HEAT TRANSFER THROUGH INCOMPRESSIBLE LAMINAR BOUNDARY LAYERS

Dissertation submitted in partial fulfilment of the requirements for the Degree of Master of Engineering in Applied Thermodynamics (Refrigeration and Air- Conditioning)

By

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February, 1962.

CERTIFICATE

CERTIFIED that the dissertation entitled... Heat Transfer through Incompressible Laminar Boundary ....

which is being submitted by Sri... D. R. Sood. .... in partial fulfilment for the award of the Degree of Master of Engineering in... Applied Thermodynamics (Refrigeration & Airconditioning) ....

of University of Roorkee is a record of student's own work carried out by him under my/our supervision and guidance. The matter embodied in this dissertation has not been submitted for the award of any other Degree or Diploma.

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A C K N O W L E D G E M E N T S

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NOTATION

a	- Parameter of Joukovsky Transformation.
b	- Function.
c	- Chord, Characteristic length of the body.
cp	- Specific heat at constant pressure Kilo Cal/Kilogram/^oC
e	- Increment (Eq. 3-19)
f	- Any function.
g	- Acceleration due to gravity. Meters/hour^2
h	- Heat transfer coefficient. Kilo Cals/hour/M/^oC
i	- $\sqrt{-1}$
k	- Parameter in Cascade Flow,
	Thermal Conductivity Kilo Cals/hr/M/^oC
ℓ	- Length.
m	- Parameter in wedge flows.
p	- Pressure. Kilo Gram/Meter^2.
q	- Heat Transfer/Unit area/ Unit time Specific rate of heat flow
s	- Gap (in cascade)
s/c	- Solidity ratio.
u	- Velocity component along the wall. Meter/hour.
v	- Velocity component normal to the wall. Meters/hour
x	- Coordinate along the wall-meters.
y	- Coordinate normal to the wall-meters.

A - Area $(\text{Meter})^2$

B - Breadth - Meter.

C - Constant.

F - Function.

Q - Heat Flow/ Unit Time.
Kilo Cals/hr.

T - Temperature °Kelvin.

T_1 - Temperature outside the boundary layer. °K

T_w - Temperature of the Wall. °K

T_∞ - Temperature of the free stream. °K

$U(x)$ - Velocity in Potential Flow,
Outside the boundary layer.

U_2 Velocity at exit from Cascade.

U_∞ - Free Stream Velocity.

α - Angle,
Thermal diffusivity = $\frac{k}{\rho c_p}$ $\frac{(\text{Meter})^2}{\text{hour}}$

β - Angle,
$\frac{2m}{m+1}$ in wedge flows.

γ - Parameter in wedge flows.

δ - Any velocity boundary layer thickness.

$\delta_1 = \int_0^\infty \left(1 - \frac{u}{U}\right) dy$ Displacement thickness

$\delta_2 = \int_0^\infty \frac{u}{U}\left(1 - \frac{u}{U}\right) dy$ Momentum thickness.

$\delta_\tau = U \Big/ \left(\frac{\partial u}{\partial y}\right)_0$ Shear Thickness.

Δ - Any Temperature boundary layer thickness.

$\Delta_1 = \int_0^\infty \left(\frac{T-T_1}{T_w-T_1}\right) dy$ Temperature displacement thickness.

$\Delta_2 = \int_0^\infty \frac{u}{U} \left(\frac{T-T_1}{T_w-T_1}\right) dy$ Enthalpy flux thickness.

$\Delta_4 = -\left(T_w - T_1\right) / \left(\frac{\partial T}{\partial y}\right)_0$ Conduction thickness.

ζ - Ratio of the thicknesses of the thermal and velocity boundary layers,

- ($\xi + i\eta$) Transformed plane.

η - New Variable (defined in text)

θ - Dimensionless Temperature Parameter,

- Angle.

λ - Angle of Stagger.

μ - Dynamic Viscosity $\frac{\text{Kilogram hour}}{\text{Meter}^2}$

ν - Kinematic viscosity $\frac{\text{Meter}^2}{\text{Hour.}}$

ρ - Density $\frac{\text{kilogram}}{(\text{Meter})^3}$

τ - Shearing Stress.

ϕ - Heat dissipation function, ~~angkex~~

- angle

ψ - Stream function.

Non-dimensional Numbers.

P - Prandtl Number $\frac{\mu c_p}{k}$

Re - Reynolds Number $\frac{Uc}{\nu}$

Nu - Nusselt Number $\frac{hc}{k}$

ABSTRACT

This report is an attempt to present the available techniques for calculating heat transfer coefficients through incompressible laminar boundary layers. Some of the methods developed for general purpose have been outlined. Two of these have been used to predict the distribution of heat transfer coefficients around a cascade of gas turbine blades of known pressure distribution.

After a brief introduction giving the significance of the problem, the equations of boundary layer are stated and a criterion for neglecting frictional heating derived.

This is followed by the solution of the problem for the case of a flat plate. Pohlhausen's exact solutions for the case of an isothermal wall and adiabatic wall have been discussed and subsequently approximate simpler solutions have been presented. The case of arbitrarily varying wall temperatures is dealt with next.

In the next Chapter, the methods for calculating

the heat transfer coefficients for a general two dimensional body are given. Starting with the well known wedge flows, the methods of Squire (1942), Lighthill (1950), Smith and Spalding (1958) and Spalding (1958), ~~Smith and Spalding (1958) and Spalding (1958)~~, which is an attempt to improve the Lighthill method, are presented.

In order to evaluate the heat transfer coefficients, the potential velocity distribution is required. Two methods, typical of present day approach, are, therefore, given. The problem of theoretically predicting the potential flow, outside the boundary layers is complicated and the theory formulated is still far from sound. The methods, however, bring out the possibility of prediction of the heat transfer coefficients, from entirely theoretical analysis. Due to the reason stated above, the experimental velocity distribution data, published by Pope and Wilson, for a cascade of gas turbine blades, has been used to calculate the heat transfer coefficients by the methods of Smith and Spalding and Spalding. This also facilitates comparison with the published data for heat transfer coefficients by the same authors.

The last Chapter deals with these calculations and the conclusions which point towards need for comprehensive experimentation to evaluate the relative accuracy of the methods.

CHAPTER I.

INTRODUCTION.

1.1 Significance of the problem.

The transfer of heat from a solid boundary to a fluid stream or vice-versa is appreicably effected by the character of the stagnant fluid layer called "Boundary layer" formed on the solid surface. The heat transfer through this layer is by conduction. The problem is of pertinent interest to the designers of -

(a) Turbo-machines, where the cooling of blades holds out a promise of increased plant efficiency.

(b) Supersonic Aircraft, where kinetic heating may result in unduly high skin temperatures.

(c) and to numerous other applied fields.

It has been recognised for quite a time that the efficiency of the gas turbine plant can be appreciably improved by increasing temperatures at the turbine inlet. This necessiates a two pronged drive, one metallurgical for better materials capable of withstanding these temperatures and the other thermodynamic, for artificially cooling the surfaces to safe temperature levels. The metallurgical limitations have encouraged. the exploration of thermodynamic means. The knowledge of the distribution of heat transfer coefficients along the blade surface, is of paramount importance for the design of any cooling system. Similarly for supersonic aircraft,the skin protection demands knowledge of the heat generated and its distribution along the surface. The problem has, however, not arisen solely with the advent of supersonic aircraft and the high efficiency gas turbine plants. The formation of ice on the lifting surfaces of the subsonic aircraft impairing the performance and endangering stability called for an estimate of heat transfer coefficients to devise a suitable method for removing the ice by melting.

This report is an attempt to present the available techniques for predicting heat transfer coefficients. The actual phenemenon is quite complicated and presents insurmountable mathematical difficulties because of the turbulent nature of the boundary layer, compressibility effects, dissociation and variable properties etc. This brings out the desirability of understanding the basic phenemenon associated with two dimensional incompressible laminar flows. Extensions can then be made to approximate the actual operating conditions taking into account the compressibility effectsetc. There is strong evidence that even at high Mach numbers, the front portion of the boundary layer is always laminar. This report is, therefore, restricted to the study of heat transfer coefficients through incompressible laminar boundary layers.

1.2 Historical Review.

Pohlhausen[1] in 1921 solved the temperature distribution equation for laminar flow along a flat isothermal plate using Blasius' velocity distribution. Frick and Mccullough[2] in 1943 extended Pohlhausen's solution to the case of an aerofoil. They assumed the temperature and the velocity boundary layers to be related to each other in the same way as for a flat plate. However, to account for the pressure gradient, the thickness parameter of the velocity boundary layer was calculated by the method of Jackob and Von Donehoff. Allen and Look[3] in 1943 developed an analogous method based on Reynold's analogy which relates the temperature gradient and the velocity gradient at the wall. The anology holds strictly for fluids of Prandtl number unity.

Squire[4] in 1942, assumed that the temperature distribution every where is proportional to the velocity distribution i.e.,

$$\frac{T - T_1}{T_w - T_1} = \frac{u}{U}$$

By assuming u/U to be given by the Blasius profile, he deduced an expression for the thickness of the temperature boundary layer Δ_1. The relative thicknesses of the two layers were then determined by satisfying the integrals of the equation of motion and the equation of energy across the layer. The assumption of similar velocity and temperature profiles means that the pressure gradients affect the temperature and velocity distribution in exactly the same manner.

This method differed from those described previously in the manner in which the pressure gradient was taken into account. The previous methods used the equations of flat plate and took account of the pressure gradient in calculating the thickness parameter of the boundary layer, from the Karman momentum relation, which includes the terms involving pressure gradient. Squire, on the other hand deduced the relation between temperature and velocity boundary layer thicknesses by satisfying the integrals of the equations of motion and energy which contain pressure gradient terms. The methods of Allen and Look, Frick and Mccullough failed to take into account the variations in the velocity profiles in the boundary layer along the aerofoil surface. In fact, the profile varies from $u = cx$ at the stagnation point to the separation profile at the point of separation. Blasius profile holds only where the velocity is more or less constant. Squire's method, though more accurate, is extremely laborious.

Lal[5] in 1949 suggested the use of Thwaite's[6] quadrature technique for calculating the boundary layer thickness parameter. Thwaite's method takes into account the variations in the shape of the boundary layer profiles. He used Thwaite's method to calculate the value of the velocity gradient at the wall and then calculated the rate of heat transfer, On the assumption that the velocity and temperature boundary layers were related to each other in the same way as in the Pohlhausen's solution for the flat plate.

The importance of the problem has focused the need for understanding the basic phenemenon. In this connection, it may be noted that whereas only one boundary condition, namely absence of slip can be prescribed at the wall for the velocity boundary layer equation, a number of boundary conditions can be prescribed for the temperature boundary layer equation, at the wall. Methods have been developed for evaluating the rate of heat transfer in the case of, isothermal walls, adiabatic walls, stepwise and uniform temperature variations along the wall. In some cases the heat flux to the wall may be prescribed. The multiplicity of the boundary conditions has highlighted the difficulties encountered in evolvig exact analytic solutions. This has encouraged the development of approximate solutions. The names of Lighthill, Smith, Spalding, Schuch, Eckert, Tribus, Thampman etc. may be mentioned as the typical among those who have contributed to the better understanding of the problem.

1.3 The Boundary Layer Equations.

The incompressible, laminar, boundary layer equations for the steady, two dimensional fluid flow are

Continuity

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad \text{....} \quad (1\text{-}1)$$

Momentum

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{\mu}{\rho}\left(\frac{\partial^2 u}{\partial y^2}\right) \qquad \text{....} \quad (1\text{-}2)$$

$$0 = \left(\frac{\partial p}{\partial y}\right) \qquad \text{....} \quad (1\text{-}3)$$

Energy

$$\rho C_p\left[u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y}\right] = k\frac{\partial^2 T}{\partial y^2} + \mu\left(\frac{\partial u}{\partial y}\right)^2 \qquad \text{....} \quad (1\text{-}4)$$

The equations were first derived by Prandtl in 1905 by the application of the famous order of magnitude arguments, to the Navier-Stokes equations of motion and to the equation of en rgy derived from the first law of thermodynamics.

Equation (1-3) states that the pressure does not vary across the boundary layer and is, therefore, constant for any station x along the wall. Besides the usual assumptions, the curvature of the wall is neglected and the properties like conductivity, viscosity etc. are considered independent of temperature. An examination of the above equations reveals the non-linear character of the equation (1-2), whereas energy equation (1-4) is linear in temperature. Moreover, the assumption of independence of properties with respect to temperature makes it possible to solve the equations of motion and energy independently. From the solution of velocity boundary layer equation, u and v are supposed to be known parameters in the equation (1-4).

The second term on the right hand side of the equation (1-4) represents the contribution of the viscous stresses. It is important to devise a criterion for deciding when this can be neglected.

1.4 Criterion for Neglecting Dissipation.

Let us measure the temperatures with a unit which is of the order of temperature difference imposed on the problem i.e. the difference between the wall temperature and the stream temperature, ($T_w - T_1$). Velocities will be measured with stream velocity as unity. Then from the equation of energy

$$\rho C_p \left[u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} + w \frac{\partial T}{\partial z} \right] = K \nabla^2 T + \mu (\phi) \quad \text{.. (1-5)}$$

the boundary layer form of equation given by (1-4) is derived. The first term on the right hand side of equation (1-4) is of the order of $\delta^2 \Delta T_0 / \delta^2$ and the second term of the order of $\delta^2 U^2/\delta^2$. Both these are of order 1 when $U^2/\Delta T_0$ is of the order 1. This ratio can be divided by C_p to make it dimensionless $U^2/C_p \Delta T_0$ When this number is small compared with unity, the dissipation term may be neglected. For gases, at low Mach numbers, i.e. incompressible case, usually diss-ipation may be neglected.

CHAPTER 2

SOLUTION OF BOUNDARY LAYERS FOR FLAT PLATES.

2·1 Exact Solutions.

As already mentioned the solution of the boundary layer equations prosents considerable difficulties. This is primarily because of the non-linear character of the velocity boundary layer equations. A few exact solutions for some simple cases have been worked out. The solutions are refered to as exact when they are derived from the boundary layer equations as contrasted with the solutions derived from the Karman momentum relation and the heat flow equation which are termed as approximate solutions.

2.1(i) Pohlhausen's Solution.

Pohlhausen[1] used Blasius velocity distribution to solve the temperature boundary layer equation along a flat plate. The governing equations for the case of uniform, steady, two dimensional flow past a flat plate are:-

$$\left(\frac{\partial u}{\partial x}\right) + \left(\frac{\partial v}{\partial y}\right) = 0 \qquad \cdots\cdots (2\text{-}1)$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\left(\frac{\partial^2 u}{\partial y^2}\right) \qquad \cdots\cdots (2\text{-}2)$$

$$\rho C_p \left(u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} \right) = k \frac{\partial^2 T}{\partial y^2} \quad \cdots\cdots (2-3)$$

Blasius introduced new variables for the solution of the flow equation so as to transform equation (2-2) into an ordinary differential equation through (2-1). The new variables are

$$\eta = y \sqrt{\frac{U_\infty}{\nu x}} \quad ; \quad \psi = \sqrt{\nu x U_\infty}\, f(\eta) \quad \cdots (2-4)$$

whence $u = U_\infty f'(\eta)$ and $v = \frac{1}{2} \sqrt{\frac{\nu U_\infty}{x}} (\eta f' - f)$

The differential equation for $f(\eta)$ becomes

$$f f'' + 2 f''' = 0 \quad \cdots\cdots\cdots (2-5)$$

where prime denotes differentiation with respect to η. With the boundary conditions

$$\eta = 0 \quad : \quad f = f' = 0$$
$$\eta = \infty \quad : \quad f' = 1$$

the solution of the equation (2-5) was obtained by Blasius in the form of power series about $\eta = 0$ and an assymptotic expansion at $\eta = \infty$, the two being joined at a suitable point. Introducing the dimensionless parameter Θ_1 to describe the temperature within the boundary layer and defining

$$\Theta_1 = \frac{T - T_w}{T_\infty - T_w} \quad \cdots\cdots\cdots (2-6)$$

the energy equation can be transformed to

$$\frac{d^2 \Theta_1}{d\eta^2} + \frac{P}{2} f(\eta) \frac{d\Theta_1}{d\eta} = 0 \quad \cdots\cdots\cdots (2-7)$$

With the boundary conditions

$$\eta = 0 \quad : \quad \theta_1 = 0$$

$$\eta = \infty \quad : \quad \theta_1 = 1$$

equation (2-7) was integrated by Pohlhausen to give

$$\theta_1 = \frac{\int_0^{\eta} \left[e^{-\frac{P}{2}\int_0^{\eta} f(\eta)\,d\eta} \right] d\eta}{\int_0^{\infty} \left[e^{-\frac{P}{2}\int_0^{\eta} f(\eta)\,d\eta} \right] d\eta} \quad \ldots (2\text{-}8)$$

This can also be expressed as

$$\theta_1 = \frac{\int_0^{\eta} \left[f''(\eta) \right]^P d\eta}{\int_0^{\infty} \left[f''(\eta) \right]^P d\eta} \quad \ldots (2\text{-}9)$$

The temperature gradient at the wall

$$-\left(\frac{d\theta_1}{d\eta}\right)_{\eta=0} = \frac{\left[f''(0) \right]^P}{\int_0^{\infty} \left[f''(\eta) \right]^P d\eta}$$

This relation was approximated by Pohlhausen by the formula

$$-\left(\frac{d\theta_1}{d\eta}\right)_{\eta=0} = (0.332)^P \Big/ \int_0^{\infty} \left[f''(\eta) \right]^P d\eta \quad \ldots (2\text{-}10)$$

$$\simeq 0.332 \sqrt[3]{P}$$

which gives fairly good accuracy

Pohlhausen also gave a solution of the problem when the viscous heating term could not be neglected.

In this case the energy equation transforms into

$$\frac{d^2T}{d\eta^2} + \frac{P}{2} f \frac{dT}{d\eta} = -\frac{P U_\infty^2}{g c_p} \left[f''(\eta) \right]^2 \qquad \ldots\ldots (2\text{-}12)$$

Introducing the dimensionless temperature parameter

$$\Theta_2 = \frac{T(\eta) - T_\infty}{U_\infty^2 / g c_p} \qquad \ldots\ldots (2\text{-}13)$$

the above equation can be written as

$$\frac{d^2\Theta_2}{d\eta^2} + \frac{P}{2} f \frac{d\Theta_2}{d\eta} = -\frac{P U_\infty^2}{g c_p} f''^2 \qquad \ldots\ldots (2\text{-}14)$$

With the boundary conditions

$$\eta = 0 \quad : \quad \Theta_2' = 0$$

$$\eta = \infty \quad : \quad \Theta_2 = 0$$

A particular solution of the above non-homogeneous equation for an adiabatic wall was given by Pohlhausen using the method of 'Variation of Constant'. The complete solution of the equation (2-14) can be obtained by adding a particular solution of (2-14) to the general solution of the homogeneous equation. The particular solution for the case of adiabatic wall is

$$\Theta_2(\eta, P) = 2P \int_\eta^\infty \left[f''(\eta) \right]^P \times \left(\int_0^\eta \left[f''(\eta) \right]^{2-P} d\eta \right) d\eta$$

The temperature assumed by the surface owing to frictional heating, i.e. the adiabatic wall temperature is

$$T_{2w} - T_\infty = T_a - T_\infty = \frac{U_\infty^2}{2gc_p} b(P)$$

where

$$b(P) = \Theta_2(0, P)$$

Pohlhausen has shown that for moderate Prandtl numbers

$$b(P) = \sqrt{P} \qquad \ldots\ldots (2\text{-}15)$$

is a good approximation.

Heat Transfer.

(a) Neglecting Frictional Heating.

$$q(x) = -k\sqrt{\frac{U_\infty}{\nu x}}\left(\frac{dT}{d\eta}\right)_{\eta=0} \qquad \ldots\ldots (2\text{-}16)$$

From the equation (2-11)

$$\left(\frac{dT}{d\eta}\right)_{\eta=0} = -0.332\sqrt[3]{P}\,(T_w - T_\infty)$$

$$\therefore \quad q(x) = 0.332\,k\sqrt[3]{P}\sqrt{\frac{U_\infty}{\nu x}}\,(T_w - T_\infty) \qquad \ldots (2\text{-}17)$$

The overall heat transfer Q can be computed by integrating $q(x)\,dx$

$$Q = 2B\int_0^l q(x)\,dx$$

$$= 4B \times 0.332\, k\, \sqrt[3]{P}\, \sqrt{\frac{U_\infty l}{\nu}}\,(T_w - T_{ad})$$

$$= 1.328\, B\, k\, \sqrt[3]{P}\, \sqrt{R_l}\,(T_w - T_\infty) \qquad \cdots\cdots (2\text{-}18)$$

(b) <u>With Frictional Heating.</u>

The general solution for the prescribed temperature difference between the wall and free stream can be obtained as

$$T(\eta) - T_\infty = \left[(T_w - T_\infty) - (T_a - T_\infty)\right]\Theta_1(\eta, P) + \frac{U_\infty^2}{g\,c_p}\,\Theta_2(\eta, P) \qquad \cdots\cdots (2\text{-}19)$$

$$\therefore \left(\frac{dT}{d\eta}\right)_{\eta=0} = -\,0.332\,\sqrt[3]{P}\,(T_w - T_a)$$

and therefore as before

$$q(x) = 0.332\, k\, \sqrt[3]{P}\, \sqrt{\frac{U_\infty}{\nu x}}\,(T_w - T_a) \qquad \cdots (2\text{-}20)$$

$$Q = 1.328\, B\, k\, \sqrt[3]{P}\, \sqrt{R_l}\,(T_w - T_a) \qquad \cdots (2\text{-}21)$$

In the absence of frictional heating, heat flows from the plate to the stream when $T_w > T_\infty$ while with frictional heating heat flows when $T_w > T_a$. Thus, the condition for heat flow from the wall into the fluid becomes

$$(T_w - T_\infty) > (T_a - T_\infty) > \sqrt{P}\,\frac{U_\infty^2}{g\,C_p}$$

2.2 Approximate Methods.

Exact solutions of the boundary layers are extremely laborious and are rarely attempted. The quest for methods, which yield accuracy commensurate with the effort, has led to the development of approximate methods of varying accuracy. Karman and Pohlhausen(7) devised a mothod where the differential equations of the boundary layer are satisfied only average in and over the boundary layer thickness, rather than satisfying the boundary conditions for individual particles. This mean value function is refered to as Momentum Theorem and is obtained as an integral of the equation of motion over the boundary layer thickness. The equation of motion can be expressed as

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = U\frac{dU}{dx} + \frac{\mu}{\rho}\left(\frac{\partial^2 u}{\partial y^2}\right) \quad \cdots (2\text{-}22)$$

Integrating over the boundary layer thickness

$$\int_0^\delta \left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - U\frac{dU}{dx}\right)dy = -\frac{\tau_w}{\rho}$$

which can be put down as

$$U^2\frac{d\delta_2}{dx} + (2\delta_2 + \delta_1)\,U\frac{dU}{dx} = \frac{\tau_w}{\rho} \quad \cdots (2\text{-}23)$$

This gives an ordinary diffe:ential equation for the boundary layer thickness, provided a suitable form is assumed for the velocity profile. Similarly on integrating the equation of energy from $y = 0$ to ∞ and neglecting frictional heating we obtain the following heat flow equation.

$$\frac{d}{dx}\int_0^\infty u(T - T_1)\, dy = -\alpha \left(\frac{\partial T}{\partial y}\right)_{y=0} \quad \ldots\ldots (2\text{-}24)$$

It is now possible to devise approximate methods of solution based on the momentum and the heat flow equations.

a) Profile Assumption.

