatic digital computers, shall be indicated at the end of this chapter.

4.2 Crout's Auxiliary Matrix Method.

The work of solving a system of equations is largely concentrated in the determination of an "auxiliary matrix" and is roughly half that required by a matrix multiplication. The process is particularly adapted for use with a computing machine, for each element is determined by one continuous machine operation (sum of products with or without a final division).

The setting down of this matrix and of the final solution is the only writing required by the process. The work involved is cut almost in half if the coefficient matrix of the given matrix equation is symmetrical, as always happens with a structural problem. A "check column" can be carried along if desired.

The amount of work required to obtain a solution is considerably less than that required by the Gauss's method of successive elimination, even when there is symmetry and the coefficients are real, in which case Gauss's method has been considerably refined by Doolittle. (Gauss's method is much shorter than a solution by determinants.)

The method as given by Crout is applicable to m equations in n unknowns, there being no restriction on the rank of the matrix of the coefficients. But in a structural problem the coefficient matrix is essentially a square matrix and hence the method will be illustrated for n equations in n unknowns - n can be any natural number.

4.2.1 Description of the method.

Let the given system of equations be specified by its given matrix, thus

x_1	x_2	x_3	x_4	=
1	4	1	3	2
0	-1	3	-1	1
3	1	0	2	-1
1	-2	5	1	3

(4.1)

the first equation being

$$x_1 + 4x_2 + x_3 + 3x_4 = 2$$

The solution requires the formation of one matrix and a set of final results; thus we have an auxiliary matrix

x_1	x_2	x_3	x_4	=	
1	4	1	3	2	
0	-1	-3	1	-1	(4.2)
3	-11	-36	-0.1111	0.5000	
1	-6	-14	+2.4446	0.81812	

and a final matrix

$$x_1 = -0.86345$$
$$x_2 = -0.04545 \qquad (4.3)$$
$$x_3 = 0.59089$$
$$x_4 = 0.81812$$

The procedure for obtaining the auxiliary matrix from the given matrix is contained in the following rules.

(1) The various numbers or elements are obtained in the following order: elements of first column, then elements of first row to the right of the first column; elements of second column below first row, then elements of second row to the right of second column; elements of third column below the second row, then elements of third row to the right of third column; and so on until all elements are determined.

(2) The first column is identical with the first column of the given matrix. Each element of the first row except the first is obtained by dividing the corresponding element of the given matrix by that first element.

(3) Each element on or below the principal diagonal is equal to the corresponding element of the given matrix minus the sum of those products of elements in its row and corresponding elements in its column (in the auxiliary matrix) which involve only previously computed elements.

(4) Each element to the right of the principal diagonal is given by a calculation which differs from rule (3) only in that there is a final division by its diagonal element (in the auxiliary matrix).

As examples we have the following typical calculations made in obtaining (4.2), the letters R and C representing the words "row" and "column" respectively.

$R_1\ C_3$	1	=	$1 \div 1$
$R_2\ C_2$	-1	=	$-1 - 0 \times 4$
$R_4\ C_2$	-6	=	$-2 - 1 \times 4$
$R_2\ C_5$	-1	=	$(1 - 0 \times 2) \div (-1)$
$R_3\ C_3$	-36	=	$0 - 3 \times 1 - (-11) \times (-3)$
$R_4\ C_3$	-14	=	$5 - 1 \times 1 - (-6) \times (-3)$
$R_3\ C_4$	-0.1111	=	$\{2 - 3 \times 3 - (-11) \times (+1)\} \div (-36)$
$R_4\ C_4$	2.4446	=	$1 - 1 \times 3 - (-6) \times 1 - (-14) \times (-0.1111)$
$R_4\ C_5$	0.81812	=	$(3 - 1 \times 2 - 6 \times 1 + 14 \times 0.5) \div 2.4446$

Since an electric desk calculator gives in one continuous operation a sum or difference of products with or without a final division, we see that each element of the auxiliary matrix is given by a single machine operation.

The procedure for obtaining the one columned final matrix from the auxiliary matrix is contained in the following rules.

(1) The elements are determined in the following order: last, next to last, second from last, third from last etc.

(2) The last element is equal to the corresponding element in the last column of the auxiliary matrix.

(3) Each element is equal to the corresponding element of the last column of the auxiliary matrix minus the sum of those products of elements in its row in the auxiliary matrix and corresponding elements in its column in the final matrix which involve only previously computed elements.

We see that in forming products only those elements of the auxiliary matrix are used which lie to the right of the principal diagonal and to the left of the last column. The calculations made in obtaining (4.3) are

$$R_3 \quad C_1 \quad 0.59089 = +0.5000 + 0.1111 \times 0.81812$$

$$R_2 \quad C_1 \quad -0.04545 = -1.00 - 1 \times 0.81812 + 3 \times 0.59089$$

$$R_1 \quad C_1 \quad -0.86345 = 2 - 3 \times 0.81812 - 1 \times 0.59089 + 4 \times 0.04545$$

It may be noted that each element of the final matrix is given by a single machine operation.

It is not necessary but is strongly recommended that the values of the unknowns, which compose the final matrix, be substituted in each of the given equations, the result being a number of checks equal to the number of equations. Since the satisfaction of these checks guarantees the correctness of the solution, it is not necessary to check the calculations which gave the auxiliary matrix and the final matrix. The first of the four checks obtained from (4.1) and (4.3) is

$$-1 \times 0.86345 - 4 \times 0.04545 + 1 \times 0.59089$$
$$+ 3 \times 0.81812 = 2.00$$

Evidently each check requires but one machine operation.

4.2.2. Systems having symmetrical coefficient matrix.

If there is symmetry (as is the usual case with structural problems, since both the stiffness matrix or the flexibility matrix of the problem have to be symmetrical), the work of computing the auxiliary matrix is cut almost in half by the fact that if the coefficients of the unknowns are symmetrical about the principal diagonal; each element of the auxiliary matrix below the principal diagonal gives, if divided by its diagonal element, the symmetrically opposite element above this diagonal. Elements below the principal diagonal of the auxiliary matrix are thus obtained as bye products of calculations made in determining elements above this diagonal.

As an example, the symmetrical set of equations

x_1	x_2	x_3	x_4	x_5	=	
0.55777	0.017888		0.01183	-0.02683	40	
0.17888	0.71554	0.17888			0	
	0.17888	0.55777	-0.02683	0.01183	-40	(4.4)
0.01183		-0.02683	0.00686	-0.00536	-24	
-0.02683		0.01183	-0.00536	0.00686	24	

has the auxiliary matrix,

x_1	x_2	x_3	x_4	x_5	=	x	
0.55777	0.30270	0	0.02120	0.04810		71.7141	
0.17888	0.65817	0.27178	-0.00576	0.01307		-19.4907	
0	0.17888	0.50915	-0.05067	0.01864		-17.7146	(4.5)
0.01183	-0.00379	-0.02580	0.00528	-0.80871		-5070.54	
-0.02683	0.00860	0.00949	-0.00427	0.00182		2813.77	

and the final matrix

$$
\begin{aligned}
x_1 &= +266.378 \\
x_2 &= -0.13040 \\
x_3 &= -265.787 \\
x_4 &= -2795.02 \\
x_5 &= +2813.77
\end{aligned}
\qquad (4.6)
$$

In the auxiliary matrix the lement in row 3 and column 4 is

$$(-0.02683 - 0 \times 0.02120 + 0.17888 \times 0.00576) \Big/ 0.50915$$

$$= \frac{-0.02580}{0.50915} = -0.05067$$

the numerator - 0.02580 being recorded in the symmetrically opposite position before the final division by the diagonal element 0.50915 is carried out. The final matrix is obtained in the usual manner.

If , now, we change only the last column of the given matrix, the solution to this new set of matrix equation shall be very readily obtained. The first five columns of the auxiliary matrix shall remain unaltered and only the last column has to be re-calculated. The final matrix is calculated in the usual manner from this new column of the auxiliary matrix.

For example, let us now recalculate the value of 10 unknowns with the changed last column of the given matrix as

$$\begin{bmatrix} -15 \\ -5 \\ 0 \\ 10.5 \\ 1.5 \end{bmatrix}$$

the last column of the auxiliary matrix becomes

$$\begin{bmatrix} -26.8928 \\ -0.28779 \\ 0.10110 \\ 2049.17 \\ 5236.22 \end{bmatrix}$$

The final matrix is given by

$$x_1 = 119.773$$

$$x_2 = -92.5657$$

$$x_3 = 220.895$$

$$x_4 = 6283.75$$

$$x_5 = 5236.22$$

This saving in calculation work is of great significance if a structure has to be analysed for a number of loading conditions.

4.2.3 Continuous Check on Calculation.

If desired, a "check column" may be written at the right of the given matrix, each element of this column being the sum of the elements of the corresponding row in the matrix. This column is now treated in exactly the same manner as the last column of the given matrix, the calculations being carried along with those for the other columns, and the result being the addition of corresponding " check columns" to the auxiliary matrix and the final matrix. The check columns thus obtained for (4.1), (4.2), and (4.3) are, respectively

$$\begin{matrix} 11 \\ 2 \\ 5 \\ 8 \end{matrix}, \quad \begin{matrix} 11 \\ -2 \\ 1.3889 \\ 1.81812 \end{matrix} \quad \text{and} \quad \begin{matrix} 0.13655 \\ 0.95455 \\ 1.59089 \\ 1.81812 \end{matrix} \qquad (4.7)$$

These columns provide checks at all stages of the computation, because

1. In the auxiliary matrix any element in the check column is equal to one plus the sum of the other elements in its row which lie to the right of the principal diagonal.

2. In the final matrix any element in the check column is equal to one plus the sum of the other elements in its row.

For example noting (4.2), (4.3), and (4.7), two of the checks are

$$1 - 3 + 1 - 1 = -2$$

$$1 - 0.04545 = 0.95455$$

The above statements are true and the procedure is the same for any number of equations and unknowns.

4.2.4 Improvement in Accuracy.

Since the number of decimal places in the computations is limited, the values obtained for the unknowns are in general not exact. However, if they are placed in the given equations and the differences between the two sides are obtained, and if these differences are then inserted in place of the right hand sides of the given equations, the resulting equations have as their solution the corrections to the values first obtained. Noting that the above differences are obtained in applying the final checks (that of substituting the computed values of unknowns in the given set of equations), and that the auxiliary matrix for the modified equations is the same as that for the original equations except for the last column, it follows that if the column of the differences obtained in in applying the final checks be annexed to the given matrix and then treated in the same manner as the last column, the corresponding column obtained in the final matrix is composed of the required corrections.

Since the problem of solving the modified equations is similar to the original problem, the above process may be repeated; thus the final checks on the corrections give data for another column in the given matrix, which leads to a column in the final matrix composed of corrections to the first corrections, etc. In the usual case each application of this process increases the number of significant figures in the results by approximately the same number obtained with the original solution, the data in the given equations being considered exact.

But in problems of structural analysis results correct upto three significant places of decimal are usually accepted for subsequent design work. By using the available Marchant Electrical desk calculating machine and working with five decimal places, fairly exceptable correct results are being obtained in the first solution. Hence, the labour involved in calculating the corrections is not justified for our purpose by obtaining a little more accuracy in results over those obtained in the first set of calculations.

4.2.5 Mathematical Proof.

The mathematical proofs which established the method have been omitted in describing the technique. The method, in essence, is a combination of various processes which compose Gauss's elimination method, and adapting them for use with a computing machine. However, direct proofs of the induction type have been given by Crout in his original paper and the reader, if interested, may make a reference of the same. (See 4 'References').

4.3 Matrix inversion by Doolittle technique.

Large size matrix equations are most conveniently solved on automatic digital computers with a very high speed and degree of accuracy. The technique of inverting large size matrices, which is generally adopted for use on such giant size machines, is explained below in a tabular form. Let the coefficient matrix K of a set of simultaneous equations be given by

$$K = \begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14} \\ k_{21} & k_{22} & k_{23} & k_{24} \\ k_{31} & k_{32} & k_{33} & k_{34} \\ k_{41} & k_{42} & k_{43} & k_{44} \end{bmatrix}$$

		GIVEN MATRIX				
	(1)	k_{11}	k_{12}	k_{13}	k_{14}	
	(2)	k_{21}	k_{22}	k_{23}	k_{24}	
	(3)	k_{31}	k_{32}	k_{33}	k_{34}	(
	(4)	k_{41}	k_{41}	k_{43}	k_{44}	(
	(5)	$\frac{k_{11}}{k_{14}}$	$\frac{k_{12}}{k_{14}}$	$\frac{k_{13}}{k_{14}}$	1	1/
	(6)	$\frac{k_{21}}{k_{24}}$	$\frac{k_{22}}{k_{24}}$	$\frac{k_{23}}{k_{24}}$	1	c
	(7)	$\frac{k_{31}}{k_{34}}$	$\frac{k_{32}}{k_{34}}$	$\frac{k_{33}}{k_{34}}$	1	c
	(8)	$\frac{k_{41}}{k_{44}}$	$\frac{k_{42}}{k_{44}}$	$\frac{k_{43}}{k_{44}}$	1	c
) - (8)	(9)	l_{11}	l_{12}	l_{13}		1/
(6) - (8)	(10)	l_{21}	l_{22}	l_{23}		c
(7) - (8)	(11)	l_{31}	l_{32}	l_{33}		c
	(12)	$\frac{l_{11}}{l_{13}}$	$\frac{l_{12}}{l_{13}}$	1		$\frac{1}{k_{14}}$
	(13)	$\frac{l_{21}}{l_{23}}$	$\frac{l_{22}}{l_{23}}$	1		c
	(14)	$\frac{l_{31}}{l_{33}}$	$\frac{l_{32}}{l_{33}}$	1		c
(12) - (14)	(15)	m_{11}	m_{12}			$-\frac{1}{k_{14}}$
(1) - (14)	(16)	m_{21}	m_{22}			c
	(17)	$\frac{m_{11}}{m_{2}}$	1			$\frac{1}{k_{14}}$
	(18)	$\frac{m_{21}}{m_{22}}$	1			c
((18)	(19)	n_{11}				
	(20)	1				C

TAB

Similarly from row (12), c_{31}, c_{32}, c_{33}, c_{34}, are obtained

$$c_{11} \frac{l_{41}}{l_{13}} + c_{21} \frac{l_{12}}{l_{13}} + c_{31} = \frac{1}{k_{14}\, l_{13}}$$

and from (5) we have c_{41}, c_{42}, c_{43}, c_{44}.

$$c_{11} \frac{k_{11}}{k_{14}} + c_{21} \frac{k_{12}}{k_{14}} + c_{31} \frac{k_{13}}{k_{14}} + c_{41} = \frac{1}{k_{14}}$$

Thus, the elements of the inverted matrix in rows other than the first are obtained by a process of back substitution.

To illustrate the above procedure we shall solve a problem of matrix inversion on the desk calculator as follows. With the aid of automatic digital computers, the same technique is extended and coded for inverting large size square matrices.

Table 4.2

	Given matrix				Unit matrix			
(1)	1	4	1	3	1	0	0	0
(2)	0	-1	3	-1	0	1	0	0
(3)	3	1	0	2	0	0	1	0
(4)	1	-2	5	1	0	0	0	1
(5)	0.3333	1.3333	0.3333	1	0.3333	0	0	0
(6)	0	1	-3	1	0	-1	0	0
(7)	1.5	0.5	0	1	0	-0	0.5	0
(8)	1	-2	5	1	0	0	0	1

		GIVEN MATRIX				UNIT MATRIX			
	(1)	k_{11}	k_{12}	k_{13}	k_{14}	1	0	0	0
	(2)	k_{21}	k_{22}	k_{23}	k_{24}	0	1	0	0
	(3)	k_{31}	k_{32}	k_{33}	k_{34}	0	0	1	0
	(4)	k_{41}	k_{42}	k_{43}	k_{44}	0	0	0	1
	(5)	$\frac{k_{11}}{k_{14}}$	$\frac{k_{12}}{k_{14}}$	$\frac{k_{13}}{k_{14}}$	1	$1/k_{14}$	0	0	0
	(6)	$\frac{k_{21}}{k_{24}}$	$\frac{k_{22}}{k_{24}}$	$\frac{k_{23}}{k_{24}}$	1	0	$1/k_{24}$	0	0
	(7)	$\frac{k_{31}}{k_{34}}$	$\frac{k_{32}}{k_{34}}$	$\frac{k_{33}}{k_{34}}$	1	0	0	$1/k_{34}$	0
	(8)	$\frac{k_{41}}{k_{44}}$	$\frac{k_{42}}{k_{44}}$	$\frac{k_{43}}{k_{44}}$	1	0	0	0	$1/k_{44}$
)-(8)	(9)	l_{11}	l_{12}	l_{13}		$1/k_{14}$	0	0	$-1/k_{44}$
(6)-(8)	(10)	l_{21}	l_{22}	l_{23}		0	$1/k_{24}$	0	$1/k_{44}$
(7)-(8)	(11)	l_{31}	l_{32}	l_{33}		0	0	$1/k_{34}$	$-1/k_{44}$
	(12)	$\frac{l_{11}}{l_{13}}$	$\frac{l_{12}}{l_{13}}$	1		$\frac{1}{k_{14}\,l_{13}}$	0	0	$-\frac{1}{k_{44}\,l_{13}}$
	(13)	$\frac{l_{21}}{l_{23}}$	$\frac{l_{22}}{l_{23}}$	1		0	$\frac{1}{k_{24}\,l_{23}}$	0	$-\frac{1}{k_{44}\,l_{23}}$
	(14)	$\frac{l_{31}}{l_{33}}$	$\frac{l_{32}}{l_{33}}$	1		0	0	$\frac{1}{k_{34}\,l_{33}}$	$-\frac{1}{k_{44}\,l_{33}}$
(12)-(14)	(15)	m_{11}	m_{12}			$\frac{1}{k_{14}\,l_{13}}$	0	$-\frac{1}{k_{34}\,l_{33}}$	$\frac{1}{k_{44}\,l_{33}} - \frac{1}{k_{44}\,l_{13}}$
(1)-(14)	(16)	m_{21}	m_{22}			0	$\frac{1}{k_{24}\,l_{23}}$	$-\frac{1}{k_{34}\,l_{33}}$	$\frac{1}{k_{44}\,l_{33}} - \frac{1}{k_{44}\,l_{23}}$
	(17)	$\frac{m_{11}}{m_{12}}$	1			$\frac{1}{k_{14}\,l_{13}\,m_{12}}$	0	$-\frac{1}{k_{34}\,l_{33}\,m_{12}}$	$\frac{1}{m_{12}\,k_{44}}\left(\frac{1}{l_{33}} - \frac{1}{l_{13}}\right) = a$
	(18)	$\frac{m_{21}}{m_{22}}$	1			0	$\frac{1}{k_{24}\,l_{23}\,m_{22}}$	$-\frac{1}{k_{34}\,l_{33}\,m_{22}}$	$\frac{1}{m_{22}\,k_{44}}\left(\frac{1}{l_{33}} - \frac{1}{l_{23}}\right) = b$
((18)	(19)	n_{11}				(a) - (b)			
	(20)	1				C_{11}	C_{12}	C_{13}	C_{14}

TABLE 4·1

For ease in explaining we have chosen a 4 x 4 matrix, although the method can be extended to any order. Table 4.1 shows step by step the Doolittle technique. On the left hand side is the given matrix and on the right hand side is the unit matrix. The table is self-explanatory.

The last four elements, i.e., c_{11} , c_{12} , c_{13} , c_{14} , obtained from the unit matrix, form the first row of elements of the inverse matrix, given below.

$$K^{-1} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & c_{14} \\ c_{21} & c_{22} & c_{23} & c_{24} \\ c_{31} & c_{32} & c_{33} & c_{34} \\ c_{41} & c_{42} & c_{43} & c_{44} \end{bmatrix}$$

To obtain the other elements of the inverse, multiplying row (17) with K^{-1} i.e., $\left[\frac{m_{11}}{m_{12}} K^{-1}\right] \left[I\right]$, we have

$$c_{11}\frac{m_{11}}{m_{12}} + c_{21} = \frac{1}{k_{14}\,l_{13}\,m_{12}}$$

$$c_{12}\frac{m_{11}}{m_{12}} + c_{22} = 0$$

$$c_{13}\frac{m_{11}}{m_{12}} + c_{23} = \frac{1}{k_{34}\,l_{33}\,m_{12}}$$

$$c_{14}\frac{m_{11}}{m_{12}} + c_{24} = \left(\frac{1}{k_{44}\,l_{33}} - \frac{1}{k_{44}\,l_{13}}\right)\frac{1}{m_{12}}$$

which give the values of c_{21} , c_{22} , c_{23} , c_{24} .

Similarly from row (12), c_{31} , c_{32}, c_{33}, c_{34} , are obtained

$$c_{11}\frac{l_{41}}{l_{13}} + c_{21}\frac{l_{12}}{l_{13}} + c_{31} = \frac{1}{k_{14}\,l_{13}}$$

and from (5) we have c_{41} , c_{42} , c_{43} , c_{44} .

$$c_{11}\frac{k_{11}}{k_{14}} + c_{21}\frac{k_{12}}{k_{14}} + c_{31}\frac{k_{13}}{k_{14}} + c_{41} = \frac{1}{k_{14}}$$

Thus, the elements of the inverted matrix in rows other than the first are obtained by a process of back substitution.

To illustrate the above procedure we shall solve a problem of matrix inversion on the desk calculator as follows. With the aid of automatic digital computers, the same technique is extended and coded for inverting large size square matrices.

Table 4.2

	Given matrix				Unit matrix			
(1)	1	4	1	3	1	0	0	0
(2)	0	-1	3	-1	0	1	0	0
(3)	3	1	0	2	0	0	1	0
(4)	1	-2	5	1	0	0	0	1
(5)	0.3333	1.3333	0.3333	1	0.3333	0	0	0
(6)	0	1	-3	1	0	-1	0	0
(7)	1.5	0.5	0	1	0	-0	0.5	0
(8)	1	-2	5	1	0	0	0	1

		Given matrix			Unit matrix			
(5)-(8)	(9)	-0.6667	3.3333	-4.667	0.3333	0	0	-1
(6)-(8)	(10)	-1	3	-8	0	~~-0.125~~ -1.00	0	-1
(7)-(8)	(11)	0.5	2.5	-5	0	0	0.5	-1
	(12)	0.1428	-0.7143	1	-0.0714	0	0	0.2143
	(13)	0.125	-0.375	1	0	0.125	0	0.125
	(14)	-0.1	-0.5	1	0	0	-0.1	0.2
(12)-(14)	(15)	0.2428	-0.2143		-0.0714	0	0.1	0.0143
(13)-(14)	(16)	0.225	0.125		0	0.125	0.1	-0.075
	(17)	-1.1329	1		0.3332	0	-0.4666	-0.0667
	(18)	1.8	1		0	1	0.8	-0.6
(17)-(18)	(19)	-2.9329			0.3332	-1	-1.2666	0.5333
	(20)	1			-0.1136	0.3409	0.4318	-0.1818

Row (20) of table (4.2) gives the values of the elements in the first row of the inverse matrix. Elements of the second row are obtained from equations formed with row (17) as follows.

$$1.1329 \times 0.1136 + c_{21} = 0.3332$$

$$-1.1329 \times 0.3409 + c_{22} = 0$$

$$-1.1329 \times 0.4318 + c_{23} = -0.4666$$

$$1.1329 \times 0.1818 + c_{24} = -0.0667$$

which in turn give

$$c_{21} = 0.2045$$
$$c_{22} = 0.3862$$
$$c_{23} = 0.0226$$
$$c_{24} = -0.2726$$

Similarly, from row (12) we have

$$-0.1428 \times 0.1136 - 0.7143 \times 0.2045 + c_{31} = -0.0714$$
$$0.1428 \times 0.3409 - 0.7143 \times 0.3862 + c_{32} = 0$$
$$0.1428 \times 0.4318 - 0.7143 \times 0.0226 + c_{33} = 0$$
$$-0.1428 \times 0.1818 + 0.7143 \times 0.2726 + c_{34} = 0.2143$$

giving,

$$c_{31} = 0.0909$$
$$c_{32} = 0.2272$$
$$c_{33} = -0.0455$$
$$c_{34} = 0.0455$$

and from row (5)

$$-0.3333 \times 0.1136 + 1.3333 \times 0.2045 + 0.3333 \times 0.0909 + c_{41} = 0.3333$$
$$0.3333 \times 0.3409 + 1.3333 \times 0.3862 + 0.3333 \times 0.2272 + c_{42} = 0$$
$$0.3333 \times 0.4318 + 1.3333 \times 0.0226 - 0.3333 \times 0.0455 + c_{43} = 0$$
$$-0.3333 \times 0.1818 - 1.3333 \times 0.2726 + 0.3333 \times 0.0455 + c_{44} = 0$$

which gives,

$$c_{41} = 0.0682$$
$$c_{42} = -0.7043$$
$$c_{43} = -0.1589$$
$$c_{44} = 0.3678$$

The inverted matrix, thus becomes

$$K^{-1} = \begin{bmatrix} -0.1136 & 0.3409 & 0.4318 & -0.1818 \\ 0.2045 & 0.3862 & 0.0226 & -0.2726 \\ 0.0909 & 0.2272 & -0.0455 & 0.0455 \\ 0.0682 & -0.7043 & -0.1589 & 0.4089 \end{bmatrix}$$

CHAPTER 5

FLEXIBILITY MATRIX METHOD

5.1 General

The method to be adopted for the analysis of a statically indeterminate structure is the removal of the redundant reaction components or the member forces by introducing a number of releases in the structure so that it is transformed to a statically determinate one. The solution to the problem is obtained by determining what values of the redundants at the positions of, and of the kind corresponding to, the releases will enable geometrical continuity to be re-established at all the releases whene the structure is loaded. The redundant reaction components and the number forces are taken as the arbitrary constants of the problem since it is the values of these which have to be determined in solving the structure by satisfying the boundary conditions of continuity.

