ON THE MULTIPLE SCATTERING OF WAVES BY SLAB SCATTERERS

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By
UMENDRA PRASAD

DEPTT. OF ELECTRONICS & COMMUNICATION ENGINEERING
UNIVERSITY OF ROORKEE
ROORKEE
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CERTIFICATE

CERTIFIED that the dissertation entitled " ON THE MULTIPLE SCATTERING OF WAVES BY [illegible] SCATTERERS" which is being submitted by Sri UPENDRA PRASAD in partial fulfilment for the award of the Degree of Master of Engineering in Electronics and Communication Engineering (Advanced Electronics) of University of Roorkee, Roorkee is a record of student's own work carried out by him under my supervision and guidance. The matter embodied in this dissertation has not been submitted for the award of any other Degree or Diploma.

This is further to certify that he has worked for a period of 7 months from Jan. 1966 to August 1966 for preparing dissertation for Master of Engineering Degree at the University.

N C Mathur

(N. C. MATHUR)
Reader
Department of Electronics & Communication Engg.,
University of Roorkee.
Roorkee.

Dated August 16, 1966.

A C K N O W L E D G E M E N T

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C O N T E N T S

Page.

ABSTRACT

This report considers the propagation of electromagnetic waves in a random medium. When the randomness is caused by the presence of discrete, identical scatterers embedded in a homogeneous medium, the problem is formulated in terms of multiply scattered fields. This type of formulation was first given in 1945, by Foldy(3), who introduced the concept of the configurational average. Since then much work has been done by Lax, (5,6) Twersky (9,10), Waterman and Truell (11), on this subject. The present thesis confines the main discussion to only the scalar problem where the vector nature of pertinent field quantities is neglected.

The problem has been formulated using a self consistent approach. This approach leads to equations governing the expectation value of the total field and exciting field which are quite general and can be used for scatterers of any shape or size. They are written in terms of the scattering properties of a single, isolated scatterer.

The problem of scattering by plates has been considered in detail. In the Born approximation, which is quite adequate in the case of weakly random media, the

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Figures written in parenthesis denote the serial number of bibliography given at the end.

Results show that the distribution of scatterers is equivalent to a modified homogeneous medium of propagation constant K_1 which is a function of the size, density and properties of the slabs. It is found that when the right half space is completely filled with the scatterers, the Foldy approximation does not remain valid. In the case of multiple scattering, the average total field propagates in an equivalent homogeneous medium of propagation constant K_1. A dispersion relation has been obtained which governs the propagation constant corresponding to the mode of propagation. Cases of interest e.g., thin slab approximation and sparse concentration of scatterers, have been considered and the equations so obtained show that the complex propagation constant K_1, may be specified explicitly in terms of the number of scatterers per unit length and the forward and backward scattering amplitudes obtained for a single, isolated scatterer. The average total field, when the point of observation is in $x < 0$, is the sum of incident and a reflected field.

In the case of multiple mode propagation, it is found that the average total field propagates as a collection of plane wave modes. The propagation constants of the various modes are determined by the dispersion relation. In case of thin slab approximation and sparse concentration only one mode propagates.

1. INTRODUCTION

The phenomena of wave propagation through a random medium has, historically speaking, two main origins. One stems from statistical mechanics, and is capable in principle at least, of explaining all electromagnetic phenomena in a material medium. In practice this approach tends to be limited to one single aspect of 'randomness' viz., configurations of randomly placed scatterers, and to the determination of one single parameter, usually the average value of one field quantity associated with the incident plus scattered fields.

The other approach, essentially macroscopic in character, emerged from the problem of scattering of acoustic and electromagnetic radiation by turbulent media.

A random medium can be defined as a medium some properties of which are random functions of position or time or both. Such a definition obviously includes almost all physical media due to its generality. Since only macroscopic quantities can be measured experimentally in most of the cases, we usually assume that the medium can be treated as a continuum. The continuum theory has been successful for a large class of physical problems.

and its use is very desirable as long as it is valid.

When the properties of a medium do not depart appreciably from the average value, the medium is said to be weakly random. In such cases a perturbation technique, such as the well known Born solution, can usually be employed in theoretical investigations. If, however, the properties of the medium are allowed to change appreciably in some manner, the perturbation technique is useless and some new approach must be used. In the present investigation both the perturbation method and the more exact formulation are used.

This thesis is concerned with the propagation of electromagnetic waves in a medium in which are embedded randomly positioned, identical scatterers with similar orientation. The value of the electric field for a given configuration of scatterers is not usually of interest. We are more interested in the statistical expectation of the field for all admissible configurations. The positions of these scatterers are governed by the joint probability density function. It is assumed that the scattering properties of each scatterer are known. For a given configuration of scatterers, the total field at a point is given, according to the self-consistent approach, by the sum of the incident field and the fields scattered from all the scatterers. Therefore, the total field depends upon the knowledge of the exciting fields at the scatterers. A similar self-consistent approach can be used to write equations for the exciting fields. In principle, these equations are to be solved to get the total field for a particular configuration. The ensemble average of the total field would then give the expectation value of the total field. Unfortunately, these equations are extremely complicated in practice and it is impossible to solve them directly.

The alternate route is to average the equations as they stand. In doing so we obtain a system of equations. The first equation involves the average total field and the first partial average of the exciting field. This first partial average is obtained in terms of the second partial average of the exciting field, which is the average taken with two scatterers held fixed.

On continuing the procedure, one obtains a hierarchy of equations. Since this chain of equations is not closed it is still impossible to solve, unless the chain can be broken by introducing valid approximations. These approximations and criteria of their validity are discussed elsewhere (9), (5), (10). Here we approximate the exciting field on a scatterer by the total field there, then the interaction is removed. It should be noted that this approximation is still much better than the single scattering approximation. Using this approximation, the average exciting field equation is closed and becomes a genuine integral equation which, if solved, determines the total field. The formulation is based on the work of Waterman and Truell (12).

The special case of slabs is next considered in detail. We consider a perfectly random distribution. Furthermore, we consider a constant density n_0 of scatterers per unit length confined to half space. The exact solution of scattering of plane incident wave by a single slab situated at an arbitrary point is obtained next. This is used in first obtaining expressions for total average field in the Born approximation. This approximation is essentially the first order iteration of the multiple field equation in which all scatterers are assumed to be excited by the incident field alone. For cases where Born approximation holds, the medium with scatterers behaves like an equivalent homogeneous medium with a modified propagation constant.

In strongly random media, the Born approx. is not valid and effects of multiple scattering have to be taken into account. For this purpose we assume some form of the average exciting field and obtain the total field. The total field equation shows that the field propagator is an equivalent medium with modified propagation constant. We have also found the dispersion relation which governs all modes that can propagate in the equivalent medium. Some special cases of interest, such as, thin slab approximation and sparse concentration of scatterers have also been considered.

2. HISTORICAL SURVEY

The problem of wave propagation in a medium containing a distribution of obstacles has been studied extensively due to its practical importance. The earliest studies were concerned with light and acoustic waves. Maxwell's work in the nineteenth century led to the identification of light as a form of electromagnetic radiation and laid the foundations for the modern approach to the subject. This was followed by Lord Rayleigh's classical work in 1899 on scattering by random distributions which explained the colour of the sky. Extensive computations for scattering by single objects have been carried out recently at the University of Michigan, U.S.A. A comprehensive review of the subject has been given by Twersky (9).

In recent years, statistical methods have come to play an important part in the study of propagation in random media. The statistical properties of radio signals received from radio stars and artificial earth satellites as affected by the fluctuations of density and refractive index of the upper atmosphere have been studied by numerous workers such as Booker (1), Chernov (2), Keller (4) and Yeh (13). The same problem can be treated from the point of view of a distribution of discrete scatterers in a homogeneous medium. Regular, periodic distributions have been studied as boundary value problems using Fourier analysis. But this is not applicable to random distributions.

The literature on multiple scattering is not very extensive. The first systematic treatment of multiple scattering of waves by a random distribution of isotropic point scatterers was given by Foldy (8) in 1945. He used the field- consistent approach to obtain expressions for the expectation values of the coherent and incoherent fields. This procedure was later generalized by Lax (9) to include point scatterers with quite general scattering properties using a quantum - mechanical formulation. One of the main difficulties in studying multi-scatterer problems lies in the estimation of the exciting field on a scatterer which is part of a configuration of scatterers. Various approximations in this connection are discussed by Foldy and Lax. In a comprehensive paper on multiple scattering, Waterman and Truell (12) have derived a criterion for the validity of these approximations. In an extension of their work, Fikioris and Yeh (7) have considered two or three and spherical scatterers of arbitrary size and have derived an expression for the dispersion relation. In this thesis equations have been reduced to one-dimensional case of slabs and both single order and multiple scattering effects are considered.

3. FORMULATION OF THE PROBLEM

3.1. THE HALF-SPACE [illegible]

Let us consider a collection of N identical scatterers of arbitrary size, shape and scattering properties, distributed randomly in the half-infinite space $z > 0$. Let the various configurations of scatterers be governed by the probability density distribution $p(r_1, r_2, \ldots, r_N)$. Thus $p(r_1, r_2, \ldots, r_N)\, dv_1 dv_2 \ldots dv_N$ is the probability of finding the first scatterer in the volume dv_1 centered at r_1, the second scatterer in dv_2 centered at r_2 and so on. Since all scatterers are identical, a configuration is specified by the scatterer positions alone. We shall place two restrictions on this distribution:

i. The scatterers are confined in the half-space $z > 0$. Therefore, $p(r_1, r_2, \ldots r_N) = 0$ whenever any position vector r_j lies in the space $z < 0$.

ii. Interpenetration of scatterers is excluded. Therefore, $p(r_1, r_2, \ldots r_N) = 0$ whenever any two position vectors r_j, r_k are such that

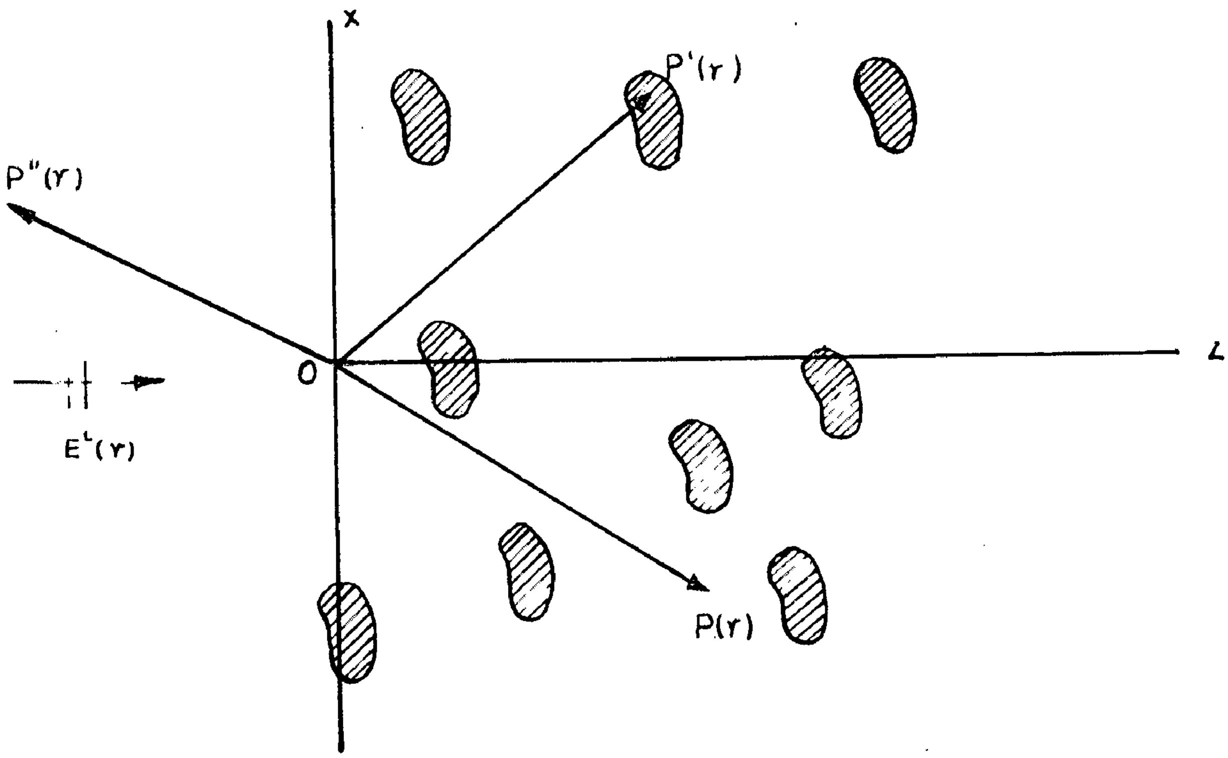


Fig. 1 GEOMETRY OF THE PROBLEM

the scatterers centered at r_j and r_k will overlap.

