MATHEMATICAL MODELLING OF GROUNDWATER SYSTEM

A THESIS

Submitted in fulfilment of the requirements for the award of the degree

of

DOCTOR OF PHILOSOPHY

in

HYDROLOGY

By

DEEPAK KASHYAP

SCHOOL OF HYDROLOGY
UNIVERSITY OF ROORKEE
ROORKEE-247 672 (INDIA)

July, 1981

CANDIDATE'S DECLARATION

I hereby certify that the work which is being presented in the thesis entitled MATHEMATICAL MODELLING OF GROUNDWATER SYSTEM in fulfilment of the requirement of the award of the Degree of Doctor of Philosophy, submitted in the School of Hydrology of the University is an authentic record of my own work carried out during the period from December 1977 to July 1981 under the supervision of Dr. Satish Chandra.

The matter embodied in this thesis has not been submitted by me for the award of any other degree.

Deepak Kashyap

Deepak Kashyap

This is to certify that the above statement made by the candidate is correct to the best of my knowledge.

Satish Chandra

Satish Chandra
Professor
School of Hydrology
University of Roorkee,
Roorkee (U.P.)

ROORKEE

Dated July 24, 1981

ACKNOWLEDGEMENTS

I express my deep sense of gratitude to Dr. Satish Chandra, Professor of Hydrology for his keen interest, guidance and encouragement throughout the course of the present study. This venture of research could be taken up and sustained only because of the excellent research facilities made available by him.

I also wish to express my gratitude to Dr. G.C. Mishra, Reader in Hydrology for his keen interest in the study.

I am indebted to Shri Prakash Bahadur for providing the financial support from the funds of Groundwater Investigation Organisation, U.P., during the earlier part of the study. The financial support provided by Central Groundwater Board in the subsequent period is also gratefully acknowledged.

I am thankful to Dr. B.S. Mathur, Coordinator, School of Hydrology for making available facilities of the School.

I wish to express my sincere thanks and a sense of gratitude to my friend Dr. Susanta Basu, Reader in Earthquake Engineering for his keen interest in the work, meaningful discussions and very substantial help in the development and operation of computer codes, many a times putting himself into considerable personal inconvenience.

I am thankful to Mr. G.G. Dikshit for making available the pumpage data of Daha area, very painstakingly derived from the original records. I am thankful to him and Mr. A.R. Bakshi for many meaningful discussions and ideas.

I feel grateful to my wife Neena for encouraging to take up the research career and for her support and patience throughout the study.

I wish to thank Mr. Prakash Lal and Mr. Shri Chand Sharma for their expert typing and patience. Thanks are also due to Mr. S.C. Sharma and Mr. Vimal Chand for making drawings.

Deepak Kashyap

ABSTRACT

Mathematical models of groundwater system have become an important component of water resources planning. These models assist the planners in arriving at optimal groundwater development policies either by studying large number of alternatives and choosing the best one or by more objective optimisation methods. Inspite of intensive research in recent years, the use of groundwater modelling techniques in real situations is ridden with many problems relating either to an inadequacy of data or to the unrealistic assumptions. In the present work an attempt has been made to provide solutions to a few of these problems.

The spacing and orientation of the observation wells are very rarely adequate to meet the data input requirements of a distributed model directly. In addition to this, the irregularly spaced data points render the estimation of the hydraulic gradients and the second spatial derivatives of piezometric head, almost unacceptably subjective. The Lagrangian methods of functional approximation are generally not suitable due to the requirement of a high degree polynomial. A least square approximation can cut down the degree and the size (number of terms) of the polynomial. In the present study the capability of a least square polynomial to attenuate the data noise and a need to restrict its degree and size have been demonstrated by a simulation study. The use of the statistical tests of significance

for arriving at an optimal form of the approximating polynomial has been suggested. These polynomials are amenable to the differentiation and integration, necessary for the estimation of spatial derivative of piezometric heads and the ground water storages.

The estimation of aquifer parameters by solving inverse problem generally requires a prior knowledge of the directions of principal permeabilities and the distribution of the net vertical accretion in space and time. The available field data are generally too inadequate to provide a direct estimation of the directions of principal permeabilities. The rainfall recharge, an important component of the net vertical accretion is generally estimated from the rainfall records employing certain empirical or semi-empirical relations. These relations involve certain parameters which are not directly measurable quantities. In the present work an inverse problem model has been developed which affords an explicit estimation of principal permeability directions and the recharge parameters.

The linear programming based model for arriving at the optimal cropping and groundwater withdrawal patterns neglects the distributed nature of groundwater system. It can in no way incorporate the constraints of restricting the watertable elevations at all the space points during all the periods, within an acceptable range. In the present work, a model has been developed which overcomes these limitations, by incorporating a spatially distributed aquifer response model in the scheme of computations and solving the problem by nonlinear optimisation.

The applicability of these models to field situations has been demonstrated by using the data of Daha area(India).

I N D E X

Index(Contd.)

Index(contd.)

LIST OF FIGURES

LIST OF TABLES

CHAPTER - 1

INTRODUCTION

For the survival of human race it is necessary that the rapid population growth being experienced these days, is matched by a corresponding expansion of agricultural production. This realisation has led to the development of high yielding varieties of crops, increased reliance on chemical fertilizers and more intensive irrigation. All these measures have greatly increased the water requirement for irrigation. To meet this requirement, a large number of major water projects incorporating a dam or a barrage and a network of canals have been implemented. These projects increase the groundwater recharge and in the absence of proper drainage measures result in rise of water table. The excessive rise of water table has severe consequences in the form of waterlogging and salt accumulation. These consequences are indeed so grave that they not only threaten the viability of such projects but endanger the very existence of the civilisation. In poorly drained canal irrigated areas, the groundwater development apart from providing additional water, can also assist in arresting the rise of watertable. The minimum permissible level of pumping in such areas is governed by the requirement of maintaining a minimum depth to watertable. The additional pumped water can be utilised in improving the cropping pattern of the area.

The irrigation requirements of the areas not served by a canal system, are almost entirely met by the groundwater withdrawals. However, large groundwater withdrawals can lower the watertable/piezometric elevations excessively. This excessive lowering of watertable can render many shallow wells dry, reduce the base flows in the hydraulically connected rivers and cause salt water intrusion in coastal aquifers. The reduction of base flow in rivers can adversely affect the ecology of area, operation of the downstream surface water projects and the quality of river water. Therefore, the base flows in these rivers should not be allowed to fall below a minimum level. In such areas artificial recharge may have to be resorted to.

The aquifer response models can be employed to determine quantitatively the extent of pumping required to affect the drainage or the artificial recharge to maintain a predecided minimum watertable elevation . These models are almost always distributed in time and may be lumped or distributed in space. The lumped models can be employed to estimate the average response of the aquifers to a known pumping or recharge pattern. Lumped models, however, do not account adequately for the stream aquifer interaction. Apart from this, the distribution of watertable/piezometric elevations in space and time is determined not only by the total volume of withdrawals or recharge, but also by the way these volumes are distributed in time as well as in space. This calls for the use of spatially distributed models. Although there has been a considerable progress in the development of such models, their applicability to the physical problems is restricted because of the enormity of the data requirement.

The primary data requirement for the distributed models is the historical watertable/piezometric elevations of 'nodal' points. The observation wells very rarely coincide with these points. Thus, the watertable/piezometric elevations of nodal points need to be interpolated from the available data. The graphical procedure of contour drawing for interpolation may not be always dependable. In this work, the use of least square polynomials for approximating the spatial variation of watertable/piezometric elevation has been demonstrated. The least square polynomials apart from interpolation, also assist the estimation of groundwater storages, subsurface horizontal flows and the second spatial derivatives of piezometric head. The second spatial derivatives of piezometric head are required for the estimation of aquifer parameters by inverse problem.

The other major input requirements of distributed aquifer response model ~~is~~ are the distributed aquifer parameters. The current method of evaluating these parameters is based upon the analysis of test pumping data. The procedure of test pumping is far too expensive and elaborate to be carried out at large enough number of locations. In addition to this, the conventional analysis of test pumping data is based upon many assumptions which may not be always realistic. The solution of inverse problem is another viable method of aquifer parameter estimation, and is based upon the use of historical data of piezometric elevations, pumpage, recharge and boundary conditions. One of the major difficulties in the application of these methods is the estimation of historical recharge data. In addition to this, most of the proposed models of inverse problem require a prior knowledge of the principal permeability directions. In this work a digital model of solving inverse problem has been developed, which affords the estimation of aquifer parameters, principal permeability directions and the recharge parameters. The model employs least square polynomial approximation described earlier, for the estimation of spatial derivatives of piezometric head.

The aquifer response models can provide information relating to the adequacy of a pumping pattern in affecting necessary drainage in canal irrigated areas. However, the pumping, apart from providing the drainage, can also help in improving the cropping pattern of the area. Thus, the pumping should be so decided that it, in conjunction with the surface-water supplies can support the most rewarding cropping pattern, and at the same time can provide the necessary drainage. The most rewarding or optimal cropping pattern can be defined as one, which yields the maximal return from the agricultural activity. The current model for arriving at such an optimal pattern makes use of lumped aquifer response equations for checking the adequacy of a given pumping pattern in providing drainage (Roger and Smith, 1970). The lumped approach has many limitations described earlier. In this work, a model has been developed which makes use of a distributed aquifer response to ascertain the adequacy of a pumping pattern, thereby, ensuring that the watertable/piezometric elevations at all the space points during all the periods are within the permissible range of variation. Apart from this the model also provides information relating to the best way, in which the crops can be distributed within the area. The model is essentially based upon linking of an aquifer response model to a non-linear optimisation complex.

The details of the models developed in the present work, along with the relevant literature review are given in the following chapters. The applicability of these models to solve realistic problems has been demonstrated by applying them to a field situation of Krishni-Hindon interbasin(India).

CHAPTER-2

REVIEW OF LITERATURE

2.1 GENERAL

Almost all the equations of the modern saturated groundwater flow theory owe their existance to the pioneering work of Henry Darcy reported in 1856. By means of experiments, he established a linear relation betwoen one-dimensional seepage velocity and the hydraulic gradient in a saturated porous medium (Remson et al, 1971). The equation in its simplest form is given by

$$v = -k \frac{\partial h}{\partial x} \qquad \ldots (2.1)$$

where v is seepage velocity, h is hydraulic head, x is the distance in the direction of flow and k is a constant known as permeability.

Subsequently the continuity equation for steady state two-dimensional flow of incompressible fluids and the Darcy's law were incorporated in a single equation. This introduced the Laplace equation in the literature of saturated flow. The theoretical work leading to this work was done independently by Jules Dupuit, P.Ferchheimer and Charles Slichter (Remson et al, 1971). First unsteady state equation for axisymmetric radial flow towards a fully penetrating discharging well of infinitesimal diameter, in an infinite confined aquifer was derived by Theis (1935). The equation is as follows.

$$s_d = \frac{114.6\ Q}{T}\ W(u)$$

$$u = \frac{1.87\ r^2\ S}{T\ t} \qquad \ldots (2.2)$$

where s_d is the drawdown (in feet) at a distance of r (in feet) from a well from which water is being abstracted at a uniform rate Q (in gpm), t is the time (in days) after the pumping started and W(u) is the well function.

The equation for two-dimensional unsteady state flow in a confined aquifer, accounting for the deformity of the aquifer and compressibility of fluid, was derived by Jacob (1950) and was further modified by DeWiest (1966) and Cooper (1966). The equation in its presently accepted form is given by

$$\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} = \frac{\rho g}{k}\,(SP.EW + ES)\,\frac{\partial h}{\partial t} \qquad \ldots (2.3)$$

where SP is the porosity of aquifer, EW is the compressibility of water, ES is the vertical-aquifer compressibility, ρ is the mass density of water and t is the time coordinate. This equation can be rewritten in the following form

$$T(\partial^2 h/\partial x^2 + \partial^2 h/\partial y^2) = S\ \ \partial h/\partial t$$

$$T = k\ b_s \text{ and } S = \rho g\ b_s(SP.EW + ES) \qquad \ldots (2.4)$$

where T is the transmissibility and S is the storage coefficient of the aquifer and b_s is the saturated thickness.

This equation can be modified to account for anisotropy, heterogeneity and vertical accretion as follows (Jacob, 1950).

$$\frac{\partial}{\partial x}\left(T_{xx}\frac{\partial h}{\partial x}\right) + \frac{\partial}{\partial y}\left(T_{yy}\frac{\partial h}{\partial y}\right) + Q = S\frac{\partial h}{\partial t} \quad \ldots (2.5)$$

where T_{xx} and T_{yy} are the transmissibilities in the directions of principal permeabilities colinear with x and y and Q is the net vertical accretion rate per unit area.

For unconfined aquifers the solutions are greatly facilitated if Dupuit-Forchheimer's assumptions hold good. With these assumptions the governing differential equation for two dimensional transient groundwater flow in an isotropic, homogeneous, unconfined aquifer, with no vertical accretion is as follows (Remson et al, 1971)

$$k\left[\frac{\partial}{\partial x}\left(h\frac{\partial h}{\partial x}\right) + \frac{\partial}{\partial y}\left(h\frac{\partial h}{\partial y}\right)\right] = S\frac{\partial h}{\partial t} \quad \ldots (2.6)$$

In this case, S, refers to the specific yield and is analogous to the storage coefficient defined in equation 2.4. This equation can be modified as follows to account for anisotropy, nonhomogeneity and the vertical accretion (Hall and Dracup, 1970)

$$\frac{\partial}{\partial x}\left(k_{xx}\, h\frac{\partial h}{\partial x}\right) + \frac{\partial}{\partial y}\left(k_{yy}\, h\frac{\partial h}{\partial y}\right) + Q = S\frac{\partial h}{\partial t} \quad \ldots (2.7)$$

x and y refer to the principal permeability directions, and k_{xx} and k_{yy} are the permeabilities in direction x and y respectively.

The governing differential equation for three dimensional groundwater flow can be written as follows :

$$\frac{\partial}{\partial x}(k_{xx}\frac{\partial h}{\partial x}) + \frac{\partial}{\partial y}(k_{yy}\frac{\partial h}{\partial y}) + \frac{\partial}{\partial z}(k_{zz}\frac{\partial h}{\partial z}) = s\frac{\partial h}{\partial t} \quad \ldots (2.8)$$

where x, y, z are the coordinates in principal permeability directions ; k_{xx}, k_{yy}, k_{zz} are the principal permeabilities and s is the specific storage (L^{-1}).

A brief review of the literature pertaining to the present study is given in the following paragraphs.

2.2 AQUIFER RESPONSE

The present state of knowledge in groundwater hydrology provides adequate tools to study the impact of a deterministic groundwater withdrawal and/or recharge pattern on the piezometric elevations through lumped models (Sokolov and Chapman, 1974, Chandra et al, 1975) or distributed models (Remson et al, 1971; Pinder and Gray, 1977). The water balance equation proposed by Sokolov and Chapman (1974) can be written as follows -

Inflows - Outflows - Increase in storage + Error term = 0 ... (2.9)

The inflows include the rainfall recharge, recharge from rivers, subsurface horizontal inflow, artificial recharge and inflow from the other aquifers (overlying or underlying). The outflows include base flow to the rivers, outflow of groundwater into the zone of aeration for moisture recovery lost by evapotranspiration, outflow to overlying or underlying aquifers, subsurface horizontal outflow, groundwater outflow through springs, and groundwater pumped from artisian aquifers. The lumped aquifer response to known inflows or outflows can be obtained by defining the groundwater storage fluctuations in terms of the piezometric head fluctuations and the storage coefficient.

The distributed parameter groundwater flow models are based upon the solutions of differential equations governing two dimensional(equations 2.5 and 2.7) or three dimensional (equation 2.8) transient groundwater flows in saturated zone. Closed form or series solutions of governing differential equations are available only for idealised boundary and recharge conditions (Bear, 1967). These are generally based upon the assumptions of homogeneity and isotropy. For predicting the aquifer response to spatially and temporally varying recharge and pumpage under realistic conditions the governing differential equations have to be solved by appropriate computer assisted numerical methods.

Two distinct types of digital computer based methods are available for obtaining the numerical solution of groundwater flow problems - finite difference and finite element. Finite difference methods employ finite difference approximation to convert the partial differential equation into a determinate set of linear algebraic equations. The discretisation of space, necessary for finite difference approximations, can be based upon either any convenient pattern (Tyson and Webor, 1964) or a more stringent rectangular grid pattern (figure 2.1). The former is generally known as Tyson Webor model and the latter as finite difference method.

In the Tyson Webor model the nodal points need not follow any symmetrical pattern and can be made to coincide with watertable observation points. This can eliminate the need for interpolation to a large extent. This model, though relatively simpler to implement, has limitations relating to the stability and convergence. The finite difference approach is a general method for calculating approximate solution of partial differential equations and was introduced by Richardson (1910). This method has been programmed to solve two dimensional (Bittinger et al,1967; Pinder and Bredehoeft, 1968; Pinder, 1970; Prickett and Lonnquist, 1971; Bedinger et al, 1973; Trescott et al 1976) and three dimensional (Bredehoeft and Pinder, 1970; Trescott, 1975) transient groundwater flow problems.

The implicit form of the finite difference equation will generally require the solution of a large number of linear simultaneous equations and the storage requirement to store all the elements of the coefficient matrix will be prohibitive. This difficulty is overcome by employing various iterative procedures which primarily bank upon the sparseness of the coefficient

a _ TYSON WEBOR MODEL

b _ RECTANGULAR GRID

FIG. 2·1 _ FINITE DIFFERENCE METHODS

matrix. Some of the more commonly used procedures are alternate direction implicit scheme (Peaceman and Rachford, 1955), Crank-Nicolson scheme (Crank and Nicolson, 1947), line successive over relaxation (Trescott et al, 1976) and strongly implicit procedure (Stone, 1968). The numerical properties of these methods have been a subject of large number of reported studies (Rushton, 1974; Tomilson and Rushton, 1975; Trescott and Larson, 1977). Murray and Johnson (1977) have demonstrated the second method of linearisation of Boussinesq's equation for simulating the response of unconfined aquifers. Recently the methodology of finite elements for the numerical solution of differential equations has been increasingly put to use for the evaluation of aquifer response. This followed the development of variational principles for linear initial value problems by Gurtin (1964). The finite element method is reported to circumvent many difficulties relating to irregular geometry of area, hetrogeneity and boundary conditions (France, 1974), in addition to giving results of higher order accuracy (Javandal and Witherspoon, 1968). Other significant contributions are by Neuman and Witherspoon (1970), Pinder and Frind (1972), Neuman, (1973-a), Pinder (1973), Pinder and Gray, (1977).

2.2.1 Recharge

Recharge rate, an input requirement of all aquifer response models, is one of the important factors controlling the amount of water that may be pumped from an aquifer without depleting it. Recharge is a process by which water is added into an aquifer by percolation through unsaturated zone. There is experimental evidence due to Richards, to conclude that

Darcy's law can be applied to unsaturated zone also subject to certain restriction on seepage velocity and with the permeability being a function of moisture content (Remson et al 1971). Richard's equation can be solved by finite difference methods to estimate the groundwater recharge (Krishnamurthy, 1977; Kafri and Asher, 1978). The input data requirement would, amongst others, include transient soil moisture measured as a function of vertical position, permeability - soil moisture curves and evapotranspiration. The other approach of recharge estimation is based upon water budgeting and field capacity

$$P = ET + SRF + \Delta SM \qquad \ldots (2.10)$$

where P is precipitation, ET is actual evapotranspiration, SRF is the surface runoff and ΔSM is the increase of water availability in root zone.

$$\begin{aligned} \text{Recharge} &= \Delta SM - DFCT \quad &\text{if } DFCT < \Delta SM \\ &= 0 &\text{if } DFCT \geq \Delta SM \end{aligned} \qquad \ldots (2.11)$$

where DFCT is the current soil moisture deficiency below field capacity.

Lately, some doubts have been expressed regarding the validity of this conventional method (Watson et al, 1976; Rushton and Ward, 1979). Rushton and Ward (1979) have proposed alternative recharge mechanism which allows recharge to occur even when a soil moisture deficit exists. Morel-Seytoux (1976) has derived formulae for prediction of ponding time and cumulative infiltration, under the influence of rainfall. The derivation is based upon, amongst other things, viscous flow of air. Reeves (1975) has tested a method of recharge estimation, which assumes that the maximum infiltration rate is simply a function of cumulative rainfall regardless of rainfall vs time history. A few reported studies make use of the watertable data for the estimation of recharge (Venetis, 1971; Chandra et al, 1975).

The experimental methods for the measurement of recharge include tritium tracer technique (Dincer et al, 1974; Vogel et al 1974; Allison and Hughes, 1975; Sukhija, et al 1976), gamma ray transmission (Singh and Chandra, 1977). An empirical relation between the rainfall and recharge was given by Chaturvedi (Chaturvedi and Chandra, 1961).

2.2.2 Interpolation of Piezometric Head

The need of interpolation arises because the location of piezometric data points is very rarely decided on the basis of the requirements of the digital models. Thus, in most of the cases the piezometric heads at the nodal points have to be estimated by interpolating the recorded heads. The conventional graphical and Lagrangian methods of interpolation (Ralston, 1965) may not be always suitable because of the subjective bias and artificial undulations associated with the high degree polynomials respectively. Esler et al (1968) reported the use of least square polynomials for getting the spatial variation of piezometric heads. Sagar et al (1973, 1975) demonstrated the use of spline functions to approximate the spatial variation of piezometric heads. The closed form functions so obtained can assist in interpolation. Delhomme (1978) has reported the possible use of Kriging in providing spatial variation of piezometric head. Britles and Morel (1979) proposed a least-square based method of interpolation wherein the unsteady state flow equation is reduced to a steady state by assuming zonewise spatial uniformity of certain quantities.

2.3 INVERSE PROBLEM

One of the most difficult steps in mathematical modelling of groundwater flow is the collection of adequate data relating to the spatial distribution of storage coefficient and transmissibility. The difficulty arises on a number of counts, the principal one being that these properties are not physically measurable quantities (Yeh, 1975). These are in fact parameters appearing in the differential equations governing the aquifer response (piezometric head fluctuations) to the aquifer excitation (pumpage, recharge or change in boundary conditions). The inclusion of transmissibility and storage coefficient in these differential equations is based upon Darcy's law and the continuity equation for unsteady state flow in porous media (Todd, 1959) respectively. The estimation of these parameters is thus, an inverse process wherein these are calculated from the historical excitation response and the associated initial & boundary conditions data. The system in this case is represented by the equation correlating the response to excitation (figure 2.2). Since the same parameters are employed to estimate the aquifer

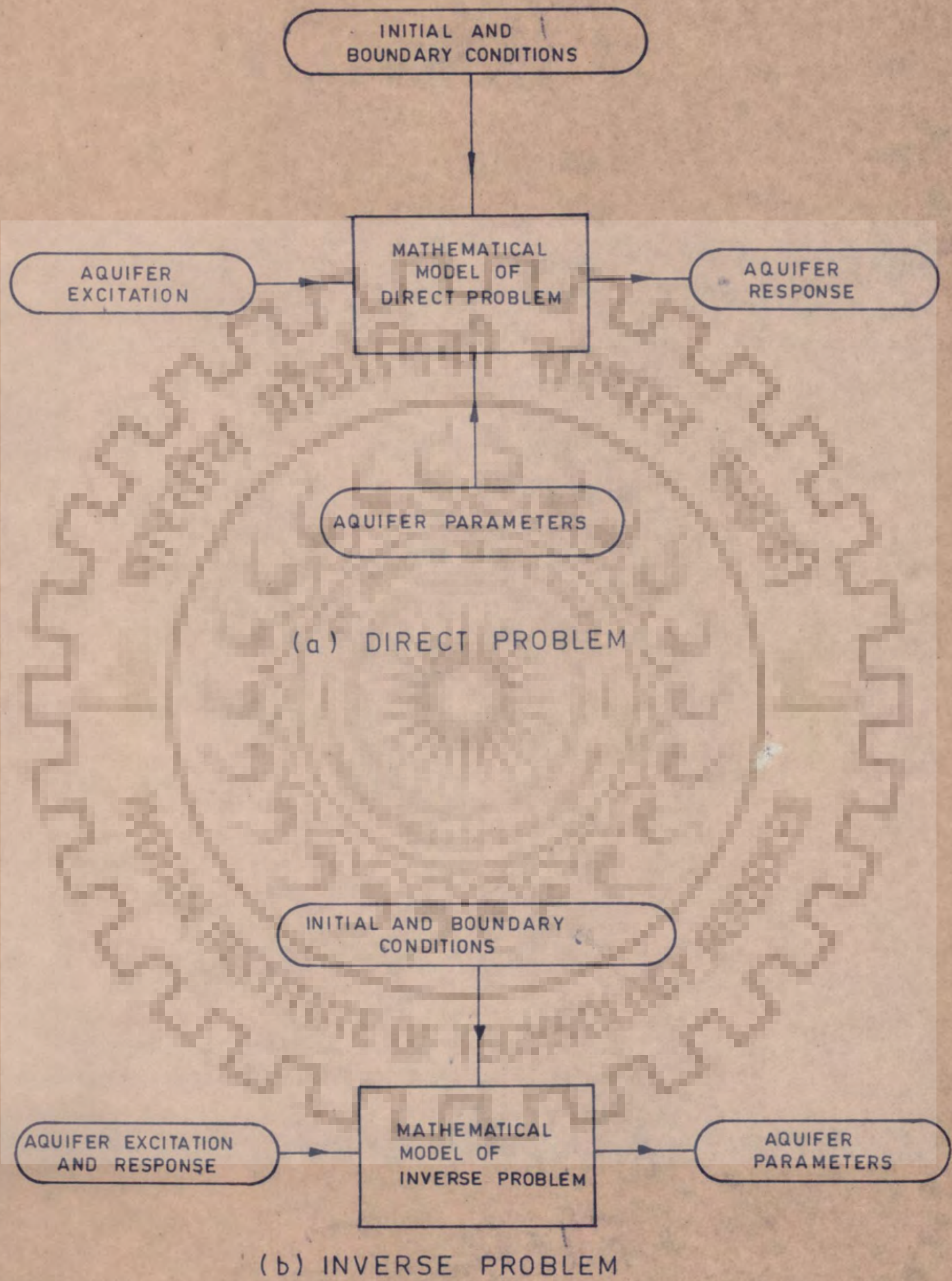


FIG. 2.2 _ DIRECT AND INVERSE PROBLEMS IN GROUNDWATER HYDROLOGY

response to the different forms of excitations and under varying conditions/assumptions, a variety of methods exist for their estimation.

The most widely used method is generally known as test pumping (Kruseman and Ridder, 1970). This method essentially consists of generating and recording the aquifer response under a controlled excitation of the form of pumping from a single well. The generated data are employed to estimate parameters based upon the criterion of 'closest match' between the recorded response and the response as given by the governing equation. In the conventional approach, Theis equation (2.2) which relates the piezometric head response to pumping in a single well under the unsteady state conditions, is adopted as the governing equation; and the closest match is decided rather subjectively by the 'type curve' methods. This approach has two serious limitations; the first being that Theis equation is applicable in case of radial flow in a confined homogenous, isotropic aquifer towards a fully penetrating well of infinitesimal size from which water is abstracted at a constant rate. These requirements are very rigorous and are very rarely fulfilled in real-world situations. The second limitation relates to the inherent subjective bias in the graphical procedure of type curves.

The first limitation can be taken care of to a large extent by modifying the governing equation to account for the violation of some of the restrictive assumptions like fully penetrating well, isotropy and infinite aquifer (Walton, 1970; Lakshminarayana and Rajagopalan, 1978). Yeh and others (Yeh and Tauxe, 1971; Marino and Yeh, 1973) have proposed viable methods for the analysis of test pumping data of unconfined and leaky confined aquifers. The proposed approach employs the technique of quasilinearisation (Bellman and Kalaba, 1965) to linearise the governing nonlinear differential equation; and the least square criterion to define the closest match. A few other studies involving the analysis of test pumping data more objectively have been reported (Pinder and Bredehoeft, 1968, Saleem, 1970).

The other form of the aquifer excitation, which alongwith the corresponding response of piezometric elevations, can be employed to estimate transmissibility and storage coefficient, is the fluctuations of waterlevels in hydraulically connected waterbodies. This can be viewed as excitations induced by change in boundary conditions. Pinder and others (1969) employed iterative procedures to determine aquifer diffusivity from the historical data relating to the watertable elevations and river stage. Yeh (1975) employed quasilinearisation (Bellman and Kalaba, 1965) to estimate diffusivity of an unconfined aquifer by employing historical data of stream-aquifer interaction. Singh and Sagar (1977) developed an analytical method to determine aquifer diffusivity using the Green's function of linearised Boussinesq's equation and specifying an extra boundary condition at the stream aquifer interface. In a subsequently reported study (Sagar and Singh, 1979) they studied the effect of observational errors in the raw data on the values of estimated diffusivity. Almost all the proposed methods have many built-in assumptions, the more restrictive amongst these are one dimensional flow, absence of vertical accretion and the existence of impervious boundary condition at the other end of the aquifer. These methods can thus, be employed to estimate aquifer parameters of flood plains. The study reported by Venetis (1971) overcomes the assumption of 'no vertical accretion' in a somewhat limited way by assuming uniform accretion. Other reported contributions are by Ferris (1952) and Rowe (1960).

A more general form of the excitation-response, in the context of parameter estimation of groundwater system, will be the piezometric head fluctuations in space and time as a response to known-pumping/recharge pattern, boundary conditions (incorporating the fluctuations of waterlevel in hydraulically connected rivers, if any) and initial conditions. The estimation of aquifer parameters from such excitation-response data has to be essentially based upon Boussinesq's equation since closed form analytical solutions for such complex system are not available. This approach has been in general termed as 'inverse problem' in groundwater hydrology. The prevalent methods of estimating aquifer parameters by inverse problem can be classified in two categories i.e. indirect and direct (Neuman, 1973-b).

The indirect methods essentially consist of repetitive numerical solution of the Boussinesq's equation with successively modified aquifer parameter values, to affect the closest reproduction of recorded aquifer response under the recorded-excitations (withdrawals, recharge), initial conditions and boundary conditions. The modification in the aquifer parameter values at the end of each iteration can be either based upon mathematically rigorous procedures (Knowles et al, 1972; Evenson and Johnson, 1975), or upon subjective judgement. The latter approach is also known as 'calibration' or 'subjective optimisation'.

In the direct methods of solving inverse problem, Boussinesq's equation is directly solved for the aquifer parameters. The equation (2.5) when rewritten with transmissibility as the unknown, becomes a first order partial differential equation and can be written as follows :

$$\frac{\partial h}{\partial x} \frac{\partial T_{xx}}{\partial x} + \frac{\partial h}{\partial y} + \frac{\partial T_{yy}}{\partial y} + (T_{xx} \frac{\partial^2 h}{\partial x^2} + T_{yy} \frac{\partial^2 h}{\partial y^2} + Q) = S\frac{\partial h}{\partial t} \quad \ldots (2.12)$$

The boundary conditions necessary for solving this partial differential equation directly for transmissibility, consist of transmissibility values at the boundaries, also known as Cauchy data (Emsellem and DeMarsily, 1971). This approach can afford the estimation of spatially distributed transmissibility values from the piezometric and vertical accretion data of a single time period, provided the data are elaborate enough to permit the estimation of spatially distributed spatial and temporal derivatives of the piezometric head. Since a single uncoupled differential equation can yield the solution of one variable only, the method can not be used to estimate both transmissibility and the storage coefficient. Either the storage coefficient should be an input data-or the piezometric data should correspond to steady-state conditions.This 'differential equation' approach is serously handicapped by the fact that in most of the real situations the necessary 'Cauchy data' are not available. This problem can be overcome by employing multiple-period data. The overdetermined system, so obtained, can be employed to arrive at such values of the parameters which minimise a predecided functional of the residues in the Boussinesq's equation. This approach affords the 'smoothening' of the effects of discrepancies in the raw data. Various forms of functionals can be adopted. It can be proved that 'least square' yields the most likelyhood solution when the errors in raw data are normally distributed (Beck and Arnold,1977).This least-square form of the residue functional, necessitates the use of nonlinear programming.Cooley (1977)proposed the method of regression for the estimation of parameters of a two-dimensional steady-state flow domain. In a subsequent paper (Cooley,1979) he analysed the goodness of fit and significance of computed parameters. Kleinecke (1971) adopted 'mini-max'and'the sum of the modulii' functionals and used linear programming to obtain optimal values of aquifer parameters. The main drawback of this method is that the final solution may not include the parameters of all the grid points. In the aforesaid study, out of 144 nodes, 44 did not produce an estimate of storage coefficient. Transmissibility results were even more sparse; of the 676 estimates of transmissibility, 56 percent were not in the solution.Subs quently, Nevarro (1977) modified the method of Kleinecke to make use of field information relating to the values of storage coefficient and transmissibility.

Sagar and others (1973, 1975) proposed the 'algebraic equation' method which dispenses with the requirement of boundary conditions and is not based upon the discretisation of space by nodal points. The spatial derivatives of hydraulic head are estimated by one dimensional spline functions fitted to the observed hydraulic head data. The boundary conditions are eliminated by studying the water budget at each individual space point and the parameters are estimated by minimising the sum of the squares of the residues (i.e. the imbalances in the microlevel water budgets). The proposed method affords the estimation of transmissibility only.

Emsellem and DeMarsily (1971) for the first time demonstrated the use of multiple objectives, in the context of aquifer parameter estimation. One of the objectives was to minimise the residue functional and the other was to affect 'smoothness' with respect to the variation of transmissibility in space. The suggested approach is to reduce gradually the residue functional by relaxing the 'smoothness' until any further relaxation does not affect the residue functional significantly. The method applies to steady state conditions and thus, can yield only transmissibility estimates. The idea of multiple objectives has been further extended by Neuman, (1973b, 1975). He has proposed a number of criteria to affect 'smoothness' of solution. The problem has been solved by applying parametric linear programming to a finite element based model of groundwater flow.

The finite element method was also used by Frind and Pinder (1973) for solving inverse problem. Haimes and others (1968) employed decomposition and multilevel optimisation, to arrive at optimal aquifer parameters. The method assumes an infinite aquifer. Labadie (1975) proposed the term 'Surrogate parameters' meaning that the estimated parameters may be quite different from the aquifer properties, because these parameters represent a sort of lumped effect of aquifer properties and many other unaccounted local features like heterogeneity, complex boundary conditions etc. This concept was first explicitly expressed in this study, though it is implicit in almost all the inverse problem methods.

Nutbrown (1975) proposed another viable method of estimating transmissibility and storage coefficient. Transmissibility is estimated by studying the waterbalance of a stream tube for a period of zero storage change. The stream tube is extended to a 'turning point' (water divide) and thus, the transmissibility at the other end is determined. Storage coefficient is found by employing the data of period displaying storage changes. This approach was further extended by Britles and Morel (1979) who incorporated 'equivalent steady state' concept for interpolation, in the aforesaid approach. Other significant contributions are - Energy dissipation method by

Nelson (1968), use of hybrid computations by Vemuri and Karplus (1969), study on impact of additional data on calibration by Gates and Kisiel (1974), regression based drawdown model by Maddock (1976), use of subjective information by Lovel and others (1972). Though most of the proposed inverse problems models are aimed at the estimation of aquifer parameters, in a few reported studies the aquifer excitation has also been estimated (Tison, 1965, Monech and Kisiel, 1970, Venetis,1971, Sagar et al,1975).

2.4 OPTIMAL GROUNDWATER WITHDRAWALS AND CONJUNCTIVE USE

It was recognised by Banks in 1953 (Young and Bredehoeft, 1972) that aquifers can serve as water storage reservoirs, and the conjunctive use of releases from aquifer and the surface water supplies can go a long way in increasing the monetary returns.

Buras (1963) solved the problem of arriving at optimal operating policies of the surface and groundwater reservoirs by employing dynamic programming. In this study optimality is defined as the maximised net returns on the basis of certain assumed benefit functions of quadratic and parabolic type for surface and groundwaters respectively. Pumpage is limited by the water available in the aquifer and surface water releases limited by the amount of water stored behind the dam plus the quantity flowing into the reservoir in the current period. The operating policy is defined with respect to a lumped system, thereby ignoring the distributed nature of the groundwater system.

Young and Bredehoeft (1972) proposed a simulation model for solving the management problems of conjunctive use. The model accounts for the stream aquifer interaction and the distributed nature of groundwater flow. The stream aquifer interaction is restricted by a penalty function. The model, being simulation based, does not directly provide the optimal solution but can yield the value of objective function for each of a number of alternative courses of action. From the array of computed outcomes approximate optimum may be identified.

Yu and Haimes (1974) suggested an analysis in which multilevel formulation is used for explicitly coupling the distributed groundwater flow model with a management scheme to optimise conjunctive use of ground and surface waters. Maddock and Haimes (1975) demonstrated the use of 'algebraic technological functions' for coupling a distributed groundwater system with a tax quota management scheme. These functions essentially relate the aquifer response at any point and time to the pumping at all the space point upto the time period under consideration (Maddock, 1972) and unit response functions. These unit response functions can be obtained numerically or analytically and are stored in the computer memory. This approach, quite similar to the discrete kernel approach proposed by Morel-Seytoux (1975), can cut down the computer time significantly but will require large storage especially in case of unsymmetrical large areas. Haimes and Dreizin(1977) used decomposition for analysing the conjunctive use problem. A constant return per unit volume of water anywhere in space and time is considered. The study is extended in a subsequent paper (Dreizin and Haimes, 1977).

