STATISTICAL DESIGN OF SAMPLED DATA CONTROL SYSTEMS

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By

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ABSTRACT

Analytical technique for the design of control systems is adopted to overcome the drawbacks of classical trial and error approach which is based on the system response to a selected input in absence of noise. Statistical properties of noise and input signals are used to make the system design more realistic.

In this dissertation, analytical design of sampled data systems, using statistical properties of signals, is carried out, extending the technique for design of continuous data systems. Optimization of sampled-data systems is carried out in time-domain, minimizing the mean-square value of error sequence. By Z-transform, optimum-system pulse-transfer function is obtained in terms of pulse-spectral density of input signals and cross-spectral density of input, ideal output signals. Optimization is done for free and semi-free configurations.

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CHAPTER 1

1.1 INTRODUCTION

Sampled data systems have been in vogue recently. Their use is increasing rapidly with advances in other fields of science and technology. Sampled data systems are the systems where the signal is sampled at one or more points.

The sampling of the signal may be inherent. The input to a radar tracking system is in the form of a pulse train. In time multiplexed control or communication systems, data from several channels is sampled and multiplexed for transmission over the same channel.

Many times the sampling of the continuous signal is introduced intentionally to improve the performance of the system. Continuous system with transportation lag can be stabilized by introducing sampling systems with sampled data can, in general, facilitate the realization of adaptive principles. Sampled data systems are also used for improved sensitivity, by sampling the low power signal, the sampling device can be made extremely sensitive in terms of power gain. The sampling is also introduced, because of the ease and accuracy with which the digital signals can be stored, transmitted and processed.

Some of the applications of sampled data systems

1 Semi-Automatic Ground Environment (SAGE) system used by U.S. Department of Defence, a large complex system consisting of telemetry link for weapon guidance.

2 Pulse control of low power motors.

3 Digital control in controlling machine tools, for precision components.

4 Communication and control systems for aerospace communication use sampled data signals.

1.2. [illegible] OF SAMPLED-DATA CONTROL [illegible]

Control system transfer function can be represented either in time domain or in the frequency domain. The methods of design for control systems can also be classified along these lines.

In time domain, transient and steady state behavior of the system are of interest. System specifications, e.g., maximum overshoot, time of first overshoot, maximum settling time etc., can be expressed in terms of the damping ratio and undamped natural frequency of the system. Transient behaviour of a system is determined by the roots of the characteristic equation,

$$1 + G(s)H(s) = 0$$

where $G(s)H(s)$ is open-loop transfer function of the system.

Stability roots are key to the dynamic performance of a system. Root solution of the characteristic equation by direct analytical method is laborious and impractical for design purposes. A graphical approach using plotting of root locus practical for complex systems. This makes available a complete picture of stability changes due to the effect of individual elements. Original root locus is modified by insertion of compensation element that places the roots of characteristic equation at a more favourable point.

Root locus approach is essentially an analytical approach, where the characteristic equation of the system must be known.

Other method of control system design is that in frequency domain. The system can be represented by its response to a sinusoidal signal of constant amplitude. This is essentially a graphical method of system design. Frequency response transfer function of a system can be represented by Nyquist plot, Bode plots or magnitude versus phase shift plot. The frequency response design methods are preferable, because the experimental data is in the form of frequency response and final design can also be checked by frequency analysis.

Various frequency domain specifications are the system bandwidth, resonance peak, resonance frequency, cutoff rate , gain margin , phase margin etc.

Bode plot method of control system design and compensation is preferred, because the effect of compensation is easily obtained by adding magnitude and phase angle curves of individual elements.

1.3. APPLICATION OF CONVENTIONAL DESIGN TECHNIQUES TO SAMPLED- DATA SYSTEMS

Compensation and design techniques for the continuous data control systems also extended to the design of sampled-data systems (7) . But the sampling operation makes the design of feedback system compensation more difficult. Compensation of sampled-data systems may be effected by two general methods.

1. Compensation by continuous devices, making use of continuous data compensation networks in series with other components of the system.

2. Compensation by pulsed data devices whose output is sampled in synchronism with its input at a constant rate.

When the transfer function of a system is in factorized form, it is preferable to work with Bode- diagrams, because of the ease and simplicity with which the asymptotic Bode- plot of a transfer function can be plotted and reshaped.

Characteristic equation of a sampled data control system is a transcendental equation and open loop transfer function of the system is a transcendental function in s. Difficulties encountered in plotting of transcendental functions, discourage the application of Bode-plot techniques for the design of sampled data control systems. This difficulty is overcome by converting transcendental function in s into a rational function in z, by the transformation

$$z = e^{sT}$$

This process maps the primary and complementary strips of left half of s - plane into unit circle in z plane. Bilinear transformation,

$$z = \frac{1 + w}{1 - w}$$

maps, unit circle in z - plane onto imaginary axis of another plane w, and interior of the unit circle into entire left-half of w - plane.

In the extension of root-locus techniques to the design of sampled data control systems, numerous difficulties are encountered in construction of root-loci from a starred transfer function, because of infinite number of poles and zeroes. Complicated nature of root-locus plot in s plane makes it difficult to study the effect of added compensation.

These difficulties are overcome by the use of Z - transform technique. The overall system characteristic equation is transformed to

$$1 + \Delta(z) = 0$$

where, $\Delta(z)$ is commonly called transfer function, a rational function of z containing a finite number of poles and zeroes. Stable operation of the system requires that the roots of sampled data system be confined to the unit circle in z plane.

1.4. DRAWBACKS OF CONVENTIONAL DESIGN APPROACH

Classical design techniques suffer from several drawbacks, that hamper the systematic design of control systems and compensating networks:

1 Compensating networks and control systems are designed, assuming that a known input like transient like step, or sinusoidal, is available. Such signals do not occur in general, and the design becomes unrealistic.

2 System is designed and only for processing the signal. No consideration is taken of uncertainty of noise in the system due to physical nature of the system, e.g., shot noise in the tubes, nonlinearity, tolerances or other disturbances.

3. Classical design techniques are based on trial and error approach. The designer makes some changes in the system, studies the changes in response, again makes changes in system parameter until system performance is within the satisfactory limits. The system designer, except experience does not have any means to recognize an incompatible set of specifications. Trial and error cycles may be repeated one after the other, without achieving the desired results.

1.3 ANALYTICAL [illegible]

To overcome the drawbacks of repeated trials and error considerations, analytical design approach is adopted. In analytical procedure the design of control system begins, with the specification of system input and desired output. By analytical approach, an incompatible set of specifications can be recognized, and either a new set of specification is prescribed or the design is given up as not being feasible.

Considerable work has been done on the analytical design of a control system with continuous signals, based on the mean square error criterion and integral square error criterion (9,10,11,13). Analytical design procedure developed by Newton, Gould and Kaiser (14), requires the user specification that is not explicitly used for trial and error procedure. This requires the [illegible] in computation. The system to be designed or fixed configuration, according to , if there are no constraints of the system configuration or fixed elements, constraints of fixed elements only, and both the system configuration and the fixed elements being specified. Attempts have also been made for the analytical design of systems with constraints like saturation(12) or system bandwidth (13).

1.6. ANALYTICAL DESIGN METHODS RELATED TO SAMPLED- DATA CONTROL SYSTEMS

Some work has been done on the analytical design of sampled-data systems. In a paper concerning the statistical treatment of sampled data control systems for random signals(11), Lord , deals with the correlation function of time series, and pulsed-spectral densities for sampled data control systems. This paper also deals with modified Z- transforms, when the signals are considered at sampling instants and during intervals between sampling instants.

Work has been done on the statistical design of sampled data control systems, utilizing statistical properties of input and output signals (2,3,16). The design of control system is carried in Z- domain, minimizing the mean-square error between desired and actual outputs of the system. Tou has also considered compensation of sampled-data control system in Z- domain (17). Tou has.

also considered design procedure for discrete-data control system subject to power limitation(17). Kalman (1) has considered statistical design of sampled-data systems with randomly varying sampling.

The statistical design procedure has its own limitations. For example, design of systems using statistical procedure is unrealistic, since the signal would require a [illegible] and [illegible] of the [illegible] system to be [illegible] [illegible] [illegible] and [illegible] [illegible] [illegible] [illegible] [illegible] [illegible] by the equations.

Statistical method of analytical design is highly restrictive on the [illegible] inputs are concerned. The inputs are assumed to be stationary and ergodic. In actual the conditions of ergodicity are satisfied, it may be rather difficult to deduce from the actual data whether the assumption of ergodicity is true or not.

2.2. PROBLEM FORMULATION

The problem under investigation is to analytically design the sampled-data system, utilizing the statistical properties of the desired signal, control signal [illegible], and the noise.

A control system can be classified as a time, [illegible] or fixed [illegible] system, depending whether, system performance specification alone, or system performance specifications alongwith some elements of the system, or system performance specifications alongwith the system configuration, are specified.

In the present work, the design of sampled-data control system, has been attempted in the time domain.

In the second chapter of this dissertation, statistical nature of input signals, correlation techniques utilizing the statistical properties of the signals, appropriateness of mean-square error as performance index are considered for the statistical design procedure.

In the third chapter, time-domain equation is developed for the optimum sampled - data system weighting sequence that gives minimum mean-square error, this equation is similar to Wiener - Hopf equation for the continuous data systems,

$$\int_{-\infty}^{\infty} dt_2\, W_m(t_2) \cdot \phi_{ii}(t_1 - t_2) - \phi_{id}(t_1) = 0 \text{ for } t_1 \geq 0$$

where

$W_m(t)$ = weighting function of optimum linear control system.

$R_{xx}(t_1 - t_2)$ = auto correlation function of input signal.

$R_{xd}(t_1, t_2)$ = Cross-correlation of function of input, ideal output signals.

An explicit solution is found for the pulse transfer-function of the optimum system, with mean-square value of error minimum.

This Wiener solution, is extended to the multiple system, in mean-square error sense, with fixed elements, or multiloop configuration system.

Analytical design of sampled-data systems with deterministic input signals is also attempted.

CHAPTER [illegible]

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2.1 INTRODUCTION

Inputs to the control systems are of random nature. These random signals can be represented statistically, in terms of probability density function, average value and moments of random signal. Mean-square value of the system error may be taken as a measure of criterion of system performance. Integral-square error for a [illegible] [illegible] [illegible] system [illegible] [illegible] [illegible].

[illegible] [illegible] the [illegible] [illegible], of the [illegible] [illegible] [illegible] [illegible]. [illegible] [illegible] of the [illegible] [illegible] [illegible] of the [illegible] system [illegible] [illegible] of [illegible] [illegible] of [illegible] in the system. [illegible] to the [illegible] of [illegible] [illegible] systems is also [illegible] in the [illegible] chapter.