In addition, only elastic scattering will be considered i.e., the scatterers are in no way affected by the incident field. Also, it is assumed that the motion of scatterers, if any, is too slow to be of significance.

Let an electromagnetic field $E^i(r,t)$ be incident from the left. We shall consider only the forced oscillation case with time dependence $e^{-i\omega t}$. For simplicity we shall usually suppress the time dependence. Our object is to find the total field at a point r. For the configuration $r_1, r_2, \ldots, r_N$, we shall denote the total field at r by $E(r; r_1, r_2, \ldots, r_N)$. Clearly if r lies in $z > 0$, it may lie outside all scatterers (as at P in Fig. 1) or it may lie within some scatterer at any r_j (as at P'). However, if r lies in $z < 0$ (as at P''), it must lie outside all scatterers. We shall consider the two cases separately.

3.2. POINT OF OBSERVATION INSIDE THE SCATTERING MEDIUM

Let $E^s(r, r_j; r_1, r_2, \ldots, r_N)$ denote the scattered field at r from the scatterer at r_j for the configuration $r_1, r_2, \ldots, r_N$. This is governed by the exciting field of the scatterer at r_j, denoted by $E^e(r, r_j; r_1, r_2, \ldots, r_N)$

and by the scattering properties of the scatterer which we shall denote by the operator $T(r, r_j)$. This operator operates on the exciting field at the scatterer at r_j to give the scattered field at r . We thus have

$$u^{s}(r, r_j; r_1, r_2, \ldots, r_n) = T(r, r_j)\, u^{E}(r, r_j; r_1, r_2, \ldots, r_n)$$

Here we shall assume that the scattering properties of a single scatterer are known so that $T(r, r_j)$ is known. We shall denote the total field at r , when r is inside the scatterer at r_j , by $T^{I}(r, r_j)\, u^{E}(r, r_j; r_1, \ldots r_n)$ and shall assume, likewise, that $T^{I}(r, r_j)$ the interior scattering operator, is known. We shall further take $T(r, r_j) = 0$ whenever r is inside the scatterer at r_j and $T^{I}(r, r_j) = 0$ whenever r is outside the scatterer at r_j . The total field at r for a fixed configuration $r_1 \ldots r_n$ of scatterers is given by the sum of the incident field $u^{i}(r)$ and the scattered fields from all n scatterers,

$$u(r; r_1, \ldots r_n) = u^{i}(r) + \sum_{j=1}^{n} T(r, r_j)\, u^{E}(r, r_j; r_1, r_2, \ldots r_n)$$

when r is outside all scatterers. When r is inside the scatterer at r_j , the total field is given by

$$u(r; r_1, r_2, \ldots r_n) = T^{I}(r, r_j)\, u^{E}(r, r_j; r_1, r_2, \ldots r_n)$$

These two equations can be combined into one by the following device used by Waterman and Truell. Let us define $\alpha(r, r_k)$ as follows :

$$\alpha(r, r_k) = \begin{cases} 0 & \text{when } r \text{ is inside the scatterer at } r_k \\ 1 & \text{when } r \text{ is outside the scatterer at } r_k \end{cases}$$

The total field may now be written

$$\psi(r|r_1, r_2, \ldots, r_n) = \left[\prod_{k=1}^{n} \alpha(r, r_k)\right] \left[\psi^i(r) + \sum_{j=1}^{n} u(r, r_j)\, \psi^E(r, r_j|r_1, \ldots, r_n)\right] + \sum_{k=1}^{n} \left[1 - \alpha(r, r_k)\right] \left[\psi^I(r, r_k)\, \psi^E(r, r_k|r_1, r_2, \ldots, r_n)\right] \quad ..(3.1)$$

For a given type of scatterers, the total field cannot be evaluated for an arbitrary configuration. Therefore, we proceed to take the ensemble average of the total field. The statistical expectation value of the total field, called the average total field, is given by

$$\langle \psi(r) \rangle = \int dv_1 \int dv_2 \ldots \int dv_n\, p(r_1, r_2, \ldots, r_n)\, \psi(r|r_1, r_2, \ldots, r_n)$$

Each integration is carried out over the whole volume accessible to the scatterers. From equation (3.1) we get

$$\langle E(r)\rangle = \int d v_1 \int dv_2 \,..\int dv_N \; P(r_1, r_2, \ldots r_N)\Big[\prod_{k=1}^{N} \alpha(r,r_k) E^A(r)\Big]$$

$$+ \int dv_1 \int dv_2 \ldots \int dv_N \; P(r_1 \ldots r_N)\Big[\prod_{k=1}^{N} \alpha(r,r_k)\Big]\Big[\sum_{j=1}^{N} T(r,r_j)$$

$$E^D(r,r_j;\, r_1 \ldots r_N)\Big] + \int dv_1 \int dv_2 \ldots \int dv_N \; P(r_1 \ldots r_N)$$

$$\Big[\sum_{k=1}^{N} \big[1 - \alpha(r,r_k)\big] \; T^I(r,r_k) E^D(r,r_k;\, r_1 \ldots r_N)\Big] \qquad \ldots(3.2)$$

This equation on simplification gives, the average total field (Mathur and Rao, (7)).

$$E(r) = E^A(r)\Big[1 - \int_{|r-r'|<a} dv' \, n(r')\Big] + \int_{|r-r'|>a} dv' \, n(r') T(r,r') \langle E^D(r;r')\rangle$$

$$+ \int_{|r-r'|>a} dv' n(r') \int_{\substack{|r-r''|<a \\ |r'-r''|>2a}} dv'' n(r''|r') \, T(r,r') \, \langle E^D(r;r',r'')\rangle$$

$$+ \int_{|r-r'|\leq a} dv' \, n(r') \;\; T^I(r,r') \;\; \langle E^D(r;r')\rangle \qquad ..(3.3)$$

Where $\langle E^D(r;r')\rangle$ is the first partial average of the exciting field on the scatterer at r' when it is held fixed. Similarly $\langle E^D(r;\, r',\, r'')\rangle$ is the second partial average of the emitting field with scatterers at r' and r'' held fixed.

Also $n(r')$ is the number density of scatterers at r' and is given by

$$n(r') = np(r')$$

Similarly $n(r''|r') = (n-1)\, p(r''|r')$

It may be noted that if there are no scatterers in the medium then equation (5.5) reduces to

$$\langle E(r) \rangle = E^0(r)$$

as would be expected. On the other hand, if the number density of scatterers is so large that the entire right half space is filled with scatterers, equation (5.5) shows that the incident field is extinguished. This is because in this case the scatterer density is constant and we have

$$\int_{|r-r'|<a} dv'\, n(r') = n_0 \int_{|r-r'|<a} dv' = V_0$$

where V_0 is the fractional volume occupied by the scatterers. When scatterers occupy the entire right half space, the fractional volume is unity. Therefore, we have

$$\psi^{i}(r)\left[1 - \int_{|r-r'|<a} dv'\, n(r')\right] = 0$$

This is consistent with the extinction theorem.

3.3. FORM OF [illegible] OUTSIDE THE SCATTERING REGION

When r lies in the space n = 0 , the total field equation can easily be derived as follows :

$$\psi(r; r_1,\dots,r_N) = \psi^{i}(r) + \sum_{j=1}^{N} s(r,r_j)\, \psi^{E}(r_j; r_1, r_2,\dots,r_N)$$

Therefore, the average value is

$$\langle\psi(r)\rangle = \int dv_1 \int dv_2 \dots \int dv_N\; p(r_1,r_2,\dots,r_N)\; \psi^{i}(r)$$

$$+ \int dv_1 \int dv_2 \dots \int dv_N\, p(r_1,\dots,r_N) \left[\sum_{j=1}^{N} s(r,r_j)\, \psi^{E}(r_j; r_1,\dots,r_N)\right]$$

This can be simplified to (3.4)

$$\langle\psi(r)\rangle = \psi^{i}(r) + \int_{|r-r'|>a} dv'\, n(r')\, s(r,r')\, \langle\psi^{E}(r; r')\rangle \qquad \dots(3.4)$$

This is in the form of a sum of the incident and the reflected fields.

3.4. THE EXCITING FIELD

The exciting field on a scatterer at r_1 is given by the self-consistent equation

$$E^e(r, r_1, r_2 \ldots r_n) = E^i(r_1) + \sum_{j=2}^{n} T(r_1, r_j)\, E^e(r, r_j, r_1 \ldots r_n)$$

...(3.9)

These relations account completely for the effect on each scatterer due to the presence of other scatterers. Because of the complicated nature of equation (3.9) it does not appear feasible to attempt to invert them to obtain explicit expressions for the exciting fields. Instead, the equations will be averaged as they stand. To get the first partial average when the scatterer at r_1 is held fixed, we use $p(r_2, \ldots r_n \mid r_1)$ and integrate over all positions except r_1 to get

$$\langle E^e(r, r_1)\rangle = \int dv_2 \ldots \int dv_n\, p(r_2, r_3, \ldots r_n \mid r_1)\, E^i(r_1)$$

$$+ \int dv_2 \ldots \int dv_n\, p(r_2 \ldots r_n \mid r_1) \Big[\sum_{j=2}^{n} T(r_1, r_j)\, E^e(r, r_j, r_1 \ldots , r_n) \Big]$$

$$= E^i(r_1) + \int_{|r_1 - r'| > 2a} dv'\, n(r' \mid r_1)\, T(r_1, r')\, \langle E^e(r, r', r_1)\rangle$$

...(3.10)

We have put $p(r_2 | r_1) = \dfrac{n(r_2 | r_1)}{n-1}$

and have replaced sum of $(n-1)$ terms by $(n-1)$ times one term. The domain of integration is such that the scatterer at r_1 is not within scatterer at r'.

This lack of completeness, the fact that the exciting field with one scatterer fixed is given in terms of the field with two scatterers fixed, is the basic difficulty encountered in the explicit approach to multiple scattering. In computing higher partial averages of the exciting field, one new feature enters. Consider the partial average of E_1 with r_1 and r_2 fixed, for example. This is given by

$$\langle E^E(r;\, r_1, r_2)\rangle = E^i(r_1) + T(r_1, r_2)\, \langle E^E(r_2;\, r_1, r_2)\rangle + \int_{\substack{|r_1 - r'| > 2a \\ |r_2 - r'| > 2a}} dv'\, n(r' | r_1, r_2)\, T(r_1, r')\, \langle E^E(r;\, r', r_1, r_2)\rangle \qquad \ldots(3.7)$$

We notice that, since one term in the summation is a scattered wave from r_2, after averaging this term stands apart from the others, outside of the integral sign. Additional terms of this form appear as more scatterers are held fixed. Thus we obtain the hierarchy of equations

$$\langle \psi^{n}(r\,;\, r_1, r_2, \ldots, r_n)\rangle = \psi^{i}(r_1) + \left[\sum_{j=2}^{n} T(r_1, r_j)\,\langle \psi^{n}(r, r_j\,;\, r_1, \ldots, r_n)\rangle\right] + \left[\int dr'\, n(r'\,;\, r_1, \ldots, r_n)\; T(r_1, r')\,\langle \psi^{n}(r\, r'\,;\, r_1, \ldots\, r_n)\rangle\right]$$

The last sum of these n equations will be the required integral equation for a fixed configuration which is the analogue of equation (3.3) above.