The symbol we shall use for the arbitrary constants is x and there will be as many of them as the structure has degrees of statical indeterminacy.

The most convenient way of arriving at the flexibility matrix method of structural analysis is through the concept of strain energy principles.

5.2 Flexibility Influence Coefficient Equations.

Let us consider the application of the theorem of least work to a simple problem; a three span continuous beam on rigid supports (Fig. 5.1).

The structure is released by removing the two intermediate supports, so that the support reactions at B and C become

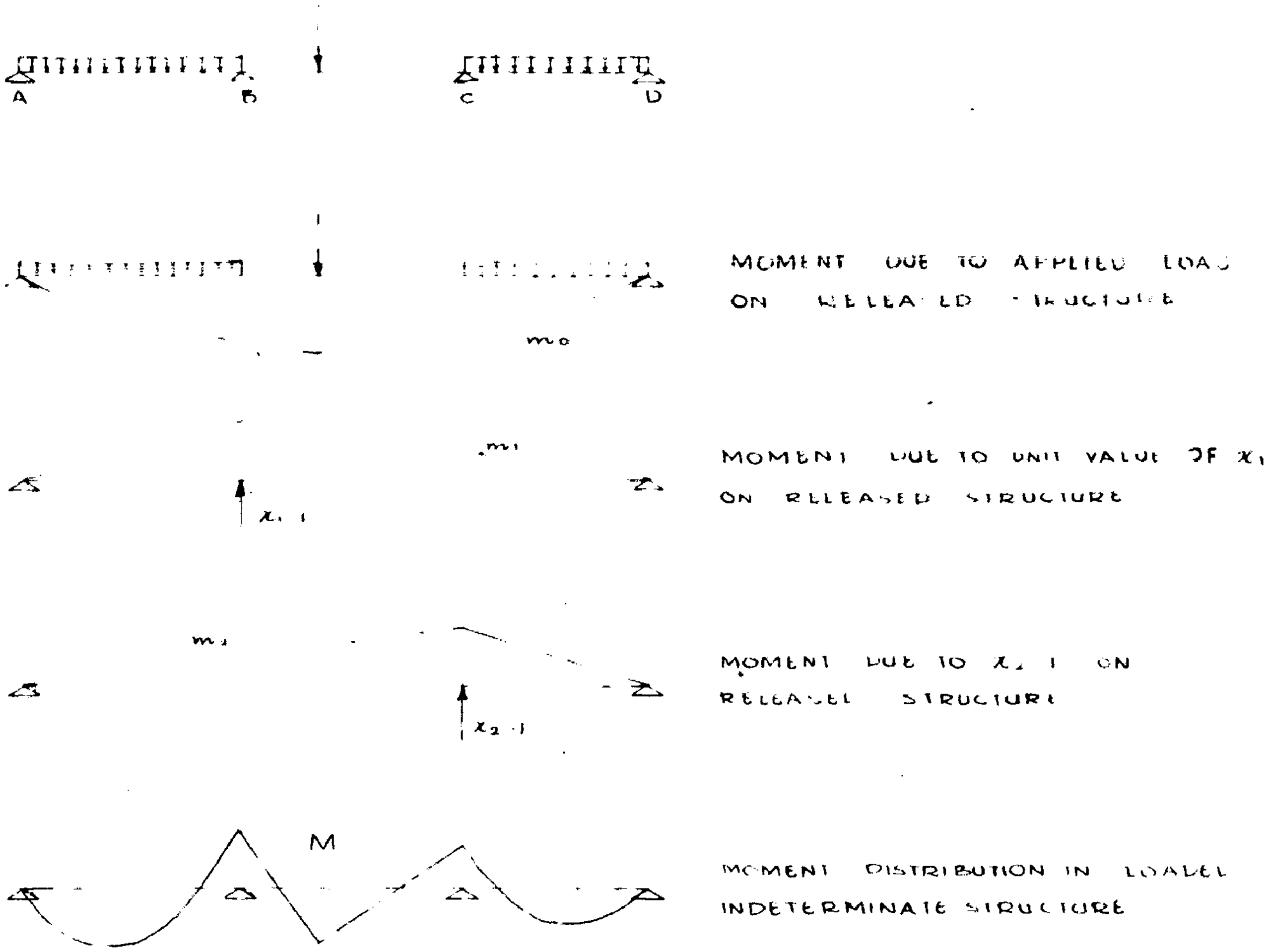
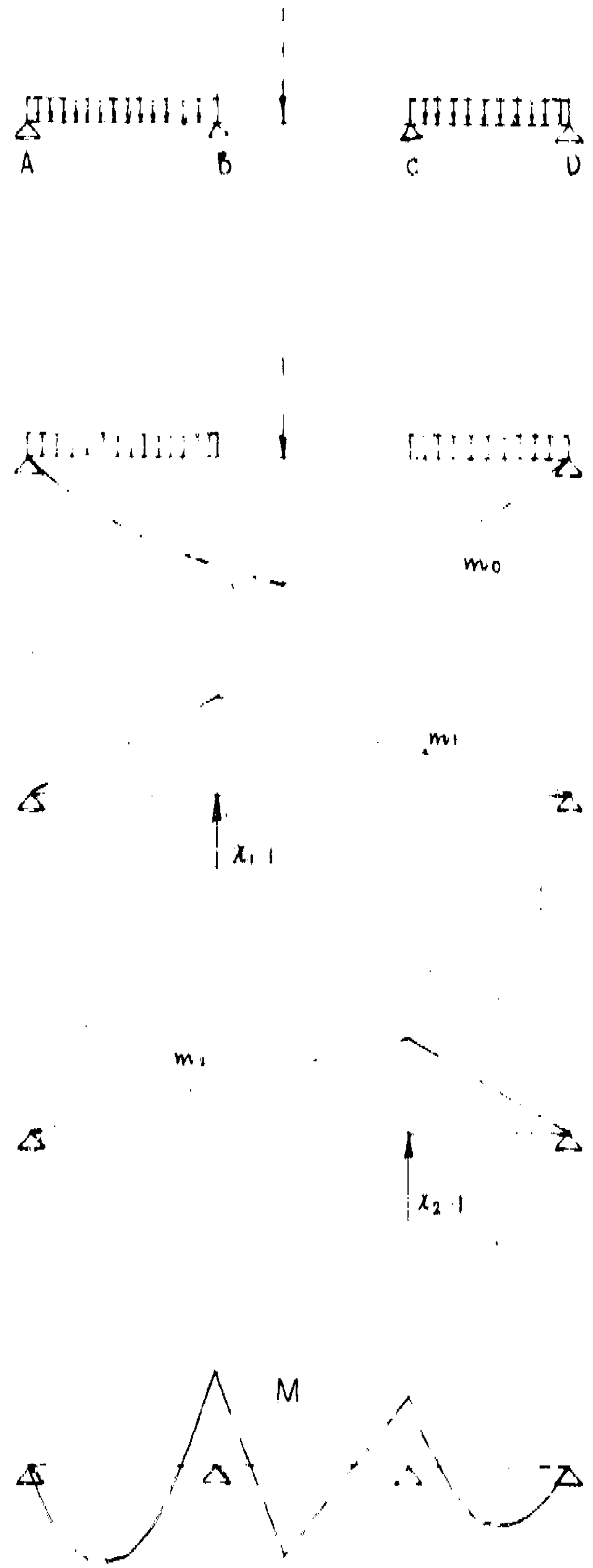


FIGURE 5·1

the arbitrary constants, x_1 and x_2, of the problem. We shall consider only the flexural strain energy. By the principle of superposition the total moment distribution M can be considered as composed of two parts:

$$M = m_0 + (m_1 x_1 + m_2 x_2) \qquad (5.1)$$

(a) m_0, the moment distribution due to the applied loads only acting on the released structure. This is also called the "particular solution" of the problem. It satisfies



MOMENT DUE TO APPLIED LOAD ON RELEASED STRUCTURE

MOMENT DUE TO UNIT VALUE OF X_1 ON RELEASED STRUCTURE

MOMENT DUE TO $X_2 = 1$ ON RELEASED STRUCTURE

MOMENT DISTRIBUTION IN LOADED INDETERMINATE STRUCTURE

FIGURE 5·1

the arbitrary constants, x_1 and x_2, of the problem.
We shall consider only the flexural strain energy. By the principle of superposition the total moment distribution M can be considered as composed of two parts:

$$M = m_0 + (m_1 x_1 + m_2 x_2) \qquad (5.1)$$

(a) m_0, the moment distribution due to the applied loads only acting on the released structure. This is also called the "particular solution" of the problem. It satisfies the conditions of equilibrium but not the geometrical compatibility of the problem.

(b) $(m_1 x_1 + m_2 x_2)$, the moment distribution due to the action of x_1 and x_2 acting on the released structure whose job is ~~thus~~ to satisfy the compatibility conditions. This is also termed as the " complementary function".

The total strain energy is given by

$$U = \int_S \frac{M^2}{2EI}\, ds$$

the integration extending over the whole length of the structure.

Applying the theorem of least work we obtain two equations for the determination of x_1 and x_2. Thus ,

$$\frac{\partial U}{\partial x_1} = \int_S \frac{\partial}{\partial x_1}\left(\frac{M^2}{2EI}\right) ds = 0$$

$$\frac{\partial U}{\partial x_2} = \int_S \frac{\partial}{\partial x_2}\left(\frac{M^2}{2EI}\right) ds = 0 \qquad (5.2)$$

Substituting for M from Eq. (5.1), Eqs. (5.2) become

$$\frac{\partial U}{\partial x_1} = \int_S \frac{m_1}{EI}\left(m_0 + m_1 x_1 + m_2 x_2\right) ds = 0$$

$$\frac{\partial U}{\partial x_2} = \int_S \frac{m_2}{EI}\left(m_0 + m_1 x_1 + m_2 x_2\right) ds = 0$$

which can also be expanded as

$$\delta_1 = x_1 \int_S \frac{m_1^2}{EI}\, ds + x_2 \int_S \frac{m_1 m_2}{EI}\, ds + \int_S \frac{m_1 m_0}{EI}\, ds = 0$$

$$\delta_2 = x_1 \int_S \frac{m_2 m_1}{EI}\, ds + x_2 \int_S \frac{m_2^2}{EI}\, ds + \int_S \frac{m_2 m_0}{EI}\, ds = 0 \qquad (5.3)$$

Since, Castigliano's second theorem gives

$$\frac{\partial U}{\partial x_1} = \delta_1$$

and
$$\frac{\partial U}{\partial x_2} = \delta_2$$

δ_1 and δ_2 being the deflections at the position and in the direction of x_1 and x_2 respectively.

Supposing,

(a) $m_o = x_2 = 0$, the first equation of (5.3) becomes

$$\delta_1 = x_1 \int_s \frac{m_1^2}{EI} ds = x_1 f_{11} \quad \text{(say)}$$

Thus f_{11} = the deflection of the released structure at the position and in the direction of x_1 for a unit value of x_1 acting alone (Figure 5.2).

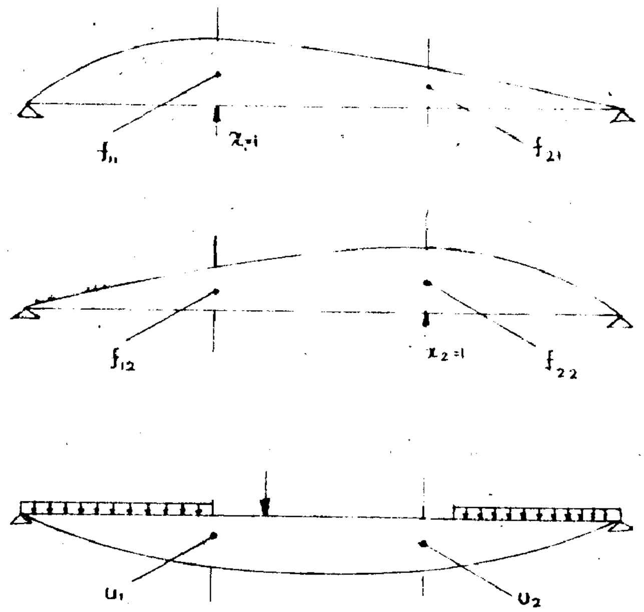


FIGURE 5·2

(b) $m_o = 0$ and $x_1 = 0$, then

$$\delta_1 = x_2 \int_s \frac{m_1 m_2}{EI} ds = x_2 f_{12} \quad \text{(say)}$$

f_{12} = the deflection of the released structure at the position and in the direction of x_1 for a unit value of x_2 acting alone (Fig. 5.2).

(c) $x_1 = x_2 = 0$, then

$$\delta_1 = \int_S \frac{m_1 m_0}{EI} ds = u_1 \quad (say)$$

u_1 = the deflection of the released structure at the position and in the direction of x_1 due to the applied loads (Fig. 5.2).

With these definitions for the integrals we can rewrite the first equation of (5.3) in the form

$$f_{11} x_1 + f_{12} x_2 = -u_1 \qquad (5.4\ a)$$

Similarly, we can write the second equation, which relates to the boundary condition at the support c , as

$$f_{21} x_1 + f_{22} x_2 = -u_2 \qquad (5.4\ b)$$

where f_{21} , f_{22} and u_2 have meanings similar to f_{12} , f_{11} and u_1 , but related to x_2 .

It has to be noted that

$$f_{12} = f_{21} = \int_S \frac{m_1 m_2}{EI} ds$$

which also follows from Maxwell's reciprocal theorem due to the physical meanings of f_{12} and f_{21} .

In matrix notations Eqs. (5.4) can be written as,

$$\begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = - \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \qquad (5.5)$$

The solution to the structural problem, the result the engineer requires, is the distribution of the bending moment M in the statically indeterminate structure which is computed by solving the matrix Eq. (5.5) for x_1 and x_2 and inserting these values in the expression

$$M = m_0 + m_1 x_1 + m_2 x_2$$

The above example is one in which there are two arbitrary constants, the redundant reactions x_1 and x_2. In general, the solution of a structure with n arbitrary constants (the redundants) will lead to a matrix equation

$$\begin{bmatrix} f_{11} & f_{12} \cdots\cdots & f_{1j} \cdots\cdots & f_{1n} \\ f_{21} & f_{22} \cdots\cdots & f_{2j} \cdots\cdots & f_{2n} \\ \vdots & & & \\ f_{i1} & f_{i2} \cdots\cdots & f_{ij} \cdots\cdots & f_{in} \\ \vdots & & & \\ f_{n1} & f_{n2} \cdots\cdots & f_{nj} \cdots\cdots & f_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_j \\ \vdots \\ x_n \end{bmatrix} = - \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_i \\ \vdots \\ u_n \end{bmatrix} \qquad (5.6)$$

where,

$$f_{ij} = \int_s \frac{m_i m_j}{EI} ds \qquad (5.7)$$

and

$$u_i = \int_s \frac{m_i m_0}{EI} ds \qquad (5.8)$$

still assuming that flexural energy alone is significant.

The meanings of the f's and u's remain as displacements which may include deflections and rotations according to whether the corresponding arbitrary constants are forces or moments.

Writing Eq. (5.6) as

$$FX = -U \qquad (5.9)$$

F is the " flexibility matrix" of the structure which is composed of the flexibility influence coefficients of the type given in Eq. (5.7). It is to be noted that the matrix F is a symmetrical matrix.

The matrix Eq. (5.6) is solved for the values of the elements of the vector x by our excepted methods of computation and the final distribution of moment will be determined from the expression

$$M = m_0 + m_1 x_1 + m_2 x_2 + \ldots\ldots m_n x_n \quad (5.10)$$

5.3 Other Strain Energies.

If it is necessary to consider the effect of shear and direct forces also in contributing to the strain energy, then

$$U_{total} = U_{bending} + U_{shear} + U_{direct\ force}$$

for plane frames.

The expressions for the influence coefficients f_{ij} and that for u_i , will, in the complete plane frame case, become

$$\left.\begin{aligned} f_{ij} &= \int_s \frac{m_i m_j}{EI} ds + k \int_s \frac{s_i s_j}{GA} ds + \int_s \frac{n_i n_j}{EA} ds \\ &= f_{ij}^m + f_{ij}^s + f_{ij}^n \\ u_i &= \int_s \frac{m_i m_0}{EI} ds + k \int_s \frac{s_i s_0}{GA} ds + \int_s \frac{n_i n_0}{EA} ds \\ &= u_i^m + u_i^s + u_i^n \end{aligned}\right\} \quad (5.11)$$

where s and n are the " unit shear force and direct force diagrams" defined in the same way as the m diagrams.

It is to be pointed out that, although a problem may be solved by considering only one, or some, of the stress resultants contributing significantly to the total strain energy, it is of course still possible to determine the distribution of all the other stress resultants. Thus, in the case of the three-span beam discussed above, the total shear distribution S is determined by adding the contributions due to the applied loading so, due to the arbitrary constants $s_1 x_1 + s_2 x_2$, so that

$$S = s_0 + s_1 x_1 + s_2 x_2$$

5.4 <u>Evaluation of the Integrals.</u>

The various quantities whose values have to be evaluated appear as the integrals of the products of the ordinates of diagrams of moment (or direct force, shear etc.) and of variations in structural properties ($\frac{1}{EI}$, $\frac{1}{EA}$,etc). A visual representation of the stress resultant and structural property distributions will be valuable and we shall, wherever possible, draw diagrams. However, in order to draw diagrams, we shall require sign conventions.

(a) Bending Moments.

We shall adopt the convention of drawing the positive ordinate on the side of the member under tensile bending stress.

(b) Direct force.

Direct forces will be denoted positive if they are tensile.

(c) Shear force.

We put an arrow on each member to indicate the sense of increasing length coordinate. Then sitting on the member behind the arrow and cutting it in front of the arrow we draw the shear diagram on the side which the portion of the member in front of the cut would appear to us to move.

The calculation of integrals of the type

$$\int_s \frac{m_i m_j}{EI} ds$$

is often considerably simplified if we remember that they are always zero if one is symmetrical ($) and the other is antisymmetrical (A/S) i.e. if m_i is $ and m_j is A/S and $\frac{1}{EI}$ is $

$$\int_s \frac{m_i m_j}{EI} ds = 0$$

In carrying out product integrations in the general case we shall frequently have to use an approximate method of integration. Of the many formulae which exist for this purpose that known as Simpson's first rule is probably the most convenient since it is easy to apply and is precise upto third degree curves.

In the case of simple geometrical figures we can determine useful expressions for the product integrals. A number of these are given in Table 5.1.

PRODUCT INTEGRALS $\int_0^1 m_i m_j \, ds$

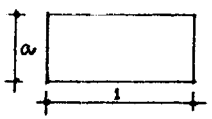
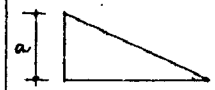
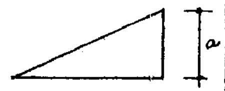
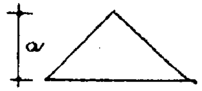
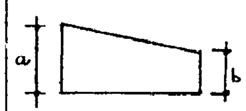
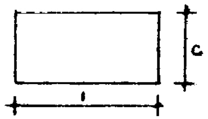
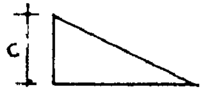
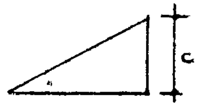
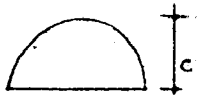
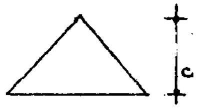
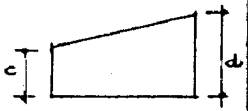
m_j \ m_i						
	ac	$\frac{1}{2} ac$	$\frac{1}{2} ac$	$\frac{2}{3} ac$	$\frac{1}{2} ac$	$\frac{1}{2}(a+b)c$
	$\frac{1}{2} ac$	$\frac{1}{3} ac$	$\frac{1}{6} ac$	$\frac{1}{3} ac$	$\frac{1}{4} ac$	$\frac{1}{6}(2a+b)c$
	$\frac{1}{2} ac$	$\frac{1}{6} ac$	$\frac{1}{3} ac$	$\frac{1}{3} ac$	$\frac{1}{4} ac$	$\frac{1}{6}(a+2b)c$
	$\frac{2}{3} ac$	$\frac{1}{3} ac$	$\frac{1}{3} ac$	$\frac{8}{15} ac$	$\frac{5}{12} ac$	$\frac{1}{3}(a+b)c$
	$\frac{1}{2} ac$	$\frac{1}{4} ac$	$\frac{1}{4} ac$	$\frac{5}{12} ac$	$\frac{1}{3} ac$	$\frac{1}{4}(a+b)c$
	$\frac{1}{2} a(c+d)$	$\frac{1}{6} a(2c+d)$	$\frac{1}{6} a(c+2d)$	$\frac{1}{3} a(c+d)$	$\frac{1}{4} a(c+d)$	$\frac{1}{6}\left\{ a(2c+d) + b(2d+c) \right\}$

TABLE 5·1

5.5 Deflections at any point in a structure.

Using Castigliano's second theorem, it is easy to show that the deflection δ at the point of application and in the direction of a load P acting on a structure is given by

$$\delta = \int_s \frac{m_\delta M}{EI}\, ds \qquad (5.12)$$

where m_δ is the bending moment on the released structure due to P = 1 and M the total bending moment on the loaded statically indeterminate structure. The deflection δ will correspond to displacement if P is a force and to rotation if it is a moment.

We may use expression (5.12) to determine deflections and it is to be noted that the computation of the integral is dependent upon the information already determined in solving the structure plus a moment m_δ, due to a unit load acting on the released structure at the position and in the direction of the required deflection.

In the case of a structure in which the strain energy includes contributions from other stress resultants in addition to bending moment the expression for deflection will include extra terms, e.g.

$$\delta = \int_s \frac{m_\delta M}{EI}\, ds + k\int_s \frac{s_\delta S}{GA}\, ds + \int_s \frac{n_\delta N}{EA}\, ds \qquad (5.13)$$

Example 5.1

Let us now examine the application of the foregoing analysis, in numerical terms, to a four span continuous beam loaded as shown below. The problem itself is almost trivial but will serve to illustrate the process.

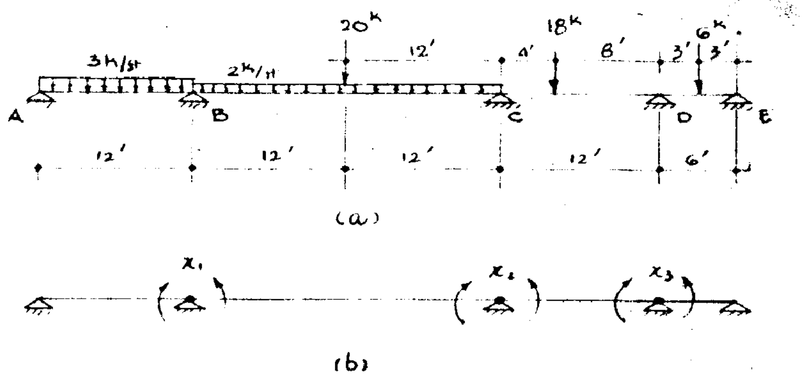


FIGURE 5·3

We choose to produce the statically determinate system by inserting hinges at the intermediate supports (Fig. 5.3 b) so that the arbitrary constants of the problem, x_1 , x_2 , and x_3 , become the moments in the beam at these points. If flexural energy only is considered we shall require for the analysis only the bending moment diagrams. A tabular method is suggested in order to represent the calculations in a symmetrical way.

