FLOW TO A WELL IN MULTIPLE AQUIFER SYSTEM

A THESIS

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requirements for the award of the degree
of
DOCTOR OF PHILOSOPHY
in
HYDROLOGY

By

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This work is dedicated
to the memory
of my parents

CANDIDATE'S DECLARATION

I hereby certify that the work which is being presented in the thesis entitled 'FLOW TO A WELL IN MULTIPLE AQUIFER SYSTEM' in fulfilment of the requirement for the award of the Degree of Doctor of Philosophy, submitted in the School of Hydrology of the University is an authentic record of my own work carried out during a period from June 1979 to May 1984 under the supervision of Dr. Satish Chandra and Dr. G.C. Mishra.

The matter embodied in this thesis has not been submitted by me for the award of any other degree.

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This is to certify that the above statement made by the candidate is correct to the best of our knowledge.

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ABSTRACT

Water wells generally tap more than one aquifer. The mathematical solutions developed so far for determining drawdown and individual aquifer's contribution during the unsteady state flow to a multiaquifer well are intractable. Therefore, only a few numerical results are available so far for a multiaquifer well system. In the present study using discrete kernel approach, complete analytic solutions have been developed for the following problems of unsteady flow to a multiaquifer well :

a) Unsteady flow to a well tapping two confined aquifers separated by an aquiclude ;
b) Unsteady flow to a well tapping more than two aquifers which are separated by aquicludes ;
c) Unsteady flow to a well tapping two aquifers separated by an aquitard.

For a well tapping two aquifers the studies have been extended when the top aquifer is unconfined and has delayed yield characteristics. The two aquifers may either be separated by an aquiclude or aquitard.

Discrete kernel coefficients for drawdown in an unconfined aquifer have been evaluated using Boulton's solution. An efficient method has been found to compute

the discrete kernel coefficients for any value of η, [$\eta = (\phi + \phi_y)/\phi$, where the storage coefficients ϕ and ϕ_y correspond to early and later part of time drawdown curve of an unconfined aquifer].

With the method of analysis developed in the thesis, it is easy to find the discharge contributions of each of the aquifers when a multiaquifer well is pumped. When the well is tapping a two aquifer system separated by an aquitard, the discharge contributions by each of the aquifers and the exchange of flow taking place between the two aquifers through the intervening aquitard have been evaluated. The variations of each aquifer's contribution to well discharge with time have been presented in non dimensional form for various values of aquifer parameters. The following conclusions have been drawn from the present study.

In a multiaquifer well when pumping is started, the aquifer with lowest hydraulic diffusivity contributes maximum to the discharge. However, as the pumping continues its contribution decreases with time. At nearly steady state condition i.e. after a prolonged constant pumping, contributions by each of the aquifers are proportional to their respective transmissivity values.

When the aquifers tapped have equal hydraulic diffusivity values, their contributions to well discharge are independent of time and are proportional to their respective transmissivity values. It is true for both the cases of the aquifers separated by aquiclude or aquitard. In such a case when the two aquifers are separated by aquitard no exchange of flow takes place through the aquitard irrespective of the magnitude of the leakage factor and the drawdown at any section in both the aquifers are same.

When the two aquifers are separated by an aquitard and the well taps both the aquifers, the leakage factor may be defined as $L = \sqrt{\bar{T}\, C}$ where $\bar{T}$ is the mean value of the transmissivities. The mean transmissivities may either be harmonic, geometric or arithmetic mean value of the transmissivities of the two aquifers tapped.

In case of two aquifers separated by aquitard, the near steady state conditions are attained comparatively at shorter time for lower values of leakage factor.

NOTATIONS

The following notations have been used in this thesis (except in chapter 2 which deals with review of literature, where original notations have been used)

Notation	Description	Dimension
B_1	Thickness of the aquitard	l
C	Hydraulic resistance of aquitard (B_1/K_1)	t^{-1}
c, [illegible]	Time step	t
K_1	Hydraulic conductivity of the aquitard	lt^{-1}
L	Leakage factor $(\sqrt{T\ C})$	l
M	Total number of aquifers	
m, n	Time steps	t
Q_p	Constant well discharge	l^3t^{-1}
$Q_1(n)$, $Q_2(n)$, $Q_M(n)$	Discharge contributions by individual aquifers at nth time step	l^3t^{-1}
$Q_r(i,j,n)$	Recharge taking place through the area of influence of node (i,j) at nth time step	l^3t^{-1}
$Q_R(n)$	Total recharge taking place from one aquifer to the other	l^3t^{-1}

Notation	Description	Dimension
r	Distance of observation well from the pumped well	l
r_w	Radius of well	l
s	Drawdown at distance r from the pumping well at time t after the onset of pumping	l
T	Transmissivity	l^2t^{-1}
$\bar{T}$	Harmonic mean transmissivity	l^2t^{-1}
t, τ	time	t
Δx	Grid size	l
x, y	Cartesian coordinates	l, l
$J_o(\ )$	Bessel function of first kind and zero order	
$J_1(\ )$	Bessel function of first kind and first order	
α	Reciprocal of Boulton's delay index	t^{-1}
β	Hydraulic diffusivity (T/ϕ)	l^2t^{-1}
ϕ	Volume of water intantaneously released from aquifer storage per unit drawdown per unit horizontal area (storage coefficient)	
ϕ_y	Total volume of delayed yield from storage per unit drawdown per unit horizontal area which is commonly referred as specific yield	
$\partial(n)$	Discrete kernel coefficient	$1/(l^3/t)$

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CONTENTS

CHAPTER 1

INTRODUCTION

Ground water constitutes more than one fifth of the world's fresh water resources. It plays an important role in the development of a region. The occurrence, movement and development of ground water has primarily been studied because of its importance as a resource. Although the origin of ground water had been understood centuries ago, the understanding of the behaviour of water bearing formations (aquifers) when pumped is relatively of recent times. Dupuit (1863) is the first scientist to analyse steady state flow of ground water to a well. Flow towards wells and galleries was analysed by A. Thiem (1870). G. Thiem (1906) developed a field method for determining permeability of aquifer using a pumping well and the resultant drawdowns in observation wells. De Glee (1930) studied the steady state flow towards a well in leaky confined aquifer replenished by an overlying formation.

A need was felt by ground water hydrologists for solving the hydraulics of well under unsteady state conditions. A bench mark study was conducted by Theis (1935) who gave the solution for unsteady flow to a well in confined aquifer. Hantush and Jacob (1955) incorporating De Glee's concept of recharge to the pumped aquifer from another aquifer through intervening semipermeable layer (aquitard), analysed the unsteady flow to a well in a leaky confined aquifer. The other important study in the field of well hydraulics is that of Boulton (1963) who gave a mathematical solution for evaluation of drawdown due to pumping of an unconfined aquifer having delayed yield characteristics.

Under field conditions the aquifer geometry rarely conforms to the concept of one aquifer system. In a borehole it is common to identify number of aquifers. Often the aquifer pumped is part of a complex aquifer system. A multiple aquifer system generally consists of a series of aquifers separated from each other by confining layers. The confining layers may have negligible permeability (aquiclude) or low permeability (aquitard). When the aquifers are separated by aquicludes interaction between the aquifers is only through the well screens. However, when the aquifers are separated by aquitards, interaction between the aquifers takes place through the aquitard besides through the well screens.

Generally to get dependable yield,wells are constructed tapping more than one aquifer. In ground water exploration it is possible to evaluate hydrogeological parameters of individual aquifers. With the known values of hydrogeological parameters, Sokol (1963) derived a simple steady state equation relating water level fluctuation in a non pumping multiaquifer well to head change in any one aquifer penetrated by the well. Papadopulos (1966), Khader and Veerankutty (1975) have studied unsteady flow to a well tapping two aquifers, the aquifers separated by an aquiclude.

Ground water hydrology is a quantitative science and mathematics is its important dialect. Mathematical tools have enabled analysis of many complex ground water flow problems. Discrete kernel approach is comparatively new within its ambit. Using discrete kernel approach intricate stream-aquifer-well interaction problems have been analysed extensively by Morel-Seytoux (1975a). The discrete kernels are the properties of a linear system. The discrete kernels for drawdown are the response of an aquifer due to unit pulse excitation. Using the discrete kernel approach unsteady flow to a well in multiple aquifer system has been studied and the results are presented in the thesis. The scheme of presentation of the thesis is as follows :

Chapter 2 deals with the review of literature pertaining to flow to a multiaquifer well and application of discrete kernel approach to ground water flow problems. In chapter 3 an efficient method has been described to generate discrete kernels for drawdown in an unconfined aquifer having delayed yield. Unsteady flow to a well tapping two confined aquifers separated by an aquiclude has been analysed for a continuous, constant pumping rate and the analysis is presented in chapter 4. Chapter 5 deals with solution of unsteady flow to a well tapping multiple (more than 2) aquifers separated by aquicludes. In chapter 6 the case of pumping of a well tapping two aquifers separated by an aquitard has been studied for unsteady condition. The general conclusions are brought out in chapter 7.

CHAPTER 2

REVIEW OF LITERATURE

INTRODUCTION

The study of aquifers when pumped is an important aspect of ground water hydrology. Many research workers have contributed to this study. In this chapter literature review has been done pertaining to flow to a well with emphasis on multiple aquifer well interaction and application of discrete kernel to ground water flow problems.

WELL TAPPING A SINGLE AQUIFER

Studies prior to the work of Theis (1935) were dealing with the steady state flow towards a well. A need was felt for analysis of unsteady flow towards a well and a solution was given by Theis (1935) which is based on the solution given by Carslaw and Jaeger (1959) for an analogous problem of conduction of heat in solids. The solution is given as

$$s = \frac{Q}{4\pi T} \int_{u}^{\infty} \frac{e^{-x}}{x} dx \qquad \ldots(2.1)$$

Where

s = drawdown at a distance r from the pumping well at time t after the onset of pumping,

$u = \frac{\phi r^2}{4Tt}$,

r = distance of observation well from the pumped well,

ϕ = storage coefficient,

T = transmissivity,

t = time and

Q = pumping rate.

Eq.(2.1) is known as non equilibrium formula for unsteady flow to a well.

The assumptions made in the analysis are :

a) the aquifer is infinite, homogeneous, isotropic and of uniform thickness over the area of influence of pumping ,

b) pumping is continued at a constant rate ,

c) prior to pumping the water level is nearly horizontal over the area influenced by pumping ,

d) the well fully penetrates the aquifer and receives water from the entire thickness of the aquifer by horizontal flow ,

e) the well is of infinitesimal diameter , and

f) the aquifer is confined and release of water from storage is instantaneous.

The non equilibrium formula has extensively been applied to the analysis of test pumping data (Todd 1959 , Walton 1970, Kruseman and De Ridder 1970). However when analysing the time drawdown data of unconfined aquifers composed of stratified sediments, it is observed that the time drawdown curve deviates from Theis type curve. A plausible explanation for the behaviour of unconfined aquifers has been given by Boulton (1954). Boulton introduced the concept of delayed yield which envisages the effect of gravity drainage on time drawdown curve of an unconfined aquifer. The gravity drainage of water through stratified sediments is not instantaneous (as presumed in Theis solution).

The differential equation which governs an axially symmetric radial unsteady ground water flow in unconfined aquifer with delayed yield is (Boulton, 1954)

$$T\left(\frac{\partial^2 s}{\partial r^2} + \frac{1}{r}\frac{\partial s}{\partial r}\right) = \phi \frac{\partial s}{\partial t} + \alpha \phi_y \int_0^\infty \frac{\partial s}{\partial c} e^{-\alpha(t-c)} dc \qquad ...(2.2)$$

The solution of Eq.(2.2) for constant pumping rate given by Boulton (1963) is

$$s = \frac{Q}{4\pi T}\int_0^\infty \frac{2}{x}\left[1 - e^{-\mu_1}\left(\text{Cosh } \mu_2 + \frac{\alpha t \eta(1-x^2)}{2\mu_2}\text{Sinh } \mu_2\right)\right] J_0\left(\frac{rx}{\gamma D}\right) dx \qquad ...(2.3)$$

Where

$$\eta = 1 + \frac{\phi_y}{\phi} ,$$

ϕ_y = total volume of delayed yield from storage per unit drawdown per unit horizontal area which is commonly referred as specific yield,

ϕ = volume of water instantaneously released from storage per unit drawdown per unit horizontal area which is the effective early time storage coefficient,

$\gamma = \sqrt{(\frac{\eta-1}{\eta})}$,

$\frac{1}{\alpha}$ = Boulton's delay index,

$D = \sqrt{T/(\alpha\phi_y)}$,

$\mu_1 = \frac{\alpha t \eta (1+x^2)}{2}$,

$\mu_2 = \frac{\alpha t \sqrt{\eta^2 (1+x^2)^2 - 4\eta x^2}}{2}$,

$J_o(\ )$ = Bessel function of first kind, zero order,
T, s, r, t have already been defined.

The Boulton's solution of the Eq. (2.2) is based on the assumptions outlined in Theis solution. Besides those, the drawdown in the aquifer is small in comparison to the saturated thickness of the aquifer.

WELL TAPPING A SINGLE AQUIFER IN A MULTIPLE AQUIFER SYSTEM

The solutions given by Theis (1935) and Boulton (1963) deal with pumping of a single aquifer. De Glee (1930) was the first scientist to visualise the contribution of adjacent aquifer to the discharge well through leakage. He studied the steady state flow towards a well in leaky confined aquifer replenished by an overlying formation. The analysis is based on the assumptions that a) flow is vertical in the aquitard and horizontal in the aquifer, b) there is no drawdown in the bed source, and c) the leakage through the confining aquitard takes place in proportion to the drawdown in piezometric level; besides the assumptions made in Theis (1935) analysis.

De Glee (1930) obtained the following solution for the steady state condition :

$$S_m = \frac{Q}{4\pi T} K_o \left(\frac{r}{L}\right) \qquad \ldots(2.4)$$

Where

S_m = steady state (maximum) drawdown in the piezometer at distance r from the pumped well,

L = $\sqrt{TC}$ = leakage factor,

C = B_1/K_1 = hydraulic resistance of aquitard,

B_1 = thickness of the aquitard,

K_1 = hydraulic conductivity of the aquitard, and

$K_o(\ )$= modified Bessel function of second king and zero order.

Other notations have been defined earlier.

Jacob (1946) used the same assumptions as that of De Glee (1930) to develop a partial differential equation for unsteady flow in a leaky aquifer.

The equation for axially symmetric and radial flow in polar coordinate notations is given as follows :

$$\frac{\partial^2 s}{\partial r^2} + \frac{1}{r}\frac{\partial s}{\partial r} - \frac{s}{L^2} = \frac{\phi}{T}\frac{\partial s}{\partial t} \qquad \ldots(2.5)$$

The equation is derived assuming no release from the aquitard storage.

Solution to the Eq.(2.5) as given by Hantush and Jacob(1955) is

$$s = \frac{Q}{4\pi T}\int_u^\infty \frac{e^{-\left(y + \frac{r^2}{4L^2 y}\right)}}{y}\,dy \qquad \ldots(2.6)$$

Eq.(2.6) is generally written as

$$s = \frac{Q}{4\pi T} W(u, \frac{r}{L})$$

Where, $W(u, \frac{r}{L})$ stands for the integral in Eq.(2.6) and is known as well function for leaky confined aquifer with fully penetrating well without water released from aquitard storage and no bed source drawdown.

Tabulated values of $W(u, \frac{r}{L})$ for the practical ranges of u and $\frac{r}{L}$ have been given by Hantush (1956). Based on these values, type curves have been prepared by Walton (1960) which are widely used in the analysis of the pumping test data of leaky aquifers.

Subsequently Hantush (1960) presented a modified approach to obtain the solution of unsteady flow to a well in a leaky confined aquifer taking effect of aquitard storage into consideration and supposing no drawdown in the bed source. However to suppose no drawdown in the bed source may not be valid as there may be signigicant decline in the piezometric surface of the unpumped aquifer in case of prolonged pumping. Taking bed source drawdown into consideration, steady flow to a well has been analysed by Spiegel (1962) and Polubarinova-Kochina (1962). For the same case, the unsteady state flow has been investigated by Hantush (1967). Neuman and Witherspoon (1969a, 1969b) have analysed flow to a well tapping an aquifer in a two aquifer-aquitard system. The analysis takes into consideration the bed source drawdown and the water released from aquitard storage.

A comprehensive review of flow to a well in leaky artesian aquifer has been made by Walton (1979).

Numerical methods have also been used for the analysis of flow to a well in a multiple aquifer system. The well taps only one of the aquifers. Using backward difference implicit method, Mucha and Kaergaard (1982) have proposed a numerical model for aquifer test in multi layered aquifer-aquitard system.

WELL TAPPING MORE THAN ONE AQUIFER

Sokol (1963) has derived a simple steady state equation relating water level fluctuation in a nonpumping well tapping multiple aquifer to a change of head in any one of the aquifers penetrated by the well. Sokol found that the ratios of the water level fluctuation in the well to the head change is equal to the ratio of the transmisivity of the aquifer in which the head change occurs to the sum of the transmissivities of all the aquifers penetrated by the well.

Papadopulos (1966) has obtained solution for the nonsteady flow to multiaquifer well open to two confined aquifers of infinite areal extent. Papadopulos found that exact solution to the problem are intractable for numerical calculations. Asymptotic solutions amenable to computation and which yield results accurate enough for practical application have been developed by him. The solutions for $t \leq t_0$ are

$$H_1 - h_1 = \frac{H_1 - H_2}{1 + \partial} A(\tau/\epsilon^2, \rho/\epsilon) \quad \text{...(2.7)}$$

$$H_2 - h_2 = \frac{-\partial(H_1 - H_2)}{1 + \partial} A(\tau/\epsilon^2, \alpha\rho/\epsilon) \quad \text{...(2.8)}$$

$$Q_1(t) = 2\pi T_1(H_1 - H_2)\, G(\tau/\epsilon^2)/(1 + \partial) \quad \text{...(2.9)}$$

$$Q_2(t) = -Q_1(t) \quad \text{...(2.10)}$$

These solutions are for the boundary value problem where the aquifers remain unpumped for a period t_o during which flow occurs from one aquifer to the other through the well screen owing to the difference in initial heads in upper and lower aquifers.

For $t > t_o$

$$H - h_1 = \frac{H_1 - H_2}{1+\delta} A(\tau/\epsilon^2, \rho/\epsilon) + \frac{Q}{4\pi T_1(1+\delta)} [W(\rho^2/4\tau^*) - \frac{1}{(1+\delta)} (\ln \alpha^2) A(\tau^*/\epsilon^2, \rho/\epsilon)] \qquad ...(2.11)$$

$$H_2 - h_2 = -\frac{\delta(H_1 - H_2)}{1+\delta} A(\tau/\epsilon^2, \alpha\rho/\epsilon) + \frac{Q}{4\pi T_2(1+\delta)} [W(\alpha^2 \rho^2/4\tau^*) + \frac{\delta}{(1+\delta)} (\ln \alpha^2) A(\tau^*/\epsilon^2, \alpha\rho/\epsilon)] \qquad ...(2.12)$$

$$Q_1(t) = \frac{2\pi T_1(H_1 - H_2) G(\tau/\epsilon^2)}{(1+\delta)} + \frac{Q\delta}{2(1+\delta)} [2e^{-1/4\tau^*} - \frac{1}{(1+\delta)} (\ln\alpha^2) G(\tau^*/\epsilon^2)] \qquad ...(2.13)$$

$$Q_2(t) = Q - Q_1(t) \qquad ...(2.14)$$

Where

$$G(x) = \frac{4x}{\pi} \int_0^\infty e^{-xu^2} [\frac{\pi}{2} + \tan^{-1} \frac{y_o(u)}{J_o(u)}] u \, du$$

h_1, h_2 = heads at any distance r and time t

T_1, T_2 = transmissivities of upper and lower aquifers,

ϕ_1, ϕ_2 = storage coefficients of upper and lower aquifers,

H_1, H_2 = initial heads in upper and lower aquifers,

$Q_1(t), Q_2(t)$ = discharges from upper and lower aquifers at time (t),

J_o, Y_o = zero order Bessel functions of first and second. kind respectively,

t = time since the well is completed,

t_o = time at which pumping started,

r = radial distance to any point from the axis of the well ,

r_w = radius of the well ,

ν_1, ν_2 = hydraulic diffusivities of upper and lower, aquifers ($\nu = T/\phi$),

α = $\sqrt{\nu_1/\nu_2}$,

∂ = T_1/T_2 ,

ϵ = $\alpha^{[\partial/(1+\partial)]}$,

ρ = r/r_w ,

τ = ${}_1 t/r_w^2$,

τ^* = ${}_1(t-t_o)/r_w^2$,

Q = constant discharge from the well ,

$$A(x,y) = 1 - \frac{2}{\pi}\int_0^\infty \frac{e^{-xu^2}}{u} \cdot \frac{J_o(u)\,Y_o(u.y) - Y_o(u)J_o(u.y)}{J_o^2(u) + Y_o^2(u)}\,du,$$

$$W(x) = \int_x^\infty \frac{e^{-u}}{u}\,du,$$ an exponential integral.

Problem tackled by Papadopulos has also been solved by Khader and Veerankutty (1975). The problem has been solved by using Schapery's (1962) approximate method of inversion of Laplace transform. The expressions derived by Khader and Veerankutty for drawdown around a well penetrating two confined aquifers (separated by aquiclude) with identical initial heads for constant pumping rate and the discharge contributions by individual aquifers are :

$$s_1 = \frac{Q \frac{T_1}{T_2} K_o(\sqrt{2u}) . B(T, \rho\alpha, F)}{2\pi T_1 [B(T,\alpha, F_w) + (T_1/T_2) K_o(\sqrt{2u})]} \quad ...(2.15)$$

$$s_2 = \frac{Q . B(T,\alpha,F_w) . K_o(\rho\sqrt{2u})}{2\pi T_2 [B(T ,\alpha,F_w) + (T_1/T_2) K_o(\sqrt{2u})]} \quad ...(2.16)$$

$$Q_1(t) = \frac{T_1/T_2 \; Q \; K_o(\sqrt{2u})}{B(T,\alpha,F_w) + (T_1/T_2) K_o(\sqrt{2u})} \quad ...(2.17)$$

$$Q_2(t) = \frac{Q \; B(T,\alpha,F_w)}{B(T,\alpha,F_w) + (T_1/T_2) K_o (\sqrt{2u})} \quad ...(2.18)$$

Where

s_1 = drawdown in first aquifer at any distance r from the centre of the well, time t and height z measured from the bottom of the first aquifer,

s_2 = drawdown in the second aquifer at any distance r from the c entre of the well at time t,

H_1, H_2 = initial heads in first and second aquifers respectively, measured from the bottom of the first aquifer,

K_1, K_2 = hydraulic conductivities of the first and second aquifers respectively,

ϕ_y = specific yield of unconfined aquifer,

T_1 = $K_1 H_1$ transmissivity of unconfined aquifer,

T_2 = $K_2 b_2$ transmissivity of second aquifer,

b_2 = thickness of the second aquifer,

ϕ_2 = storage coefficient of the second aquifer,

ν = T_2/ϕ_2 hydraulic diffusivity of second aquifer,

r_w = effective radius of the well,

h_w = water level in the well measured from bottom of unconfined aquifer at any time t,

$Q_1(t)$, $Q_2(t)$ = discharge contributions of first and second aquifers respectively at any time t,

Q = constant rate of pumping,

$$B(T,\alpha,F) = \int_0^\infty \frac{J_o(\alpha\theta)}{\theta}\left[\frac{\text{Cosh } \theta + T\theta \text{ Sinh } \theta - \text{Cosh } (F\theta)}{\text{Cosh } \theta + T\theta \text{ Sinh } \theta}\right] d\theta,$$

$$u = \frac{r_w^2}{4\nu t}, \quad \alpha = \frac{r_w}{H_1},$$

$$T = \frac{2K_1 t}{H_1 \phi_y}, \quad \rho = \frac{r}{r_w},$$

$$F = \frac{h}{H_1}, \quad F_w = \frac{h_w}{H_1},$$

$K_o(\ )$ = Modified Bessel function of second kind and zero order.

Numerical results have been given by the authors for some values of aquifer parameters.

APPLICATION OF DISCRETE KERNEL APPROACH

Instead of predicting the hydrologic behaviour of a system in response to a particular set of numerical values of excitation, Maddock (1972) has suggested for finding out a functional relationship between the excitation and response. Using linear system theory and Green's function, Maddock has obtained the expression for drawdown at a point due to pumping of number of wells. The expression given by Maddock is :

$$s(k,n) = \sum_{j=1}^{M} \sum_{i=1}^{n} \partial[k,j,(n-i+1)]q(j,i) \qquad \ldots(2.19)$$

Where

s(k,n) is the drawdown at kth well at nth time period; M is the total number of wells; q(j,i) is the discharge from the jth well in ith time period; the coefficients ∂(k,j,i) are known as algebraic technological functions.

The above expression has been derived with the assumption that the aquifer had no previous development i.e., s(x,y,0) = 0.

The same approach has also been developed by Morel-Seytoux (1975a) for ground water problems with and without stream interaction. He designated the coefficients as discrete kernels.

For a homogeneous isotropic aquifer of infinite areal extent the pumping kernel is given by (Carslaw and Jaeger 1959)

$$k(r,t) = \frac{e^{-\frac{r^2}{4\beta t}}}{4\pi T t}, \qquad \ldots(2.20)$$

Where

k(r,t) is the drawdown at time t at a distance r from the pumping well when unit impulse quantity is withdrawn at time t = 0. β = T/∅ (T and ∅ have already been defined in this chapter).

Using the above relation the discrete kernel coefficients can be written as (Morel-Seytoux 1975a) :

$$\partial(n) = \frac{1}{4\pi T} \int_0^1 \frac{e^{\frac{-r^2}{4\beta(n-c)}}}{(n-c)} dc$$

$$= \frac{1}{4\pi T} [E_1 (\frac{r^2}{4\beta n}) - E_1 (\frac{r^2}{4\beta(n-1)})] \qquad \ldots(2.21)$$

Where

$$E_1(x) = \int_x^\infty \frac{e^{-y}}{y} dy ;$$

r = distance of the observation well (response point) from the pumping well (excitation point).

Morel-Seytoux (1975a) has highlighted the advantages of the discrete kernel approach, some of which are as follows :

With the use of the discrete kernel approach it is possible to solve problems of optimal management through the efficient techniques of mathematical programming rather than through the use of successive trial and error required in simulation. The Mathematical Programming problem is considerably reduced in size compared to a formulation that incorporates the finite difference equations of the hydrologic model.

Morel-Seytoux (1975a, 1975b), Morel-Seytoux and Daly (1975) have developed efficient and accurate stream-aquifer interaction models by using discrete pumping kernel and discrete reach kernel. The discrete reach kernel for drawdown at the centre of a reach due to unit withdrawl has been derived by Morel-Seytoux et al (1975) and is given by

$$\partial_{rr}(m) = \int_0^1 \frac{1}{\phi ab}\, \mathrm{erf}\left[\frac{a}{4\sqrt{\beta(m-c)}}\right] \mathrm{erf}\left[\frac{b}{4\sqrt{\beta(m-c)}}\right] dc \qquad \ldots(2.22)$$

Where a and b are respectively length and width of the reach and erf( ) is the error function.

Morel-Seytoux and Daly (1975) have given a complete description of discrete kernel generator including truncation error propogation, accuracy and run cost while analysing stream aquifer interaction problem. The aquifer response i.e. return flow to a given reach for a given week has been expressed as an explicit function of the pumping rate.

Using the discrete kernel coefficients Heidari (1982) has studied ground water management model to find the optimal pumpage policies, subject to physical and institutional constraints.

Basu (1980) has applied discrete kernel approach to study the unsteady flow to a leaky confined aquifer with bed source drawdown. The study concluded that the aquitard resistance governs the total quantity of recharge from source bed and its temporal and spatial distribution. The assumption that the drawdown in the bed source does not change with respect to time is valid only for high aquitard resistance.

Patel and Mishra (1983) have analysed unsteady flow to a large diameter well using discrete kernel approach. They have found the approach to be simple, less time consuming in comparison with that of the solution given by Papadopulos and Cooper (1967).

CONCLUSIONS

From the literature review it may be concluded that only few results have been given in case of a well tapping two aquifers. No solution is available for unsteady flow to a well tapping more than two aquifers. Also solution is not available when the well taps two aquifers which are separated by an aquitard. The integral transform method applied so far to arrive at the solution of unsteady flow to multiaquifer well are intractable. Thus, there is need to develop simple but accurate mathematical procedure to analyse unsteady flow to multiaquifer well.

CHAPTER 3

DISCRETE KERNEL FOR AN UNCONFINED AQUIFER WITH DELAYED YIELD

INTRODUCTION

Generation of discrete kernels for drawdown in a confined aquifer has been described by Morel-Seytoux (1975b). The discrete kernels can be regarded as the properties of a linear system. The discrete kernels for drawdown are response (i.e. drawdowns in piezometric surface at a point in the aquifer) of an aquifer initially at rest condition due to an unit pulse excitation (withdrawl of unit quantity of water in the 1st unit time period and no pumping afterwords). Using Boulton's solution for unsteady flow to a well in an unconfined aquifer having delayed yield characteristics, generation of discrete kernels for drawdown in an unconfined aquifer with delayed yield has been described in this chapter.

GENERATION OF DISCRETE KERNEL

The equation governing an axially-symmetric radial flow in an unconfined aquifer having delayed yield characteristics has been described in Chapter 2 [Eq.(2.2)]. The solution given at Eq.(2.3) has been obtained by

Boulton with the assumption (besides other usual assumptions) that the drawdown is very small in comparison to the thickness of the aquifer. Eq.(2.2) being linear, method of superposition and proportionality are applicable. If Q = 1.0 and pumping continues indefinitely , Eq.(2.3) gives the response of a linear system due to unit step excitation. Designating K(m) as the unit step kernel (response due to an unit step excitation), which is the drawdown at the end of time step m due to continuous pumping at unit quantity per unit time period, the discrete kernel coefficients $\delta(m)$ can be expressed as

$$\delta(m) = K(m) - K(m-1) \qquad \ldots(3.1)$$

Substituting m for t in Eq.(2.3) and replacing Cosh μ_2 and Sinh μ_2 by $(e^{\mu_2} + e^{-\mu_2})/2$ and $(e^{\mu_2} - e^{-\mu_2})/2$ respectively and rearranging, the unit step kernel is written as

$$K(m) = \frac{1}{4\pi T} \int_0^\infty \frac{2}{x} [1 - \frac{1}{2} (e^{-(\mu_1 - \mu_2)} (1 + \frac{\alpha m \eta (1-x^2)}{2\mu_2}) + e^{-(\mu_1 + \mu_2)} (1 - \frac{\alpha m \eta (1-x^2)}{2\mu_2}))] J_o(\frac{r}{\gamma}\frac{x}{D}) \, dx \qquad \ldots(3.2)$$

The integral appearing in Eq.(3.2) is an improper integral as one of the limits of integration is infinite. For finite values of η the numerical integration of the improper integral takes considerable computer time to obtain results of reasonable accuracy. The following is an efficient method for evaluation of K(m) for any value of η. For given values of aquifer parameters it is found that the limit of the term

$$[1 - \frac{1}{2} (e^{-(\mu_1-\mu_2)} (1 + \frac{\alpha m\eta(1-x^2)}{2\mu_2}) + e^{-(\mu_1+\mu_2)}(1- \frac{\alpha m\eta(1-x^2)}{2\mu_2}))]$$

in Eq.(3.2) tends to 1 as the dummy variable x increases.

Let beyond $x=x_1$ this term has a value equal to $1-\epsilon$, where ϵ is as small as .000001.

Eq.(3.2) can be written as

$$K(m) = \frac{1}{4\pi T} \int_0^{x_1} \frac{2}{x}[1 - \frac{1}{2}(e^{-(\mu_1-\mu_2)}(1 + \frac{\alpha m\eta(1 - x^2)}{2\mu_2})$$

$$+ e^{-(\mu_1+\mu_2)}(1 - \frac{\alpha m\eta(1-x^2)}{2\mu_2}))] J_o(\frac{r\ x}{\gamma D})\ dx$$

$$+ \frac{1}{4\pi T} \int_{x_1}^{\infty} (1 - \epsilon) \frac{2}{x}\ J_o(\frac{rx}{\gamma D})\ dx \qquad ...(3.3)$$

$$= I_1 + I_2 \qquad ...(3.4)$$

For evaluation of the proper integral I_1, numerical integration is carried out assuming dx = .001. This value of dx has been adopted after studying the effect of dx on the accuracy of the results.

The integration

$I_2 = \int_{x_1}^{\infty} \frac{2}{x}\ J_o(\frac{rx}{\gamma D})\ dx$ is carried out as follows :

Let

$$y = \frac{r}{\gamma D} x$$

Then

$$I_2 = \int_{\frac{rx_1}{\gamma D}}^{\infty} \frac{2}{y} J_o(y)\, dy \qquad \ldots(3.5)$$

Depending upon the numerical values of $\frac{r}{\gamma D} x_1$ the following approximations can be used for evaluation of the improper integral I_2.

For $\frac{r}{\gamma D} x_1 < 2$ (Abramowitz and Stegun 1970, pp.481)

$$\int_{\frac{rx_1}{\gamma D}}^{\infty} \frac{J_o(y)}{y}\, dy = -\,0.5772156 - \log_e\left(\frac{rx_1}{2\gamma D}\right) - \sum_{p=1}^{\infty} \frac{(-1)^p \left(\frac{rx_1}{2\gamma D}\right)^{2p}}{2p(p!)^2} \qquad \ldots(3.6)$$

The series appearing in Eq.(3.6) is a rapidly converging one.

For $5 \leq \frac{rx_1}{\gamma D} \leq \infty$ (Abramowitz and Stegun 1970, pp.482)

$$\int_{\frac{rx_1}{\gamma D}}^{\infty} \frac{J_o(y)}{y}\, dy$$

$$= \frac{2g_1\left(\frac{rx_1}{\gamma D}\right) J_o\left(\frac{rx_1}{\gamma D}\right)}{\left(\frac{rx_1}{\gamma D}\right)^2} - \frac{g_o\left(\frac{rx_1}{\gamma D}\right) J_1\left(\frac{rx_1}{\gamma D}\right)}{\left(\frac{rx_1}{\gamma D}\right)} \qquad \ldots(3.7)$$

Where $J_o(\ )$ and $J_1(\ )$ are Bessel functions of first kind of zero and first order respectively ;

$$g_o\left(\frac{rx_1}{\gamma D}\right) = \sum_{p=0}^{9} (-1)^p a_p \left(\frac{rx_1}{5\gamma D}\right)^{-2p} + \left(\frac{rx_1}{\gamma D}\right) ;$$

and

$$g_1\left(\frac{rx_1}{\gamma D}\right) = \sum_{p=0}^{9} (-1)^p b_p \left(\frac{rx_1}{5\gamma D}\right)^{-2p} + \left(\frac{rx_1}{\gamma D}\right) .$$

$$\left|\epsilon\left(\frac{rx_1}{\gamma D}\right)\right| \leq 2 \times 10^{-7}$$

The values of a_p and b_p are as follows :

p	a_p	b_p
0	1.0	1.0
1	0.159992815	0.319985629
2	0.101619385	0.304858155
3	0.130811585	0.523246341
4	0.207404022	1.037020112
5	0.283300508	1.699803050
6	0.279029488	1.953206413
7	0.178915710	1.431325684
8	0.066228328	0.596054956
9	0.010702234	0.107022336

For $2 < \frac{rx_1}{\gamma D} < 5$,

the integral $\int_{x_1}^{\infty} \frac{2}{x} J_o(\frac{rx}{\gamma D})\, dx$ is evaluated in the following manner :

$$\int_{x_1}^{\infty} \frac{2}{x} J_o(\frac{rx}{\gamma D})dx = \int_{x_1}^{x_2} \frac{2}{x} J_o(\frac{rx}{\gamma D})dx + \int_{x_2}^{\infty} \frac{2}{x} J_o(\frac{rx}{\gamma D})dx \quad \ldots(3.8)$$

Evaluation of

$\int_{x_1}^{x_2} \frac{2}{x} J_o(\frac{rx}{\gamma D})dx$ is done numerically and

$\int_{x_2}^{\infty} \frac{2}{x} J_o(\frac{rx}{\gamma D})\, dx$ is done using Eq.(3.7)

because value of x_2 is such that $\frac{rx_2}{\gamma D} \geq 5$.

RESULTS AND DISCUSSION

Let,

$$\text{TERM} = [1 - \tfrac{1}{2}(e^{-(\mu_1-\mu_2)}(1 + \frac{\alpha m\eta(1-x^2)}{2\mu_2}) + e^{-(\mu_1+\mu_2)}(1 - \frac{\alpha m\eta(1-x^2)}{2\mu_2}))].$$

The value of TERM at different values of x are presented in Table (3.1) for aquifer parameters : $T = 350\ m^2/day$, $\phi = .003$, $\phi_y = .1$, $\alpha = 13.8/day$. As seen from the table, beyond x = 0.999, TERM = 1 - ϵ and $\epsilon < 0.000001$. Hence when x > 0.999 the value of TERM can be taken as 1. x_1 for the above set of aquifer parameters is therefore equal to 0.999.

Discrete kernel coefficients are generated for the following sets of aquifer parameters

T m^2/day	ϕ	ϕ_y	α 1/day	η
350.0	0.001	.03	20.0	31.0
700.0	0.001	0.03	20.0	31.0

Discrete kernel coefficients are generated when excitation and observation points are different. The generated discrete kernel coefficients are presented in Figs. 3.1 and 3.2. In Table (3.2) discrete kernel coefficients for drawdown in unconfined aquifers without and with delayed yield characteristics having the following parameters: $T = 700.0\ m^2/day$, $\phi = 0.031$; and $T = 700.0\ m^2/day$, $\phi = 0.001$, $\phi_y = .03$, $\alpha = 20.0/day$ respectively have been presented for the purpose of comparison.

The procedure described here can also be extended to evaluate the discrete kernel coefficients when the excitation and response points are same. Fig. 3.3 shows a square grid from which unit quantity of water is withdrawn during the first unit time period (and pumping stopped). In order to find the response at the centre of the grid due to the pulse excitation the grid is divided into 36 equal units as shown. It is envisaged that 36 wells are operating one at a time at the centre of each unit. Using method of superposition the drawdown at the centre of grid when all the 36 wells are operating simultaneously is obtained. Sum of the drawdowns is divided by 36 to arrive at the response due to unit withdrawl from the grid. The discrete kernel coefficient generated is designated as $\delta_{rr}(m)$. The $\delta_{rr}(m)$ values have been plotted in Fig. 3.4.

Using the present procedure the well function $W(u_{ay}, \frac{r}{D})$, [$W(u_{ay}, \frac{r}{D})$ is the well function of an unconfined aquifer having delayed yield characteristics], has been evaluated for $\eta = 10.0$, $\frac{r}{D} = 2.0$ for different values of u_a and u_y ($u_a = \frac{r^2 \phi}{4Tt}$, $u_y = \frac{r^2 \phi y}{4Tt}$) and the same has been plotted in Fig.3.5. Also ,the results obtained by Boulton (1964) for these aquifer paramters have been plotted in the same figure.

In order to compare the well function for finite and infinite values of η,the results obtained by Boulton (1963) for a large value of η ($\eta > 100$) have also been presented in Fig.3.5. It may be seen that the type curve for $\eta > 100$ deviates appreciably from the curve for η=10.0.

CONCLUSIONS

a) An efficient method to evaluate type curves for drawdown in an unconfined aquifer with delayed yield for finite value of η has been described.

b) The discrete kernel coefficientsfor drawdown in an unconfined aquifer with delayed yield have been obtained.

