OPTIMIZATION OF RELIABILITY OF A SERIES SYSTEM

By

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C E R T I F I C A T E

Certified that the thesis entitled "OPTIMIZATION OF RELIABILITY OF A SERIES SYSTEM" which is being submitted by Mr. Jaydev Sharma in fulfilment of the requirements for the degree of Doctor of Philosophy (Electrical Engineering) of the University of Roorkee, is a record of the student's own work carried out by him under our supervision and guidance. The matter embodied in this thesis has not been submitted for the award of any other degree or diploma.

This is further to certify that he has worked for a period of two years and three months, from December 1971 to February 1974 for preparing this thesis for the Doctor of Philosophy Degree, at the University.

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A B S T R A C T

The present work deals with the optimal design of a system by using structural redundancy. A basic consideration in the design of a complex system is the reliability which should be very high. Generally, the reliability of the constituent components is not sufficient to meet the system reliability requirement. One way of enhancing the system reliability is to curtail the complexity of the system which may result in poor stability and transient response of the system and degradation in the quality of product. The other practical way is to introduce structural redundancy at the subsystem level. The amount of redundancies to be employed depend on the resources available which are usually limited and pose a problem to the system designers. Therefore, in the optimal design of a system, the problem of optimal allocation of redundancies to optimize reliability subject to the multiple constraints such as cost, weight, power consumption etc., arises. An attempt has been made to solve this problem in the present work. In the interest of generality, any particular system is not considered in this study.

This thesis embodies the mathematical modelling of the optimal design problem of a system having active or dynamic redundancy. The active redundancy includes parallel, series, series parallel, majority voting and multiple-line redundancy while dynamic redundancy comprises standby and hybrid redundancy. Generalized expressions are derived for the models suggested. The effect of switch failures, i.e. false switching, gradual failure and failure to operate, and dormancy in the dynamic redundancy are considered in the mathematical modelling. The systems having standby redundancy with spare and repair facilities are also considered. These models result only in partial optimization of the design problem. A true optimal design requires optimal allocation of reliability as well as redundancy in a system. Considering this fact, reliability problem is formulated. It takes the final form of nonlinear mixed integer programming problem.

These nonlinear integer programming problems are linearized by using the bivalent variables. The linearized reliability problem has same feasible solution region as the original one but the number of variables are increased.

The nonlinear integer programming reliability problem is converted into the Geometric Programming formulation by assuming variables to be continuous which leads it to a system of nonlinear simultaneous equations with variables one less than the number of constraints. When the system has only one constraint, expressions are derived to get optimal number of redundant components required in terms of resources available. These expressions are very useful to the system designer.

An algorithm is devised for solving reliability problem by using SUMT formulation. The constrained problem is solved by steepest ascent and tree search method. This algorithm is effective when system is subjected to multiple constraints and provides an exact solution.

The use of nonbinary tree search based on graphy theory is made to solve the linearized reliability problem. The method is computationally efficient than the other available zero-one programming methods as it requires only few branching and less computer sorage. The same method is modified to avoid the calculation of external stable set to find upper bound on the objective function.

The linearized reliability problem is solved by the flexible enumeration scheme which allows a great deal of flexibility in the backtracking process and thus improving the efficiency of the search procedure. This method requires simple algebraic computation and provides accurate results.

The multiple constraints linearized reliability problem is converted into an equivalent knapsack type problem having a single constraint by aggregating the constraints. This equivalent problem is easier to solve than the original problem. A Branch and Bound method is brought out to solve the equivalent problem.

A very efficient method is developed to solve nonlinear integer programming reliability problem. The method is based on the fact that for maximizing the system reliability one component must be added sequentially to that particular stage which has lowest reliability. As the method needs only simple calculations and very little memory, it can be used to solve large systems.

The optimal allocation of reliability and redundancy problem is solved by using SUMT formulation with discretization penalty function.

The computer programs are developed and have been applied to solve various problems with success. To illustrate the methods of attack, numerical examples are incorporated. These methods can be used for the reliability-based design of the system such as control system, digital system
At the end, the various methods discussed in this thesis are compared so that a system designer may know their limitations and advantages. Future avenues of research are also discussed.

In short, the mathematical models have been presented for the optimal reliability design problem. Various types of redundancies are considered and methods to solve the reliability problem are discussed.

ACKNOWLEDGEMENTS

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SYMBOLS

The following symbols are those which have a specific meaning throughout the text. The other symbols used may have somewhat different meaning which are defined separately in the text.

a_{ij} - resources requirements associated with each component of j^{th} stage.

b_i - total amount of resources available for the i^{th} type of constraint.

k - number of stages or subsystems in a system.

m_j - number of standy components.

n_j - number of redundant components in j^{th} stage

$\bar{N}_j$ - upper bound on n_j

p_i - the probability of occurrence of i^{th} event.

q_j - unreliability of the j^{th} type component.

q_s - unreliability of the switching device.

Q_s - system unreliability.

r_j - reliability of the j^{th} type component

R_j - reliability of the j^{th} stage.

R_s - system reliability

R_v - reliability of the voter

s - number of constraints on the system.

t - mission time

X - binary variable

Z - binary variable

λ_j - failure rate of j^{th} type component

μ_j - repair rate of j^{th} type component

γ_j - standby failure rate of j^{th} type component

Chapter 1

INTRODUCTION

Reliability of a system is defined as the probability of achieving the required input-output function within specified limits throughout the whole mission under given environment. Therefore, reliability of a nonredundant system is a decreasing function of the failure rates of the constituent components, the size of the system and the time for which the system is designed to operate. Due to increased complexity, sophistication and automation in modern systems, system reliability always tends to decrease. An interruption in the operation of the system has consequences in terms of cost, time wasted, the psychological effect of inconvenience and in certain instances personnel and national security. In some cases, the cost associated with the failure of a component is not only its cost due to a complete curtailment of the whole system, but also cost due to the deterioration in the quality of manufactured product. For example, an interruption in the power supply to the electric arc furnace will result in damage to the furnace as well as will deteriorate the quality of steel to be produced. Due to some remote large-scale failures in nuclear power plant, large quantity of radio-active material may be released and may provide risk to human life. Modern process plants are quite complex and involve high capital cost. In order to increase the efficiency of the process, moderate values of process parameters are used. For example, in chemical plants, processes are performed at high pressure and temperature with higher concentrations of reactive chemical for increasing its effectiveness. On the occurrence of a fault in these processes, there are possibilities of great damage to the plant as well as to the operating personnel.

Generally, the reliability of the constituent components of the system is not sufficient to meet the system reliability goal. Therefore, some means must be employed to enhance the system reliability. It can be increased by incorporating the following methods:

1. reducing the complexity of the system
2. increasing the reliability of the components by product improvement program
3. using structural redundancy

Curtailment of system complexity may yield in poor stability and transient response of the system and reduced accuracy and degradation in the quality of product. The part improvement program demands the use of improved package and shielding techniques, derating etc. Although these techniques result in reduced failure rate of the component, but require more time for design and special state of art of production. Therefore the cost of part improvement program is higher as against the cost of a redundant component. The employment of structural redundancy at subsystem level, keeping specific system topology, can provide theoretically unity system reliability. When there are many similar components in the system, this method provides very effective results. Structural redundancy may involve use of two or more identical components, so that when one fails, others are available in such a way that the system is able to perform the specified task in the presence some faults in the components. The use of four engines in an aeroplane is one of the examples of the redundant system.

The various types of redundancy schemes can be grouped into two categories: active redundancy and dynamic redun- . In the active redundancy, all the redundant components are kept in the operating condition. On the failure of one component, others will be able to perform the system task. In the dynamic redundant system, only one component (called basic component) is kept in operating condition while others are kept in an inactive state. These are put on sequentially only when the basic component fails. The use of redundancy provides quickest solution if time is main consideration, easiest solution if component is already designed, cheapest method if cost of redesign of a component is too high and the only solution if improvement in the component reliability is not possible.

It is definite that the use of redundancy increases the system reliability, but on the other hand, system weight, cost, power-requirement etc. increase. But these are usually limited and such constraints pose a problem to the system designer. Therefore, in the optimal design of a system, the problem of maximizing system reliability by using structual redundancy subject to the multiple constraints arises.

Various methods are available on the active parallel redundancy case with one or more specific constraints. Moskowitz Mclean [21] considered the problem of maximizing reliability with a cost constraint using a variational method. Proschan and Bray [22] extended Kettelle's [23] computational method for maximizing reliability subject to the cost constraint to multiple constraints. A dynamic programming approach was suggested by Bellman and Dreyfus [24]. A modified dynamic programming formulation of reliability problem was developed by Misra [25]. Fan et al. [26] used the discrete maximum principle for maximizing reliability. Tillman and Liittschwager [27] developed a method for maximizing reliability or minimizing cost subject to several constraints by using an integer programming formulation. Mizukami [28] used a convex integer programming method for maximizing reliability with multiple linear constraints. Federowicz and Mazumdar [29] formulated the redundancy allocation problem in the form of geometric programming problem, to obtain approximate solutions. Ghare and Taylor [30] maximized the reliability of parallel redundant systems by a branch and bound procedure. Misra [31] used a binary algorithm to optimize system reliability or cost subject to multiple constraints. Lambert et al. [32] used maximum principle approach for maximizing availability subject to cost constraint. Misra [33] used least square formulation for maximizing system reliability. Banerjee and Rajamani [34] used the parametric approach to solve reliability problem.

All the above papers considered active parallel redundancy case. Messinger and Shooman [35] used generalized Lagrange multiplier and dynamic programming approach for finding the optimum number of spare components in a system. Burton and Howard [36] presented a dynamic programming method for allocating standby components to maximize system reliability subject to cost and weight constraints.

In the present study, various types of redundancies are considered. Different methods are developed for finding optimal allocation of redundancies to maximize system reliability subject to multiple constraints. All these methods can be grouped into two categories:

a. Method which provide approximate results, which, in some cases, are also true optimum.

b. Methods which give true optimum solution.

The procedure to be adopted for solution of reliability problem, depends on the accuracy of the results and cost of obtaining them. The system designer has to seek several alternatives and alteration in design parameter from other technical considerations.

The redundancy allocation problem is a sub-optimization problem. If the components of different reliabilities are available, the true optimal solution requires the optimum selection of number of redundant components as well component reliability. This problem is formulated as a mixed integer programming problem and solved by using sequential unconstrained minimization technique .

When cost of repair in money as well as in time is less in comparison with the cost of equipment, it is economical to consider system repair. It may be possible that at a time more than one component fail simultaneously. This requires more than one crew in order to increase the operating time of the equipment. But in case of non-redundant system, repair will not help in the sense of increasing the system reliability. It can be enhanced by providing spare components. Both the use of multiple repair facilities and spare components require additional resources. For optimal design, a mathematical model is developed which is solved by Lexicographic Enumeration technique.

The aim of present work is to present the mathematical formulation of the optimal design problem of a system from reliability consideration and the techniques to solve them. These methods can be used for fault-tolerant optimal design of control systems, digital systems, communication systems, etc. In the interest of generality, any particular system is not considered in this study.

Chapter 2

PROBLEM FORMULATION

A complex system consists of many functional units. They can be grouped into various modules or stages. The size and complexity of the modules rely on the volume of irredundant structure, degree of logical branching of the functional units, easiness of replacement and checking etc. After decomposing the complete system into modules, it is necessary to draw the logic diagram of the system for reliability analysis. A logic diagram gives an idea that which components must operate failure-free for performing the intended job. If a complex system is partitioned into k modules and failure of any module results in loss or premature termination of the job or mission, the logic diagram of such type of system will have k modules in series. If the failure of a module does not result in system shutdown, it will be represented by a parallel element in the logic circuit. Consider a digital system shown in Fig. 2.1. All the ten components are partitioned into seven modules. If all the components are required for successful operation of the system, the logic diagram will be a series system as shown in Fig. 2.2.

2.1. ASSUMPTIONS

After drawing the logic diagram, mathematical model of the system can be developed. The various assumptions which are to be made for reliability analysis are

1. The inputs to the system from outside world are all perfect

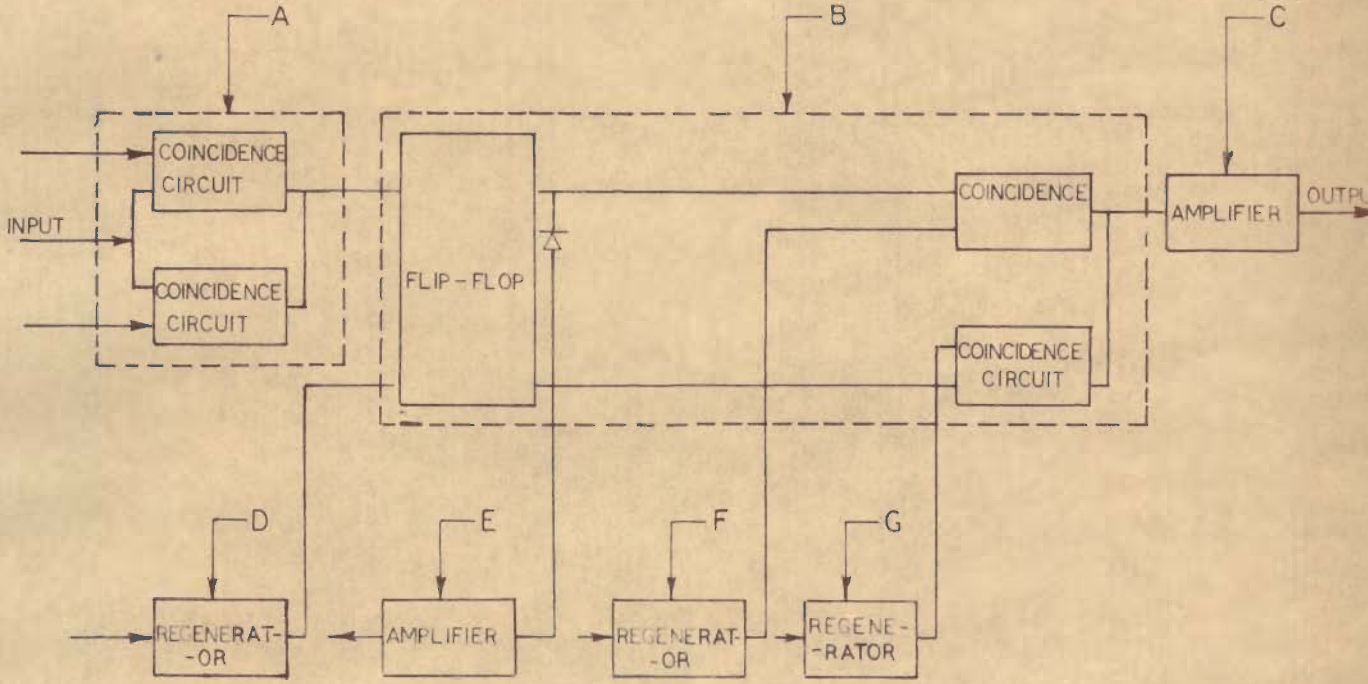


FIG.2.1 A DIGITAL SYSTEM

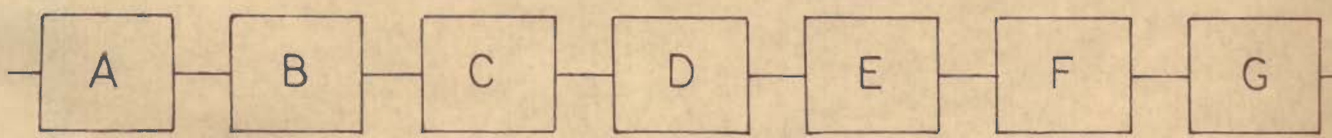


FIG.2.2 LOGIC DIAGRAM OF SYSTEM SHOWN IN FIG. 2.1

i.e. highly reliable.

2. The failure of any subsystem or module results in system failure.

3. The failures of the subsystems are statistically independent.

4. The arrangement of the function in the system is given.

5. The failure distribution of the component is exponential with failure rate as λ_j

The first assumption is made for the ease of calculations and can be considered in the mathematical formulation. Second assumption is generally true for the system which is in the design phase. The system in which this assumption is not valid are considered in Chapt. 4. The failure independence assumption in calculating system reliability results in a slight underestimate of the system reliability. This error (less than 10% normally) is negligible. Single component failure in a series system greatly outnumber overlapping failures. Also, as soon as a single component fails, the system is at least partially de-energized, thus resulting in a reduction in the probability of subsequent overlapping failures. The actual value of the error caused by the statistical independence consideration can be estimated [37] and it can be shown that dependence has at the most a second-order effect. From field data it is found that times to failure of electrical and electronic components are generally exponentially distributed. Therefore, fifth assumption seems to be reasonable.

2.2. GENERAL PROBLEM

Consider a system having k subsystems or modules or stages. With the above-mentioned assumptions, the system reliability can be given by [9]

$$R_s(\bar{n}) = \prod_{j=1}^{k} R_j(n_j) \tag{2.1}$$

where,

$R_j(n_j)$ = reliability of j^{th} stage

n_j = number of redundant components in j^{th} stage

Since the use of redundancy is limited by the availability of resources, the optimal design problem can be stated as

Maximize system reliability.

$$R_s(\bar{n}) = \prod_{j=1}^{k} R_j(n_j) \tag{2.2}$$

subject to the constraints

$$\sum_{j=1}^{k} G_{ij}(n_j) \leq b_i \tag{2.3}$$

$n_j \geq 1$ and integer; $i = 1, 2, \ldots . s$

where $G_{ij}(n_j)$ is the i^{th} type resources requirement for j^{th} stage and b_i is the total amount of resources available for the i^{th} type of constraint. Mathematically, the problem can be stated as: the selection of non-negative integer vector n such that $R_s(\bar{n})$ is maximum subject to the constraints given as above. As the formulation shows it is a nonlinear integer programming problem. For solving this problem, the expression for reliability of j^{th} stage is required. This expression depends on the type of redundancy which is to be employed for enhancing the system reliability. It may be either active or dynamic redundancy or hybrid redundancy.

2.3. ACTIVE REDUNDANCY

The use of active redundancy results in less stresses in the components if load sharing exists. and thereby provides higher system reliability than the dynamic redundancy. But special care should be taken to impedance levels, power, signal gains and linearity etc. In some cases, active redundancy provides better performance than dynamic redundancy. For example, in a time-sharing system, a user may have devoted considerable efforts at console, which can be destroyed if a system failure occurs. The use of active redundant console can save his efforts, even when one console fails. The active redundancy can be classified as parallel, series-parallel, parallel-series, majority voting, and multiple line redundancy.

2.3.1. Parallel Redundancy:

A parallel redundant system as shown in Fig. 2.3 is defined as the system in which failure of one or more paths still allows the remaining path or paths to perform the intended function. An example of such type of system is two transmitters A and B connected in parallel. Even on the failure of transmitter A, transmitter B will perform the job. If mode of component failures is fail safe this type of reuun-dancy provides an easiest method of improving the system reliability.

Consider a system having k stages connected in cascade. Let stage j have a set of n_j components connected in parallel,

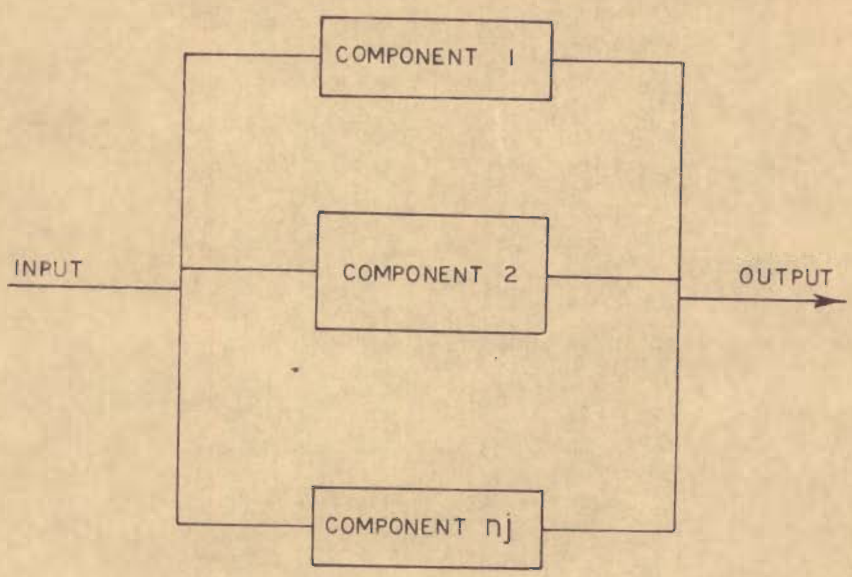


FIG. 2.3a A PARALLEL REDUNDANT SUB-SYSTEM

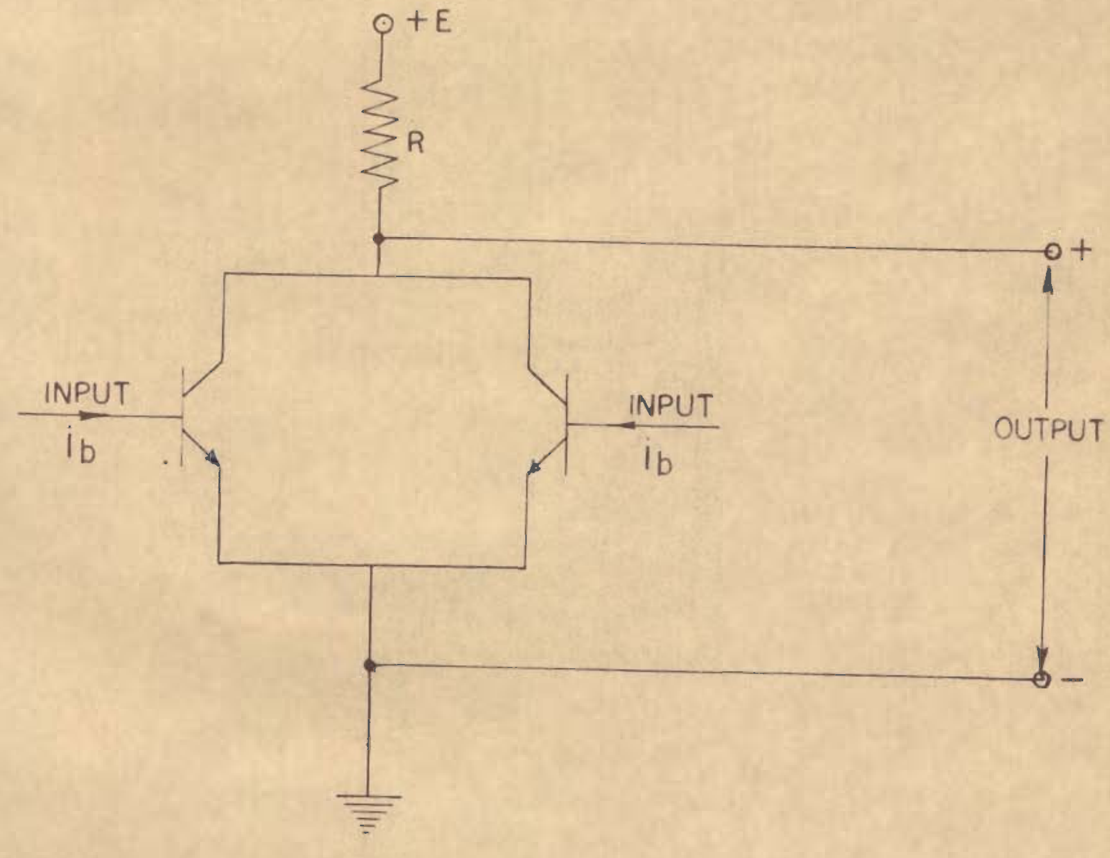


FIG. 2.3 b A PARALLEL REDUNDANT TRANSISTOR AND GATE

each having probability of failure q_j. With the assumption made in 2.1, the system reliability can be written as [1]

$$R_s(n) = \prod_{j=1}^{k} R_j(n_j) \qquad (2.4)$$

where,

$$R_j(n_j) = 1 - q_j^{n_j} \qquad (2.5)$$

In the above derivation, it is assumed that the failures of n_j components in the j^{th} stage are statistically independent. But failure of any redundant element connected in parallel causes changes in the technical characteristics of the redundant stage. For example, change in the resistance or capacitance of a circuit may cause it to operate in an unstable and irregular manner. Considering that due to the failure of a component, redistribution of loads or voltages occur in the parallel connected elements and thereby there is a change in their failure rates. Let there be n_j diodes connected in parallel in the j^{th} stage and total load on the j^{th} stage is equal to the rated load of a single diode, the failure rate of the stage when exactly i diodes have failed can be given by

$$\bar{\lambda}_i = (n_j - i)^{-d} \lambda_{jr} \qquad (2.6)$$

where λ_{jr} is the failure rate of the diode when operating at full load and d is a constant.

This subsystem fails, when all n_j components fail. The possible states which this subsystem can have are $0, 1, 2, .. n_j - 1$. Assuming that at time t subsystem is in state i and after an infinitesmal interval Δt, it changes to $i+1^{th}$ state. The characteristic equation defining the state of the subsystem at any time t can be written as

$$p_i'(t) = -\lambda_{jr}(n_j-i)^{1-d}p_i(t)+\lambda_{jr}(n_j-i+1)^{1-d}p_{i-1}(t) + O(\Delta t) \qquad (2.7)$$

$$i = 0, 1, \ldots n_j-1$$

with initial conditions

$$p_i(0) = \begin{cases} 1 & \text{for } i = 0 \\ 0 & \text{else} \end{cases}$$

where $p_i(t)$ is the probability of subsystem being in i^{th} state and probabilities of more than one transition are negligible.

Solving the above differential equations with specified initial conditions

$$p_i(t) = \left[\prod_{l=0}^{i-1} (n_j-l)^{1-d}\right]\left[\sum_{l=0}^{i} \frac{\exp\left[-\lambda_{jr}(n_j-l)^{1-d}t\right]}{\prod_{\substack{\omega=0 \\ \omega\neq l}}^{i}\left[(n_j-\omega)^{1-d}-(n_j-l)^{1-d}\right]}\right]$$

$$i = 1, 2, \ldots n_j-1$$

and

$$p_o(t) = \exp\left[-\lambda_{jr}\, n_j^{1-d}\, t\right] \qquad (2.8)$$

Therefore, reliability of j^{th} subsystem can be given by

$$R_j(n_j) = \sum_{l=0}^{n_j-1} p_i(t)$$

Therefore, j^{th} subsystem reliability can be written as

$$R_j(n_j) = \exp\left[-\lambda_{jr}\, n_j^{1-d}\, t\right] + \sum_{i=1}^{n_j-1}\left[\prod_{l=0}^{i-1}(n_j-l)\sum_{l=0}^{i}\frac{\exp\left[-\lambda_{jr}(n_j-l)^{1-d}t\right]}{\prod_{\substack{\omega=0 \\ \omega\neq l}}^{i}\left[(n_j-\omega)^{1-d}-(n_j-l)^{1-d}\right]}\right] \qquad (2.9)$$

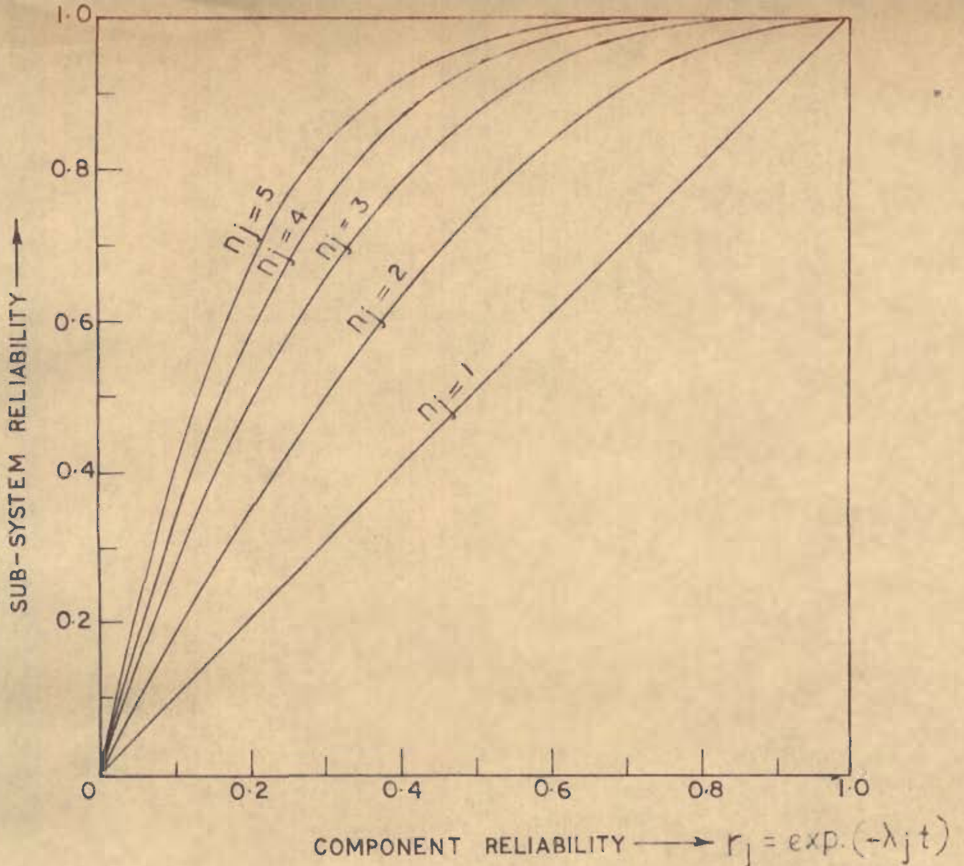


FIG. 2.4 PARALLEL REDUNDANT SYSTEM RELIABILITY VS COMPONENT RELIABILITY

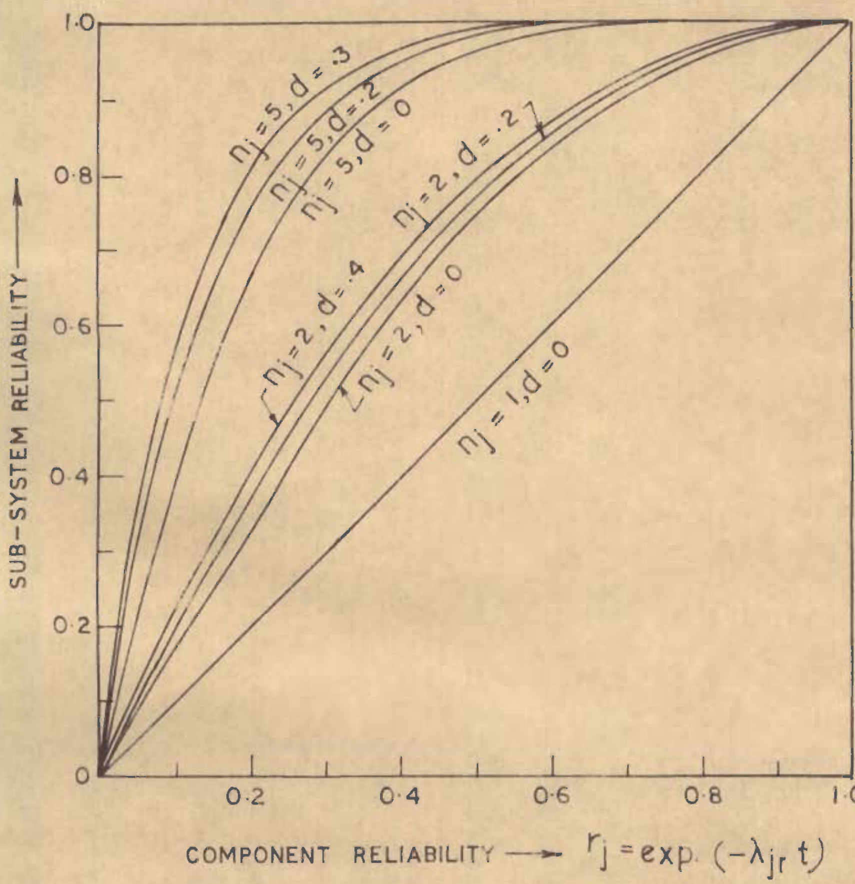
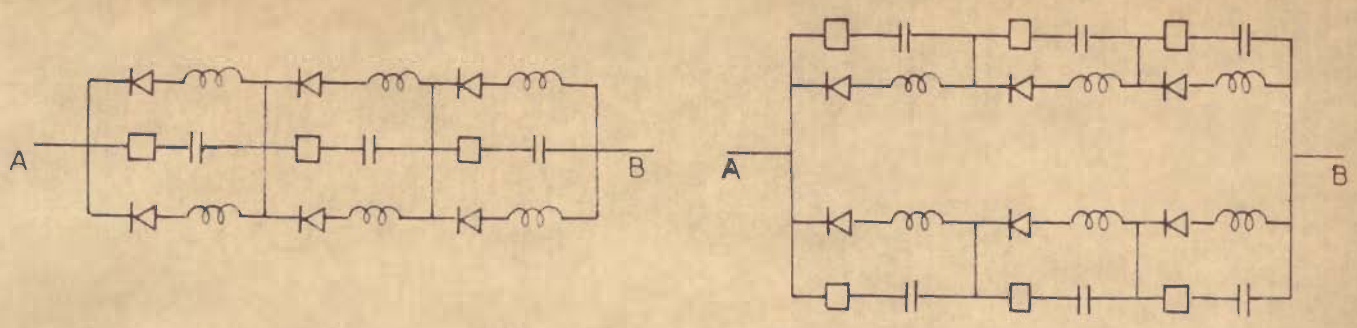


FIG. 2.5 PARALLEL REDUNDANT SYSTEM RELIABILITY VS COMPONENT RELIABILITY CONSIDERING DEPENDENCY

The reliability expressions for a parallel redundant subsystem without dependency (2.5) and with load dependency (2.9) are shown in Fig. 2.4 and Fig. 2.5, respectively. From Fig. 2.5, it is clear that if load sharing between parallel redundant components is possible, this type of redundancy will provide higher improvement in subsystem reliability than standby redundancy (d = 0).