One of the earliest reported studies incorporating the distributed aquifer response and the benefits from agricultural activity, is that of Young and Bredehoeft (1972). It explicitly incorporates the areas under different crops as decision variables, though within the frame work of simulation rather than optimisation model. An optimisation model, in this context, was proposed by Roger and Smith (1970) (Figure 2.3). The model is based upon linear programming and the groundwater response is considered as lumped. Linear programming though numerically very elegant is rather restrictive, in the sense that it requires the net benefits and the associated constraints to be expressed as linear functions of the decision variables. This is quite frequently employed because of its computational ease rather than its capability to represent the real systems; which more often than not are nonlinear. In the aforesaid model the constraint of restricting the watertable elevations has been linearised by lumping the system in space. Similarly, the nonlinear nature of the objective function has been linearised by assuming amongst other things lift-independent cost of pumping.

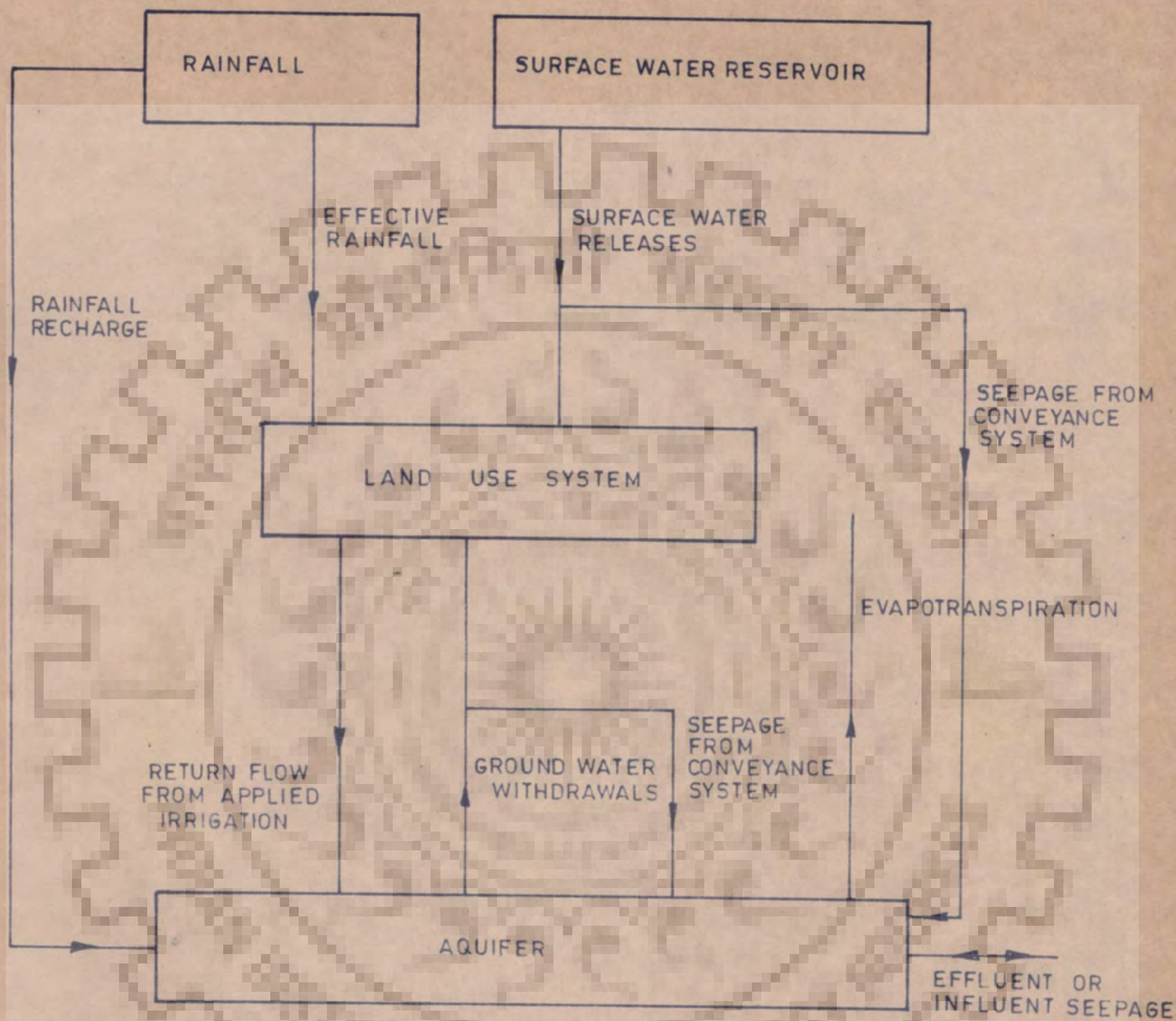


FIG. 2.3 – CONJUNCTIVE USE OF GROUND AND SURFACE WATERS

The distributed nature of the groundwater system can be accounted for in a linear programming model on the lines of the procedure suggested by Eduardo and others (1974). While solving the problem of dewatering with minimum pumping, they imposed the constraints of restricting the spatially distributed watertable elevations below a predecided level, within the framework of linear programming. This was accomplished by treating the steady state nodal watertable elevations as decision variables and including the finite difference forms of the governing differential equation as equality constraints. This approach is feasible when the groundwater withdrawals are uniform in time and the constraints of restricting watertable elevations within a range can be defined in terms of steady state conditions. This may be valid when the water is primarily pumped for use in some perennial industry. In case of time variant periodic withdrawals for agricultural activity the number of decision variables to define aforesaid constraints for all times will blow up to an unmanageable limit. The other notable studies related to the inclusion of the distributed character of groundwater system are by Pinder and Bredehoeft (1968) and Bredehoeft and Pinder (1970).

CHAPTER - 3

LEAST SQUARE POLYNOMIAL APPROXIMATION FOR SPATIAL VARIATION OF PIEZOMETRIC HEAD

3.1 PROBLEM IDENTIFICATION

A historical record of spatial and temporal variation of piezometric head is a prerequisite for carrying out studies for quantitative evaluation of groundwater potential. The available records usually consist of piezometric head measured periodically in observation wells (open wells or piezometers). These discrete point data can be employed to generate continuous functions which approximate the true functional relations between the piezometric head and the spatial coordinates, in the domain bounded by the observation wells. These approximate functional relations can greatly facilitate the subsequent analysis of the data. In the following paragraphs a deterministic method for generating such approximate functional relations has been developed. The method considers the available piezometric data to be made up of the true values and an unmeasured Gaussian data noise.

3.2 PROCESSING OF PIEZOMETRIC DATA

The observation well data can be processed to yield a variety of information relating to the groundwater movement, storages and the piezometric elevations at other locations in space. The data processing requirements are :

(i) Differentiation : The differentiation of the piezometric head with respect to space is necessitated by the linear velocity - hydraulic gradient relation incorporated in Darcy's law (equation 2.1). Thus, the first derivative of piezometric head with respect to space i.e. the hydraulic gradient needs to be calculated for the estimation of subsurface velocities and horizontal recharge. Similarly the second spatial derivatives of piezometric head assist in the calculation of the imbalance between the horizontal inflows and the outflows at any space point.

(ii) Integration : The integration of piezometric head in space is required to be carried out for estimating the water released from or taken into groundwater storage in a given area during a certain duration.

(iii) Interpolation : The interpolation becomes necessary when the nodal points of a distributed parameter groundwater flow model do not coincide with the observation points. The available data of the observation points are employed to assign piezometric heads at the nodal points.

3.3 PREVALENT TECHNIQUES

The aforesaid mathematical operations can be performed by arriving at a spatial variation of piezometric head using stochastic, deterministic or graphical method. Kriging, a stochastic method (Delhomme, 1978) assumes that the groundwater system is too complex to be described by analytical expressions. For a deterministic system analysis, the spatial variation can be arrived at by graphical procedures or functional approximation. Alternatively finite differences can be directly employed to perform the interpolation, differentiation or integration.

The graphical procedures essentially involve the drawing of contours of equal piezometric heads by visual inspection of the piezometric elevations at the observation points. This procedure, though employed quite frequently, has two major handicaps i.e. the strong subjective bias and a subconscious assumption of linear variation of piezometric head between two observation points. These handicaps become all the more pronounced when the data points are sparse or nonuniformly distributed over the space. The finite difference methods of differentiation, integration and interpolation are procedures which in most of the cases, afford the estimation of error bounds. However, the requirement of data positioning along two orthogonal directions is very rarely satisfied.

The 'functional approximation' approach consists of approximating the true functional relation betwen the piezometric head and space coordinates by an explicit function which is directly amenable to the mathematical operations. The functional approximation can either be of 'exact' type i.e. no residues at the observation points or the 'least-square' type (Ralston, 1965). These procedures require the total number of coefficients in the approximating function to be equal to or less than the total number of data points respectively. In case , the general form of the functional relation is not known, a polynomial may be used to approximate the function in a finite domain provided it is known a priori that the function is continuous. This approach is validated by the well known theorem due to Wieirstrass (Ralston, 1965). The Lagrangian methods of 'exact' functional approximation in one dimension by a polynomial can be suitably extended to a two-dimensional situation. This would however, require a polynomial of very high degree which, apart from increasing the computational efforts involved in the numerical operation of the function, can also cause large artificial undulations of the functional estimate in be-between the observation points.

The spline function approximation (Sagar, et al, 1973, 1975) involves passing piecewise continuous polynomial functions through the known functional values at the observation points with the compatibility conditions at the interfaces of the adjacent polynomials satisfied. This approach is a numerical equivalent of fitting by french-curves, figure (3.1). It overcomes the necessity of employing high degree polynomial, but may still cause artificial undulations if the observational data are not completely error-free, as they would very rarely be. The differentiation of these approximating functions to arrive at the derivatives of the true function, may involve large errors originating from even small amplitude 'noise' present in the raw piezometric head data (figure 3.2).

The magnitudes of the 'noise' are unknown but their frequency distribution will be near-Gaussian if the sample size is large enough and there are no systematic errors (Beck and Arnold, 1977). In such situations the smoothening of 'noise' in the raw data can be accomplished by adopting a 'least square polynomial' rather than an 'exact' polynomial, provided the

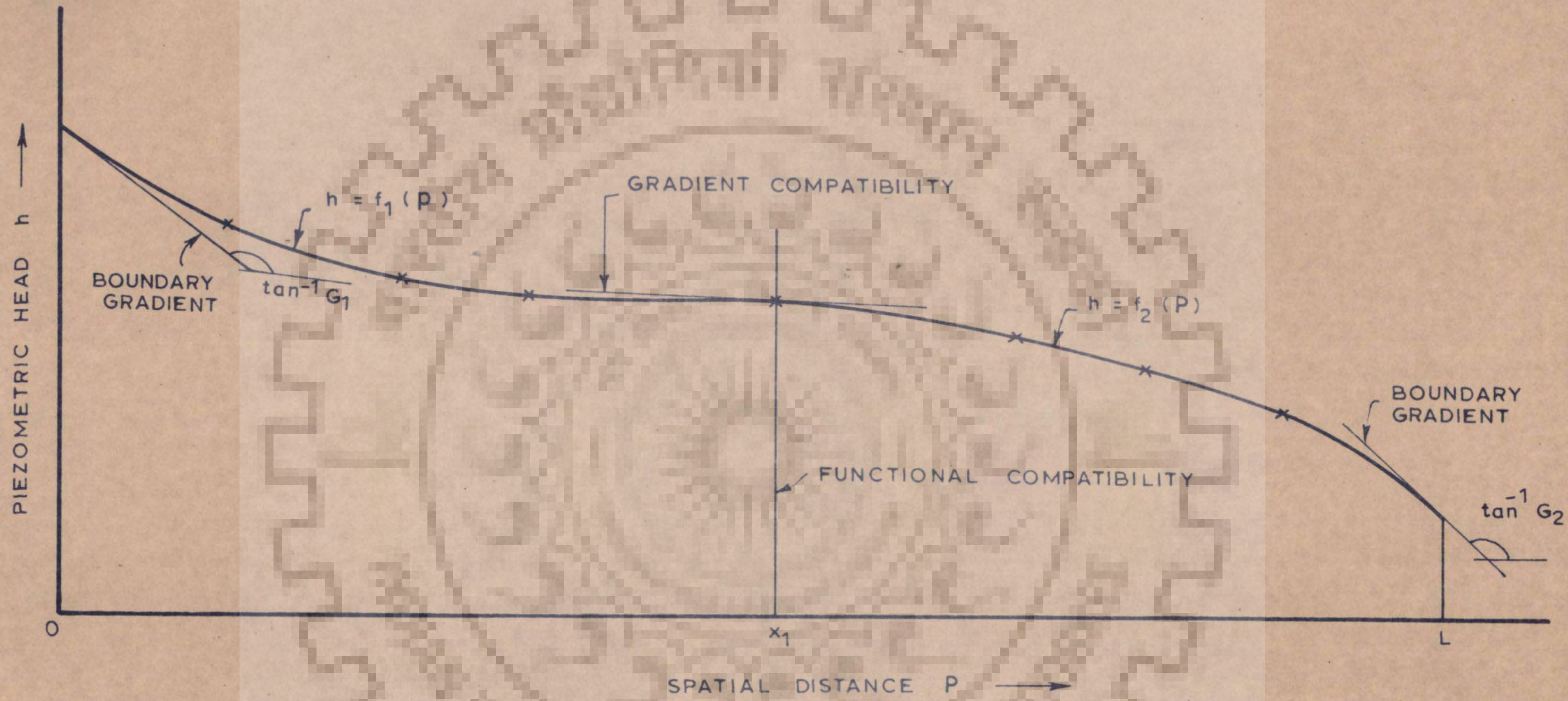


FIG. 3·1 – SPLINE FUNCTION NUMERICAL FRENCH CURVE FITTING

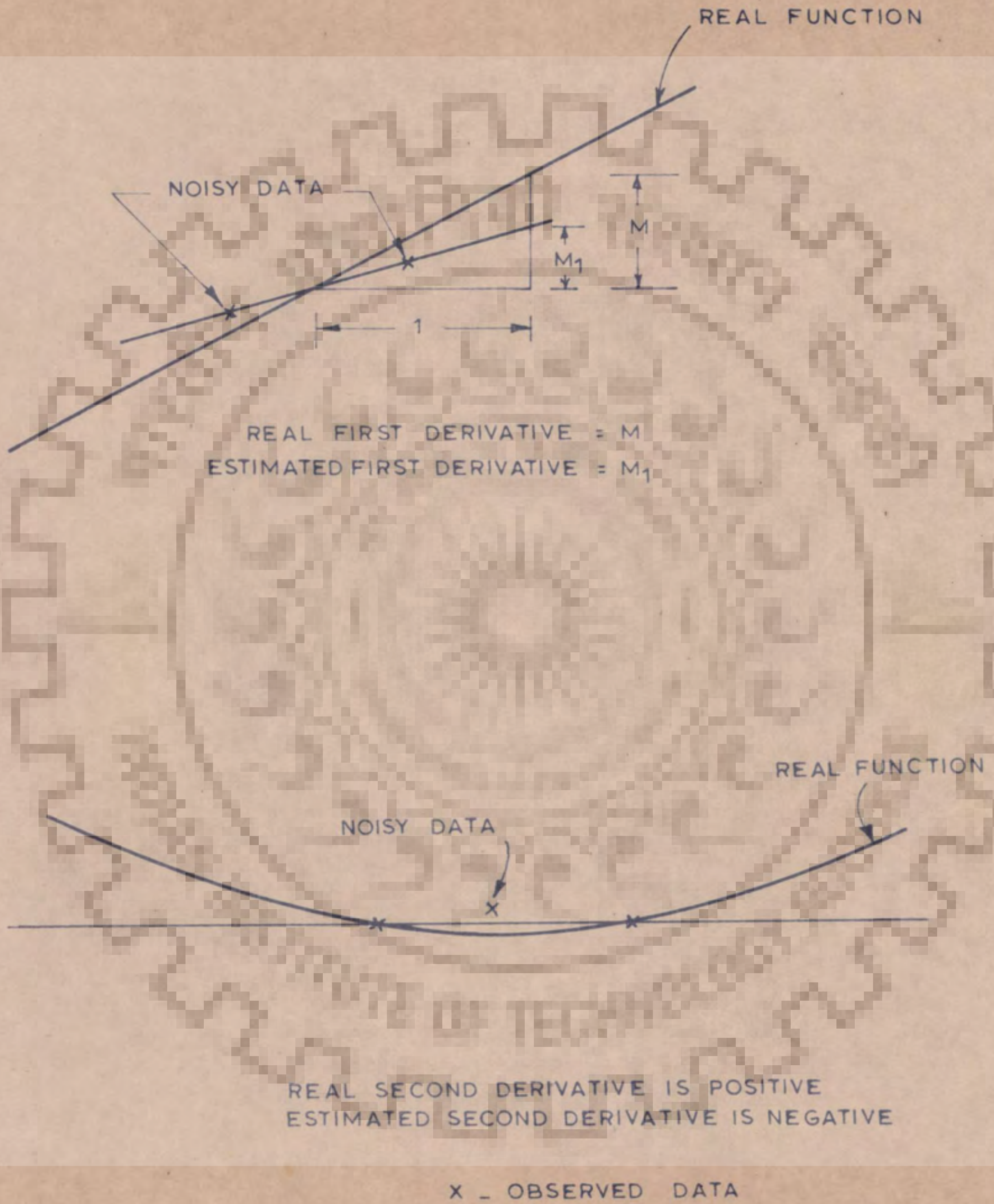


FIG. 3·2 _ SENSITIVITY OF NUMERICALLY ESTIMATED DERIVATIVES TO DATA 'NOISE' adopted polynomial closely approximates the true functional relation between the piezometric head and the spatial coordinates in the given domain of space. The relevence of the smoothening process can be directly inferred from the 'smooth' watertable or piezometric surfaces encountered in physical situations of alluvial aquifers (Neuman, 1973; Sagar et al, 1975).

3.4 LEAST SQUARE POLYNOMIAL APPROXIMATION

The spatial and temporal variation of piezometric head in a groundwater aquifer is governed by the diffusion type of equation. Therefore, one may consider the observed piezometric values available at finite number of points in space and time to be made up of two components i.e.

$$h^*_{ik} = h_k\ (p_i,\ q_i) + \epsilon \qquad \begin{matrix} i=1,\ldots.n \\ k=1,\ldots.m \end{matrix} \qquad \ldots (3.1)$$

where h^*_{ik} is the observed piezometric head at i^{th} observation point $(p_i,\ q_i)$ and k^{th} time interval; $h_k\ (p_i, q_i)$ is the true value of piezometric head at (p_i, q_i) and k^{th} discrete time $h_k(p, q)$ a function of space coordinates p and q being the solution of groundwater flow equation at k^{th} discrete time ;(ϵ) is random error associated with observations and assumed to be statistically uncorrelated; n is the number of observation points and m is the number of time points.

For realistic conditions it may not be possible to know functions $h_k\ (p, q)$ explicitly. However, it is known a priori that these functions are continuous in space. The continuity of these functions can be directly inferred from the continuity of watertable or the piezometric surfaces which do not show any spatial discontinuity in aquifers free of hydrogeological structures. Thus, the functions $h_k(p,q)$ can be approximated by polynomials of the spatial coordinates p and q, $H_k(p,q)$. By choosing an appropriate degree of the approximating polynomial, the difference between the true function and the approximating function can be restricted to a small but non-zero value i.e.

$$\max_{p,q} \left[|h_k(p,q) - H_k(p,q)| \right] \leq \delta_e \qquad \ldots (3.2)$$

$$k = 1, \ldots m \text{ and } \delta_e > 0$$

For obtaining a least square solution, the total number of the coefficients in chosen polynomial must be less than the total number of data points. The coefficients of the approximating polynomials are estimated employing the observed piezometric head data (h^*_{ik}), by the least square criterion which implies the estimation of the coefficients in the chosen equation such that the sum of the squares of the deviations of the observed values from those predicted by the equation is minimised i.e.

$$\text{Minimise} \sum_i \left[h^*_{ik} - H_k(p_i, q_i) \right]^2 \qquad \ldots (3.3)$$

The residues at the i^{th} observation point (at k^{th} time point) are given by

$$e_i = h^*_{ik} - H_k(p_i, q_i) \qquad \ldots (3.4)$$

A complete attenuation of the errors (ϵ) and a one to one representation of $h_k(p,q)$ by $H_k(p,q)$ can be ensured only if the residues (e_i) are identical to the errors in the raw data (ϵ). However this identity would generally not be occuring because of the following reasons :

(i) The true form of the function $h_k(p,q)$ is not known and it is being approximated by $H_k(p,q)$. According to the Wierstrass theorem the minimum difference between the two functions within the domain of the observation points, will always be greater than zero.

(ii) Even if the true form of function $h_k(p,q)$ were known, (e_i) will be identical to (ϵ) only if the latter are normally distributed. Because of the finite sample size the frequency distribution of the errors may be normal only approximately.

3.5 SIMULATION STUDIES

Simulation studies were carried out to determine the extent to which the residues of the least square polynomials are identical to the errors in the raw data. The degree of similarity between the true function $h_k(p,q)$ and the approximating function $H_k(p,q)$; and the level of error attenuation are directly dependent upon it. This exercise relates to one dimensional flow domain and demonstrates the utility of least square polynomial approximation in attenuating the unmeasured data noise. The study was carried out by simulating monthly watertable elevations of 24 months at 40 equally spaced points in a one-dimensional flow domain with constant head boundaries (Figure 3.3).

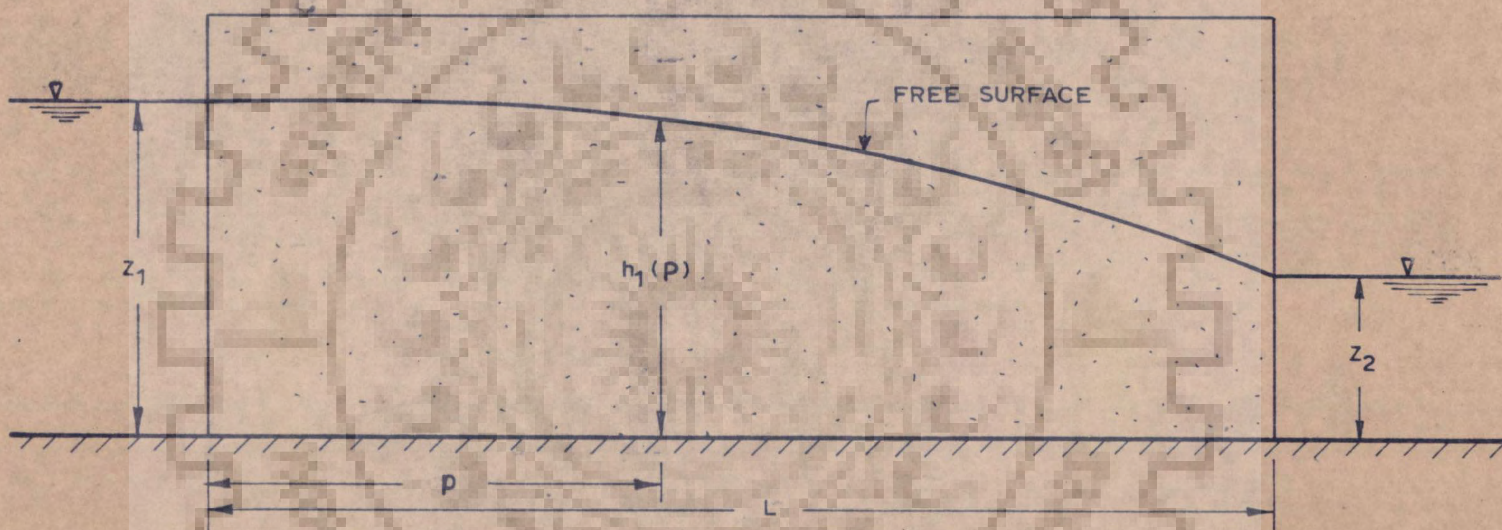


FIG. 3·3 _ INITIAL CONDITIONS OF SIMULATION STUDY

The 'observed watertable elevation' at i^{th} observation point (p coordinate = p_i) and k^{th} time point, h_{ik}^* is given by

$$h_{ik}^* = h_k(p_i) + \epsilon\ (0,\sigma^2) \qquad \ldots (3.5)$$

where $h_k(p)$ is the function describing the spatial variation of hydraulic head at k^{th} time point and ϵ are normally distributed error. The true watertable elevations were generated by the numerical solution of the partial differential equation governing one dimensional groundwater flow. The initial condition i.e. $h_1(p)$ are simulated by Dupuit-Forchhiemer equation for steady state one-dimensional groundwater flow in homogenous unconfined aquifer with no vertical accretion (figure 3.3).

$$h_1(p) = \left[z_1^2 - \frac{z_1^2 - z_2^2}{L}\, p \right]^{1/2} \qquad \ldots (3.6)$$

Constant head boundary conditions are imposed by stipulating $h_k(0)$ and $h_k(L)$ as constants for all k's i.e. $h_k(0) = z_1$ and $h_k(L) = z_2$ (figure 3.3).

The following differential equation is solved numerically to generate true watertable elevations for the remaining 23 months (i.e. $h_k(p)$, $k > 1$). The numerical solution is obtained by finite difference (implicit) scheme.

$$\frac{\partial}{\partial p} (T_{pp}\ \partial h/\partial p) + Q = S\ \partial h/\partial t \qquad \ldots (3.7)$$

This is the governing differential equation for one-dimensional unsteady groundwater flow in confined aquifer with vertical accretion Q. T_{pp} and S are the aquifer characteristics. In the present study this equation has been solved to simulate the response of unconfined aquifer by varying the transmissibility at each time step to account for the varying saturated thickness caused by fluctuating watertable. The entire domain was discretised by equally spaced 40 nodal points and each month was discretised by two 15 days' periods. Uniform transmissibility of 2000 m^2/day was assigned for each node. Normally distributed random numbers were generated to simulate the following elements. The stipulated global mean (μ) and variance (σ), and the range of variation at 95 percent level of confidence ($\mu \pm 2\sigma$) for each element are also given.

i) Nodal storage coefficient values. $\mu = 0.1$, $\sigma = 0.01$ Range = 0.08 to 0.12. Thus the simulated range for diffusivity (T/S) is 1.6×10^4 to 2.4×10^4 m^2/day

ii) Nodal vertical accretion for k^{th} period (k = 1 to 24) $\mu = \bar{Q}_{k'}$, $\sigma = 0.1\ \bar{Q}_{k'}$, Range = $0.8\ \bar{Q}_{k'}$ to $1.2\ \bar{Q}_{k'}$

$$k' = k \quad \text{for} \quad k \leq 12 \qquad \ldots (3.8)$$
$$\quad = k-12 \quad \text{for} \quad k > 12$$

Thus, k' varies from 1 to 12, $\bar{Q}_{k'}$ are assigned as monthly mean accretions of Ganga basin for twelve months (table 3.1).

iii) Observational errors $\mu = 0.0$ with two levels of standard deviation $\sigma_1 = 0.15$, Range = -0.30 to + 0.30 (medium errors); $\sigma = 0.30$, Range = -0.60 to + 0.60 (large errors). Different sets of random numbers were generated for each period.

Table 3.1 Aquifer excitations for simulation study

Month (k)	Rainfall (mm)	Withdrawals(mm)
1	200.0	8.0
2	200.0	8.0
3	200.0	8.0
4	200.0	8.0
5	0.0	12.0
6	20.0	8.0
7	20.0	8.0
8	0.0	16.0
9	0.0	32.0
10	0.0	28.0
11	0.0	28.0
12	20.0	28.0

Recharge coefficient for rainfall = 0.18

Two observed hydraulic head matrices (h_{ik}^*) were simulated employing the simulated values of $h_k(p_i)$ and ε for two different levels of σ. A third set was generated by assuming no observational errors. Two different situations i.e. sparse and dense network of data collecting points were simulated by utilizing the (h_{ik}^*) data of alternate nodes and of all the nodes for fitting the least square polynomials. The polynomial degree was restricted to four. The simulation procedure was repeated with different sets of random numbers. Three replications were carried out in the present study. Results of a typical run are shown in figure 3.4.

The following summary statistics were calculated for each of the periods. Additional variance caused by errors at k^{th} time point in r^{th} replication, V_{kr} is given by

$$V_{kr} = \sum_i \left[h_{ik}^* - h_k(p_i) \right]^2 \qquad \ldots (3.9)$$

The part of the additional variance which is unexplained even after 'smoothening', U_{kr} is given by

$$U_{kr} = \sum_i \left[H_k(p_i) - h_k(p_i) \right]^2 \qquad \ldots (3.10)$$

Here (h_{ik}^*), $h_k(p)$ and $H_k(p)$ pertain to the r^{th} replication. Therefore, the fraction of the additional variance which is attenuated by 'smoothening' the data of k^{th} time is given by

$$E_{kr} = \frac{V_{kr} - U_{kr}}{V_{kr}} \qquad \ldots (3.11)$$

FIG. 3·4 - ATTENUATION OF GAUSSIAN 'NOISE'

Thus the mean error attenuation for k^{th} time is given by

$$\bar{E}_k = \frac{\sum_{r=1}^{n_r} E_{kr}}{n_r} \qquad \dots (3.12)$$

where n_r is the total number of replications.

The confidence limits of $\bar{E}_k$ are given by

$$\bar{E}_k \pm \frac{\sigma_E \, t}{\sqrt{n_r}} \qquad \dots (3.13)$$

$$\text{where } \sigma_E^2 = \frac{\Sigma \, (E_{kr} - \bar{E}_k)^2}{(n_r - 1)} \qquad \dots (3.14)$$

The mean $(\bar{E}_k)$ along with the envelops of confidence limits at 95 percent level of confidence were calculated for the combinations of the following -

i) Two different levels of observational errors i.e. σ = 0.15 and 0.30 meters

ii) Two different levels of data point spacings i.e. 1 and 2 kilometers.

Following statistics were estimated for the least square polynomials fitted to the true data i.e. the case of no observation errors.

$$\beta_{kr}^2 = \frac{U_{kr}}{(n-1)} \qquad \dots (3.15)$$

The mean residual error for k^{th} period is given by

$$\bar{\beta}_k = \frac{\sum_r \beta_{kr}}{n_r} \qquad \ldots (3.16)$$

The confidence limits of $\bar{\beta}_k$ are given by

$$\bar{\beta}_k + \frac{\sigma_\beta \, t}{\sqrt{n_r}} \qquad \ldots (3.17)$$

where $$\sigma_\beta^2 = \frac{\Sigma \, (\beta_{kr} - \bar{\beta}_k)^2}{n_r - 1} \qquad \ldots (3.18)$$

$(\bar{\beta}_k)$ alongwith their envelops of confidence were estimated for the two levels of data spacing.

The plots of $(\bar{E}_k)$ are given in figure 3.5. The confidence intervals are given in table 3.2. These results indicate that a considerable part of the additional variance (V_{kr}) in the raw data is eliminated by the smoothening provided by the least square polynomial approximation. The error attenuation expressed as a fraction of the additional variance, is more pronounced and certain for large errors and dense data (σ = 0.30 meters, spacing=1 km). For small errors the attenuation is effective only if the data are dense. In case of completely error free data the inadequacy of least square fits in providing zero residues at the observation points is simulated by $(\bar{\beta}_k)$. The plots of $(\bar{\beta}_k)$ are given in figure 3.6. The confidence intervals are given in table 3.3. As can be expected the estimates of $(\bar{\beta}_k)$ for dense data are smaller than those for sparse data.

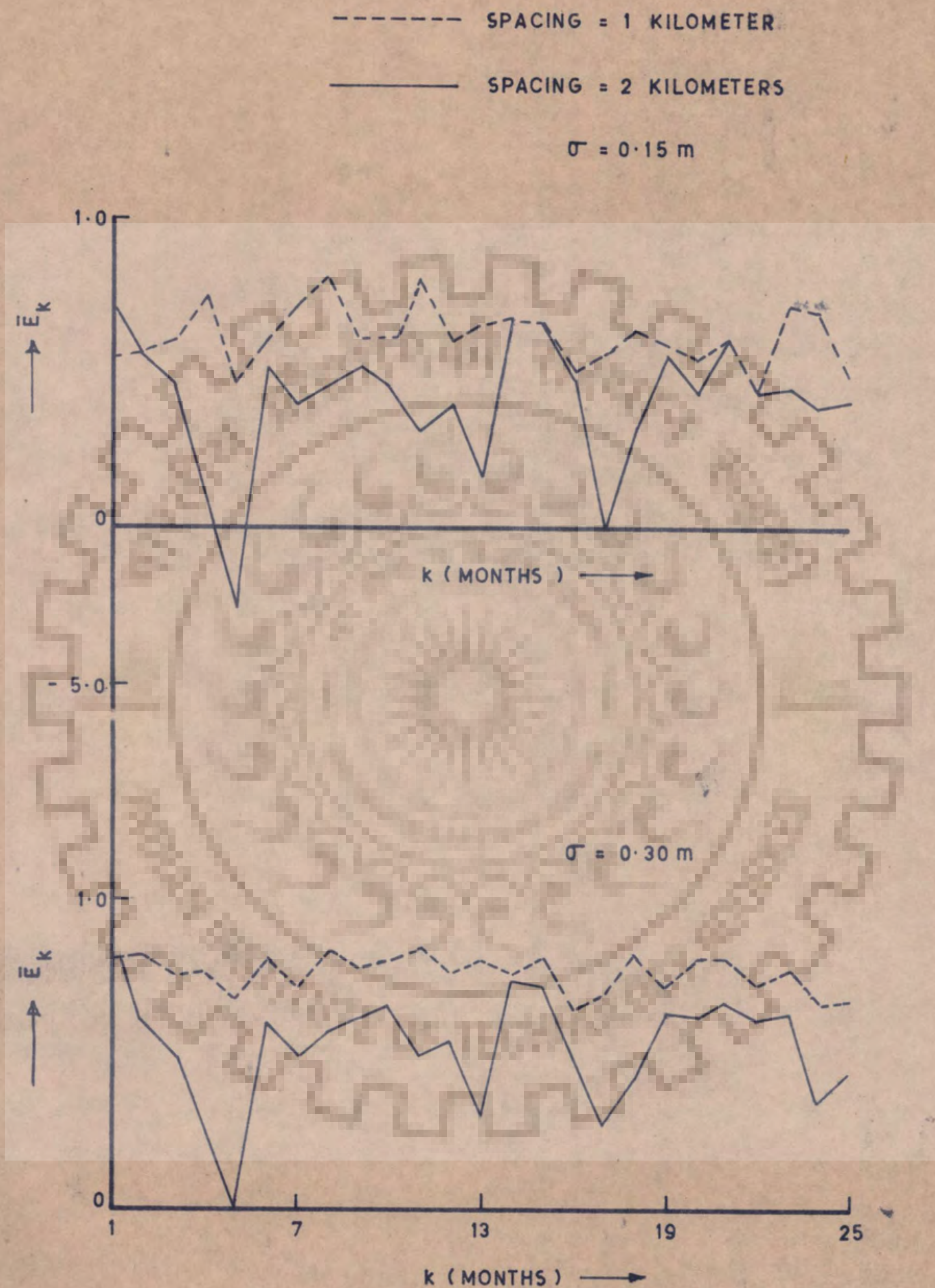


FIG.3·5 - ERROR ATTENUATION BY LEAST SQUARE APPROXIMATION

Table 3.2 - Confidence Interval of $\overline{E_k}$ at 95 percent level

Period k	Confidence Intervals σ = 0.30 m Spacing=1 km	σ = 0.30 m 2 km	σ = 0.15 m 1 km	σ = 0.15 m 2 km
1	0.63-0.99	0.82-0.96	-0.10-1.00	0.48-1.00
2	0.80-0.85	0.30-0.95	0.18-0.95	0.30-0.81
3	0.68-0.84	0.04-0.97	-0.04-1.00	-0.01-0.94
4	0.65-0.93	-0.17-0.74	0.57-0.91	-0.70-0.91
5	0.54-0.82	-1.30-1.0	0.33-0.60	-0.71-0.24
6	0.60-0.98	0.47-0.71	0.03-1.00	0.17-0.87
7	0.53-0.92	0.11-0.90	0.34-1.00	-0.17-0.98
8	0.69-1.00	0.35-0.80	0.73-0.89	0.13-0.80
9	0.71-0.85	0.43-0.83	0.26-0.96	0.33-0.71
10	0.55-1.00	0.50-0.82	0.10-1.00	0.26-0.69
11	0.65-1.00	0.40-0.61	0.61-0.98	-0.11-0.75
12	0.61-0.91	0.12-0.95	0.28-0.93	0.19-0.63
13	0.71-0.91	0.21-0.39	0.57-0.78	0.05-0.27
14	0.61-0.90	0.60-0.89	0.47-0.86	0.55-0.83
15	0.74-0.90	0.47-0.97	-0.22-1.00	0.20-1.00
16	0.38-0.92	-0.07-1.00	0.30-0.72	-0.00-0.96
17	0.57-0.84	-0.28-0.84	0.25-0.89	-1.33-1.00
18	0.64-1.00	0.24-0.62	0.11-1.00	-0.25-0.89
19	0.54-0.93	0.43-0.84	0.12-1.00	0.07-1.00
20	0.71-0.90	0.46-0.81	0.17-0.93	0.02-0.86
21	0.60-1.00	0.47-0.89	00.17-1.00	0.37-0.84
22	0.50-0.96	0.45-0.80	-0.81-1.00	0.15-0.72
23	0.64-0.92	0.55-0.75	0.35-1.00	0.18-0.71
24	0.46-0.87	0.14-0.55	0.37-1.00	0.07-0.31
25	0.32-1.00	-0.32-1.00	-0.04-1.00	-1.08-1.00