2.2 [illegible]

[illegible] [illegible] [illegible] system [illegible] [illegible] [illegible] to [illegible], which [illegible] be [illegible] by [illegible] [illegible]. A control system is [illegible] to [illegible] [illegible] [illegible] [illegible] random from a range of possible tasks, but within certain limits. Many performance characteristics are essentially random functions of time and can be described in statistical sense only. Because of the manufacturing tolerances in a system the sensitivity and other fixed parameters of operating components are subject to random fluctuations. When examined in detail all physical processes are discontinuous and indeterminate. The voltage output of a vacuum tube oscillator is considered as a continuous smooth function. But, on microscopic examination, the curve is found to be relatively rough because of shot noise.

Auto-and- cross correlation functions, and spectral densities for random signals are given by

autocorrelation function of signal $x(t)$

$$\phi_{xx}(\tau) = \lim_{T \to \infty} \frac{1}{2T}\int_{-T}^{T} x(t)\, x(t+\tau)\, dt \qquad ..(1)$$

Cross correlation function of signals $x(t)$ and $i(t)$

$$\phi_{xi}(\tau) = \lim_{T \to \infty} \frac{1}{2T}\int_{-T}^{T} x(t)\, i(t+\tau)\, dt \qquad ..(2)$$

Spectral density of signal $x(t)$

$$\Phi_{xx}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \phi_{xx}(\tau)\, e^{-j\omega\tau}\, d\tau \qquad ..(3)$$

$$\Phi_{xi}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \phi_{xi}(\tau)\, e^{-j\omega\tau}\, d\tau \qquad ..(4)$$

2.3 THE CORRELATION SEQUENCE AND THE PULSE SPECTRAL DENSITY FOR SAMPLED SIGNALS

Correlation sequences for sampled, random signals (11) are defined as follows :

Auto correlation sequence,

$$\phi_{xx}(kT) = \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=-N}^{N} x(nT)\, x(n+k)T \qquad ..(1)$$

Cross correlation sequence

$$\phi_{xy}(kT) = \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=-N}^{N} x(nT)\, y(n+k)T \qquad ..(2)$$

Pulse spectral density is the Fourier transform of the correlation function.

$$\Phi_{xx}(w) = T \sum_{k=-\infty}^{\infty} \phi_{xx}(kT)\, e^{-jwkT} \qquad ..(3)$$

Cross pulse spectral density,

$$\Phi_{xy}(w) = T \sum_{k=-\infty}^{\infty} \phi_{xy}(kT)\, e^{-jwkT} \qquad ..(4)$$

Correlation sequences and pulsed-spectral density of pulsed data signals are characterized by

$$\phi_{ii}(\omega) = \phi_{ii}(-\omega) \quad \ldots\ldots \quad \ldots(5)$$

$$\phi_{io}(\omega) = \phi_{oi}(-\omega) \quad \ldots\ldots \quad \ldots(6)$$

$$\phi'_{ii}(s^{-1}) = \phi'_{ii}(s) \quad \ldots \quad \ldots(7)$$

$$\phi'_{io}(s^{-1}) = \phi'_{oi}(s) \quad \ldots\ldots \quad \ldots(8)$$

If the response of the linear data system $G(s)$ to an input $x^{*}(t)$ is $e^{*}(t)$, then the response of this system to an input $\phi_{ii}(\omega)$ is $\phi_{io}(\omega)$ and the response of this system to an input $\phi_{oi}(\omega)$ is $\phi_{oo}(\omega)$

$$\sum_{n=-\infty}^{\infty} g(nT)\, \phi_{ii}(\omega - nT) = \phi_{io}(\omega) \quad \ldots \quad \ldots(9)$$

$$\sum_{n=-\infty}^{\infty} g(nT)\, \phi_{oi}(\omega - nT) = \phi_{oo}(\omega) \quad \ldots \quad \ldots(10)$$

$$\phi'_{io}(s) = G(s)\, \phi'_{ii}(s) \quad \ldots \quad \ldots(11)$$

$$\phi'_{oo}(s) = G(s)\, \phi'_{oi}(s) \quad \ldots \quad \ldots(12)$$

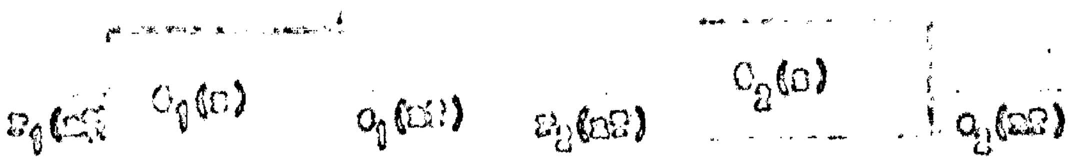


FIG. 1

With reference to Fig. 1 cross correlation sequence and also cross spectral density for the output sequences of sampled data system $C_1(z)$ and $C_2(z)$ possess following characteristics :

$$\rho_{x_2c_1}(kT) = \sum_{n=-\infty}^{\infty} c_1(nT)\,\rho_{x_2x_1}(kT-nT) \qquad \ldots(13)$$

$$\rho_{c_1c_2}(kT) = \sum_{n=-\infty}^{\infty} c_2(nT)\,\rho_{c_1x_2}(kT-nT) \qquad \ldots(14)$$

$$\rho_{x_1c_2}(kT) = \sum_{n=-\infty}^{\infty} c_2(nT)\,\rho_{x_1x_2}(kT-nT) \qquad \ldots(15)$$

$$\rho_{c_2c_1}(kT) = \sum_{n=-\infty}^{\infty} c_1(nT)\,\rho_{c_2x_1}(kT-nT) \qquad \ldots(16)$$

$$D_{x_2c_1}(z) = C_1(z)\,D_{x_2x_1}(z) \qquad \ldots(17)$$

$$D_{c_1c_2}(z) = C_2(z)\,D_{c_1x_2}(z)$$

$$D_{c_1c_2}(z) = C_1(z^{-1})\,C_2(z)\,D_{x_1x_2}(z) \qquad \ldots(18)$$

$$D_{x_1c_2}(z) = C_2(z)\,D_{x_1x_2}(z) \qquad \ldots(19)$$

$$D_{c_2c_1}(z) = C_1(z)\,D_{c_2x_1}(z)$$

$$D_{c_2c_1}(z) = C_1(z)\,C_2(z^{-1})\,D_{x_2x_1}(z) \qquad \ldots(20)$$

2.4 [illegible]

Performance index is defined as some mathematical function of measured error, the function being chosen to give emphasis to system specifications of interest(5,6). Principles for selection of a performance index are:

1. Reliability : Performance index should measure the quality of performance as closely as possible.
2. Selectivity : Optimum value of system parameters should be clearly discernible from some characteristic, such as a minimum value of the performance index versus system parameters.
3. Performance index should be easily calculable from existing techniques.
4. Performance index should be unaffected by unlikely short-lived deviations from the mean, or shifts in time-origin; instead it should be a measure of the average behaviour of control system.

The mean-square error criterion is widely used. It is defined as follows :

$$\overline{e^2} = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} e^2(t)\, dt$$

$$= \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} \left[c(t) - i(t) \right]^2 dt$$

where

$c(t)$ = Actual output of the system.

$i(t)$ = Ideal output of the system

$e(t)$ = Error in the output

One of the main reasons for wide usage of mean-square error criterion stems from its mathematical convenience. A different error criterion may be preferable except for attendant mathematical complication. Mean square error criterion is adequate whenever the undesirability of an error grows with its magnitude.

Mean square value of a random process is one of the easier parameters to evaluate experimentally. Mean-square error together with the mean value of a random process yields information about the process when it is Gaussian. Central limit theorem is often invoked to assume Gaussian process.

Integral square error is a measure of transient response of the system. It was first applied by A.C. Hall. ISE is defined as :

$$ISE = \int_{-\infty}^{\infty} |e(t)|^2 \, dt$$

Both integral-square error and mean-square error can be represented in terms of the system impulse response and correlation functions amongst system input and desired output. Both ISE and MSE criteria can be extended to the sampled data systems also the systems being charac-terized as having sampled inputs and outputs.

Mean - square error.

$$i \quad \overline{e^2(nT)} = \overline{\left[i(nT) - o(nT) \right]^2}$$

$$= \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=-N}^{N} e^2(nT)$$

Total square error

$$= \sum_{n=-\infty}^{\infty} \left[i(nT) - o(nT) \right]^2$$

$$= \sum_{n=-\infty}^{\infty} e^2(nT)$$

2.5 WIENER-HOPF SOLUTION FOR CASE OF LINEAR SYSTEM

In Truxal, Gould and Kaiser approach (14) to the design of optimum linear systems, a system is classified as free, semi-free or fixed configuration system, depend-ing on the restrictions placed on configuration.

In the following section, the design of an optimum free configuration, continuous - data system with minimum mean square error criterion is given (14).

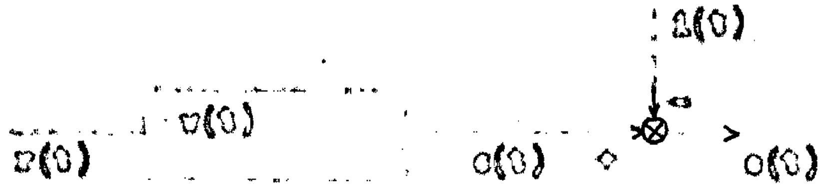


Fig. 2

The error signal at the output of the system, with reference to Fig. 2, is defined as

$$e(t) = c(t) - i(t) \qquad \ldots(1)$$

where $r(t)$ = actual random input to the system

$i(t)$ = ideal output of the system

$c(t)$ = actual output of the system

$h(t)$ = impulse response of optimum linear system

The mean square error expressed in terms of correlation functions of system input and ideal output is given by

$$\overline{e^2(t)} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\tau_1)\, h(\tau_2)\, \phi_{rr}(\tau_1-\tau_2)\, d\tau_1\, d\tau_2$$

$$- 2\int_{-\infty}^{\infty} h(\tau_1)\, \phi_{ri}(\tau_1)\, d\tau_1 + \phi_{ii}(0) \qquad \ldots(2)$$

The bar indicates time average of the function.