2.3.2. Group Redundancy:

Some components such as diodes, relays, transistors, vacuum tubes etc. fail in two modes, i.e. open circuit and short circuit. If such components are connected in parallel, failure of any one due to short circuit will result in complete system failure. Similarly, if they are connected in series, an open circuit failure will also result in system failure. To increase the reliability of such components, it is necessary to reduce the probability of open and short circuit failure simultaneously. This can be achieved by using group redundancy. Group redundancy or mixed redundancy can be realized by two types of arrangements, i.e. series-parallel redundancy and parallel-series redundancy as shown in Fig. 2.6a and 2.6b, respectively. Which type of mixed redundancy to be used, depends on the technical characteristics of the components. For example, let parallel-series redundancy be used to increase the reliability of a thyrister valve in a convertor circuit. When one of the valves arcs back or fire through, a voltage rise occurs across the remaining valves. But in case of series-parallel arrangement, voltage across all the remaining parallel connected valves group rises. If one of the valves in parallel series



(a) SERIES — PARALLEL REDUNDANCY

(b) PARALLEL — SERIES REDUNDANCY

FIG. 2.6 GROUP REDUNDANCY

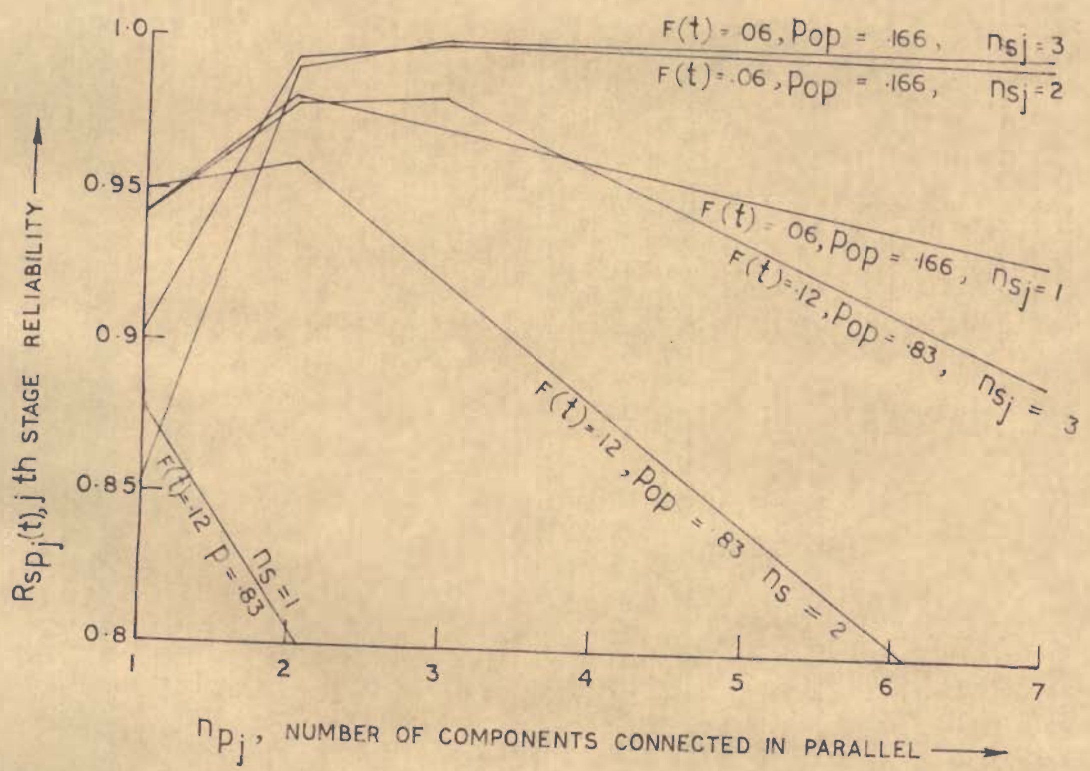


FIG. 2.7 RELIABILITY FUNCTION FOR SERIES-PARALLEL REDUNDANT SYSTEM

arrangement fails to fire, the current in that link of valves will fail to flow and load will be carried by other links connected in parallel. This may cause current overload in the convertor circuit, while in series parallel arrangement current overload takes place only in the valves connected in parallel with the one which fails to fire. The other factors on which the arrangement to be chosen depends are, the possibility to disconnect the faulty valve without disturbing the operation of convertor circuit, the method of feeding the control pulses to the valve grid and the layout of valves at the convertor station. Simultaneous firing of the series connected valves is necessary when parallel-series arrangement is used. The simultaneous firing of valves in series-parallel arrangement is not strictly required, but in case of considerable lag in the firing, the parallel connected valves lose uniformity in the current distribution. The replacement of the valve in this type of arrangement can easily be done without interrupting the operation of convertor system. When failure rate of a component is a function of the load that it is carrying, the parallel-series arrangement is preferred.

Consider a j^{th} stage having n_{pj} components connected in parallel and n_{sj} components connected in series. Let p_{op} be the conditional probability of an open circuit failure of a component, given that a failure has occurred, $\bar{p}_{op}=1-p_{op}$, denotes the conditional probability of a short circuit,given that a failure has occurred and $F_j(t)$ is the failure time distribution functionof the j^{th} type component. The j^{th} stage will fail when all n_{pj} components in any unit fail by opencircuiting or when at least one component in each unit fails by short-circuiting if series-parallel redundancy is used. Assuming that a short circuit failure cannot occur after an open circuit failure, the stage reliability (using series-parallel redundancy) can be given by [10]

$$R_{spj}(t) = \left[1-(p_{op}F_j(t))^{n_{pj}}\right]^{n_{sj}} - \left[1-(1-\bar{p}_{op}F_j(t))^{n_{pj}}\right]^{n_{sj}} \quad (2.10)$$

If parallel-series redundancy is used to increase the stage reliability, the stage will fail when at least one component in each parallel connected chain fails or when all components in a parallel connected chain fail. [10]

$$R_{psj}(t) = \left[1-(\bar{p}_{op}F_j(t))^{n_{sj}}\right]^{n_{pj}} - \left[1-(1-p_{op}F_j(t))^{n_{sj}}\right]^{n_{pj}} \quad (2.11)$$

The reliability expression for series-parallel redundant system is plotted in Fig. 2.7 which shows that if the probability of open circuit of a component is low, series-parallel redundancy is preferred. Beside this, stage reliability increases with number of components to be connected in parallel upto a point after that it decreases with increase in n_{pj}.

2.3.3. Majority Voting Redundancy:

Use of majority voting redundancy is the most effective method of improving the reliability of the digital system, when mission time is short and repair is not possible. It does not require error-detecting and switching device and is, therefore, ineffective by the random transient failures which generally occur in the digital computer systems. This type of redundancy is also called as N modular redundancy (NMR). In NMR system each stage has $(2n_j+1)$ identical components connected in parallel whose outputs are fed into a majority voter as shown in Fig. 2.8. The output of the voter is the majority of its input signal. The j^{th} stage will fail when (n_j+1) components fail, if R_v is the voter reliability, the j^{th} stage reliability can be expressed as [9]

$$R_j(n_j) = R_v \sum_{i=n_j+1}^{2n_j+1} \left[\frac{(2n_j+1)!}{(2n_j+1-i)!\,i!} \; r_j^{i} \; (1-r_j)^{2n_j+1-i} \right] \tag{2.12}$$

where $n_j=1, 2, \ldots$

In the above analysis it is assumed that a component has an equal probability of failure with output 0 and with output 1, which is not always valid. Beside this, one may intentionally design a component to fail in a given output state. Consider a Triple modular redundant (TMR) system where one component failure is tolerated. If second component fails to the opposite logic level (0 or 1), thus neutralizing the voting effect of the first failed component, and resulting the output of the TMR system same as the good component signal.

If p_{1j} is the conditional probability of the component of j^{th} stage failing to logical one and $(1-p_{1j})$ is the conditional probability of the component failing to logically zero, the j^{th} stage reliability can be given by

$$R_j(n_j) = R_v \left[3r_j^2 - 2r_j^3 + 6p_{1j}(1-p_{1j})r_j(1-r_j)^2 \right] \tag{2.13}$$

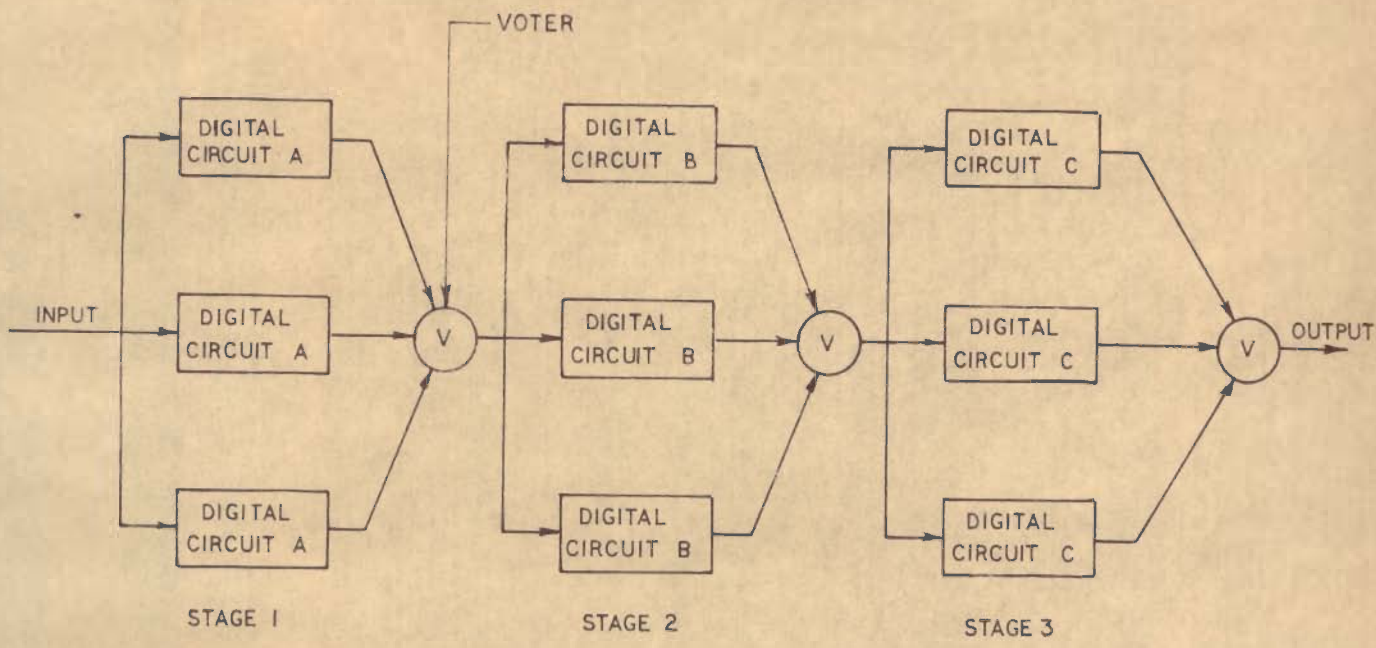


FIG. 2.8 TRIPLE MODULAR REDUNDANT SYSTEM

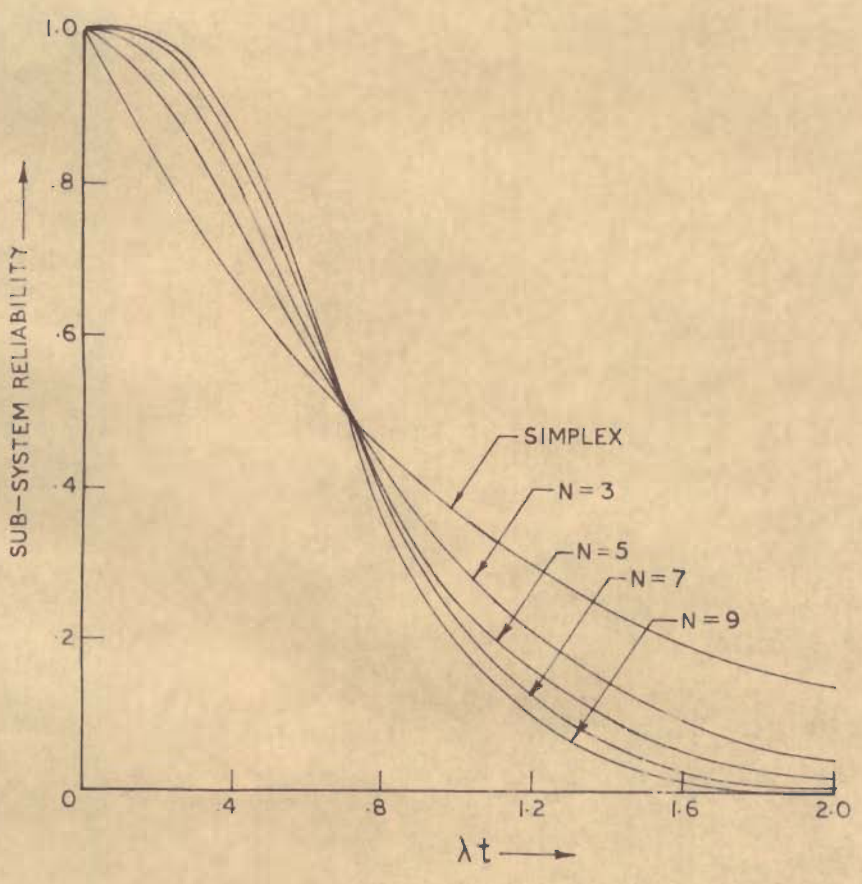


FIG. 2.9 RELIABILITY FUNCTION OF NMR SYSTEM VS NORMALIZED TIME

For the components having symmetrical failures, the choice of voter is a majority element. If the components have asymmetrical probability of failure, the majority voter will not be a best choice. For example, if a component always fails with zero output, an OR gate is a best choice for voting elements. Reliability expression (2.12) has been plotted with respect to normalized time $\lambda_j t$ and is shown in Fig. 2.9 which shows that if a component has normalized time greater than 0.65, the use of NMR system will provide higher system reliability.

2.3.4. Multiple Line Redundancy:

It can be shown that total triplication is superior to partial triplication for a system having component unreliability less than 0.25. A system in which total triplication is done, is called as multiple line redundant system. The reliability of such a system can further be increased by providing three voters per stage, as failure of a voter in a single voter system, which is simple to design, brings about the failure of the complete system. The various factors which affect the number and placement of the voters are

1. availability of resources such as cost, weight etc.
2. the voter circuit delay and drive requirement.
3. the testing facilities.
4. the trouble shooting time and logistic requirement
5. number of signals to be transferred out of a component
6. the reliability of the voter

Consider a multiple line redundant system having k .-

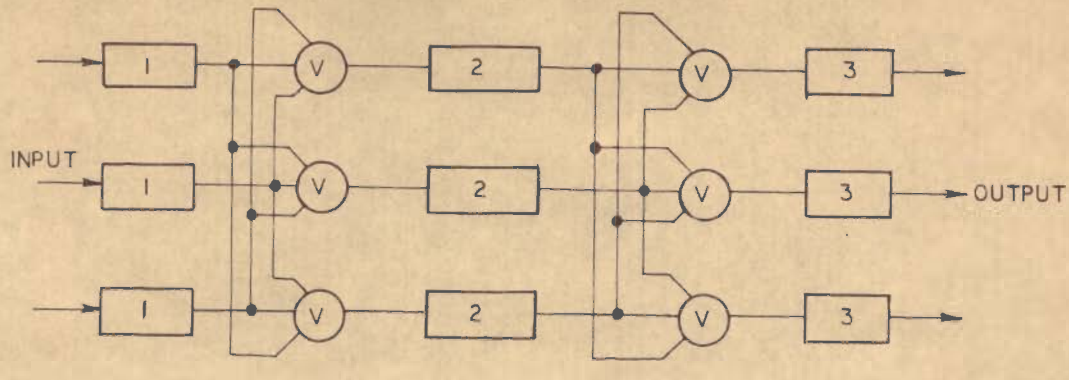


FIG. 2.10 MULTIPLE LINE REDUNDANT SYSTEM

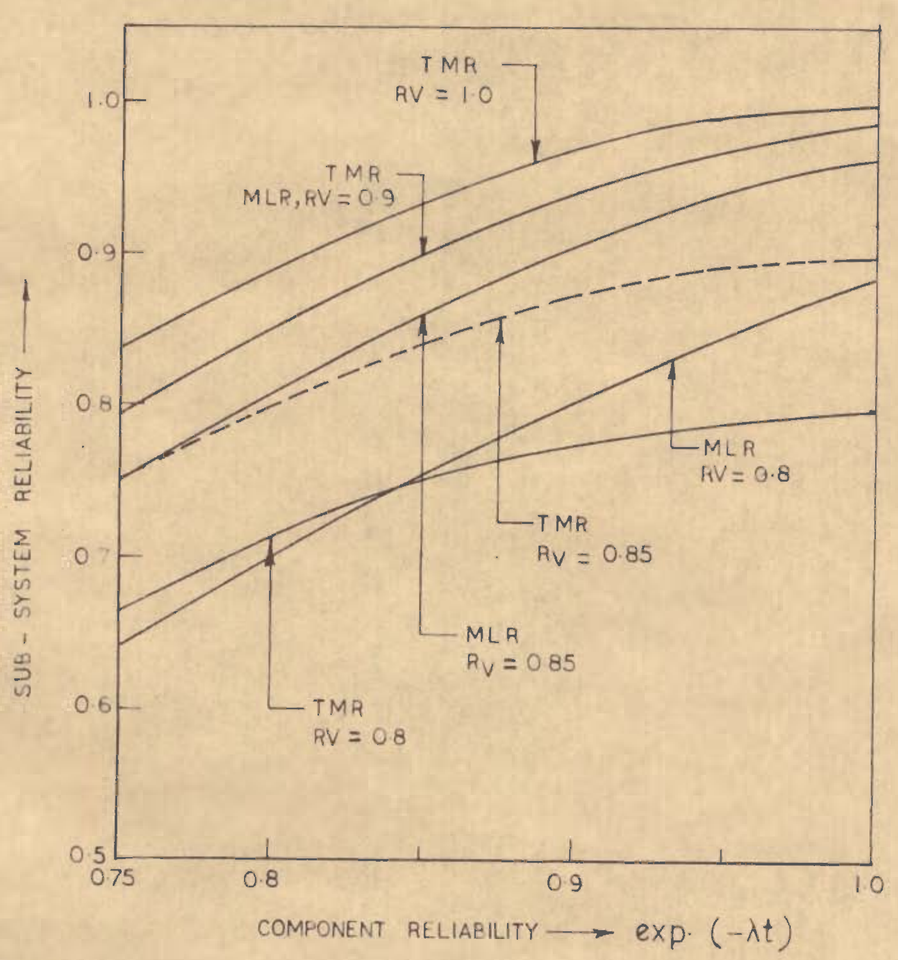


FIG. 2.11 RELIABILITY EXPRESSION FOR MULTIPLE LINE REDUNDANT SYSTEM (MLR)

independent stages or subsystems, the system reliability can be given by

$$R_s = \left[\prod_{j\in I} (3r_j^2R_v^2 - 2r_j^3R_v^3) \right] \left[3(\prod_{l\notin I} r_l)^2 - 2(\prod_{l\notin I} r_l)^3 \right] \qquad (2.14)$$

where I is the set of stages having majority structure. A family of curves illustrating the behaviour of this type of redundancy is shown in Fig. 2.11.

2.4. DYNAMIC REDUNDANCY

This type of redundancy is also called as standby redundancy. Realization of standby redundancy requires a fault detecting and switching device, which makes it possible to locate the faulty component and replace it by the standby component. If the fault detecting and switching device is perfect, i.e. highly reliable, theoretically it enables to achieve system reliability close to unity. Such type of redundant system is shown in Fig. 2.12. Its operation can be explained as follows. Consider that the j^{th} stage has n_j redundant components. Initially, the basic component is only kept in energized condition and others are kept standby. When the basic component fails, a standby component is switched-in to take its place. The failure of the stage occurs when n_j components fail. Assuming that the fault detecting and switching device is perfect and requires no time for operation, spare components do not age while waiting for replacement and the distribution of the number of failures of the components upto and including time t is poisson with mean as $\lambda_j t$, the j^{th} stage reliability can be given by [9]

$$R_j(n_j) = e^{-\lambda_j t} \sum_{l=0}^{n_j-1} \frac{(\lambda_j t)^l}{l!} \tag{2.15}$$

The fault detection and switching operation can be performed by a human being. But it requires considerable time in locating the fault and in the replacement of faulty component. If t_{frj} is the time required in the fault detection and replacement in j^{th} stage, then j^{th} stage reliability can be given by [10]

$$R_j(n_j) = \exp\left[\lambda_j \left\{t-(n_j+1)t_{frj}\right\}\right] \cdot \left[\sum_{l=0}^{n_j-1} \frac{\lambda_j^l \left\{t-(n_j+1)t_{frj}\right\}^l}{l!}\right] \tag{2.16}$$

Above expression shows that there is a considerable improvement in the system reliability if fault detection and replacement are instantaneous. This can be achieved by using automatic switches. The expression (2.15) is only valid when switches are highly reliable. Generally, switches remain in inoperative condition; they are only required to operate when a fault has occurred. It might be possible that under this condition switch may fail to operate. Therefore, the possibility of a stage failure may occur due to shortage of any spare component and failure of fault detecting and switching device when it is called for operation. For analysing system under these conditions, let state i denote that the i^{th} standby component is in operation; and n_j^{th} state denote the failure of the j^{th} stage. If q_s is the probability of failure of fault detecting and switching device and during infinitesi-

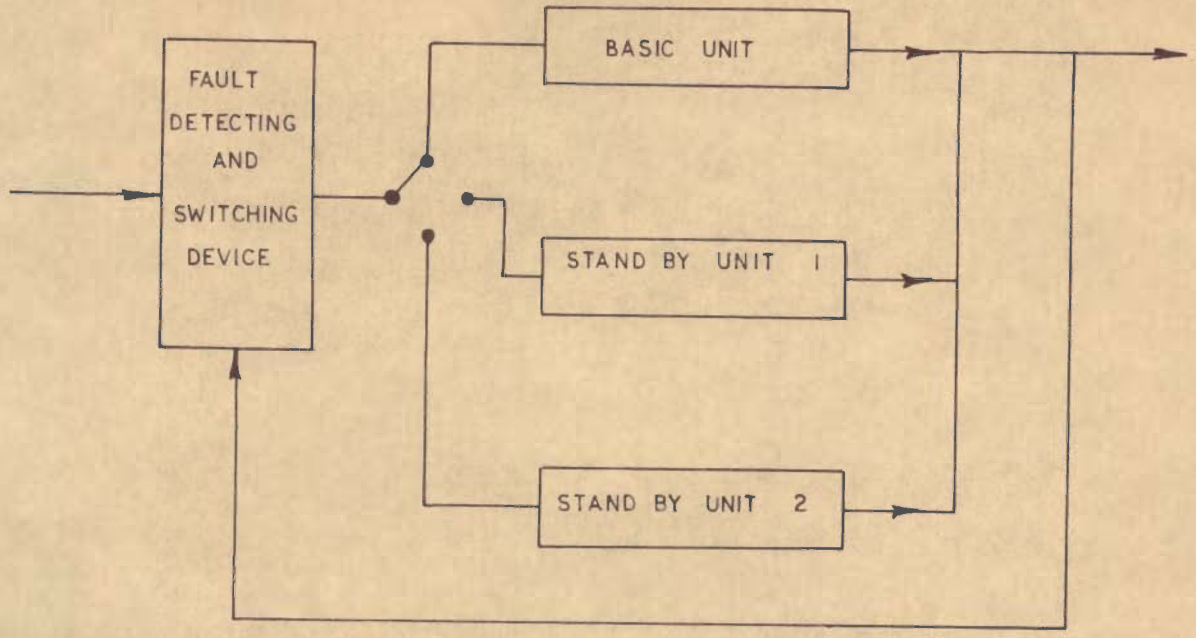


FIG 2.12 A SUB-SYSTEM WITH STAND BY REDUNDANCY

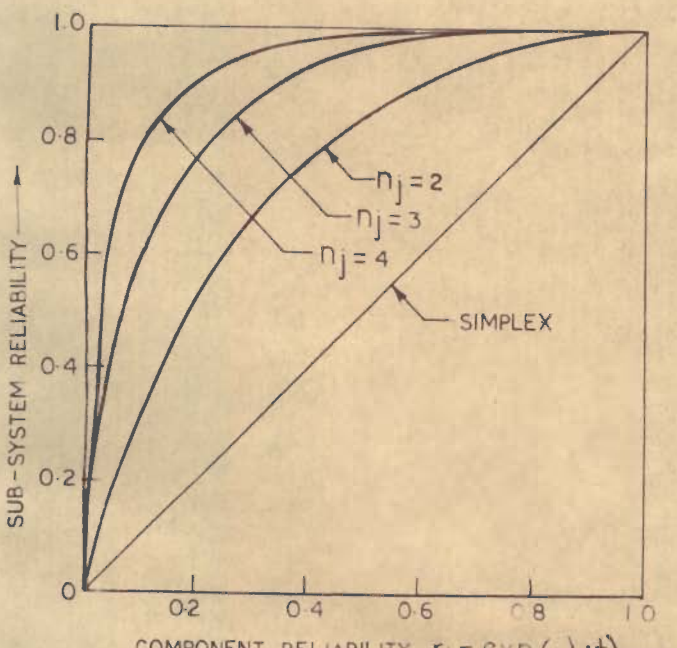


FIG. 2.13a RELIABILITY FUNCTION OF STANDBY REDUNDANT SUB-SYSTEM WITH PERFACT SWITCH

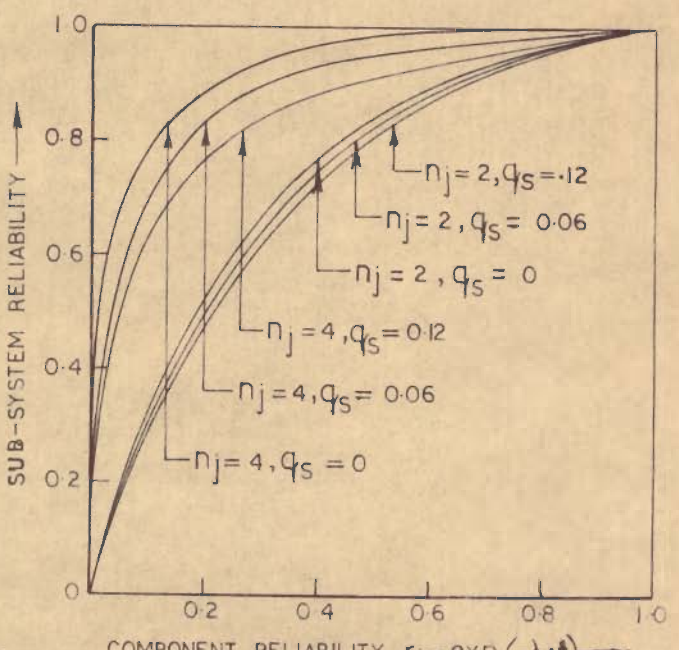


FIG. 2.13b RELIABILITY FUNCTION OF STANDBY REDUNDANT SUB-SYSTEM WITH SWITCH FAILURE

-mal interval Δt, the j^{th} stage changes from state i to state $i+1$ & probabilities of more than one transition are negligible, the differential equations defining the various states of the system can be written as

$$p'_o(t) = -\left[\lambda_j \Delta t\right] p_o(t) + 0\,(\Delta t) \qquad (2.17)$$

$$p'_i(t) = \left[\lambda_j (1-q_s)\Delta t\right] p_{i-1}(t) - \lambda_j \Delta t p_i(t) + 0(\Delta t) \qquad (2.18)$$

$$i = 1, 2, \ldots n_j - 1$$

$$p'_{n_j}(t) = -\left[\lambda_j q_s \Delta t\right] p_i(t) \qquad (2.19)$$

$$i = 1, 2, \ldots n_j - 1$$

with initial conditions as

$$p_i(o) = \begin{cases} 1 & i=0 \\ 0 & i=1,\ldots n_j-1 \end{cases}$$

From (2.17), $p_o(t)$ can be calculated as

$$p_o(t) = \exp(-\lambda_j t)$$

and from (2.18), $p_i(t)$ can be calculated as

$$p_i(t) = \frac{\left[\lambda_j (1-q_s) t\right]^i}{i!} e^{-\lambda_j t}$$

Therefore, the j^{th} stage reliability can be given by

$$R_j(n_j) = \sum_{i=0}^{n_j-1} p_i(t) = \exp(-\lambda_j t) \sum_{i=0}^{n_j-1} \frac{\left[\lambda_j (1-q_s) t\right]^i}{i!} \qquad (2.20)$$

A family of curves are plotted in Fig. 2.13 for various values of switching device reliability, which show that switch device should have high reliability if standby redundancy is to be used for enhancing the system reliability.

In the above analysis it is assumed that failure rate of the standby components when unpowered, is zero. But, generally, the ratio of failure rates of the components with power on, to that with power off, ranges from 1.21 to 2.16, depending upon the type of component, environment and packaging. Beside this switch may fail in more than one mode. Considering more complex situation, the various modes of failures of the standby system can be categorised as -

1. Gradual failure of the components: The gradual failure in the components occurs while they are kept as standby or in operating condition. Let the time upto first power-on and power-off failures be distributed according to exponential law with parameters, as λ_j and γ_j respectively.
2. Static failure of switch: The switch operates when it is not called for operation. This may be due to false sensing or due to some failure in switching mechanisms or external conditions such as vibrations etc. This will cause unnecessary switching of one standby component. Let the probability distribution of static failure be poisson with parameter as α_j.
3. Dynamic failure of switch: If switch fails to operate when it is called for operation, the failure of switch is called as dynamic failure. This failure may occur due to jamming of contacts or failure of the switching mechanism. For analysis, it is assumed that probability of such type of failure is q_s.