Table 3.3 Confidence Interval of $\bar{\beta}_k$ at 95 percent level

Period k	Confidence Interval (both limits are in metres) Spacing = 1 km	Spacing = 2 km
1	0.00 - 0.00	0.00 - 0.00
2	0.05 - 0.06	0.05 - 0.12
3	0.88 - 0.89	0.09 - 0.10
4	0.09 - 0.13	0.09 - 0.15
5	0.10 - 0.16	0.11 - 0.19
6	0.07 - 0.12	0.08 - 0.13
7	0.06 - 0.10	0.06 - 0.11
8	0.05 - 0.09	0.05 - 0.10
9	0.04 - 0.08	0.04 - 0.08
10	0.05 - 0.08	0.05 - 0.08
11	0.06 - 0.09	0.07 - 0.09
12	0.08 - 0.10	0.09 - 0.11
13	0.09 - 0.11	0.10 - 0.12
14	0.03 - 0.07	0.03 - 0.08
15	0.02 - 0.08	0.02 - 0.08
16	0.17 - 0.13	0.02 - 0.13
17	0.09 - 0.12	0.10 - 0.13
18	0.07 - 0.06	0.07 - 0.10
19	0.05 - 0.08	0.05 - 0.08
20	0.05 - 0.08	0.05 - 0.07
21	0.05 - 0.06	0.05 - 0.06
22	0.06 - 0.15	0.05 - 0.08
23	0.06 - 0.10	0.07 - 0.11
24	0.09 - 0.11	0.10 - 0.12
25	0.10 - 0.12	0.10 - 0.13

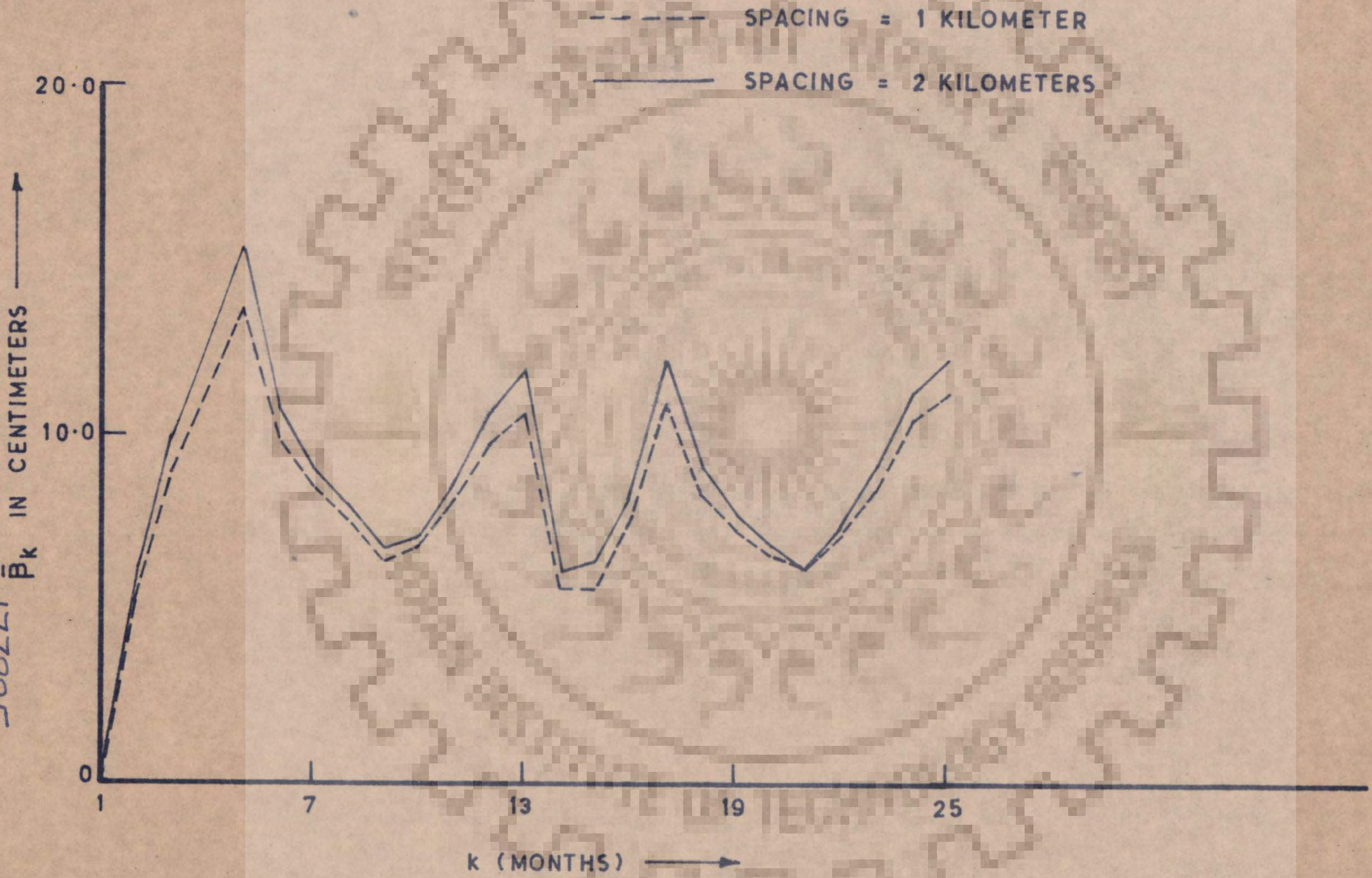


FIG.3·6 - RESIDUES IN LEAST SQUARE APPROXIMATION

3.6 SELECTION OF LEAST SQUARE POLYNOMIAL

As the degree of the least square polynomial is increased, it may get closer and closer to the true functional relation between the piezometric head and the space coordinates in the given domain. However, beyond a certain limit, any further increase in the degree of the polynomial may only induce the approximating least square polynomial to conform to 'noise' pattern of the raw data instead of providing a better approximation to the true piezometric surface. Obviously beyond this limit, the higher degree polynomials start providing a poorer approximation to the true piezometric surface. Similarly partial polynomials may provide better approximation than a full polynomial. This effect has been demonstrated by repeating the third replication of the simulation study (described in previous paragraph) with a sixth degree polynomial. By increasing the degree of polynomial from four six, E_{kr} registered a decline instead of a rise (figures 3.7 and 3.8). Thus, it is desirable to arrive at an optimal form of polynomial i.e. a partial polynomial of minimum possible degree and terms. This, apart from providing a better approximation, also minimises the computational efforts. The optimal form of the approximating function can be arrived at by employing the statistical tests of significance (Daniel and Wood, 1971).

The standard tests of significance used in the context of multiple regression, can be directly applicable to least square polynomials provided the 'independent variables' (i.e. the terms containing different exponents of the spatial coordinates) are not correlated to each other(Dooge,1973). This will hold if the spatial coordinates and hence the 'independent variables' are known exactly (without any error) and are thus,

FIG.3.7- 4th. DEGREE POLYNOMIAL

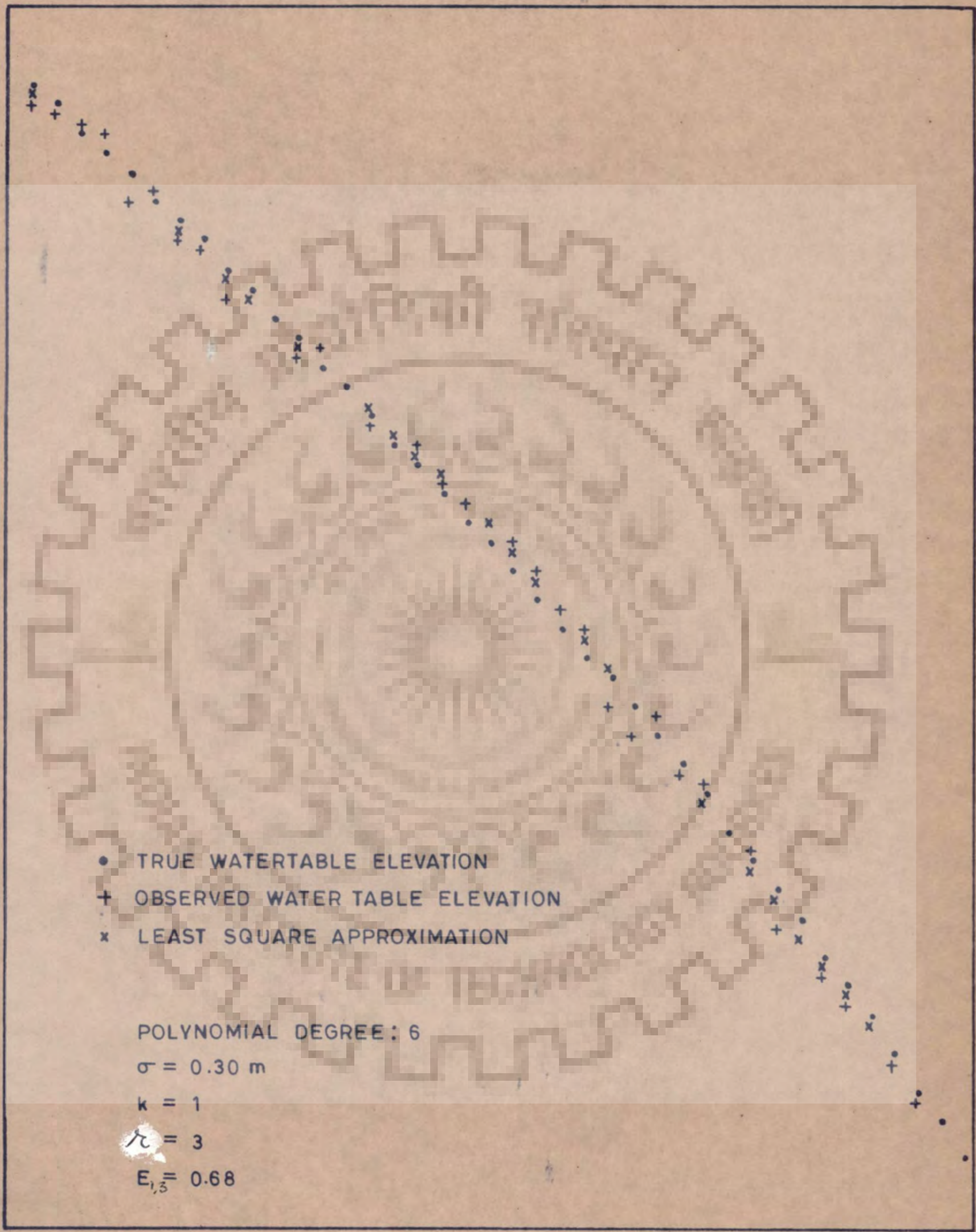


FIG.3.8-6th. DEGREE POLYNOMIAL

deterministic in nature. The statistical tests that can be used, to arrive at the polynomial, are given in the following paragraphs.

3.6.1 Degree of Polynomial

The degree of polynomial can be decided on the basis of criterion of minimum standard error s.

$$s^2 = \sum_i e_i^2 / (n - n_p) \qquad \text{... (3.19)}$$

$$n > n_p$$

where e_i is the residue at i^{th} observation point i.e.

$$e_i = h_{ik}^* - H_{ik}(p_i, q_i) \qquad \text{... (3.20)}$$

Σe_i^2 is the sum of the squares of the residues (SSR); n is the total number of data points and n_p is the number of coefficients in the polynomial; $(n-n_p)$ is known as the degree of freedom (df) of the proposed least square polynomial. As the degree of polynomial is increased, there will be a reduction in SSR but at the same time (df) will also decrease. Thus, s^2 may decrease with increasing degree upto a certain limit beyond which it may start increasing. The polynomial which gives minimum standard error is chosen tentatively as the approximating function (Ralston, 1965).

3.6.2 Truncation of Polynomial

The confidence limit for each of the estimated coefficient can be defined for a given level of confidence. This criterion can be used to delineate the terms which contribute significantly towards the explaining power of the polynomial model.

The confidence limits of estimate for coefficient of i^{th} term b_i is given by

$$b_i \pm t\,(n - n_p, \tfrac{\alpha}{2})\; s.e\,(b_i) \qquad \ldots (3.21)$$

where $t(n-n_p, \frac{\alpha}{2})$ is $(1-\alpha)$ percentile of t distribution with $(n-n_p)$ degrees of freedom, $s.e(b_i)$ is the standard error of b_i and is given by

$$s.e(b_i) = s\sqrt{C_{ii}} \qquad \ldots (3.22)$$

where C_{ii} is the i^{th} diagonal element of the corrected sum of squares and product matrix.

The statistics for testing $(b_i = 0)$ is $t = \dfrac{b_i}{s.e.\,(b_i)}$

Null hypothesis is accepted if the computed value of t is lower than the critical value of t for $(n-n_p)$, d.f. at some chosen level of confidence (generally 95 percent). Acceptance of the Null hypothesis implies that the i^{th} term does not have significant explaining power and may be dropped. This way a few terms of the polynomial may be dropped.

Students t test indicates the terms which may be dropped. However, it has to be tested if the reduced model (RM) gives as good a fit as the full model (FM). Coefficients of the polynomial with reduced number of terms and SSR are reestimated. If k_d is the number of terms dropped then F statistic is estimated.

$$F = \frac{[\text{SSR (RM)} - \text{SSR (FM)}]\ /k_d}{\text{SSR(FM)}/(n-n_p)} \qquad \ldots (3.23)$$

If the calculated F is large in comparison to the tabulated value of F with k_d and $(n - n_p)$ degrees of freedom at α percentile level of confidence, the result is significant at level α i.e. the reduced model is unsatisfactory. This implies that though each of the terms dropped invidually does not have significant explaining power, but collectively these terms explain a significant part in of the variation in hydraulic head.

Terms of the initially proposed polynomial to be retained in the approximating function can be decided on the basis of criteria given above.

3.7 ACCEPTABILITY OF MODEL

Subsequent to the estimation of coefficients it is desirable to make an assessment of the goodness of the fit. The most widely used index is the multiple correlation coefficient R defined by

$$R^2 = 1 - \frac{\text{SSR}}{\sum_{i=1}^{n} (h^*_{ik} - \bar{h}_k\ )} \qquad \ldots (3.24)$$

$$\bar{h}_k = \sum_{i=1}^{n} h^*_{i,k}/n$$

R^2 represents the fraction of the initial variance explained by the model.

The adequacy of the proposed fit to approximate the true function may be checked by examining the residues. The standard residue at i^{th} station es_i is given by

$$es_i = e_i/s \qquad \ldots (3.25)$$

The standard residues have zero mean and unit standard deviation. In general when the model is correct, the standard residues tend to fall between ± 2 and their plots against independent variables i.e. p and q do not exhibit any trend.

3.8 EVALUATION OF GROUNDWATER SYSTEM ELEMENTS

As has been pointed out earlier, the approximating least square functions can be employed to obtain system elements relating to the groundwater studies. These are described in the following paragraphs.

3.8.1 Groundwater Storage

The groundwater storage in k^{th} time can be estimated by integrating the function $H_k(p,q)$ over the space defined by the study area. Thus, the groundwater storage SG_k above datum ZD is given by

$$SG_k = \int\int_{AG} S \cdot \left[H_k(p,q) - ZD \right] dp\, dq \qquad \ldots (3.26)$$

where AG is the study area. The integration over irregularly shaped area can be carried out by dividing the area into finite number of strips extending along one of the two directions, p or q. The groundwater storage in i^{th} strip(figure 3.9) is given by

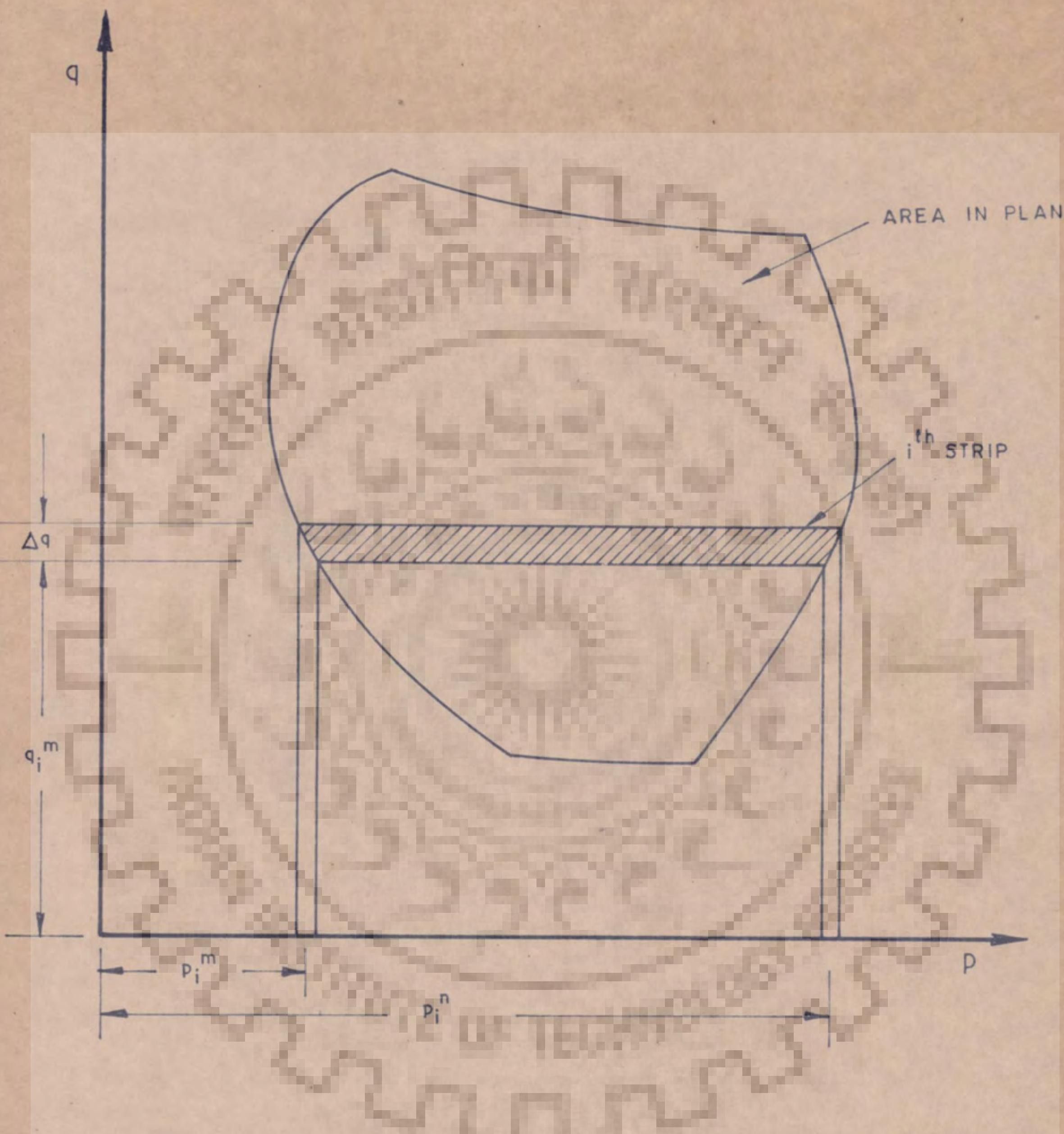


FIG. 3.9 _ ESTIMATION OF GROUNDWATER STORAGE

$$\Delta S_i = \int_{p=p_i^m}^{p_i^n} \int_{q=q_i^m}^{q_i^m+\Delta q} S \cdot [H_k(p,q) - ZD] \, dp \, dq \qquad \ldots (3.27)$$

Total groundwater storage is given by

$$SG_k = \sum_{i=1}^{n_s} \Delta S_i \qquad \ldots (3.28)$$

where n_s is the total number of strips. The function $| H_k(x,y) - ZD |$ can be integrated analytically. Thus ΔS_i can be evaluated directly if the storage coefficient S is uniform in i^{th} strip. If S is varying in i^{th} strip, then SG_k may have to be evaluated by the numerical solution of equation (3.26).

3.8.2 Boundary Recharge

Total subsurface inflow rate from across the boundary of the study area at k^{th} time point is written as

$$I_k = \int_C - \frac{\partial}{\partial n^c} H_k(p,q) \, T \, dc \qquad \ldots (3.29)$$

where C is the boundary contour and n^c is its inner normal. The integration can be performed by dividing the boundary contour into finite number of elementary lengths (figure 3.10). Then the subsurface flow from i^{th} elementary length is given by

$$\Delta I_i = - \frac{\partial}{\partial n^c} H_k(p_i^*, q_i^*) \, \Delta L_i \, T_i \qquad \ldots (3.30)$$

where p_i^* and q_i^* are the coordinates of the central point of i^{th} elementary length, ΔL_i is the elementary length and T_i is the average transmissibility of i^{th} elementary length.

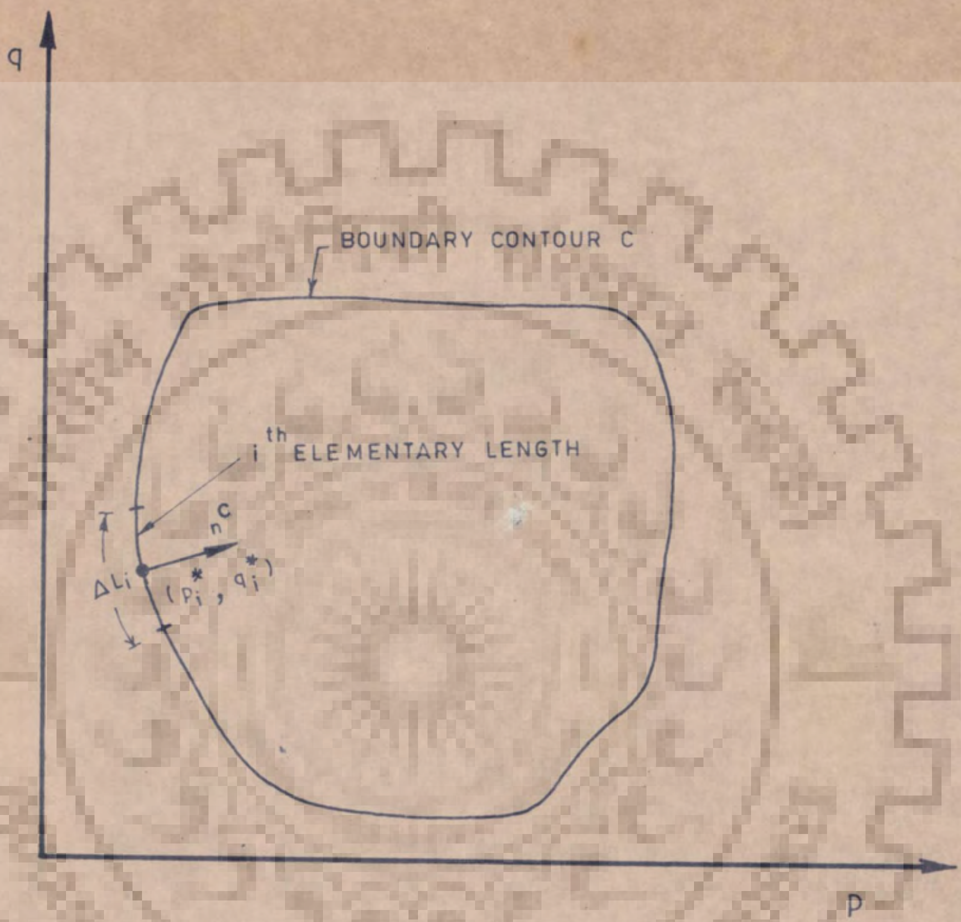


FIG. 3.10 – ESTIMATION OF SUBSURFACE HORIZONTAL RECHARGE

$$I_k = \sum_{i=1}^{n_e} \Delta I_i$$

where n_e is the total number of elementary lengths. In equation 3.30, the hydraulic gradient in the direction n^c can be evaluated directly, knowing the average orientation of the elementary length with respect to p and q directions.

3.8.3 Estimation of Second Spatial Derivatives

The second spatial derivatives of piezometric head can yield the information relating to the imbalance between the subsurface horizontal inflows and outflows at a given space point. These derivatives can be directly obtained by differentiating the approximating polynomial twice. Second spatial derivatives are generally not needed in the conventional groundwater studies. Their estimation is necessary for estimating transmissibility by the direct methods of inverse problem (Sagar et al 1973,1975). This aspect has been discussed in Chapter 4.

3.8.4 Hydraulic Continuity of Aquifer

Hydraulic discontinuities in the aquifer can originate from geological structures like folds, faults or dykes. As a result of these geological structures, the piezometric surface will show discontinuity displayed by marked differences in piezometric elevations. Such piezometric surfaces can not be adequately approximated by least square polynomials.

If the least square polynomial provides a good fit to the observed data, the hydraulic continuity of the aquifer is established. The 'goodness' of a least square fit is characterised by high correlation coefficient (low standard errors), random distribution of standard residues in space and absence of outliers. The orientation and distribution of outliers can give some idea about the location and delineation of discontinuities if present. However, the occurrence of outliers without any systematic orientation in space can be due to some localised phenomena like incomplete recoupment at the time of observation, inadequate hydraulic connection between the well and the aquifer.

3.8.5. Interpolation and Contour Drawing

Piezometric head at any space point (p_i, q_i) in the k^{th} time point $h_k (p_i, q_i)$ can be approximated by the corresponding functional value of $H_k (p,q)$.

$$h_k (p_i, q_i) \simeq H_k(p_i, q_i) \qquad \ldots (3.31)$$

This equation can be employed to interpolate the piezometric head at any space point. However, the least square polynomial $H_k(p,q)$ provides a good approximation of $h_k (p,q)$ in a domain covered by the observation points. Thus, the extrapolation of head should be avoided.

The contours of equal piezometric head can be drawn by locating adequate number of space points lying within the area of interest, where the functional value of the approximating polynomial equals the desired level of the contour. Thus, the contour of piezometric head = h_c at time k, is defined by the following equation

$$H_k(p,q) = h_c \qquad \ldots (3.32)$$

A finite number of real solutions of this equation can yield the points of equal piezometric head ($=h_c$) which can be joined to get the contour of h_c. This procedure can however be very elaborate since it requires the repeated calculation of real and complex roots of possibly large degree polynomials.

The other relatively simpler procedure involves the evaluation of the approximating function at large number of closely spaced points. The entire range of variation of piezometric head can be discretised by a finite number of intervals. The space points where the functional value changes from one interval to another, will roughly represent the contour of the common elevation of these two successive intervals.

Both these procedures can be programmed on a digital computer to accomplish computer assisted contour drawing.

3.9 COMPUTER CODE

A general computer code has been developed to calculate the optimal form and the coefficients of the least square polynomials. The code has the following salient features:

i) It normalises the coordinates prior to the calculation of the coefficients of least square polynomial.

ii) It automatically detects and deletes (if so instructed) the terms of the polynomial which do not contribute significantly towards the explaining power of the polynomial. This is done on the basis of students t test (Daniel and Wood, 1971). The instruction relating to the deletion of terms is conveyed through an input integer variable.

iii) It automatically detects and deletes (if so instructed) the 'outliers'. The detection of the outliers is based upon the standard residues. The instruction relating to the deletion of the outliers is again conveyed through an input integer variable.

iv) Subsequent to the deletion of the terms or the 'outliers' the least square polynomial is recomputed with revised form of the polynomial or with reduced number of data points. The process is continued till t test does not warrant any further deletion of terms and all the standard residues/residues are acceptable.

v) After deciding the polynomial of minimum degree with minimum number of terms and the corresponding least square coefficients, the computer plots the watertable contours, if so instructed (through an input integer variable). The necessary input data for this, is the geometry of the area. Plotting of the contours is followed by the computation of mean watertable elevation (by the process of integration) and mean boundary gradient (through the process of differentiation). If so desired, the contours can also be plotted after each deletion of the polynomial terms or the data points.

vi) The form of the least square polynomial and the co-efficients are stored in disk/tape for subsequent calculations (if any).

The step wise procedure is shown in figure 3.9.

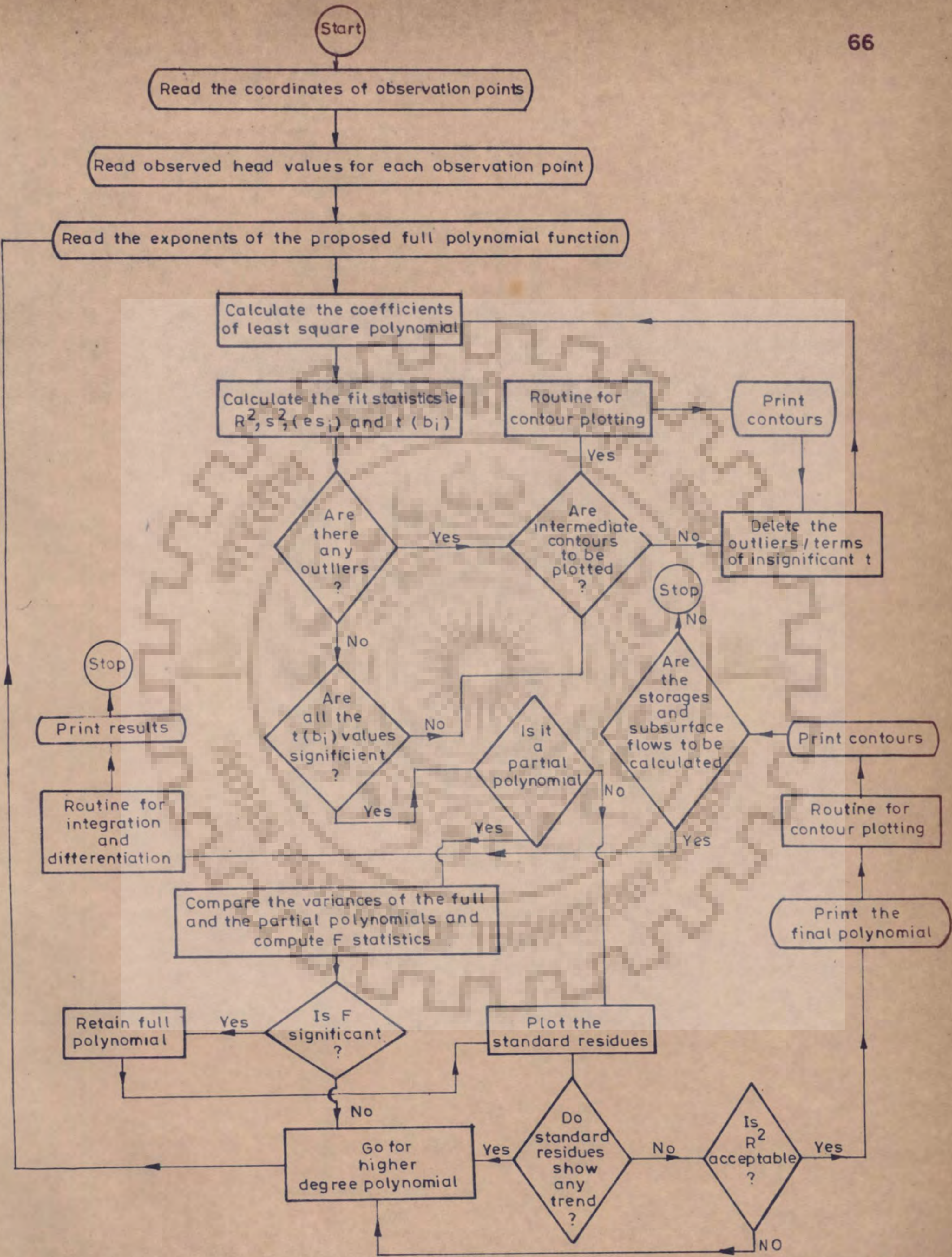


Fig.3.11- Flow chart for least square polynomial approximation

CHAPTER - 4

AQUIFER PARAMETER ESTIMATION BY INVERSE PROBLEM

4.1 PROBLEM IDENTIFICATION

Simulation of the response of aquifer to a deterministic pattern of groundwater withdrawals and recharge is known as 'direct problem' in groundwater hydrology and is of immense interest to groundwater planners. The data requirement for carrying out such a study includes the spatially distributed estimates of the insitu transmissibility (T) and storage coefficient/specific yields (S).

Test pumping, the most popular method of estimating these parameters, involves generating the aquifer response to the pumping in a single well. The generated data are analysed to arrive at the estimates of aquifer properties. The analysis is usually based upon Theis equation (Theis, 1935) which is the governing equation for radial-unsteady groundwater flow in a confined, homogenous, isotropic aquifer towards a fully penetrating well of infinitesimal size from which water is abstracted at a constant rate. Apart from the uncertainties associated with these assumptions, the test pumpings have to be generally restricted to a few number due to high cost involved. The results of a few spatial points are generally employed to assign nodal values in a distributed model.

The other approach for estimating S and T, known as inverse problem, is to employ the historical data of aquifer response and the corresponding aquifer excitations. The aquifer excitation can be either of the form of vertical accretions (Kleinecke, 1971; Sagar et al, 1973, 1975), or change in the boundary conditions like river stage (Singh and Sagar, 1977). In the following paragraphs a numerical scheme for estimating aquifer parameters by this approach has been developed. The proposed scheme is based upon analysis of historical aquifer response to vertical accretions. The vertical accretions may be resulting from human-activity (e.g. pumping) and/or natural phenomena (e.g. rainfall).

4.2 PROBLEM DEFINITION

The application of inverse problem in groundwater hydrology has been mostly confined to the estimation of storage coefficient and transmissibility. The solution generally requires a prior knowledge of the directions of principal permeabilities and the distribution of net vertical accretion in space and time.

The available field data are generally not adequate to afford a direct estimation of the directions of principal permeabilities. Historical Records of rainfall-recharge - an important component of the net accretion in tropics, are almost never available because rainfall-recharge can not be directly measured at field level. The recharge data are indirectly derived from the rainfall data. The differential equation, approach of estimating the recharge from rainfall (Krishnamurthy, 1977) requires enormous data relating to amongst others, the soil moisture distribution and permeability - soil moisture curves. The enormity of data requirement prohibits the application of this approach to field problems. The most widely used approach is to employ certain semi-empirical (Rushton and Ward, 1979) or empirical (Chaturvedi and Chandra, 1961) relations between rainfall and the recharge. These relations incorporate certain parameters like field-capacity, recharge coefficient, threshhold rainfall.

These parameters may vary considerably in space and therefore, should be evaluated for the study area. However, their evaluation can be very elaborate affair, since these are not physically measurable quantities. In most of the studies, certain values for these parameters are assumed on the basis of estimated parameters of neighbouring or even distant places.

Thus the applicability of the inverse problem for the estimation of aquifer parameter can be greatly enhanced by incorporating the estimation of the orientation of principal permeability directions and the recharge parameters amongst its objectives.

4.3 DIFFERENTIAL EQUATION

The governing differential equation of two dimensional transient groundwater flow in a heterogenous and anisotropic confined aquifer (equation 2.5) can be rewritten as

$$T_{xx} \frac{\partial^2 h}{\partial x^2} + T_{yy} \frac{\partial^2 h}{\partial y^2} + \frac{\partial T_{xx}}{\partial x} \frac{\partial h}{\partial x} + \frac{\partial T_{yy}}{\partial y} \frac{\partial h}{\partial y} + Q = S \frac{\partial h}{\partial t} \quad ..(4.1)$$

This equation is valid for confined aquifers only since it assumes a time invariant thickness of saturated flow and uniform horizontal velocity over the entire depth of flow. However, the equation may be nearly valid for unconfined aquifers as well provided

i) Dupuit Forchheimer's assumptions are satisfied.

ii) Temporal fluctuations of watertable are small in comparison to the mean saturated thickness. The aquifer parameters will thus, correspond to the mean saturated thickness.

Assuming that the hydraulic gradients ($\partial h/\partial x$, $\partial h/\partial y$) and the transmissibility gradients ($\partial T_{xx}/\partial x$, $\partial T_{yy}/\partial y$) are small their products will be negligibly small in comparison to other terms of equation 4.1. Neglecting these terms, the equation reduces to the following form -

$$T_{xx}\, \partial^2 h/\partial x^2 + T_{yy}\, \partial^2 h/\partial y^2 + Q = S\, \partial h/\partial t \qquad \dots (4.2)$$

$\partial^2 h/\partial x^2$ and $\partial^2 h/\partial y^2$ are the second spatial derivatives of the piezometric head in the directions of the principal permeabilities. These can be defined in terms of the corresponding derivatives D_p and D_q in any two arbitrarily decided orthogonal direction p and q.

$$\partial^2 h/\partial x^2 = D_p \cos^2 \theta + D_q \sin^2 \theta \qquad \dots (4.3)$$

and $$\partial^2 h/\partial y^2 = D_p \sin^2 \theta + D_q \cos^2 \theta \qquad \dots (4.4)$$

In these expressions the terms containing $\frac{\partial h}{\partial p}.\frac{\partial \theta}{\partial x}$, $\frac{\partial h}{\partial q}.\frac{\partial \theta}{\partial x}$, $\frac{\partial h}{\partial p}.\frac{\partial \theta}{\partial y}$ and $\frac{\partial h}{\partial q}.\frac{\partial \theta}{\partial y}$ have been neglected, on account of their small order of magnitude.

where θ is the angle between the two sets of the orthogonal axes (figure 4.1).

Alternatively, the governing differential equation can also be written in terms of T_{pp}, T_{qq} and T_{pq} (Sagar, 1975) with the associated non-equality constraint $T_{pp}\,T_{qq} - (T_{pq})^2 \geq 0$. The proposed approach, apart from eliminating this constraint, affords an explicit estimation of the orientation of principal-permeability directions.