The method will be illustrated by application to uniform flow past an isothermal plate. Karman-Pohlhausen postulated

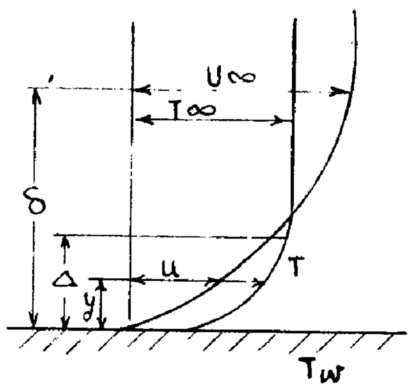

FIG.1. TEMPERATURE & VELOCITY PROFILES ON A WELL

that the velocity profile in the boundary layer can be approximated by a polynomial with a number of free constants, to be determined from the conditions, which the profile is known to fullfil exactly or approximately. Such conditions can be specified at the wall and at the outer edge of the boundary layer.

$$y = 0 \; : \; u = 0 \quad \ldots\ldots (2\text{-}25)$$

$$y = \delta \; : \; u = U_\infty \quad \ldots\ldots (2\text{-}26)$$

The equation of motion written for $y = 0$, becomes

$$y = 0 \; : \; \left(\frac{\partial^2 u}{\partial y^2}\right) = 0 \quad \ldots\ldots (2\text{-}27)$$

Furthermore, the profile in the layer must join the potential flow solution smoothly.

Therefore,

$$y = \delta \; : \quad \left(\frac{\partial u}{\partial y}\right) = 0 \quad ; \quad \left(\frac{\partial^2 u}{\partial y^2}\right) = 0 \quad \ldots\ldots (2\text{-}28)$$

One can demand, even, that the third and higher derivatives be zero. Introducing the dimensionless

$$\eta = y/\delta(x)$$

$$\frac{u}{U_\infty} = a + b\eta + c\eta^2 + d\eta^3 \quad \ldots\ldots (2\text{-}29)$$

The coefficients in equation (2-29) can be determined from the above conditions (2.25 - 2.28) and the final form is

$$\frac{u}{U_\infty} = \frac{3}{2}\eta - \frac{1}{2}\eta^3 \quad \ldots\ldots (2\text{-}30)$$

The momentum equation for flat plate becomes

$$U_\infty^2 \frac{d}{dx}\int_0^\delta \frac{u}{U_\infty}\left(1 - \frac{u}{U_\infty}\right) dy = \nu\left(\frac{\partial u}{\partial y}\right)_{y=0} \quad \ldots\ldots (2\text{-}31)$$

The integral can be evaluated by substituting the expression for velocity profile and this gives

$$U_\infty^2 \cdot \frac{39}{280}\frac{d\delta}{dx} = \frac{3}{2}\nu\frac{U_\infty}{\delta} \quad \ldots\ldots (2\text{-}32)$$

$$\text{or} \quad \delta\, d\delta = \frac{140}{13}\frac{\nu}{U_\infty}dx \quad \ldots\ldots (2\text{-}33)$$

Integrating

$$\delta = 4 \cdot 64 \sqrt{\frac{\nu x}{U_\infty}} \quad \ldots\ldots (2\text{-}34)$$

Now consider the case of a flat plate which may not be heated upto x_0. This is the case of non-coincidental start of the hydrodynamic and thermal boundary layers. Analogous to the solution of velocity boundary layer, a profile is assumed for the temperature boundary layer, such that it satisfies the boundary conditions known to hold good, i.e.

$$y = 0 \; : \; T = T_w \quad \ldots\ldots\ldots\ldots (2\text{-}35)$$

$$y = \delta_T \; : \; T = T_\infty \; ; \; \left(\frac{\partial T}{\partial y}\right) = 0 \quad \ldots (2\text{-}36)$$

and the energy equation written for $y = 0$ becomes

$$y = 0 \; : \; \left(\frac{\partial^2 T}{\partial y^2}\right) = 0 \quad \ldots\ldots (2\text{-}37)$$

Introducing the dimensionless parameter

$$\eta_T = y/\delta_T(x) \quad \ldots\ldots (2\text{-}38)$$

the thermal boundary layer profile can be approximated by

$$\frac{T - T_w}{T_\infty - T_w} = \frac{\theta}{\theta_\infty} = a_1 + b_1\eta_T + c_1\eta_T^2 + d_1\eta_T^3 \quad \ldots (2\text{-}39)$$

which on simplification gives

$$\frac{\theta}{\theta_\infty} = \frac{3}{2}\eta_T - \frac{1}{2}(\eta_T)^3 \quad \ldots\ldots (2\text{-}40)$$

The integral in the heat flow equation (2-24) can now be evaluated.

$$\int_0^l u(\theta - \theta_\infty)\,dy = -\theta_\infty U_\infty \int_0^l \frac{u}{U_\infty}\left(1 - \frac{\theta}{\theta_\infty}\right) dy$$

Substituting the expressions for $\frac{u}{U_\infty}$ & $\frac{\theta}{\theta_\infty}$, the integral transforms to

$$- \Theta_\infty U_\infty \int_0^{\delta_T} [1 - \tfrac{3}{2}\eta_T + \tfrac{1}{2}(\eta_T)^3][\tfrac{3}{2}\eta - \tfrac{1}{2}\eta^3] \, dy$$

$$= - \Theta_\infty U_\infty \, \delta \left(\tfrac{3}{20} \zeta^2 - \tfrac{3}{280} \zeta^4 \right) \quad \ldots\ldots (2-41)$$

where $\zeta = \eta / \eta_T$. The integral has been evaluated upto δ_T only for it is presumed that the thermal boundary layer is smaller than hydrodynamic layer. As ζ is less than 1, the ζ^4 term can be neglected. Substituting in the heat flow equation

$$\frac{3}{20} \Theta_\infty U_\infty \frac{d}{dx} (\zeta^2 \delta) = \frac{3}{2} \alpha \frac{\Theta_\infty}{\zeta \delta} \quad \ldots (2-42)$$

$$\text{or} \quad \frac{1}{10} \Theta_\infty U_\infty \left(\zeta^3 \delta \frac{d\delta}{dx} + 2 \zeta^2 \delta^2 \frac{d\zeta}{dx} \right) = \alpha \quad \ldots (2-43)$$

Introducing the values of $\delta \frac{d\delta}{dx}$ and δ from (2-33) and (2-34) respectively,

$$\zeta^3 + \frac{4}{3} x \frac{d}{dx} (\zeta^3) = \frac{13}{14} P \quad \ldots\ldots (2-44)$$

The solution of the above equation is seen to be of the form

$$\zeta^3 = \frac{13}{14} P + C x^{-3/4} \quad \ldots\ldots (2-45)$$

at $\quad x = x_0 \quad : \quad \zeta = 0$

$$\therefore \zeta = \frac{1}{1 \cdot 026 \sqrt[3]{P}} \sqrt[3]{1 - \left(\frac{x_0}{x}\right)^{3/4}} \quad \ldots\ldots (2-46)$$

Heat transfer coefficient is introduced by

$$q = -k \left(\frac{\partial T}{\partial y} \right)_{y=0} = - h \Theta_\infty \quad \ldots (2-48)$$

$$h = \frac{k}{\Theta_\infty}\left(\frac{\partial T}{\partial y}\right)_{y=0}$$

$$= \frac{3}{2} k \frac{1}{\zeta \delta}$$

$$= \frac{0.332 k \sqrt[3]{P}}{\sqrt[3]{1-(x_0/x)^{3/4}}} \sqrt{\frac{U_\infty}{\nu x}} \quad \ldots\ldots\ (2\text{-}48)$$

If the plate is heated over the entire section

$$h = 0.332 k \sqrt[3]{P} \sqrt{\left(\frac{U_\infty}{\nu x}\right)} \quad \ldots\ldots\ (2\text{-}48a)$$

The close agreement of this result with Pohlhausen's exact solutions is, of course, purely coincidental, but the example brings out the potentialities of the procedure.

2.3. The Plane Plate with Arbitrary Varying Wall Temperatures.

In engineering applications, the varying wall temperatures are, often, of importance. Only isothermal and adiabatic walls have been considered till now. The variation of temperature along the surface effects the shape as well as the thickness of the boundary layer. The first effect becomes evident when the energy equation is differentiated w.r.t. x and written for $y = 0$

$$\alpha \left(\frac{\partial^3 T}{\partial y^3} \right)_{y=0} = \left(\frac{\partial u}{\partial y} \right)_0 \frac{\partial T_w}{\partial x} \quad \cdots\cdots (2-49)$$

This gives a relation between the third derivative of the temperature profile and the wall temperature gradient. Temperature difference θ is now a function of x . The above relationship can be used to determine the constants in the assumed polynomial.

Rubasin[(8)] makes use of Duhamel's superposition theorem for obtaining a solution for the temperature distribution for the case of stepped variation in wall temperature. The energy equation being linear, a solution of the temperature field can be built up by superposing a large number of stepwise variations. If we have a number of particular solutions (T_i) of the energy equation, then it can be shown that

$$T = \sum_{i=1}^{m} C_i T_i \quad \cdots\cdots\cdots\cdots (2-50)$$

is also a solution. The constants C_i can be used to adjust this new solution to the required boundary conditions.

a) Stepwise Variations.

Let each particular solution correspond to a condition in which the wall temperature is equal to the stream temperature upto a certain location and then suddenly changes by an amount ΔT_{wi} . For each particular solution, a heat transfer coefficient can be defined by

$$q = -k\left(\frac{\partial T_i}{\partial y}\right)_{y=0} = h_i \, \Delta T_{wi} \quad \cdots\cdots\cdots (2\text{-}51)$$

where ΔT_{wi} is the jump in the wall temperature.

Heat flow at the surface is

$$q = -k \sum_{i=1}^{n} C_i \left(\frac{\partial T_i}{\partial y}\right)_{y=0}$$

Then for the temperature field

$$q = \sum_{i=1}^{n} C_i \, h_i \, \Delta T_{wi} \quad \cdots\cdots\cdots (2\text{-}52)$$

From the variations it is obvious that the constant C_i's are all equal to 1.

Figure (2) shows the temperature variations in a stepwise fashion with steps $\Delta T_{w1}, \Delta T_{w2} \cdots\cdots\cdots \Delta T_{wi}$ occuring at the locations $\xi_1, \xi_2 \cdots\cdots\cdots \xi_i$

Then heat flow at the wall at the location x is given by

$$q = \sum_{i=1}^{n} h(x, \xi_i)\, \Delta T_{wi} \quad \cdots\cdots\cdots (2\text{-}53)$$

whereas before h is given by

$$h = 0.332\, k \sqrt[3]{P} \, \frac{1}{\sqrt[3]{1 - (\xi_i/x)^{3/4}}} \quad \cdots\cdots (2\text{-}47)$$

In the above analysis it is assumed that the plate has unheated section upto ξ_1, otherwise an additional temperature step $\Delta T_{w0} = (T_{w0} - T_\infty)$ at $\xi = 0$ will have to be added.

b) Continuously Varying Wall Temperature.

Continuously varying wall temperature can be treated by replacing the series in the above equation by an integral.

$$q = \int_0^x h(x,\xi)\, d\,T_w(\xi) \quad \cdots\cdots\cdots (2\text{-}54)$$

which can also be expressed as

$$q = \int_0^x h(x,\xi)\, \frac{d\,T_w(\xi)}{d\,\xi}\, d\,\xi \quad \cdots\cdots (2\text{-}55)$$

If continuous temperature variations and stepwise variations occur simulataneously

$$q = \int_0^x h(x,\xi)\, \frac{d\,T_w(\xi)}{d\,\xi}\, d\xi + \sum_0^m h(x,\xi_i)\, \Delta T_w(\xi_i) \quad \cdots\cdots (2\text{-}56)$$

By summation and integration, the heat flow can be evaluated for a variety of the situations.

CHAPTER 3.

SOLUTIONS FOR GENERAL TWO DIMENSIONAL BODIES.

3.1 Wedge Flows.

In the last chapter, the solutions of the boundary layers for flat plate were discussed. The solutions of the general two dimensional boundary layers with pressure gradient, will ~~xxxxx~~ now be presented.

Exact solutions for the case of two-dimensional flow over a surface, having stream velocity variations as

$$U = cx^{m} \qquad \text{(3-1)}$$

have been obtained. Such a velocity distribution exists along the surface of an infinite wedge, symmetrical with respect to its apex, having an opening angle $\alpha = \frac{2m}{m+1}\pi = \beta\pi$ placed in an incompressible fluid stream. Accordingly these solutions are refered to as "Wedge Flow Solutions". Transformation of the independent variable y, which leads to an ordinary differential equation is

$$\eta = y\sqrt{\frac{m+1}{2}}\sqrt{\frac{U}{\nu x}} \qquad \text{(3-2)}$$

The equation of continuity is integrated by the introduction of the stream function

$$\psi = \sqrt{\frac{2}{m+1}}\sqrt{\nu c}\, x^{\frac{m+1}{2}} f(\eta) \quad \cdots\cdots (3\text{-}3)$$

Introducing these into the equation of motion

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = U\frac{dU}{dx} + \nu\left(\frac{\partial^2 u}{\partial y^2}\right) \quad \cdots\cdots (3\text{-}4)$$

the following differential equation is obtained

$$f''' + ff'' + \beta(1 - f'^2) = 0 \quad \cdots\cdots (3\text{-}5)$$

the boundary condition for this being the usual i.e.

$$\left.\begin{aligned} \eta &= 0\ ; \quad f = 0\ ; \quad f' = 0 \\ \eta &= \infty\ ; \quad\quad\quad\quad\ \ f' = 1 \end{aligned}\right\} \cdots\cdots (3\text{-}6)$$

The solution of equation(3-5) was first given by Falkner and Skan and latter investigated in detail by Hartree.(9) The corrosponding energy equation was solved by Fage and Falkner,(10) who showed that it can be transformed into an ordinary differential equation when the difference between the wall and stream temperature varies according to the following law

$$T_w - T_1 = Ct\, x^{\gamma} \quad \cdots\cdots (3\text{-}7)$$

The transformation yields the following ordinary differential equation for the temperature distribution.

$$\frac{d^2\theta}{d\eta^2} + Pf\frac{d\theta}{d\eta} - \gamma(2-\beta)Pf'\theta = 0 \quad \cdots\cdots (3\text{-}8)$$

where
$$\theta = \frac{T - T_1}{T_w - T_1}$$

having the following boundary conditions

$$\left.\begin{aligned} \eta &= 0\ ; \quad \theta = 1 \\ \eta &= \infty\ ; \quad \theta = 0 \end{aligned}\right\} \cdots\cdots (3\text{-}9)$$

This equation has been investigated by several authors. The value $\gamma = 0$ corresponds to constant temperature difference and hence constant wall temperature. Pohlhausen and Eckert[11] have considered this case. For non-isothermal wedges, the solution can be obtained numerically except for the special case in which the factor $(2-\beta)\gamma = -1$. In this case the equation is exact and hence a simple solution can be obtained.

a) Isothermal Wedge.

In this case both the temperature and velocity layers are similar, that is, they do not alter in shape along the surface. Eckert has tabulated the shapes and thickness of the velocity and temperature layers. It is easy to show that

$$U \frac{d}{dx} \left(\Delta_m^2 \right) = 2 Z_m^2 \left(\frac{1-m}{1+m} \right) \qquad \ldots\ldots (3-10)$$

and

$$\frac{\Delta_m^2}{\nu} \frac{dU}{dx} = 2 Z_m^2 \left(\frac{m}{1+m} \right) \qquad \ldots\ldots (3-11)$$

where Δ_m is any temperature layer thickness. Eckert has tabulated Z_1, Z_2, Z_4 against β for various Prandtl numbers corresponding to the temperature displacement am, the enthalpy flux thickness and heat flux thickness respectively, where

$$\Delta_1 = \int_0^\infty \left(\frac{T-T_1}{T_w-T_1} \right) dy \qquad \ldots\ldots (3-12)$$

$$\Delta_2 = \int_0^\infty \frac{u}{U}\left(\frac{T-T_1}{T_w-T_1}\right)dy \qquad \ldots\ldots\ldots\ldots (3\text{-}13)$$

$$\Delta_4 = -\frac{(T_w-T_1)}{\left(\frac{\partial T}{\partial y}\right)_{y=0}} \qquad \ldots\ldots\ldots\ldots (3\text{-}14)$$

The relationship between $U\frac{d}{dx}(\Delta_4)^2$ and $\frac{\Delta_4^2}{\nu}\frac{dU}{dx}$ enables Δ_4 and hence the heat transfer coefficient $h(x)$ to be calculated since

$$h(x) = k/\Delta_4(x) \qquad \ldots\ldots\ldots (3\text{-}15)$$

b) <u>Particular Case. $(2-\beta)\gamma = -1$</u>

For this case the solution is

$$\Theta = \frac{T-T_1}{T_w-T_1} = e^{-\int_0^\eta Pf\,d\eta} \qquad \ldots\ldots\ldots (3\text{-}16)$$

In this case $\left(\frac{d\Theta}{d\eta}\right)_{\eta=0} = 0$ and the temperature gradient at the wall

$$\left(\frac{\partial T}{\partial y}\right)_{y=0} = C^{1/2}C_t\left(\frac{d\theta}{d\eta}\right)_{\eta=0}\frac{x^{-1}}{\sqrt{\nu(2-\beta)}} \qquad \ldots\ldots (3\text{-}17)$$

which is zero everywhere except at $x = 0$. The heat input, impulsively, at this point can be determined from the heat flow into the boundary layer at any point x.

$$Q = b\int_0^x q(x)\,dx$$

$$= \rho c_p U(T_w-T_1)\int_0^\infty \Theta\cdot\frac{u}{U}\,dy$$

$$= c_r \rho c_p \sqrt{\nu c(2-\beta)} \int_0^\infty \theta f'(\eta) d\eta \quad \cdots\cdots (3\text{-}18)$$

c) <u>Other Values of γ</u>

For other values of γ, numerical methods have to be used. S.Levvy(12) has put the differential equation in the form of a difference equation

$$\frac{\theta_n - 2\theta_{n-1} + \theta_{n-2}}{e^2} + P f_{n-1} \frac{\theta_n - \theta_{n-1}}{e}$$
$$- \gamma P(2-\beta) f'_{n-1} \theta_{n-1} = 0 \quad \cdots\cdots (3\text{-}19)$$

the increment in θ being e apart and n denoting the number of increments. He has solved the equation using a high speed ~~speed~~ electronic computer. The temperature gradient at the wall which gives the heat flux is

$$\left(\frac{d\theta}{d\eta}\right)_{\eta=0} = -\frac{(1-\theta_1)}{e} \quad \cdots\cdots (3\text{-}20)$$

As the boundary condition $\theta_n \to 0$ as $n \to \infty$ is to be satisfied, the value θ_{61} was put equal to 0 and then the values of θ_1 etc were evaluated. Knowing θ_1, $\left(\frac{d\theta}{d\eta}\right)_0$ could be found and hence the heat flux evaluated.

3.2 Frossling Approach.(13)

Blasius in (1908) indicated a method for obtaining the velocity distribution in the boundary layer for an arbitrarily shaped two dimensional body.

For the cases both symmetrical and assymetrical with respect to an axis of the cylinder which is parallel to the free stream the velocity of the potential flow is assumed to be a power series in terms of the arc length measured from the stagnation point along the contour. The velocity profiles in the boundary layer are then obtained as a power series in arc length with coefficients dependent on y , the distance normal to the body. Howarth[14] (1935) by a suitable assumption regarding the power series succeeded in obtaining the y -dependent coefficients, independent of the contour configuration. The coefficients can be tabulated and the computations are considerably simplified.

Frossling (1940) utilised these functions to calculate new ones giving the temperature distribution in the boundary layer, provided wall temperature is assumed constant. The method has been recently extended by Guha and Chia-Shun Yih[15] (1957) for the case of variable wall temperatures. A brief resume of the method is presented below.

Let the potential flow for symmetrical cases be given by the series

$$U(x) = u_1 x + u_3 x^3 + u_5 x^5 + \cdots\cdots \tag{3-21}$$

The coefficients u_1, u_3, u_5 being known from the body shape

Howarth assumed the stream function to be ~~taken~~

$$\psi = \psi_1 x + \psi_3 x^3 + \psi_5 x^5 + \cdots\cdots \quad (3\text{-}22)$$

in which

$$\left.\begin{aligned} \psi_1 &= f_1(u_1) \quad ; \quad \psi_3 = \frac{4u_3}{\sqrt{u_1}} f_3 \\ \psi_5 &= \frac{6u_5}{\sqrt{u_1}}\left(g_5 + \frac{u_3^2}{u_1 u_5} h_5\right) \end{aligned}\right\} \cdots (3\text{-}23)$$

and so on.

$f_1, f_3 \cdots$ etc. being functions of the new variable $\eta = y/\sqrt{u_1}$. The equation of motion can be written as

$$\frac{\partial\psi}{\partial y}\cdot\frac{\partial^2\psi}{\partial x\partial y} - \frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial y^2} = U\frac{dU}{dx} + \frac{\partial^3\psi}{\partial y^3} \quad \cdots\cdots (3\text{-}24)$$

Substituting equations(3-23) into the above equation a series of ordinary differential equations for $f_1, \cdots$ $\cdots\cdots$ etc. is obtained.

For the unsymmetric case, if the expression for $U(x)$ is:

$$U(x) = u_1 x + u_2 x^2 + u_3 x^3 + \cdots\cdots (3\text{-}25)$$

the stream function, can be taken to be

$$\psi(x) = \psi_1 x + \psi_2 x^2 + \psi_3 x^3 + \cdots\cdots (3\text{-}26)$$

in which

$$\left.\begin{aligned} \psi_1 &= f_1 \sqrt{u_1} \quad ; \quad \psi_2 = \frac{3u_2}{\sqrt{u_1}} f_2 \\ \psi_3 &= \frac{4u_3}{\sqrt{u_1}} \left(g_3 + \frac{u_2^2}{u_1 u_3} h_3 \right) \text{ etc} \end{aligned} \right\} \cdots\cdots (3\text{-}27)$$

The series of ordinary differential equations obtained by substituting the above in the equation of motion have been solved by Howarth upto an index of three.

Frossling used the velocity distribution obtained from the Blasius-Howarth expansion to calculate the temperature distribution. For a symmetric case if T_1 is the ambient temperature

$$\Theta = \frac{T - T_1}{T_1} = C_0 + C_2 x^2 + C_4 x^4 + \cdots\cdots \quad (3\text{-}28)$$

c's are taken to be

$$\left.\begin{aligned} C_0 &= F_0 \quad ; \quad C_2 = \frac{4u_3}{u_1} F_2 \\ C_4 &= \frac{6u_5}{u_1} \left(G_4 + \frac{u_3^2}{u_1 u_5} H_4 \right) \end{aligned} \right\} \cdots\cdots (3\text{-}29)$$

Substituting into the energy equation

$$u \frac{\partial \theta}{\partial x} + v \frac{\partial \theta}{\partial y} = \frac{1}{P} \frac{\partial^2 \theta}{\partial y^2} \quad \cdots\cdots (3\text{-}30)$$

a series of ordinary differential equations is obtained. Frossling solved these differential equations with consistent boundary conditions of constant wall temperature and for Prandtl number 0.7. The tabulated functions can be used readily to calculate the temperature distribution and hence the heat transfer. As pointed out earlier Guha and Yih have extended the work and tabulated functions enabling the temperature distribution to be determined for the case of variable wall temperatures. They resolve the series of differential equations obtained into homogeneous and non-homogeneous parts and thus take into account the variable wall temperatures.