4. Gradual failure of the switch occurs during the operating time of the active component. Let the time upto such failure be distributed exponentially with parameter β_j.

For the analysis, it is assumed that these failures are statistically independent. Consider the j^{th} stage of the system. Let the state i ($i=1,2,...n_j$) denote that i^{th} component is in operating condition and switches are working properly. Let state n_j+i $(i=1,2,...n_j)$ denote that i^{th} component is working but switch is not working properly and $(2n_j+1)^{th}$ state denotes that j^{th} stage failed. The probability that the j stage changes from i^{th} state to $i+1^{th}$ state during an infinitesimal time Δt can be given by

$$P_r\left\{i \longrightarrow i+1, \Delta t\right\} = \begin{cases} \left[\lambda_j+(n_j-i)\gamma_j+\alpha_j\right] \bar{q}_s \Delta t + O(\Delta t) & 1 \leq i \leq n_j \\ \beta_j \Delta t + O(\Delta t) & n_j < i \leq 2n_j \end{cases}$$

$$P_r\left\{i \longrightarrow i\right\} = \begin{cases} \left[\lambda_j+(n_j-i)\gamma_j+\alpha_j+\beta_j\right] \Delta t+O(\Delta t) & 1 \leq i \leq 2n_j \\ \lambda_j \Delta t + O(\Delta t) & n_j < i \leq 2n_j \end{cases} \quad (2.21)$$

where $\bar{q}_s = 1-q_s$

It is assumed that probability of more than one transition is zero. The differential equations describing the behaviour of the j^{th} stage can bewritten as

$$p_1'(t) = -\left[\lambda_j+(n_j-1)\gamma_j+\alpha_j+\beta_j\right]p_1(t) \quad (2.22)$$

$$p_i'(t) = -\left[\lambda_j+(n_j-i)\gamma_j+\alpha_j+\beta_j\right]p_i(t) + \left[\lambda_j+(n_j-i+1)\gamma_j+\alpha_j\right]\bar{q}_s\, p_{i-1}(t)$$

$$i=2,\ldots n_j \tag{2.23}$$

$$p_{n_j+i}'(t) = \beta_j p_i(t)-\lambda_j p_{n_j+i}(t)$$

$$i=1,2,\ldots n_j \tag{2.24}$$

with initial conditions as

$$p_i(0) = \begin{cases} 1 & \text{for } i=1 \\ 0 & \text{for } 1 < i \leq 2n_j \end{cases} \tag{2.25}$$

Taking Laplace transform of both sides of above equations and solving for $p_i(S)$, it will result in

$$p_1(S) = \frac{1}{\left[S+\lambda_j+(n_j-1)\gamma_j+\alpha_j+\beta_j\right]} \tag{2.26}$$

$$p_i(S) = \frac{\prod_{l=1}^{i-1}\left[\lambda_j+(n_j-l+1)\gamma_j+\alpha_j\right]\bar{q}_s}{\prod_{l=1}^{i}\left[S+\lambda_j+(n_j-l)\gamma_j+\alpha_j+\beta_j\right]}$$

$$i=2,3,\ldots n_j \tag{2.27}$$

$$p_{n_j+i}(S) = \frac{\beta_j \prod_{l=1}^{i-1}\left[\lambda_j+(n_j-l+1)\gamma_j+\alpha_j\right]\bar{q}_s}{(S+\lambda_j)\prod_{l=1}^{i}\left[S+\lambda_j+(n_j-l)\gamma_j+\alpha_j+\beta_j\right]}$$

$$i=1,2,\ldots n_j \tag{2.28}$$

The probability that j^{th} stage will be in state i at time t can be calculated by taking Laplace transform of (2.26), (2.27) and (2.28), which results in

$$p_1(t) = \exp\left[-\left\{\lambda_j+(n_j-1)\,\gamma_j+\alpha_j+\beta_j\right\}t\right] \qquad (2.29)$$

$$p_i(t) = \bar{q}_s^{\,i-1}\left[\prod_{l=1}^{i-1}\left\{\lambda_j+(n_j-l+1)\,\gamma_j+\alpha_j\right\}\right] .$$

$$\left[\sum_{\omega=1}^{i}\frac{\exp[-\{\lambda_j+(n_j-\omega)\gamma_j+\alpha_j+\beta_j\}\,t]}{[\,\gamma_j^{i-1}\prod_{l=1}^{i}(\omega-l)]}\right]$$

$$i=2,3,\ldots,n_j \qquad (2.30)$$

$$p_{n_j+i}(t)=\beta_j\bar{q}_s^{\,i-1}\left[\prod_{l=1}^{i-1}[\lambda_j+(n_j-l+1)\,\gamma_j+\alpha_j]\right] .$$

$$\left[\frac{\exp(-\lambda_j t)}{\prod_{l=1}^{i}\left\{(n_j-l)\,\gamma_j+\alpha_j+\beta_j\right\}} + \sum_{\omega=1}^{i}\frac{\exp[-\,\lambda_j+(n_j-\omega)\gamma_j+\alpha_j+\beta_j\;t]}{\gamma_j^{i-1}\prod_{\substack{l=1\\ l\neq\omega}}^{i}[\omega-l].[-\{(n_j-\omega)\gamma_j+\alpha_j+\beta_j\}]}\right]$$

$$i = 1,2,\ldots,n_j \qquad (2.31)$$

The j^{th} stage reliability can be given by

$$R_j(n_j) = \sum_{i=1}^{2n_j} P_i(t) \qquad (2.32)$$

Therefore the stage reliability is a function of failure rate of the component with power on and power off, and switch reliability. A family of curves are plotted between stage reliability and component reliability with different values of α_j , β_j , and q_s as shown in Fig. 2.14.

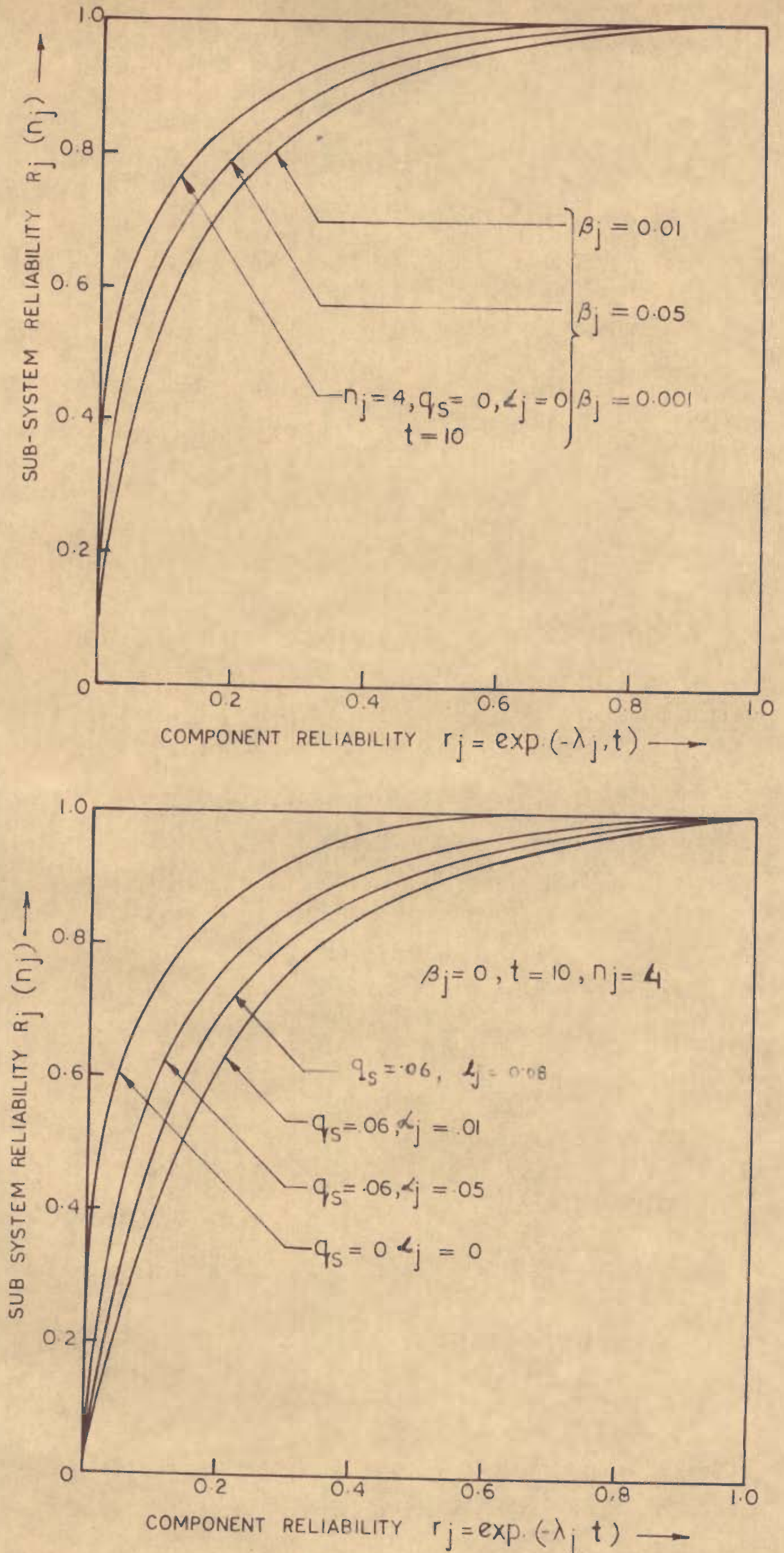


FIG.2.14 PLOT OF SUB-SYSTEM RELIABILITY VS COMPONENT RELIABILITY FOR STAND BY REDUNDANCY FOR THREE TYPES OF SWITCH FAILURES

2.5. HYBRID REDUNDANCY

Hybrid redundancy consists of the combination of an NMR with majority voting and standby redundancy. This type of redundancy is superior than NMR due to higher improvement of system reliability, specially when component reliability is very low and provides larger mean life than the nonredundant system. The operation of the hybrid redundant system (N, m) as shown in Fig. 2.15, can be explained as follows. This hybrid system has NMR core with 2 spare components. When any component in the NMR core fails, it is detected by disagreement detector by comparing each input to the voter with its output. The failed component is then disconnected from NMR core by a switching device and a spare component is switched in, if available, thus restoring the NMR in the system. When all spare components are exhausted, the hybrid system operates as a NMR system.

For analysis purpose, consider a j^{th} subsystem or stage having N_j ($N_j=2n_j+1$, where $n_j=1, 2, \ldots$) fold active redundant components forming the NMR system and m_j spare components. Any component in the subsystem may be either operating or waiting in spare storage. The subsystem fails when (m_j+n_j+1) components fail. Therefore, the possible states of the subsystem are 0, 1, 2, ..(m_j+n_j+1). Assuming that the active and standby components have constant failure rate as λ_j and γ_j respectively; during a small interval of time Δt, the subsystem state changes from state i to i+1. The probability of transition from state i to i+1 can be given by

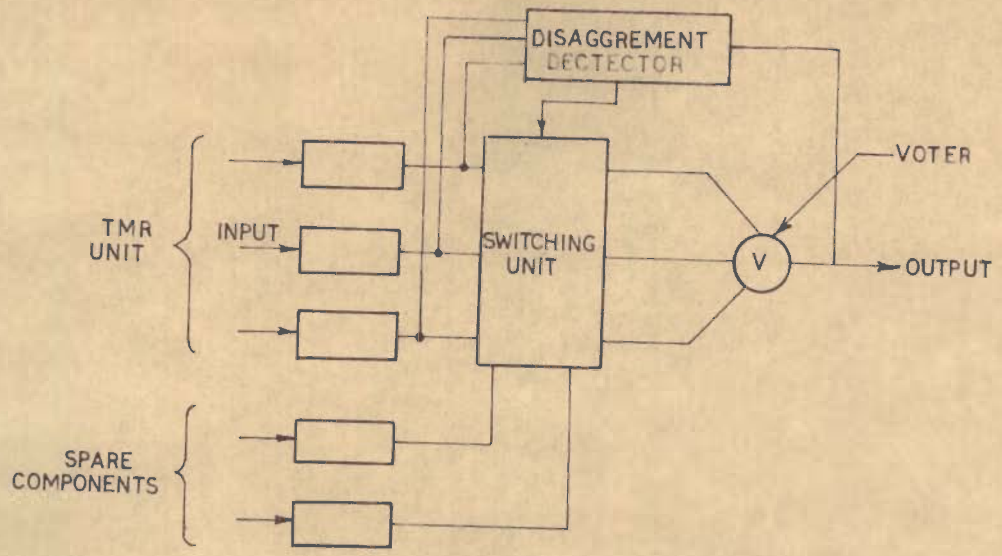


FIG. 2.15 HYBRID REDUNDANT SYSTEM

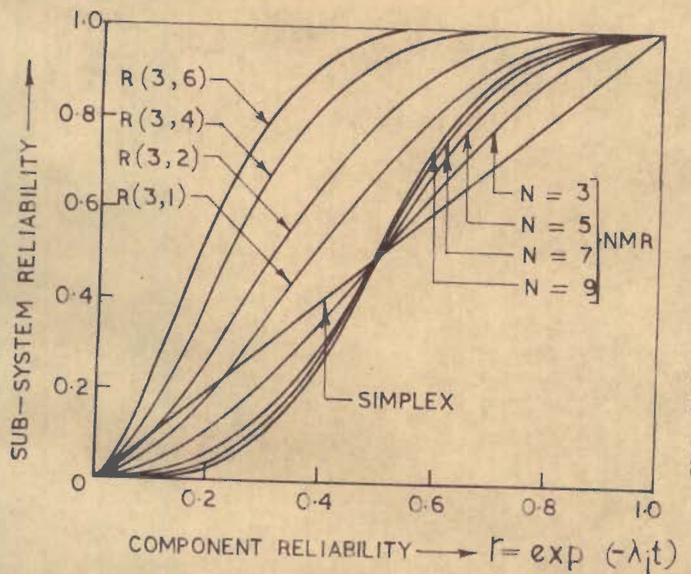


FIG. 2.16 a RELIABILITY FUNCTION OF HYBRID AND NMR SYSTEM VS COMPONENT RELIABILITY ($\lambda/\gamma = 1, q_s = 0$)

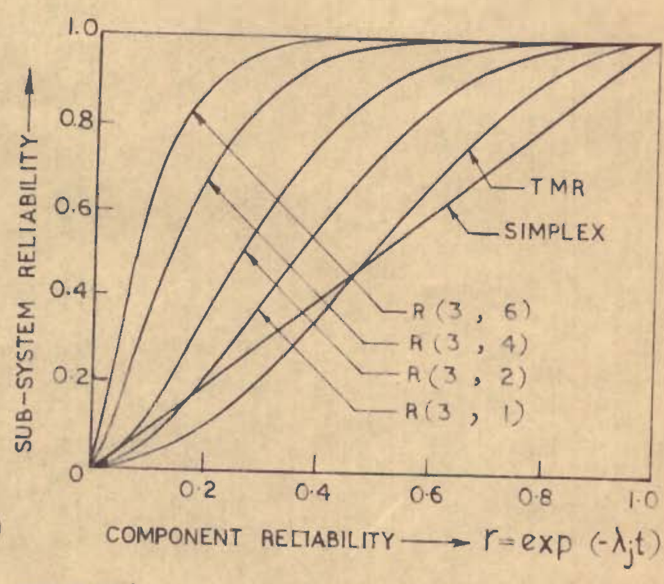


FIG. 2.16 b RELIABILITY FUNCTION OF HYBRID REDUNDANT SYSTEM VS COMPONENT RELIABILITY ($\lambda/\gamma = 10, q_s = 0$)

$$
P_r\left\{i \longrightarrow i+1,\ \Delta t\right\} = \begin{cases} \left[N_j\lambda_j+(m_j-1)\gamma_j\right]\Delta t+O(\Delta t) & 0 \le i \le m_j \\ \binom{N_j+m_j-i}{1}\lambda_j\Delta t(1-\Delta_j\Delta t)^{N_j+m_j-1-i} + O(\Delta t) & \\ \approx \left[N_j+m_j-i\right]\lambda_j\Delta t & m_j < i \le n_j+m_j \\ 0 \quad \text{otherwise} & \end{cases} \tag{2.33}
$$

For simplicity defining a new variable $\bar{\lambda}_i$ as,

$$
\bar{\lambda}_i = \begin{cases} \left[N_j\lambda_j+(m_j-i)\gamma_j\right] & 0 \le i \le m_j \\ \left[N_j+m_j-i\right]\lambda_j & m_j < i \le n_j+m_j \\ 0 \quad \text{otherwise} & \end{cases} \tag{2.34}
$$

The probability of more than one transition in infinitesimal interval Δt can be neglected as it is very small. If $p_i(t)$ is the probability of the system being in state i at time t, the differential equations characterising the state of the system can be written as

$$
p_i'(t) = -\bar{\lambda}_i p_i(t)+\bar{\lambda}_{i-1}\bar{q}_s p_{i-1}(t) \qquad i = 0,1,\ldots n_j+m_j \tag{2.35}
$$

The initial conditions of the system are $p_o(0)=1$ and $p_i(0)=0$ for $i\neq 0$. Taking Laplace transform of (2.35) it results in

$$
p_i(S) = \frac{\bar{\lambda}_{i-1}\bar{q}_s}{(S+\bar{\lambda}_i)}\, p_{i-1}(S) \tag{2.36}
$$

Since

$$p_o(S) = \frac{1}{S+\bar{\lambda}_o} \tag{2.37}$$

expression (2.36) can be written as

$$p_i(S) = \frac{\bar{q}_s^{\,i-1} \prod_{l=0}^{i-1} \bar{\lambda}_l}{\prod_{l=0}^{i} (S+\bar{\lambda}_l)} \tag{2.38}$$

Taking Laplace inverse of (2.38), we have

$$p_i(t) = \left[\bar{q}_s^{\,i-1} \prod_{l=0}^{i-1} \bar{\lambda}_l \right] \sum_{l=0}^{i} \frac{\exp(-\bar{\lambda}_l t)}{\prod_{\substack{\omega=0 \\ \omega\neq l}}^{i} (\bar{\lambda}_\omega - \bar{\lambda}_l)} \tag{2.39}$$

$$i=1\ldots m_j+n_j$$

Therefore, the j^{th} subsystem reliability can be given by

$$R_j(n_j+m_j) = R_v \sum_{i=0}^{n_j+m_j} \left[\prod_{l=0}^{i-1} \bar{\lambda}_l \right] \left[\sum_{l=0}^{i} \frac{\exp(-\bar{\lambda}_l t)}{\prod_{\substack{\omega=0 \\ \omega\neq l}}^{i} (\bar{\lambda}_\omega - \bar{\lambda}_l)} \right] \tag{2.40}$$

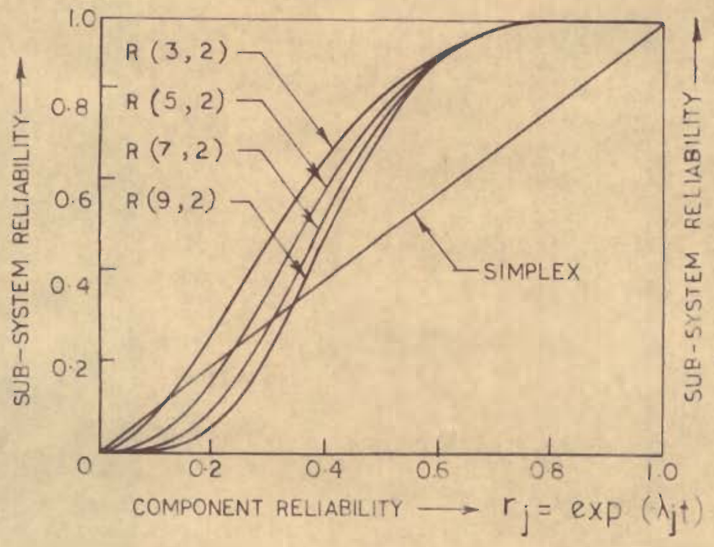
if $m_j \geq 1$

and if $m_j=0$

$$R_j(n_j\ ,\ 0) = R_v \sum_{i=0}^{n_j} \binom{N_j}{i} \left[\exp(-\lambda_j t)\right] \left[1-\exp(-\lambda_j t)\right]^i \tag{2.41}$$

where R_v is the voter reliability

The reliability expression for NMR and Hybrid (3, m) is plotted in Fig. 2.16a which shows that when component reliability is less than 0.5, the use of NMR decreases the subsystem reliability. The larger value of N_j further makes the



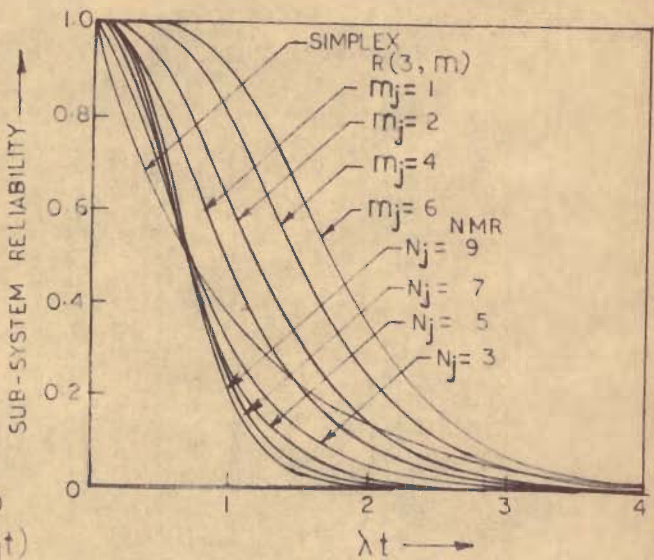
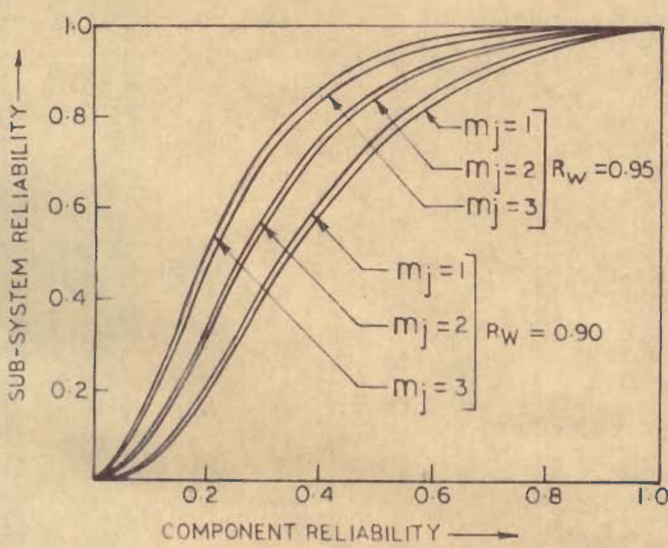


FIG.2.16c RELIABILITY FUNCTION OF R(N, m) SYSTEM VS COMPONENT RELIABILITY r_j

FIG 2.16d RELIABILITY COMPARISION OF A R(3 m) AND NMR SYSTEM VS NORMALIZED TIME λt



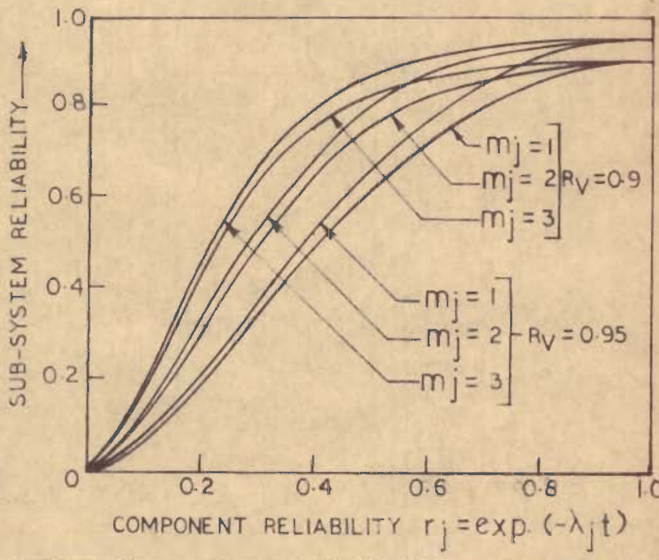


FIG.2.16e RELIABILITY FUNCTION OF R(3, m) SYSTEM ($\lambda/\gamma = 10$, $R_W = 1.0$)

FIG.2.16f RELIABILITY FUNCTION OF R(3, m) SYSTEM ($\lambda/\gamma = 10$, $R_V = 1.0$)

subsystem worst, while use of hybrid redundancy results in appreciable shift of the well-known cross-over point as indicated in Fig. 2.16b. The shift of the cross-over point is effected by the ratio of γ_j/λ_j, N_j and m_j. As shownin Fig. 2.16d, for $m_j=1$, increase in N_j will improve the subsystem reliability only when $0.58 < r_j < 1$. Even in these ranges, larger value of N_j do not provide significant increase in the subsystem reliability. Therefore, N_j is kept as 3 and m_j is varied. The plot of reliability function when m_j isa variable is shown in Fig. 2.16a which shows that any desired level of system reliability can be achieved by increasing m_j. The effect of switch failure is to reduce system reliability as shown in Fig. 2.16c.

2.6. STANDBY REDUNDANCY WITH REPAIR FACILITIES

When a fault in a system is nonrecoverable the failed equipment is disconnected from the system and repair is performed. It may be possible that at a time more than one component can fail simultaneously. This requires more than one repair crew in order to increase the operating time of the system. But in case of irredundant system, repair will not help in the sense of increasing the system reliability. It can be enhanced by providing spare components. The behaviour of such type of system can be explained as follows. Initially, one component is kept in operation and others are kept as spare. When a component fails, it is replaced by a spare component and the failed component is sent for repair. When repaired, it is kept as a standby component.

Consider a j^{th} subsystem having N_j (N_j=1, 2, ..) components connected in series and m_j components as spare. Let the repair time have exponential distribution with parameter as μ_j and replacement time is very small and can be neglected. Any component in the system may be in one of the states -

a) operating in the subsystem

b) waiting in the standby

c) waiting for or receiving the repair facilities

The transition diagram for system having two spare components is shown in Fig. 2.17. The probability of transition during infinitesimal time interval (Δt) can be expressed as

$$p_r \left\{ i \longrightarrow i+1 \right\} = \bar{\lambda}_i (\Delta t) + o(\Delta t)$$

$$p_r \left\{ i \longrightarrow i-1 \right\} = \bar{\mu}_i (\Delta t) + o(\Delta t) \qquad (2.42)$$

where $\bar{\lambda}_i$ and $\bar{\mu}_i$ can be defined as

$$\bar{\lambda}_i = \begin{cases} N_j \lambda_j & \text{if } 0 \leq i \leq m_j \\ 0 & \text{otherwise} \end{cases} \qquad (2.43)$$

and

$$\bar{\mu}_i = \begin{cases} i\mu_j & \text{if } 1 \leq i \leq r_{cj} \\ r_{cj}\mu_j & \text{if } r_{cj} \leq i \leq m_j \\ 0 & \text{otherwise} \end{cases} \qquad (2.44)$$

where r_{cj} is the number of repair crew for j^{th} stage.

If $p_1(t)$ is the probability of the system being in state i at time t, the probability that system will be in state i after time $t+\Delta t$ will be

$$p(t+\Delta t) = (1-\bar{\lambda}_i\Delta t)(1-\bar{\mu}_i\Delta t)p_i(t)+\bar{\lambda}_{i-1}\Delta t(1-\bar{\mu}_{i-1}\Delta t)p_{i-1}(t)$$

$$+(1-\bar{\lambda}_{i+1}\Delta t)\bar{\mu}_{i+1}\Delta t p_{i+1}(t)+o(\Delta t) \qquad (2.45)$$

Therefore,

$$p_i'(t) = (\bar{\lambda}_i+\bar{\mu}_i)p_i(t)+\bar{\lambda}_{i-1}p_{i-1}(t)+\bar{\mu}_{i+1}p_{i+1}(t) \qquad (2.46)$$

with initial conditions as

$$p_i(0) = 0 \qquad \text{for } 0 < i$$

and $\quad p_o(0) = 1$

From (2.46), the set of differential equations describing the behaviour of the subsystem can be described as

$$p_o'(t) = -N_j\lambda_j p_o(t) + \mu_j p_1(t) \qquad (2.47)$$

$$p_i'(t) = -(N_j\lambda_j+i\mu_j)p_i(t)+N_j\lambda_j p_{i-1}(t)+(i+1)\mu_j p_{i+1}(t)$$

$$1 \leq i \leq r_{cj} \qquad (2.49)$$

$$p_i'(t) = -(N_j\lambda_j+r_{cj}\mu_j)p_i(t)+N_j\lambda_j p_{i-1}(t)+r_{cj}\mu_j p_{i+1}(t)$$

$$r_{cj} < i \leq m_j \qquad (2.49)$$

The steady state reliability of the subsystem can be found by setting up lefthand side of (2.47), (2.48) and (2.49) to zero and solving for $p_i(t \longrightarrow \infty)$

$$p_i = \frac{1}{i\,!}\left(\frac{N_j\lambda_j}{\mu_j}\right)^i p_o \qquad i \leq i \leq r_{cj} \qquad (2.50)$$

$$p_i = \left[\frac{1}{r_{cj}\,!}\left(\frac{N_j\lambda_j}{\mu_j}\right)^{r_{cj}}\left(\frac{N_j\lambda_j}{r_{cj}\mu_j}\right)^{i-r_{cj}}\right]p_o \qquad (2.51)$$

Since $\sum_{i=1}^{m_j+1} p_i=1$, the probability that system will be

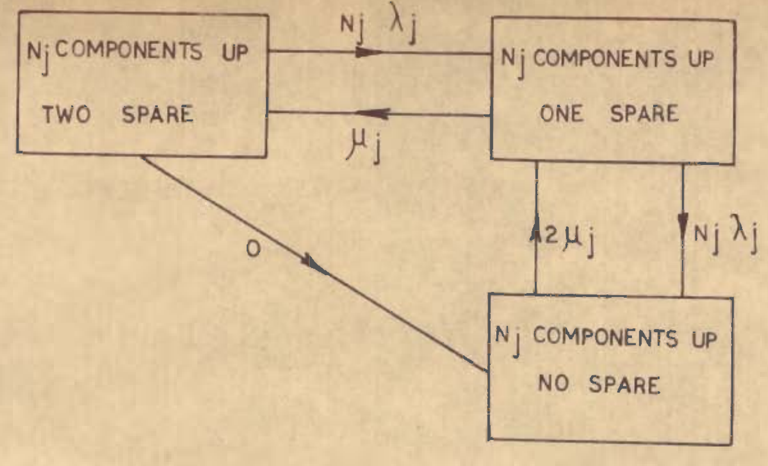


FIG. 2.17 TRANSITION DIAGRAM FOR A SYSTEM HAVING N_j COMPONENTS WITH TWO SPARE

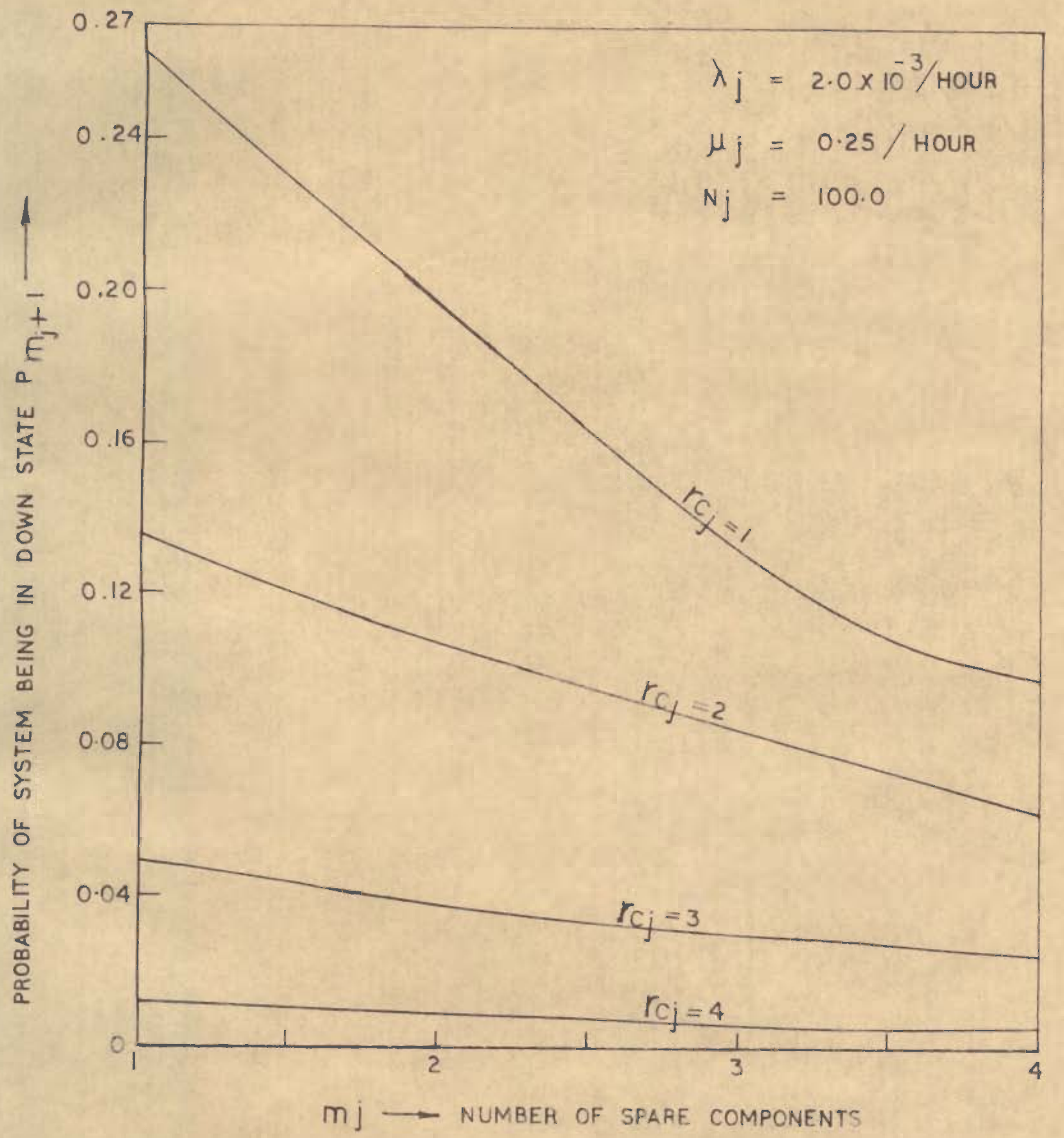


FIG. 2.18 RELIABILITY EXPRESSION FOR SYSTEM WITH m_j SPARE AND r_{cj} REPAIR CREWS

in zero state can be written as

$$P_o = \left[1 + \sum_{i=1}^{r_{cj}} \frac{1}{i!} \left(\frac{N_j \lambda_j}{\mu_j} \right)^i + \sum_{i=r_{cj}+1}^{m_j+1} \frac{1}{i!} \left(\frac{N_j \lambda_j}{\mu_j} \right)^{r_{cj}} \left(\frac{N_j \lambda_j}{r_c \mu_j} \right)^{i - r_{cj}} \right]^{-1} \tag{2.52}$$

and the probability that the system will be in down state p_{m_j+1} can be given by

$$p_{m_j+1} = \frac{1}{r_{cj}!} \left(\frac{N_j \lambda_j}{\mu_j} \right)^{r_{cj}} \left(\frac{N_j \lambda_j}{r_{cj} \mu_j} \right)^{m_j - r_{cj} + 1} \tag{2.53}$$

The reliability of the subsystem can be written as

$$R_j(r_{cj} m_j) = 1 - p_{m_j+1} \tag{2.54}$$

The expression (2.54) is plotted between probability of subsystem being in down state, number of repair crews and standby by component as shown in Fig. 2.18.