(a) Determination of the elements of flexibility matrix F.

$$f_{11} = \int_s \frac{m_1^2}{EI} ds$$

Using Table 5.1, we have

$$= \frac{1}{3x3} \text{ x } 12 + \frac{1}{10x3} \text{ x } 24 \qquad (a = c = 1)$$

$$= 1.3334 + 0.8 \qquad = 2.1334$$

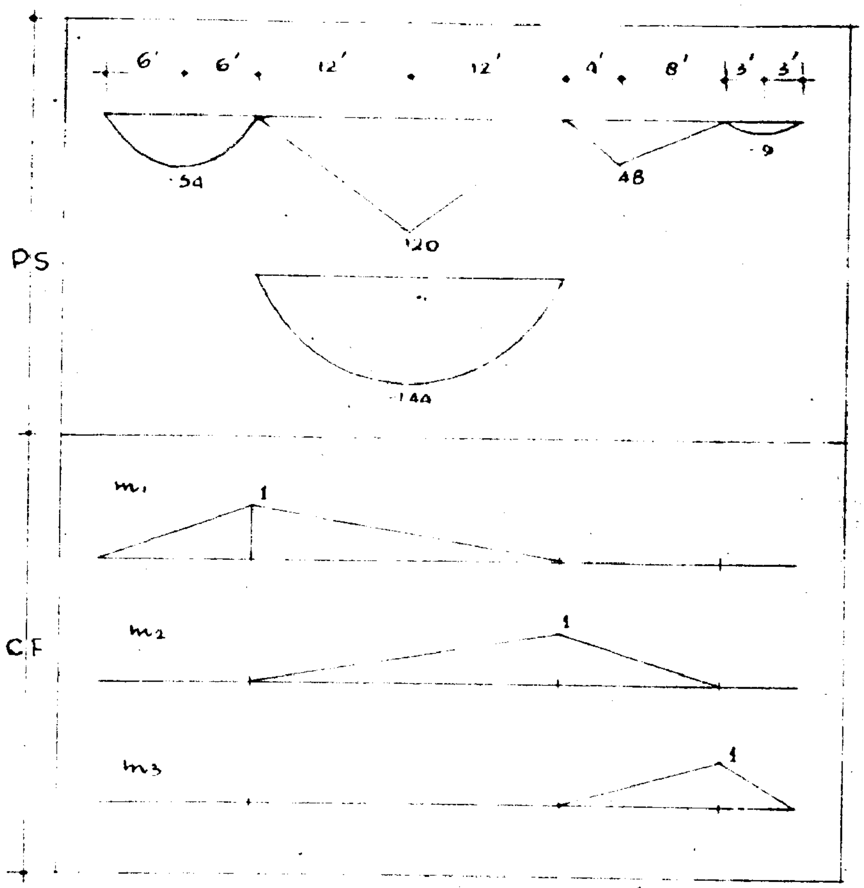


TABLE 5·2

$$f_{22} = \frac{1}{10x3} \text{ x } 24 + \frac{1}{2x3} \text{ x } 12 = 2.8$$

$$f_{33} = \frac{1}{2x3} \text{ x}12 + \frac{1}{2x3} \text{ x } 6 = 3.0$$

$$f_{12} = f_{21} = \int_s \frac{m_1 \; m_2}{EI} ds = \frac{1}{10 \text{ x } 6} \text{x } 24 = 0.4$$

$$f_{13} = f_{31} = \int_s \frac{m_1 \; m_3}{EI} ds = 0$$

$$f_{23} = f_{32} = \int_s \frac{m_2 \; m_3}{EI} ds = \frac{1}{2x6} \text{ x } 12 = 1.00$$

Hence, the flexibility matrix F is

$$\begin{bmatrix} 2.1334 & 0.4 & 0 \\ 0.4 & 2.8 & 1.0 \\ 0 & 1.0 & 3.0 \end{bmatrix}$$

(b) Determination of the elements of vector U.

$$u_1 = \int_s \frac{m_1 \ m_o}{EI} \ ds$$

$$= - \frac{1}{3 \times 3} \times 54 \times 12 - \frac{1}{10 \times 4} \times 120 \times 24 - \frac{1}{10 \times 3} \times 144 \times 24$$

$$= - 72.0 - 72.0 - 115.2 \quad = - 259.2$$

$$u_2 = - 72.0 - 115.2 - \frac{1}{2 \times 6} \times 4 \left(1 + \frac{4}{3}\right) 48 - \frac{1}{2 \times 3} \times 8 \times \frac{2}{3} \times 48$$

$$= - 277.2$$

$$u_3 = - \frac{1}{2 \times 3} \times 4 \times \frac{1}{3} \times 48 - \frac{1}{2 \times 6} \times 8 \left(\frac{2}{3} + 1\right) \times 48 - \frac{1}{3 \times 2} \times 6 \times 9$$

$$= -73.0$$

(c) Solution of the Matrix Equation.

Hence, the matrix equation of the problem is

$$\begin{bmatrix} 2.1334 & 0.4 & 0 \\ 0.4 & 2.8 & 1.0 \\ 0 & 1.0 & 3.0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 259.2 \\ 277.2 \\ 73.0 \end{bmatrix}$$

The computed auxiliary matrix (Crout's method)

x_1	x_2	x_3	=
2.1334	0.1874	0	121.49
0.4	2.7250	0.3669	83.891
0	1.0000	2.6331	-4.1361

and finally,

$$x_1 = 105.76$$

$$x_2 = 85.408$$

$$x_3 = -4.1361$$

(d) Solution of the problem

The distribution of bending moment in the statically indeterminate structure is given by the expression

$$M = m_0 + 105.76\, m_1 + 85.408\, m_2 - 4.1361\, m_3$$

This is a very simple job of superposition of bending moment diagrams m_0 , m_1 , m_2 , and m_3 according to the above expression, and get the net moment distribution diagram of the indeterminate beam. In the same way we may draw the individual S.F. diagrams and combine them according to

$$S = s_0 + 105.76\, s_1 + 85.408\, s_2 - 4.1361\, s_3$$

to obtain the S.F. distribution in the beam, even though the shear forces were ignored in the computation.

Example 5.2

Consider the encastered plane portal of Fig. (5.4a) with the released structure shown in Fig. (5.4 b) along with the chosen arbitrary constants.

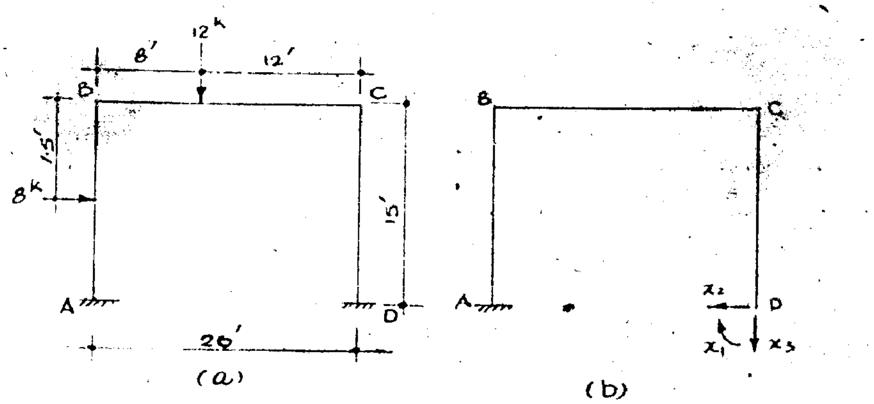


FIGURE 5·4

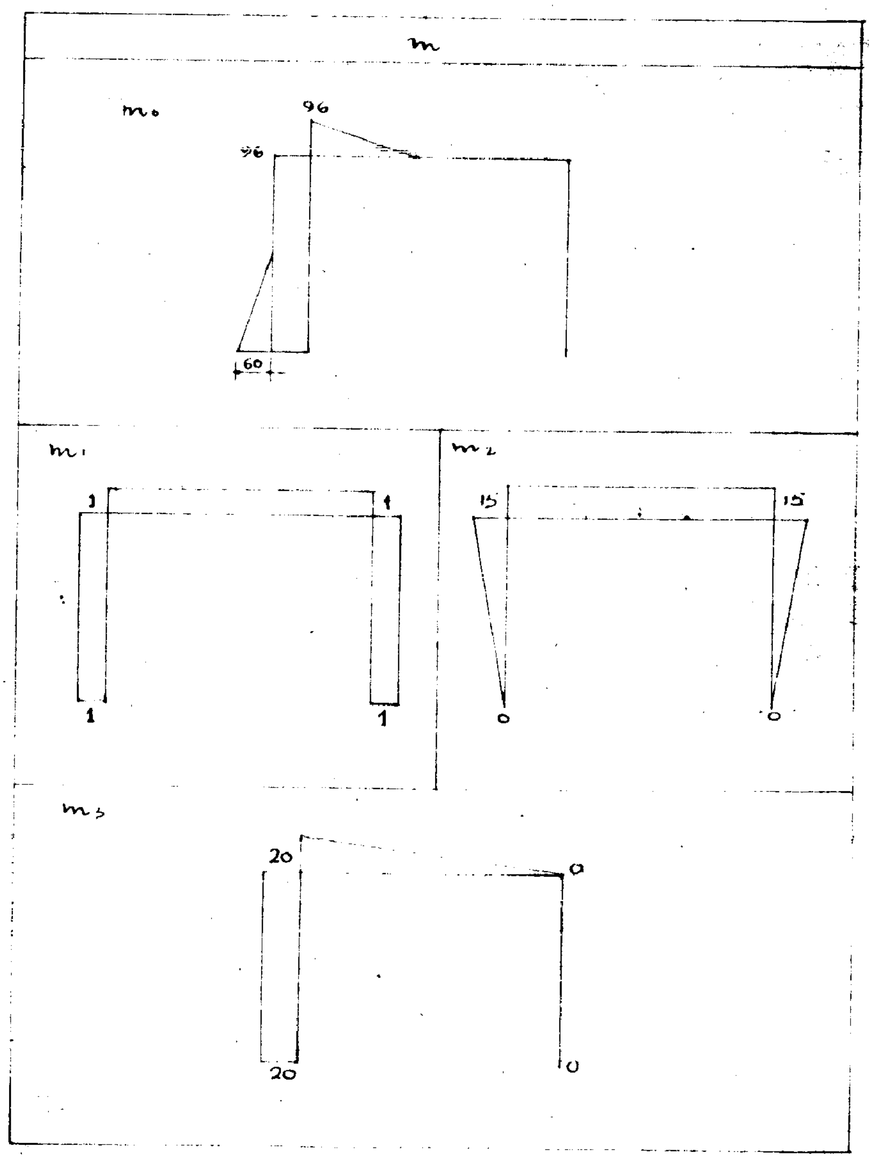


TABLE 5·3

(a) Calculation of F.

The elements of the flexibility matrix F are computed in the usual way from the expression

$$f_{ij} = \int_s \frac{m_i m_j}{EI} ds$$

where only the flexural strain energy is being considered.

Let EI = 100 units

Thus, the computed flexibility matrix

$$F = \begin{bmatrix} 0.50 & 5.25 & 5.00 \\ 5.25 & 67.50 & 52.50 \\ 5.00 & 52.50 & 86.67 \end{bmatrix}$$

(b) Calculation of u's

$$u_i = \int_s \frac{m_i \ m_o}{EI} ds$$

Thus,

$$u = \begin{bmatrix} 20.49 \\ 279.22 \\ 570.56 \end{bmatrix}$$

(c) Solution of the matrix equation

x_1	x_2	x_3	=
0.50	5.25	5.00	-20.49
5.25	67.50	52.50	-279.22
5.00	52.50	86.67	-570.56

The auxiliary matrix

x_1	x_2	x_3	=
0.50	10.50	10.00	-40.98
5.25	12.73	0	- 5.18
5.00	0	36.67	- 9.97

and the final matrix

$$x_1 = +113.11$$

$$x_2 = -5.18$$

$$x_3 = -9.97$$

which gives the values of the redundant reaction components and hence the solution of the problem.

Example 5.3

Consider the three bay pin-jointed plane truss shown in Fig. (5.5a) carrying the loads indicated. The structure carries the loads by a system of direct forces only in the members and is therefore three times statically indeterminate.

The structure is made statically determinate by cutting one diagonal member in each bay (Fig. 5.5b), so that it becomes a straight-forward process to write down the values of the forces in each of the members. Since the forces are constant along the members it is sufficient in this case to indicate the value and sign (Table 5.4). Following the

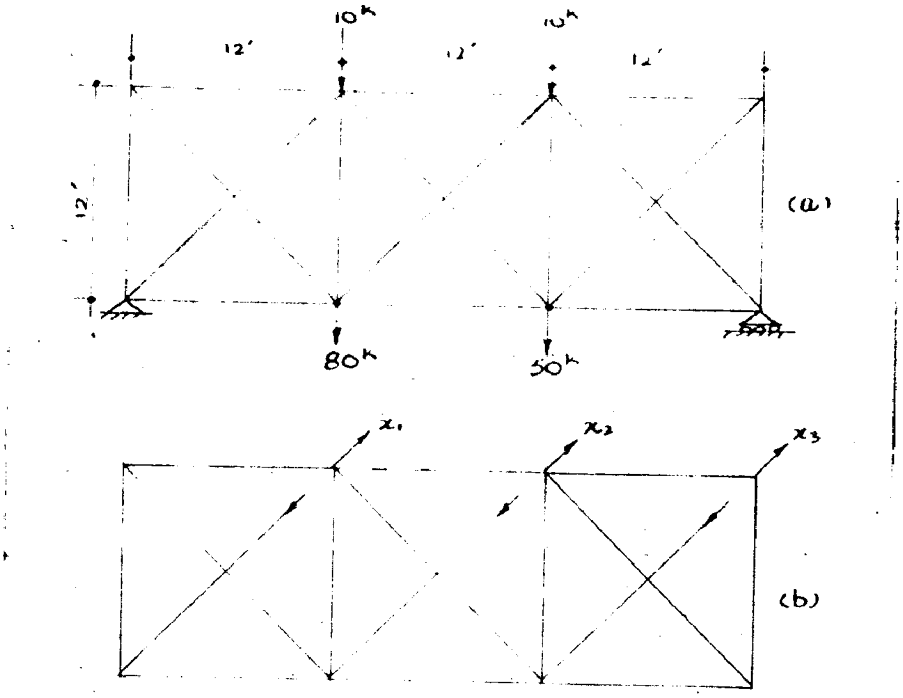


FIGURE 5·5

usual procedure we next determine the stress distribution due to the application of unit values of the arbitrary constants which effects are again indicated by value and sign (Table 5.4).

(a) Determination of F.

The elements of the flexibility matrix F will be determined by consideration of direct strain energy only,

i.e. $$f_{ij} = \int_s \frac{n_i n_j}{EA} ds$$

Let the frame be assumed to be made of members of similar cross-section with EA = 1.

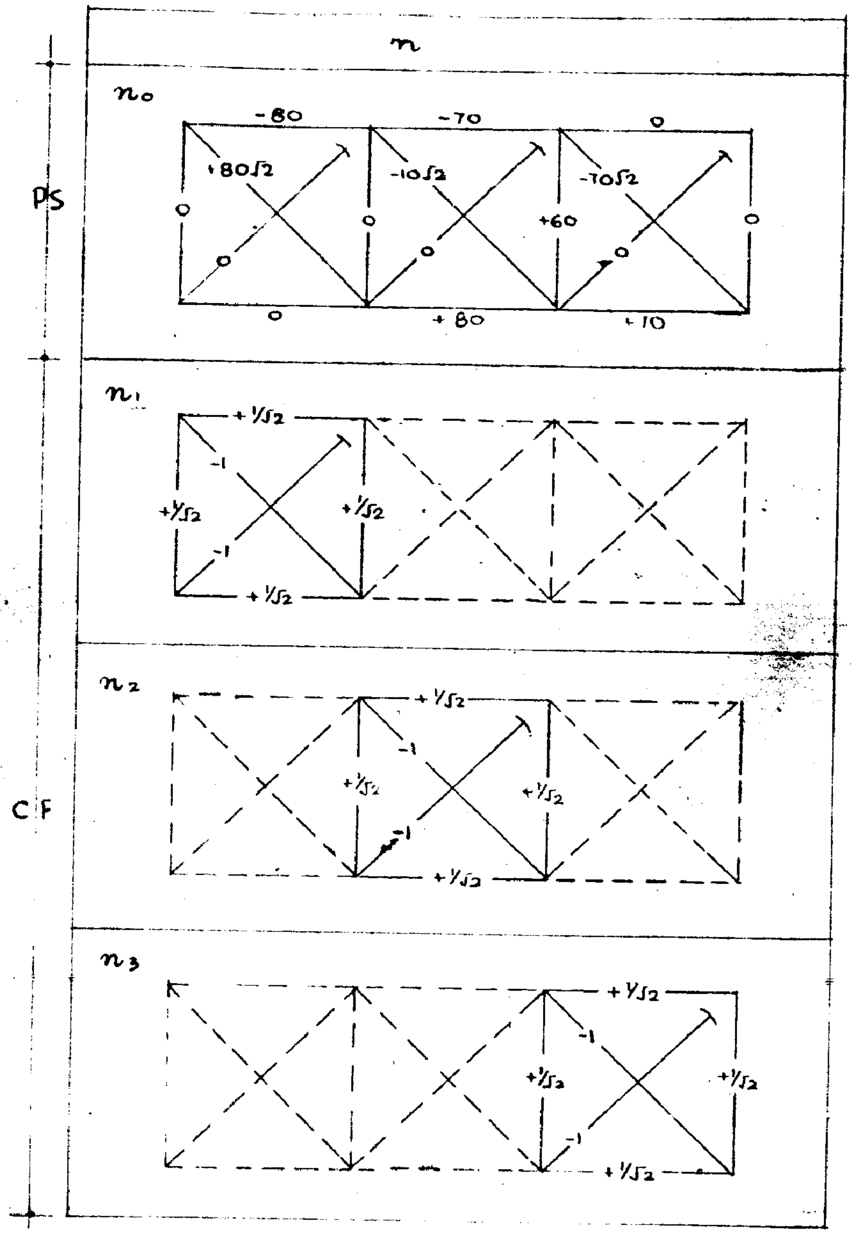


TABLE 5·4

Thus

$$F = \begin{bmatrix} 24(1+\sqrt{2}) & 6 & 0 \\ 6 & 24(1+\sqrt{2}) & 6 \\ 0 & 6 & 24(1+\sqrt{2}) \end{bmatrix}$$

(b) Determination of ω's

$$u_i = \int_s \frac{n_i n_o}{EA} ds$$

and we calculate

$$U = \begin{bmatrix} -(1920 + 960\sqrt{2}) \\ 240 + 420\sqrt{2} \\ 1680 + 780\sqrt{2} \end{bmatrix}$$

The matrix equation of the problem becomes

x_1	x_2	x_3	=
57.940	6.0	0	3277.63
6.0	57.940	6.0	- 833.96
0	6.0	57.940	-2783.07

It has the auxiliary matrix

x_1	x_2	x_3	=
57.940	0.1035	0	56.569
6	57.319	0.1046	-20.470
0	6.0	57.312	-46.416

and the final matrix

$$x_1 = +58.185$$
$$x_2 = -15.615$$
$$x_3 = -46.416$$

(c) Solution of the problem

In the usual way, the solution to the problem is obtained by determining the final stresses in members from the expression

$$N = n_0 + n_1 X_1 + n_2 X_2 + n_3 X_3$$

This gives the final force distribution shown in fig. 5.6.

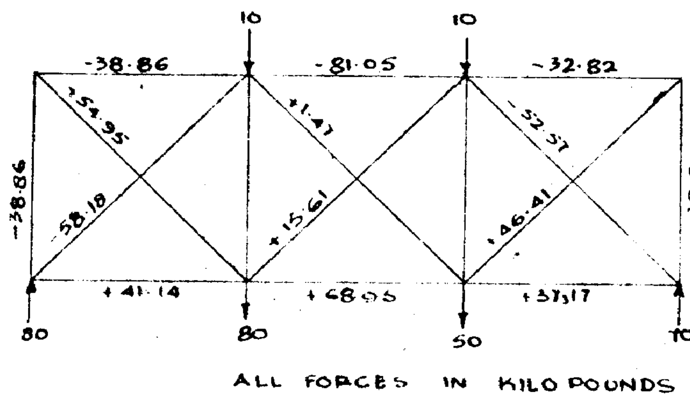


FIGURE 5·6

The above problems belong to the very elementary class of indeterminate structures, but have been solved here only to illustrate the application of the flexibility method. For more complex problems the evaluation of the flexibility influence coefficients becomes too cumbersome and laborious task, in which case the stiffness method (to be discussed in details in the next chapter) proves to be most convenient, since the stiffness influence coefficients are very readily obtained. However, the flexibility method is of particular significance for solution of indeterminate pin-jointed trusses having few degrees of redundancy, for, in such cases the flexibility matrix is easier to assemble and also it is of a much smaller order than that of the corresponding stiffness matrix of the problem.

CHAPTER 6

STIFFNESS MATRIX METHOD

The techniques described in the present chapter are methods, in which all internal forces and moments are expressed in terms of joint displacements and rotations, and the latter are found by solving the matrix equation which is obtained by considering the equilibrium of joints. This group of methods, however, possesses certain advantages, since the stiffness matrix equation of a structural problem is comparatively much easier to assemble.

We shall consider, for our present discussion, only the structures composed of straight uniform members joined at their ends (although much of the analysis is applicable to non-uniform cases also) and the analysis will be restricted to problems in which the external loading consists of forces and moments applied at the joints. This involves no real loss of generality, since any loading of a member at points between its ends may be replaced by equivalent "fixed-end" forces and moments at the joints themselves, without affecting the stresses in the rest of the framework. When the displacements and stresses due to this equivalent loading have been evaluated, the actual stresses in the loaded member may be found by simple super-position.

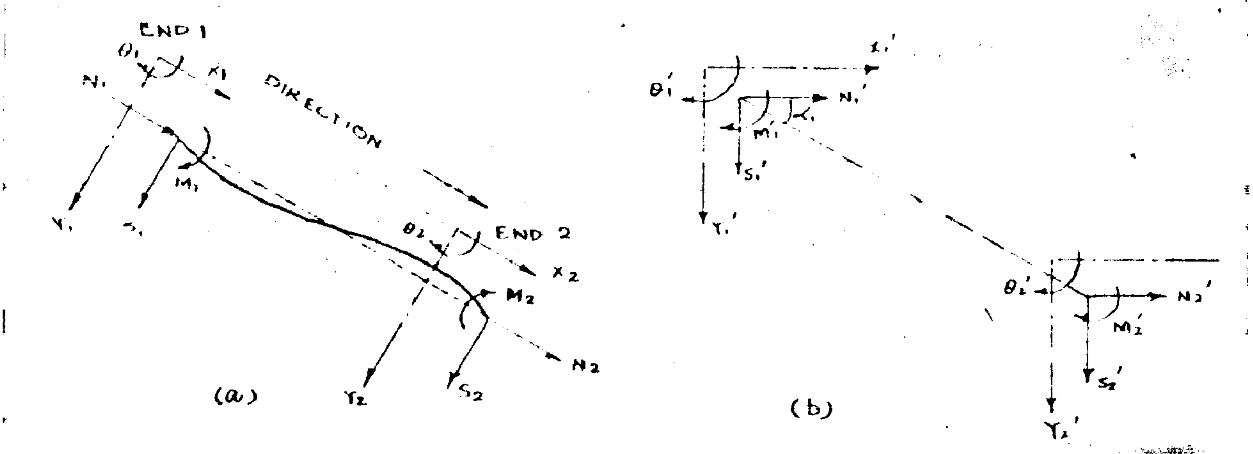


FIGURE 6·1 CHANGE FROM "MEMBER" TO "SYSTEM" COORDINATES

6.1 Notation

The basic unit of a plane rigid frame is a straight uniform member of length L, cross-sectional area A, and flexural rigidity EI. It is convenient in diagrams to place an arrow on the member to denote a specific direction, which may be chosen arbitrarily, and the two ends are then denoted by suffices 1 and 2. The positive directions of the three displacement coordinates x, y and θ at each end are shown in Fig. (6.1 a). The displacements of the two ends are then denoted by single symbols, D_1 and D_2, where

$$D_1 = \begin{bmatrix} x_1 \\ y_1 \\ \theta_1 \end{bmatrix} \quad \text{and} \quad D_2 = \begin{bmatrix} x_2 \\ y_2 \\ \theta_2 \end{bmatrix}$$

Similarly, the loads applied at the ends of the members are represented by single symbols, F_1 and F_2, where

$$F_1 = \begin{bmatrix} N_1 \\ S_1 \\ M_1 \end{bmatrix} \quad \text{and} \quad F_2 = \begin{bmatrix} N_2 \\ S_2 \\ M_2 \end{bmatrix}$$

In the rest of the discussion of this chapter the terms "displacements" and "loads" will be taken to mean column vectors of the above type.