An alternate approach to the problem is the use of iteration technique. Equation (3.3) can be written in terms of a multiple - order of scattering, namely, those primary scattering in the ? the incident wave alone, secondary scattering represents the scattering of the primary waves, and so on. Thus employing repeated iteration to obtain the infinite series

$$\psi^{n}(r, r_1\,;\, r_1, r_2, \ldots, r_n) = \psi^{i}(r_1) + \sum_{j=2}^{n} T(r_1, r_j)\,\psi^{i}(r_j)$$

$$+ \sum_{j=2}^{n} T(r_1, r_j)\left[\sum_{\substack{k=1\\ k\neq j}}^{n} T(r_j, r_k)\,\psi^{i}(r_k)\right]$$

$$+ \sum_{j=2}^{n} T(r_1, r_j)\left[\sum_{\substack{k=1\\ k\neq j}}^{n} T(r_j, r_k)\left[\sum_{\substack{l=1\\ l\neq k}}^{n} T(r_k, r_l)\,\psi^{i}(r_l)\right]\right]$$

$$+ \ldots \qquad \ldots(3.[illegible])$$

Here the single summation gives the primary scattered terms, the double summation the secondary terms, and so on. If either equation (3.9) or (3.8) could be solved, the result could be substituted in the equation for the (n-1)th partial average and, we could solve the equation . By successive solutions and substitutions we could ultimately solve the equation for the first partial average. In practice this is impossible due to large number of scatt-erers. Therefore, some approximations are necessary.

3.3 APPROXIMATIONS IN MULTIPLE SCATTERING

In order to formulate the many-body scattering problem in a form that can be solved for specific cases, we have to consider some approximations.

One approach is to look at multiple scattering from the point of view of successive orders of scattering as expressed in equation (3.8) . The first approximation would be to consider the first term alone and replace the exciting field by the incident field itself. This is called the Born approximation. This approximation is good enough when the average separation of scatterers is large compared to their size. In the second approximation, each scatterer would be excited by the incident plus the once-scattered field. Such successive approximations can be made to get results to any desired accuracy if one can evaluate the integrals involved.

Another approach is to express the exciting field incident on the jth scatterer as the total field that would exist in that neighbourhood if the jth scatterer were not present. When the scatterer is then 'inserted' in position, additional terms must be added to the total field, representing scattering from all other scatterers of the collection from the 'inserted' scatterer, and so on. Thus we have

$$\psi^{j}(r, r_j; r_1, r_2, \ldots, r_N) = \phi(r, r_j; r_1, r_2, \ldots, r_N) + \sum_{\substack{k=1 \\ k \neq j}}^{N} g(r_j, r_k) \left[g(r_k, r_j)\, \phi(r, r_j; r_1, \ldots, r_N) \right] + \ldots \quad ..(3.9)$$

The approximation consists in neglecting the second and higher order terms on the right hand side. For dense systems in which multiple scattering effects are most important, this is a much better approximation than the Born approximation. A comparison of the magnitude of the second term with that of the first has been made by Waterman and Truell (12) by considering point scatterers and scalar waves. They have developed a criterion according to which the second term is much smaller than the first if

$$\frac{n_0 f}{k^2} \ll 1$$

where n_0 is the constant number density of scatterers, σ_0 is the scattering cross section of a single scatterer and Z is the propagation constant of the medium in which the scatterers are located. This criterion is shown to be quite generally valid for most of the physical situations.

Finally, Lax (9) has suggested breaking off the hierarchy at each stage by arbitrarily replacing the exciting field in an integrand by the corresponding field with one less scatterer held fixed, that is taking

$$\langle \psi^E(r : r_1, r_2, \ldots, r_s, r_j) \rangle \approx \langle \psi^E(r : r_1, r_2, \ldots, r_s) \rangle$$

for each s and j. If we truncate hierarchy at the first equation itself then we have

$$\langle \psi^E(r : r_j, r_k) \rangle \approx \langle \psi^E(r : r_j) \rangle \quad ..(3.10)$$

This approximation has been designated as the "quasi-crystalline" approximation by Lax, since it holds exactly in the case of crystals.

We shall use the last two approaches to simplify our equations.

3.6. [illegible]

Let us approximate the exciting field at the scatterer at r_j by the total field at r_j when it is removed from the configuration, we get

$$E^{n}(r, r_1 ; r_1, r_2, \ldots r_n) \approx E(r_1, r_2, \ldots, r_n)$$

$$= E^{0}(r_1) + \sum_{j=2}^{n} T(r_1, r_j) E^{n}(r, r_j, r_2, r_3 \ldots r_n)$$

Using this equation and averaging we get an expression for the exciting field with an [illegible] held fixed as follows (7).

$$\langle E^{n}(r ; r_1)\rangle = E^{0}(r) + \int_{|r_1 - r'| > 2a} dv' \, n(r'|r_1) T(r_1, r') \langle E^{n}(r;r')\rangle_{n-1} \qquad \ldots(3.11)$$

Here the notation $\langle E^{n}(r ; r')\rangle_{(n-1)}$ indicates the first partial average of the exciting field with $(n-1)$ scatterers. It is obvious that for a sufficiently large number of scatterers,

$\langle E^{n}(r; r')\rangle_{n-1} \approx \langle E^{n}(r;r')\rangle$. The term R is given by

$$R = \sum_{j=2}^{n} \int dv_2 \ldots \int dv_n \; p(r_j|r_1) \left[p(r_2, \ldots, r_n | r_j) - p(r_2, \ldots, r_n | r_j, r_1) \right] T(r_1, r_j) E^{n}(r, r_j, r_2, \ldots, r_n)$$

In the case of perfectly random distributions, the scatterers are statistically independent and

$$p(r_1, r_2, \ldots, r_n) = p(r_1)\, p(r_2) \ldots p(r_n)$$

In this case

$$\left[p(r_2, \ldots, r_n | r_1) - p(r_2, \ldots, r_n | r_1, r_1) \right] = 0$$

and therefore, $B = 0$. Also $p(r' | r_1) = p(r')$

The equation (3.11) therefore, becomes

$$\langle E^D(r\,;\,r) \rangle = E^A(r) + \int_{|r-r'| > 2a} dv' \; n(r')\, T(r,r') \; \langle E^D(r;\,r') \rangle \qquad \ldots(3.12)$$

A comparison with equation (3.6) shows that for statistically independent scatterers, the approximation

$$E^D(r,\, r_1,\, r_1,\, r_2,\, \ldots,\, r_n) \approx E(r_1,\, r_2,\, \ldots,\, r_n)$$

is equivalent to the 'quasi-crystalline' approximation

$$\langle E^D\,(r\,;\, r_1,\, r_2\,) \rangle \approx \langle E^D(r\,;\, r_1) \rangle$$

We shall now use this approximation to simplify the total field equation.

The average total field is given by the equation (3.3) which with the above approximation, becomes

$$\langle E(r) \rangle = E^A(r) \left[1 - \int_{|r-r'|<a} dv' n(r') \right] + \int_{|r-r'| > a} dv' n(r') T(r,r') \langle E^D(r;r') \rangle$$

$$\left[1 - \int_{\substack{|r-r''| < a \\ |r'-r''| > 2a}} dv'' n(r'') \right] + \int_{|r-r'| < a} dv' n(r') \; T^B(r,r') \; \langle E^D(r;r') \rangle$$

In the second term we have put

$$\int\limits_{|r-r'|>a} dV' n(r') \int\limits_{\substack{|r-r''|<a \\ |r'-r''|>2a}} dV'' n(r''|r')\, \Theta(r,r') \left\langle E^{D}(r; r', r'')\right\rangle$$

$$= \int\limits_{|r-r'|>a} dV'\, n(r')\, \Theta(r,r') \left\langle E^{D}(r; r')\right\rangle \int\limits_{\substack{|r-r''|<a \\ |r'-r''|>2a}} dV''\, n(r'')$$

We note that except for the case when r' is near r the r''-integration can be carried out over the domain $|r-r''|<a$ since no chance of overlapping of the molecules at r' and r'' will arise. Since the r'-integration is over the entire hollow-space such that r is outside the molecules of r' and the r''-integration is over a small volume of the size of a single molecule such that r is always within the molecule at r'', no significant error will be involved in replacing

$$\int\limits_{\substack{|r-r''|<a \\ |r'-r''|>2a}} dV''\, n(r'') \quad \text{by} \quad \int\limits_{|r-r''|<a} dV''\, n(r'')$$

Then the average total field at a point r in the space $S \neq 0$ because

$$\langle E(r)\rangle = \Big[1 - \int\limits_{|r-r'|<a} dV'\, n(r')\Big]\Big[E^{0}(r) + \int\limits_{|r-r'|>a} dV'\, n(r')\, \Theta(r,r') \left\langle E^{D}(r; r')\right\rangle\Big] + \int\limits_{|r-r'|<a} dV'\, n(r')\, E^{D}(r,r') \left\langle E^{D}(r; r')\right\rangle \quad ..(3.19)$$

We shall now proceed to solve the problem for the case of dielectric media in the following chapter.

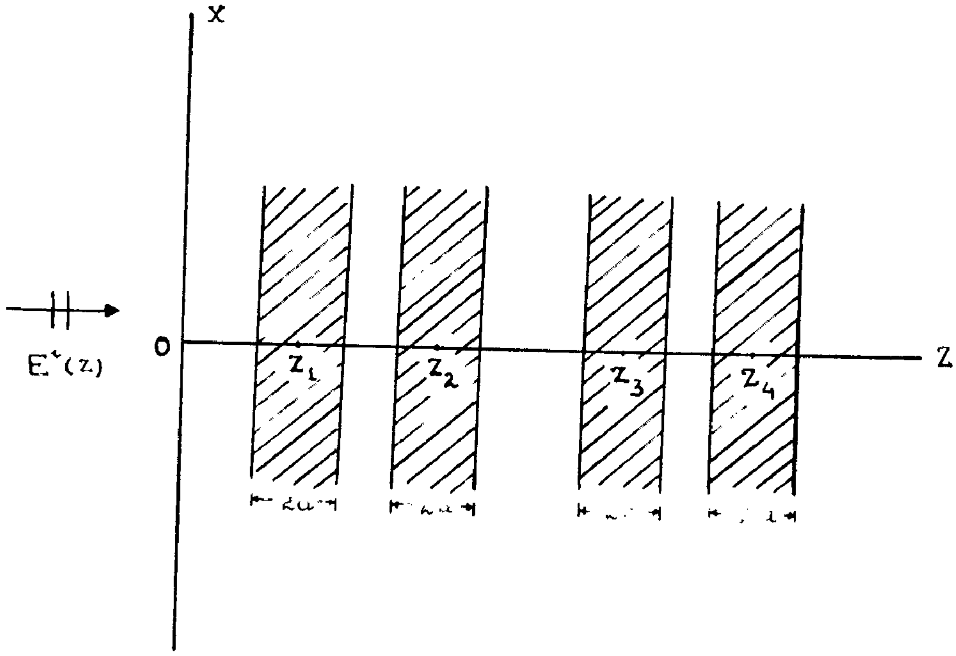


Fig. 2 THE SCATTERING MEDIUM
(SCHEMATIC)

4. FIELD SCATTERED BY SLABS

4.1. FIELD EQUATIONS IN ONE DIMENSION

In this paper, we treat the multiple scattering of incident plane waves by dielectric slabs, positioned randomly in the right half-space. There are n identical slabs, all of the same thickness 2a. Choose a cartesian system of coordinates, and let the x- axis point in the direction normal to the slabs. The slabs are regarded as infinite in the y and z directions. The configuration is illustrated in Fig. 2.