By calculus of variations, equation for optimum system minimizing the mean square error is obtained. The equation, known as Wiener - Hopf equation is as follows :

$$\int_{-\infty}^{\infty} h(\tau_2)\, \phi_{rr}(\tau_1-\tau_2)\, d\tau_2 - \phi_{ri}(\tau_1) = 0 \quad \text{for } \tau_1 \geq 0 \qquad \ldots(3)$$

Transfer function of the optimum system, in terms of spectral - and cross-spectral density of system input and ideal output is given by the equation

$$Y_b(s) = \frac{\left[\dfrac{\phi_{xd}(s)}{\phi_{xx}^{-}(s)}\right]_{+}}{\phi_{xx}^{+}(s)} \qquad ..(4)$$

Where

$Y_b(s)$ = Transfer function of the optimum system

$\phi_{xx}^{+}(s)$ = Factor of $\phi_{xx}(s)$ which includes all the poles and zeros of $\phi_{xx}(s)$ in the left half plane.

$\phi_{xx}^{-}(s)$ = Factor of $\phi_{xx}(s)$ which includes all the poles and zeros of $\phi_{xx}(s)$ in the right half plane.

$\left[\dfrac{\phi_{xd}(s)}{\phi_{xx}^{-}(s)}\right]_{+}$ = Component of $\dfrac{\phi_{xd}(s)}{\phi_{xx}^{-}(s)}$, which has all its poles in the left half plane.

$\left[\dfrac{\phi_{xd}(s)}{\phi_{xx}^{-}(s)}\right]_{-}$ = Component of $\dfrac{\phi_{xd}(s)}{\phi_{xx}^{-}(s)}$, which has all its poles in the right half plane.

2.5 [illegible] APPLICATION TO DESIGN OF [illegible] [illegible] DATA OF [illegible] SYSTEM

According to J.R. Ragazzini(16), the optimum design of a sampled-data system is obtained for sampled stochastic control signal $r_s(nT)$ and sampled noise $n_s(nT)$. Both the signal and noise are

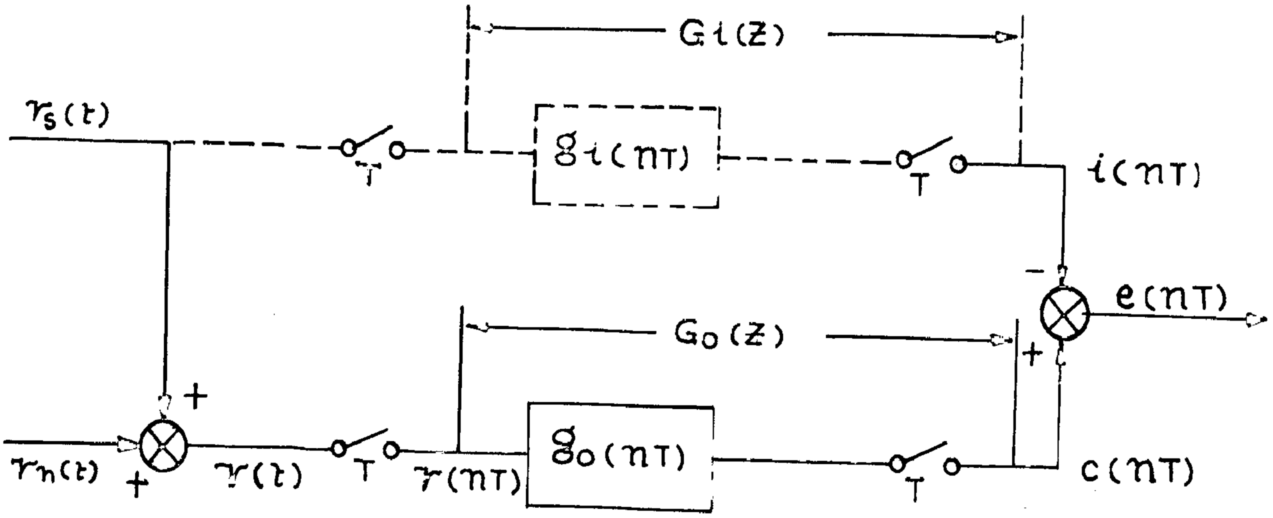


Figure 5.

assumed to be stationary random functions.

For the system shown in Fig. 5, $G_o(z)$ is the pulse transfer function of optimum control system, $G_i(z)$ is pulse transfer function of the ideal system when, there is no noise present.

Error sequence of the system is,

$$e(nT) = c(nT) - i(nT) \qquad ..(1)$$

where

T = sampling period.

Error sequence of the system may be written as

$$e(nT) = c_s(nT) + c_n(nT) - i(nT) \qquad ..(2)$$

where,

$c_s(nT)$ = System response to control signal $r_n(nT)$

$q_3(nT)$ = [illegible] $q_3(nT)$)

The mean square value of error sequence of the system is defined as

$$\sigma^2(nT) = \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=-N}^{N} \sigma^2(nT) \qquad ..([illegible])$$

$$= \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=-N}^{N} [illegible]$$

[illegible]

..([illegible])

= [illegible]

[illegible]

[illegible] ..([illegible])

[illegible] the equations [illegible] to [illegible] the mean square value of error sequence may be written in terms of correlation functions, assuming a stationary condition.

$$\sigma^2(nT) = \phi_{q_1q_1}(0) + \phi_{q_2q_2}(0) + \phi_{q_3q_3}(0) [illegible]$$

[illegible] ...([illegible])

$$S^2(\omega) = \frac{1}{2\pi j} \oint_{\Gamma} \Big[\Phi_{e_0e_0}(z) + \Phi_{e_1e_1}(z) + \Phi_{e_2e_2}(z)$$

$$+ \Phi_{e_0e_1}(z) + \Phi_{e_1e_0}(z) - \Phi_{e_1e_2}(z) - \Phi_{e_2e_1}(z)$$

$$- \Phi_{e_0e_2}(z) - \Phi_{e_2e_0}(z) \Big] . z^{-1} dz \qquad ..(7)$$

where, contour of integration Γ is the unit circle in the z plane.

Since cross spectral densities may be expressed in terms of cross spectral densities of input signals and pulse transfer functions with the help of equations ([illegible]) to ([illegible]).

$$\Phi_{e_0e_0}(z) = \Phi_{r_0r_0}(z)\, G_0(z)\, G_0(z^{-1}) \qquad ..([illegible])$$

$$\Phi_{e_1e_1}(z) = \Phi_{r_1r_1}(z)\, G_1(z)\, G_1(z^{-1}) \qquad ..([illegible])$$

$$\Phi_{e_2e_2}(z) = \Phi_{r_2r_2}(z)\, G_2(z)\, G_2(z^{-1}) \qquad ..([illegible])$$

$$\Phi_{e_0e_1}(z) = \Phi_{r_0r_1}(z)\, G_0(z)\, G_1(z^{-1}) \qquad ..([illegible])$$

$$\Phi_{e_1e_0}(z) = \Phi_{r_1r_0}(z)\, G_1(z)\, G_0(z^{-1}) \qquad ..([illegible])$$

$$\Phi_{e_1e_2}(z) = \Phi_{r_1r_2}(z)\, G_1(z)\, G_2(z^{-1}) \qquad ..([illegible])$$

$$\Phi_{e_0e_2}(z) = \Phi_{r_0r_2}(z)\, G_0(z)\, G_2(z^{-1}) \qquad ..([illegible])$$

[illegible] $= [illegible]$ ...[illegible]

[illegible] $= [illegible]$ ...[illegible]

[illegible]

$[illegible] = \frac{1}{2\pi j} \oint_{\Gamma} D_{\infty}(z)\, [illegible]$ ...[illegible]

[illegible]

$D_{\infty}(z) = [illegible]$

$+ [illegible]$

$+ [illegible]$

$+ [illegible]$ ...[illegible]

[illegible]

[illegible] $\eta(z)$ [illegible]

[illegible]

$[illegible] = -\frac{1}{2\pi j} \oint_{\Gamma} \eta(z) [illegible]$

$+ [illegible] + \frac{1}{2\pi j} \oint_{\Gamma} \eta [illegible]$

$+ [illegible]$ ...[illegible]

Where

$$D(z) = D_{x_0x_0}(z) + D_{x_1x_1}(z) + D_{x_0x_1}(z) + D_{x_1x_0}(z) \qquad ..(12)$$

For reasons of stability all the poles and zeros of $G_0(z)$ and $\eta(z)$ lie inside the unit circle in z plane, and those of $G_0(z^{-1})$ and $\eta(z^{-1})$ lie outside the unit circle. From equations 2.3-3 to 2.3-6

$$D(z) = D(z^{-1}) \qquad ..(13)$$

and D (z) may be written as

$$D(z) = D^{+}(z)\, D^{-}(z) \qquad ..(14)$$

where $D^{+}(z)$ is a function of z with poles and zeros lying inside the unit circle in z plane, and $D^{-}(z)$ is a function with poles and zeros lying outside the unit circle.

[illegible] ..(15)

[illegible] ..(16)

Second term in braces of equation (2.6-16) contains poles inside the unit circle in z-plane as well as outside the unit circle, this may be written as

$$\frac{G_d(z)\{G_{r_f r_n}(z)+G_{n_f n_n}(z)\}}{G^-(z)} = \left[\frac{G_d(z)\;G_{r_f r_n}(z)+G_{n_f n_n}(z)}{G^-(z)}\right]_+ + \left[\frac{G_d(z)\;G_{r_f r_n}(z)+G_{n_f n_n}(z)}{G^-(z)}\right]_-$$

Symbol $[\;]_+$ implies operation of picking part of a function of z with poles outside unit circle in z-plane and $[\;]_-$ implies the operation of picking part of a function of z with poles inside unit circle in z-plane. The contour integral vanishes, if the integrand has its poles either all inside the unit circle or all outside the unit circle. Hence

$$\frac{1}{2\pi j}\oint_\Gamma \eta(z)\,G^+(z)\;G_d(z^{-1})\left[\frac{G_{r_f r_n}(z)+G_{[illegible]}(z)}{G^+(z)}\right] z^{-1}dz = 0 \qquad \ldots(13A)$$

$$\frac{1}{2\pi j}\oint_\Gamma \eta(z)\,G^-(z)\;G_d(z^{-1})\left[\frac{G_{r_f r_n}(z)+G_{n_f r_n}(z)}{G^-(z)}\right]_- z^{-1}dz = 0 \qquad \ldots(13B)$$