3.3 Lighthill's Method.

Lighthill[16] has formulated an approximate method for the calculation of heat transfer across an incompressible laminar boundary layer for the case of arbitrary distribution of main stream velocity and of wall temperature. The flow parameter used is the arbitrary skin friction. The method is, rather, a generalisation of the method of Fage and Falkner. Energy equation is used in Von Mises[17] form, where x and ψ are the independent variables, and operational methods are used to solve the differential equation for the temperature distribution. The energy equation (1-4) is transformed to

$$\frac{\partial T}{\partial x} = \frac{\mu \rho U}{C_p}\left(\frac{\partial u}{\partial \psi}\right)^2 + \frac{1}{P}\frac{\partial}{\partial \psi}\left(\mu \rho U \frac{\partial T}{\partial \psi}\right) \qquad \ldots\ldots (3-31)$$

and the equation of motion to

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial \psi}\left(\mu \rho u \frac{\partial u}{\partial \psi}\right) \qquad \ldots\ldots (3-32)$$

An examination of the above equation reveals an interesting fact. The influence of compressibility is reflected by the variation in the property values μ and ρ and since in the above equations both enter only as product, therefore, if the variation of viscosity with temperature is postulated to be linear, $(\mu \propto T)$ the heat transfer for a main stream with uniform speed but arbitrary Mach number is given by the low speed equations.

The energy equation for the incompressible case neglecting frictional heating can be written as

$$\frac{\partial T}{\partial x} = \frac{\mu \rho}{P} \frac{\partial}{\partial \psi} \left(u \frac{\partial T}{\partial x} \right) \quad \cdots\cdots (3\text{-}33)$$

Following Fage and Falkner, the value of u is approximated by

$$u = \frac{\tau_w(x)}{\mu} y \quad \cdots\cdots (3\text{-}34)$$

where $\tau_w(x)$ represents the shear stress at a distance x from the start of the layer. The approximation is closely accurate near the surface,

Now $\psi = \rho \int_0^y u \, dy$

$$= \frac{\rho \, \tau_w(x)}{2\mu^2} y^2 \quad \cdots\cdots (3\text{-}35)$$

So that

$$u = \left[\frac{2 \tau_w(x) \psi}{\mu \rho} \right]^{1/2} \quad \cdots\cdots (3\text{-}36)$$

Then the energy equation becomes

$$\frac{\partial T}{\partial x} = \frac{1}{P}\left[2\mu\rho\tau_w(x)\right]^{1/2} \times \frac{\partial}{\partial\psi}\left[\psi^{1/2}\frac{\partial T}{\partial\psi}\right] \quad \ldots\ldots\ldots (3\text{-}37)$$

which is solved under the boundary conditions

$$\left.\begin{array}{l} y = 0 \;;\quad T = T_w(x)\;;\; k\left(\frac{\partial T}{\partial y}\right) = q(x) \\ T = 0 \quad \text{at } x = 0 \quad \text{and at } y = \infty \end{array}\right\} \cdot (3\text{-}38)$$

Thus, the scale chosen is with the main stream temperature value as zero.

Lighthill with thehelp of the operational methods solves the above equation to give

$$q(x) = -k\left(\frac{P\rho}{9\mu^2}\right)^{1/3}\frac{J\{\tau_w(x)\}}{(1/3)!} \times \int_0^x\left[\int_\xi^x J\{\tau_w(\xi)\}\,d\xi\right]^{-1/3} dT_w(\xi) \quad (3\text{-}39)$$

which is the required relation between $q(x)$, $\tau_w(x)$ and $T_w(\xi)$

The total heat transfor rate for a surface of unit breadth between $x = 0$ and $x = l$ can then be calculated by evaluating

$$\int_0^l q(x)\,dx$$

or

$$Q(x) = -\frac{k}{(1/3)!}\left(\frac{P\rho}{9\mu^2}\right)^{1/3}\int_0^l J\{\tau_w(x)\}\,dx \times \int_0^x\left[\int_\xi^x J\{\tau_w(\xi)\}\,d\xi\right]^{-1/3} dT_w(\xi)$$

$$= -\frac{k}{(1/3)!}\left(\frac{\rho c}{9\mu^2}\right)^{1/3}\int_0^l d\tau_w(\xi)\ \times$$

$$\int_\xi^l \sqrt{\{\tau_w(x)\}}\left[\int_\xi^x \sqrt{\{\tau_w(\xi)\}}\,d\xi\right]^{-1/3} dx$$

$$= -\frac{\frac{1}{2}k}{(1/3)!}\left(\frac{3\rho c}{\mu^2}\right)^{1/3}\int_0^l \left[\int_x^l \sqrt{\{\tau_w(\xi)\}}\,d\xi\right]^{2/3} d\tau_w(x) \qquad \ldots\ldots(3\text{-}40)$$

Liepmann[18] has recently given an alternative derivation of the equation (3-39). His method is based upon the use of the energy equation in the integral form. He further shows, how the approach can be modified for application of the formula to the case of flow near separation point where τ_w is zero.

Lighthill's method is assymptotically exact, when the thermal boundary layer is much thinner than that of the velocity. The formula is accurate enough for large Prandtl number but for small Prandtl numbers of the order of 0.7, the results may be in error. For low Prandtl number Lighthill suggests the replacement of the constant $\frac{\frac{1}{2}\cdot 3^{1/3}}{(1/3)!} = 0.807$, by 0.73 to improve the accuracy of the results.

3.4 Squire's Method.

Squire[4] (1942) outlined a method for the calculation of heat transfer in the laminar flow region of aerofoils. He assumes a standard shape for the velocity and temperature distribution across the layer, i.e. that of Blasius distribution for a flat plate.

Young and Winterbottom[19] gave a solution of the momentum equation as

$$\delta_2^2 = \frac{0.441\nu}{U^6}\int_0^x U^5 dx \quad \ldots\ldots\ldots (3\text{-}41)$$

For Blasius distribution $\delta_1/\delta_2 = 2.591$

$$\therefore \delta_1^2 = \frac{2.960\nu}{U^6}\int_0^x U^5 dx \quad \ldots\ldots (3\text{-}42)$$

The velocity distribution given by Blasius can be expressed as

$$\frac{u}{U} = f'(\eta) = f'(0.8604\, y/\delta_1)$$

where the similarity transform

$$\eta = \frac{1}{2} y \sqrt{\frac{U}{\nu x}}$$ is replaced by

$$\delta_1 = \int_0^\infty (1-\frac{u}{U})dy = 1.7208\sqrt{\frac{\nu x}{U}}$$

For the thermal boundary layer, we define a displacement thickness by the equation

$$\Delta_1 = \int_0^\infty (\frac{T-T_1}{T_w-T_1})\, dy \quad \ldots\ldots\ldots (3\text{-}12)$$

If it is assumed that the temperature distribution is similar to the velocity distribution, then,

$$(\frac{T-T_1}{T_w-T_1}) = 1 - f'(0.8604\, y/\Delta_1) \quad \ldots (3\text{-}43)$$

Neglecting dissipation energy equation in the integral form is

$$\frac{d}{dx}\left[\int_0^\infty u(T-T_1)\,dy\right] = -\alpha\left(\frac{\partial T}{\partial y}\right)_{y=0} \quad \cdots\cdots (2\text{-}24)$$

Now

$$\frac{-1}{(T_w-T_1)}\left(\frac{\partial T}{\partial y}\right)_{y=0} = 0.8604/\Delta_1\, f''(0)$$

$$= \frac{0.5715}{\Delta_1} \quad \cdots\cdots\cdots (3\text{-}44)$$

$$\text{since } f''(0) = 0.6641$$

The integral on the left hand side of the energy equation can be expressed as

$$\int_0^\infty u(T-T_1)\,dy = U(T_w-T_1)\,\Delta_1\,\phi\left(\frac{\Delta_1}{\delta_1}\right)$$

where

$$\phi\left(\frac{\Delta_1}{\delta_1}\right) = \frac{1}{0.8604}\int_0^\infty f'\left(\frac{\Delta_1\eta_2}{\delta_1}\right)\left[1-f'(\eta_2)\right]d\eta_2$$

where

$$\eta_2 = \frac{0.8604\,y}{\Delta_1}$$

Substituting in the energy equation

$$\frac{d}{dx}\left[U\Delta_1\,\phi\left(\frac{\Delta_1}{\delta_1}\right)\right] = \frac{0.5715\,\alpha}{\Delta_1}$$

or

$$U\phi\,\Delta_1\frac{d\Delta_1}{dx} + \Delta_1^2\frac{d}{dx}(U\phi) = 0.5715\alpha$$

or

$$2(U\phi)^2\Delta_1\frac{d}{dx}(\Delta_1) + 2\Delta_1^2\,U\phi\frac{d}{dx}(U\phi)$$

$$= 1.143\alpha\,U\phi$$

or $$\frac{d}{dx}\left(U\phi \Delta_1^2 \right) = 1{\cdot}143 \alpha U\phi$$

$$\left(U\phi \Delta_1 \right)^2 = 1{\cdot}143 \alpha \int_0^x U\phi \, dx \quad \ldots\ldots\ldots (3-45)$$

From the known values of $f(\eta)$, Squire has tabulated the values of $\phi\left(\frac{\Delta_1}{\delta_1}\right)$, for the values of $\frac{\Delta_1}{\delta_1}$ in the range 0.5 to 2.0.

Dividing equation (3-45) by equation (3-42) gives

$$\frac{\Delta_1^2}{\delta_1^2}\,\phi\left(\frac{\Delta_1}{\delta_1}\right) = \frac{0{\cdot}3861}{P}\;\frac{U^4 \int_0^x U\phi\, dx}{\phi \int_0^x U^5 dx} \quad \ldots\ldots (3-46)$$

For computational purposes, the first approximation is obtained by omitting ϕ from the right hand side of the above equation. The valuo of ϕ thus obtained can then be used to get the second approximation. From Δ_1, the gradient at the surface and hence heat transfer rate can be calculated

$$h = \frac{-k}{(T_w - T_1)}\left(\frac{\partial T}{\partial y}\right)_{y=0}$$

$$= \frac{0{\cdot}5715}{\Delta_1} \quad \ldots\ldots\ldots (3-47)$$

Quadrature Techniques.

The methods outlined earlier have highlighted the computational labour involved. The quest for approximate methods, which will give results of accuracy compatiable with the effort has led to the development of methods where integration is performed by a quadrature, which is similar to that used for evaluating momentum thickness of the velocity boundary layer by Walz, Young and Winterbottom,[19] Thwaites[6] etc.

The procedures fall in two categories. In the first, limited to uniform wall temperatures and coincidental start of the boundary layers, are the methods of Ambrok, Eckert etc. Eckert's method has been recently simplified and extended by Smith and Spalding.[20] The procedures in the second category are applicable to the cases where the wall temperature varies or where the heated section does not start on the leading edge itself. Methods of Spalding,[21] Schuch[22] and Lighthill[16] are typical, the last of which has already been described.

3.5 Smith and Spalding[20] Method.

Standard methods of dimensional analysis can be used to set up a continuation equation for the growth of any temperature boundary layer thickness Δ in the form

$$\frac{U}{\nu}\frac{d}{dx}(\Delta^2) = f\left(\frac{\Delta^2}{\nu}\frac{dU}{dx}\right) \quad \ldots\ldots(3\text{-}48)$$

provided

i) Prandtl number is constant

ii) The rate of growth depends only on local conditions.

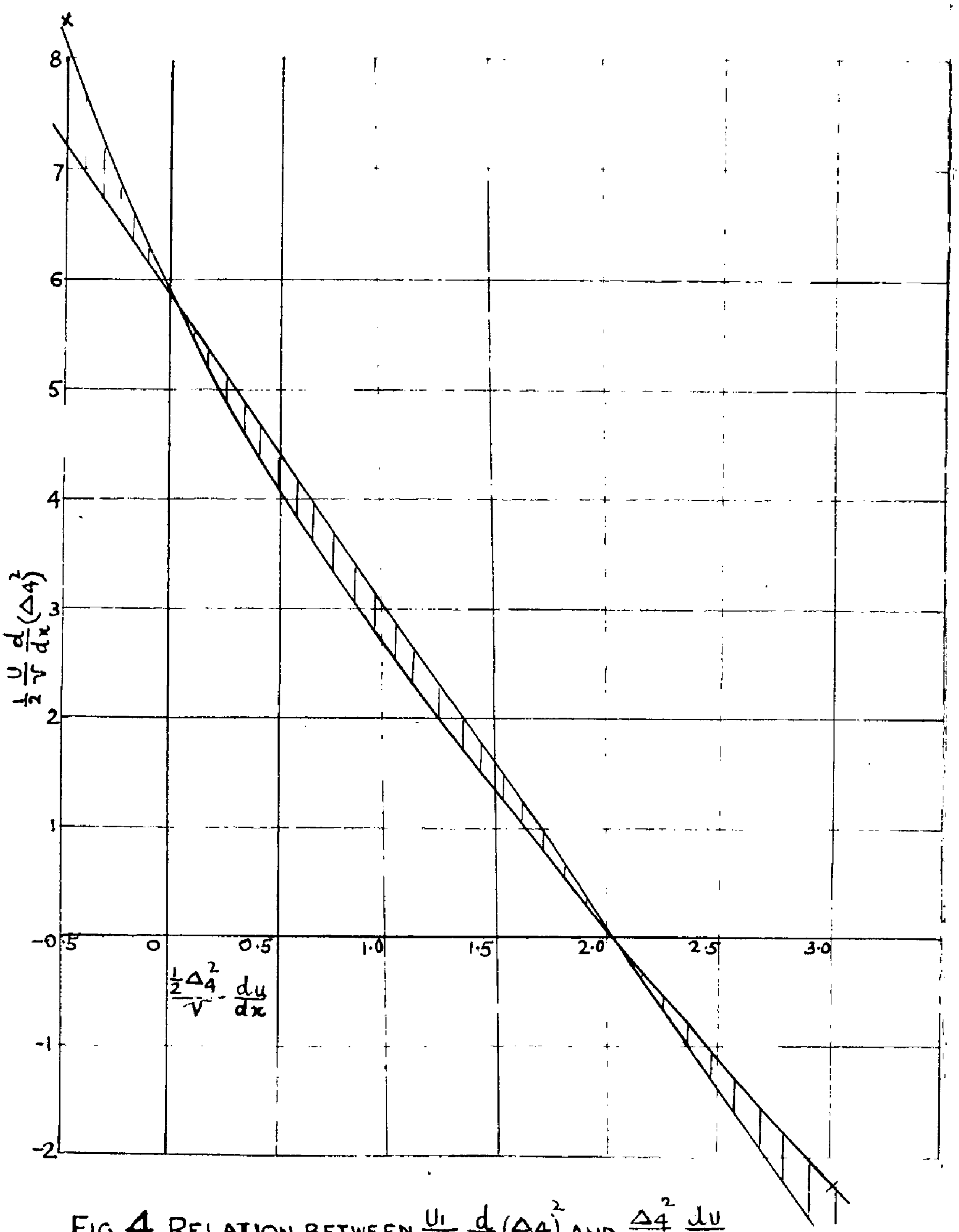

FIG. 4 RELATION BETWEEN $\frac{U_1}{\nu}\frac{d}{dx}(\Delta_4)^2$ AND $\frac{\Delta_4^2}{\nu}\frac{du}{dx}$
$P = 0.1$

iii) Any dependence of the rate of growth on the velocity layer shape or thickness is ignored.

The underlying assumption in the method is that the unknown function in the above relationship can be taken from the exact wedge solutions for uniform surface temperatures. This implies that the relationships which hold good for flows with $U \propto x^m$ holds as well when U is an arbitrary function of x.

From Eckert's[11] data, it is found that the relationship is nearly a straight line, so that (3-48) can be written as

$$\frac{U}{\nu}\frac{d}{dx}(\Delta_4)^2 = A - B\left(\frac{\Delta_4^2}{\nu}\frac{dU}{dx}\right) + E_4\left(\frac{\Delta_4^2}{\nu}\frac{dU}{dx}\right) \quad \ldots (3\text{-}49)$$

where E_4 is the error of non-linearity and Δ_4 is conduction thicknes. The equation can be integrated with the help of integrating factor U^{B-1} and we obtain

$$\frac{U^B}{\nu}(\Delta_4)^2\Big|_0^x = A\int_0^x U^{B-1}dx + \int_0^x U^{B-1}E_4\left(\frac{\Delta_4^2}{\nu}\frac{dU}{dx}\right) \quad \ldots (3\text{-}50)$$

values of A and B can be obtained from Eckert's tabulated data. For P = 0.7 the values are A =11.68 and B = 2.87. From the equation (3-50). Δ_4 may be calculated ignoring E_4 term as a first approximation.

The value obtained can then be relied upon to give second approximation which is close enough for most of the purposes. For computational purposes the demensionless forms of the above equation are more convenient.

$$\left(\frac{\Delta_4}{c}\right)^2 \frac{U_2 c}{\nu} = \frac{11.68}{(U/U_2)^{2.87}} \int_0^{x/c} (U/U_2)^{1.87} d(x/c)$$

$$+ \frac{1}{(U/U_2)^{2.87}} \int_0^{x/c} (U/U_2)^{1.87} E_4(\quad) d(x/c) \quad \ldots\ldots (3\text{-}51)$$

and

$$\frac{hc}{k} \Big/ \sqrt{\frac{U_2 c}{\nu}} = \frac{1}{\frac{\Delta_4}{c} \sqrt{\frac{U_2 c}{\nu}}} \quad \ldots\ldots (3\text{-}52)$$

At the front stagnation point where

$$(U/U_2) = (x/c)\left[\frac{d(U/U_2)}{d(x/c)}\right]_0 \quad \ldots\ldots (3\text{-}53)$$

the equation (3-51) becomes

$$\left(\frac{\Delta_4}{c}\right)^2 \frac{U_2 c}{\nu} = \frac{4.07}{\left[\frac{d(U/U_2)}{d(x/c)}\right]_0} \quad \ldots\ldots (3\text{-}54)$$

The method has been employed to calculate the coeff-icient of heat transfer from the surface of a gas turbine blade. Table (4) shows the computational scheme and the results. The method has the advantage of being computationally simpler but as pointed out earlier, its applications are limited to the cases of Isothermal walls.

3.6 Spalding's[21] Method.

This is typical of the methods grouped under the second category. This method is applicable to a general case where the wall temperature is arbitrary.

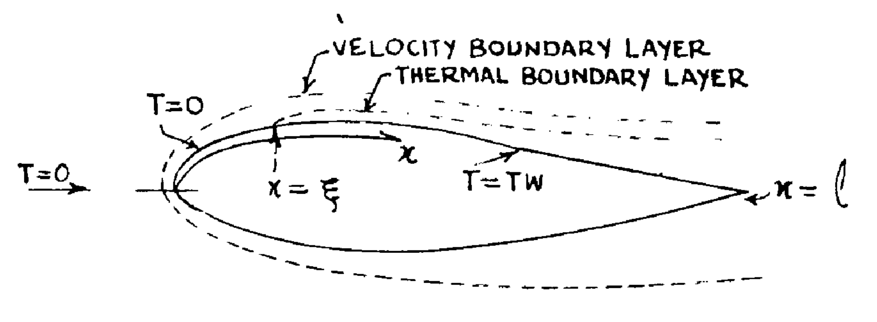

FIG. 5. VELOCITY & THERMAL BOUNDARY LAYER ON AN AEROFOIL.

The figure shows an aerofoil. Measured above the free stream temperature, the wall temperature is zero upto a distance $x = \xi$ from the leading edge and thereon has a value T_0. The figure illustrates the growth of the boundary layers.

As in the previous method, a continuation equation is set up with the help of standard methods of dimensional analysis. Shear thickness δ_4 is supposed to be known and the attention is focussed on the conduction thickness. Δ_4. We assume that the rate of growth of Δ_4 along the x direction depends only on the local value of the stream velocity,

velocity gradient, kinematic viscosity, ν, thermal diffusivity α and the thicknesses of the boundary layers. The equation can be written as

$$\frac{U}{\nu}\frac{d}{dx}(\Delta_4)^2 = f\left(\frac{\delta_4^2}{\nu}\frac{dU}{dx}, \frac{\Delta_4}{\delta_4}, P\right) \qquad (3\text{-}55)$$

This is a first order differential equation, usually non-linear. Once the function on the right hand side is evaluated. Conduction thickness can be determined as a function of the other parameters. The reason for choosing conduction thickness as the dependent variable is, that it is very simply related to the heat transfer coefficient h. i.e,

$$\Delta_4 = k/h \qquad (3\text{-}56)$$

where k is the thermal conductivity of the fluid.

A continuation equation for the velocity boundary layer can also be set up by means of dimensional analysis in the form

$$\frac{U}{\nu}\frac{d}{dx}(\delta_4)^2 = f\left(\frac{\delta_4^2}{\nu}\frac{dU}{dx}\right) \qquad (3\text{-}57)$$

Spalding's method aims at improving the accuracy of the Lighthill method presented earlier in Section (3.3). It has been pointed out that the method of Lighthill is assymptotically exact when the thermal boundary layer is much thinner than the velocity boundary layer. In such a case the thermal boundary layer lies wholly within the velocity boundary layer. In order to proceed in the direction of Lighthill's results using dimensional analysis, it may be noted that U and δ_4 can enter only in the form of gradient U/δ_4 and further that ν must have no influence. The form of the function in (3-55) can then be determined and the following compact continuation equation written

$$\frac{1}{\alpha}\left(\delta_4/U\right)^{1/2}\frac{d}{dx}\left[\Delta_4^{3}\left(U/\delta_4\right)^{3/2}\right] = \text{Constant} \quad \ldots (3\text{-}58)$$

Recalling Lighthill's solution the heat flux $q(x)$ at any station x was given by

$$q(x) = -k\left(\frac{P\rho}{9\mu^2}\right)^{1/3}\frac{J\{\tau_w(x)\}}{(1/3)!}\times$$
$$\times\int_0^x\left[\int_{\xi}^{x} J\{\tau_w(\xi)\}\,d\xi\right]^{-1/3} d\,T_w(\xi) \quad \ldots (3\text{-}39)$$

$q(x)$ is related to the heat transfer coefficient $h(x)$ by

$$q(x) = \int_{\xi}^{x} h(\xi, x)\,d\,T_w(\xi)$$

$$\therefore\ h(\xi) = -k\left(\frac{P\rho}{9\mu^2}\right)^{1/3}\frac{J\{\tau_w(x)\}}{(1/3)!}\times$$
$$\times\left[\int_{\xi}^{x} J\{\tau_w(\xi)\}\,d\xi\right]^{-1/3} \quad \ldots\ldots (3\text{-}59)$$

By definition of δ_4,

$$\tau_w = \mu\left(\frac{U}{\delta_4}\right)$$

$$\therefore\ h(\xi) = -k\left(\frac{P\rho}{9\mu^2}\right)^{1/3}\frac{\mu^{1/2}\left(U/\delta_4\right)^{1/2}}{(1/3)!}\times$$
$$\times\left[\int_{\xi}^{x}\mu^{1/2}\left(U/\delta_4\right)^{1/2}dx\right]^{-1/3}$$

$$\text{or} \quad \frac{1}{\Delta_4} = \left(\frac{1}{\alpha}\right)^{1/3} \cdot \frac{1}{3^{2/3} \cdot (1/3)!} \left(\frac{U}{\delta_4}\right)^{1/2} \left[\int_{\xi}^{x} (U/\delta_4)^{1/2} dx \right]^{-1/3}$$

$$= \left(\frac{1}{\alpha}\right)^{1/3} \cdot \frac{1}{(6.41)^{1/3}} \left(\frac{U}{\delta_4}\right)^{1/2} \left[\int_{\xi}^{x} (U/\delta_4)^{1/2} dx \right]^{-1/3} \quad \ldots\ldots (3-39a)$$

where

$$(3)^{2/3} \cdot (1/3)! = 6.41$$

The equation (3-58) can be expressed as

$$(\Delta_4)^3 (U/\delta_4)^{3/2} = \text{constant} \cdot \alpha \int_{\xi}^{x} (U/\delta_4)^{1/2} dx$$

$$\text{Or} \quad \frac{1}{\Delta_4} = \left(\frac{1}{\text{Constant}}\right)^{1/3} \left(\frac{1}{\alpha}\right)^{1/3} (U/\delta_4)_x^{1/2} \times \left[\int_{\xi}^{x} (U/\delta_4)^{1/2} dx \right]^{-1/3} \quad \ldots (3-58a)$$

Comparing the above equation (3.58 a) with the modified form of Lighthill equation (3-39 a) similarity between the two can be observed. Thus, the value of the constant in (3-58 a) can be taken to be the same as in (3-39 a). The value of the constant was obtained by Lighthill by solving the energy equation of the boundary layer in Von Mises form.

a) Correction in Lighthill's Method.

