UNIVERSITY OF ROORKEE,
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Certified that the attached dissertation on "HIGHER ORDER NONLINEAR SYSTEMS"

was submitted by

V. L. ABHYANKER

and accepted for the award of Degree of Master of Engineering in

APPLIED ELCETRONICS AND SERVOMECHANISM

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[signature]

(S.S.SRIVASTAVA)
Assistant Registrar (Exam.)
S.S. 2.XII.1966

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Higher Order Nonlinear Systems

A Dissertation
submitted in partial fulfilment
of the requirements for the Degree
of

MASTER OF ENGINEERING

in

ELECTRONICS AND COMMUNICATION ENGINEERING
(APPLIED ELECTRONICS AND SERVOMECHANISM)

By
V L. ABHYANKER

DEPARTMENT OF ELECTRONICS AND
COMMUNICATION ENGINEERING
UNIVERSITY OF ROORKEE
ROORKEE

July, 1966

C E R T I F I C A T E

Certified that the dissertation entitled 'HIGHER ORDER NONLINEAR SYSTEMS', which is being submitted by Shri Vasant Laxman Abhyanker in partial fulfilment for the award of the Degree of Master of Engineering in APPLIED ELECTRONICS AND SERVOMECHANISM of the University of Roorkee is a record of student's own work carried out by him under my supervision and guidance. The matter embodied in this dissertation has not been submitted for the award of any other Degree or Diploma.

This is to further certify that he has worked for a period of more than six months from 1.1.1966 to 14.7.1966 for preparing this dissertation for Master of Engineering Degree at the University.

Manoharlal

(Manohar Lal)
Reader,
Electronics and Communication
Engineering Department,
University of Roorkee,
Roorkee.

Dated July 14,1966.

C_O_N_T_E_N_T_S

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V.L. Abhyanker

A B S T R A C T

The problem of extending phase-space techniques to higher order nonlinear systems has been investigated in this dissertation. A new technique has been developed which is shown to be applicable to higher order nonlinear systems.

The technique suggested is based on the fact that the higher order derivatives in a nonlinear differential equation are related to the first order derivative through the slope of the phase trajectory (in the x-$\dot{x}$ plane) and its derivatives. In this technique, increment in the slope at the end of each small segment of the trajectory is calculated and this incremental value is added to the previous slope to get a new value as the slope for the next trajectory segment. The technique is discussed in detail and is illustrated with the help of a few examples of linear, nonlinear, autonomous and nonautonomous systems. The use of computer for this technique is also discussed and a Flow chart for use in computer programme is given.

Some new problems such as use of improved techniques for plotting are discussed briefly and it is hoped that further work in this direction will lead to more worthwhile results.

Chapter 1.

INTRODUCTION AND STATEMENT OF THE PROBLEM

INTRODUCTION

1.1 The behaviour of a physical dynamical system can be described by differential equation(s). Hence the analysis and design ofa physical system is concerned directly or indirectly with the solution of the differential equation(s) which characterise the system. If the differential equation describing the system is nonlinear, the system is said to be nonlinear. Most of the systems, though inherently nonlinear, can be approximated by linear models. However, there are situations, when it is essential to study the systems without making any approximations. The degree of accuracy demanded of a modern control system requires a rigorous analysis and design of the systems based on nonlinear theory.

In general the following methods are in use for solving nonlinear problems.

(a) Perturbation methods.

(b) Isocline method and other graphical methods.

(c) Linearization methods.

(d) Phase space method and the use of analogue and digital computers.

The perturbation method was developed by Poincare(12) and Lindsteht (1) primarily for astronomical purposes. Poincare' and Liapounov published their treatise in 1892-1893. Lord Rayleigh's (2) theory of sound was published in 1893-1896. The isocline method was used by Van der Pol (3) to find limit cycle. The other graphical methods have been developed by Lienard (4) and others. The phase space method has been applied to control systems by Ku (5) and others. Whitebeck (6) developed a graphical approach of obtaining both the phase trajectory and the time solution of a nonlinear time varying system fora given set of initial conditions.

A great amount of work has been done in the field of second order nonlinear systems using phase-space method. For a second order autonomous system, method of isoclines (7) is readily applicable and this is because the slope of the phase trajectory in the phase plane is uniquely defined at every point. Deekshatulu (8) has used certain transformations which aid in the plotting of phase trajectories.

The application of the phase trajectory concept to practical problems is a subject which is still in the early stages of its evolution. Only the two dimensional case has been studied extensively. For higher order systems, Ku (9) was the first investigator to analyse nonlinear systems. Deekshatulu (10) suggested the use of R^n-X trajectories to solve higher order nonlinear system equations. He also suggested another method (11) using $\ddot{X} - \dot{X}$ and $\dot{X} - X$ planes.

1.2 STATEMENT OF THE PROBLEM

Practical handling of the problem by phase-space concept for higher order systems requires plotting in a plane which may be called a phase plane. The problem of extending phase-space techniques to higher order nonlinear systems has been investigated in this dissertation.

In particular, a new technique for solving higher order systems (linear, nonlinear, time varying and time invariant) for a given set of initial conditions is developed. It is shown with the help of a few examples (linear, nonlinear, time varying and time invariant) that this technique can be successfully applied to higher order nonlinear systems.

The technique is based on the fact that the higher order derivatives in a nonlinear differential equation are related to the first order derivative through the slope of the phase trajectory and its derivaties. In this technique, increment in the slope at the end of each small segment of the trajectory is calculated and this incremental value is added to the previous slope to get a new value as the slope for the next trajectory segement.

In the second chapter, n^{th} order nonlinear differential equations are taken up and various phase space techniques which are available for solving such equations are described briefly.

In the third chapter, a new technique for solving higher order nonlinear systems is developed. It is shown with the help of a few examples that this technique can be successfully applied to higher order nonlinear systems. Various examples of higher order systems are solved by this method and results are found to be in close agreement with those obtained by other methods.

In the final chapter, the results are summarised and further problems suggested by this investigation are discussed briefly.

Chapter II

HIGHER ORDER NONLINEAR SYSTEMS

2.1 INTRODUCTION

In this chapter a nonlinear higher order system is defined first. It is pointed out further that because of practical considerations and difficulty in visualising the space trajectories, their projections, in general, are studied for higher order systems.

The present techniques are listed and briefly reviewed. Starting with a third order system, generalisation is made for an n-th order system. In particular techniques due to Ku and Deekshatulu are discussed and critically examined.

2.2 PRESENT TECHNIQUES

Let us consider a differential equation

$$\frac{d^n x}{dt^n} = f\left(t, x, \dot{x}, \ddot{x}, \ldots\ldots \frac{d^{n-1}x}{dt^{n-1}}\right) \qquad (2\text{-}1)$$

This is an n^{th} order differential equation and is linear or nonlinear, depending on the nature of the coefficients of $x, \dot{x}, \ddot{x} \ldots\ldots\ldots \frac{d^{n-1}x}{dt^{n-1}}$ i.e. if any of these coefficients are function of depended variable X, the equation is nonlinear otherwise it is linear. If an analytic solution of such an equation exists, the solution of equation (2-1) can be represented by

$$X = f(t) \qquad (2\text{-}2)$$

In general, solution of the form of nonlinear equation (2-2) either cannot be obtained or requires too much of labour. Atleast theoretically for a given set of initial conditions a phase-space curve can be constructed in n - dimensional space. Such curves can be constructed for two and three - dimensional space, but for fourth and other higher order spaces, practical considerations require that the curves be projected onto a plane.

For nonlinear higher order systems, the present techniques available are :-

i) Ku's method (5) (using phase-planes)

ii) Deekshatulu's method (10) of R^n - x trajectories

iii) Deekshatulu's (using phase-planes) (11)

i) Ku's method:

Ku was the first investigator to analyse nonlinear system in the $\ddot{x}$-x and $\ddot{x}$-$\dot{x}$ (also termed as acceleration - position and acceleration velocity planes).

Thus a second order nonlinear system can be analysed on the $\ddot{x}$-x and $\ddot{x}$-$\dot{x}$ planes, while a third order autonomous system will require use of any two of the $\dddot{x}$-$\dot{x}$ $\dddot{x}$-x and $\dot{x}$-x planes. Trajectories on these planes are the projections of the space trajectory. Thus a general n^{th} order nonlinear autonomous system will require (n-1) planes to solve the equation. Ku was the first to analyse a number of third order systems. Such systems include problems in electrical circuits, mechanical systems and in fluid mechanics. Full details of the method can be obtained from (9).

ii) R-x, R^2 -x Planes

Until recently Ku's method was the only graphical method of solving higher order nonlinear systems. Deekshatulu suggested the use of R^n - x planes (R being the radius vector in the n-dimensional space with vectors as x, $\dot{x}$, $\ddot{x}$, $\dddot{x}$, ...$x^{(n-1)}$). These planes are as useful as the phase plane. For certain classes of nonlinear systems, they have proved to be simpler in determining the general behaviour of the system.