Table 3.1 Values of 'TERM' for different values of x

x	TERM
$.9900001\text{x}10^{-1}$	.1267259
.1990000	.4213664
.2990000	.7086827
.3989998	.8882457
.4989996	.9671634
.5989998	.9925374
.6990000	.9986624
.7990002	.9998028
.8990004	.9999737
.9990006	.9999961
.1099001x10	.9999992
.1199001x10	.9999997
.1299001x10	.9999999
.1399001x10	.9999999
.1499002x10	.9999999
.1599002x10	1.0000000
.1699002x10	1.0000000

Table 3.2 Discrete kernel coefficients for drawdown in an unconfined aquifer

Time in days	∂_{rp} (with delayed yield) m/(m³/day) *		∂_{rp} (with out delayed yield) m/(m³/day) **	
	r = 300m	r = 600m	r = 300m	r = 600m
1	$.2711\times10^{-4}$	$.8445\times10^{-6}$	$.2510\times10^{-4}$	$.4372\times10^{-6}$
2	$.3763\times10^{-4}$	$.5498\times10^{-5}$	$.3879\times10^{-4}$	$.5177\times10^{-5}$
3	$.3023\times10^{-4}$	$.9030\times10^{-5}$	$.3064\times10^{-4}$	$.9120\times10^{-5}$
4	$.2434\times10^{-4}$	$.1022\times10^{-4}$	$.2451\times10^{-4}$	$.1036\times10^{-4}$
5	$.2021\times10^{-4}$	$.1029\times10^{-4}$	$.2029\times10^{-4}$	$.1040\times10^{-4}$
6	$.1722\times10^{-4}$	$.9925\times10^{-5}$	$.1728\times10^{-4}$	$.1001\times10^{-4}$
7	$.1500\times10^{-4}$	$.9415\times10^{-5}$	$.1502\times10^{-4}$	$.9472\times10^{-5}$
8	$.1326\times10^{-4}$	$.8861\times10^{-5}$	$.1329\times10^{-4}$	$.8910\times10^{-5}$
9	$.1189\times10^{-4}$	$.8335\times10^{-5}$	$.1191\times10^{-4}$	$.8370\times10^{-5}$
10	$.1077\times10^{-4}$	$.7843\times10^{-5}$	$.1078\times10^{-4}$	$.7870\times10^{-5}$
11	$.9841\times10^{-5}$	$.7385\times10^{-5}$	$.9853\times10^{-5}$	$.7409\times10^{-5}$
12	$.9062\times10^{-5}$	$.6973\times10^{-5}$	$.9070\times10^{-5}$	$.6992\times10^{-5}$

* $T = 700.0\ m^2/day$, $\phi = 0.001$, $\phi_y = .03$, $\alpha = 20.0/day$

** $T = 700.0\ m^2/day$, $\phi = .031$.

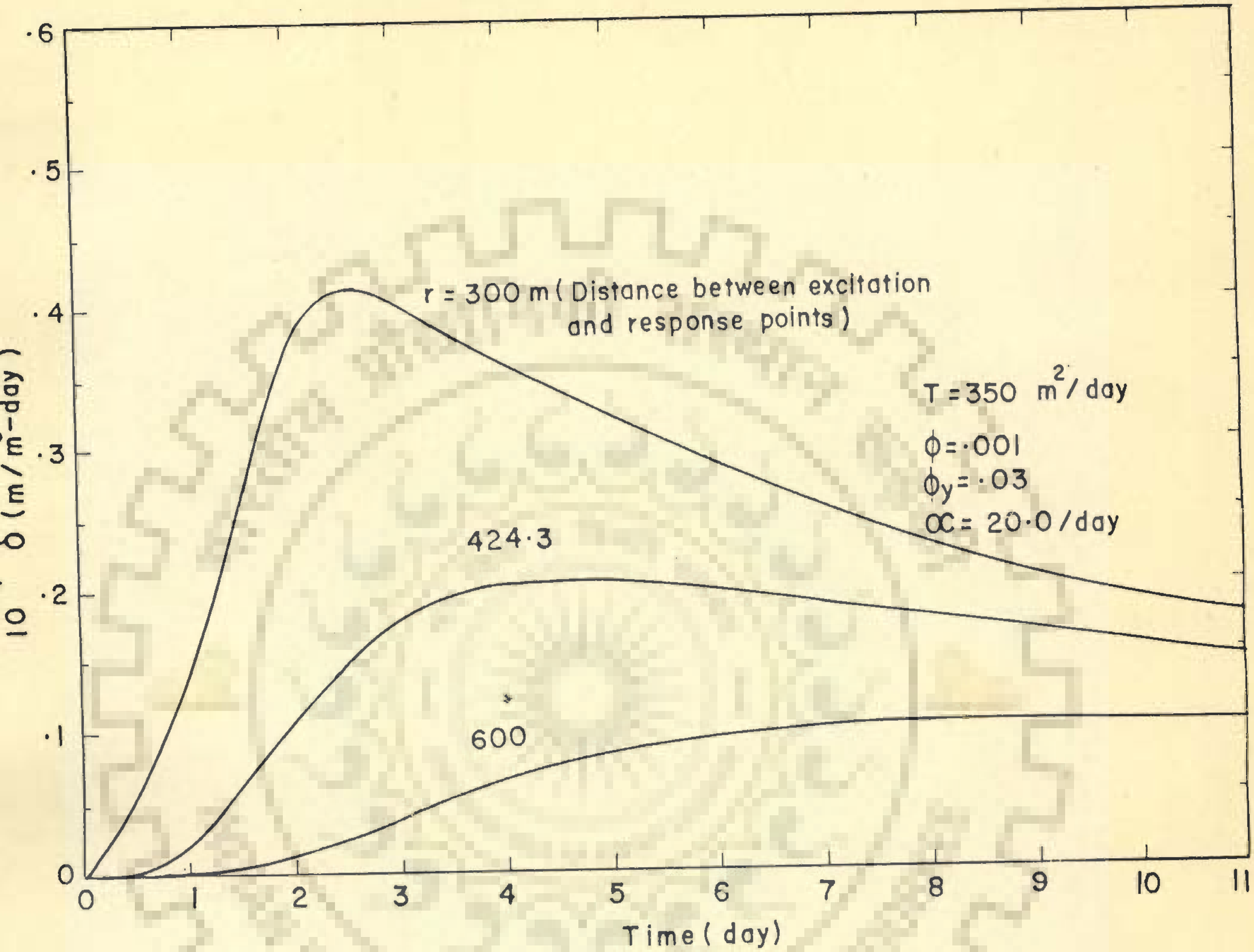

Fig. 3·1 Discrete kernel coefficients for drawdown in an unconfined aquifer having delayed yield ; excitation and response points are different.

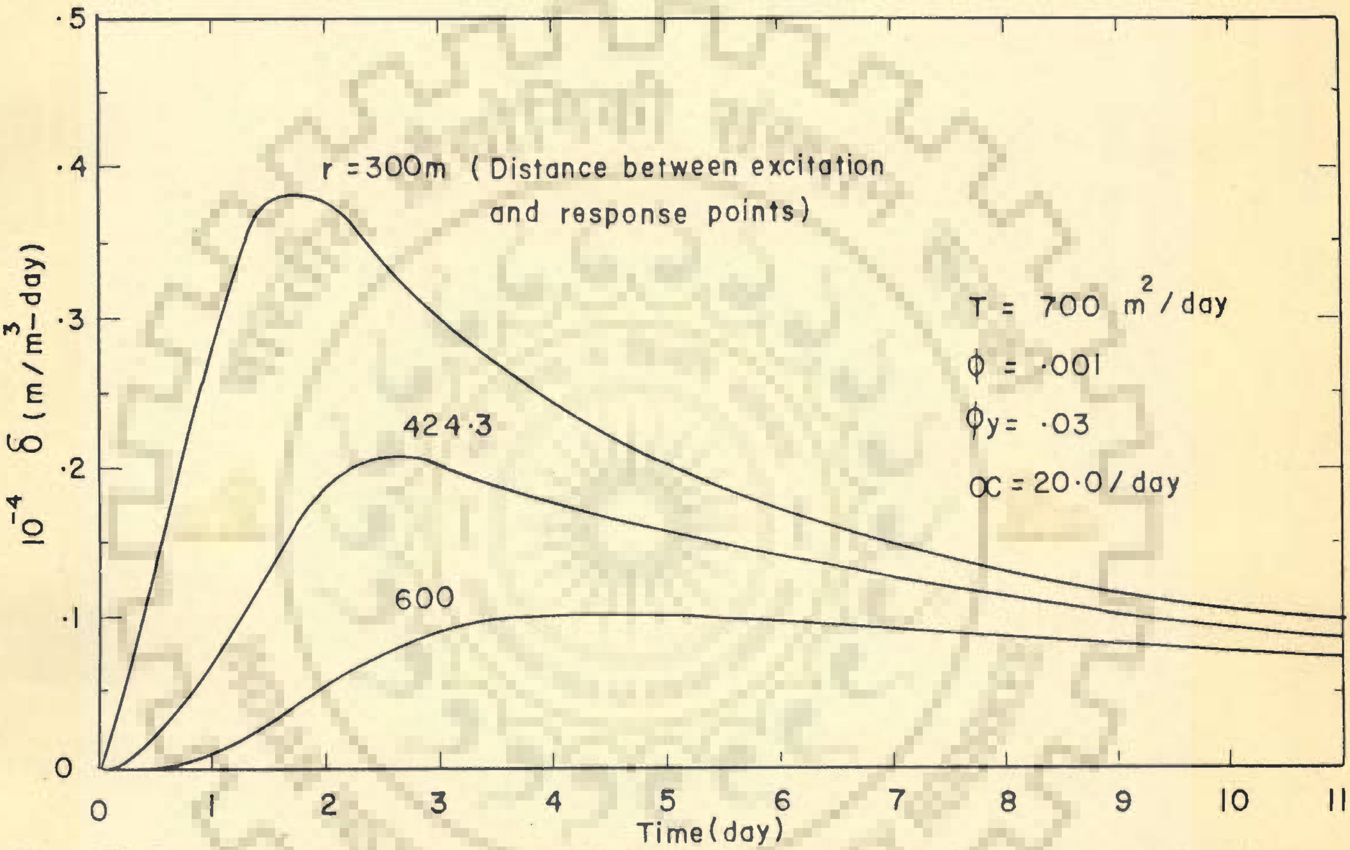

Fig. 3·2 Discrete kernel coefficients for drawdown in an unconfined aquifer having delayed yield; excitation and responce points are different.

Fig. 3·3 Division of a grid into 36 units for evaluation of response when the excitation and observation points are same.

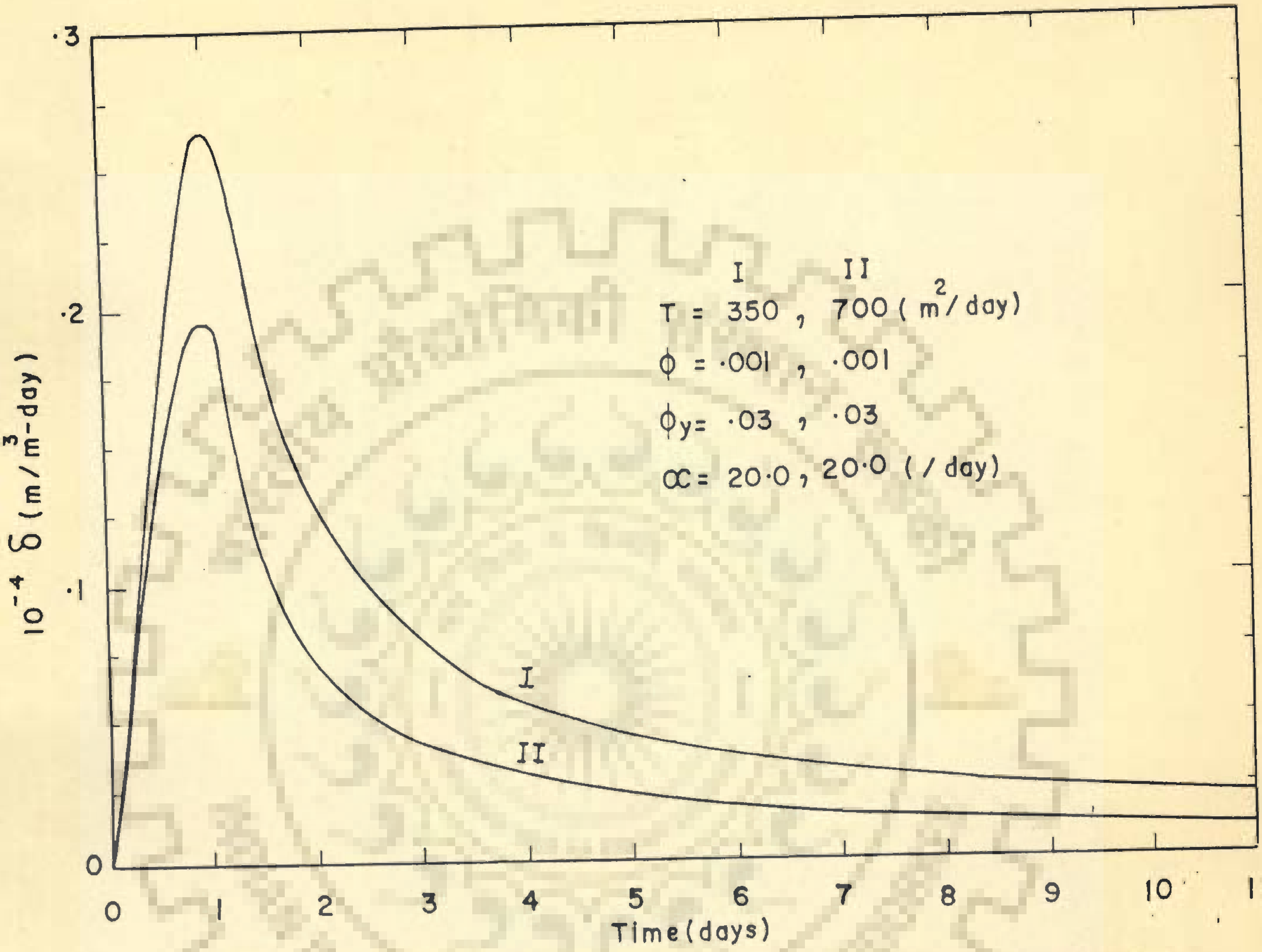

Fig. 3·4 Discrete kernel coefficients for drawdown in an unconfined aquifer having delayed yield ; excitation and response points are same.

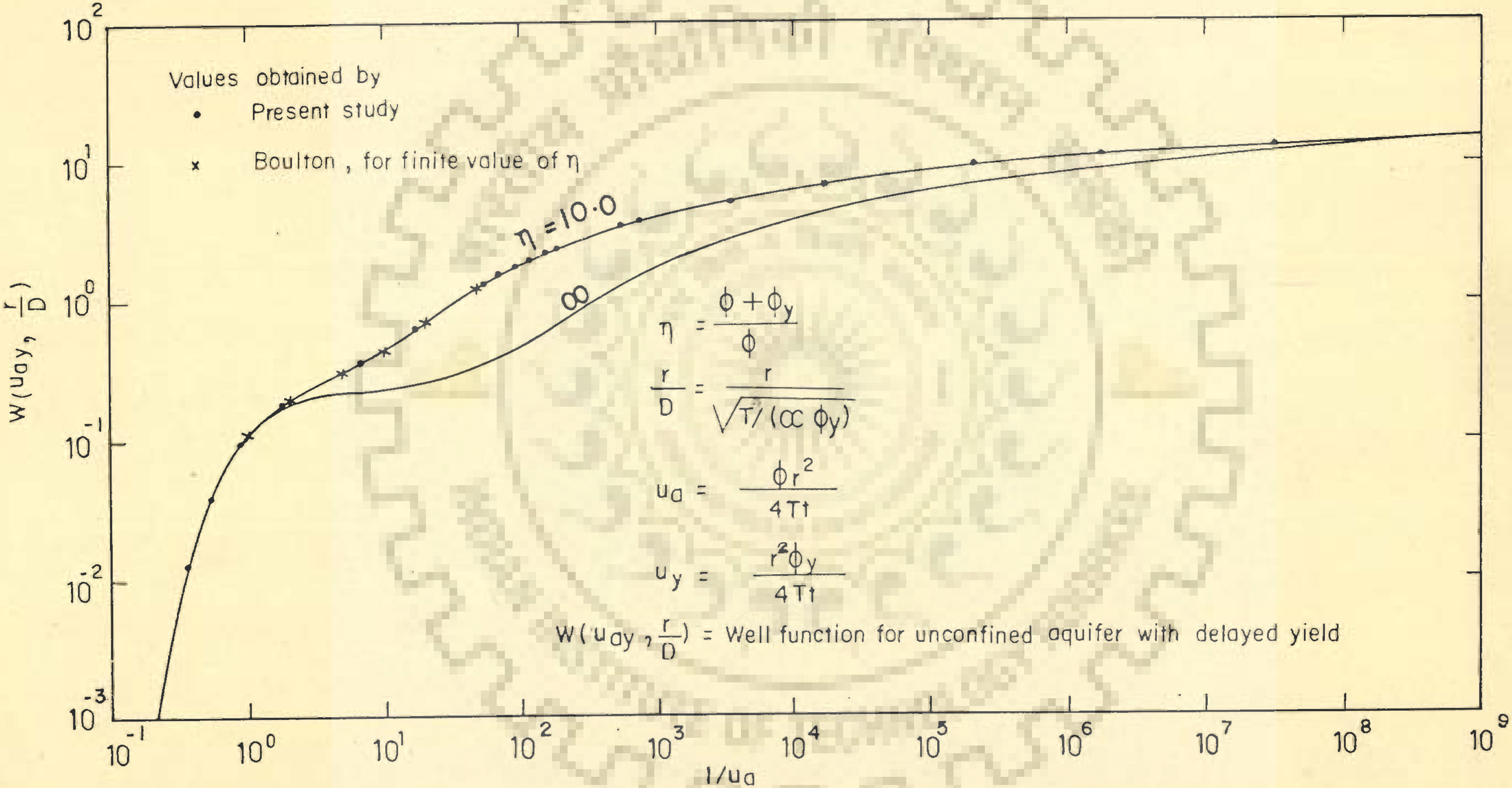

Fig. 3·5 Type curve for an unconfined aquifer with delayed yield for $\frac{r}{D} = 2{\cdot}0$ and $\eta = 10{\cdot}0$.

CHAPTER 4

UNSTEADY FLOW TO A WELL TAPPING TWO AQUIFERS SEPARATED BY AN AQUICLUDE

INTRODUCTION

Water wells are generally constructed tapping more than one aquifer in order to have dependable yield. It may also be worthwhile to evaluate the necessity or otherwise of tapping deeper aquifers of low transmissivity. Analysis of unsteady flow to a well tapping two aquifers separated by an aquiclude has been carried out by Papadopulos (1966) and Khader and Veerankutty (1975) who have used integral transform technique. In this chapter unsteady flow to a well tapping two aquifers separated by an aquiclude has been carried out using a discrete kernel approach.

STATEMENT OF THE PROBLEM

A schematic cross section of a well tapping two confined aquifers is shown in Fig.(4.1). The aquifers are separated by an aquiclude. Therefore no exchange of flow takes place between the two aquifers through the intervening layer. Each of the aquifers is homoneneous, isotropic, infinite in areal extent and is of uniform thickness. Drawdown in the piezometric surfacesare caused by discharge from the

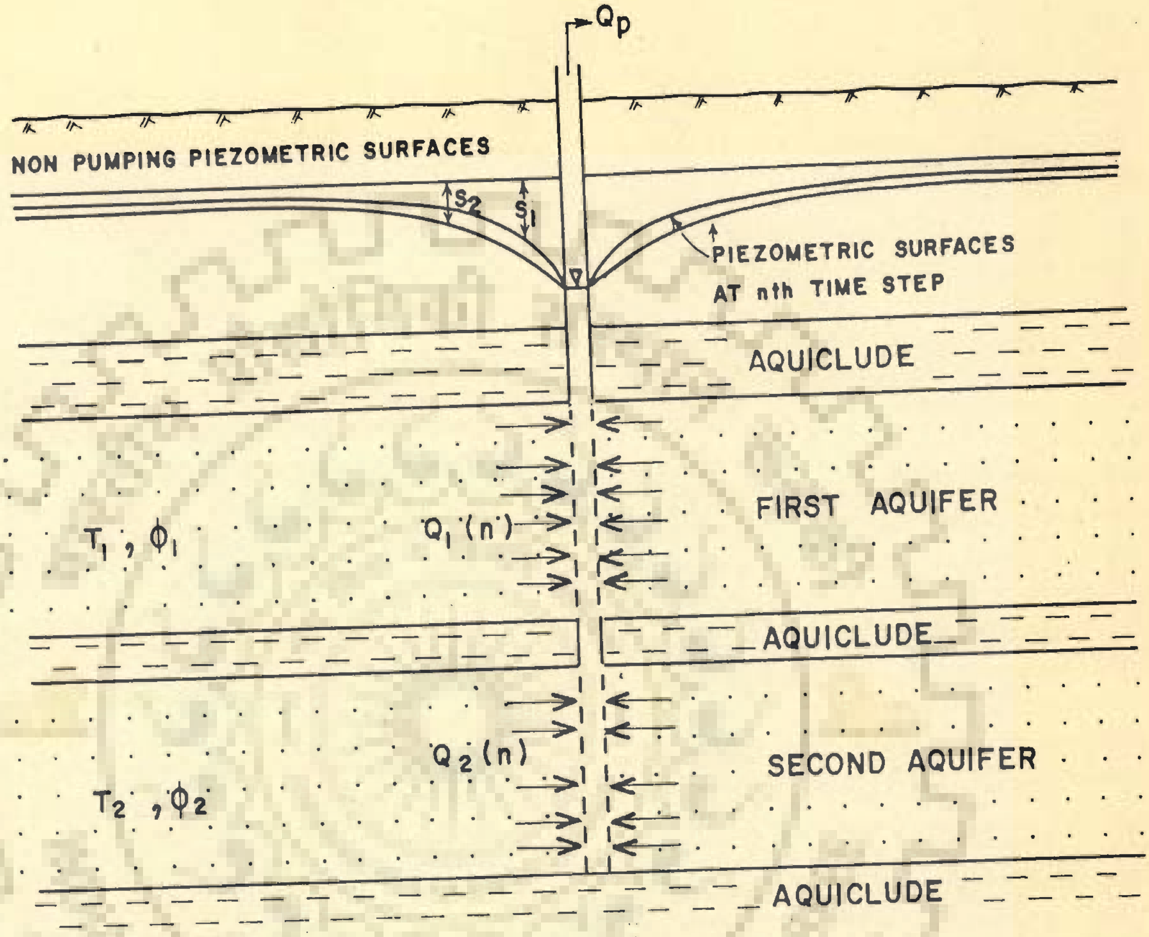

Fig. 4·1 Schematic section of a well tapping two confined aquifers separated by an aquiclude.

aquifers. It is required to find the contributions of each of the aquifers to pumping and drawdown in the piezometric surfaces of each aquifer in response to a uniform rate of pumping.

ANALYSIS

The following assumptions have been made in the analysis :

(a) Both the aquifers are initially at rest condition prior to pumping.

(b) The well discharges at a constant rate.

(c) At any time the drawdowns in both the aquifers at the well face are same but vary with time.

(d) The time parameter is discrete. Within each time step, the abstraction rates of water derived from each of the aquifers are separate constants.

(e) The radius of the well is small and hence the well storage is neglected.

The differential equation which describes the axially symmetric, radial, unsteady flow in each aquifer is given by

$$\frac{\partial^2 s_i}{\partial r^2} + \frac{1}{r}\frac{\partial s_i}{\partial r} = \frac{\phi_i}{T_i}\frac{\partial s_i}{\partial t} \qquad (4.1)$$

Where

s_i = drawdown in piezometric surface in the ith aquifer,

r = radial distance,

t = time,

ϕ_i = storage coefficient, and

T_i = transmissivity of the ith aquifer.

Had the aquifers been tapped separately, for the initial condition $s_i(r,o) = 0$, and boundary condition $s_i(\infty,t) = 0$, solution to differential Eq.(4.1) when unit impulse quantity of water is withdrawn from the aquifer 'i' is (Carslaw and Jaeger, 1959)

$$s_i(r,t) = \frac{1}{4\pi T_i} \frac{e^{-\frac{r^2}{4\beta_i t}}}{t} \; ; \; \beta_i = \frac{T_i}{\phi_i} . \qquad (4.2)$$

Defining an unit impulse kernel

$$k_i(t) = \frac{e^{-\frac{r^2}{4\beta_i t}}}{4\pi T_i t} \qquad (4.3)$$

drawdown for variable withdrawal from the aquifer i can be written in the form

$$s_i(r,t) = \int_o^t Q_i(c)\, k(t-c)\, dc \qquad (4.4)$$

where $Q_i(c)$ is variable discharge rate from the aquifer i at time c. Dividing the time span into discrete time steps and assuming that the aquifer discharge is constant within each timestep but varies from time step to time step, the drawdown at the end of time step n can be written as (Morel-Seytoux, 1975)

$$s_i(r,n) = \sum_{\gamma=1}^{n} \delta_{r,i}(n-\gamma+1)\, Q_i(\gamma) \qquad \ldots(4.5)$$

where the discrete kernel coefficient $\delta_{r,i}(m)$ is defined as

$$\delta_{r,i}(m) = \int_0^1 k_i(m-c)\, dc$$

$$= \frac{1}{4\pi T_i} \left[E_1\left(\frac{r^2}{4\beta_i m}\right) - E_1\left(\frac{r^2}{4\beta_i (m-1)}\right)\right] \qquad \ldots(4.6)$$

in which $E_1(x)$ is an exponential integral (Abramowitz and Stegun 1970) defined as

$$E_1(x) = \int_x^\infty \frac{e^{-u}}{u}\, du\,.$$

The discrete kernel coefficient $\delta_{r,i}(m)$ is the drawdown at the end of m^{th} time step at distance r from the pumping well in response to withdrawl of unit quantity of water from the storage of i^{th} aquifer during the 1st time period. A unit time step may be 0.1 day, 1 day or 1 week etc. The transmissivity T_i to be used to evaluate the discrete kernel coefficients has the dimension of $length^2$ per unit time period.

When the two aquifers are tapped by a single well and the well is pumped, there is contribution from each aquifer to the pumping through the respective well screen. Let $Q_1(n)$ and $Q_2(n)$ be the contributions from aquifer 1 and 2 respectively at time step n. Since pumping rate is constant therefore,

$$Q_1(n) + Q_2(n) = Q_p \qquad ...(4.7)$$

The drawdown at the well face at the end of time step n in aquifer 1 is given by

$$s_{1w}(n) = \sum_{\gamma=1}^{n} Q_1(\gamma)\, \partial_{rw1}(n-\gamma+1) \qquad ...(4.8)$$

where

$$\partial_{rw1}(m) = \frac{1}{4\pi T_1} [E_1(\frac{r_w^2}{4\beta_1 m}) - E_1(\frac{r_w^2}{4\beta_1 (m-1)})] \qquad ...(4.9)$$

Similarly the drawdown at the well face at the end of time step n in aquifer 2 is given by

$$s_{2w}(n) = \sum_{\gamma=1}^{n} Q_2(\gamma)\, \partial_{rw2}(n-\gamma+1) \qquad ...(4.10)$$

where

$$\partial_{rw2}(m) = \frac{1}{4\pi T_2} [E_1(\frac{r_w^2}{4\beta_2 m}) - E_1(\frac{r_w^2}{4\beta_2 (m-1)})] \qquad ...(4.11)$$

Since

$s_{1w}(n) = s_{2w}(n)$, therefore,

$$\sum_{\gamma=1}^{n} Q_1(\gamma)\, \partial_{rw1}(n-\gamma+1) = \sum_{\gamma=1}^{n} Q_2(\gamma)\, \partial_{rw2}(n-\gamma+1) \qquad ...(4.12)$$

Rearranging,

$$Q_1(n)\partial_{rw1}(1)-Q_2(n)\partial_{rw2}(1)=\sum_{\gamma=1}^{n-1} Q_2(\gamma)\partial_{rw2}(n-\gamma+1)-\sum_{\gamma=1}^{n-1} Q_1(\gamma)\partial_{rw1}(n-\gamma+1) \quad ...(4.13)$$

Eqs.(4.7) and (4.13) can be written in the following matrix form

$$\begin{bmatrix} 1 & , 1 \\ \partial_{rw1}(1), & -\partial_{rw2}(1) \end{bmatrix} \cdot \begin{bmatrix} Q_1(n) \\ Q_2(n) \end{bmatrix} = \begin{bmatrix} Q_p \\ \sum_{\gamma=1}^{n-1} Q_2(\gamma)\partial_{rw2}(n-\gamma+1)-\sum_{\gamma=1}^{n-1} Q_1(\gamma)\partial_{rw1}(n-\gamma+1) \end{bmatrix} \quad ...(4.14)$$

Hence

$$\begin{bmatrix} Q_1(n) \\ Q_2(n) \end{bmatrix} = \begin{bmatrix} 1 & , 1 \\ \partial_{rw1}(1), & -\partial_{rw2}(1) \end{bmatrix}^{-1} \cdot \begin{bmatrix} Q_p \\ \sum_{\gamma=1}^{n-1} Q_2(\gamma)\partial_{rw2}(n-\gamma+1)-\sum_{\gamma=1}^{n-1} Q_1(\gamma)\partial_{rw1}(n-\gamma+1) \end{bmatrix} \quad ...(4.15)$$

Thus $Q_1(n)$ and $Q_2(n)$ can be solved in succession starting from time step 1.

In particular for time step 1

$$\begin{bmatrix} Q_1(1) \\ Q_2(1) \end{bmatrix} = \begin{bmatrix} 1 & , 1 \\ \partial_{rw1}(1), & -\partial_{rw2}(1) \end{bmatrix}^{-1} \cdot \begin{bmatrix} Q_p \\ 0 \end{bmatrix} \quad ...(4.16)$$

Once $Q_1(n)$ and $Q_2(n)$ values are solved, the drawdown at any distance r in aquifer 1 and 2 can be found using Eq.(4.5).

RESULTS AND DISCUSSION

The discrete kernel values are generated for known values of transmissivities, storage coefficients and radius of the well. The radius of the well has been assumed to be 0.1 m. Using the discrete kernel coefficients $Q_1(n)$ and $Q_2(n)$ are found in succession starting from time step 1. The variation of $Q_1(n)/Q_p$ with non dimensional factor $u_1 = \frac{r_w^2}{4\beta_1 n}$ is graphically shown in Figs.4.2 through 4.7 for ratios of T_1/T_2 = 0.125,0.5,1,2,10,100 and ϕ_1/ϕ_2 = 1,5,10,25,100,250, 500,2500. The curves have been presented for the non-dimensional factor u_1, in the range of 10^{-6} to 3×10^{-10}. However, to study the contribution of individual aquifer to discharge at short times after pumping, results are presented only for T_1/T_2 = 10,1,.1 and ϕ_1/ϕ_2=100.

The contribution of each of the aquifers is controlled by its hydraulic diffusivity value i.e., $\beta = \frac{T}{\phi}$. As observed from the figures, $Q_1(n)$ decreases with increasing time if $\beta_1 < \beta_2$. Conversely, the contribution of the aquifer with higher value of hydraulic diffusivity increases with time.

As seen from the Figs. 4.4,4.6,4.7 when $\beta_1=\beta_2$ the contributions from individual aquifers are independent of time. The same fact can also be proved in the following manner :

Using Eqs.(4.7) and (4.13)

$$Q_1(n) = \frac{1}{1+\frac{\partial_{rw1}(1)}{\partial_{rw2}(1)}}\left[Q_p - \frac{1}{\partial_{rw2}(1)}\sum_{\gamma=1}^{n-1} Q_1(\gamma)\partial_{rw1}(n-\gamma+1)\right.$$

$$\left. + \frac{1}{\partial_{rw2}(1)}\sum_{\gamma=1}^{n-1} Q_2(\gamma)\partial_{rw2}(n-\gamma+1)\right] \qquad \text{...(4.17)}$$

and

$$Q_2(n) = Q_p - Q_1(n) \qquad \text{...(4.18)}$$

For time step 1

$$\frac{Q_1(1)}{Q_p} = \frac{1}{1+\frac{\partial_{rw1}(1)}{\partial_{rw2}(1)}} \qquad \text{...(4.19)}$$

Substituting the values of $\partial_{rw1}(1)$ and $\partial_{rw2}(1)$ by Eq.(4.6)

$$\frac{Q_1(1)}{Q_p} = \frac{T_1}{T_1+T_2} \qquad \text{...(4.20)}$$

Therefore,

$$\frac{Q_2(1)}{Q_p} = \frac{T_2}{T_1+T_2} \qquad \text{...(4.21)}$$

Similarly for the time step 2

$$\frac{Q_1(2)}{Q_p} = \frac{1}{1 + \frac{\partial_{rw1}(1)}{\partial_{rw2}(1)}} \left[1 - \frac{Q_1(1)}{Q_p}\frac{\partial_{rw1}(2)}{\partial_{rw2}(1)} + \frac{Q_2(1)}{Q_p}\frac{\partial_{rw1}(2)}{\partial_{rw2}(1)}\right] \qquad \ldots(4.22)$$

Substituting the expression for discrete kernels Eq.(4.22) simplifies to

$$\frac{Q_1(2)}{Q_p} = \frac{T_1}{T_1+T_2} \; . \qquad \ldots(4.23)$$

Hence $\frac{Q_1(n)}{Q_p}$ is independent of time when $\beta_1=\beta_2$. However the contributions of the aquifers to well discharge are proportional to their respective transmissivity values.

Using Thiem (1906) equation it can be proved that when a well taps two aquifers and is located at the centre of a circular island under steady state condition

$$2\pi T_1 \; (h_e-h_w) = Q_1 \log \frac{R}{r_w} \qquad \ldots(4.24)$$

and

$$2\pi T_2 \; (h_e-h_w) = Q_2 \log \frac{R}{r_w} \qquad \ldots(4.25)$$

where

h_e is the elevation of piezometric surface at island boundary and h_w is the elevation of piezometric surface at well face, R = radius of island, r_w = radius at well face.

Therefore under steady state condition the contributions by individual aquifers are in proportion to their respective transmissivity values.

It is seen from Figs. 4.2 to 4.7 that as the non dimensional factor u_1 decreases i.e. when time increases the limit of $Q_1(n)/Q_2(n)$ tends to T_1/T_2 .

As seen from Fig.4.8, at $\frac{\phi_1 r_w^2}{4T_1 n} = 10$, $\frac{Q_1(n)}{Q_p} \simeq 1$. Thus during the beginning of pumping all the water is withdrawn from the aquifer having the lower hydraulic diffusivity.

In order to compare the results obtained by discrete kernel approach with the results given by Khader and Veerankutty (1975) variation of $Q_1(n)/Q_p$ with $\log[(T_1+T_2)n/(\phi_1+\phi_2)r_w^2]$ has been plotted in Fig.4.9. As seen from the figure the results obtained by both the approaches match only for large time [ie $\log[(T_1+T_2)n/(\phi_1+\phi_2) r_w^2]>5$]. The deviation of the results given by Khader and Veerankutty (1975) from the results obtained by discrete kernel approach may be due to the numerical integration of an improper integral involving

Bessel functions of first and modified second kind, zero order in Khader and Veerankutty's approach. Needless to say that the discrete kernel approach is simple and less time consuming in comparison to integral transform method.

With the method of analysis developed it is easy to compute drawdowns at any point in both the aquifers. The drawdowns computed at the well face are given in Table 4.1. It is to be noted that the drawdowns at the well face in both the aquifers are the same.

In order to show the versatility and simplicity of discrete kernel approach, discrete kernels for drawdown are generated for an unconfined aquifer with delayed yield characteristic. Using these discrete kernel coefficients, $Q_1(n)/Q_p$ have been obtained for $T_1/T_2=0.5$, 1,10; $\phi_1/\phi_2=1,10,100$; $\eta = 4$ and $\alpha = 20$/day and are presented in Figs.4.10 through 4.12.

CONCLUSIONS

(a) The contributions to well discharge by each of the aquifers is controlled by its hydraulic diffusivity value. $Q_1(n)$, i.e. contribution by first aquifer decreases with increasing time if $\beta_1<\beta_2 (\beta = \frac{T}{\phi})$. Conversely, the contribution of the aquifer with higher value of hydraulic diffusivity increases with time.

(b) When the hydraulic diffusivity values of both the aquifers are equal ($\beta_1=\beta_2$), contributions by each of the aquifers are independent of time, and proportional to the respective transmissivity values.

(c) At early stage of pumping i.e. for high values of $\frac{r_w^2 \phi_1}{4T_1 n}$, major contribution is by the aquifer whose diffusivity is lower.

(d) At large values of time i.e. at near steady state conditions the contribution by each aquifer is in proportion to its transmissivity value i.e. the limit of $Q_1(n)/Q_2(n)$ tends to T_1/T_2 as n increases.

(e) In case of pumping a well tapping two aquifers piezometers may be placed in each of the aquifers and drawdowns observed. Using the recorded drawdowns in each aquifer, transmissivity and storage coefficient of each of the aquifers may be computed using the present analysis by minimising the error (i.e. sum of the square of difference between observed and calculated drawdowns).

Table 4.1 Drawdown at the well face for $T_1/T_2=0.5$, $\phi_1/\phi_2=100$, $r_w=0.1$ m, $Q_p=1000$ m^3/day.

Time in days	Drawdown in metres
1	1.444217
6	1.582555
11	1.629235
16	1.658066
21	1.678980
26	1.695399
31	1.708918
36	1.720408
41	1.730399
46	1.739238
51	1.747163

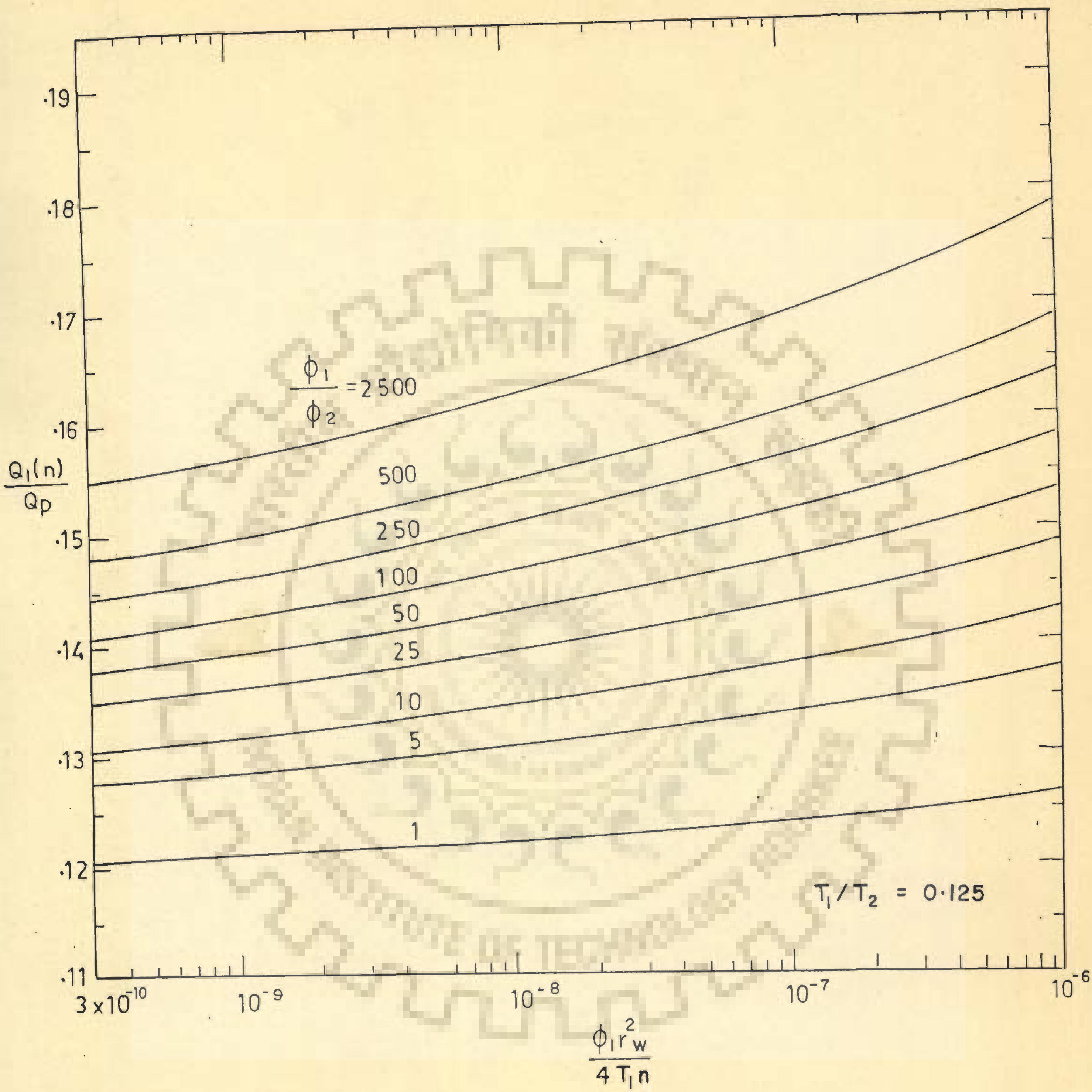

Fig. 4·2 Contribution of top aquifer to discharge at various time steps due to pumping of a well tapping two confined aquifers separated by an aquiclude.