The equation (2.7-9) is reduced to

[illegible]

$$[illegible] = \frac{[illegible]}{[illegible]\pi[illegible]} \oint_{\Gamma} \eta \; [illegible]$$

$$= \frac{[illegible]}{[illegible](\omega)} \; [illegible]$$

$$\frac{[illegible]}{[illegible]\pi[illegible]} \oint_{\Gamma} \eta \; [illegible]$$

$$= \frac{[illegible]}{[illegible]} \; [illegible] \quad \ldots ([illegible])$$

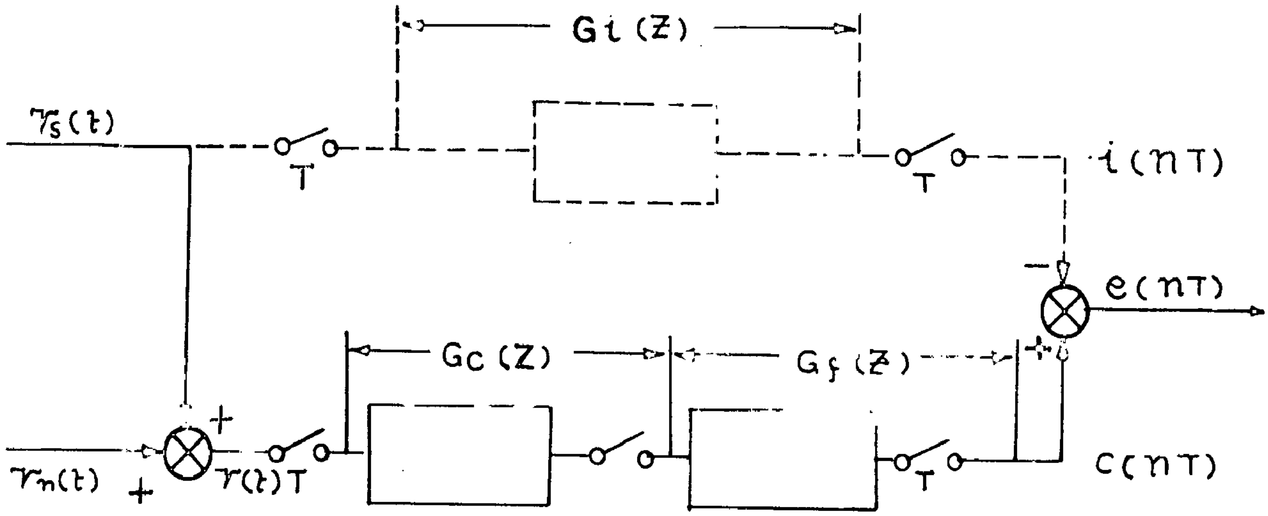
[illegible] δ [illegible] η [illegible]

$$[illegible] = \frac{[illegible]}{[illegible](\omega)} \quad [illegible] \ldots ([illegible])$$

[illegible]

$$[illegible](\omega) = \frac{[illegible]}{[illegible](\omega)} \quad [illegible] \quad \ldots ([illegible])$$

[illegible]



[illegible]

[illegible] is subjected to a sampled stochastic control signal and a sampled random noise, both being stationary functions.

For the sampled - data system, Fig. 6, the symbols used are as follows:

[illegible] = sampled stochastic control signal

[illegible] = sampled random noise

[illegible] = pulse transfer function of ideal system

[illegible] = pulse transfer function of fixed part of the system [illegible]

[illegible]

$$[illegible] \qquad \ldots(1)$$

[illegible]

$$\sigma^2(\omega) = \phi_{xx}(0) - \phi_{xy}(0) - \phi_{yx}(0) + \phi_{yy}(0) \qquad \ldots(2)$$

[illegible] $\sigma^2(\omega)$ [illegible]

$$\sigma^2(\omega) = \frac{1}{2\pi j}\oint_{\Gamma} [illegible] \qquad \ldots(3)$$

$$[illegible] \qquad \ldots(4)$$

[illegible]

$$[illegible]$$

$$[illegible] \qquad \ldots(5)$$

[illegible] $\eta(z)$ [illegible] $\eta(z^{-1})$ [illegible]

[illegible] $\delta\,\overline{\sigma^2(\omega)}$ [illegible]

$$\sigma^{2}(ns) = \frac{1}{2\pi j}\oint_{\Gamma}\eta(z)\left[G_{x}(z)D_{x}(z^{-1})D_{0}(z^{-1})G(z)-G_{x}(z^{-1})D_{x}(z)\right]\left[\Phi_{x_0 x_0}(z)+\Phi_{x_0 x_n}(z)\right]z^{-1}\,dz + \frac{1}{2\pi j}\oint_{\Gamma}\eta(z^{-1})$$

$$\left[G_{x}(z^{-1})D_{x}(z)D_{0}(z)G(z)-G_{x}(z)D_{x}(z^{-1})\right]\left[\Phi_{x_n x_0}(z)+\Phi_{x_n x_0}(z)\right]z^{-1}dz \quad ..(7)$$

where $\quad G(z) = \Phi_{x_0 x_0}(z) + \Phi_{x_0 x_n}(z)+\Phi_{x_n x_0}(z)+\Phi_{x_n x_n}(z) \quad ..(8)$

Defining $G(z) = G^{+}(z)\,G^{-}(z) \quad ......(9)$

$$\left[G_{x}(z)D_{x}(z^{-1})\right] = \left[G_{x}(z)D_{x}(z^{-1})\right]^{+}\left[G_{x}(z)\,G_{x}(z^{-1})\right]^{-} \quad ..(10)$$

$G^{+}(z)$ indicates collecting portion of function $G(z)$ that lie within the unit circle and $G(z)^{-}$ indicates collecting the portion of function $G(z)$ that lie outside the unit circle in z-plane.

$$\sigma^{2}(ns) = \frac{1}{2\pi j}\oint_{\Gamma}\eta(z)G^{+}(z)\left[G_{x}(z)D_{x}(z^{-1})\right]^{+}\left[G_{x}(z)D_{x}(z^{-1})\right]^{-}$$

$$D_{0}(z^{-1})\,G^{-}(z) - \frac{G_{x}(z^{-1})D_{x}(z)\left[\Phi_{x_n x_0}(z)+\Phi_{x_n x_n}(z)\right]}{G^{+}(z)\;G_{x}(z)\,G_{x}(z^{-1})^{+}}\; z^{-1}dz$$

$$+ \frac{1}{2\pi j}\oint_{\Gamma} \eta(z^{-1}) G^{-1}(z) \left[G_2(z)G_2(z^{-1}) \right]^{-} \left[G_2(z)G_2(z^{-1}) \right]^{+} G^{+}(z)\, G_0(z) - \frac{G_1(z)\, G_2(z^{-1}) \left[\Phi_{x_0x_0}(z) + \Phi_{n_0x_0}(z) \right]}{G^{-}(z)\, \left[G_2(z)\, G_2(z^{-1}) \right]^{-}} \; z^{-1}\, dz \qquad \ldots(11)$$

The second term in the bracket contains poles inside the unit circle in the z- plane, as well as outside the unit circle. The contour integral vanishes if the integrand has its poles either all inside the unit circle or all outside the unit circle. Thus,

$$\frac{1}{2\pi j}\oint_{\Gamma} \eta(z^{-1}) \cdot G^{-}(z) \left[G_2(z)\, G_2(z^{-1}) \right]^{-} \left[\frac{G_1(z)G_2(z^{-1}) \left[\Phi_{x_0x_0}(z) + \Phi_{n_0x_0}(z) \right]}{G^{-}(z) \left[G_2(z)\, G_2(z^{-1}) \right]^{-}} \right]^{-} = 0 \qquad \ldots(12),$$

$$\frac{1}{2\pi j}\oint_{\Gamma} \eta(z) G^{+}(z) \left[G_2(z)G_2(z^{-1}) \right]^{+} \left[\frac{G_1(z)G_2(z^{-1}) \left[\Phi_{x_0x_0}(z) + \Phi_{n_0x_0}(z) \right]}{G^{+}(z) \left[G_2(z)G_2(z^{-1}) \right]^{+}} \right]^{-} z^{-1}\, dz = 0 \qquad \ldots(13)$$

where, symbol $[\;]_{+}$ implies the operation of picking the part of a function of z with poles inside the unit circle in z- plane, and the symbol $[\;]_{-}$ implies the operation of picking the part of a function of z with poles outside the unit circle in z plane.

Hence equation (2.7-11) reduces to :

$$e^2(nT) = \frac{1}{2\pi j}\oint_{\Gamma} \eta(z)\, G^{+}(z)\left[G_f(z)G_f(z^{-1})\right]^{+}\Big[\left[G_f(z)\, G_f(z^{-1})\right]^{-} G_c(z^{-1})G^{-}(z) - \left[\frac{G_1(z^{-1})G_f(z)\left[\Phi_{r_c r_n}(z)+\Phi_{r_n r_n}(z)\right]}{G^{+}(z)\left[G_f(z)\, G_f(z^{-1})\right]^{+}}\right]_{-}\Big] z^{-1}dz$$

$$+ \frac{1}{2\pi j}\oint_{\Gamma}\eta(z^{-1})G^{-}(z)\left[G_f(z)G_f(z^{-1})\right]^{-}\Big[\left[G_f(z)G_f(z^{-1})\right]^{+} G_c(z)\, G^{+}(z) - \left[\frac{G_f(z)\, G_f(z^{-1})\left[\Phi_{r_c r_n}(z) + \Phi_{r_n r_n}(z)\right]}{G^{-}(z)\left[G_f(z)\, G_f(z^{-1})\right]^{-}}\right]_{+}\Big] z^{-1}\, dz \qquad ..(14)$$

For the compensation to be optimum, giving minimum mean square error, $\delta e^2(nT)$ should vanish for arbitrary $\eta(z)$. Optimum compensation is given by

$$G_c(z)\left[G_f(z)G_f(z^{-1})\right]^{+} G^{+}(z) - \left[\frac{G_f(z^{-1})G_1(z)\,\Phi_{r_c r_n}(z) + \Phi_{r_n r_n}(z)}{G^{-}(z)\; G_f(z)\, G_f(z^{-1})^{-}}\right]_{+} = 0 \qquad ...(15)$$

$$G_c(z) = \left[\frac{G_f(z^{-1})\, G_1(z)\left[\Phi_{r_c r_n}(z)+\Phi_{r_n r_n}(z)\right]}{G^{-}(z)\left[G_f(z)G_f(z^{-1})\right]^{-}}\right]_{+} \frac{1}{\left[G_f(z)G_f(z^{-1})\right]^{+} G^{+}(z)} \qquad ...(16)$$

Equation (2.7-16), gives the pulse-transfer function of the compensation that will optimize the system in the minimum mean square error sense. This can be obtained in terms of pulse-transfer function of fixed components and pulse spectral density of input signals.

CHAPTER - IX

[illegible]

9.1 INTRODUCTION

In the previous chapter, optimum design for the systems, whose input and output are sampled in synchronism, is carried out in time-domain. The optimum system equation for sampled-data systems with random input signals, is obtained by minimization of the mean-square value of error sequence. The system equation obtained is as follows :

$$\sum_{m=-\infty}^{\infty} g(mT)\ \phi_{rr}(k-m)T - \phi_{rd}(kT) = 0 \qquad \text{for } k \geqslant 0 \ldots \quad (i)$$

where

$g(mT)$ - weighting sequence of the optimum sampled data system.