Method of Plotting the $R-x, \ldots\ldots R^n-x$ Trajectory:-

Starting with a third order system, techniques have been generalised upto n^{th} order system.

A general third order nonlinear autonomous system may be represented by the differential equation

$$\dddot{x} = f(x, \dot{x}, \ddot{x}) \quad \ldots\ldots\ldots\ldots\ldots\ldots \ (2\text{-}3)$$

R, the radius vector in the three dimensional space is given by

$$R = \sqrt{x + \dot{x}^2 + \ddot{x}^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots \ (2\text{-}4)$$

$$N_x = dR/dx = (x\dot{x} + \dot{x}\ddot{x} + \ddot{x}\dddot{x})/R\,\dot{x} = f_1(x,\dot{x},\ddot{x})$$

$$= F_1(x,\dot{x},\sqrt{R^2 - \dot{x}^2 - x^2}) \quad \ldots\ldots\ldots\ldots\ldots \ (2\text{-}5)$$

$$N_{\dot{x}} = dR/d\dot{x} = (x\dot{x} + \dot{x}\ddot{x} + \ddot{x}\dddot{x})/R\,\ddot{x} = f_2(x,\dot{x},\ddot{x})$$

$$= F_2(x,\dot{x},\sqrt{R^2 - x^2 - \dot{x}^2}) \quad \ldots\ldots\ldots\ldots\ldots \ (2\text{-}6)$$

In general,

$$dR^n/dx = n\,R^{n-1}\,(x\dot{x} + \dot{x}\ddot{x} + \ddot{x}\dddot{x})/R\,\dot{x} \quad .(2\text{-}7)$$

$$dR^n/d\dot{x} = n\,R^{n-1}\,(x\dot{x} + \dot{x}\ddot{x} + \ddot{x}\dddot{x})/R\,\ddot{x} \quad .(2\text{-}8)$$

The initial points in the two planes $R-x$ and $R-\dot{x}$ are (R_0, x_0) and $(R_0, \dot{x}_0)$ respectively. The initial slopes corresponding to these points in the two planes are obtained from equations (2-5) and (2-6). For a small increment ΔR, new values of x and $\dot{x}$ are found out.

Thus,

$$R_1 = R_0 + \Delta R \quad \ldots\ldots\ldots\ldots\ldots \ (2\text{-}9)$$

$$x_1 = x_0 + \Delta R/N_{x_0} \quad \ldots\ldots\ldots\ldots\ldots \ (2\text{-}10)$$

$$\dot{x}_1 = \dot{x}_0 + \Delta R/N_{\dot{x}_0} \quad \ldots\ldots\ldots\ldots\ldots \ (2\text{-}11)$$

The process is repeated for successive points.

In general, for an n^{th} order system, one gets the equations:

$$dR/dx = \frac{x\dot{x} + \dot{x}\ddot{x} + \ddot{x}\dddot{x} + \ldots\ldots \overset{(n-1)}{x}\ \overset{(n)}{x}}{R\dot{x}} \quad ..(2\text{-}12)$$

$$dR/d\dot{x} = \frac{x\dot{x} + \dot{x}\ddot{x} + \ldots\ldots\ldots\ldots \overset{(n-1)}{x}\ \overset{(n)}{x}}{R\ddot{x}} \quad ..(2\text{-}13)$$

$$d\,R/d\,\dot{x}^{(n-2)} = \frac{x\dot{x} + \dot{x}\ddot{x} + \ldots\ldots\ \dot{x}^{(n-1)}\,\dot{x}^{(n)}}{R\,\dot{x}^{(n-1)}} \ldots\ldots \quad (2\text{-}14)$$

In each of the above equations,

$$R = \sqrt{x^2 + \dot{x}^2 + \ldots\ldots\ldots\ (\dot{x}^{(n-1)})^2} \ldots\ldots\ldots\ldots\ldots\ldots \quad (2\text{-}15)$$

and $\dot{x}^{(n)}$ is to be determined from the given differential equation.

The direction of motion is determined by the fact that a positive acceleration causes and increase in velocity and displacement should increase as long as the velocity ~~and~~ is positive and vice versa. An additional plane d R/dt facilitates the solution of time varying systems.

The accuracy of this method depends upon the length of the interval ΔR. This interval need not be same for all steps. Smaller length of ΔR when d R/dx or d R/dx is small gives better results.

Evaluation of time from an R-x trajectory

The method would be to approximate an R-x trajectory by straight lines. The slope of each such straight line is known already since it has been calculated while plotting the trajectory.

For a portion of the R-x trajectory in the interval $x = x_1$ to $x = x_2$, let the equation be given by

$$R = K\,x + c \qquad \ldots \qquad \ldots \qquad (2\text{-}16)$$

K being the slope and c the R-axis intercept of the straight line. If t is the time taken during this interval, the expression would be,

$$t_{12} = \int_{x_1}^{x_2} \frac{dx}{\dot{x}}$$

$$= \int_{x_1}^{x_2} \frac{dx}{\sqrt{(kx+c)^2 - x^2}}$$

$$= \int_{x_1}^{x_2} dx/\sqrt{ax^2+bx+d} \qquad \ldots \qquad \ldots \qquad (2\text{-}17)$$

where $a = K^2 - 1$, $b = 2\,K\,c$, $d = c^2$

Integrating, we get

$$t_{12} = \frac{1}{\sqrt{a}} \log_e \frac{2ax_2 + b + 2\sqrt{a(R_2^2 - x_2^2)}}{2ax_1 + b + 2\sqrt{a(R_1^2 - x_1^2)}}, \text{ if } K > 1 \ldots\ldots (2\text{-}18$$

and,

$$= \frac{1}{\sqrt{-a}} \quad \sin^{-1} \frac{-2ax - b}{\sqrt{b^2 - 4ad}} \qquad \ldots\ldots\ldots\ldots (2\text{-}18$$

if $K < 1$

iii) <u>Deekshatulu's method (Using phase planes)</u>

The method is numerical and is developed in the following manner.

$$\dddot{x} = f\,(\,x,\,\dot{x},\,\ddot{x},\,t\,) \qquad \ldots \qquad \ldots \qquad (2\text{-}19)$$

Equation (2-19) in its general form is a nonlinear time varying equation of third order. Here we define K_1 and K_2 as the slopes of the trajectories on the two planes, $\dot{x} - x$ and $\ddot{x} - x$ planes, such that,

$$K_1 = \ddot{x}/\dot{x} \qquad \ldots \qquad \ldots \qquad \ldots \qquad (2\text{-}20)$$

$$K_2 = \dddot{x}/\dot{x} \qquad \ldots \qquad \ldots \qquad \ldots \qquad (2\text{-}21)$$

With initial conditions $x_o, \dot{x}_o$, and $\ddot{x}_o$, the initial slopes are

$$K_{2o} = \dddot{x}_o/\dot{x}_o, \text{ where } \dddot{x}_o = f(\ddot{x}_o, \dot{x}_o, x_o, t_o) \quad \ldots \quad (2\text{-}22)$$

$$K_{1o} = \ddot{x}_o/\dot{x}_o \quad \ldots \quad (2\text{-}23)$$

In case of previous methods, a third plane (x-t) is necessary to obtain x_1, $\dot{x}_1$, $\ddot{x}_1$ for the preceeding point. In this case, Δt is fixed and may be taken as equal to 0.01, 0.02 etc.

The successive points are obtained using the relations

$$\dot{x}_{n+1} = \dot{x}_n \exp K_{1n} \Delta t \quad \ldots \quad (2\text{-}24)$$

$$x_{n+1} = x_n + (\dot{x}_{n+1} - \dot{x}_n) K_{1n} \quad \ldots \quad (2\text{-}25)$$

$$\ddot{x}_{n+1} = \ddot{x}_n + (x_{n+1} - x_n) K_{2n} \quad \ldots \quad (2\text{-}26)$$

$$\dddot{x}_{n+1} = f(x_n, \dot{x}_n, \ddot{x}_n, t_n)$$

$$t_n = n \Delta t \quad \ldots \quad (2\text{-}27)$$

The process is repeated.