When considering a structure formed of connected members, the displacements and loads in the final set of equations will be expressed in a single reference frame x', y', θ' chosen arbitrarily for the whole structure (normally we take axes x' and y' along the horizontal and vertical). Primes will be used to denote such overall " system coordinates". It could easily be seen by a simple geometrical argument that D_1 and D_2 can be expressed in terms of D_1' and D_2' as

$$D_1 = T D_1' , \quad D_2 = T D_2' \qquad (6.1)$$

Similarly the end loads, F_1 and F_2 , are expressed in system coordinates by F_1' and F_2' as

$$F_1 = T F_1' , \quad F_2 = T F_2' \qquad (6.2)$$

where T is the orthogonal transformation matrix given by

$$T = \begin{bmatrix} \text{Cos}\,\alpha & \text{Sin}\,\alpha & 0 \\ -\text{Sin}\,\alpha & \text{Cos}\,\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

α being the inclination of the member to the x'-axis.

The inverse of T

$$T^{-1} = \begin{bmatrix} \text{Cos}\,\alpha & -\sin\alpha & 0 \\ \text{Sin}\,\alpha & \text{Cos}\,\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

enables us to carry out the reverse transformation e.g.,

$$D_1' = T^{-1} D_1 \qquad \text{etc.}$$

It will be noticed that the inverse of the orthogonal transformation matrix is the same as its 'transpose', -

the matrix formed by interchanging rows and columns.

When discussing a structure composed of several members, an additional suffix is required to denote a particular member. Thus, the displacement of end 1 of member (3) is written D_{13} . In order to avoid confusion, square matrices, which already ahve two suffices, are placed in brackets before the member suffix is attached; that is, $(K_{12})_3$ is a matrix associated with the third member. It has also been found convenient to mark the joints separately. Joint loads and displacements are represented by the symbol P and Δ, respectively, these, of course, being measured in the general reference frame (primes are discarded here since there is no possible ambiguity.) Thus if members (1) and (2) meet at joint B, the end 2 of member (1) being rigidly attached to the end 1 of member (2) , the equations of compatibility and equlibrium at the joint B will be written

$$\Delta_B = D'_{21} = D'_{12} \text{ , and}$$

$$P_B = F'_{21} + F'_{12}$$

6.2 Load-Displacement Equations for a Single Member.

Using the above notation, the equations connecting the quantities D_1 , D_2 and F_1 and F_2 now have to be established. In order to obtain the equations for a member with arbitrary loads acting at each end (these loads, however, being such as to keep the member in equilibrium), consider the two following cases and then apply the principle of superposition.

Clamp end 2 of the member shown in Fig. (6.1), (i.e. $D_2 = 0$), and then apply a load Q_1 to end 1; the single theory of bending gives

$$Q_1 = K_{11} D_1 ,$$

$$\text{where } K_{11} = \begin{bmatrix} \frac{EA}{L} & 0 & 0 \\ 0 & \frac{12EI}{L^3} & \frac{6EI}{L^2} \\ 0 & \frac{6EI}{L^2} & \frac{4EI}{L} \end{bmatrix}$$

Also, from the conditions of equilibrium,

$$Q_2 = \lambda_{21} Q_1$$

$$\text{Where} \quad \lambda_{21} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ L & L & -1 \end{bmatrix}$$

Similarly, clamping end 1 ($D_1 = 0$) and applying a load R_2 to end 2, gives

$$R_2 = K_{22} D_2 ,$$

Where

$$K_{22} = \begin{bmatrix} \frac{EA}{L} & 0 & 0 \\ 0 & \frac{12EI}{L^3} & -\frac{6EI}{L^2} \\ 0 & \frac{-6EI}{L^2} & \frac{4EI}{L} \end{bmatrix}$$

and from the conditions of equilibrium,

$$R_1 = \lambda_{12} R_2$$

where

$$\lambda_{12} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & -L & -1 \end{bmatrix}$$

Combination of the two cases gives

$$F_1 = Q_1 + R_1 = Q_1 + \lambda_{12} R_2 = K_{11} D_1 + \lambda_{12} K_{22} D_2$$

and, $$F_2 = Q_2 + R_2 = \lambda_{21} Q_1 + R_2 = \lambda_{21} K_{11} D_1 + K_{22} D_2$$

If K_{12} and K_{21} are now defined by the equations

$$K_{12} = \lambda_{12} K_{22}$$
$$K_{21} = \lambda_{21} K_{11}$$

then $$\begin{bmatrix} F_1 \\ \hline F_2 \end{bmatrix} = \left[\begin{array}{c|c} K_{11} & K_{12} \\ \hline K_{21} & K_{22} \end{array}\right] \begin{bmatrix} D_1 \\ \hline D_2 \end{bmatrix} \qquad (6.3)$$

This matrix equation gives the end loads in terms of the end displacements for a single member in its own reference axes. This cannot be solved for D_1 and D_2, since the matrix formed by the four K- matrices is singular; this is to be expected, since it is obviously possible to give the member an arbitrary rigid-body displacement.

In order to write down conditions of compatibility and equilibrium at the joints, it is necessary to express

all end displacements and end loads in the general reference frame of the system. The next step is therefore to determine the form taken by equation (6.3) in system coordinates.
If x', y', are the system coordinates, as shown in Fig.(6.1b), substituting from eqns. (6.1) and (6.2), eqn. (6.3) becomes

$$\begin{bmatrix} T\,F_1' \\ T\,F_2' \end{bmatrix} = \begin{bmatrix} K_{11}\,T & K_{12}\,T \\ K_{21}\,T & K_{22}\,T \end{bmatrix} \begin{bmatrix} D_1' \\ D_2' \end{bmatrix}$$

and, therefore,

$$\begin{bmatrix} F_1' \\ F_2' \end{bmatrix} = \begin{bmatrix} T^{-1}K_{11}\,T & T^{-1}K_{12}\,T \\ T^{-1}K_{21}\,T & T^{-1}K_{22}\,T \end{bmatrix} \begin{bmatrix} D_1' \\ D_2' \end{bmatrix}$$

or, defining K_{ij}' by the relationship

$$T^{-1}K_{ij}\,T = K_{ij}' \quad , \quad (i,\ j,\ = 1,2) \qquad (6.4)$$

we have

$$\begin{bmatrix} F_1' \\ F_2' \end{bmatrix} = \begin{bmatrix} K_{11}' & K_{12}' \\ K_{21}' & K_{22}' \end{bmatrix} \begin{bmatrix} D_1' \\ D_2' \end{bmatrix} \qquad (6.5)$$

Using Eq. (6.4) the four K'-matrices of Eq.(6.5) are obtained, the coefficients of which have been tabulated and given in Table (6.1).

[illegible] in [illegible]

	$\lambda^2 \frac{EA}{L} + \mu^2 \frac{12EI}{L^3}$	$\mu\lambda\left(\frac{EA}{L} \quad \frac{12EI}{L^3}\right)$	$\mu \frac{6EI}{L^2}$
K_{11}	$\mu\lambda\left(\frac{EA}{L} \quad \frac{12EI}{L^3}\right)$	$\mu^2 \frac{EA}{L} + \lambda^2 \frac{12EI}{L^3}$	$\mu \frac{6EI}{L^2}$
	$\mu \frac{6EI}{L^2}$	$\mu \frac{6EI}{L^2}$	$\frac{4EI}{L}$
	$\left(\lambda^2 \frac{EA}{L} + \mu^2 \frac{12EI}{L^3}\right)$	$\mu\lambda\left(\frac{EA}{L} \quad \frac{12EI}{L^3}\right)$	$\mu \frac{6EI}{L^2}$
K'_{12}	$\mu\lambda\left(\frac{EA}{L} \quad \frac{12EI}{L^3}\right)$	$\left(\mu^2 \frac{EA}{L} + \lambda^2 \frac{12EI}{L^3}\right)$	$\lambda \frac{6EI}{L^2}$
	$\mu \frac{6EI}{L^2}$	$\lambda \frac{6EI}{L^2}$	$\frac{2EI}{L}$
	$\left(\lambda \frac{EA}{L} + \mu^2 \frac{12EI}{L^3}\right)$	$\mu\lambda\left(\frac{EA}{L} \quad \frac{12EI}{L^3}\right)$	$\mu \frac{6EI}{L^2}$
K'_{21}	$\mu\lambda\left(\frac{EA}{L} \quad \frac{12EI}{L^3}\right)$	$\left(\mu \frac{EA}{L} + \lambda^2 \frac{12EI}{L^3}\right)$	$\lambda \frac{6EI}{L^2}$
	$\mu \frac{6EI}{L^2}$	$\lambda \frac{6EI}{L^2}$	$\frac{2EI}{L}$
	$\lambda^2 \frac{EA}{L} + \mu^2 \frac{12EI}{L^3}$	$\mu\lambda\left(\frac{EA}{L} - \frac{12EI}{L^3}\right)$	$\mu \frac{6EI}{L^2}$
K'_{22}	$\mu\lambda\left(\frac{EA}{L} - \frac{12EI}{L^3}\right)$	$\mu^2 \frac{EA}{L} + \lambda^2 \frac{12EI}{L^3}$	$\lambda \frac{6EI}{L^2}$
	$\mu \frac{6EI}{L^2}$	$\lambda \frac{6EI}{L^2}$	$\frac{4EI}{L}$

TABLE 6·1

6.3 Analysis of a Plane Rigid-Jointed Framework-

Consider the structure shown in Fig.(6.2) which is acted upon by loads P_B and P_C, as shown. Eq.(6.5) can be written down for each member as follows -

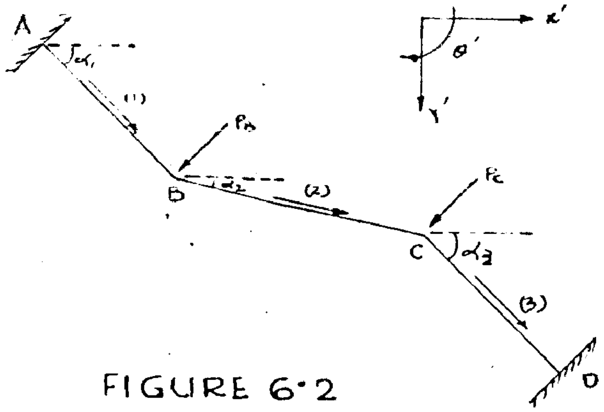


FIGURE 6·2

$$\begin{bmatrix} F'_1 \\ F'_2 \end{bmatrix}_i = \begin{bmatrix} K'_{11} & K'_{12} \\ K'_{21} & K'_{22} \end{bmatrix}_i \begin{bmatrix} D'_1 \\ D'_2 \end{bmatrix}_i \qquad (6.6)$$

where i may denote member 1, 2 or 3.

The conditions of compatibility and equilibrium at the joints are

$$D'_{11} = 0 \;, \quad D'_{12} = D'_{21} = \Delta_B$$

$$D'_{22} = D'_{13} = \Delta_C \;, \quad D'_{23} = 0$$

and

$$F'_{21} + F'_{12} = P_B \;, \quad F'_{22} + F'_{13} = P_C \;.$$

Substituting these expressions in Eqs. (6.6), we obtain

$$F'_{11} = (K'_{12})_1 \Delta_B \qquad (6.7)$$

$$\begin{bmatrix} P_B \\ P_C \end{bmatrix} = \begin{bmatrix} (K'_{22})_1 + (K'_{11})_2 & (K'_{12})_2 \\ (K'_{21})_2 & (K'_{22})_2 + (K'_{11})_3 \end{bmatrix} \begin{bmatrix} \Delta_B \\ \Delta_C \end{bmatrix} \quad (6.8)$$

also written as, $P = K \Delta$ (6.9)

$$F'_{23} = (K'_{21})_3 \Delta_C \quad (6.10)$$

The Eq. (6.8) is called the stiffness matrix equation of the structure , the matrix of coefficients being called the stiffness matrix of the structure. The elements of this matrix are often referred to as the stiffness influence coefficients.

The matrix equation (6.8) forms a set of six linear equations for the six degrees of freedom of the structure. The stiffness matrix is non-singular, and the Eq.(6.8) can therefore be solved for Δ_B and Δ_C . When this has been done, Eqs. (6.7) and (6.10) give the redundant reactions at A and D , or, alternatively, equations (6.6) enable all the internal forces to be determined.

6.4 Some Practical Details of the Method-

Table 6.1 gives the coefficients of the four K'-matrices for a member inclined at angle α to the x'-axis. In the majority of our structural problems, a large proportion of the members will have the simpler matrices associated with the values $\alpha = 0$, $\alpha = 90^\circ$.

Referring to Eq. (6.5), the matrices K'_{11} , K'_{22} , may be described as "direct"- stiffness matrices, relating the load at one end to the displacement thereat, while the matrices K'_{12} , K'_{21} , are "cross"-stiffness matrices, relating the load at one end of a member to the displacement at the other. Using these ideas, it is quite easy to build up the matrix equation (6.8) for the structure, directly from the stiffness matrices of individual members, without first writing down the Eqs (6.6).

Considering the stiffness matrix K of Eq. (6.8), the leading diagonal coefficient matrices are found to be the sums of the direct-stiffness matrices of the members meeting at the respective joints.

The term $(K'_{22})_1 + (K'_{11})_2$, for instance, represents the sum of the direct-stiffness matrices of members (1) and (2). The matrix $(K'_{22})_1$ appears because member (1) has end at the joint B; member (2) has end 1 at the joint and is therefore represented by $(K'_{11})_2$. Similarly, the coefficient matrices away from the leading diagonal represent cross-stiffness effects due to the deflections of the other joints.

For a structure with any number of joints, the matrix equation takes a form similar to (6.8). In the equation for a given joint, the coefficient matrix on the leading diagonal is the sum of the direct-stiffness matrices of the members meeting at that joint, the matrices chosen being either K'_{11}, if the joint is at end 1 , or K'_{22} if at end 2 of the member. Coefficient matrices away from the leading diagonal represent the effects of all other joints directly coupled to the one considered.

It will be found in practice that, if the diagram is first labelled with arrows showing the direction of all the members, the synthesis of the stiffness matrix K for the whole structure, from the K'-matrices of the individual members is a straightforward process. As a check, it is useful to remember that the elements of the stiffness matrix K, for any structure, must always be symmetrical about the leading diagonal.

6.5 Space Frames.

The above analysis may easily be extended to cover the problem of rigidly jointed frameworks in three dimensions. Each joint will have six degrees of freedom- three components of displacement and three of rotation. The load and displacement column-vectors will have their full six omponents, while the four K-matrices of Eqn. (6.3) will be square matrices having six rows and six columns. The orthogonal transformation matrix T will also be more complicated, taking the general form

$$\begin{bmatrix} l_{ij} & 0 \\ 0 & l_{ij} \end{bmatrix},$$

where l_{ij} represents the matrix of the direction cosines of the new axes referred to the old. With these extensions, the analysis becomes identical with that presented for planar structures.

Matrix methods provide a general approach to the analysis of complex space frames, although it must be admitted that the computational work is extremely laborious when carried out by hand. In practice the large scale matrix operations, involved in the analysis, are normally carried out on automatic digital computers. Keeping in view the limited scope of the present work we shall not take up the detailed discussion of such complex problems.

6.6 Modification for Hinged Supports-

In order to lessen the redundancy of a rigid structure, it is often fixed to its foundations by pin joints. For instance, the effect of pin-jointing the ends A and D of the structure represented in Fig. (6.2) would be to alter the compatibility equations at those points. If the structure is connected to the hinged support by a single member, it is possible to modify the stiffness matrices of the member, and then consider it as if it were rigidly anchored at the hinged end. This is a more general case of the modification of the "carry-over" factor for a pin-supported member in the moment-distribution method.

Consider a member hinged to a rigid foundation at end 1 ; then Eq. (6.3) may be written in expanded form as

$$F_1 = \begin{bmatrix} N_1 \\ S_1 \\ 0 \end{bmatrix} = K_{11} \begin{bmatrix} 0 \\ 0 \\ \theta_1 \end{bmatrix} + K_{12} \; D_2 \qquad (6.11)$$

$$F_2 = K_{21} \begin{bmatrix} 0 \\ 0 \\ \theta_1 \end{bmatrix} + K_{22} \; D_2 \qquad (6.12)$$

Eq. (6.11) gives, considering the last of the three scalar equations,

$$0 = \begin{bmatrix} 0 & \frac{6EI}{L^2} & \frac{4EI}{L} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \theta_1 \end{bmatrix} + \begin{bmatrix} 0 & \frac{-6EI}{L^2} & \frac{2EI}{L} \end{bmatrix} D_2$$

which, by simple manipulation, gives

$$\begin{bmatrix} 0 \\ 0 \\ \theta_1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \frac{3}{2L} & -\frac{1}{2} \end{bmatrix} D_2 = X_2 \; D_2$$

where x_2 is defined by this equation. Substituting this in Eqs. (6.11) and (6.12) we obtain

$$\begin{bmatrix} F_1 \\ F_2 \end{bmatrix} = \begin{bmatrix} (K_{12} + K_{11} X_2) \\ (K_{22} + K_{21} X_2) \end{bmatrix} D_2 \qquad (6.13)$$

or, defining modified matrices as K''_{11} and K''_{22}, Eq. (6.13) can be re-written as

$$\begin{bmatrix} F_1 \\ F_2 \end{bmatrix} = \begin{bmatrix} K''_{12} \\ K''_{22} \end{bmatrix} D_2 \qquad (6.14)$$

in which form they are similar to the equation for a rigidly-encastered member.

For a member, with end 2 hinged, the analysis is similar. The matrix X_1 comes out to be

$$X_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -\frac{3}{2L} & -\frac{1}{2} \end{bmatrix}$$

and the modified matrices K''_{11}, K''_{21}, are defined by the equations

$$K''_{11} = K_{11} + K_{12} X_1$$

$$K''_{21} = K_{21} + K_{22} X_1$$

The components of these modified matrices have been evaluated and are being given in Table 6.2, herewith. Transformations of these matrices follow the normal rules.

In case of a joint where two or more members are rigidly attached to each other and the joint as a whole is pin-anchored (as, for instance, at the end of a bridge truss), modifications on the above lines become impracticable, and the method suggested here should be adopted. If the joint displacement is Δ'_s, where x'_s, y'_s take prescribed zero values, and the applied moment M_s is also zero, the three scalar equations for the joint should be constructed as if it were a normal one. The first two equations (containing the unknown reactions) should then be discarded, together with all coefficients of x'_s and y'_s occurring in the other equations. Thus, one equation will be added to the set of load-displacement equations, corresponding to the extra variable θ_s. This problem shall be discussed later in this chapter, when we shall solve numerical example of a bridge truss.

END 1 HINGED	K''_{12}	EA/L	0	0
		0	$-3EI/L^3$	$3EI/L^2$
		0	0	0
	K''_{22}	EA/L	0	0
		0	$3EI/L^3$	$-3EI/L^2$
		0	$-3EI/L^2$	$3EI/L$
END 2 HINGED	K''_{11}	EA/L	0	0
		0	$3EI/L^3$	$3EI/L^2$
		0	$3EI/L^2$	$3EI/L$
	K''_{21}	EA/L	0	0
		0	$-3EI/L^3$	$-3EI/L^2$
		0	0	0

TABLE 6.2 (MODIFIED MATRICS)

6.7 Modifications of the General Theory.

The analysis so far has applied to any structure with rigid joints. Its application to certain classes of problems more common to a structural engineer, will now be considered.

6.7.1 Continuous Beams.

In the case of a horizontal continuous beam with only vertical loading, Eq. (6.3) may be applied directly, since all the member and system coordinates may be made to coincide. Furthermore, all deflections and forces in the x-direction will be zero, and hence the first scalar equation of each of the matrix equations (6.2) becomes identically zero. Hence, the K-matrices become 2 x 2 matrices, and D and F vectors will have two components. Equation (6.3) for a member of a horizontal continuous beam can be written out in full as follows.

$$\begin{bmatrix} S_1 \\ M_1 \\ S_2 \\ M_2 \end{bmatrix} = \left[\begin{array}{cc|cc} \frac{12EI}{L^3} & \frac{6EI}{L^2} & -\frac{12EI}{L^3} & \frac{6EI}{L^2} \\ \frac{6EI}{L^2} & \frac{4EI}{L} & -\frac{6EI}{L^2} & \frac{2EI}{L} \\ \hline -\frac{12EI}{L^3} & -\frac{6EI}{L^2} & \frac{12EI}{L^3} & -\frac{6EI}{L^2} \\ \frac{6EI}{L^2} & \frac{2EI}{L} & \frac{6EI}{L^2} & \frac{4EI}{L} \end{array}\right] \begin{bmatrix} y_1 \\ \theta_1 \\ y_2 \\ \theta_2 \end{bmatrix} \qquad (6.15)$$

It may be pointed out that, by this method of analysis, an allowance can easily be made for the effect of elastic supports. To illustrate this effect, let us consider the beam on elastic supports shown in Fig.6.3.

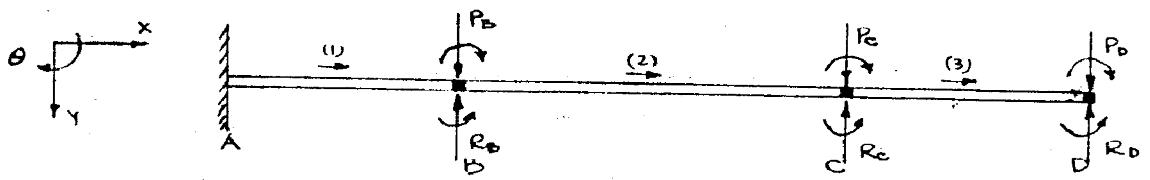


FIGURE 6·3

The applied joint loads (which are the same as fixed-end loads with opposite sign) each comprise of a force and a moment, and are represented by the symbols P_B , P_C and P_D , while the symbols R_B , R_C and R_D represent the forces and moments applied by the elastic supports located, in the present problem, at points B, C and D respectively. We shall assume that at each of the elastic supports the vertical reaction is k_y times the vertical displacement of the joint, while the reactive moment is k_θ times the angular rotation.

We can write down the Eqs. (6.3) for each span. If the displacements of the joints B, C , D are Δ_B , Δ_C, Δ_D respectively, we have the following compatibility equations

$$D_{21} = D_{12} = \Delta_B ,$$

$$D_{22} = D_{13} = \Delta_C ,$$

and $$D_{23} = \Delta_D$$

We also have the following joint equilibrium equations

$$F_{21} + F_{12} = P_B - R_B$$

$$F_{22} + F_{13} = P_C - R_C$$

$$F_{23} = P_D - R_D$$

the reactions R_B , R_C , R_D , being given by

$$R_B = Z\Delta_B , R_C = Z\Delta_C , R_D = Z\Delta_D$$

where Z is the stiffness matrix of the supports, defined by

$$Z = \begin{bmatrix} k_y & 0 \\ 0 & k_\theta \end{bmatrix}$$

By usual substitution, we get the load-displacement equations in matrix form as follows,

$$\begin{bmatrix} P_B \\ P_C \\ P_D \end{bmatrix} = \begin{bmatrix} Z + (K_{22})_1 + (K_{11})_2 & (K_{12})_2 & 0 \\ (K_{21})_2 & Z + (K_{22})_2 + (K_{11})_3 & (K_{12})_3 \\ 0 & (K_{21})_3 & Z + (K_{22})_3 \end{bmatrix} \begin{bmatrix} \Delta_B \\ \Delta_C \\ \Delta_D \end{bmatrix} \quad (6.16)$$

In the above matrix equation, the stiffness matrix for the complete beam could easily be obtained by direct synthetic process and the equation (6.16) solved for the unknowns Δ_B, Δ_C, Δ_D. The effect of the elasticity of the support B for instance, on the stiffness matrix of the complete structure is to add, to the diagonal term for the joint, a term Z which is the direct stiffness of the support at B.