We shall consider the number density of scatterers to be constant so that $n(x') = n_0$ for $x > 0$ and $n(x') = 0$ and for $x < 0$. Let the incident wave be a linearly polarized plane wave, incident normally from the left. The averaged field equation (3.13) reduces to

$$\langle u(x)\rangle = \left[1 - n_0 \int_{|x-x'|<a} dx'\right]\left[u^i(x) + n_0 \int_{|x-x'|>a} dx'\, T(x,x') \langle u^E(x|x')\rangle\right]$$

$$+ n_0 \int_{|x-x'|<a} dx'\, T^E(x,x') \quad \langle u^E(x|x')\rangle \quad \text{for } x \geq 0 \qquad (4.1)$$

The volume integration is replaced by the linear integration on the x-axis. Similarly we have other equations,

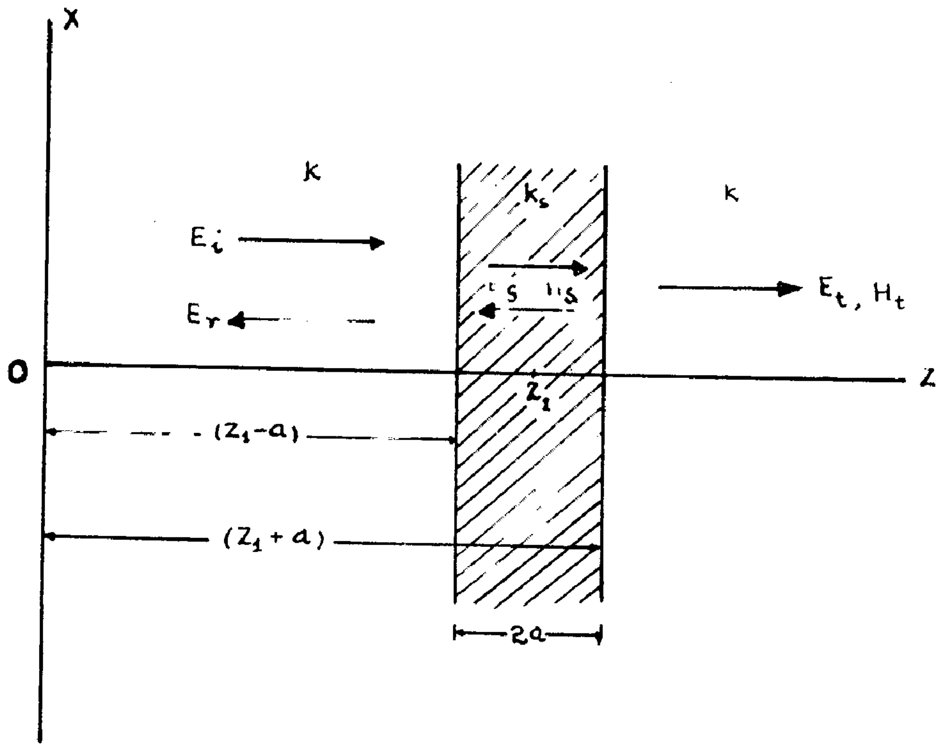


Fig. 3

$$\langle E(x)\rangle = E^0(x) + k_0 \int_{|x-x'|>a} dx'\, G(x,x')\, \langle E^e(x; x')\rangle \quad \text{for } x < 0 \qquad ..(4.2)$$

The exciting field satisfies the equation

$$\langle E^e(x; a)\rangle = E^0(x) + k_0 \int_{|x-x'|>2a} dx'\, G(x,x')\, \langle E^e(x; x')\rangle \qquad ..(4.3)$$

4.2. [illegible]

We shall consider a slab of thickness [illegible] in contact at any arbitrary point x_1 on the x-axis (as shown in Fig. 3) the propagation constant of the homogeneous medium is k and that of the material of the slab is k_0. The incident and reflected waves on the left of the slab are given by [Section (3)].

$$E_i = E_0\, e^{ikx} \quad , \qquad H_i = \frac{k}{\omega\mu} E_i \, .$$

$$E_r = E_1\, e^{-ikx} \quad , \qquad H_r = -\frac{k}{\omega\mu} E_r$$

Here we have suppressed the time dependence of the field. The field within the slab consists of a forward going wave and a backward going wave. It can be expressed as follows

$$E_s = (E_2^{+}\, e^{ik_0 x} + E_2^{-}\, e^{-ik_0 x})$$

$$H_2 = \frac{k_2}{\omega \mu_2} \left(A_2^+ e^{ik_2 x} - A_2^- e^{-ik_2 x} \right)$$

While the transmitted wave is

$$E_3 = A_3 e^{ikx} \qquad , H_3 = \frac{k}{\omega \mu} E_3$$

It will be convenient to employ here the intrinsic impedance concept. For a plane wave in a homogeneous, isotropic medium,

$$E_j = Z_j H_j \quad , \quad Z_j = \frac{\omega \mu_j}{k_j}$$

We shall now define the impedance ratios

$$Z_{jk} = \frac{Z_j}{Z_k} = \frac{\mu_j k_k}{\mu_k k_j}$$

Applying boundary conditions at $x = (x_1 - a)$ and $x = (x_1 + a)$ leads to four relations, in terms of the impedance ratios.

$$A_1 e^{ik(x_1-a)} + B_1 e^{-ik(x_1-a)} = A_2^+ e^{ik_2(x_1-a)} + A_2^- e^{-ik_2(x_1-a)}$$

$$A_1 e^{ik(x_1-a)} - B_1 e^{-ik(x_1-a)} = Z_{12} \left(A_2^+ e^{ik_2(x_1-a)} - A_2^- e^{-ik_2(x_1-a)} \right)$$

$$A_2^+ e^{ik_2(x_1+a)} + A_2^- e^{-ik_2(x_1+a)} = A_3 e^{ik(x_1+a)}$$

$$A_2^+ e^{ik_2(x_1+a)} - A_2^- e^{-ik_2(x_1+a)} = Z_{23} A_3 e^{ik(x_1+a)}$$

where
$$S_{12} = \frac{k_c\mu}{k\mu_c} \quad \text{and} \quad S_{21} = \frac{k\mu_c}{k_c\mu} = \frac{1}{S_{12}}$$

Adding and subtracting equations ([illegible]) and ([illegible]), we get respectively.

$$2A_0\, e^{ik(a_1-a)} = A_2^{+}(1+S_{12})\,e^{ik_c(a_1-a)} + A_2^{-}(1-S_{12})\,e^{-ik_c(a_1-a)}$$

$$2A_1\, e^{-ik(a_1-a)} = A_2^{+}(1-S_{12})\,e^{ik_c(a_1-a)} + A_2^{-}(1+S_{12})\,e^{-ik_c(a_1-a)}$$

Similarly adding equations ([illegible]) and ([illegible]), we get

$$2A_2^{+}\, e^{ik_c(a_1+b)} = A_3(1+S_{21})\,e^{ik(a_1+b)}$$

$$2A_2^{-}\, e^{-ik_c(a_1+b)} = A_3(1-S_{21})\,e^{ik(a_1+b)}$$

Now, substituting for A_2^{+} and A_2^{-} in terms of A_3 in the above equations we get

$$2A_0\, e^{ik(a_1-a)} = \frac{A_3}{2}(1+S_{12})(1+S_{21})\,e^{ika_1+i(kb-ak_c)} + \frac{A_3}{2}(1-S_{12})(1-S_{21})\,e^{ika_1+i(kb+ak_c)}$$

$$2A_1\, e^{-ik(a_1-a)} = \frac{A_3}{2}(1-S_{12})(1+S_{21})\,e^{ika_1+i(kb-ak_c)} + \frac{A_3}{2}(1+S_{12})(1-S_{21})\,e^{ika_1+i(kb+ak_c)}$$

Dividing second relation by first we get

$$\frac{E_1\, e^{-ikz_1}}{E_0\, e^{ikz_1}} = e^{-2ikz_1}\,\frac{(1-r_{12})(1+r_{21})+(1+r_{12})(1-r_{21})e^{2ik_2 d}}{(1+r_{12})(1+r_{21})+(1-r_{12})(1-r_{21})e^{2ik_2 d}} \qquad ..(4.4)$$

We define E_1/E_0 as the "Reflection Coefficient". The third relation in E_0 and E_3 gives

$$\frac{E_3}{E_0} = \frac{4\, e^{-i(k-k_2)d}}{(1+r_{12})(1+r_{21})+(1-r_{12})(1-r_{21})\, e^{2ik_2 d}} \qquad ..(4.5)$$

$\frac{E_3}{E_0}$ is defined as "Transmission Coefficient", T.

The reflection coefficient is the ratio of the reflected intensity to the incident intensity. Since due to reflection at the boundary the phase of the wave is reversed, the phase term in the ratio E_1/E_0 does not get cancelled, instead it is added and we get a term like e^{2ikz_1}. But in the case of transmission the phase of the transmitted wave remains the same as that of the incident wave at the boundary, the phase term gets cancelled in the ratio E_3/E_0. The reflection and transmission coefficients define the scattering parameter $S(k,k')$. Next we substitute for E_3 and get

$$\frac{U_1^{\circ}}{U_0} = \frac{2(1+s_{21})e^{i(k-k_0)(a_1-a)}}{(1+s_{12})(1+s_{21})+(1-s_{12})(1-s_{21})e^{2ikb}} \qquad ..(4.6)$$

and

$$\frac{U_2^{\circ}}{U_0} = \frac{2(1-s_{21})\,e^{i(k+k_0)(a_1-a)}\,e^{2ikb}}{(1+s_{12})(1+s_{21})+(1-s_{12})(1-s_{21})e^{2ikb}} \qquad ..(4.7)$$

The equations (4.6) and (4.7) define the internal reflecting parameters $G^n(a,a')$ for an isolated medium.

We shall now define the complex ratios

$$r_{jk} = \frac{1-s_{jk}}{1+s_{jk}} = \frac{k_k - k_j}{k_k + k_j} = -r_{kj}$$

The quantity r_{jk} is the complex ratio of the amplitudes of reflected and incident waves, at normal incidence for the plane interface dividing two semi-infinite media.

In terms of these r_{jk} we have

$$\frac{U_1}{U_0} = e^{2ik(a_1-a)}\,\frac{r_{12}+r_{21}e^{2ikb}}{1+r_{12}r_{21}e^{2ikb}}$$

$$\frac{U_2}{U_0} = \frac{4}{(1+s_{12})(1+s_{21})}\cdot\frac{e^{-i(k-k_0)b}}{1+r_{12}r_{21}e^{2ikb}}$$

$$\frac{U_2^{\circ}}{U_0} = \frac{2}{(1+s_{12})}\,\frac{e^{i(k-k_0)(a_1-a)}}{1-r_{12}r_{21}e^{2ikb}}$$

and

$$\frac{E_2^+}{E_0} = \frac{2}{(1+\varepsilon_{12})} \quad \frac{r_{12}-r_{12}\,e^{2ik_2 d}}{1+r_{12}\,r_{21}\,e^{2ik_2 d}} \quad e^{i(k+k_0)(n_1-n)}$$

4.3. AVERAGE TOTAL FIELD IN BORN APPROXIMATION

Now we shall consider scattering by slabs and shall consider the first order scattering only. This is called the Born approximation and consists in replacing the exciting field $\langle E^e(r\,rr')\rangle$ by $E^i(r)$ on the right hand side of equations (4.1) and (4.2). When the point of observation lies in $z > 0$, the average total field becomes

$$\langle E(r)\rangle = \Big[1 - n_0 \int_{|r-r'|<a} dr'\Big].\Big[E^i(r) + n_0 \int_{|r-r'|>a} dr'\, g(r,r')\, E^i(r')\Big]$$

$$+\; n_0 \int_{|r-r'|<a} dr'\, g^s(r,r')\, E^i(r') \qquad \ldots(4.3)$$

The incident plane wave is given by

$$E^i(z) = e^{ikz}$$

Therefore, substituting for the limits of integration and scattered fields etc. we get, from equation (4.3)

$$\langle E(z)\rangle = \Big[1 - n_0 \int_{z-a}^{z+a} dz'\Big]\Big[e^{ikz} + n_0 f \int_{z+a}^{\infty} e^{ikz'}\; e^{-ikz}\, dz'$$

$$+\; n_0 b \int_{0}^{z-a} e^{ikz}\, dz'\Big] + n_0 \int_{z-a}^{z+a} \Big[g_2^{+} e^{ikz} + g_2^{-} e^{-ikz}\Big]\, dz'$$

$$= [1-2m_0]\left[e^{ikz} + a_0 k e^{-ikz} \left\{ \frac{e^{2ikz'}}{2ik} \right\}_{z-a}^{\infty} + a_0 B e^{ikz} \left\{ z' \right\}_{0}^{z-a} \right]$$

$$+ a_0 \int_{z=a}^{z-a} \left[\xi_1 e^{ik_0 z} e^{i(k-k_0)z'} + \xi_2 e^{-ik_0 z} e^{i(k+k_0)z'} \right] dz'$$

where $$\xi_1 = \frac{2}{1+r_{12}} \quad \frac{e^{-i(k-k_0)a}}{1+r_{12}r_{21}e^{2ik_0 a}}$$

and $$\xi_2 = \frac{2}{1+r_{12}} \quad \frac{r_{12}r_{21}e^{-i(k+k_0)a}}{1+r_{12}r_{21}e^{2ik_0 a}}$$