$\phi_{rr}(k-m)T$ - Autocorrelation sequence of the input signal

$\phi_{rd}(kT)$ - Input-ideal output cross-correlation sequence.

which is quite similar to Wiener-Hopf equation for optimum continuous data systems. A solution of the equation is suggested and is carried out in z-domain. The equation is modified as

$$\sum_{m=-\infty}^{\infty} g(mT)\ \phi'_{rr}(q-m)T = \beta_o(qT)$$

where $\beta_o(qT)$ is defined by equations (9.3-4) and (9.3-[illegible])

Pulse transfer function of the optimum system is obtained by taking z-transforms of both sides of the modified system equation. Pulse transfer function of the optimum multiple-rate system in terms of spectral densities of system signals, is given by

$$G_o(z) = \frac{1}{\Phi_{ii}^{+}(z)} \times \left[\frac{\Phi_{id}(z)}{\Phi_{ii}^{-}(z)} \right] .$$

where,

$G_o(z)$ Optimum system pulse-transfer function

$\Phi_{ii}(z)$ Auto-spectral density of input signal

$\Phi_{id}(z)$ Cross-spectral density of input and desired output signals.

Optimization of the multiple-rate with deterministic input signals, minimizing total square error, has also been discussed and it is shown that an optimum finite settling time system with random input signals satisfying minimum mean square error criterion can be obtained.

It is further shown that when the control signal component and the noise enter the system at different points, minimization of mean square error yields the optimum multiple-rate system equation.

$$\sum_{j=0}^{\infty} g_o(jT)\, \phi_{ii}[(n-j)T] + g_1(jT)\, \phi_{in}[(n-j)T] - \phi_{id}(nT) = 0 \quad \text{for } n \geq 0$$

where,

$w(nT)$ weighting sequence for deterministic control signal component.

$C_n(nT)$ weighting sequence for random noise component

$r_s(nT)$ input control signal component sequence

$r_n(nT)$ input noise sequence

Optimization of sampled - data system has been extended to [illegible] configuration systems. For random input signals, it is shown that the minimization of mean square error yields the optimum system equation,

$$\sum_{\nu_1=-\infty}^{\infty} G_f(\nu_1 T)\,\phi_{ri}(k_1+\nu_1)T - \sum_{\nu_1=-\infty}^{\infty}\ \sum_{k_2=-\infty}^{\infty}\ \sum_{\nu_2=-\infty}^{\infty} G_f(\nu_1 T)$$

$$G_f(\nu_2 T)\, G_c(k_2 T)\,\phi_{rr}(k_2+\nu_2-k_1-\nu_1)T \quad = 0 \ \text{ for } k_1 \geqslant 0$$

Where

$G_f(\nu T)$ = Weight sequence of fixed elements

$G_c(\nu T)$ = Weighting sequence of compensation elements.

Pulse transfer function of the compensation elements in terms of pulse spectral densities of input and ideal output signals and the pulse transfer function of fixed elements is obtained and the expression obtained is as follows:

$$G_c(z) = \left[\frac{G_f(z^{-1})\,\Phi_{ri}(z)}{G_f(z)\,G_f(z^{-1})^{-}\,\Phi_{rr}^{-}(z)}\right]_{+} \frac{1}{G_f(z)G_f(z^{-1})^{+}\,\Phi_{rr}^{+}(z)}$$

Next it is shown that when fixed element pulse-transfer function has all its poles and zeros within the unit circle in z plane, the pulse transfer function of compensation system is the ratio of pulse-transfer function of a free configuration system, with same performance specifications, and the pulse transfer function of fixed elements.

The expression for minimum mean square error of the sampled-data system, has been obtained in terms of the optimum system pulse transfer function and spectral densities of the system signals.

5.2 [illegible]

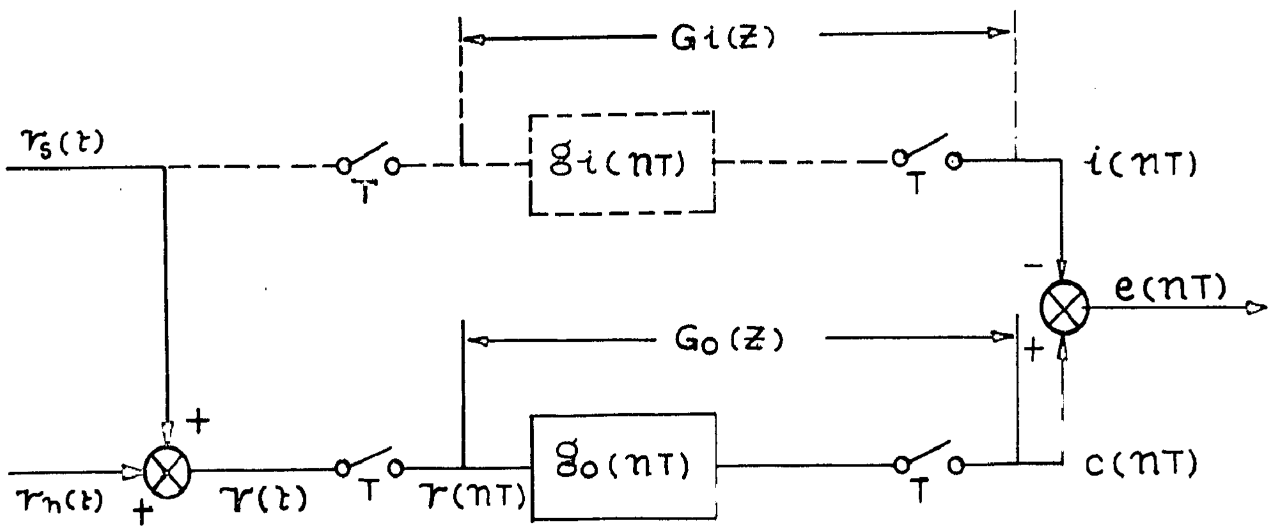


Fig. 9

In free-configuration case, there are no constraints, either on the form of the system configuration or of the fixed elements and the designer has to choose both the form and elements. In the present section, time-domain equation of the optimum sampled data system weighting sequence, in the mean square sense has been derived for the free configuration case.

The output of a system, Fig. 5, input and output of which are sampled in synchronism, will be given by

$$c(nT) = \sum_{k=0}^{n} g(kT)\, r(nT-kT) \qquad ..(1)$$

where,

$c(nT)$ = Sampled output of the system at nth sampling instant

$r(nT)$ = Sampled input of the system at nth sampling instant

$g(kT)$ = Weighting sequence of sampled-data system.

T = Sampling period

Difference between optimum system output $c_o(nT)$ and ideal output sequence, i.e. system error sequence Equation $c_e(nT)$ will be given by

$$c_e(nT) = c(nT) - i(nT) \qquad ..(2)$$

$$\text{i.e.} \quad c_e^2(nT) = c^2(nT) - 2c(nT)\, i(nT) + i^2(nT) \qquad ..(3)$$

$\overline{c_e^2}$ being mean square value of error sequence

$$\overline{c_e^2}(nT) = \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=-N}^{N} c_e^2(nT)$$

* Weighting sequence $g(kT)$ is zero for negative values of k.

$$\sigma_e^2(nT) = \sigma^2(nT) - 2c(nT)\,g(nT) + g^2(nT) \qquad ..(4)$$

$$\sigma^2(nT) = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} c(nT)\, c(nT)$$

$$= \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} \sum_{k=-\infty}^{\infty} c(kT)x(n-k)T \sum_{m=-\infty}^{\infty} c(mT)x(n-m)T$$

Since k, n & m are dummy variables interchanging them

$$\sigma^2(nT) = \sum_{k=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} c(kT)c(mT) \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} x(n-k)T\,x(n-m)T$$

From equation (2.3-1)

$$\sigma^2(nT) = \sum_{k=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} c(kT)\, c(mT)\, \phi_{xx}\,(k-m)T \qquad ...(5)$$

Similarly from equation (2.3-2)

$$c(nT)g(nT) = \sum_{k=-\infty}^{\infty} c(kT)\, \phi_{xg}(kT) \qquad .... \qquad ..(6)$$

and

$$g^2(nT) = \phi_{gg}\,(0) \qquad ... \qquad ... \quad ..(7)$$

Hence, from equations (3.2-4),(3.2-5), (3.2-6),(3.2-7)

$$\sigma_e^2(nT) = \phi_{gg}(0) - 2\sum_{k=-\infty}^{\infty} c(kT)\phi_{xg}(kT) + \sum_{k=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} c(kT)c(mT)\phi_{xx}(k-m)T \qquad ..(8)$$

If the optimum system c(nT) is replaced by another system c(nT) + ε h(nT), where h(nT) is any arbitrarily realizable weighting sequence, ε is a parameter, that is varied to test the optimality of c(nT). Hence with

[illegible]

[illegible]

$$\frac{\partial^2}{\partial \epsilon^2} \; [illegible]$$

[illegible]

$$\frac{\partial}{\partial \epsilon} \; [illegible] \Big|_{\epsilon = 0} = 0 \qquad \ldots [illegible]$$

[illegible]

$$\sum [illegible] \left[\sum [illegible] \right] = 0 \qquad \ldots [illegible]$$

[illegible]

$u(m) = 0$ [illegible] < 0 or $n < 0$ ...(iii)

[illegible] $(3.[illegible])$ [illegible]

$$\sum u(m) \sum_{m=0} [illegible] \quad \text{for } n \geq 0 \qquad \ldots(iv)$$

Using (iii), [illegible] expression can be modified for $n \geq 0$. So for [illegible] the formula [illegible] $n \geq 0$

$$\sum_{m=0}^{\infty} u(m) A_{mn}([illegible]) = 0 \quad \text{for } n \geq 0 \qquad \ldots(v)$$

Obviously [illegible] $u(m)$ is the [illegible] by the above equations.

3.3. [illegible]

[illegible] $u(m)$, of the [illegible] given by :

$$\sum_{m=0}^{\infty} u(m) A_{mn}([illegible]) = 0 \quad \text{for } n \geq 0 \qquad (1)$$

The above equation holds good for $n \geq 0$, but need not necessarily hold good for $n<0$ also, since $A_{mn}([illegible])$ and $A_{mn}([illegible])$ in general will not be zero for $n<0$.

