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**STATISTICAL DESIGN
OF
SAMPLED DATA CONTROL SYSTEMS**

A Dissertation
submitted in partial fulfilment
of the requirements for the Degree
of
MASTER OF ENGINEERING
in
ADVANCED ELECTRONICS

By
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**CHECKED
1995**

**DEPT. OF ELECTRONICS & COMMUNICATION ENGINEERING
UNIVERSITY OF ROORKEE
ROORKEE
September, 1966**

ACADEMIC ACHIEVEMENT

During the year to which this report refers
of students to Dr. J. H. Lee, Director of Instruction
and Educational Administration, University of
Houston, Houston, Texas, the recipient, the recipient,
continued to receive the valuable guidance
throughout the preparation of this report, which
which is hereby set forth as being for the benefit of the
the general community.

UNIVERSITY OF TORONTO

OFFICE OF THE CHANCELLOR

Reference is made to the letter of the 10th inst. from the
President of the Board of Governors of the University of Toronto
concerning the proposed changes in the structure of the
Faculty of Education. The Board of Governors has approved the
proposed changes and has authorized the Chancellor to
execute the necessary arrangements. The Board of Governors
has also authorized the Chancellor to execute the necessary
arrangements for the transfer of the property of the
Faculty of Education to the University of Toronto.

The Board of Governors has also authorized the Chancellor
to execute the necessary arrangements for the transfer of
the property of the Faculty of Education to the University
of Toronto.

Yours faithfully,
1953, December 9, 1953.

Hubert
(S. 101)

1953, December 9, 1953
The Chancellor
University of Toronto
Toronto

ABSTRACT

Analytical techniques for the design of control systems is adopted to overcome the drawbacks of classical trial and error approach which is based on the system response to a selected input in absence of noise. Statistical properties of noise and input signals are used to make the system design more realistic.

In this dissertation, analytical design of sampled data system, using statistical properties of signals, is carried out, extending the technique for design of continuous data system. Optimization of sampled-data system is carried out in time-domain, minimizing the mean-square value of error sequence. In conclusion, optimal-system pulse-transfer function is obtained in terms of pulse-spectral density of input signal and cross-spectral density of input, ideal output signals. Optimization is done for free and anti-free configurations.

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CHAPTER I

1.1 Introduction

Encoded data systems have been in vogue recently. Their use is increasing rapidly with advances in other fields of science and technology. Encoded data systems are the systems where the signal is encoded at one or more points.

The encoding of the signal may be inherent. The input to a carrier tracking system is in the form of a pulse train. In the multiplexed control or communication systems, data from several channels is encoded and multiplexed for transmission over the same channel.

Even when the encoding of the continuous signal is introduced intentionally to improve the performance of the system, continuous system with transportation lag can be stabilized by introducing encoding systems with encoded data can, in general, facilitate the realization of adaptive regulation. Encoded data systems are also used for improved connectivity. By encoding the low power signal, the encoding device can be made extremely sensitive in terms of power gain. The encoding is also introduced, because of the accuracy and accuracy with which the digital signals can be stored, transmitted and received.

Some of the applications of encoded data systems

1. Anti-aircraft ground environment (AAG) system used by U.S. Department of Defense, a large complex system consisting of telemetry link for weapon guidance.
2. Audio control of low power notes.
3. Signal control in controlling machine tools, for production components.
4. Communication and control systems for aerospace applications use sampled data signals.

1.2. CHARACTERIZATION OF SYSTEMS IN THE TIME AND FREQUENCY DOMAINS

Control system transfer functions can be represented either in the time domain or in the frequency domain. The methods of design for control systems can also be described along these lines.

In the time domain, transient and steady state behavior of the system are of interest. System specifications, e.g., maximum overshoot, time to start overshoot, maximum settling time etc., can be expressed in terms of the damping ratio and undamped natural frequency of the system. Steady state behavior of a system is determined by the roots of the characteristic equation.

$$1 + G(s)H(s) = 0$$

where $G(s)H(s)$ is open-loop transfer function of the

system.

Stability roots are key to the dynamic performance of a system. Root calculation of the characteristic equation by direct analytical method is laborious and impractical for design purposes. A graphical approach makes plotting of root locus practical for complex systems. This makes available a complete picture of stability changes due to the effect of individual elements. Original root locus is modified by insertion of compensation element that places the roots of characteristic equation at a more favorable point.

Root locus approach is essentially an analytical approach, where the characteristic equation of the system must be known.

Other method of control system design is that in frequency domain. The system can be represented by its response to a sinusoidal signal of constant amplitude. This is essentially a graphical method of system design. Frequency response transfer function of a system can be represented by Bode plot, Nichols plot or magnitude versus phase shift plot. The frequency response design methods are preferable, because the experimental data is in the form of frequency response and final design can also be checked by frequency analysis.