If Lighthill's method is to be improved, then the correction must take into account the relative thicknesses of the two layers. It may be assumed that the correction required to account for the influence of $\left(\frac{\Delta_4}{\delta_4}\right)$, $\frac{\delta_4^2}{\nu}\frac{dU}{dx}$ and P is a function solely of the extent of the thermal boundary layer into the region where the velocity profile in the boundary layer is curved. A measure of the curvature can be obtained by writing the velocity distribution close to the wall.

$$u = \left(\frac{\partial u}{\partial y}\right)_{y=0} y + \frac{1}{2}\left(\frac{\partial^2 u}{\partial y^2}\right)_{y=0} y^2 \quad \ldots\ldots (3\text{-}59)$$

The equation can be put in a more appropriate form by writing the equation of motion of the boundary layer

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = U\frac{dU}{dx} + \nu\frac{\partial^2 u}{\partial y^2} \quad \ldots\ldots (1\text{-}2)$$

At the boundary, applying no slip condition

$$\left(\frac{\partial^2 u}{\partial y^2}\right)_{y=0} = -\frac{U}{\nu}\frac{dU}{dx}$$

$$\therefore \quad u = \left(\frac{U}{\delta_4}\right)y - \frac{1}{2}\frac{U}{\nu}\frac{dU}{dx}\cdot y^2 \quad \ldots\ldots (3\text{-}60)$$

At $y = \Delta_4$, the ratio of the second degree to the first degree terms is

$$\frac{U}{2\nu}\frac{dU}{dx}\Delta_4^2 \Big/ \left(\frac{U}{\delta_4}\right)\cdot\Delta_4 = \frac{\Delta_4\delta_4}{2\nu}\frac{dU}{dx} \quad \ldots$$

If the correction is solely a function of this term, then the equation (3-58) can be written as

$$\frac{1}{\alpha}\left(\frac{\delta_4}{U}\right)^{1/2}\frac{d}{dx}\left[\Delta_4^3\cdot\left(\frac{U}{\delta_4}\right)^{3/2}\right] = 6.41 + F\left(\frac{\Delta_4\delta_4}{\nu}\frac{dU}{dx}\right)$$

$$\ldots\ldots (3\text{-}61)$$

where F is the unknown correction function. Eckert has tabulated datas for Isothermal wedges in the form:-

$$U \frac{d}{dx} (\Delta_4)^2 = 2 Z_4^2 \left(\frac{1-m}{1+m} \right)$$

$$\frac{\Delta_4^2}{\nu} \frac{dU}{dx} = 2 Z_4^2 \left(\frac{m}{1+m} \right)$$

Therefore,

$$U \frac{d}{dx} (\Delta_4)^2 = f\left(\frac{\Delta_4^2}{\nu} \frac{dU}{dx} \right)$$

Similarly from the known solutions of the velocity boundary layer for wedge flows given by Hartree

$$\frac{u}{U} = f'(\eta)$$

where η is the similarity transform

$$\eta = y \sqrt{\frac{m+1}{2}} \sqrt{\frac{U}{\nu x}} \quad \ldots\ldots (3\text{-}2)$$

a relationship between $\frac{U}{\nu} \frac{d}{dx} (\delta_4)^2$ and $\frac{\delta_4^2}{\nu} \frac{dU}{dx}$.

in Equation (3-57) can be obtained.

Equation (3-61) can be put in the form

$$\frac{3}{2\alpha} \left[\frac{\Delta_4}{\delta_4} U \frac{d}{dx} (\Delta_4)^2 + \frac{\Delta_4}{\delta_4} \left(\Delta_4^2 \frac{dU}{dx} \right) - \frac{1}{2} \left(\frac{\Delta_4}{\delta_4} \right)^3 U \frac{d}{dx} (\delta_4)^2 \right] = 6.41$$

$$+ F \left(\frac{\Delta_4}{\delta_4} \cdot \frac{\delta_4^2}{\nu} \frac{dU}{dx} \right)$$

$\ldots\ldots (3\text{-}62)$

Therefore the above equation can be plotted

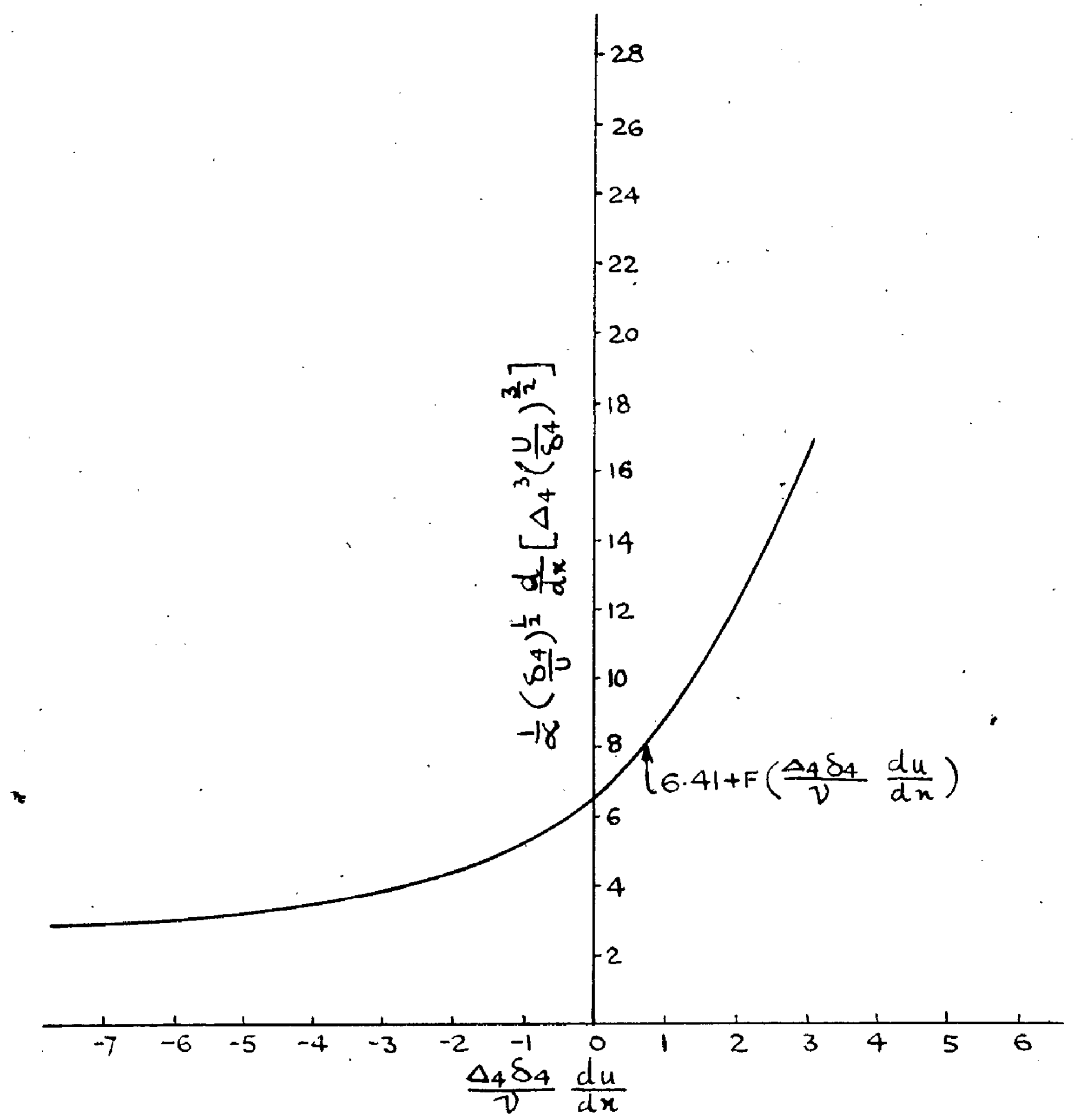

FIG.6. DEDUCTION OF THE CONTINUATION FUNCTIONS

The supposition about the form of the correction function is well founded, as we obtain only a single curve in place of one parametric family of curves. Some scatter, however, is present and has to be tolerated.

b) **Integration of Equation (3.61)**

The equation (3-61) can be integrated numerically to yield Δ_4 as a function of the independent variable x . U, δ_4, $\frac{dU}{dx}$ being the known functions of x . Since the correction term is normally small, therefore, integration can be done by means of a quadrature.

$$\frac{1}{\alpha}\left(\frac{\delta_4}{U}\right)^{1/2}\frac{d}{dx}\left[(\Delta_4)^3\left(\frac{U}{\delta_4}\right)^{3/2}\right] = 6.41 + F\left(\frac{\Delta_4\,\delta_4}{\nu}\,\frac{dU}{dx}\right)$$

or
$$(\Delta_4)^3\left(\frac{U}{\delta_4}\right)^{3/2}\Bigg|_{\xi}^{x} = 6.41\,\alpha\int_{\xi}^{x}\left(\frac{U}{\delta_4}\right)^{1/2}dx + \alpha\int_{\xi}^{x}\left(\frac{U}{\delta_4}\right)^{\frac{1}{2}}F(\;)\,dx$$

or writing $\Delta_4(x)$ as $\Delta_4(\xi) = 0$

$$\Delta_4(x) = \left(\frac{\delta_4}{U}\right)^{1/2}\left[6.41\,\alpha\int_{\xi}^{x}\left(\frac{U}{\delta_4}\right)^{1/2}dx + \alpha\int_{\xi}^{x}\left(\frac{U}{\delta_4}\right)^{1/2}F(\;)\,dx\right]^{1/3} \quad \ldots\ (3\text{-}63)$$

Iterative calculations are indicated as Δ_4 occurs in the argument of the correction term F. As a first approximation the second integral is neglected and the value of Δ_4 so obtained is used in the argument of F to get better approximation.

c) **Evaluation of δ_4**

Uptill now δ_4 has been supposed to be known as a function of x. δ_4 is evaluated indirectly by invoking, the use of unique relationship which exists between shear thickness δ_4 and the momentum thickness δ_2. For wedge flows δ_4/δ_2 is a unique function of the pressure gradient parameter

$$\delta_2^2/\nu \; dU/dx$$

δ_2 can be evaluated by means of an approximate ~~g~~ quadrature similar to the one used by Walz-Thwaites etc. For the momentum thickness

$$\frac{U}{\nu}\frac{d}{dx}(\delta_2)^2 = f\left(\frac{\delta_2^2}{\nu}\frac{dU}{dx}\right) \quad \ldots\ldots (3-64)$$

The unknown function is taken from the known wedge solutions. The relationship is approximately linear. Hence,

$$\frac{U}{\nu}\frac{d}{dx}(\delta_2)^2 = A - B\left(\frac{\delta_2^2}{\nu}\frac{dU}{dx}\right) - e_2\left(\frac{\delta_2^2}{\nu}\frac{dU}{dx}\right) \quad \ldots\ldots (3-65)$$

where e_2 is the error of non-linearity. ~~Thm~~
The above equation can be integrated with the aid of integrating factor U^{B-1}.

$$\frac{U^B}{\nu}\frac{d}{dx}(\delta_2)^2 + BU^{B-1}\frac{\delta_2^2}{\nu}\frac{dU}{dx} = AU^{B-1} - U^{B-1}e_2(\quad)$$

$$\text{Or} \quad \delta_2^2 = \frac{A\nu}{U^B}\int_0^x U^{(B-1)}dx - \frac{\nu}{U^B}\int_0^x U^{(B-1)}e_2(\quad)\,dx \quad \ldots\ldots (3-66)$$

Spalding gives the values of A and B in Eq. (3.66) as 0.4418 and 5.17 respectively. δ_2 can be obtained in the same manner as outlined for Δ_4 . e_2 is again a small correction term. Refering to the tabulated values δ_4 can be obtained from δ_2. Spalding has tabulated some values based on the work of Hartree. While calculating, it is more convenient to use the dimensionless form of the above equations.

The whole calculations can be carried out in three stages which are summarised below.

1. Evaluation of Momentum Thickness.

The equation (3.66) can be put in a non-dimensional form as below.

$$\left(\frac{\delta_2}{c}\right)^2 \frac{U_2 c}{\nu} = \frac{0.4418}{(U/U_2)^{5.17}} \int_0^{x/c} (U/U_2)^{4.17} d(x/c)$$
$$- \frac{1}{(U/U_2)^{5.17}} \int_0^{x/c} (U/U_2)^{4.17} e_2 (\quad) d(x/c)$$

where e_2 is a tabulated function of

$$\left[\left(\frac{\delta_2}{c}\right)^2 \frac{U_2 c}{\nu} \frac{d(U/U_2)}{d(x/c)} \right]$$

2. Reference to the Auxiliary Function.

δ_4 is obtained from δ_2 by refering to the tables of δ_4/δ_2 with the argument $\frac{\delta_2^2}{\nu}\frac{dU}{dx}$ or non-dimensionally to $\left(\frac{\delta_2}{c}\right)^2 \frac{U_2 c}{\nu} \frac{d(U/U_2)}{d(x/c)}$ Having obtained δ_4, Δ_4 can be evaluated.

3. Evaluation of Δ_4

Δ_4 is given by the equation (3-63). The equation can be non-dimensionalised and put as below.

$$\frac{1}{P}\left[\left(\frac{\Delta_4}{c}\right)^2 \frac{U_2 c}{\nu}\right]^{3/2} = \left\{\left(\frac{\delta_4}{c}\right)^2 \frac{U_2 c}{\nu}\right\}^{3/4} \times$$

$$\times \left[6.41 \int_{\xi/c}^{x/c} \left(\frac{U}{U_2}\right)^{1/2} \frac{1}{\left[\left(\frac{\delta_4}{c}\right)^2 \frac{U_2 c}{\nu}\right]^{1/4}} d(x/c)\right.$$

$$\left. + \int_{\xi/c}^{x/c} \left(\frac{U}{U_2}\right)^{1/2} \frac{1}{\left[\left(\frac{\delta_4}{c}\right)^2 \frac{U_2 c}{\nu}\right]^{1/4}} F(\quad) d(x/c)\right]$$

where F is a function of $\frac{\Delta_4 \delta_4}{\nu} \frac{dU}{dx}$ or non-dimensionally of

$$\left[\left\{\left(\frac{\Delta_4}{c}\right)^2 \frac{U_2 c}{\nu}\right\}\left\{\left(\frac{\delta_4}{c}\right)^2 \frac{U_2 c}{\nu}\right\}\right]^{1/2} \frac{d(U/U_2)}{d(x/c)}$$

3.7 Schuch's[22] Method.

This method also involves steps similar to those of Spalding method, namely-

i) Determination of $\delta_2(x)$

ii) Reference to auxiliary function to obtain δ_4.

iii) Evaluation of Δ_4 by iteration from the quadrature formula

$$\Delta_4 = \left(\frac{3}{2p}\right)^{1/3} \left(\frac{\delta_2}{U z_i}\right)^{1/2} \frac{1}{G^{3/2}} \left[\int_{\xi}^{x} \left(\frac{U z_i}{\delta_2}\right)^{1/2} G^{3/2} \, dx \right]$$

where

$$z_i = z_i \left(\frac{\delta_2^2}{\nu} \frac{dU}{dx} \right)$$

$$G = G \left(\frac{\Delta_4}{\delta_2}, \frac{\delta_2^2}{\nu} \frac{dU}{dx} \right)$$

Tabulated values of both these functions are available in Schuch's paper.(22)

CHAPTER 4

POTENTIAL FLOW AROUND AEROFOILS IN CASCADE.

4.1 Introduction.

In order to calculate the local heat transfer coefficient and the overall heat transfer using the methods described earlier, potential velocity distribution around the aerofoil is required. Extensive work has been done for the case of isolated aerofoils. The analysis can be extended to the case of aerofoils in cascade. Whereas initially, the approach of the turbo-machine designer was to treat passages between blades as channels of rectangular, or some such simple geometry, now the emphasis is on the analysis of the cascade of aerofoils in order to choose an optimum arrangement of blades and the most suitable blade profile. The extensive experimentation, as was resorted to in the case of isolated aerofoils, must be ruled out, because of the larger number of parameters involved. Comprehensive theoretical analysis, supported with limited experimentation seems to be the only way out.

4.2 Parameters of Flow.

Purely geometric parameters are those which refer to the blades and their arrangements. The profile of the blade is characterised by thickness and camber distribution along the chord and some other minor details like the leading edge radius and the ordinate at the trailing edge. The parameters depending on the geometry of the cascade are:-

1) Solidity ratio s/c.
2) Angle of stagger λ .

Aerofoil parameters depend on the blade chosen but the cascade parameters depend on a particular arrangement of the blades.

Aerodynamic parameters are the various velocities, flow deviation angles, pressure difference across the cascade, loss coefficient and the chordwise pressure distribution on the blade surfaces.

The purpose of the analysis is to establish a relationship between the geometric and the aerodynamic parameters. In the case of isolated aerofoils, comparatively sounder theories exist which corelate the above parameters. The problem is, however, immensely complicated in the case of the cascades. Currently, attempts are being made to formulate a sound theory for the basic, two dimensional incompressible potential flow around a cascade of aerofoils. The effect of various other factors such as,

a) Compressibility,

b) Three dimensional effects,

c) Variation of solidity with radius,

d) Centrifugal forces in rotating cascades.

can then be incorporated by improvising the basic solution. Hence the importance of investigating the basic 2-dimensional incompressible potential flow can not be over-emphasised.

In the potential flow analysis around a cascade of aerofoils, two types of problems occur.

1) The direct problem in which the geometric parameters are known and the resulting aerodynamic parameters are to be found out as function of the known inflow conditions.
2) The indirect problem, in which the aerodynamic parameters are known from design considerations and a cascade geometry to satisfy these is to be found.

The solutionsfor the above problems have been recently attempted. The solution procedures follow either the method of conformal transformation or the method of the distribution of singularities.

It is well known that the two dimensional irrotational flow of an incompressible fluid can be represented by an analytic function of a complex variable. The use of complex variables permits us to use the powerful tool of conformal transformation. The determination of a suitable mapping function, which will transform the flow past the cascades into flow past a known obstacle, will enable us to predict the flow past cascades.

The method of the distribution of the singularities, though classical in origin, has recently been revived for application to both isolated aerofoils and the cascades. This has been necessitated by the inherent limitation of the method of conformal mapping i.e. inability to extend the results to three dimensional flows. Recently number of papers have appeared, extonding the application of this method to the cascades of aerofoils.

4.3 Method of Conformal Transformation.

(a) General Approach- Fanti, Kemp and Nilson (23) have outlined an approach typical of the methods of conformal transformation. The objective is to develop a pair of equations which relate the cartesian components of velocity u, v of the two-dimensional incompressible potential flow past the aerofoil on the aerofoil surface. The thin-aerofoil theory approximation i.e., the aerofoil being replaced by the chord line, makes the transformation a simple one. However, it limits the analysis to ~~thin~~ aerofoils of small camber.

The problem of determining potential flow past an aerofoil, a boundary value problem of the potential theory, is transfered to the chord line $(-c/2 \leqslant x \leqslant c/2)$, $(y = 0)$. On the chord line, then the condition of continuity, irrotationality and tangency must be satisfied. The first two conditions imply that the complex velocity $(u - iv)$ is an analytic function of the complex variable z. This fact yields one relation between u and v. Tangency condition yields the second relationship, enabling u and v to be expressed entirely in terms of the aerofoil coordinates.

b) The case of Isolated Aerofoil.

Consider first the case of an isolated aerofoil. The transformation scheme is shown in Fig 7

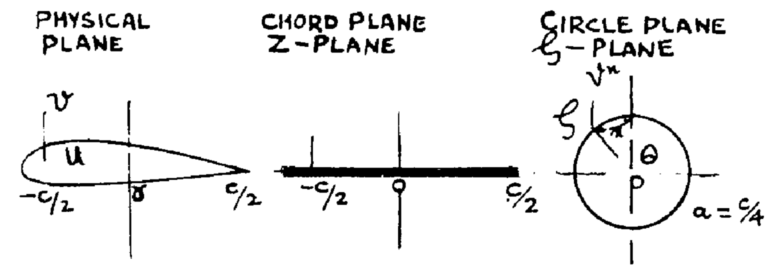

FIG.7. ISOLATED AEROFOIL WITH CHORD AND CIRCLE PLANES

Aerofoil having been mapped into the slit plane (this is an approximation), the mapping into the circle plane is affected by Joukovsky ~~and~~ transformation. The velocities on the two planes are related by

$$(u^* - i v^*) = (u - i v)\frac{dz}{d\zeta} \qquad \ldots\ldots (4\text{-}1)$$

or

$$(u^* - i v^*)\zeta = (u - i v)\, 2ai\, g(\theta) \qquad \ldots (4\text{-}2)$$

where $\qquad g(\theta) = \sin\theta$

The physical significance of $(u^* - i v^*)\zeta$ lies in the fact that its integral round the circle has the imaginary term equal to the circulation and the real term equal to the net outflow of fluid from the circle. The function $(u^* - i v^*)\zeta$ is analytic outside the circle except at the pole $\zeta = \infty$. If the principal part $(u^*_\infty - i v^*_\infty)\zeta$ at this pole is subtracted from the function, then the difference is analytic. By applying the fundamental theorem of Hilber and Demov to

$$f(\zeta) = \left[(u^* - u^*_\infty) - i(v^* - v^*_\infty)\right]\zeta \qquad \ldots (4\text{-}3)$$

the integral equation is obtained.

(c) The fundamental theorem.