2.7. EQUIVALENT LINEAR PROBLEM

From the reliability expressions and (2.2) and (2.3), it is clear that reliability problem is a nonlinear integer programming problem having integer variables. It can be linearized by using bi-valent variables. Taking logarithm of (2.2), it will result in a separable function as

$$\text{Ln } R_s(n) = \sum_{j=1}^{k} \text{Ln } R_j (n_j) \tag{2.55}$$

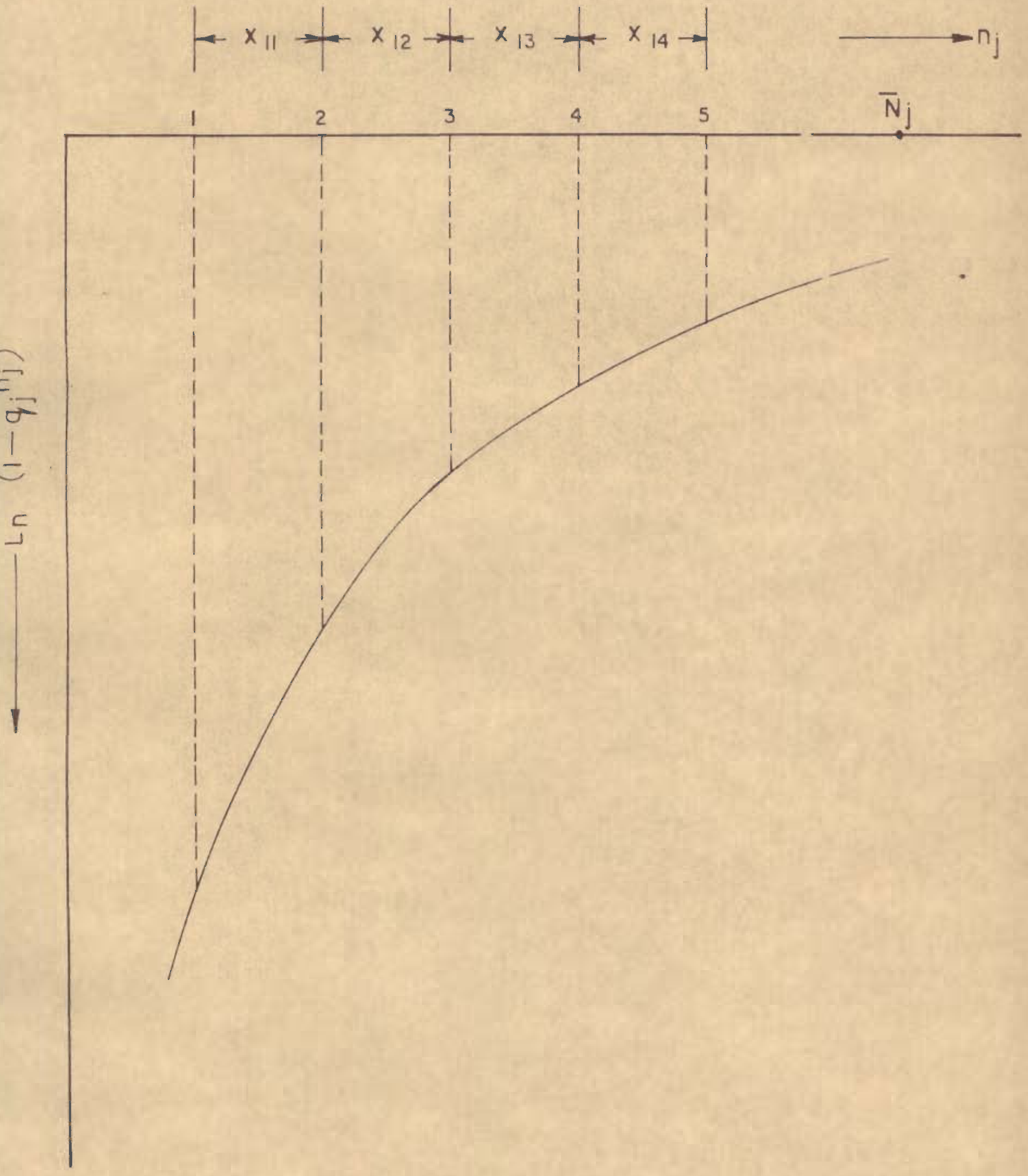


FIG. 2 19 PLOT OF RELIABILITY FUNCTION FOR j th STAGE

The separable reliability expression can be linearized by approximating it to be astraight line between two values of n_j as shown in Fig. 2.19. Let $x_{jl}=1$, $l=1,2,\ldots\bar{N}_j$ be the increment in variable n_j between interval l+1 and l, the linearized reliability expressionfor j^{th} stage can be written as

$$\text{Ln } R_j(n_j) = \sum_{l=1}^{\bar{N}_j} c_{jl}\, x_{jl} + \text{Ln } R_j(1) \tag{2.56}$$

where c_{jl} is the slope of the l^{th} segment and can be given by

$$c_{jl} = \text{Ln}\left[\frac{R_j(l+1)}{R_j(l)}\right] \tag{2.57}$$

$\bar{N}_j$ is the upper bound on n_j. Assuming that the constraints on the system are linear, $\bar{N}_j$ can be calculated from constraints set as

$$\bar{N}_j = \min_i \left[\frac{b_i}{a_{ij}}\right] \qquad i=1,2,\ldots s \tag{2.58}$$

where a_{ij} and b_i are the resources requirements associated with each component of j^{th} stage and the total amount of resources availability for the i^{th} type of constraint, respectively.

Similarly, with some manipulations constraints can also be written in terms of x_{jl} variable. Therefore, the nonlinear reliability problem is transferred into an equivalent linear problem with x_{jl} variables as

Maximize

$$\phi(x) = \sum_{j=1}^{k} \sum_{l=1}^{\bar{N}_j} c_{jl} \, x_{jl} + \sum_{j=1}^{k} \text{Ln } R_j(1) \qquad (2.59)$$

subject to the constraints

$$\sum_{j=1}^{k} \sum_{l=1}^{\bar{N}_j} a_{ij} \, x_{jl} \leq b'_i \qquad i=1,2,\ldots,s \qquad (2.60)$$

$$x_{jl} = 0 \text{ or } 1$$

where,

$$b'_i = b_i - \sum_{j=1}^{k} a_{ij} \qquad (2.61)$$

As the objective function is concave and monotone increasing (except mixed redundancy), it ensures that $c_{j1} > c_{j2} > c_{j\bar{N}_j}$, which indicates that for $p > 1$, the variable x_{jp} can be one only when $x_{j1}=1$, i.e. the variable x_{j1} will always enter the solution before x_{jp}. Therefore, the linearized reliability problem has same optimal solution as original one.

The optimal solution of the original problem defined by (2.2) and (2.3) can be obtained from the optimal solution of the above equivalent problem. Let X be a feasible solution to problem given by (2.59) and (2.60), then

$$\sum_{j=1}^{k} \sum_{l=1}^{\bar{N}_j} a_{ij} \, x_{jl} = b'_i$$

$$\sum_{j=1}^{k} a_{ij} \sum_{l=1}^{\bar{N}_j} x_{jl} = b_i - \sum_{j=1}^{k} a_{ij}$$

$$\sum_{j=1}^{k} a_{ij} \left\{ 1 + \sum_{l=1}^{\bar{N}_j} x_{jl} \right\} = b_i$$

hence,

$$n_j = \sum_{l=1}^{\bar{N}_j} x_{jl} \tag{2.62}$$

This maximization problem can be converted into a minimization problem by replacing x_{jl} by $(1-\bar{x}_{jl})$. The resulting problem can be stated as

Maximize $\psi(\bar{x})$

$$\psi(\bar{x}) = - \sum_{j=1}^{k} \sum_{l=1}^{\bar{N}_j} c_{jl} \bar{x}_{jl} + g_o \tag{2.63}$$

subject to the constraints

$$\sum_{j=1}^{k} \sum_{l=1}^{\bar{N}_j} a_{ij} \bar{x}_{jl} > e_i \tag{2.64}$$

where,

$$e_i = \sum_{j=1}^{k} a_{ij} \bar{N}_j - b_i' \tag{2.65}$$

and

$$g_o = \sum_{j=1}^{k} \sum_{l=1}^{\bar{N}_j} c_{jl} + \sum_{j=1}^{k} \ln R_j(1) \tag{2.66}$$

Since term g_o is constant, less than or equal to zero, maximization of the function $\psi(\bar{x})$ is same as minimization of $\sum_{j=1}^{k} \sum_{l=1}^{\bar{N}_j} c_{jl} \bar{x}_{jl}$. The equivalent reliability problem can be rewritten as

Minimize $F(\bar{x})$

$$F(\bar{x}) = \sum_{j=1}^{k} \sum_{l=1}^{\bar{N}_j} c_{jl} \bar{x}_{jl} \tag{2.67}$$

subject to the constraints

$$\sum_{j=1}^{k} \sum_{l=1}^{\bar{N}_j} a_{ij} \bar{x}_{jl} \geq e_i \tag{2.68}$$

$$i = 1, 2, \ldots s$$

An optimal solution to a problem defined by (2.2) and (2.3) can be obtained from the optimum solution of the equivalent minimization problem with the help of the following relation

$$n_j = 1 + \sum_{l=1}^{\bar{N}_j} (1-\bar{x}_{jl}) \tag{2.69}$$

For easy handling of the problem, the above formulation can be expressed in terms of single subscripted variables, viz.

Minimize F(z)

$$F(z) = \sum_{j=1}^{w} g_j z_j \tag{2.70}$$

subjeç to

$$\sum_{j=1}^{w} h_{ij} z_j \geq e_i \tag{2.71}$$

$$i = 1, 2, \ldots s$$

$$z_j = 0 \text{ or } 1$$

where,

$$w = \sum_{j=1}^{k} \bar{N}_j \tag{2.72}$$

g_j and h_{ij} are related to c_{jl} and a_{ij} respectively by the following relations:

$$\begin{cases} c_{11} & \left[\begin{array}{ll} \text{for} & j=1, 2, \ldots \bar{N}_1 \\ & l=1, 2, \ldots \bar{N}_1 \end{array} \right. \end{cases}$$

$$g_j = \begin{cases} c_{21} & \left[\begin{array}{l} \text{for } j=(\bar{N}_1+1)\ldots(\bar{N}_1+\bar{N}_2) \\ l=1,2,\ldots,\bar{N}_2 \end{array}\right. \\ \vdots & \\ c_{kl} & \text{for } j=\sum_{p=1}^{k-1} \bar{N}_p+1,\ldots,w \end{cases} \tag{2.73}$$

and

$$h_{ij} = \begin{cases} a_{i1} & \text{for } j=1,\ldots,\bar{N}_1 \\ a_{i2} & \text{for } j=\bar{N}_1+1,\ldots,(\bar{N}_1+\bar{N}_2) \\ \vdots & \\ a_{ik} & \text{for } j=\sum_{p=1}^{k-1} \bar{N}_p+1,\ldots,w \end{cases} \tag{2.74}$$

If constraints on the system are nonlinear, they can be linearized in the same way as the objective function.

Chapter 3

TECHNIQUES OF RELIABILITY OPTIMIZATON

In the previous chapter, reliability problem has been formulated as a nonlinear integer programming and linear integer programming problem. This problem can also be solved by assuming n_j to be a continuous variable and thereby the solution obtained will be an approximate one. In this chapter, methods are given for solving this problem by using three types of formulations:

1. Nonlinear programming formulation assuming n_j to be continuous variables
2. Linear integer programming formulation
3. Nonlinear integer programming formulation

3.1. GEOMETRIC PROGRAMMING FORMULATION

A new formulation for the problem of system reliability optimization when constrained by some linear constraints is presented. This formulation is applicable to the systems in which the active parallel redundancy can be used for enhancing the system reliability. The formulation provided is easily adaptable to Geometric Programming form. The problem is further reduced to that of an optimization of an unconstrained objective function with variables one less than the number of constraints, when its dual is defined.

Reliability optimization problem of a system using parallel redundancy can be expressed as (2.2, 2.3, 2.4)

$$R_s(n) = \prod_{j=1}^{k} (1-q_j^{n_j}) \tag{3.1}$$

subject to the constraints

$$\sum_{j=1}^{k} a_{ij}\, n_j \leq b_i \qquad i=1,2,\ldots,s \tag{3.2}$$

Since $q_j << 1$, the expression (3.1) for unreliability of the system can be approximated as the sum of the unreliabilities of the stages. Therefore, the reliability problem can be reformulated as

Problem 1

Minimize the system unreliability

$$\phi(n) = \sum_{j=1}^{k} q_j^{n_j} \tag{3.3}$$

subject to the constraints given by (3.2).

To obtain the geometric programming formulation of the reliability problem we define Ω_j in terms of n_j as

$$n_j = \frac{\operatorname{Ln} \Omega_j}{\operatorname{Ln} q_j} \qquad j=1,2,\ldots,k \tag{3.4}$$

Substituting Ω_j in (3.3) and by exponentiating (3.2), the geometric programming formulation of (3.3) and (3.2)

Problem 2

Minimize

$$\sum_{j=1}^{k} \Omega_j \tag{3.5}$$

subject to the constraints

$$\exp(-1) \prod_{j=1}^{k} Q_j^{y_{ij}} \qquad (3.6)$$

where,

$$y_{ij} = \frac{a_{ij}}{b_i . \ln q_j} \qquad \begin{matrix} i=1,2,\ldots,s \\ j=1,2,\ldots,k \end{matrix} \qquad (3.7)$$

Assuming n_j to be continuous variables, the dual geometric programming formulation of problem 2 is

<u>Dual problem</u>

Maximize

$$\prod_{j=1}^{k} \left(\frac{1}{\delta_{s+j}}\right)^{\delta_{s+j}} \prod_{i=1}^{s} \left[\frac{\exp(-1)}{\delta_i}\right]^{\delta_i} \prod_{i=1}^{s} \left[\delta_i\right]^{\delta_i} \qquad (3.8)$$

subject to

$$\sum_{j=1}^{k} \delta_{s+j} = 1$$

and

$$\delta_{s+j} - \sum_{i=1}^{s} y_{ij}\, \delta_i = 0 \qquad j=1,2,\ldots,k \qquad (3.9)$$

$$\delta_i \geq 0 \quad i=1,2,\ldots,s+k$$

where δ_i [i=1,2,...,s] are the dual variables corresponding to (3.6) and δ_{s+j} [j=1,2,...,k] are the dual variables corresponding to (3.5).

Expressing

$$e_i = \sum_{j=1}^{k} y_{ij} \qquad (3.10)$$

$$\left.\begin{aligned} z_j &= y_{ij}/e_s \\ \text{and } L_{ij} &= y_{ij} - z_j e_i \quad \begin{matrix} i=1,2,\ldots,s \\ j=1,2,\ldots,k \end{matrix} \end{aligned}\right] \qquad (3.11)$$

Substituting the above-defined constraints in (3.8) and (3.9), and taking the logarithm of them, we have an unconstrained problem of S-1 variables, which is

Minimize

$$\sum_{i=1}^{s-1} \delta_i \left[(1-\frac{e_i}{e_s}) + \sum_{j=1}^{k} L_{ij}\, Ln(z_j + \sum_{l=1}^{s-1} L_{lj}\delta_l) \right]$$

$$+ \sum_{j=1}^{k} z_j\, Ln(z_j + \sum_{l=1}^{s-1} L_{lj}\,\delta_l) + \frac{1}{e_s} \qquad (3.12)$$

Differentiating (3.12) with respect to δ_i $|i=1,2,\ldots,(s-1)|$ and equating to zero, we have

$$(1-\frac{e_i}{e_s}) + \sum_{j=1}^{k} L_{ij}\, Ln(z_j + \sum_{l=1}^{s-1} L_{lj}\,\delta_l) +$$

$$\sum_{\omega=1}^{s-1} \delta_\omega \left[\sum_{j=1}^{k} \frac{L_{\omega j}\, L_{ij}}{z_j + \sum_{l=1}^{s-1} L_{lj}\,\delta_l} \right]$$

$$+ \sum_{j=1}^{k} \frac{z_j\, L_{ij}}{z_j + \sum_{l=1}^{s-1} L_{lj}\,\delta_l} = 0$$

$$i=1,2,\ldots,(s-1) \qquad (3.13)$$

(3.13) is a system of (s-1) nonlinear equations which can be solved by Newton's method or by the subrelaxation method.

Let us consider the case when there is only one linear constraint on the system, i.e. s=1 or a constraints set in which the active constraint is known. From (3.9) and (3.10) we have

$$\delta_1 = 1/\sum_{j=1}^{k} y_{1j} = \frac{1}{\epsilon_1}$$

and $\delta_{j+1} = y_{1j}/\epsilon_1$

Using primal and dual relationship of Geometric Programming and (3.4), we have

$$n_j = \left[\sum_{i=1}^{k} y_{1i} \, Ln(\epsilon_1/y_{1i}) \quad \epsilon_1 - \frac{1}{\epsilon_1} + Ln(y_{1j}/\epsilon_1)\right] \quad Lnq_j \tag{3.15}$$

$$j=1,2,\ldots,k$$

By substituting various constants defined by (3.7) and (3.10), n_j , the optimum number of parallel components in each stage can be calculated, and the optimum reliability of the system can be given by R_s

$$R_s = 1 - \sum_{j=1}^{k} \exp\left[\sum_{i=1}^{k} y_{1i} \, Ln(\epsilon_1/y_{1i}) \quad \epsilon_1 + Ln(y_{1j}/\epsilon_1) - 1/\epsilon_1\right] \tag{3.16}$$

The above expression does not give the exact system reliability due to the assumption made in deriving expression (3.3). This gives 0.09% error in the calculation, which is very small. From (3.16), the expression for reliability in terms of resources allocated can be derived which may be very useful to the system designer.

Numerical Example

A system consists of four stages, each having reliability, cost and weight as tabulted below (Table 3.1). It is required to find the optimum number of redundant components so that the system reliability is maximized with cost and weight constraints as 56 and 30 units, respectively.

Table 3.1

Stage number	Reliability	Cost	Weight
1	0.80	1.2	1.0
2	0.70	2.3	1.0
3	0.75	3.4	1.0
4	0.85	4.5	1.0

The above data were substituted in (3.15) by making use of (3.14) and (3.7). The results obtained are -

$n_1 = 4.8997$

$n_2 = 6.4941$

$n_3 = 5.2417$

$n_4 = 3.9415$

Rounding off to the nearest integer value we get an optimal allocation as given in Table 3.2.

Table 3.2

Stage number	Number of parallel components
1	5
2	6
3	5
4	4

System Reliability = 0.99809 [From (3.10)]

Actual System Reliability = 0.99713

The proposed approach of solving the reliability problem is practical method due to its simplicity and less algebraic calculation. The problem with nonlinear constraints can also be tackled by this approach after transforming nonlinear constraints into a posynomial form as required by Geometric Programming. Computation time depends only on the number of constraints on the system.

3.2. PENALTY FUNCTION METHOD

A method is developed in this section in which the use of penalty function is made to convert the constrained reliability problem into an equivalent unconstrained problem. The latter is solved by the steepest ascent method by assuming n_j to be continuous variable.

The reliability problem can be written as

Minimize

$$-\text{Ln } R_s(n) = -\sum_{j=1}^{k} \text{Ln } R_j(n_j) \tag{3.17}$$

subject to the constraints

$$\sum_{j=1}^{k} G_{ij}(n_j) \leq b_i \qquad i=1,2,\ldots,s \tag{3.18}$$

$n_j > 0$ and integer

The equivalent unconstrained problem can be written as

Minimize

$$f(n, r) = -\text{Ln } R_s(n) + r_p \sum_{i=1}^{s} \left[b_i - \sum_{j=1}^{k} G_{ij}(n_j) \right]^{-1} \tag{3.19}$$

where r_p is a parameter called as penalty factor. A sequence of positive value of r_p which are strictly decreasing to zero, are used for minimizing (3.19). It results in a sequence of minimum points which converges to the constrained minimum of the $-\mathrm{Ln}\, R_s(n)$. If the optimal solution is integral, then problem is solved. Otherwise, a non-integer variable (let it be n_j) is chosen which has highest fractional part (dn_j). A new constraint is incorporated in the original problem which can be written as

$$n_j \geq |n_j| + 1 \qquad (3.20)$$

where $|n_j|$ is the integral portion of the n_j. The new problem is again solved in the similar way as original unconstrained problem. If new problem converges, n_j is set as $|n_j|+1$; otherwise, as $|n_j|$. The same procedure is repeated for other variables. The stepwise procedure can be summarised as follows. [Fig. 3.1]

ALGORITHM

1. Select an initial value of $r_p > 0$ and an interior point n°. Set $l=0$.

2. If n^l nearly minimizes $f(n, r_p)$, go to step 6, otherwise calculate direction vectors d_j

$$d_j = \frac{\partial F(n, r_p)}{\partial n_j} \qquad j=1, 2, \ldots, k$$

3. Choose stepsize t_l that minimizes $f(n^l + t_l d^l, r_p)$

4. Calculate new trial point

$$n_j^{l+1} = n_j^l + t_l d_j^l \qquad j=1, 2, \ldots, k$$

5. Set $l=l+1$ and go to step 2.

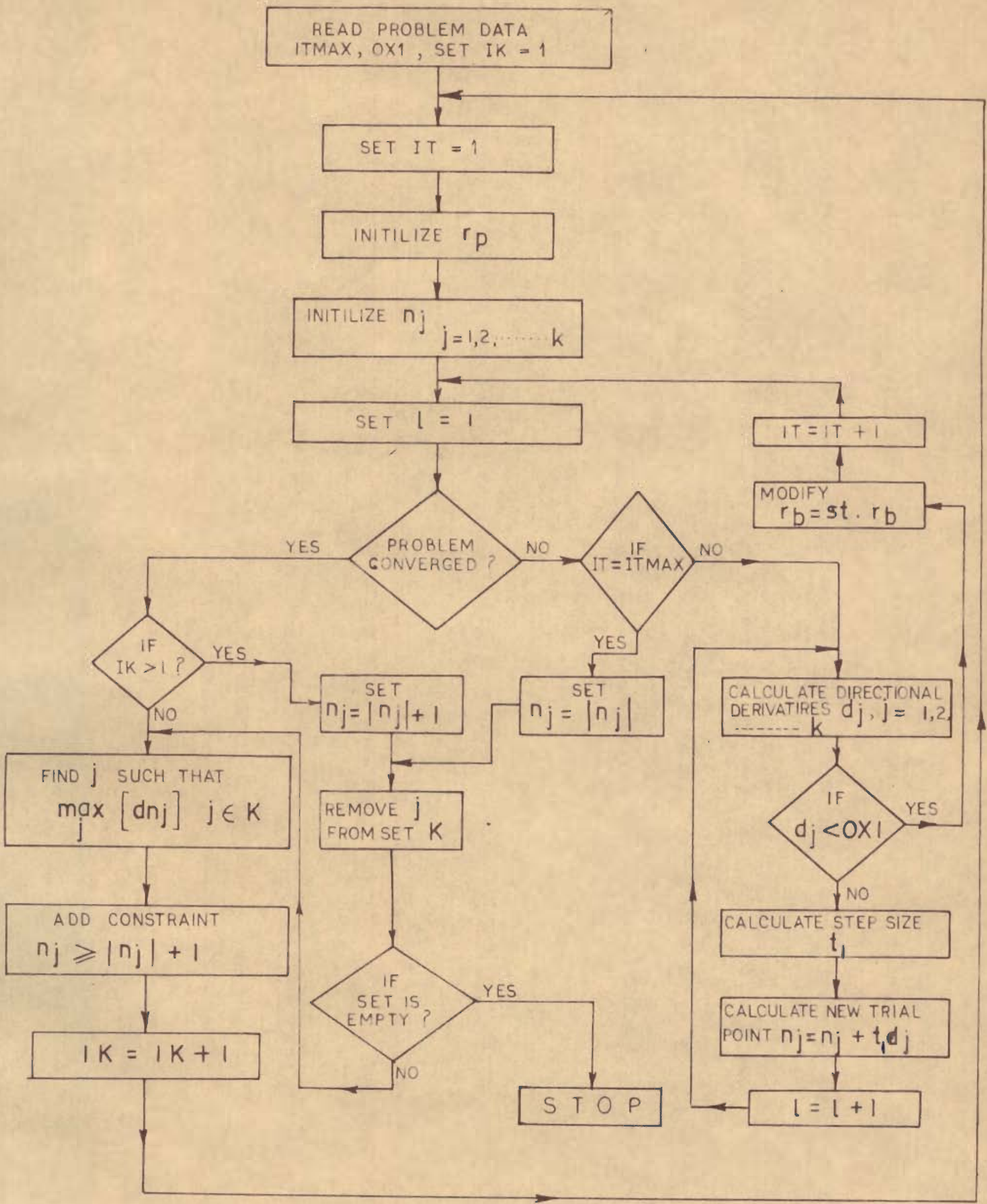


FIG. 3.1 FLOW CHART FOR PENALTY FUNCTION METHOD

6. Check convergence. If solution is optimal go to step 7; else replace r_p by str_p. where $0 < st \leq 1$ and go to step 2 with $l=0$.

7. Choose that variable which has greatest dn_j and add the following constraint in the problem

$$n_j \geq |n_j| + 1$$

8. Repeat step 2 - 5. If problem converges set $n_j = |n_j| + 1$; otherwise, $n_j = |n_j|$ and remove j^{th} stage from calculation.

9. If all variables are tried, stop; else, go to step 7.

The initial value of r_p should be such that

$$r_p^\circ = F_c \frac{-\text{Ln } R_s(n)}{\sum_{i=1}^{s} \left[b_i - \sum_{j=1}^{k} G_{ij}(n_j) \right]^{-1}} \tag{3.21}$$

where F_c is $0.01 < F_c < 1$. Various problems were solved on IBM1620 by using this method and satisfactory results were obtained. The use of numerical differentiation is made in case of standby and hybrid redundancy.

NUMERICAL EXAMPLE.

An electric power system in an aeroplane consists of three stages: I.C. engine, generator, and a frequency convertor connected in series. The cost, weight, volume and reliability data for these equipments are tabulated in table 3.3. The problem is to maximize system reliability by using parallel redundancy such that cost, weight and volume of the complete system do not exceed 50, 52, and 65, respectively.

Initially, r_p is assumed as 0.8 which gave a minimum point as (1.9481, 1.6929, 1.3573). In the next iteration, r_p is reduced by 0.35 and again a minimum is obtained. This procedure is repeated until an optimal solution to problems (3.17) and (3.18) is achieved. The complete results are tabulated in table 3.4.

Table 3.3

	I.C. Engine	Generator	Frequency convertor
Reliability	0.86	0.91	0.96
Cost	4.00	8.00	6.00
Weight	6.00	6.00	10.00
Volume	10.00	5.00	10.00

Table 3.4

Iteration No.	r_p	n_1	n_2	n_3
1	-	1.0	1.0	1.0
2	0.8	1.948	1.692	1.357
3	.28	2.197	1.891	1.517
4	.098	2.431	2.070	1.647
5	.0343	2.668	2.203	1.710
Optimum solution		3	2	2

3.3. FLEXIBLE TREE SEARCH METHOD

This enumerative procedure [38] allows a great deal of flexibility in the backtracking steps which improves the efficiency of the search procedure.

The linearized reliability problem can be written as

Mi imize

$$F(z) = \sum_{j=1}^{w} g_j z_j \tag{3.22}$$

subject to the constraints

$$\sum_{j=1}^{w} h_{ij} z_j \geq e_i \qquad i=1,2,\ldots,s \tag{3.23}$$

The stepwise method for solving above problem by flexible tree search can be described as

ALGORITHM.

1. Start. Set all variables free and r=1.

2. Forward move - Pick out a variable z_f from the set of free variables which has maximum u_f, where u_f is

$$u_f = \max_{j=1,\ldots,w} \left[\sum_{i=1}^{s} d_i + h_{ij} \right] \tag{3.24}$$

and d_i for i^{th} constraint is defined as

$$d_i = \sum_{j\in S} h_{ij} z_j - e_i \qquad i=1,2,\ldots,s \tag{3.25}$$

S is the set of variables specified at any iteration and z_j is the value assigned to a variable. Set z_f=1. If there is a tie, choose that variable which has minimum g_j. Label this variable as assigned and put it in the list of specified variable of rank r. If this set is feasible, check whether it is optimal; if yes, record it and go to step 4, else go to next step.