Various frequency domain specifications are the system bandwidth, resonance peak, resonance frequency, cutoff

rate, gain margin, phase margin etc.

Bode plot method of control system design and compensation is preferred, because the effect of compensation is easily obtained by adding magnitude and phase plots curves of individual elements.

1.5. ANALOGUE CONTROL SYSTEMS WITH SAMPLED-DATA

Compensation and design techniques for the continuous data control systems also extended to the design of sampled-data systems (7). But the sampling operation makes the design of feedback system compensation more difficult. Compensation of sampled-data systems may be effected by two general methods.

1. Compensation by continuous devices, making use of continuous data compensation networks in series with other components of the system.
2. Compensation by pulsed data devices whose output is sampled in synchronism with its input at a constant rate.

When the transfer function of a system is in factored form, it is preferable to work with Bode diagrams, because of the ease and simplicity with which the asymptotic Bode plot of a transfer function can be plotted and redrawn.

Characteristic equation of a sampled data control system is a transcendental equation and open loop transfer function of the system is a transcendental function in z . Difficulties encountered in plotting of transcendental functions, discourage the application of Bode-plot techniques for the design of sampled data control systems. This difficulty is overcome by converting transcendental function in z into a rational function in s , by the transformation

$$s = \frac{1 - z^{-1}}{T}$$

This process maps the primary and complementary origin of left half of $z =$ plane into unit circle in s plane. Bilinear transformation,

$$s = \frac{1 + v}{1 - v}$$

maps, unit circle in $z =$ plane into imaginary axis of another plane v , and interior of the unit circle into entire left-half of $v =$ plane.

In the extension of root-locus techniques to the design of sampled-data control systems, numerous difficulties are encountered in construction of root-locus from a starred transfer function, because of infinite number of poles and zeroes. Complicated nature of root-locus plot in z plane makes it difficult to study the effect of added compensation.

These calculations are avoided by the use of s -transfer techniques. The overall system characteristic equation is transformed to

$$1 + \Delta(s) = 0$$

where, $\Delta(s)$ is complex plane transfer function, a rational function of s containing a finite number of poles and zeros. Stable operation of the system requires that the root locus of sampled data system be confined to the unit circle in s plane.

1.4. DESIGN OF GENERAL SYSTEMS

Classical design techniques suffer from several limitations, that impede the systematic design of control systems and compensating networks:

- 1. Compensating elements and control systems are designed, assuming, that a known input like sinusoidal like etc., or disturbance, is available. Such signals do not occur in general, and the design becomes unrealistic.
- 2. System is designed only for processing the signal. No consideration is taken of inevitability of noise in the system due to physical nature of the system, e.g., shot noise in the tubes, manufacturing tolerances or other disturbances.

3. Generalized design techniques are based on control and error concepts. The designer makes some changes in the system, studies the changes in response, again makes changes in system parameters until system performance is within the satisfactory limits. The system designer, using conventional design not know any means to recognize in advance that out of specifications, trial and error system may be repeated and stop the other, without obtaining the desired results.

1.3. ANALYTICAL DESIGN METHOD

To overcome the drawbacks of repeated trials and error conventional, analytical design method is adopted. In analytical procedure the design of control system begins with the specification of system input and desired output. By analytical approach, an immediate out of specifications can be recognized, and either a new set of specifications is prescribed or the design is given up as not being feasible.

Specifications such as low error in the analytical design of control systems with continuous control, based on the zero steady state error criterion and integral error curve criterion (9, 10, 14, 15). Analytical design method was introduced by Barden, Condit and others (14), specifies (10) that specifications that do not explicitly need the trial and error procedure. This includes the design of transfer function in approximation. The system to be designed

or fixed configuration, according to , if there are no constraints of the system configuration or fixed elements, constraints of fixed elements only, and both the system configuration and the fixed elements being specified. Attempts have also been made for the analytical design of systems with constraints like saturation(12) or system bandwidth (13).

9.6. ANALYTICAL DESIGN METHODS RELATING TO SAMPLED-DATA CONTROL SYSTEMS

Some work has been done on the analytical design of sampled-data systems. In a paper concerning the statistical treatment of sampled data control systems for random signals(11), Nori , deals with the correlation function of the error, and power-spectral densities for sampled data control systems. This paper also deals with modified z - transforms, when the signals are considered at sampling instants and during intervals between sampling instants.