Equation 4.2 can be rewritten as

$$T_{xx}(D_p \, Cos^2\theta + D_q \, Sin^2\theta) + T_{yy}(D_p \, Sin^2\theta + D_q Cos^2\theta) + Q = S \; \partial h/\partial t \quad \ldots (4.5)$$

The solution of direct problem for an anisotropic aquifer with spatially varrying θ, can be obtained through the solution of governing differential equation by finite element method (Freeze and Cherry, 1979; Pinder and Gray, 1977)

4.4 ANALYSIS OF RECHARGE

In equation 4.5, Q is the spatially and temporally varying net vertical recharge rate per unit area and equals the algebraic sum of the following inflow and outflow components.

i) Inflows (Positive recharge)
   a) Recharge due to rainfall
   b) Return flows from applied irrigation
   c) Artificial recharge
   d) Seepage from canals

ii) Outflows (negative recharge)
   a) Groundwater pumpage
   b) Evapotranspiration

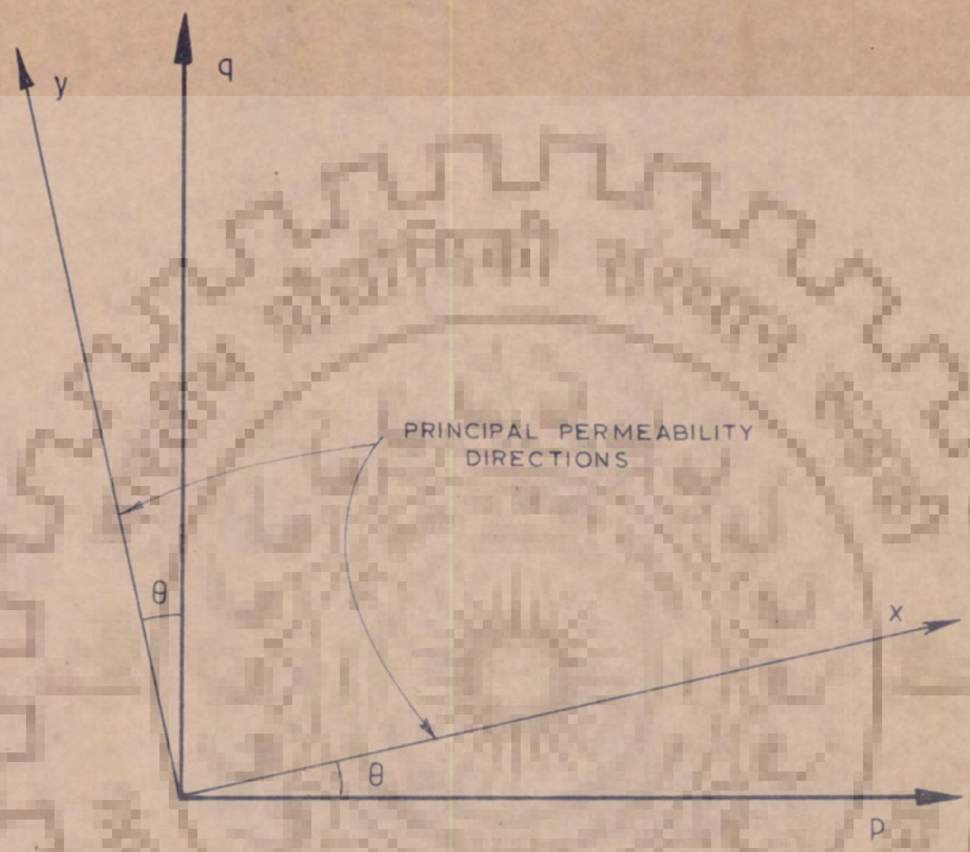


FIG. 4·1 _ DIRECTIONS OF PRINCIPAL PERMEABILITIES

In most of the aquifers, the rainfall recharge and the groundwater pumpage, are the predominant components of the net-recharge. Thus Q, for i^{th} space point and k^{th} time period can be written as

$$Q_{ik} = R_{ik} - W_{ik} + X_{ik} \qquad \ldots (4.6)$$

R_{ik} is the rainfall recharge, W_{ik} is the effective withdrawal (i.e. the withdrawal adjusted for the return flow) and X_{ik} is the algebraic sum (recharge taken positive and discharge negative) of all other components of Q_{ik}.

The rainfall recharge R_{ik} is an important component of groundwater balance in the tropic regions and the available rainfall records are usually adequate to estimate the spatial and temporal distribution of rainfall. Therefore, conceptual or physical-process rainfall-recharge relations have to be incorporated in equations 4.5 and 4.6. with the aim of identifying the parameters involved in these relations.

The general form of these relations can be written as

$$R_{ik} = f_r\,(P_{ik}, P_{i,k-1}, \ldots P_{i,k-m_e}, \alpha^{i}) \qquad \ldots (4.7)$$

where P_{ik} is the rainfall at i^{th} space point during k^{th} period m_e is the number of preceeding periods' rainfalls affecting the recharge in the current period, α^{i} is a row matrix containing n_c number of parameters. In the soil moisture budgeting procedure (equation 2.10) the parameter to be determined will be the field capacity. In case the elaborate data required for this procedure are not available simpler locally applicable functional forms can be employed and the built-in parameters estimated. The simplest functional form could be as follows -

$$R = K_r P \qquad \ldots(4.8)$$

where K_r is known as recharge coefficient. Chandra et al (1975, 1979) have estimated K_r for different interbasins of Indo-Gangetic plains by carrying out seasonal waterbalance studies. The studies indicate small temporal variability of K_r for a given basin. For shorter time periods the antecedent soil moisture conditions can be accounted for by incorporating a parameter thresh-hold rainfall, (α_3^i) - the minimum rainfall required in one time period to fill in the soil moisture deficiency. Thus, the rainfall-recharge relation can be written as follows -

$$R_{ik} = \alpha_1^i \left[\alpha_2^i P_{ik} + (1-\alpha_2^i) P_{i,k-1} - \alpha_3^i \right] \text{ if } P_{i,k-1} < \alpha_3^i \quad \ldots(4.9a)$$

$$= 0 \text{ if } \alpha_3^i > \alpha_2^i P_{ik} + (1-\alpha_2^i) P_{i,k-1} \text{ and } P_{i,k-1} < \alpha_3^i$$

$$R_{ik} = \alpha_1^i \left[\alpha_2^i P_{ik} + (1-\alpha_2^i) P_{i,k-1} \right] \text{ if } P_{i,k-1} \geq \alpha_3^i \qquad \ldots(4.9b)$$

where α_1^i, α_2^i and α_3^i are the recharge parameters of i^{th} point. The α_1^i corresponds to the recharge coefficient and α_2^i accounts for the delayed aquifer response.

The approach of incorporating the rainfall-recharge parameters in the inverse problem formulation not only provides consistent estimates of historical rainfall recharge, but also effectively calibrates the locally acceptable functional forms of the rainfall recharge relations.

4.5 DISCRETISATION OF BOUSSINESQ'S EQUATION

Equation 4.5 is the continuous form of the governing differential equation. Its discretisation will involve writing down the following terms for discrete space and time points.

i) Recharge i.e. Q

ii) Spatial and temporal derivatives i.e. $\partial^2 h/\partial p^2$, $\partial^2 h/\partial q^2$ and $\partial h/\partial t$.

iii) Aquifer parameters

Employing the discretised form of Q as arrived in equations (4.6) and (4.7), equation (4.5) can be rewritten as

$$TX_i(DP_{ik}\,Cos^2\theta_i + DQ_{ik}\,Sin^2\theta_i) + TY_i(DP_{ik}\,Sin^2\theta_i + DQ_{ik}\,Cos^2\theta_i) + f_r(P_{ik}, P_{i,k-1}, \ldots P_{i,k-m_e}, \alpha^i) - W_{ik} + X_{ik} - S_i\,DT_{ik} = 0 \qquad \ldots(4.10)$$

where TX_i, TY_i, θ_i, S_i are the aquifer properties of i^{th} space point corresponding to T_{xx}, T_{yy}, θ and S; DP_{ik}, DQ_{ik} and DT_{ik} are the discretised forms of D_p, D_q and $\partial h/\partial t$ for i^{th} space point and k^{th} time period.

Equation 4.10 can be rewritten as

$$f(TX_i, TY_i, \theta_i, S_i, \alpha^i, f_r, DP_{ik}, DQ_{ik}, DT_{ik}, W_{ik}, X_{ik}, P_{ik}, P_{i,k-1}, \ldots P_{i,k-m_e}) = 0 \qquad \ldots (4.11)$$

Equations 4.10 and 4.11 express the governing differential equation in terms of

i) The system properties: TX_i, TY_i, θ_i, S_i, α^i, f_r

ii) Information derived from piezometric head data : (DP_{ik}), (DQ_{ik}), (DT_{ik})

iii) Information derived from rainfall records : P_{ik}, $P_{i,k-1}$, ... $P_{i,k-m_e}$

iv) Information derived from hydrologic, agricultural and irrigation records : (W_{ik}), (X_{ik}).

4.6 ESTIMATION OF DERIVATIVES

The terms DP_{ik}, DQ_{ik} and DT_{ik} appearing in equations 4.10 and 4.11 are the estimates of the second derivatives of the piezometric head in two arbitrarily decided orthogonal directions p and q (refer figure 4.1) and the first temporal derivative respectively. These terms are to be directly estimated from the available historical piezometric data. The least square polynomial approximation as described in Chapter 3 can be employed to arrive at the derivatives of the piezometric head. The least square polynomial , apart from smoothening the data, also provides a closed form differentiable functional relation between the piezometric head and the spatial coordinates. Thus if $H_k(p,q)$ and $H_{k+1}(p,q)$ are the least square polynomial approximations for the functional relation between the piezometric head and the space coordinates p and q at beginning and at the end of k^{th} time interval respectively then

$$DP_{ik} = \frac{1}{2} \left[\frac{\partial^2}{\partial p^2} H_k(p_i, q_i) + \frac{\partial^2}{\partial p^2} H_{k+1}(p_i, q_i) \right] \qquad \ldots (4.12)$$

$$DQ_{ik} = \frac{1}{2} \left[\frac{\partial^2}{\partial q^2} H_k(p_i, q_i) + \frac{\partial^2}{\partial q^2} H_{k+1}(p_i, q_i) \right] \qquad \ldots (4.13)$$

$$DT_{ik} = \frac{H_{k+1}(p_i, q_i) - H_k(p_i, q_i)}{\Delta t_k} \qquad \ldots (4.14)$$

where (p_i, q_i) are the coordinates (p, q) of the i^{th} space point and Δt_k is the span of k^{th} time interval.

4.7 DETERMINATE SYSTEM OF EQUATIONS

Equation 4.10 is the trigno-algebraic form of equation 4.2. The estimates of the derivatives (DP_{ik}), (DQ_{ik}) and (DT_{ik}) can be obtained from the historical piezometric data, and the estimates of (W_{ik}) and (X_{ik}) from the available historical data relating to the groundwater withdrawals, canal supplies, evapotranspiration etc. Substitution of these estimates in equation 4.10 for any given values of i and k, yields an equation in terms of the system parameters. The equation involves $(4 + n_c)$ unknowns and is thus, indeterminate. However, it can be converted into a determinate system by using multiple period data since the unknowns are time-invariant. The determinate system, when designed to estimate the properties of a single space point, will consist of equation 4.10 written for i^{th} space point and $(4 + n_c)$ different time periods. i.e.

$$TX_i(DP_{ik}\,Cos^2\theta_i + DQ_{ik}\,Sin^2\theta_i) + TY_i(DP_{ik}\,Sin^2\theta_i + DQ_{ik}Cos^2\theta_i)$$

$$+ f_r(P_{ik}, P_{i,k-1}, \cdots P_{i,k-m_e}, \alpha^i) - S_i \cdot DT_{ik} = W_{ik} - X_{ik} \quad \cdots (4.15)$$

$$k = 1, \ldots, (4 + n_c)$$

The resulting system of non-linear simultaneous equations can be solved for $(4+n_c)$ unknown variables. The procedure can be ridded with many problems since there are no fool-proof methods for the solution of non-linear simultaneous equations. The system of equations can be converted into a linear system by assuming linear rainfall-recharge relation and assigning some constant value to θ (probably from the known hydrogeological conditions of the aquifer under study). To the latter there is an alternative of writing the governing differential equation in terms of T_{pp}, T_{qq} and T_{pq}. This approach is likely to give mathematically incompatible solution (i.e. computed $T_{pp}\ T_{qq}$ may be less than T_{pq}^2).

The transformation of equation 4.1 which is a first order partial differential equation in T_{xx} and T_{yy} into an trigno-algebraic equation in T_{xx} and T_{yy} has been accomplished by neglecting the third and fourth terms of the left hand side of equation 4.1. The trigno-algebraic character of the governing equation can be maintained even while retaining these terms by treating $\partial T_{xx}/\partial x$ and $\partial T_{yy}/\partial y$ as just another two unknowns (Sagar, 1975). This however can lead to incompatible solution unless the parameters at all the space points are estimated jointly and the compatibility conditions relating to the computed transmissibility gradients and the computed spatial variation of transmissibility are imposed. No significant improvement in the solution may be obtained by retaining the terms unless these conditions are imposed. The inclusion of these conditions makes the problem computationally almost infeasible. Thus these terms which anyway are of small order of magnitude, can be neglected.

4.8 OPTIMISATION

Equation 4.10is the discretised form of equation 4.2 and in addition incorporates functional relationship for rainfall recharge. The right hand side of this equation may not be exactly zero under real world field conditions because of the one or more of the following reasons.

i) There are many built-in assumptions in equation 4.2 which may not be always realistic. Neuman (1973) has pointed out a large number of physical situations which may violate many of these assumptions. These situations include existence of fractures, unsaturated flow, compressibility, three dimensional geometry, changes in fluid density, non-Darcian flow in the areas of high Reynold's number.

ii) Numerical errors associated with the estimation of the spatial and temporal derivatives of piezometric head i.e. (DP_{ik}), (DQ_{ik}) and (DT_{ik}). These errors can originate from the numerical algorithms adopted to arrive at these derivatives.

iii) The assumptions incorporated in the adopted rainfall-rainfall recharge relation (equations 4.7, 4.9) may not be always valid.

Because of these possible discrepancies, the solution of the system of simultaneous equations may yield grossly misleading values of the parameters. Therefore, the determinate system can not be used to estimate the parameters in most of the real-world situations.

4.8.1 Residue Functional

Keeping this in view the equation 4.10 may be rewritten as

$$TX_i(DP_{ik}\, Cos^2\theta_i + DQ_{ik}\, Sin^2\theta_i) + TY_i(DP_{ik} Sin^2\theta_i + DQ_{ik} Cos^2\theta_i) +$$
$$f_r(P_{ik}, P_{i,k-1}, \cdots P_{i,k-m_e}, \alpha^i) - W_{ik} + X_{ik} - S_i DT_{ik} = \epsilon_{ik} \quad \cdots(4.16)$$

where ϵ_{ik} is the residue in governing differential equation when written in the discrete form as given in 4.10, for the i^{th} space point and k^{th} time period. The exact magnitudes of (ϵ_{ik}) are not known. However, these residues will have certain frequency distribution. One of the most obvious frequency distribution for a large sample is normal distribution with zero mean. The frequency distribution is as follows

$$f(\xi) = \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left[-\frac{1}{2}\left(\frac{\xi}{\sigma}\right)^2\right] \qquad \ldots (4.17)$$

If this assumption relating to the normal distribution of residues holds good, then the minimisation of the sum of squares of the residues yields the most likelihood estimates, i.e. minimise Y_1 given by

$$Y_1 = \sum_{i=1}^{n} \sum_{k=1}^{m} \varepsilon_{ik}^2 \qquad \ldots (4.18)$$

Subjectively speaking, the assumption of normal distribution of ε will be generally valid if sample size is large and there are no systematic errors.

The other commonly used residue functional is the sum of the modulii of the residues. The minimisation of this functional yields the most likelihood solution if the residues are distributed as per the following distribution.

$$f(\xi) = \frac{1}{2\alpha_p} \exp\left[\frac{-|\xi|}{\alpha_p}\right] \qquad \ldots (4.19)$$

The most likelihood estimate corresponds to

$$\text{Minimise} \sum_{i=1}^{n} \sum_{k=1}^{m} |\varepsilon_{ik}| \qquad \ldots (4.20)$$

The objective function to be minimised can incorporate residue functional (square of the residue or the modulus of the residue) summed up over time domain or over both time and space domains. Thus the objective function can assume one of the following forms, in addition to the those given in (4.18) and (4.20).

$$Y_2 = \sum_{k=1}^{m} e_{ik}^2 \qquad \ldots(4.21)$$

and $$Y_2 = \sum_{k=1}^{m} | e_{ik} | \quad \text{for a given } i \qquad \ldots(4.22)$$

where n is the total number of space points at which the parameters are to be estimated and m is the total number of time periods for which the historical data are available.

Y_2 is a function of the aquifer and hydrologic properties (i.e. parameters) of the space point under consideration. The total number of decision variables involved in the minimisation of Y_2 equals $(4+n_c)$. Thus, the parameters of each space point are estimated in seperate optimisation runs. This approach restricts the number of decision variables to a managable limit, but does not permit the inclusion of the constraints relating to the spatial variation of the parameters. These type of constraints may be necessary from the viewpoint of getting 'smooth solution' (Emsellem and DeMarsily, 1971). These can be incorporated in the parameter estimation provided the parameters of all the space points are estimated by the minimisation of a single objective function. This is permitted by minimisation of Y_1 (eqs.4.18,4.20) which is a function of the parameters of all the space points. The total number of decision variables involved in the minimisation of Y_1 is $n(4+n_c)$. This may increase the number of decision variables enormously which inturn may aggravate uncertainities generally associated with nonlinear programming.

4.8.2 Constraints

The actual frequency distribution of the residues though following the general pattern of the normal or exponential distribution, may not strictly follow the functional relations as stipulated in these distributions (equations 4.17 and 4.19). As a result the residues corresponding to the minimised residue functionals will seldom be identical to the actual residues. Thus, there may be an infinite number of near optimal solutions which may vary considerably from the optimal solution but may lead to a much better predictive model (Neuman, 1973b). Knowledge relating to the geohydrology of the aquifer can be employed to choose the best set of parameters for getting the near optimal solution. The relevant information after judicious quantification can be incorporated in the parameter estimation programme in the form of appropriate constraints. The constraints, apart from ensuring non-negativity of parameters whereever required, may also stipulate the permissible range of variation for each or a few of the parameters. The stipulated ranges may be much more restrictive in nature that the ones derived purely from the theoretical conditions. The range for a storage coefficients (specific yield) of unconfined alluvial aquifers may be stipulated as 0.05 to 0.30. Similarly constraints can be imposed to restrict the spatial variation of aquifer parameters and to obtain a 'smooth' solution (Emsellem and DeMarsily, 1971).

4.8.3. Optimisation Algorithms

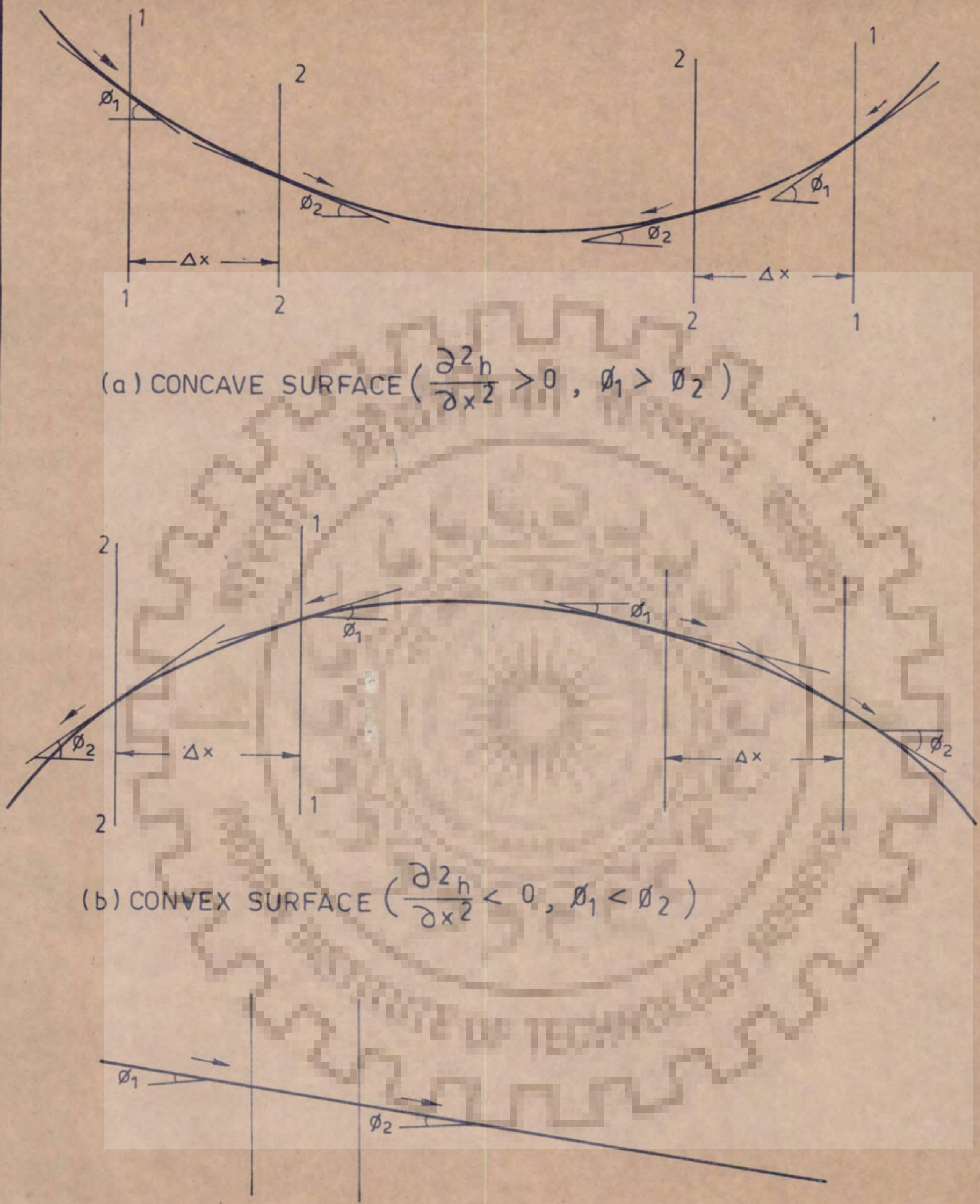
The algorithms that can be employed to arrive at the optimal solution depends primarily upon the nature of the residue functional and the constraints. The unconstrained minimisation of the sum of the squares of the residues can be accomplished by the classical least square approach. This approach is mathematically elegant and generally involves the solution of a set of linear simultaneous equations. The constrained minimisation of the sum of the squares of the residues or the sum of the modulii of the residues can be carried out employing with necessary modification, one of the standard nonlinear optimisation algorithms e.g. SUMT - Sequential unconstrained minimisation technique (McCormic and Fiacco, 1968). These techniques employ 'search' method and are iterative in nature. The unconstrained least square procedure involves far less programming and computational efforts but is seldom of much use in parameter estimation, as some of the parameters may be bounded.

4.8.4 Water Balance

The procedures of parameter estimation by optimisation essentially involves arriving at such estimates of the parameters which consistently minimise the residues of the Boussinesq's equation, given by equation ~~4.10~~ 4.16. These residues, can be expressed in their simplest form as follows.

$$\varepsilon = \left(T_{xx} \frac{\partial^2 h}{\partial x^2} + T_{yy} \frac{\partial^2 h}{\partial y^2}\right) + Q - S \frac{\partial h}{\partial t} \qquad \ldots(4.27)$$

The two terms in the parenthesis on the right hand side of the equation 4.27 represents the net excess horizontal inflow rate per unit area. Positive values of second derivatives indicate a concave surface and hence an excess of inflows(figure 4.2). Similarly negative second derivatives indicate convex surface and the resultant net horizontal outflow. A zero second derivative indicates a plane surface with a complete balance between horizontal inflows and outflows. The term Q represents the net vertical accretion rate per unit area. The term $S \frac{\partial h}{\partial t}$ represents the rate of storage increase per unit area. Thus, ϵ denotes the net imbalance between the total inflows (horizontal and vertical) and the increase of groundwater storage. As per the continuity equation this imbalance should be zero for all time periods. However the residues may never vanish completely because of inadequate quantification. These residues are functions of T_{xx}, T_{yy}, S and other parameters representing the orientation of principal permeability directions and the rainfall-recharge relations (refer equations 4.7 to 4.9). Since these parameters are known/ assumed to be time invariant, the objective is to arrive at such estimates of parameters for a specific space point which consistently minimises as much as possible this imbalance at the corresponding space point for all time periods. The minimisation of the residue functional can be viewed as multiple period water balance with time invariant parameters representing aquifer and hydrologic properties.



(c) FLAT SURFACE $\left(\frac{\partial^2 h}{\partial x^2} = 0\ ,\ \phi_1 = \phi_2\right)$

FIG. 4·2 _ SECOND SPATIAL DERIVATIVES OF PIEZOMETRIC HEAD FOR DIFFERENT SHAPES OF PIEZOMETRIC SURFACES

4.9 COMPUTER CODE

The algorithm described in the preceeding paragraphs has been programmed for arriving at digital computer assisted solutions. The code permits the estimation of parameters of one space point at a time.

The computer code makes use of the following data of the least square polynomial approximations of spatial variation of piezometric head, for each of the temporal points.

i) Number of terms in the approximating polynomial

ii) Exponents of the two coordinates in each term

iii) Coefficient of each term

iv) The mean and standard deviation of the coordinates of observation points.

The computer code described at the end of Chapter 3 provides these data for each temporal point. These data assist in the computation of the spatial and temporal derivatives of the piezometric head at the space point under consideration (i.e. the space points for which the parameters are to be estimated) in each of the time periods. The following steps are involved.

i) Evaluate the normalised coordinates of the space point employing (a) mean and standard deviation of the coordinates of observation points and (b) the coordinates of the space point.

ii) Employing the normalised coordinates calculate the functional values and the second spatial derivatives of the least square polynomials at the beginning and end of the period.

iii) Calculate the second spatial and temporal derivatives as per equations 4.12 to 4.14.

These derivatives which correspond to all the periods and the given space point, are stored as subscripted variables.

The corresponding rainfall data of different raingauging stations for the same periods are read as follows :

i) The coordinates of each of the raingauge stations.
ii) Rainfall values of each of the raingauge stations for all the periods.

The rainfall values to the space point under consideration are assigned in the following steps :

i) Calculate the distance of the space point from each of the raingauge stations. Find out the raingauge station with the shortest distance.
ii) Assign the rainfall values to the space point corresponding to data of the nearest raingauge station.

This is the numerical equivalent of the Theissen's polygon procedure. The rainfall values are stored as subscripted variables.

The groundwater withdrawal and recharge data are read for each period. The available data are generally not of continuous form but are available zone wise. The serial number of the corresponding zone is read along with the coordinates of the space point. The data of the zone are assigned to the space point under consideration. These are stored as subscripted variables.

A subroutine is written which computes the objective function and the constraints for the current values of the decision variables. The adopted form of function f_r (equation 4.7) and the residue functional (equation 4.21 or 4.22) have to be incorporated in this subroutine. This subroutine is called from the optimisation routine. The following data are transferred to this subroutine from the main programme through COMMON blocks.

i) Initial values of the system parameters.

ii) The data stored in the subscripted variables, described in the preceeding paragraphs.

iii) The data relating to the constraints (like minimum and maximum permissible values for each parameter).

Subsequent to making these information available in the subroutine described earlier, the nonlinear optimisation routine is called. The returned optimal values are printed. Then it is checked if the parameters of any more space point are to be estimated. If yes, the procedure is repeated for the next space point. If no, the instruction is given to stop.

The flow chart of the computation is given in Figure 4.3.

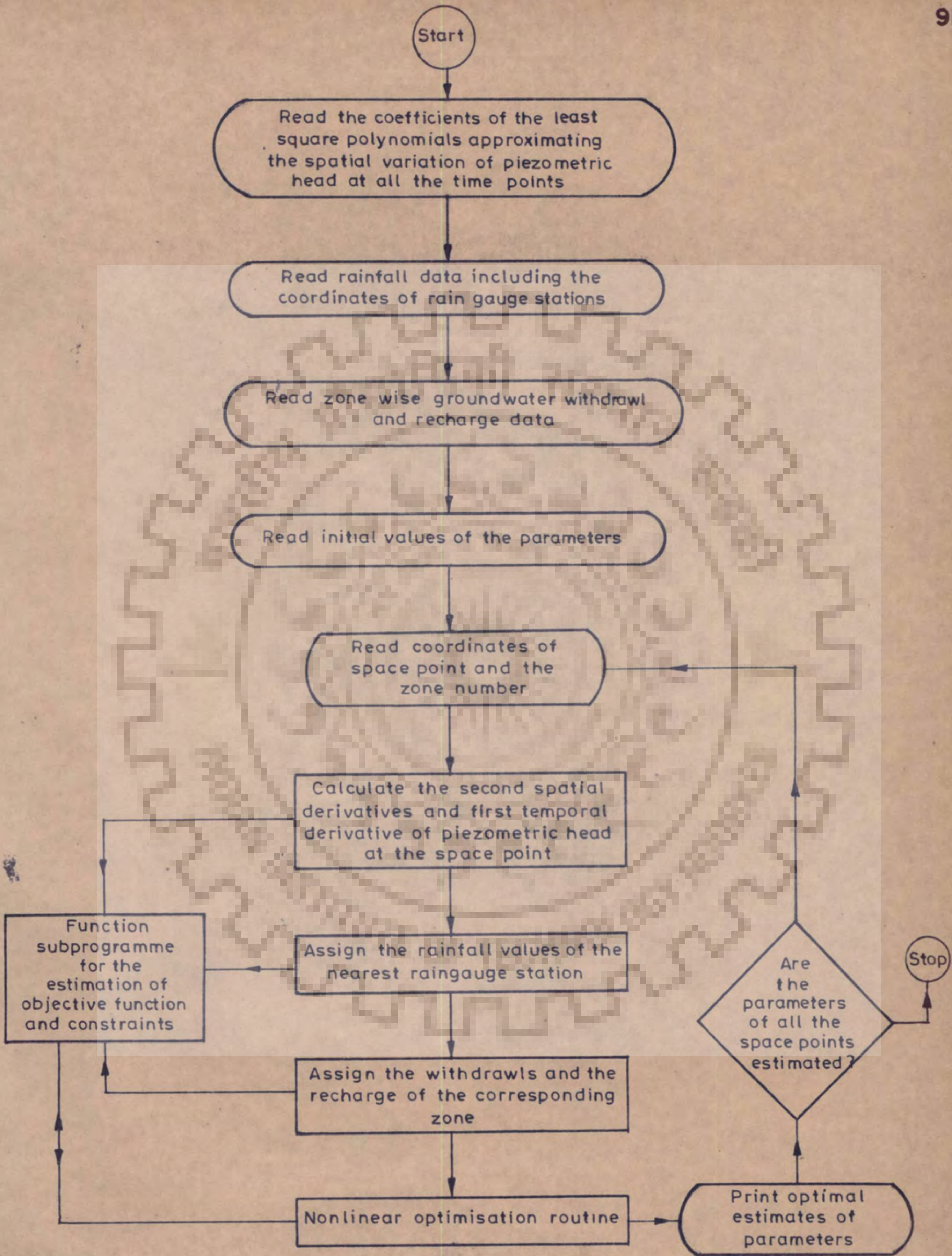


Fig.4.3-Flowchart for the aquifer parameter estimation

CHAPTER - 5

A DISTRIBUTED GROUNDWATER RESPONSE AND PLANNING MODEL FOR OPTIMAL DEVELOPMENT

5.1 PROBLEM IDENTIFICATION

An acceptable or feasible groundwater withdrawal policy can be defined as one which restricts the watertable elevations within an acceptable range and does not have any adverse socio-economic implications. The optimal withdrawal policy, apart from being feasible, also maximises the net benefits derived from the use of groundwater in conjunction with surface water. In case the water is to be predominantly used for agricultural activity, the net benefits are linked with the net returns from the crops. This approach affords the inclusion of areas under different crops amongst the decision variables. Roger and Smith (1970) proposed a linear programming based optimisation model for arriving at optimal cropping pattern and the releases of water from ground and surface water reservoirs. In this model the groundwater response is considered as lumped. Young and Bredehoeft (1972) proposed a simulation model for analysing such a system. The model accounts for stream-aquifer interaction and the distributed nature of groundwater response. However, it does not directly provide the optimal solution but can yield the value of objective function for each of a number of alternative courses of action. From the array of outcomes approximate optimum may be identified.

In the following paragraphs an optimisation based distributed model for evolving optimal groundwater withdrawals and the cropping patterns, for a predefined pattern of surface water availability, has been proposed. The model permits the inclusion of constraints of restricting the watertable elevations within a prescribed range.

5.2 PROBLEM DEFINITION

Groundwater withdrawals enhance the net returns from the agricultural activity by supplementing the rainfall and/or surface water supplies for an improved cropping pattern. However, the groundwater withdrawal activity can not be unbounded because of the necessity of restricting the watertable elevations within an acceptable range. Therefore, the feasibility of a given groundwater withdrawal policy is governed by the watertable fluctuations in space and time, consequent to the pumping. It is necessary to develop a model which apart from maximising the net benefits from agricultural activity, also accounts for the spatial variability of the watertable elevations arising out of spatially varying withdrawals and/or boundary conditions. The spatially varying groundwater withdrawals are desirable because of the spatial variability of rainfall, surface water supplies and the aquifer response characteristics. A judicious distribution of groundwater withdrawals and the cropping pattern in space and time, as per the meteorological and geohydrological conditions, can improve the net benefits from the agricultural activity. This can be accomplished by adopting a model which permits the calculation of spatially and temporally varying optimal cropping pattern and the groundwater withdrawals, and can ensure the confinement of watertable elevations within an acceptable range. Thus, the model should have following characteristics.

i) The decision variables should include spatially varying cropping pattern and spatially and temporally varying groundwater withdrawals.

ii) The feasibility with respect to restricting the water table elevations within a prescribed limit, should be checked by a distributed model of groundwater response. This constraint renders the problem nonlinear.

5.3 LUMPED MODEL

The lumped conjunctive use model (Figure 2.3) proposed by Roger and Smith (1970) can be, with suitable modifications, used to arrive at optimal policy relating to the temporal releases of groundwater and the cropping pattern, for given temporal distribution of mean rainfall and the surfacewater supplies.

The objective function to be maximised, representing the net benefits from the agricultural activity, can be written as follows -

$$Y = \sum_{j=1}^{NCR} (CC_j \cdot CY_j - C_j)\, A_j - CG \cdot \sum_{k=1}^{m} W_k - CS \qquad \ldots (5.1)$$

where CC_j, CY_j and C_j are the market price (per unit weight), yield (weight/unit area) and cost of inputs excluding water(per unit area) for j^{th} crop; A_j is the area under j^{th} crop; CG is the cost of a unit volume of groundwater; W_k is the groundwater withdrawal (volume) in k^{th} period; NCR is the total number of feasible crops; m is the number of periods discretising twelve months and CS (a constant) is the total cost of surfacewater as per its availability in the area. The magnitude of CS will not affect the optimal values of (A_j) and (W_k) but will merely influence the optimal 'net benefits'.

The optimal policy of crop areas (A_j) and the groundwater releases (W_k) would be one which maximises Y. However the maximisation has to be carried out within a feasible region bounded by the following constraints.

i) Crop water requirement : If δ_{jk} is the net irrigation requirement of j^{th} crop during k^{th} period then the following constraints must be satisfied.

$$\eta_g W_k + \eta_s \cdot SW_k \geq \sum_j \delta_{jk} A_j \qquad \ldots (5.2)$$

where η_g and η_s are the efficiencies of irrigation for ground and surfacewaters respectively and SW_k is the surfacewater availability during k^{th} period. Total number of constraints = m

ii) Available land areas for irrigation : In any time period the sum of the areas under all the crops must not exceed the culturable command area. Thus, if

$$\xi^{c}_{jk} = \frac{\text{Area under } j^{th} \text{ crop in } k^{th} \text{ period}}{\text{Culturable command area}}$$

then $\sum_{j} \xi^{c}_{jk} \leq 1.0$ ... (5.3)

Total number of constraints = m

iii) Maximum and minimum areas under each crop : If $AMAX_j$ and $AMIN_j$ are the maximum and minimum permissible areas under j^{th} crop then,

$$AMAX_j \geq A_j \geq AMIN_j \qquad \text{... (5.4)}$$

Total number of constraints = 2.NCR

iv) Permissible number of groundwater structures per unit area : If q_w is the withdrawal capacity of a single groundwater structure, NMAX is the maximum permissible number of structures per unit area and AG is the total geographical area then

$$\frac{W_k}{AG \cdot q_w} \leq NMAX \qquad \text{... (5.5)}$$

Total number of constraints = m.

v) Total groundwater withdrawal : If R^a is the total annual groundwater recharge from all the sources and M the permissible annual mining, then

$$0.0 \leq \sum_{k} W_k - R^a \leq M \qquad \text{... (5.6)}$$

This constraint is equivalent to using a lumped model for ascertaining the feasibility of a pumping pattern.

The maximisation of the objective function described in equation (5.1) with respect to (W_k) and (A_j), subject to the constraints expressed in (5.2) to (5.6) can yield optimal policy incorporating the following :

i) The total groundwater withdrawals in the entire study area in different time periods.

ii) Total area under each crop.

5.4 APPLICABILITY OF LUMPED MODEL

The applicability of lumped model in arriving at physically realistic policies is hampered by the fact that it tends to ignore the distributed character of the groundwater system. Thus it yields the areas under different crops and the total groundwater withdrawals for different discrete periods without incorporating their spatial distribution. It assumes that the constraint (5.6) will always ensure the conditions of 'no-rise' or 'permissible-decline', for any spatial distribution of crop areas and the consequent spatial distribution of groundwater withdrawals. This presumption has the following limitations.

i) Time domain : The constraint 5.6 ensures that in a calender year the change of groundwater storage is either zero or is negative to a permissible level. In a spatially lumped system this only ensures that the change in watertable elevation at the end of 12 months' period is either zero or negative, without any reference whatsoever to the history of fluctuations. The history of fluctuations is quite important from the view point of restricting the stream-aquifer interaction to a permissible level.

ii) Space domain : The position is all the more complicated in space domain. For isolated aquifers, the constraint (5.6) ensures no-rise or marginal-decline conditions at the end of 12 months, on an average (with respect to space) basis. This can be quite deceptive since the watertable elevations may vary considerably in space and the average conditions, in no way, ensure that the watertable elevations are within permissible limits over the entire area. In case of continuous aquifers the aforesaid constraint does not account for the change in subsurface inflows or outflows due to redistribution of the watertable elevations in space and time.