The first term is

$$= (1-2m_0)e^{ikz}\left[1 + a_0 B(z-a) - a_0 k \frac{e^{2ika}}{2ik}\right]$$

The second term is

$$= a_0 \left[\frac{\xi_1 e^{ik_0 z}}{i(k-k_0)} \left(e^{i(k-k_0)(z-a)} - e^{i(k-k_0)(z-a)} \right) \right.$$

$$\left. + \frac{\xi_2 e^{-ik_0 z}}{i(k+k_0)} \left(e^{i(k+k_0)(z-a)} - e^{i(k+k_0)(z-a)} \right) \right]$$

$$= 2 a_0 e^{ikz} \left[\xi_1 \frac{\sin(k-k_0)a}{(k-k_0)} + \xi_2 \frac{\sin(k+k_0)a}{(k+k_0)} \right]$$

The average total field equation is, therefore,

$$\langle E(z)\rangle = e^{ikz}\left[(1-2\pi n_0)\left\{1+n_0 f(0-0)-n_0 B\,\frac{e^{2ikz}}{2ik}\right\}\right.$$

$$\left.+\,n_0\left\{\xi_1\frac{\sin(k-k_0)z}{(k-k_0)}+\xi_2\frac{\sin(k+k_0)z}{(k+k_0)}\right\}\right]$$

..(4.10)

The average total field when $z<0$, is given by

$$\langle E(z)\rangle = e^{ikz} + n_0 B\int_0^{\infty} e^{2ikz'}\; e^{-ikz}\,dz'$$

$$= e^{ikz} - \frac{n_0 B\, e^{-ikz}}{2ik} \qquad ..(4.11)$$

An important result is seen from equation (4.11) which is of the form

$$\langle E(z)\rangle = e^{ikz} + R_1^T e^{-ikz}, \quad z<0$$

This shows that the right half space containing the scatterers acts like a modified medium which reflects part of the incident field. The reflection coefficient R_1^T is determined by the size and density of scatterers and the wave length. The behaviour of the right half space as a modified homogeneous medium is also seen from equation (4.10) which can be written as

$$\langle E(z)\rangle = T_1^0\, e^{ikz}\,(1+i\,\delta\, z), \quad z\geq 0$$

If δ is small, we can write

$$e^{ikn}(1+i\delta n) \approx e^{ik_0 n}$$

where $k_0 = k + \delta$

Thus the modified medium has a propagation constant k_0 and a 'transmission coefficient' T_0^0. Within this medium the incident field is extinguished as would be expected.

When the right half space is completely filled with scatterers, the number density of scatterers $\rho_0 = 1/a$ (2.10). In that case, the average total field equation (4.10) reduces to

$$\langle u(x) \rangle = e^{ikx}\left[\xi_1 \frac{\sin\,(k - k_0)a}{(k-k_0)a} + \xi_2 \frac{\sin\,(k + k_0)a}{(k+k_0)a}\right]$$

which implies that the average total field in the right half space propagates with the propagation constant k of free space. This violates the rule of transmission of waves into a semi-infinite medium, because the incident field must be extinguished in the medium. Hence, we can say that, the Foldy approximation is valid only for the case of sparse concentration of scatterers.

[illegible]

Stop band and pass band characteristics.

Neglecting second order terms in $\alpha_0 a$ from equation (4.10) we have the average total field expression as

$$\langle E(z) \rangle = e^{ikz} \Big[1 - [illegible] \frac{e^{[illegible]}}{[illegible]} + [illegible] $$

$$ [illegible] \Big\{ \xi_1 \frac{[illegible]}{[illegible]} + \xi_2 \frac{[illegible]}{[illegible]} \Big\} \Big] \quad ..(4.11)$$

and $E_0 = 1 - [illegible]$

$$\Big[\xi_1 \frac{[illegible]}{[illegible]} + \xi_2 \frac{[illegible]}{[illegible]} \Big]$$

and $E_1 = [illegible]$

Thus equation (4.11) can, therefore, be written as

$$\langle E(z) \rangle = e^{ikz} (E_0 + E_1 z)$$

$$= E_0 \, e^{ikz} e^{i\delta z} \quad \text{if } E_1/E_0 \ll 1, \; E_1/E_0 \triangleq \delta$$

$$= E_0 \, e^{i([illegible]\delta)z} = E_0 \, e^{ik_m z}$$

where $k_m = ([illegible]\delta)$ is the modified propagation constant.

We shall now consider propagation of low frequency waves for which the wavelength is much larger than the width of the slab. In this case (ka) and $(k_0 a)$ is very small compared to unity. For weak concentration case, the fractional length $(n_0 a)$ occupied by the slabs is very small compared to unity. Next we expand the coefficients C_0 and C_1, term by term, assuming $(k_0 a) \ll 1$, $(ka) \ll 1$ and $(2a\, n_0) \ll 1$, and taking only first order terms into account

$$n_0 C_0\, a = n_0 a \frac{4\, e^{i(k-k_0)a}}{(1+r_{12})(1+r_{21}) - (1-r_{12})(1-r_{21})\, e^{2ik_0 a}}$$

$$= n_0 a \frac{r_{12}\left[1 + i\,(k - k_0)\,a\right]}{r_{12} - i\,(1-r_{12})^2\, k_0 a}$$

Since $r_{12} = 1/r_{21}$

$$= n_0 a \left[1 + i(k-k_0)a\right] \left[1 - i\,\frac{(\mu_0 k - \mu k_0)^2}{\mu\, \mu_0\, k}\, a\right]^{-1}$$

Since $r_{12} = \mu k_0 / \mu_0 k$

$$= n_0 a \left[1 + ia\left(2k - \frac{\mu_0}{\mu} k - \frac{\mu k_0^2}{\mu_0 k}\right)\right]$$

$$n_0 C_1 \frac{e^{i2ka}}{2ik} = n_0 \frac{e^{i2ka}}{2ik} \frac{(1-r_{12})(1+r_{21}) - (1+r_{12})(1-r_{21})\, e^{2ik_0 a}}{(1+r_{12})(1+r_{21}) - (1-r_{12})(1-r_{21})\, e^{2ik_0 a}}$$

$$\cdot\, e^{-i2ka}$$

$$= \frac{n_0}{2k} \cdot \frac{(1-r_{12}^2)(-2k_0 c)}{r_{12} + 2(1-r_{12})^2 k_0 c}$$

$$= \frac{n_0 c}{2}\left(\frac{\mu k_0^2}{\mu_0 k^2} - \frac{\mu_0}{\mu}\right)\left[1 + 2c\left(\frac{\mu_0 k}{\mu k_0} + \frac{\mu k_0^2}{\mu_0 k^2} - 2 k_0\right)\right]$$

$$2n_0 c\, \xi_1 \frac{i \sin (k - k_0)c}{(k - k_0)c} = 2 n_0 c\, \xi_1$$

$$\lim \frac{\sin (k - k_0)c}{(k - k_0)c} \to 1 \quad c\,(k - k_0)c \to 0$$

$$= 2 n_0 c \frac{2\, e^{-2(k - k_0)c}\,(1 + r_{12})}{(1 + r_{12})^2 - (1 - r_{12})^2\, e^{2ik_0 c}}$$

$$= n_0 c\left(\frac{\mu_0 k}{\mu k_0} + 1\right)\left[1 - 2c\left(k - \frac{\mu_0 k}{\mu} - \frac{\mu k_0^2}{\mu_0 k} + k_0\right)\right]$$

$$2n_0 c\, \xi_2 \frac{i \sin (k + k_0)c}{(k + k_0)c} = 2 n_0 c\, \xi_2$$

$$= 2n_0 c \frac{2(r_{12} - 1)\, e^{-2(k+k_0)c}}{(1 + r_{12})^2 - (1 - r_{12})^2\, e^{2ik_0 c}}$$

$$= n_0 c\left(1 - \frac{\mu_0 k}{\mu k_0}\right)\left[1 - 2c\left(k + k_0 - \frac{\mu_0 k}{\mu} - \frac{\mu k_0^2}{\mu_0 k}\right)\right]$$

If we further assume that $\mu_0 = \mu$, then we have.

$$G_0 = (1-2m_0) - m_0\left[1-2a\left(k-\frac{k_0^2}{k}\right)\right] - \frac{m_0}{2}\left[\left\{1+2a\left(k-2k_0+\frac{k_0^2}{k}\right)\right\}\left(\frac{k_0^2}{k^2}-1\right)\right] + m_0\left(\frac{k}{k_0}+1\right)\left[1-2a\left(k_0-\frac{k_0^2}{k}\right)\right] + m_0\left(1-\frac{k}{k_0}\right)\left[1+2a\left(k_0+\frac{k_0^2}{k}\right)\right]$$

$$= 1-\frac{m_0}{2}\left(1+\frac{k_0^2}{k^2}\right)+2m_0a^2\left(\frac{k_0^2}{k}+\frac{k_0^3}{k^2}-\frac{k_0^4}{2k^3}-k_0-\frac{k}{2}\right)$$

and
$$G_1 = m_0\left[1-2a\left(k-\frac{k_0^2}{k}\right)\right]$$

Therefore,

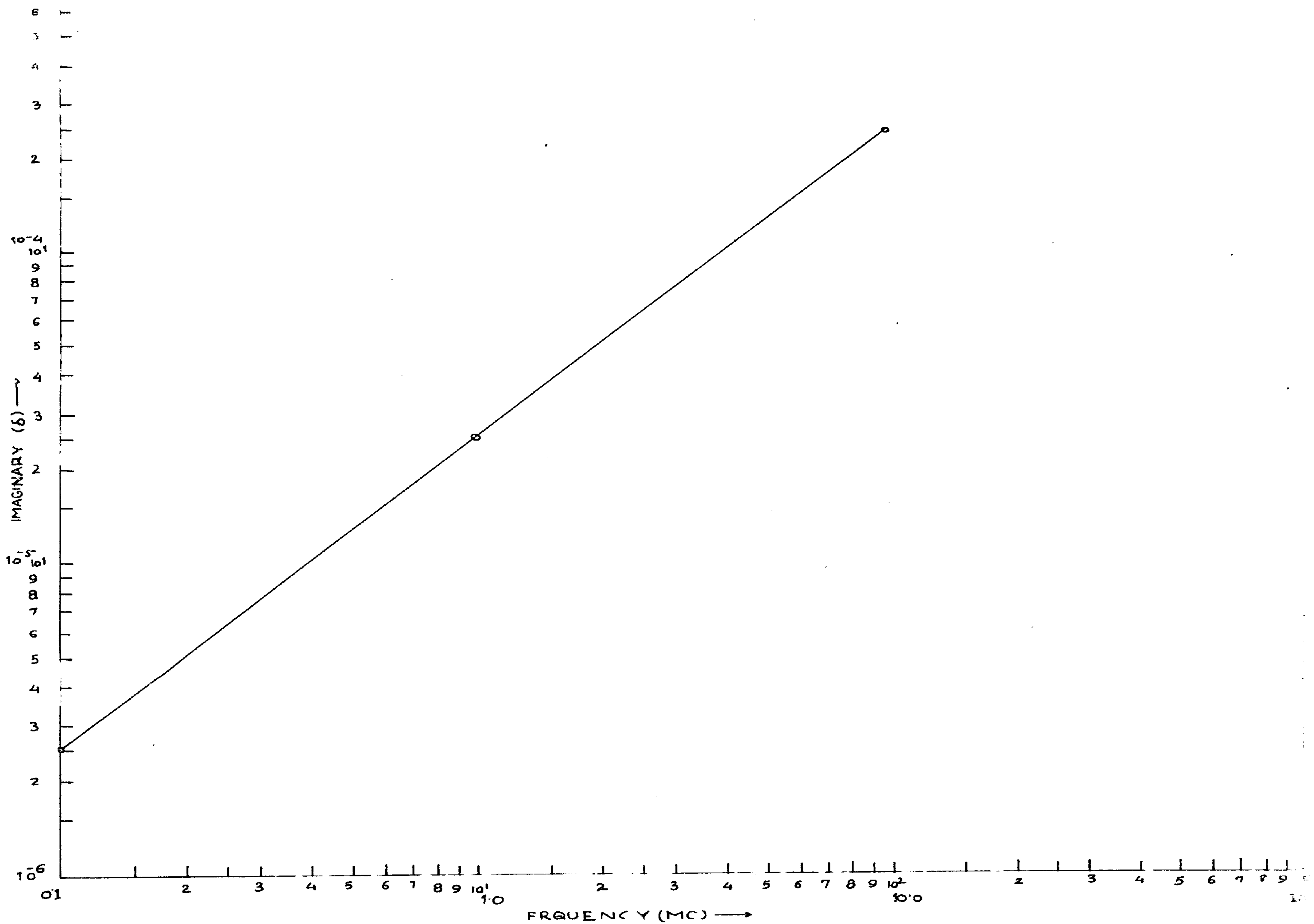
$$\delta = \frac{m_0\left[1-2a\left(k-\frac{k_0^2}{k}\right)\right]}{1-\frac{m_0}{2}\left(1+\frac{k_0^2}{k^2}\right)+2m_0a^2\left(\frac{k_0^2}{k}+\frac{k_0^3}{k^2}-\frac{k_0^4}{2k^3}-k_0-\frac{k}{2}\right)}$$

The real part of δ corresponds to attenuation in the scattering medium and the imaginary part corresponds to the phase current. Their values for different frequencies have been given in tables below.