3. Test for next move - If this set has interesting solution,

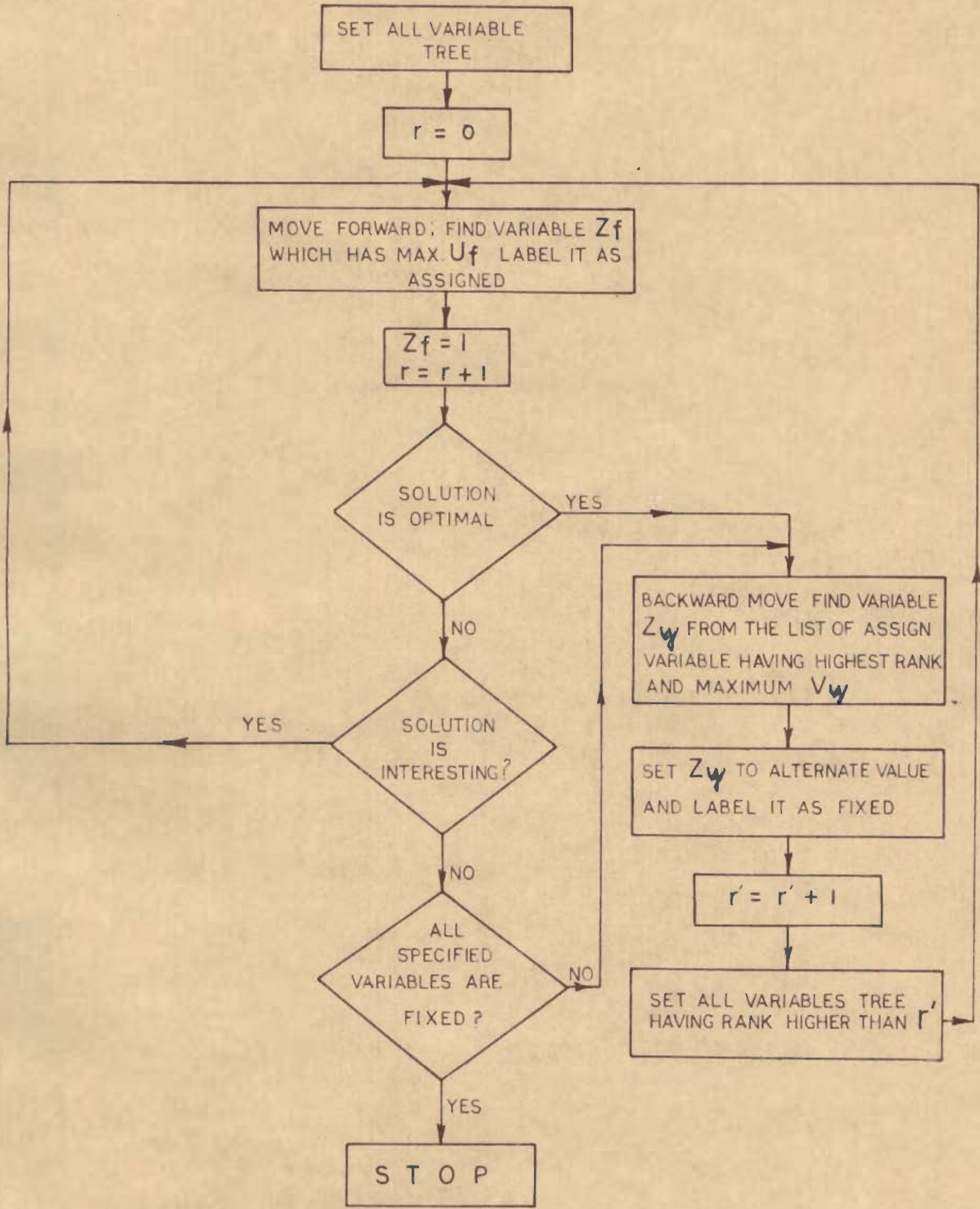


FIG. 3.2 FLOW CHART FOR FLEXIBLE TREE SEARCH METHOD

go to step 2, else go to step 5.

4. Backward move - Pick out a variable z_y from the list of assigned variables which has the highest rank and maximum v_y. In case of a tie, choose that variable which has maximum g_j where

$$v_y = \bar{\sum_i} t_{iy} \qquad (3.26)$$

and

$$t_{iy} = d_i - h_{iy} \qquad i=1,2,\ldots,s \qquad (3.27)$$

($\bar{\sum_i}$ represents only these t_{iy}'s are to be added which have negative sign.)

Set z_y to alternate value and label it as fixed. Assign rank $r' = r+1$ to variable z_y and to all the variables from the list, which have higher rank than r are set free. Go to step 2.

5. Test for termination - All specified variables are fixed, go to step 6; else go to 4.

6. Stop.

An Example

The reliability of a system, consisting of three stages in series is to be maximized through the use of parallel redundant components. The reliability cost and weight of each component type are tabulated in table 3. The entire cost and weight of the system should not exceed 50 and 60 units, respectively. From the design consideration, it is known that the maximum number of redundant components at each stage can at the best be three.

Table 3.5

Stage	Reliability	Cost	Weight
1	0.75	6	10
2	0.85	6	5
3	0.90	10	10

The linearized problem with single subscripted variables can be given by

Minimize f(z)

$$f(z) = 0.223132z_1+0.04879z_2+0.01184z_3+0.13076z_4 +0.01937z_5+0.00287z_6+0.09531z_7+0.00904z_8 +0.0009z_9 \quad (3.28)$$

subject to

$$6z_1+6z_2+6z_3+6z_4+6z_5+6z_6 10z_7+10z_8+10z_9-38 \geq 0$$
$$10z_1+10z_2+10z_3+5z_4+5z_5+5z_6+10z_7+10z_8+10z_9-40 \geq 0 \quad (3.29)$$
$$z_j=0 \text{ or } 1.$$

Initially, all variables are assumed to be free, i.e. $z_j=0$, $\left[j=1,...,9\right]$ and u_j are calculated. The complete procedure is tabulated in table 3.6. The optimal solution obtained is

$$z_9=z_8=z_3=z_6=z_5=1$$
$$z_1=z_2=z_4=z_7 \quad =0 \quad (3.30)$$

or, in other words, the number of redundant components in stage one, two and three are three, two and two, respectively.

Various reliability problems were solved on IBM1620 using this approach. In all cases, the exact optimal solutions were obtained. In the enumeration methods available so far,

Table 3.6 - Details of Stepwise Solution

Step	u_1 v_1	u_2 v_2	u_3 v_3	u_4 v_4	u_5 v_5	u_6 v_6	u_7 v_7	u_8 v_8	u_9 v_9	Variable to be set to 0 or 1	Objective function	d_1	d_2	List of specified variable	Feasible Solution	Remarks
Start	-	-	-	-	-	-	-	-	-		0.0	-38	-40	Empty	-	-
Move forward	-62	-62	-62	-67	-67	-67	-58	-58	-58	$z_9=1$	.0009	-28	-30	z_9	-	-
Move forward	-42	-42	-42	-47	-47	-47	-38	-38	-	$z_8=1$	.00994	-18	-20	z_9, z_8	-	-
Move forward	-22	-22	-22	-27	-27	-27	-18	-	-	$z_7=1$	.10525	- 8	-10	z_9, z_8, z_7	-	-
[illegible]	-2	-2	-2	-7	-7	-7	-	-	-	$z_3=1$	.11709	- 2	0	z_9, z_8, z_7, z_3	-	-
Move forward	0	0	0	-	0	0	-	-	-	$z_6=1$	.11996	4	5	z_9, z_8, z_7, z_3, z_6	$z_9=z_8=z_7=$ $z_3=z_6=1$ $z_1=z_2=z_4=$ $z_5=0$	
Move backward	-	-	-7	-	-	-2	-7	-7	-7	[illegible]	[illegible]709	- 2	0	z_9, z_8, z_7, z_3, z_6	-	Setting to 1 other variables will give $F(z) > 0.11996$. Next move will be backward

Table 3.6 (continued)

Step	u_1	u_2	u_3	u_4	u_5	u_6	u_7	u_8	u_9	Variable to be set	Objective function	d_1	d_2	List of specified variable	Feasible Solution	Remarks
Move backward	-,	-,	-18	-,	-,	-,	-22	-22	-22	$z_3=0$	.10525	-8	-10	$z_9, z_8, z_7, \bar{z}_3$		Setting to 1 other variables will give $F(z) > 0.11996$, the known feasible solution, next move backwards
Move backward	-,	-,	-,	-,	-,	-,	-38	-38	-38	$z_7=0$	.00994	-18	-20	$z_9, z_8, \bar{z}_7$	-	-
Move forward	-22	-22	-22	-27	-27	-27	-,	-,	-,	$z_3=1$	.02178	-12	-10	$z_9, z_8, \bar{z}_7, z_3$	-	-
Move forward	-6	-6	-	-11	-11	-11	-,	-,	-	$z_2=1$	.07057	-6	0	$z_9, z_8, \bar{z}_7, z_3, z_2$	-	-
Move forward	0	-	-	0	0	0	-,	-,	-,	$z_6=1$	.07344	0		$z_9, z_8, \bar{z}_7, z_3, z_2, z_6$	$z_9=z_8=z_3=z_2=z_6=1$ $z_7=z_1=z_4=z_5=0$	
Move backward	-,	-11	-11	-,	-,	-5	-15	-15	-15	$z_6=0$	.07057	-6	0	$z_9, z_8, \bar{z}_7, z_3, z_2, z_6$		Setting to 1 other variables will give $F(z) > 0.07344$, next move backward.
Move backward	-,	-22	-22	-,	-,	-,	-,	-26	-26	$z_2=0$	.02178	-12	-10	$z_9, z_8, \bar{z}_7, z_3, \bar{z}_2$	-	-

Table 3.6 (continued)

Step	u_1	u_2	u_3	u_4	u_5	u_6	u_7	u_8	u_9	Variable to be set to 0 or 1	Objective function	d_1	d_2	List of specified variable	Feasible Solution	Remarks
Move forward	-6	-,	-,	-11	-11	-11	-,	-,	-,	$z_1=1$	.24491	-6	0	$z_9, z_8, \bar{z}_7, z_3, \bar{z}_2, z_1$		Setting to any other variable to 1 will give $F(z) > 0.0734$. Next move backward.
Move Backward	-22	-,	-22	-,	-,	-,	-26	-26		[illegible]	.02178	-12	-10	$z_9, z_8, \bar{z}_7, z_3$		
Move forward	-,	-,	-,	-11	-11	-11	-,	-,	-,	$z_6=1$	.02465	-6	-5	$z_9, z_8, z_7, z_3, \bar{z}_2, \bar{z}_1, z_6$		
Move forward	-	-,	-,	0	0	-,	-	-,	-	[illegible]	.04402	0	0	$z_9, z_8, \bar{z}_7, z_3, \bar{z}_2, \bar{z}_1, z_6, z_5$	$z_9=z_8=z_3=z_6=z_5=1$ $z_7=z_2=z_1=z_4=0$	
Move backward	-,	-,	-11	-,	-6	-6	-,	-15	-15	$z_5=0$	.02465	-6	-5	$z_9, z_8, \bar{z}_7, z_3, \bar{z}_2, \bar{z}_1, z_6, z_5$		By setting $z_4=1$, $F(z) > 0.04402$. Next move backward.
Move backward	-,	-,	-22	-	-	-17	-,	-26	-26	$z_6=0$	.02178	-12	-10	$z_9, z_8, \bar{z}_7, z_3, \bar{z}_2, \bar{z}_1, \bar{z}_6$		By setting z_4 and z_5 to 1 $F(z) > 0.04402$. Next move backward.
Move back-	-,	-,	-38	-,	-,	-,	-,	-42	-42	$z_3=0$	.00994	-18	-20	$z_9, z_8, \bar{z}_7, \bar{z}_3$		By setting variables to 1, results in $F(z) > 0.04402$. Next move backward.

Table 3.6 (continued)

Step	u_1	u_2	u_3	u_4	u_5	u_6	u_7	u_8	u_9	Variable to be set to 0 or 1	Objective function	d_1	d_2	List of specified variable	Feasible Solution	Remarks
Move backward	-,	-,	-,	-,	-,	-,	-,	-58	-58	$z_8=0$	.0009	-28	-30	z_9, $\bar{z}_8$		By setting variables to 1, results in $F(z) > 0.04402$. Next move backwards.
Move backward	-,	-,	-,	-,	-,	-,	-,	-,	-,	$z_9=0$	0	-38	-40	$\bar{z}_9$		By setting variables to 1, results in $F(z) > 0.04402$. All variables are fixed.
Stop																

last variable, in order, is set to zero for backtracking. In this method, a more flexible rule is used for backtracking which improves the efficiency of search procedure. This approach requires simple calculation and less computational effort and memory.

3.4. ZERO-ONE PROGRAMMING METHOD

This method makes use of zero-one programming [40]. This method depends on the non-binary tree search, where upper bound is calculated by making use of graphy theory. All the tree-search methods for 0-1 problem, available to-date, are binary. They can be divided into two subproblems, firstly a variable is set to one and search is made for the remaining free variables, and secondly set the same variable equal to zero and again search is made. While in this method, the use of tree search is made to calculate only the lower bound to the objective function at the nodes.

The linearized reliability problem given by (2.70) and (2.71) can be solved by this method. The stepwise procedure for solving the reliability problem can be summarised as -

(1) Consider node A where all variables z_j are free and set $T = A$.

(2) $e' > 0$ $e' = e_i - \sum_{z_j \in H} h_{ij}$, where H is the set of variable which has been set to 1 find reduce set $|S_{Ti}|$, otherwise set all variables $z_j = 0$ which will give optimum solution and stop.

(3) Calculate external stable set.

(4) Find lower bound on objective function.

(5) Find $|S^*_{Ti}|=\min_i|S_{Ti}|$. Form tree by branching r nodes from node T by setting $z_{ji}=1$ for each node. Each branch can be treated as one subproblem.

(6) Repeat step (2) for each node.

(7) Repeat step (3) for each node.

(8) Repeat step (4) for each node.

(9) Find the node L which has lowest lower bound on objective function set $z_{jL}=1$.

(10) If for this L^{th} node all $|S_{Li}|=\emptyset$ then current partial solution is the solution to the problem and stop, otherwise set T=L and go to step (5).

CALCULATION OF REDUCE SET [40]

If $e_i' > 0$ for i^{th} constraint, then h_{ij} of the free variables are arranged in the descending order to get a table of h_{ij1}, h_{ij2}, ..., h_{ijF}. A variable z_{jl} is the member of reduce set S_i if and only if

$$\sum_{p=1}^{m} h_{ijp} \geq e_i$$

for any q such that $1 \leq q \leq F$

It is clear that if Z_{jl} does not satisfy the above condition, $Z_{j,l+1}$, $Z_{j,l+2}$ ... will also not be a member of this reduce set.

CALCULATION OF MINIMAL EXTERNAL STABLE SET [15]

For calculating minimal external stable set for a system, logic expression is tob formed for each vertex x, in which eithe x or one of the elements Υ_x is to be included. The associate properties and law of absorption is used to simplify logic expression and remove redundancies. The resultant expression gives the number of minimal external stable sets.

LOWER BOUND ON OBJECTIVE FUNCTION:

Setting each $Z_j \in S_i$ to one lower bound on objective function is calculated by expression

$$Z = Z_o + \min_p \left[Z_{j \in Z_p} \Sigma g_j \right]$$

where Z_o is the value of the partial solution.

NUMERICAL EXAMPLE

The reliability of a system consisting of three stages having reliabiloty cost, weight as tabulated in Table 3.5, is to be maximized by using parallel redundant stages. The cost and weight of the system must no exceed 50 and 60 units. From a design consideration it is known that the maximum number of redundant components which each stage can have is three, i.e. $\bar{N}_j=3$.

The linearized reliability problem is

Minimize

$$0.223132z_1+0.04879z_2+0.01184z_3+0.13976z_4+0.01937z_5 +0.00287z_6+0.09531z_7+0.00904z_8+0.0009z_9 \quad (3.31)$$

subject to

$$6z_1+6z_2+6z_3+6z_4+6z_5+6z_6+10z_7+10z_8+10z_9 \geq 38$$

$$10z_1+10z_2+10z_3+5z_4+5z_5+5z_6+10z_7+10z_8+10z_9 \geq 40$$

$$z_j=0 \text{ or } 1 \quad \text{for } j=1,\dots,9 \quad (3.32)$$

Assuming all variables are zero at node A (Fig. 3.3), reduced set for this node are $|S_{A1}| = (z_7, z_8, z_9)$, $|S_{A2}| = (z_1, z_2, z_3, z_7)$ and the minimal external stable set are

(z_1, z_8), (z_2, z_8), (z_3, z_8), (z_1, z_9), (z_2, z_9), (z_3, z_9), (z_7).

The lower bound L=0.01274 is produced by set (z_3, z_9). The lower bound on the objective function is 0.01274. Since the reduced set S_{A1} has minimum number of variables, therefore we continue the branching by setting either $z_7=1$ (node B_1), $z_8=1$ (node B_2), $z_9=1$ (node B_3) in Fig. 3.3. Now we start with node B_1. The complete calculations are tabulated in Table 3.7.

The optimum solution of the problem is given by (3.22) and (3.23) is

$$z_1=z_2=z_4=z_7=0$$

and

$$z_3=z_5=z_6=z_8=z_9 = 1$$

or solution to the primal 0-1 programming solution is

$$x_{11}=x_{12}=x_{21}=x_{31}=1,$$

$$x_{13}=x_{22}=x_{23}=x_{32}=x_{33}=0$$

From (2.62) the optimum number of redundant components to be employed in stage one, two and three are three, two and two, respectively.

A number of problems were solved by this method and it is found that this approach requires fewer iterations than the other available zero-one programming algorithms.

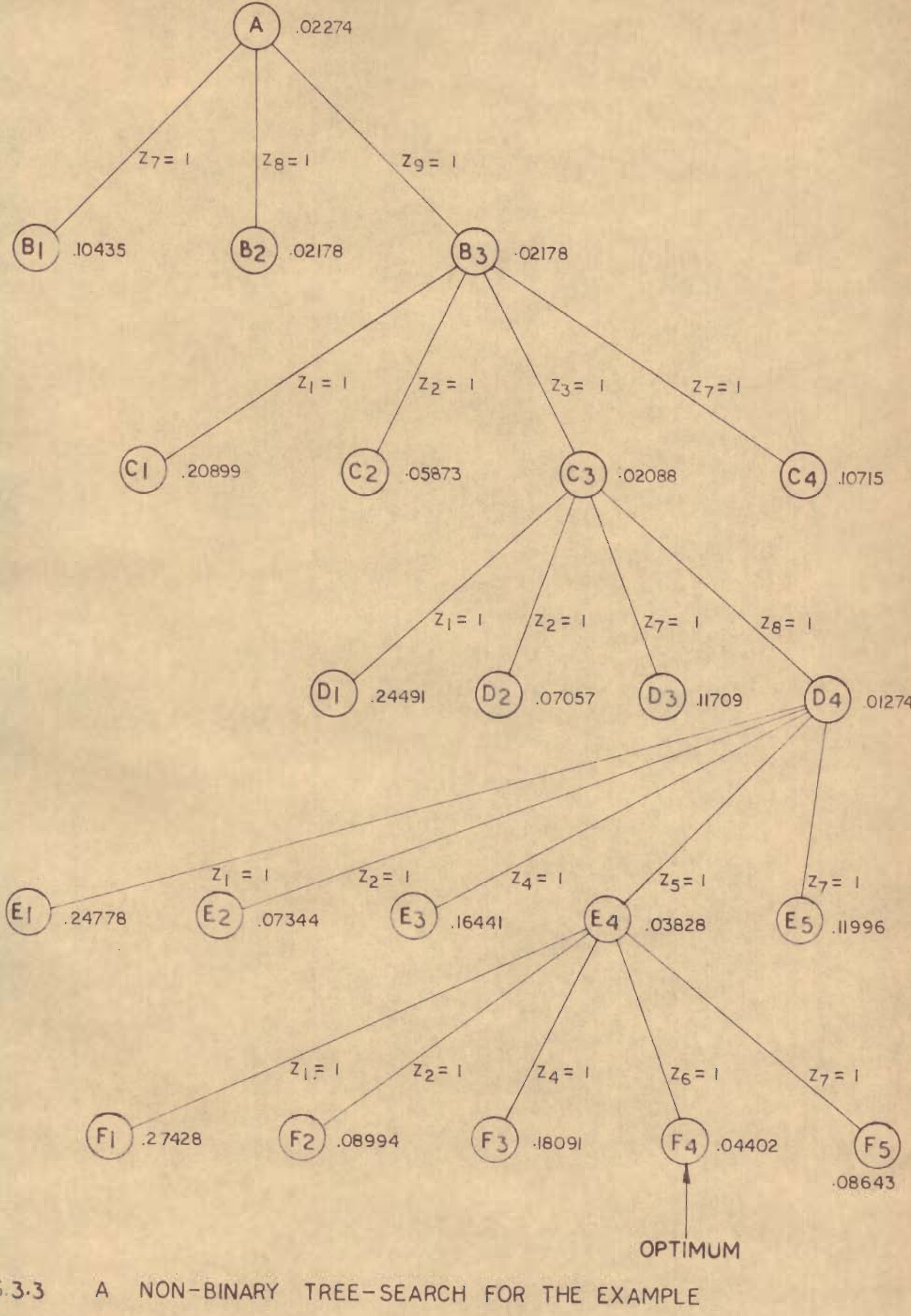


FIG. 3.3 A NON-BINARY TREE-SEARCH FOR THE EXAMPLE

Table 3.7

Node Name	Z_j	Reduce Set	Minimal external stable set	Lower bound on objective function	Partial solution	Branching from
A	Z	$S_1=(Z_7, Z_8, Z_9)$ $S_2=(Z_1, Z_2, Z_3, Z_7)$	$(Z_1, Z_8), (Z_2, Z_8), (Z_3, Z_8)$ $(Z_1, Z_9), (Z_2, Z_9), (Z_3, Z_9)$, (Z_7)	0.02274		A
B_1	$Z_7=1$	$S_1=(Z_8, Z_9, Z_1, Z_2)$ $S_2=(Z_1, Z_2, Z_3, Z_8)$	$(Z_1), (Z_2), (Z_8), (Z_9, Z_3)$	0.10435		
B_2	$Z_8=1$	$S_1=(Z_7, Z_9, Z_1, Z_2)$ $S_2=(Z_1, Z_2, Z_3, Z_7)$	$(Z_1), (Z_2), (Z_7), (Z_9, Z_3)$	0.02178	$Z_9=1$	B_3
B_3	$Z_9=1$	$S_1=(Z_7, Z_8, Z_1, Z_2)$ $S_2=(Z_1, Z_2, Z_3, Z_7)$	$(Z_1), (Z_2), (Z_7), (Z_8, Z_3)$	0.02178		
C_1	$Z_1=1$	$S_1=(Z_2, Z_3, Z_7, Z_8)=S_2$	$(Z_2), (Z_3), (Z_7), (Z_8)$	0.20899		
C_2	$Z_2=1$	$S_1=(Z_1, Z_3, Z_7, Z_8)$ $S_2=(Z_1, Z_3, Z_7, Z_3)$	$(Z_1), (Z_3), (Z_7), (Z_8)$	0.05873	$Z_9=1$ $Z_3=1$	C_3
C_3	$Z_3=1$	$S_1=(Z_1, Z_2, Z_7, Z_8)$ $S_2=(Z_1, Z_2, Z_7, Z_9)$	$(Z_1), (Z_2), (Z_7), (Z_8)$	0.02088		

(continued)

C_4	$Z_7=1$	$S_1=(Z_1, Z_2, Z_3, Z_4, Z_8)$ $S_2=(Z_1, Z_2, Z_3)$	$(Z_1), (Z_2), (Z_3)$	0.10715		C_3
D_1	$Z_1=1$	$S_1=(Z_2, Z_4, Z_7, Z_8)$ $S_2=(Z_2, Z_4, Z_5, Z_7, Z_8)$	$(Z_2), (Z_4), (Z_7), (Z_8)$	0.24491		
D_2	$Z_2=1$	$S_1=(Z_1, Z_4, Z_7, Z_8)$ $S_2=(Z_1, Z_4, Z_5, Z_7, Z_8)$	$(Z_1), (Z_4), (Z_7), (Z_8)$	0.07057	$Z_9=1$	
D_3	$Z_7=1$	$S_1(Z_1, Z_2, Z_4, Z_5, Z_8)$ $S_2=(Z_1, Z_2, Z_4, Z_5, Z_7)$	$(Z_1), (Z_2), (Z_4), (Z_5), (Z_8)$	0.11709	$Z_3=1$ $Z_8=1$	D_4
D_4	$Z_8=1$	$S_1=(Z_1, Z_2, Z_4, Z_5, Z_7)$ $S_2=(Z_1, Z_2, Z_4, Z_5, Z_7)$	$(Z_1), (Z_2), (Z_4), (Z_5), (Z_7)$	0.01274		

(continued)

Table 3.7 (contd.)

E_1	$Z_1=1$	$S_1=(Z_2, Z_4, Z_5, Z_6, Z_7)$ $S_2=\Phi$	$(Z_2), (Z_4), (Z_5), (Z_6), (Z_7)$	0.24778		
E_2	$Z_2=1$	$S_1=(Z_1, Z_4, Z_5, Z_6, Z_7)$ $S_2=\Phi$	$(Z_1), (Z_4), (Z_5), (Z_6), (Z_7)$	0.07344	$Z_9=1$	
E_3	$Z_4=1$	$S_1=(Z_1, Z_2, Z_5, Z_6, Z_7)$ $S_2=\Phi$	$(Z_1), (Z_2), (Z_5), (Z_6), (Z_7)$	0.16441	$Z_3=1$ $Z_8=1$	E_4
E_4	$Z_5=1$	$S_1=(Z_1, Z_2, Z_4, Z_6, Z_7)$ $S_2=\Phi$	$(Z_1), (Z_2), (Z_4), (Z_6), (Z_7)$	0.03828	$Z_5=1$	
E_5	$Z_7=1$	$S_1=(Z_1, Z_2, Z_4, Z_5, Z_6)$ $S_2=\Phi$	$(Z_1), (Z_2), (Z_4), (Z_5), (Z_6)$	0.11996		
F_1	$Z_1=1$	$S_1=\Phi$, $S_2=\Phi$		0.27428	$Z_9=1$	
F_2	$Z_2=1$	$S_1=\Phi$, $S_2=\Phi$		0.08994	$Z_3=1$	
F_3	$Z_4=1$	$S_1=\Phi$, $S_2=\Phi$		0.18091	$Z_8=1$	No branching
F_4	$Z_6=1$	$S_1=\Phi$, $S_2=\Phi$		0.04402	$Z_5=1$	
F_5	$Z_7=1$	$S_1=\Phi$, $S_2=\Phi$		0.08643	$Z_6=1$	

3.5. MODIFIED NON-BINARY TREE SEARCH METHOD

A method is proposed to solve the linearized reliability problem. A simple rule of branching is given, which reduces the computation time and memory requirement considerably.

Since all the coefficients of the linearlized reliability expression are positive, the smallest lower bound can be obtained by setting a variable which has smallest c_{jl} and is a member of a reduce set having minimum number of elements.

The stepwise procedure for solving the linearized problem (2.67) and (2.68) by this method can be described as

(1) Set all variables ($\bar{x}_{jl}$; $j=1,2,\ldots,k,$ $l=1,2,\ldots,\bar{N}_j$), free.

(2) Calculate e_i' , $(i=1,2,\ldots,m)$, whose e_i' is given by

$$e_i' = e_i - \sum_{j=1}^{k} \sum_{l=1}^{\bar{N}_j} a_{ij} \bar{x}^*_{jl}$$

$\bar{x}^*_{jl}$ are the variables which are assigned as 1.

(3) If $e_i' > 0$, find the reduced set S_i , else go to step 6.

(4) Find the set S_{min} , where S_{min} is given by

$$S_{min} = \min_i S_i \qquad i=1,2,\ldots,s$$

i.e. S_{min} is that reduced set which has lowest number of elements or variables.

(5) Find the variable $\bar{x}_{jl}$ from the set S_{min} which has the lowest c_{jl} and assign this variable $\bar{x}_{jl}=1$. Go to step 2.

(6) Set all free variables to zero and the resulting solution will be optimum.

(7) Stop.

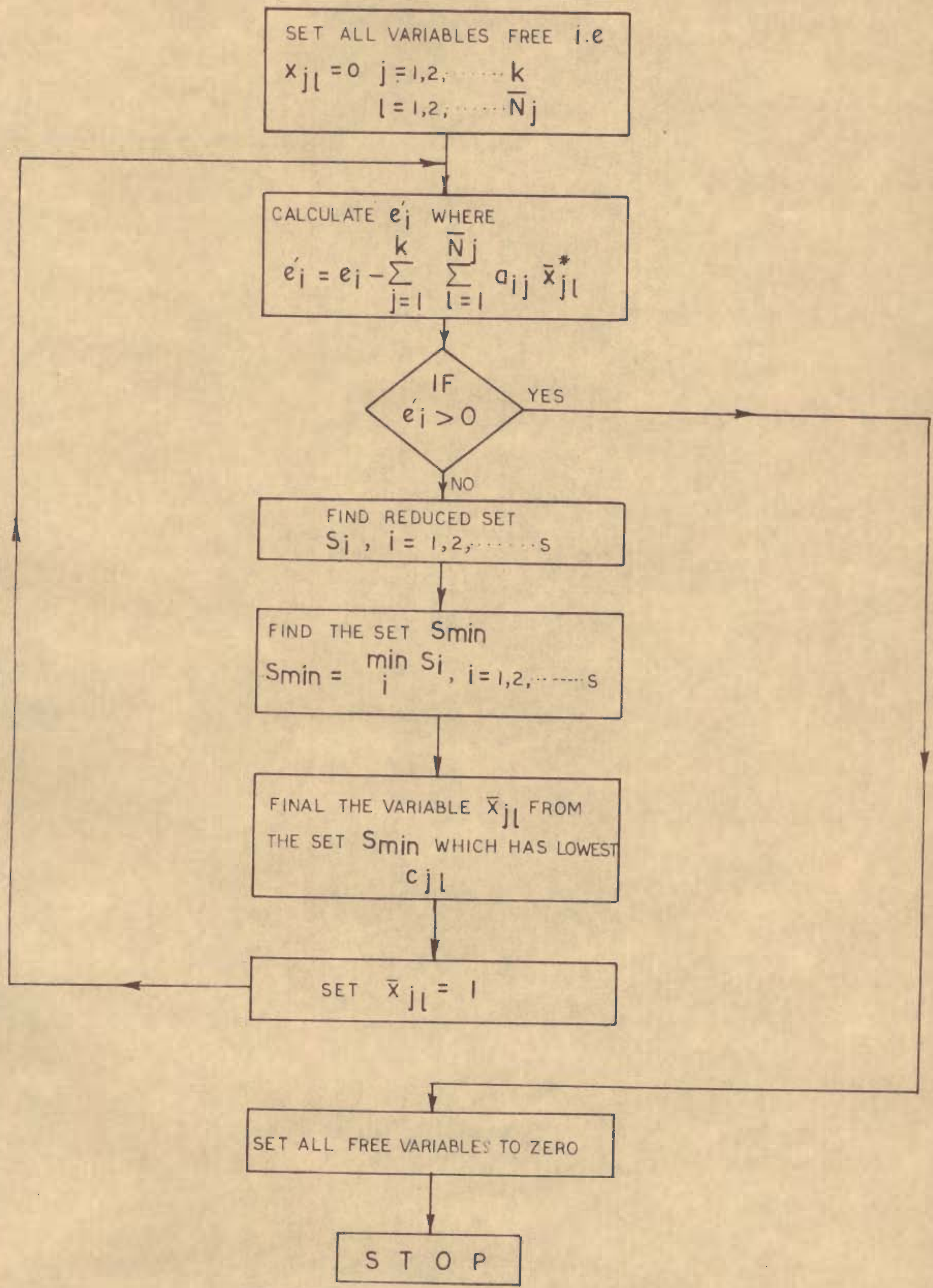


FIG. 3.4 FLOW CHART FOR MODIFIED NON-BINARY TREE-SEARCH METHOD

The flow chart for this method is shown in Fig. 3.4. Various reliability problems were solved on IBM1620 by using this method and exact results were obtained.