Thus, it can be inferred that the constraint (5.6) is of limited scope because it tends to restrict the watertable fluctuations within a permissible range by defining upper and lower bounds on total annual withdrawals. A more realistic approach would be to impose constraints on global maximum and minimum watertable elevations directly. This approach, though physically more realistic, is more tedious since it introduces constraints which are not only nonlinear but also implicit.

5.5 DISTRIBUTED MODEL

The limitations of the lumped model described in the previous section can be taken care of in the following ways :

i) Considering the cropping areas, groundwater withdrawals, surfacewater supplies and the net irrigation requirements as spatially varying quantities.

ii) By imposing the constraints on maximum and minimum depths to watertable.

A model incorporating these features is described in the following paragraphs.

5.5.1 Objective function and Constraints

The entire study area is divided into finite number of zones of uniform or near-uniform surfacewater supplies and net irrigation requirements. The discretisation so affected is only with respect to cropping areas and the groundwater withdrawals. The decision variables, in the distributed model, will include areas under different crops in different zones, (A_{jl}) and the groundwater withdrawals in different periods in different zones, (W_{kl}), the subscript l representing the zone number. The objective function can be written in terms of these decision variables as follows -

$$Y = \sum_j \left[(CC_j.CY_j - C_j) \sum_{l=1}^{NZ} A_{jl} \right] - CG \sum_l \sum_k W_{kl} - CS \quad ...(5.8)$$

where NZ is the total number of zones.

Other constraints can be written as follows.

(i) Cropwater requirement constraints : These constraints are imposed to ensure that the net irrigation requirements of each zone is met at a given level (δ^m). In the context of distributed model the net irrigation requirement is dependent upon the crop, period as well as the zone (refer 5.2).

$$(\eta_g W_{kl} + \eta_s SW_{kl}) \geq \delta^m \sum_j \delta_{jkl} A_{jl} \qquad \ldots (5.9)$$

$k = 1, \ldots m$ and $l = 1, \ldots NZ$

(ii) Land availability : These constraints are imposed to ensure land availability at the zonal level (refer 5.3). Thus if

$$\xi^c_{jkl} = \frac{\text{Area under } j^{th} \text{ crop in } l^{th} \text{ zone in } k^{th} \text{ period}}{\text{Culturable command area of } l^{th} \text{ zone}}$$

$$\sum_j \xi^c_{jkl} \leq 1.0 \qquad \ldots (5.10)$$

$k = 1, \ldots m$ and $l = 1, \ldots NZ$

(iii) Maximum and minimum areas under each crop : The total area under each crop is obtained by summing up the areas in each zone and the constraints can be written as follows (refer 5.4).

$$AMAX_j \geq \sum_l A_{jl} \geq AMIN_j \qquad \ldots (5.11)$$

$j = 1, \ldots NCR$

(iv) Permissible number of structures per unit area : The constraint requires that the maximum number of structures required in each zone should be less than the maximum permissible structures. The following linear constraints can be written(refer 5.5) to ensure this :

$$\frac{W_{kl}}{AG_l \, q_w} \leq NMAX \qquad \ldots (5.12)$$

$$k = 1, \ldots, m \text{ and } l = 1, \ldots NZ$$

The total number of constraints required to impose this condition is M.NZ. This number can be reduced to one as follows :

$$\max_{k,l} \left(\frac{W_{kl}}{AG_l \, q_w}\right) \leq NMAX \qquad \ldots (5.13)$$

(v) Maximum and minimum depths to watertable : The constraints relating to the maximum and minimum permissible depths to watertable are imposed by expressing the depths to watertable, under the conditions of dynamic equilibrium, as functions of the zonal withdrawal patterns (W_{kl}), boundary conditions, aquifer parameters, recharge and the groundwater elevations. Out of these only (W_{kl}) are the decision variables and the rest are the input data. For realistic conditions these depths will be implicit functions of (W_{kl}). Thus, for given (W_{kl}) these depths can be evaluated at discrete space points only. The procedure will consist of the numerical solution of Boussinesq's equation(2.7) to estimate dynamic equilibrium watertable elevations corresponding to the known boundary conditions, recharge, aquifer parameters and withdrawal patterns.

Boussinesq's equation can be solved numerically by discretising the entire space by a finite number of nodes. This discretisation is necessary for numerical evaluation of aquifer response to the known trial patterns of the zonal pumpings. Thus, if the space is discretised by n number of nodes and h_{ik} and G_i are the watertable elevation at i^{th} node in k^{th} time and the ground elevation of i^{th} node respectively, then,

$$\max_{i,k} (G_i - h_{ik}) < dmax$$

$$\min_{i,k} (G_i - h_{ik}) > dmin \quad \ldots (5.14)$$

(5)

where dmin and dmax are the permissible minimum and maximum depths to watertable.

5.5.2 Zoning and Discretisation

In the preceeding paragraphs, cropping pattern and the groundwater withdrawals have been distributed in space by dividing the area into a finite number of zones. Each zone has uniform cropping pattern, rainfall, surface water supplies and hence groundwater withdrawals. Thus, the permissible level of pumping in a given zone will be limited by the maximum drawdown occuring anywhere (i.e. at any node) in the zone. To minimise the 'loss' in optimal returns due to this process of discretisation, the zones should be demarcated in such a manner that each zone, apart from having near uniform surface water supplies and rainfall, also displays near uniform drawdown/

discharge characteristics. This can be ensured by selecting zones of nearly same geohydrologic characteristics and the boundary effects. The size and the number of zones is primarily governed by the necessity of restricting the number of decision variables to a managable limit. Apart from this, the number of zones should not be too large to permit the implementation of the evolved policy.

The discretisation of area by a finite number of nodes for obtaining numerical solution of Boussinesq's equation is governed by entirely different set of conditions relating mainly to the properties of algorithm adopted for the numerical solution. The finite difference algorithms require the nodes to be positioned along two orthogonal directions coinciding with the major and minor permeability directions (figures 5.1). The spacing of nodes is governed by the consideration of stability, convergence and truncation errors. For restricting truncation errors, a much smaller nodal spacing (as compared to the size of a zone) will be required.

While calculating the aquifer response, all the nodes lying in a given zone are governed by the current trial groundwater withdrawal pattern of the zone (refer figure 5.1). Thus, the feasibility with respect to the constraint given in (5.14) (5) is checked at the nodal level by studying the nodal-response of the aquifer to the zonal-groundwater withdrawal pattern.

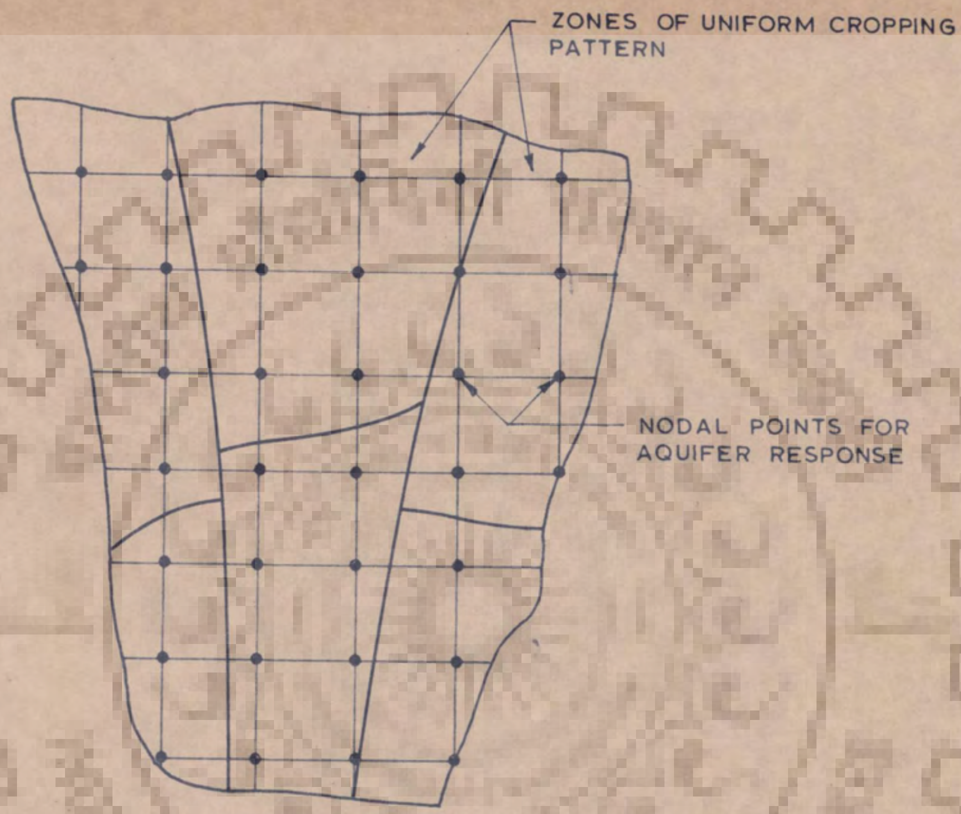


FIG. 5·1-ZONING AND DISCRETISATION

5.5.3 Maximisation of the Objective Function

Unlike the lumped model, the distributed model as described in (5.8) to (5.14) can not be analysed by linear programming since the constraints given in (5.14) are nonlinear implicit functions of the decision variable (W_{kl}). The non-linear and implicit nature of these functions point towards the necessity of employing non-linear optimisation methods for maximising the net benefits. These methods permit the implicit evaluation of objective function, constraints and their derivatives. This calls for linking an aquifer response code as a subroutine to an optimisation complex incorporating the formulation of objective function and other constraints given in (5.1) to (5.13)

5.6 COMPUTER CODE

The algorithm described in the preceeding paragraphs has been programmed on a digital computer. The code makes use of a finite difference model for evaluation of distributed aquifer response (Prickett and Lonnquist, 1971).

5.6.1 Data

The code reads the following data

i) Number of feasible crops; market price, input cost, yield, maximum and minimum areasfor each crop. Consumptive use requirement of each crop in different periods. Costs of ground and surface waters.

ii) System of discretisation for aquifer response : number of rows, columns, boundary conditions initial conditions; nodal values of transmissibility, storage coefficient, recharge and ground deviation.

iii) Maximum and minimum permissible depths to water-table.

iv) Number of zones of uniform cropping pattern and groundwater withdrawals.

v) Effective rainfall (utilisable by plants) and surfacewater supplies for each zone in different periods. Geographical area and culturable command area in each zone.

vi) Coordinates (row and column numbers) of nodes lying in each zone.

5.6.2 Initial Values

Subsequent to reading the data the code assigns initial values to the decision variables in the following manner:

i) Calculate the cropping area variables as per the minimum permissible arearequirements in the following way

$$A_{jl} = \frac{AG_l}{\sum_l AG_l} AMIN_j$$

$j = 1, \ldots NCR$

$l = 1, \ldots NZ$

ii) For each zone calculate the net groundwater irrigation, requirement of all the crops in all the periods, employing the data of consumptive use requirements; zonal rainfall and surfacewater supplies.

iii) Calculate the total groundwater withdrawal requirement of each zone in all the periods i.e. (W_{kl}).

iv) From the arrived at values of (W_{kl}) and the number of nodes in each zone, assign nodal pumpage values for each zone in each period.

v) Operate aquifer response model and calculate the watertable elevations at all the nodes in all the periods, corresponding to the state of dynamic - equilibrium. This is accomplished by comparing the current values of the watertable elevations with the corresponding simulated values of preceeding year. The simulation is terminated if this difference is less than a stipulated values or if the simulation has proceeded for a stipulated number of years, whichever is earlier.

vi) Calculate the maximum and minimum depths to watertables. Check if these depths are within the feasible region (i.e. dmin to dmax). If yes, the initial feasible solution has been reached. If no, stop, giving a massage 'initial solution not found'.

5.6.3 Evaluation of Constraints and Objective Function

A subroutine is written which computes the objective function and the constraints for the current values of decision variables. For evaluating the constraints relating to the maximum and minimum depths to watertable, it calls the subroutine of distributed aquifer response. The data mentioned in 5.6.1 are made available in this subroutine through COMMON blocks. The data mentioned in (ii) of 5.6.1 are made available in the aquifer response routine as well. The nodal pumpage values are computed in the subroutine from the current values of zonal pumpages. These are transferred to the aquifer response routine through the arguments of the corresponding CALL statement.

5.6.4 Nonlinear Optimisation

Subsequent to making the aforesaid information available in the corresponding subroutines, the nonlinear optimisation routine is called from the main programme. The subroutine mentioned in the previous paragraph is repeatedly called from this subroutine for the evaluation of objective function, constraints and the consequent modifications in the decision variables. The returned optimal values are printed.

The stepwise procedure is given in figure 5.2.

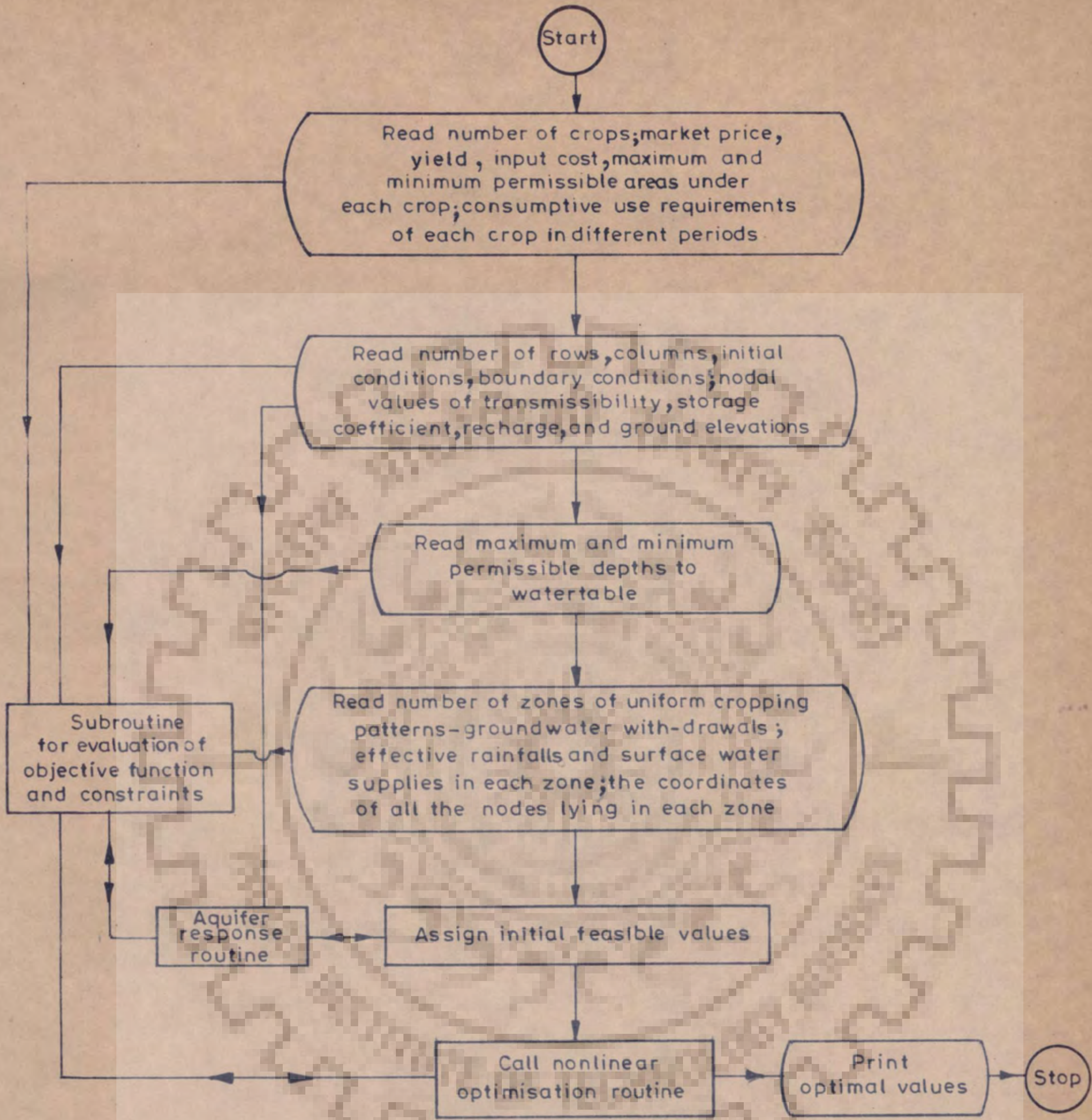


Fig.5.2-Flow chart for optimal cropping patterns and groundwater withdrawals

CHAPTER - 6

PARAMETER ESTIMATION FOR DAHA AREA

6.1 VIABILITY OF A HYDROLOGIC MODEL

The ultimate aim of development of a hydrologic model is to solve real world problems. The data requirements of groundwater models are in general quite rigorous because of the distributed nature of the groundwater system. Many of the input data are not physically measurable quantities. Such data are estimated indirectly from related directly measured quantities, making use of certain empirical or analytical equations. The groundwater models should be flexible and robust enough to deal with situations of inadequacy of data. The applicability of the models described in Chapters 3 and 4, under these situations, has been demonstrated by taking up case study of a moderately monitored area (Daha Area).

6.2 STUDY AREA

The study area lies towards north-west of district Meerut of U.P. (India) and is bounded by river Hindon on the east and river Krishni on the west (figure 6.1). It is about 21 kilometers in length from north to south and 24.3 kilometers in width from east to west at its maximum sections. It lies between longitudes 77^{o}-19' E to 77^{o}-32' and longitudes 29^{o}-05' N to 29^{o}-17' N. The elevation of above mean sea level varies from 238.205 meters in the north to 225.055 meters in south. The terrain in general slopes from north to south which is

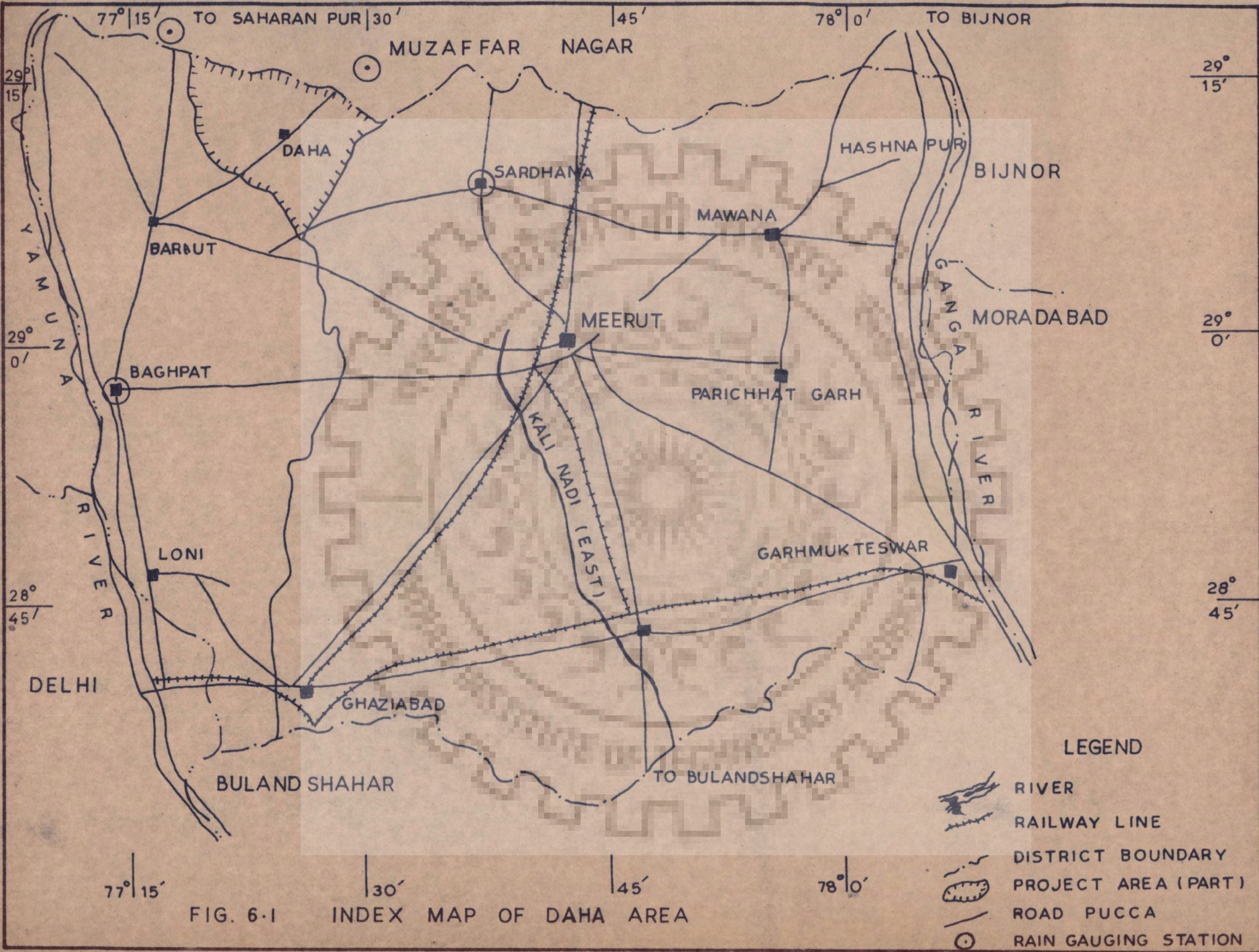


FIG. 6·1 INDEX MAP OF DAHA AREA

also the direction of main drainages in the area. The area experiences extreme type of climate with very hot summers and cold winters. The rainy season sets in by the middle of June and lasts till the end of September. The rains in winter months (November to February) are low. The rainfall in the rest of the months is almost negligible. The area is fairly fertile and is suitable for raising sugarcane, cereals and other cash crops. Presently there is no surfacewater input and the irrigation requirements are met through intensive groundwater development. The area falls in the Indo-Gangetic plain and is formed of unconsolidated fluviatile formations comprising of sand, silt, clay and kankar. From geological consideratio sediments are favourably combedded for the occurence of groundwater.

6.3 DATA UTILISED FOR THE STUDY

The data utilised for this study have been drawn from the project report (1976) and dissertations (Dikshit, 1978 and Desai, 1978). A brief description of the data is as follows :

i) Watertable data : The data used for the study comprise of monthly watertable elevations of twenty one observation wells for the period June 1972 to July 1975. The observation wells are fairly well distributed in the area and roughly fall along five cross-sections across east-west (figure 6.2).

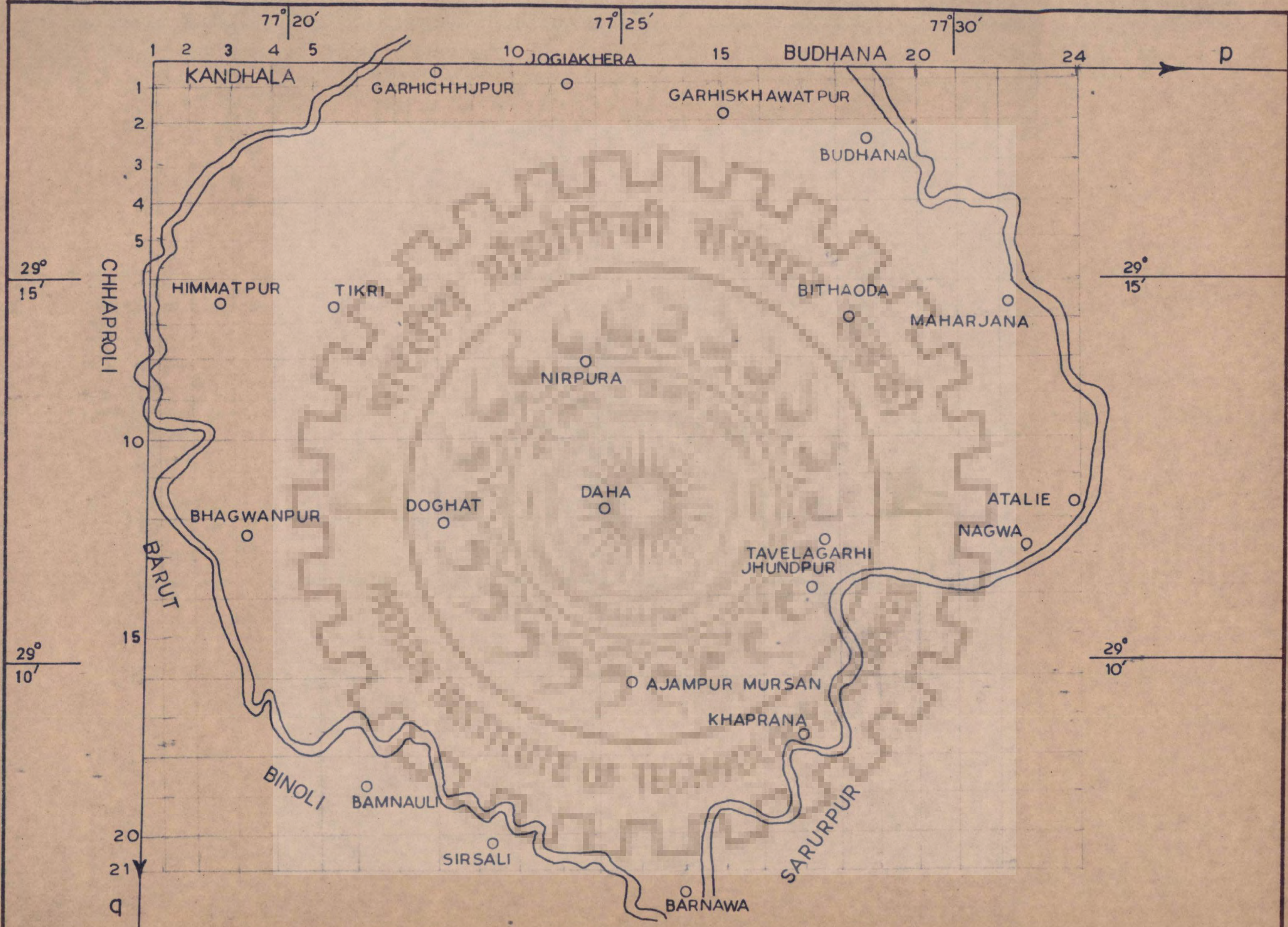
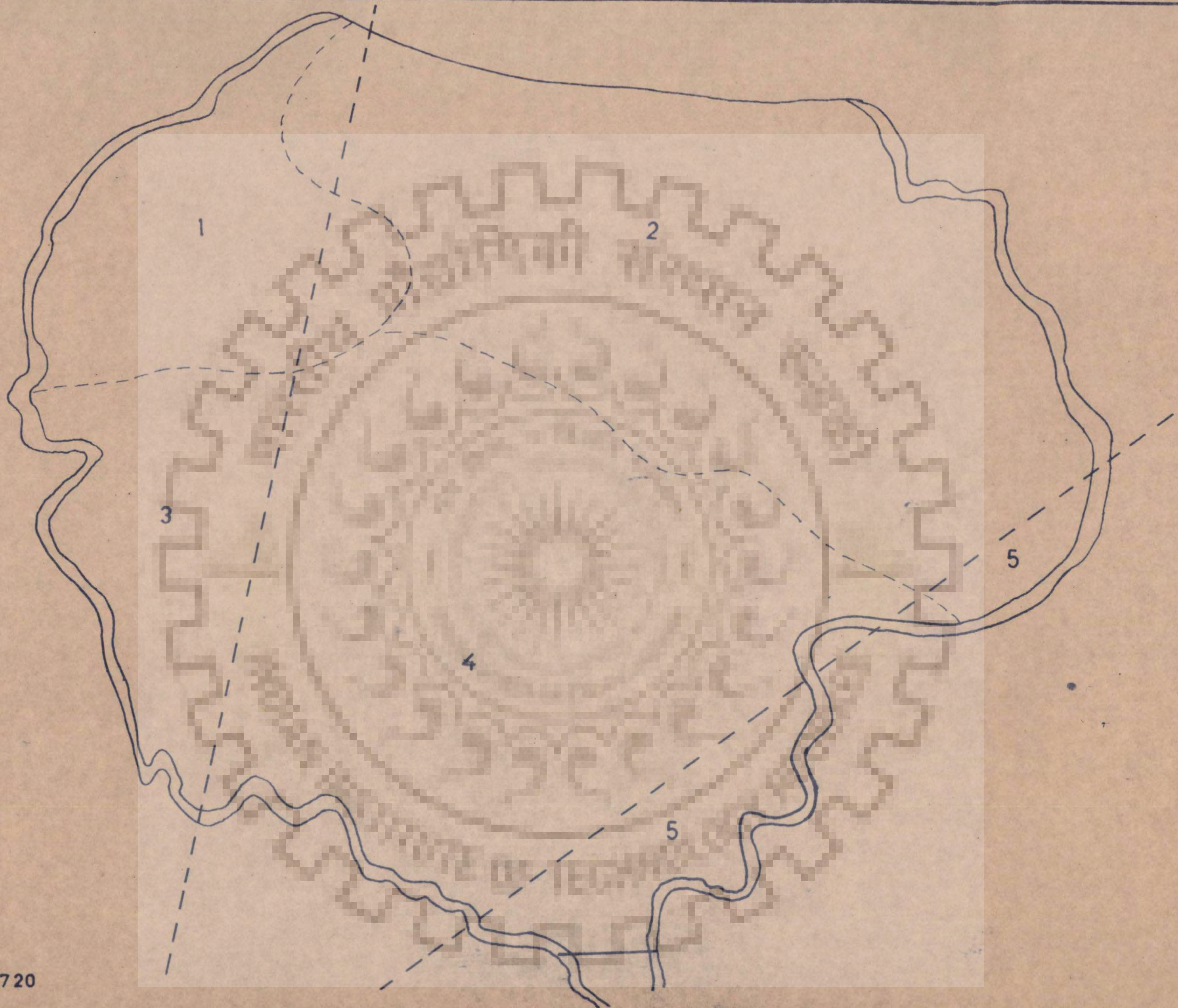


FIG. 6.2 - LOCATION OF OBSERVATION WELLS IN DAHA AREA

ii) Rainfall data : Monthly rainfall data of four raingauge stations for the corresponding period were used for the study. Apart from this, the average annual rainfall for the period 1961-1971 were also available.

iii) Groundwater withdrawal data : Elaborate groundwater withdrawal data are required for the estimation of geohydrological parameters as per the procedure listed in Chapter 4. The data requirement incorporated in (W_{ik}) pertains to the spatial and temporal distribution of groundwater withdrawals. The available data comprised of the number of different types of structures, their distribution in space and the average discharge and annual running hours, during the period 1972-75. These data could provide only the distribution of withdrawals in space. The temporal distribution of these withdrawals as obtained by Dikshit (1978) and Desai (1978) through a lumped area waterbalance study, was adopted for the study. The details of the methodology adopted by them is given in the section 6.3.1. The data available from these studies comprised of the monthly groundwater withdrawals (adjusted for the return flows) in five zones (figure 6.3) of uniform withdrawals, for the period June 1972 to June 1975).



SCALE:-1:126720

FIG. 6·3 - FIVE ZONES OF GROUNDWATER ABSTRACTIONS

iv) Initial values of the parameters : Initial values of the parameters are necessary for parameter estimation by non-linear optimisation. The solution of nonlinear optimisation problems is greatly facilitated if the initial values, apart from being feasible are also close to the real values. This can, to some extent, dispense with the requirement of large number of trials with different initial values, to take care of multimodal objective functions. The initial values of transmissibility and storage coefficient assigned on the basis of values adopted in the aforesaid project report are 2000 m^2/day and 0.11 respectively.

6.3.1 Groundwater Balance Study

The aforesaid groundwater balance study conducted by Dikshit (1978) and Desai (1978) considers the following equation (Chandra et al 1975).

$$R + I_g + S^i = S^e + O_g + ET + W + \Delta S^g \qquad \ldots (6.1)$$

where R is the rainfall recharge, I_g is the subsurface inflows, S^i is the influent seepage, S^e is the efluent seepage, O_g is subsurface outflows, ET is the evapotranspiration, W is the groundwater withdrawals and ΔS^g is the change of groundwater storage. There is no surface water irrigation in the area.

In the said study the annual groundwater withdrawals were distributed in time on the basis of the consumptive use requirements of the historical cropping pattern. The consumptive use of different crops during different months were estimated on the basis of available pan-evaporation and the crop consumptive use coefficient. The total area was divided into five zones of uniform cropping patterns and rainfall(figure 6.3). For each zone the net irrigation requirements in different months were estimated by subtracting the effective monthly rainfall from the corresponding consumptive use requirement. Monthwise gross irrigation requirements of each zone were estimated by dividing the corresponding net irrigation requirements by the efficiency of irrigation (assumed as 0.65). Annual gross irrigation requirement was estimated by adding up the monthly gross irrigation requirements. The annual gross irrigation requirement was compared with the annual groundwater withdrawals, computed from the available data. It has been reported that the annual groundwater withdrawals were only about 58.5 percent of the total gross irrigation requirement. The monthly groundwater withdrawals for each zone were calculated assuming that only 58.5 percent of the net irrigation requirement in each month, is met by the groundwater withdrawals. The effective monthly withdrawals for each month were estimated by assuming that 40.5 percent of the applied irrigation goes back to the groundwater reservoir as return flows.

The rainfall recharge coefficient was determined by the waterbalance study of the monsoon periods (June to October) of the years 1972 to 1974. Groundwater withdrawals for these periods were estimated as per the procedure outlined in the previous paragraph. Other terms I_g, S^i, S^e, O_g, ET and ΔS^g were estimated from the available records. These estimates yield the value of R which was divided by the corresponding rainfall to get the recharge coefficient (K_r). The following values were obtained.

Table 6.1 - Recharge Coefficient by Water balance

Period	Recharge coefficient (K_r)
1.6.1972 to 31.10.72	0.22
1.6.1973 to 31.10.73	0.22
1.6.1974 to 31.10.74	0.18

Therefore, the initial value of recharge coefficient K_r was adopted as 0.18 for the aquifer parameter estimation in the distributed model.

6.4 LEAST SQUARE POLYNOMIAL APPROXIMATION

Least square polynomials were evolved to approximate the spatial variation of watertable elevation in thirty six sequential months from June 1972 to May 1975. The coefficient matrix so generated was employed to estimate spatial and temporal derivatives of hydraulic head.

The origin along with the axes of coordinate system is shown in figure 6.2 The coordinates of each observation point were measured in kilometers. However, the origin and the scale of coordinates are immaterial, since the coordinates are subsequently normalised.

The degree of the polynomial is decided on the basis of minimum standard error described in Chapter 3(paragraph 3.6.1). However, while dealing with observed data, it is desirable to restrict the degree of polynomial to three or four. Therefore, to start with a third degree polynomial of the following form containing ten terms was selected as a starting full polynomial for each period.

$$H_k(p,q) = b_1p^3+b_2q^3+b_3p^2q+ b_4pq^2+ b_5p^2+b_6q^2 + b_7pq + b_8p + b_9q + b_{10} \quad \ldots(6.2)$$

Corresponding to this full polynomial, he coefficients and the summary statistics were estimated for all the thirty six months. The space points which displayed a standard residue outside the range $\pm$ 2 as outliers. The approximating least square polynomials were re-estimated from the partial data exclusive of the outliers, if any. The computed modulii of t values of the coefficients were compared with the critical values of t at 95 percent level of confidence and degree of freedom equalling number of data points retained

Table 6.2 Least square polynomials - 1

Month	No.of terms retain-ed	No.of data points retain-ed	S.No.of data points dropped	R^2		s(m)	
				Initial	Final	Initial	Final
1	2	3	4	5	6	7	8
6,1972	7	17	2,6,9,20	0.9259	0.9702	0.9827	0.5523
7,1972	7	18	2,6,13	0.9342	0.9746	0.9716	0.5307
8,1972	8	16	2,6,8,9,13	0.9301	0.9563	0.9623	0.6019
9,1972	7	19	2,6	0.9413	0.9678	0.8643	0.5322
10,1972	7	19	2,6	0.9422	0.9726	0.8613	0.4898
11,1972	6	19	2,3	0.9458	0.9501	0.8449	0.6208
12,1972	7	19	2,6	0.9513	0.9732	0.8263	0.5118
1,1973	7	18	2,6,18	0.9466	0.9673	0.8745	0.5623
2,1973	7	18	2,6,18	0.9457	0.9746	0.8809	0.4949
3,1973	7	18	2,6,18	0.9474	0.9678	0.8679	0.5578
4,1973	7	18	2,6,18	0.9407	0.9700	0.9130	0.5339
5,1973	7	18	2,6,18	0.9469	0.9683	0.8662	0.5519
6,1973	6	14	1,2,6,9, 13,14,18	0.9176	0.9476	1.0867	0.5726
7,1973	8	18	2,3,4	0.9358	0.9525	0.9575	0.6592
8,1973	6	14	2,3,6,7,8, 9,18	0.9289	0.9846	1.0219	0.4205
9,1973	5	16	2,3,4,10,18	0.9240	0.9521	1.0395	0.5541
10,1973	7	17	2,3,9,18	0.9301	0.9565	1.0026	0.6480
11,1973	6	17	2,3,9,18	0.9333	0.9600	0.9899	0.6063
12,1973	6	17	2,3,13,18	0.9455	0.9679	0.9102	0.5510

Table 6.2 (contd.)