The fundamental theorem states that if $f(\zeta)$ is (i) analytic exterior to the circular contour $|\zeta| = a$, (ii) is continuous on the circle and (iii) is bounded as $\zeta \to \infty$, then

$$f(\zeta) = \frac{1}{2\pi i}\int_0^{2\pi} f(ae^{i\phi}) \cot\frac{\theta-\phi}{2}\, d\phi + \text{Constant } C \quad \ldots (4\text{-}4)$$

where

$$C = \frac{1}{2\pi}\int_0^{2\pi} f(ae^{i\phi})\, d\phi = f(\infty) \quad \ldots (4\text{-}5)$$

Application of the fundamental theorem and likewise its corrollary which holds in the interior of the circle, results in the following velocity relationships

$$v(\theta) = \frac{V\cos(\theta-\alpha)}{\sin\theta} + \frac{1}{2\pi\sin\theta}\int_0^{2\pi} g(\phi)\,u(\phi)\cot\frac{\theta-\phi}{2}\, d\phi \quad \ldots (4\text{-}6)$$

$$u(\theta) = V\left[\frac{1-\cos\theta}{\sin\theta}\sin\alpha + \cos\alpha\right]$$

$$-\frac{1}{2\pi\sin\theta}\int_0^{2\pi} g(\phi)\,v(\phi)\left[\cot\frac{\theta-\phi}{2} + \cot\frac{\phi}{2}\right] d\phi \quad \ldots (4\text{-}7)$$

Now the tangency condition

$v = u\dfrac{dy}{dx}$ can be applied, where y is a double valued function. The substitution of boundary condition results in singular integral equations of the form

$$v(\theta) = \frac{V\cos(\theta-\alpha)}{\sin\theta} + \frac{1}{2\pi\sin\theta}\int_0^{2\pi} v(\phi)\,\frac{1}{\frac{dy}{dx}}\,g(\phi)\cot\frac{\theta-\phi}{2}\,d\phi \qquad \ldots\ldots(4\text{-}8)$$

$$u(\theta) = V\left[\frac{1-\cos\theta}{\sin\theta}\sin\alpha + \cos\alpha\right] - \frac{1}{2\pi\sin\theta}\int_0^{2\pi} u(\phi)\,\frac{dy}{dx}\,g(\phi)\left[\cot\frac{\theta-\phi}{2} + \cot\frac{\phi}{2}\right]d\phi \;. \qquad \ldots(4\text{-}9)$$

where dy/dx is evaluated at x associated with $x = c/2\cos\phi$. The solutions of the above equations have been carried out on the digital computers. The numerical solution is found to converge sufficiently rapidly, three or four iterations being sufficient.

d) Extension to Cascades.

The blades in cascades are replaced by their chord lines. The figure shows a cascade of staggered blades. The transformation to the circle plane is carried out by means of the mapping function

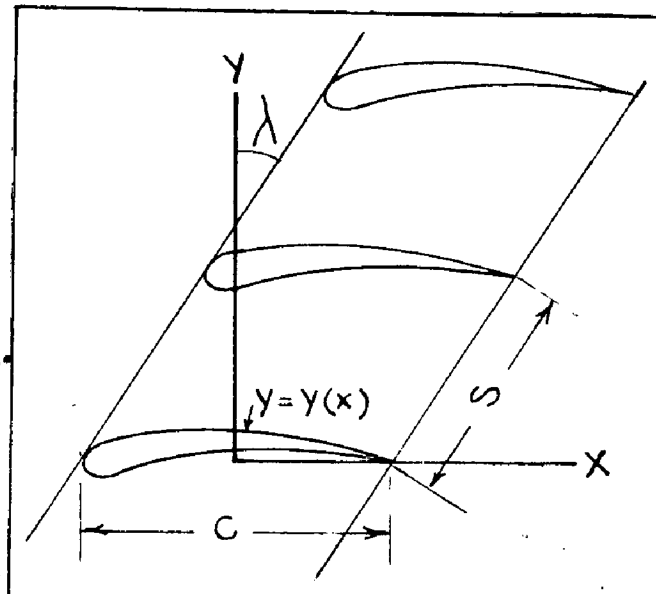

FIG.8. CASCADE OF AEROFOILS

$$z = \frac{s}{2\pi}\left[e^{-i\lambda}\ln\frac{\frac{1}{k}+\frac{\zeta}{a}}{\frac{1}{k}-\frac{\zeta}{a}} + e^{i\lambda}\ln\frac{\frac{\zeta}{a}+k}{\frac{\zeta}{a}-k}\right] \qquad \ldots(4\text{-}10)$$

Logarithmic nature of the transformation makes z, a multiple valued function, so that the circle $\zeta = ae^{i\theta}$ is transformed into the infinite set of line segments.

$$z = x + imse^{-i\lambda} \qquad m = 0, \pm 1, \pm 2 \cdots \qquad \ldots(4\text{-}11)$$

where X denotes the position of the line segment w ich lies on the x axis. On the circle the relation between Z and θ is given by the equation (4-11) where the function $X(\theta)$ is

$$X(\theta) = \frac{S}{\pi}\left[\cos\lambda \operatorname{arc\,tanh}\frac{\cos\theta}{K} + \sin\lambda \operatorname{arc\,tan}\frac{\sin\theta}{K^*}\right] \qquad \ldots\ldots (4\text{-}12)$$

where $K = \dfrac{k + 1/k}{2}$ and $K^* = \dfrac{1/k - k}{2}$. The derivative of the transformation is

$$\frac{dz}{d\zeta} = \frac{S}{2\pi a}\left[e^{-i\lambda}\frac{2/k}{\frac{1}{k^2} - (\zeta/a)^2} - e^{i\lambda}\frac{2k}{(\zeta/a)^2 - (1/k)^2}\right] \qquad \ldots\ldots (4\text{-}13)$$

which on the circle becomes

$$\frac{dz}{d\zeta} = \frac{dx}{iae^{i\theta}d\theta} = 2ie^{-i\theta}G(\theta) \qquad \ldots\ldots (4\text{-}14)$$

where

$$G(\theta) = \frac{S}{2\pi a}\,\frac{K\sin\theta\cos\lambda - K^*\cos\theta\sin\lambda}{K^2 - \cos^2\theta} \qquad \ldots (4\text{-}15)$$

The leading edge and the trailing edge of the straight line segment correspond to the points $ae^{i\theta_T}$ and $ae^{i\theta_L}$ on the circle. A useful relation between λ and θ_T is

$$\cos\theta_T = K\cos\lambda / J \qquad \ldots\ldots (4\text{-}16)$$

where $J = \sqrt{K^2 - \sin^2\lambda}$

and $\theta_L = \theta_T + \pi \qquad \ldots\ldots (4\text{-}17)$

Velocity in the circle plane is related to the velocity in the cascade plane by

$$(u^* - iv^*) = (u - iv)\frac{dz}{d\zeta} \quad \cdots\cdots (4\text{-}18)$$

On the circle,

$$\zeta(u^* - iv^*) = 2iaG(\theta)(u - iv) \quad \cdots (4\text{-}19)$$

The function $\zeta(u^*-iv^*)$ is not analytic everywhere outside the circle. Eq. (4-13) shows that $dz/d\zeta$ has poles at the points $\zeta = \pm a/h$. At these points let the velocity be $V_1 e^{-i\alpha_1}$ and $V_2 e^{-i\alpha_2}$ respectively. Then from equation (4-13) and (4-18) follow the relations

$$(u^* - iv^*) \simeq \frac{Se^{-i\lambda}}{2\pi a}\,\frac{V_1 e^{-i\alpha_1}}{(1/h) + (\zeta/a)} \quad \text{as } \zeta \to -\frac{a}{h}$$

$$(u^* - iv^*) \simeq \frac{Se^{-i\lambda}}{2\pi a}\,\frac{V_2 e^{-i\alpha_2}}{(1/h) - (\zeta/a)} \quad \text{as } \zeta \to \frac{a}{h}$$

In view of these the function

$$F(\zeta) = \zeta\left[u^* - iv^* - \frac{Se^{-i\lambda}}{2\pi a}\left(\frac{V_1 e^{-i\alpha_1}}{1/h + \zeta/a} + \frac{V_2 e^{-i\alpha_2}}{1/h - \zeta/a}\right)\right] \quad \cdots\cdots (4\text{-}20)$$

is both analytic outside the circle and bounded as $\zeta \to \infty$. The fundamental principle applied to $F(\zeta)$ yields relationship for the velocity components

$$2aG(\theta)\left[v(\theta) + iu(\theta)\right] - 2ae^{i\theta}H(\zeta)$$

$$=$$

$$= \frac{1}{2\pi i} \int_0^{2\pi} 2a\, G(\phi)\left[v(\phi) + i\, u(\phi)\right] \cot\frac{\theta-\phi}{2}\, d\phi + C \qquad \ldots\ldots (4\text{-}21)$$

where

$$H(\zeta) = \frac{S e^{-i\lambda}}{2\pi a}\left[\frac{V_1 e^{-i\alpha_1}}{1/k_2 + \zeta/a} + \frac{V_2 e^{-i\alpha_2}}{1/k_2 - \zeta/a}\right]$$

The constant C is related to the total circulation by the fact that

$$C = \frac{i\Gamma}{2\pi} \qquad \ldots\ldots (4\text{-}22)$$

Separating (4-21) into real and imaginary parts,

$$G(\theta)\,v(\theta) = \frac{1}{2\pi}\int_0^{2\pi} G(\phi)\, u(\phi) \cot\frac{\theta-\phi}{2}\, d\phi$$

$$- \frac{\Gamma}{8\pi a}\frac{\sin 2\theta}{K^2 - \cos^2\theta} + \frac{V_S}{2\pi a}\left[\frac{K^* \cos\theta \cos(\alpha+\lambda)}{K^2 - \cos^2\theta}\right.$$

$$\left. + \frac{K \sin\theta\, \sin(\alpha+\lambda)}{K^2 - \cos^2\theta}\right] \qquad \ldots\ldots (4\text{-}23)$$

and

$$G(\theta)\,u(\theta) = -\frac{1}{2\pi}\int_0^{2\pi} G(\phi)\, v(\phi) \cot\frac{\theta-\phi}{2}\, d\phi$$

$$+ \frac{\Gamma}{4\pi a}\frac{K K^*}{K^2 - \cos^2\theta} + \frac{V_S}{2\pi a} \times$$

$$\times \frac{K \sin\theta \cos(\alpha+\lambda) - K^* \cos\theta \sin(\alpha+\lambda)}{K^2 - \cos^2\theta} \qquad \ldots\ldots (4\text{-}24)$$

Circulation is determined from Kuttas condition of finite velocity at the trailing edge. Since at θ_T

$G(\theta_T) = 0$ therefore the right hand side of (4-24) must vanish at Θ_T

$$\therefore \Gamma = \frac{2V_s \sin\alpha}{J} + \frac{2KK^*}{J^2}\int_0^{2\pi} a\,G(\phi)\,v(\phi)\cot\frac{\theta-\phi}{2}\,d\phi \qquad \text{....(4-25)}$$

From equations (4-23), (4-24) and (4-25) and with the help of the boundary condition

$$v = u\frac{dy}{dx} \qquad \text{..............(4-26)}$$

the desired singular integral equations for u and v can be obtained. These on simplification give

$$\frac{u(\theta)}{V} = -\frac{1}{2\pi}\int_0^{2\pi}\left\{\frac{\sin(\phi-\theta_T)}{\sin(\theta-\theta_T)}\left[\cot\frac{\theta-\phi}{2}(K^2-\cos^2\theta)\right.\right.$$

$$\left.\left. - \cot\frac{\theta_T-\phi}{2}(K^2-\cos^2\theta_T)\right]\right\}\frac{u(\phi)}{V}\frac{dy}{dx}\,d\phi$$

$$+ \frac{KK^*\sin\alpha + KJ\sin\theta\cos(\alpha+\lambda) - K^*J\cos\theta\sin(\alpha+\lambda)}{J^2\sin(\theta-\theta_T)}$$

$$\text{......(4-27)}$$

and for $v(\theta)$

$$\frac{v(\theta)}{V} = \frac{a}{SV}\frac{K^2-\cos^2\theta}{K\sin\theta\cos\lambda - K^*\cos\theta\sin\lambda}\int_0^{2\pi}\left\{G(\phi)\right.$$

$$\left. v(\phi)\left[\frac{1}{dy/dx}\cot\frac{\theta-\phi}{2} - \frac{KK^*\sin 2\theta}{2J^2(K^2-\cos^2\theta)}\cot\frac{\theta_T-\phi}{2}\right]\right\}d\phi$$

$$-\left(K\sin\theta\cos\lambda - K^*\cos\theta\sin\lambda\right)\left[\frac{a\sin\alpha\sin 2\theta}{2J}\right.$$

$$\left. - K^*\cos\theta\cos(\alpha+\lambda) + K\sin\theta\sin(\alpha+\lambda)\right]$$

(4-28)

These equations can be solved on a high speed computer using iterative procedure.

4.4 Singularities Distribution Method.

The blade is replaced by a distribution of vortices and sources and sinks along the camber line. The proper distribution which will make the blade, a stream line in the flow is sought. Several simplifying assumptions have to be made in order to reduce the computational labour involved.

The aerofoil, itself may be described by camber distribution and the thickness distribution.

$$\frac{y_c}{c} \equiv C_b f_c(x/c) \quad \ldots\ldots (4-29)$$

$$\frac{y_t}{c} \equiv \frac{t}{c} f_t(x/c) \quad \ldots\ldots (4-30)$$

t is the maximum thickness of the aerofoil and C_b is a constant defined by the equation.

$$C_b \equiv \int_0^{2\pi} \frac{dy_c}{dx} \cos\theta \, d\theta \quad \ldots\ldots (4-31)$$

Equations (4-29) and (4-30) describe a family of aerofoils where the constants (t/c) and C_b are used to scale up or down the camber and thickness of the individual member. The physical significance of C_b is that at the design angle of attack, it gives the lift coefficient in the case of isolated aerofoil.

a) Induced Velocity Equations.

Figure (9) shows the contributions of vorticity and sources and sink, placed at x on the mth blade, to the velocity at x_0 on the oth blade.

Summing up all the blades from $-\infty$ to $+\infty$ and integrating them all at once from $x/c = 0$ to $x/c = 1$ yields.

$$\frac{u_0}{V_m} = -\frac{c/s}{2}\int_0^1 \frac{\gamma}{V_m} I\left(\frac{x_0-x}{c}, \frac{c}{s}, \lambda\right) d\left(x/c\right)$$

$$+ \frac{1}{2\pi}\int_0^1 \frac{q}{V_m} \frac{d\left(x/c\right)}{\frac{x_0}{c}-\frac{x}{c}}$$

$$+ \frac{c/s}{2}\int_0^1 \frac{q}{V_m} R\left(\frac{x_0-x}{c}, \frac{c}{s}, \lambda\right) d\left(x/c\right) \qquad \ldots (4\text{-}32)$$

$$\frac{v_0}{V_m} = -\frac{1}{2\pi}\int_0^1 \frac{\gamma}{V_m} \frac{d\left(x/c\right)}{x_0/c - x/c}$$

$$- \frac{c/s}{2}\int_0^1 \frac{\gamma}{V_m} R\left(\frac{x_0-x}{c}, \frac{c}{s}, \lambda\right) d\left(x/c\right)$$

$$- \frac{c/s}{2}\int_0^1 \frac{q}{V_m} I\left(\frac{x_0-x}{c}, \frac{c}{s}, \lambda\right) d\left(x/c\right) \qquad \ldots (4\text{-}33)$$

where R and I are called cascade influence functions and are given by

$$R\left(\frac{x_0-x}{c}, \frac{c}{s}, \lambda\right)$$

$$= \frac{1}{\pi}\left[\sum_{1}^{\infty} + \sum_{-\infty}^{-1}\right] \frac{\frac{x_0-x}{c}\cdot\frac{c}{s} - n\sin\lambda}{\left[\frac{x_0-x}{c}\cdot\frac{c}{s} - n\sin\lambda\right]^2 + n^2\cos^2\lambda} \qquad \ldots\ldots (4\text{-}34)$$

and

$$I\left(\frac{x_0-x}{c},\frac{c}{s},\lambda\right)$$

$$=\frac{1}{\pi}\left[\sum_{1}^{\infty}+\sum_{-\infty}^{-1}\right]\frac{n\cos\lambda}{\left[\frac{x_0-x}{c}\cdot\frac{c}{s}-n\sin\lambda\right]^2+n^2\cos^2\lambda} \quad \ldots\ldots(4\text{-}35)$$

R and I can be expressed in closed form and are actually real and imaginary parts of a complex functions. Scholz[24] has tabulated these functions at various $\frac{x_0-x}{s}$ and for considerable number of angles of stagger λ.

b) Tangential boundary condition.

The aerofoil is made a streamline by replacing it with a source-sink and vorticity distribution. Therefore, the resultant velocity is tangential to the blade camber. This condition of flow tangency may be written as

$$\frac{V_m\sin\alpha+v_0}{V_m\cos\alpha+u_0}=\left(\frac{dy_c}{dx}\right)=C_b f_c'(x_0/c) \quad \ldots\ldots(4\text{-}36)$$

Kutta Joukovsky condition stipulates that there should be no discontinuity in the velocity field at the trailing edge. Since the strength of the vortex sheet is given by the jump in velocity, it follows that

$$\gamma\Big|_{T.E}=0 \quad \ldots\ldots\ldots(4\text{-}37)$$

In case this was not zero, there would be velocity discontinuity at the trailing edge. Thus any solution for vorticity distribution must satisfy the above condition.

For source-sink distribution, assuming a closed profile, the fundamental condition to be satisfied is:

$$\int_0^1 q\left(\tfrac{x}{c}\right) d\left(x/c\right) = 0 \quad \ldots\ldots (4\text{-}38)$$

Applying the equation of continuity to a segment of the profile, (Fig 10)

$$(V_m + u)\,y_t + q\,dx = \left(V_m + u + \frac{du}{dx}dx\right)\left(y_t + \frac{dy_t}{dx}dx\right)$$

$$\text{or} \quad q = V_m \frac{dy_t}{dx} + \frac{d}{dx}\left(u \cdot y_t\right)$$

$\frac{d}{dx}\left(u \cdot y_t\right)$ is small and therefore

$$\frac{q}{V_m} = \frac{dy_t}{dx} = \frac{t}{c} f_t'\left(x/c\right) \quad \ldots\ldots (4\text{-}39)$$

This gives the solution of the source sink distribution. It can be seen that this satisfies the fundamental condition

$$\int_0^1 q\left(x/c\right) d\left(x/c\right) = 0$$

For the vortex sheet, we assume a Fourier expansion in the form

$$\frac{\gamma}{V_m} = 2A_0 \frac{1+\cos\theta}{\sin\theta} + 4\sum_{n=1}^{\infty} A_n \sin n\theta \quad \ldots\ldots (4\text{-}40)$$

Substituting (4-40) and (4-39) for $\frac{\gamma}{V_m}$ and q/V_m in the induced velocity equations (4-32) and (4-33) and then substituting in equation (4-36), gives:-

$$\sum_0^{\infty} A_n g_n = \sin\alpha - C_b f_c'(\theta_0) + C_b \sum_0^{\infty} A_n h_n - \frac{t}{c} C_b B - \frac{t}{c} T \quad \ldots\ldots (4\text{-}41)$$

where

$$g_0(\theta_0, c/s, \lambda) = 1 + \tfrac{1}{2}\, c/s \int_0^{\pi} R\,(1+\cos\theta)\, d\theta$$

$$g_n(\theta_0, c/s, \lambda) = -2\cos n\theta + \tfrac{1}{2}\, c/s \int_0^{\pi} 2R \sin n\theta \sin\theta\, d\theta \quad \ldots\ n>0$$

$$h_0(\theta_0, c/s, \lambda) = \tfrac{1}{2}\, c/s \int_0^{\pi} f_c'(\theta_0)\, I\,(1+\cos\theta)\, d\theta$$

$$h_n(\theta_0, c/s, \lambda) = \tfrac{1}{2}\, c/s \int_0^{\pi} 2 f_c'(\theta_0)\, I \sin n\theta \sin\theta\, d\theta \quad \ldots\ n>0$$

$$B(\theta_0, c/s, \lambda) = \tfrac{1}{2} \int_0^{\pi} f_c'(\theta_0)\, f_t'(\theta_0) \left[\frac{1/\pi}{\cos\theta - \cos\theta_0} + \frac{c}{s}\frac{R}{2}\right] \sin\theta\, d\theta$$

$$T(\theta_0, c/s, \lambda) = \frac{c/s}{2} \int_0^{\pi} f_t'(\theta_0)\, I\, \frac{\sin\theta}{2}\, d\theta$$

Schlichting's(25) basis of analysis was to tabulate the functions similar to g, h, B and T for every c/s and λ at many chord intervals θ_0. Applying boundary condition (4.36) at three chord-wise positions he obtains three simultaneous equations and solves for A_0, A_1 and A_2.

Mellor(26) suggests proceeding with Fourier analysis analogous to that used in the thin aerofoil theory. If we now multiply every term of (4.41) by $\cos k\theta_0$ and integerate from 0 to π, we have

$$\sum_{m=0}^{\infty} A_m g_{mk} = \sin\alpha \cdot S_{ok} - Cb \cdot \frac{1}{\pi}\int_0^{\pi} f_c'(\theta_0)\cos k\theta_0\, d\theta_0$$
$$+ Cb\sum_0^{\infty} A_m h_{mk} - \frac{t}{c} Cb B_k - \frac{t}{c} T_k \qquad \ldots\ldots(4-42)$$

where

$$g_{ok}(c/s,\lambda) = S_{ok} + \frac{c/s}{2\pi}\int_0^{\pi}\int_0^{\pi} R(1+\cos\theta)\cos k\theta_0\, d\theta\, d\theta_0$$

$$g_{mk}(c/s,\lambda) = -S_{mk} + \frac{c/s}{2\pi}\int_0^{\pi}\int_0^{\pi} 2R\sin m\theta \times \sin\theta\cos k\theta_0\, d\theta\, d\theta_0 \qquad m>0$$

$$h_{ok}(c/s,\lambda) = \frac{c/s}{2\pi}\int_0^{\pi}\int_0^{\pi} f_c'(\theta_0)\, I(1+\cos\theta)\cos k\theta_0\, d\theta\, d\theta_0$$

$$h_{mk}(c/s,\lambda) = \frac{c/s}{2\pi}\int_0^{\pi}\int_0^{\pi} f_c'(\theta_0)\, 2I\sin m\theta \times \sin\theta\cos k\theta_0\, d\theta\, d\theta_0 \qquad m>0$$

$$B_k(c/s,\lambda) = \frac{1}{2\pi}\int_0^{\pi}\int_0^{\pi} f_c'(\theta_0)\, f_t'(\theta_0) \left[\frac{1/\pi}{\cos\theta-\cos\theta_0} + \frac{c}{s}\cdot\frac{R}{2}\right]\sin\theta\cos k\theta_0\, d\theta\, d\theta_0$$

$$T_k(c/s,\lambda) = \frac{c/s}{2\pi}\int_0^{\pi}\int_0^{\pi} f_t'(\theta_0)\, I\cos k\theta_0 \frac{\sin\theta}{2}\, d\theta\, d\theta_0$$

$S_{mk} = 1.0$ when $m = k$

$= 0.0$ when $m \neq k$

Mellor has indicated a method of evaluating the double Fourier integral and then tabulated the values of the function like g_{00}, g_{02}, g_{11}, g_{20}, g_{22}, h_{00}, h_{01}, h_{10}, B_0, B_1, T_0 & T_1

Mellor suggests several simplifications in equation (4.42). These are:-

(a) Diagonal terms g_{mk} are very nearly equal to 1 for $m > 2$, $k > 2$

(b) Non-diagonal terms g_{mk} may be neglected for $m > 2$, $k > 2$

(c) h_{mk} may be neglected for $m > 1$, $k > 1$.