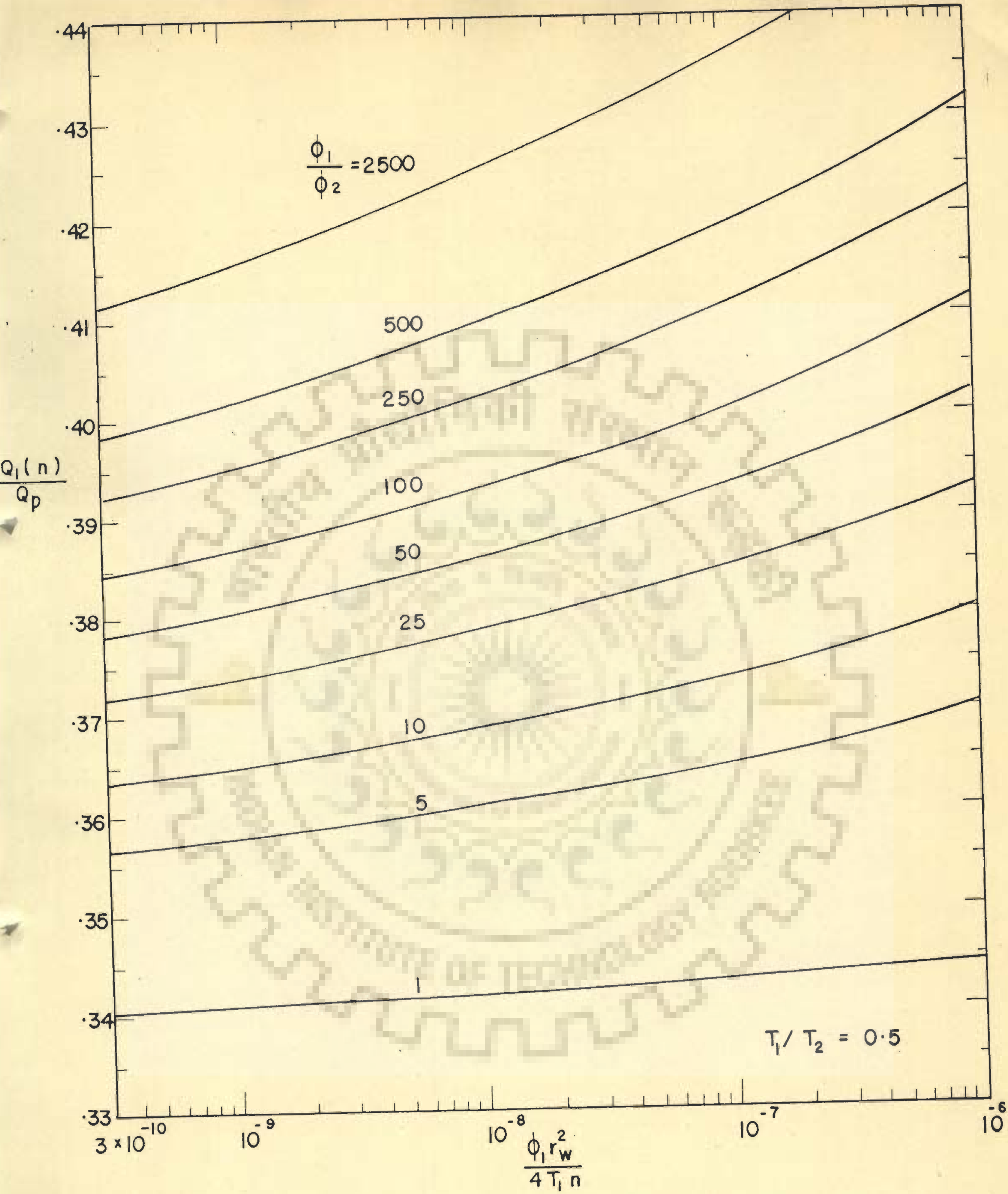

Fig. 4·3 Contribution of top aquifer to discharge at various time steps due to pumping of a well tapping two confined aquifers separated by an aquiclude.

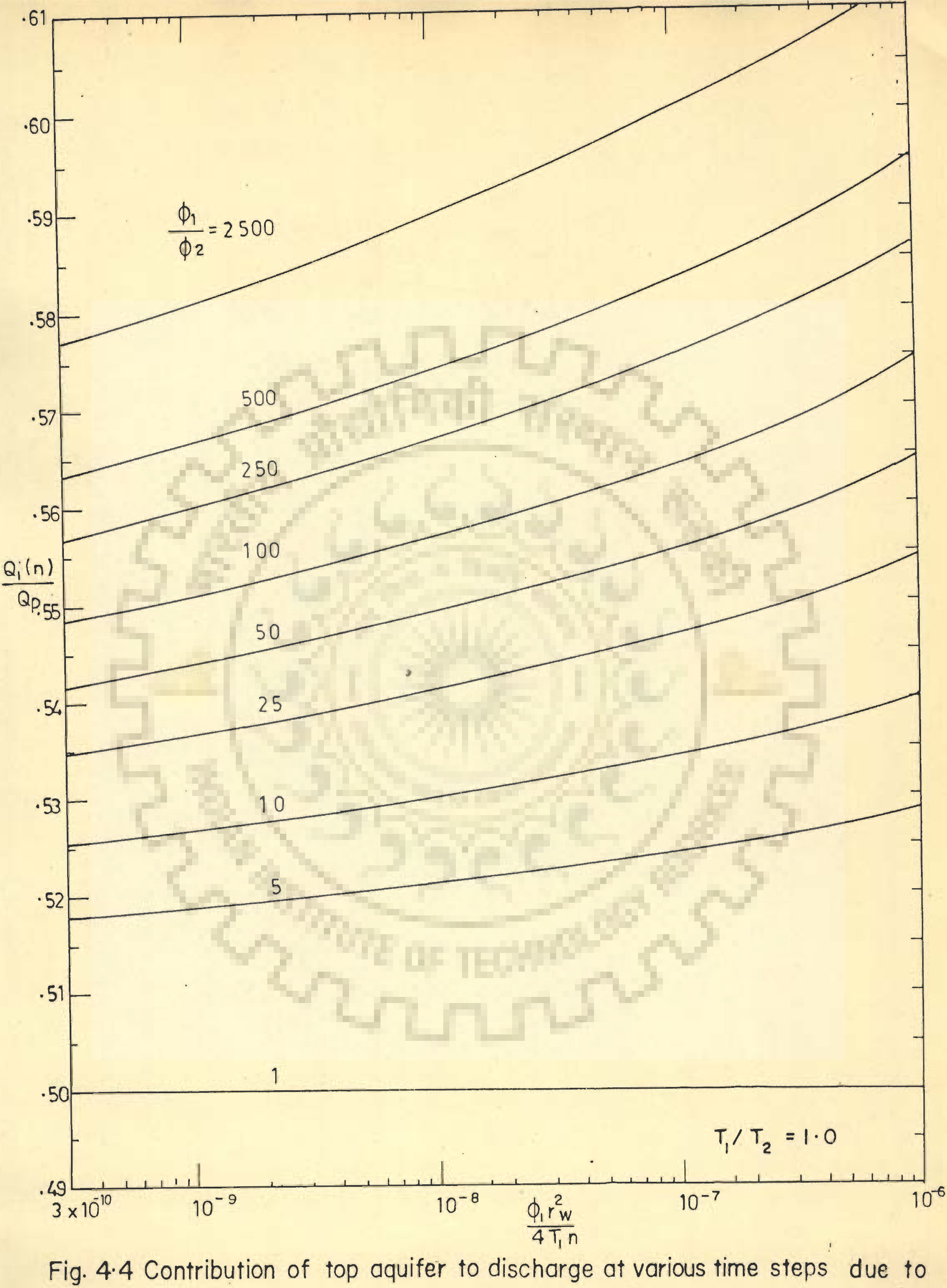

Fig. 4·4 Contribution of top aquifer to discharge at various time steps due to pumping of a well tapping two confined aquifers separated by an aquiclude.

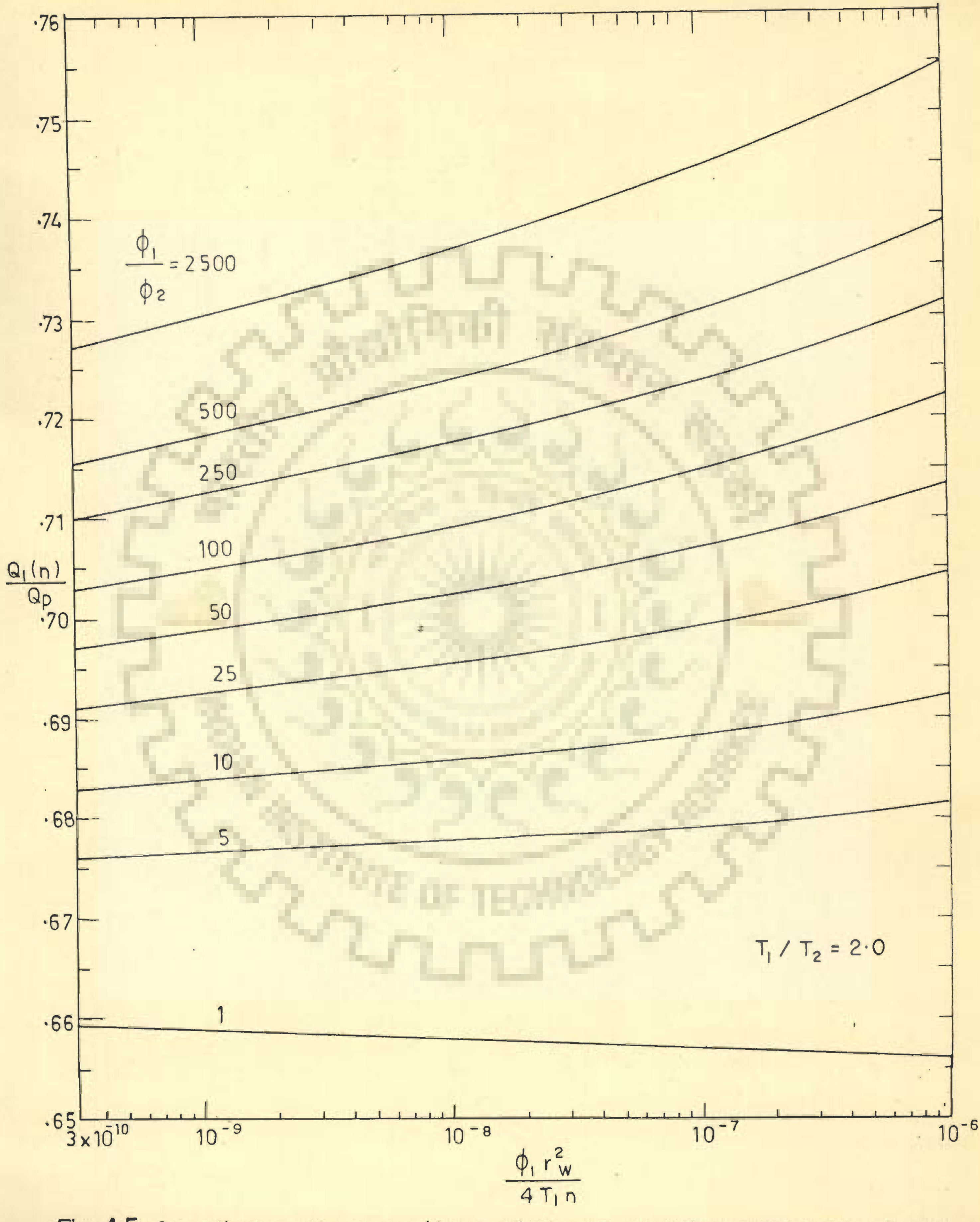

Fig. 4·5 Contribution of top aquifer to discharge at various time steps due to pumping of a well tapping two confined aquifers separated by an aquiclude.

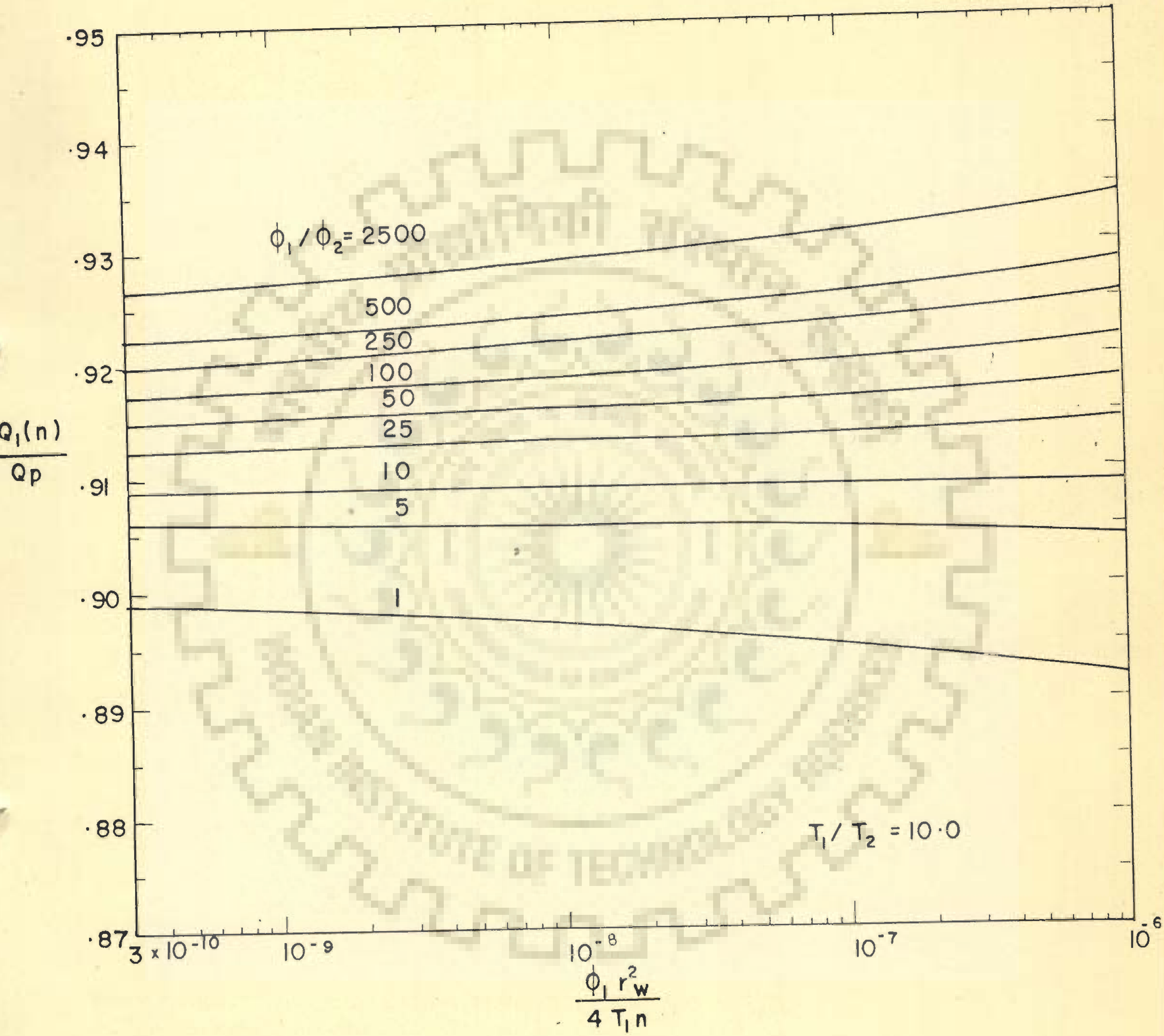

Fig. 4·6 Contribution of top aquifer to discharge at various time steps due to pumping of a well tapping two confined aquifers separated by an. aquiclude

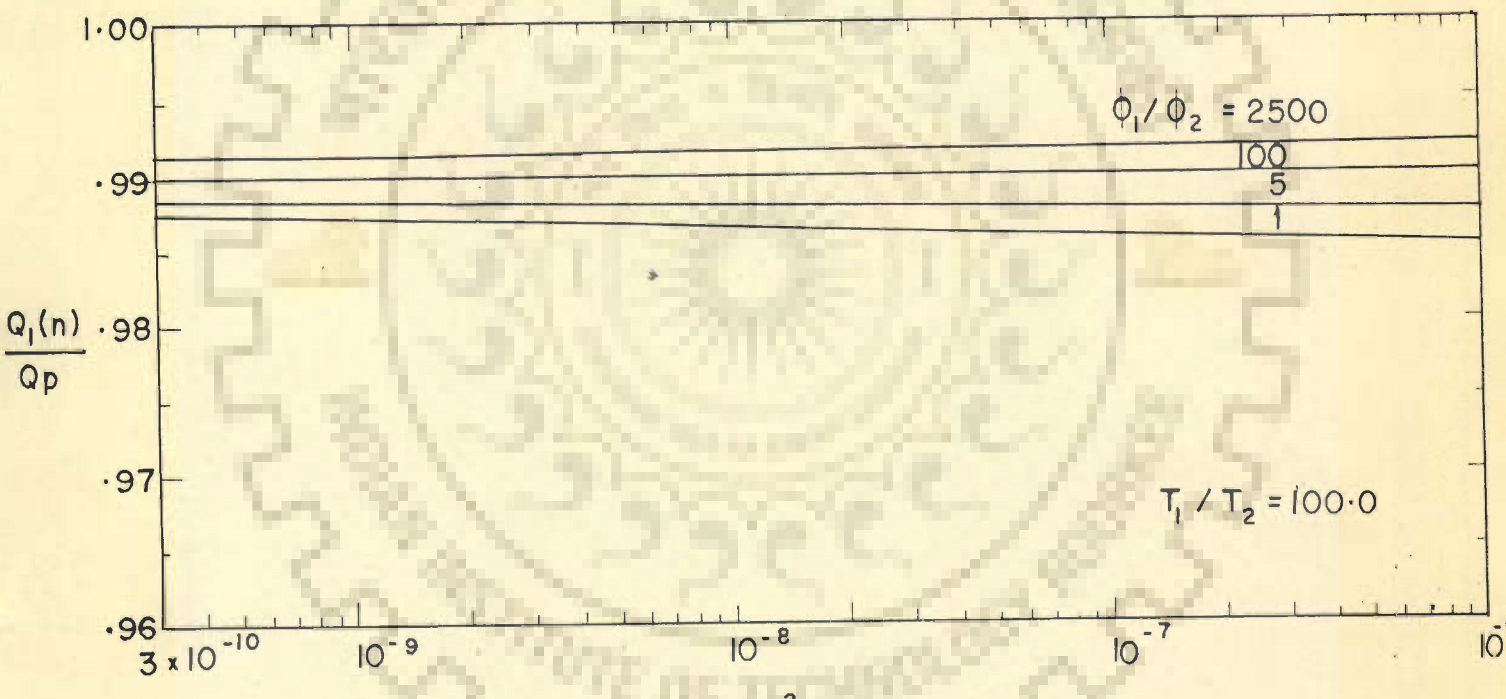

Fig. 4·7 Contribution of top aquifer to discharge at various time steps due to pumping of a well tapping two confined aquifers separated by an aquiclude.

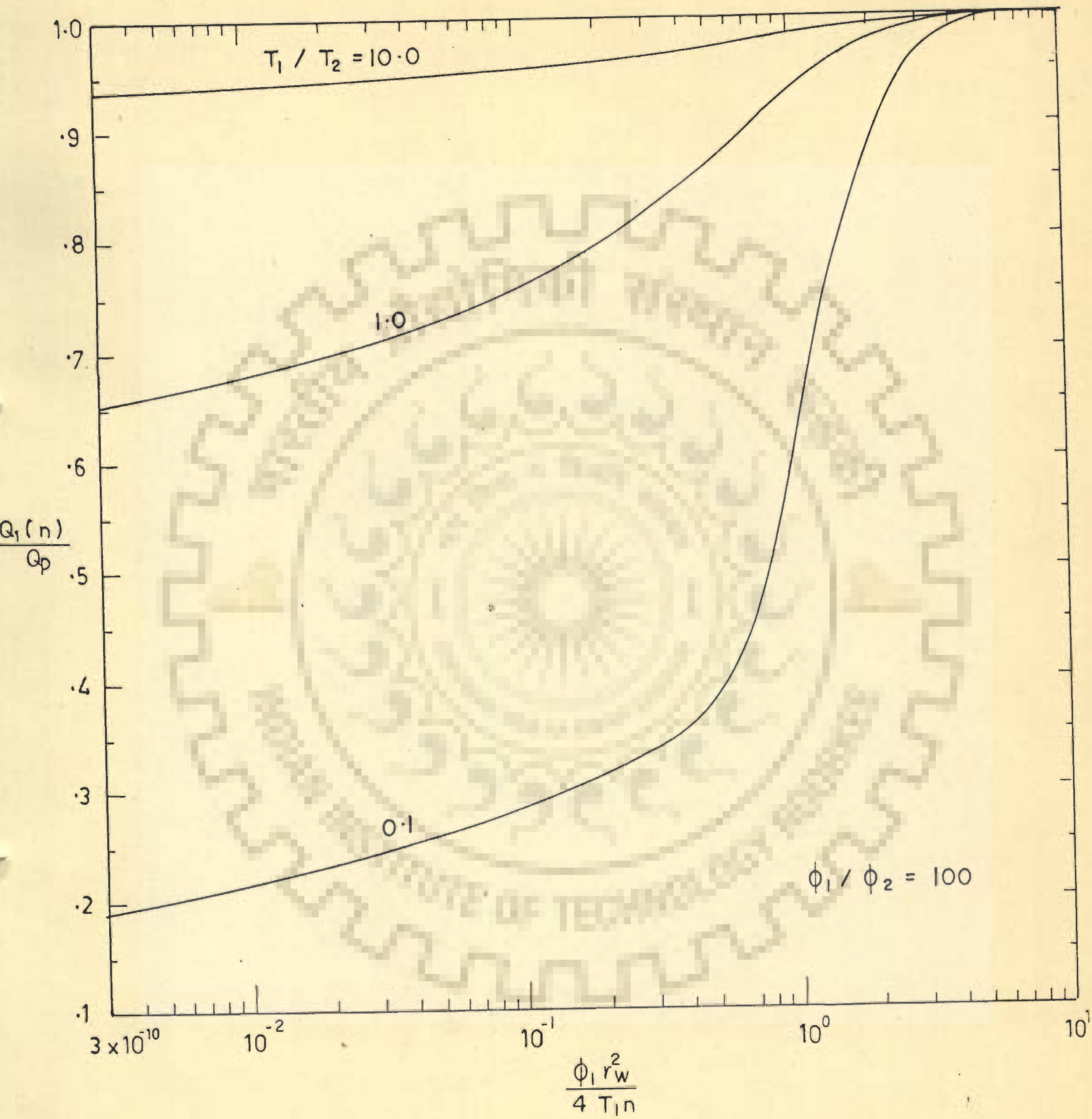

Fig. 4·8 Contribution of top aquifer to discharge at various time steps due to pumping of a well tapping two confined aquifers separated by an aquiclude.

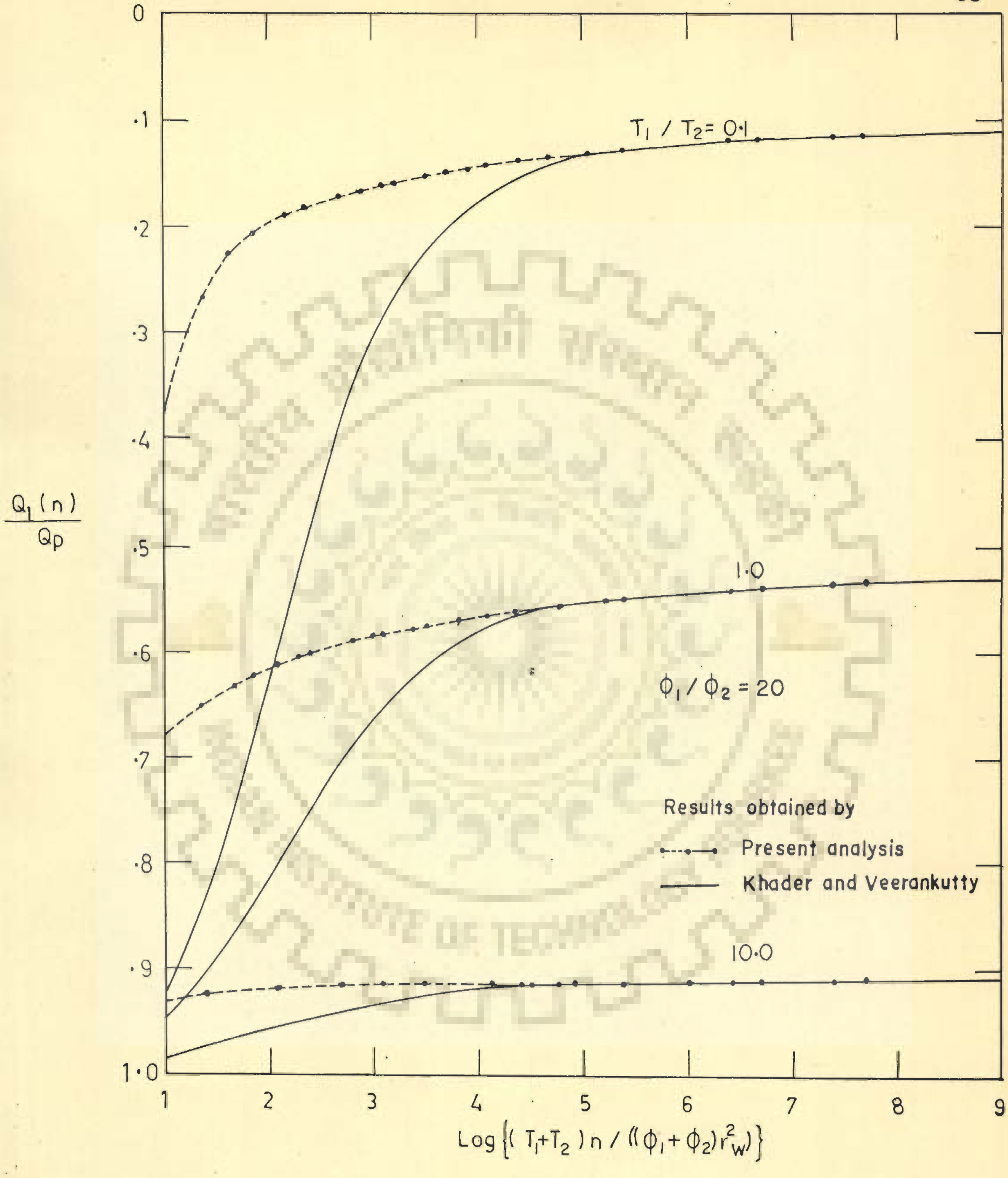

Fig. 4·9 Contribution of top aquifer to discharge at various time steps due to pumping of a well tapping two confined aquifers separated by an aquiclude.

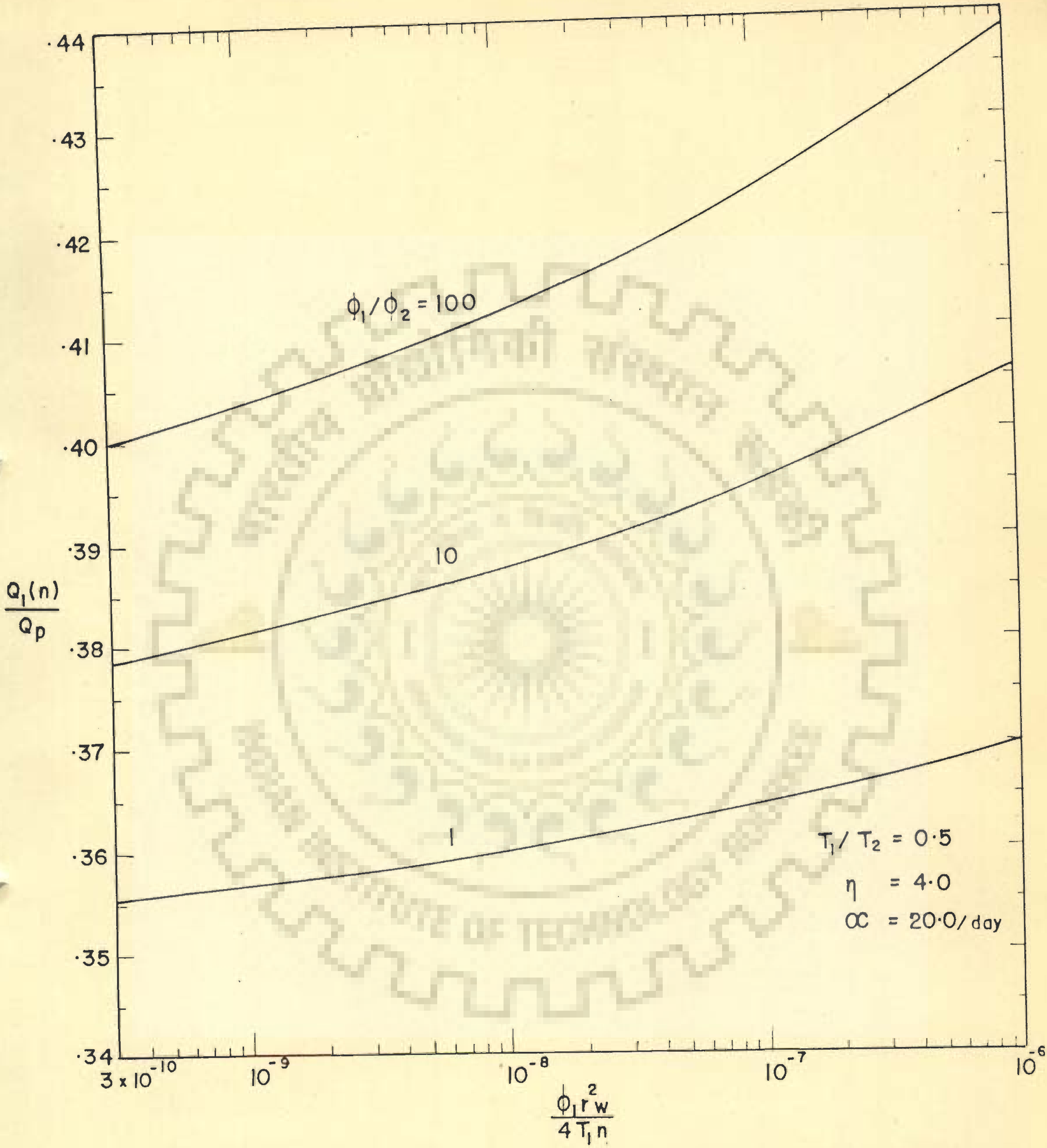

Fig. 4·10 Contribution of top aquifer to discharge at various time steps due to pumping of a well tapping an unconfined and a confined aquifer separated by an aquiclude.

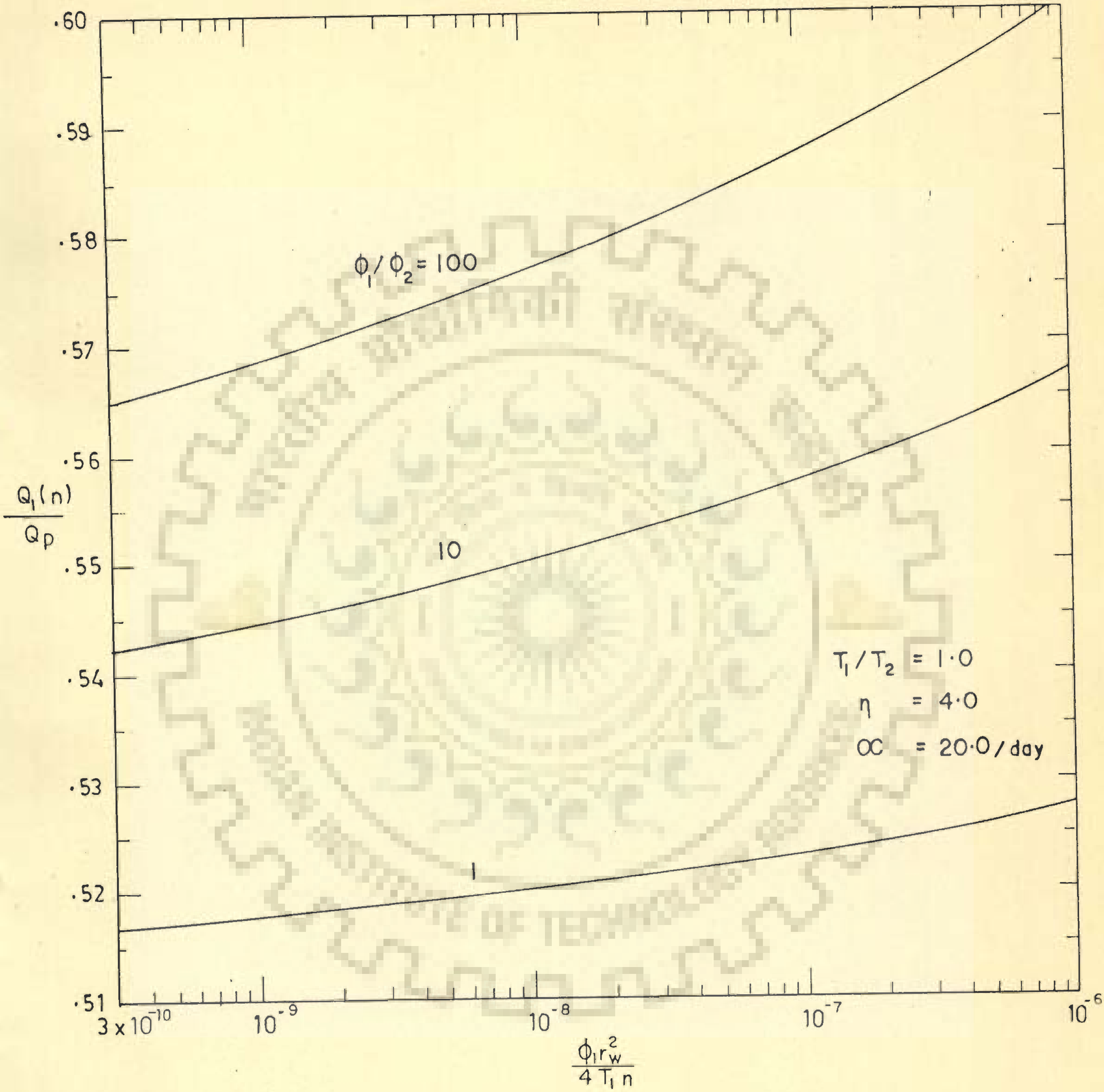

Fig. 4·11 Contribution of top aquifer to discharge at various time steps due to pumping of a well tapping an unconfined and a confined aquifer separated by an aquiclude.

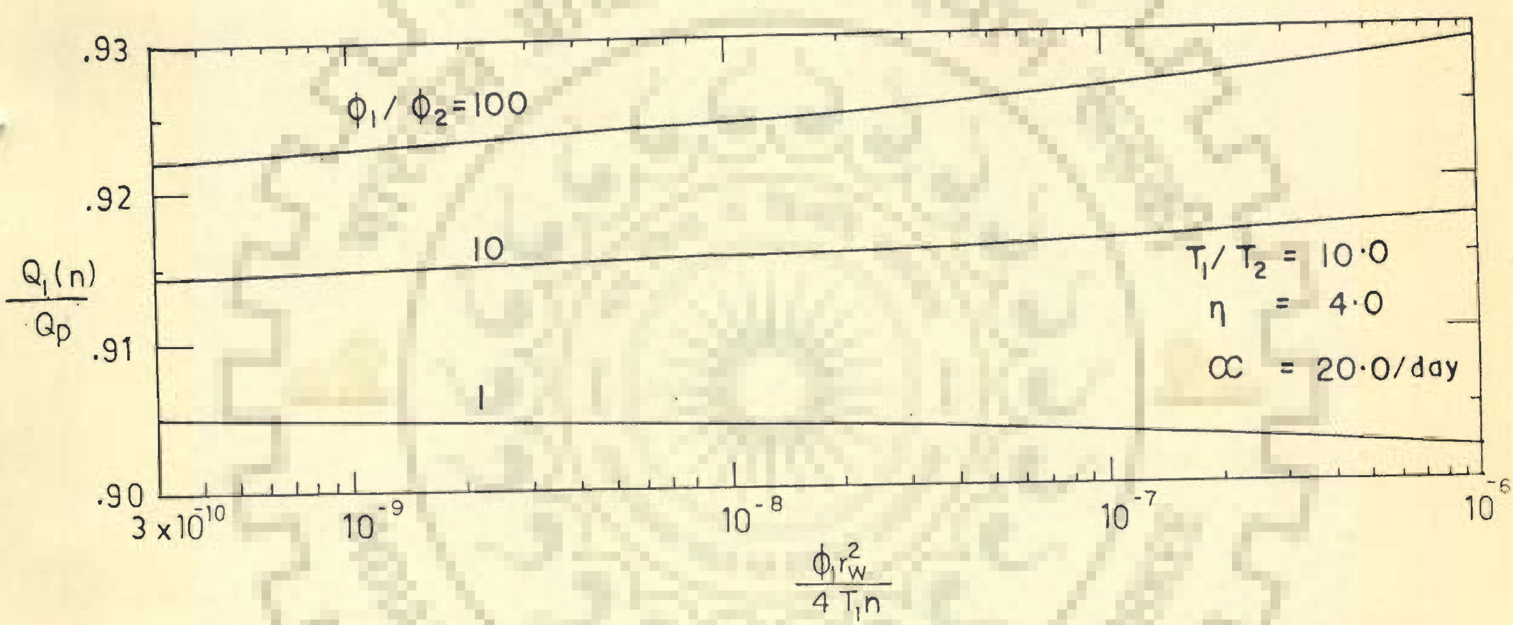

Fig. 4·12 Contribution of top aquifer to discharge at various time steps due to pumping of a well tapping an unconfined and a confined aquifer separated by an aquiclude.

CHAPTER 5

UNSTEADY FLOW TO A WELL TAPPING MULTIPLE (MORE THAN TWO) AQUIFERS SEPARATED BY AQUICLUDES

INTRODUCTION

The application of discrete kernel theory is not limited to two aquifers system only. Unsteady flow to a well tapping several aquifers which are separated by aquicludes can also be analysed with ease by the discrete kernel method.

STATEMENT OF THE PROBLEM

Fig. 5.1 shows a schematic cross section of a well tapping several confined aquifers which are separated by aquicludes. Each of the aquifers is homogeneous, isotropic, infinite in areal extent and initially at rest condition. Water is being pumped at a constant rate. It is required to find the contribution of each individual aquifer to pumping.