1	2	3	4	5	6	7	8
1 1974	9	17	3,7,17,18	0.9223	0.9888	0.9730	0.3808
2,1974	9	18	3,7,18	0.9413	0.9869	0.8772	0.4328
3,1974	8	19	17,18	0.9212	0.9613	0.9008	0.5751
4,1974	7	17	1,2,7,11	0.9312	0.9785	0.9294	0.4563
5,1974	8	16	1,4,8,17,18	0.8992	0.9776	1.1233	0.5175
6,1974	5	14	1,2,4,8,11, 17,18	0.8798	0.9501	1.1677	0.5241
7,1974	4	12	1,2,4,6,8, 10,11,18,20	0.8482	0.9649	1.3254	0.4999
8,1974	8	18	4,7,8	0.8969	0.9716	1.0827	0.5372
9,1974	8	14	4,8,11,16, 17,18,20	0.9117	0.9940	1.1202	0.3088
10,1974	7	14	1,4,7,8, 16,17,20	0.8991	0.9807	1.1756	0.5013
11,1974	6	16	2,4,7,8,17	1.1596	0.9492	0.8938	0.5981
12,1974	4	16	4,7,8,11, 17	0.8925	0.9731	1.1850	0.4488
1,1975	5	16	4,8,11,16, 17	0.9050	0.9848	1.1225	0.3519
2,1975	7	15	4,7,8,9, 11,16	0.8200	0.9832	1.4888	0.4040
3,1975	7	12	1,2,3,4,6, 7,8,11,16	0.8649	0.9655	1.4753	0.4903
4,1975	10	10	2,5,10,12, 13,14,15, 17,19,21	0.8280	1.000	1.5745	na
5,1975	9	13	3,4,6,7, 8,9,11,17	0,8722	0.9934	1.3638	0.3663

minus the number of terms. It was found that the values of computed t of almost all the terms other than the constant, were lower than the corresponding critical t values. In addition to this, the standard error was found to be large, thereby implying large residues. The deletion of the polynomial terms on the basis of t statistics further worsens the residues and standard error. This situation of large standard error was tackled by including the data points displaying residues outside the range ± 1.0 meters, amongst the outliers. Subsequent to the deletion of these data points, the standard error fell significantly and computed t values registered a rise. Still most of the computed t values were lower than critical t values. Therefore, only the terms having computed t value less than 1.0 were deleted (Chatterjee, 1977). This implies that for a term to be retained, it's coefficient must be atleast as large as its standard error. This criterion is less stringent than the one given in paragraph 3.6.2 and permits the retention of the larger number of terms. This was found to limit the increase in the sum of the squares of the residues as a result of the curtailment, within the permissible limits defined by F test (paragraph 3.6.2) in addition to restricting the increase in standard error as a result of curtailment. The results of typical period (July, 1972) are given in Table 6.3.

Table - 6.3

Typical Sequential Least-Square Fits

Computed t statistics, with all the data points retained

term	p^3	q^3	p^2q	pq^2	p^2	q^2	pq	p	q	constant
t	-.93	-.19	.33	-1.98	.51	-1.88	-1.46	1.57	-2.94	455.2

Standard error = 0.9716 meter

All the standard residues were within the range ± 2.0, but three residues at data points 2, 6 and 13 were outside the range ± 1.0 meters. The statistics of the fit, subsequent to deletion of these points are as follows.

Computed t statistics subsequent to deletion of outliers

term	p^3	q^3	p^2q	pq^2	p^2	q^2	pq	p	q	constant
t	-1.68	-.83	-.31	-3.22	1.31	-.38	-3.76	2.71	-3.55	720.19

Standard error = 0.5803 meters

Three terms displaying t statistics less than 1.0 i.e. q^3, p^2q and q^2 were deleted.

Standard error of partial polynomial fit = 0.5307 meter.

Subsequent to the curtailment of the polynomials based upon the above described criterion, the coefficients and the summary statistics of the partial polynomials were recomputed. The details of the terms retained and the corresponding coefficients for each month are given in tables 6.2 and 6.4 respectively. On the examination of results it was found that for a few months the number of outliers is quite excessive. This indicates the indequacy of third degree polynomials in approximating the spatial variation of watertable during these months. Thus, the least square polynomials were recomputed for the months which display more than five outliers, starting this time from a fourth degree polynomial containing fifteen terms. The form of the full polynomial is as follows :

$$H_k(p,q) = b_1p^4 + b_2q^4 + b_3p^3q + b_4pq^3 + b_5p^2q^2 + b_6p^3 + b_7q^3 + b_8p^2q + b_9pq^2 + b_{10}p^2 + b_{11}q^2 + b_{12}pq + b_{13}p + b_{14}q + b_{15} \quad \ldots(6.3)$$

This polynomial was truncated on the lines described earlier. The truncated polynomials displayed a better spatial distribution displayed by a fewer number of outliers.

The effects of dropping the space points or the terms of the polynomial on the spatial variation of the watertable elevations have been demonstrated by getting the computer assisted plots of watertable contours for each of the intermediate stages. To restrict the size of the output this

Table 6.4 Least square polynomials - 2

Month	Coefficient of least square polynomials									
	p^3	q^3	p^2q	pq^2	p^2	q^2	pq	p	q	Constant
1	2	3	4	5	6	7	8	9	10	11
6,1972	-0.4974	x	0.6977	-0.6718	x	-0.3480	x	1.5884	-2.8488	220.0766
7,1972	-0.3246	x	x	-0.8821	0.2556	x	-0.7980	1.4245	-2.5418	219.6667
8,1972	-0.2507	x	x	-0.8619	x	x	-0.7525	1.1952	-2.5568	219.9401
9,1972	-0.7275	-0.4000	x	-1.1071	x	x	-0.6908	2.2605	-1.5554	219.8586
10,1972	-0.7416	-0.4128	x	-1.1594	x	x	-0.6565	2.3219	-1.5214	219.7764
11,1972	-0.6109	x	x	-1.1247	x	x	-0.5543	1.9719	-2.4489	219.8520
12,1972	-0.6673	-0.3673	x	-1.2017	x	x	-0.7533	2.1886	-1.7756	219.7454
1,1973	-0.5167	x	0.4321	-1.1133	x	x	-0.5144	1.8039	-2.6885	219.8514
2,1973	-0.5134	-0.4117	x	-1.0972	x	x	-0.7008	1.8067	-1.6282	219.8489
3,1973	-0.5488	x	0.4455	-1.1625	x	x	-0.4744	1.9065	-2.6871	219.8748
4,1973	-0.5729	-0.3751	x	-1.1116	x	x	-0.7175	1.9089	-1.6766	219.7398
5,1973	-0.5790	x	0.4530	-1.1509	x	x	-0.4808	1.9771	-2.6758	219.7487
6,1973	x	x	0.3576	-0.7160	x	x	-0.2821	0.4530	-2.8311	219.8004
7,1973	-0.4637	0.4109	0.6564	-1.4719	x	-0.4928	x	1.7220	-3.3090	219.8666
8,1973	-0.4910	x	x	-1.0446	x	x	-1.0711	1.9987	-2.7773	220.0054

Table 6.4(contd.)

1	2	3	4	5	6	7	8	9	10	11
9,1973	-0.2676	x	x	-1.1494	x	x	x	1.4790	-2.1259	219.6813
10,1973	-0.3262	x	0.5034	-0.7034	x	x	-0.4758	1.1794	-2.8482	220.1953
11,1973	-0.3262	x	x	-0.8680	x	x	-0.6793	1.3037	-2.5973	220.1276
12,1973	-0.3760	x	x	-0.9700	x	x	-0.7471	1.3995	-2.6458	220.0029
1,1974	-0.6509	0.9748	1.6699	-0.8734	-0.3929	-0.1609	x	1.7997	-5.5377	220.9914
2,1974	-0.7067	1.1949	1.9539	-1.1173	-0.3829	-0.4176	x	2.0450	-6.2553	221.0827
3,1974	-0.2406	0.3331	0.9582	x	x	0.3718	0.3901	0.7970	-3.2500	220.1139
4,1974	-0.2779	-0.3436	x	-0.9179	x	x	0.6470	1.3030	-1.9933	220.0815
5,1974	-0.4631	x	1.5649	-1.3371	x	0.2844	1.0778	1.3153	-3.1456	219.1655
6,1974	x	x	1.2955	-0.8182	x	x	1.0043	x	-2.7187	219.4619
7,1974	x	x	x	-1.3086	x	x	x	0.8741	-2.5708	219.2735
8,1974	-0.3275	0.2241	x	-1.1130	x	-0.2870	-0.5657	1.6014	-2.9491	220.0330
9,1974	-0.5759	x	1.2302	x	-0.2387	-0.1388	1.0980	1.1648	-3.0154	219.6841
10,1974	-0.5465	-0.6457	x	-1.1815	-0.3157	x	x	1.6693	-1.5145	219.8132
11,1974	-0.5595	x	x	-1.0358	x	x	-0.4845	1.9678	-2.3209	219.3120
12,1974	x	-0.5303	x	x	x	x	x	0.3646	-1.1911	219.7497
1,1975	x	-0.5106	x	x	-0.2665	-0.2045	x	x	-1.2107	219.9879

Table 6.4 (contd.)

1	2	3	4	5	6	7	8	9	10	11
2,1975	-0.5970	-0.2725	x	-1.0492	x	x	-0.9432	1.5196	-1.6547	219.2362
3,1975	-0.7755	-0.4192	-0.9169	x	x	x	-2.3985	2.1122	-1.5336	219.5633
4,1975										
5,1975	-0.3315	-1.3198	-1.1162	0.8586	0.4765	0.9051	-1.6825	1.2439	x	217.7699

measure was adopted for a few months data only and for the rest the contours with respect to the finally retained form of the polynomial only were plotted. The intermediate results and the contours of the period July 1972 are given in tables 6.5 and 6.6 and figures 6.4 to 6.6.

Table 6.5 Intermediate approximating polynomials

Sl. No.	Number of terms in the polynomial	Number of data points	R^2	Contours given in figure	Groundwater storages per unit area per unit sp. yield above msl (in meters)	Space points/terms being deleted
1	10	21	0.9342	6.3	220.3757	Three points being deleted
2	10	18	0.9779	6.4	220.0570	Three terms being deleted
3	7	18	0.9746	6.5	220.0591	Nil

Finally retained polynomial with seven coefficients is of the following form :

$$H_2(p,q) = b_1p^3 + b_2pq^2 + b_3p^2 + b_4pq + b_5p + b_6q + b_7$$

The deleted space points are 2, 6 and 13 (refer figure 6.4). On a close examination it is found that the watertable data

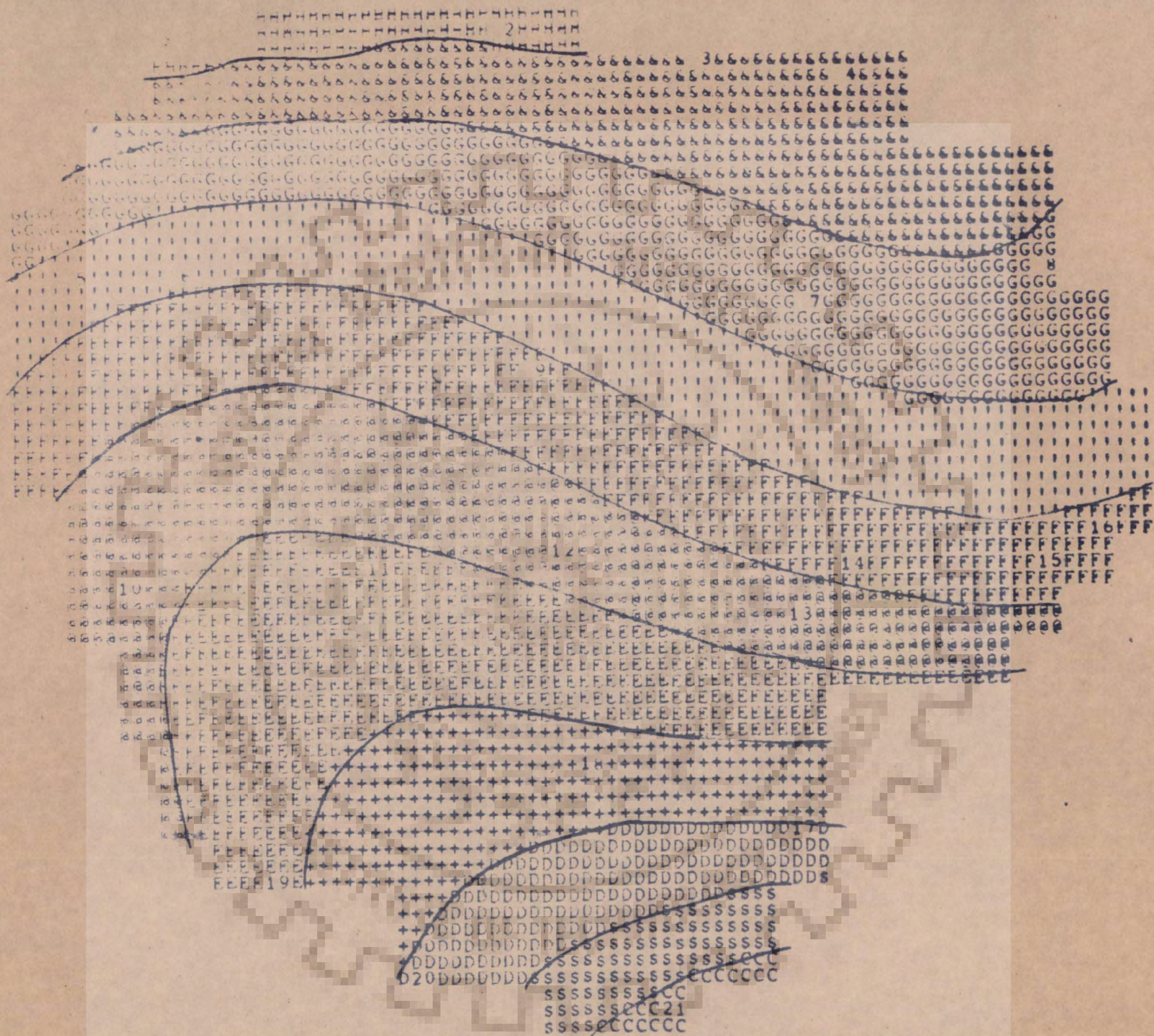


FIG. 6·4 _ LEAST SQUARE POLYNOMIAL SOLUTION _ 1

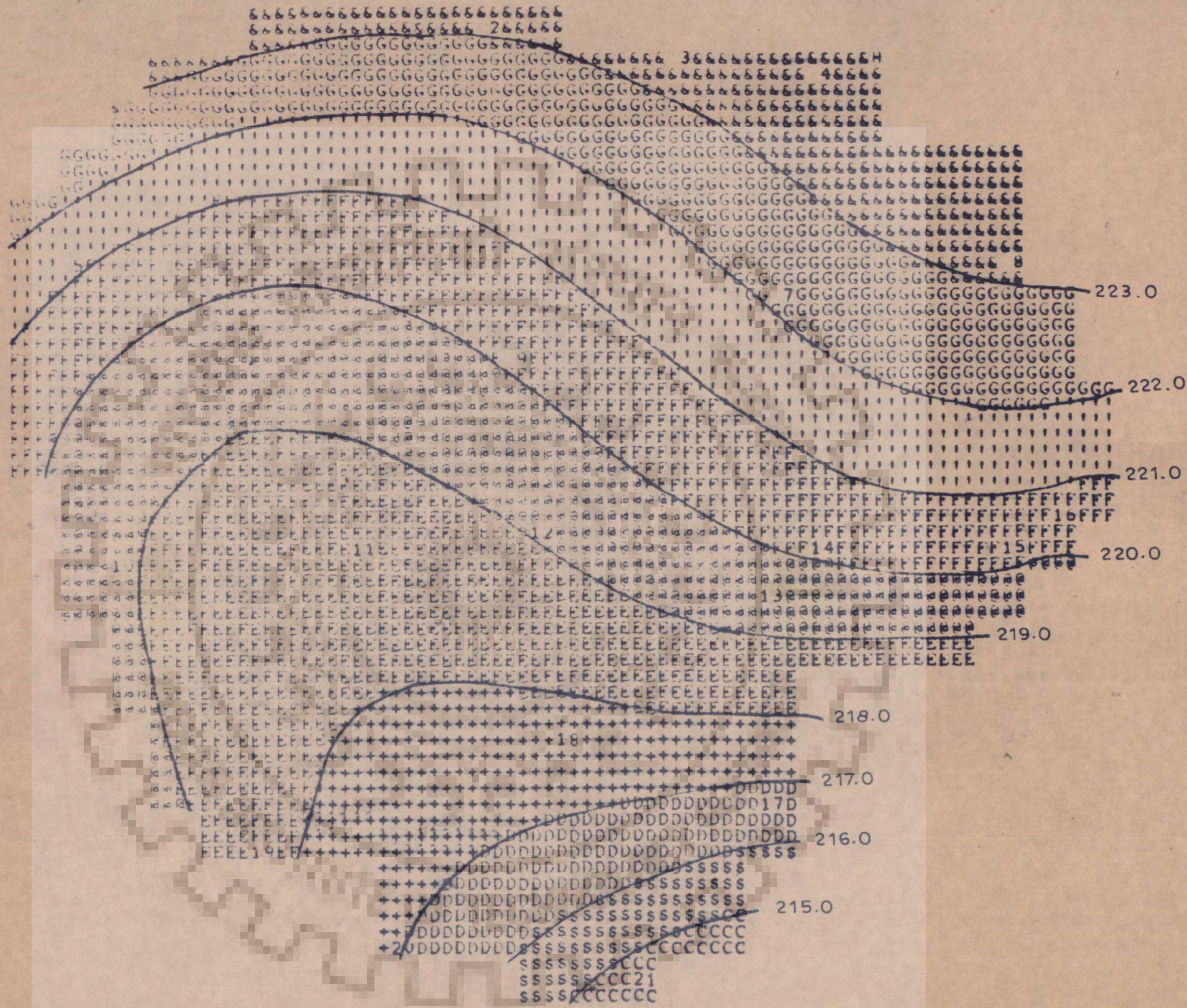


FIG. 6·5 _ LEAST SQUARE POLYNOMIAL SOLUTION _ 2

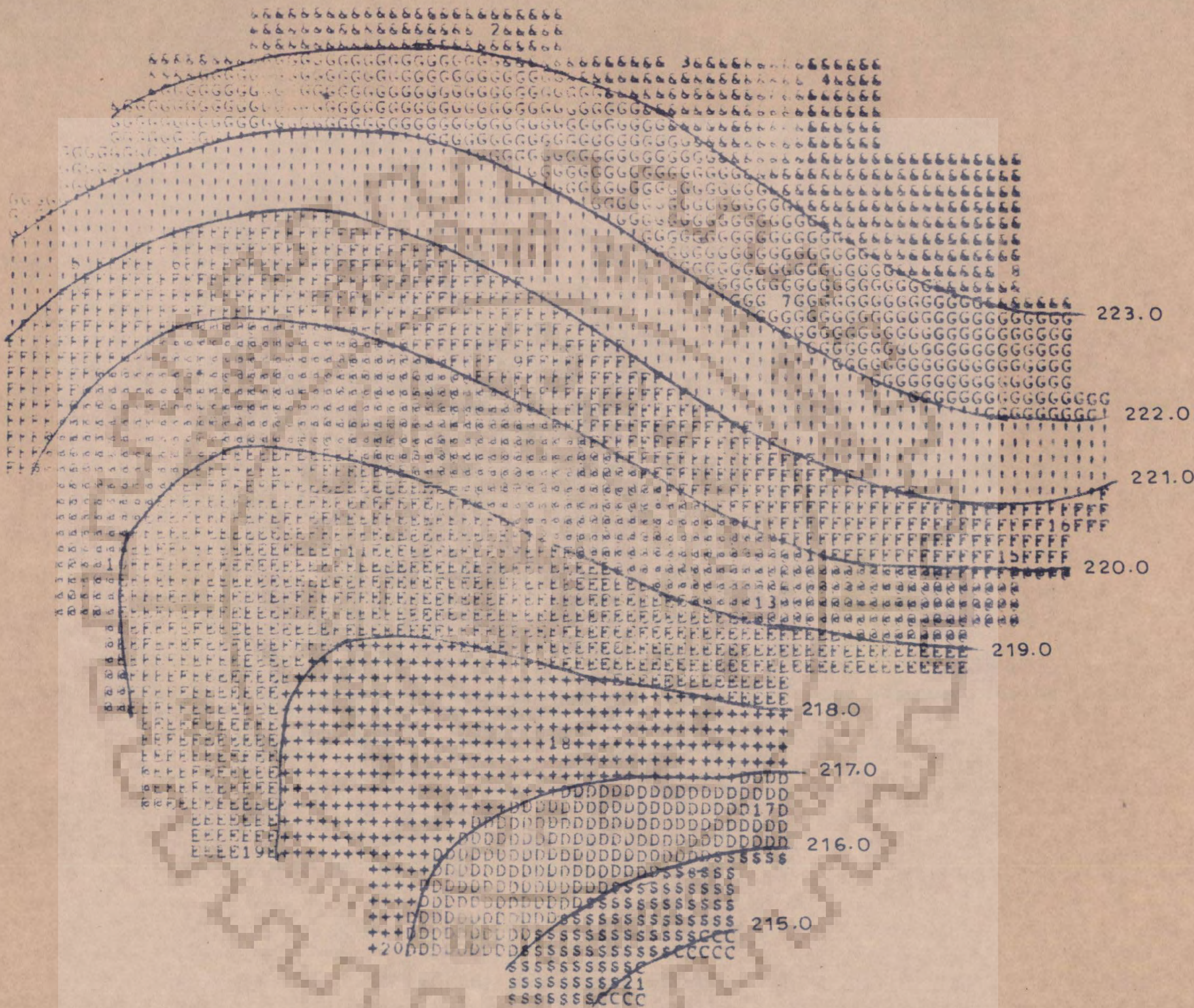


FIG. 6·6 _ LEAST SQUARE POLYNOMIAL SOLUTION _ 3

at these points are inconsistent to varying degrees with those of the surrounding ones. The relative magnitudes are given in table 6.6. While in case of points 2 and 6 there are somewhat abrupt changes in watertable elevations, in case of point 13, there is an abrupt change of gradients (refer figure 6.4). Points 14 and 18 lie on the two sides of point 13 and are almost colinear.

Table 6.6 Watertable elevations at the outliers and surrounding points

Outlier	Watertable elevation	Surrounding space points		
		Point	distance from outlier in kilometers	Watertable elevation in meters
2	225.7530	3	3 km	223.1130
6	222.1940	5	1.0 km	220.6670
13	220.5800	18	4 km	216.9100
		14	1.5 km	220.6500

Subsequent deletion of 3 out of 10 terms affected R^2 only marginally (i.e. from 0.9779 to 0.9746). This demonstrates the adequacy of t test in pin-pointing the terms which may be deleted without any significant loss in the goodness of the fit.

6.5 AQUIFER PARAMETER ESTIMATION

The aquifer parameter estimation was carried out by considering the following data of 35 discrete sequential periods of one month duration each.

i) The coefficients of the least square polynomial approximating the spatial variation of watertable elevation at the beginning and at the end of each period.

ii) The rainfall figures for all the raingauge stations, during each period.

iii) The effective groundwater withdrawals in five different zones during each period.

These data are employed to generate the following column matrices for a given i^{th} space point.

$$(DP_{ik}),\ (DQ_{ik}),\ (DT_{ik}), (P_{ik})\ \text{and}\ (W_{ik})$$

Each matrix has 35 elements corresponding to the 35 periods.

(X_{ik}) were taken to be zero since there is no surfacewater irrigation scheme in the area. Marginal evapotranspiration near the river banks have been incorporated in (W_{ik}). The following initial values were adopted for the decision variables.

$$S_i = 0.11,\ TX_i = 2000.0\ m^2/day,\ TY_i = 2000.0\ m^2/day,$$
$$\theta_i = 0.0,\ \alpha_1^i = 0.18,\ \alpha_2^i = 0.80\ \text{and}\ \alpha_3^i = 30\ mm/month. \quad \ldots (5.4)$$

The initial values of α_2^i was assigned assuming a 20 percent carryover effect from the previous month. The decision regarding the initial value of α_3^i was made on the basis of Chaturvedi's formula (Chaturvedi and Chandra 1961).

Thus, the residues of discretised Bousinesq's equation (4.16) for the i^{th} space point and all the thirty five periods are completely defined by the input data matrices and the current values of the seven decision variables. The following residue functional was adopted for the estimation of optimal parameters.

$$Y_i = \sum_{k=1}^{35} \varepsilon_{ik}^2$$

where $Y_i = Y\,(S_i,\ TX_i,\ TY_i,\ \theta_i,\ \alpha_1^i,\ \alpha_2^i,\ \alpha_3^i\ )$

The minimisation of Y_i with respect to these seven decision variables was carried out subject to the following constraints :

$$0.18 > S_i > 0.08$$

$$5000.0 > TX_i, TY_i > 500.0$$

$$\pi/2 > \theta_i > 0.0$$

$$0.30 > \alpha_1^i > 0.0$$

$$1.0 > \alpha_2^i > 0.75$$

$$80.0 > \alpha_3^i > 0.0$$

Total number of constraints = 14.

The parameters of fifteen space points (i.e. i = 1, ... 15) were estimated by minimising Y_i subject to the aforesaid constraints. The results were not very encouraging since for all the space points S_i were touching the lower limit (0.05) and and TX_i, TY_i the upper limits (5000 m^2/day). Subsequent relaxation of the lower limit of S_i to 0.0, too did not yield interior solutions since all the storage coefficients started touching zero. A detailed examination revealed the following possible explaination for the anomaly .

The net vertical outflows i.e. (W_{ik}) are far greater than the net vertical inflows (R_{ik}). The excess of (W_{ik}) over (R_{ik}) are met through the releases from the storages (i.e. $-S_i\ DT_{ik}$) and the net horizontal inflows directly related to the transmissibilities TX_i and TY_i. This is clarified by writing equation 4.10 in the following form

$$W_{ik}-R_{ik}= (-S_i DT_{ik})+ \left[TX_i(DP_{ik} \cos^2\theta_i+ DQ_{ik} \sin^2\theta_i) + TY_i(DP_{ik} \sin^2 \theta_i + DQ_{ik} \cos^2 \theta_i) \right] \quad ...(6.6)$$

In the optimal solution the excesses (W_{ik} - R_{ik}) are divided amongst the storage releases and the net horizontal inflows in such a way that the difference between these excesses and the source terms on right hand side is reduced to the minimum level consistently for all the time periods with the same time invariant values of storage coefficients and transmissibilities. This optimal division is affected by assigning the appropriate values to the storage coefficients and transmissibilities. In the present study it was found that the data $(W_{ik}-R_{ik})$ and (DT_{ik}) are quite inconsistent to each other over k. The term 'inconsistent' implies that the large excess withdrawals $(W_{ik}-R_{ik})$ do not display corresponding large negative temporal derivative (DT_{ik}). Also the periods of negative excess (i.e. during the rainy period when rainfall recharge is higher than (W_{ik})) do not display positive temporal derivatives. Thus, the releases from storage $(-S_i\ DT_{ik})$ are assigned minimal value in the optimisation since this term is incapable of contributing towards $(W_{ik}-R_{ik})$ consistently for all the time periods with time invariant value of S_i. The minimal role of this term has to be followed by the largest possible role of horizontal flows to get the closest match between the left and right hand sides of equation 6.6. This explains the lowest permissible value of S_i and excessively large values of transmissibilities.

This discrepancy between the temporal fluctuations of watertable and the estimated monthly withdrawals could have been occuring due to the assumption made in the process of distributing the annual groundwater withdrawals amongst various months. In the process it has been assumed that the groundwater withdrawals are affected in proportion of the consumptive use requirements. This assumption may provide fairly good distribution for routine analysis, but may not provide adequately accurate monthwise distribution for solving inverse problem.

This problem of inadequate means to get the monthwise distribution of the annual groundwater withdrawals has been taken care of by treating (W_{ik}) as the decision variables alongwith the seven aforesaid system parameters. This approach, however renders the problem non-unique since any values of storage coefficient and transmissibility can reduce the residues to zero for all the periods. This can be done by assigning the following value to the decision variable W_{ik}

$$W_{ik} = R_{ik} - S_i DT_{ik} + TX_i (DP\ Cos^2\theta_i + DQ\ Sin^2\theta_i) + TY_i\ (DP\ Sin^2\theta_i + DQ\ Sin^2\theta_i) \qquad \ldots (6.7)$$

The extent of non-uniqueness can be limited by imposing constraints on (W_{ik}). The constraints have to be severe enough in restricting the feasible regions for these variables.

In this study, the variation of these variables has been constrained by assuming that though the consumptive use requirement concept does not provide accurate enough monthwise distribution of groundwater withdrawals, it does provide adequate season wise distribution. Thus, the thirty five months' period was divided into six cropping seasons and it was stipulated that the sum of withdrawals in each season equals the corresponding total seasonal withdrawal estimated from the consumptive use requirement concept plus minus certain fraction to account for the uncertainties associated with return flows. Thus, the following constraints were imposed.

$$1.15 \sum_{k=T_{s,1}}^{T_{s,2}} W_{ik} > \sum_k \bar{W}_{ik} > 0.85 \sum_k W_{ik}$$

$$s = 1, \ldots\ldots 6$$

$$W_{ik} > 0.0$$

where $T_{s,1}$ and $T_{s,2}$ are the values of k corresponding to the beginning and end of s^{th} season and $(\bar{W}_{ik})$ are the distribution of annual groundwater withdrawals by the consumptive use requirement. These constrainsts provide the necessary flexibility to (W_{ik}) without rendering the solution completely non-unique.

In addition to this, the data of few months, which showed conspicuously large inconsistencies, were neglected. The parameters were reestimated with the scheme mentioned above. The parameters were estimated for all the fifteen points and the storage coefficient values in the new scheme of optimisation did not nosedive to the lowest stipulated limits. The initial values of the decision variables were assigned as per the prior knowledge relating to the hydrogeology and hydrology of the aquifer. The optimal solutions, thus obtained correspond to the global minima or a local minima closest to the starting solution in case of multimodal functions. This closest minima may be the desired solution if the initial values are assigned judiciously. A few trials were carried out to find the lowest minima within the feasibl domain. Number of trials had to be restricted because of the limitation on availability of computer time. The results are given in figures 6.7 to 6.13 and table 6.7.

Table 6.7 - Estimated Parameters

Parameter	Range of variation	Mean	Standard Deviation	Spatial variation shown in figure
S(percent)	8.00 to 11.26	9.61	1.12	6.7
T_{xx}(m^2/day)	1599 to 2459	1995.0	208.0	6.8
T_{yy}(m^2/day)	502 to 2075	1715.0	436.0	6.9
θ(radians)	0.09 to 0.39	0.26	0.085	6.10
α_1^i(percent)	10.00 to 22.55	14.75	4.3	6.11
α_2^i(fraction)	0.754 to 0.999	0.852	0.088	6.12
α_3^i(mm/month)	30.12 to 79.77	48.7	17.62	6.13

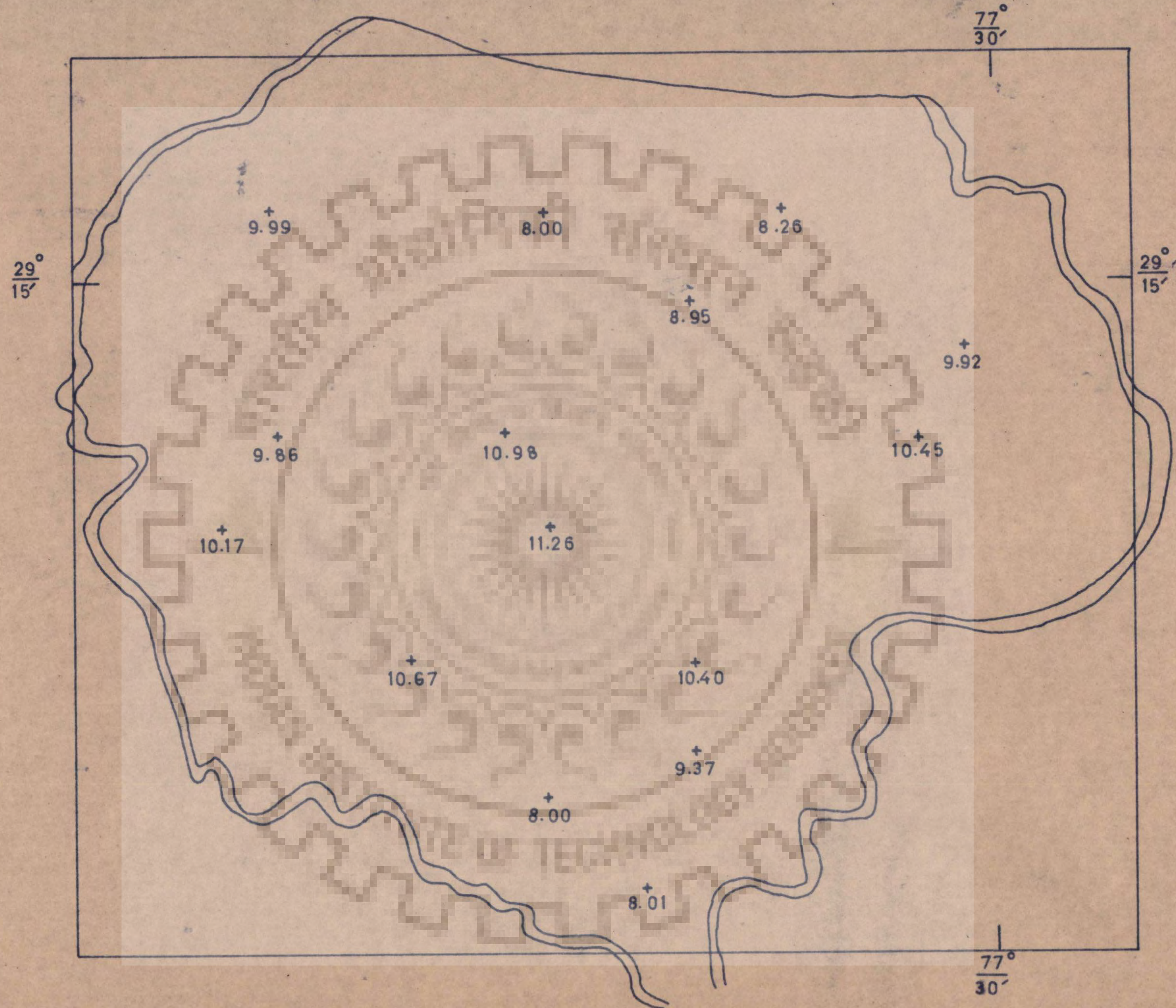
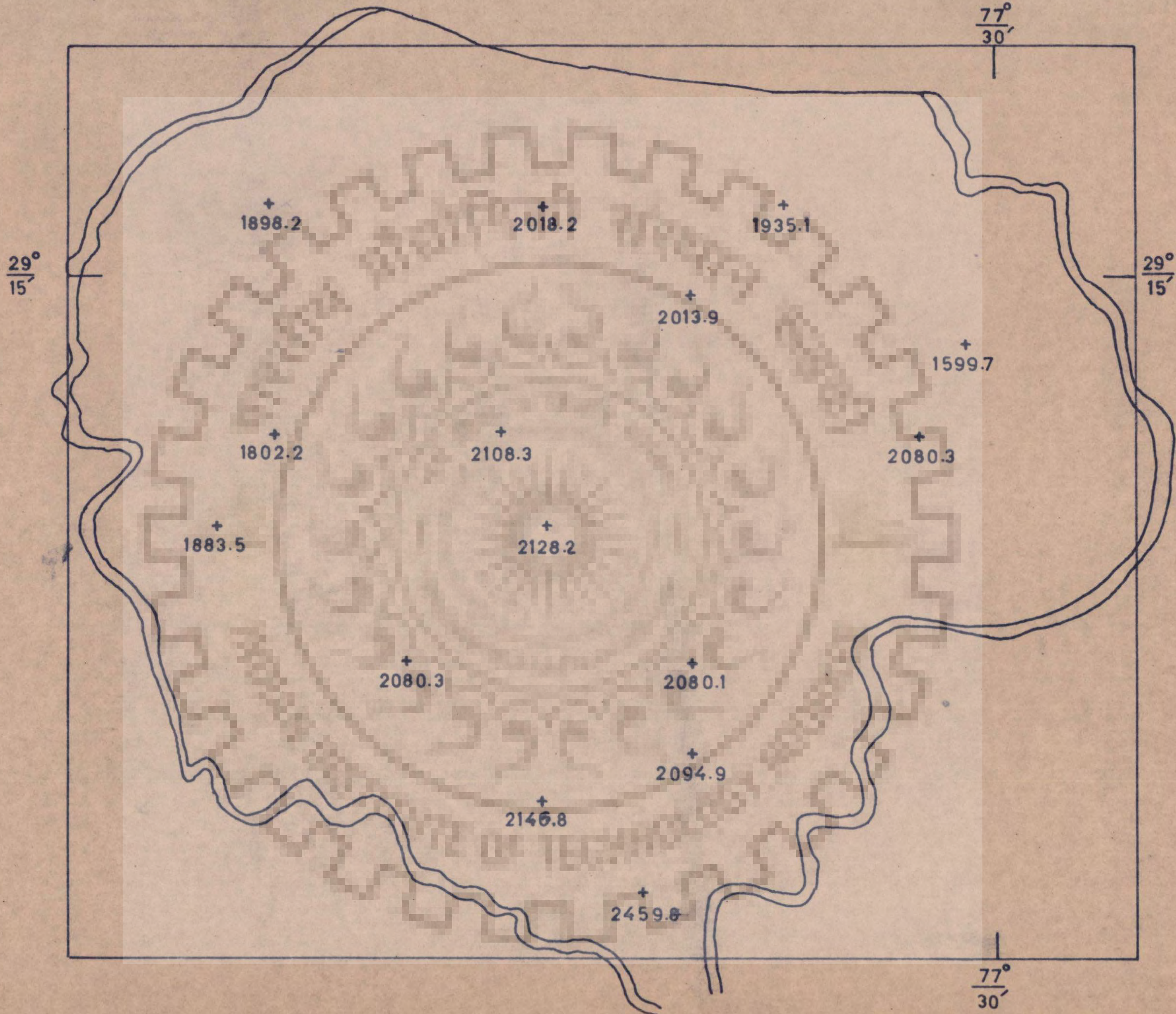
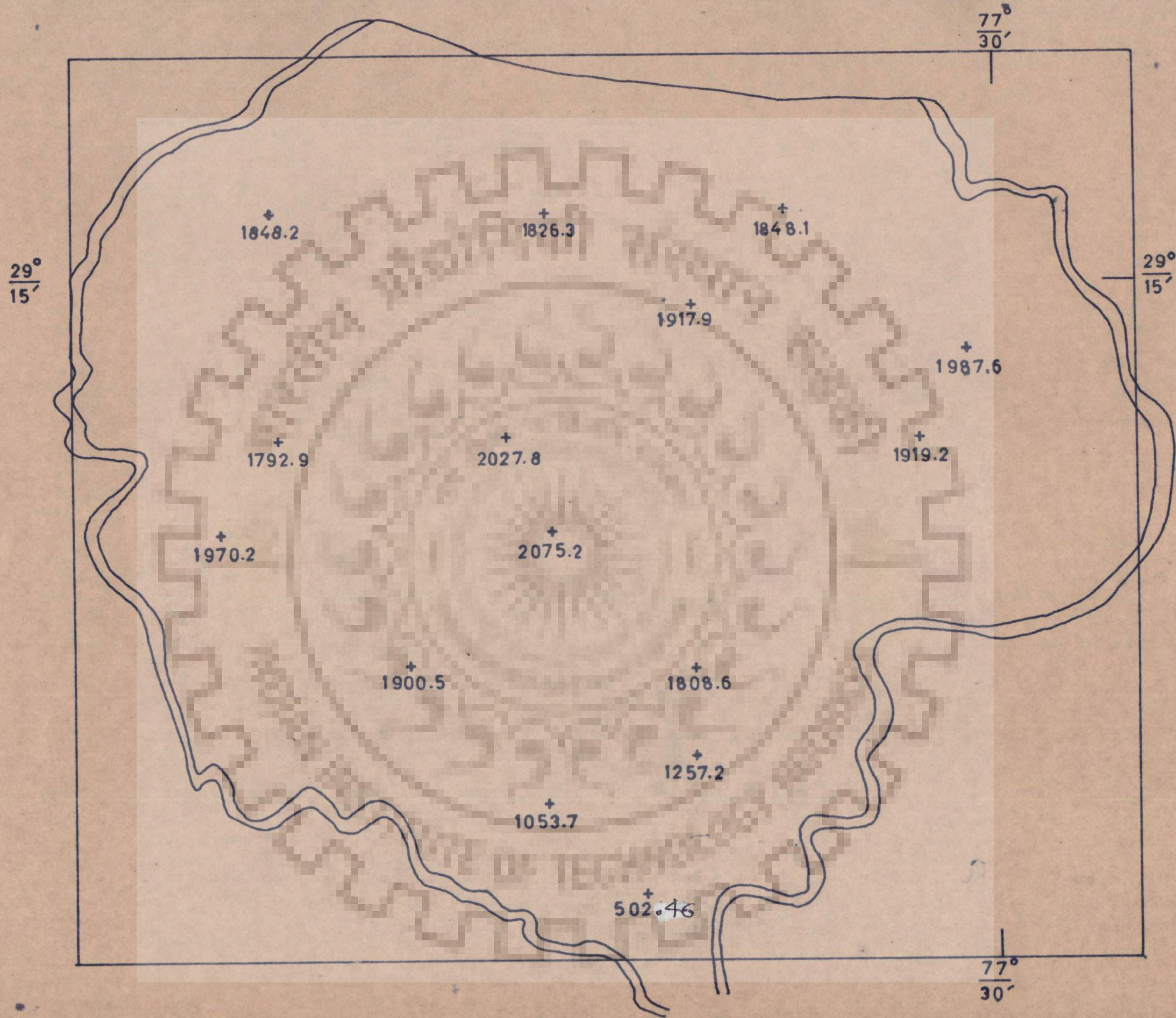