Work has been done on the statistical design of sampled data control systems, utilizing statistical properties of input and output signals (2,3,16). The design of control system is carried in z - domain, minimizing the mean-square error between desired and actual outputs of the system. One has also considered compensation of sampled-data control system in z - domain (17). See also.

also considered design procedures for discrete-time control systems subject to some limit (17). Design (1) has considered statistical design of sampled-data systems with randomly varying sampling.

The statistical design procedure has the same limitations. For example, design of systems with statistical procedure is unrealistic, since the plant would require a priori and a priori of the desired system to be available, as well as the plant and system, and of the plant and system available themselves by the system.

Statistical method of analytical design is highly restrictive as far as inputs are concerned. The inputs are assumed to be stationary and ergodic. In general the conditions of ergodicity are not known, so any to define themselves to define how the actual data whether the conditions of ergodicity is true or not.

1.1. ANALYTICAL DESIGN

The problem under investigation is to analytically design the sampled-data system, utilizing the statistical properties of the desired signal, control signal component, and the noise.

A control system can be considered as a free, non-free or fixed configuration system, depending whether,

system performance specifications alone, or system performance specifications alongwith some elements of the system, or system performance specifications alongwith the system configuration, are specified.

In the present work, the design of sampled-data control system, has been attempted in the time domain.

In the second chapter of this dissertation, statistical nature of input signals, correlation techniques utilizing the statistical properties of the signals, requirements of mean-square error as performance index are considered for the statistical design procedure.

In the third chapter, time-domain equation is developed for the optimum sampled-data system weighting sequences that gives minimum mean-square error, this equation is similar to Wiener - type equation for the continuous data system.

$$\int_{-\infty}^{\infty} \psi_2^*(\tau) \cdot \psi_1(\tau - \tau_1 - \tau_2) = \psi_{\tau_2}(\tau_1) = 0, \text{ for } \tau_2 > 0$$

where

$\psi_1(\tau)$ = weighting function of optimum linear control system.

$R_{xx}(0, 0)$ = auto-correlation function of input signal.

$R_{yy}(0, 0)$ = cross-correlation of functions of input, ideal output signals.

An explicit solution is found for the pulse transfer-function of the optimum system, with non-square value of error sequence.

Steady-state solution, is extended to the discrete system, in non-square error case, with ideal elements, or continuous configuration system.

Analytical design of sampled-data systems with continuous signals is also attempted.

CONTROL SYSTEMS

CHAPTER 1. (CONTINUOUS)

1.1 INTRODUCTION

Inputs to the control systems are of random nature. These random signals can be represented statistically, in terms of probability density functions, average values and moments of random signal. The average value of the system output may be found by a statistical estimation of system performance. Statistical analysis can be a powerful method to analyze the behavior of a control system with random inputs.

Transfer function of a linear time-invariant system is a useful measure of the performance of the system. In the frequency domain, the characteristics of the transfer function transfer function of the control system is obtained in terms of system transfer function of the system. The transfer function of a system is a function of frequency in the system. The transfer function of a system is a function of frequency in the system. The transfer function of a system is a function of frequency in the system.

1.2 CONTINUOUS-TIME SYSTEMS

In continuous-time systems, the input signal is a continuous function of time, which may be completely random or partially random. A control system is designed to produce a desired output response.

random from a range of possible values, but within certain limits. Many performance inaccuracies are essentially random functions of time and can be prescribed in statistical sense only. Because of the manufacturing tolerances in a system the sensitivity and other fixed parameters of operating components are subject to random fluctuations. When examined in detail all physical processes are discontinuous and indeterminate. The voltage output of a vacuum tube oscillator is considered as a continuous smooth function. But, on microscopic examination, the wave is found to be relatively rough because of shot noise.

Auto-and- cross correlation functions, and spectral densities for random signals are given by

autocorrelation function of signal $x(t)$

$$R_{xx}(\tau) = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T x(t) x(t+\tau) dt \quad \dots(1)$$

cross correlation function of signals $x(t)$ and $y(t)$

$$R_{xy}(\tau) = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T x(t) y(t+\tau) dt \quad \dots(2)$$

spectral density of signal $x(t)$

$$S_{xx}(v) = \frac{1}{2\pi} \int_{-\infty}^{\infty} R_{xx}(\tau) e^{-jv\tau} d\tau \quad \dots(3)$$

$$S_{xy}(v) = \frac{1}{2\pi} \int_{-\infty}^{\infty} R_{xy}(\tau) e^{-jv\tau} d\tau \quad \dots(4)$$

2.3 CORRELATION FUNCTIONS FOR REAL AND COMPLEX VALUED RANDOM SIGNALS

Correlation sequences for complex, random signals (11) are defined as follows:

Auto correlation sequence,

$$R_{xx}(k) = \lim_{N \rightarrow \infty} \frac{1}{2N+1} \sum_{n=-N}^N x(n) x^*(n+k) \quad \dots(1)$$