ANALYSIS

The differential equations which govern the radial axis-symmetric flow in the aquifers are given by

$$\frac{\partial^2 s_i}{\partial r^2} + \frac{1}{r}\frac{\partial s_i}{\partial r} = \frac{\phi_i}{T_i}\frac{\partial s_i}{\partial t} \qquad \ldots(5.1)$$

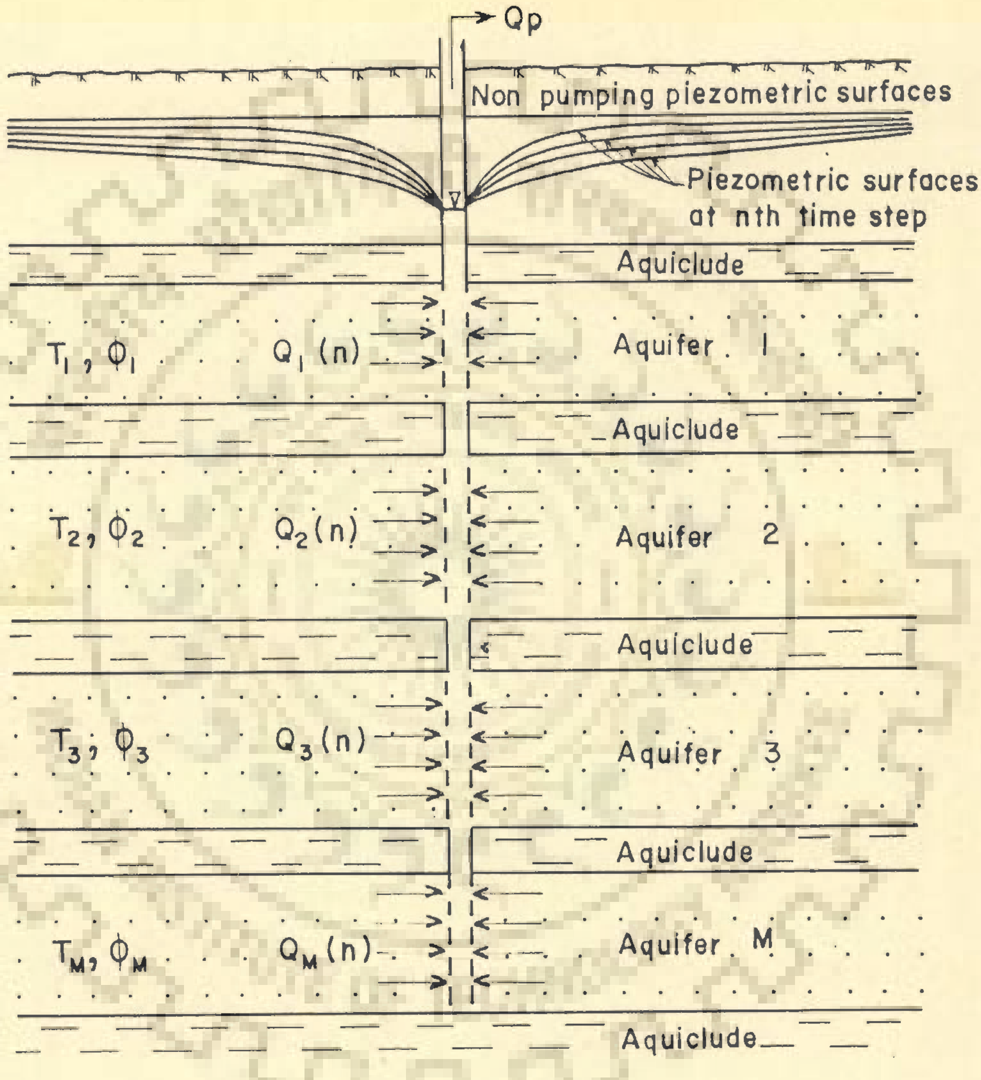

Fig. 5·1 Schematic section of a well tapping multiple aquifers separated by aquicludes.

$i = 1,2,\ldots\ldots,M$

Where

M = total number of aquifers tapped by the well ;

s_i = drawdown at distance r from the well at time t in the i^{th} aquifer; T_i and ϕ_i are the transmissivities and storage coefficients of the i^{th} aquifer.

Solutions to the equations are to be found for the initial conditions

$s_i(r,0) = 0$, $i = 1,2,\ldots\ldots,M$ and for the boundary condition

$$\sum_{i=1}^{M} 2\pi r_w T_i \left.\frac{\partial s_i}{\partial r}\right|_{r=r_w} = \text{pumping rate ;}$$

$$s_1(r_w,t) = s_2(r_w,t) = \ldots\ldots\ldots\ldots = s_M(r_w,t)$$

Let $Q_i(n)$ be the contribution by the i^{th} aquifer during the n^{th} unit time period and let water be pumped from the well at a rate equal to Q_p. The sum of contributions by each of the aquifers should be equal to the pumping rate. Hence,

$$Q_1(n) + Q_2(n) + \ldots\ldots\ldots\ldots + Q_M(n) = Q_p \qquad \ldots(5.2)$$

If $Q_i(\gamma)$ are the contributions by the i^{th} aquifer, drawdown at the well face in the i^{th} aquifer at the end of time step n is given by

$$s_i(r_w,n) = \sum_{\gamma=1}^{n} Q_i(\gamma)\, \delta_{rwi}(n-\gamma+1) \qquad \ldots(5.3)$$

in which the discrete kernel coefficient $\partial_{rwi}(m)$ is difined as:

$$\partial_{rwi}(m) = \frac{1}{4\pi T_i}\left[E_i\left(\frac{r_w^2}{4\beta_i m}\right) - E_i\left(\frac{r_w^2}{4\beta_i (m-1)}\right)\right]$$

Since the drawdown at the well face in all the aquifers are equal, therefore,

$$\sum_{\gamma=1}^{n} Q_1(\gamma)\,\partial_{rw1}(n-\gamma+1) = \sum_{\gamma=1}^{n} Q_2(\gamma)\,\partial_{rw2}(n-\gamma+1)$$

$$= \sum_{\gamma=1}^{n} Q_i(\gamma)\,\partial_{rwi}(n-\gamma+1) = \ldots\ldots = \sum_{\gamma=1}^{n} Q_M(\gamma)\partial_{rwM}(n-\gamma+1) \qquad \ldots(5.4)$$

The above set of equations can be written as

$$Q_1(n)\,\partial_{rw1}(1) + \sum_{\gamma=1}^{n-1} Q_1(\gamma)\,\partial_{rw1}(n-\gamma+1)$$

$$= \quad Q_2(n)\,\partial_{rw2}(1) + \sum_{\gamma=1}^{n-1} Q_2(\gamma)\,\partial_{rw2}(n-\gamma+1)\ ;$$

$$Q_1(n)\,\partial_{rw1}(1) + \sum_{\gamma=1}^{n-1} Q_1(\gamma)\,\partial_{rw1}(n-\gamma+1)$$

$$= \quad Q_3(n)\,\partial_{rw3}(1) + \sum_{\gamma=1}^{n-1} Q_3(\gamma)\,\partial_{rw3}(n-\gamma+1)\ ;$$

⋮

and

$$Q_1(n)\,\partial_{rw1}(1) + \sum_{\gamma=1}^{n-1} Q_1(\gamma)\,\partial_{rw1}(n-\gamma+1)$$

$$= \quad Q_M(n)\,\partial_{rwM}(1) + \sum_{\gamma=1}^{n-1} Q_M(\gamma)\,\partial_{rwM}(n-\gamma+1) \qquad \ldots(5.5)$$

In matrix notation the M equations can be written as

$$\begin{bmatrix} 1 & , 1 & , 1 & , 1 & ,\ldots, 1 \\ \partial_{rw1}(1), & -\partial_{rw2}(1), & 0 & , 0 & ,\ldots, 0 \\ \partial_{rw1}(1), & 0 & ,-\partial_{rw3}(1), & 0 & ,\ldots, 0 \\ \partial_{rw1}(1), & 0 & , 0 & ,-\partial_{wr4}(1), & \ldots, 0 \\ \partial_{rw1}(1), & 0 & , 0 & , 0 & ,\ldots,-\partial_{rwM}(1) \end{bmatrix} \cdot \begin{bmatrix} Q_1(n) \\ Q_2(n) \\ Q_3(n) \\ Q_4(n) \\ \vdots \\ Q_M(n) \end{bmatrix}$$

$$= \begin{bmatrix} Q_p \\ -\sum_{\gamma=1}^{n-1} Q_1(\gamma)\partial_{rw1}(n-\gamma+1) + \sum_{\gamma=1}^{n-1} Q_2(\gamma)\partial_{rw2}(n-\gamma+1) \\ -\sum_{\gamma=1}^{n-1} Q_1(\gamma)\partial_{rw1}(n-\gamma+1) + \sum_{\gamma=1}^{n-1} Q_3(\gamma)\partial_{rw3}(n-\gamma+1) \\ \vdots \\ -\sum_{\gamma=1}^{n-1} Q_1(\gamma)\partial_{rw1}(n-\gamma+1) + \sum_{\gamma=1}^{n-1} Q_M(\gamma)\partial_{rwM}(n-\gamma+1) \end{bmatrix} \quad \ldots(5.6)$$

In particular, for time step 1

$$\begin{bmatrix} 1 & , 1 & , 1 & , 1,\ldots, 1 \\ \partial_{rw1}(1), & -\partial_{rw2}(1), & 0 & , 0,\ldots, 0 \\ \partial_{rw1}(1), & 0 & ,-\partial_{rw3}(1), & 0,\ldots, 0 \\ \vdots & \vdots & \vdots & \vdots \\ \partial_{rw1}(1), & 0 & , 0 & , 0,\ldots -\partial_{rwM}(1) \end{bmatrix} \cdot \begin{bmatrix} Q_1(1) \\ Q_2(1) \\ Q_3(1) \\ Q_4(1) \\ \vdots \\ Q_M(1) \end{bmatrix} = \begin{bmatrix} Q_p \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

...(5.7)

Hence,

$$\begin{bmatrix} Q_1(n) \\ Q_2(n) \\ Q_3(n) \\ \cdot \\ \cdot \\ Q_i(n) \\ \cdot \\ Q_M(n) \end{bmatrix} = \begin{bmatrix} 1 & , 1 & , 1 & , \ldots & ., 1 \\ \partial_{rw1}(1), & -\partial_{rw2}(1), & 0 & , \ldots & ., 0 \\ \partial_{rw1}(1), & 0 & , -\partial_{rw3}(1), & \ldots & ., 0 \\ \cdot & \cdot & \cdot & , \ldots & ., 0 \\ \cdot & \cdot & \cdot & \ldots & ., 0 \\ \partial_{rw1}(1), & 0 & , 0 & , \ldots -\partial_{rwi}(n), & ., 0 \\ \cdot & \cdot & \cdot & \ldots & . \\ \partial_{rw1}(1), & 0 & , 0 & , \ldots & ., -\partial_{rwM}(1) \end{bmatrix}^{-1}$$

$$\cdot \begin{bmatrix} Q_p \\ -\sum_{\gamma=1}^{n-1} Q_1(\gamma)\partial_{rw1}(n-\gamma+1) + \sum_{\gamma=1}^{n-1} Q_2(\gamma)\partial_{rw2}(n-\gamma+1) \\ -\sum_{\gamma=1}^{n-1} Q_1(\gamma)\partial_{rw1}(n-\gamma+1) + \sum_{\gamma=1}^{n-1} Q_3(\gamma)\partial_{rw3}(n-\gamma+1) \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ -\sum_{\gamma=1}^{n-1} Q_1(\gamma)\partial_{rw1}(n-\gamma+1) + \sum_{\gamma=1}^{n-1} Q_M(\gamma)\partial_{rwM}(n-\gamma+1) \end{bmatrix} \quad \ldots(5.8)$$

Thus

$Q_1(n), Q_2(n), \ldots\ldots, Q_M(n)$ can be solved in succession starting from time step 1.

RESULTS AND DISCUSSION

Results have been obtained for a case when the well taps three confined aquifers separated by aquicludes. For assumed values of T_1, T_2, T_3 and ϕ_1, ϕ_2, ϕ_3 discrete kernel coefficients are generated for r_w = 0.1 m. Using these discrete kernel coefficients $Q_1(n)$, $Q_2(n)$ and $Q_3(n)$ have been solved in succession starting from time step 1.

In Figs.(5.2) through (5.5), the variations of $Q_1(n)/Q_p$ and $Q_2(n)/Q_p$ with time are presented for various ratios of transmissivity and storage coefficient values.

In Figs.(5.2) and (5.3), the graphs for $\phi_1/\phi_2=1$ correspond to the case where all the aquifers have equal hydraulic diffusivity values. As seen from these two curves the variations of $Q_1(n)$ and $Q_2(n)$ and hence $Q_3(n)$ are independent of time when the aquifers have equal diffusivity values. Also when the aquifers have equal diffusivities their contributions during pumping are proportional to their respective transmissivity values. If $T_1=T_2=T_3$ and $\phi_1 = \phi_2 = \phi_3$ then individual aquifer should contribute one third of the discharge (Q_p) of the well. The same can be observed in Figs.(5.2) and (5.3).

In Figs. 5.6 and 5.7 the variation of $Q_1(n)/Q_p$ and $Q_2(n)/Q_p$ with $\frac{\phi_1 r_w^2}{4T_1 n}$ have been plotted for $T_1 : T_2 : T_3 = 1 : 2 : 4$ and $\phi_1 : \phi_2 : \phi_3 = 1 : 2 : 4$. This case also corresponds to a situation where the aquifers have equal hydraulic diffusivity values. As seen from the figures, $Q_1(n):Q_2(n):Q_3(n) = 1:2:4$.

In table 5.1 the drawdowns at r = 10 m, ($Q_p = 1000\ m^3/day$) in first, second and third aquifer having equal hydraulic diffusivity values, have been tabulated. As seen from the table, when the aquifers have equal hydraulic diffusivity values, drawdowns at any section in all the aquifers are same.

The aquifer whose hydraulic diffusivity is lowest its contribution to discharge is the highest in the beginning of pumping and as time increases its contribution decreases. However, at large time the aquifer contributions are proportional to their respective transmissivity values.

CONCLUSIONS

(a) When all the aquifers tapped have equal diffusivity values, their contributions are proportional to the respective transmissivity values.

(b) That aquifer whose hydraulic diffusivity is lowest contributes more than other aquifers during the beginning of pumping. When the time is very large i.e. at nearly steady state condition, the contributions are proportional to transmissivity values only.

(c) Using the methodology developed contributions of individual aquifers to pumping when the well is tapping several aquifers separated by aquicludes can also be evaluated.

Table 5.1 Drawdowns in aquifers having equal hydraulic diffusivities.

Time in days	Aquifer 1 $\beta_1 = \frac{T_1}{\phi_1} = \frac{70}{.001} = 70000$ m^2/day	Aquifer 2 $\beta_2 = \frac{T_2}{\phi_2} = \frac{140}{.002} = 70000$ m^2/day	Aquifer 3 $\beta_3 = \frac{T_3}{\phi_3} = \frac{280}{.004} = 70000$ m^2/day
	Drawdown in metre	Drawdown in metre	Drawdown in metre
1	0.1195370	0.1195370	0.1195370
2	0.1307910	0.1307910	0.1307910
3	0.1373749	0.1373749	0.1373749
4	0.1420465	0.1420465	0.1420465
5	0.1456701	0.1456701	0.1456701
6	0.1486309	0.1486309	0.1486309
7	0.1511342	0.1511342	0.1511342
8	0.1533027	0.1533027	0.1533027
9	0.1552154	0.1552154	0.1552154
10	0.1569265	0.1569265	0.1569265
11	0.1584743	0.1584743	0.1584743
12	0.1598873	0.1598873	0.1598873

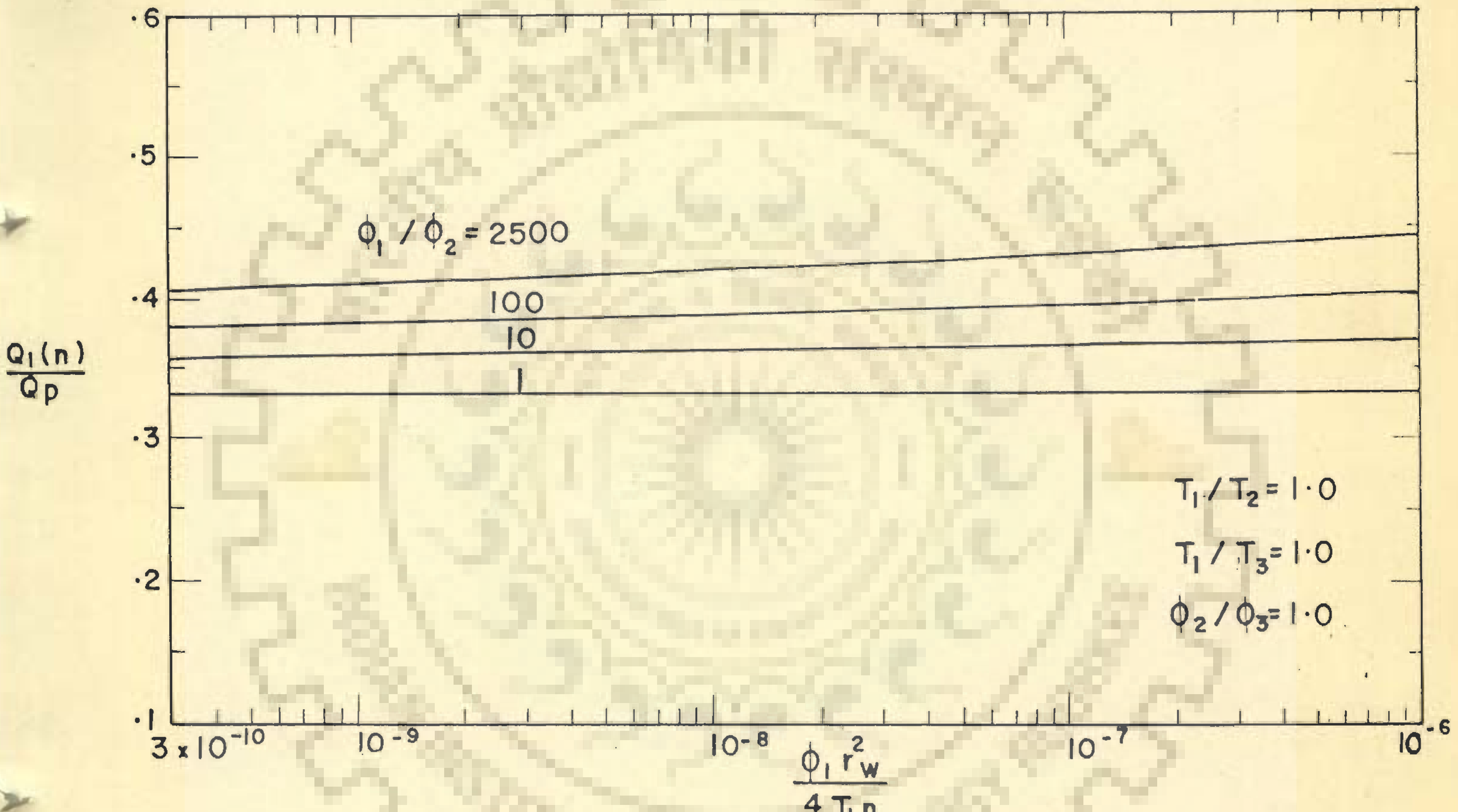

Fig. 5·2 Contribution of top aquifer to discharge at various time steps due to pumping of a well tapping three confined aquifers separated by aquicludes.

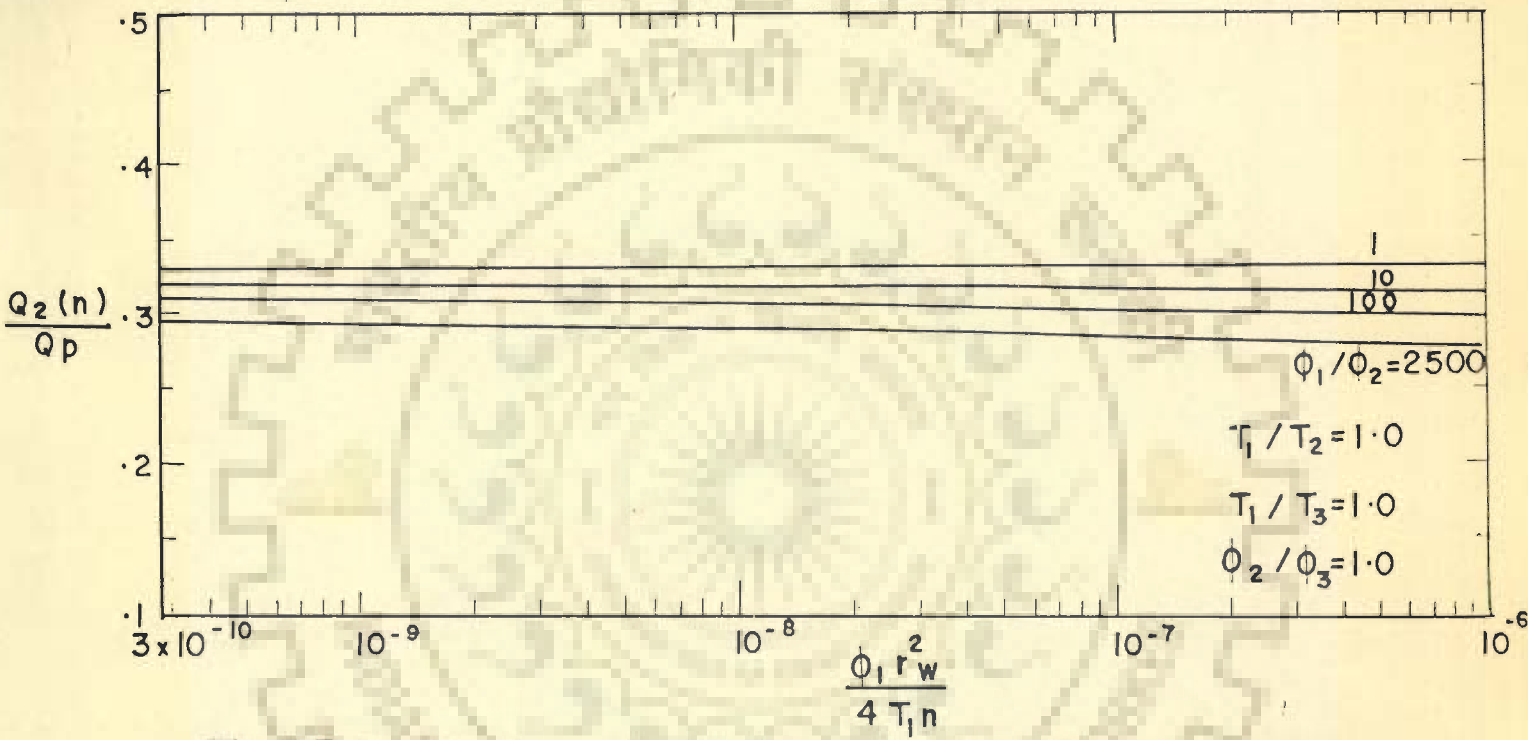

Fig. 5·3 Contribution of middle aquifer to discharge at various time steps due to pumping of a well tapping three confined aquifers separated by aquicludes.

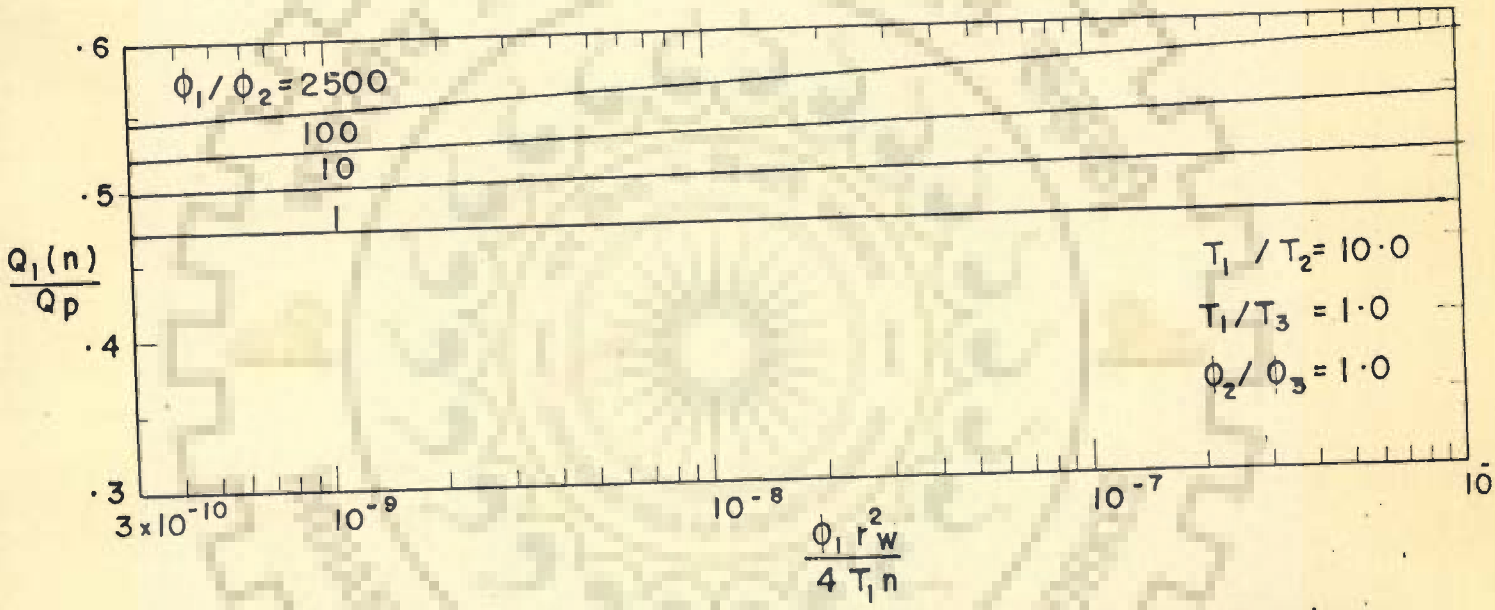

Fig. 5·4 Contribution of top aquifer to discharge at various time steps due to pumping of a well tapping three confined aquifers separated by aquicludes.

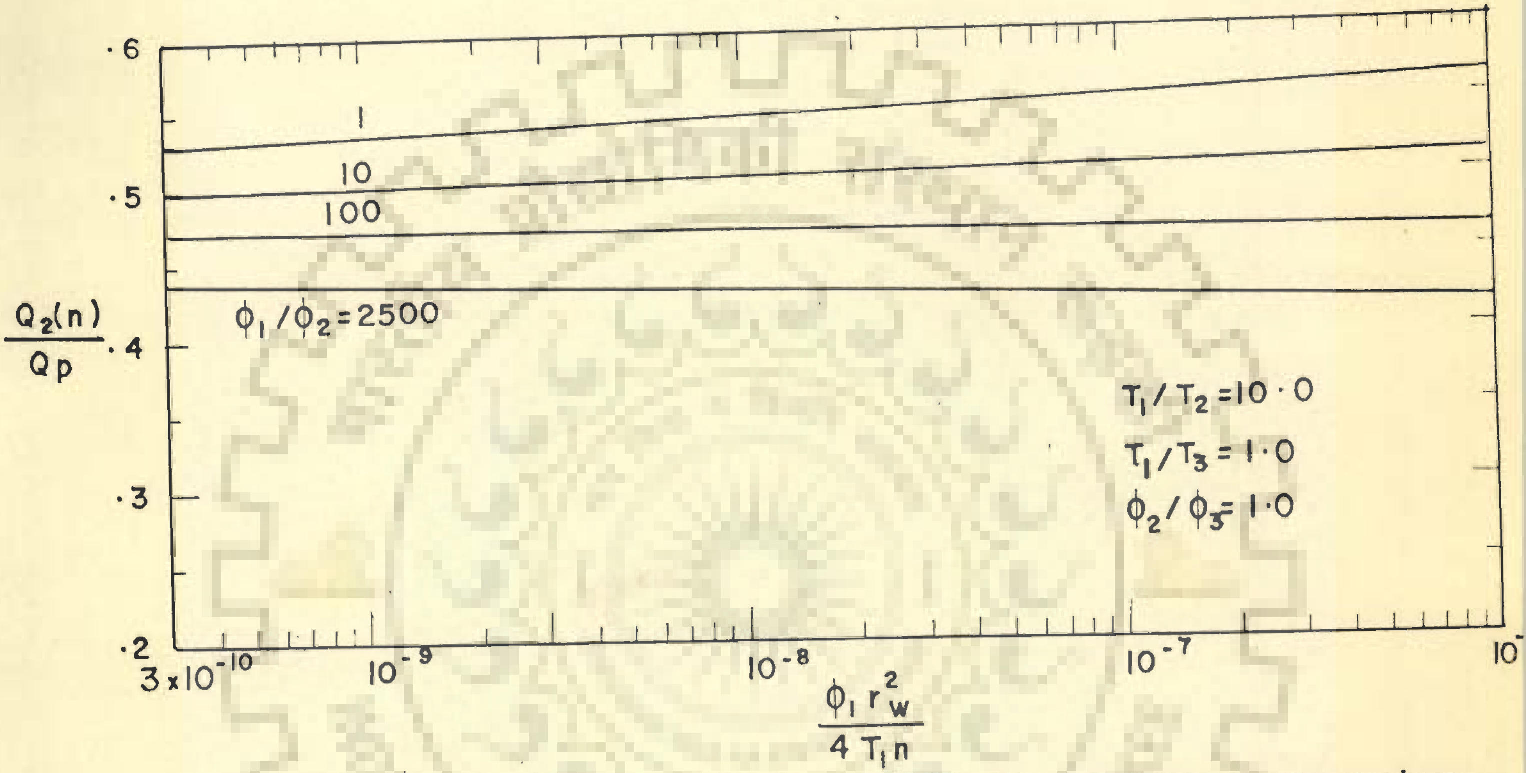

Fig. 5·5 Contribution of middle aquifer to discharge at various time steps due to pumping of a well tapping three confined aquifers separated by aquicludes.

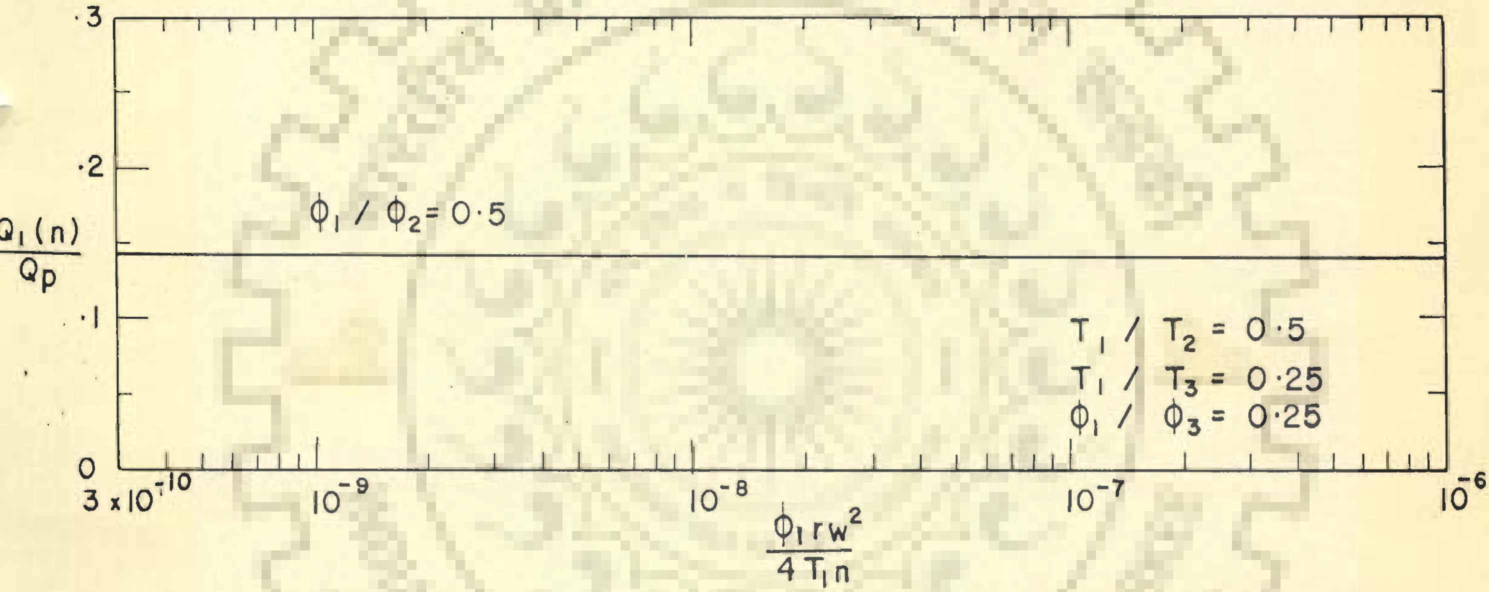

Fig. 5·6 Contribution of top aquifer to discharge at various time steps due to pumping of a well tapping three confined aquifers separated by aquicludes.

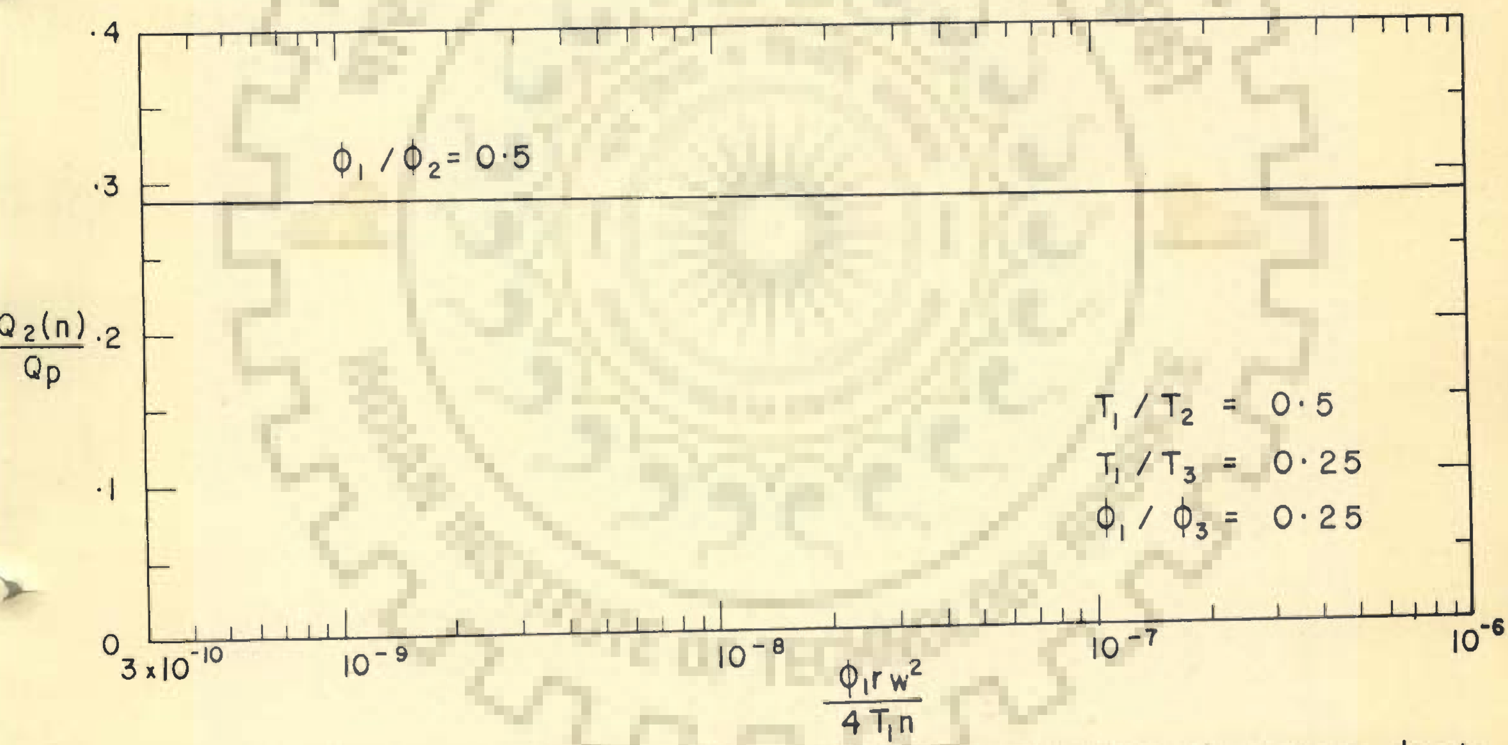

Fig. 5·7 Contribution of middle aquifer to discharge at various time steps due to pumping of a well tapping three confined aquifers separated by aquicludes.

CHAPTER 6

UNSTEADY FLOW TO A WELL TAPPING TWO AQUIFERS SEPARATED BY AN AQUITARD

INTRODUCTION

Interactions of multiple aquifers, where a single well taps only one of the aquifers have been studied by several investigators (De Glee 1930; Hantush 1956, 1960, 1964; Hantush and Jacob 1955; and Neuman and Witherspoon 1969a, 1969b etc.). Situation where a single well taps several aquifers which are separated by aquitard is not uncommon. In this chapter the contribution by individual aquifer to well discharge through the respective screen and the exchange of flow between the aquifers through the intervening aquitard have been quantitatively determined in response to a constant rate of pumping of a well tapping two aquifers. The analysis has been done using hydrologic decomposition technique and discrete pumping kernel coefficients.

STATEMENT OF THE PROBLEM

Fig.6.1 shows schematic cross section of a well tapping two aquifers. The well completely penetrates the top and the bottom aquifers. The two aquifers are

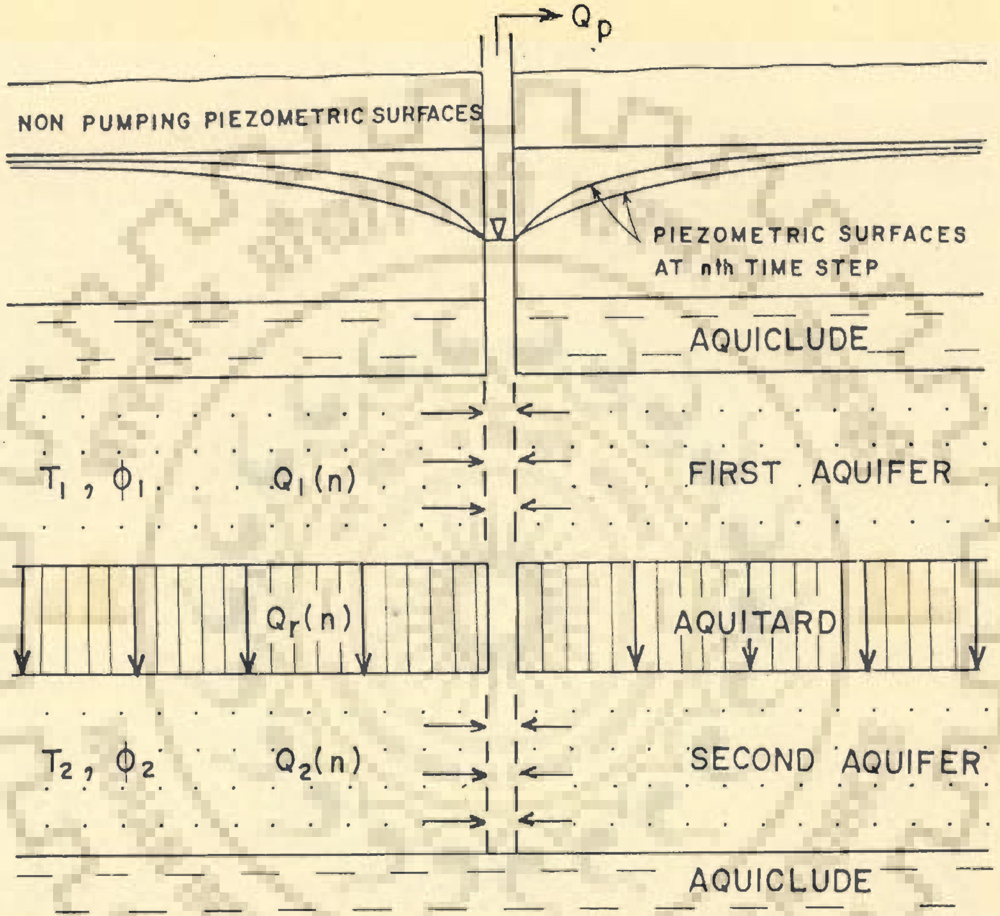

Fig. 6·1 Schematic section of a well tapping two confined aquifers separated by an aquitard.

separated by an incompressible aquitard of uniform thickness B_1. The aquifers and the aquitard are homogeneous, isotropic and infinite in areal extent. The aquifers are initially at rest condition. It is required to find the contribution of each aquifer to well discharge through the screen, the exchange of flow between the aquifers through the aquitard and the drawdown in the piezometric surface in response to a uniform rate of pumping.

ANALYSIS

The following assumptions have been made in the analysis :

i) The diameter of the well is very small and accordingly the well storage has been neglected.

ii) The aquitard is assumed to be incompressible so that no water is released from the aquitard storage.

iii) The flow is assumed to be in vertical direction in aquitard and radial in the aquifers.

The Boussinesq equation which governs the two dimensional unsteady flow in an isotropic aquifer is given by

$$\frac{\partial}{\partial x}\left[T\frac{\partial s}{\partial x}\right] + \frac{\partial}{\partial y}\left[T\frac{\partial s}{\partial y}\right] = \phi\frac{\partial s}{\partial t} - Q_w\delta_w \qquad (6.1)$$

Where

s = drawdown (measured positive downward from a horizontal datum located at the level of initial piezometric surface),

ϕ = storage coefficient (drainable or effective porosity),

T = transmissivity of aquifer,

x,y=horizontal cartesian co-ordinate,

t = time,

Q_w= instantaneous abstraction or recharge through a well (+ve for abstraction and -ve for recharge), and

δ_w= Dirac Delta function singular at well point at time τ

For a homogeneous aquifer of infinite areal extent and with no previous development the solution for drawdown at a distance r from the well due to pumping at a rate of $Q_p(c)$, is given by (Carslaw and Jaeger, 1959)

$$s(r,t) = \int_0^t \frac{Q_p(c)\, e^{-\frac{r^2}{4\beta(t-c)}}}{4\pi T\,(t-c)}\, dc; \beta = \frac{T}{\phi} \qquad (6.2)$$

Eqs. (6.1) and (6.2) are applicable for both the aquifers shown in Fig.6.1.