The slabs are considered to be made of paraffin, which is which is a good dielectric and the scattering medium is free space.

For paraffin,

Specific inductive capacity $k_0 = \frac{\epsilon_D}{\epsilon_0} = 2.1$


IMAGINARY (б)
FRQUENCY (MC)
0.1
1.0
10.0

Conductivity $\sigma = 10^{-14} - 10^{-16}$ mhos/meter

The propagation constant, therefore, is

$$k_D = \sqrt{\mu_0 \epsilon_0} \sqrt{K_D} \cdot \omega$$

and hence depends on the frequency of the wave.

For Free Space:

Conductivity $\sigma = 0$

Therefore, the propagation constant is given by

$$k = \sqrt{\mu_0 \epsilon_0} \quad \omega$$

Now,

the number density $n_0 = 10^{-1}$ particles/meter

and thickness of the slab is $= 2 \times 10^{-2}$ meter

$\therefore$ $2 \omega_0 = 2 \times 10^{-3} = 0.002$

TABLE I

Frequency (c/s)	k	k_D	Real part of δ	Imaginary part of δ
0.1	0.002	0.003	0.1	2.9×10^{-6}
1.0	0.02	0.03	0.1	2.9×10^{-5}
10.0	0.2	0.3	0.1	2.9×10^{-4}
100.0	2.0	3.0	0.1	2.9×10^{-3}

Next we consider thicker slabs.

$$n_0 = 10^{-1} \text{ scatterer / meter}$$

$$2a = 2 \times 10^{-1} \text{ meter}$$

$$\therefore \ 2an_0 = 2 \times 10^{-2} = 0.02$$

TABLE II

Frequency (Mc)	n	n_0	Real part of δ	Imaginary part of δ
0.1	0.002	0.003	0.101	2.54×10^{-5}
1.0	0.02	0.03	0.101	2.54×10^{-4}
10.0	0.2	0.3	0.101	2.54×10^{-3}
100.0	2.0	3.0	0.101	2.54×10^{-2}

The above tables show that the real part of which corresponds to attenuation remains constant over a very wide range of frequencies. A plot of imaginary part of δ against frequency has been obtained as in Fig. 4 which shows that the $I_m(\delta)$ increases linearly with frequency.

—

9. MULTIPLE SCATTERING BY SLABS

It has been pointed out earlier that instead of solving the integral equation (4.3) for the exciting field, we can obtain average total field to various degrees of accuracy by successive iteration. The first iteration, which is the Born approximation, has been considered in Chapter 4 and expressions for the average total field have been obtained. For the second and higher iterations, the complexity of the integrals involved increases very rapidly. In this Chapter we shall take some form of the exciting field satisfying certain conditions and consider the multiple scattering effects.

9.1. PLANE SLAB GEOMETRY

The exciting field satisfies the equation

$$\langle \psi^E(r|z_1)\rangle = \psi^i(r) + n_0 \int dr'\, G(r,r')\, \langle \psi^E(r|r')\rangle \qquad ..(9.1)$$

$$|r-r'| > 2a$$

$$z' \geq 0$$

The exciting field must satisfy the following conditions i.e. Ref (14).

(i) Due to absence of interpenetration of slabs, in the neighbourhood of the given scatterer, the exciting field must satisfy

$$(\nabla^2 + k^2) \quad \langle E(z, z') \rangle = 0$$

(ii) Since the exciting field is the total field minus the field scattered from the given scatterer, $\langle E(z, z') \rangle$ must be regular at $z = z'$.

(iii) In parallel slab case there is planar symmetry.

The most general form of a field that satisfies these conditions is

$$\langle E(z, z') \rangle = A_1(z')\, e^{ik(z-z')} + A_2(z')\, e^{-ik(z-z')}$$

The scattering operator S operates on z for a fixed scatterer at z' and hence commutes with $A_1(z')$ and $A_2(z')$. Therefore,

$$S(z,z')\langle E(z,z') \rangle = \begin{cases} [A_1(z')\, f^- + A_2(z')\, f^+]\, e^{-ik(z-z')}, & z \le z'-a \\ [A_1(z')\, f^+ + A_2(z')\, f^-]\, e^{ik(z-z')}, & z \ge z'+a \end{cases}$$

where f^+ & f^- are respectively the forward and the backward scattering amplitudes for an isolated slab scatterer centred at the origin. Now take a trial solution

$$A_1(z') = B_1\, e^{ik_1 z'}, \quad A_2(z') = B_2\, e^{ik_2 z'}$$

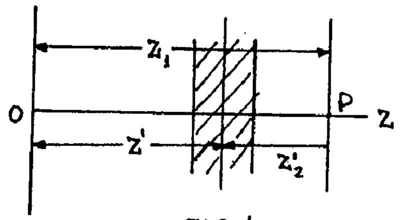


Fig. 4.

[illegible]

[illegible]

[illegible]

$$= \begin{cases} [\text{illegible}], & z_2' \geq a \\ [\text{illegible}], & z_2' \leq -a \end{cases}$$

[illegible]

[illegible]

Substituting back to equation (4) we get

$$A_1 e^{ik_1 x_1} e^{ik(x-x_1)} + A_2 e^{ik_2 x_1} e^{-ik(x-x_1)} = e^{ikx} \psi_0 I \qquad \ldots\ldots(5.2)$$

Equating coefficients of e^{ikx} on both sides of equation (5.2) we get

$$A_1 e^{i(k_1-k)x_1} = i\psi_0 \left[-\frac{A_1 e^{+}}{i(k_1-k)} - \frac{A_2 e^{-}}{i(k_2-k)} + \frac{A_1 e^{+}}{i(k_1-k)} e^{i(k_1-k)(x_1-x)} + \frac{A_2 e^{-}}{i(k_2-k)} e^{i(k_2-k)(x_1-x)} \right]$$

Now, equating coefficients of e^{0} on both sides, we get

$$i\psi_0 \left[-\frac{A_1 e^{+}}{i(k_1-k)} - \frac{A_2 e^{-}}{i(k_2-k)} \right] = 0$$

The extinction theorem is verified because the incident wave is cancelled by waves generated at the boundary. Again, equating the coefficients of e^{-ik_1} we get,

$$A_1 e^{ik_1 x_1} = \psi_0 \left[\frac{A_1 e^{+}}{i(k_1-k)} e^{ik_1(x_1-x)} e^{ikx} + \frac{A_2 e^{-}}{i(k_2-k)} e^{ik_2(x_1-x)} e^{ikx} \right]$$

This equation states that $k_1 = k_2$

$$1 = \frac{a_0}{2(k_1-k)}\left(a_1\Gamma^+ + a_2\Gamma^-\right) \qquad ..(5.10a)$$

$$A_1 = \frac{a_0}{2(k_1-k)}\left(a_1\Gamma^+ + a_2\Gamma^-\right) e^{i(k-k_1)d} \qquad ..(5.10b)$$

Substituting (a) and (b) gives,

$$\boxed{A_1 = e^{i(k-k_1)d}} \qquad ..(5.10c)$$

Similarly applying continuity of e^{-ikx} in equation (5.8), we get

$$A_2\, e^{ik_1 d_1}\, e^{ikd_1} = \frac{a_0}{2(k_1+k)}\left(-a_1\Gamma^- - a_2\Gamma^+\right) e^{i(k_1+k)(d_1+d)}$$

$$\text{or } A_2 = -\frac{a_0}{2(k_1+k)}\left(a_1\Gamma^- + a_2\Gamma^+\right) e^{i(k_1+k)d}$$

$$\therefore A_2 = -\frac{a_0\,\Gamma^-\, e^{ikd}}{2(k_1+k) + a_2\Gamma^+ e^{i(k_1+k)d}} \qquad ..(5.11)$$

Subtracting (a) and (d) from (c), we get

$$\boxed{k_1^2 - a_2\Gamma^+ k_1\left(e^{ik_1 d} - e^{-ik_1 d}\right) e^{ikd} = k^2 - a_2\Gamma^+ k e^{ikd} \left(e^{ik_1 d} + e^{-ik_1 d}\right) - \left(a_2\Gamma^+\right)^2 e^{2ikd} + \left(a_2\Gamma^-\right)^2 e^{2ikd}} \qquad ..(5.12)$$

This is the dispersion relation giving all modes that can propagate in the equivalent medium. Two special cases are of interest.

(i) Thin slab approximation i.e. when a → 0,

$$K_1^2 = (k - i n_0 f^+)^2 + (n_0 f^-)^2 \qquad \ldots(9.5)$$

where K_1 is the propagation constant for the scattering medium, its real and imaginary parts related to modified phase velocity and attenuation, respectively.

(ii) Sparse concentration: To assume the number density of scatterers to be small and so neglect higher order terms in n_0 in equation (9.4)

To zero th order in n_0

$$K_1^2 = k^2 + O(n_0) \qquad \ldots(9.6)$$

To first order in n_0

$K_1^2 = k^2 - 2i n_0 f^+ K_1$ [illegible] $= k^2 - 2i n_0 f^+ k$ [illegible]

or $K_1^2 = k^2 - 2i n_0 f^+ (\cos 2K_1 a - i \sin 2K_1 a)\, e^{i2ka}$

$= k^2 - 2i\, n_0 f^+ k\, e^{i2ka} (\cos 2K_1 a - i \sin 2K_1 a)$ [illegible]

$= k^2 - 2i\, n_0 f^+ k\, e^{-2i(K_1 - k)a}$

$$= k^2 - 2i\, n_0 f^+ k \qquad \ldots(9.7)$$

To second order in n_0

$$K_1^2 = k^2 - 2i n_0 f^+ (\cos 2K_1 a - i \sin 2K_1 a) - (n_0 f^+)^2 e^{4ika} + (n_0 f^-)^2 e^{4ika}$$

Substitute for K_1 from equation (9.7) in the right hand side of the above expression

$$K_1 = K\left(1 - \frac{2i\, n_0 f^+}{K}\right)^{1/2}$$

$$= (K - i n_0 f^+) \quad \text{neglecting higher terms.}$$

$$\therefore K_1^2 = K^2 - 2i\, n_0 f^+ \left[K \cos 2K_1 a - i(K - i n_0 f^+) \sin 2K_1 a\right] e^{i2Ka}$$
$$- (n_0 f^+)^2 e^{4iKa} + (n_0 f^-)^2 e^{4iKa}$$

$$= K^2 - 2i n_0 f^+ K e^{-i(K_1 - K)2a} + 2i (n_0 f^+)^2 e^{i2Ka} \sin 2K_1 a$$
$$- (n_0 f^+)^2 e^{4iKa} + (n_0 f^-)^2 e^{4iKa}$$

$$e^{i2Ka} \sin 2K_1 a = \frac{1}{2i}\left(e^{i(K_1 + K)2a} - e^{-i(K_1 - K)2a}\right)$$

$$= \frac{1}{2i}\left[e^{4iKa}(1 + 2a n_0 f^+ + \ldots) - (1 - 2a n_0 f^+ + \ldots)\right]$$

$$\therefore K_1^2 = K^2 - 2i n_0 f^+ K(1 - 2a n_0 f^+) - (n_0 f^+)^2 + (n_0 f^-)^2 e^{4iKa}$$

$$= (K - i n_0 f^+)^2 + 4iKa (n_0 f^+)^2 + (n_0 f^-)^2 e^{4iKa} \qquad ..(9.8)$$

All the equations, namely (9.5), (9.6), (9.7) and (9.8) show that the behaviour of the scattering medium characterised by the complex propagation constant K_1, may be specified explicitly in terms of the number of scatterers per unit length and the forward and the backward scattering amplitudes obtained for a single scatterer.