Cross correlation sequence

$$R_{xy}(k) = \lim_{N \rightarrow \infty} \frac{1}{2N+1} \sum_{n=-N}^N x(n) y^*(n+k) \quad \dots(2)$$

Auto spectral density is the Fourier transform of the correlation function.

$$S_{xx}(\omega) = \sum_{k=-\infty}^{\infty} R_{xx}(k) e^{-j\omega k} \quad \dots(3)$$

Cross auto spectral density,

$$S_{xy}(\omega) = \sum_{k=-\infty}^{\infty} R_{xy}(k) e^{-j\omega k} \quad \dots(4)$$

Correlation sequences and auto-spectral density of real data signals are characterized by

$$P_{22}(z) = P_{11}(-z) \quad \dots \quad \dots (5)$$

$$P_{20}(z) = P_{02}(-z) \quad \dots \quad \dots (6)$$

$$P_{11}(z^{-1}) = P_{11}(z) \quad \dots \quad \dots (7)$$

$$P_{20}(z^{-1}) = P_{02}(z) \quad \dots \quad \dots (8)$$

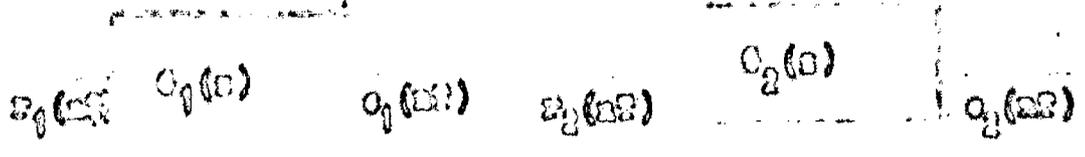
If the response of the reduced data system $G(z)$ to an input $x^T(z)$ is $y^T(z)$, then the response of this system to an input $P_{11}(z)$ is $P_{20}(z)$ and the response of this system to an input $P_{02}(z)$ is $P_{11}(z)$.

$$\sum_{k=-\infty}^{\infty} c(kz) P_{11}(z^{-1} - kz) = P_{20}(z) \quad \dots \quad \dots (9)$$

$$\sum_{k=-\infty}^{\infty} c(kz) P_{02}(z^{-1} - kz) = P_{11}(z) \quad \dots \quad \dots (10)$$

$$P_{20}(z) = c(z) P_{11}(z) \quad \dots \quad \dots (11)$$

$$P_{11}(z) = c(z) P_{02}(z) \quad \dots \quad \dots (12)$$



With reference to Fig. 1 cross correlation sequences and also cross spectral density for the output sequences of sampled data systems $C_1(z)$ and $C_2(z)$ possess following characteristics :

$$R_{z_2 c_1}(kz) = \sum_{n=-\infty}^{\infty} c_1(nz) R_{z_2 z_1}(kz - nz) \quad \dots(13)$$

$$R_{c_1 c_2}(kz) = \sum_{n=-\infty}^{\infty} c_2(nz) R_{c_1 z_2}(kz - nz) \quad \dots(14)$$

$$R_{z_1 c_2}(kz) = \sum_{n=-\infty}^{\infty} c_2(nz) R_{z_1 z_2}(kz - nz) \quad \dots(15)$$

$$R_{c_2 c_1}(kz) = \sum_{n=-\infty}^{\infty} c_1(nz) R_{c_2 z_1}(kz - nz) \quad \dots(16)$$

$$D_{z_2 c_1}(z) = C_1(z) D_{z_2 z_1}(z) \quad \dots(17)$$

$$D_{c_1 c_2}(z) = C_2(z) D_{c_1 z_2}(z)$$

$$D_{c_1 z_2}(z) = C_1(z^{-1}) C_2(z) D_{z_1 z_2}(z) \quad \dots(18)$$

$$D_{z_1 c_2}(z) = C_2(z) D_{z_1 z_2}(z) \quad \dots(19)$$

$$D_{c_2 c_1}(z) = C_1(z) D_{c_2 z_1}(z)$$

$$D_{c_2 z_1}(z) = C_1(z) C_2(z^{-1}) D_{z_2 z_1}(z) \quad \dots(20)$$

2.4 PERFORMANCE INDEX

Performance index is defined as some mathematical function of measured response, the function being chosen to give emphasis to system specifications of interest (3,6) Principles for selection of a performance index are:

1. Reliability : Performance index should measure the quality of response as closely as possible.
2. Selectivity : Optimum value of system parameters should be clearly distinguishable from zero characteristic, such as a minimum value of the performance index versus system parameters.
3. Performance index should be easily calculable from existing techniques
4. Performance index should be unaffected by uniformly short-lived deviations from the mean, or shifts in its location; instead it should be a measure of the average behaviour of control system.