(d) $$g_{10} = g_{21} = g_{12} = h_{11} = 0$$
$$g_{01} = 1 - g_{00}$$

for symmetrically cambered aerofoils

$\int_0^{\pi} f_c'(\theta_0) \cos k\theta_0 \, d\theta_0$ is zero for even values of k. The equation (4-42) then reduces to the following system of equations.

$$A_0 g_{00} + A_2 g_{20} = \sin\alpha + C_b (A_0 h_{00} + A_1 h_{10}) - \frac{t}{c}(C_b B_0 - T_0)$$

$$A_0 (1 - g_{00}) + A_1 g_{11} = \frac{C_b}{2\pi} + C_b A_0 h_{01} - \frac{t}{c}(C_b B_1 + T_1)$$

$$A_0 g_{02} + A_2 g_{22} = 0$$

These equations may be solved for A's and give

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$$A_0 = \frac{\sin\alpha}{P_0 + \frac{C_b h_{10}}{g_{11}} P_1} + \frac{C_b}{2\pi} \, \frac{\frac{C_b h_{10}}{\cancel{h_{10}}\, g_{11}}}{P_0 + \frac{C_b h_{10}}{g_{11}} P_1} + \frac{t}{c}\left[\frac{-C_b B_0 - T_0}{P_0 + \frac{C_b h_{10}}{g_{11}} P_1} - \frac{C_b B_1 + T_1}{P_0 + \frac{C_b h_{10}}{g_{11}} P_1} \cdot \frac{C_b h_{10}}{g_{11}} \right] \qquad \ldots (4\text{-}43)$$

where $P_0 = g_{00} - \frac{g_{20} \cdot g_{02}}{g_{22}} - C_b h_{00}$

and $P_1 = 1 - g_{00} - C_b h_{01}$

Mellor puts these in the form

$$A_0 = A_{0\alpha} \cdot \sin\alpha + A_{0c} \cdot \frac{C_b}{2\pi} + A_{0t} \cdot \frac{t}{c}$$

where the coefficients $A_{0\alpha}$, A_{0c}, A_{0t} have the values indicated above. Similarly

$$A_1 = A_{1\alpha} \cdot \sin\alpha + A_{1c} \cdot \frac{C_b}{2\pi} + A_{1t} \cdot \frac{t}{c}$$

where

$$A_{1\alpha} = -\frac{P_1}{g_{11}} A_{0\alpha} \quad ; \quad A_{1c} = \frac{1}{g_{11}} - \frac{P_1}{g_{11}} A_{0c}$$

$$A_{1t} = \frac{-T - C_b B_1}{g_{11}} - \frac{P_1}{g_{11}} A_{0t}$$

Thus A_{0c} .... A_{0t} etc. can be tabulated and the values of the coefficients A_0, A_1 etc. determined for particular situation.

c) <u>Blade Surface Velocity.</u>

$$\frac{V_0}{V_m} = \left(\cos\alpha + \frac{u_0}{V_m}\right) \pm \frac{\gamma/2}{V_m} \qquad \ldots (4\text{-}44)$$

Substitution of $\frac{u_0}{V_m}$ from (4.32) and for γ from (4.40) gives

$$\frac{V_o}{V_m} = \cos\alpha - \frac{c/s}{2}\int_0^1 \frac{\gamma}{V_m} I \, d(x/c)$$

$$+ \frac{1}{2\pi}\int_0^1 \frac{q}{V_m} \frac{d(x)}{x_0 - x}$$

$$+ \frac{c/s}{2}\int_0^1 \frac{q}{V_m} R \, d(x/c)$$

$$\pm A_0 \frac{1+\cos\theta}{\sin\theta} \pm 2\sum_{n=1}^{\infty} A_n \sin n\theta \qquad \ldots\ldots (4-45)$$

Substituting for $\frac{q}{V_m}$ from (4.39) and ~~su~~ simplifying, the results can be put in the following form

$$\frac{V_o}{V_m} = \cos\alpha \pm 2\pi A_0 \Gamma_{00} \pm 2\pi \Gamma_{01}$$

$$\pm C b \Gamma_{0\Sigma} + \frac{t}{c} Q_0 + \frac{c}{s}\cdot\frac{t}{c} Q_\Sigma$$

$$- 2\pi A_0 \frac{c}{s} \sin\lambda \, \Gamma_{\Sigma 0} - 2\pi A_1 \sin\lambda \, \Gamma_{\Sigma 1}$$

$$\ldots\ldots (4-46)$$

where

$$\Gamma_{00} = \frac{1+\cos\theta}{2\pi\sin\theta} \quad ; \quad \Gamma_{01} = \frac{\sin\theta}{2\pi}$$

$$\Gamma_{0\Sigma} = \sum_{n=2}^{\infty} \frac{2}{\pi}\int_0^{\pi} f_c'(\theta_0) \cos n\theta_0 \sin n\theta \, d\theta$$

$$\Gamma_{\Sigma 0} = \frac{1}{4\pi\sin\lambda}\int_0^{\pi} I (1+\cos\theta) \, d\theta$$

$$\Gamma_{\Sigma 1} = \frac{1}{2\pi\sin\lambda}\int_0^{\pi} I \sin^2\theta \, d\theta$$

$$Q_0 = \frac{1}{2\pi}\int_0^{\pi} \frac{f_t'}{x_0 - x} dx \; ; \; Q_\Sigma \cdot \frac{1}{2}\int_0^{\pi} R f_t' d(x/c)$$

Harmonics of higher degree ($n > 2$) do not appreciably affect the results. Single function $\Gamma_{0\Sigma}$ takes into account the effect of the higher harmonics. Mellor has tabulated the above functions and hence the velocity distribution may be computed readily.

The methods outlined above present formidable computational difficulties. Some of the functions depend on the blade parameters and thus have to be computed afresh for every blade profile. In order to estimate the heat transfer coefficient the velocity distribution outside the boundary layer is required. Therefore, the boundary layer characteristics have to be determined and new set of calculations performed in each case. The methods are presented here to show that the heat transfer coefficients may be predicted entirely from theoretical calculations. However, in the next Chapter for calculation purposes data for pressure distribution around a cascade of gas turbine blades published by Pope and Wilson[27] has been used to calculate the heat transfer coefficients. This also facilitates the comparison of the theoretical results with the experimental data given by the authors.

CHAPTER 5

HEAT TRANSFER CALCULATIONS RESULTS AND DISCUSSION

5.1 General.

Pope and Wilson have experimentally tested a cascade of gas turbine blades and published data regarding the distribution of the heat transfer coefficients around the blade surface. Instead of calculating the potential flow around the blade profile theoretically by the methods described in Cahpter 4. The measured pressure distribution of Pope and Wilson has been used to predict theoretically the distribution of heat transfer coefficient. Calculations have been made by, (i) Smith and Spalding (Section 3- 5) and (ii) Spalding (Section 3- 6) methods. Figure (11) shows the T_6 blade profile and the cascade arrangement. This blade profile being chosen as this is typical of the present day gas turbine practice.

Laminar flow can only support very small adverse pressure gradient without separation. The equilibrium of the fluid in the boundary layer is determined by three causes. It is retarted by friction at the solid boundary, pulled forward by the viscous action of the stream above and is retarted by the adverse pressure gradient. At a

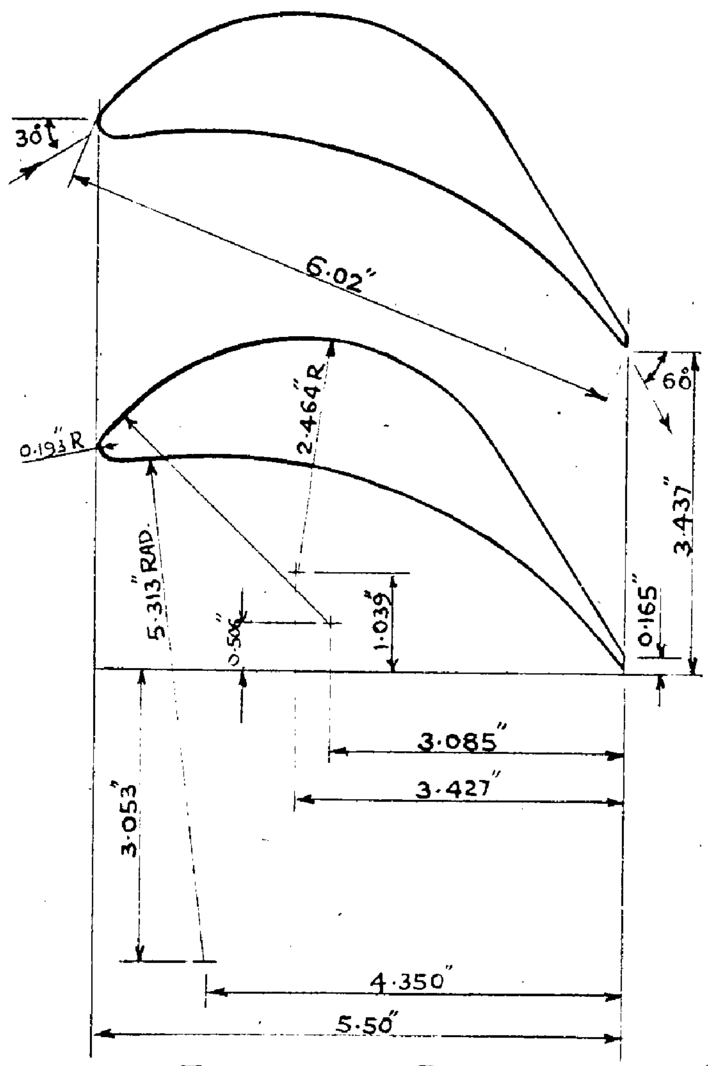


FIG. 11 BLADE PROFILE

point where the energy and the momentum of the entrained fluid are insufficient to overcome the adverse pressure gradient, back flow may actually set in. The forward stream, then leaves the surface and separation is said to have occured. The point of separation may be said to be at the point where $(\partial u/\partial y)_0 = 0$ The position of the point of separation may be determined from the known pressure distribution outside the boundary layer. Turbulent boundary layers on the other hand can support considerable adverse pressure gradients. The transition may take place due to instability resulting from the growth in the thickness of the laminar boundary layer. Several criteria are available to determine the point of transition from laminar to turbulent flows. When a definite pressure minimum occurs, the point of transition may be expected to be located there. The calculations have therefore been performed on the convex side upto $x/c = 0.6$ only. On the concave side, the laminar boundary layer is expected to exist only for a negligible distance. Thus, the prediction of the distribution of heat transfer coefficients, over the complete blade surface involves analysis of turbulent heat transfer besides the analysis of the phenemenon occuring in transition zone.

5.2 Calculations by Smith and Spalding Method.

Table (4) shows the computational scheme and the details of calculations. The numerical integrations were performed by Simpson's rule taking an interval of 0.025 from 0 to 0.1 and then an interval of 0.05. The correction term was taken from the graph in Smith and Spalding's paper reproduced in the figure (4). Figure (15) shows the results plotted. It may be observed from the tabulated values that the correction term is sufficiently small. Hence only one iteration has been done.

5.3 Calculations by Spalding Method.

Tables (7) and (8) detail the computational scheme and the calculations for this method. The first step is the evaluation of the momentum thickness of the velocity boundary layer . Table (7) shows that the correction term e_2 introduced to take into account non-linearity in equation (3.65) is negligibly small. Spalding in his paper has given only a representative table for the correction term and the ratio δ_4/δ_2 . Therefore, table (5 & 6) had to be improvised from Hartree's wedge solutions reproduced by Hunter Rouse.[28] Hunter Rouse has tabulated for the various values of m, the values of the parameters η_1, η_2, H, E etc defined as follows.

$$\eta_1 = \delta_1 \sqrt{\frac{U}{\nu x}}$$

$$\eta_2 = \delta_2 \sqrt{\frac{U}{\nu x}}$$

$$H = \delta_1/\delta_2$$

$$E = \tau_w . \delta_2 / \mu U$$

A relationship of the following form is required.

$$\frac{U}{\nu} \frac{d}{dx} (\delta_2)^2 = 0.4418 - 5.17 \left(\frac{\delta_2^2}{\nu} \frac{dU}{dx} \right) - e_2 \left(\frac{\delta_2^2}{\nu} dU/dx \right) \quad \cdots (5\text{-}1)$$

Writing the momentum equation of the boundary layer

$$U^2 \frac{d\delta_2}{dx} + (2\delta_2 + \delta_1) U \frac{dU}{dx} = \frac{\tau_w}{\rho}$$

$$\frac{d\delta_2}{dx} = \frac{\tau_w}{\rho U^2} - (2\delta_2 + \delta_1) \frac{1}{U} \frac{dU}{dx}$$

$$= \frac{\nu}{U} . \frac{E}{\delta_2} - (2 + H) \frac{\delta_2}{U} \frac{dU}{dx}$$

or

$$\frac{U}{\nu} \delta_2 \frac{d\delta_2}{dx} = E - m \eta_2^2 (2 + H) \quad \cdots (5\text{-}2)$$

Now the equation (5.1) can be integrated to yield

$$\frac{1}{\nu U^{4.17}} \frac{d}{dx} \left(U^{5.17} \delta_2^2 \right) = 0.4418 - e_2 (\quad)$$

or

$$\frac{1}{\nu U^{4.17}} \left[\delta_2^2 \cdot 5.17\, U^{4.17} \frac{dU}{dx} + U^{5.17}\, 2 \delta_2 \frac{d\delta_2}{dx} \right] = 0.4418 - e_2$$

or

$$5.17 \frac{\delta_2^2}{\nu} \frac{dU}{dx} + \frac{U}{\nu} 2 \delta_2 . \frac{d\delta_2}{dx} = 0.4418 - e_2 (\quad)$$

Substituting from (5.2)

$$5.17\, m \eta_2^2 + 2E - 2m\, \eta_2^2 (2 + H) = 0.4418 - e_2 (m \eta_2^2)$$

or

$$m \eta_2^2 (1.17 - 2H) + 2E = 0.4418 - e_2 (m \eta_2^2)$$

$$\therefore \; e_2 = 0.4418 - m\eta_2^2(1.17 - 2H) - 2E \qquad \ldots. (5-3)$$

and

$$\delta_2/\delta_4 = E \qquad \ldots. (5-4)$$

With these relationships tables (5) and (6) have been computed and δ_4 evaluated.

In the evaluation of Δ_4 the correction term F was obtained from Spalding's paper. The integerations were done as before by Simpson's rule employing the interval 0.025 from 0 to 0.1 and the interval 0.05 from 0.1 onwards. These results along with the results obtained for Smith and Spalding method and the experimental data of Pope and Wilson are shown in Fig.(15)

5.4 Remarks on the Results.

Results of calculations of heat transfer coefficients for a gas turbine blade, by i) Smith and Spalding Method ii) Spalding method discussed in Sections (3-5) and (3-6) respectively, have been plotted in figure 15. Fig.15 shows the distribution of the parameter $Nu/\sqrt{Re_2}$ around the blade surface. For the sake of comparison, experimental results of Pope and Wilson for the same blade profile for R = 3.02 x 10^5 have been superimposed. Fig. 16 shows the experimental curves obtained by Pope and Wilson at various Reynold Numbers.

The experimental curves of fig. 16 show that the coefficient $Nu/\sqrt{Re_2}$ is a function of Reynolds Number as in the plot of $Nu/\sqrt{Re_2}$ versus x/c a family of curves with Reynolds Number as parameter is obtained. The experimental curves however are very close together in the low Reynolds Number rang (1.84x10^5 to 5.17x10^5). The analytical methods on the other hand, show that $Nu/\sqrt{Re_2}$ is not dependent on Reynold Number and therefore admit the possibility of only a single curve. For higher Reynolds Numbers, the experiment curve deviates considerably from the curves for low Reynold Numbers. The theoretical curves are more or less in agreeme with the experimental curves for low Reynolds Numbers. Therefore, the theoretical results are likely to be in erro for high Reynolds Numbers.

At $\frac{x}{c} \simeq 0.16$ a fluctuation in the experimental values of heat transfer coefficient is indicated. The theoretical curves are, however, found to be more or less smooth. Because of the small thickness of the layer at this point, the surface roughness at this point may have predominant effect on the boundary layer characteristics. The fluctuation in the experimental values may be due to the surface roughness of the blades.

The experimental curves record a rapid fall in the value of the heat transfer coefficient at $\frac{x}{c} \simeq 0.6$. This rapid fall can only be due to the onset of separation. Therefore, the assumption that the point of separation is at $\frac{x}{c} \simeq 0.6$ is well founded. As pointed out earlier, the laminar boundary layer exists for a negligible distance on the concave side of the blade and therefore calculations have not been performed for the concave side.

Regarding the relative merits of the two theoretical methods, Smith and Spalding method is computationaly simpler of the two. The calculations have been performed only for the coincidental start of the thermal and hydro-dynamic boundary layers. The close agreement of the results of Smith and Spalding method with the experimental curve as is evident from fig. 15 may be fortuitous. The results given by Spalding method follow the variations in the experimental curves more closely. The underlying assumption in Smith and Spalding method is that the relationships between some parameters which hold good for wedge flows, hold also for arbitrary flows. This may result in considerable inaccuracies. Spalding method is an attempt to improve Lighthillis method. The implicit assumption in Smith and Spalding method which emperically relates velocity and thermal boundary layer thicknesses is done away with and instead an ordinary differential equation is set up for each boundary layer thickness. This method aims at improving the accuracy of Lighthill's method. However, the crucial assumption is that the correction term is solely a function of $\frac{\Delta_4 \delta_4}{\nu} \frac{dU}{dx}$. This method becomes difficult to apply at the separation point where $\delta_4 \rightarrow \infty$. The difference in the results may be partly due to deficiencies in theoretical analysis and partly due to errors in experimental results subject as they are, to a number of experimental errors. It will be too much to base the conclusions entirely on one set of the results. The important question, however, is that which method gives results approaching those obtainable under actual operating conditions. Pope and Wilson estimate their results to be within plus or minus 10%. The figure has been questioned and deviations as high as 40% reported.

Only comprehensive experimentation can point out the deficiencies in analytical methods and help in evaluating their relative reliability.

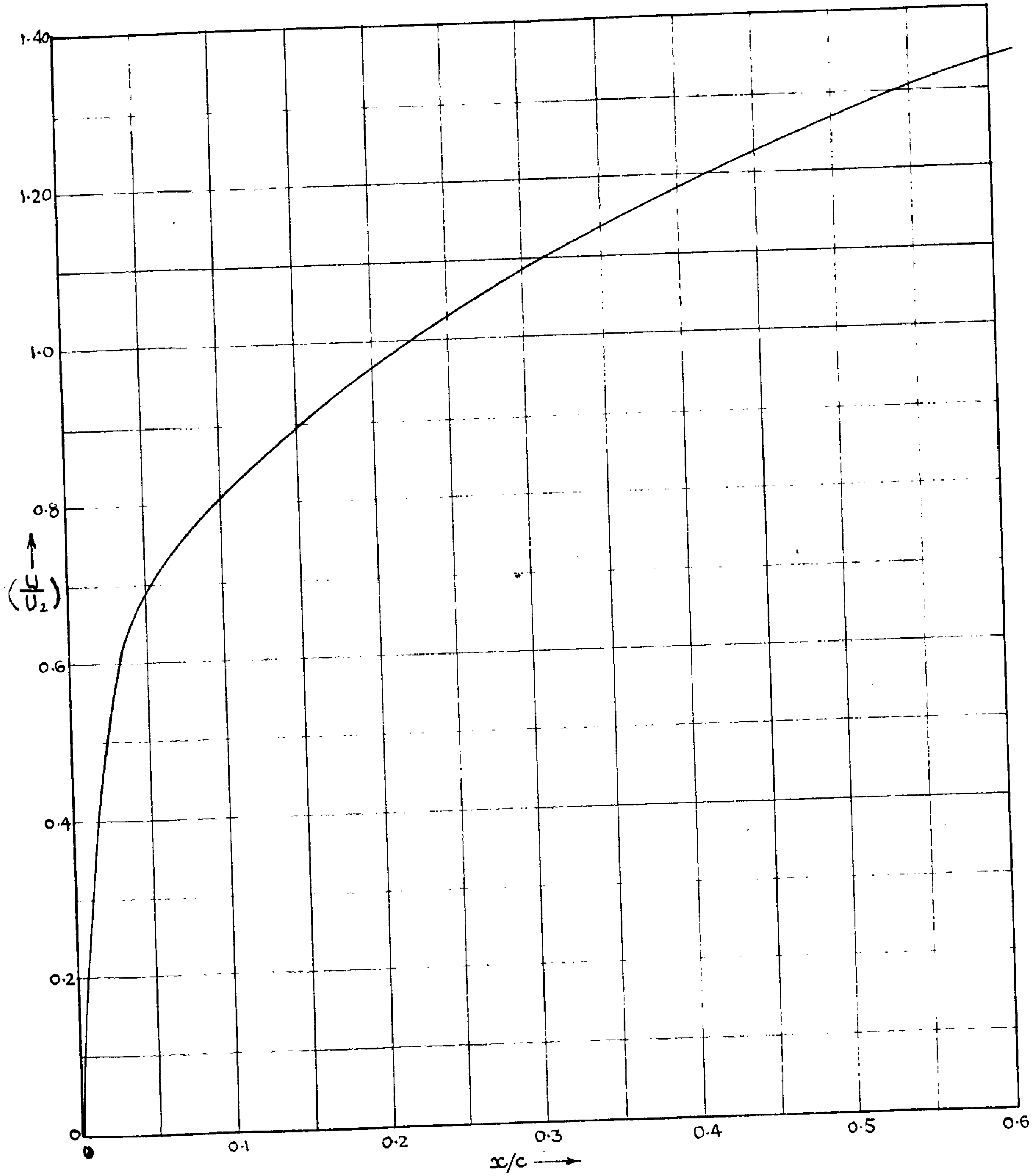
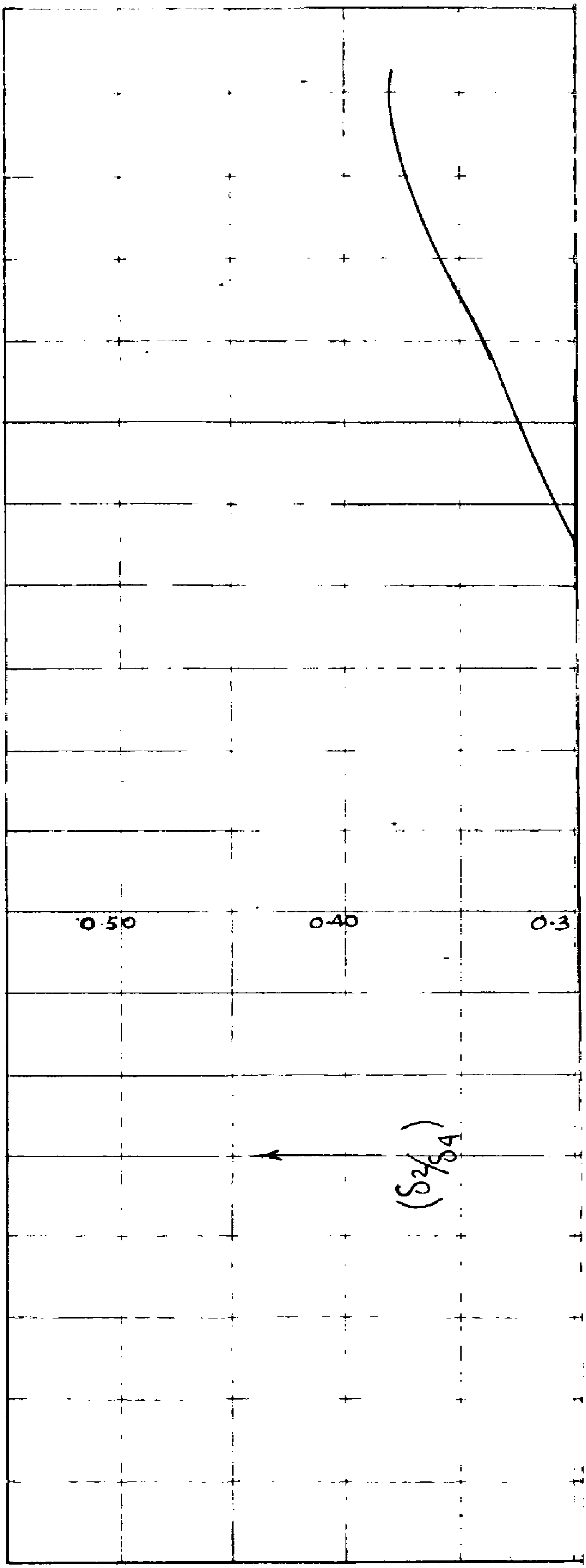