FIG. 6·7 -ESTIMATED SPECIFIC YIELDS (%)



SCALE –
1 : 126720

FIG. 6·8 - ESTIMATED TRANSMISSIBILITIES, T_{XX} (m^2/day)



SCALE –
1 : 126720

FIG. 6-9 - ESTIMATED TRANSMISSIBILITIES, T_{yy} (m^2/day)

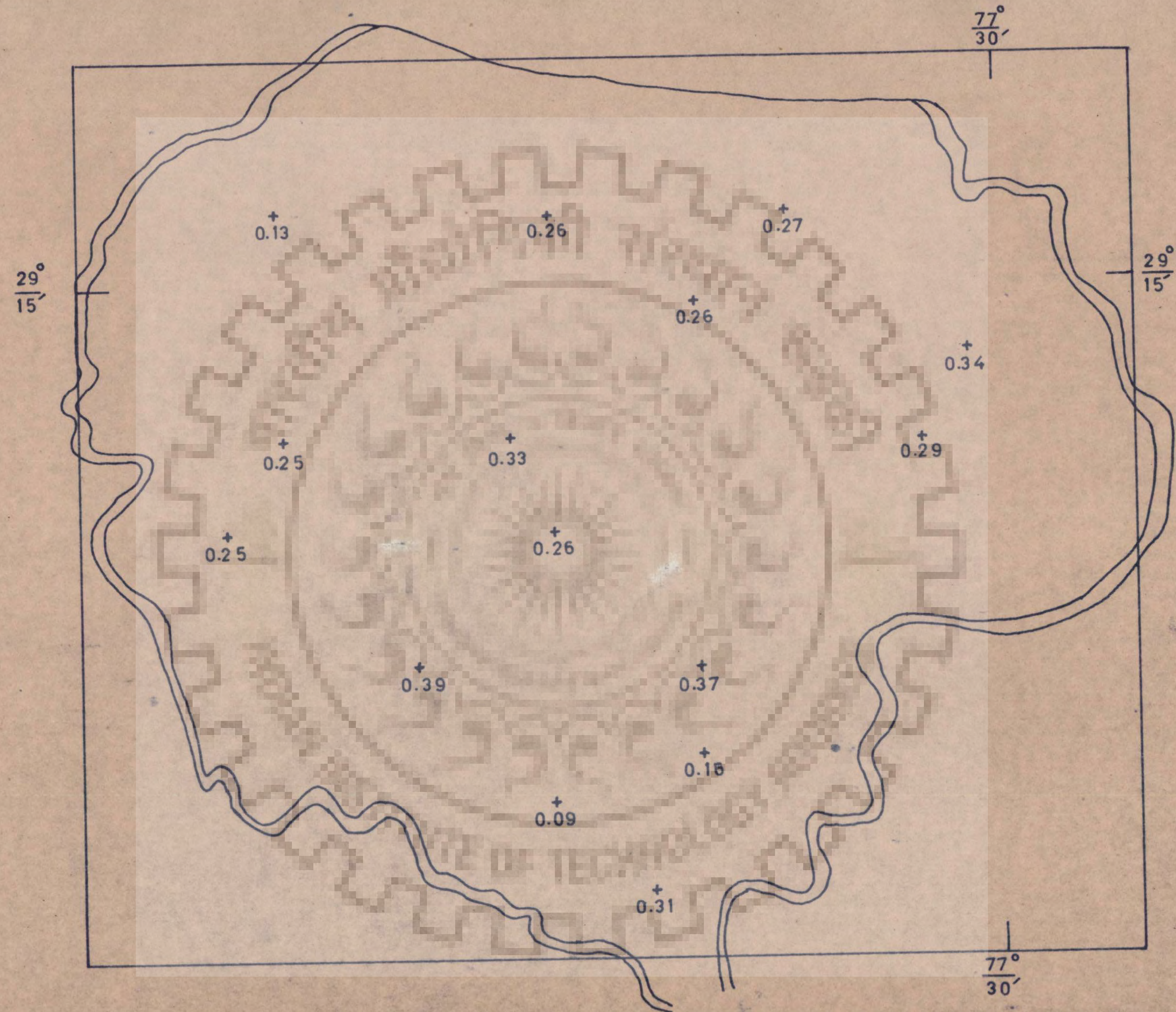


FIG. 6·10 - ESTIMATED ORIENTATION OF PRINCIPAL PERMEABILITY DIRECTIONS (radians)

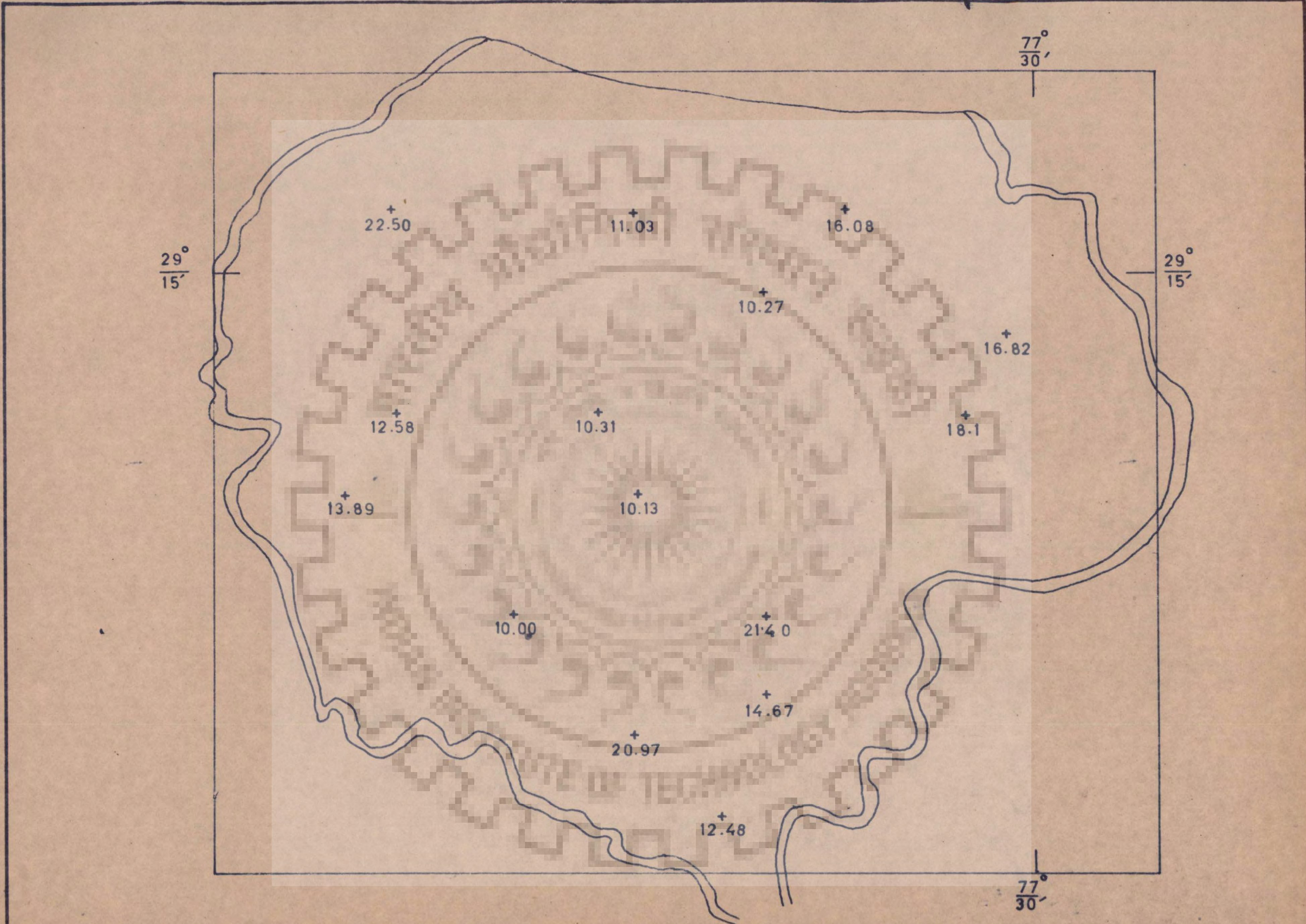


FIG. 6·11 - ESTIMATED RECHARGE COEFFICIENTS (%)

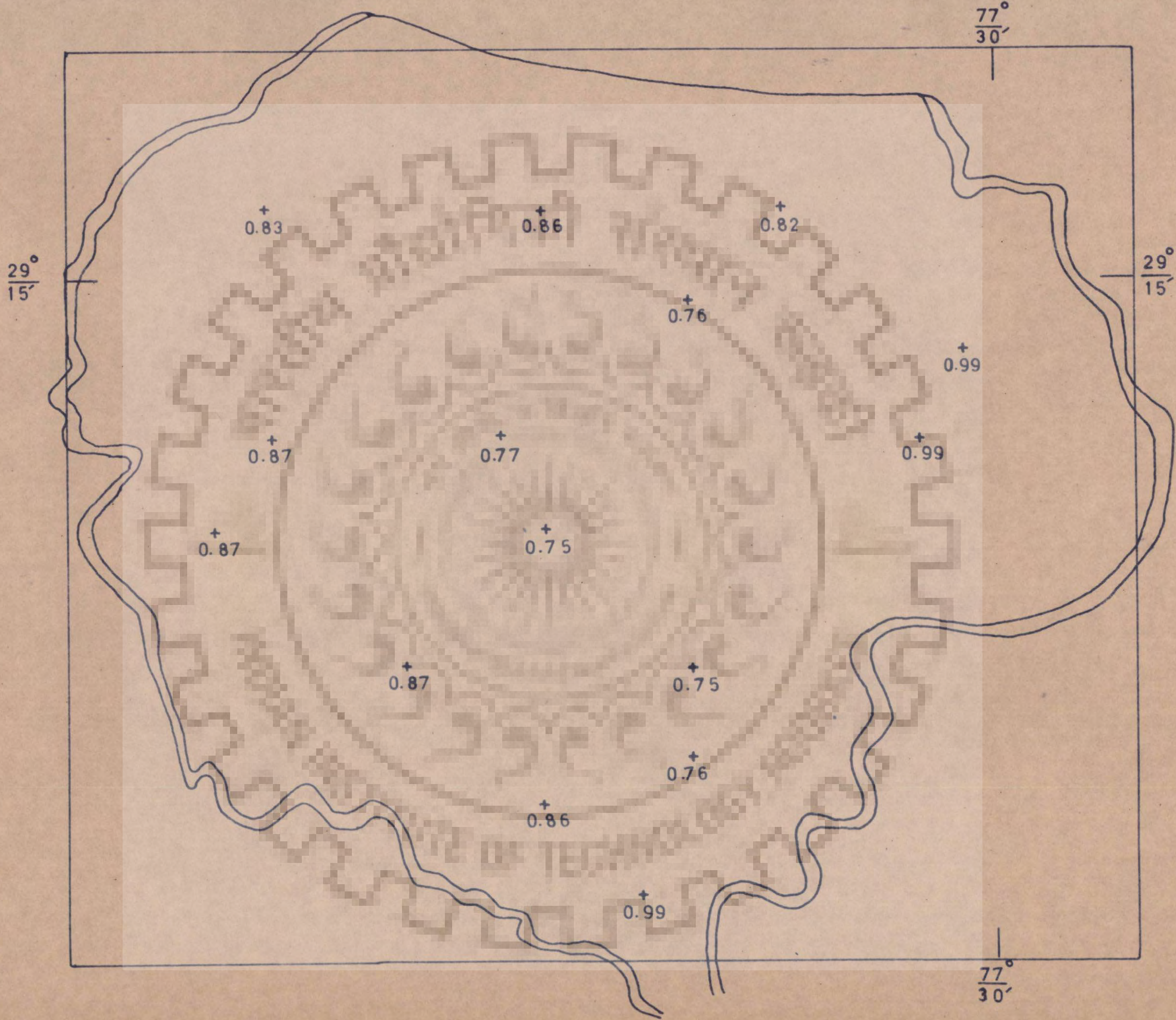


FIG. 6·12 - ESTIMATED TIME LAG FACTORS

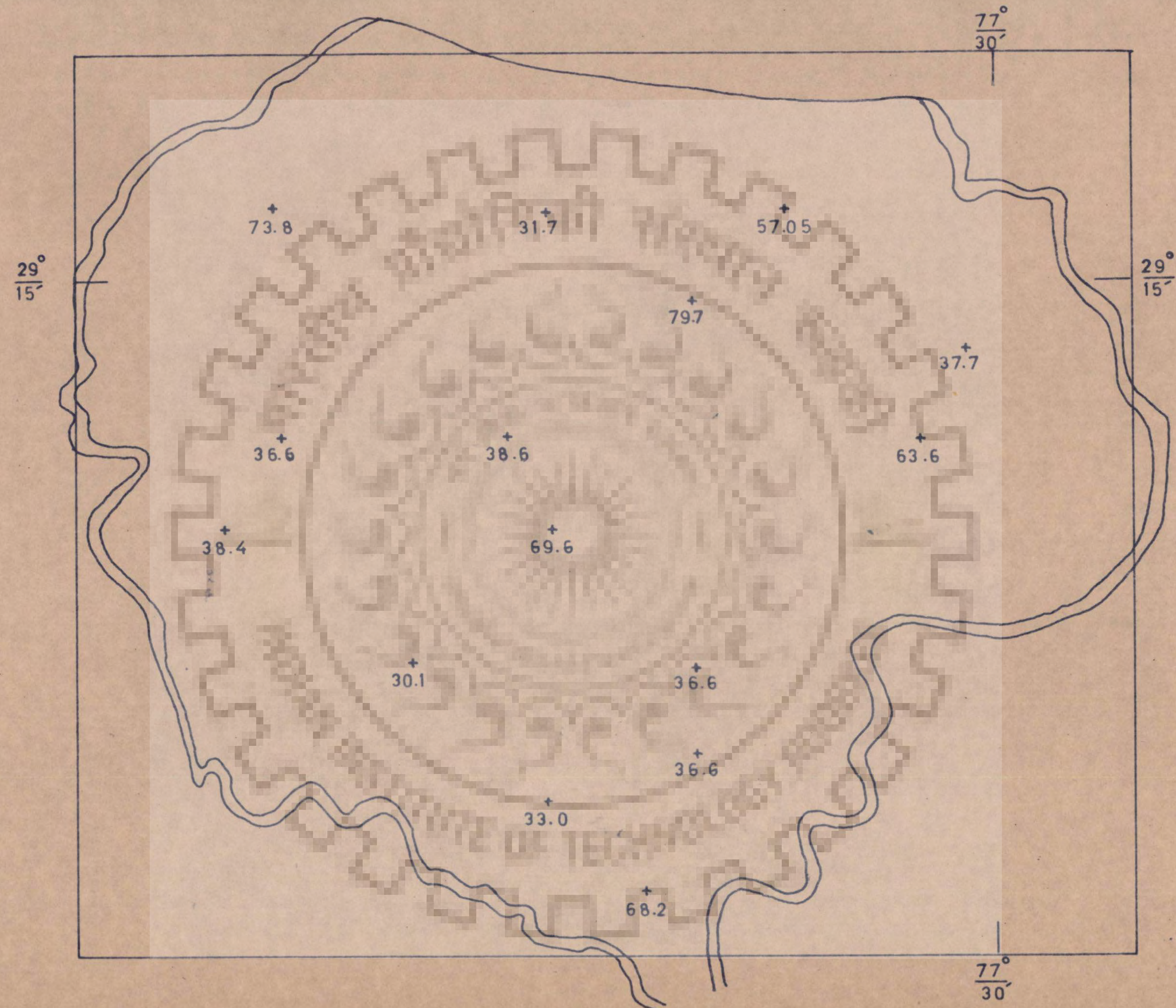


FIG. 6·13 - ESTIMATED THRESHOLD RAIN FALLS (mm / month)

CHAPTER-7

APPLICATION OF DISTRIBUTED CONJUNCTIVE USE MODEL FOR DAHA AREA

7.1. GENERAL

The model developed in Chapter-5 was employed to arrive at the optimal cropping and groundwater withdrawal patterns for Daha area under the condition of the availability of additional water from a proposed canal network. This study follows the aquifer parameter estimation for the same area described in Chapter-6.

7.2. STUDY AREA

An optimal pattern of groundwater development was evolved for Daha area employing the distributed model developed in Chapter-5. The general features of Daha area have been discussed in Chapter-6 (paragraph 6.2). The area at present, is not served by any surface water irrigation system and the water requirements are being met exclusively by intensive groundwater development in the area. This has resulted in a sustained lowering of water table. The areas lying towards east and west of the study area, across the rivers Krishni and Hindon (Figure 6.1) are being served by perennial canal systems. These canal systems are providing groundwater recharge in these areas. The aquifers of these areas are hydraulically connected to the aquifer of the study area since the rivers are partially penetrating. Therefore, the lowering of water table in the study area is being accompanied by increased subsurface horizontal inflows from aquifers across the rivers. These aquifers, on account of being heavily recharged, are serving almost as sources of abundant water supply. The occurrence of large subsurface inflow from east and west has been confirmed by the water balance study conducted by Dikshit (1978) and Desai (1978).

The irrigation department of Uttar Pradesh intends to construct a river fed canal system in the area. These canals will be operational during the months of July to October when the discharge in the two rivers is adequate. The canal system though intended primarily for checking the lowering of water table by way of providing additional recharge, can also be utilised in improving the cropping pattern of the area. The expected month wise discharge of the canal system is given in Table 7.1.

7.3. GROUNDWATER RESOURCES OF THE AREA

The groundwater resources of an area are essentially governed by the recharge (vertical as well as horizontal) received by the aquifer. The total recharge of Daha area is made up of the vertical component (rainfall recharge) and the horizontal component (subsurface recharge).

The horizontal component, unlike in the normal situations, is not insignificant as compared to rainfall recharge. The extent of this component is not constant and is primarily governed by the existing watertable elevations. As the water table elevations go down, the subsurface recharge from outside region increases due to the increased gradients. However,

Table 7.1. - Surface water Supplies and Recharge
All figures in mm/month

Month	Surface water supplies	Surface water recharge
January	0	0
February	0	0
March	0	0
April	0	0
May	0	0
June	0	0
July	65.0	26.0
August	65.0	26.0
September	65.0	26.0
October	21.67	8.64
November	0	0
December	0	0

the extent to which watertable can be permitted to go down is limited by the following factors.

i) The maximum permissible depth to water

ii) Maximum permissible subsurface inflows.

The first factor is controlled by the economics of pumping and depth of the shallow wells in the area. The second factor assumes importance since the aquifers across the rivers can serve as sources of abundant water supply only upto a certain level of recharge. The level depends upon the extent of groundwater development in the adjoining aquifers. If the subsurface recharge exceeds a certain limit, then any further lowering of watertable in the area may result in a significant lowering of watertable of the adjoining aquifers alongwith the increase in subsurface recharge. In the present sudy the horizontal component of groundwater resource of the area has been limited by the first consideration, since adequate data relating to the recharge patterns of the adjoining aquifers were not available. The maximum permissible depth to watertable has been fixed at 18.0 meters. This figure has been arrived at with a view to limit the drying up of shallow open wells and tubewells. This constraint can restrict the extent of subsurface inflows to the level of the year 1975 when the maximum depth to watertable was of the order of 18.0 meters, though strictly speeking the subsurface recharge would depend upon the watertable elevations at the boundary of the area rather than upon the maximum depth to watertable occuring any where in space.

The vertical component of the groundwater resources has been fixed according to the recharge parameters of the area, as arrived in Chapter-6 (Table 6.7).

7.4. FEASIBLE CROPS AND CULTURABLE COMMAND AREA

The seven crops considered likely to be grown in the area even with increased water availability are wheat, peas and grams, other rabi, sugarcane, other kharif, maize and paddy. All the crops other than sugarcane are seasonal in nature while sugarcane is perennial. Wheat, peas, other rabi and grams are cultivated during the period November to April, and other kharif, maize and paddy during the period May to October. The net irrigation requirements of these crops in different months and their net return per unit area (Project report, 1976) as assumed in the study are given in Tables 7.2&7.3 The culturable command area has been taken as 0.85 of the geographical area (Project report, 1976). It has been found that only 58.5 per cent of the net irrigation requirements are met (Dikshit, 1978).

7.5. DISTRIBUTED AQUIFER RESPONSE

The estimation of distributed aquifer response has been incorporated in the scheme of computation by discretising the entire area into a finite number of nodes at spacing of 2 kilometers in both the directions (figure 7.1). Dirichlet boundary conditions oorresponding to the watertable elevations of year 1975 have been assigned at the boundary nodes. This is based upon the assumption of the insensitivity

Table 7.2 - Crop data used for study

Crop	Net irrigation requirements in mm											
	Jan.	Feb.	March	April	May	June	July	Aug.	Sept.	Oct.	Nov.	Dec.
Wheat	46.7	107.6	203.6	12.14	0.0	0.0	0.0	0.0	0.0	7 6	2.73	23.6
Peas and grams	86.1	35.9	0.0	0.0	0.0	0.0	0.0	0.0	0.0	7.7	105.4	113.6
Other rabi	81.7	12.96	0.0	0.0	0.0	0.0	0.0	0.0	0.0	13.0	64.0	83.95
Sugarcane	0.0	76.0	138.0	218.0	306.0	216.0	0.0	27.6	49.0	130.2	103.4	47.4
Other kharif	0.0	0.0	0.0	75.0	5.0	5.0	0.0	27.6	25.0	8.9	0.0	0.0
Maize	0.0	0.0	0.0	0.0	72.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
Paddy	0.0	0.0	0.0	0.0	164.2	48.4	198.0	247.0	315.0	68.8	0.0	0.0

Table 7.3. - Crop areas and returns

Crop	Existing area (percent)	Assigned minimum permissible area (percent)	Assigned maximum permissible area (percent)	Net returns (Rs./hectare)
Wheat	35.0	30.0	40.0	2960.0
Peas and grams	3.0	3.0	9.0	815.0
Other Rabi	3.0	3.0	9.0	690.0
Sugarcane	30.0	20.0	30.0	8050.0
Other kharif	15.0	10.0	20.0	1680.0
Maize	10.0	5.0	15.0	1436.0
Paddy	3.0	3.0	25.0	3750.0

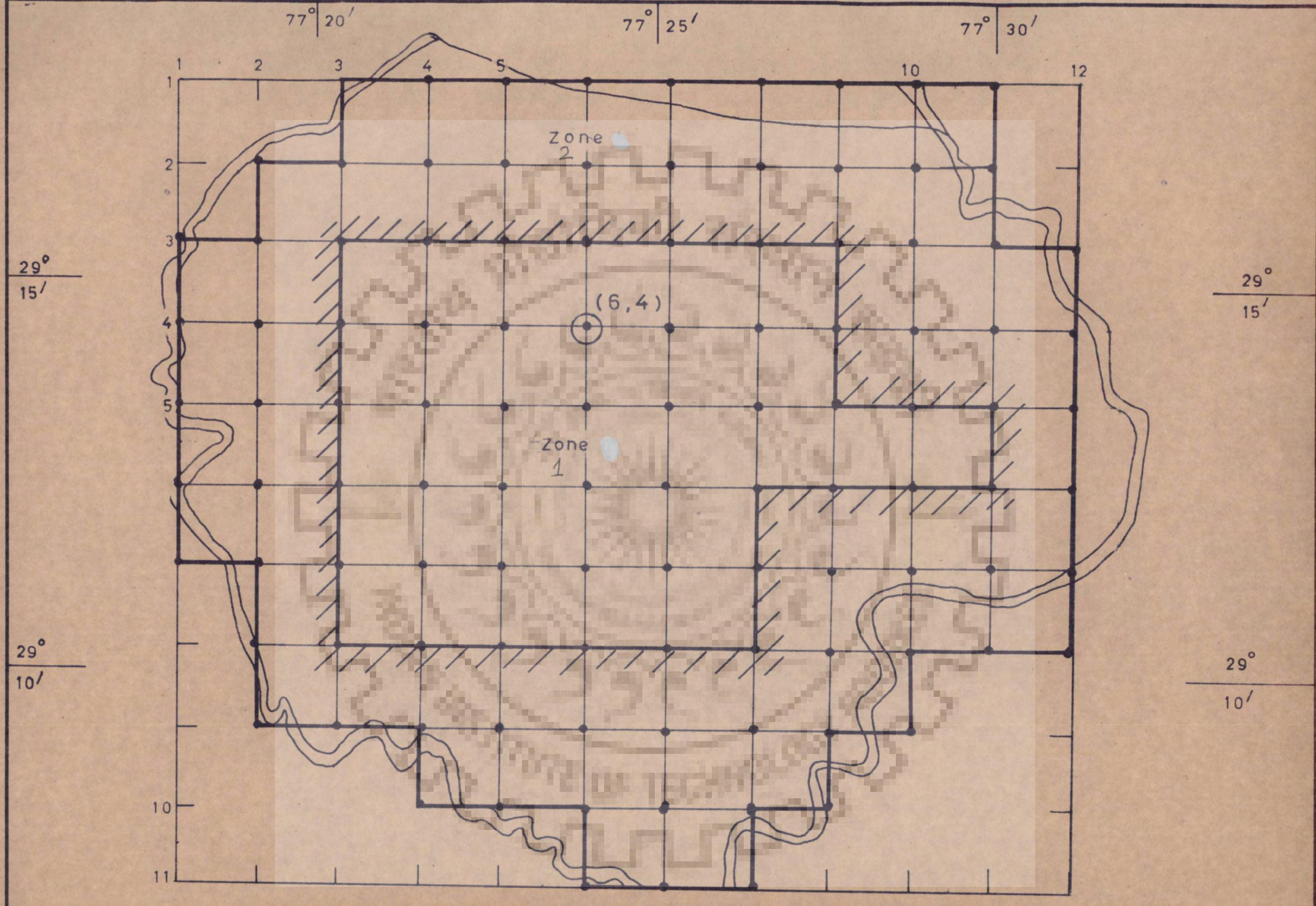


FIG. 7·1 ZONING AND DISCRETISATION OF DAHA AREA of the watertable elevations in the adjoining aquifers and the river discharge to the pumping or recharge pattern in the study area. The aquifer parameters have been assigned as per the estimates obtained by the solution of inverse problem (table 6.7). The nodal values of the aquifer excitation are made of the following components.

i) Current trial values of groundwater withdrawals

ii) Rainfall recharge

iii) Surfacewater recharge during the months July to October.

iv) Return flows from the applied irrigation, assumed as 35 percent and 40 percent for ground and surface waters respectively.

The initial values are assigned by the coefficients of least square polynomial corresponding to the month January 1975. The initial conditions are not important since the constraints relating to the maximum depth to water table should be evaluated when the state of dynamic equilibrium for the current trial values of ground water withdrawals has been reached. The state of dynamic equilibrium is dependent only upon the boundary conditions, aquifer parameters and the periodic recharge and discharge patterns.

Thus, starting from any initial conditions the simulation is to be continued till the following conditions are satisfied in k^{th} month.

$$h_{i,k-12} + \epsilon \geq h_{i,k} \geq h_{i,k-12} - \epsilon \qquad \ldots(7.1)$$

$$i = 1, \ldots\ldots, n$$

where h_{ik} is the simulated value of water table elevation at i^{th} node in k^{th} month, n is the total number of nodes and ϵ is a tolerance factor. This simulation is to be carried out large number of times during optimisation computations for evaluation of the maximum depths to water table for different trial values of withdrawals. The computer time requirement for these computations will be too large. To tackle this situation the condition incorporated in equation 7.1. has been relaxed in the following manner.

The initial values of the cropping patterns and groundwater withdrawals are decided as close as possible to the optimal values by judgement. The state of near-dynamic equilibrium for the initial values of withdrawals are estimated by carrying out simulation for five years. The simulated values at the end of five years period are fed as initial conditions for subsequent simulations with changed withdrawals. In the optimisation computations, the repeated simulations are carried out by changing the groundwater withdrawals of a month by one percent to calculate the derivatives of the constraints numerically. At the end of each iteration the ground water withdrawals are changed gradually to reach the optimal solution. Thus, the change of dynamic equilibrium of current simulation will be only marginally different from the state of dynamic equilibrium of the preceeding simulation. Keeping this in view, the simulations subsequent to the first one are carried out till near dynamic equilibrium is achieved, or twentyfive months, whichever occurs earlier. This approach cuts down computer time requirement enormously.

7.6 ZONING OF THE AREA

For obtaining the spatial distribution of crop areas and groundwater withdrawals, the entire area is divided into zones of uniform cropping patterns and groundwater withdrawals. This process is quite different from the discretisation for computing distributed aquifer response, for reasons indicated in para 5.5.2. The decision variables include 7 crop areas and 12 monthly withdrawals for each zone. Thus, the total number of decision variables will be nineteen times the number of zones. To restrict the number of variable to a manageable limit the number of zones should be kept as small as possible. In the present study the zoning has been affected from the view point of maintaining near uniform aquifer response characteristics in each zone. The other criteria like uniform rainfall and surface water supplies do not become effective since the area is small and the spatial variability of these may be marginal. The entire area has been divided into two zones, one near the boundaries and other interior (figure 7.1). The areas near the boundaries will show smaller drawdown for the same pumping because of the existence of constant head boundaries in the near vicinity. The interior is sandwitched within the exterior zone. This will restrict the total number of decision variables to 38.

7.7 MAXIMISATION OF NET BENEFIT

The aim of the study is to arrive at such values of zonal cropping patterns and groundwater withdrawals which maximise the net benefits from agricultural activity subject to the constraints given in equations 5.9 to 5.14.

The objective function representing the net benefits is given in equation 5.8. The data adopted for evaluating this function for the known values of crop areas and withdrawals are as follows.

i) Net returns from each crop as given in Table 7.3

ii) Cost of pumping a unit volume of water has been assumed as Rs.1500/- hectare meter.

iii) Annual cost of surface water input to the area: Rs.8.145×10^6.

The constraints of meeting the crop water requirements (equation 5.9) were incorporated in the computations by adopting the following data.

i) Net irrigation requirements of the crops in different months as given in table 7.2.

ii) Surface water availability in the two zones in different months as given in table 7.1. Uniform surface water availability in the entire area was assumed since adequate data for getting spatial distribution were not available.

iii) The efficiencies of ground and surface water use assumed as 65 percent and 45 percent respectively.

iv) Initially ξ^m, the level at which the net irrigation requirement is met, was stipulated as 0.75. Subsequently the results were obtained for two other levels 0.65 and 0.70.

The constraints for restricting the total cropped area to the culturable command area (equation 5.10) were imposed by dividing the seven crops amongst two groups of concurrent crops. The constraints are as follows -

$$A_{1,l} + A_{2,l} + A_{3,l} + A_{4,l} \leq 0.85\, AG_l \quad \ldots \quad 7.1$$

$$A_{4,l} + A_{5,l} + A_{6,l} + A_{7,l} \leq 0.85\, AG_l \quad \ldots \quad 7.2$$

$$l = 1,2$$

where suffixes 1,2,3,4,5,6 and 7 refer to wheat, peas and grams, other rabi, sugarcane, other kharif, maize and paddy respectively and suffix l stands for the zone number.

The total areas under each crop were restricted within a range by assigning minimum and maximum permissible areas for each crop (equation 5.11) as given in table 7.3. The table also contains the existing areas under each crop. The maximum and minimum limits have been adopted with a view to implement the current policy in the area to increase the area under paddy and reduce the sugarcane area. The surface water supplies can provide water during the period July to October and paddy requires heavy irrigation during these months. Therefore the maximum permissible area under paddy was kept at 25 percent, a level significantly higher than the existing one. The scope of increasing the areas under crops requiring irrigation during other months is rather limited. The only way these areas could be increased is by the way of increasing groundwater pumpage during November and June to the extent permitted by the extra rise of watertable during July to October.

The constraints relating to the permissible depths to watertable (equation 5.14) were imposed by stipulating the maximum permissible depth to watertable as 18 meters. The necessary ground elevations of all the nodal points of the finite difference grid were computed from the available topographical data. The storage coefficient and transmissibility data have been generated at fifteen points by the solution of inverse problem (figures 6.7 to 6.10). The transmissibility and θ values indicate the presence of anisotropy with spatially varying directions of principal permeabilities. However, the level of anisotropy is low. T_{xx}/T_{yy} varies between 1.00 to 1.05 for about 50 percent of the area and between 1.05 to 1.10 for about 40 percent of the area. Keeping this in view, the anisotropy of the aquifer was neglected. Equivalent isotropic transmissibility values ($\sqrt{T_{xx}.T_{yy}}$) were calculated for each point and finite difference method was adopted for the solution of the governing differential equation. The nodal values of transmissibility and storage coefficient for the finite difference grid were estimated by employing the discrete point values of storage coefficient and equivalent isotropic transmissibility. This estimation was accomplished by approximating the spatial variation of these parameters by least square polynomials of spatial coordinates. The following form of the least square polynomial, which was generally found to be adequate to represent the spatial variation of watertable elevations (table 6.4), was adopted.

$$T(p,q) = b_{11}p^3 + b_{21}\ pq^2 + b_{31}pq + b_{41}p + b_{51}q + b_{61}$$
$$S(p,q) = b_{12}p^3 + b_{22}\ pq^2 + b_{32}pq + b_{42}p + b_{52}q + b_{62} \quad ...(7.2)$$

The generated storage coefficient and equivalent isotropic transmissibility data were employed to arrive at the coefficients of these equations (table 7.4).

Table 7.4

Coefficients* for spatial variation of storage coefficient and Transmissiblity b_{ij}

j	Term i =	1	2	3	4	5	6
2	S(%)	-0.51×10^{-5}	-0.37×10^{-2}	0.93×10^{-1}	-0.83×10^{-1}	.51	10.95
1	T(1000 m^2/day)	-0.12×10^{-3}	-0.67×10^{-3}	0.11×10^{-1}	0.97×10^{-2}	0.96×10^{-2}	1.52

* All figures are rounded off to second decimal place.