The mean-square error criterion is widely used, it is defined as follows :

$$\begin{aligned}
 \overline{e^2} &= \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T e^2(t) dt \\
 &= \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T [e(t) - d(t)]^2 dt
 \end{aligned}$$

where

- $e(t)$ = Actual output of the system.
- $S(t)$ = Ideal output of the system
- $e(t)$ = Error in the output

One of the main reasons for this type of non-square error criterion stems from the mathematical convenience. A different error criterion may be preferable once, it is obtained with mathematical difficulties. This square error criterion is adopted because the unacceptability of an error grows with the magnitude.

The square value of a random process is one of the easier parameters to estimate experimentally. Furthermore, when together with the mean value of a random process, yields information about the process that is in convenient. Central Limit theorem is often invoked to assume Gaussian processes.

Integral square error is a measure of transient response of the system. It was first applied by A.S. Holt.

ISE is defined as :

$$ISE = \int_{-\infty}^{\infty} e(t)^2 dt$$

Both Integral-square error and non-square error can be represented in terms of the system impulse response and correlation functions amongst system input and desired output. Both ISE and MSE criteria can be extended to

the sampled data systems also the systems being characterized as having sampled inputs and outputs.

Mean square error,

$$\begin{aligned}
 \sigma^2(nT) &= [d(nT) - o(nT)]^2 \\
 &= \lim_{T \rightarrow 0} \frac{1}{2T} \sum_{n=0}^N o^2(nT)
 \end{aligned}$$

$$\begin{aligned}
 \text{Total square error} &= \sum_{n=0}^N [d(nT) - o(nT)]^2 \\
 &= \sum_{n=0}^N o^2(nT)
 \end{aligned}$$

2.5 CONTINUOUS-TIME APPROXIMATION FOR SAMPLED DATA SYSTEMS

In Section 2.4, the error approach (14) to the design of optimum linear systems, a system is classified as free, semi-free or fixed configuration system, depending on the restrictions placed on configuration.

In the following section, the design of an optimum free configuration, continuous-time system with minimum mean square error criterion is given (14).

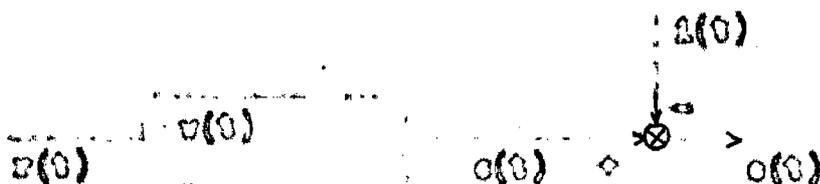


Fig. 2

The error signal at the output of the system, with reference to Fig. 2, is defined as

$$e(t) = c(t) - z(t) \quad \dots(1)$$

where $F(t)$ = desired system input to the system

$z(t)$ = ideal output of the system

$c(t)$ = actual output of the system

$\Psi_2(t)$ = transfer function of optimum linear system

The mean square error expressed in terms of cross-correlation function of system input and ideal output is given by

$$\begin{aligned} \overline{e^2(t)} &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Psi_2(\tau_1) \Psi_2(\tau_2) \rho_{FF}(\tau_1 - \tau_2) d\tau_1 d\tau_2 \\ &= 2 \int_{-\infty}^{\infty} \Psi_2(\tau) \rho_{FF}(\tau) d\tau = \rho_{EE}(0) \quad \dots(2) \end{aligned}$$

The bar indicates time average of the function.

By calculus of variations, equation for optimum

system minimizing the mean square error is obtained. The optimum mean square error is given by the following equation

$$\int_{-\infty}^{\infty} \Psi_2(\tau_2) \rho_{FF}(\tau_1 - \tau_2) d\tau_2 - \rho_{Fz}(\tau_1) = 0 \quad \text{for } \tau_1 \geq 0 \quad \dots(3)$$

Transfer function of the optimum system, in terms of spectral and cross-spectral density of system

input and output are given by the equation

$$Y(s) = \frac{\frac{P_{21}(s)}{P_{11}(s)}}{P_{12}(s)}}{P_{22}(s)} \quad \dots(4)$$

where

$Y(s)$ = transfer function of the system

$P_{11}(s)$ = factor of $P_{11}(s)$ which includes all the poles and zeros of $P_{11}(s)$ in the left half plane.

$P_{22}(s)$ = factor of $P_{22}(s)$ which includes all the poles and zeros of $P_{22}(s)$ in the right half plane.