In general, for an m^{th} - order Servo system,

$$d^m x/dt^m = \dot{x}^{(m)} = f(x, \dot{x}, \ddot{x}, \ldots \dot{x}^{(m-1)}, t) \ldots (2\text{-}28)$$

with the given set of initial conditions $x_o, \dot{x}_o \ldots, \dddot{x}_o \ldots$ $\ldots \dot{x}_o^{(m-1)}$, one can plot the x-t curve using the relations

$$K_{(m-1)n} = \dot{x}_n^{(m)} / \dot{x}_n$$

$$K_{(m-2)n} = \dot{x}_n^{(m-1)} / \dot{x}_n$$

$$\ldots\ldots\ldots\ldots\ldots\ldots$$

$$K_{2n} = \dddot{x}_n/\dot{x}_n \text{ and } K_{1n} = \ddot{x}_n/\dot{x}_n \quad \ldots \quad (2\text{-}29)$$

Also,

$$\dot{x}_{n+1} = \dot{x}_n \exp K_{1n} \Delta t$$

$$t_n = n \Delta t$$

$$x_{n+1} = x_n + (\dot{x}_{n+1} - \dot{x}_n) K_{1n}$$

$$\dot{x}^{(m-1)}_{n+1} = \dot{x}^{(m-1)}_n + K_{(m-1)n} (x_{n+1} - x_n)$$

$$\dot{x}^{(m)}_n = f\left(x_n, \dot{x}_n, \ldots\ldots\ \dot{x}^{(m-1)}_n, t_n\right) \quad \ldots\ldots \quad (2\text{-}30)$$

Thus the present techniques for higher order nonlinear systems were reviewed in this chapter. Use of a particular plane, say R^2-x or R^3-x may prove superior for a restricted class of nonlinear systems.

Equations (2-5) and (2-6) show that for a third order nonlinear system, with the initial conditions as $\dot{x}$ = o and $\ddot{x}$ = o, the quantities n_x and $n_{\dot{x}}$ both become o/o, a quantity which is indeterminate. No mention was made about how to proceed in such cases.

Deekshatulu compared his method (11) with that of Ku (5) and error analysis showed favourable results in case of Deekshatulu's method of solving higher order nonlinear systems.

As in case of R-x plane method, a problem arises while calculating the value of K_{10} in equation (2-23). Again, K_{10} becomes 0/0 which is indeterminate. The method to proceed under such circumstances was not explained.

Chapter III

A NEW TECHNIQUE FOR HIGHER ORDER NONLINEAR SYSTEMS

3.1 A NEW TECHNIQUE FOR SOLVING HIGHER ORDER SYSTEMS

It has been seen already that a few methods are available at present for solving higher order systems. Two of them, one by Ku and the other by Deekshatulu make use of phase planes- their number equal to the order of the system if the system is nonauthonomous and one less than the order of the differential equation if the system is autonomous. The other method by Deekshatulu makes use of some new planes R^n-x and the stability conditions in the phase plane have been transformed to the stability conditions in these new planes.

A new technique developed under the present investigation first transforms all the derivatives $\ddot{x}, \dddot{x} \ldots . x^{(n)}$ into $\dot{x}, N, \dot{N}, \ldots\ldots N^{(n-2)}$ (where N is the slope of the trajectory in the x-$\dot{x}$ plane) and a systematic approach is suggested which leads to the solution of third and higher order system equations.

First the technique is shown to be applicable to third order nonlinear systems. It is generalised further upto n^{th} order.

THIRD ORDER NONLINEAR SYSTEMS:

Let the general form of time varying nonlinear equation of third order be

$$\dddot{x} = f(\ddot{x}, \dot{x}, x, t) \quad \ldots \quad \ldots \quad (3\text{-}1)$$

If N is the slope of the phase trajectory in the $\dot{x}$-x plane, we have,

$$\ddot{x} = d^2x/dt^2 = d\dot{x}/dt = (d\dot{x}/dx).(dx/dt) = N\dot{x} \quad \ldots \quad (3\text{-}2)$$

$$\dddot{x} = d\ddot{x}/dt \quad \dddot{x} = d/dt(N\dot{x})$$

$$\dddot{x} = \dot{N}\dot{x} + N\ddot{x}$$

$$= \dot{N}\dot{x} + N^2\dot{x} \qquad \dots \qquad \dots \qquad (3\text{-}3)$$

Values of $\dddot{x}$, $\ddddot{x}$ etc. have been found out in terms of N, $\dot{x}$ and derivatives of N are presented in Table - 1.

Substitution of equations (3-2) and (3-3) in equation (3-1) will give,

$$\dot{N}\dot{x} + N^2\dot{x} = f\,(N\dot{x},\ \dot{x},\ x,\ t)$$

$$\dot{N} = 1/\dot{x}\left[-N^2 x + f\,(N\dot{x},\ \dot{x},\ x,\ t)\right]$$

$$= f\,(N,\ \dot{x},\ x,\ t) \qquad \dots \qquad \dots \qquad (3\text{-}4)$$

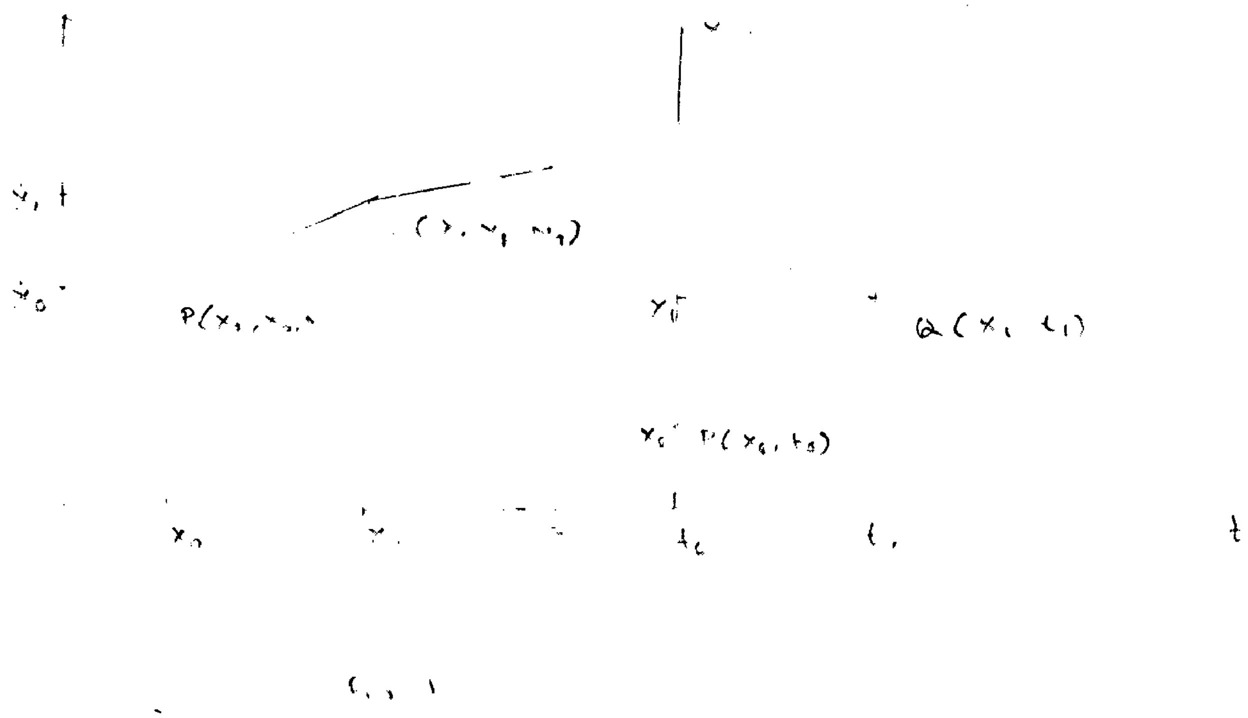
Table - 1

$x^{(n)}$	$x^{(n)} = f\,(N,\ \dot{N},\ \ddot{N},\ \dots\ N^{(n-2)},\ \dot{x})$
$\ddot{x}$	$N\dot{x}$
$\dddot{x}$	$\dot{N}\dot{x} + N^2\dot{x}$
$x^{(4)}$	$\ddot{N}\dot{x} + (N^3 + 3N\dot{N})\dot{x}$
$x^{(5)}$	$\dddot{N}\dot{x} + (N^4 + 6N^2\dot{N} + 3\dot{N}^2 + 4N\ddot{N})\dot{x}$
$x^{(6)}$	$\ddddot{N}\dot{x} + (N^5 + 10N^3\dot{N} + 15N\dot{N}^2 + 10\dot{N}\ddot{N} + 10N^2\ddot{N} + 5N\dddot{N})x$
$x^{(m)}$	$N^{(m-2)}\dot{x} + f\,(N,\ \dot{N},\ \ddot{N},\ \dots N^{(m-3)},\ \dot{x})$

Equation (3-1) has to be solved for a certain set of initial conditions. Let the initial conditions be

$$\ddot{x} = \ddot{x}_0,\ \dot{x} = \dot{x}_0,\ x = x_0 \text{ at } t = t_0 \qquad \dots \qquad (3\text{-}5)$$

The procedure for plotting the phase trajectory side by side with the solution in the x - t plane is as follows.