The nodal values of rainfall recharge were estimated by assigning the mean computed values of recharge parameters (table 6.7) for all the nodes. The ADI method of finite differences was employed for simulation. The constraint relating to the minimum depth to watertable was not imposed since in this over drafted area, the conditions of water logging are not anticipated. A considerable computer time is saved by not incorporating this constraint, which any way is not going to affect the optimal policy. This constraint if imposed will result into an increased number of groundwater simulations, during the search for optimal solution.

The constraint given in equation 5.12 was not imposed since the optimal solution is not likely to incorporate any significant increase in groundwater withdrawals. The existing withdrawals are quite large and the constraint of the maximum permissible depth to water table does not leave much scope for their significant increase.

The constraints and the objective function are completely defined by the input data (described in the preceeding paragraphs) and the following decision variables -

Cropping areas (A_{jl}) $j = 1, \ldots 7$ and $l = 1,2$

Groundwater withdrawals (W_{kl}) $k=1, \ldots 12$ and $l=1,2$

Total number of decision variables = 38

The total number of constraints equals 43 as per the following breakup.

Table 7.5 - Number of constraints

Sl.No.	Constraint	Number
1.	Crop water requirements must be met (equation 5.9)	2 x 12 = 24
2.	CCA must not be exceeded (equation 5.10)	2 x 2 = 4
3.	Permissible range for areas under different crops (equation 5.11)	2 x 7 = 14
4.	Maximum depth to water table (equation 5.14)	1

The maximisation of the objective function was carried out within the frame work of nonlinear programming since the constraint given in equation 5.14, relating to the maximum permissible depth to water table is a non-linear implicit function of the decision variables (W_{kl}).

Sequential unconstrained minimisation technique was adopted to arrive at the optimal values of the decision variables i.e. areas under seven crops and monthly groundwater withdrawals for the two zones. The initial values of the thirty eight variables were assigned by fixing the crop areas in each zone at levels 2% higher than the minimum prescribed (table 7.3). The groundwater withdrawals were calculated for these areas as per the net irrigation requirements (table 7.2), surface water supplies (table 7.1) and assuming that the net irrigation requirement is met at 75% level. The maximum depth to water table corresponding to this pumping pattern was calculated as 16.7 meters, thus establishing the feasibility of the initial solution. The optimal values of groundwater withdrawals and the crop areas are given tables 7.6 and 7.7 respectively. The crop areas have been given at two additional levels of irrigation. The maximum depth to water table for the optimal withdrawals (table 7.6) equals 17.99 meters, occuring at the node (6,4) (figure 7.1) during the month of June. The minimum depth to water table was calculated as 7.32 meters during the month of September. The maximised benefits have been calculated as Rs.2936.93 per hectare.

The optimal policy assures maximised return from agricultural activity under the constraint that the maximum depth to watertable does not exceed 18.0 at any space or temporal point. The minimum depth to water table is well above the critical depth for water logging conditions. The optimal cropping pattern has in general shown perference for zone-2 lying near the recharge boundaries. This is because of relatively more favourable discharge/drawdown characteristics of this zone. This will generally be the feature of optimal distribution of cropping areas in space, if the groundwater withdrawals are restricted by the constraint of maximum depth to water table. The trend may be reversed if the groundwater activity is limited by the permissible level of exchange of flow between the aquifer and the recharge boundary.

Table 7.6 - Optimal groundwater withdrawals
All figures are in millimeters

Month	Existing withdrawal pattern	Optimal withdrawal pattern	
		Zone 1	Zone 2
January	19.23	24.65	33.47
February	55.72	62.46	76.99
March	101.38	117.65	136.71
April	62.67	64.37	72.51
May	82.61	93.58	108.94
June	92.61	97.62	110.29
July	1.306	0.0	0.0
August	16.52	5.94	7.99
Sept.	23.26	18.49	26.11
October	69.35	106.89	124.95
November	33.33	53.26	61.52
December	25.5	31.02	40.64

Table 7.7 - Optimal Cropping Pattern

Crop	Crop areas (% of geographical area)					
	Irrigation level 0.75		0.70		0.65	
	Zone 1	2	1	2	1	2
Wheat	33.15	34.39	35.51	36.831	38.24	39.68
Peas and grams	4.47	6.94	4.787	7.432	5.15	8.03
Other Rabi	5.47	6.98	4.783	7.475	5.15	8.05
Sugarcane	25.36	24.94	27.160	26.710	29.26	28.78
Other Kharif	12.79	14.72	13.698	15.765	14.75	16.98
Maize	7.68	9.72	8.225	10.410	8.86	11.21
Paddy	15.63	17.31	16.953	18.539	18.26	19.97

The existing and optimal cropped areas in the two seasons of Rabi and Kharif are given in table 7.8. The existing cropped areas are irrigated at a level of 58.5% only. The cropped area as per the optimal policy at 75% level of irrigation shows a decline from 71% to 67.45% during the Rabi season in zone 1. This policy may be difficult to implement. Therefore, the crop areas at two other levels of irrigation i.e. 70% and 65% were calculated and are given in table 7.7. The seasonal areas corresponding to these levels are given in table 7.8.

Table 7.8 - Seasonal Cropped areas

Season	Existing cropped Areas(% of geographical area)	Optimal Cropped Areas (% of geographical area)					
		Irrigation level 0.75		0.70		0.65	
		Zone 1	2	1	2	1	2
Rabi Crops included 1,2,3,4	71.00	67.45	73.25	72.23	78.44	77.82	84.55
Kharif Crops included 4,5,6,7	58.00	61.46	66.69	66.02	71.42	71.13	76.65

CHAPTER-8

CONCLUSIONS

The present study has been mainly related to the least square polynomial approximation for the spatial variation of piezometric head, the inverse problem for aquifer-hydrologic parameter estimation and the distributed aquifer response based conjunctive use planning. The prominent conclusions of the study are as follows:

i) The least square polynomials can be employed to approximate the spatial variation of piezometric head. These functions apart, from providing closed form relations between piezometric head and spatial coordinates, can also attenuate unmeasured Gaussian data noise, to a varying degree depending upon the initial noise level and spacing of data points. The attenuation is maximum in case of large levels of noise and dense spacing, it can be as high as 90% of the variance of the noise.

ii) A higher degree polynomial need not always provide a better approximation to the spatial variation of piezometric head. The criterion of minimum standard error can be employed to decide upon the degree of polynomial. The truncation of the full polynomial can be affected on the basis of t statistics of the coefficients. The acceptability of the partial polynomial can be checked by computing F statistics.

iii) The polynomials for approximating the spatial variation of piezometric head in aquifer of Daha area, were arrived at by employing the tests of significance. Commencing from third degree polynomials containing ten terms, the polynomials were truncated by comparing the computed t statistics of the coefficients with the critical value at 95% level of confidence. However, it was found that partial polynomials were significantly inferior to the full polynomials, as per computed F statistics at 95% level of confidence. It was found that the t statistics of the coefficients of the full polynomials show a significant improvement on dropping the data points showing standard residues outside the range ± 2 or residues outside the range ± 1 meter. Subsequent deletion of terms with computed t statistics less than 1.0 was found to be acceptable as per F statistics. In general the finally adopted polynomials were of third degree, truncated to six terms and explained more than 90% of initial variance (tables 6.2 and 6.4). These functions provided the closed form relations between the water table elevations at dififerent temporal points and the spatial coordinates.

iv) The approximate closed form relations provided by the least square polynomials can be employed to estimate groundwater storages, subsurface flows, piezometric heads at non-tabular points and the first and higher derivatives of piezometric head with respect to time and space. The relations can also be used for computer assisted drawing of contours.

(v) The approximate closed form relations between water table elevations and space coordinates for Daha area were employed to estimate the second spatial and first temporal derivatives of the piezometric head for subsequent computations of inverse problem. In addition to this the ground water storages were computed and computer assisted contours drawn (figures 6.4 to 6.6).

(vi) The solution of inverse problem, which is essentially based upon micro level consistent water balance for a large number of periods, can be formulated to yield information relating to the monthly rainfall recharge and orientation of principal permeability directions, in addition to the aquifer parameters S and T.

(vii) A linear form of the rainfall recharge with a built in threshold value of rainfall and an accounting for the delayed response of water table can be considered as a possible monthly rainfall recharge relation.

(viii) A direct method of solving inverse problem for the estimation of aquifer and hydrologic parameters mentioned in vi) and vii) has been developed. The method is based upon minimisation of residue functional by constrained minimisation.

(ix) The aquifer and recharge parameters of Daha area were estimated by the proposed inverse problem model, employing monthly data. The distribution of the total ground water withdrawals amongst months, on the basis of consumptive use requirement was not found to be accurate enough for inverse problem computations. The inverse problem model was adapted to include amongst its objectives the estimation of monthwise distribution of seasonal withdrawals.

(x) The estimated storage coefficients varied between 0.08 and 0.1126 with a mean 0.0961 and standard deviation 0.012. The angular distance between the arbitrarily assumed orthogonal axes and the principal permeability directions, varied between 0.09 to 0.39 radians with mean and standard deviations 0.26 and 0.085 radians respectively. The transmissibilities in the principal permeability directions varied between 1599 to 2459 m^2/day, 502 to 2075 m^2/day with means 1995.0 and 1714.0 m^2/day and standard deviations 208.0 and 436.0 m^2/day. The mean values of storage coefficient and equivalent isotropic transmissibility compare well with the corresponding estimates found acceptable in a reported spatially lumped model study.

(xi) The linear rainfall recharge relation was implicitly calibrated along with the parameter estimation. The estimated recharge coefficients varied between 0.10 to 0.225 with a mean of 0.1475 and standard deviation 0.043. The threshold rainfalls varied between 30.12 to 79.77 mm/day with mean and standard deviation as

48.7 and 17.62 mm/day respectively. The coefficient representing the delayed response varied between 0.754 to 1.0 (1.0 implies no delay in response) with a mean of 0.852 and standard deviation 0.088. The rainfall recharge expressed as percentage of rainfall varied between zero (for nonmonsoon periods) to 14.4. This compares well with the average seasonal estimates of rainfall recharge for rainy periods, reported in a different study.

(xii) The conjunctive use model can be formulated to include a) the estimation of spatial as well as temporal variations of cropping pattern and ground water withdrawals, and b) confinement of water table elevations in a stipulated range, amongst its objectives. This way, the groundwater resource of an area can be defined in terms of the maximum permissible depth to water table. Similarly the minimum permissible level of pumping can be defined in terms of the minimum permissible depth to water table.

(xiii) The features listed in xii) have been incorporated in the conjunctive use model by integrating it with a distributed aquifer response model and solving the problem by objective nonlinear optimisation techniques. The resultant model can afford the imposition of constraints of restricting the stream aquifer interaction.

(xiv) The applicability of the proposed distributed model of conjunctive use to field situations has been demonstrated by employing the data of Daha area.

(xv) The imposition of the constraints on the maximum depth to water table,has resulted in the optimal policy incorporating more intensive cropping patterns in the zone situated near recharge boundaries.

(xvi) The proposed additional surface water recharge in Daha area, in addition to stabilising the maximum depth to water table at 18 meters, can also improve the cropping pattern of the area and can assist in meeting the net irrigation requirement at levels higher than the prevalent level of 58.5%.

LIST OF VARIABLES

Variable	Description
A_j	Area under j^{th} crop
A_{jl}	Area under j^{th} crop in l^{th} zone
AG_l	Geographical area of l^{th} zone
AG	Total geographical area
$AMAX_j$	Maximum permissible area under j^{th} crop
$AMIN_j$	Minimum permissible area under j^{th} crop
b_i	Coefficient of i^{th} term of the least square polynomial
b_s	Saturated thickness
c_{ii}	i^{th} diagonal element of corrected sum of squares and product matrix
C	Boundary contour
C_j	Cost of inputs other than water per unit area of j^{th} crop
CC_j	Market price of j^{th} crop
CG	Cost of unit volume of groundwater
CS	Total annual cost of surfacewater supplies
CY_j	Crop yield of j^{th} crop
d.f.	degree of freedom
dmax	Maximum permissible depth to watertable
dmin	Minimum permissible depth to watertable
D_p, D_q	Second spatial derivatives of piezometric head in directions p and q
DFCT	Soil moisture deficit
DP_{ik}, DQ_{ik}	Second spatial derivatives of piezometric head at i^{th} space point and k^{th} time point in directions p and q.

DT_{ik}	First temporal derivative of piezometric head at i^{th} space point and k^{th} time point.
e_i	Difference between the observed head and the least square approximation, for i^{th} space point.
(e_{ik})	Minimised residue corresponding to i^{th} space point and k^{th} time point
es_i	Standard residue of i^{th} space point
$\bar{E}_k$	Mean fraction of the additional variance attenuated by least square polynomial of k^{th} period
E_{kr}	Fraction of the additional variance attenuated by least square polynomial of k^{th} period in r^{th} replication of the simulation run.
ES	Compressibility of aquifer
ET	Evapotranspiration
EW	Compressibility of water
$f(\xi)$	Frequency distribution
FM	Full model
G_1, G_2	Boundary gradients
G_i	Ground elevation of i^{th} space point
h	Piezometric head
h_c	Level of contour
$h_k(p,q)$	True piezometric head at space point (p,q) in k^{th} time point
$\bar{h}_k$	Mean piezometric head in k^{th} time point
h^*_{ik}	Observed piezometric head at i^{th} space point in k^{th} time point
$H_k(p,q)$	Least square polynomial solution for k^{th} period
i	Subscript for space point, node, elementary length, strip and the coefficients of least square polynomials
Ig	Subsurface inflows

I_k	Subsurface horizontal inflows during k^{th} time period
j	Subscript for crop number
k, k'	Subscripts for time point/period
k_d	Number of terms of the full least square polynomial dropped on the basis of t test
K	Permeability
K_r	Recharge coefficient
K_{xx}, K_{yy}	Principal permeabilities
l	Subscript for zone number
m	Number of time points/periods
m_e	Number of the previous periods' rainfall affecting the recharge in the current month
M	Permissible annual mining
n	Number of space points
n_c	Number of rainfall-recharge parameters
n_e	Number of elementary lengths in which the boundary contour is subdivided for estimating subsurface flows
n_p	Number of terms in the least square polynomial approximation
n_r	Number of replications in simulation study
n_s	Number of strips for the estimation of groundwater storages
n^c	Inner normal
NCR	Number of feasible crops
NMAX	Maximum permissible number of structures per unit area
NZ	Number of zones of uniform cropping and groundwater withdrawal patterns.
O_g	Subsurface outflows
p, q	A system of orthogonal axes with any orientation

p_i	p coordinate of i^{th} space point
p_i^m, p_i^n	Minimum and maximum p coordinates of i^{th} strip (for the estimation of groundwater storage)
p_i^*	p coordinate of the central point of i^{th} elementary length (for the estimation of subsurface recharge)
q, p	A system of orthogonal axes with any orientation
q_i^*	q coordinate of i^{th} space point
q_i^m	q coordinate of i^{th} strip (for the estimation of groundwater storage)
q_i	q coordinate of the central point of i^{th} elementary length (for the estimation of subsurface recharge)
q_w	Withdrawal capacity of a single groundwater structure
Q	Net vertical accretion (recharge positive)
Q_{ik}	Net vertical accretion at i^{th} space point during k^{th} period
$\bar{Q}_k$	Mean vertical accretion during k^{th} period
r	Radial distance
r(subscript)	Subcript for replication number in simulation study
R	Rainfall recharge
R_{ik}	Rainfall recharge at i^{th} space point during k^{th} period
R^a	Annual groundwater recharge
R^2	Coefficient of correlation
RM	Reduced model
s	Standard error of least square polynomial approximation
s(subscript)	Crop season number
s_d	Drawdown

s.e.(b_i)	Standard error of b_i, i^{th} coefficient of least square polynomial
S	Storage coefficient/specific yield
S_i	Storage coefficient/specific yield of i^{th} space point
S^e	Efluent seepage
S^i	Influent seepage
SG_k	Groundwater storage in k^{th} time point
SP	Porosity
SRF	Surface runoff
SSR	Sum of the squares of the residues
SW_k	Surfacewater availability during k^{th} period
t	Time coordinate
T	Transmissibility
T_i	Transmissibility of i^{th} elementary length(for the estimation of subsurface recharge)
T_{pp}, T_{qq}	Transmissibilities in any two orthogonal directions
T_{xx}, T_{yy}	Transmissibilities in principal permeability directions
TX_i, TY_i	Transmissibilities in principal permeability directions at i^{th} space point
u	Argument of well function
U_{kr}	Unexplained variance for k^{th} period in r^{th} replication of the simulation
v	Seepage velocity
V_{kr}	Additional variance caused by the errors, for k^{th} period in r^{th} replication of the simulation
W	Groundwater withdrawals

$W_{i,k}$	Groundwater withdrawal at i^{th} space point during k^{th} period
$\overline{W}_{ik}$	Groundwater withdrawal at i^{th} space point during k^{th} period calculated from consumptive use requirement.
W_k	Groundwater withdrawal during k^{th} period
W_{kl}	Groundwater withdrawal in l^{th} zone during k^{th} period
x, y	Principal permeability axes
x_i, y_i	Coordinates of i^{th} space point
X_{ik}	Recharge due to sources other than rainfall adjusted for abstractions other than pumpage, at i^{th} space point during k^{th} period
y, x	Principal permeability axes
y_i, x_i	Coordinates of i^{th} space point
Y, Y_1, Y_2	Objective function
ZD	Datum for the estimation of groundwater storage
Z_1, Z_2	Boundary conditions
α	Confidence level parameter
(α^i)	A matrix containing n_c number of recharge parameters of i^{th} space point
β^2_{kr}	Residual variance of least square approximation for k^{th} period of i^{th} replication
$\overline{\beta}_k$	Mean of β_{kr} over all the replications for k^{th} period
δ_e	Maximum residue in the polynomial approximation
δ_{jk}	Net irrigation requirement of j^{th} crop in k^{th} period
δ_{jkl}	Net irrigation requirement of j^{th} crop in l^{th} zone during k^{th} period
ΔI_i	Subsurface inflow from i^{th} elementary length
ΔL_i	Length of i^{th} elementary length(for the estimation of subsurface recharge)

ΔS_i	Groundwater storage in i^{th} strip
ΔSM	Increase of water availability in the root zone
ΔS^g	Increase of groundwater storage
Δt_k	Duration of k^{th} time period
η_g	Efficiency of groundwater irrigation
η_s	Efficiency of surfacewater irrigation
ϵ	Data 'noise' or residues in Boussinesq's equation
θ	Angular distance between the axes p, q and principal permeability directions
θ_i	Angular distance between the axes p,q and principal permeability directions at space point i.
μ	Mean for generating normally distributed random numbers
ξ	Frequency distribution
ξ^c_{jkl}	Area factor of j^{th} crop
ρ	Mass density
σ	Standard deviation
ϕ_1, ϕ_2	Hydraulic gradients at the boundary

REFERENCES

Allison, G.B. and Hughes, M.W.,, 1975. The use of environmental tritium to estimate recharge to a South Australian aquifer. Journal of Hydrology, Vol.26, 245-254.

Bear, Jacob, 1972. Dynamics of Fluids in Porous media, American Elesvier Publishing Co., Inc., 764 pp.

Beck, V.James and Arnold, J.Kenneth, 1977. Parameter Estimation in Engineering and Science. Wiley Interscience, 501 pp.

Bedinger, M.S., Reed, J.E. and Griffin, J.D., 1973. Digital Computer programs for analysis of groundwater flow. U.S.Geol. Survey open file report, 85 pp.

Bellman, R. and Kalaba, R., 1965. Quasilinearization and non-linear boundary value problems. American Elsevier Publishing Co., Inc., New York.

Bittinger, M.W., Duke, H.R. and Longenbaugh, R.A., 1967. Mathematical simulation for better aquifer management. Publication no.72 of IASH Symposium of Haifa, 1967 : 509-519.

Bredehoeft, J.D. and Pinder, G.F., 1970. Digital analysis of areal flow in multiaquifer ground water systems : a quasi three dimensional model. Water Resources Research, Vol.6, No.3 : 883-888.

Britles, A.B. and Morel, E.H., 1979. Calculation of aquifer parameters from sparse data. Water Resources Research, Vol.15, No.4 : 832-844.

Buras, N., 1963. Conjunctive operation of dams and aquifers. Journal of Hydraulic Div., ASCE, Vol.89, No.6: 111-131.

Chandra, Satish and Pande, P.K., 1975. Recharge studies for a basin using mathematical model. Proceedings Second World Congress, IWRA, New Delhi.

Chandra, Satish, Sinha, C.P. and Mohan, Jagdish, 1979. Study of changes in water balance of Gomti Kalyani Interbasin due to Sarda Sahayak Waters. International Symposium on Specific Aspects of Hydrological Computations for Water Projects, Leningrad, (U.S.S.R.).

Chaturvedi, R.S. and Chandra, S., 1961. Analytical approach towards solving water logging problems. Research Journal, Vol.IV, No.1, University of Roorkee, Nov.1961.

Chatterjee, Samprit, 1977. Regression Analysis by Example. John Wiley and Sons, 228 pp.

Cooley, Richard L., 1977. A method of estimating parameters and assessing reliability for models of steady state groundwater flow. 1. Theory and numerical properties. Water Resources Research Vol.13 No.2 : 318-324

Cooley, Richard L., 1979. A method of estimating parameters and assessing reliability for models of steady state groundwater flow. 2. Application of statistical analysis. Water Resources Research, Vol.15, No.3 : 603-617

Cooper, H.H., 1966. The equation of groundwater flow in fixed and deforming co-ordinates. Journal Geophysical Research, Vol.71, No.20 : 4785-4790.

Grank, J. and Nicolson, P., 1947. A practical method for numerical evaluation of solutions of partial differential equations of the heat conduction type. Proc. Camb.Phil.Soc.Vol.43, 50-67.

Daniel, C. and Wood, F.S., 1971. Fitting equations to data. Wiley Interscience, 342 pp.

Buras, N., 1963. Conjunctive operation of dams and aquifers. Journal of Hydraulic Div., ASCE, Vol.89,No.6:111-131.

Chandra, Satish and Pande, P.K., 1975. Recharge studies for a basin using mathematical model, Proceedings Second World Congress, IWRA, New Delhi.

Chandra, Satish, Sinha, C.P. and Mohan, Jagdish, 1979. Study of changes in water balance of Gomti Kalyani Interbasin due to Sarda Sahayak Waters. International Symposium on Specific Aspects of Hydrological Computations for Water Projects, Leningrad, (U.S.S.R.).

Chaturvedi, R.S. and Chandra, S., 1961. Analytical approach towards solving water logging problems. Research Journal, Vol.IV, No.1, University of Roorkee, Nov.1961.

Chatterjee, Samprit, 1977. Regression Analysis by Example. John Wiley and Sons, 228 pp.

Cooley, Richard L., 1977. A method of estimating parameters and assessing reliability for models of steady state groundwater flow. 1. Theory and numerical properties. Water Resources Research Vol.13 No.2 : 318-324

Cooley, Richard L., 1979. A method of estimating parameters and assessing reliability for models of steady state groundwater flow. 2. Application of statistical analysis. Water Resources Research, Vol.15, No.3 : 603-617

Cooper, H.H., 1966. The equation of groundwater flow in fixed and deforming co-ordinates. Journal Geophysical Research, Vol.71, No.20 : 4785-4790.

Grank, J. and Nicolson, P., 1947. A practical method for numerical evaluation of solutions of partial differential equations of the heat conduction type. Proc. Camb.Phil.Soc.Vol.43, 50-67.

Daniel, C. and Wood, F.S., 1971. Fitting equations to data. Wiley

Delhomme, J.P., 1978. Kringing in Hydrosciences. Advances in Water Resources, Vol.1, No.5.

Desai, R.C., 1978. Mathematical model studies of ground water by R-C method for Daha area. M.E.Thesis, University of Roorkee (India).

DeWiest, R.J.M., 1966, On the storage coefficient and the equations of groundwater flow. Journal Geophysical Research, Vol.71, No.4.

Dikshit, G.G., 1978. Mathematical model studies by finite difference method for Daha area. M.E.Thesis, University of Roorkee, (India).

Dincer, T., A.Al-Mugrin and U.Zimmermann, 1974. Study of infiltration and recharge through the sand dunes in arid zones with special reference to the stable isotopes and thermonuclear tritium. Journal of Hydrology, Vol.23, 79-109.

Dooge, J.C.I., 1973. Linear theory of hydrological systems. Agricultural Research Service. United States Department of Agriculture. Technical Bulletin No.1468 : 227 pp.

Dreizin, Y.C. and Haimes, Y.Y., 1977. A hierarchy of response functions for groundwater management. Water Resources Research Vol.13, No.1 : 78-86.

Eduardo, A., Remson, I., Pikul, M.F. and Thomas, W.A., 1974. Optimal pumping for aquifer dewatering. Journal of Hydraulic Division, ASCE, HY7, 869-877.

Emsellem, Y. and De Marsily, G., 1971. An automatic solution for inverse problem. Water Resources Research, Vol.7,No.5, 1264-1283.

Esler, J.E., Smith, P.F. and Davis, J.C., 1968. A fortran program for multiple regression and geological trend analysis. Geological Survey Computer Contribution 28, University of Kansas, Lawrence.

Evenson, Donald. E. and Johnson, Allen E., 1975. Parameter Estimation Program. Water Resources Engineers, Inc.

Ferris, J.G., 1952. Cyclic fluctuations of water levels as a basis for determining aquifer transmissibility. U.S.Geological Survey Groundwater, Note 1. April 1952.

France, P.W., 1974. Finite element analysis of three dimensional groundwater flow problems. Journal of Hydrology, Vol.21, 381-398.

Freeze, R. Allen and Cherry, John A., 1979, Groundwater. Prentice-Hall Inc., 704 pp.

Frind, E.O. and Pinder, G.F., 1973. Finite elements in the solution of inverse problem in groundwater flow. Finite elements in flow problems, 559-572.

Gates, J.S. and Kisiel, C.C., 1974. Worth of additional data to a digital computer model of groundwater basin. Water Resources Research, Vol. 10, No.5 : 1031-1038.

Gurtin, M.E., 1964. Variational principles for linear initial value problems. Quarterly Applied Mechanics, Vol.22, 2522 pp.

Haimes, Y.Y. and Dreizin, Y.C., 1977. Management of groundwater and surface water through decomposition. Water Resources Research, Vol.13, No.1 : 69-77.

Haimes, Y.Y., Perrine, R.L. and Wismer, D.A., 1968. Identification of aquifer parameters by decomposition and multilevel optimisation. Contribution No.123, University of California, Water Resources Centre.

Hall, W.A. and Dracup, J.A., 1970. Water Resources Systems Engineering. McGraw-Hill Book Co., New York, 313 pp.

Jacob, C.E., 1950. Flow of groundwater. Engineering Hydraulics. H.Rouse (Ed.), New York, John Wiley and Sons, 321-386.

Javendel, I. and Witherspoon, P.A., 1968. Analysis of transient fluid flows in multi-layered systems. Water Resources Centre, Contribution No.124, University of California, Berkeley, California.

Kafri, U. and Asher, J.Ben, 1978. Computer estimates of natural recharge through soils in southern Arizona, U.S.A. Journal of Hydrology, No.38 : 125-138.

Kleincke, D., 1971. Use of linear programming for estimating geohydrological parameters of ground water basins. Water Resources Research, Vol.7, No.2 : 367-374.

Knowles, T.R., Calborn, B.J. and Wells, D.M., 1972. A computerised procedure to determine aquifer characteristics. Department of Civil Engineering, Texas Tech.Univ., Lubbock, Texas, WRC-72-5.

Krishnamurthy, N., 1977. Mathematical modelling of natural ground water recharge. Water Resources Research, Vol.13,No.4 : 720-724.

Kruseman, G.P. and De Ridder, N.A., 1970. Analysis and evaluation of test pumping data. International Institute for land reclamation and improvement. Wageninger, The Netherlands, 200 pp.

Labadie, J.W., 1975. A surrogate parameter approach to modelling ground water basins. Water Resources Bulletin,Vol.11, No.1 : 97-113.

Lakshminarayana, V. and Rajagopalan, S.P., 1978. Digital model studies of unsteady state radial flow to partially penetrating wells in unconfined anisotropic aquifers. Journal of Hydrology, Vol.38, 249-262.

Lovel, R.E., Duckstein, L. and Kisiel, C.C., 1972. Use of subjective information in estimation of aquifer parameters. Water Resources Research, Vol.8, No.3 : 680-690.

McCormick, G. and Fiacco, A., 1968. Non-linear programming sequential unconstrained minimisation. Wiley, New York.

Maddock, T., III, 1972. Algebraic technological function from a simulation model. Water Resources Research, Vol.8, No.1 : 129-134.

Maddock, T., 1976. A drawdown prediction model based on regression analysis. Water Resources Research Vol.12,No.4: 818-822.

Maddock, T., III and Haimes, Y.Y., 1975. A tax system for the planning and management of groundwater. Water Resources Research, Vol. II, No.1 : 7-14.

Marino, M.A. and Yeh, W.W.G., 1973. Identification of parameters in finite leaky aquifer system. Journal of Hydr.Division, ASCE 99 (HY2) : 319-336.

Monech, A.F. and Kisiel, C.C., 1970. Application of convolution relation to estimating recharge from an ephemeral stream. Water Resources Research, Vol.6, No.4: 1087-1094.

Morel-Seytoux,H.J., 1975. A combined model of water table and river stage evolution. Water Resources Research,Vol.11,No.6.

Morel Seytoux,H.J.,1976. Derivation of equation for rainfall infiltration. Journal of Hydrology,No.31: 203-219.

Murray, W.A. and Johnson,R.L.,1977. Modelling of unconfined groundwater systems. Groundwater, Vol.15,No.4: 308-312.

Nelson,R.W., 1968. Inplace determination of permeability distribution for heterogeneous porous media through analysis of energy dissipation. Society of Petroleum Engineers Journal,33-42.

Neuman, S.P., 1973-a. Saturated-unsaturated seepage by finite elements. Journal of Hydr.Div.ASCE,Vol.99,HY12:2233-2251.

Neuman, S.P., 1973-b. Calibration of distributed parameters groundwater flow models viewed as multiple objective decision process under uncertainty. Water Resources Research Vol.9,No.4: 1006-1021.

Neuman, S.P., 1975. Role of subjective value judgement in parameter identification. In G.C.Van Steenkiste,(ed.), Modelling and simulation of water resources systems, North Holland, Amstredam, 59-84.

Neuman, S.P. and Witherspoon, P.A., 1970. Variational principles for confined and unconfined flow of groundwater. Water Resources Research,Vol.6,No.5: 1376-1382.

Nevarro, A., 1977. A modified optimisation method of estimating aquifer parameters. Water Resources Research,Vol.13,No.6: 935-937.

Nutbrown, D.A., 1975. Identification of parameters in linear equation of groundwater flow. Water Resources Research,Vol.11, No.4.

Peaceman,D.W. and Rachford, H.H., 1955. The numerical solution of parabolic and elliptic differential equations. Journal SIAM,Vol.3,28-41.

Pinder, G.F., 1970. A digital model for aquifer evaluation automated data processing and computations. U.S.Geol.Survey Techniques. Water Resources, Inv.,Book 7,Chap.C1,18 pp.

Pinder,G.F., 1973. A Galerkin infinite element simulation of groundwater contamination on Long Island, New York. Water Resources Research Vol.9, No.6: 1657-1669.

Pinder, G.F. and Bredehoeft, J.D., 1968. Application of the digital computer for aquifer evaluation. Water Resources Research, Vol.4, No.5: 1069-1109.

Pinder, G.F. and Frind, E.O., 1972. Application of Galerkin's procedure to aquifer analysis. Water Resources Research, Vol.8, No.1 : 108-120.

Pinder, G.F. and Gray, W.G., 1977. Finite Element Simulation in Surface and Subsurface Hydrology. Academic Press, Inc. (London) Ltd. 295 pp.

Pinder, G.F., Bredehoeft, J.D. and Cooper, H.H.Jr., 1969. Determination of aquifer diffusivity from aquifer response to fluctuation in river stage. Water Resources Research, Vol.5, No.4:850-855.

Prickett, T.A. and Lonnquist, C.G., 1971. Selected digital computer techniques for groundwater resource evaluation. Ill. State Water Survey Bulletin, Vol.55, 62 pp.

Project Report, 1975-76. Evaluation of groundwater resources and planning for artificial recharge for optimal cropping pattern in Daha area. UNESCO sponsored, 4th International Post-Graduate Course in Hydrology. University of Roorkee, (India).

Ralston, Anthony, 1965. A First Course in Numerical Analysis. McGraw-Hill Inc., 578 pp.

Reeves, M., 1975. Estimating infiltration for erratic rainfall. Water Resources Research, Vol.11, No.1.

Remson, Irwin, Hornberger, G.M. and Molz, F.J., 1971. Numerical Methods in Subsurface Hydrology. Wiley Interscience, 389 pp.

Richardson, L.F., 1910. The approximate arithmetical solution by finite differences of physical problems involving differential equations with an application to the stresses in a masonry dam. Phil. Trans. Royal Soc. A 210, 307-357.

Roger , P. and Smith, D.V., 1970. An algorithm for Planning irrigation projects. Bulletin of I.C.I.D. Jan. 1970.

Rowe, P.P., 1960. An equation for estimating transmissibility and coefficient of storage from river level fluctuations. Journal of Geophysical Research, Vol.65, No.10: 3419-3424.

Rushton, K.R., 1974. Critical analysis of alternate direction implicit method of aquifer analysis. Journal of Hydrology, Vol.21, 153-172.

Rushton, K.R. and Ward, C., 1979. The estimation of groundwater recharge. Journal of Hydrology, Vol.41, 345-361.

Sagar, B. and Singh, S.R., 1979. Aquifer diffusivity from noisy boundary data. Journal of Hydr.Div. ASCE 105(HY8), 943-954.

Sagar, B., Kisiel, C.C. and Duckstein, L., 1973. Estimation of aquifer properties from field data. Proceedings International Symposium on Development of Groundwater Resources, Madras, India. No.1, 26-29, Vol.I.

Sagar, B., Yakowitz, S. and Duckstein,L., 1975. A direct method for identification of parameters of dynamic non-homogeneous aquifers. Water Resources Research,Vol.II,No.4 : 563-570.

Saleem,Z.A., 1970. A computer method for pumping test analysis. Groundwater,Vol.8, No.5 : 21-24.

Singh,B.P. and Chandra,S., 1977. Evaluation of the optimal thickness of soil between source and detector in the gamma ray transmission method. Journal of Hydrology,Vol.32, 189-191.

Singh,S.R. and Sagar, B., 1977. Estimation of aquifer diffusivity in stream-aquifer systems. Journal of Hydr.Div.ASCE,103, (HY11) : 1293-1302.

Sokolov, A.A. and Chapman,T.G. (edited by), 1974. Methods of Water Balance Computations - An International Guide for Research and Practice. The UNESCO Press, 116 pp.

Stone,H.K., 1968. Iterative solution of implicit approximation of multidimensional partial differential equations. Society of Industrial applied Mathematics. Anal.Vol.5,No.3: 530-558.

Sukhija, B.S. and Shah, C.R., 1976. Conformity of ground water recharge by tritium method and mathematical modelling. Journal of Hydrology,Vol.30,167-178.

Theis, C.V., 1935. The relation between lowering of piezometric surface and the rate and duration of a well using ground water storage. Transactions Am.Geophy.Union,Vol.16,par 2 : 519-524.

Tison, G. Jr., 1965. Fluctuations of ground water levels, Advances in Geophysics Vol.2, 303-325.

Todd, D.K., 1959. Ground Water Hydrology, John Wiley and Sons, N.Y., 336 pp.

Tomilson, L.M. and Rushton, K.R., 1975. The alternating direction explicit method for analysing ground water flow. Journal of Hydrology, Vol.27, 267-274.

Trescott, P.C., 1975. Documentation of finite-difference model for simulation of three dimensional ground water flow. U.S. Geol. Survey Openfile report 75-438, 32 pp.

Trescott, P.C. and Larson, S.P., 1977. Comparison of iterative methods of solving two-dimensional ground water flow. Water Resources Research, Vol.13, No.1: 125-136.

Trescott, P.C., Pinder, G.F. and Larson, S.P., 1976. Finite difference model for aquifer simulation in two dimensions, with results of numerical experiments, U.S. Geol. Survey Techniques Water Resources Inv., book 8, Chap. C1.

Tyson, H.N. and Weber, E.M., 1964. Ground water management for national future- Computer simulation of ground water basins. Journal of Hydraulic Div. Proc. ASCE Vol.90, No.4: 59-77.

Vemuri, V. and Karplus, W.J., 1969. Identification of non-linear parameters of ground water basins by hybrid computations. Water Resources Research, Vol.5, No.1: 172-185.

Venetis, C., 1971. Estimating infiltration and/or the parameters of unconfined aquifers from groundwater level observations. Journal of Hydrology, Vol.12, 161-169.

Vogel, J.C., Thilo, L. and Van Dijken, M., 1974. Determination of ground water recharge with tritium. Journal of Hydrology, Vol.23, 131-140.

Walton, W.C., 1970. Ground Water Resources Evaluation, McGraw Hill, 664 pp.

Watson, Philip, Sinclair, P. and Waggoner, Ray, 1976. Quantitative evaluation of a method for estimating recharge to the desert basins of Navado. Journal of Hydrology, Vol.31, 335-357.

Yeh, W.W.G., 1975. Aquifer parameter estimation, Journal of Hydr. div. ASCE 101 (HY 9): 1197-1209.

Yeh, W.W.G. and Touxe, G.W., 1971. A proposed technique for unconfined aquifer parameters. Journal of Hydrology, Vol.12, No.2: 117-128.

Young, R.A. and Bredehoeft, J.D., 1972. Digital computer simulation for solving management problems of conjuctive groundwater and surface water systems. Water Resources Research, Vol.8, No.3: 533-556.

Yu, W. and Haimes, Y.Y., 1974. Multiple Optimisation for conjuctive use of ground and surface water, Water Resources Research, Vol.10, No.4: 625-636.