$\frac{P_{21}(s)}{P_{11}(s)}$ = component of $\frac{P_{21}(s)}{P_{11}(s)}$, which has all the poles in the left half plane.

$\frac{P_{12}(s)}{P_{22}(s)}$ = component of $\frac{P_{12}(s)}{P_{22}(s)}$, which has all

the poles in the right half plane.

2.6 NEW'S APPROACH TO DESIGN OF CONTROL SYSTEMS (PART 2) DATA CONTROL SYSTEM

According to S.S. New's approach (16), the optimal design of a multi-rate system is obtained for sampled stochastic control signal $U_j(z)$ and sampled noise $W_j(z)$. Both the signal and noise are

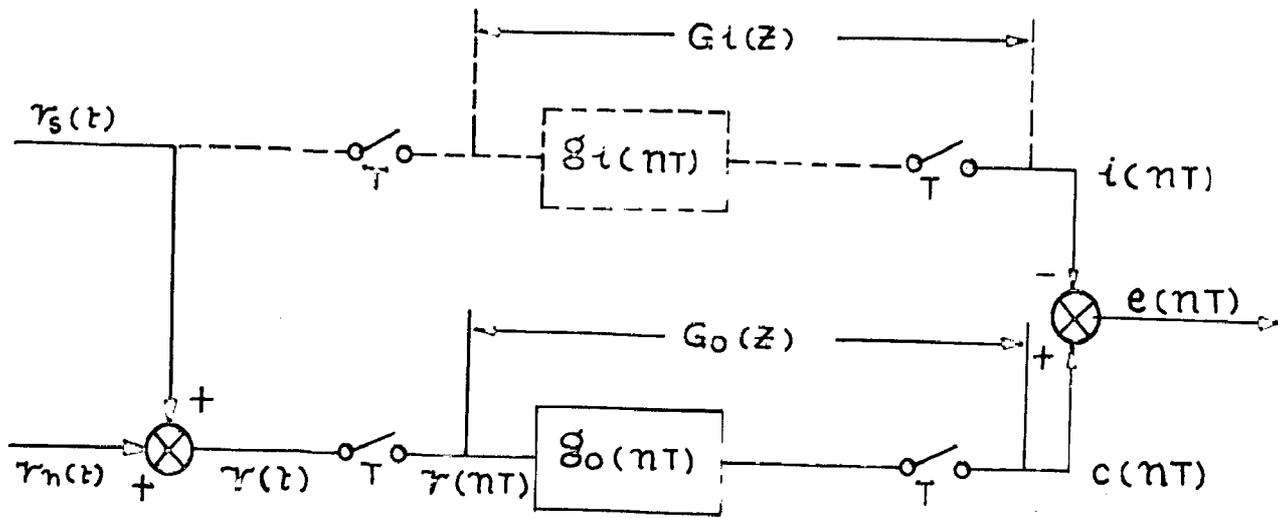


Figure 9.

assumed to be stationary random functions.

For the system shown in Fig. 9, $G_o(z)$ is the pulse-transfer function of system control system, $G_i(z)$ is pulse transfer function of the ideal system then, there is no noise present.

Error sequence of the system is,

$$o(nT) = o(nT) - g(nT) \quad \dots(1)$$

where

T = sampling period.

Error sequence of the system may be written

$$o(nT) = o_o(nT) + o_n(nT) = g(nT) \quad \dots(2)$$

where,

$o_o(nT)$ = System response to control signal $r(nT)$

$q_j(t)$ = (total number of control points $q_j(t)$)

The LHS = square value of error required of the system for defined as

$$e^2(t) = \sum_{j=1}^N \frac{1}{2} q_j^2(t) \quad \dots(5)$$

$$= \frac{1}{2} \sum_{j=1}^N (q_1^2(t) + q_2^2(t) + q_3^2(t) + q_4(t)q_5(t) + q_5(t)q_6(t) + q_6(t)q_7(t) + q_7(t)q_8(t) + q_8(t)q_9(t))$$

$$= \frac{1}{2} (q_1^2(t) + q_2^2(t) + q_3^2(t) + q_4(t)q_5(t) + q_5(t)q_6(t) + q_6(t)q_7(t) + q_7(t)q_8(t) + q_8(t)q_9(t)) \quad \dots(6)$$

From the equations 2.3-1 to 2.3-4 the error square value of error can be written in terms of control point equations and this = control condition.