AN ILLUSTRATIVE EXAMPLE:

Consider a feedback control system as shown in Fig. 3.5 consisting of an input transducer with three function groups denoted by $G_1(S)$, $G_2(S)$ and $G_3(S)$ and two feedback loops. The major and minor loops have output transducer and feedback position denoted as $H_1(S)$. For successful operation of the control system, each component must be in proper working condition. Reliability of the error detector is assumed as unity. The unreliability cost, weight, and power consumption for each component are given in Table 3.8. It is required to maximize the reliability of the control system by using redundant components. The incremental cost, weight, and power consumption of the system must not exceed 43, 35 and 90 units, respectively. From design consideration, it is known that at the most, each stage may have two redundant components.

Table 3.8 - Parameters for a feedback control system shown in Fig. 3.5

Component	Unreliability	Cost	Weight	Power consumption
1. Input Transducer	0.06	15.0	2.0	10.0
2. Function Group $G_1(S)$	0.08	5.0	4.0	15.0
3. Function Group $G_2(S)$	0.05	8.0	8.0	20.0
4. Function Group $G_3(S)$	0.03	6.0	6.0	15.0
5. Feedback position $H_1(S)$	0.10	5.0	3.0	5.0
6. Output Transducer	0.09	10.0	4.0	5.0

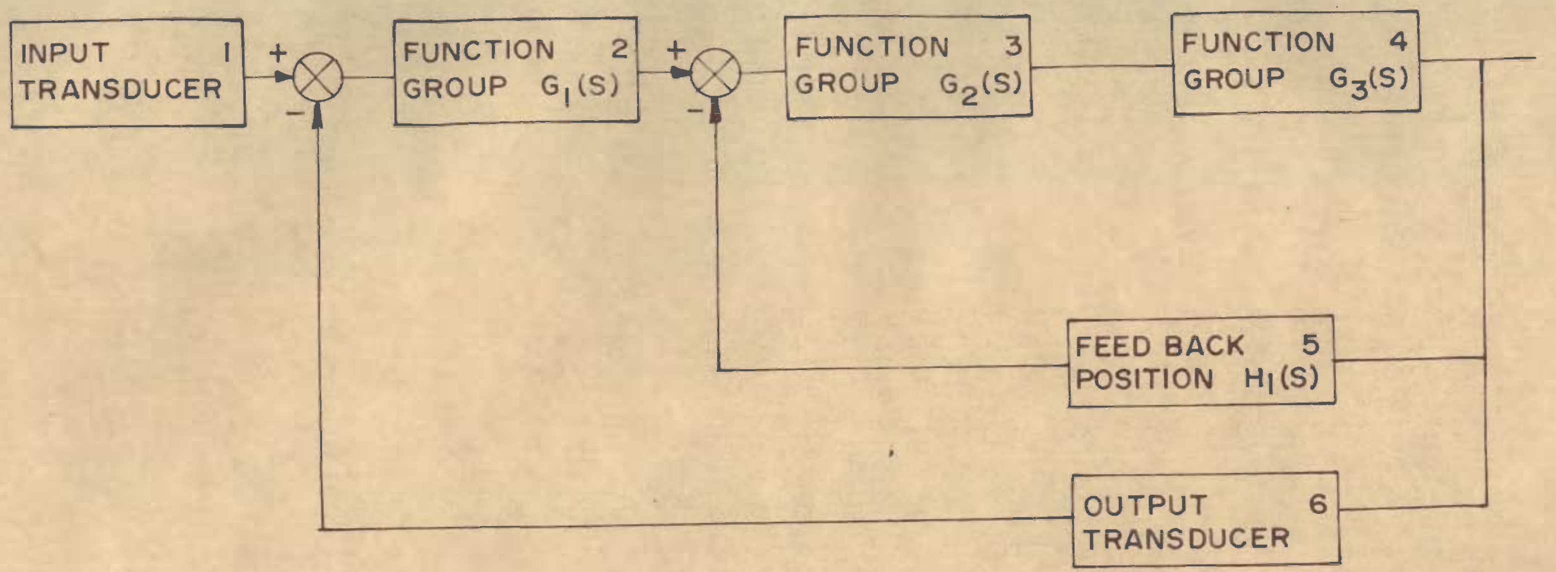


FIG.3.5 A FEED-BACK CONTROL SYSTEM.

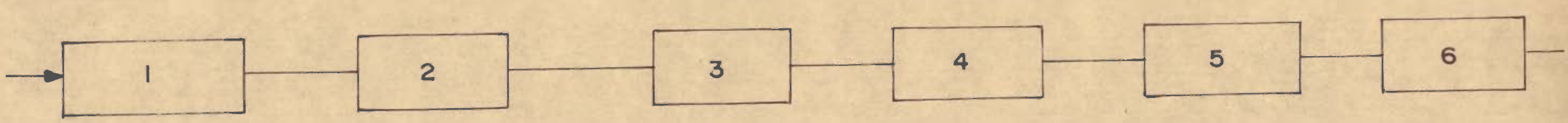


FIG.3.6 A LOGIC DIAGRAM OF THE FEED-BACK CONTROL SYSTEM SHOWN IN FIG.3.5

The functional diagram of the feedback control system shown in Fig. 3.1 will be a series system as shown in Fig. 3.6. The linearized reliability problem for this control system will be

Minimize $F(\bar{x})$

$$0.05827\bar{x}_{11}+0.00399\bar{x}_{12}+0.07696\bar{x}_{21}+0.00591\bar{x}_{22}+0.04879\bar{x}_{31}$$
$$+0.00237\bar{x}_{32}+0.02956\bar{x}_{41}+0.00087\bar{x}_{42}+0.09531\bar{x}_{51}$$
$$+0.00905\bar{x}_{52}+0.08618\bar{x}_{61}+0.0074\bar{x}_{62} \qquad (3.34)$$

subject to the constraints

$$15\bar{x}_{11}+15\bar{x}_{12}+5\bar{x}_{21}+5\bar{x}_{22}+8\bar{x}_{31}+8\bar{x}_{32}+6\bar{x}_{41}+6\bar{x}_{42}+5\bar{x}_{51}+5\bar{x}_{52}$$
$$+10\bar{x}_{61}+10\bar{x}_{62} \geq 55 \qquad (3.35)$$

$$2\bar{x}_{11}+2\bar{x}_{12}+4\bar{x}_{21}+4\bar{x}_{22}+8\bar{x}_{31}+8\bar{x}_{32}+6\bar{x}_{41}+6\bar{x}_{42}+3\bar{x}_{51}+3\bar{x}_{52}$$
$$+4\bar{x}_{61}+4\bar{x}_{62} \geq 19 \qquad (3.36)$$

$$10\bar{x}_{11}+10\bar{x}_{12}+15\bar{x}_{21}+15\bar{x}_{22}+20\bar{x}_{31}+20\bar{x}_{32}+15\bar{x}_{41}+15\bar{x}_{42}$$
$$+5\bar{x}_{51}+5\bar{x}_{52}+5\bar{x}_{61}+5\bar{x}_{62} \geq 50 \qquad (3.37)$$

where $\bar{x}_{jl}=0$ or 1 ; $j=1,\ldots,6$; $l=1,\ldots,2$

1. Assuming all variables free,

2. The reduced sets are: $S_1 = (\bar{x}_{11}, \bar{x}_{12}, \bar{x}_{61}, \bar{x}_{62})$,

   $S_2 = (\bar{x}_{22}, \bar{x}_{31}, \bar{x}_{32}, \bar{x}_{41}, \bar{x}_{42}, \bar{x}_{61}, \bar{x}_{62})$,

   $S_3 = (\bar{x}_{21}, \bar{x}_{22}, \bar{x}_{31}, \bar{x}_{32}, \bar{x}_{41}, \bar{x}_{42})$.

   The S_{min} is S_1 and the variable to be assigned as 1 is $\bar{x}_{12}$ as it has smallest c_{jl} in set S_1. The modified d_i',s are $d_1' = 40$, $d_2' = 17$, $d_3' = 40$.

3. The reduced sets are $S_1 = (\bar{x}_{11}, \bar{x}_{31}, \bar{x}_{32}, \bar{x}_{61}, \bar{x}_{62})$,

$S_2 = (\bar{x}_{22}, \bar{x}_{31}, \bar{x}_{32}, \bar{x}_{41}, \bar{x}_{42}, \bar{x}_{62})$,

$S_3 = (\bar{x}_{21}, \bar{x}_{22}, \bar{x}_{31}, \bar{x}_{32}, \bar{x}_{41}, \bar{x}_{42})$,

and the variable to be assigned as 1 is $\bar{x}_{32}$. Modified $d'_1=32$, $d'_2=9$, $d'_3=20$.

4. The reduced sets are $S_1 = (\bar{x}_{11}, \bar{x}_{31}, \bar{x}_{42}, \bar{x}_{61}, \bar{x}_{62})$,

   $S_2 = (\bar{x}_{21}, \bar{x}_{22}, \bar{x}_{31}, \bar{x}_{41}, \bar{x}_{42}, \bar{x}_{61}, \bar{x}_{62})$,

   $S_3 = (\bar{x}_{11}, \bar{x}_{21}, \bar{x}_{22}, \bar{x}_{31}, \bar{x}_{41}, \bar{x}_{42}, \bar{x}_{62})$.

   The variable to be assigned as 1 is $\bar{x}_{42}$. The resulting d'_1 's are $d'_1=26$, $d'_2=3$, $d'_3=5$.

5. The reduced sets are, $S_1 = (\bar{x}_{11}, \bar{x}_{31}, \bar{x}_{41}, \bar{x}_{61}, x_{62})$,

   $S_2 = (\bar{x}_{21}, \bar{x}_{22}, \bar{x}_{31}, \bar{x}_{41}, \bar{x}_{42}, \bar{x}_{51}, \bar{x}_{52}, \bar{x}_{61}, \bar{x}_{62})$,

   $S_3 = (\bar{x}_{11}, \bar{x}_{21}, \bar{x}_{22}, \bar{x}_{31}, \bar{x}_{41}, \bar{x}_{51}, \bar{x}_{52}, \bar{x}_{61},$

   $\bar{x}_{62})$ and the variable to be assigned as 1 is $\bar{x}_{62}$.

   Modified d_i 's are $d'_1=16$, $d'_2=-$, $d'_3=0$.

6. The reduced sets are S_1 $(\bar{x}_{11}, \bar{x}_{22}, \bar{x}_{31}, \bar{x}_{41}, \bar{x}_{61})$,

   $S_2=\emptyset$**, $S_3=\emptyset$. The variable to be assigned as 1 is $\bar{x}_{22}$.

   Modified d_i's are $d'_{11}=11$, -, -.

7. The reduced sets are $S_1 = (\bar{x}_{11}, \bar{x}_{31}, \bar{x}_{41}, \bar{x}_{52}, \bar{x}_{61})$,

   $S_2 = \emptyset$, $S_3 = \emptyset$. The variable to be assigned as 1 is $\bar{x}_{52}$.

   Modified d_i's are $d'_1=6$, -, -.

8. The reduced sets are $S_1 = (\bar{x}_{11}, \bar{x}_{21}, \bar{x}_{31}, \bar{x}_{41}, \bar{x}_{61})$,

   $S_2 = \emptyset$, $S_3 = \emptyset$. Variable to be assigned as 1 is $\bar{x}_{41}$.

   Modified d_i's are $d'_1=0$, -, -.

9. The reduced sets are $S_1 = \emptyset$, $S_2=\emptyset$, $S_3=\emptyset$.

** When $e_i \leq 0$, the reduced set S_i is empty and is denoted by $\emptyset$.

10. Set all free variables to zero. The optimum solution is

$$\bar{x}_{12} = \bar{x}_{22} = \bar{x}_{32} = \bar{x}_{41} = \bar{x}_{42} = \bar{x}_{52} = \bar{x}_{62} = 1$$

$$\bar{x}_{11} = \bar{x}_{21} = \bar{x}_{31} = \bar{x}_{51} = \bar{x}_{61} = 0$$

By (2.69), the optimum number of redundant components to be used are

$$n_1 = 1, \quad n_2 = 1, \quad n_3 = 1, \quad n_4 = 0, \quad n_5 = 1, \quad n_6 = 1$$

Optimum structure of the feedback control system with redundant components is shown in Fig. 3.7.

AN EXAMPLE:

Consider a control system as shown in Fig. 3.8 consisting of a measuring element, amplifier, comparator and an actuator in series. Their parameters are tabulated in Table 3.9. It is needed that the system reliability is to be maximized by using spare standby components. The maximum number of the spare components for each stage may be assumed as three and the replacement time is to be neglected in comparison with the life time of the system. The constraints on the system are

$$\sum_{j=1}^{k} a_{ij}\, n_j^2 \leq 36 \qquad (3.38)$$

$$\sum_{j=1}^{k} f_j\, n_j \exp\,(n_j/4) \leq 150 \qquad (3.39)$$

The life time of the system is 10 years and the reliability of the switch is 0.999.

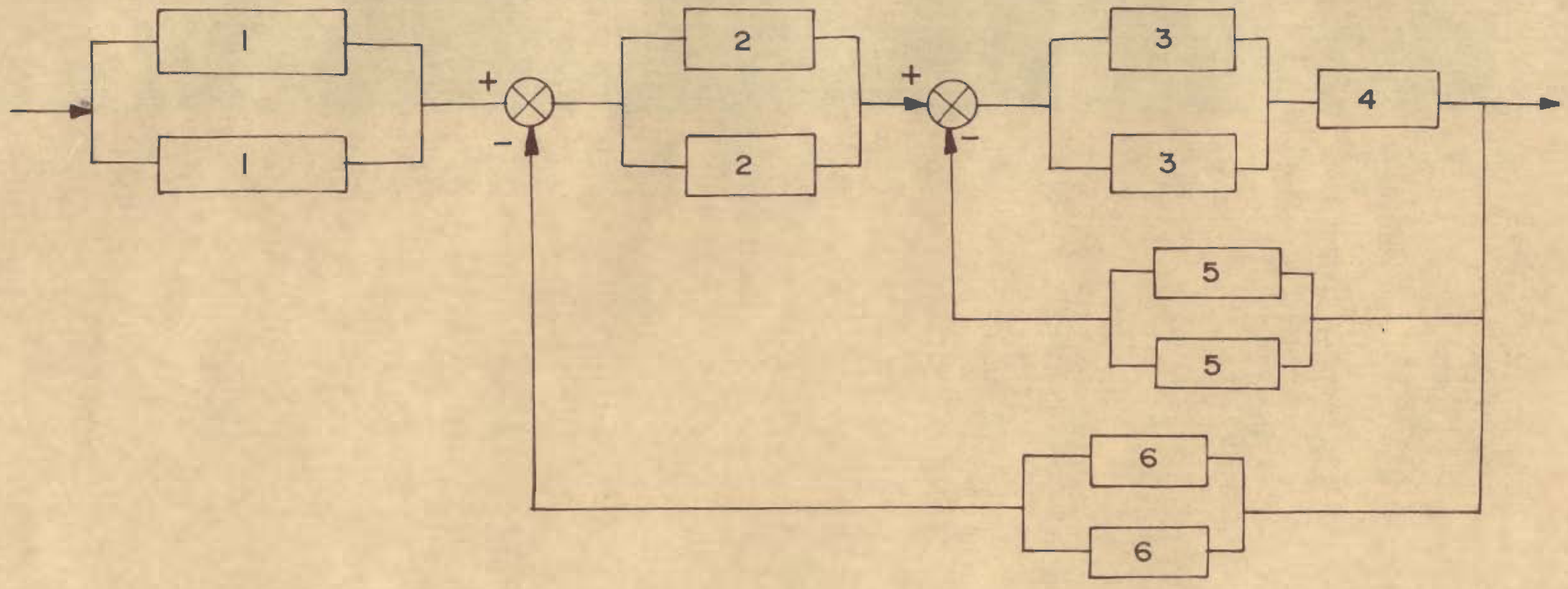


FIG. 3.7 REDUNDANT FEED-BACK CONTROL SYSTEM.

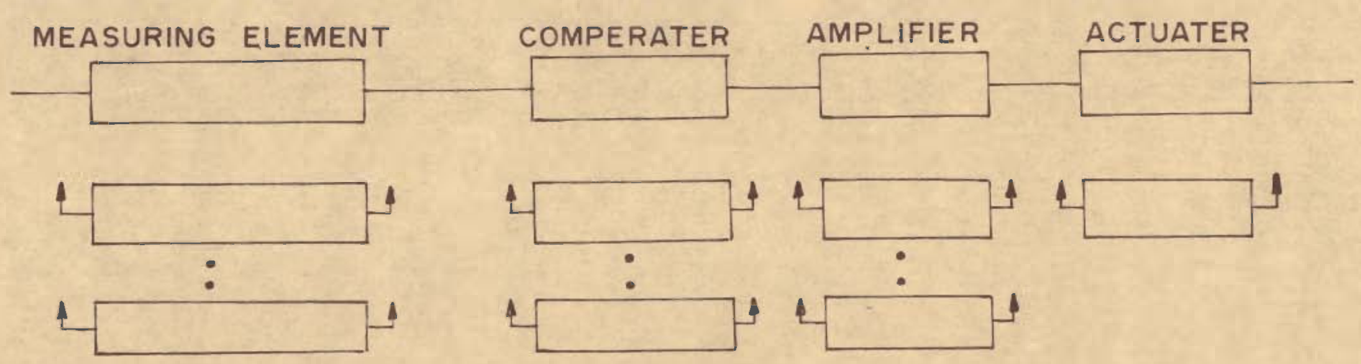


FIG. 3.8 CONTROL SYSTEM WITH SPARE STAND BY REDUNDANT COMPONENTS.

Table 3.9

S t a g e	λ_j failure rate per year	a_j	f_j
1. Measuring element	0.0798	1.0	7.0
2. Amplifier	0.0328	2.0	8.0
3. Comparator	0.0066	3.0	6.0
4. Actuator	0.026	4.0	9.0

The linearized reliability problem can be written as

Minmize

$$0.58623\bar{x}_{11}+0.16281\bar{x}_{12}+0.03915\bar{x}_{13}+0.28343\bar{x}_{21}+0.03064\bar{x}_{22}$$
$$+0.00423\bar{x}_{23}+0.06385\bar{x}_{31}+0.00204\bar{x}_{32}+0.00004\bar{x}_{33}$$
$$+0.23091\bar{x}_{41}+0.02642\bar{x}_{42}+0.00221\bar{x}_{43} \qquad (3.40)$$

subject to the constraints

$$\bar{x}_{11}+3\bar{x}_{12}+5\bar{x}_{13}+2\bar{x}_{21}+6\bar{x}_{22}+10\bar{x}_{23}+3\bar{x}_{31}+9\bar{x}_{32}+15\bar{x}_{33}$$
$$+4\bar{x}_{41}+12\bar{x}_{42}+20\bar{x}_{43} \geq 54 \qquad (3.41)$$

$$8.988\bar{x}_{11}+14.094\bar{x}_{12}+21.375\bar{x}_{13}+10.272\bar{x}_{21}+16.107\bar{x}_{22}$$
$$+24.428\bar{x}_{23}+7.704\bar{x}_{31}+12.08\bar{x}_{32}+18.321\bar{x}_{33}+11.556\bar{x}_{41}$$
$$+18.12\bar{x}_{42}+27.482\bar{x}_{43} \geq 40.530 \qquad (3.42)$$

$$x_{jl} = 0 \text{ or } 1 \ , \ j=1,..,4 \ , \ l=1,...,3$$

(1) Set all variables free.

(2) The reduced sets are $S_1 = (\bar{x}_{33}, \bar{x}_{42}, \bar{x}_{43})$,

$S_2 = (\bar{x}_{12}, \bar{x}_{13}, \bar{x}_{22}, \bar{x}_{23}, \bar{x}_{32}, \bar{x}_{33}, \bar{x}_{42}, \bar{x}_{43})$,

Variable to be assigned as 1 is $\bar{x}_{33}$. Resulting state

$d'_1 = 39$, $d'_2 = 22.206$.

(3) The reduced sets are $S_1 = (\bar{x}_{23}, \bar{x}_{42}, \bar{x}_{43})$,

$S_2 = (\bar{x}_{12}, \bar{x}_{13}, \bar{x}_{22}, \bar{x}_{23}, \bar{x}_{32}, \bar{x}_{41}, \bar{x}_{42}, \bar{x}_{43})$.

Variable to be assigned as 1 is $\bar{x}_{43}$. Rêsulting state

$d'_1 = 19$, -.

(4) The reduced sets are $S_1 = (\bar{x}_{22}, \bar{x}_{23}, \bar{x}_{32}, \bar{x}_{42})$,

$S_2 = \emptyset$. Variable to be set to 1 is $\bar{x}_{32}$. Resulting state

$d'_1 = 10$, -.

(5) The reduced sets are $S_1 = (\bar{x}_{13}, \bar{x}_{22}, \bar{x}_{23}, \bar{x}_{41}, \bar{x}_{42})$,

$S_2 = \emptyset$. Variable to be set to 1 is $\bar{x}_{23}$. Resulting state

$d'_1 = 0$, -.

(6) The optimum solution is

$$\bar{x}_{23} = \bar{x}_{32} = \bar{x}_{33} = \bar{x}_{43} = 1$$

$$\bar{x}_{1} = \bar{x}_{12} = \bar{x}_{13} = \bar{x}_{21} = \bar{x}_{22} = \bar{x}_{31} = \bar{x}_{41} = \bar{x}_{42} = 0$$

and the optimum solution of the original reliability problem is $n_1=3$, $n_2=2$, $n_3=1$, $n_4=2$.

3.6. BRANCH AND BOUND METHOD

The linearized reliability problem (2.70 and 2.71) can be converted into an equivalent knapsack problem by aggregating the constraints which have integer coefficients. When constraints have rational coefficients, they can be converted into integer coefficients by multiplying constraints by a suitable multiplier. Consider a set of two constraints

$$\sum_{j=1}^{w} h_{ij}\ z_j + z_{w+i} = e_i \qquad i = 1, 2, \ldots \tag{3.43}$$

where z_{w+i} are the slack variables. These two equations can be combined by choosing two suitable multipliers t_1 and t_2 such that, one of the following conditions holds good [41]

(a) $t_1 \geq u_2+1$ and $t_2 \geq u_1+1$

(b) $t_1 \geq -L_2+1$ and $t_2 \geq -L_1+1$

(c) $t_1 > 0$ arbitrary and $t_2 \geq \max\left[u_1+1,\ -L_1+1\right]$

(d) $t_1 \geq \max\left[u_2+1,\ -L_2+1\right]$ and $t_2 > 0$ arbitrary (3.44)

where,

$$L_i = -e_i \quad i=1,2,\ldots,s$$

$$u_i = \sum_{j=1}^{w} h_{ij} \tag{3.45}$$

The single equivalent constraint which has the same solution as the original constraints (3.43) can be written as

$$t_1\left[\sum_{j=1}^{w} h_{1j}\ z_j+z_{w+1}-e_1\right]+t_2\left[\sum_{j=1}^{w} h_{2j}\ z_j+z_{w+1}-e_2\right] = 0 \tag{3.46}$$

Recursively using the above construction for all constraints, it results in a single equivalent constraint which has same feasible solution as the original problem. Therefore, the reliability problem can be converted into an equivalent Knapsack type problem. The equivalent Knapsack type problem for (2.70) and (2.71) can be written as

Maximize F(Z)

$$F(Z) = \sum_{j=1}^{w} g_j z_j$$

subject to the equivalent constraint

$$\sum_{j=1}^{w} y_j z_j \leq v \tag{3.47}$$

$$z_j = 0 \text{ or } 1$$

The stepwise procedure for solving the above problem by Branch and Bound method can be described as follows:

(1) List the variables such that their coefficients g_j/y_j [j=1, 2,...,w] are in descending order.

(2) Consider node A where all variables are free. Set N=A and l=1.

(3) From tree by branching two nodes B_l and C_l from N by C_l from N by setting $z_j^l=1$ and $z_j^l=0$, respectively.

(4) Find the upper bound on the feasible solution at node B_l and C_l.

(5) If B_l has greater upper bound on F(Z) than at C_l, assign z_j^l as 1 and $N=B_l$, otherwise assign z_j^l as 0 and $N=C_l$.

(7) If l>w go to next step, else set l=l+1 and go to step 3.

(8) Find the corresponding index j of l from list and stop.

If z_j are set to 1 in the sequence as given in the list [l=1, 2,...], while satisfying the constraint, will give an upper bound on the feasible solution. If z_j having l^{th} index in the list breaks a constraint or constraints, then z_j corresponding to $(l+1)^{th}$ index is tried.

AN EXAMPLE:

Consider a system of an aerospace computer consisting of coincidence circuit, amplifiers, regenerators and flipflops as dhown in Fig. 3.9. The complete system is divided into seven subsystems. The reliability cost, weight and power requirement for each subsystem is tabulated in Table 3.10. The reliability of the majority voter is 0.999. It is required to increase the reliability of the system by using triple modular redundancy. The cost, weight and power consumption of the system must not exceed 66, 60 and 70 units, respectively. The cost, weight and power consumption of the voter is 3, 4 and 2 units, respectively.

The reliability problem in the form of (2.70) and (2.71) can be written as

Table 3.10 - The Parameters of the System

Stage Number	No. of circuits	r_j	p_j	Cost	Weight	Power consumption
1	1, 2	0.99	0.6	5.0	2.0	4.0
2	3, 4, 5	0.99	0.2	10.0	7.0	10.0
3	6	0.97	0.6	2.0	1.0	1.0
4	7	0.92	0.2	4.0	4.0	5.0
5	8	0.94	0.6	3.0	3.0	4.0
6	9	0.94	0.6	3.0	3.0	4.0
7	10	0.95	0.6	5.0	1.0	2.0

Maximize F(Z)

$$F(Z) = 0.00849z_1+0.00984z_2+0.02807z_3+0.06078z_4+0.055369z_5 +0.055369z_6+0.046452z_7 \quad (3.48)$$

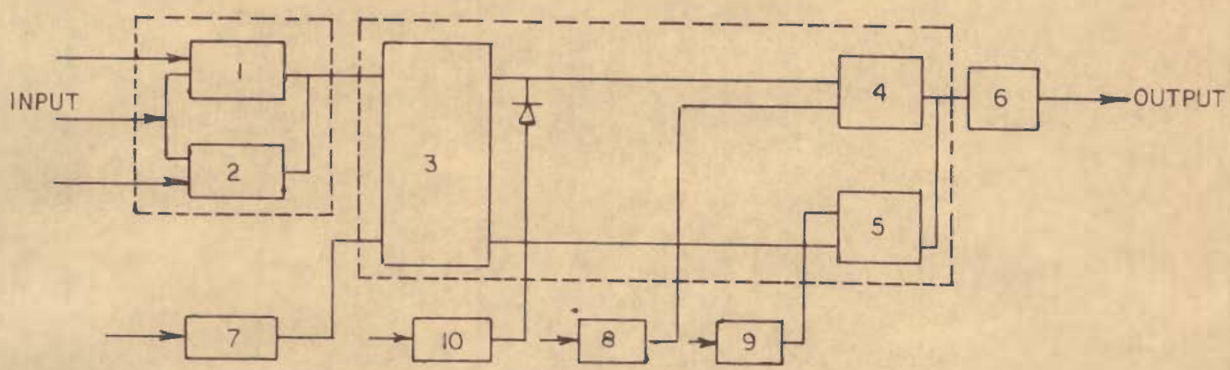


FIG 3.9a A SYSTEM OF A COMPUTER

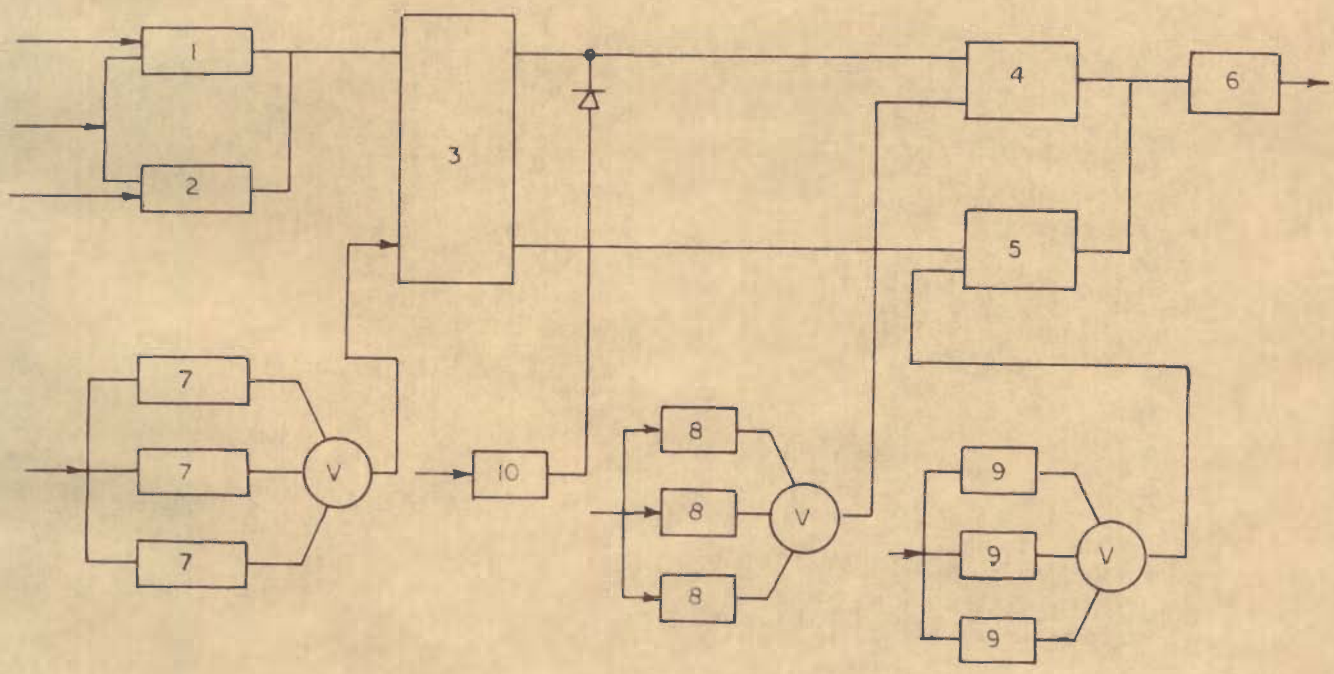


FIG.3.9b OPTIMUM STRUCTURE OF THE SYSTEM SHOWN IN FIG.3.9a

subject to the constraints

$$13z_1+23z_2+7z_3+11z_4+9z_5+9z_6+13z_7 \quad \leq \quad 34 \tag{3.49}$$

$$8z_1+18z_2+6z_3+12z_4+10z_5+10z_6+6z_7 \quad \leq \quad 39 \tag{3.50}$$

$$10zz_1+22z_2+4z_3+12z_4+10z_5+10z_6+6z_7 \quad \leq \quad 40 \tag{3.51}$$

$$z_j=0 \quad \text{or} \quad 1$$

Constraints (3.49) and (3.50) are combined by choosing t_1 and t_2 as 72 and 1 according to (3.44). The equivalent constraint is

$$944z_1+1674z_2+510z_3+810z_4+658z_5+658z_6+$$
$$+942z_7+72z_8+z_9 \quad = \quad 2478 \tag{3.52}$$

(3.51) and (3.52) can be combined by using $t_1=1$ and $t_2=76$ according to (3.44) resulting in

$$71754z_1+127246z_2+38764z_3+61112z_4+50018z_5+50018z_6$$
$$+71598z_7+5472z_8+76z_9+z_{10} \quad = \quad 189052$$

Dropping slack variables z_8, z_9 and z_{10}, the equivalent constraint is

$$71754z_1+127246z_2+38764z_3+61112z_4+50018z_5+50018z_6$$
$$+71598z_7 \quad \leq \quad 189052 \tag{3.53}$$

The equivalent reliability problem is given by (3.48) and (3.53). Arranging z_j in the order of descending coefficients g_j/y_j as given in Table 3.11.