FIG. 12. VELOCITY DISTRIBUTION (CONVEX SIDE)
AFTER POPE & WILSON



FIG

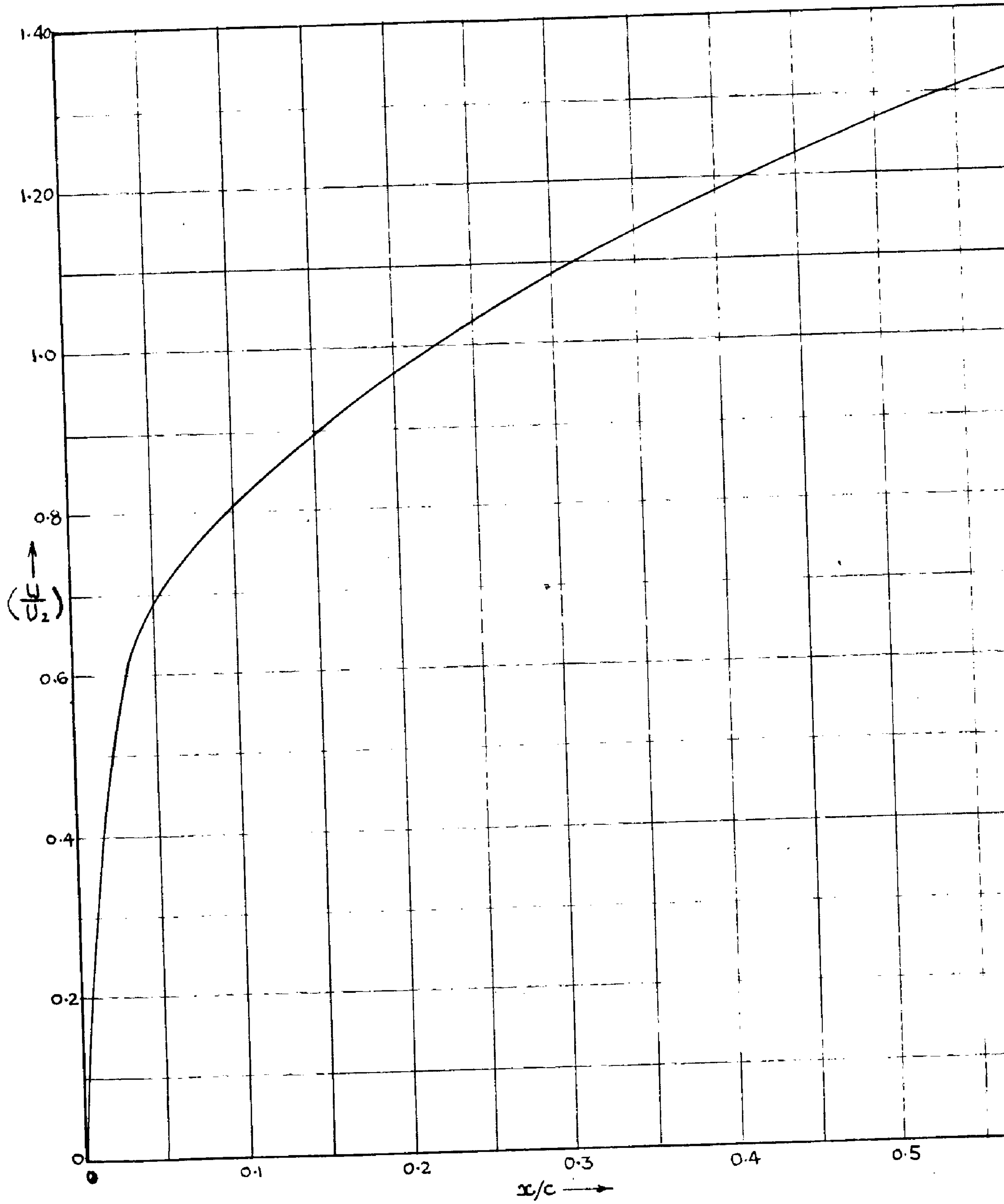


FIG. 12. VELOCITY DISTRIBUTION (CONVEX SIDE)
AFTER POPE & WILSON

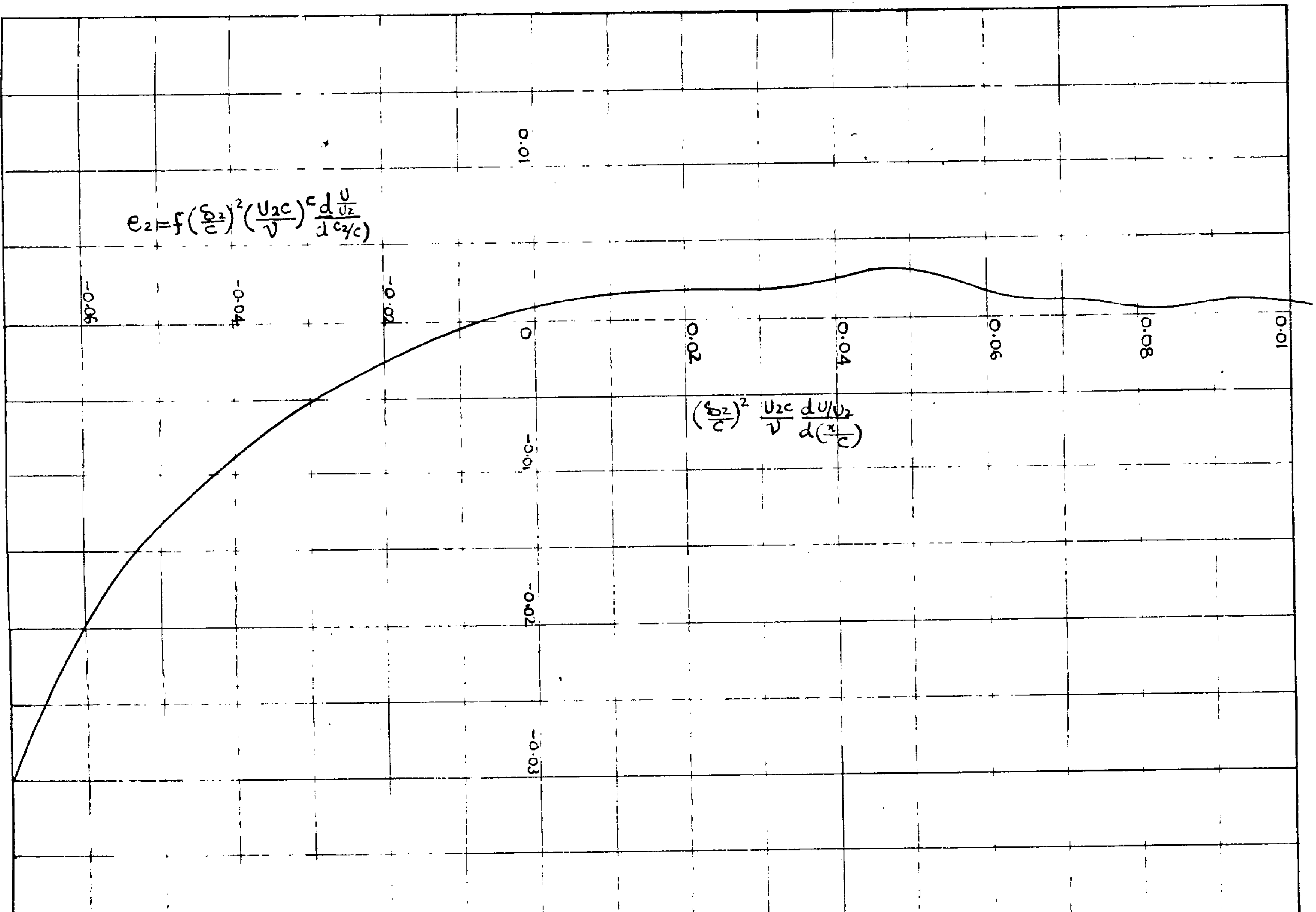
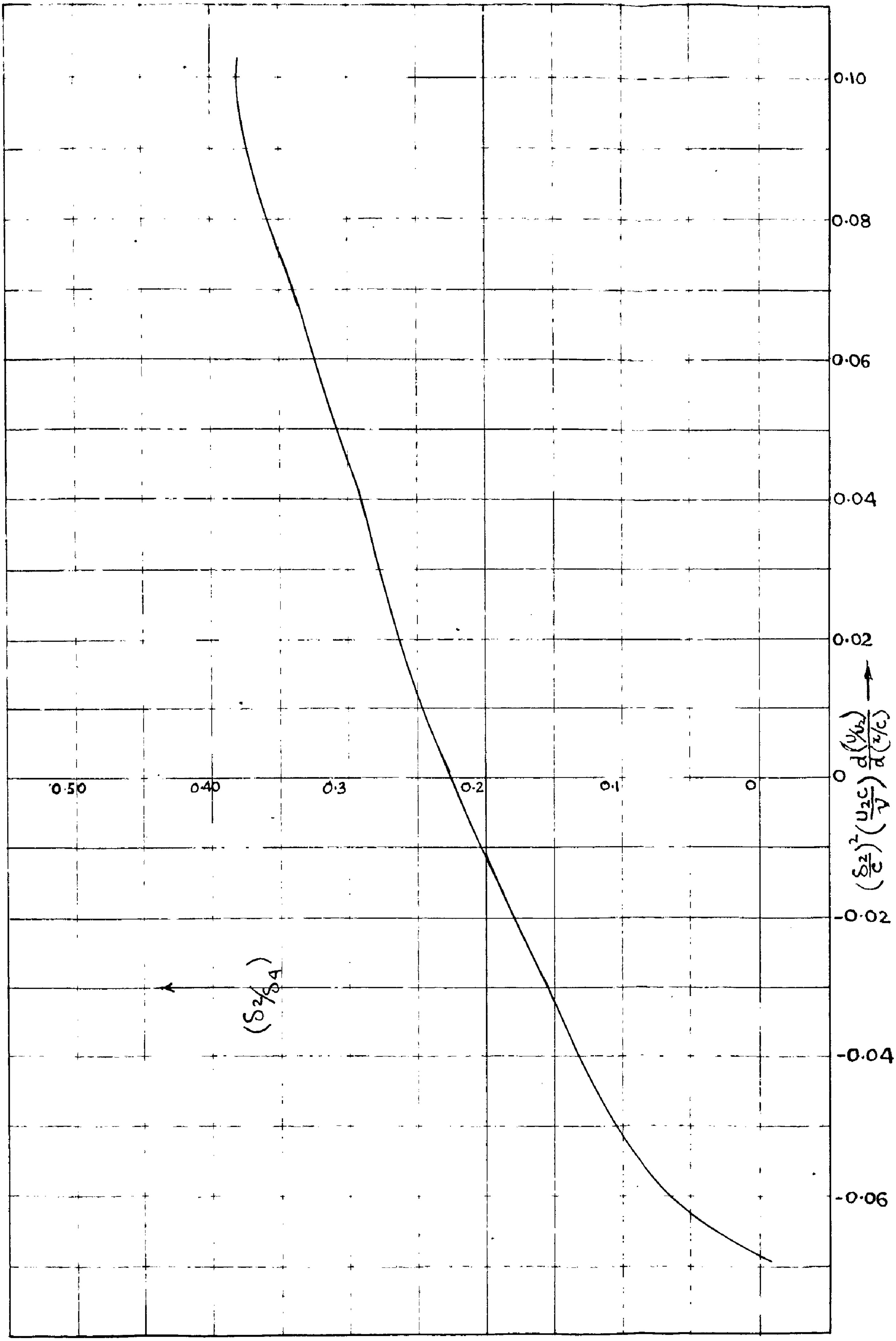

$e_2 = f\left(\frac{\delta_2}{c}\right)^2\left(\frac{U_2c}{\nu}\right)^c\frac{d\frac{U}{U_2}}{d\,c_2/c)}$
0.01
-0.06
-0.04
-0.02
0
0.02
0.04
0.06
0.08
0.01
$\left(\frac{\delta_2}{c}\right)^2\ \frac{U_2c}{\nu}\ \frac{d\,U/U_2}{d\left(\frac{x}{c}\right)}$
-0.01
-0.02
-0.03

FIG. 13



0.10
0.08
0.06
0.04
0.02
0
-0.02
-0.04
-0.06
0.50
0.40
0.3
0.2
0.1
0
$\left(\frac{\delta_2}{c}\right)^2\left(\frac{U_2 c}{\nu}\right)\frac{d(U/U_2)}{d(x/c)}$
(δ_2/δ_4)

FIG. 14

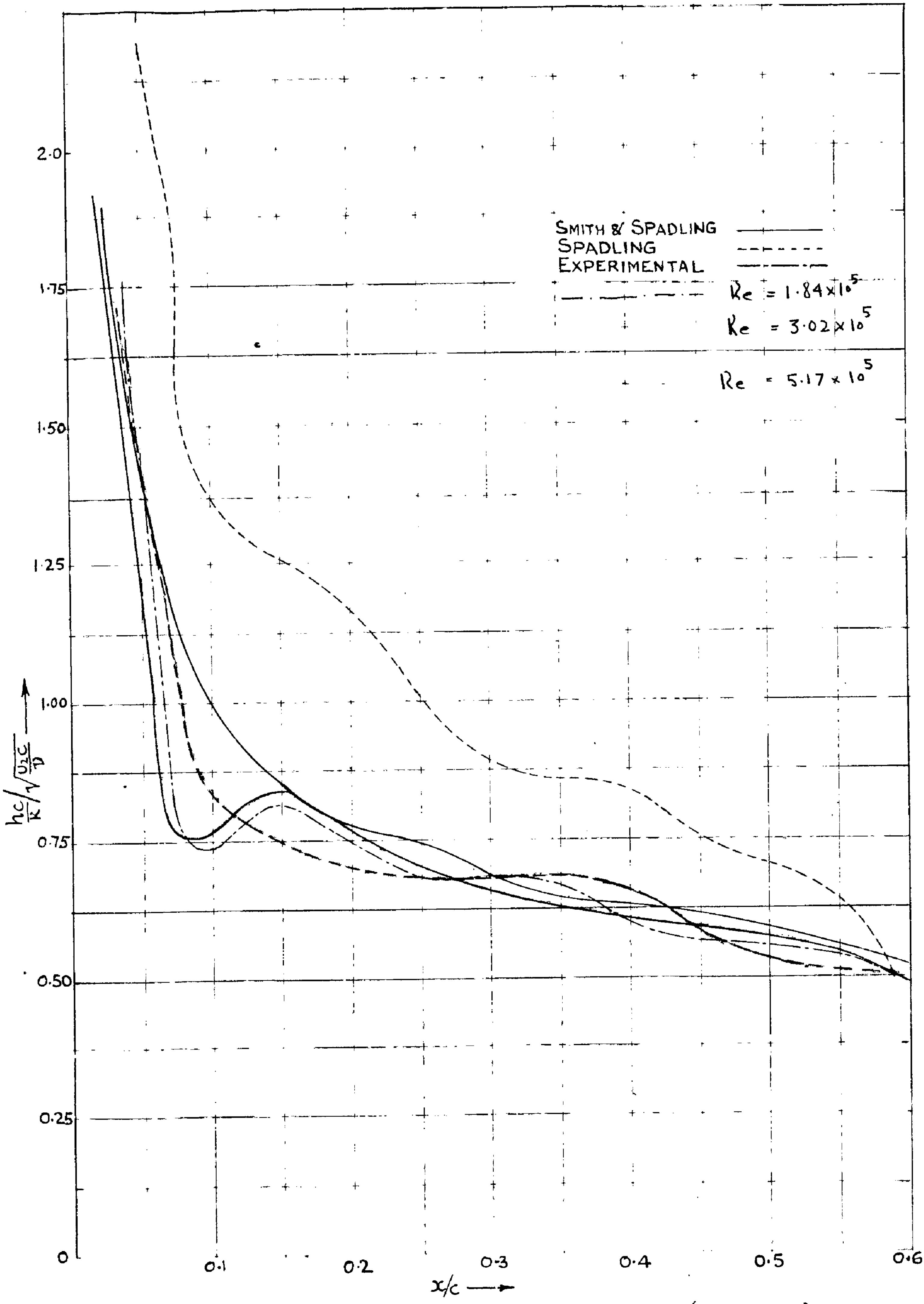


FIG. 15. DISTRIBUTION OF HEAT TRANSFER COEFFICIENT (CONVEX SIDE)

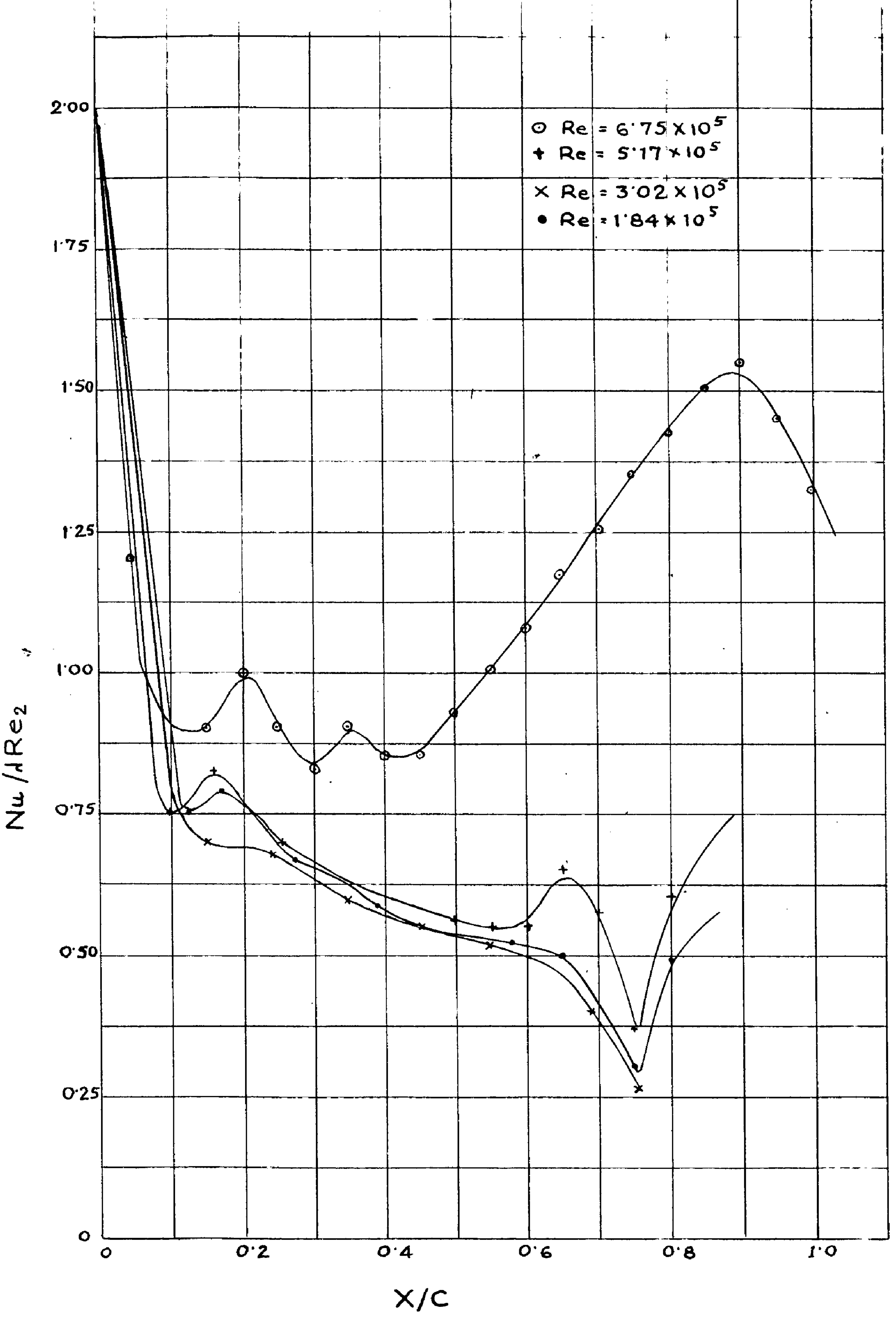


FIG. 16 - DISTRIBUTION OF HEAT TRANSFER CO-EFFICIENT

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A P P E N D I C E S

Table I

Velocity Distribution (Convex Side)

$\frac{x}{c}$	$\frac{p-p_2}{\frac{1}{2}\rho U_2^2}$	$\left(\frac{U}{U_2}\right)^2$	$\left(\frac{U}{U_2}\right)$
0	1.00	0	0
0.025	0.70	0.3	0.547,7
0.050	0.50	0.50	0.707,1
0.075	0.40	0.60	0.774,6
0.10	0.375	0.625	0.790,5
0.15	0.25	0.75	0.866,0
0.20	0.075	0.925	0.961,7
0.25	-0.05	1.05	1.024,7
0.30	-0.15	1.15	1.072,4
0.35	-0.25	1.25	1.118
0.40	-0.4	1.40	1.183,2
0.45	-0.50	1.50	1.229,7
0.50	-0.60	1.60	1.2649
0.55	-0.67	1.67	1.292,3
0.60	-0.7	1.70	1.303,8

Table 2.