If we consider, in the above example, that the supports are rigid (k_y , $k_\theta \rightarrow \infty$) in which the displacements of the supports all tend to zero, the joints B, C, D have only rotational movement θ_B, θ_C, θ_D. The applied joint loads will only comprise of the equivalent fixed-end moments M_B , M_C , M_D . Eq.(6,3) will adopt the simple form

$$\begin{bmatrix} M_1 \\ M_2 \end{bmatrix} = \begin{bmatrix} \frac{4EI}{L} & \frac{2EI}{L} \\ \frac{2EI}{L} & \frac{4EI}{L} \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} \qquad (6.17)$$

Eq.(6.17) is seen to be the slope-deflection equation for a span of a continuous beam with rigid supports, — $\frac{2EI}{L}$ being known to be the cross-stiffness coefficient for a member of uniform cross-section.

6.7. Pin-connected Frames.

The analysis of redundant pin-connected frames is a straight-forward special case of the general theory. It is clear that a pin-jointed frame is equivalent (for stress analysis purposes) to a rigid-jointed frame whose members have zero moments of inertia. In analysing such a frame, the third scalar equation at each joint may be discarded, since no moments can be applied to the frame and the joint rotations are irrelevant. Hence, end-loads F', and displacement-vectors D', become 2-vectors, and the K-matrices are reduced to the simple form

$$K_{11} = K_{22} = \begin{bmatrix} \frac{EA}{L} & 0 \\ 0 & 0 \end{bmatrix}$$

$$K_{21} = K_{12} = \begin{bmatrix} \frac{-EA}{L} & 0 \\ 0 & 0 \end{bmatrix}$$

In view of this simplification,

$$K_{11} = K_{22} = K \quad , \quad K_{21} = K_{12} = -K.$$

The transformation matrix T is now given by

$$T = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} \qquad (6.18)$$

and Eq. (6.5) becomes

$$\begin{bmatrix} F_1' \\ F_2' \end{bmatrix} = \begin{bmatrix} K' & -K' \\ -K' & K' \end{bmatrix} \begin{bmatrix} D_1' \\ D_2' \end{bmatrix} \qquad (6.19)$$

where

$$K' = T^{-1} K T = \frac{EA}{L} \begin{bmatrix} \mathrm{Cos}^2\alpha & \mathrm{Cos}\alpha\ \mathrm{Sin}\alpha \\ \mathrm{Cos}\alpha\,\mathrm{Sin}\alpha & \mathrm{Sin}^2\alpha \end{bmatrix} \qquad (6.20)$$

Replacing $\mathrm{Cos}\alpha$ by λ and $\mathrm{Sin}\alpha$ by μ, for a pin-ended truss member the matrix equation (6.19) can be written down, in full , as

$$\begin{bmatrix} N_1' \\ S_1' \\ N_2' \\ S_2' \end{bmatrix} = \frac{EA}{L} \begin{bmatrix} \lambda^2 & \lambda\mu & -\lambda^2 & -\lambda\mu \\ \lambda\mu & \mu^2 & -\lambda\mu & -\mu^2 \\ -\lambda^2 & -\lambda\mu & \lambda^2 & \lambda\mu \\ -\lambda\mu & -\mu^2 & \lambda\mu & \mu^2 \end{bmatrix} \begin{bmatrix} x_1' \\ y_1' \\ x_2' \\ y_2' \end{bmatrix} \qquad (6.21)$$

The method of solving any (determinate or indeterminate) pin-connected truss will be presented and discussed in details with reference to a simple example of a pin-jointed frame. The method is essentially tabular in nature and consists in setting up the 'complete' matrix of stiffness influence coefficients relating the joint forces and displacements. As soon as this matrix has been framed, the solution for node deflections, external reactions, and separate member forces proceeds from routine matrix operations, already described in Chapters 2 and 4.

Analysis of a Simple Truss.

When the stiffness expressions (like Eq. 6.21) for individual members of a structure are known, the stiffness of an assemblage of such members may be formed. A simple example is represented by the truss in Fig. 6.4. The arrows on the members in the figure are inserted to show their positive x-direction which is a very convenient way of distinguishing ends 1 and 2 of a member in a diagram.

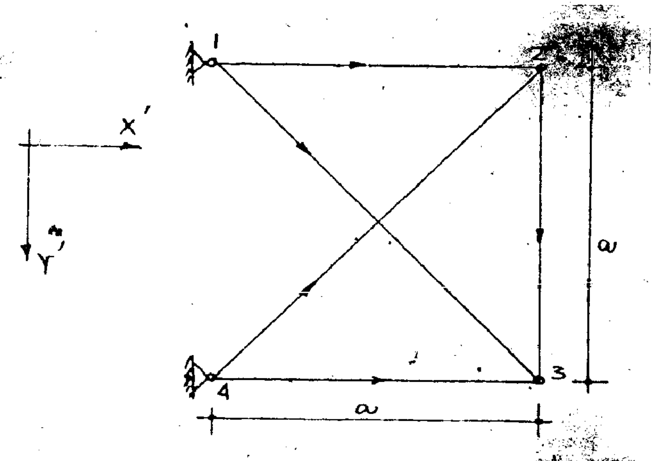


FIGURE 6'4.

For simplicity it is assumed that all members have equal values of A and E. The stiffness matrix can be developed by first determining λ^2, μ^2 and $\lambda\mu$ and then $\bar{\lambda}^2$, $\bar{\mu}^2$ and $\bar{\lambda}\bar{\mu}$ for each member - this is done in Table 6.3.

$$\bar{\lambda}^2 = \frac{\lambda^2}{L/a}$$

$$\bar{\mu}^2 = \frac{\mu^2}{L/a}$$

$$\bar{\lambda}\bar{\mu} = \frac{\lambda\mu}{L/a}$$

for each member.

MEMBER	L	λ	μ	λ^2	μ^2	$\lambda\mu$	$\bar{\lambda}^2$	$\bar{\mu}^2$	$\bar{\lambda}\bar{\mu}$
1 2, 4 3	a	1	0	1	0	0	1	0	0
2 3	a	0	1	0	1	0	0	1	0
1 3	$a\sqrt{2}$	$1/\sqrt{2}$	$1/\sqrt{2}$	$1/2$	$1/2$	$1/2$	$1/2\sqrt{2}$	$1/2\sqrt{2}$	$1/2\sqrt{2}$
4 2	$a\sqrt{2}$	$-1/\sqrt{2}$	$1/\sqrt{2}$	$1/2$	$1/2$	$-1/2$	$1/2\sqrt{2}$	$1/2\sqrt{2}$	$-1/2\sqrt{2}$

TABLE 6.3

It is now desirable to form the stiffness matrix for the complete truss. This is the principal task that the engineer must perform in the analysis, and, as will be seen, it is routine.

Eq. (6.21) for the complete truss can be written directly from Table 6.3. The result will be presented first and then explained in some detail.

		x_1'	y_1'	x_2'	y_2'	x_3	y_3'	x_4'	y_4'
N_1'		$1+1/2\sqrt{2}$	$1/2\sqrt{2}$	1	0	$1/2\sqrt{2}$	$1/2\sqrt{2}$	0	0
S_1'		$1/2\sqrt{2}$	$1/2\sqrt{2}$	0	0	$1/2\sqrt{2}$	$1/2\sqrt{2}$	0	0
N_2'		1	0	$1+1/2\sqrt{2}$	$-1/2\sqrt{2}$	0	0	$1/2\sqrt{2}$	$1/2\sqrt{2}$
S_2		0	0	$-1/2\sqrt{2}$	$1+1/2\sqrt{2}$	[illegible]	[illegible]	$1/2\sqrt{2}$	$1/2\sqrt{2}$
N_3'	$= \frac{EA}{a}$	$-1/2\sqrt{2}$	$1/2\sqrt{2}$	0	0	$1+1/2\sqrt{2}$	$1/2\sqrt{2}$	1	0
S_3'		$1/2\sqrt{2}$	$1/2\sqrt{2}$	0	-1	$1/2\sqrt{2}$	$+1/2\sqrt{2}$	0	0
N_4'		0	0	$-1/2\sqrt{2}$	$1/2\sqrt{2}$	1	0	$1+1/2\sqrt{2}$	$1/2\sqrt{2}$
S_4		0	0	$1/2\sqrt{2}$	$-1/2\sqrt{2}$	0	0	$1/2\sqrt{2}$	$1/2\sqrt{2}$

EQUATION (6.22)

The first element in the first column in the square stiffness matrix of Eq. (6.22) represents force N_1' due to x_1' , and the second element represents force S_1' due to x_1' . Similar explanations apply to the remainder of the column. Similarly, the second column represents these forces due to displacement y_1' , and so on for other coluns. Two checks can be applied to the stiffness matrix. First, it must be symmetrical. Second, for each column the sum of the N'-forces must vanish as must the sum of the S'-forces.

In our problem, nodes 1 and 4 are fixed so as to prevent any rigid-body motion of the frame, whereas nodes 2 and 3 are kept free. Eq.(6.22) can be written as

$$\begin{bmatrix} N_2' \\ S_2' \\ N_3' \\ S_3' \\ \hline N_1' \\ S_1' \\ N_4' \\ S_4' \end{bmatrix} = \left[\begin{array}{c|c} K_{11} & K_{12} \\ \hline K_{21} & K_{22} \end{array}\right] \begin{bmatrix} x_2' \\ y_2' \\ x_3' \\ y_3' \\ \hline x_1' = 0 \\ y_1' = 0 \\ x_4' = 0 \\ y_4' = 0 \end{bmatrix} \qquad (6.23)$$

in which $K_{11} \ldots\ldots K_{22}$ are the sub-matrices of (K) and are obtained from the K -values of Eq.(6.22). For example,

$$K_{11} = \frac{EA}{a}\begin{bmatrix} 1+\frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} & 0 & 0 \\ -\frac{1}{2\sqrt{2}} & 1+\frac{1}{2\sqrt{2}} & 0 & -1 \\ 0 & 0 & 1+\frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} \\ 0 & -1 & \frac{1}{2\sqrt{2}} & 1+\frac{1}{2\sqrt{2}} \end{bmatrix}$$

$$K_{21} = \frac{EA}{a}\begin{bmatrix} -1 & 0 & -\frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} \\ 0 & 0 & -\frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} \\ -\frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & -1 & 0 \\ \frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} & 0 & 0 \end{bmatrix}$$

In Eq. (6.23), N_2' , S_2' , N_3' , S_3' , are applied loads at the free nodes 2 and 3 of the structure, whereas the other forces are the unknown external reactions at the support points 1 and 4 , and x_2' , y_2' , x_3' , y_3' , are the unknown displacements. Solution for the unknown quantities results from expanding Eq. (6.23) into the following two sets of equations:

$$\begin{bmatrix} N_2' \\ S_2' \\ N_3' \\ S_3' \end{bmatrix} = \left[K_{11}\right] \begin{bmatrix} x_2' \\ y_2' \\ x_3' \\ y_3' \end{bmatrix} \qquad (6.24\ a)$$

and

$$\begin{bmatrix} N_1' \\ S_1' \\ N_4' \\ S_4' \end{bmatrix} = [K_{21}] \begin{bmatrix} x_2' \\ y_2' \\ x_3' \\ y_3' \end{bmatrix} \qquad (6.24\ b)$$

The values of unknown deflection components are computed from Eq. (6.24 a) which when substituted in Eq. (6.24 b) give the values of the support reactions.

The final step is that of determining truss member forces. Let us consider a general member ij of the frame having end loads and displacements as shown in Fig. 6.5 below.

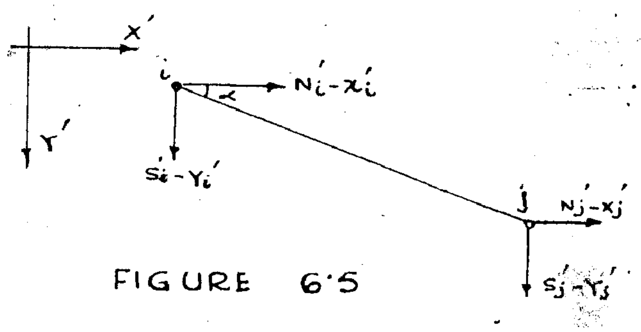


FIGURE 6·5

It is easy to show that direct force in the member is given by

$$P_{ij} = N_i - N_j = \frac{EA}{L}\left\{\lambda\,(x_i' - x_j') + \mu\,(y_i' - y_j')\right\} \qquad (6.25)$$

Compression is denoted by a positive sign. Such a relationship applies for each member of the truss. Because displacements are known in terms of applied loads from Eq. (6.24 a), the member forces can be computed from Eq. (6.25).

In the present problem, if a downward load of 5 kips. is applied at the node 3, the Eq. (6.24 a) becomes,

x_2'	y_2'	x_3'	y_3'		
1.3535	-0.3535	0	0	=	0
-0.3535	1.3535	0	- 1.00		0
0	0	1.3535	0.3535		0
0	- 1.00	0.3535	1.3535		5.00

which has its auxiliary matrix

x_2'	y_2'	x_3'	y_3'	=	
1.3535	-0.2611	0	0		0
-0.3535	1.2612	0	-0.7928		0
0	0	1.3535	0.2611		0
0	-1.00	0.3535	0.4684		10.6746

and the final matrix

$$x_2' = 2.2096 \frac{a}{EA}$$

$$y_2' = 8.4628 \frac{a}{EA}$$

$$x_3' = -2.7871 \frac{a}{EA}$$

$$y_3' = 10.6746 \frac{a}{EA}$$

Member forces as determined from Eq. (6.25) are

$$P_{12} = -2.2096 \text{ kips.}$$

$$P_{43} = +2.7871 \text{ kips.}$$

$$P_{23} = -2.2118 \text{ kips.}$$

$$P_{13} = \tfrac{1}{2}\,(2.7871 - 10.6746) = -3.9437 \text{ kips.}$$

$$P_{42} = \tfrac{1}{2}\,(-2.2096 + 8.4628) = +3.1266 \text{ kips.}$$

Eq. (6.246) on substitution becomes

$$\begin{matrix} & & 2.2096 & 8.4628 & -2.7871 & 10.6746 \end{matrix}$$

$$\begin{bmatrix} N_1' \\ S_1' \\ N_4' \\ S_4' \end{bmatrix} = \begin{bmatrix} -1.00 & 0 & -0.3535 & -0.3535 \\ 0 & 0 & -0.3535 & -0.3535 \\ -0.3535 & 0.3535 & -1.00 & 0 \\ 0.3535 & -0.3535 & 0 & 0 \end{bmatrix}$$

from which

$$N_1' = -4.9979^K \qquad N_4' = +4.9976^K$$

and

$$S_1' = -2.7883^K \qquad S_4' = -2.2105^K$$

These are the four reaction components at the support points 1 and 4.

The method described above is most useful in dealing with highly redundant trusses, where the number of members is large and the number of joints small. It is obviously not suitable for analysing statically determinate trusses.

6.7.3 Rigidly Jointed Frames.

It is obvious that a direct application of the general method, described under Art. 6.3 and 6.4, always gives three equations for each joint, corresponding to two degrees of freedom in displacement and one in rotation. The strains produced by axial forces are automatically included whether they are important or not. This may be desirable in analysing rigidly jointed trusses (which gives primary and secondary stresses combined), while in other frameworks, it may be an unnecessary refinement. In the latter case when the framework is composed of inclined members also, the general theory will be applied. The only modification needed is that the term $\frac{EA}{L}$ is put equal to zero wherever it occurs in the K'-matrices of Table 6.1.

In the case of rectangular building frames, if the axial strains are to be ignored in the analysis, the vertical movements of all the joints will be neglected and the horizontal movements of joints at one storey level will be equal. The modification in the general theory needed for analysing such frames and the procedure for obtaining the stiffness matrix of the complete frame directly from the stiffness matrices of the individual members shall be discussed below.

Such frames are composed of horizontal beam members and vertical columns. Since the vertical movement of the joints is neglected, the beam members will have their stiffness matrix equations of the form

$$\begin{bmatrix} M_1 \\ M_2 \end{bmatrix} = \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

where $$\begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} = \begin{bmatrix} \frac{4EI}{L} & \frac{2EI}{L} \\ \frac{2EI}{L} & \frac{4EI}{L} \end{bmatrix} \qquad (6.2.6)$$

the member being of a uniform cross-section throughout its length.

Let us consider a column member 12 whose upper end 2 has a positive horizontal displacement relative to the lower end 1. The matrix equation of the member will be of the form

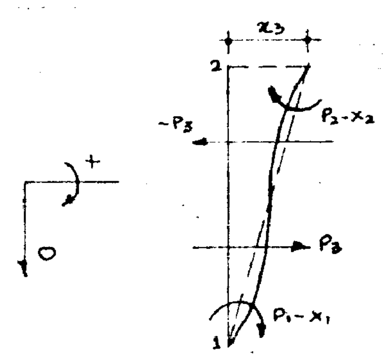


FIGURE 6'6

$$\begin{bmatrix} P_1 \\ P_2 \\ P_3 \end{bmatrix} = \begin{bmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & k_{23} \\ k_{31} & k_{32} & k_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

where the elements of the stiffness matrix are the coefficients in the slope-deflection equations for the member. Thus,

$$\begin{bmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & k_{23} \\ k_{31} & k_{32} & k_{33} \end{bmatrix} = \begin{bmatrix} \frac{4EI}{L} & \frac{2EI}{L} & \frac{-6EI}{L^2} \\ \frac{2EI}{L} & \frac{4EI}{L} & -\frac{6EI}{L^2} \\ \frac{6EI}{L^2} & \frac{6EI}{L^2} & \frac{-12EI}{L^3} \end{bmatrix} \quad (6.27)$$

Shear at the end 2 is $-P_3$.

The method of obtaining the stiffness influence coefficients for the complete structure will be clear from the following simple example of a two-storeyed, single bay portal frame shown in Fig. (6.7 a).

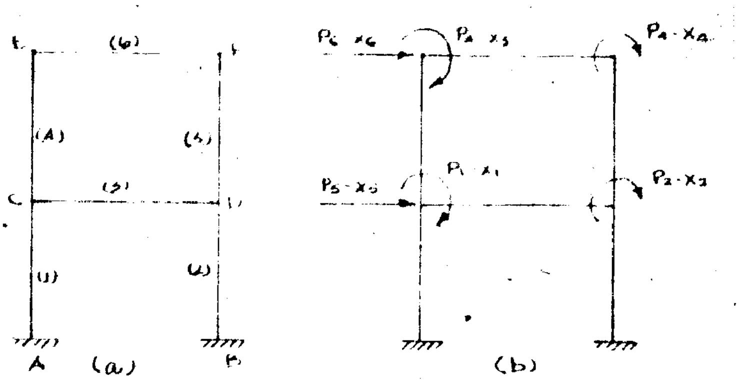


FIGURE 6·7

The stiffness matrices of beam members (3) and (6) will be of the form given in Eq. (6.26) and those of the column members (1), (2), (4) and (5) being given by Eq. (6.27). The load-displacement equations in matrix form P = KX are readily obtained from the stiffness influence coefficients of individual members. This is given by Eq. (6.28)

$$
\begin{array}{cc}
X_1 & X_2 \\
(k_{22})_1+(k_{11})_3+(k_{11})_4 & (k_{12})_3 \\
(k_{21})_3 & (k_{22})_3+(k_{22})_2+(k_{11})_5 \\
(k_{21})_4 & 0 \\
0 & (k_{21})_5 \\
-(k_{32})_1+(k_{31})_4 & -(k_{32})_2+(k_{31})_5 \\
-(k_{31})_4 & -(k_{31})_5
\end{array}
$$

As a check, it should be noted that the matrix K is symmetrical.

6.8 Procedure.

In applying the stiffness matrix technique, described in the preceding articles, to the analysis of indeterminate structures, one proceeds as follows.

1. Assume unknown deflections (angular rotations, horizontal, linear and vertical linear) at all,

Joints that can have deflections. This includes assumed sidesway deflections in rigid frames. There will be n unknown deflections.

(We do not assume deflections at points that cannot deflect. Thus a hinged end can have only an angular deflection and not linear. A rigidly encastred end cannot have deflection of any kind. This assumption is necessary, since otherwise the determinant of the stiffness matrix will be zero and the inverse of the matrix undefined.)

2. Assume a load acting at each joint corresponding to the assumed deflection - moment for angular deflection, force for linear one. There will be n loads, all of which are evaluated as the equivalent fixed-end loads.

3. Compute the elements of the stiffness matrix K for the complete structure from the stiffness matrices of the individual members. These will be the elements of an n x n symmetrical matrix.

4. Solve the matrix equation (6.9) for the unknown deflections.

5. Knowing all the deflections and using the relation (6.6) for every member, determine the internal stresses at all key points in the structure.

6. The solution obtained in step 5 is then superimposed over the fixed-end solution of the structure.

6.9 Miscellaneous Problems.

The stiffness matrix technique described in the preceding articles can be extended to other classes of structural problems also. These problems being of highly complex nature can be attempted with this technique only when automatic digital computer facilities are available to the structural engineer. Two of the more common types of structural problems are considered below.

(a) Arch Rib Analysis.

Consider the arch rib shown in Fig. (6.8 a). The arch structure may be represented by a series of straight beam segments between the load points.

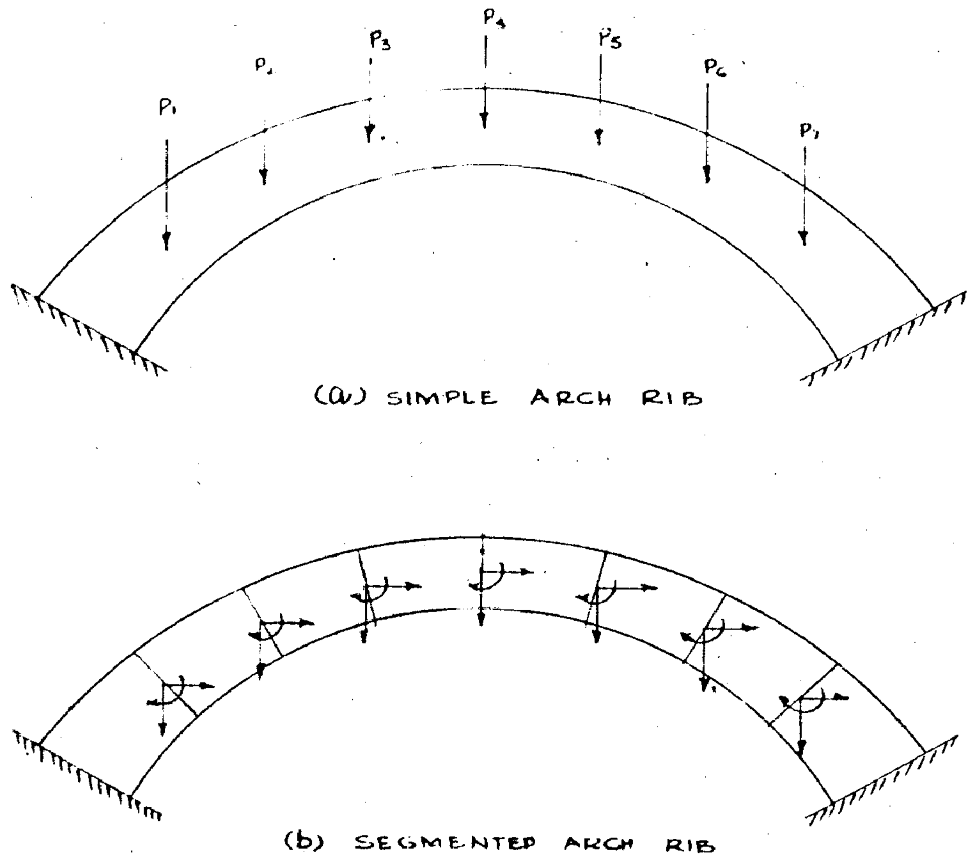


FIGURE 6·8

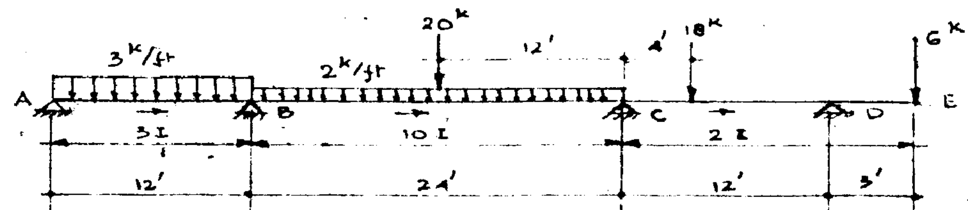
The deflection and loading (i.e., P-X diagram) of the structure may then be represented by the system of coordinates shown in Fig. (6.8 b). 21 coordinates are required for the stiffness matrix analysis. Moments, shears and axial thrusts may be computed at each end of each segment.