$$e^2(t) = f_{q_1 q_1}(t) + f_{q_2 q_2}(t) + f_{q_3 q_3}(t) + f_{q_4 q_5}(t) + f_{q_5 q_6}(t) + f_{q_6 q_7}(t) + f_{q_7 q_8}(t) + f_{q_8 q_9}(t) \quad \dots(7)$$

$$\begin{aligned}
 \phi^2(u) &= \frac{1}{2\pi j} \int_{\Gamma} \left[\mathcal{F}_{c_1 c_1}(u) + \mathcal{F}_{c_2 c_2}(u) + \mathcal{F}_{c_3 c_3}(u) \right. \\
 &\quad \left. + \mathcal{F}_{c_1 c_2}(u) + \mathcal{F}_{c_2 c_1}(u) + \mathcal{F}_{c_1 c_3}(u) + \mathcal{F}_{c_3 c_1}(u) \right. \\
 &\quad \left. + \mathcal{F}_{c_2 c_3}(u) + \mathcal{F}_{c_3 c_2}(u) \right] \cdot z^{-1} dz \quad \dots(7)
 \end{aligned}$$

where, contour of integration Γ is the unit circle in the z plane.

These auto spectral densities may be expressed in terms of auto spectral densities of input signals and auto transfer functions with the help of equations (2.9-19) to (2.9-20).

- $S_{c_1 c_1}(u) = D_{F_1 F_1}(u) G_1(u) G_1^*(u^{-1}) \dots(8)$
- $S_{c_2 c_2}(u) = D_{F_2 F_2}(u) G_2(u) G_2^*(u^{-1}) \dots(9)$
- $S_{c_3 c_3}(u) = D_{F_3 F_3}(u) G_3(u) G_3^*(u^{-1}) \dots(10)$
- $S_{c_1 c_2}(u) = D_{F_1 F_2}(u) G_1(u) G_2^*(u^{-1}) \dots(11)$
- $S_{c_2 c_1}(u) = D_{F_2 F_1}(u) G_2(u) G_1^*(u^{-1}) \dots(12)$
- $S_{c_1 c_3}(u) = D_{F_1 F_3}(u) G_1(u) G_3^*(u^{-1}) \dots(13)$
- $S_{c_3 c_1}(u) = D_{F_3 F_1}(u) G_3(u) G_1^*(u^{-1}) \dots(14)$
- $S_{c_2 c_3}(u) = D_{F_2 F_3}(u) G_2(u) G_3^*(u^{-1}) \dots(15)$
- $S_{c_3 c_2}(u) = D_{F_3 F_2}(u) G_3(u) G_2^*(u^{-1}) \dots(16)$

$$D_{\frac{1}{2}, \frac{1}{2}}(z) = D_{\frac{1}{2}, \frac{1}{2}}(z) D_0(z) D_0(z^{-1}) \quad \text{--- (1)}$$

$$D_{\frac{1}{2}, \frac{1}{2}}(z) = D_{\frac{1}{2}, \frac{1}{2}}(z) D_0(z) D_0(z^{-1}) \quad \text{--- (2)}$$

From the above we can deduce that

$$D^2(z) = \frac{1}{2\pi} \int_{\pi} D_{\infty}(z) z^{\theta} dz \quad \text{--- (3)}$$

where

$$D_{\infty}(z) = D_0(z) \left[D_0(z) \left(D_0(z^{-1}) \right) \right]^{-1} D_{\frac{1}{2}, \frac{1}{2}}(z)$$

- $D_0(z) \left(D_0(z^{-1}) \right) = D_0(z^{-1}) \quad D_{\frac{1}{2}, \frac{1}{2}}(z)$
- $D_0(z^{-1}) \left(D_0(z) \right) = D_0(z) \quad D_{\frac{1}{2}, \frac{1}{2}}(z)$
- $D_0(z) \left(D_0(z^{-1}) \right) = D_{\frac{1}{2}, \frac{1}{2}}(z) \quad \text{--- (4)}$

Consequently we can deduce that the above equation

can be written in the following form: $D^2(z) = \frac{1}{2\pi} \int_{\pi} \eta(z) D_{\infty}(z) z^{\theta} dz$ where $\eta(z) = D_0(z) \left(D_0(z^{-1}) \right) \left(D_0(z) \right)^{-1} D_{\frac{1}{2}, \frac{1}{2}}(z)$. This is the same as the above equation (3) where $\eta(z) = D_0(z) \left(D_0(z^{-1}) \right) \left(D_0(z) \right)^{-1} D_{\frac{1}{2}, \frac{1}{2}}(z)$.