Writing $\phi_{rr}(kT)$ as

$$\phi_{rr}(kT) = \sum_{j=-\infty}^{\infty} \phi_{rr}^{+}((k-j)T)\, \phi_{rr}^{-}(jT) \qquad \ldots \quad (2A)$$

where $\phi_{rr}^{+}(jT) = 0$ for $j<0$ ... (2B)

$\phi_{rr}^{-}(jT) = 0$ for $j>0$ ... (2C)

Multiplying both sides of equation (5.5-2A by z^{-k} and summing for all values of k from $-\infty$ to $+\infty$.

$$\sum_{k=-\infty}^{\infty} \phi_{rr}(kT)\, z^{-k} = \sum_{k=-\infty}^{\infty} z^{-k} \left[\sum_{j=-\infty}^{\infty} \phi_{rr}^{+}((k-j)T)\, \phi_{rr}^{-}(jT) \right]$$

Changing order of summation,

$$\Phi_{rr}(z) = \sum_{j=-\infty}^{\infty} \phi_{rr}^{-}(jT)\, z^{-jT} \sum_{k=-\infty}^{\infty} \phi_{rr}^{+}((k-j)T)\, z^{-(k-j)T}$$

$$\Phi_{rr}(z) = \Phi_{rr}^{-}(z)\, \Phi_{rr}^{+}(z) \qquad \ldots (3)$$

$\Phi_{rr}^{+}(z)$ will have poles and zeroes inside the unit circle in z-plane only, since $\phi_{rr}^{+}(kT)$ is zero for negative values of k. Similarly, $\Phi_{rr}^{-}(z)$ will have poles and zeroes outside the unit circle in z-plane only since $\phi_{rr}^{-}(kT)$ is zero for positive values of k.

By equation (5.5-3), spectral density $\Phi_{rr}(z)$ can be expressed as product of two parts, one containing poles and zeroes inside the unit circle in z-plane, while the other has poles and zeroes outside the unit circle only.

Cross correlation sequence $\phi_{rd}(kT)$ can be written as

$$p_m(n) = \sum_{\nu=-\infty}^{\infty} \beta(n-\nu)\, p^{*}_{m}(\nu) \qquad ..(4)$$

Substituting (9.5-4), (9.5-5) in (9.5-1), we get

$$\sum_{k=-\infty}^{\infty} a(k) \sum_{\nu=-\infty}^{\infty} \beta^{*}_{m}(n-\nu-k)\, p^{*}_{m}(\nu) - \sum_{\nu=-\infty}^{\infty} \beta(n-\nu)\, p^{*}_{m}(\nu) = 0$$

$$\text{for } n \geq 0 \qquad ..(5)$$

Interchanging terms,

$$\sum_{\nu=-\infty}^{\infty} p^{*}_{m}(\nu) \left[\sum_{k=-\infty}^{\infty} a(k)\, \beta^{*}_{m}(n-\nu-k) - \beta(n-\nu) \right] = 0 \quad \text{for } n \geq 0 \qquad ..(6)$$

From equation (9.5-5)

$$p^{*}_{m}(\nu) = 0 \qquad \text{for } \nu > 0$$

Hence, the equation will hold good for $\nu \leq 0$, if the expression within the bracket is zero,

$$\text{or} \quad \sum_{k=-\infty}^{\infty} a(k)\, \beta^{*}_{m}(n-\nu-k) - \beta(n-\nu) = 0 \quad \text{for } n \geq 0,\ \nu < 0 \qquad ..(7)$$

The above equation holds good for $n \geq 0$, hence, it will hold for $n-\nu \geq 0$ also, where ν is a negative number. The equation (9.5-7) may be written as, with a change of variable

$$q = n - \nu$$

$$\sum_{k=-\infty}^{\infty} a(k)\, \beta^{*}_{m}(q-k) - \beta(q) = 0, \text{ for } q \geq 0 \qquad ..(8)$$

For a realizable weighting sequence,

$$a(k) = 0 \qquad \text{for } k < 0$$

$$\text{also} \quad \beta^{*}_{m}(k) = 0 \qquad \text{for } k < 0$$

[illegible] (9.3?) [illegible] $q < 0$ [illegible] < 0 [illegible] $q < 0$. [illegible] $\beta(q)$ [illegible]

$$\beta(q) = \beta_{+}(q) + \beta_{-}(q) \qquad ..([illegible])$$

[illegible]

$$\beta_{+}(q) = 0 \qquad \text{for } q < 0 \qquad ..([illegible])$$

$$\beta_{-}(q) = [illegible] \qquad \text{for } q > 0 \qquad ..([illegible])$$

[illegible] (9.3?) [illegible]

$$\sum_{[illegible]}^{\infty} [illegible]\, \rho_{[illegible]}^{+}([illegible]) = \beta_{+}(q) \quad \text{for all } [illegible] \qquad ..([illegible])$$

[illegible] (9.3?) [illegible] of q [illegible] (9.3?) [illegible] of q [illegible] $\beta_{-}(q)$.

[illegible] (9.3?) by z^{-q} [illegible] of q [illegible] $-\infty$ to ∞ [illegible]

$$\sum_{[illegible]}^{\infty} z^{-q} \sum_{[illegible]}^{\infty} [illegible]\, \rho_{[illegible]}^{+}([illegible]) = \sum_{[illegible]}^{\infty} \beta_{+}(q)\, z^{-q}$$

$$= \sum_{[illegible]}^{\infty} [illegible] \sum_{[illegible]}^{\infty} \rho_{[illegible]}^{+}([illegible])\, z^{-([illegible])} = \sum_{[illegible]}^{\infty} \beta_{+}(q)\, z^{-q}$$

$$[illegible](z)\, \rho_{[illegible]}^{+}(z) = \beta(z) \qquad ..([illegible])$$

[illegible]

$$\therefore\ [illegible](z) = \frac{[\beta(z)]_{+}}{\rho_{[illegible]}^{+}(z)} \qquad ..([illegible])$$

[illegible] (9.3?)

$$[illegible] = \beta(\omega)\ [illegible] \qquad ..([illegible])$$

[illegible] ([illegible]) [illegible] ([illegible]) [illegible]
[illegible] ([illegible]) [illegible] by .

$$[illegible] \qquad ..([illegible])$$

[illegible]

$$[illegible] \qquad ..([illegible])$$

$$[illegible] \qquad ..([illegible])$$

$$[illegible] \qquad ..([illegible])$$

A relatively shorter proof can also be given as follows :

If the optimum smoothed data system equation 3.2-16 is

$$\sum_{n=-\infty}^{\infty} h(nT)\,\phi_{ss}(k-n)T - \phi_{ds}(kT) = 0 \qquad \text{for } k \geq 0$$

and

$$\sum_{n=-\infty}^{\infty} h(nT)\,\phi_{ss}(k-n)T - \phi_{ds}(kT) = q(kT) \qquad \text{for } k<0 \qquad ...(18)$$

$q(kT)$ is in general nonzero for $k<0$, but vanishes for $k \geq 0$. The optimum smoothed data system equation can be written as

$$\sum_{n=-\infty}^{\infty} h(nT)\,\phi_{ss}(k-n)T - \phi_{ds}(kT) = q(kT) \qquad ...(19)$$

Taking z-transform of $q(kT)$

$$Q(z) = \sum_{k=-\infty}^{\infty} q(kT)\, z^{-k}$$

$$Q(z) = \sum_{k=-\infty}^{0} q(kT)\, z^{-k} \qquad ...(20)$$

$Q(z)$ has poles outside the unit circle in z plane only.

Taking double sided z transform of equation 3.4-19.

$$H_o(z)\,\Phi_{ss}(z) - \Phi_{ds}(z) = Q(z) \qquad ..(21)$$

$\Phi_{ss}(z)$ is an even function and can be separated by equation (3.4-5) as the product of $\Phi_{ss}^{+}(z)$ and $\Phi_{ss}^{-}(z)$. Equation (3.4-21) can be written as

$$H_o(z)\,\Phi_{ss}^{+}(z)\,\Phi_{ss}^{-}(z) - \Phi_{ds}(z) = Q(z)$$

$$H_o(z)\,\Phi_{ss}^{+}(z) - \frac{\Phi_{ds}(z)}{\Phi_{ss}^{-}(z)} = \frac{Q(z)}{\Phi_{ss}^{-}(z)} \qquad ...(22)$$

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

$$G_o(z)\,\Phi_{rr}^{+}(z) - \left[\frac{\Phi_{ri}(z)}{\Phi_{rr}^{-}(z)}\right]_{+} = 0 \qquad ..(25)$$

Hence the optimum sampled data system pulse transfer function is given by

$$G_o(z) = \frac{1}{\Phi_{rr}^{+}(z)} \cdot \left[\frac{\Phi_{ri}(z)}{\Phi_{rr}^{-}(z)}\right]_{+} \qquad ..(26)$$

When $\Phi_{ri}(z)/\Phi_{rr}^{-}(z)$ is not a rational function of z

$$G_o(z) = \frac{1}{\Phi_{rr}^{+}(z)} \left[\sum_{n=0}^{\infty} z^{-n} \frac{1}{2\pi j} \oint_{\Gamma} \frac{\Phi_{ri}(z)}{\Phi_{rr}^{-}(z)} z^{n-1}\, dz\right] \qquad ..(27)$$

9.4 [illegible]

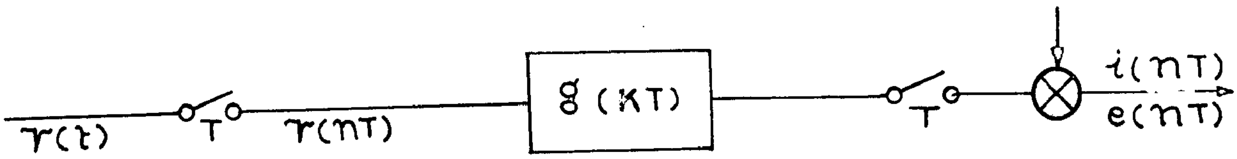


Fig. 6

Optimum design of a sampled data system for deterministic signals can be achieved on the basis of "total square error" criterion in a way analogous to the design of continuous data control systems with integral square error criterion.