5.2. THE AVERAGED NOISE FIELD FOR z ≥ 0

$$\langle E(z)\rangle = (1-2\pi n_0)\left[E^i(z) + n_0 \int\limits_{\substack{|z-z'|>a \\ z'>0}} G(z,z')\langle E^D(z;z')\rangle\, dz'\right.$$

$$\left. + \, n_0 \int\limits_{|z-z'|<a} G^R(z,z')\langle E^D(z;z')\rangle\, dz' \right] \qquad \ldots(5.9)$$

Now we have found

$$G(z,z')\langle E^D(z;z')\rangle = \begin{cases} (A_1 I^- + A_2 I^+)e^{ik_1 z'}\; e^{-ik(z-z')}, & z \le z'-a \\ (A_1 I^+ + A_2 I^-)e^{ik_1 z'}\; e^{ik(z-z')}, & z \ge z'+a \end{cases}$$

and

$$G^R(z,z')\langle E^D(z;z')\rangle = \Big[(D_1 I^+ + D_2 I^-)e^{ik_0(z-z')} + (D_1 I^- + D_2 I^+)e^{-ik_0(z-z')}\Big]_{|z-z'|<a} \cdot e^{ik_1 z}$$

The integral $\int\limits_{\substack{|z-z'|>a \\ z'>0}} dz'$ is the same as I if z is replaced by z and z_1 is replaced by z and $k_1 = k_2$

The integral $\int\limits_{|z-z'|<a} dz'$ can be integrated. Let $z'-z = z_1'$

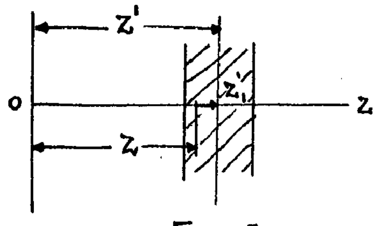


Fig. 5.

$$\int\limits_{|t-t'|<\tau} dt'\, T^x(t,t')\, \langle E^y(t-t')\rangle$$

$$= \int_{-\tau}^{\tau} \left[(a_1 E^+ + a_2 E^-) e^{-i\omega_0 t_1'} + (a_1 E^- + a_2 E^+) e^{i\omega_0 t_1'} \right] e^{i\omega_1 (t - t_1')} dt_1'$$

$$= e^{i\omega_1 t} \left[\frac{2(a_1 E^+ + a_2 E^-)}{(\omega_1 - \omega_0)} \sin(\omega_1 - \omega_0)\tau + \frac{2(a_1 E^- + a_2 E^+)}{(\omega_1 + \omega_0)} \sin(\omega_1 + \omega_0)\tau \right]$$

Putting back to equation (5.9), we get the averaged total field

$$\langle E(t)\rangle = (1 - 2\tau n_0) \left[e^{i\omega t} + n_0 e^{i\omega t} \left[- \frac{a_1 E^+ + a_2 E^-}{i(\omega_1 - \omega)} \right] \right.$$

$$+ n_0 e^{i\omega_1 t} \left[\frac{a_1 E^+ + a_2 E^-}{i(\omega_1 - \omega)} \right] e^{i(\omega - \omega_1)\tau} + n_0 e^{i\omega_1 t} \left[- \frac{a_1 E^- + a_2 E^+}{i(\omega_1 + \omega)} \right] e^{i(\omega + \omega_1)\tau} \Big]$$

$$+ n_0 e^{i\omega_1 t} \left[\frac{2(a_1 E^+ + a_2 E^-)}{(\omega_1 - \omega_0)} \sin(\omega_1 - \omega_0)\tau + \frac{2(a_1 E^- + a_2 E^+)}{(\omega_1 + \omega_0)} \sin(\omega_1 + \omega_0)\tau \right]$$

Making use of the convolution theorem and relations 5.22, we get

$$\langle E(t)\rangle = \Big\{ (1 - 2\tau n_0) \ldots \text{[illegible]}$$

$$+ 2n_0 \left[\frac{(a_1 E^+ + a_2 E^-)}{(\omega_1 - \omega_0)} \sin(\omega_1 - \omega_0)\tau + \frac{(a_1 E^- + a_2 E^+)}{(\omega_1 + \omega_0)} \sin(\omega_1 + \omega_0)\tau \right] \Big\} e^{i\omega_1 t}$$

Hence, the average total field propagates in an equivalent medium with propagation constant k_1, where D_1 and D_2 are given by equations (5.9 c) and (5.9d). The internal scattering coefficients X^{+} and X^{-} are the same as given by equation, (4.8)

When the point of observation is in the left half space, the total average field is given by

$$\langle E(z)\rangle = E^{i}(z) + a_0 \int_{|z-z'|>0} dz'\, G(z,z') \langle E^{E}(z\,;\,z')\rangle$$

$$= e^{ikz} + a_0 \int_0^{\infty} (D_1 X^{-} + D_2 X^{+})\, e^{-ikz}\, e^{i(k_1+k)z'}\, dz'$$

$$= e^{ikz} + e^{-ikz} \left[R_2\, e^{-i(k_1+k)2a} \right] \qquad \ldots(5.10)$$

Thus on the left of the scattering region, the total field is the sum of the incident field and a reflected field. The reflection coefficient is determined by the properties of the scatterers.

Some numerical examples have been considered below for thin slab approximation and sparse concentration case.

1. The dispersion relation for thin slab approximation reduces to

$$k_1^2 = (k - i\, a_0 X^{+})^2 + (a_0 X^{-})^2$$

Since $G^* = 1$ and $F^* = 0$ as $a \to 0$

Hence $k_1^2 = (k - i\,n_0)^2$ $\therefore$ $k_1 = k - in_0$

The real and imaginary parts of k_1 for different frequencies are given below in table III.

TABLE III

Frequency (c/s)	k	Real part of k_1	Imaginary part of k_1
0.1	0.002	0.002	0.1
1.0	0.02	0.02	0.1
10.0	0.2	0.2	0.1

(ii) Large conductivities. Assuming the number density of scatterers to be small and plate to be thin so that $|a n_0| \ll 1$ and (ka) and $(k_0 a) \ll 1$, the dispersion relation reduces to

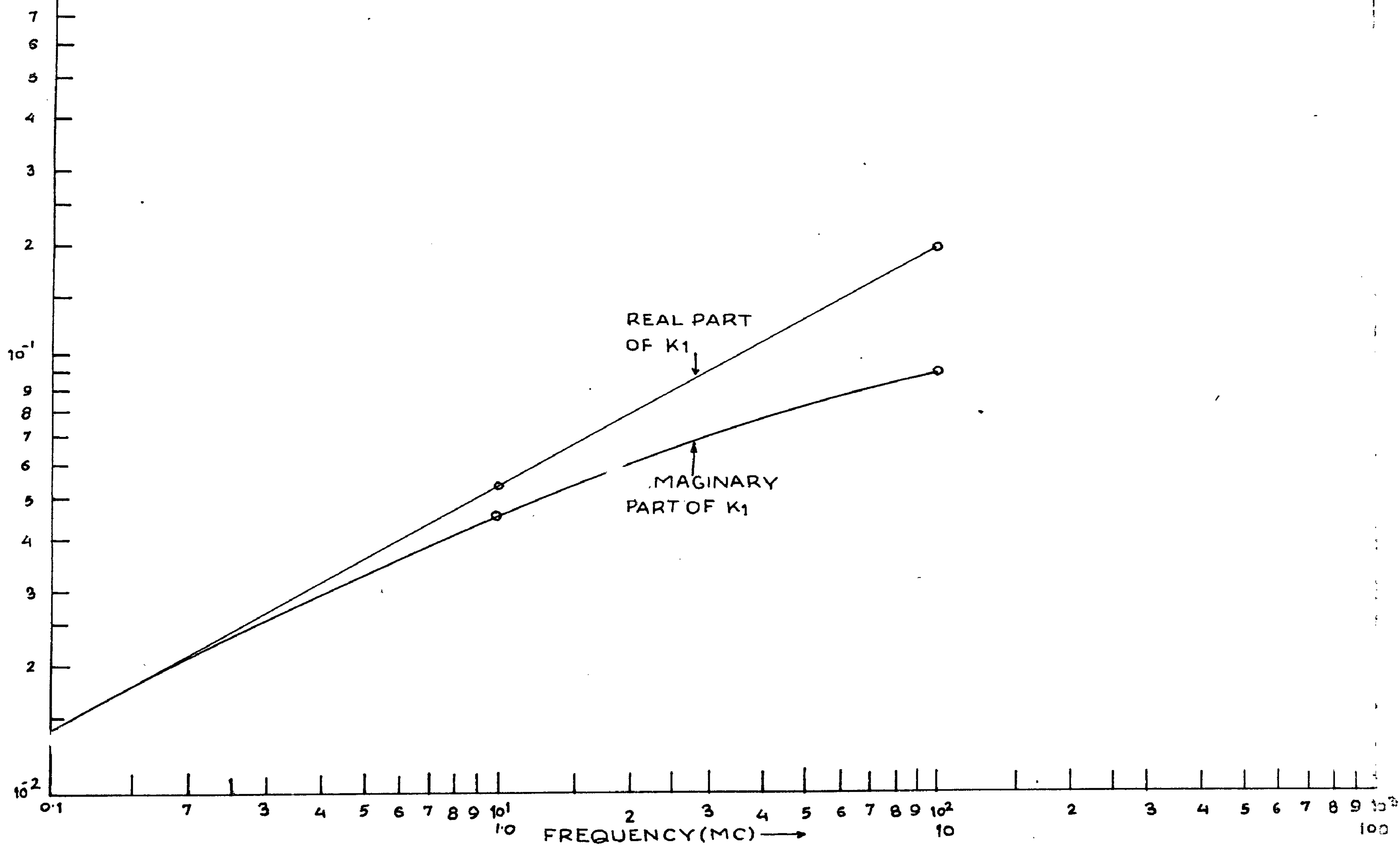
$$k_1^2 = k^2 - 2in_0 G^* k$$

where

$$G^* = \left[1 - ia\left(k - \frac{k_0^2}{k}\right)\right]$$

$$\therefore k_1^2 = k^2 - 2in_0 k \left[1 - ia\left(k - \frac{k_0^2}{k}\right)\right]$$

$$\approx k^2 - 2in_0 k \quad \text{neglecting the second term}$$


REAL PART
OF K1
MAGINARY
PART OF K1
FREQUENCY (MC) →
0·1
1·0
10
100

The real and imaginary parts of k_1 for different frequencies are given below in Table IV.

Number density n_0 = 0.1 scatterers/meter.

TABLE IV

Frequency (Hz)	k	Real part of k_1	Imaginary part of k_1
0.1	0.002	1.48×10^{-2}	1.01×10^{-2}
1.0	0.02	4.7×10^{-2}	4.39×10^{-2}
10.0	0.2	20.6×10^{-2}	10.0×10^{-2}

This shows an increase in both the phase constant and the attenuation constant with frequency as shown in the graph.

5.5. [illegible] PROPAGATION

We assume the trial solution of the form

$$A_1(x') = \sum_l B_{1l}\, e^{ik_{1l}x'} \quad , \quad A_2(x') = \sum_l B_{2l}\, e^{ik_{2l}x'}$$

which represents a number of plane waves propagating with propagation constants given by k_{1l}'s and k_{2l}'s.