Setting all variables free at node A as shown in Fig. 3.11, node B_1 and C_1 can be branched by setting $z_5=1$ and $z_5=0$, respectively. At node B_1 the feasible solution is $z_5=z_6=z_4=0$,

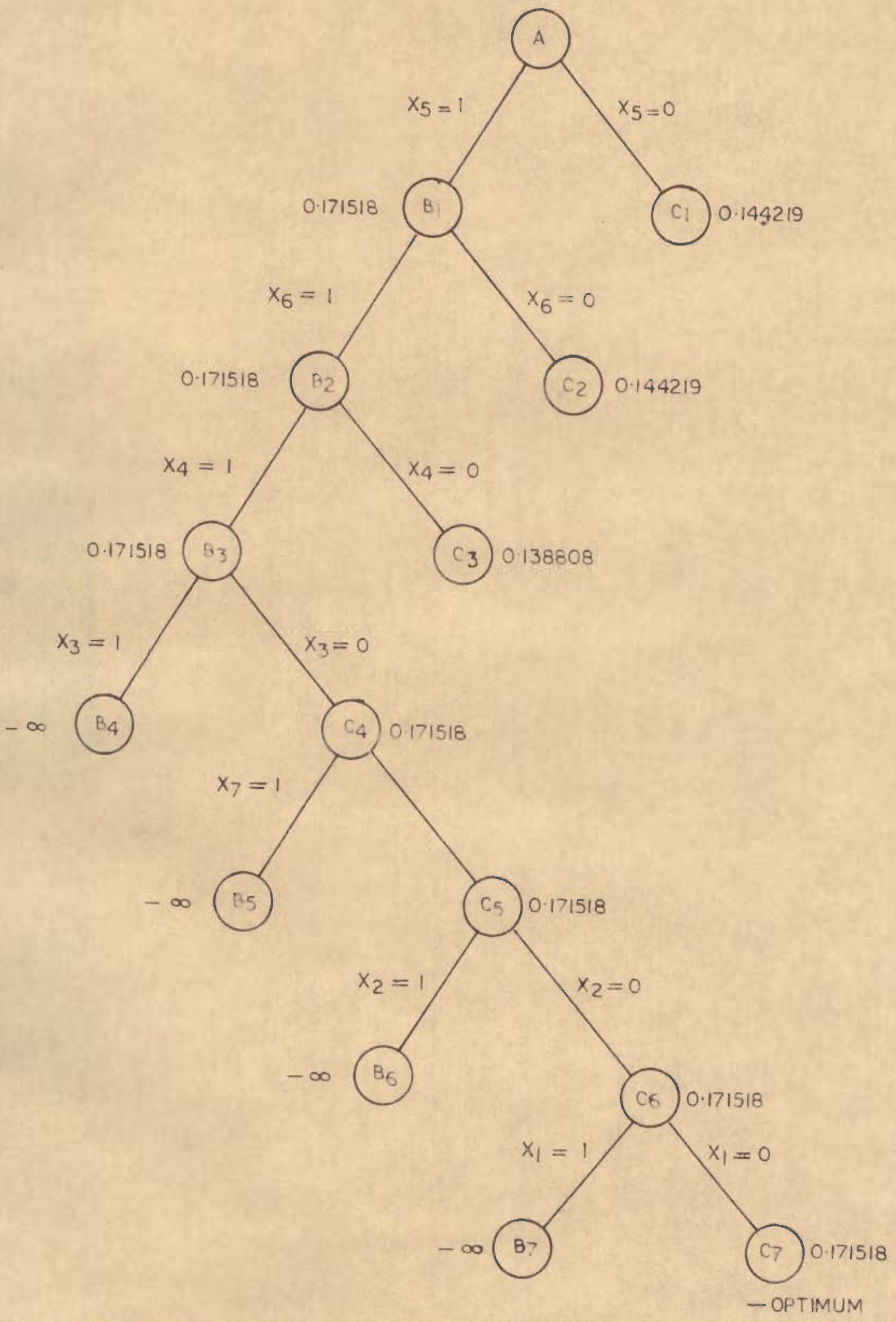


FIG. 3.10 A TREE DIAGRAM OF NUMERICAL EXAMPLE

Table 3.11

Index Number l	1	2	3	4	5	6	7
g_j	.055369	.055369	.06078	.02807	.046452	.00984	.00844
y_j	51334	51334	62724	39784	73638	130596	73642
Stage Number j	5	6	4	3	7	2	1

giving an upper bound on objective as 0.171518. At node C_1 the feasible solution is $z_6=z_4=z_3=1$, $z_5=z_1=z_2=z_7=0$ resulting in an upper bound on objective function as 0.144219. Therefore, further branching is to be done from node B_1. The complete calculations are shown in Fig. 3.1. The optimum solution obtained is $z_5=z_6=z_4=1$ and $z_1=z_2=z_3=z_7=0$, giving system reliability as 0.9257. The optimum structure of the system (Fig.3.9) is shown in Fig. 3.10.

3.7. A DIRECT SEARCH METHOD

A simple computational procedure is developed in this section. It can be used to solve reliability problem having parallel, standby, majority voting, hybrid redundancy. By taking the logarithm of the expression (2.1), it changes to

$$\text{Ln } R_s(n) = \sum_{j=1}^{k} \text{Ln } R_j(n_j)$$

or
$$\frac{\partial R_s(n)}{\partial R_j(n_j)} = \frac{R_s(n)}{R_j(n_j)}$$

that is, increment in the system reliability will be maximum if stage l satisfies the following condition:

$$R_l(n_l) = \min_{1 \leq j \leq k} R_j(n_j) \qquad (3.54)$$

Therefore for maximizing the system reliability, one component must be added to the l^{th} stage. Intuitively, it can be said that if we go on adding one component, i.e. increasing the decision variable by one, to that particular stage which satisfies the condition given by (3.54) without violating the constraints, total increment in the system reliability will be maximum. When decision variables reach in the neighbourhood of the boundary of its feasible region, active constraint is found out by calculating the slack. From active constraint, a feasible set of stages (J) is calculated, in which the increment in the stage reliability is possible. Again, test (3.54) is made for finding the candidate stage (i.e. the stage in which one more component can be added), from set (J). This procedure is to be repeated until set (J) becomes empty. If more than one constraint are active, the candidate stage must be common to each set calculated from each active constraint. The complete procedure can be explained stepwise by dividing it into two phases as given below.

Algorithm:

Phase I:

(1) Initially set $n_j=1$ for all j $(1 \leq j \leq k)$, that is, system is considered to be irredundant.

(2) Find the stage l which satisfies the following condition

$$R_l(n_l) = \min_{1 \le j \le k} R_j(n_j)$$

In case of tie, evaluate S_{min} and select that stage which has lowest $a_{min,j}$, where S_{min} is given by

$$S_{min} = \min_{i} \left[b_i - \sum_{j=1}^{k} (a_{ij} n_j) \right] \qquad (3.55)$$

(3) Assign $n_l = n_l + 1$

(4) Check constraints, if not violated go to step 2, else go to next step.

(5) Set $n_l = n_l - 1$, which will be the optimal number of redundant components to be allocated to the current l^{th} stage.

Phase II:

(6) Evaluate S_{min}.

(7) If $S_{min} = 0$, stop; else find out set (J) which is defined as

$$j \in J \text{ if } y_j \ge 1$$

where $y_j = S_{min}/a_{min,j}$ (3.56)

(8) If set is empty stop, else go to next step.

(9) Select stage which satisfies the condition

$$R_l(n_l) = \min_{j \in J} R_j(n_j)$$

In case of tie, choose that stage which has lowest $a_{min,j}$.

(10) Set $n_l = n_l + 1$ and go to step 6. When same constraint is active in the next iteration of Phase II, corresponding new set (J) can be calculated from the old set (J) cal-

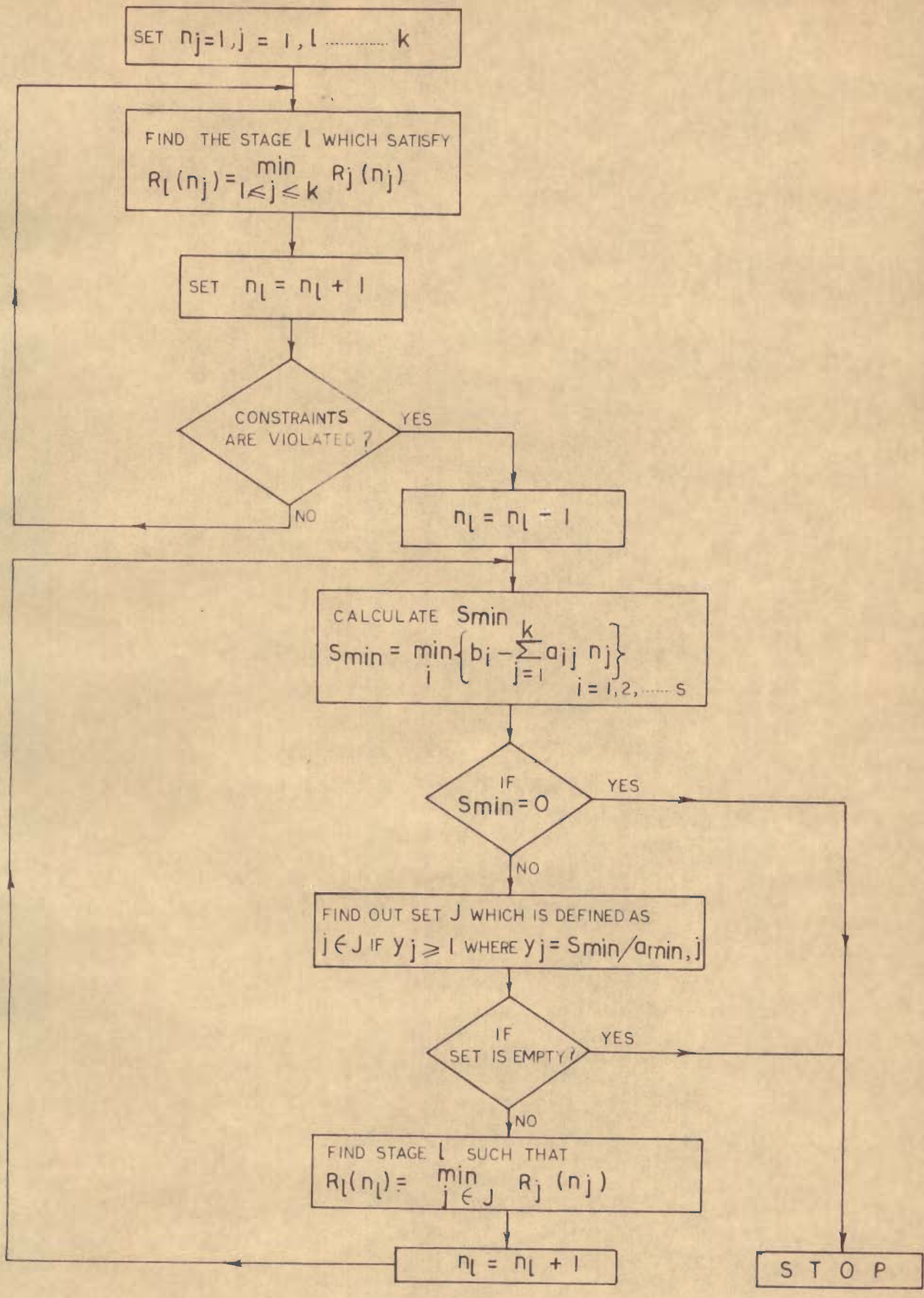


FIG. 3.11 FLOW CHART FOR A DIRECT SEARCH METHOD

-culated in the previous iteration.

A flow chart for this method is shown in Fig. 3.11. A number of problems were solved by this method and satisfactory results were obtained.

EXAMPLE 1:

Consider a digital system shown in Fig. 3.12, where different blocks represent the logic elements. All blocks are required for the successful operation of the system. The reliability, cost, power consumption of each stage or block are given in Table 3.12.

The system reliability is to be maximized by using majority voter redundancy, while total cost, volume and power consumption of the system must not exceed 125, 350 and 100 units, respectively. It is assumed that external inputs to the system are perfectly reliable.

Table 3.12

Stage number	Reliability	Cost	Volume	Power consump -tion
1	0.900	10.0	16.0	4.0
2	0.99	16.0	21.0	2.0
3	0.880	18.0	30.0	6.0
4	0.980	3.0	25.0	12.0
5	0.950	8.0	30.0	15.0
Voter	0.99	5.0	10.0	8.0

The reliability expression for a k-stages system having majority voting redundancy can be expressed as

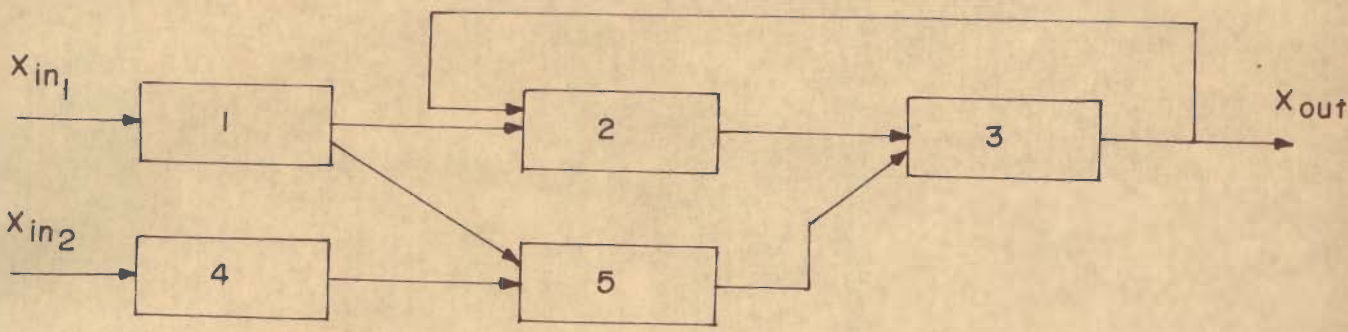


FIG.3.12a BLOCK DIAGRAM OF A DIGITAL SYSTEM.

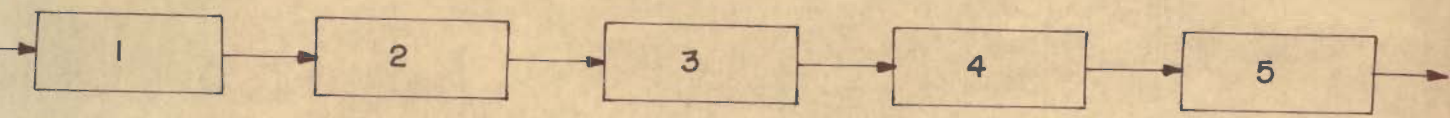


FIG.3.12b LOGIC DIAGRAM OF THE SYSTEM SHOWN IN FIG.3.12a

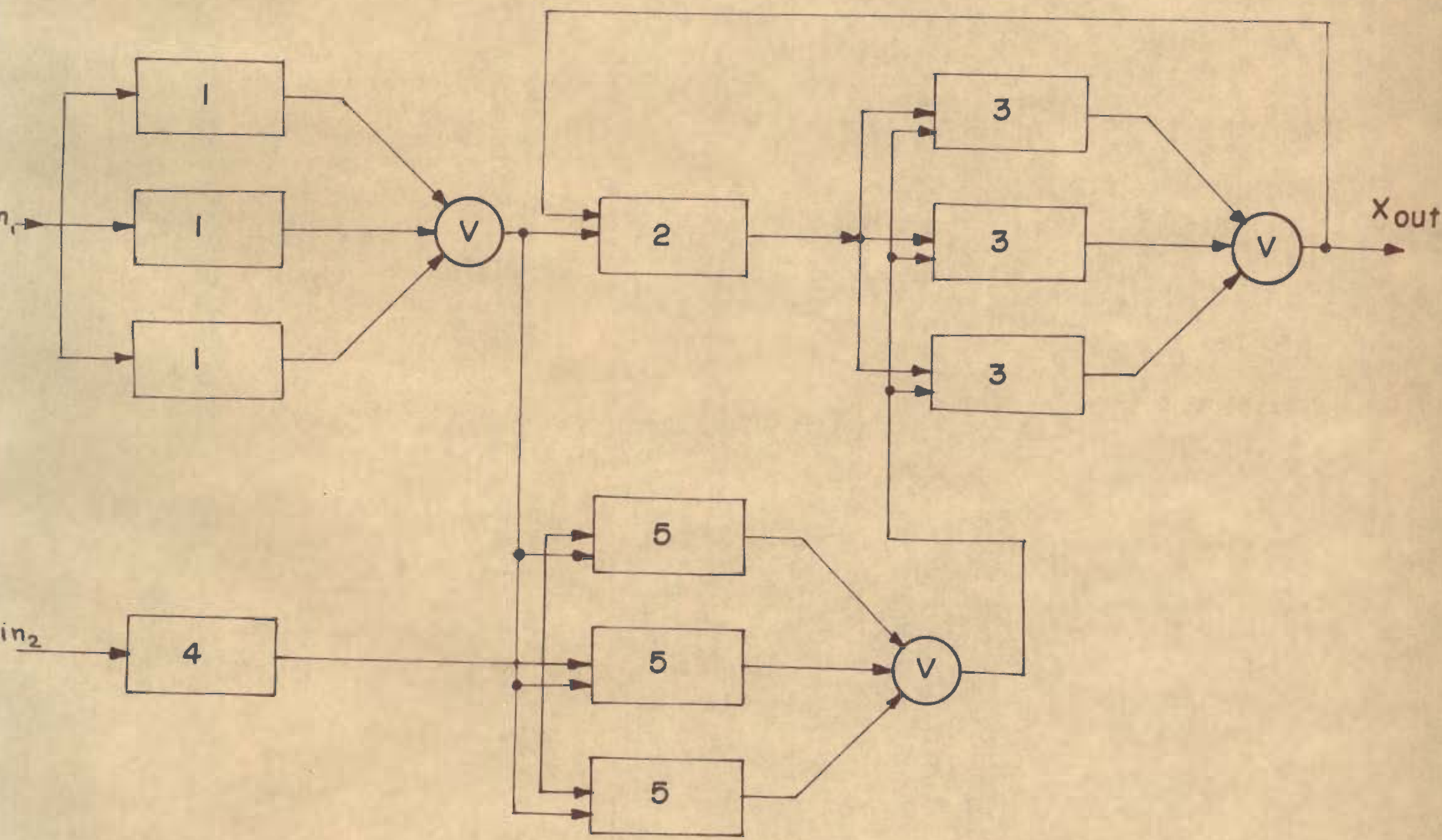


FIG.3.13 OPTIMUM REDUNDANT STRUCTURE OF THE SYSTEM SHOWN IN FIG.3.12a

$$R_s(n) = \prod_{j=1}^{k} R_j(n_j)$$

$$\text{or} \quad Ln\, R_s(n) = \sum_{j=1}^{k} R_j(n_j)$$

$$\text{where} \quad R_j(n_j) = R_v \sum_{i=n_j+1}^{2n_j+1} \frac{(2n_j+1)!}{(2n_j+1-i)!\, i!}$$

The n_j will have value equal to zero if j^{th} stage is non-redundant. Therefore, initially all n_j are set to zero. The candidate stage is 3. With $n_3=1$ and other $n_j=0$, constraints are checked which are within limit. The next candidate stage is 1 and again constraints are checked with $n_3=n_1=1$ and $n_2=n_4=n_5=0$. We proceed in the similar way and finally get $n_1=n_3=n_5=1$ and $n_2=n_4=0$ as an optimal solution. The optimum redundant structure is shown in Fig. 3.12.

EXAMPLE 2:

The use of parallel redundancy is to be made for maximizing the system reliability with three nonlinear constraints. The system is shown in the following table.

Stage	1	2	3	4	5
Element Reliability	0.80	0.85	0.90	0.65	0.75

The constraints are

$$g_1(n) = n_1^2+2n_2^2+3n_3^2+4n_4^2+2n_5^2 \leq 110$$

$$g_2(n) = 7(n_1+e^{n_1/4})+7(n_2+e^{n_4/4})+5(n_3+e^{n_3/4})$$

$$+9(n_4+e^{n_4/4})+4(n_5+e^{n_5/4}) \leq 175$$

Table 3.13

Number of Components in Stage					Unreliability of Stage					$g_1(n)$	$g_2(n)$	$g_3(n)$
n_1	n_2	n_3	n_4	n_5	1	2	3	4	5			
1	1	1	1	1	0.2	0.15	0.1	0.35[a]	0.25	12	73.1	48.8
1	1	1	2	1	0.2	0.15	0.1	0.1225	0.25	24	85.4	60.8
1	1	1	2	2	0.2[a]	0.15	0.1	0.1225	0.025	30	90.8	79.0
2	1	1	2	2	0.04	0.15[a]	0.1	0.1225	0.0625	33	100.4	93.3
2	2	1	2	2	0.04	0.0225	0.1	0.1225[a]	0.0625	39	109.9	109.2
2	2	1	3	2	0.04	0.0225	0.1[a]	0.042875	0.0625	59	123.1	127.5
2	2	2	3	2	0.04	0.0225	0.01	0.042875	0.0625[a]	68	130.0	143.6
2	2	2	3	3	0.04[a]	0.0225	0.01	0.042875	0.015625	78	136.0	171.1
3	2	2	3	3	0.008	0.0225	0.01	0.042875	0.015625	83	146.1	192.5

[a]This is the stage to which a redundant component is to be added.

$$g_3(n) = 7n_1 e^{n_1/4} + 8n_2 e^{n_2/4} + 8n_3 e^{n_3/4} + 6n_4 e^{n_4/4} + 9n_5 e^{n_5/4} \leq 200$$

Starting with n=(1,1,1,1,1), add one element at a time as shown in Table 3.13, hence we obtain the optimum number of redundant components

$$n = (3, 2, 2, 3, 3)$$

Many problems were tried and exact results were received. Due to less memory requirement and computation effort this method is suitable for optimal design of a large system from reliability consideration.

3.8. A SIMPLE METHOD

A simple rule is usedin this section to find an equivalent problem having only one constraint. This equivalent problem has the same number of variables and feasible solutions as the original problem. It is easier to solve an equivalent problem rather than to solve the original problem with many constraints, which is generally computationally tedious for a practical system with many stages. A simple method is developed to solve this equivalent problem.

AGGREGATING CONSTRAINTS:

By adding the slack variables n_{k+1} [i=1,2,...s] the inequalities (3.2) are transformed into the equalities, as

$$\sum_{j=1}^{k} a_{ij} n_j + n_{k+i} = b_i \qquad i=1,2,\ldots,s \tag{3.57}$$

Consider the first two constraints, i.e. $i = 1,2$,

$$\sum_{j=1}^{k} a_{1j}\, n_j + n_{k+1} = b_1 \tag{3.58}$$

and

$$\sum_{j=1}^{k} a_{2j}\, n_j + n_{k+2} = b_2 \tag{3.59}$$

They can be combined to form a new constraint by using multipliers t_1 and t_2 satisfying the following conditions as derived by Glover and Woolsey [42].

(1) t_1 and t_2 should be relatively prime.

(2) t_1 does not divide b_2 and t_2 does not divide b_1.

(3) $t_1 > b_2 - a_2$ and $t_2 > b_1 - a_1$ where a_i represents the smallest of the positive a_{ij}. (3.60)

Then equivalent constraint which has the same solution as the constraints (3.58) and (3.59), can be written as

$$t_1 \sum_{j=1}^{k} a_{1j}\, n_j + n_{k+1} + t_2 \sum_{j=1}^{k} a_{2j}\, n_j + n_{k+2} = t_1 b_1 + t_2 b_2 \tag{3.61}$$

Recursively using the construction (3.61) for all constraints, the reliability problem reduces to the maximization of the reliability function subject to a single constraint, i.e.

Maximize system reliability

$$\ln R_s(n) = \sum_{j=1}^{k} \ln R_j(n_j) \tag{3.62}$$

subject to the constraint

$$\sum_{j=1}^{k} y_j \, n_j \leq b' \tag{3.63}$$

SOLUTION PROCEDURE:

The reliability problem given by (3.62, 3.63) can be solved by any standard reliability optimization method in which the presence of single constraint is advantageous. Here, a simple method is developed for finding the optimal solution of (3.63).

ALGORITHM:

(1) Calculate the derivatives of (3.62), d_j with respect to n_j at $n_j=1$.

(2) Replace n_j in (3.63) by $(1+st.d_j)$ and solve it for st.

(3) Select the stage which has lowest reliability. Let l^{th} be the stage satisfying this condition. Set n_l equal to $n_l+\Delta n_l$, where Δn_l is the integral part of the $st d_l$. Modify the resources and remove this stage from calculations.

(4) If all stages are removed from calculations, stop; else go to step 2.

EXAMPLE:

Consider a system having two stages in series. The component reliability, cost, weight, volume and power consumption data are as follows:

Stage	Component reliability	Cost	Weight	Volume	Power consumption
1	0.99	1	2	3	5
2	0.98	3	4	4	2

Find the optimum allocation of the redundancy for maximizing the system reliability. Total cost, weight, volume and power consumption must not exceed 8, 10, 15 and 10 units, respectively.

Adding slack variables n_3, n_4, n_5 and n_6, the equality constraint on the system can be written as

$$n_1+3n_2+n_3+0n_4+0n_5+0n_6 = 8 \quad (3.64)$$

$$2n_1+4n_2+0n_3+0n_4+0n_5+0n_6 = 10 \quad (3.65)$$

$$3n_1+4n_2+0n_3+0n_4+0n_5+0n_6 = 15 \quad (3.66)$$

$$5n_3+2n_2+0n_3+0n_4+0n_5+0n_6 = 10 \quad (3.67)$$

Combining the constraints (3.64) and (3.65) by choosing suitable values of the multipliers, to form a new constraint, $t_1=11$ and $t_2=9$ satisfy the conditions given by (3.60). The new constraint is

$$29n_1+69n_2+11n_3+9n_4 = 178 \quad (3.68)$$

Now the constraints (3.68) and (3.66) are combined by choosing $t_1=16$ and $t_2=170$, to form an equivalent constraint which can be written as

$$974n_1+1784n_2+176n_3+144n_4+170n_5 = 5398 \quad (3.69)$$

For combining (3.69) and (3.67), the suitable values of t_1 and t_2 are 11 and 5255, respectively, giving

$$36989n_1+30134n_2+1936n_3+1584n_4+1370n_5 +5255n_6 = 111928 \quad (3.70)$$

Dropping slack variables the equivalent inequality constraint on the system is

$$36989n_1 + 3013n_2 \leq 111928 \qquad (3.71)$$

The original reliability problem reduces to the maximization of the system reliability $R_s(n)$ subject to the single constraint (3.71). Derivatives of $1-(1-r_j)^{n_j}$ are calculated with respect to n_1 and n_2 and n_1 and n_2 in (3.71) are replaced by (1+.22·st) and (1+.139·st). The st is obtained as 3.6. As described above, the stage to be selected is 2, as it has lowest reliability. Therefore, n_2 is set as 2. Same procedure is repeated for n_1 and n_1 is found out to be one. Therefore, the optimum number of components to be used in stage 1 and 2 are one and two, respectively.

The solution of the reliability problem is obtained by solving an equivalent problem having only one constraint. The generality of this method is not bound by the requirement of the integer coefficients of the constraints, as any irrational number can be approximated by a rational number, which in turn can be converted into an integer form by multiplying the inequality by a suitable factor. Reliability problem with nonlinear constraints can also be solved by linearizing them.

3.9. LEXICOGRAPHIC ENUMERATION TECHNIQUE

The nonlinear integer reliability problem can be converted into zero-one nonlinear programming problem by replacing n_j by binary vector x_{jl} having numerical ordering as

$$n_j = 1 + x_{j1} + 2^1 x_{j2} + \ldots + \ldots 2^{l-1} x_{jl}$$

where x_{jl}=0 or 1. l is chosen to be sufficiently large for 2^{l-1} to be an upper bound on the value of n_j. But the condition is that the objective function and the constraints should be monotone non-increasing in each of the variables x_{jl}. The reliability problem can be stated as

Maximize system reliability

$$\text{Ln } R_s(n) = \sum_{j=1}^{k} \text{Ln } R_j(n_j) \tag{3.72}$$

subject to the constraints

$$\sum_{j=1}^{k} G_{ij}(n_j) - b_i \leq 0 \qquad i=1,2,\ldots,s \tag{3.73}$$

n_j should be integer

In terms of the binary variables x_{jl}, the reliability problem can be expressed as

Maximize

$$\text{Ln } R_s(X) = \sum_{j=1}^{k} \text{Ln } R_j(x_{jl}) \tag{3.74}$$

subject to the constraints

$$\sum_{j=1}^{k} G_{ij}(x_{jl}) - b_i \leq 0 \qquad i=1,2,\ldots,s \tag{3.75}$$

x_{ji} =0 or 1 $\qquad$ i=1,2,...,l

Usually in the reliability problem, objective function and the constraints are increasing function of the variables x_{jl}. This can be converted into non-increasing function by replacing x_{jl} by $(1-\bar{x}_{jl})$. That is: n_j can be given by

$$n_j = 1 + \sum_{i=1}^{l} 2^{i-1} (1 - \bar{x}_{ji}) \qquad (3.76)$$

Therefore the reliability problem can be restated as

Maximize

$$G_o(\bar{x}) = Ln\ R_s(\bar{x}) = \sum_{j=1}^{k} R_j(\bar{x}_{jl})$$

subject to

$$\sum_{j=1}^{k} G_{ij} (\bar{x}_{jl}) - b_i \leq 0 \qquad (3.77)$$

$$i = 1, 2, \ldots, s$$

$$\bar{x}_{jl} = 0 \text{ or } 1$$

If the objective function $G_o(\bar{x})$ is not monotone non-increasing as in case of systems having mixed redundant components, a new constraint is added. The reliability problem in this case will result in

Maximize

$$-x_o$$

subject to

$$-x_o - G_o(\bar{x}) \leq 0$$

$$\sum_{j=1}^{k} G_{ij} (\bar{x}_{jl}) - b_i \leq 0 \qquad (3.78)$$

$$i = 1, 2, \ldots, s$$

$$\bar{x}_{jl} = 0 \text{ or } 1$$

The above problem can be solved by total enumerating of the binary vector in lexicographically increasing order. The best suited numerical ordering is

$$t(\bar{y}) = (\bar{x}_{k1}\ldots\bar{x}_{21}\bar{x}_{11}\ldots\ldots\bar{x}_{k2}\ldots\bar{x}_{22}\bar{x}_{12}\bar{x}_{k1}\ldots\bar{x}_{21}\bar{x}_{11}) \quad (3.78)$$

Using this ordering and some other skipping rules faster convergence is achieved than [43] and [31]. In order to avoid total enumeration, certain skipping rules can be used. If the current binary vectors $\bar{x}$ are ordered by $t(\bar{x})$, the skipping will result the next vector to be enumerated as x^*. For given vector $(\bar{x})$, x^* can be found out [14] by the following method.