(b) Natural Mode Analysis.

Most civil engineering structures are not analysed to determine their response to dynamic forces, there being two exceptions. First the structure which is analysed to determine the effect of earthquake forces upon it and second, the structure which is designed to resist the effect of a bomb blast. In either case, the calculation of the dynamic response of the structure to a transient load is easily performed if one first obtains the natural frequencies and mode shapes for the structure. The mode shape analysis is easily carried out using the K matrix obtained for the deflection calculation. Iterative procedures which converge on the lowest mode give best results. The computations must be performed on Automatic Digital Computers, which will be dealt with in the next chapter.

Example 6.1

We shall now analyse the continuous beam shown in Fig. 6.9. It will illustrate the stiffness matrix technique described in Art. 6.7.1.



	A–B		B–C		C–D		D–E
RELATIVE STIFFNESS	$\frac{3}{12}\times 12 = (3)$		$\frac{10}{24}\times 12 = (5)$		$\frac{2}{12}\times 12 = (2)$		
FIXED END MOMENTS	−36	+36	−156	+156	−32	+16	−18
UNBALLANCED JOINT MOMENT	+36 (A)		+120 (B)		−124 (C)		+2 (D)

FIGURE 6·9

From matrix Eqs. (6.17) for individual spans, the complete set of equilibrium equations for the entire beam is obtained as given below.

$$
\begin{array}{cccccc}
\theta_A & \theta_B & \theta_C & \theta_D & = & \\
12 & 6 & 0 & 0 & & +36 \\
6 & 12+20 & 10 & 0 & & +120 \\
0 & 10 & 20+8 & 4 & & -124 \\
0 & 0 & 4 & 8 & & +2
\end{array}
$$

The auxiliary matrix is computed to be

12	0.5	0	0	=	3.00
6	29	0.3448	0		3.5172
0	10	24.552	0.1629		-6.483
0	0	4	7.3484		3.801

and finally the values of θ's as

$$+0.017 \qquad +5.966 \qquad -7.102 \qquad +3.801$$

Once the joint deflections have been evaluated thus, the calculation of moments and shear forces are quite simple. The calculations for end moments are given below.

kips-ft.

$M_{AB} = -36 + 3(\ 4 \times 0.017 + 2 \times 5.966) \qquad = \ 0$

$M_{BA} = +36 + 3(\ 2 \times 0.017 + 4 \times 5.966) \qquad = +107.69$

$M_{BC} = -156 + 5(\ 4 \times 5.966 - 2 \times 7.102) \qquad = -107.70$

$M_{CB} = +156 + 5(\ 2 \times 5.966 - 4 \times 7.102) \qquad = +73.62$

$M_{CD} = -32 + 2(\ -4 \times 7.102 + 2 \times 3.801) \qquad = -73.61$

$M_{DC} = +16 + 2\ (-2 \times 7.102 + 4 \times 3.801) \qquad = +18$

$M_{DE} = -18$

Equilibrium of the joints is the check on the accuracy of the analysis. It will be noted that the sum of the moments at all joints vanishes.

Alternatively, let us solve the above problem by modifying the stiffnesses of the end spans AB and CD according to the modification given in Art. 6.6. After making this modification the ends A and D shall be treated as fixed. But the fixed-end moments will also have to be modified.

Joint.	A	B	B	C	C	D	D
Normal fixed-end moments.	-36	+36	-156	+156	-32	+16	-18
	+36	+18			+1	+2	
Modified fixed-end moments.	0	+54	-156	+156	-31	+18	-18
Unbalanced moments.		+102			-125		

The modified equilibrium equations now become

$$\begin{matrix} \theta_B & \theta_C & & \\ 9+20 & 10 & & +102 \\ & & = & \\ 10 & 20+6 & & -125 \end{matrix}$$

which has its auxiliary matrix

$$\begin{matrix} \theta_B & \theta_C & = \\ 29 & 0.3448 & 3.5172 \\ 10 & 22.552 & -7.1023 \end{matrix}$$

and finally

$\theta_B = + 5.9661$

$\theta_C = - 7.1023$

End moments are calculated as usual

$M_{AB} = 0$

$M_{BA} = +54 + 3 \ (3 \times 5.9661) = 107.69$

$M_{BC} = - 156 + 5(4 \times 5.9661 - 2 \times 7.1023) = -107.70$

$M_{CB} = + 156 + 5 \ (2 \times 5.9661 - 4 \times 7.1023) = + 73.615$

$M_{CD} = - 31 + 2 \ (-3 \times 7.1023) = -73.614$

$M_{DC} = + 18$

$M_{DE} = -18$

The values of end moments are found identical with those obtained by the previous analysis.

If now the end A of the above beam is assumed to be fixed, the equilibrium equations will be obtained by deleting the first row and column from the original matrix equation. Thus,

θ_B	θ_C	θ_D	=
12 + 20	10	0	+ 120
10	20 + 8	4	- 124
0	4	8	2

The auxiliary matrix is computed to be

32	0.3125	0	3.75
10	24.875	0.1608	-6.4924
0	4	7.3568	3.8018

which gives

$$\theta_B = +5.9699$$

$$\theta_C = -7.1037$$

$$\theta_D = +3.8018$$

As usual, the computation of end moments from these values of joint rotations follows from the slope-deflection equations for various spans. The computed values are given below.

(K-ft)

$M_{AB} = -0.1806$

$M_{BA} = +107.638$

$M_{BC} = -107.639$

$M_{CB} = +73.625$

$M_{CD} = -73.622$

$M_{DC} = +17.999$

$M_{DE} = -18$

For each subsequent loading condition, only the last column in the auxiliary matrix has to be re-calculated and from it the final matrix, giving the values of the unknown joint rotations. The end moments are then calculated with these values.

Example 6.2

Analyse the simple building frame loaded as shown in Fig. 6.10 (a) below.

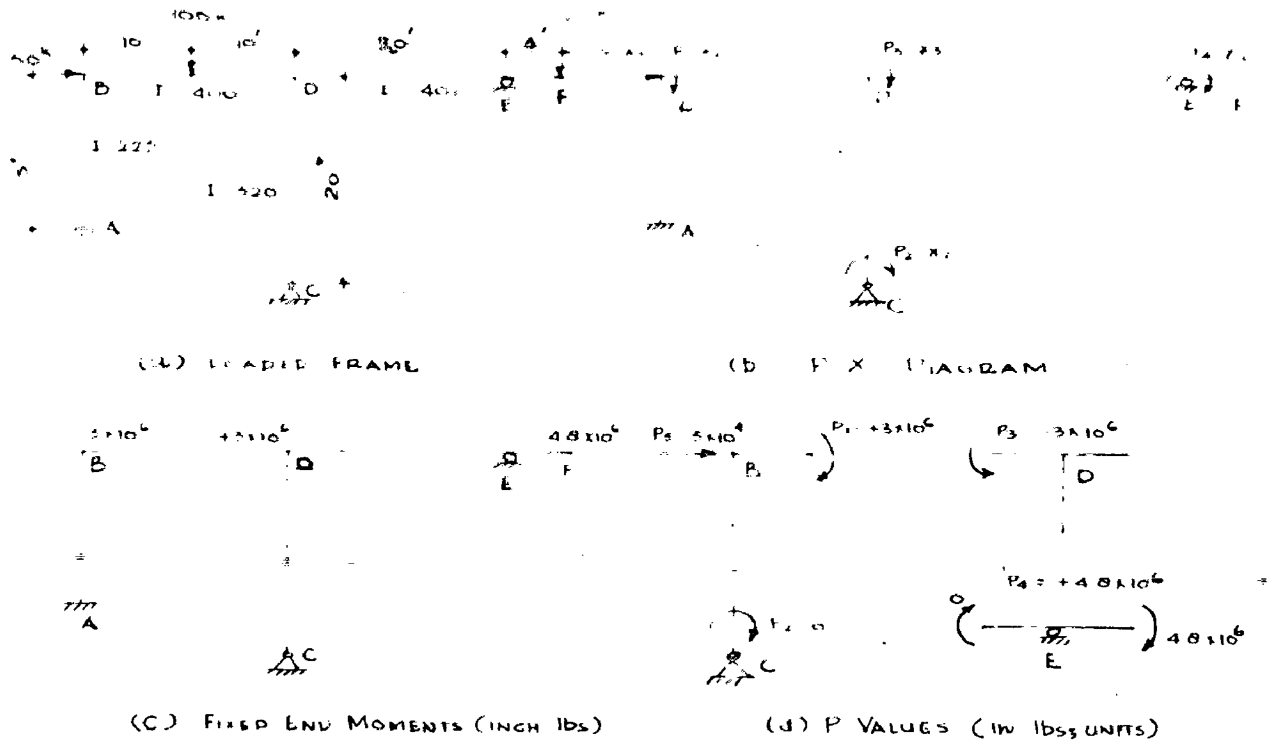


FIGURE 6·10

Following the method given in Art. 6.7.3, the stiffness matrix of the complete frame, corresponding to P-x values assumed as shown above, is assembled as given below.

x_1	x_2	x_3	x_4	x_5	=
$(k_{22})_{AB}+(k_{11})_{BD}$	0	$(k_{12})_{BD}$	0	$(k_{23})_{AB}$	P_1
0	$(k_{11})_{CD}$	$(k_{12})_{CD}$	0	$(k_{13})_{CD}$	P_2
$(k_{21})_{BD}$	$(k_{21})_{CD}$	$\left\{ \begin{matrix} (k_{22})_{BD}+(k_{22})_{CD} \\ +(k_{11})_{DE} \end{matrix} \right\}$	$(k_{12})_{DE}$	$(k_{23})_{BD}$	P_3
0	0	$(k_{21})_{DE}$	$(k_{22})_{DE}$	0	P_4
$-(k_{32})_{AB}$	$(k_{31})_{CD}$	$(k_{32})_{CD}$	0	$-(k_{33})_{AB}$ $(k_{33})_{CD}$	P_5

The given matrix of the problem thus becomes

X_1	X_2	X_3	X_4	X_5	=
0.35×10^9	0	0.1×10^9	0	-0.125×10^7	0.3×10^7
0	0.16×10^9	0.8×10^8	0	-0.1×10^7	0
0.1×10^9	0.8×10^8	0.56×10^9	0.1×10^9	-0.1×10^7	-0.3×10^7
0	0	0.1×10^9	0.2×10^9	0	0.48×10^7
-0.125×10^7	-0.1×10^7	-0.1×10^7	0	0.222×10^5	0.5×10^5

and the auxiliary matrix is computed to be

x_1	x_2	x_3	x_4	x_5	=
0.35×10^9	0	0.28571	0	-0.35714×10^{-2}	0.85714×10^{-2}
0	0.16×10^9	0.500	0	-0.625×10^{-2}	0
0.1×10^9	0.8×10^8	0.49143×10^9	0.20348	-0.29070×10^{-3}	-078487×10^{-2}
0	0	0.1×10^9	0.17965×10^9	0.16181×10^{-3}	0.31087×10^{-1}
-0.125×10^7	-0.1×10^7	-0.14286×10^6	0.29070×10^5	0.11461×10^5	5.12075

which gives the final matrix

X_1	X_2	X_3	X_4	X_5
0.30431×10^{-1}	0.38254×10^{-1}	-0.0125	0.30258×10^{-1}	5.12075

From these values of the unknown displacement components, internal stresses in various members can be calculated from

Eqs. (6.26) and (6.27). The computed values of the end-moments are given below.

$M_{AB} = -4118.25$ (kip-inches)

$M_{BA} = -1836.00$ "

$M_{BD} = +1836.20$ "

$M_{DB} = +3543.10$ "

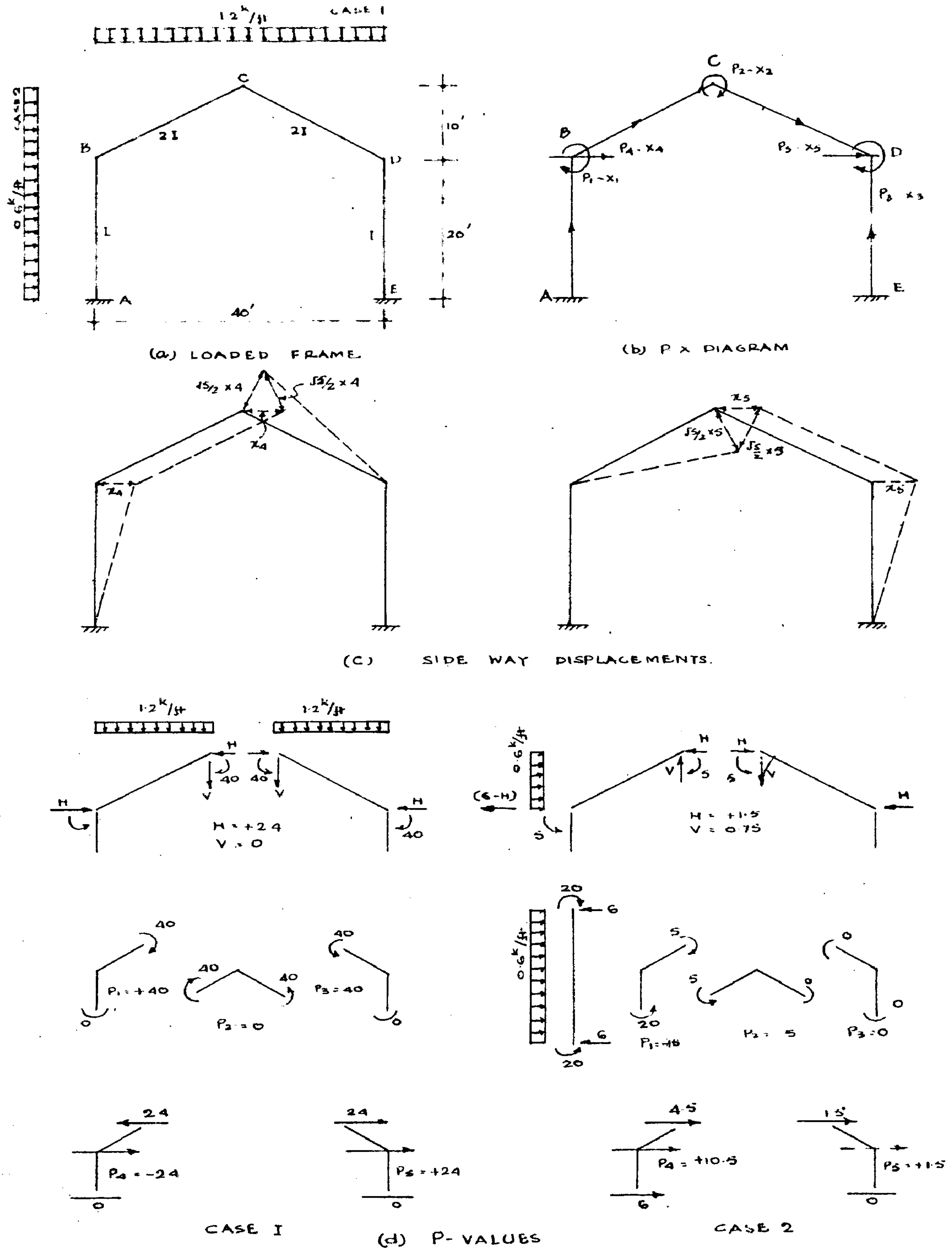
$M_{DC} = -4060.00$ "

$M_{CD} = 0$ "

$M_{DE} = +525.80$ "

$M_{ED} = +4801.60$ "

$M_{EF} = -4800.00$ "



1.2k/ft
CASE 1
CASE 2
0.6k/ft
C
B
D
A
E
2I
2I
I
I
10'
20'
40'
(a) LOADED FRAME
P2 - X2
P4 - X4
P5 - X5
P1 - X1
P3 - X3
(b) P X DIAGRAM
√5/2 x 4
√5/2 x 4
x4
x5
√5/2 x5
√5/2 x5
(C) SIDE WAY DISPLACEMENTS.
H = +24
V = 0
H = +1.5
V = 0.75
(6-H)
P1 = +40
P2 = 0
P3 = 40
P3 = 0
P4 = -24
P5 = +24
P4 = +10.5
P5 = +1.5
CASE I
(d) P- VALUES
CASE 2

FIGURE 6·11

Example 6.3.

Let us analyse by stiffness matrix method, the gable frame shown in Fig. (6.11 a). Effect of axial strains is to be neglected. Thus, the frame will have three unknown joint rotations and two sidesway displacements as shown in Fig.(6.11.b).

Stiffness matrix for members AB and ED is given by

$$\begin{bmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & k_{23} \\ k_{31} & k_{32} & k_{33} \end{bmatrix} = EI \begin{bmatrix} 0.2 & 0.1 & -0.15\times10^{-1} \\ 0.1 & 0.2 & -0.15\times10^{-1} \\ 0.15\times10^{-1} & 0.15\times10^{-1} & -0.15\times10^{-2} \end{bmatrix}$$

and that for BC and DC will be

$$EI \begin{bmatrix} 0.35777 & 0.17888 & -0.24\times10^{-1} \\ 0.17888 & 0.35777 & -0.24\times10^{-1} \\ 0.24\times10^{-1} & 0.24\times10^{-1} & -0.21466\times10^{-2} \end{bmatrix}$$

The assembled stiffness matrix of the complete frame is given as below.

	X_1	X_2
P_1	$(k_{21})_{AB} + (k_{11})_{BC}$	$(k_{12})_{BC}$
P_2	$(k_{21})_{BC}$	$(k_{22})_{BC} + (k_{2}$ …
P_3	0	$(k_{12})_{DC}$
P_4	$-(k_{32})_{AB} + \frac{\sqrt{5}}{2}(k_{31})_{BC}$	$-\frac{\sqrt{5}}{2}(k_{32})_{BC} - \frac{\sqrt{5}}{2}($ …
Pc	$-\frac{\sqrt{5}}{2}(k_{31})_{BC}$	$-\frac{\sqrt{5}}{2}(k_{32})_{BC} + \frac{\sqrt{5}}{2}(k$ …

(Multiplier EI)

1.	0.55777	0.17888	0	0.11833×10^{-1}	-0.26832×10^{-1}
2	0.17888	0.71554	0.17888	0	0
3	0	0.17888	0.55777	-0.26832×10^{-1}	0.11832×10^{-1}
4	0.11832×10^{-1}	0	-9.26832×10^{-1}	0.68665×10^{-2}	-0.53665×10^{-2}
5	-0.26832×10^{-1}	0	0.11832×10^{-1}	-0.53665×10^{-2}	0.68665×10^{-2}

Auxiliary matrix is computed to be

1.	0.55777	0.32070	0	0.21213×10^{-1}	-0.48106×10^{-1}
2.	0.17888	0.65817	0.27178	-0.57653×10^{-2}	0.13074×10^{-1}
3.	0	0.17888	0.50915	-0.50674×10^{-1}	0.18645×10^{-1}
4.	0.11832×10^{-1}	-0.37946×10^{-2}	-0.25801×10^{-1}	0.52862×10^{-2}	-0.80713
5.	-0.26832×10^{-1}	0.86052×10^{-2}	0.94933×10^{-2}	-0.42666×10^{-2}	0.18425×10^{-2}

P-values as found from Fig. (6.11 d), for the two cases of loading, are

	Case 1	Case 2.
P_1	40	-15
P_2	0	-5
P_3	-40	0
P_4	-24	10.5
P_5	24	1.5

(kip-ft units)

The corresponding last columns of the auxiliary matrix are computed to be

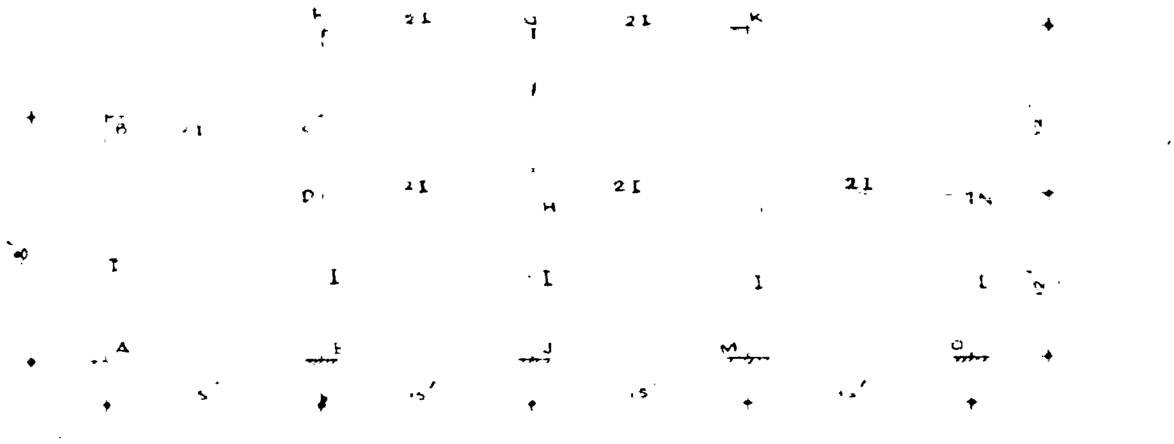
	Case 1	Case 2.
1.	0.71714×10^{2}	-0.26893×10^{2}
2.	-0.19491×10^{2}	-0.00288×10^{2}
3.	-0.71714×10^{2}	0.00101×10^{2}
4.	-0.50646×10^{4}	0.20468×10^{4}
5.	0.28027×10^{4}	0.51629×10^{4}

and the corresponding final matrices are

	Case 1	Case 2	
X_1	0.26599×10^{3}	0.11897×10^{3}	
X_2	0	-0.91406×10^{2}	
X_3	-0.26599×10^{3}	0.21872×10^{3}	(multiplier $\frac{1}{EI}$)
X_4	-0.28027×10^{4}	0.62139×10^{4}	
X_5	0.28027×10^{4}	0.51629×10^{4}	

Moments at the ends of all members for both loading conditions can now be calculated and the final values are give below.