Therefore,

$$S(u,u')\langle U^2(u;u')\rangle$$

$$= \begin{cases} \left[\sum_l A_{1l}\, e^{-ik_{1l}u'} + \sum_l A_{2l}\, e^{+ik_{2l}u'}\right] e^{-ik(u-u')} & u \leq u' < \infty \\ \left[\sum_l A_{1l}\, e^{+ik_{1l}u'} + \sum_l A_{2l}\, e^{-ik_{2l}u'}\right] e^{ik(u-u')} & u \geq u' > -\infty \end{cases}$$

Translating the coordinates as before (in Fig. 4) i.e. putting $u' = u_1 + u_2'$ to get the correlated fields as

$$S(u,u')\langle U^2(u;u')\rangle$$

$$= \begin{cases} e^{-iku}\left[\sum_l A_{1l}\, e^{-i(k_{1l}+k)(u_1+u_2')} + \sum_l A_{2l}\, e^{+i(k_{2l}+k)(u_1+u_2')}\right] & u_2' \geq 0 \\ e^{iku}\left[\sum_l A_{1l}\, e^{+i(k_{1l}-k)(u_1+u_2')} + \sum_l A_{2l}\, e^{-i(k_{2l}-k)(u_1+u_2')}\right] & u_2' \leq 0 \end{cases}$$

For u in the neighbourhood of u_1, the integral

$$I = \int_{\substack{|u_1-u'| > 2a \\ u' \geq 0}} S(u,u')\, \langle U^2(u;u')\rangle \, du'$$

$$= \int_{-u_1}^{-2a} e^{iku}\left[\sum_l A_{1l}\, e^{+i(k_{1l}-k)(u_1+u_2')} + \sum_l A_{2l}\, e^{-i(k_{2l}-k)(u_1+u_2')}\right] du_2'$$

$$+ \int_{2a}^{\infty} e^{-iku}\left[\sum_l A_{1l}\, e^{-i(k_{1l}+k)(u_1+u_2')} + \sum_l A_{2l}\, e^{+i(k_{2l}+k)(u_1+u_2')}\right] du_2'$$

$$= e^{ikx} \sum_{l} \Big[- \frac{A_{1l} C^{+}}{i(k_{1l}-k)} - \frac{A_{2l} C^{-}}{i(k_{2l}-k)} + \frac{A_{1l} C^{+} e^{i(k_{1l}-k)(x_1-2a)}}{i(k_{1l}-k)}$$

$$+ \frac{A_{2l} C^{-} e^{i(k_{2l}-k)(x_1-2a)}}{i(k_{2l}-k)} \Big] + e^{-ikx} \sum_{l} \Big[- \frac{A_{1l} C^{-} e^{i(k_{1l}+k)(x_1+2a)}}{i(k_{1l}+k)}$$

$$- \frac{A_{2l} C^{+} e^{i(k_{2l}+k)(x_1+2a)}}{i(k_{2l}+k)} \Big] \qquad ..(5.11)$$

Substituting back in equation (1) we get

$$\sum_{l} A_{1l} e^{ik_{1l}x_1} e^{ik(x-x_1)} + \sum_{l} A_{2l} e^{ik_{2l}x_1} e^{-ik(x-x_1)}$$

$$= e^{ikx} + A_0 I \qquad ...(5.12)$$

Equating coefficients of e^{ikx} on both sides of (5.12) we get

$$\sum_{l} A_{1l} e^{ik_{1l} x_1} e^{-ikx_1} = 1 + A_0 \sum_{l} \Big[- \frac{A_{1l} C^{+}}{i(k_{1l}-k)}$$

$$- \frac{A_{2l} C^{-}}{i(k_{2l}-k)} + \frac{A_{1l} C^{+} e^{i(k_{1l}-k)(x_1-2a)}}{i(k_{1l}-k)}$$

$$+ \frac{A_{2l} C^{-} e^{i(k_{2l}-k)(x_1-2a)}}{i(k_{2l}-k)} \Big]$$

Now equating the coefficients of e^0 on both sides, gives

$$1 = a_0 \sum_l \left[\frac{B_{12}\, C^+}{i(k_{12}-k)} + \frac{B_{22}\, C^-}{i(k_{22}-k)} \right] \qquad (5.13)$$

This verifies the extinction theorem as seen before.

Also equating the coefficients of e^{-ikx_1} on both sides gives

$$\sum_l B_{12}\, e^{ik_{12}x_1} = a_0 \sum_l \left[\frac{B_{12}C^+ e^{ik_{12}(x_1-2a)}}{i(k_{12}-k)} + \frac{B_{22}C^- e^{ik_{22}(x_1-2a)}}{i(k_{22}-k)} \right] \cdot e^{2ika}$$

This equation shows that $k_{12} = k_{22} = k_l$

Comparing this equation with equation (5.13), therefore gives

$$\sum_l B_{12}\, e^{ik_l 2a} = e^{2ika} \qquad ..(5.14)$$

Similarly equating coefficients of e^{-ikx} on both sides of equation (5.11) gives,

$$\sum_l B_{22}\, e^{ik_l x_1}\, e^{ikx_1} = a_0 \sum_l \left[\frac{B_{12}\, C^- + B_{22}\, C^+}{i(k_l+k)}\, e^{i(k_l+k)(x_1+2a)} \right]$$

Equating coefficients of e^{ik_1x} on both sides of this equation, gives

$$\sum_l B_{2l}\, e^{ik_1 x} = -a_0 \sum_l \left[\frac{B_{1l}C^- + B_{2l}C^+}{i(k_1+k)}\, e^{ik_1(x+2a)} \cdot e^{i2ka} \right]$$

$$\text{or } \sum_l B_{2l} = -a_0 \sum_l \left[\frac{B_{1l}C^- + B_{2l}C^+}{i(k_1+k)}\, e^{i(k_1+k)2a} \right]$$

$$\therefore\; B_{2l} = -\frac{a_0}{i(k_1+k)}\left(B_{1l}C^- + B_{2l}C^+\right) e^{i(k_1+k)2a} \qquad \ldots(5.15)$$

Solving for B_{2l} explicitly from equation (5.15) we get

$$B_{2l} = -\frac{a_0 C^- e^{i4ka}}{i(k_1+k) + a_0 C^+ e^{i(k_1+k)2a}}$$

Substituting (5.15) and (5.14) into equation (5.13) we get the Dispersion relation, as

$$k_1^2 + 2a_0C^+(k_1 \sin 2k_1a - ik \cos 2k_1a)e^{i2ka} - k^2 + \left[(a_0C^+)^2 - (a_0C^-)^2\right] e^{i4ka} = 0 \qquad (5.16)$$

The dispersion relation remains unchanged.

The Total Average Field:

The total average field for $x > 0$ is given by

$$\langle E(z)\rangle = (1-2an_0)\, E^I(z) + n_0 \int_{\substack{|z-z'|>a \\ z'>0}} S(z,z') \langle E^E(z|z')\rangle \, dz'$$

$$+\, n_0 \int_{|z-z'|<a} S^E(z,z') \langle E^E(z|z')\rangle \, dz'$$

Here the integral $\int_{\substack{|z-z'|>a \\ z'>0}} dz'$ is the same as I #1

δ is replaced by a and a_1 is replaced by a and

$n_{12} = n_{22} = n_2$ in equation (5.11)

Let the internal scattered fields be given by

$$S^E(z,z') \langle E^E(z|z')\rangle = \sum_{l} \left[(a_{12}R^{+} + a_{22}R^{-})\, e^{ik_0(z-z')} + (a_{12}R^{-} + a_{22}R^{+})\, e^{-ik_0(z-z')} \right] e^{ik_2 z'}$$

The integral $\int_{|z-z'|<a} dz'$ can be integrated. Also let

$z'-z = z_1'$ (Fig. 5). Therefore,

$$\int_{|z-z'|<a} S^E(z,z') \langle E^E(z|z')\rangle = \int_{-a}^{a} \sum_{l} \left[(a_{12}R^{+} + a_{22}R^{-})\, e^{-ik_0 z_1'} + (a_{12}R^{-} + a_{22}R^{+})\, e^{ik_0 z_1'} \right] e^{ik_2(z+z_1')} \, dz_1'$$

$$= \sum_{l} \left[\frac{2(D_{1l}X^{+} + D_{2l}X^{-})}{(k_l - k_0)} \sin (k_l - k_0)h + \frac{2(D_{1l}X^{-} + D_{2l}X^{+})}{(k_l + k_0)} \sin (k_l + k_0)h \right] e^{ik_l z}$$

Putting back to the total average field equation and making use of the extinction theorem and relations (5.13) & (5.14) and (5.15), we get

$$\langle E(z) \rangle = \left\{ (1 - 2i\, n_0) \sum_{l} \left[D_{1l}\, e^{i(k_l - k)h} + D_{2l}\, e^{-i(k_l + k)h} \right] + i n_0 \sum_{l} \left[\frac{(D_{1l}X^{+} + D_{2l}X^{-})}{(k_l - k_0)} \sin (k_l - k_0)h + \frac{(D_{1l}X^{-} + D_{2l}X^{+})}{(k_l + k_0)} \sin (k_l + k_0)h \right] \right\} e^{ik_l z}$$

$$= \sum_{l} \text{Const.}\ e^{ik_l z}$$

Therefore, the total average field propagates as a collection of plane waves whose wave numbers are determined by the dispersion relation (5.16).

In the following a few special cases will be considered.

1. Thin Slab Approximation: The dispersion relation reduces to

$$k_l^2 = (k + 2n_0 f^{+})^2 + (n_0 f^{-})^2$$

Where C^+ and C^- are equal to 1 and 0 respectively

as $a \to 0$

Hence $k_2 = (k - 2n_0)$

Therefore, equation (5.14) becomes

$$\sum_l a_{l2} \; e^{i(k-2n_0)2a} = e^{i2ka}$$

$$\sum_l a_{l2} = 1$$

$$\text{or} \quad a_1 = 1 \qquad ..(5.17)$$

The equation (5.16) becomes

$$a_2 = - \frac{n_0 C^-}{i(k'-k) + n_0 C^+} \qquad ..(5.18)$$

and the dispersion relation becomes

$$k'^2 = (k - n_0 C^+)^2 + (n_0 C^-)^2 \qquad ..(5.19)$$

This shows that in this case there is only one pole.

If we expand to the second order in $(k_2 a)$ then equation (5.16) gives

$$k_2^2 + 2n_0 C^+ k_2 (e^{-i2k_2 a} - e^{i2k_2 a}) e^{i2ka} + 2n_0 C^+ k (e^{i2k_2 a} + e^{-i2k_2 a}) e^{i2ka}$$

$$= k^2 - (n_0 C^+)^2 e^{i4ka} - (n_0 C^-)^2 e^{i4ka}$$

$$\therefore \; k'^2 = \frac{(k - 2n_0 C^+ e^{i2ka})^2 + (n_0 C^- e^{i2ka})^2}{1 + 4 n_0 C^+ a (1-2ka) e^{i2ka}}$$

2. [illegible] Concentration: In this case n_0 is the expansion parameter. Expanding to the second order in n_0 as before (9.9) we get

$$\omega^2 = \text{[illegible]} \qquad ..(9.10)$$

Again we obtain just one mode.

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6. DISCUSSION

The problem of wave propagation in a random medium is a very important problem and has received much attention in the literature. In this paper we have established a method useful for problems involving multiple scattering of waves. To study the problem theoretically one requires a mathematical model describing the properties of the random medium. One model is that in which some properties of the medium, such as density, refractive index etc., are the random functions of position. Another model considers the randomness as being due to the presence of distinct scatterers which have different electromagnetic properties from the medium in which they are placed. We have considered the second type of model. In this case the problem is formulated in terms of multiply scattered fields which satisfy a wave equation and boundary conditions on the surface of the scatterers.

We have next confined the treatment to the case of slabs. Previous work on multiple scattering is restricted to the three dimensional problems considering spheres etc., as the scattering objects, but we have considered slabs of finite width, and thus the problem has been reduced to the case of one-dimension only. Our results, in case of the

— Born approximation, tally with those obtained by Waterman and Truell.

While considering multiple scattering of waves, we have obtained the average total field, for both the cases, when the point of observation is inside or outside of the scattering medium. It is found that the right half space is equivalent to a homogeneous medium of propagation constant K_1. A dispersion relation has also been obtained for the scattering medium, giving all modes of propagation in the equivalent medium.

In closing, extensions are listed that are in need of further investigation.

1. Removing the restriction of identical scatterers. This has been considered by Foldy and Lax.

2. The theory needs to be generalized to oblique incidence. It appears that the integrals involved can be solved along the same lines of this treatment.

3. The full significance of the theory can be appreciated only when it is applied to get numerical results for specific cases.

4. Finally, the computation of power and energy appears to be straightforward expansion of this work.

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