Let P be the initial point with coordinates $(x_0, \dot{x}_0)$ and (x_0, t_0) in the two planes x-$\dot{x}$ and x-t respectively. The initial slope of the trajectory N_0 in the x-x plane will be given by

$$N_0 = \ddot{x}_0 / \dot{x}_0 \qquad \ldots \qquad \ldots \qquad (3\text{-}6)$$

A straight line segment with this slope is drawn corresponding to a known interval Δx along the x-axis in the phase plane. A point is obtained with x_1 and $\dot{x}_1$ as its coordinates along x and $\dot{x}$ axis respectively.

$$x_1 = x_0 + \Delta x$$
$$\dot{x}_1 = \dot{x}_0 + N_0 \Delta x \qquad \ldots \qquad \ldots \qquad (3\text{-}7)$$

The interval of time correspinding to Δx will be given by

$$\Delta t_0 = \Delta x / \dot{x}_{av} \quad = \Delta x / (\dot{x}_0 + \dot{x}_1)/2 \ \ldots \qquad (3\text{-}8)$$

To get slope N, at the new point , use is made of equation (3-4).

$$\dot{N} = \Delta N/\Delta t = f\ (N,\ \dot{x},\ x,\ t)$$

$$\dot{N}_0 = \Delta N_0/\Delta t_0 = f\ (N_0,\ \dot{x}_0,\ x_0,\ t_0)$$

$$N_0 = f\ (N_0,\ \dot{x}_0,\ x_0,\ t_0)\ \Delta t$$

ΔN_0 is the increment in N_0 when the point P travels to Q.

Hence, $N_1 = N_0 + \Delta N_0$ ... ... (3-9)

Thus, all the necessary information at point Q has been obtained which is required to find out next point following the same process. It should be noted here that accuracy of the result will depend on how small the interval Δx is taken.

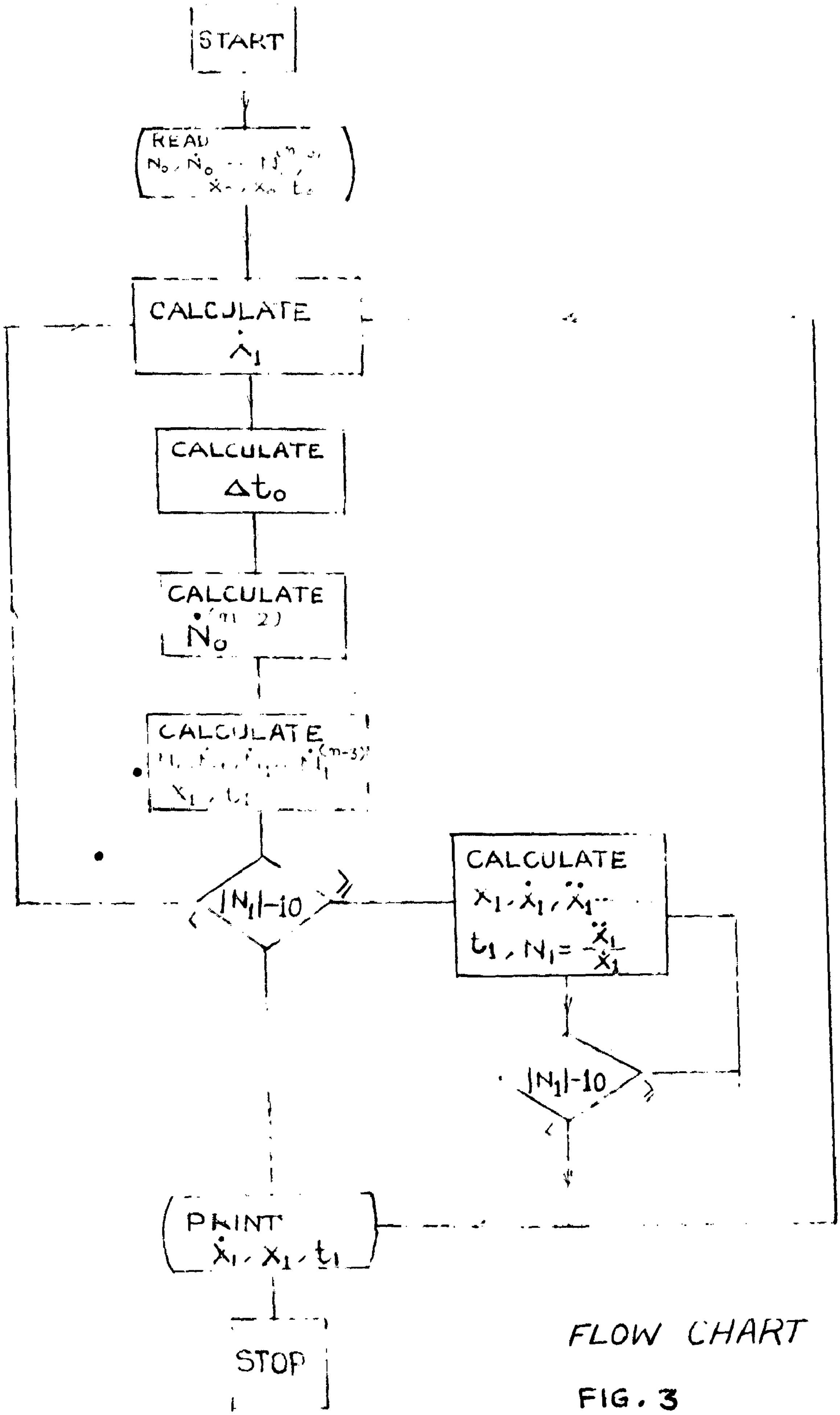
GENERAL N-ORDER SYSTEMS.

The method can easily be extended to fourth, fifth, and in general n^{th} order. In each case, the derivative terms in x are to be replaced by the first derivative of x, the slope N and the derivatives of N with respect to time. Table 1 shows these transformations. Let equation (3-10) represent a general form of n^{th} order differential equation.

$$\dot{x}^{(n)} = f\ (\dot{x}(n-1),\ \dot{x}^{(n-2)},\ \ldots\ \ddot{x},\ \ddot{x},\ \dot{x},\ x,\ t)\ ..(3\text{-}10)$$

Use of Table 1 transforms equation (3-10) into equation (3-11)

$$\dot{N}^{(n-2)} = f\ (\dot{N}^{(n-3)}, \dot{N}^{(n-4)}\ \ldots .\ \dot{N},\ N,\ \dot{x},\ x,\ t)\ \ldots(3\text{-}11)$$



FLOW CHART

FIG. 3

Let the initial conditions be

$t = t_0,\ \dot{x} = \dot{x}_0,\ \ddot{x} = \ddot{x}_0,\ x = x_0,\ \ldots\ x^{(n-1)}$

Use of table 1 gives

$t = t_0,\ N = N_0,\ \dot{N} = \dot{N}_0,\ \ldots\ \dot{N}^{(n-3)} = \dot{N}_0^{(n-3)}$

The successive operations to be performed to obtain co-ordinates of the ~~preceding~~ succeeding points in the phase plane $(\dot{x} - x)$ side by side with the solution plane $(x - t)$, are as given below.

$$\dot{x}_1 = x_0 + N_0 \Delta x \quad \ldots \quad \ldots \quad (3\text{-}12)$$

$$\Delta t_0 = 2 \Delta x / (\dot{x}_0 + \dot{x}_1) \quad \ldots \quad \ldots \quad (3\text{-}13)$$

$$N_0^{(n-2)} = F\,(N_0^{(n-3)},\ N_0^{(n-2)},\ \ldots\ldots\ N_0,\ N_0,\ N_0,\ x_0,\ x_0,\ t_0)/x \quad (3\text{-}14)$$

$$N_1 = N_0 + \Delta t_0\, \dot{N}_0 \quad \ldots \quad \ldots \quad (3\text{-}15)$$

$$\dot{N}_1 = \dot{N}_0 + \Delta t_0\, \ddot{N}_0 \quad \ldots \quad \ldots \quad (3\text{-}16)$$

$$\ddot{N}_1 = \ddot{N}_0 + \Delta t_0\, \dddot{N}_0 \quad \ldots \quad \ldots \quad (3\text{-}17)$$

$$\ldots \quad \ldots \quad \ldots$$

$$\ldots \quad \ldots \quad \ldots$$

$$\dot{N}_1^{(n-4)} = \dot{N}_0^{(n-4)} + \Delta t_0 \dot{N}_0^{(n-3)} \quad \ldots \quad (3\text{-}18)$$

$$\dot{N}_1^{(n-3)} = \dot{N}_0^{(n-3)} + \Delta t_0 \dot{N}_0^{(n-2)} \quad \ldots \quad (3\text{-}19)$$

$$x_1 = x_0 + \Delta x \quad \ldots \quad \ldots \quad (3\text{-}20)$$

$$t_1 = t_0 + \Delta t \quad \ldots \quad \ldots \quad (3\text{-}21)$$

Flow chart of these operations is given in Figure .