	Case 1	Case 2.	
M_{AB}	+68.64	-101.312	
M_{BA}	+ 95.24	- 49.415	
M_{BC}	- 95.24	+ 49.413	(Units
M_{CB}	- 62.82	+ 21.778	kips.ft)
M_{CD}	+ 62.82	- 21.780	
M_{DC}	+ 95.24	+ 33.700	
M_{DE}	- 95.239	-33.700	
M_{ED}	- 68.64	- 55.573	

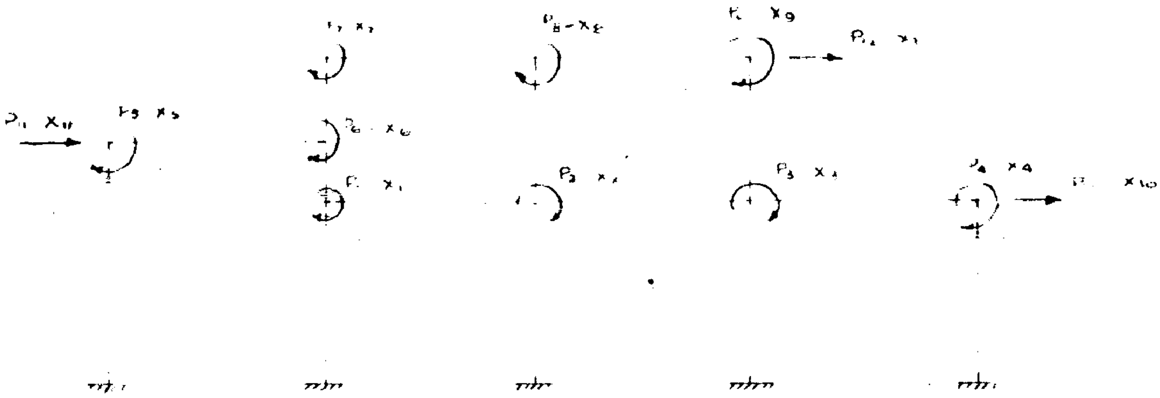


LOADING

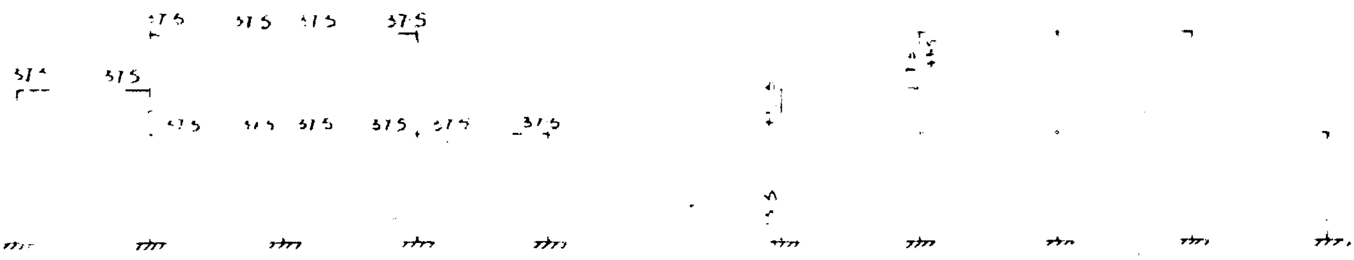
CASE 1 ALL BEAM MEMBERS LOADED WITH 2.0 k/ft.

CASE 2 HORIZONTAL WIND LOAD 0.5 k/ft FROM LEFT TO RIGHT.

(a) LOADED BUILDING FRAME



(b) P - X DIAGRAM



CASE 1. CASE 2

(c) FIXED END MOMENTS (in k ft)

FIGURE 6.12

Example 6.4

Let us now analyse a portal type broken storey building frame for the two cases of loading condition as shown in Fig. (6.12 a).

Stiffness matrices of individual members are as given below.

All beam members

$$EI \begin{bmatrix} 0.53333 & 0.26666 \\ 0.26666 & 0.53333 \end{bmatrix}$$

$$K_{AB} = EI \begin{bmatrix} 0.22222 & 0.11111 & -0.18518 \times 10^{-1} \\ 0.11111 & 0.22222 & -0.18518 \times 10^{-1} \\ 0.18518 \times 10^{-1} & 0.18518 \times 10^{-1} & -0.20576 \times 10^{-1} \end{bmatrix}$$

$$K_{DC,CF} = EI \begin{bmatrix} 0.66666 & 0.33333 & -0.16666 \\ 0.33333 & 0.66666 & -0.16666 \\ 0.16666 & 0.16666 & -0.55555 \times 10^{-1} \end{bmatrix}$$

Other column members,

$$EI \begin{bmatrix} 0.33333 & 0.16666 & -0.41666 \times 10^{-1} \\ 0.16666 & 0.33333 & -0.41666 \times 10^{-1} \\ 0.41666 \times 10^{-1} & 0.41666 \times 10^{-1} & -0.69444 \times 10^{-2} \end{bmatrix}$$

From these matrices the stiffness matrix K for the complete frame is assembled, using the technique described in Art. 6.7.3. The equation P = KX is given in Table 6.4.

Table 6.4

Given matrix (Multiplier H)

	X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8	X_9	X_{10}	X_{11}	X_{12}
P_1	1.53332	0.26666	0	0	0	0.33333	0	0	0	0.125	-0.16666	0
P_2	0.26666	1.73332	0.26666	0	0	0	0	0.16666	0	0	0	-0.41666×10^{-1}
P_3	0	0.26666	1.73332	0.26666	0	0	0	0	0.16666	0	0	-0.41666×10^{-1}
P_4	0	0	0.26666	0.86666	0	0	0	0	0	-0.41666×10^{-1}	0	0
P_5	0	0	0	0	0.75555	0.26666	0	0	0	0	-0.18518×10^{-1}	0
P_6	0.33333	0	0	0	0.26666	1.86665	0.33333	0	0	0.16666	0	-0.16666
P_7	0	0	0	0	0	0.33333	1.19999	0.26666	0	0	0.16666	-0.16666
P_8	0	0.16666	0	0	0	0	0.26666	1.39999	0.26666	0.41666×10^{-1}	0	-0.41666×10^{-1}
P_9	0	0	0.16666	0	0	0	0	0.26666	0.86666	0.41666×10^{-1}	0	-0.41666×10^{-1}
P_{10}	0.125	0	0	-0.41666×10^{-1}	0	0.16666	0	0.41666×10^{-1}	0.41666×10^{-1}	0.97221×10^{-1}	-0.55555×10^{-1}	-0.13888×10^{-1}
P_{11}	-0.16666	0	0	0	-0.18518×10^{-1}	0	0.16666	0	0	-0.55555×10^{-1}	1.13167×10^{-1}	-0.55555×10^{-1}
P_{12}	0	-0.41666×10^{-1}	-0.41666×10^{-1}	0	0	-0.16666	-0.16666	-0.41666×10^{-1}	-0.41666×10^{-1}	-0.13888×10^{-1}	-0.55555×10^{-1}	0.69444×10^{-1}

Table 6.5

Auxiliary matrix.

	X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8	X_9	X_{10}	X_{11}	X_{12}	
1.	1.53332	0.17391	0	0	0	0.21739	0	0	0	0.81522×10^{-1}	-0.10869	0	
2.	0.26666	1.68694	0.15807	0	0	-0.34363×10^{-1}	0	0.98794×10^{-1}	0	-0.12886×10^{-1}	0.17181×10^{-1}	-0.24699×10^{-1}	
3.	0	0.26666	1.69117	0.15767	0	0.54183×10^{-2}	0	-0.50372×10^{-1}	0.98547×10^{-1}	0.20318×10^{-2}	-0.27091×10^{-2}	-0.20743×10^{-1}	
4.	0	0	0.26666	0.82461	0	-0.17521×10^{-2}	0	0.50372×10^{-2}	-0.31868×10^{-1}	-0.51185×10^{-1}	0.87606×10^{-3}	0.67078×10^{-2}	
5.	0	0	0	0	0.75555	0.35293	0	0	0	0	-0.24509×10^{-1}	0	
6.	0.33333	-0.57969×10^{-1}	0.91632×10^{-2}	-0.14448×10^{-2}	0.26666	1.69808	0.19630	0.34611×10^{-2}	-0.55891×10^{-3}	0.81651×10^{-1}	0.25787×10^{-1}	-0.98874×10^{-1}	
7.	0	0	0	0	0	0.33333	1.13456	0.23402	0.1642×10^{-3}	-0.23988×10^{-1}	0.13932	-0.11784	
8.	0	0.16666	-0.26344×10^{-1}	0.41537×10^{-2}	0	0.58770×10^{-2}	0.26551	1.32098	0.20390	0.37827×10^{-1}	-0.30342×10^{-1}	-0.47353×10^{-2}	
9.	0	0	0.16666	-0.26278×10^{-1}	0	-0.94905×10^{-3}	0.1863×10^{-3}	0.26935	0.79448	0.37604×10^{-1}	0.1088×10^{-1}	-0.46356×10^{-1}	
10.	0.125	-0.21738×10^{-1}	0.34362×10^{-2}	-0.42208×10^{-1}	0	0.13865	-0.27217×10^{-1}	0.49969×10^{-1}	0.29875×10^{-1}	0.69596×10^{-1}	-0.57678	-0.27979×10^{-1}	
11.	-0.16666	0.28983×10^{-1}	-0.45815×10^{-2}	0.72241×10^{-3}	-0.18518×10^{-1}	0.43787×10^{-1}	0.15806	-0.40081×10^{-1}	0.86456×10^{-2}	-0.40141×10^{-1}	0.46475×10^{-1}	-0.70782	
12.	0	-0.41666×10^{-1}	-0.35079×10^{-1}	0.55313×10^{-2}	0	-0.16789	-0.13370	-0.62552×10^{-2}	-0.36829×10^{-1}	-0.19472×10^{-2}	-0.32896×10^{-1}	0.10219×10^{-1}	Final matrix (multiplier I/EI)
	3.32963x10	-8.23574	1.16727x10	-4.52996x10	6.72583x10	-5.05350x10	3.86873x10	-0.52973	-4.99793x10	3.24857x10	-0.85577x10	-5.69312x10	Case 1
	2.52875x10	7.54658	9.65346	1.16711x10	-1.27376x10	2.12769x10	-2.30579	5.46678	8.71587	3.04541×10^{2}	5.15706×10^{2}	5.59435×10^{2}	Case 2

The last column in the given matrix (the P-values) for the two cases and the corresponding check columns are as given below.

	P-values.		Check Column	
	Case 1	Case 2	Case 1	Case 2
1.	+ 37.5	0	39.59165	2.09165
2.	0	0	2.39163	2.39163
3.	0	0	2.39163	2.39163
4.	- 37.5	0	-36.40834	1.09165
5.	+ 37.5	-13.5	38.50369	-12.49631
6.	- 37.5	+ 1.5	-34.70003	4.29997
7.	+ 37.5	- 1.5	39.29998	0.29998
8.	0	0	2.09997	2.09997
9.	- 37.5	0	-36.20002	1.29998
10.	0	0	0.36110	0.36110
11.	0	+ 6.0	-0.016461	5.98354
12.	0	+ 1.5	-0.49998	1.00002

The auxiliary matrix is computed in Table 6.5 with the columns corresponding to above given below.

1.	2.44567x10	0	2.58208x10	1.36413
2.	-3.86594	0	-2.66384	1.20210
3.	-0.60957	0	1.83421	1.22464
4.	-4.56732x10	0	-4.47453x10	0.92781
5.	4.96327x10	-1.78678x10	5.09611x10	-1.65393x10
6.	-3.48538x10	3.68935	-3.36461x10	4.89711
7.	4.32924x10	-2.40602	4.45241x10	-1.17435
8.	-7.90297	0.46718	-6.69636	1.67380
9.	-4.62117x10	- 0.15342	-4.52095x10	0.84871
10.	3.90145x10	- 8.56046	3.94098x10	-8.16529
11.	3.17393x10	1.19727x10^{2}	3.20315x10	1.20019x10^{2}
12.	-5.69312x10	5.59435x10^{2}	-5.59313x10	5.6043x10^{2}

With these values and the auxiliary matrix of Table 6.5 the final matrices for the two cases are computed which give the values of the unknown X's. These are at the bottom of the Table 6.5 .

Joint displacements having been evaluated thus, the calculation of final end-moments follow from the stiffness matrix equations for individual members, superimposed with the fixed-end moments. The computed values of the end moments for the two cases of loading condition are listed as below. Equilibium of joints will check the accuracy of the analysis.

Moments.	Case 1.	Case 2.	(Values in kps.ft)
M_{AB}	7.6315	-24.4651	
M_{BA}	15.1046	1.1196	
M_{BC}	-15.1048	-1.1196	
M_{CB}	28.4833	7.9510	
M_{CD}	-15.7507	-12.5792	
M_{DC}	12.1928	-11.2424	
M_{DE}	9.7451	-4.2600	
M_{ED}	4.1956	-8.4746	
M_{CF}	-12.7321	4.6280	
M_{FC}	17.0084	-0.2328	
M_{FG}	-17.0082	0.2300	
M_{GF}	47.5338	2.3007	

Moments.	Case 1	Case 2
M_{GH}	2.1765	-7.5405
M_{HG}	0.8921	-7.1938
M_{GK}	-51.1100	5.2398
M_{KG}	10.7033	6.1062
M_{KL}	-10.9886	-6.1063
M_{LK}	-0.7130	-5.9500
M_{DH}	-21.9382	15.4989
M_{HD}	41.9864	10.7680
M_{HL}	-38.7797	6.5990
M_{LH}	41.5292	7.1608
M_{LN}	-43.3542	8.2607
M_{NL}	16.4530	8.7987
M_{JH}	-2.7261	-11.4313
M_{HJ}	-4.0987	-10.1735
M_{ML}	0.5918	-11.0802
M_{LM}	2.5373	-9.4712
M_{ON}	-8.9032	-10.7439
M_{NO}	-16.4533	-8.7986

Example 6.5

A redundant pin-connected truss shown in Fig.6.13, will be analysed in this example. The truss is loaded as shown in the figure. The nodes of the truss are numbered 1 through 8. Values of A and E will be assumed as the same for each member. With the supports, as shown, the truss has three internal redundant members and five redundant components of external reactions.

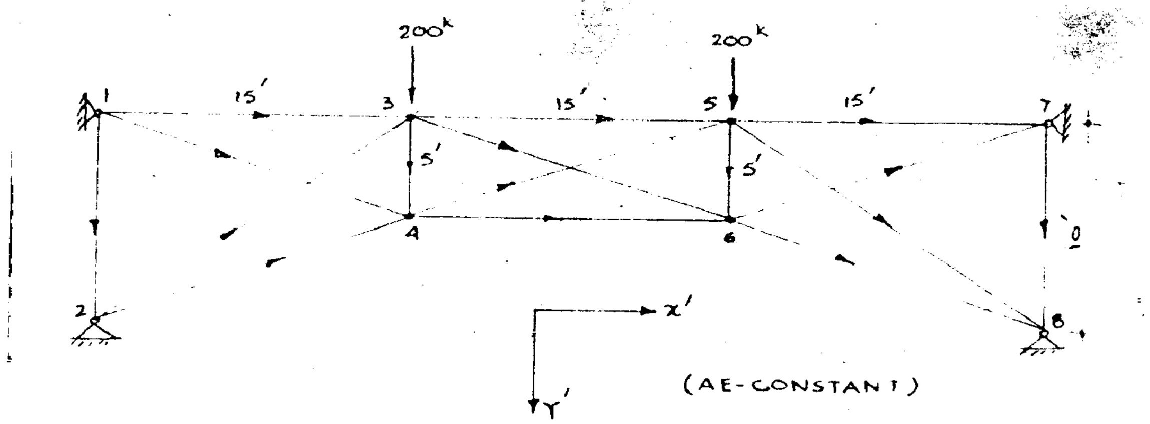


FIGURE 6.13

The stiffness matrix can be developed by first determining λ^2, μ^2 and $\lambda\mu$ and then $\bar{\lambda}^2$, $\bar{\mu}^2$ and $\bar{\lambda}\bar{\mu}$ for each member - this is done in Table 6.6.

Arrows have been marked on each member in Fig. 6.13 to show their positive x - directions.

MEMBER	L	λ	μ	λ^2
1-2, 7-8	10	0	1	0
3-4, 5-6	5	0	1	0
1-3, 3-5, 4-6, 5-7	15	1	0	1
2-4, 4-5, 6-7	$5\sqrt{10}$	$3/\sqrt{10}$	$-1/\sqrt{10}$	0.9
1-4, 3-6, 6-8	$5\sqrt{10}$	$3/\sqrt{10}$	$1/\sqrt{10}$	0.9
2-3	$5\sqrt{13}$	$3/\sqrt{13}$	$-2/\sqrt{13}$	9/13
5-8	$5\sqrt{13}$	$3/\sqrt{13}$	$2/\sqrt{13}$	9/13

TABLE

From values given in Table 6.6, the stiffness matrix of the complete truss is assembled. This matrix is given in Table 6.7, along with the last column of applied joint loads and the check column. The corresponding auxiliary matrix is obtained in Table 6.8, at the bottom of which is also given the final matrix.

Having, thus, obtained the values of unknown joint displacements, the axial forces in the individual members can easily be computed from Eq. (6.25). These are computed and the values with sign are shown in the following Fig.6.14.

Table 6.7.

Given matrix (Multiplier EA/5)

	X'_3	y'_3	x'_4	y'_4	x'_5	y'_5	x'_6	y'_6	=	Check column
N'_3	1.1432	-0.03313	0	0	-0.3333	0	-0.2846	-0.09487	0	0.3973
S'_3	-0.03313	1.1169	0	-1	0	0	-0.09487	-0.03162	200	199.9573
N'_4	0	0	1.1871	-0.09487	-0.2846	0.09487	-0.3333	0	0	0.5692
S'_4	0	-1	-0.09487	1.09486	0.09487	-0.03162	0	0	0	0.06324
N'_5	-0.3333	0	-0.2846	0.09487	1.1432	0.03313	0	0	0	0.6533
S'_5	0	0	0.09487	-0.03162	0.03313	1.1169	0	-1	200	200.2133
N'_6	-0.2846	-0.09487	-0.3333	0	0	0	1.1871	0.09487	0	0.5692
S'_6	-0.09487	-0.03162	0	0	0	-1	0.09487	1.09486	0	0.06324

Table 6.8

Auxiliary Matrix

x_3'	y_3'	x_4'	y_4'	x_5'	y_5'	x_6'	y_6'	=	Check column
1.1432	-0.02898	0	0	-0.2915	0	-0.2489	-0.08299	0	0.3475
-0.03313	1.1159	0	-0.8961	-0.8654×10^{-2}	0	-0.9241×10^{-1}	-0.3080×10^{-1}	1.7923×10^{2}	1.7920×10^{2}
0	0	1.871	-0.7992×10^{-1}	-0.2397	0.7992×10^{-1}	-0.2808	0	0	0.4795
0	-1.0	-0.9487×10^{-1}	0.1912	0.3320	-0.1257	-0.6226	-0.1611	9.3739×10^{2}	9.3781×10^{2}
-0.3333	-0.9657×10^{-2}	-0.2846	0.6347×10^{-1}	0.9567	0.6674×10^{-1}	-0.1299	-0.1853×10^{-1}	-6.0380×10^{2}	-5.9461×10^{2}
0	0	0.9487×10^{-1}	-0.2404×10^{-1}	0.6385×10^{-1}	1.1020	0.1812×10^{-1}	-0.9099	2.0543×10^{2}	2.0554×10^{2}
-0.2846	-0.1031	-0.3333	-0.1190	-0.1242	0.1996×10^{-1}	0.9226	0.7020×10^{-1}	1.2836×10^{2}	1.2943×10^{2}
-0.09487	-0.3437×10^{-1}	0	-0.3080×10^{-1}	-0.1773×10^{-1}	-1.0027	0.647×10^{-1}	0.1637	1.41498×10^{3}	1.41599×10^{3}
1.2999×10^{2}	1.4916×10^{3}	-0.2926×10^{2}	1.41417×10^{3}	-1.29992×10^{2}	1.4924×10^{3}	2.9028×10	1.41498×10^{3}	Final matrix (multiplier $\frac{5}{EA}$)	

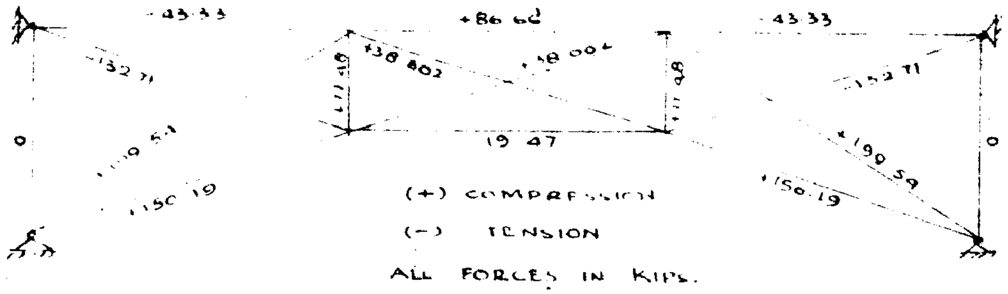

43.33
+86.62
43.33
0
0
19.47
+150.19
+199.59
(+) COMPRESSION
(−) TENSION
ALL FORCES IN KIPS.

FIGURE 6'14

Example 6.6

In this example we propose to find the secondary stresses in the members of the truss shown in Fig. 6.13, if the joints were rigid instead of being hinges. For this purpose, the equivalent fixed-end moments developed at the joints due to their known linear displacements (as computed in Example 6.5) will be considered as known - joint loads and the consequent joint rotations will be computed. In this stage the joints will be assumed to be rotation-free only, and the stiffness matrix assembled accordingly.

For a general member ij , shown in Fig. 6.5, if the joints were rigid, the fixed-end moments due to the linear joint displacements will be given by ,

$$(M_F)_{ij} = \frac{6EI}{L^2}\begin{bmatrix} -\mu & \lambda \end{bmatrix}\begin{bmatrix} (x_i' - x_j') \\ (y_i' - y_j') \end{bmatrix}$$

Using this equation, the fixed-end moments for all members will be computed as given below.

$A = 10$ sq. in.

$I = 120 \text{ in}^4$ for all members.

$$(M_F)_{13} = -\frac{1}{10 \times 9} \times 1491.6$$

$$= -16.57333 \quad \text{k-ft.}$$

$$(M_F)_{14} = -13.51195 \quad \text{k-ft.}$$

$$(M_F)_{36} = 0.41566 \quad \text{k-ft.}$$

$(M_F)_{24} = -13.32731$ k-ft.

$(M_F)_{45} = -0.41566$ k-ft.

$(M_F)_{35} = 0$ k-ft.

$(M_F)_{46} = 0$ k-ft.

$(M_F)_{23} = -10.10404$ k-ft.

$(M_F)_{34} = -15.91350$ k-ft.

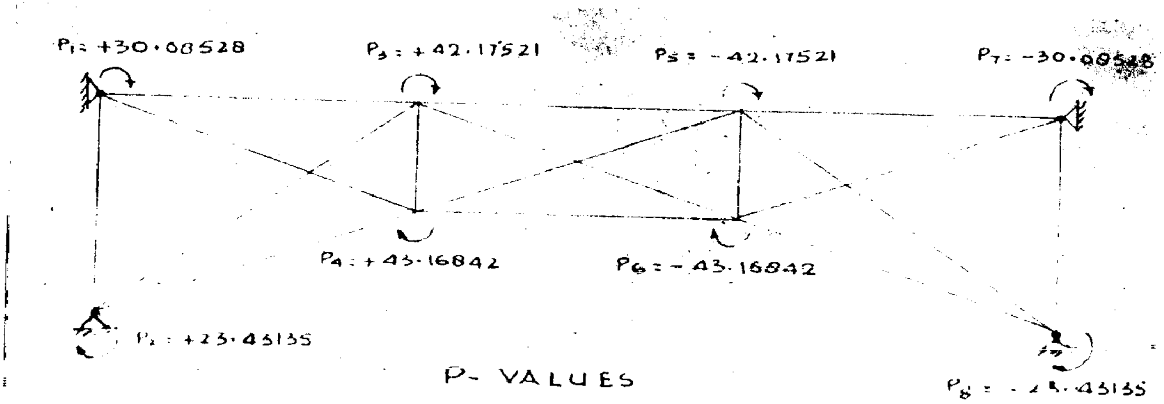


FIGURE 6·15

Stiffness matrices for individual members are given in Table 6.9.

Table 6.9.

Member	Stiffness Matrix (multiplier EI)	
12, 78	0.4	0.2
	0.2	0.4

Member	Stiffness Matrix (Multiplier EI)	
34,56	0.8	0.4
	0.4	0.8
13,35, 46, 57	0.26666	0.13333
	0.13333	0.26666
14, 36, 45, 67, 24,68	0.25298	0.12649
	0.12649	0.25298
23,58	0.22188	0.11094
	0.11094	0.22188

From these, the stiffness matrix for the entire frame is assembled, which is given in Table 6.10.