The composite two aquifers system has been divided into three independent subsystems with appropriate boundary conditions as shown in Fig.6.2. While decomposing the complex system to three subsystems the following assumptions have been made :

i) Beyond a sufficient distance from the well point the difference in drawdowns of piezometric surfaces is negligible. Therefore the exchange of flow between the two aquifers through the aquitard at large distance is negligible and assumed to be zero. The distance beyond which aquifers' interaction is negligible can only be ascertained after obtaining some trial numerical results.

ii) Two identical uniform square grid net works, one for each of the aquifers are established symmetrically around the pumping well. The size of each grid is $\Delta X.\Delta X$. The grid nodes are represented in a two dimensional coordinate system (p,q). The well position is defined by $p=i_o$ and $q=j_o$. A particular node is identified by p=i, q=j. An area of magnitude $(\Delta X)^2$ around any node i,j is regarded as the area of influence for the node i,j. The exchange of flow between the aquifers per unit time per unit area through the aquitard

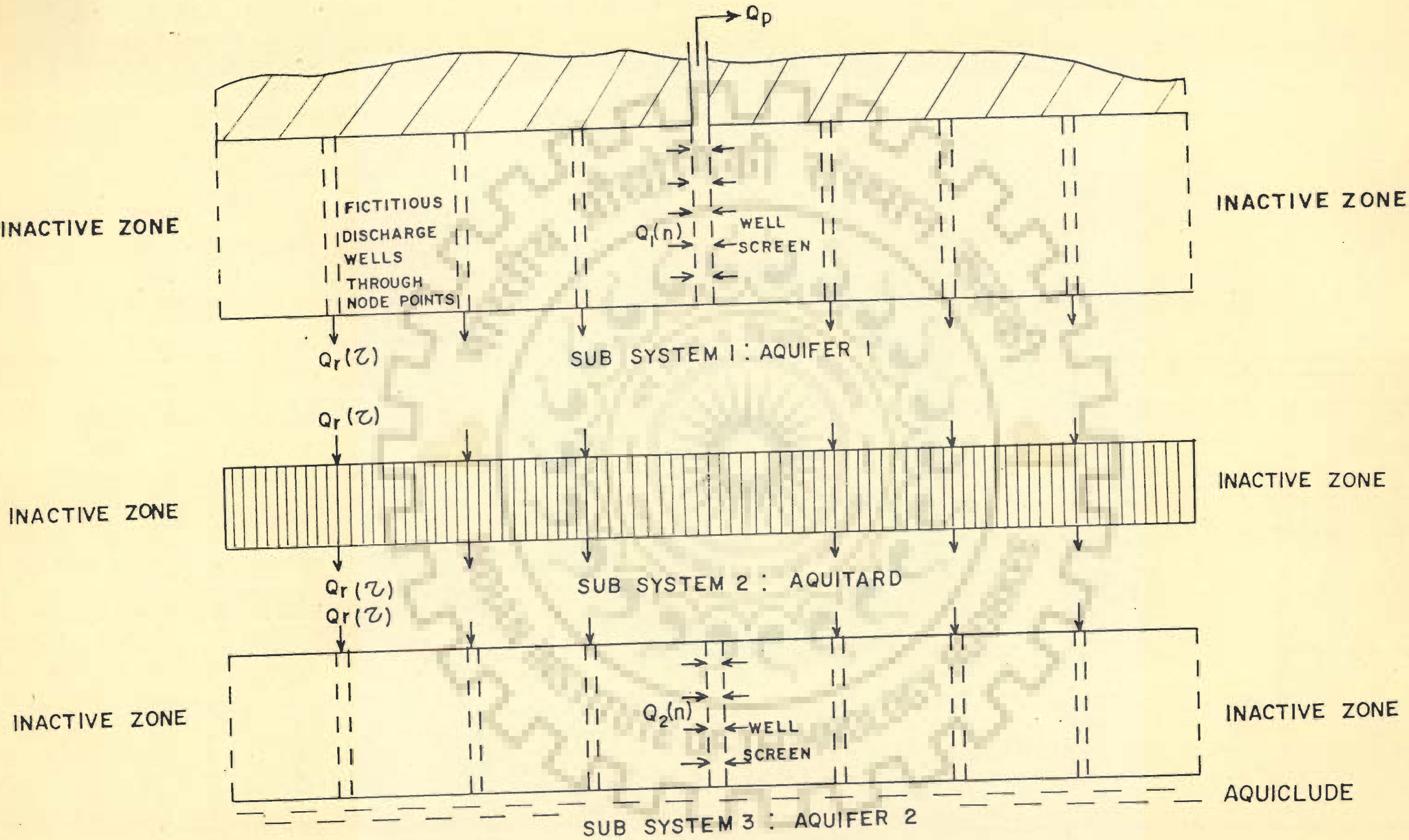

Fig. 6·2 Decomposition of the system into three subsystem.

at node i,j at time t, is directly propotional to the difference in the piezometric surfaces at node i,j at time t and inversely propotional to the thickness of the aquitard; the constant of propotionality being the vertical hydraulic conductivity of the aquitard.

iii) Formulation of the problem has been done with the assumption that recharge takes place from aquifer 1 to aquifer 2. Therefore, in case the calculated recharge from aquifer 1 at any node has a -ve sign, it is to be regarded that recharge is taking place from aquifer 2 to aquifer 1.

In formulating time drawdown equation for a node i,j in aquifer 1, it is assumed that the discharge is taking place uniformly from the zone of influence of node i,j and at all other nodes the discharge is affected through fictitious discharge wells, one operating at each nodal point. Similarly for aquifer 2, it is assumed that the recharge is uniformly distributed over the area of influence of node i,j and at all other nodes the recharge is taking place by fictitious recharge wells, one operating at each of the nodal points. The drawdown in piezometric surface at time t in aquifer 1 depends on the dischargestaking place from all nodes of aquifer 1 upto time t besides on its own contribution to discharge of the well through the screen up to time t. Similarly the drawdown in piezometric surface in aquifer 2 at time t, depends on all the recharges taking place at all nodes of aquifer 2 and on its own contribution to discharge of the well through the screen up to time t.

Boussinesq equation being linear, the drawdown at a point due to excitation at number of wells is equal to sum of the drawdowns due to excitation at each individual well. Making use of Eq.(6.2) and the principle of superposition, drawdown in aquifer 1 at node i,j at time t can be written as

$$s_1(i,j,t) = \left[\int_0^t \frac{Q_1(c)\, e^{-\frac{((i-i_0)^2+(j-j_0)^2)(\Delta X)^2}{4\beta_1(t-c)}}}{4\pi T_1 (t-c)}\, dc\right]$$

$$+\left[\sum_{q=1}^{J}\sum_{\substack{p=1 \\ p,q \neq i,j}}^{I} \int_0^t \frac{Q_r(p,q,c)}{4\pi T_1(t-c)}\, e^{-\frac{((i-p)^2+(j-q)^2)(\Delta X)^2}{4\beta_1(t-c)}}\, dc\right]$$

$$+\left[4\int_0^t \int_0^{\frac{\Delta X}{2}} \int_0^{\frac{\Delta X}{2}} \frac{Q_r(i,j,c)\, e^{-\frac{(x^2+y^2)}{4\beta_1(t-c)}}}{4\pi T_1(\Delta X)^2(t-c)}\, dc\, dx\, dy\right] \qquad (6.3)$$

Where

$s_1(i,j,t)$ = drawdown in piezometric surface in aquifer 1 at node i,j at time t,

$Q_1(c)$ = discharge per unit time from aquifer 1 through well screen at time c ,

$Q_r(p,q,c)$ = recharge per unit time from aquifer 1 to aquifer 2 at node p,q at time c ,

$Q_r(i,j,c)$ = recharge per unit time from aquifer 1 to aquifer 2 at node i,j at time c ,

T_1 = transmissivity of aquifer 1,

ϕ_1 = storage coefficient of aquifer 1,

β_1 = T_1/ϕ_1,

ΔX = grid size,

t, = time measured from onset of pumping,

x,y = dummy variables,

I = maximum value of p, and

J = maximum value of q.

In Eq.(6.3) the expression on right hand side within 1st square bracket represents part of drawdown due to discharges of aquifer 1 through well screen. The expression within the 2nd square bracket represents part of drawdown due to discharge taking place from aquifer 1 to aquifer 2 through the intervening aquitard at all nodes but at node i,j. The expression within third square bracket represents part of the drawdown due to uniform discharge taking place from aquifer 1, from the area of influence of node i,j through the intervening aquitard. To obtain the component of drawdown due to discharge through the area of influence of node i,j the origin of x,y co-ordinate axes is chosen at the center of the grid. $\frac{Q_r(i,j,c)}{(\Delta X)^2}$ dx dy is the discharge per unit time taking place from an elemental area of dx.dy at time c .

Similarly drawdown in aquifer 2 at node i,j at time t is given by

$$s_2(i,j,t) = \left[\int_0^t \frac{Q_2(c)}{4\pi T_2} \frac{e^{-\frac{((i-i_o)^2+(j-j_o)^2)(\Delta X)^2}{4\beta_2(t-c)}}}{(t-c)} dc \right]$$

$$-\left[\sum_{q=1}^{J} \sum_{p=1}^{I} \int_0^t \frac{Q_r(p,q,c)\, e^{-\frac{((i-p)^2+(j-q)^2)(\Delta X)^2}{4\beta_2(t-c)}}}{4\pi T_2(t-c)} dc \right]$$

$$p,q \neq i,j$$

$$-\left[4 \int_0^t \int_0^{\frac{\Delta X}{2}} \int_0^{\frac{\Delta X}{2}} \frac{Q_r(i,j,c)\, e^{-\frac{(x^2+y^2)}{4\beta_2(t-c)}}}{4\pi T_2(\Delta X)^2(t-c)} dx.dy.dc \right] \qquad (6.4)$$

Where

$s_2(i,j,t)$ = drawdown in piezometric surface in aquifer 2 at node i,j at time t,

$Q_2(c)$ = discharge per unit time from aquifer 2 through well screen at time c,

T_2 = transmissivity of aquifer 2,

ϕ_2 = storage coefficient of aquifer 2 and

β_2 = T_2/ϕ_2.

In Eq.(6.4) the expression on right hand side within 1st square bracket represents drawdown due to aquifer's discharge through well screen. The expression within 2nd square bracket represents rise in piezometric surface due to recharge from aquifer 1 taking place at all nodal points but for recharge at node i,j. The expression within 3rd square bracket represents rise in piezometric surface due to uniform recharge taking place through the area of influence of node i,j.

As it is assumed that the flow is radial in both the aquifers, the equipotential lines are therefore vertical in both the aquifers. In other words respective hydrostatic conditions prevail in vertical directions at a section in both the aquifers. Due to difference in hydraulic heads, across the aquitard, flow takes place from point of higher head to point of lower head through the aquitard. Applying Darcy's law the quantity of flow passing through the area of influence of node i,j at time t can be expressed as

$$Q_r(i,j,t) = \frac{K_1}{B_1} [s_2(i,j,t)-s_1(i,j,t)] (\Delta X)^2 \qquad (6.5)$$

Where

K_1 = coefficient of permeability of the aquitard in vertical direction, and

B_1 = thickness of the aquitard.

Substituting the expression for $s_1(i,j,t)$ and $s_2(i,j,t)$ given by Eqs.(6.3) and (6.4) respectively in Eq.(6.5)

$$Q_r(i,j,t) = \frac{K_1(\Delta X)^2}{B_1} \Bigg[\int_0^t \frac{Q_2(c)\; e^{-\frac{((i-i_o)^2+(j-j_o)^2)(\Delta X)^2}{4\beta_2(t-c)}}}{4\pi T_2(t-c)} dc$$

$$- \sum_{\substack{q=1 \\ p,q\neq i,j}}^{J} \sum_{p=1}^{I} \int_0^t \frac{Q_r(p,q,c)\; e^{-\frac{((i-p)^2+(j-q)^2)(\Delta X)^2}{4\beta_2(t-c)}}}{4\pi T_2(t-c)} dc$$

$$-4 \int_0^t \int_0^{\frac{\Delta X}{2}} \int_0^{\frac{\Delta X}{2}} \frac{Q_r(i,j,c)\; e^{-\frac{(x^2+y^2)}{4\beta_2(t-c)}}}{(\Delta X)^2 4\pi T_2 (t-c)} dc\, dx.dy$$

$$- \int_0^t \frac{Q_1(c)\; e^{-\frac{((i-i_o)^2+(j-j_o)^2)(\Delta X)^2}{4\beta_1(t-c)}}}{4\pi T_1(t-c)} dc$$

$$- \sum_{\substack{q=1 \\ p,q\neq i,j}}^{J} \sum_{p=1}^{I} \int_0^t \frac{Q_r(p,q,c)\; e^{-\frac{((i-p)^2+(j-q)^2)(\Delta X)^2}{4\beta_1(t-c)}}}{4\pi T_1(t-c)} dc$$

$$- 4 \int_0^t \int_0^{\frac{\Delta X}{2}} \int_0^{\frac{\Delta X}{2}} \frac{Q_r(i,j,c)}{(\Delta X)^2 4\pi T_1(t-c)} e^{\frac{-(x^2+y^2)}{4\beta_1(t-c)}} dx.dy.dc \qquad (6.6)$$

Eq.(6.6) is a linear integral equation involving the unknowns $Q_1(t)$, $Q_2(t)$, $Q_r(p,q,t)$. Dividing the time span into discrete time steps and assuming that within each time step, the recharge rate from aquifer 1 and the aquifers' contributions through well screens are separately constant but vary from time step to time step, Eq.(6.6) can be written as

$$Q_r(i,j,n) = \frac{K_1(\Delta X)^2}{B_1} \Big[\sum_{\gamma=1}^{n} Q_2(\gamma)\, \partial_2(i,j;\ i_o,j_o;\ n-\gamma+1)$$

$$- \sum_{\gamma=1}^{n} \sum_{q=1}^{J} \sum_{\substack{p=1 \\ p,q\neq i,j}}^{I} Q_r(p,q,\gamma)\, \partial_2(i,j;\ p,q;\ n-\gamma+1)$$

$$- \sum_{\gamma=1}^{n} Q_r(i,j,\gamma)\, \partial_2(i,j;\ i,j;\ n-\gamma+1)$$

$$- \sum_{\gamma=1}^{n} Q_1(\gamma)\, \partial_1(i,j;\ i_o,j_o;\ n-\gamma+1)$$

$$- \sum_{\gamma=1}^{n} \sum_{q=1}^{J} \sum_{\substack{p=1 \\ p,q\neq i,j}}^{I} Q_r(p,q,\gamma)\, \partial_1(i,j;\ p,q;\ n-\gamma+1)$$

$$- \sum_{\gamma=1}^{n} Q_r(i,j,\gamma)\ \partial_1(i,j;\ i,j;\ n-\gamma+1)] \qquad (6.7)$$

Where

$Q_r(i,j,n)$ = recharge rate through the area of influence of node i,j during the nth unit time period;

$$\partial_2(i,j;\ i_o,j_o;\ m) = \frac{1}{4\pi T_2}[E_1(\frac{R^2}{4\beta_2 m}) - E_1(\frac{R^2}{4\beta_2(m-1)})]$$

= discrete pumping kernel for draw-down in aquifer 2 ;

$$R^2 = [(i-i_o)^2+(j-j_o)^2]\ (\Delta X)^2 ;$$

$$\partial_2(i,j;\ i,j;\ m) = \int_0^1 \frac{1}{\phi_2(\Delta X)^2}[\ \mathrm{erf}\ (\frac{\Delta X}{4\sqrt{\beta_2(m-c)}})]^2 dc ;$$

$$\partial_2(i,j;\ p,q;\ m) = \frac{1}{4\pi T_2}[E_1(\frac{R_1^2}{4\beta_2 m}) - E_1(\frac{R_1^2}{4\beta_2(m-1)})] ;$$

$p,q \neq i,j$

$$R_1^2 = ((i-p)^2+(j-q)^2)\ (\Delta X)^2 ;$$

$Q_2(\gamma)$ = discharge from aquifer 2 through the well screen during the γth unit time period,

$Q_1(\gamma)$ = discharge from aquifer 1 through the well screen during the γth unit time period,

$$\partial_1(i,j;\ i_o,j_o;\ m) = \frac{1}{4\pi T_1}[E_1(\frac{R^2}{4\beta_1 m})-E_1(\frac{R^2}{4\beta_1(m-1)})]$$

= discrete pumping kernel for drawdown in aquifer 1 ;

$$\partial_1(i,j;\ p,q;\ m) = \frac{1}{4\pi T_1}[E_1(\frac{R_1^2}{4\beta_1 m})- E_1(\frac{R_1^2}{4\beta_1(m-1)})] ;$$

$$\partial_1(i,j;\ i,j;\ m) = \int_o^1 \frac{1}{\phi_1(\Delta X)^2}[\mathrm{Erf}\ (\frac{\Delta X}{4\sqrt{\beta_1(m-c)}})]^2 dc ;$$

$E_1(X) = \int_X^\infty \frac{e^{-u}}{u} du$ = an exponential integral and

$\mathrm{erf}(X) = \frac{2}{\sqrt{\pi}} \int_o^X e^{-u^2} du$ = error function.

Eq.(6.7) can be expanded and written in the following form :

$$\frac{B_1}{K_1(\Delta X)^2} Q_r(i,j,n) - Q_2(n)\ \partial_2(i,j;\ i_o,j_o;\ 1)$$

$$+ \sum_{\substack{q=1\\p,q \neq i,j}}^{J} \sum_{p=1}^{I} Q_r(p,q,n)\ \partial_2(i,j;\ p,q;\ 1)$$

$$+ Q_r(i,j,n)\ \partial_2(i,j;\ i,j;\ 1)$$

$$+ Q_1(n)\ \partial_1(i,j;\ i_o,j_o;\ 1)$$

$$+ \sum_{\substack{q=1\\p,q \neq i,j}}^{J} \sum_{p=1}^{I} Q_r(p,q,n)\ \partial_1(i,j;\ p,q;\ 1)$$

$$+ Q_r(i,j,n)\, \partial_1(i,j;\ i,j;\ 1)$$

$$= [\sum_{\gamma=1}^{n-1} Q_2(\gamma)\, \partial_2(i,j;\ i_o,j_o;\ n-\gamma+1)$$

$$- \sum_{\gamma=1}^{n-1} \sum_{q=1}^{J} \sum_{p=1}^{I} Q_r(p,q,\gamma)\, \partial_2(i,j;\ p,q;\ n-\gamma+1)$$

$$p,q \neq i,j$$

$$- \sum_{\gamma=1}^{n-1} Q_r(i,j,\gamma)\, \partial_2(i,j;\ i,j;\ n-\gamma+1)$$

$$- \sum_{\gamma=1}^{n-1} Q_1(\gamma)\, \partial_1(i,j;\ i_o,\ j_o;\ n-\gamma+1)$$

$$- \sum_{\gamma=1}^{n-1} \sum_{q=1}^{J} \sum_{p=1}^{I} Q_r(p,q,\gamma)\, \partial_1(i,j;\ p,q;\ n-\gamma+1)$$

$$p,q \neq i,j$$

$$- \sum_{\gamma=1}^{n-1} Q_r(i,j,\gamma)\, \partial_1(i,j;\ i,j;\ n-\gamma+1)] \qquad (6.8)$$

Similar I x J number of equations can be written, one for each of the nodal points. The total number of unknowns during any unit time period are the quantities of recharge at I x J nodal points and the aquifers' discharges through their respective well screens. The recharges at the nodal points and the aquifers' discharges through well screens are known for all previous time steps. Thus there are I x J + 2 number of unknowns during any unit time period. Two more equations can be written considering the facts that at any time the pumping rate is equal to sum of the rate of discharge by the aquifers through the respective well screen, and the drawdown in piezometric surfaces in both the aquifers at the well point are equal. Thus

$$Q_1(n) + Q_2(n) = Q_p \;, \tag{6.9}$$

Let r_w is the radius of the well. Drawdown in the piezometric surface in aquifer 1 at the well face at the end of nth unit time can be expressed as :

$$\begin{aligned} s_1(r_w,n) = & \sum_{\gamma=1}^{n} Q_1(\gamma)\, \partial_{1w}(n-\gamma+1) \\ & + \sum_{\gamma=1}^{n} \sum_{q=1}^{J} \sum_{\substack{p=1 \\ p,q\neq i_o,j_o}}^{I} Q_r(p,q,\gamma)\, \partial_1(i_o,j_o;\, p,q,\, n-\gamma+1) \\ & + \sum_{\gamma=1}^{n} Q_r(i_o,j_o,\gamma)\, \partial_1(i_o,j_o;\, i_o,j_o;\, n-\gamma+1) \end{aligned} \tag{6.10}$$

where $\partial_{1w}(m) = \frac{1}{4\pi T_1}[E_1(\frac{r_w^2}{4\beta_1 m})-E_1(\frac{r_w^2}{4\beta_1(m-1)})]$

Similarly drawdown in the piezometric surface in aquifer 2 at the end of nth unit time can be expressed as :

$$\begin{aligned} s_2(r_w,n) = & \sum_{\gamma=1}^{n} Q_2(\gamma)\, \partial_{2w}(n-\gamma+1) \\ & - \sum_{\gamma=1}^{n} \sum_{q=1}^{J} \sum_{\substack{p=1 \\ p,q\neq i_o,j_o}}^{I} Q_r(p,q,\gamma)\, \partial_2(i_o,j_o;\, p,q;\, n-\gamma+1) \end{aligned}$$

$$- \sum_{\gamma=1}^{n} Q_r(i_o,j_o,\gamma)\ \partial_2(i_o,j_o;\ i_o,j_o;\ n-\gamma+1) \qquad (6.11)$$

Where

$$\partial_{2w}(m) = \frac{1}{4\pi T_2}\left[E_1\left(\frac{r_w^2}{4\beta_2 m}\right)-E_1\left(\frac{r_w^2}{4\beta(m-1)}\right)\right]$$

Since,

$$s_1(r_w,n) = s_2(r_w,n)$$

Therefore,

$$\sum_{\gamma=1}^{n} Q_1(\gamma)\quad \partial_{1w}(n-\gamma+1)$$

$$+ \sum_{\gamma=1}^{n}\sum_{\substack{q=1\\ p,q\neq i_o,j_o}}^{J}\sum_{p=1}^{I} Q_r(p,q,\gamma)\ \partial_1(i_o,j_o;\ p,q;\ n-\gamma+1)$$

$$+ \sum_{\gamma=1}^{n} Q_r(i_o,j_o,\gamma)\ \partial_1(i_o,j_o;\ i_o,j_o;\ n-\gamma+1)$$

$$= \sum_{\gamma=1}^{n} Q_2(\gamma)\ \partial_{2w}(n-\gamma+1)$$

$$- \sum_{\gamma=1}^{n}\sum_{\substack{q=1\\ p,q\neq i_o,j_o}}^{J}\sum_{p=1}^{I} Q_r(p,q,\gamma)\ \partial_2(i_o,j_o;\ p,q;\ n-\gamma+1)$$

$$- \sum_{\gamma=1}^{n} Q_r(i_o,j_o,\gamma)\ \partial_2(i_o,j_o;\ i_o,j_o;\ n-\gamma+1) \qquad (6.12)$$

Expanding and rearranging Eq.(6.12) is simplified to

$$Q_1(n)\ \partial_{1w}(1)$$

$$+ \sum_{q=1}^{J} \sum_{\substack{p=1 \\ p,q \neq i_o,j_o}}^{I} Q_r(p,q,n)\ \partial_1(i_o,j_o;\ p,q;\ 1)$$

$$+ \ Q_r(i_o,j_o,n)\ \partial_1(i_o,j_o;\ i_o,j_o;1)$$

$$- \ Q_2(n)\ \partial_{2w}(1)$$

$$+ \sum_{q=1}^{J} \sum_{\substack{p=1 \\ p,q \neq i_o,j_o}}^{I} Q_r(p,q,n)\ \partial_2(i_o,j_o;\ p,q;\ 1)$$

$$+ \ Q_r(i_o,j_o,n)\ \partial_2(i_o,j_o;\ i_o,j_o;\ 1)$$

$$= - \sum_{\gamma=1}^{n-1} Q_1(\gamma)\ \partial_{1w}(n-\gamma+1)$$

$$- \sum_{\gamma=1}^{n-1} \sum_{q=1}^{J} \sum_{\substack{p=1 \\ p,q \neq i_o,j_o}}^{I} Q_r(p,q,\gamma)\ \partial_1(i_o,j_o;\ p,q;\ n-\gamma+1)$$

$$- \sum_{\gamma=1}^{n-1} Q_r(i_o,j_o,\gamma)\ \partial_1(i_o,j_o;\ i_o,j_o;\ n-\gamma+1)$$

$$+ \sum_{\gamma=1}^{n-1} Q_2(\gamma)\ \partial_{2w}(n-\gamma+1)$$

$$- \sum_{\gamma=1}^{n-1} \sum_{q=1}^{J} \sum_{\substack{p=1 \\ p,q \neq i_o,j_o}}^{I} Q_r(p,q,\gamma)\ \partial_2(i_o,j_o;\ p,q;\ n-\gamma+1)$$

$$- \sum_{\gamma=1}^{n-1} Q_r(i_o,j_o,\gamma)\ \partial_2(i_o,j_o;\ i_o,j_o;\ n-\gamma+1) \qquad (6.13)$$

Solving the (I x J) + 2 set of algebraic linear equations i.e. Eqs.(6.8),(6.9) and (6.13) in succession starting from time step 1, the recharges at (I x J) nodal points and the aquifers' contributions $Q_1(n)$ and $Q_2(n)$ during time step n can be solved.

RESULTS AND DISCUSSION

An uniform square grid net work having 49 nodes as shown in Fig.6.4 has been adopted. The nodes are numbered as shown in the figure. Let the discrete kernel be designated as $\delta_i(c,e,n)$ where i stands for the ith aquifer, c is the observation point, e is the excitation point and n is time step. A unit pulse excitation has been given at node 1. At all the nodes lying along and below the diagonal the responses have been recorded for n number of time steps for assumed values of aquifer parameters. Making use of these values the discrete kernels $\delta_i(o,e,n)$,c = 1,49 and e = 1,49 have been obtained for different values of n. This could be done as the aquifer is assumed to be homogeneous and isotropic. This procedure of generating the discrete kernel coefficients which takes minimum computer time has been adopted from Basu (1980).

The well being located at the centre of the square grid network i.e. node 25, there are nine sets of identical nodes. The quantities of recharge at any time step from these identical nodes in a set are equal. The identical values recognised are as follows :

$Q_r(1,n) = Q_r(7,n) = Q_r(43,n) = Q_r(49,n)$;

$Q_r(2,n) = Q_r(6,n) = Q_r(8,n) = Q_r(14,n) = Q_r(36,n) =$
$Q_r(42,n) = Q_r(44,n) = Q_r(48,n)$;

$Q_r(3,n) = Q_r(5,n) = Q_r(15,n) = Q_r(21,n) = Q_r(29,n) =$
$Q_r(35,n) = Q_r(45,n) = Q_r(47,n)$;

$Q_r(14,n) = Q_r(22,n) = Q_r(28,n) = Q_r(46,n)$;

$Q_r(9,n) = Q_r(13,n) = Q_r(37,n) = Q_r(41,n)$;

$Q_r(10,n) = Q_r(12,n) = Q_r(16,n) = Q_r(20,n) = Q_r(30,n) =$
$Q_r(34,n) = Q_r(38,n) = Q_r(40,n)$;

$Q_r(11,n) = Q_r(23,n) = Q_r(27,n) = Q_r(39,n)$;

$Q_r(17,n) = Q_r(19,n) = Q_r(31,n) = Q_r(33,n)$;

$Q_r(18,n) = Q_r(24,n) = Q_r(26,n) = Q_r(32,n)$.

Thus it is only necessary to solve the recharge at nodes 1,8,15,22,9,16,23,17,24,25 besides $Q_1(n)$ and $Q_2(n)$ at any time step. The equations can be written in the following matrix form :

$$[A] \cdot \begin{bmatrix} Q_r(1,n) \\ Q_r(8,n) \\ Q_r(15,n) \\ Q_r(22,n) \\ Q_r(9,n) \\ Q_r(16,n) \\ Q_r(23,n) \\ Q_r(17,n) \\ Q_r(24,n) \\ Q_r(25,n) \\ Q_1(n) \\ Q_2(n) \end{bmatrix} = [B]$$

Where [A] is a 12x12 matrix and [B] is a 12x1 matrix. The elements of the matrix [A] and [B] are given in Appendix-I. Only by inverting the matrix [A] once and knowing the column matrix [B] at different time steps the recharge values at the above mentioned ten nodes and $Q_1(n)$ and $Q_2(n)$ can be solved in succession starting from time step 1.

In order to determine the grid size for obtaining reasonably accurate results different values of grid size (Δx) have been tried, starting from 100 m to 700 m. Table 6.1 gives the values of $Q_1(n)$ for different values of ΔX at different time steps for two values of leakage factor. If the grid size is increased indiscreminately the recharge and the drawdowns are being averaged over a larger area. Therefore a grid size of 300 m has been adopted. More accurate results can be obtained by increasing the grid points depending upon the capacity of the computer available or a variable grid size i.e. finer grid near the well point and coarser grid at farther region from the well point may be adopted.

In Fig.6.6 through 6.11 variations of $Q_1(n)/Q_p$ with the nondimensional factor u_1 ($u_1 = \frac{\phi \, r_w^2}{4T_1 n}$) have been presented for different values of leakage factor. The leakage factor L is defined here as $L = \sqrt{\bar{T} C}$ where $C = B_1/K_1$, B_1 being the thickness of the aquitard ; K_1 its vertical permeability and $\bar{T}$ is the harmonic mean transmissivity of the aquifers given by $\frac{2T_1 \times T_2}{T_1 + T_2}$. However the geometric mean or the arithmatic mean values of the transmissivities of the two aquifers could also be used to define the leakage factor.

For an aquiclude the leakage factor L tends to infinite. The variation of $Q_1(n)/Q_p$ with $\phi r_w^2/4T_1 n$ for L = infinite has also been plotted in Fig 6.6 for the purpose of comparison . This result has been obtained from Fig.4.3 of chapter 4 which deals with the situation when the aquifers are separated by aquiclude. As seen from the figure, for u_1 upto to the value of $5x10^{-9}$ i.e. during the early part of pumping an aquitard having leakage factor L = 15275 m nearly behaves like an aquiclude. It may be noted here that the results obtained by two approaches, one presented in this chapter and the other in chapter 4 compare well.

It is observed from Figs. 5.6 through 5.11 that the aquifer whose hydraulic diffusivity is lower its contribution to well discharge through the screen decreases as pumping continues. Conversely, the aquifer whose hydraulic diffusivity is higher its contribution increases with pumping.

If pumping continues at a constant rate for an indefinite period, the limit $Q_1(n)/Q_2(n)$ tends to T_1/T_2. This has been proved in chapter 4 using Thiem (1906) equation. This fact is also being observed in the figures. For given ratios of T_1/T_2 and ϕ_1/ϕ_2, as the leakage factor decreases the near steady state condition approaches comparatively at shorter time. In Fig.5.7 for $T_1/T_2 = 1$, $\phi_1/\phi_2 = 10$ and L = 5915 m ; $Q_1(n)/Q_p = 0.524$ at $\phi_1 r_w^2/4T_1 n = 3x10^{-10}$. Where as, when the leakage factor L = 19 m the value of $Q_1(n)/Q_p = 0.5$ at $\phi_1 r_w^2/4T_1 n = 2x10^{-9}$. Thus, for the lower value of leakage factor the near steady state condition has been attained earlier.

In Fig.6.8 the results have been presented for the variation of $Q_1(n)/Q_p$ with u_1. The results presented are for $T_1/T_2 = 10$, $\phi_1/\phi_2 = 10$. These results correspond to a case when the tapped aquifers have equal hydraulic diffusivity values. As seen from the figure, when the aquifers have equal hydraulic diffusivities the contributions to well discharge by the individual aquifer through their respective well screen are independent of time and are proportional to their transmissivity values. $Q_1(n)/Q_p$ is equal to 0.909 for $T_1/T_2 = 10$ irrespective of the value of leakage factor. In such cases no exchange of flow takes place through the intervening aquitard and drawdown in the piezometric surface at a particular section in both the aquifers are same at all the times during pumping. In Table 6.2 the drawdowns in aquifer 1 and 2 at r = 300 m,600 for the situation when the aquifers have equal hydraulic diffusivity values are presented.

Table 6.3 gives the recharge rates under two different hydrogeological settings. In the first case aquifer parameters have the following values :

T_1 = 700.0 m^2/day, ϕ_1 = .02

T_2 =1400.0 m^2/day, ϕ_2 = .06

K_1 = 9.333m/day, B_1 = 1.0 m

In the second case the parameters have the following values.

T_1 = 350.0 m^2/day, ϕ_1 = .01

T_2 = 700.0 m^2/day, ϕ = .03

K_1 = 4.666m/day, B_1 = 1.0 m

For both the cases Q_p = 10 m^3/day, r_w = 0.1 m, ΔX = 300.0 m, and L = 10 m.

For such coincidance in the values of aquifer parameters the recharge rates are identical. It may be noted that the corresponding hydraulic diffusivities of the aquifers are equal (i.e. T_1/ϕ_1 of case I equals to T_1/ϕ_1 of case II, T_2/ϕ_2 of case I equals to T_2/ϕ_2 of case II) besides their leakage factor values. (both have leakage factor values=10 m)

Figs.6.12 through 6.16 shows the variation of $Q_R(n)/Q_p$ with u_1 for ratios of T_1/T_2 = 0.5,1,10 and ϕ_1/ϕ_2 = 10,100 for different values of leakage factors. $Q_R(n)$ is the total recharge from an area of 4.41×10^6 sq.m.

As seen from the figures, with higher values of leakage factor the recharge from the above area stablises when $u_1 < 10^{-8}$. However for lower values of leakage factors the recharge from this area increases starting from zero to certain value and then decreases.

For leakage factor L = 1871 m/as seen in Figs.6.7, 6.13, the first aquifer contributes 52.5% of pumping through screens, besides contributing 6.5% of pumping as recharge to the second aquifer.

Results have also been presented for a two aquifer system separated by an aquitard, the top aquifer being unconfined with delayed yield characteristics.

CONCLUSIONS

a) When a well taps two aquifers separated by an aquitard the leakage factor is to be defined as $L = \sqrt{\bar{T}C}$ where $\bar{T}$ is the mean transmissivity value. The mean transmissivity may either be a geometric mean or a harmonic mean or an arithmatic mean.

In different two aquifer aquitard systems if the corresponding hydraulic diffusivity values are equal (i.e. β_1, β_2 of one set are equal β_1, β_2

of the other) and their leakage factors are same, then the recharge values are also same.

b) If the aquifers have equal diffusivity values, the contributions by each of the aquifers through the well screens during pumping at a constant rate are independent of time and proportional to the respective transmissivity values. In such a case there is no exchange of flow through the intervening aquitard irrespective of the magnitude of leakage factor and the drawdowns at any section in both the aquifers are same.

c) Aquifer whose hydraulic diffusivity is lower its contribution to well discharge through the screen decreases as pumping continues. Conversely, the aquifer whose hydraulic diffusivity is higher its contribution increases as the pumping continues.

If the pumping continues at a constant rate for an indefinite period, the limit $Q_1(n)/Q_2(n)$ tends to T_1/T_2. As the leakage factor decreases the nearly steady state condition approaches comparatively at a shorter time.

Table 6.1 Values of $Q_1(1)$ and $Q_1(12)$ for different values of ΔX and $Q_p = 10\ m^3/day$.

ΔX (m)	$Q_1(1)$ ($u_1 = 7.142857 \times 10^{-9}$)	$Q_1(12)$ ($u_1 = 5.952381 \times 10^{-10}$)
	L = 1528 m	
100	3.651314	3.601253
200	3.612369	3.536864
300	3.585746	3.482829
400	3.570740	3.444594
500	3.563343	3.419170
700	3.560817	3.339265
	L = 4030 m	
100	3.677292	3.635408
200	3.672243	3.625681
300	3.667842	3.613519
400	3.664573	3.600727
500	6.662323	3.588389
700	3.660089	3.567113

Table 6.2 Drawdown in aquifer 1 and 2 having equal hydraulic diffusivity values for $Q_p = 10\ m^3/day$.

Time in days	AQUIFER 1		AQUIFER 2	
	Drawdown in $m \times 10^{-2}$		Drawdown in $m \times 10^{-2}$	
	r = 300 m	r = 600 m	r = 300 m	r = 6000 m
1	0.29890	0.16521	0.29890	0.16521
2	0.36890	0.23051	0.36890	0.23051
3	0.41025	0.27026	0.41025	0.27026
4	0.43971	0.29890	0.43971	0.29890
5	0.46260	0.32131	0.4626	0.32131
6	0.48134	0.33972	0.48134	0.33972
7	0.49719	0.35533	0.49719	0.35533
8	0.51093	0.36890	0.51093	0.36890
9	0.52306	0.38089	0.52306	0.38089
10	0.53391	0.39163	0.53391	0.39163
11	0.54373	0.40136	0.54373	0.40136
12	0.55269	0.41025	0.55269	0.41025

Hydrogeological parameters of aquifer 1 and 2 are as follows

$T_1 = 700\ m^2/day$, $\phi_1 = .001$, $\beta_1 = 7 \times 10^5\ m^2/day$

$T_2 = 70\ m^2/day$, $\phi_2 = .0001$, $\beta_2 = 7 \times 10^5\ m^2/day$.

Table 6.3 Recharge $[Q_R(n)]$ when the corresponding aquifers of two different hydrogeological settings have equal hydraulic diffussivity values and the aquitards have equal leakage factors.

Time in days	Recharge $[Q_R(n)]$ in Case I (m^3/day)	Recharge $[Q_R(n)]$ in Case II (m^3/day)
1	-0.7221490	-0.7227677
2	-0.8933202	-0.8933399
3	-0.8281627	-0.8281772
4	-0.7970765	-0.7970880
5	-0.7793128	-0.7793370
6	-0.7646697	-0.7446786
7	-0.7501670	-0.7501877
8	-0.7353342	-0.7353396
9	-0.7202554	-0.7202582
10	-0.7052081	-0.7052290
11	-0.6904655	-0.6904707
12	-0.6761129	-0.6761399

The recharges are negative i.e. recharge is taking place from second aquifer to first aquifer.