Let the right-most position of one in $\bar{x}$ be u and the position of right-most 0 to the left of u be v. The x^* vector can be obtained from $\bar{x}$ by

1 - putting $x^*_v = 1$

2 - putting $x^*_i = 0 \quad v+1 \leq i \leq u$

3 - putting $x^*_i = \bar{x}_1 \quad 1 \leq i \leq v-1$

where u is the total length of vector $\bar{x}$. The stepwise procedure [14] for solving above problem can be explained as

ALGORITHM

(1) Set $\bar{X} = (0,\ldots 0)$. If it is feasible to (3.77), stop and it is an optimal solution. Else, set $\bar{X} = (0,0,\ldots 1)$ and $\bar{G}_o = -\infty$

(2) If $G_o(\bar{x}) \leq \bar{G}_o$, go to step 5. Else go to next step.

(3) If $\bar{x}$ is feasible to (3.77), set $\bar{G}_o = G_o(\bar{x})$ and go to step 5. Else go to next step.

(4) If x^* exist and for some i

$\sum_{j=1}^{k} G_{ij}(x^*-1) - b_i > 0$, go to step 5. Else if $\bar{x}=(1,\ldots 1)$ go to step 6; otherwise replace $\bar{x}$ by $\bar{x}+1$ and go to step 2.

(5) If x^* does not exist, go to next step. Otherwise set $\bar{x}=x^*$ and go to step 2.

(6) Terminate.

NUMERICAL EXAMPLE:

Consider a system consisting of two stages. The reliability, cost and weight parameters of the components are given below. It is required to find the optimal number of parallel components to be employed in each stage to increase the system reliability. The total cost and weight of the system must not exceed 40 and 30 units, respectively.

Stage number	one	two
Component reliability	0.91	0.96
Cost	9	6
Weight	5	8

Mathematically, the reliability problem can be written as

Maximize $\quad \text{Ln } R_s(n)$

$$\text{Ln } R_s(n) = \text{Ln}(1-0.09^{n_1}) + \text{Ln}(1-0.04^{n_2})$$

subject to

$$9n_1+6n_2-40 \le 0$$

$$5n_1+8n_2-30 \le 0$$

Let each stage not have more than three components. With the help of (3.76), the reliability problem in terms of binary variables can be written as

Maximize $\quad Ln\ R_s(\bar{X})$

$$Ln\ R_s(\bar{X}) = Ln(1-0.09^{4-\bar{x}_{11}-2\bar{x}_{12}}) + Ln(1-0.04^{4-\bar{x}_{21}-2\bar{x}_{22}})$$

subject to the constraints

$$-9\bar{x}_{11}-18\bar{x}_{12}-6\bar{x}_{21}-12\bar{x}_{22}+20 \leq 0$$

$$-5\bar{x}_{11}-10\bar{x}_{12}-8\bar{x}_{21}-16\bar{x}_{22}+22 \leq 0$$

$$\bar{x}_{jl} = 0 \text{ or } 1 \qquad j=1,2$$

$$l=1,2$$

The solution sequence is given in Table 3.14. Initially, $\bar{G}_o$ is set as $-\infty$ and $\bar{X}$ as (0,0...0).

Table 3.14

TEST VECTOR				COMMENTS
$\bar{x}_{22}$	$\bar{x}_{12}$	$\bar{x}_{21}$	$\bar{x}_{11}$	
0	0	0	1	Step 4, i=1,2, skip to x*
0	0	1	0	Step 4, i=1,2, skip to x*
0	1	0	0	Step 4 change $\bar{x}$ to $\bar{x}+1$
0	1	0	1	Step 4, i=2, skip to x*
0	1	1	0	Step 4, change $\bar{x} \rightarrow \bar{x}+1$
0	1	1	1	$\bar{G}_o$=-0.09437, skip to x*
1	0	0	0	Step 4, change $\bar{x}$ to $\bar{x}+1$
1	0	0	1	Step 4, change $\bar{x}$ to $\bar{x}+1$
1	0	1	0	Step 4, change $\bar{x}$ to $\bar{x}+1$
1	0	1	1	$\bar{G}_o$=-0.04155, skip to x*
1	1	0	0	$\bar{G}_o$=-0.00973, skip to x*

The optimal solution obtained is $\bar{x}_{22}=\bar{x}_{12}=1$ and $\bar{x}_{21}=\bar{x}_{11}=0$. From (3.76), the optimum number of redundant components employed in each stage are two and the optimum system relia-

--bility obtained is 0.9914.

Many reliability problems were solved on IBM1620 using this approach and exact results were obtained. This method provides faster convergence for small problems than the methods discussed in section 3.3 and 3.4.

3.10. MIXED INTEGER PROGRAMMING METHOD

The techniques discussed so far in this study are applicable for solving the reliability maximization problem by treating number of redundant components to be used in each stage, as variables. If the components of varied reliability are available, the true optimal reliability problem involves in finding the optimal number of redundancies as well as the component reliability [44]. Therefore the reliability problem can be stated as

Maximize system reliability

$$R_s(n, r) = \prod_{j=1}^{k} R_j(n_j , r_j)$$

subject to the constraints

$$\sum_{j=1}^{k} G_{ij} (n_j , r_j) \leq b_i \qquad i =, 1, 2, \ldots s$$

$$0 < r_j \leq 1$$

and

$$n_j \text{ should be integer} \tag{3.79}$$

It is a mixed integer nonlinear programming where r_j and n_j are continuous and integer variables, respectively.

The reliability maximization problem can be converted into the separable minimization problem by taking logarithm of the system reliability expression as

Minimize $F(n, r)$

$$F(n, r) = -\sum_{j=1}^{k} \operatorname{Ln} R_j(n_j, r_j)$$

subject to the constraints

$$g_i = b_i - \sum_{j=1}^{k} G_{ij}(n_j, r_j) \geq 0 \qquad i=1,2,\ldots s$$

$$\left.\begin{aligned} g_{s+j} = 0 < r_j \leq 1 \\ n_j \text{ is integer} \end{aligned}\right] \; j=1,2,\ldots k \tag{3.80}$$

This constrained minimization problem can be converted into an unconstrained problem by using weighing factors. The transformed problem can be defined as [45].

Minimize $F(n, r)$

$$F_l(n, r) = f(n,r) + \tau_c^{l} I_l \{g_i(n, r)\} + \tau_d^{l} M_l(n) \tag{3.81}$$

If this problem is solved sequentially, that is for a series of l, then

$$\min_{\substack{n,r \\ l \to \infty}} F_l(n,r) = \min_{n,r} \left\{ f(n,r),\ g_i \geq 0,\ M_l(n)=0 \right\} \tag{3.81}$$

For a given precision, it will result in a finite value of l. Using SUMT formulation the constraint penalty function term can be defined as

$$I_l(g_i(n,r)) = \sum_i \frac{1}{g_i} \tag{3.82}$$

and the discretization penalty function [45] can be defined as

$$M_1(n) = \sum_{j=1}^{k} \left[4\bar{n}_j(1-\bar{n}_j) \right]^{\sigma_1} \qquad (3.83)$$

where,

$$\bar{n}_j = (n_j - n_j^l)/(n_j^u - n_j^l) \qquad (3.84)$$

and

$$n_j^l \leq n_j \leq n_j^u$$

n_j^l and n_j^u are the lower bound and upper bound on the n_j. In the above problem, τ_c^l and τ_d^l are the weighing factors corresponding to the constraint penalty function and discretization penalty function. σ^l is a constant and is used to change the shape of discretization penalty function while weighting factor τ_d^l is used to change its amplitude. The value of this function will be zero at the optimum point. This unconstrained problem can be solved by Davidon-Fletcher-Powell method for sequence of τ_c and τ_d such that

$$\tau_c^{l+1} < \tau_c^l$$

$$\tau_d^{l+1} > \tau_d^l$$

$$\text{and} \quad \sigma^{l+1} < \sigma^l \qquad (3.85)$$

One serious difficulty arises in this method, that problem may converge to a false optimum point due to wrong selection of the parameters of discretization penalty function. This situation occurs when one of the discretization point happens to be in the neighbourhood of a constraint boundary on the infeasible side. A recovery procedure is applied under such circumstances. The complete procedure can be explained stepwise as follows. (Fig. 3.14)

ALGORITHM:

(1) Initiatize y° where $y_j = n_j$ and $y_{k+j} = r_j$ for $j=1,2,..,k$ and process parameter. Evaluate function $F(y^{\circ})$. Set It = 1.

(2) Set i=0.

(3) Set l=0.

Solution of unconstrained problem.

(4) Set $H^l = I$ (2k * 2k identity matrix).

(5) Evaluate the gradients $\nabla F(y^l)$ at the current point.

(6) Compute current descent direction H_1^l

$$H_1^l = -H^l \nabla F(y^l) \qquad (3.86)$$

(7) Compute the current descent step length δ^l that satisfies

$$F(y^l + \delta^l H_1^l) = \min_{\delta} F(y^l + \delta H_1^l) \qquad (3.87)$$

(8) Compute the current descent step

$$\Delta y^l = \delta^l H_1^l$$

(9) Modify the value of current vector

$$y^{l+1} = y^l + \Delta y^l$$

(10) Calculate function $F(y^{l+1})$ and gradient $\nabla F(y^{l+1})$ at modified point y^{l+1}.

(11) Calculate $\Delta F^l = \nabla F(y^{l+1}) - \nabla F(y^l)$

(12) If $\left\{ F(y^{l+1}) - F(y^l) \right\} / f.F(y^l) > \epsilon$, go to next step. Otherwise go to step 15.

(13) Modify the current approximation H^l

$$H^{l+1} = H^{l} + \frac{\Delta y^{l} (\Delta y^{l})^{T}}{(\Delta y^{l})^{T} \Delta F^{l}} - \frac{H^{l} \Delta F^{l} (\Delta F^{l})^{T} (H^{l})^{T}}{(\Delta F^{l})^{T} H^{l} (\Delta F^{l})} \tag{3.88}$$

(14) Set $l=l+1$ and go to step 6.

(15) Set $i=i+1$. If $i < i_{max}$ go to next step. Else go to step 17.

(16) Set $\tau_c^{i+1}=t_1.\tau_c^{i}$, $\tau_d^{i+1}=t_2.\tau_d^{i}$ and $\sigma^{i+1}= \sigma^{i}/t_3$. Set $i=i+1$ and go to step 3.

(17) If y_j $(j=1,2,...,k)$ are integer, stop. Otherwise go to next step.

Recovery procedure:

(18) If It=1, set $\tau_c^{\circ} = \tau_c^{i-2}$ and $\tau_d^{\circ} = 2\tau_d^{i-1}$. Set It =2 and go to step 2. Otherwise go to next step.

(19) Set $\tau_c^{\circ} = \tau_c^{i-1}$ and $\tau_d^{\circ} = 2\tau_d^{i}$. Set It=1 and go to step 2.

Th Golden section method is used for single dimension search in the step (7) of above algorithm. The initial parameter setting is the drawback of this procedure. As the initial value of the process parameters τ_c, τ_d and σ influence the convergence of the problem, the initial value of the τ_c can be found out from the following expression

$$\tau_c = F(y^{\circ})/f.\left\{ I_{\circ} \, (g_i \, (y^{\circ})\,) \right\} \tag{3.89}$$

The value of f to be chosen depends on the starting point. If starting point is very close to the optimum, the large value of f is to be used. Generally, the value of f lies between 1 and 100 [45]. When location of optimum is not known, Gisvold [45] recommended the value of f as 20. The value of reduction factor t_1 should lie between 0.2 and 0.025. It does not have any effect on the computation time. If small value of t_1 is c chosen, the problem will require less iteration to converge, but computation time per iteration will be large. Initially, the parameter τ_d is calculated by the expression

$$\tau_d = C \frac{\nabla F(y^\circ, \tau_d^\circ)}{\nabla M(y^\circ, \sigma^\circ)} \tag{3.90}$$

The values for C can be taken as $0.001 \leq C \leq 0.1$. Good results were reported for C=0.01. The **constant** t_2 is calculated from t_1 with the help of the following relation

$$t_2 = \sqrt{\frac{1}{t_1}} \tag{3.91}$$

σ should always be greater than one, to make the discretization penalty function differentiable. In the programme σ=2.17 produces good results. The constant t_3 should be greater than one. A typical value of t_3=1.25 is used to solve the reliability problem.

NUMERICAL EXAMPLE:

Consider a four-stage system whose reliability is to be maximized by using parallel redundant components. The parameters of the system are tabulated below.

Stage	1	2	3	4
a_{1j}	1.00	3.50	2.00	5.00
a_{2j}	20.00	20.00	20.00	30.00
a_{3j}	0.30	0.55	0.40	0.65
a_{4j}	0.6	0.6	0.6	0.6

TABLE 3.15

τ_c	τ_d	n_1	n_2	n_3	n_4	r_1	r_2	r_3	r_4	System Reliability
Initial Point		2.0	2.0	2.0	2.0	0.5	0.5	0.5	0.5	0.3164
0.00500	0.00002	3.4	4.25	3.73	4.06	0.763	0.670	0.749	0.623	0.9593
0.00300	0.00004	3.44	4.28	3.76	4.63	0.795	0.676	0.756	0.623	0.9721
0.00180	0.00008	3.82	4.82	4.23	5.19	0.819	0.665	0.754	0.602	0.9825
0.00080	0.00016	4.86	6.19	5.110	6.97	0.680	0.570	0.671	0.586	0.9850
0.00048	0.00032	5.21	6.88	6.15	7.46	0.670	0.561	0.624	0.541	0.9880
0.00028	0.00064	6.55	7.27	6.72	7.59	0.669	0.556	0.618	0.534	0.9920
0.00016	0.00128	7.001	8.00	7.00	8.00	0.66	0.54	0.610	0.510	0.9927
Optimum solution		7.00	8.00	7.00	8.00	0.66	0.54	0.610	0.510	0.9927

The constraints on the system are

$$\sum_{j=1}^{k} a_{1j} \exp\left[a_{3j}/(1-r_j)\right] n_j \leq 300$$

and

$$\sum_{j=1}^{k} a_{2j}\, r_j^{a_{4j}}\, n_j \leq 5000 \tag{3.92}$$

It is required to find the optimum parameters of the system, i.e. component reliability and number of redundant components in each stage.

Initially, τ_c, τ_d and σ are taken as 0.005, 0.00001 and 2.17, respectively. The initial feasible point chosen is (2.0, 2.0, 2.0, 2.0, 0.5, 0.5, 0.5, 0.5). With these values of process-parameters the reliability problem is solved on IBM 1620. The complete results are tabulated in Table 3.15.

A few reliability problems were solved by this method and satisfactory results were obtained. One blind run of the programme is required for the proper initial setting of the process parameters, as they effect the convergence of the problem. Author is trying to develop a direct search method of solving the mixed integer reliability problem and hoping to report that in near future.

Chapter IV

EVALUATION OF OPTIMIZATION METHODS

The selection of a particular technique rests on the formulation of the problem and the experience of the analyst. Specifically, in order to find, which is the best method, the following criteria are to be considered :-

(i) execution time

(ii) computer memory requirement

(iii) accuracy of solution

(iv) simplicity of use (time required by the user to prepare data)

(v) simplicity of the computer programme to execute the algorithm.

The most common criteria used to evaluate the relative effectiveness of the different methods discussed in Chapter 3, are execution time and memory requirement.

TEST PROBLEMS

Problem I (parallel redundancy)

Four stage reliability problem with linear constraint. A system consists of four stages, each having reliability and cost as tabulated below. It is required to find the optimum number of redundant components so that the system reliability is maximized with cost constraints as 56 units. Assume constraint on the system is linear

Stage number	1	2	3	4
Component reliability	0.80	0.70	0.75	0.85
Cost	1.2	2.3	3.4	4.5

Optimum solution (5, 6, 5, 4)

Problem II (Standby redundancy)

Four stage reliability problem with two nonlinear constraints. Consider a system consisting of four stages. Their parameters are tabulated below. It is needed that the system reliability is to be maximized by using spare components. The maximum number of the spare components for each stage may be assumed as three and the replacement time is to be neglected in comparison with the mission time of the system which is 10 years. The constraints on the system are

$$\sum_{j=1}^{k} a_j n_j^2 \leq 36$$

and

$$\sum_{j=1}^{k} f_j n_j \exp(n_j/4) \leq 150$$

The reliability of the switch is 0.999

Stage number j	λ_j failure rate per year	a_j	f_j
1	0.0798	1.0	7.0
2	0.0328	2.0	8.0
3	0.0066	3.0	6.0
4	0.026	4.0	9.0

Optimum solution (3, 2, 1, 2)

Problem III (Parallel redundancy)

Fifteen stage reiliability problem with four linear constraints. Consider a system consisting of fifteen stages. The parameters of the system are tabulated in table given below. The system reliability isto be increased by using parallel redundant components. The system cost, weight, volume and power consumption should not increase more than 840, 170 5200 and 450 units, respectively. Assume constraints on the system are linear.

Stage number	Component reliability	Component Cost	Component weight	Component volume	Power consumption by a component
1	0.99	80.0	2.0	100.0	10.0
2	0.86	5.0	4.0	150.0	50.0
3	0.98	90.0	5.0	130.0	14.0
4	0.87	10.0	3.0	160.0	26.0
5	0.99	80.0	10.0	140.0	18.0
6	0.97	70.0	5.0	120.0	56.0
7	0.88	15.0	6.0	155.0	14.0
8	0.98	90.0	8.0	200.0	12.0
9	0.89	20.0	4.0	150.0	8.0
10	0.96	60.0	2.0	80.0	35.0
11	0.90	30.0	15.0	500.0	16.0
12	0.92	60.0	12.0	200.0	25.0
13	0.95	40.0	16.0	600.0	10.0
14	0.93	65.0	20.0	650.0	22.0
15	0.94	45.0	18.0	600.0	16.0

Optimum solution (1,2,1,2,1,1,2,1,2,1,2,1,1,1,1)

Problem IV (Hybrid redundancy)

Consider a protective systemfor a chemical plant, consisting of six stages. Assuming the reliability of the voter and fault-detecting and switching device as unity, it is required to find the optimum number of spare components for increasing the system reliability by employing hybrid redundancy. The cost of protective system must not exceed 72 units. Neglect the fault-detection and switchover time. Mission time is one year. The parameters of the system are tabulated below.

Stage number	Component	On line failure rate/year	Off line failure rate/year	Cost
1	Pressure switch	0.01278	0.001278	3.0
2	Differential pressure transducer	0.01053	0.001053	4.0
3	Oxygen Analyser	0.00833	0.000833	6.0
4	Reactor temperature indicator	0.0356	0.00356	1.0
5	Temperature trip amplifier	0.00512	0.000512	8.0
6	Invertor	0.00833	0.000833	4.0

Optimum solution (2,1,0,2,0,1)

Problem V (Maintained system)

Consider a system consisting of two stages. Each stage has 100 identical components, which should operate failure-free for the successful operation of the stage. The failure rate and repair rate of each component is constant. The parameters of the system are tabulated below. It is required to increase the system reliability by providing the spare components and multiple repair facilities. The amount available for providing repair facilities and spare components is 48 units. Neglect the replacement time.

Stage number	Failure rate of component per hour λ $\times 10^{-3}$	Repair rate per hour μ	Cost of a component	Cost of single repair facility
1	0.5	0.192	10	8
2	1.0	0.25	5	10

Optimum solution $\begin{bmatrix} \text{repair facilities } 1,\ 1 \\ \text{spare components } 2,\ 2 \end{bmatrix}$

COMPARISON OF METHODS

(a) Robustness and Accuracy -

All the test problems are solved by the methods presented in this thesis. None of the methods failed to converge. For these test problems, by rounding off the continuous solution obtained by using geometric programming formulation, and other methods gave exact solution. But, in general, methods given in item (3.1, 3.7, 3.8) will provide near-optimal solution.

(b) Computer Storage -

The memory requirement for each method is tabulated below for fifteen stages and five constraints reliability problem.

Method	Word length
1. Geometric programming formulation	261
2. Penalty function method	1259
3. Flexible tree search method	1701
4. Zero-one programming method	2686
5. Modified non-binary tree-search method	1680
6. Branch and Bound method	1282
7. Direct search method	674
8. Simple method	496
9. Lexicographic enumeration technique	2348

If there is only one constraint on a system, then geometric programming formulation will require only 48 words length.

(c) Execution time -

The execution time on IBM1620 for each method is tabulated below.

Method	Execution time in minutes				
	Problem I	Problem II	Problem III	Problem IV	Problem V
1. Geometric programming formulation	3.0	-	10.0	-	-
2. Penalty function method	8.0	11.0	13.0	16.0	18.0
3. Flexible tree search method	16.0	10.0	20.0	20.0	15.0
4. Zero-one programming method	20.0	15.0	18.0	21.0	16.0
5. Modified non-binary tree-search method	12.0	8.0	12.0	16.0	11.0
6. Branch and Bound method	9.0	6.0	13.0	18.0	12.0
7. Direct search method	10.0	8.0	12.0	12.0	-
8. Simple method	8.0	6.0	11.0	8.0	-
9. Lexicographic enumeration technique	15.0	10.0	18.0	15.0	8.0

GRADING

S. No.	G r a d i n g		
	Based on execution time	Based on memory	Based on time and memory
1.	1	1	1
2.	8	8	8
3.	7	7	7
4.	9	2	2
5.	2	6	9
6.	5	5	6
7.	6	3	5
8.	3	9	3
9.	4	4	4

The memory requirement and execution time for mixed integer programming method for problem given on page 116 is 3150 words and 40 minutes, respectively.

Chapter 5

CONCLUSIONS

Due to increased complexity, sophistication and automation in a modern system, the system reliability always tends to decrease. The use of protective redundancies which provide the easiest and cheapest solution, is made to enhance the system reliability. But it involves extra money, weight and volume etc. Therefore, for the optimal design of a reliable system the optimal allocation of redundancies to maximize system reliability subject to multiple constraints are to be found out.

The solution of this problem requires the mathematical modelling of the system. The derivation of the mathematical model is eased by first drawing a logic or functional diagram of the system. The structure of the reliability expression rely on the type of redundancies to be employed for enhancing the system reliability. The various types of redundancies which are considered in this study are parallel, series-parallel, parallel series, majority voting, multiple line, standby and hybrid redundancy. Generalized expressions for system reliability are derived in section 2.4 for standby redundant system considering the effect of dormancy and three types of switch failures, that is static, dynamic and gradual failures. For hybrid redundant system, reliability expression is derived in section 2.5, incorporating the effect of dormancy and dynamic failure of the switching device. Maintained systems with standby redundancy are analysed in section 2.6 and steady state reliability expression is derived. The type of redundancy to be used is dictated primarily by system performance considerations. The other factors are operating conditions, power requirement, modes of failure of the components and maintainability considerations etc. Because of all this, the problem is an involved one and there does not exist a straightforward solution to the problem.

The reliability problem has the form of nonlinear integer programming problem. If the system reliability expression is separable and monotone increasing function, it can be converted into an equivalent linear programming problem having zero-one variables as explained in section 2.7. If it is not monotone increasing function with respect to the variables but separable, an equivalent linear zero-one programming problem can be formulated, which results in large number of binary variables.

A new formulation for the problem of system reliability maximization using active parallel redundancies subject to linear constraints is presented in section 3.1. The constrained reliability problem is reduced to that of an optimization of an unconstrained objective function with variables one less than the number of constraints. When there is only one linear constraint on the system or a constraints set in which the active constraint is known, expressions are derived for optimum number of parallel components in each stage and optimum system reliability in terms of the system parameters. These expressions may be useful to the system designer, as he can know with the help of these expressions that how much resources are required for achieving the desired system reliability.

Formulation given in [29] which also have used geometric programming requires one more nonlinear equation to solve, than the formulation given in the report. The error in the calculation of system reliability by [29] is 10.1% while this formulation gives only 0.09% error. The reliability problem given in [22] is solved in [29] and the result reported is 5,5,4,3. The same problem is solved by suggested formulation providing optimum solution as 5,6,5,4, which is also an optimal solution obtained in [22].

When reliability problem has a number of constraints and approximate solution is required, the use of penalty function approach can be made for solving it as explained in section 3.2. This method provides continuous solution and has fast convergence. A tree search method is developed for obtaining the integer solution from the continuous solution obtained by the penalty function method. The use of numerical differentiation is made when system reliability expression is not differentiable. This formulation is highly reliable, robust and can be used for any type of constraint set. The only limitation of this formulation is that it requires initial point to be feasible one. But in reliability problems, initial feasible point is always known.

The equivalent linear reliability problem with zero-one variables is solved by flexible tree search method in section 3.3. This method allows a great deal of flexibility in the backtracking step which improves the efficiency of the search procedure.

In all the enumeration methods available so far, for calculating upper bound on objective functions, a variable is first set to one and search is made for the remaining free variables. Secondly, the same variable is set to zero and again search is made, while in the zero-one programming method given in section 3.4, the use of tree search is made for calculating lower bound using the concept of the minimal externally stable set which reduces computation considerably. The convergence of this method to a feasible solution is faster than the previous method. In this method, the termination of the search before obtaining the optimal solution, always provides a feasible solution.

A modified non-binary tree search method is proposed to solve equivalent linear programming problem in section 3.5. A simple rule for branching which eliminates the use of external stable set for calculating lower bound to the objective function, is presented. It reduces the computation time and memory requirement considerably.

The reliability problem is converted into an equivalent Knapsack problem in section 3.6 by aggregating the constraints which have integer coefficients. It is easier to solve this equivalent problem rather than to solve original problem with multiple constraints. A branch and bound method is developed which is simple and provides exact solution. This method is found to be very efficient when constraint coefficients are small and integer. The generality of this method is not bound by the requirement of the integer coefficients of the constraints, as any irrational number can be approximated by a rational number, which in turn can be converted into an integer form by multiplying the inequality by a suitable factor.

A computational method is developed in section 3.7, which can be used for solving reliability problem of the parallel, standby and hybrid redundant system. Due to simplicity and less computational effort requirement, this method is best suited for large systems. It has not been rigorously proved that this method provides optimal solution, but atleast it will always provide a near-optimal solution. Reliability problems both with linear and nonlinear constraints, are solved by this approach and exact results were obtained. Since during initial design phase, reliability problem does not require an exact solution, as several alterations and alternatives are sought from other technical considerations, therefore this method is suitable under these conditions.

In section 3.8, a multiple linear constrained reliability problem is transformed into an equivalent single constraint problem by simple rule. The equivalent problem has same number of variables and feasible solutions as the original problem. A simple method is developed to solve the equivalent problem. This method is best suited for the system having parallel, standby and dynamic redundant components with many constraints having integer coefficients of small magnitude.

The integer nonlinear reliability problem is converted into zero-one nonlinear programming problem by using binary variables in section 3.9. An improved method of generating the skipping vector x* is used for increasing the efficiency of lexicographic enumeration. This method is quite simple and easily programmed. This approach is useful forsolving the reliability problem of the mixed redundant systems and stand-by redundant systems having multiple repair facilities.

All the above methods can be grouped into two categories: (a) methods which provide approximate results which, in some cases, are also true optimum, and (b) methods which give true optimal solution. The effectiveness of a method can be examined from theoretical pointof view and experimentation. In all cases, theoretical experimentation is not possible. Therefore, experimentation for each method is performed on IBM1620. Experimentation largely depends on the programming of the algorithm and the precision required. The details about the computational experience are given in Chapter IV. For parallel, standby and hybrid redundant system, direct search method developed in section 3.7 provides the fastest convergence while for mixed redundant system and standby redundant system with repair facilities, the lexicographic enumeration technique results in the fastest convergence. If continuous solution of the reliability problem which has multiple constraints, is required, the penalty function approach provides fast convergence. When there is only one constraint on the system, the use of geometric programming formulation provides continuous solution with least computational time.

All the methods discussed above are applicable to solve the suboptimization reliability problem, that is, the component reliability is kept constant and number of redundant components to be employed are treated as variables. But the true optimal reliability problem involves in calculating the optimal redundancy level as well as component reliability. With these as variables, the reliability problem is formulated as mixed integer programming problem as explained in section 3.10. The constrained problem is converted into an unconstrained problem by using constraint and discretization penalty functions. The unconstrained problem is later on solved by variable metric method. The problem encountered in the implementation of this method is the selection of the approximate value of the process parameters. One blind run is required for correct parameter setting of the process parameters. This method is found to be suitable for big problems having many constraints. The mixed integer reliability problem requires further exploration both in the problem formulation and solution technique. Author is already pursuing some work in this direction.

The method to be used for the solution of reliability problem depends on the accuracy of the results and the cost of obtaining them. The methodsfor which the costof obtaining the results exceeds the gain in the design are not suited from practical considerations. From computational experience, it is felt that the methods presented in this study are well comparable in this regard.

Due to high risk and cost, the fault-tolerant design of the technological systems is needed. Optimization methods will undoubtedly be required to avoid overdesigning of the technological systems. Therefore, the need of efficient, reliable and flexible computational techniques is felt. It is hoped that the present work may prove of value in this connection.

List of Publications by the Author related with this Study

1. A direct method for maximizing the system reliability. IEEE Trans. on Rel., vol. R-20, 1971.

2. Reliability optimization by dual convex and Integer programming. Proc. System Control and Application, University of Roorkee, Feb. 1972.

3. A computer method of optimal redundancy allocation in satellite communication system. JIE, vol.53 July 1972.

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7. A method of improving the reliability of a d.c. link in integrated syste. Seminar on E.H.V.D.C. system, University of Roorkee, Dec. 1973.

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11. Reliability optimization of a control system. To appear in Automatica.

12. Optimum design of an aero-space computer for reliability. To appear in IEEE Trans. on Aero-space and Electronic Devices.

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14. Optimum design of computer-communication system. Proc. 'System Reliability, I.I.T., Madras, June 1974.

15. Computation of reliability of a system using graphs. To be presented in 45th Meeting of Opns. Res. Society of America, 1974.

Being communicated -

16. Reliability optimization of a fault-tolerant computer. To IEEE Trans. on Computer.

17. A method of optimal design of a self-repairable digital computer. To IEEE Trans. on Computer.

18. Cost based optimum reliability apportionment. To IEEE Trans. on Reliability.

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APPENDIX

The maintained system problem explained on page 122 can be expressed mathematically as

Maximize

System reliability $R_s(m, r_c)$

$$= R_1(m_1, r_{c1}).R_2(m_2, r_{c2}) \qquad (A.1)$$

subject to the cost constraint

$$10m_1+8r_{c1}+5m_2+10r_{c2} \leq 48 \qquad (A.2)$$

m_j and r_{cj} are integer, j=1,2

where,

$$R_1(m_1, r_{c1}) = 1- \frac{1}{r_{c1}!} \left[\frac{100 \times 0.5 \times 10^{-3}}{0.192}\right] \cdot \left[\frac{100 \times 0.5 \times 10^{-3}}{r_{c1} \times 0.192}\right]^{m_1-r_{c1}+1}$$

and (A.3)

$$R_2(m_2, r_{c2}) = 1 - \frac{1}{r_{c2}!} \left[\frac{100 \times 1 \times 10^{-3}}{0.25}\right] \cdot \left[\frac{100 \times 1 \times 10^{-3}}{r_{c2} \times 0.25}\right]^{m_2-r_{c2}+1}$$

To convert expression (A.1) into separable functionfor simplicity, logarithm of (A.1) is taken. The reliability problem can be expressed as

Maximize Ln $R_s(m, r_c)$

$$\text{Ln } R_s(m, r_c) = \text{Ln}(m_1, r_{c1}) + \text{Ln}(m_2, r_{c2}) \qquad (A.4)$$

subject to the constraint given by (A.2).

The various interesting feasible solutions of this problem are tabulated below.

Number of repair facilities		Number of spare components		System reliability	Cost of repair & spare component
Stage 1	Stage 2	Stage 1	Stage 2		
1	1	1	1	0.8503	33
1	1	1	2	0.9081	38
1	1	1	3	0.9328	43
1	1	1	4	0.9413	48
1	1	2	1	0.8540	43
1	2	1	1	0.9204	43
2	1	1	1	0.8727	41
1	1	2	2	0.9445	48
1	2	1	2	0.9001	48
2	1	1	2	0.9321	46