Table 6.10

Given Matrix (multiplier EI)

	X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8	=	Check column.
1	0.91964	0.20000	0.13333	0.12649	0	0	0	0	30.08528	31.46474
2	0.20000	0.87486	0.11094	0.12649	0	0	0	0	23.43135	24.74364
3	0.13333	0.11094	1.80818	0.40000	0.13333	0.12649	0	0	42.17521	44.88749
4	0.12649	0.12649	0.40000	1.82560	0.12649	0.13333	0	0	43.16842	45.90682
5	0	0	0.13333	0.12649	1.80818	0.40000	0.13333	0.11094	-42.17521	-39.46294
6	0	0	0.12649	0.13333	0.40000	1.82560	0.12649	0.12649	-43.16842	-40.43002
7	0	0	0	0	0.13333	0.12649	0.91964	0.30000	-30.08528	-28.70582
8	0	0	0	0	0.11094	0.12649	0.20000	0.87486	-23.43135	-22.11906

Table 6.11

Auxiliary Matrix

	X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8 =	check column	
1	0.91964	0.21747	0.14498	0.13754	0	0	0	0	32.71419	34.21419
2	0.20000	0.83136	0.98566×10^{-1}	0.11906	0	0	0	0	20.31432	21.53194
3	0.13333	0.81944×10^{-1}	1.78077	0.20884	0.74872×10^{-1}	0.71031×10^{-1}	0	0	20.29953	21.65427
4	0.12649	0.98982×10^{-1}	0.37190	1.71875	0.57393×10^{-1}	0.62204×10^{-1}	0	0	17.14633	18.26593
5	0	0	0.13333	0.98645×10^{-1}	1.79253	0.214444	0.74381×10^{-1}	0.06189	-25.98179	-24.63108
6	0	0	0.12649	0.10691	0.38439	1.72753	0.56669×10^{-1}	0.59449×10^{-1}	-21.75480	-20.63868
7	0	0	0	0	0.13333	0.97898×10^{-1}	0.90417	0.20563	-27.08714	-25.88150
8	0	0	0	0	0.11094	0.10270	0.18593	0.82365	-16.12141	-15.12141
	23.77203	16.12148	19.04518	19.44921	-19.04514	-19.44926	-23.77209	-16.12141	Final Matrix (Multiplier $\frac{1}{E2}$)	

The values of the unknown joint rotations are computed and at the bottom of Table 6.11. Using these values and the stiffness matrices of Table 6.9 are computed the end moments of the members of the truss, which when superimposed on the fixed-end moments give the final secondary bending moments. These are computed and shown in Fig. 6.16, given below

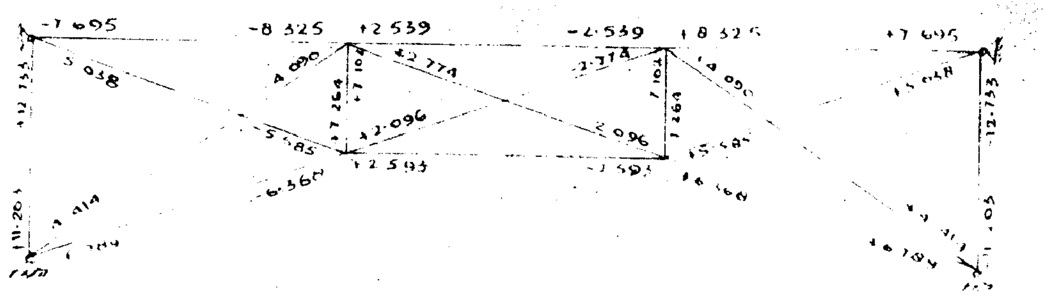


FIGURE 6·16

CHAPTER 7

AUTOMATIC DIGITAL COMPUTERS AND STRUCTURAL ANALYSIS

7.1 Introduction.

In the previous chapters we have been dealing with the techniques of setting up the relevant force-deflection equations for statically indeterminate structural problems in matrix form and also the method of solving this set of equations which is best suited to an electric desk calculator. A few typical examples have been solved on a similar machine available using the techniques suggested in the preceding work. Although these methods can be used to solve problems of any size, the problems in examples, solved for illustration purposes, have the number of unknowns (redundant forces or displacement components) limited to the order of 16 or so. This is because of the fact that the process of setting up and solving the simultaneous equations becomes quite cumbersome with the aid of the type of machine available, as the number of equations increases for more complex structures.

The invention and development of the high-speed electronic computer has now made it possible to formulate and solve many simultaneous equations in a reasonable time. An example has been cited in which a structure with 106 redundants was solved initially in about 12 hours, and each additional loading condition was solved in an additional $1\frac{1}{2}$ hours of computing time. This same problem could probably never have been solved using human labour with electric desk calculators.

The analysis of highly indeterminate structures using the electronic digital computer is really a two-part problem. The problem has first got to be set up and formulated according to the matrix techniques discussed in the previous chapters. This part of the problem lies within the scope of the structural engineer's work. The second part of the problem requires coding, or programing, and setting up the card system for the actual machine operation. This generally calls for the services of a specially trained operator, familiar with the particular computer being used. This person is often a mathematician in the field of applied mathematics, trained (usually under direction of the computer manufacturer) to set up the coding and card system for a specific type of problem. His programming can be applied to any similar future problem and is stored in a "library" and brought out and used when needed.

7.2 Functional Description of a Digital Computer.

There are two distinct types of electronic computers widely used in the solution of engineering problems, the analogue and the digital. The digital computer, as its name implies, deals directly with numbers, manipulating them much in the same way as is done with pencil and paper or a desk calculator. On the other hand, the analogue computer deals with physical quantities, such as voltages and currents, rather than with numbers, and the solution is obtained in terms of an electrical analogue of the mathematical or physical system under consideration. (A slide rule is the best known example of an analogue computer, numbers being represented by distances on the scales, and the calculations being performed by physical manipulation of these distances.) Analogue computers have many uses in solving specialised problems, but because of their greater versatility digital computers have been used in structural analysis and this discussion will be limited to that type of computers.

The procedure followed in the operation of an automatic computer may be explained most clearly by taking a very simple example. Suppose that an engineer wishes to determine the height of a mountain peak above a certain point as shown in Fig. 7.1. From a map he determines that the horizontal distance (d) from the point of observation to the peak is 20,000 ft. and with a transit he measures the vertical angle (θ) to be 15°. He knows that the height (h) will be given by the formula presented in the figure.

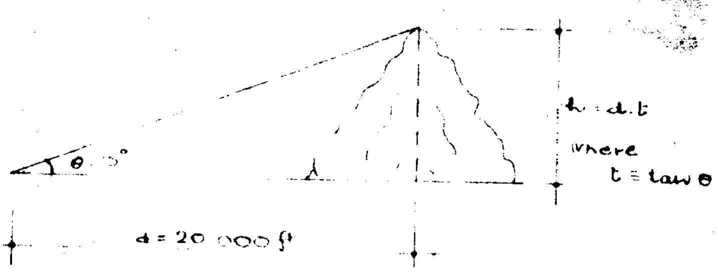


FIGURE 7·1

The important fact about an automatic digital computer is that it is not a "brain", as is sometimes remarked; it can only perform routine numerical operations as specified by a programme which has to be prepared for the particular problem under consideration. Now, in order that this simple calculation might be carried out by automatic computer, the engineer first would have to write out a calculation programme in a form which the machine could understand (usually by a code punched on cards or paper tape) and feed it into the machine. Then he would have to prepare the basic data sheet and input it similarly. Finally, pressing the start button would cause the machine to go through the operations strictly according to the instructions fed into it through the programme. The machine programme might be somewhat as shown in Fig. 7.2, which is clearly the exact equivalent of the programme which would have been followed by a computist if the calculations were to be performed by hand.

__Automatic Computer Programme__

(a) Read "d" into storage space 1.

(b) Read "θ" into storage space 2.

(c) In Table storage, locate value of "t" corresponding with number in storage space 2 and transfer to storage space 3.

(d) Transfer numbers from storage spaces 1 and 3 to arithmatic unit, multiply, and transfer product to storage space 4.

(e) Print out number contained in storage space 4

Input Data	Machine Storage
$d = 20{,}000$ feet	1. 20,000
$\theta = 15$ degrees.	2. 15
	3. 0.26795
	4. 5358.5

FIGURE 7.2

This example demonstrates the functions of the essential components of an automatic digital computer, namely:

(1) facilities for reading in instructions and data, (2) storage facilities to keep data and instructions available for use, (3) an arithmatic unit to carry out the actual numerical operations (usually limited to add, subtract, multiply and divide) and (4) a print-out device to present result in a usable form (usually typed on a sheet of paper). The additional operation involved in this problem of looking up the tangent of the angle, might be done by referring to a large scale storage unit in which a complete set of trigonometrical functions is stored for reference, or it might be handled by a special sub-programme which calculates the value of the tangent to the required accuracy by means of a series expansion.

The programme required for this example calculation is too simple to indicate the complexity of the programming problem in general. At the present time, the preparation of the programme is the principal restriction to widespread application of computing machines to structural analysis. To prepare and check out a programme for a really complex problem may take weeks or even months of concentrated effort.

For such cases, it is clear that there will be little advantage to be gained from a machine solution unless the same programme can be used time after time with different data. However, as libraries of such standard programmes are built up (where computer facilities are more commonly available to a structural engineer), more and more of structural analysis work can be done by machines with no addtional programming required.

7.3 Description of an Electronic Digital Computer.

A brief description of the basiccomponents,needed to perform the essential functions,as described in the preceding article,will be given here.

An electronic computer has two forms of storage, known as the magnetic and the electronic stores. In the former, information is retained on the surface of a rotating drum, and in the latter, it appears as a pattern of dots on the screen of a cathode ray tube. The electronic store comprises of a number of such tubes, each with a certain capacity of "lines". Dots on these lines represent digital numbers. All arithmetical operations take place in the electronic store. The machine performs these operations on lines under the control of a "routine" - or list of coded instructions- which is itself held in part of the electronic store. Instructions are obeyed at a rate of about 900 per second. Routines not in use, and other information not immediately required, are kept in the magnetic store. Information may be "read" from the magnetic store to the electronic , or "written" from the electronic to the magnetic. The magnetic store is divided into "tracks", the contents of a track corresponding to the information stored in particular tubes of the electronic store.

The reading and writing transfers mentioned above are in general part of the routines, and the instructions to perform one appears in a routine in the same way as instructions for the normal arithmetical operations. In a calculation using several routines, for instance, the last instruction in each routine will be one which results in the next routine being brought down from the magnetic stores and entered at the correct point. The detailed programme required for a given calculation is fed into the machine on a tape or cards, which must be previously punched by hand. The characters punched, are dealt with by a special input routine, which distributes the various sequences of information to their correct locations in the two stores.

In a typical calculation, the programme is first fed into the machine as described above. The tape or cards, punched with the data of the problem to be solved, are then placed in the reader unit and the machine uses the input programme to absorb the contents. The last few characters on the tape form an instruction to commence the programme of calculations already fed into the machine. The last unit is normally a printing device, which will print the results from given locations in the store.

7.4 Digital computer solution of structural problems.

Keeping in view the ease and speed in computational work afforded by an automatic digital computer, it is recommended that complex highly indeterminate problems of structural analysis be solved on such machines where available. The method most suitable for use on high speed digital computers is the stiffness matrix technique described in Chapter 6. This is because of the fact that this technique is most general in its application and also the formation of the stiffness matrix equation is a very straight forward process as compared with the complementary method - the flexibility matrix technique. As has already been pointed out while dealing with the flexibility methods - that they are convenient only for one particular type of problems - the pin connected redundant trusses, when the calculations are to be done with a small desk calculator by hand, since the number of equations is smaller in this case than if the stiffness method is employed for the same problem. But the same factor is almost immaterial when we are using giant machines - the points which matter being those which have been described as the Chief merits of the stiffness method. For this reason the adaptation of this method for use on automatic computers will be presented here.

(1) Input of Data.

(a) The first part of the data tape is concerned with forming the stiffness matrices of the individual members of the structure. For each member in turn, the quantities

E, A, I, L, α , are read into the machine, and the appropriate routine is then called down. This forms the matrices K'_{11} , K'_{12} , K'_{22} and K'_{21} , and stores them in the magnetic track assigned to that particular member. If various members of the structure possess the same stiffness matrices, it is only necessary to form and store them for one such member. This gives a useful saving both in time and storage space. For pin-jointed structures, the members are conveniently regarded as having zero flexural rigidity and merely involves punching zero for the value of I on the tape. The complete process of input of data, formation and storage of all four matrices, takes about 2½ seconds per member.

(b) When all the member-stiffness matrices have been formed, the second part of the tape is reached in which the stiffness matrix of the whole structure is assembled by considering each joint in turn. For each joint, a list is read into the store giving details of the members meeting at that joint, together with the number of the joint at the other end of each member. The appropriate routine is then called down; this forms the three equilibrium equations for that joint from the stiffness matrices already stored. Each equation of this matrix is then written up into a separate half-track of the magnetic store, together with the known elements of the external load vector on the right hand side. The time for this process depends to some extent on the number of members meeting thereat, but it is normally quoted as about 3.0 seconds per joint.

Allowances, at this stage, can easily be made for various support conditions according to modifications given, for the purpose, in Chapter 6.

(2) Solution of the Stiffness Matrix Equation.

In this stage of computations, the stiffness matrix assembled for the structure to be analysed is inverted. The method mostly adopted is the Doolittle technique of matrix inversion as has already been indicated in Chapter 4. The final solution is obtained by a matrix product of K^{-1} and the applied load vector P. The routine developed will solve a set of n simultaneous equations in about ($0.33\,n^2 + 0.8\,n + 7$) seconds (including printing the solution) but this time is very considerably reduced if many of the elements in the stiffness matrix are zero.

It is essential that a check is available on the accuracy of the solutions found. The method adopted is to substitute the computed values of the unknowns, and compare the values of the applied joint loads found with those originally taken. The time for checking is 4.0 seconds per equation of which practically the whole is spent in printing out the results.

(3) Calculation of Internal Forces-

When the joint displacements have been found and checked, the third part of the tape is read. This gives, for each member, the numbers of the joints at the two ends and the location in the store where the appropriate K'-matrices and the fixed-end load vectors may be found. The routine then selects the required displacements, stiffness matrices and the fixed-end loads and calculates and prints the end-loads N, S, M at each end of the member. The time for this routine is 12 seconds per member. The conditions of equilibrium for the individual members and the joints give a useful check on the accuracy of the whole programme.

7.4.1 Overall Time of Computation.

The times of operation for the separate routines have already been quoted. Using these, an estimate has been made of the time required to analyse a given structure. If the structure has m members and j joints, the time is approximately ($0.11\, j^2 + 22\, j + 16\, m + 28$) second; this includes all the routines described above. It may be mentioned that in the majority of cases the printing out of answers has been found to take about 60% of the whole machine time.

7.5 Matric Slope-deflection Method.

It is a general practice in analysing rigid frame problems common to a structural engineer, to ignore the effect of axial strains in the component members. This simplifies the analysis very much without any considerable loss in the accuracy of the results obtained. An interesting matrix method, suggested by C.K. Wang, is given here so that problems of complex rigid frames may be analysed quickly on digital computers. The chief merit of this method lies in the fact that it needs no special programming for a particular problem, but asks the digital computer only to produce products of matrices and inverse of a square matrix - the job which such machines can perform in a couple of minutes.

Derivation.

Let m be the total number of members, and n the total number of unknown joint deflections in a statically indeterminate structure. P is the load vector. Let M be a column matrix of 2m rows showing values of moments acting on ends of all members, exclusive of fixed-end moments. Define the statics matrix A of n rows and 2m columns as conditions expressing the elements of the load vector P in terms of those of M. Then, by definition

$$[P]_{n\times 1} = [A]_{n\times 2m}\,[M]_{2m\times 1} \qquad (7.1)$$

X is the column matrix of unknown joint deflections.

Let θ be a column matrix of 2m rows showing values of elastic r8tations at the ends of all members, as caused by the end moments M. Define the geometry matrix B of 2m rows and n columns as conditions expressing θ's in terms of X's. Then, by definition,

$$[\theta]_{2m \times 1} = [B]_{2m \times n} [X]_{n \times 1} \qquad (7.2)$$

Consider a member 12, the end 1 of which is connected to the ith joint in the structure. The end moment M_1, which acts clockwise on the member 12 and anticlockwise on the ith joint, will balance an externally applied positive moment $P_i = M_1 \; A_{i1} = M_1$ at joint i. Thus $A_{i1} = +1$. Geometrically, a clockwise rotation of X_i of joint i will cause a clockwise rotation of $\theta_1 = X_i \; B_{1i} = X_i$ at end 1 of member 12. Thus $B_{1i} = +1$ and $A_{i1} = B_{1i}$.

Next consider the effect of the end moment M_1 on sidesway. The free body for side sway is usually a joint or a horizontal member and the force acting on this free body and resulting from M_1 may be determined by considering member 12 or a group of members, as a free body. If, as an example, the positive direction of the externally applied force P_i on the ith free body for sideway is horizontal to the right, the positive direction of the balancing force $M_1 \; A_{i1}$ resulting from M_1 should be to the left on the ith free body for sidesway, but is again to the right on the free body for member 12 or the group of relevant members.

As the member 12 or the group of members is given a rigid body motion of X_i horizontal to the right at the point where the force $M_1 A_{i1}$ acts, the anticlockwise rotation at end 1 of member 12 is defined to be $\theta_1 = X_i B_{1i}$, since an anticlockwise rotation of the axis of a member will add to the elastic end rotation. During this rigid-body motion, the positive work done by the force $M_1 A_{i1}$ to the right in going through X_i to the right must be numerically equal to the negative work done by the clockwise moment M_1 in going through the anticlockwise rotation $\theta_1 = X_i B_{1i}$ or,

$$(M_1 A_{i1})(X_i) = (M_1)(X_i B_{1i})$$

which gives $A_{i1} = B_{1i}$

By virtue of both considerations discussed above it is seen that the Geometry matrix B and the statics matrix A are the transpose of each other. This is a very interesting result and can be applied as a check for the accuracy of both A and B.

$$A = B^* \qquad (7.3)$$

For a member 12,

$$\begin{bmatrix} M_1 \\ M_2 \end{bmatrix} = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} \qquad (7.4)$$

where

$$\left.\begin{aligned} K_{11} &= K_{22} = \frac{4EI}{L} \\ K_{12} &= K_{21} = \frac{2EI}{L} \end{aligned}\right\} \qquad (7,5)$$

for prismatic members.

S is a square matrix of 2m rows and 2m columns in which the end moments M are expressed in terms of the elastic end rotations θ. Entries in this matrix are those as shown in Eq. (7.4) for members with variable cross-section and in Eqs. (7.5) for prismatic members. Then, by definition,

$$[M]_{2m \times 1} = [S]_{2m \times 2m} \; [\theta]_{2m \times 1} \qquad (7.6)$$

Substituting (7.2) and (7.3) into (7.6) we get,

$$M = SA^{*} X \qquad (7.7)$$

Substituting (7.7) into (7.1)

$$P = ASA^{*} X \qquad (7.8)$$

It will be noticed that

$$ASA^{*} = K$$

where K is the stiffness matrix of the structure, same as defined in Chapter 6, but here obtained by a different process.

$$P = K X \qquad (7.9)$$

and $$X = K^{-1} P \qquad (7.10)$$

Procedure.

(1) The P-X diagram is drawn as usual.

(2) Clockwise arrows are drawn in the unloaded structure to act on ends of members and labelled $M_1-\theta_1$, $M_2-\theta_2$, etc.,

upto 2m. These end moments, as is evident, exclude fixed-end moments due to loads or support settlements.

(3) Free body diagrams of all joints and sidesway equilibrium conditions are drawn which are n in number.

(4) The statics matrix A is then constructed by observing the equilibrium of free bodies.

(5) From geometric considerations is constructed the geometry matrix B.

(6) Their accuracy is checked by B ≐ A*

(7) S matrix is now constructed.

(8) Compute all the equivalent fixed-end loads due to the known loading system and/or known settlement of supports and construct the vector P.

After having done this much of job by hand, the data tape is prepared with A, B, S and P matrices which are stored in the magnetic unit of the machine. A routine is read into the machine which asks it to perform the following matrix operations.

(9) Compute the matrix product SA* and store it for subsequent use.

(10) Compute the matrix product ASA* giving the n×n stiffness matrix of the structure.

(11) Invert the K-matrix.

(12) X-matrix is computed from the matrix product $K^{-1} P$.

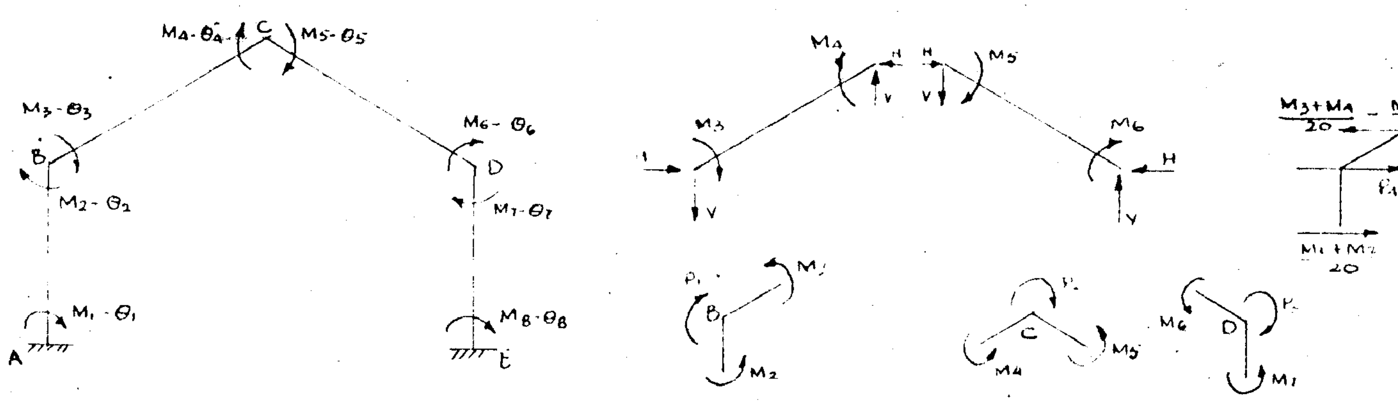
(13) M-matrix is obtained from $M = SA^{*} X$

(14) Combining the fixed-end moments with the M values obtained from step (13), the final end-moments are obtained.

The main labour involved lies in the computation of the matrix K^{-1}. Once this has been done by the computer, analysis of the structure for subsequent loading conditions requires a very little time and labour, since the construction of P-vector for each loading condition, and finding the matrix products $K^{-1} P$ and $SA^{*}X$ is a comparatively simple process.

Example

As an example, let us compute the stiffness matrix of the gable frame, of Ex.6.3, by the method suggested as above.



(a) M-Θ DIAGRAM (b) EQUILIBRIUM DIAGRAMS

FIGURE 7·3

From joint equilibrium diagrams shown in Fig. (7.3b), the statics matrix A of Eq. (7.1) may be found to be

	M_1	M_2	M_3	M_4	M_5	M_6	M_7	M_8
P_1		$+1$	$+1$					
P_2				$+1$	$+1$			
P_3						$+1$	$+1$	
P_4	$-\frac{1}{20}$	$-\frac{1}{20}$	$+\frac{1}{20}$	$+\frac{1}{20}$	$-\frac{1}{20}$	$-\frac{1}{20}$		
P_5			$-\frac{1}{20}$	$-\frac{1}{20}$	$+\frac{1}{20}$	$+\frac{1}{20}$	$-\frac{1}{20}$	$-\frac{1}{20}$

The geometry matrix B will be equal to the transpose of A. Matrix S of Eq. (7.6) will be given as below.

	θ_1	θ_2	θ_3	θ_4	θ_5	θ_6	θ_7	θ_8
M_1	0.2	0.1						
M_2	0.1	0.2						
M_3			0.35777	0.17888				
M_4			0.17888	0.35777				
M_5					0.35777	0.17888		
M_6					0.17888	0.35777		
M_7							0.2	0.1
M_8							0.1	0.2

The stiffness matrix K will be computed as a product of three matrices ASA^* , which in this example comes out as

	X_1	X_2	X_3	X_4	X_5
P_1	0.55777	0.17888	0	0.11832×10^{-1}	-0.26832×10^{-1}
P_2	0.17888	0.71554	0.17888	0	0
P_3	0	0.17888	0.55777	-0.26832×10^{-1}	0.11832×10^{-1}
P_4	0.11832×10^{-1}	0	-0.26832×10^{-1}	0.68665×10^{-2}	-0.53665×10^{-2}
P_5	-0.26832×10^{-1}	0	0.11832×10^{-1}	-0.53665×10^{-2}	0.68665×10^{-2}

It should be noted that this matrix is identical with that obtained in Example 6.3 by a direct method.

In this method the only work required to be done by hand is the formation of matrices A, S and the load vector P (for each loading condition), which evidently is quite simple a job. The rest of the procedure described above, which involves matrix operations like transposition, multiplication and inversion, can very conveniently be performed on an automatic digital computer without needing any special programming for a particular type of problem.

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