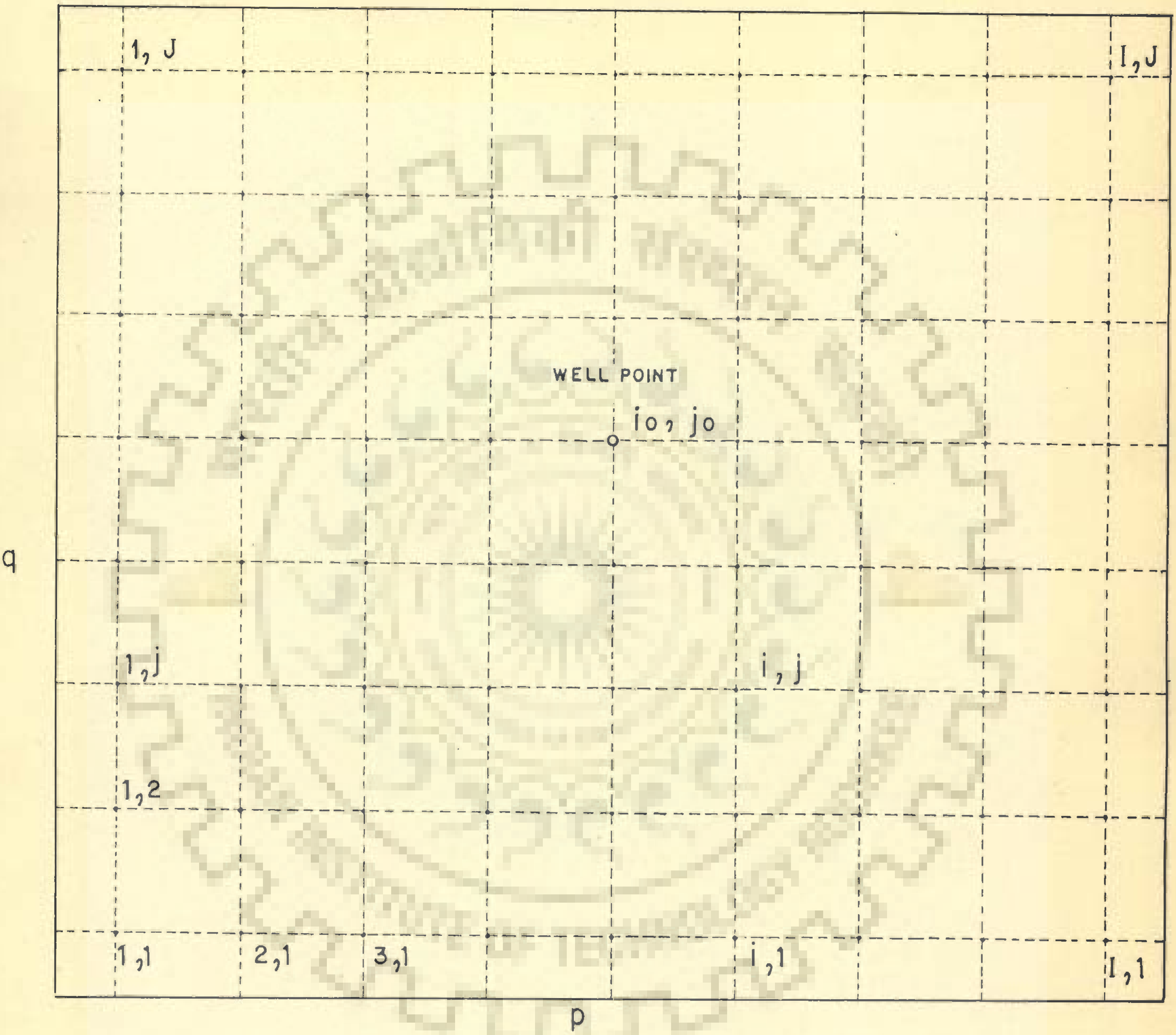

Fig. 6·3 Designation of grid nodes used in the derivations

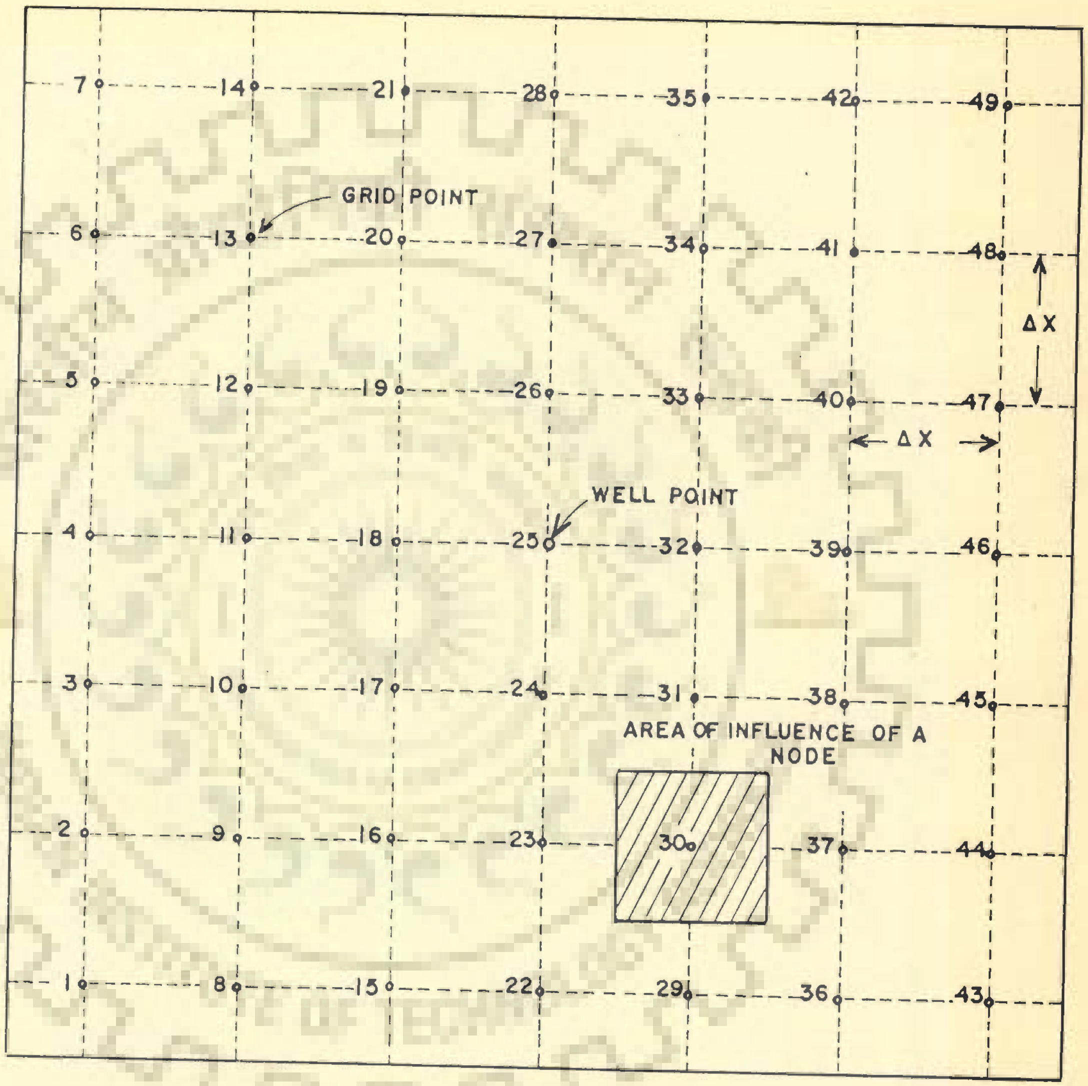

Fig. 6·4 Numbering of the grid nodes adopted in the study.

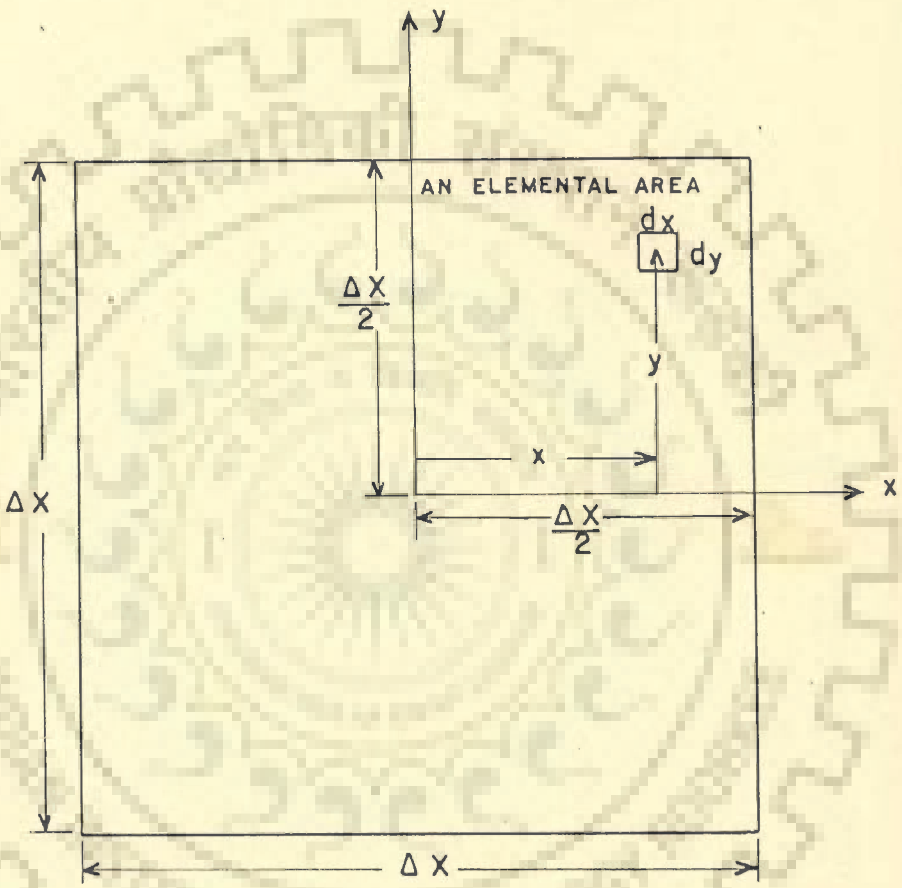

Fig. 6·5 A grid from which water is withdrawn uniformly.

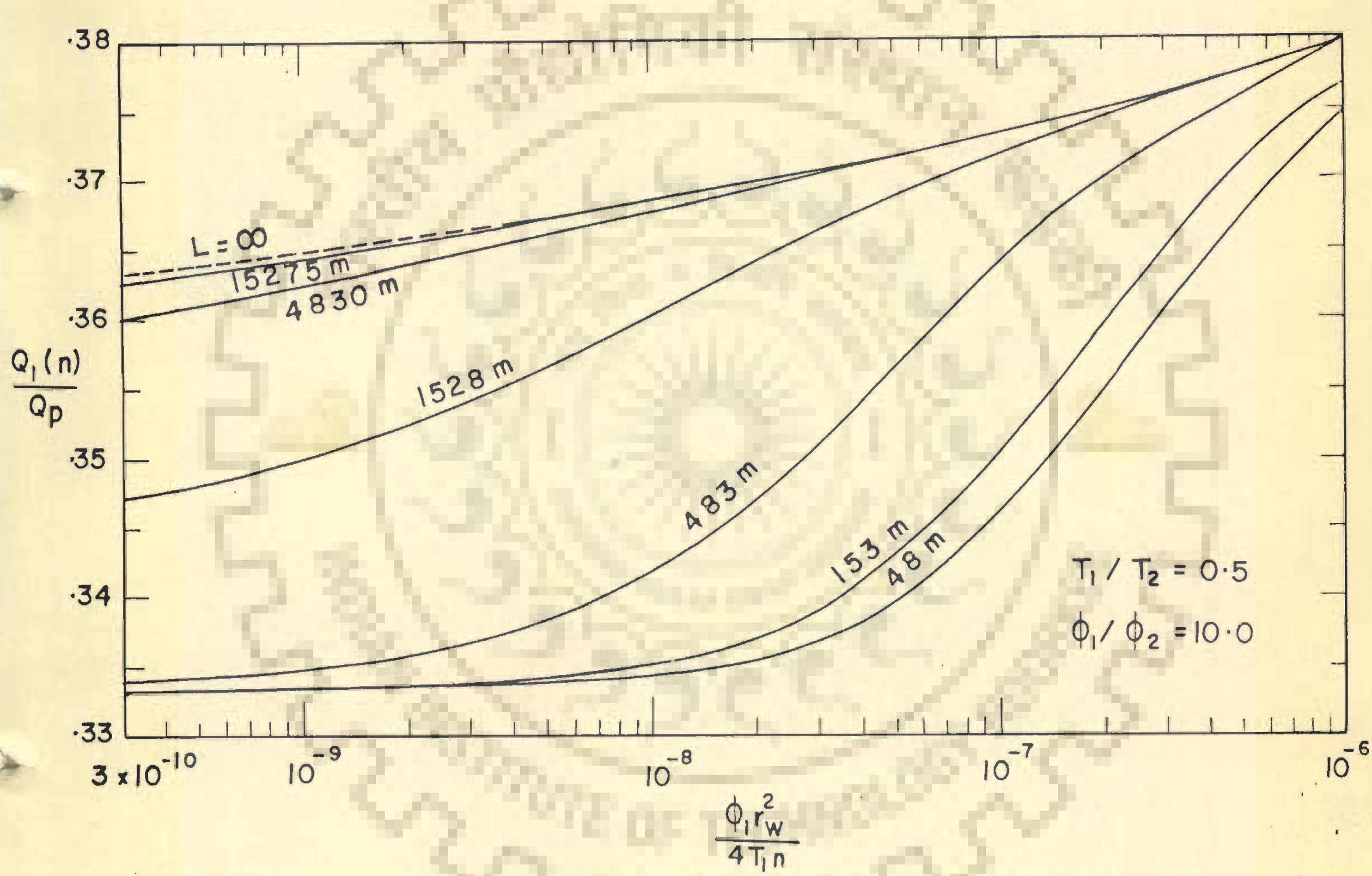

Fig. 6·6 Contribution of top aquifer to discharge at various time steps due to pumping of a well tapping two confined aquifers separated by an aquitard.

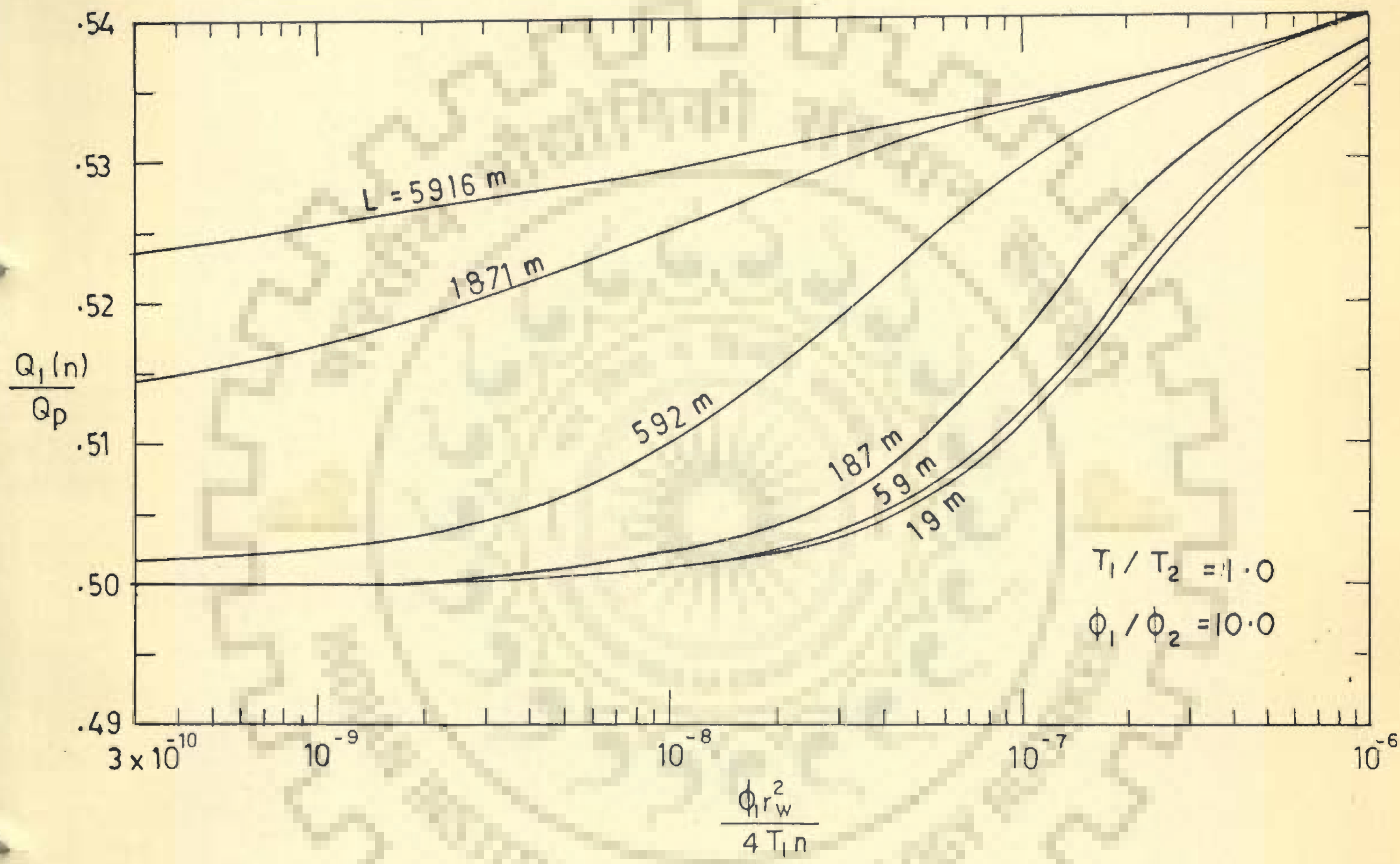

Fig. 6·7 Contribution of top aquifer to discharge at various time steps due to pumping of a well tapping two confined aquifers separated by an aquitard.

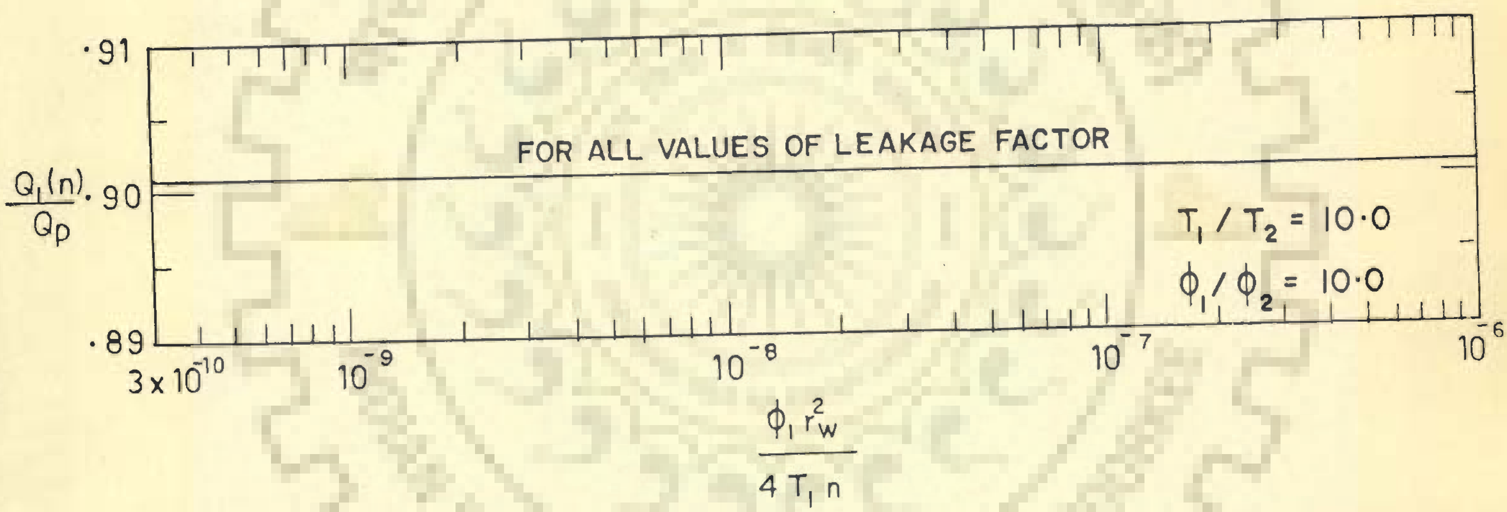

Fig. 6·8 Contribution of top aquifer to discharge at various time steps due to pumping of a well tapping two confined aquifers separated by an aquitard.

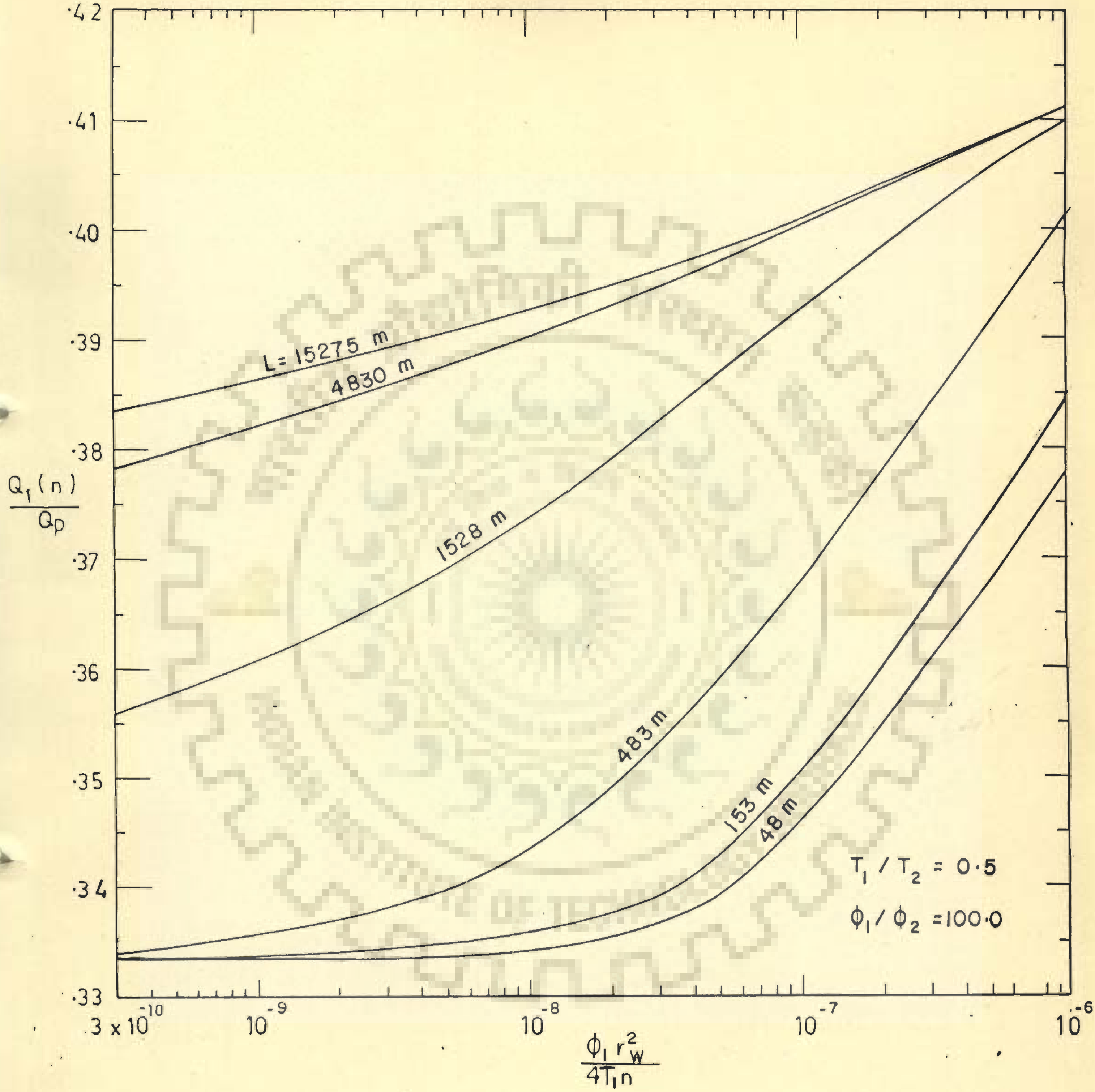

Fig. 6·9 Contribution of top aquifer to discharge at various time steps due to pumping of a well tapping two confined aquifers separated by an aquitard.

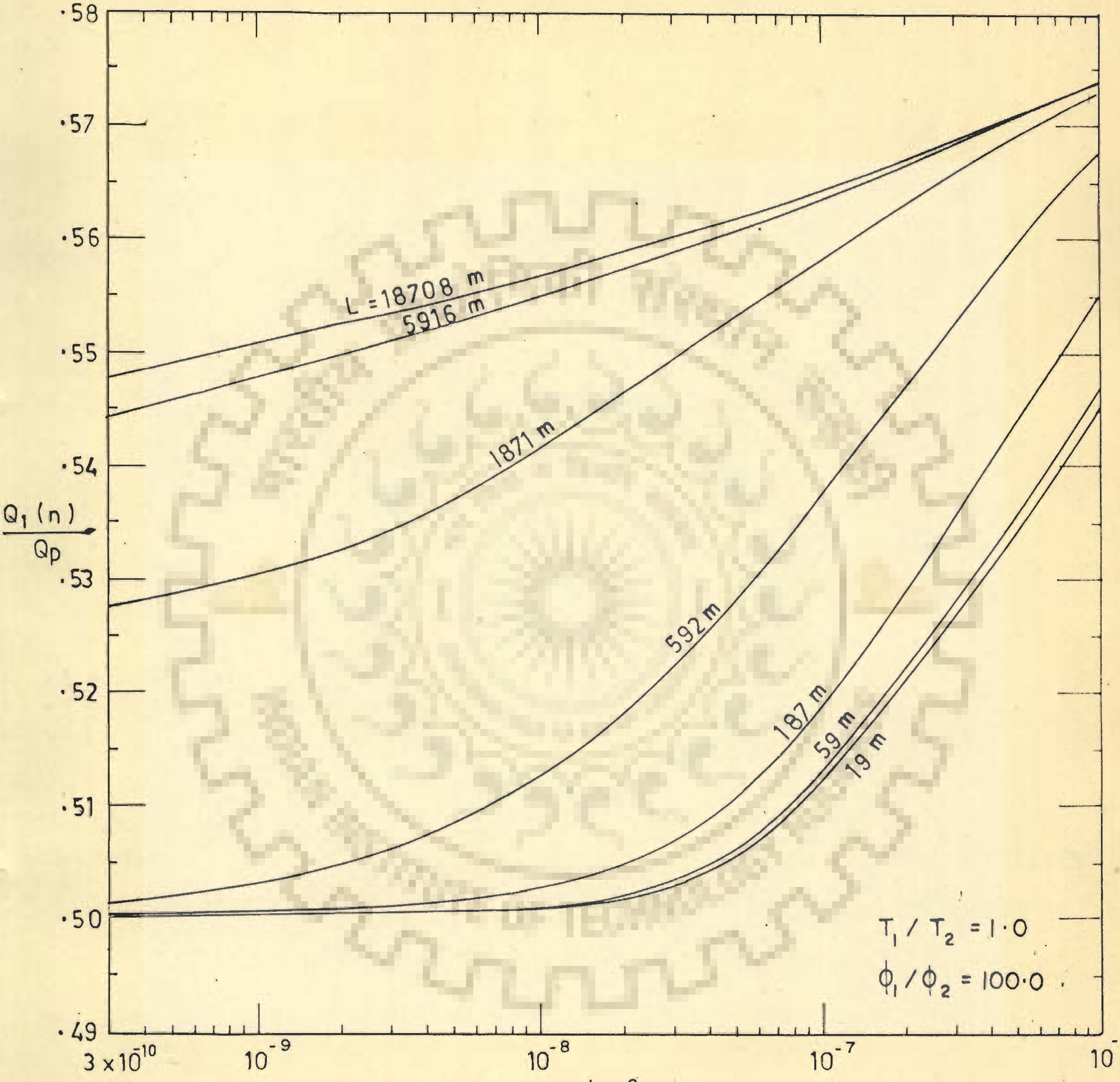

Fig. 6·10 Contribution of top aquifer to discharge at various time steps due to pumping of a well tapping two confined aquifers separated by an aquitard.

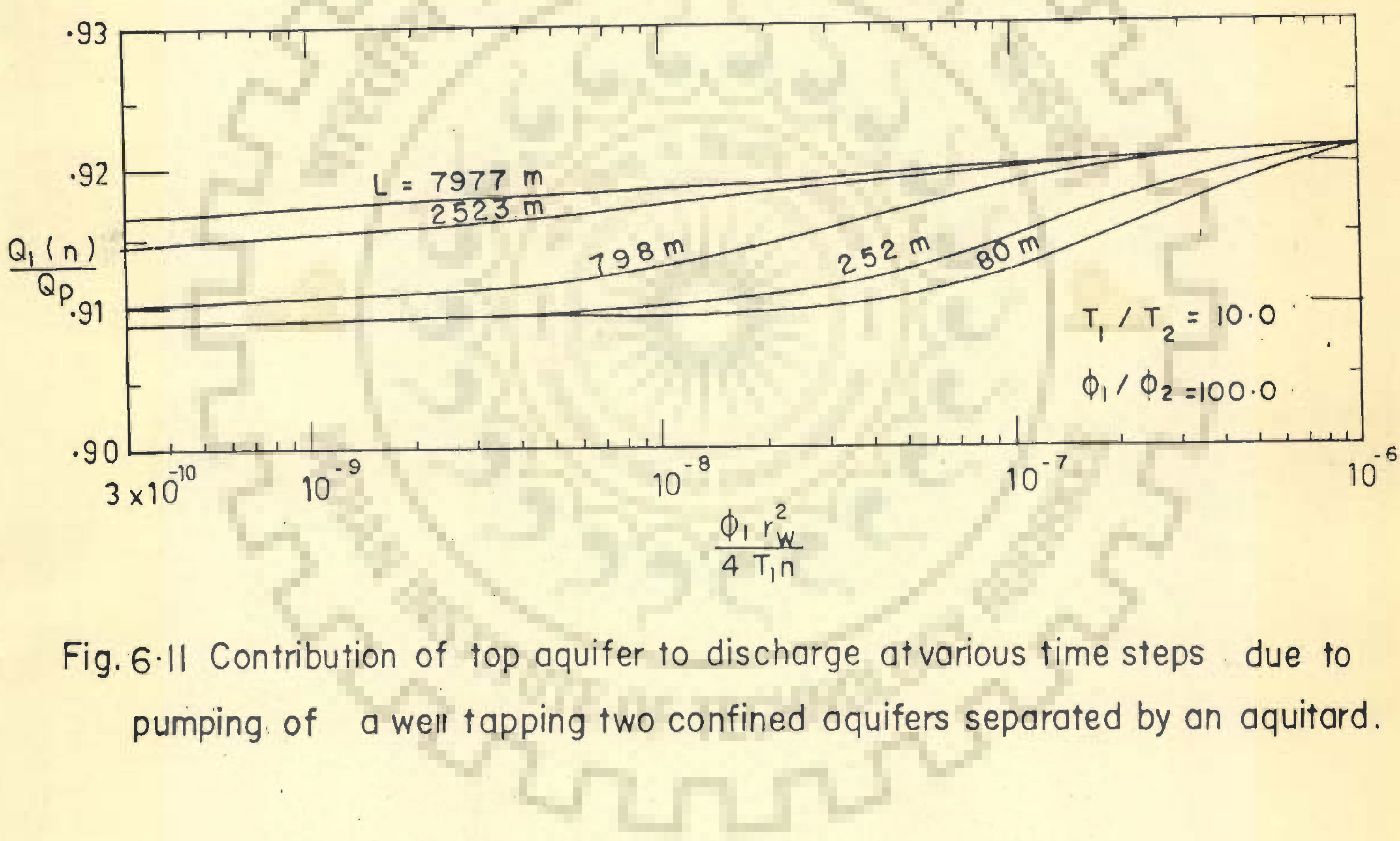

Fig. 6·11 Contribution of top aquifer to discharge at various time steps due to pumping of a well tapping two confined aquifers separated by an aquitard.

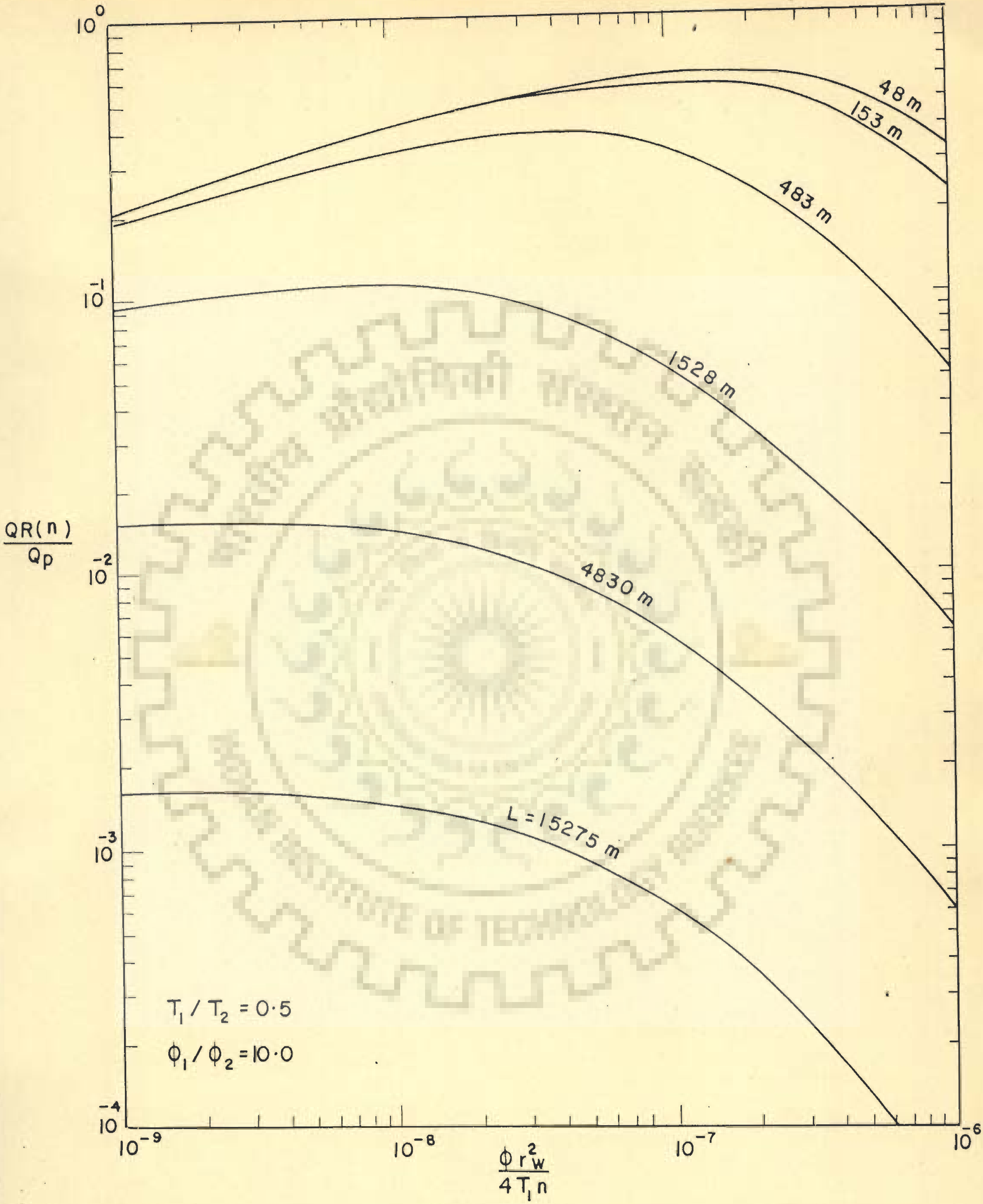

Fig. 6·12 Variation of recharge with time due to pumping of a well tapping two confined aquifers separated by an aquitard.

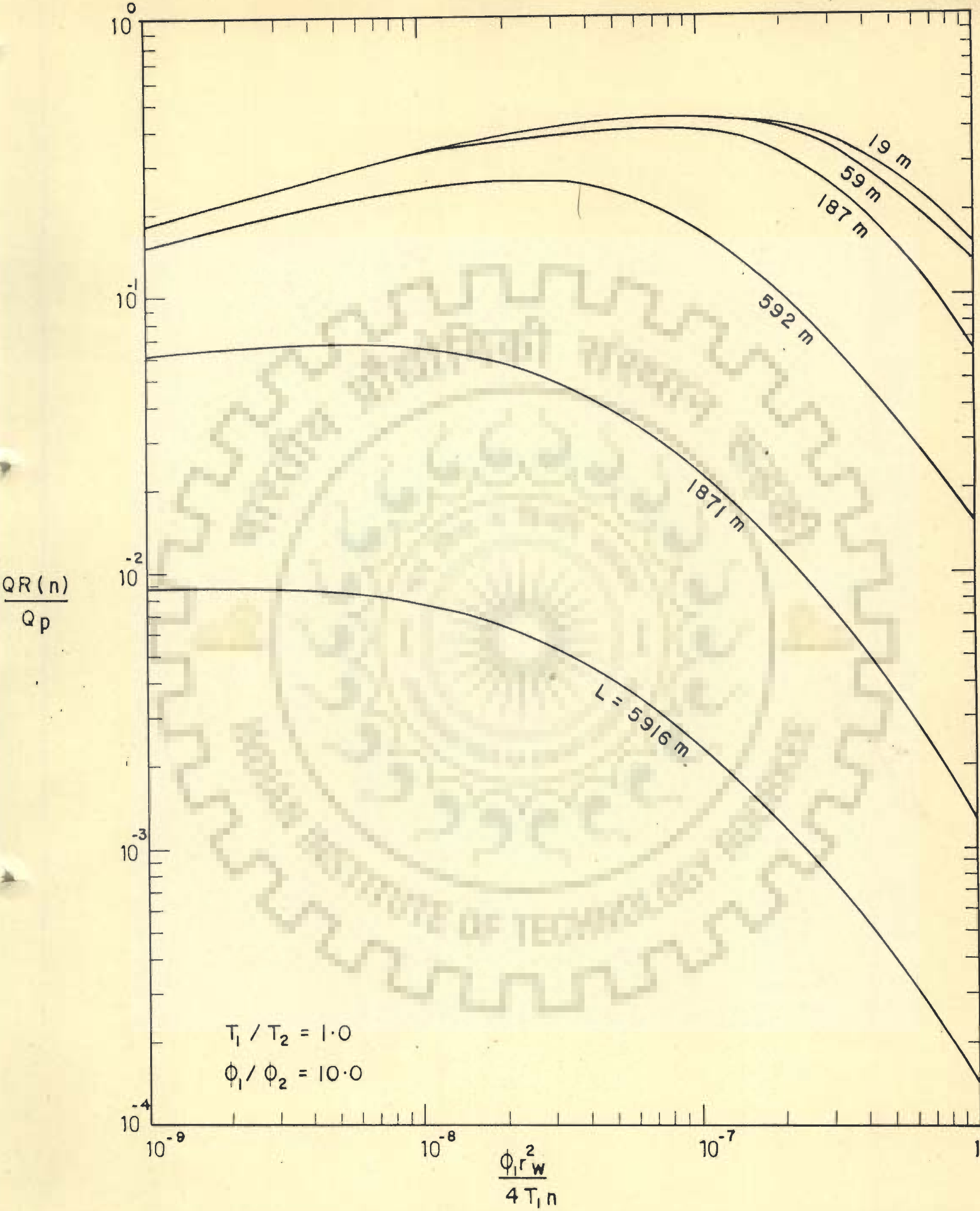

Fig.6·13 Variation of recharge with time due to pumping of a well tapping two confined aquifers separated by an aquitard.

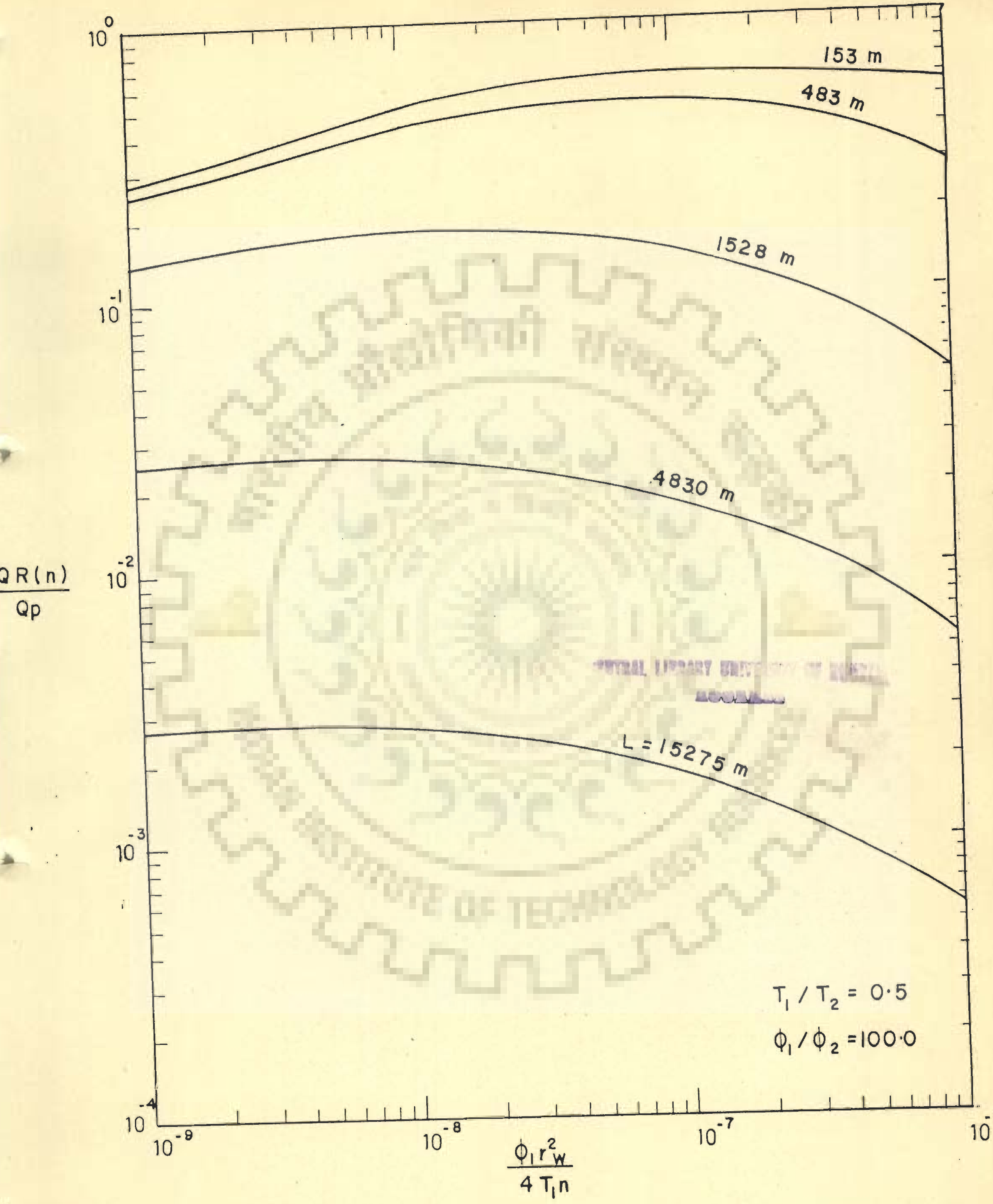

Fig.6·14 Variation of recharge with time due to pumping of a well tapping two confined aquifers separated by an aquitard.