PLOTTING OF THE TRAJECTORY AT AND NEAR $\dot{x} = 0$

By definition

$N = \ddot{x}/\dot{x}$

As $\dot{x} \to 0$, $|N| \to \infty$ Also the quantity $\dot{N}_0^{(n-2)}$ as in equation (3-14) will tend to infinity and will make the other derivative terms tend to infinity. Hence, the procedure followed uptill now will not be applicable as $|N|$ becomes large or as x gets smaller values. Hence a practical limit $|N| = 10$ can be kept upto which this method can be applied. As shown in figure 4 , let the trajectory reach the point P where $|N|$ has become approximately equal to 10. At this point all the quantities x, $\dot{x}$, N, $\dot{N}$, $\ddot{N}$, etc. are known. Again using table1, $\ddot{x}$, $\dddot{x}$ etc. are calculated.

These quantities will serve as the initial conditions for the plotting of past of the trajectory P-R. At R again $|N|$ becomes less than 10 and the method under consideration can successfully applied beyond this point. For the past P-R, any other method can be used to plot this section. One would be, (For assumed intervals of Δt)

$$x_1 = x_0 + \dot{x}_0 \Delta t_0$$

$$\dot{x}_1 = \dot{x}_0 + \ddot{x}_0 \Delta t_0$$

$$\ddot{x}_1 = \ddot{x}_0 + \dddot{x}_0 \Delta t_0$$

&c ....

$$t_1 = t_0 + \Delta t_0$$

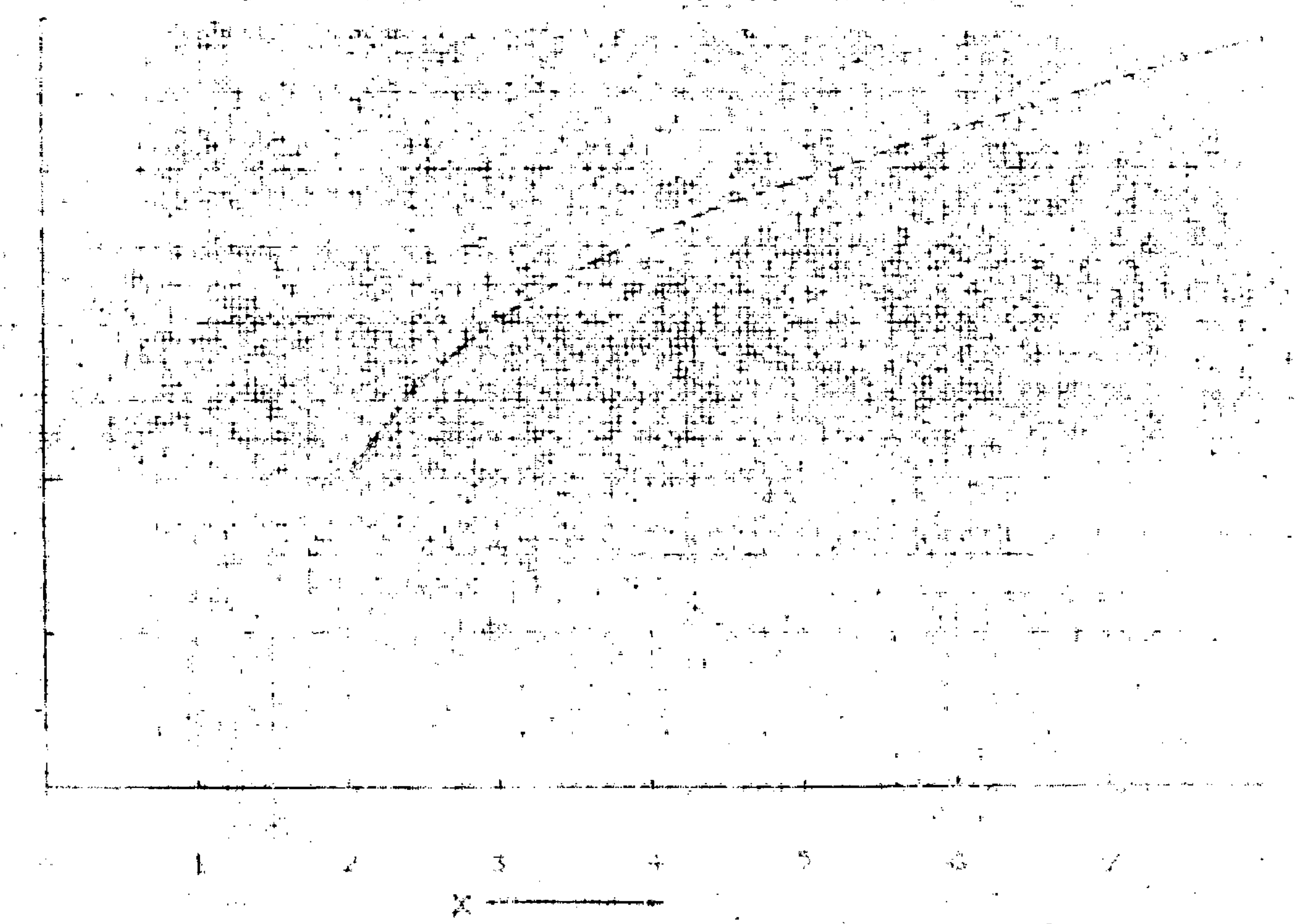
The last derivative $\dot{x}_1^{(n)}$ CAN BE FOUND OUT MAKING use of equation (3-22)