Error sequence for the sampled data system shown in Fig. 6 is given by

$$e(nT) = c(nT) - i(nT) \qquad ..(1)$$

[illegible]

[illegible] $= \sum$ [illegible] ..([illegible])

= [illegible]

[illegible] = [illegible]

[illegible] = [illegible]

[illegible] = [illegible]

[illegible] may be defined as

$$E = \sum_{[illegible]}^{\infty} [illegible]$$

$$E = \sum_{[illegible]}^{\infty} [illegible]$$ ..([illegible])

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$$[illegible] = \sum_{[illegible]}^{\infty} [illegible]$$ ..([illegible])

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[illegible]

$$E = \sum_{[illegible]}^{\infty} \sum_{[illegible]}^{\infty} [illegible] \sum_{[illegible]}^{\infty} [illegible]$$ ..([illegible])

[illegible]

$$\delta J = J_{c(kT)+\epsilon b(kT)} - J_{c(kT)}$$

$$\delta J = \sum_{k=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} \left[2\epsilon c(jT) b(kT) + \epsilon^2 b(kT) b(jT)\right] \phi_{xx}(k-j)T$$

$$- 2\epsilon \sum_{k=-\infty}^{\infty} b(kT)\, \phi_{xd}(kT) \qquad ..(7)$$

$$\therefore \frac{\partial}{\partial \epsilon}(\delta J) = 2 \sum_{k=-\infty}^{\infty} b(kT) \left[\sum_{j=-\infty}^{\infty} c(jT) \phi_{xx}(k-j)T - \phi_{xd}(kT) \right]$$

$$+ 2\epsilon \sum_{k=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} b(kT)\, b(jT)\, \phi_{xx}(k-j)T \qquad ..(8)$$

$$\frac{\partial^2}{\partial \epsilon^2}(\delta J) = 2 \sum_{k=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} b(kT)\, b(jT)\, \phi_{xx}(k-j)T \qquad ..(9)$$

For an optimum system

$$\frac{\partial}{\partial \epsilon}(\delta J)\Big|_{\epsilon=0} = 0$$

$$\text{or} \quad \sum_{k=-\infty}^{\infty} b(kT) \left[\sum_{j=-\infty}^{\infty} c(jT)\, \phi_{xx}(k-j)T - \phi_{xd}(kT) \right] = 0 \qquad ..(10)$$

Since $b(kT)$ is a realizable sequence,

$$b(kT) = 0 \qquad \text{for } k<0$$

Equation 9.4-10 will hold good for $k \geq 0$, also, if, the expression within brackets is equal to zero for $k \geq 0$

$$\sum_{j=-\infty}^{\infty} c(jT)\, \phi_{xx}(k-j)T - \phi_{xd}(kT) = 0 \qquad \text{for } k\geq 0 \qquad ...(11)$$

Equation 9.4-11 gives the optimum system, with minimum total-square error criterion, since as seen from (9.4-9) $\frac{\partial^2}{\partial \epsilon^2}(\delta E)$ is a positive quantity.

9.5 OPTIMUM DESIGN WHEN SIGNAL AND NOISE ENTER AT DIFFERENT POINTS

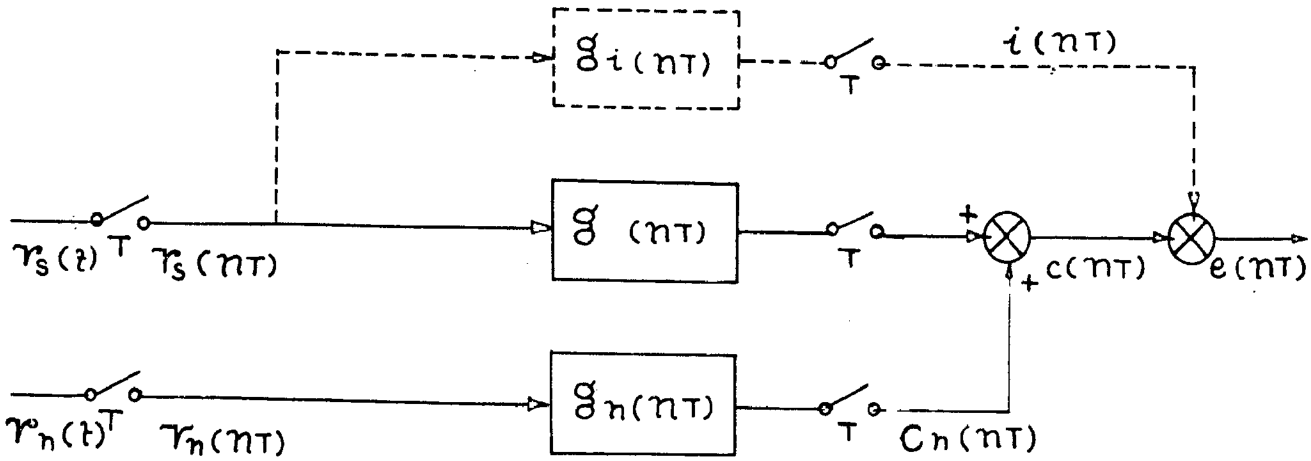


Fig. 7

In the present section, optimization of the sampled data system, Fig. 7, is considered, when noise and signal components enter the system at different points. In the block diagram shown in Fig. 7 $G_s(nT)$ is the weighting sequence for the stochastic control signal components and $G_n(nT)$ the weighting sequence for random noise samples. Both the signal and noise components are assumed to be stationary random functions.

Error sequence of the system

$$o(nT) = q_s(nT) + q_b(nT) + e(nT) \qquad ..(1)$$

where

$q_s(nT)$ = output [illegible]

$q_b(nT)$ = output [illegible]

$o(nT)$ = output [illegible]

$e(nT)$ = [illegible] output [illegible].

$q_s(nT)$ and $q_b(nT)$ can be written in terms of [illegible] and [illegible] sequences.

$$q_s(nT) = \sum_{k=0}^{n} h_s(kT)\, x_s(nT-kT) \qquad ..(2)$$

$$q_b(nT) = \sum_{k=0}^{n} h_b(kT)\, x_b(n-k)T \qquad ..(3)$$

[illegible] [illegible] 1.[illegible], 2.[illegible], 3.[illegible], 4.[illegible], 5.[illegible] will be given by

$$\overline{e^2(nT)} = \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=-N}^{N} e^2(nT)$$

$$e^2(nT) = \phi_{oo}(0) - 2\sum_{k=0}^{\infty} h_s(kT)\, \phi_{x_s o}(kT) \ldots$$

$$+ \sum_{k=0}^{\infty}\sum_{j=0}^{\infty} [\text{illegible}]$$

$$[\text{illegible}] \qquad ..(4)$$

Following [illegible] [illegible] [illegible]

[illegible]

[illegible]

optimum equation obtained is,

$$\sum_{j=0}^{\infty} g_c(jT)\,[\phi'_{rr}(n-j)T + G_f(nT)\phi_{rr}(n-k)T]' - \phi_{rs}(nT) = 0$$

for $n \geq 0$ ..(5)

When the noise and control signal component are statistically independent, cross correlation sequence is zero and optimum system equation reduces to

$$\sum_{j=0}^{\infty} g_c(jT)\,\phi_{rr}(n-j)T - \phi_{rs}(nT) = 0 \qquad \text{for } n \geq 0 \qquad ..(6)$$

9.6 [illegible]

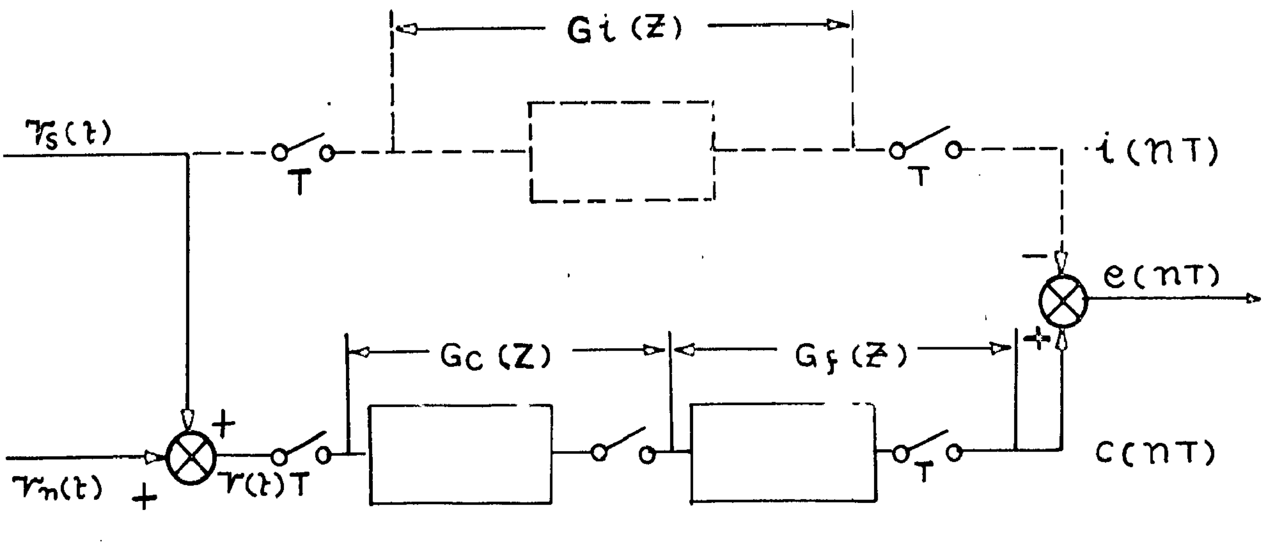


Fig. 8

In this section, the [illegible] case, is considered. Input to the system is assumed to be stationary and ergodic sampled signals. For the sampled data system block diagram of figure 8, $G_f(z)$ is the pulse transfer function of the fixed elements of the system, and $G_c(z)$ compensator pulse transfer functions are to be obtained.