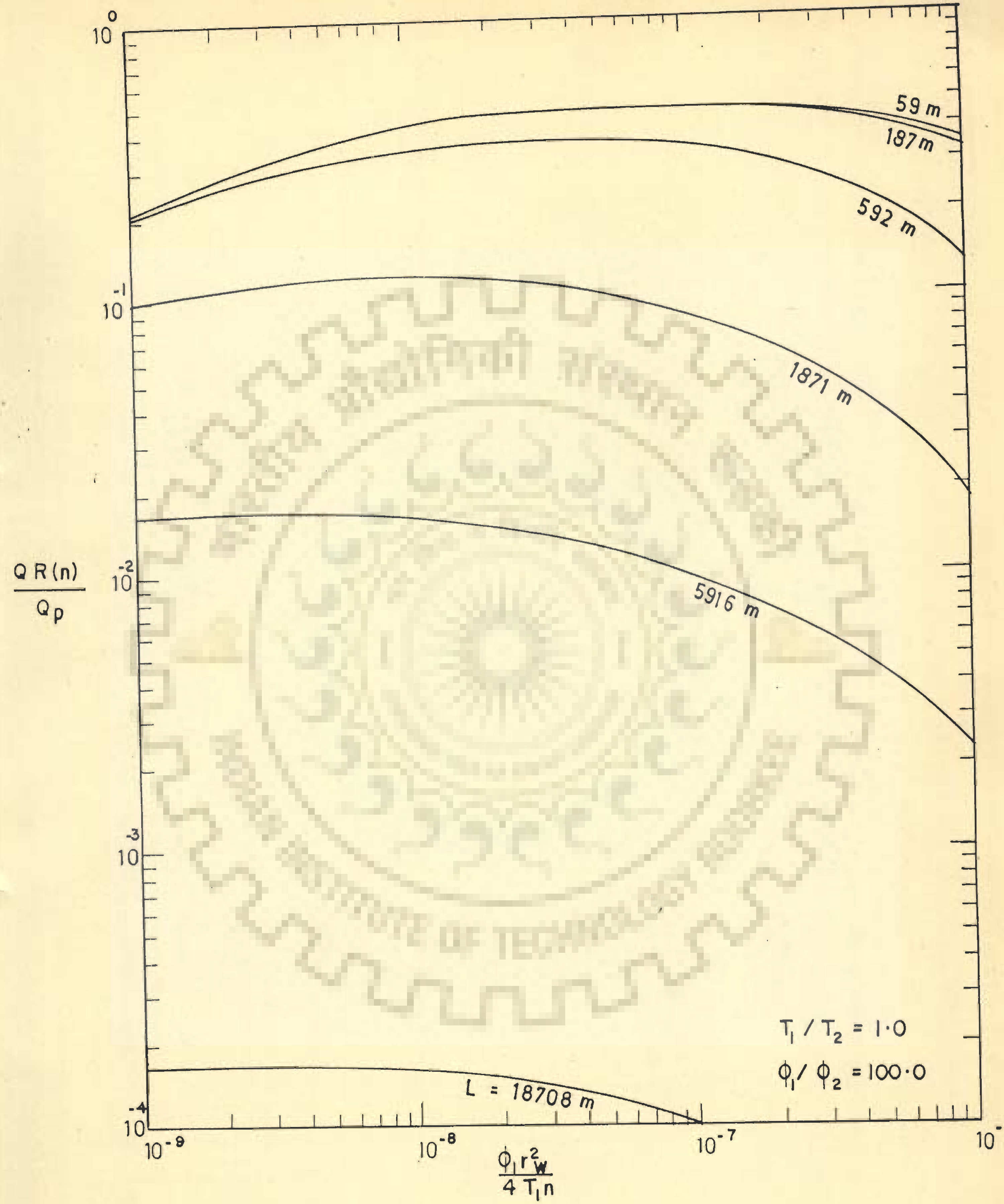

Fig.6·15 Variation of recharge with time due to pumping of a well tapping two confined aquifers separated by an aquitard.

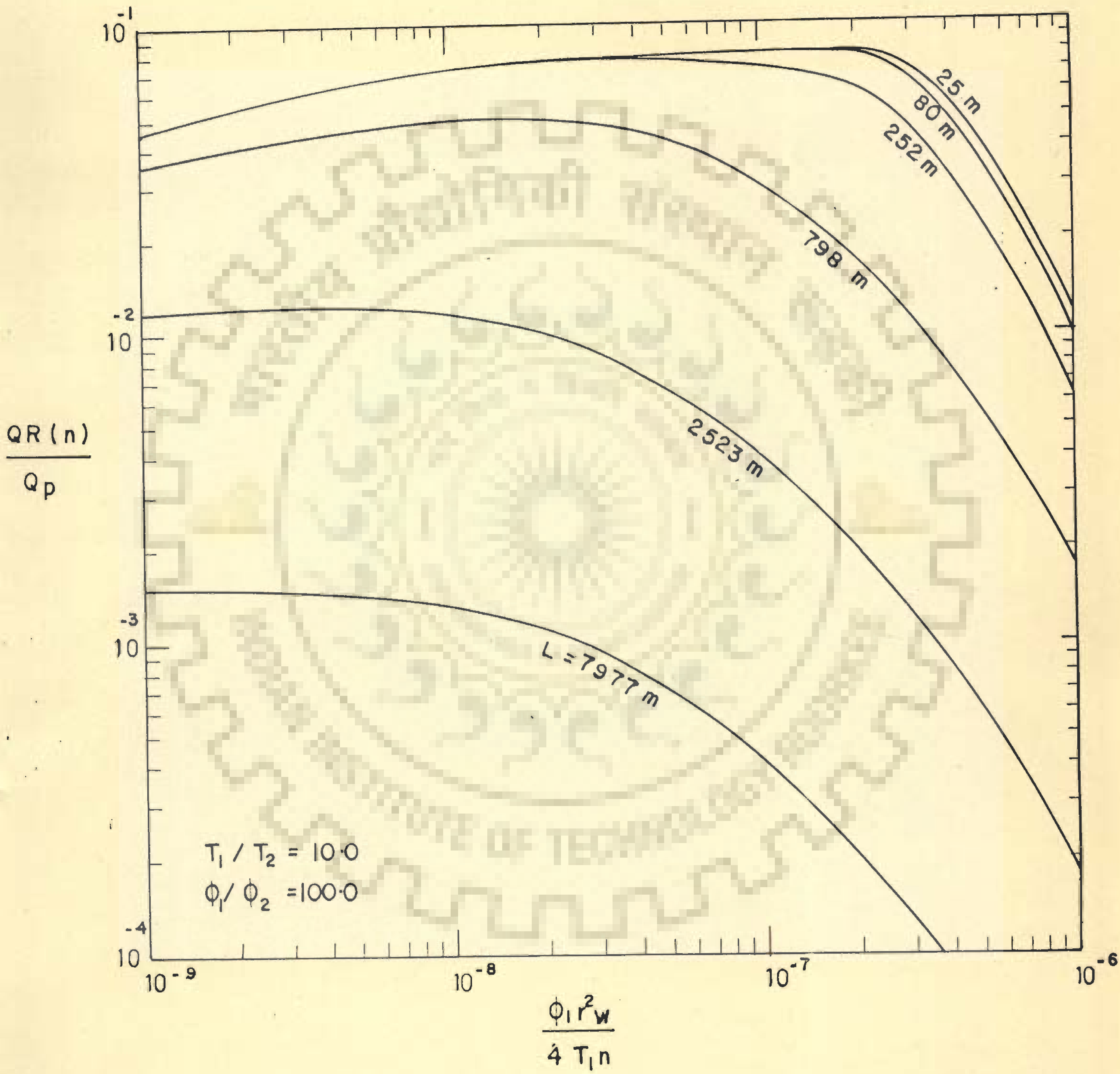

Fig. 6·16 Variation of recharge with time due to pumping of a well tapping two confined aquifers separated by an aquitard.

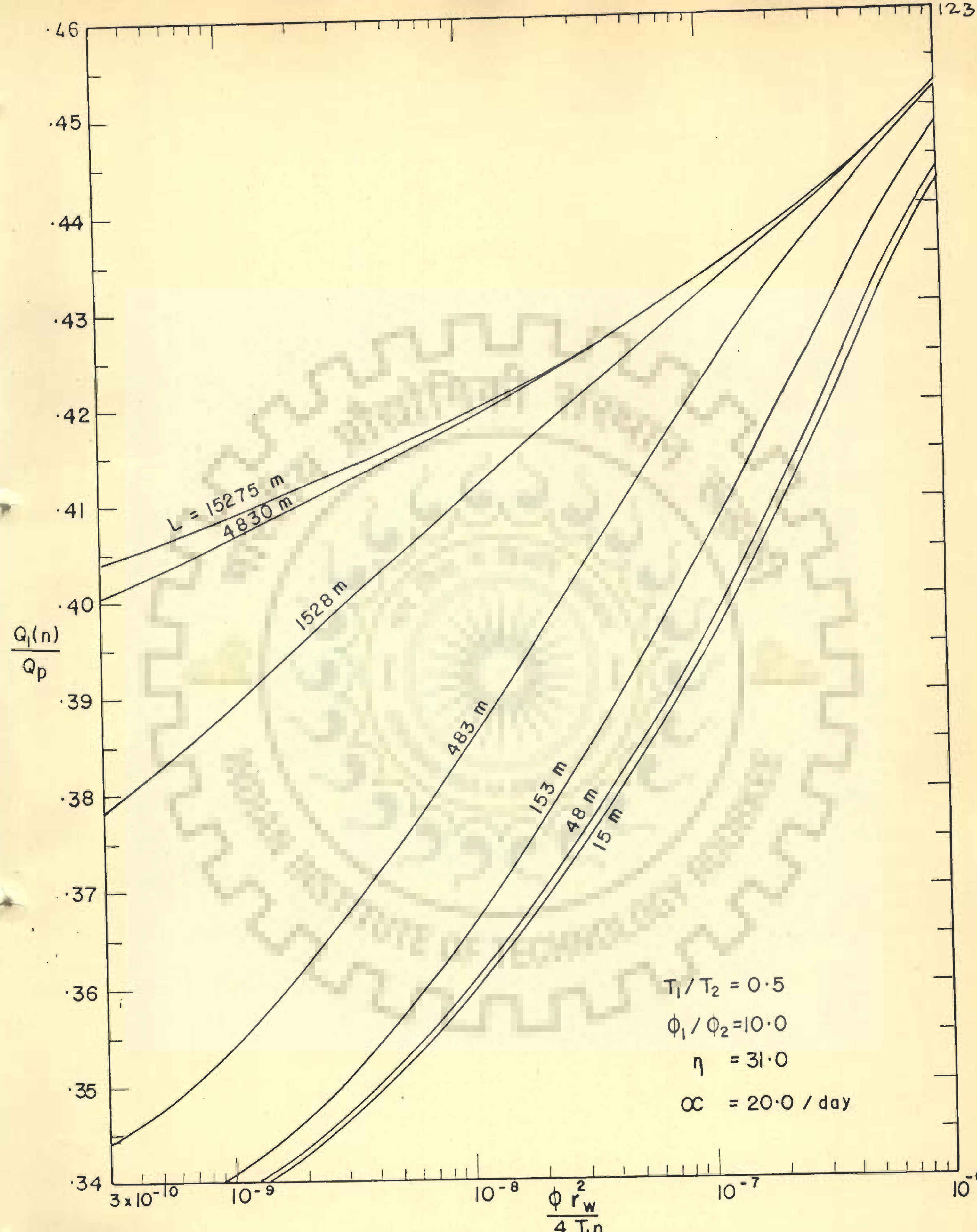

Fig. 6·17 Contribution of top aquifer to discharge at various time steps due to pumping of a well tapping an unconfined and a confined aquifer separated by an aquitard.

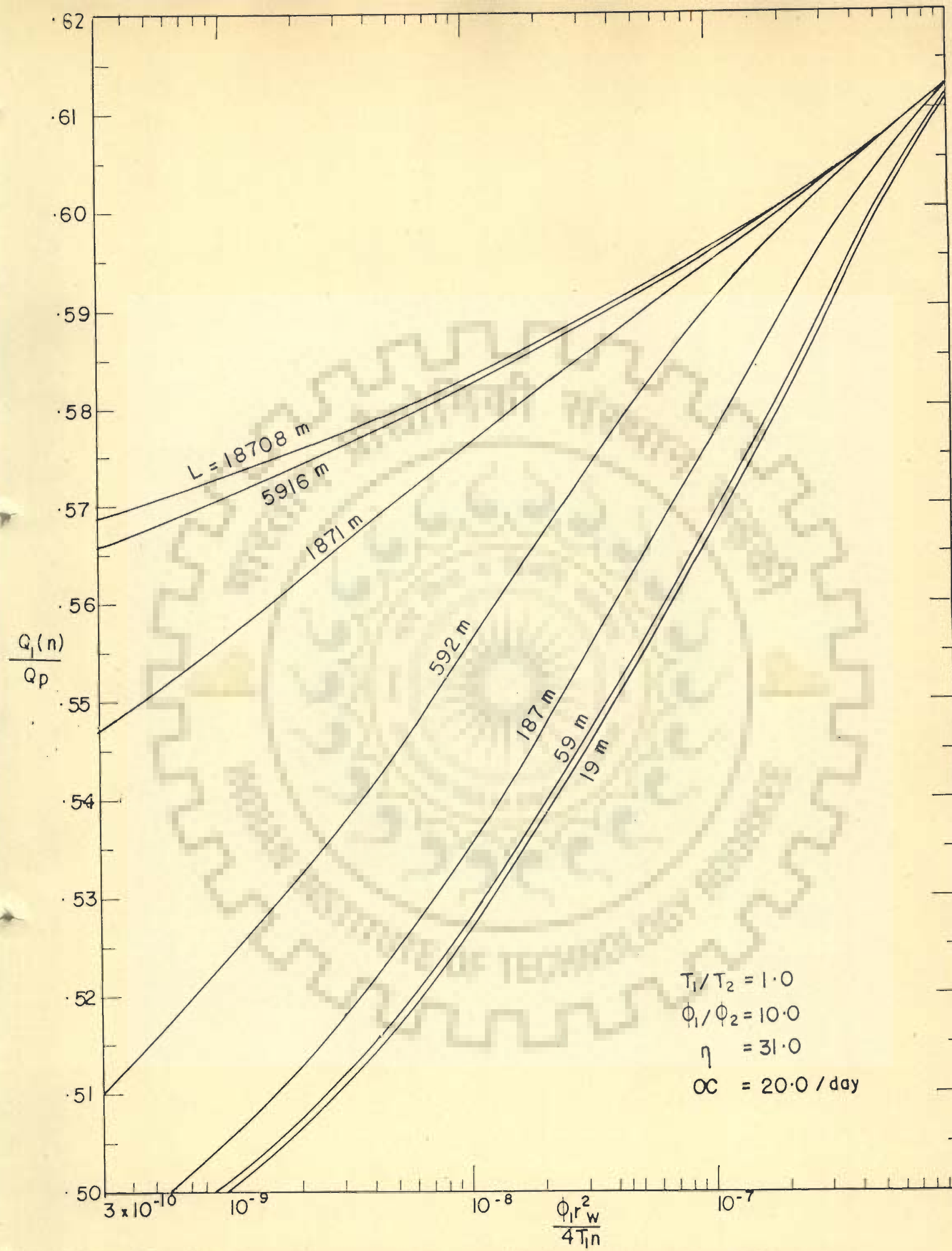

Fig. 6·18 Contribution of top aquifer to discharge at various time steps due to pumping of a well tapping an unconfined and a confined aquifer separated by an aquitard

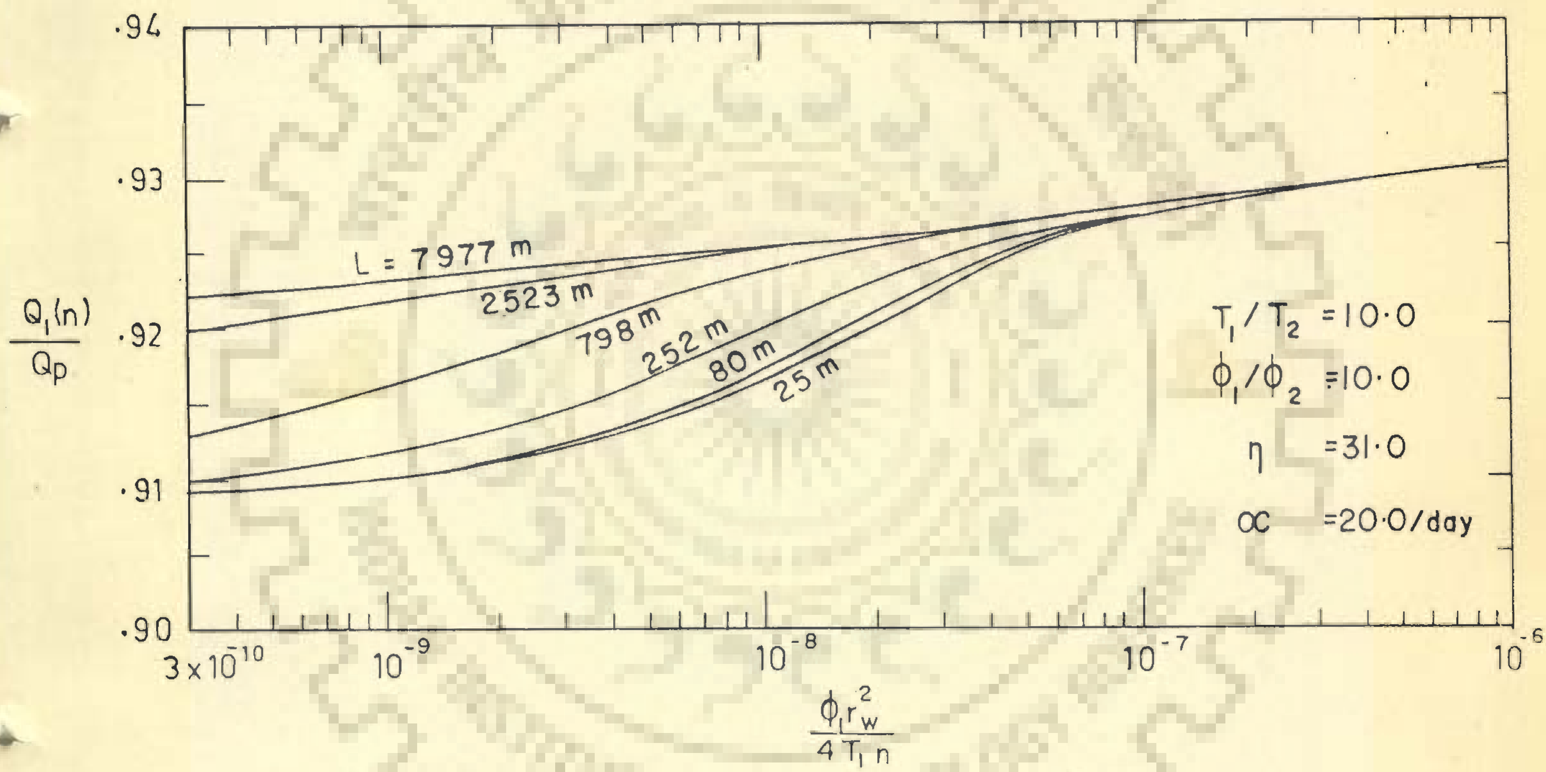

Fig.6·19 Contribution of top aquifer to discharge at various time steps due to pumping of a well tapping an unconfined and a confined aquifer separated by an aquitard.

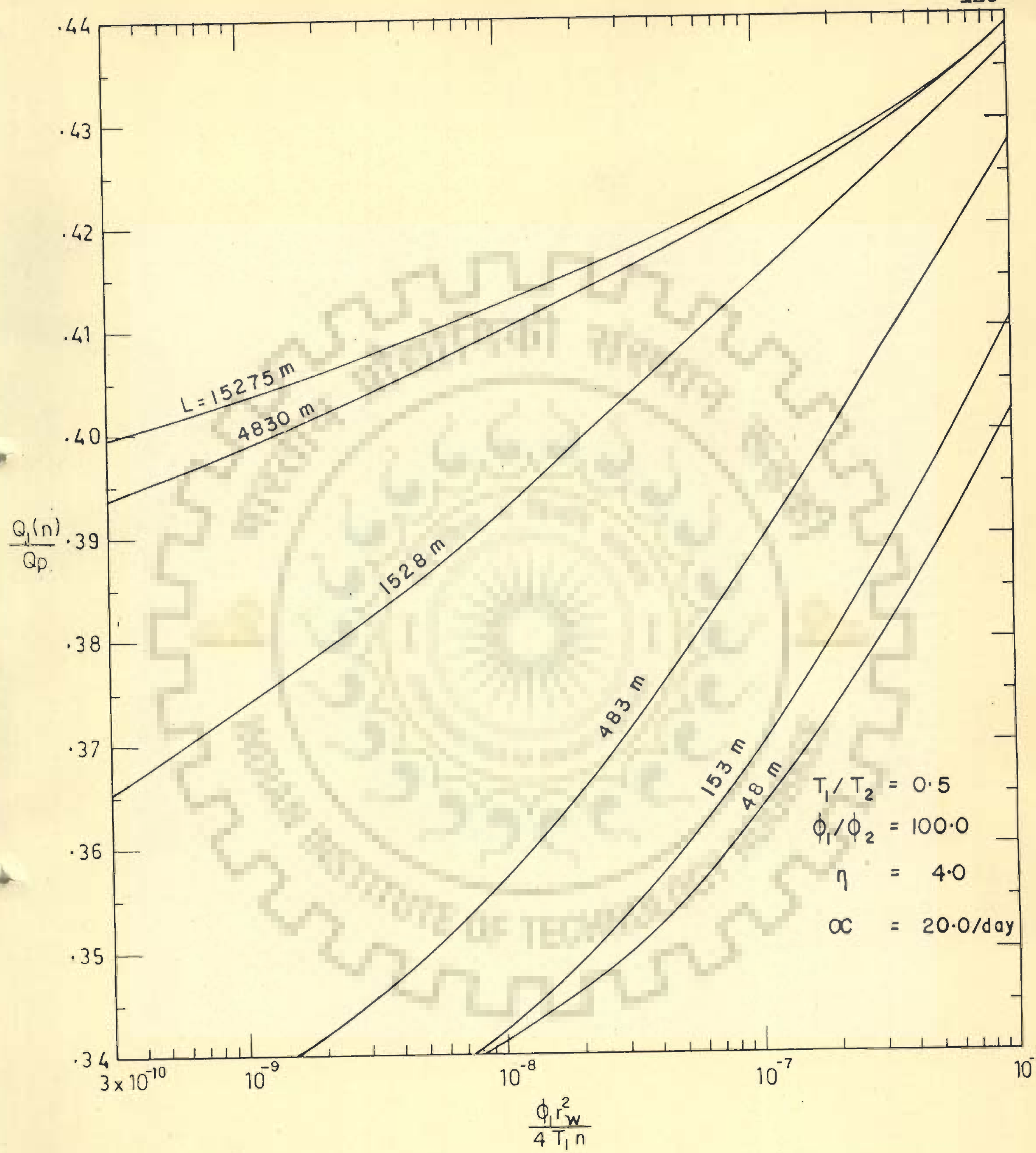

Fig. 6·20 Contribution of top aquifer to discharge at various time steps due to pumping of a well tapping an unconfined and a confined aquifer separated by an aquitard.

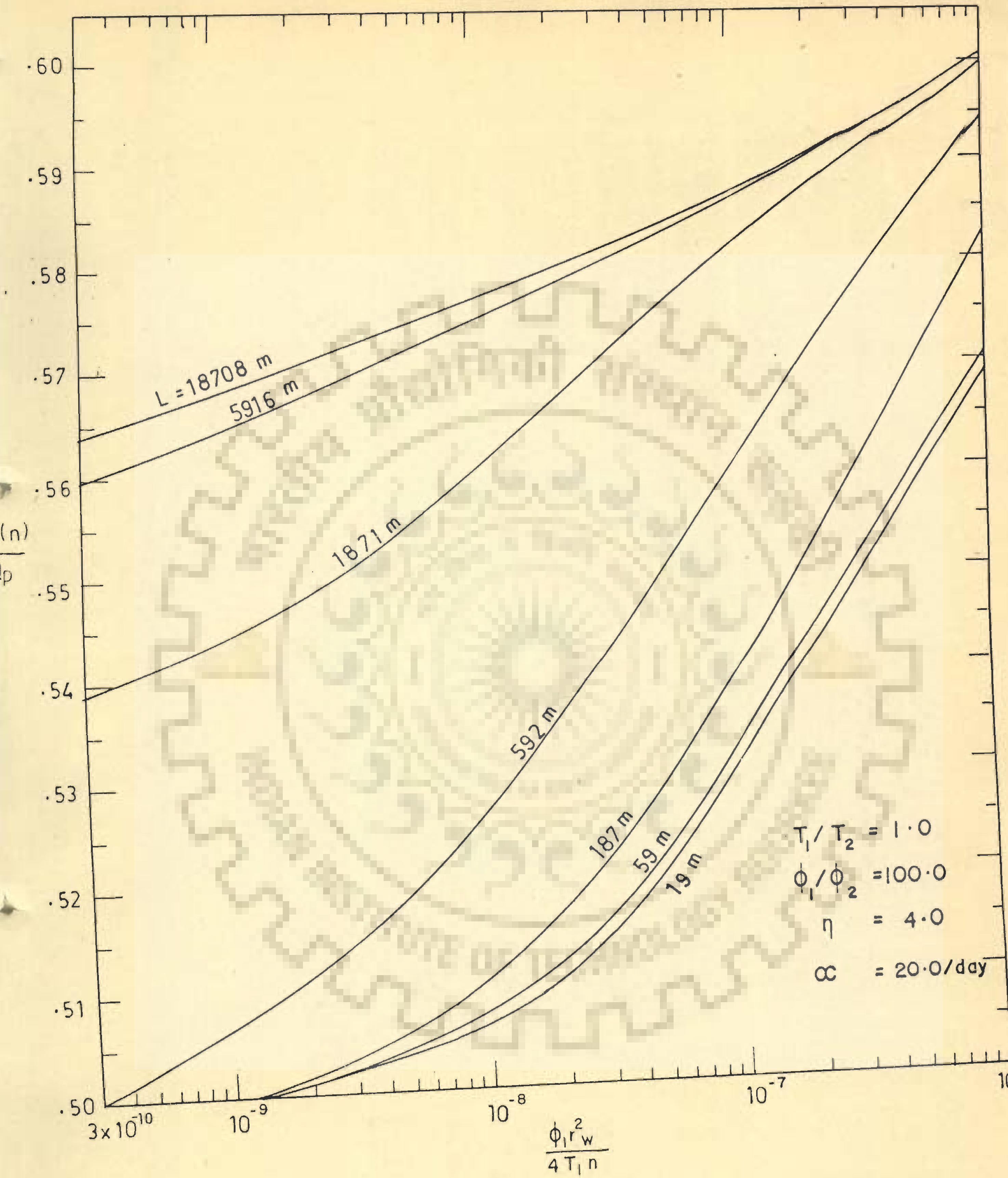

Fig. 6·21 Contribution of top aquifer to discharge at various time steps due to pumping of a well tapping an unconfined and a confined aquifer separated by an aquitard.

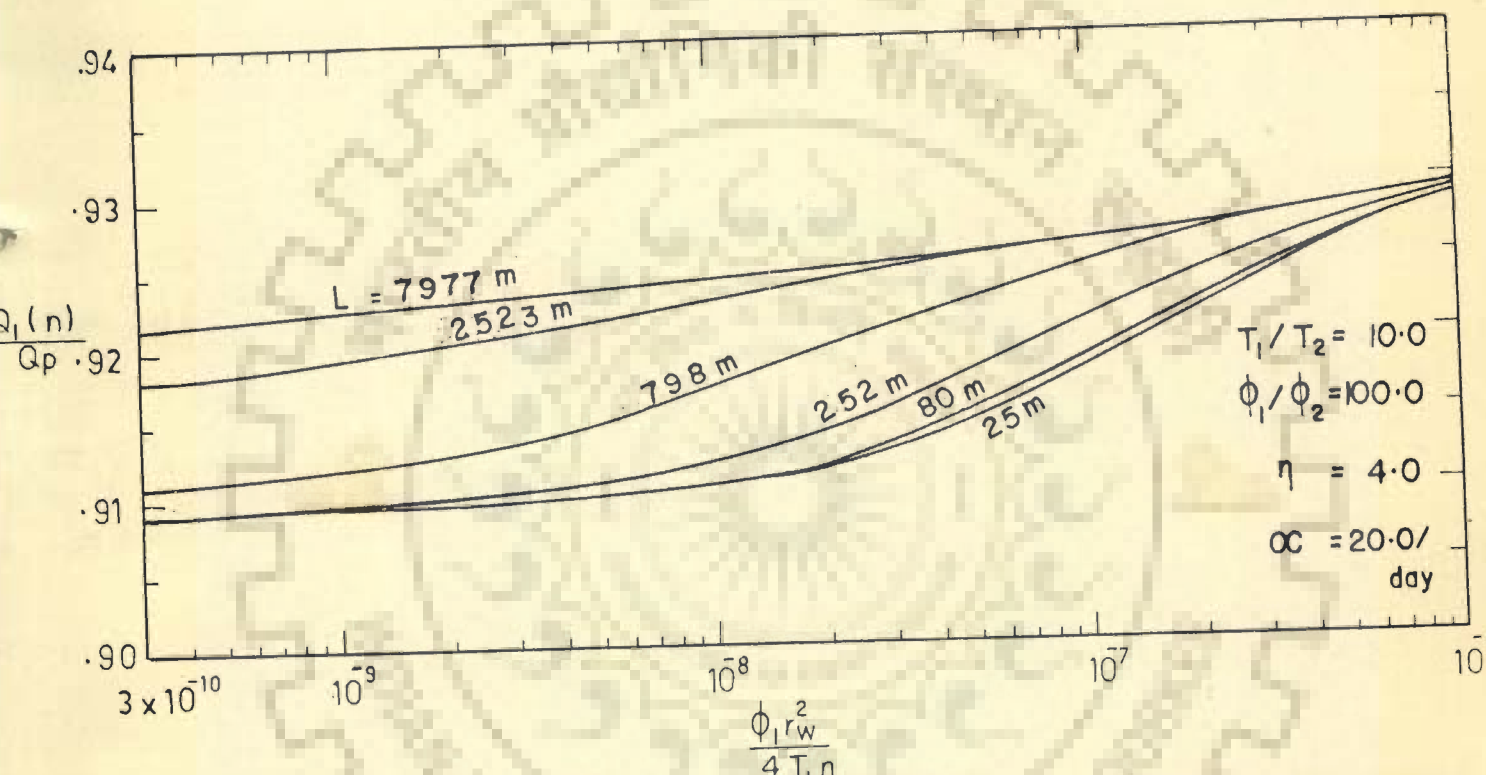

Fig.6·22 Contribution of top aquifer to discharge at various time steps due to pumping of a well tapping an unconfined and a confined aquifer separated by an aquitard.

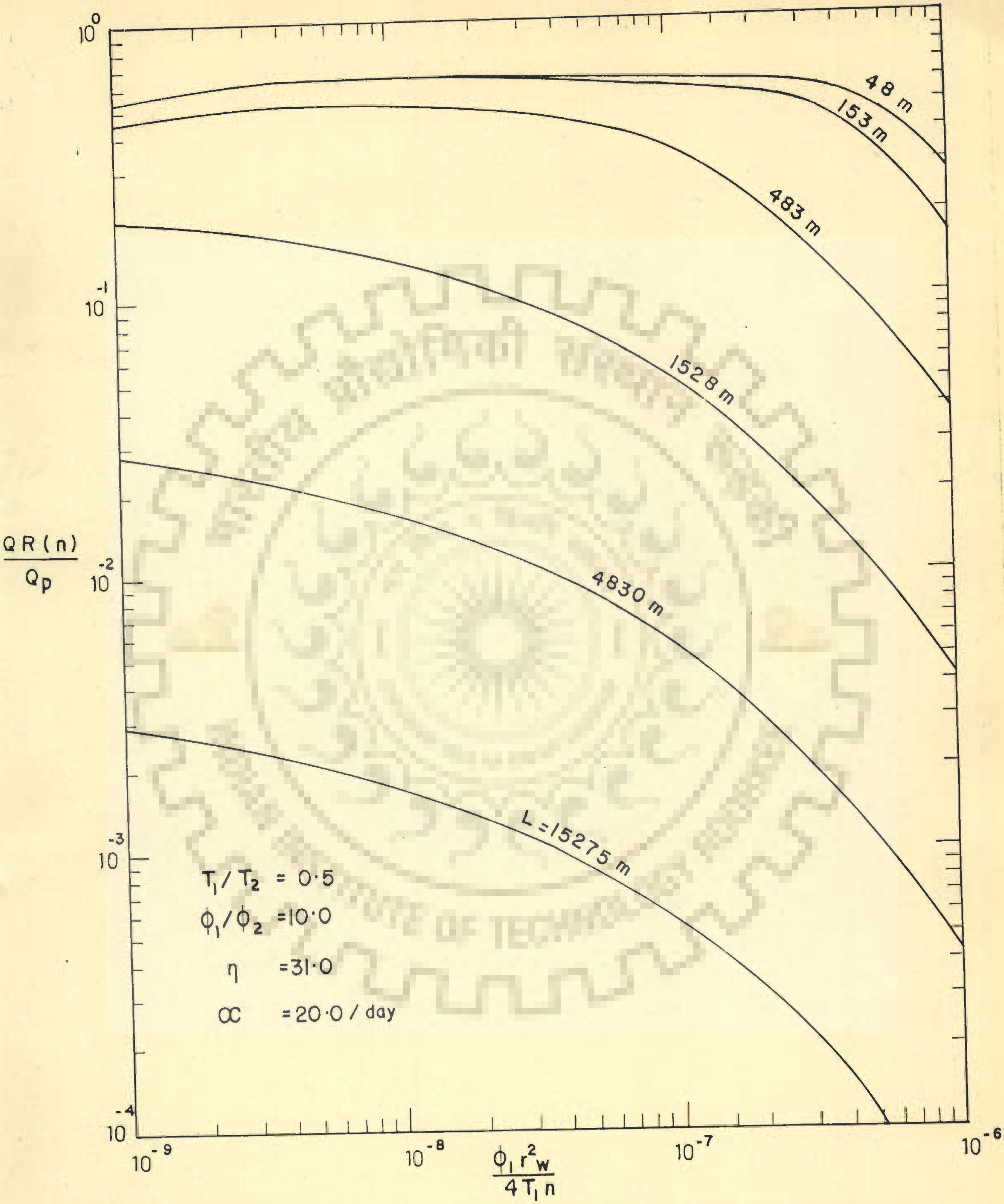

Fig. 6·23 Variation of recharge with time due to pumping of a well tapping an unconfined and a confined aquifer separated by an aquitard.

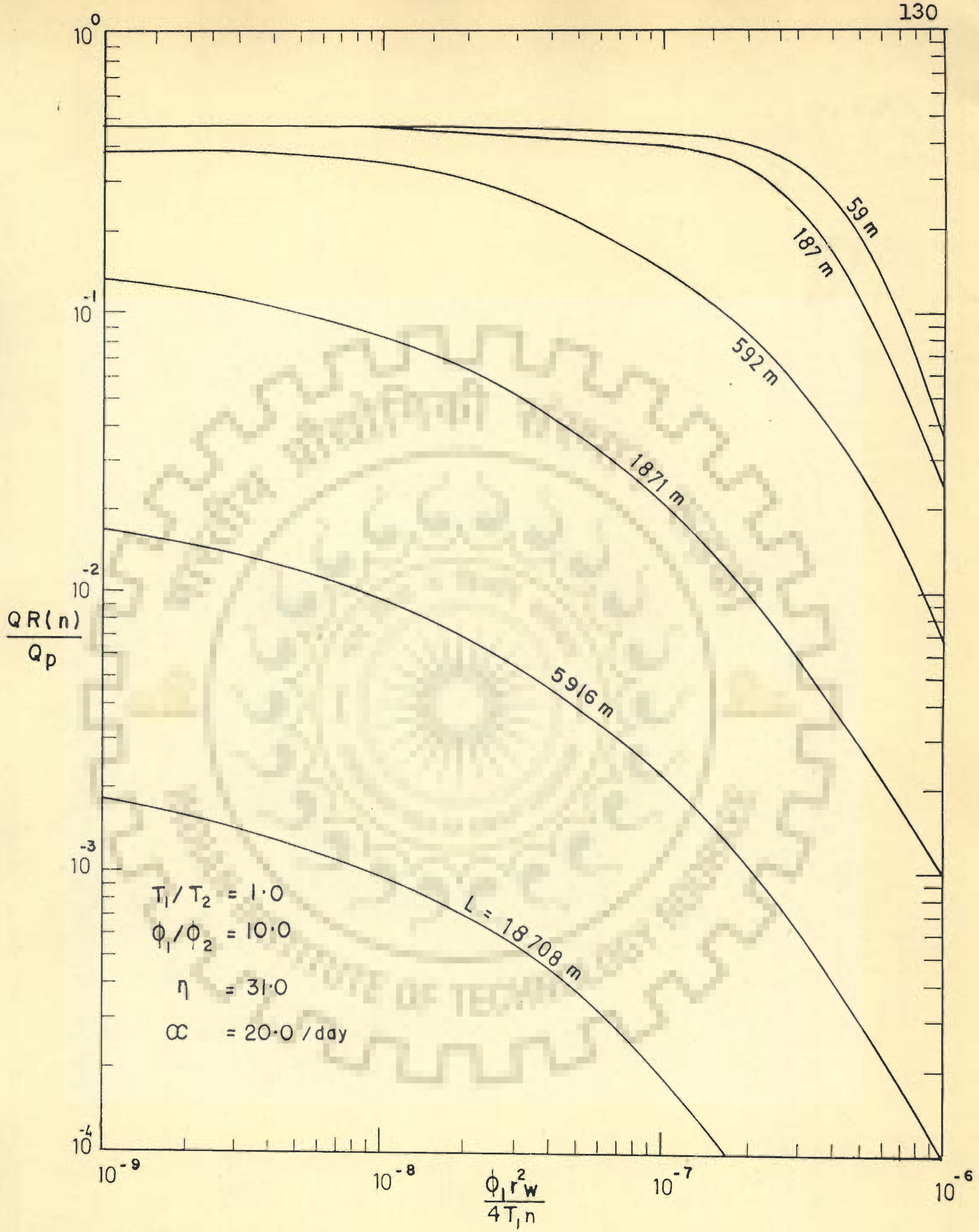

Fig.6·24 Variation of recharge with time due to pumping of a well tapping an unconfined and a confined aquifer separated by an aquitard.