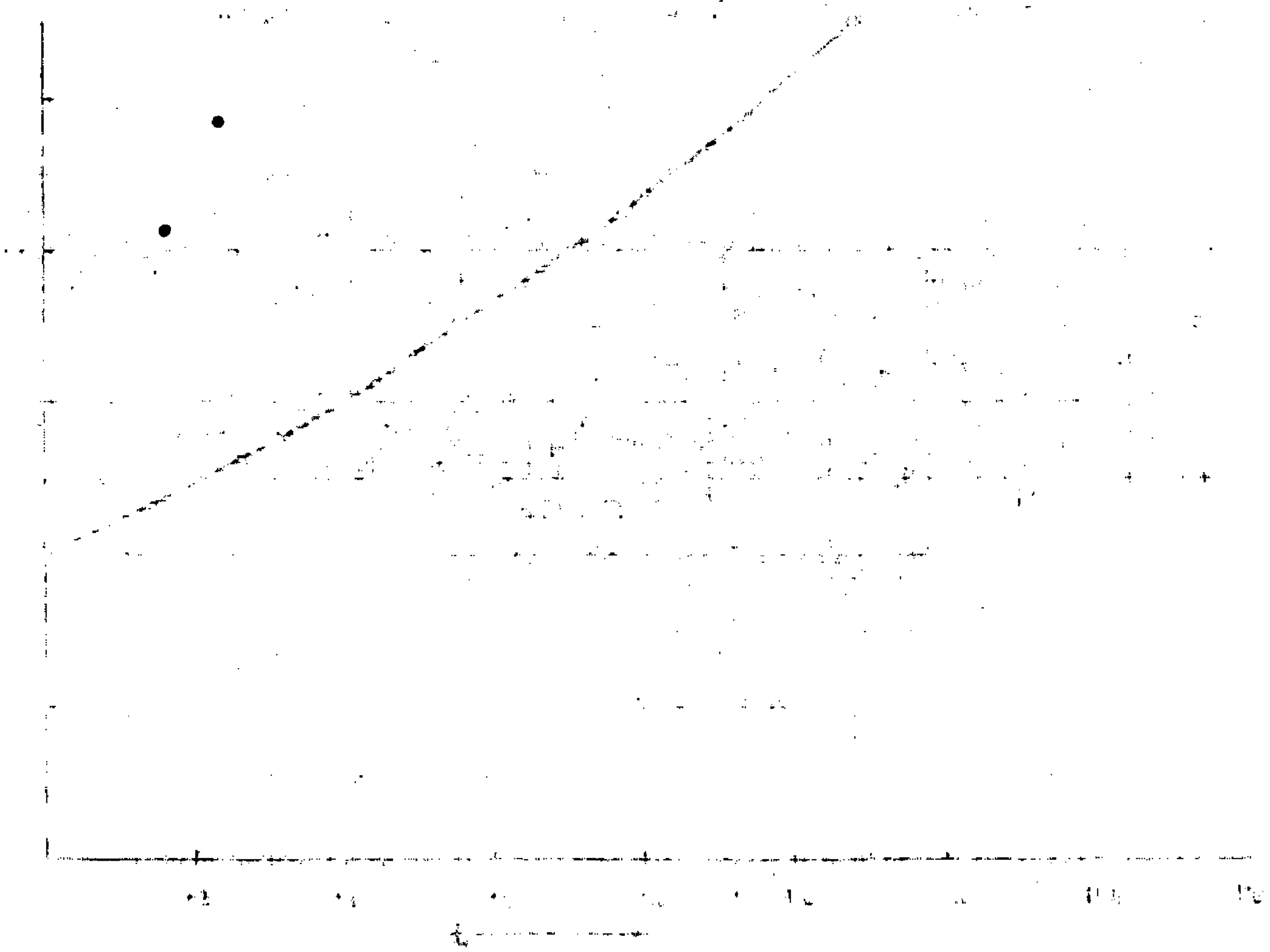
$$\dot{x}_1^{(n)} = f\left[\dot{x}_1^{(n-1)}, \dot{x}_1^{(n-2)} \ldots \ddot{x}_1, \dot{x}_1, x_1, t_1\right] \ldots (3\text{-}22)$$

The process will be repeated till we arrive at a point R where $|N| = \ddot{x}/\dot{x} \leqslant 10$.

This treatment is necessary at the points where the phase trajectory cuts the x-axis. Such points in the phase plane correspond to the maxima and minima points in the solution plane. In many cases such maxima and minima do not arise and the method can continuasly be applied for the whole range.

This method has been illustrated by a number of illustrations. It is equally well suited to the linear, nonlinear, time varying and time invariant Servo systems. A fourth order linear and, with a change in parameter, a fourth order time varying differential equation have been solved. The use has been made of the digital Computer (I BM 1620) in solving the problems.





$\ddot{X} + \dot{X} = t^{[illegible]}$ A) Phase plane B) Solution plane

Example 1

$\dddot{x} + \ddot{x} = t^2$ ... ... (3-23)

Equation (3-23) is a third order linear time varying differentialequation.

Let the initial conditions be

$\ddot{x}_0 = 3 \quad \dot{x}_0 = 2 \quad x_0 = 2 \quad t_0 = 0$

Using Table 1, equation (3-23) can be written as

$\dot{N} = (t^2/x - N^2 - N)$ ... ... (3-24)

The problem has been solved for $\Delta x = 0.06$.

$N_0 = \ddot{x}_0/\dot{x}_0 = 3/2 = 1.5$

$x_1 = x_0 + N_0 \Delta x = 2.06$ ... ... (3-25)

$\dot{x}_1 = \dot{x}_0 + \ddot{x}_0 \Delta x = 2.09$ ... ... (3-26)

$\dot{x}_{av} = (\dot{x}_0 + \dot{x}_1)/2 = 2.045$

$\Delta t_0 = \Delta x / \dot{x}_{av} = 0.02934$

$N_1 = N_0 + \dot{N}_0 \Delta t$

$= N_0 + (t_0^2/x_0 - N_0^2 - N_0)\Delta t_0$

$= 1.39$ ... ... (3-27)

$t_1 = t_0 + \Delta t_0 = 0.02934$ ... ... (3-28)

Equations (3-25) to (3-28) provide necessary information to repeat the process for obtaining point 2.

Both the phase trajectory and solution curve have been plotted in Figure 5 .

Computer Programme is given in Appendix I.

Example 2

$\ddddot{x} + \dddot{x} + 8\dot{x} + 8x = 0$ ... ... (3-29)

Equation (3-29) is a fourth order linear differential quation.

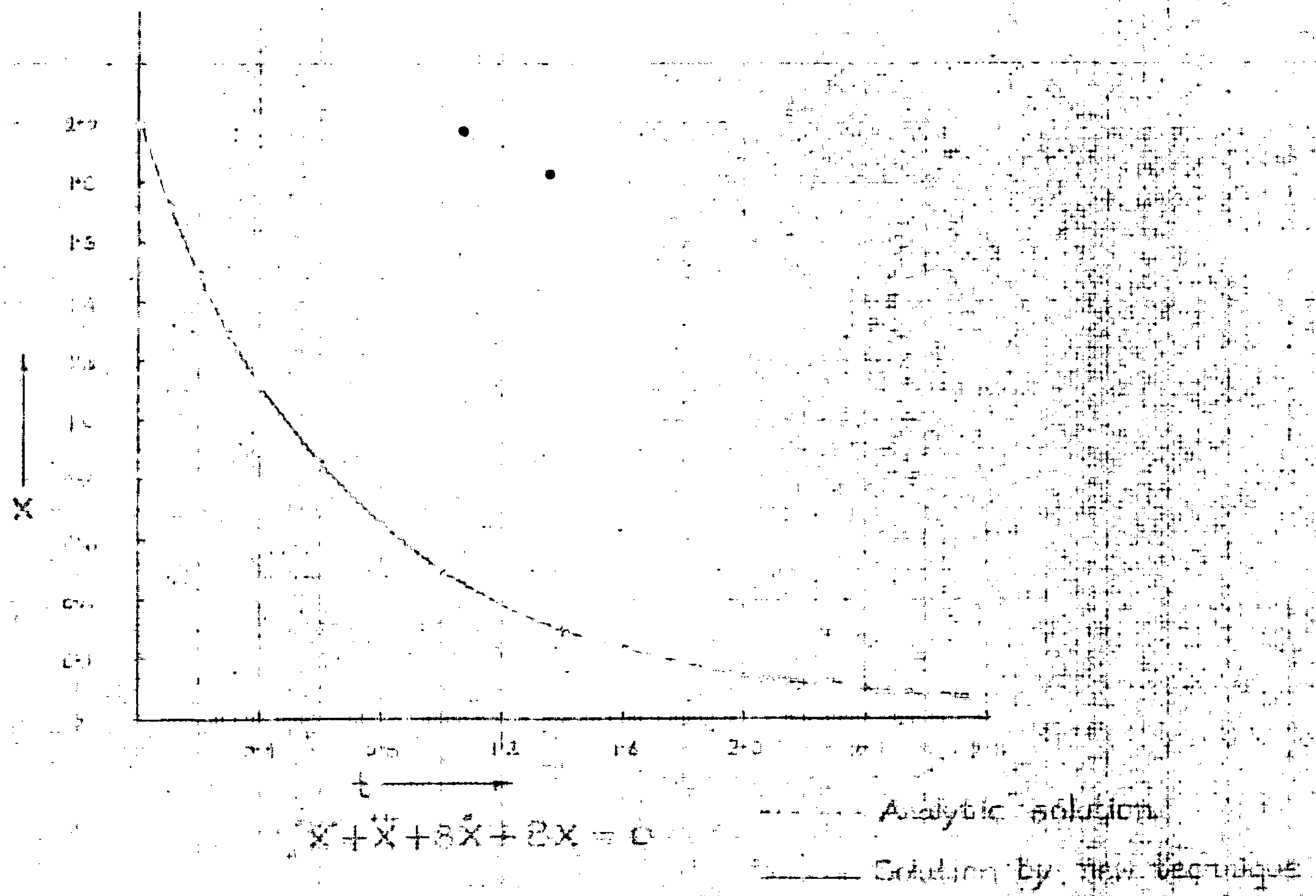

X
t
1·2
1·6
2·0
$\dddot{X} + \ddot{X} + 8\dot{X} + 2X = 0$
Analytic solution
Solution by new technique

FIG. 6

Let the initial conditions be,

$\dddot{x}_0 = -9, \quad \ddot{x}_0 = +5, \quad \dot{x}_0 = -3, \quad x_0 = 2, \quad t0 = 0$

Using table 1, equations (3-29) becomes,

$\ddot{N} = -\left[3\,N\,\dot{N} + \dot{N} + N^3 + N^2 + 8 + 8x/\dot{x}\right]$

and the initial conditions become,

$N_0 = -1.6666, \; \dot{N}_0 = 0.2222, \; \dot{x}_0 = -3, \; x_0 = 2, \; t = 0$

The problem has been solved for $\Delta x = -0.02$.