[illegible] of the [illegible] is given by

$$[illegible] = \sum [illegible] \qquad \ldots([illegible])$$

$$[illegible] = \sum \sum [illegible] \qquad \ldots([illegible])$$

where

[illegible] = [illegible]

[illegible] = [illegible]

[illegible] = [illegible]

[illegible] of the [illegible] is given by

$$[illegible] = [illegible] \qquad \ldots([illegible])$$

[illegible] is given by

$$[illegible] = [illegible] \sum [illegible]$$

$$= [illegible] \qquad \ldots([illegible])$$

[illegible]

[illegible]

[illegible]

$$[illegible] = [illegible] \sum \sum [illegible]$$

$$\sum \sum \sum \sum [illegible]$$

$$[illegible] \qquad \ldots([illegible])$$

[illegible] to [illegible] for three configurations [illegible] is given by

[illegible] ..(3)

[illegible]

[illegible] ..(4)

[illegible]

[illegible] ..(5)

[illegible] ..(6)

[illegible]

$$G_c(z) = \frac{\left[\dfrac{G_f(z^{-1})\,G_f(z)\,[\Phi_{s_i r_0}(z) + \Phi_{n_i r_0}(z)]}{G_f(z)G_f(z^{-1})\,[\Phi_{s_i s_i}(z)+\Phi_{n_i s_i}(z)+\Phi_{s_i n_i}(z)+\Phi_{n_i n_i}(z)]^{-}}\right]_+}{G_f(z)G_f(z^{-1})\,[\Phi_{s_i s_i}(z)+\Phi_{n_i s_i}(z)+\Phi_{s_i n_i}(z)+\Phi_{n_i n_i}(z)]^{+}} \qquad ...(10)$$

If $G_f(z)$, the pulse - transfer function of the fixed elements has no poles or zeros outside the unit circle in z-plane, then

$$[G_f(z)\ G_f(z^{-1})]^+ = G_f(z) \qquad ..(11A)$$

$$[G_f(z)\ G_f(z^{-1})]^- = G_f(z^{-1}) \qquad ..(11B)$$

Pulse transfer function of optimum compensation is given by

$$G_c(z) = \frac{1}{\Phi_{ii}^{+}(z)} \times \left[\frac{G_f(z)\,[\Phi_{s_i r_0}(z) + \Phi_{n_i r_0}(z)]}{\Phi_{ii}^{-}(z)}\right]_+ \qquad ..(12)$$

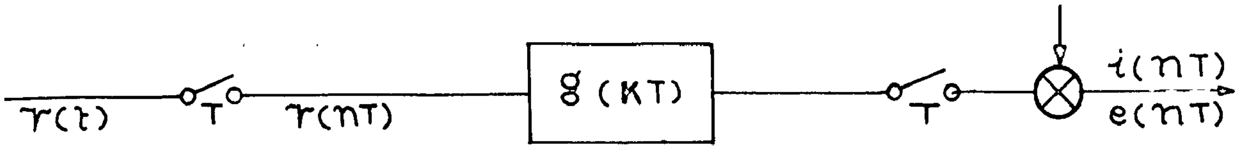
$$= \frac{\text{Pulse transfer function of optimum system when no fixed elements are present.}}{\text{Pulse transfer function of fixed elements}}$$

3.7 MEAN SQUARE ERROR

Mean square error of sampled data system is given by

$$e^2(nT) = [c(nT) - i(nT)]^2 \qquad ..(1)$$

$$\overline{e^2(nT)} = \phi_{cc}(0) + \phi_{ii}(0) - \phi_{ci}(0) - \phi_{ic}(0) \qquad ..(2)$$



r(t)
T
r(nT)
g (KT)
T
i(nT)
e(nT)

[illegible] by

$$[illegible] = \frac{1}{[illegible]} \left| \begin{matrix} [illegible] \\ [illegible] \end{matrix} \right. \qquad \ldots([illegible])$$

[illegible]

$$[illegible] = \frac{1}{[illegible]} \sum_{[illegible]}^{[illegible]} \psi([illegible])\, [illegible]$$

[illegible]

$$\psi([illegible]) = \frac{1}{2\pi i} \oint_T \frac{[illegible]}{[illegible]}\, [illegible] \qquad \ldots([illegible])$$

$$[illegible] = [illegible] \sum_{[illegible]}^{[illegible]} [illegible] \frac{1}{2\pi i} \oint_T [illegible] \qquad \ldots([illegible])$$

$$[illegible] \sum_{[illegible]}^{[illegible]} [illegible] \frac{1}{2\pi i} \oint_T [illegible] \frac{1}{[illegible]} [illegible] \qquad \ldots([illegible])$$

[illegible]

$$[illegible] = [illegible] \sum_{[illegible]}^{[illegible]} \psi([illegible]) \frac{1}{2\pi i} \oint_T \frac{[illegible]}{[illegible]} \sum_{[illegible]}^{[illegible]} [illegible] \qquad \ldots([illegible])$$

$$[illegible] \sum_{[illegible]}^{[illegible]} [illegible] \frac{1}{2\pi i} \oint_T [illegible] \frac{[illegible]}{[illegible]} \qquad \ldots([illegible])$$

$$[illegible] = [illegible] \sum_{[illegible]}^{[illegible]} \psi^{[illegible]}([illegible]) \qquad \ldots([illegible])$$

[illegible] ([illegible]), [illegible]

[illegible].

5.0 EXAMPLE

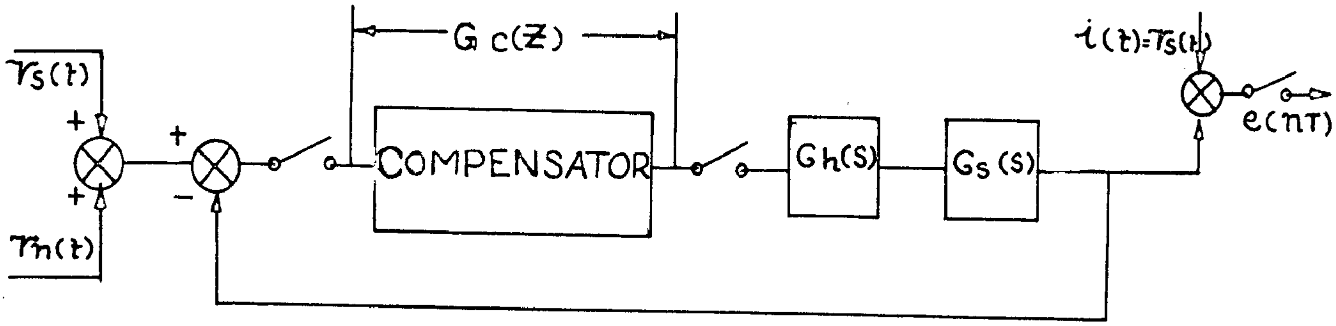


Fig. 10

A sampled data system has the configuration, as shown in Fig. 10. The transfer functions and input signals are described by

$$G_s(s) = \frac{1+0.003s}{s\left[1+\left(\frac{s}{10}\right)+\left(\frac{s}{10}\right)^2\right]} \;, \quad G_h(s) = \frac{1-e^{-Ts}}{s}$$

$$\Phi_{r_s r_s}(s) = \frac{4.5}{(0.25 - s^2)} \;, \quad \Phi_{r_n r_n}(s) = \frac{0.1}{\pi}$$

$$\Phi_{r_s r_n}(s) = 0$$

Sampling period is taken as 0.1 second. The design of sampled data compensator is to be carried out, that will minimise the mean square sampled error $\overline{e^2(nT)}$.

The block diagram of sampled data system Fig. 10, can be reduced as that of Fig. 4. Here $G_i(z)$ and $G_o(z)$ are the ideal and optimum systems respectively.

[illegible]

[illegible] $= \frac{[illegible]}{[illegible]}$ [illegible]

[illegible] $= 0$

[illegible] $= \frac{[illegible]}{[illegible]}$

[illegible] $= \frac{[illegible]}{([illegible])(z-1.0318)}$

[illegible]

[illegible] $= \frac{[illegible]}{[illegible]}$

[illegible] $= \frac{[illegible]}{[illegible]}$

[illegible] $= 1.$

[illegible]

[illegible] $= \frac{[illegible]}{[illegible]} = \frac{0.102}{[illegible]} - \frac{0.2}{[illegible]}$

[illegible]

[illegible]

$$\frac{HG_{[illegible]}(z) + HG_{[illegible]}(z) \quad G_1(z)}{HG^{*}(z)} = \frac{0.162}{z-0.9512}$$

[illegible] transfer function of system [illegible] is given by

$$G_0(z) = \frac{1}{HG^{*}(z)} \left[\frac{HG_{D}(z) + HG_{D}(z) \quad G_1(z)}{HG^{*}(z)} \right]$$

$$= \frac{0.162}{z-0.9512} \times \frac{z-0.9512}{0.970(z-0.9512)}$$

$$= \frac{0.167}{z-0.9512}$$

[illegible] transfer function of system compensator is given by

$$D(z) = \frac{1}{HG_p(z)} \left[\frac{G_0(z)}{1-G_0(z)} \right]$$

$$= \frac{(z-1)(z^2-0.7036z+0.3600)}{0.00[illegible](z^2-0.[illegible]z+1.049)} \times \frac{0.167}{z-0.[illegible]}$$

$$= \frac{99.[illegible]\,(z-1)(z^2-0.7036z+0.3600)}{(z-0.[illegible])(z^2-0.[illegible]z+1.049)}$$

CHAPTER IV

SUMMARY AND CONCLUSION

4.1 SUMMARY

In the present dissertation, statistical design of sampled-data control systems in time domain has been considered.

In the first Chapter, limitations of classical approach to the design of control systems have been pointed out, namely, this is a trial and error approach for a system with idealized signals only, disregarding random nature and the presence of noise. The limitations of classical design techniques are overcome by analytical design approach. The analytical design proceeds directly from the system specifications. It considers both control system component and noise in the system, as well as the random nature of input signals to an actual system.

The control signal component and noise component can be represented only statistically. For random signal input to a control system, mean-square value of error output is the most convenient performance index, because of its mathematical amenability, besides being selective. For deterministic signal inputs, integral square error is a convenient performance index.

In the second Chapter, [illegible] equation, [illegible]

[illegible] of coupled [illegible] systems following Bellman's [illegible]

In the third chapter of the [illegible] by Bellman's [illegible]

Optimization equation for compensator has also been obtained for the noise-free configuration. Pulse transfer function of the compensator has been obtained in terms of the pulse spectral density of input signal pulse-transfer functions of ideal system and fixed elements. If the fixed-element pulse transfer function has no poles or zeroes outside the unit circle in z- plane pulse-transfer function of the compensator, is the ratio of pulse-transfer function of the optimum system without fixed elements, and the fixed element pulse transfer function.

Optimum system equation has also been obtained for the sampled data systems, where the control signal and noise enter the system at different points.

6.2 [illegible]

The study of statistical design is restricted to the sampled-data control systems with infinitesimal sampling duration. But all the physical sampled data system have finite sampling duration. The design will be more realistic especially when the time constant of the system is not large enough compared to the sampling duration, if the finite width of sampled pulses is also considered.

Optimum sampled data system design is restricted to the cases, where the input and output signals are sampled in synchronism. It should be possible to extend the statistical design approach to the systems with input and output being sampled at different rates. The technique may also be extended to the systems where sampling itself is random.

Statistical design procedure for sampled data systems is restricted to the systems, where control signal component and noise enter the system at the same point. It should be possible to extend the concept to the systems, where control signal component and the noise enter the system at different points, as well as to the multiple input systems.

With the advancements in mathematical techniques, it should be possible to extend the statistical design approach to the nonlinear sampled-data control systems, and also the time varying systems.

In conclusion it can be said that the problem of statistical design of sampled data control systems in time domain, has been discussed briefly, in this dissertation. It is hoped that further investigations in this field will lead to more worthwhile results.

[illegible]

1. [illegible]

2. [illegible]

3. [illegible]

4. [illegible]

5. [illegible]

6. [illegible]

7. [illegible]

8. [illegible]

9. [illegible]

10. [illegible]

11. [illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]