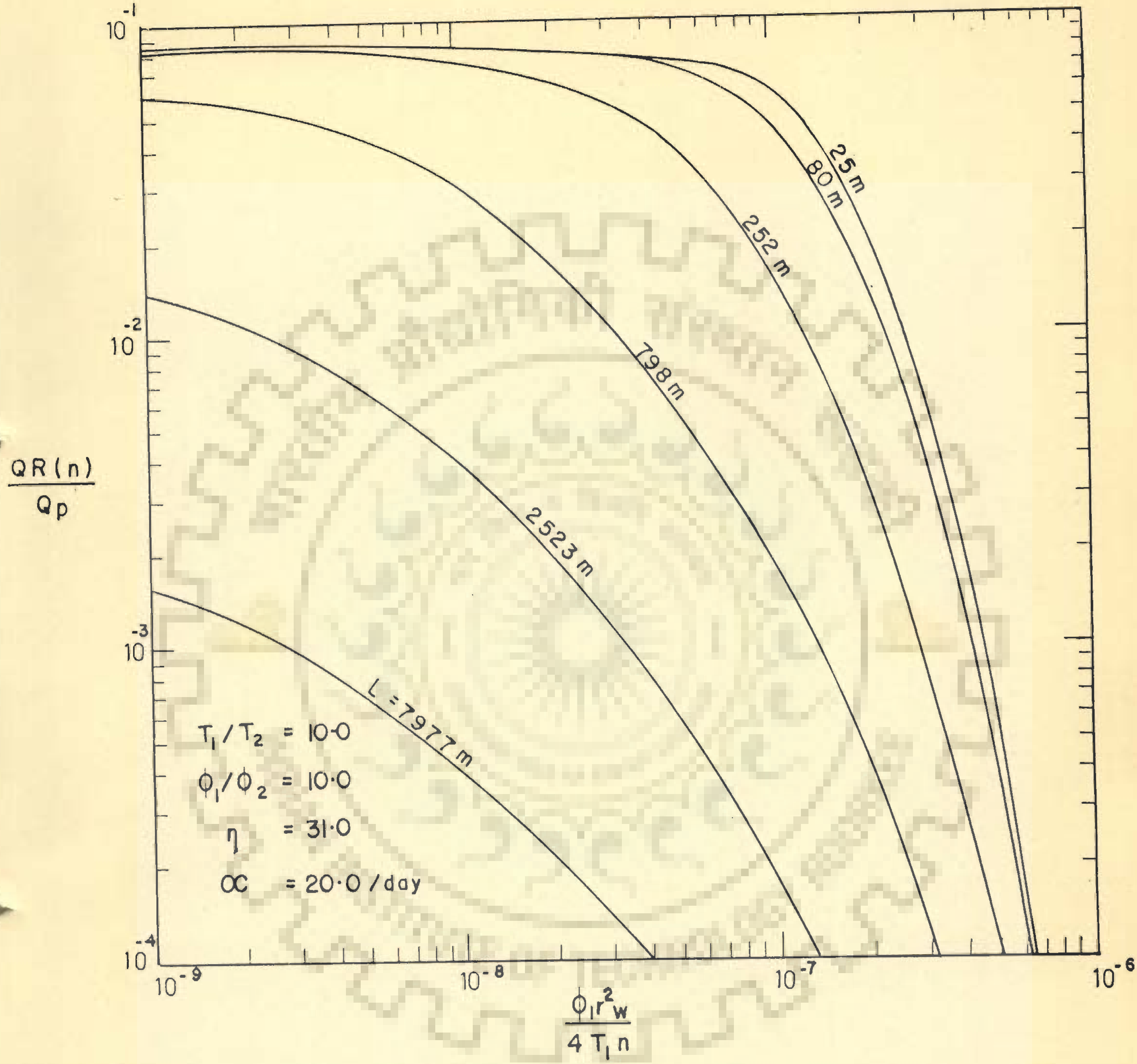

Fig. 6·25 Variation of recharge with time due to pumping of a well tapping an unconfined and a confined aquifer separated by an aquitard.

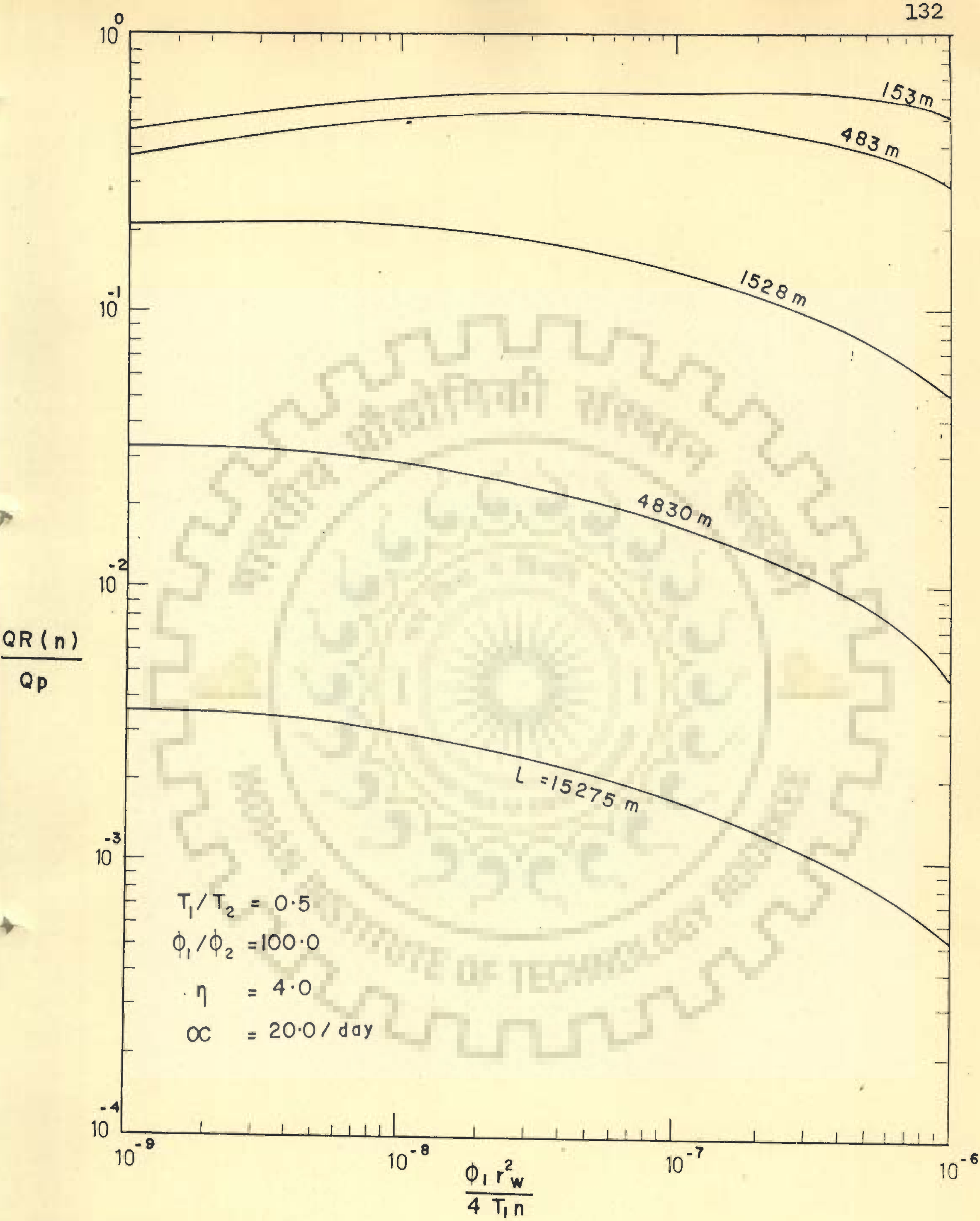

Fig. 6·26 Variation of recharge with time due to pumping of a well tapping an unconfined and a confined aquifer separated by an aquitard.

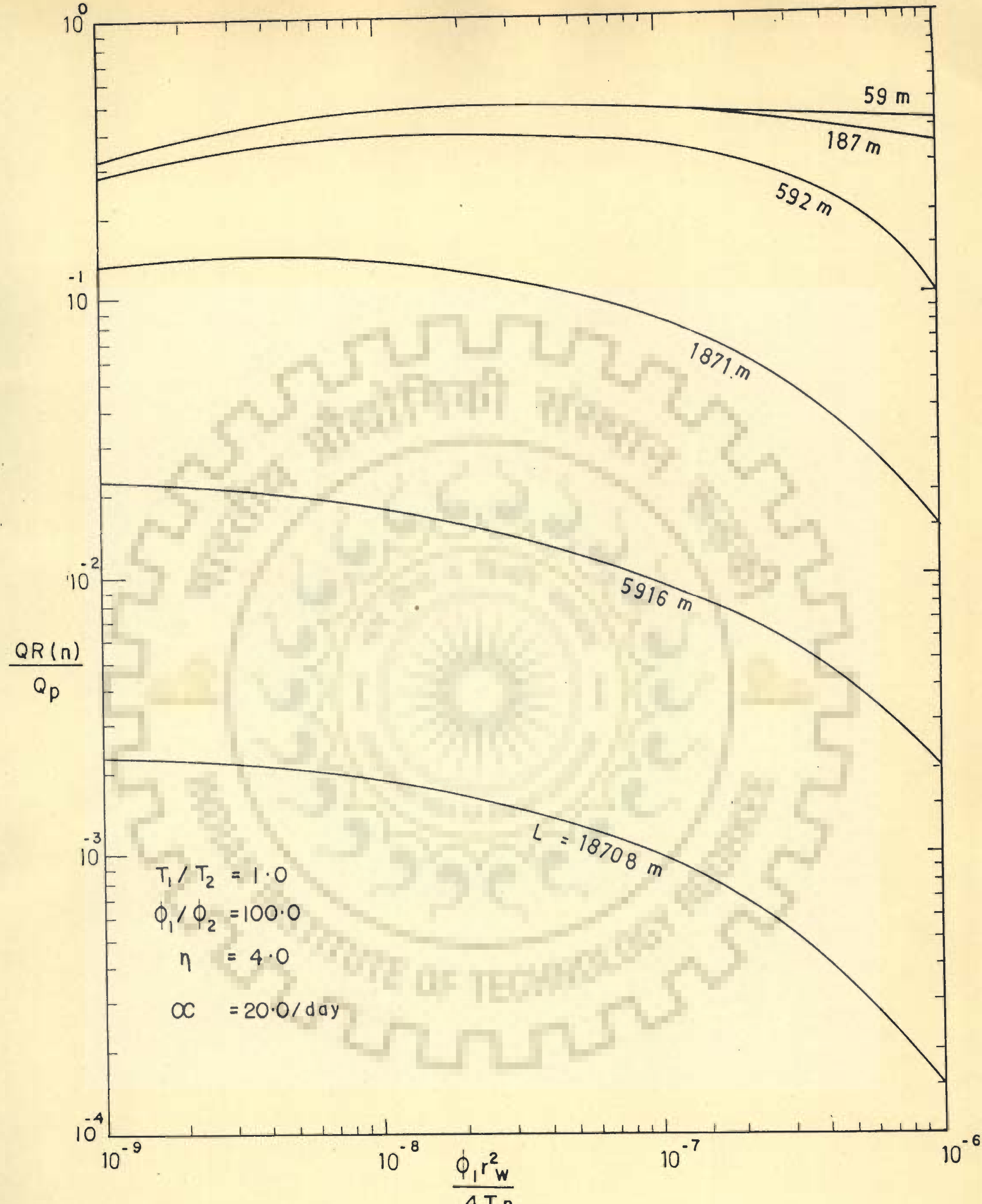

Fig. 6·27 Variation of recharge with time due to pumping of a well tapping an unconfined and a confined aquifer separated by an aquitard.

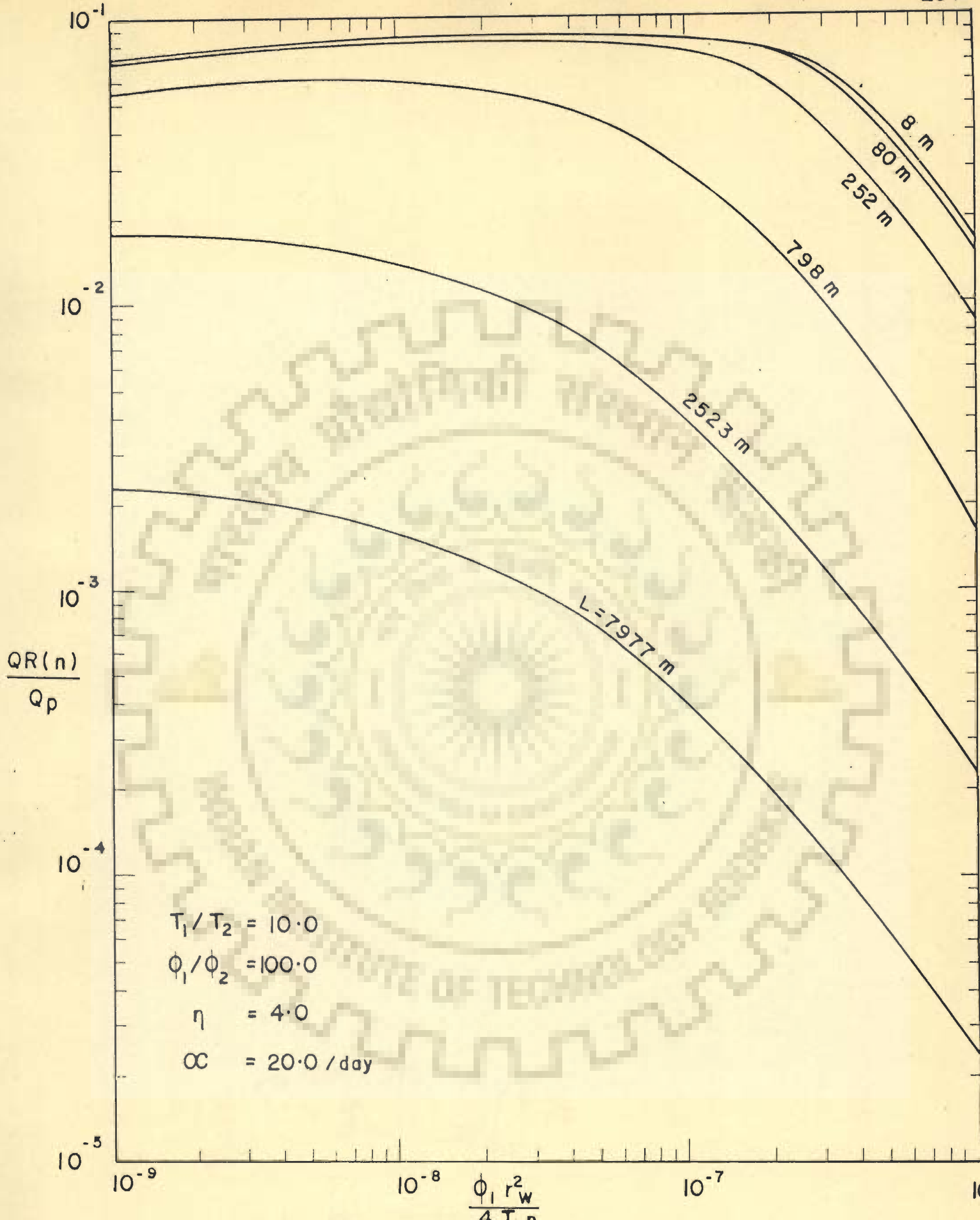

Fig. 6·28 Variation of recharge with time due to pumping of a well tapping an unconfined and a confined aquifer separated by an aquitard.

CHAPTER 7

GENERAL CONCLUSIONS

In the present study, unsteady flow to a multi-aquifer well when pumped at a constant rate has been analysed by discrete kernel approach. The discrete kernel approach is found to be quite versatile in solving multiple aquifer well interaction problems. Results for drawdowns and contributions of each of the aquifers to well discharge have been presented pertaining to unsteady flow to a well for the following cases :

When the well taps

a) two confined aquifers separated by an aquiclude,

b) multiple (more than two) aquifers separated by aquicludes,

c) two confined aquifers separated by an aquitard.

Discrete kernel coefficients for drawdown in an unconfined aquifer with delayed yield characteristics have been evaluated using Boulton's solution. An efficient method has been described to evaluate discrete kernel coefficient for drawdown for any value of η ($\eta = \frac{\phi + \phi_y}{\phi}$).

The studies have been extended when the top aquifer is unconfined and has delayed yield characteristics in a two aquifer well systems. The aquifers may be separated by an aquiclude or aquitard.

Based on the study the following conclusions are derived :

1. When the aquifers have equal hydraulic diffusivity values, the contribution by each of the aquifers under continuous constant pumping is independent of time. In such a case the contribution by each aquifer is proportional to its transmissivity value. This is true whether the aquifers are separated by an aquiclude or aquitard.

2. When the aquifers are separated by an aquitard, and both the aquifers have equal hydraulic diffusivities, the exchange of flow between the aquifers through the intervening aquitard under continuous constant pumping is zero irrespective of the magnitude of the leakage factor. The drawdown in both the aquifers at any section are same.

3. When the two aquifers are separated by an aquitard the leakage factor may be defined as $L=\sqrt{\bar{T}C}$, where $\bar{T}$ is the mean value of the transmissivities

of the aquifers tapped. The mean value may either be a geometric or a harmonic or an arithmatic mean. In different two aquifer aquitard set up if the corresponding hydraulic diffusivity values are equal (i.e., β_1, β_2 of one case equal β_1, β_2 of the other) and their leakage factors are also equal the recharge rates are identical.

4. In a two aquifer system separated by an aquiclude, the aquifer whose hydraulic diffusivity is lower, its contribution to well discharge is higher in the beginning of pumping. As pumping continues its contribution to well discharge decreases. Conversely, the contribution of the aquifer having higher value of hydraulic diffusivity increases with time.

5. If pumping continues for a long time leading to a nearly steady state condition, the contribution by each of the aquifers is in proportion to its transmissivity value. This is true whether the aquifers are separated by an aquiclude or an aquitard. However when the aquifers are separated by an aquitard, the nearly steady state condition is attained comparatively at a shorter time.

6. In case of multiple aquifers (more than two) separated by aquicludes, when the aquifers tapped have equal diffusivity values, their contributions are proportional to the respective transmissivity values. The aquifer whose hydraulic diffusivity is lowest, contributes more than other aquifers during the beginning of pumping. When the time is very large i.e. at nearly steady state condition, the contributions of the aquifers are proportional to their respective transmissivity values.

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APPENDIX-I

ELEMENTS OF MATRIX [A] AND [B]

$$A(1,1) = D(1,1,1) + D(1,7,1) + D(1,43,1) + D(1,49,1) + \frac{B_1}{K_1(\Delta X)^2},$$

$$A(1,2) = D(1,2,1) + D(1,6,1) + D(1,8,1) + D(1,14,1) + D(1,36,1) + D(1,42,1) + D(1,44,1) + D(1,48,1),$$

$$A(1,3) = D(1,3,1) + D(1,5,1) + D(1,15,1) + D(1,21,1) + D(1,29,1) + D(1,35,1) + D(1,45,1) + D(1,47,1),$$

$$A(1,4) = D(1,4,1) + D(1,22,1) + D(1,28,1) + D(1,46,1),$$

$$A(1,5) = D(1,9,1) + D(1,13,1) + D(1,37,1) + D(1,41,1),$$

$$A(1,6) = D(1,10,1) + D(1,12,1) + D(1,16,1) + D(1,20,1) + D(1,30,1) + D(1,34,1) + D(1,38,1) + D(1,40,1),$$

$$A(1,7) = D(1,11,1) + D(1,23,1) + D(1,27,1) + D(1,39,1),$$

$$A(1,8) = D(1,17,1) + D(1,19,1) + D(1,31,1) + D(1,33,1),$$

$$A(1,9) = D(1,18,1) + D(1,24,1) + D(1,26,1) + D(1,32,1),$$

$$A(1,10) = D(1,25,1),$$

$$A(1,11) = \partial_1(1,25,1),$$

$$A(1,12) = - \partial_2(1,25,1),$$

$$A(2,1) = D(8,1,1) + D(8,7,1) + D(8,43,1) + D(8,49,1),$$

$$A(2,2) = D(8,2,1) + D(8,6,1) + D(8,8,1) + D(8,14,1) + D(8,36,1) + D(8,42,1) + D(8,44,1) + D(8,48,1) + \frac{B_1}{K_1(\Delta X)^2},$$

$$A(2,3) = D(8,3,1) + D(8,5,1) + D(8,15,1) + D(8,21,1) + D(8,29,1) + D(8,35,1) + D(8,45,1) + D(8,47,1),$$

$$A(2,4) = D(8,4,1) + D(8,22,1) + D(8,28,1) + D(8,46,1),$$

$$A(2,5) = D(8,9,1) + D(8,13,1) + D(8,37,1) + D(8,41,1),$$

$$A(2,6) = D(8,10,1) + D(8,12,1) + D(8,16,1) + D(8,20,1) + D(8,30,1) + D(8,34,1) + D(8,38,1) + D(8,40,1),$$

$$A(2,7) = D(8,11,1) + D(8,23,1) + D(8,27,1) + D(8,39,1),$$

$$A(2,8) = D(8,17,1) + D(8,19,1) + D(8,31,1) + D(8,33,1),$$

$$A(2,9) = D(8,18,1) + D(8,24,1) + D(8,26,1) + D(8,32,1),$$

$$A(2,10) = D(8,25,1),$$

$$A(2,11) = \partial_1(8,25,1),$$

$$A(2,12) = -\partial_2(8,25,1),$$

$$A(3,1) = D(15,1,1) + D(15,7,1) + D(15,43.1) + D(15,49,1),$$

A(3,2) = D(15,2,1) + D(15,6,1) + D(15,8,1) + D(15,14,1)
+ D(15,36,1) + D(15,42,1) + D(15,44,1) + D(15,48,1),

A(3,3) = D(15,3,1) + D(15,5,1) + D(15,15,1) + D(15,21,1)
+ D(15,29,1) + D(15,35,1) + D(15,45,1) + D(15,47,1)
$+ \frac{B_1}{K_1 (\Delta X)^2}$,

A(3,4) = D(15,4,1) + D(15,22,1) + D(15,28,1) + D(15,46,1),

A(3,5) = D(15,9,1) + D(15,13,1) + D(15,37,1) + D(15,41,1),

A(3,6) = D(15,10,1) + D(15,12,1) + D(15,16,1) + D(15,20,1)
+ D(15,30,1) + D(15,34,1) + D(15,38,1)
+ D(15,40,1),

A(3,7) = D(15,11,1) + D(15,23,1) + D(15,27,1) + D(15,39,1),

A(3,8) = D(15,17,1) + D(15,19,1) + D(15,31,1) + D(15,33,1),

A(3,9) = D(15,18,1) + D(15,24,1) + D(15,26,1) + D(15,32,1),

A(3,10)= D(15,25,1),

A(3,11)= ∂_1(15,25,1),

A(3,12)= $-\partial_2$(15,25,1),

A(4,1) = D(22,1,1) + D(22,7,1) + D(22,43,1) + D(22,49,1),

$A(4,2) = D(22,2,1) + D(22,6,1) + D(22,8,1) + D\ (22,14,1),$
$+ D(22,36,1) + D(22,42,1) + D(22,44,1)$
$+ D(22,48,1),$

$A(4,3) = D(22,3,1) + D(22,5,1) + D(22,15,1) + D(22,21,1)$
$+ D(22,29,1) + D(22,35,1) + D(22,45,1)$
$+ D(22,47,1),$

$$A(4,4) = D(22,4,1) + D(22,22,1) + D(22,28,1) + D(22,46,1) + \frac{B_1}{K_1(\Delta X)^2},$$

$A(4,5) = D(22,9,1) + D(22,13,1) + D(22,37,1) + D(22,41,1),$

$A(4,6) = D(22,10,1) + D(22,12,1) + D(22,16,1) + D(22,20,1)$
$+ D(22,30,1) + D(22,34,1) + D(22,38,1)$
$+ D(22,40,1),$

$A(4,7) = D(22,11,1) + D(22,23,1) + D(22,27,1) + D(22,39,1),$

$A(4,8) = D(22,17,1) + D(22,19,1) + D(22,31,1) + D(22,33,1),$

$A(4,9) = D(22,18,1) + D(22,24,1) + D(22,26,1) + D(22,32,1),$

$A(4,10)= D(22,25,1),$

$A(4,11)= \partial_1(22,25,1),$

$A(4,12)= -\partial_2(22,25,1),$

$A(5,1) = D(9,1,1) + D(9,7,1) + D(9,43,1) + D(9,49,1),$

$$A(5,2) = D(9,2,1) + D(9,6,1) + D(9,8,1) + D(9,14,1) + D(9,36,1) + D(9,42,1) + D(9,44,1) + D(9,48,1),$$

$$A(5,3) = D(9,3,1) + D(9,5,1) + D(9,15,1) + D(9,21,1) + D(9,29,1) + D(9,35,1) + D(9,45,1) + D(9,47,1),$$

$$A(5,4) = D(9,4,1) + D(9,22,1) + D(9,28,1) + D(9,46,1),$$

$$A(5,5) = D(9,9,1) + D(9,13,1) + D(9,37,1) + D(9,41,1) + \frac{B_1}{K_1(\Delta X)^2}$$

$$A(5,6) = D(9,10,1) + D(9,12,1) + D(9,16,1) + D(9,20,1) + D(9,30,1) + D(9,34,1) + D(9,38,1) + D(9,40,1),$$

$$A(5,7) = D(9,11,1) + D(9,23,1) + D(9,27,1) + D(9,39,1),$$

$$A(5,8) = D(9,17,1) + D(9,19,1) + D(9,31,1) + D(9,33,1),$$

$$A(5,9) = D(9,18,1) + D(9,24,1) + D(9,26,1) + D(9,32,1),$$

$$A(5,10) = D(9,25,1),$$

$$A(5,11) = \partial_1(9,25,1),$$

$$A(5,12) = -\partial_2(9,25,1),$$

$$A(6,1) = D(16,1,1) + D(16,7,1) + D(16,43,1) + D(16,49,1),$$

$$A(6,2) = D(16,2,1) + D(16,6,1) + D(16,8,1) + D(16,14,1) + D(16,36,1) + D(16,42,1) + D(16,44,1) + D(16,48,1),$$

$$A(6,3) = D(16,3,1) + D(16,5,1) + D(16,15,1) + D(16,21,1) + D(16,29,1) + D(16,35,1) + D(16,45,1) + D(16,47,1),$$

$$A(6,4) = D(16,4,1) + D(16,22,1) + D(16,28,1) + D(16,46,1),$$

$$A(6,5) = D(16,9,1) + D(16,13,1) + D(16,37,1) + D(16,41,1),$$

$$A(6,6) = D(16,10,1) + D(16,12,1) + D(16,16,1) + D(16,20,1) + D(16,30,1) + D(16,34,1) + D(16,38,1) + D(16,40,1) + \frac{B_1}{K_1(\Delta X)^2},$$

$$A(6,7) = D(16,11,1) + D(16,23,1) + D(16,27,1) + D(16,39,1),$$

$$A(6,8) = D(16,17,1) + D(16,19,1) + D(16,31,1) + D(16,33,1),$$

$$A(6,9) = D(16,18,1) + D(16,24,1) + D(16,26,1) + D(16,32,1),$$

$$A(6,10) = D(16,25,1),$$

$$A(6,11) = \partial_1(16,25,1),$$

$$A(6,12) = -\partial_2(16,25,1),$$

$$A(7,1) = D(23,1,1) + D(23,7,1) + D(23,43,1) + D(23,49,1),$$

$$A(7,2) = D(23,2,1) + D(23,6,1) + D(23,8,1) + D(23,14,1) + D(23,36,1) + D(23,42,1) + D(23,44,1) + D(23,48,1),$$

$$A(7,3) = D(23,3,1) + D(23,5,1) + D(23,15,1) + D(23,21,1)$$
$$+ D(23,29,1) + D(23,35,1) + D(23,45,1)$$
$$+ D(23,47,1),$$

$$A(7,4) = D(23,4,1) + D(23,22,1) + D(23,28,1) + D(23,46,1),$$

$$A(7,5) = D(23,9,1) + D(23,13,1) + D(23,37,1) + D(23,41,1),$$

$$A(7,6) = D(23,10,1) + D(23,12,1) + D(23,16,1)$$
$$+ D(23,20,1) + D(23,30,1) + D(23,34,1)$$
$$+ D(23,38,1) + D(23,40,1),$$

$$A(7,7) = D(23,11,1) + D(23,23,1) + D(23,27,1) + D(23,39,1)$$
$$+ \frac{B_1}{K_1(\Delta X)^2},$$

$$A(7,8) = D(23,17,1) + D(23,19,1) + D(23,31,1) + D(23,33,1),$$

$$A(7,9) = D(23,18,1) + D(23,24,1) + D(23,26,1) + D(23,32,1),$$

$$A(7,10) = D(23,25,1),$$

$$A(7,11) = \partial_1(23,25,1),$$

$$A(7,12) = -\partial_2(23,25,1),$$

$$A(8,1) = D(17,1,1) + D(17,7,1) + D(17,43,1) + D(17,49,1),$$

$$A(8,2) = D(17,2,1) + D(17,6,1) + D(17,8,1)$$
$$+ D(17,14,1) + D(17,36,1) + D(17,42,1)$$
$$+ D(17,44,1) + D(17,48,1),$$

$$A(8,3) = D(17,3,1) + D(17,5,1) + D(17,15,1) + D(17,21,1) + D(17,29,1) + D(17,35,1) + D(17,45,1) + D(17,47,1),$$

$$A(8,4) = D(17,4,1) + D(17,22,1) + D(17,28,1) + D(17,46,1),$$

$$A(8,5) = D(17,9,1) + D(17,13,1) + D(17,37,1) + D(17,41,1),$$

$$A(8,6) = D(17,10,1) + D(17,12,1) + D(17,16,1) + D(17,20,1) + D(17,30,1) + D(17,34,1) + D(17,38,1) + D(17,40,1),$$

$$A(8,7) = D(17,11,1) + D(17,23,1) + D(17,27,1) + D(17,39,1),$$

$$A(8,8) = D(17,17,1) + D(17,19,1) + D(17,31,1) + D(17,33,1) + \frac{B_1}{K_1(\Delta X)^2},$$

$$A(8,9) = D(17,18,1) + D(17,24,1) + D(17,26,1) + D(17,32,1),$$

$$A(8,10) = D(17,25,1),$$

$$A(8,11) = \partial_1(17,25,1),$$

$$A(8,12) = -\partial_2(17,25,1),$$

$$A(9,1) = D(24,1,1) + D(24,7,1) + D(24,43,1) + D(24,49,1),$$

$$A(9,2) = D(24,2,1) + D(24,6,1) + D(24,8,1) + D(24,14,1) + D(24,36,1) + D(24,42,1) + D(24,44,1) + D(24,48,1),$$

$$A(9,3) = D(24,3,1) + D(24,5,1) + D(24,15,1) + D(24,21,1) + D(24,29,1) + D(24,35,1) + D(24,45,1) + D(24,47,1),$$

$$A(9,4) = D(24,4,1) + D(24,22,1) + D(24,28,1) + D(24,46,1),$$

$$A(9,5) = D(24,9,1) + D(24,13,1) + D(24,37,1) + D(24,41,1),$$

$$A(9,6) = D(24,10,1) + D(24,12,1) + D(24,16,1) + D(24,20,1) + D(24,30,1) + D(24,34,1) + D(24,38,1) + D(24,40,1),$$

$$A(9,7) = D(24,11,1) + D(24,43,1) + D(24,27,1) + D(24,39,1),$$

$$A(9,8) = D(24,17,1) + D(24,19,1) + D(24,31,1) + D(24,33,1),$$

$$A(9,9) = D(24,18,1) + D(24,24,1) + D(24,26,1) + D(24,32,1) + \frac{B_1}{K_1(\Delta X)^2},$$

$$A(9,10) = D(24,25,1),$$

$$A(9,11) = \partial_1(24,25,1),$$

$$A(9,12) = -\partial_2(24,25,1),$$

$$A(10,1) = D(25,1,1) + D(25,7,1) + D(25,43,1) + D(25,49,1),$$

$$A(10,2) = D(25,2,1) + D(25,6,1) + D(25,8,1) + D(25,14,1) + D(25,36,1) + D(25,42,1) + D(25,44,1) + D(25,48,1),$$

$$A(10,3) = D(25,3,1) + D(25,5,1) + D(25,15,1) + D(25,21,1) + D(25,29,1) + D(25,35,1) + D(25,45,1) + D(25,47,1),$$

$$A(10,4) = D(25,4,1) + D(25,22,1) + D(25,28,1) + D(25,46,1),$$

$$A(10,5) = D(25,9,1) + D(25,13,1) + D(25,37,1) + D(25,41,1),$$

$$A(10,6) = D(25,10,1) + D(25,12,1) + D(25,16,1) + D(25,20,1) + D(25,30,1) + D(25,34,1) + D(25,38,1) + D(25,40,1),$$

$$A(10,7) = D(25,11,1) + D(25,23,1) + D(25,27,1) + D(25,39,1),$$

$$A(10,8) = D(25,17,1) + D(25,19,1) + D(25,31,1) + D(25,33,1),$$

$$A(10,9) = D(25,18,1) + D(25,24,1) + D(25,26,1) + D(25,32,1),$$

$$A(10,10) = D(25,25,1) + \frac{B_1}{K_1(\Delta X)^2},$$

$$A(10,11) = \partial_{1w}(1),$$

$$A(10,12) = -\partial_{2w}(1),$$

$$A(11,1) = D(25,1,1) + D(25,7,1) + D(25,43,1) + D(25,49,1),$$

$$A(11,2) = D(25,2,1) + D(25,6,1) + D(25,8,1) + D(25,14,1) + D(25,36,1) + D(25,42,1) + D(25,44,1) + D(25,48,1),$$

$$A(11,3) = D(25,3,1) + D(25,5,1) + D(25,15,1) + D(25,21,1) + D(25,29,1) + D(25,35,1) + D(25,45,1) + D(25,47,1),$$

$$A(11,4) = D(25,4,1) + D(25,22,1) + D(25,28,1) + D(25,46,1),$$

$$A(11,5) = D(25,9,1) + D(25,13,1) + D(25,37,1) + D(25,41,1),$$

$$A(11,6) = D(25,10,1) + D(25,12,1) + D(25,16,1) + D(25,20,1) + D(25,30,1) + D(25,34,1) + D(25,38,1) + D(25,40,1),$$

$$A(11,7) = D(25,11,1) + D(25,23,1) + D(25,27,1) + D(25,39,1),$$

$$A(11,8) = D(25,17,1) + D(25,19,1) + D(25,31,1) + D(25,33,1),$$

$$A(11,9) = D(25,18,1) + D(25,24,1) + D(25,26,1) + D(25,32,1),$$

$$A(11,10) = D(25,25,1),$$

$$A(11,11) = \partial_{1w}(1),$$

$$A(11,12) = -\partial_{2w}(1),$$

$$A(12,1) = A(12,2) = A(12,3) = A(12,4) = A(12,5) = A(12,6) = A(12,7) = A(12,8) = A(12,9) = A(12,10) = 0,$$

$A(12,11) = 1$, and

$A(12,12) = 1$.

In above expressions ,

$$D(o,e,n) = \partial_1(o,e,n) + \partial_2(o,e,n).$$

Where o is the observation point, e is excitation point and n is time step.

For n = 1

B(1) = 0 = B(2) = B(3) = B(4) = B(5) = B(6) = B(7) = B(8) = B(9) = B(10) = B(11),

$B(12) = Q_p$.

For $n \geq 2$

$$B(1) = \sum_{\gamma=1}^{n-1} Q_2(\gamma)\, \partial_2(1,\ 25,\ n-\gamma+1)$$
$$- \sum_{g=1}^{49}\sum_{\gamma=1}^{n-1} Q_r(g,\gamma)\, [\partial_2(1,\ g,\ n-\gamma+1) + \partial_1(1,g,n-\gamma+1)]$$
$$- \sum_{\gamma=1}^{n-1} Q_1(\gamma)\, \partial_1(1,\ 25,\ n-\gamma+1),$$

$$B(2) = \sum_{\gamma=1}^{n-1} Q_2(\gamma)\, \partial_2(8,\ 25,\ n-\gamma+1)$$
$$- \sum_{\gamma=1}^{n-1} Q_1(\gamma)\, \partial_1(8,\ 25,\ n-\gamma+1)$$
$$- \sum_{g=1}^{49}\sum_{\gamma=1}^{n-1} Q_r(g,\gamma)\, [\partial_2(8,\ g,\ n-\gamma+1) + \partial_1(8,g,n-\gamma+1)],$$

$$B(3) = \sum_{\gamma=1}^{n-1} Q_2(\gamma)\, \partial_2(15,25,n-\gamma+1)$$
$$- \sum_{\gamma=1}^{n-1} Q_1(\gamma)\, \partial_1(15,25,n-\gamma+1)$$
$$- \sum_{g=1}^{49} \sum_{\gamma=1}^{n-1} Q_r(g,\gamma)\, [\partial_2(15,g,n-\gamma+1) + \partial_1(15,g,n-\gamma+1)],$$

$$B(4) = \sum_{\gamma=1}^{n-1} Q_2(\gamma)\, \partial_2(22, 25, n-\gamma+1)$$
$$- \sum_{\gamma=1}^{n-1} Q_1(\gamma)\, \partial_1(22, 25, n-\gamma+1)$$
$$- \sum_{g=1}^{49} \sum_{\gamma=1}^{n-1} Q_r(g,\gamma)\, [\partial_2(22,g,n-\gamma+1) + \partial_1(22,g,n-\gamma+1)],$$

$$B(5) = \sum_{\gamma=1}^{n-1} Q_2(\gamma)\, \partial_2(9, 25, n-\gamma+1)$$
$$- \sum_{\gamma=1}^{n-1} Q_1(\gamma)\, \partial_1(9, 25, n-\gamma+1)$$
$$- \sum_{g=1}^{49} \sum_{\gamma=1}^{n-1} Q_r(g,\gamma)\, [\partial_2(9, g, n-\gamma+1) + \partial_1(9, g, n-\gamma+1)],$$

$$B(6) = \sum_{\gamma=1}^{n-1} Q_2(\gamma)\, \partial_2(16, 25, n-\gamma+1)$$
$$- \sum_{\gamma=1}^{n-1} Q_1(\gamma)\, \partial_1(16, 25, n-\gamma+1)$$
$$- \sum_{g=1}^{49} \sum_{\gamma=1}^{n-1} Q_r(g,\gamma)\, [\partial_2(16, g, n-\gamma+1) + \partial_1(16, g, n-\gamma+1)],$$

$$B(7) = \sum_{\gamma=1}^{n-1} Q_2(\gamma)\, \partial_2(23,\ 25,\ n-\gamma+1)$$

$$- \sum_{\gamma=1}^{n-1} Q_1(\gamma)\, \partial_1(23,\ 25,\ n-\gamma+1)$$

$$- \sum_{g=1}^{49} \sum_{\gamma=1}^{n-1} Q_r(g,\gamma)\, [\partial_2(23,g,n-\gamma+1) + \partial_1(23,g,n-\gamma+1)],$$

$$B(8) = \sum_{\gamma=1}^{n-1} Q_2(\gamma)\, \partial_2(17,\ 25,\ n-\gamma+1)$$

$$- \sum_{\gamma=1}^{n-1} Q_1(\gamma)\, \partial_1(17,\ 25,\ n-\gamma+1)$$

$$- \sum_{g=1}^{49} \sum_{\gamma=1}^{n-1} Q_r(g,\gamma)\, [\partial_2(17,\ g,\ n-\gamma+1) + \partial_1(17,\ g,\ n-\gamma+1)],$$

$$B(9) = \sum_{\gamma=1}^{n-1} Q_2(\gamma)\, \partial_2(24,\ 25,\ n-\gamma+1)$$

$$- \sum_{\gamma=1}^{n-1} Q_1(\gamma)\, \partial_1(24,\ 25,\ n-\gamma+1)$$

$$- \sum_{g=1}^{49} \sum_{\gamma=1}^{n-1} Q_r(g,\gamma)\, [\partial_2(24,\ g,\ n-\gamma+1) + \partial_1(24,\ g,\ n-\gamma+1)],$$

$$B(10) = \sum_{\gamma=1}^{n-1} Q_2(\gamma)\, \partial_{rw2}\,(n-\gamma+1)$$

$$- \sum_{\gamma=1}^{n-1} Q_1(\gamma)\, \partial_{rw1}\,(n-\gamma+1)$$

$$- \sum_{g=1}^{49} \sum_{\gamma=1}^{n-1} Q_r(g,\gamma)\, [\partial_2(25,g,n-\gamma+1) + \partial_1(25,g,n-\gamma+1)],$$

$B(11) = B(10),$

$B(12) = Q_p.$