$\dot{x}_1 = \dot{x}_0 + N_0 \Delta x = -2.967$

$\Delta t_0 = 2 \Delta x/(\dot{x}_0 + \dot{x}_1) = 0.007$

$N_1 = N_0 + \dot{N}_0 \Delta t_0 = -1.665$

$\dot{N}_1 = \dot{N}_0 + \ddot{N}_0 \Delta t_0 = \dot{N}_0 - [3N_0\,\dot{N}_0 + \dot{N}_0 + N_0^3 + N_0^2 + 8 + 8x_0/\dot{x}_0]$

$x_1 = x_0 + \Delta x = 1.98$

$t_1 = t + \Delta t_0 = 0.007$

The process is repeated for point 2.

Computer Programme is given in Appendix II.

<u>Example 3</u>

$\dddot{x} + \ddot{x} + 8\dot{x} + 8t\dot{x} + 8x = 0 \quad \ldots \quad \ldots \quad (3\text{-}30)$

Equation (3-30) is a time varying differential equation of fourth order. This problem is solved for the following initial conditions.

$\dddot{x}_0 = -9, \; \ddot{x}_0 = +5, \; \dot{x}_0 = -3, \; x_0 = 2, \; t_0 = 0$

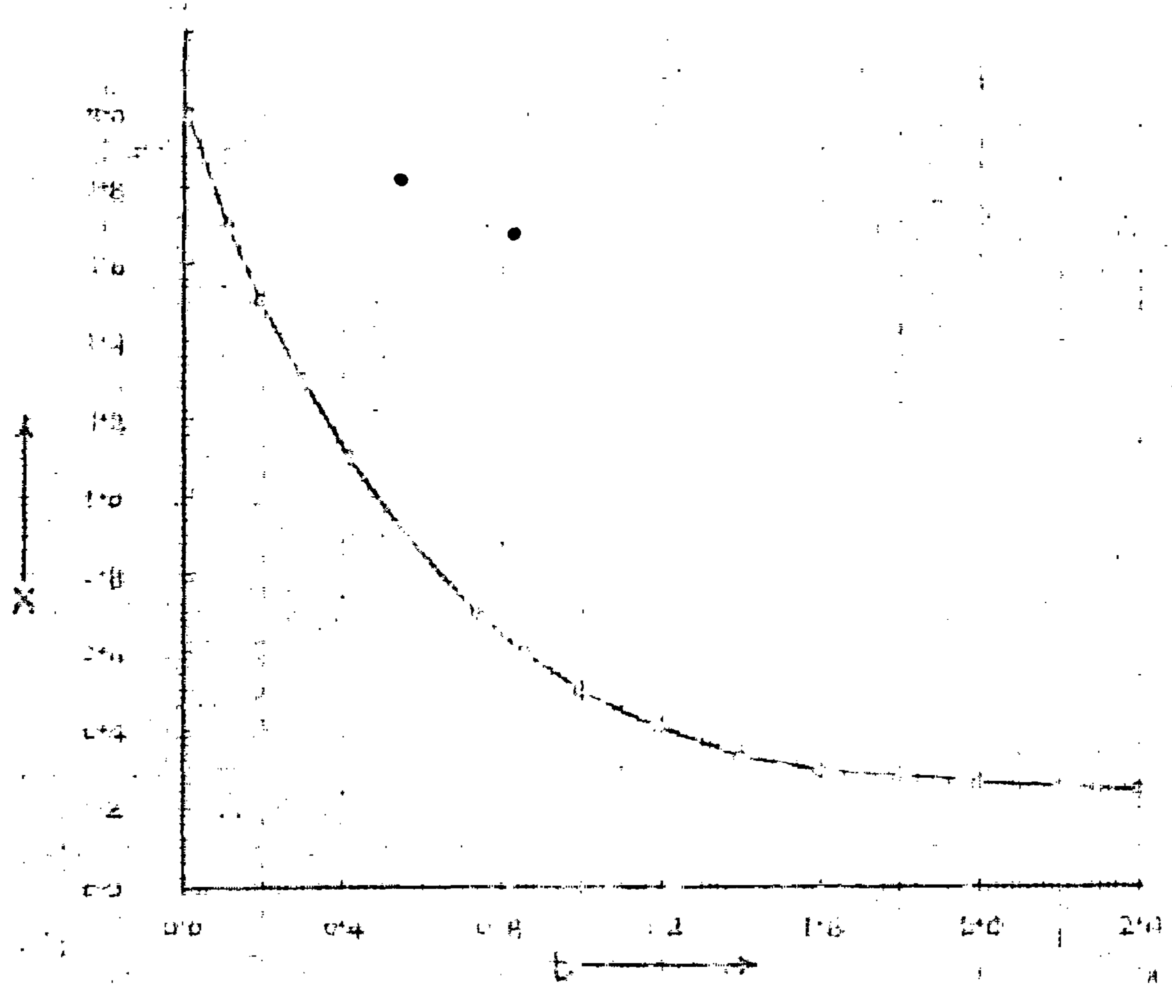
Using table 1, equation (3-30) becomes,

$\ddot{N} = -\left[3\,N\,N + N + N^3 + N^2 + 8 + 8x/x + 8t\right] .. (3\text{-}31)$

and the initial conditions become

$N_0 = -1.6666, \; \dot{N}_0 = 0.2222, \; \dot{x}_0 = -3, \; x_0 = 2, \; t_0 = 0$

This problem is similar to Example 2 except that an additional term 8t is included in equation (3-31). The