$\frac{x}{C}$	$\frac{U}{U_2}$	$\ln \frac{U}{U_2}$	$1.87 \ln \frac{U}{U_2}$	$2.87 \ln \frac{U}{U_2}$	$\left(\frac{U}{U_2}\right)^{1.87}$	$\left(\frac{U}{U_2}\right)^{2.87}$
1	2	3	4	5	6	7
0						
0.025	0.547,1	-0.261,457	-488,924	-.750,381	324,4	.177,67
0.050	0.707,1	-.150,519	-.281,470	-.431,989	.523,02	0.369,84
0.075	0.774,6	-110,923	-.207,426	-.318,349	.620,26	.480,45
0.10	0.790,5	-.102,098	-.190,923	-.293,021	.644,28	0.509,3
0.15	0.866,0	-.062,482	-116,841	-.179,323	.764,12	.661,72
0.20	0.961,7	-.016,960	-031,716	-.048,675	.929,57	0.893,97
0.25	1.024,7	.014,809	.027,692	.042,521	1.065,8	1.102,85
0.30	1.072,4	.030,393	.056,835	.087,228	1.139,8	1.222,4
0.35	1.118	.048,442	.090,586	.132,028	1.231,9	1.377,3
0.40	1.1832	.073,059	.136,620	.209,679	1.369,7	1.620,6
0.45	1.224,7	.088,026	.164,608	.252,634	1.460,8	1.789,1
0.50	1.264,9	.102,057	.190,846	.292,803	1.551,9	1.962,9
0.55	1.292,3	.111,363	.208,248	.319,611	1.615,3	2.087,4
0.60	1.3038	.115,212	.215,446	.330,658	1.642,3	2.141,2

TABLE 3

(Convox Side)

$\frac{x}{c}$	$\frac{U}{U_2}$	$\ln \frac{U}{U_2}$	$4.17 \ln \frac{U}{U_2}$	$5.17 \ln \frac{U}{U_2}$	$\left(\frac{U}{U_2}\right)^{4.17}$	$\left(\frac{U}{U_2}\right)^{5.17}$
0.025	0.5477	-.261,457	-1.090,275	-1.351,732	.081,242	.044,49
0.050	0.707,1	-.150,519	- .627,664	- .778,83	.235,69	.166,665
0.075	0.774,6	-.110,923	- .462,549	- .573,472	.344,71	.267,01
0.100	0.790,5	-.102,098	- .425,748	- .527,846	.375,19	.296,59
0.150	0.856,0	-.062,482	- .260,55	-.323,032	.548,84	.475,3
0.2[illegible]	.951,7	-.016,960	- .070,723	- .087,683	.849,72	.817,18
0.25	1.024,7	+.014,809	+ .061,753	.076,562	1.152,8	1.192,8
0.30	1.072,4	.030,393	0.126,738	.157,131	1.338,9	1.435,9
0.35	1.118 [illegible]	.048,442	0.202,003	.250,445	1.592,8	1.780,1
0.40	1.183,87	.073,059	0.304,656	.377,715	2.016,8	2.386,2
0.45	1.224,7	.088,026	.367,068	.455,024	2.328,5	2.851,6
0.50	1.264,9	0.102,057	.425,577	.527,634	2.664,3	3.370
0.55	1.292,3	.111,363	.464,383,	.575,746	2.913,3	3,756,2
0.60	1.303,8	.115,212	.480,434	.595,646	.3623	3.241,4

TABLE 4

CALCULATIONS BY SMITH AND SPALDING METHOD
CONVEX SIDE.

x/c	0.025	0.05	0.075	0.1	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50	0.55	0.60
(U/U_1)	0,547,7	0.707,1	0.774,6	0.790,5	0.866,0	0.961,7	1.024,7	1.072,4	1.118	1.1832	1.224,7	1.264,9	1.292,3	1.303,8
$I_1 = \int_0^{x/c} (U/U_1)^{1.87} d(x/c)$	.004,1	.015,2	.029,4	0.045,6	.080,8	0.122,7	0.173,2	.228,3	.287,5	.352,2	.423,7	0.498,3	0.578,4	0.659,8
$\left(\frac{\Delta_4}{c}\right)^2 \frac{U_1 c}{\nu}$	.266,6	.479,2	0.713,8	1.045	1.425,9	1.603,7	1.834,8	2.175	2.438,4	2.538,6	2.766,2	2.965,2	3.236,7	3.599,6
$\left(\frac{\Delta_4}{c}\right) \sqrt{\frac{U_1 c}{\nu}}$	0.516,3	0.692,2	0.844,8	1.022,3	1.194	1.266,4	1.354,5	1.474,1	1.561,5	1.593,3	1.663,2	1.721,9	1.799	1.897
$\frac{h_c}{k} / \sqrt{\frac{U_1 c}{\nu}}$ (1st Approximation)	1.936,8	1.444,6	1.183,6	0.880,3	.837,4	0.789,6	0.738,2	0.678,3	0.640,4	0.627,6	.601,2	0.580,7	0.555,8	0.527,1
$\left(\frac{\Delta_4}{c}\right)^2 \frac{U_1 c}{\nu} \frac{d(U/U_1)}{d(x/c)}$	5.840	3.055	1.927,2	0.664,7	2.153	3.059,5	2.312	2.073,1	2.223,8	3.310,4	2.296	2.384	1.773,7	.8279
E_4	+0.933	-0.4	-.53	-.533	-.666	-.52	-.4	-.67	-.64	-.33	-.65	-.54	-.6	-.5
$I_2 = \int_0^{x/c} (U/U_1)^{1.87} \cdot E_4 \, d(x/c)$	.003,8	.007,7	-.003,3	-0.007	-.028,5	-.054,9	-.085,4	-.104,1	-.156,5	-1.76,9	-.215,7	-.261,8	-.303,5	-.354
$I_2 / (U/U_1)^{2.87}$	.021,3	.020,7	-.006,9	-.014,2	-.043,1	-.061,4	-.076,5	-.085,2	-.113,7	-.109,2	-.120,5	-1.33,3	-1.46,4	-.165,3
$\left(\frac{\Delta_4}{c}\right)^2 \frac{U_1 c}{\nu}$	.287,8	.499,8	.706,8	1.031	1.383	1.542,2	1.758	2.087,9	2.324,7	2.429,4	2.645,716	2.832	3.091,3	3.434,
$\frac{\Delta_4}{c} \sqrt{\frac{U_1 c}{\nu}}$	0.536,5	.707	.840,7	1.015,4	1.176	1.242	1.326	1.444,9	1.524,7	1.558,6	1.626	1.682,8	1.758	1.853,1
$\frac{h_c}{k} / \sqrt{\frac{U_1 c}{\nu}}$ (IInd Approx)	1.864	1.414	1.189	.984,846	.850,387	.805,234	0.754	0.692	0.655	.641	.615	.594	.569	.539

TABLE 5.

' Wedge Flows '

$m\eta_2^2 = \frac{\delta_2^2}{\nu}\frac{dU}{dx}$	$E = \delta_2/\delta_4$
-0.068,007	0.000,0
-0.063,245	0.049,6
-0.057,900	0.072,9
-0.048,753	0.105,1
-.040,570	0.128,9
-0.026,515	0.164,3
0	0.220,5
.001,896	0.255,6
.033,354	0.280,5
.049,602	0.298,7
.053,731	0.313,0
.061,397	0.325,1
.067,755	0.334,7
.077,706	0.349,0
.085,439	0.363,0
.091,513	0.369,0
.100,108	0.380,5

TABLE 6

(Wedge Flows)

Correction Term

m	η_2^2	$m\eta_2^2$	$(1.17-2H)$	$m\eta_2^2(1.17-2H)$	$e(\ ) = 0.4410 - m\eta_2^2(1.17-2H) - 2E$
-0.0904,1	.752,209	-.068,007	-6.894,6	.468,981	-.027,8
-0.086,76	.728,974	-.063,245	-5.787,8	.366,054	-0.023,454
-0.082,51	.701,741	-.057,900	-5,427,2	.314,238	-.018,238
-0.079,07	.658,208	-.048,753	-5,016,4	.244,566	-.012,966
-0.065,42	.620,156	-.040,570	-4.761,6	.193,181	-.009,181
-0.047,62	.556,814	-.026,515	-4.436,6	.117,638	-.004,438
0.000,000	.441,028	0	-4.012,6	0	+.000,8
0.005,263	.360,240	.001.896	-3.793	-.007,191	-.062,21
0.111,11	.300,194	.033,354	-3.648,6		+.002,497
0.176,5	.252,707	.044,602	-3.558,2	-.158,705	+.003,105
0.250,0	.214,925	.053,731	-3.483,6	-.187,178	+.002,978
0.333,3	.184,212	.061,397	-3.420	-.209,980	+.001,580
0.428,6	.158,085	.067,755	-3.376,2	-.228,755	+.001,155
0.666,7	.116,554	.077,706	-3.314,8	-.257,581	+.001,381
1.000	.085,439	.085,439	-3.263,8	-.278,885	+.000,055
1.5000	.061,009	.091,513	-3.225,4	-.295,167	-.001,0
4,000	.025,027	.100,108	-3.179,0	-.318,243	-.000,99
∞	0		-3.150	0	-.333,2

TABLE 7.

EVALUATION OF SHEAR THICKNESS
SPALDING METHOD

x/c	0.025	0.05	0.075	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50	0.55	0.60
(U/U_2)	0.547,7	0.707,1	0.774,6	0.790,5	0.866,0	0.961,7	1.024,7	1.072,4	1.118	1.183,2	1.224,7	1.264,9	1.292,3	1.303,8
$= \int_0^{x/c} (U/U_2)^{4.17} d(x/c)$	.001,015	004,672	.012,420	.021,253	.044,353	.078,257	.129,361	.191,587	.264,381	.353,702	0.464,189	.586,953	.729,172	.875,961
$)^2 \frac{U_2 c}{\nu} = \frac{0.4418 I_1}{(U/U_2)^{5.17}}$	.010,079	.012,385	.020,550	.031,558	.041,227	.042,308	.047,913	.058,947	.065,616	.065,487	.072,691	.076,948	.085,764	.098,188
$)^2 \frac{U_2 c}{\nu} \frac{d(U/U_2)}{d(x/c)}$	.220,810	.078,966	.055,485	.020,070	.062,252	.080,977	.060,370	.056,235	.059,841	.085,395	.060,333	.061,866	.046,998	.022,583
e_2	0	+.001,3	+.002,5	0.002	+.0015	.000.5	.001,7	.002,2	.001,8	.000,5	.001,6	.001,7	.003	.002
$= \int_0^{x/c} (U/U_2)^{4.17} e_2 d(x/c)$	0	+.000,004	.000,017	.000,04	.000,08	.000,114	.000,154	.000,300	.000,431	.000,557	.000,608	.000,897	.001,117	.001,486
$I_2/(\frac{U}{U_2})^{5.17}$	0	.000,024	.000,063	.000,135	.000,168	.000,139	.000,129	.000,209	.000,242	.000,232	.000,213	.000,266	.000,297	.000,377
$)^2 \frac{U_2 c}{\nu}$ (IInd Approx)	.010,079	.012,361	.020,487	.031,423	.041,059	.042,169	.047,784	.058,738	.065,374	.065,255	.072,478	.076,682	.085,467	.097,811
$(\frac{\delta_2}{c})^2 \frac{U_2 c}{\nu} \frac{d(U/U_2)}{d(x/c)}$	.220,810	.078,813	.055,315	.019,985	.061,999	.080,711	.060,207	.056,036	.059,621	.085,092	.060,156	.061,652	.046,836	.022,496
$E = \delta_2/\delta_4$	.3875	.355	.316	.263	.33	.358	.326	0.32	0.324	.367	.327	.328	0.3	0.184
$\frac{\delta_4}{\delta_2} = 1/E$	2.584	2.817	3.164	3.802	3.030	2 793	3.067	3.125	3.086	2.724	3.058	3.048	3.333	5.434
$(\delta_4/\delta_2)^2$	6.677	7.935	10.010	14,455	9.181	7. 01	9.408	9.765	9,523	7.420	9.351	9.290	11.101	29.578
$(\frac{\delta_4}{c})^2 \frac{U_2 c}{\nu}$	.067,297	.098,084	.205,074	.454,219	.376,962	.3[illegible],960	.449,456	.573,576	.622,556	.484,192	.677,741	.712,375	.948,769	2.888,163
$\ln[(\frac{\delta_4}{c})^2 \frac{U_2 c}{\nu}]$	-1.172,004	-1.008,403	-.688,089	-.342,771	-.423,702	-.[illegible]2,856	-.347,312	-.241,421	-.205,821	.314,982	-.168,936	-.147,291	-.022,839	.460,621
$0.25 \ln[(\frac{\delta_4}{c})^2 \frac{U_2 c}{\nu}]$	-.293,001	-.252,101	-.172,022	-.085,692	-.105,925	-.[illegible],714	-.086,828	-.060,355	-.051,455	-.078,745	-.042,234	-.036,823	-.005,7098	.115,155
$0.75 \ln[(\frac{\delta_4}{c})^2 \frac{U_2 c}{\nu}]$	-.879,003	-.756,303	-.516,066	-.257,076	-.317,777	-.3[illegible]142	-.260,484	-.180,065	-.154,365	-.236,235	-.126,702	-.110,469	-.015,129	.345,465
$[(\frac{\delta_4}{c})^2 \frac{U_2 c}{\nu}]^{0.25}$	.509,33	.569,63	.672,94	.820,93	.783,57	.75[illegible]3	-818,79	.870,25	.888,27	.834,17	.907,33	.918,71	.986,94	1.303,6
$[(\frac{\delta_4}{c})^2 \frac{U_2 c}{\nu}]^{0.75}$	.132,13	.175,265	.304,74	.553,26	.481,09	.43[illegible]7	.548,93	.660,6	.700,86	.580,45	.746,96	.775,05	.965,76	2.215,5

TABLE 8

Evaluation of SPALDING METHOD $\frac{h_c}{k} / \sqrt{\frac{U_2 c}{\nu}}$

x/c	0.025	0.050	0.075	0.100	0.150	0.20	0.25	0.30	0.35	0.40	0.45	0.50	0.55	0.60
$\frac{1}{\left[\left(\frac{\delta}{c}\right)^2 \frac{U_2 c}{\nu}\right]^{1/4}}$	1.963,353	1.786,9	1.486,0	1.218,130	1.276,210	1.320,480	1,221,284	1.149,100	1.125,656	1.198,797	1.102,135	1.088,483	1.013,233	.767,106
$I_3 = \int_0^{x/c} \frac{(U/U_2)^{1/2} d(x/c)}{\left[\left(\frac{\delta}{c}\right)^2 \frac{U_2 c}{\nu}\right]^{1/4}}$	0.018,178	.060,997	.091,283	.123,163	.179,929	.241,970	.306,753	.336,204	.426,627	.487,114	.553,725	.610,556	.672,853	.723,546
$\left[\left(\frac{\delta}{c}\right)^2 \frac{U_2 c}{\nu}\right]^{3/4} \times 6.41\, I_3$	.015,396	.068,527	.178,310	.436,784	.554,862	.673,719	1.079,353	1.550,671	1.916,627	1.812,397	2.651,242	3.033,285	4.165,310	10.275,333
$\left[\left(\frac{\Delta}{c}\right)^2 \frac{U_2 c}{\nu}\right]^{3/2}$	.010,777	.047,969	.124,817	.306,749	.388,403	.471,603	.755,547	1.085,469	1.341,639	1.268,678	1.855,869	2.123,299	2.915,717	7.192,733
$\ln\left[\left(\frac{\Delta}{c}\right)^2 \frac{U_2 c}{\nu}\right]^{3/2}$	-1.967,502	-1.319,039	-.903,725	-.514,635	-.410,718	-.326,424	-.121,739	.035,616	.127,634	.103,351	.268,547	.327,011	.464,745	.856,894
$\ln\left[\left(\frac{\Delta}{c}\right)^2 \frac{U_2 c}{\nu}\right]$	-1.311,667	-.879,359	-.602,483	-.343,09	-.273,812	-.217,616	-.081,159	.023,744	.085,089	.068,901	.179,031	.218,073	.309,830	.571,262
$\left[\left(\frac{\Delta}{c}\right)^2 \frac{U_2 c}{\nu}\right]$	.048,79	.132,02	.249,75	.453,85	.532,34	.605,88	.829,55	1.056,2	1.216,4	1.171,9	1.510,2	1.652,2	2.041	3.726,1
$\left[\left(\frac{\Delta}{c}\right)^2 \frac{U_2 c}{\nu}\right]\left[\left(\frac{\delta}{c}\right)^2 \frac{U_2 c}{\nu}\right]$	.003,283	.012,949	.051,217	.206,147	.200,671	.199,310	.372,846	.605,811	.757,277	.567,424	1.023,524	1.175,986	1.936,437	10.761,584
$\left\{\left(\frac{\delta}{c}\right)^2 \frac{U_2 c}{\nu} \cdot \left(\frac{\Delta}{c}\right)^2 \frac{U_2 c}{\nu}\right\}^{1/2}$	.057,297	.113,798	.226,312	.454,04	.447,964	.446,418	.610,61	.778,337	.870,215	.753,276	1.012,682	1.084,428	1.391,559	3,280,268
$\left\{\left(\frac{\delta}{c}\right)^2 \frac{U_2 c}{\nu} \cdot \left(\frac{\Delta}{c}\right)^2 \frac{U_2 c}{\nu}\right\}^{1/2} \frac{d\, U/U_2}{d(x/c)}$	1.255,262	.725,576	.611,042	.288,769	.676,425	.854,444	.769,358	.742,533	.793,636	.982,272	.840,526	.871,880	.762,574	.754,461
$F(\ )$	2.84	1.49	1.29	0.59	1.39	1.74	1.65	1.5	1.70	2.19	1.72	1.84	1.63	1.56
$\frac{(U/U_2)^{1/2} F}{\left[(\delta/c)^2 \frac{U_2 c}{\nu}\right]^{1/4}}$	4.130,107	2.238,88	1.687,194	.638,993	1.650,803	2.253,113	2.049,778	1.785,102	2.023,265	2.855,084	2.097,940	2.252,559	1.877,504	1.366,375
$I_4 = \int_0^{x/c} \frac{(U/U_2)^{1/2} F\, d(x/c)}{\left[\left(\frac{\delta}{c}\right)^2 \frac{U_2 c}{\nu}\right]^{1/4}}$	.051,626	.156,327	.174,733	.236,549	.293,794	.394,804	.505,677	.598,757	.692,561	.810,976	.951,586	1.035,966	1.168,014	1.221,448
$\left[\left(\frac{\delta}{c}\right)^2 \frac{U_2 c}{\nu}\right]^{3/4} I_4 \cdot P$	.004,775	.019,179	.037,273	.091,611	.098,938	.120,043	.194,306	.276,877	.319,35	.329,511	.497,557	.562,047	.789,614	1.894,282
$\left[\left(\frac{\Delta}{c}\right)^2 \frac{U_2 c}{\nu}\right]^{3/2}$	.015,552	.067,148	.162,090	.397,360	.487,341	.591,646	.949,853	1.362,346	1.681,410	1.598,189	2.358,426	2.685,346	3.705,391	9.087,015
$\frac{1}{3}\ln\left[\left(\frac{\Delta}{c}\right)^2 \frac{U_2 c}{\nu}\right]^{3/2}$	-.602,738	-.390,989	-.263,414	-.133,605	-.104,056	-.075,980	-.007,448	.044,762	.075,223	.067,876	.123,900	.142,967	.189,611	.319,474
$\left(\frac{\Delta}{c}\right)\sqrt{\frac{U_2 c}{\nu}}$	.249,61	.405,45	.545,24	.735,18	.786,94	.839,5	.983	1.102,55	1.189,1	1.169,2	1.330,15	1.389,85	1.547,43	2.086,8
	4.008	2.460	1.834	1.360	1.270	1.191	1.017	.902	.841	.855	.751	.719	.646	.479

APPENDIX 2.

Derivation of the equations (4.32) & (4.35)

Let the source sink distribution per unit length be q and the vorticity distribution γ. Refering to fig. 9, the volocity induced by a source segment $q\,dx$ placed on x on the nth blade at a point x_0. On the 0th blade is

$$dV = q\,dx/2\pi r$$

$$du = \frac{q\,dx}{2\pi r}\cos\theta$$

$$dv = -\frac{q\,dx}{2\pi r}\sin\theta$$

From the geometry of the Cascade

$$\cos\theta = \frac{(x_0 - x) + ns\sin\lambda}{r}$$

and

$$\sin\theta = \frac{sn\cos\lambda}{r}$$

$$\therefore\quad du = \frac{q\,dx}{2\pi}\,\frac{(x_0 - x) - ns\sin\lambda}{r^2}$$

$$dv = -\frac{q\,dx}{2\pi}\,\frac{ns\cos\lambda}{r^2}$$

Similarly because of vorticity $\gamma\,dx$ at x on the nth blade, the velocity induced at x_0 on the oth blade is

$$du = -\frac{\gamma\,dx}{2\pi}\,\frac{ns\cos\lambda}{r^2}$$

$$dv = -\frac{\gamma\,dx}{2\pi}\,\frac{(x_0 - x) - ns\sin\lambda}{r^2}$$

The contribution of the source sink distribution on zeroeth blade to u, velocity at x_0 is

$$q\,dx / 2\pi (x_0 - x)$$

The vorticity distribution on 0th blade does not contribute to u, velocity. Therefore summing up from ($-\infty$ to -1) and from ($+1$ to $+\infty$) and including the contribution of zeroeth blade, u component becomes

$$du = \left[\sum_{-\infty}^{-1} + \sum_{+1}^{+\infty} \right] \left[-\frac{\gamma\,dx}{2\pi}\, ns \frac{\cos\lambda}{r^2} + \frac{q\,dx}{2\pi} \frac{(x_0 - x) - ns\sin\lambda}{r^2} \right] + \frac{q\,dx}{2\pi (x_0 - x)}$$

$$= \frac{q\,dx}{2\pi (x_0 - x)} + \frac{1}{s} \cdot \frac{q\,dx}{2\pi} \left[\sum_{-\infty}^{-1} + \sum_{1}^{\infty} \right] \times \frac{\frac{x_0 - x}{c} \cdot \frac{c}{s} - n\sin\lambda}{\left(\frac{x_0 - x}{c} \cdot \frac{c}{s} - n\sin\lambda \right)^2 + (n\cos\lambda)^2}$$

$$- \frac{1}{s} \cdot \frac{\gamma\,dx}{2\pi} \left[\sum_{-\infty}^{-1} + \sum_{1}^{\infty} \right] \frac{n\cos\lambda}{\left(\frac{x_0 - x}{c} \cdot \frac{c}{s} - n\sin\lambda \right)^2 + (n\cos\lambda)^2}$$

Integrating from $x/c = 0$ to $x/c = 1$

$$\frac{u}{V_m} = \frac{1}{2\pi} \int_0^1 \frac{q/V_m}{(x_0/c - x/c)}\, d(x/c) + \frac{c/s}{2} \int_0^1 \frac{q}{V_m} R(\,)\, d(x/c)$$

$$- \frac{c/s}{2} \int_0^1 \frac{\gamma}{V_m} I\left(\frac{x_0 - x}{c}, c/s, \lambda \right) d(x/c)$$

Similarly for v component, the contribution of vorticity distribution on 0th blade is $\dfrac{\gamma\,dx}{2\pi (x_0 - x)}$. The source, sink distribution does not con ribute to v component.

Summing up and integrating as before

$$\frac{v}{V_m} = -\frac{1}{2\pi}\int_0^1 \frac{\gamma}{V_m}\frac{d(x/c)}{(x_0/c - x/c)}$$

$$-\frac{c/s}{2}\int_0^1 \frac{q}{V_m}\, I\left(\frac{x_0-x}{c}, \frac{c}{s}, \lambda\right) d(x/c)$$

$$-\frac{c/s}{2}\int_0^1 \frac{\gamma}{V_m}\, R\left(\frac{x_0-x}{c}, \frac{c}{s}, \lambda\right) d(x/c)$$