$$\ddddot{x} + \dddot{x} + 6\ddot{x} + 2t\dot{x} + 8x = 0$$

FIG. 7

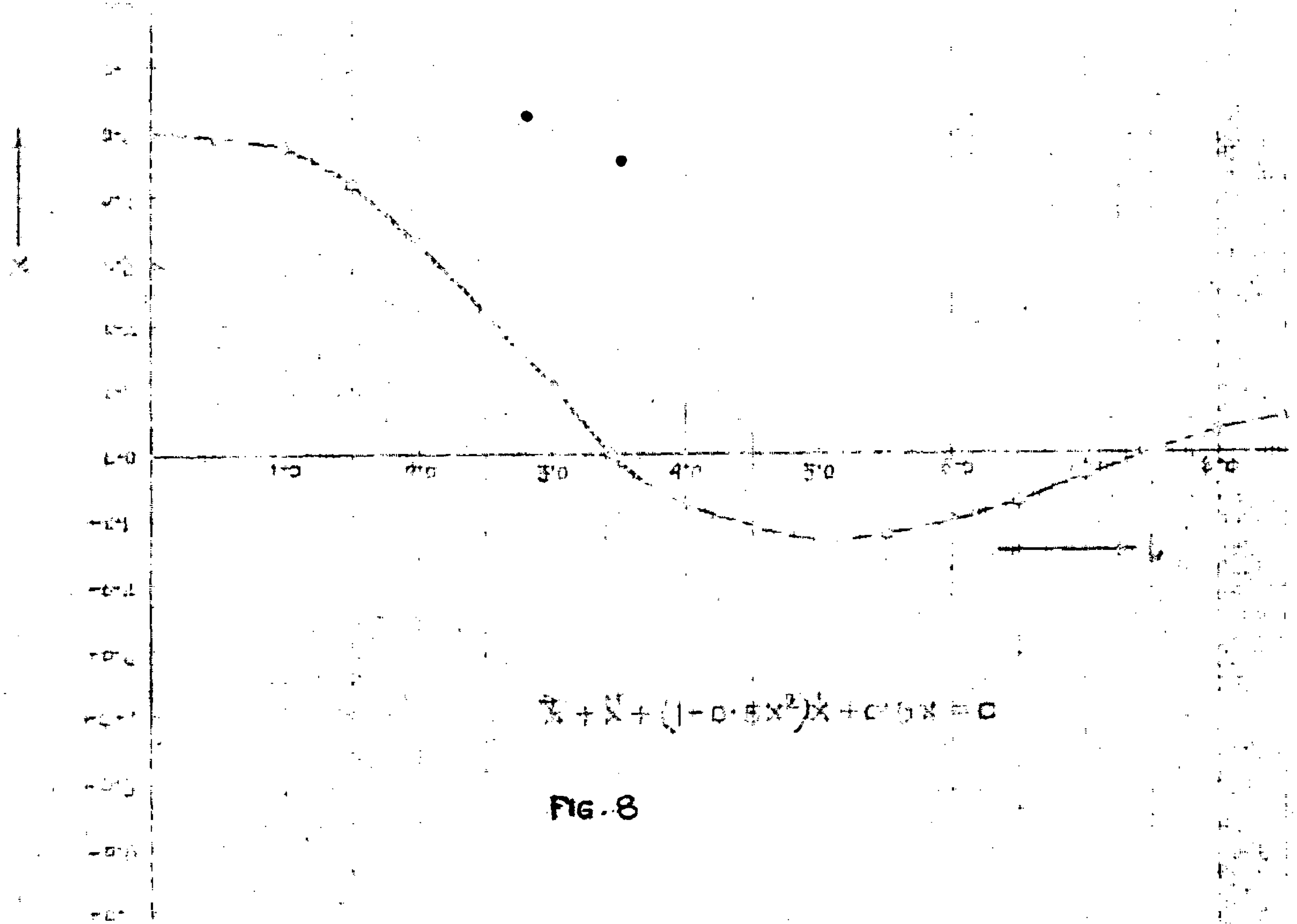


FIG. 8

problem has been solved for $\Delta x = -0.01$.

The results are plotted in Figure 7 and the programme is given in Appendix III.

Example 4

$$\dddot{x} + \ddot{x} + (1-0.5x^2)\dot{x} + 0.5x = 0 \qquad \ldots \qquad (3\text{-}32)$$

Equation (3-32) is a third order nonlinear differential equation. This problem has been solved for the initial conditions,

$$\ddot{x}_0 = 0,\ \dot{x}_0 = 0,\ x_0 = 0.5,\ t_0 = 0$$

In this case $N_0 = \infty$. Therefore till we obtain $|N| < 10$, increments of t will be taken.

Let $\Delta t = 0.1$

$$t_1 = t_0 + \Delta t$$
$$= 0 + 0.1 = 0.1$$
$$x_1 = x_0 + \dot{x}_0 \Delta t$$
$$= 0.5 + 0 = 0.5$$
$$\dot{x}_1 = \dot{x}_0 + \ddot{x}_0 \Delta t$$
$$= 0 + 0 \quad 0.1 = 0$$

From equation (3-32),

$$\dddot{x}_0 = - \ddot{x}_0 - (1 - 0.5\, x_0^2)\, \dot{x}_0 - 0.5x_0$$
$$= - 0.25$$
$$\ddot{x}_1 = \ddot{x}_0 + \dddot{x}_0 \Delta t$$
$$= 0 + (-0.25)(0.01)$$
$$= - 0.0025$$

This process is repeated till for $t = 0.3$, we get,

$\dot{x} = -0.0073$

$N = +9.25$

$x = 0.4998$

$t = 0.3$

Since $|N| < 10$, we follow the method of incremental slope to obtain successive points.

Results are plotted in Figure 8 .

Programme is given in Appendix IV.

-:o:-

Chapter IV.

SUMMARY AND CONCLUSION

4.1 SUMMARY AND CERTAIN IMPLICATIONS OF THE RESULTS

The problem of extending phase space techniques to higher order was taken up in this thesis. The previous methods for higher order were briefly reviewed and a new technique based on the increments in slopes was suggested for the analysis of higher order nonlinear systems.

The technique was illustrated with the help of a few examples and the results were plotted. In one case, the solution was compared with the one obtained analytically and the comparision showed close agreement. Flow chart for use in computer programme is given.

This investigation, however, is by no means complete. A few problems are suggested for further investigation which should make this method more accurate and applicable to a diverse class of problems.

4.2 FURTHER PROBLEMS SUGGESTED BY THIS INVESTIGATION

The first problem worth investigating further is to set up a relation which can help in ascertaining the lengths of intervals at different sections of the phase trajectory. It was pointed out in chapter III that the lengths of intervals need not remain same on the complete range of trajectory. This may be done in order to improve the accuracy for a specified total number of intervals.

The second problem is to modify the method of plotting. The method of plotting the trajectory in the phase plane consisted of finding the slope at the end of each interval and drawing another segment with that slope. Instead, the midpoint of the previous segment of the trajectory may be selected and the next segment of the trajectory may be drawn with the mean of the slopes at the end points (7) . This may prove as a better approximation.

The third problem worth investigating is the error analysis of the results obtainable by this method. A comparison of the error analyses with the other methods can judge the accuracy of this method as compared with the previous ones.

The fourth problem for further investigation is to study some aternative methods to be applied near the portion of the trajectory where $\dot{x} = 0$. It was shown that the general technique applicable for the major part of the whole range of the trajectory is not applicable for the small portion (near $\dot{x} = 0$) and a method was suggested to overcome this difficulty. Study of alternative methods is necessary to improve the technique.

APPENDIX I

PROGRAMME FOR EXAMPLE 1

THIRD ORDER TIME VARYING DIFFERENTIAL EQUATION

```
100     FORMAT  (5 F 8.5)
        READ 100,X,XO,X10,X110,TO
        X11 = X10
        XX  = XO
        ANZ = X110/X10
1       X10 = X11
        X11 = X11 + AN * X
        XX  = XX + X
        XAV = (X11 + X 10) * .5
        DT  = X/XAV
        ANZ = ANZ + (  TO*TO/X10) - ANZ*ANZ-ANZ)  DJ
        TO  = TO +DT
        PUNCH 100, XX,X11, TO
        IF (XX-8.) 1,1,2
2       STOP
        END
.06     2.        2.        3.        0.
```

APPENDIX II

PROGRAMME FOR EXAMPLE 2

FOURTH ORDER LINEAR DIFFERENTIAL EQUATION

```
1.      FORMAT  (6F 8.3)
        READ 1, XO, XDO, ANO, ANDO, DX, T
        DO  110  K = 1,200
        XD1   = XDO + ANO * DX
        DT    = 2.*DX/(XDO +XD1)
        DNO   = (3.* ANO * ANDO + ANDO + ANO * ANO * ANO
                  + ANO * ANO + 8. + 8. * XO/XDO) *DT
        ANO   = ANO + ANDO * DT
        ANDO  = ANDO + DNO
        XO    = XO + DX
        XDO   = XD1
        T     = T + DT
110     PUNCH 1, T, XO, XD1, ANO
        END
2.      -3.    -1.6667      .2222      -.02       0.
```

APPENDIX III

PROGRAMME FOR EXAMPLE 3

```
      FOURTH ORDER TIME VARYING DIFFERENTIAL EQUATION Z
1.    FORMAT ( 6 F 8.3 )
      READ 1, XO, XDO, ANO, ANDO, DX, T
      DO 110 K = 1,200
      XDI  = XDO + ANO * DX
      DT   = 2.*DX/( XDO +XD1 )
      DNO  =(3.*ANO*ANDO+ANDO+ANO*ANO*ANO+ANO*ANO+8.+8.*XO/XD(
                                                          *DT
      DNO  = DNO + T*T*DT/XDO
      ANO  = ANO + ANDO * DT
      ANDO = ANDO + DNO
      XO   = XO + DX
      XDO  = XD1
      T    = T + DT
110   PUNCH 1,T,XO,XD1, ANO
      END
2.0   -3.0      -1.66670.2222    -.01     0.0
```

APPENDIX IV

PROGRAMME FOR EXAMPLE - 4

THIRD ORDER NONLINEAR DIFFERENTIAL EQUATION

```
100     FORMAT ( 4 F 16.4 )
        T     = 0.3
        DX    = - .0002
        XD1   = - .0073
        XO    = .4998
                AN = 9.25
        DO3K = 1.500
1       XDO   = XD1S XD1=XD1+AN*DXS  DT=2.*DX/(XD1+XDO)
        DN    = -DT*(AN*AN*XDO+XDO*(1.-.5*XO*XO)+.5*XO+AN*XDO)/.
        AN    = AN + DNS  XO = XO + DX
        T     = T+DT
3       PUNCH 100, XO, XD1, AN, I
        END
```

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