$$D^2(z) = \frac{1}{2\pi} \int_{\pi} \eta(z) D_{\infty}(z) z^{\theta} dz \quad \text{--- (5)}$$

$$D_{\frac{1}{2}, \frac{1}{2}}(z) z^{\theta} = \frac{1}{2\pi} \int_{\pi} \eta(z) z^{\theta} dz$$

$$D_0(z) \left(D_0(z^{-1}) \right) \left(D_0(z) \right)^{-1} D_{\frac{1}{2}, \frac{1}{2}}(z) z^{\theta} = \frac{1}{2\pi} \int_{\pi} \eta(z) z^{\theta} dz \quad \text{--- (6)}$$

where

$$D(s) = D_{P_0 P_0}(s) \prod_{P_0} (s) \prod_{P_0} (s) \prod_{P_0} (s) \dots D_{P_0 P_0}(s) \quad \dots(12)$$

For reason of stability all the poles and zeroes of $\theta_0(s)$ and $\eta(s)$ lie inside the unit circle in a plane, and those of $\theta_0(s^{-1})$ and $\eta(s^{-1})$ lie outside the unit circle. From equation 2.9-3 to 2.9-6

$$D(s) = D(s^{-1}) \quad \dots(13)$$

and $D(s)$ may be written as

$$D(s) = D^+(s) D^-(s) \quad \dots(14)$$

where $D^+(s)$ is a function of s with poles and zeroes lying inside the unit circle in a plane, and $D^-(s)$ is a function with poles and zeroes lying outside the unit circle.

$$\frac{\delta^2}{\delta \sigma^2} (s) = \frac{1}{2\pi j} \int_{\Gamma} \eta(s) \theta_0(s^{-1}) D^+(s) D^-(s) \theta_0(s^{-1}) \frac{1}{P_0 P_0}(s)$$

$$\frac{1}{P_0 P_0}(s) \Big|_{s^{-1} \in \sigma} + \frac{1}{2\pi j} \int_{\Gamma} \eta(s^{-1}) \Big|_{\theta_0(s)}$$

$$D^+(s) D^-(s) = \theta_0(s) \frac{1}{P_0 P_0}(s) + \theta_0(s^{-1}) \frac{1}{P_0 P_0}(s) \Big|_{s^{-1} \in \sigma} \quad \dots(15)$$

$$= \frac{1}{2\pi j} \int_{\Gamma} \eta(s) D^+(s) \theta_0(s^{-1}) D^-(s) - \frac{\theta_0(s^{-1}) \frac{1}{P_0 P_0}(s)}{D^+(s)}$$

$$s^{-1} \in \sigma + \frac{1}{2\pi j} \int_{\Gamma} \eta(s^{-1}) D^-(s) \theta_0(s) D^+(s)$$

$$\frac{\theta_0(s) \frac{1}{P_0 P_0}(s)}{D^-(s)} \Big|_{s^{-1} \in \sigma} \quad \dots(16)$$

Second term in braces of equation (2.6-16) contains poles inside the unit circle in z -plane as well as outside the unit circle, this may be written as

$$\frac{O_2(z) \prod_{p=1}^n (z - p) \prod_{q=1}^m (z - q^{-1})}{z^k} = \frac{O_2(z) \prod_{p=1}^n (z - p) \prod_{q=1}^m (z - q^{-1})}{z^k} + \frac{O_2(z) \prod_{p=1}^n (z - p) \prod_{q=1}^m (z - q^{-1})}{z^k}$$

Symbol $\int_{\gamma} f(z) dz$ denotes operation of finding part of a function of z with poles outside unit circle in z -plane and $\int_{\gamma} f(z) dz$ denotes the operation of finding part of a function of z with poles inside unit circle in z -plane. The contour integral vanishes, if the integrand has its poles either all inside the unit circle or all outside the unit circle.

$$\frac{1}{2\pi j} \int_{\gamma} \gamma(z) z^k \frac{O_2(z^{-1}) \prod_{p=1}^n (z^{-1} - p) \prod_{q=1}^m (z^{-1} - q^{-1})}{z^k} z^{-l} dz = 0$$

$$\frac{1}{2\pi j} \int_{\gamma} \gamma(z) z^k \frac{O_2(z) \prod_{p=1}^n (z - p) \prod_{q=1}^m (z - q^{-1})}{z^k} z^{-l} dz = 0 \tag{2A}$$

$$\dots \tag{2B}$$

The equation (2.7-1) is reduced to

Fig. 10: A block diagram of a sampled-data control system. The input is a continuous-time signal $r_s(t)$. This signal is sampled at intervals of T to produce $r(nT)$. The sampled signal $r(nT)$ is then processed by a controller $G_c(z)$ and a feedback element $G_f(z)$ to produce a control signal $c(nT)$. The control signal $c(nT)$ is then processed by a plant $G_t(z)$ to produce the output $i(nT)$. The output $i(nT)$ is then sampled to produce $e(nT)$, which is fed back to the controller $G_c(z)$ and the feedback element $G_f(z)$.

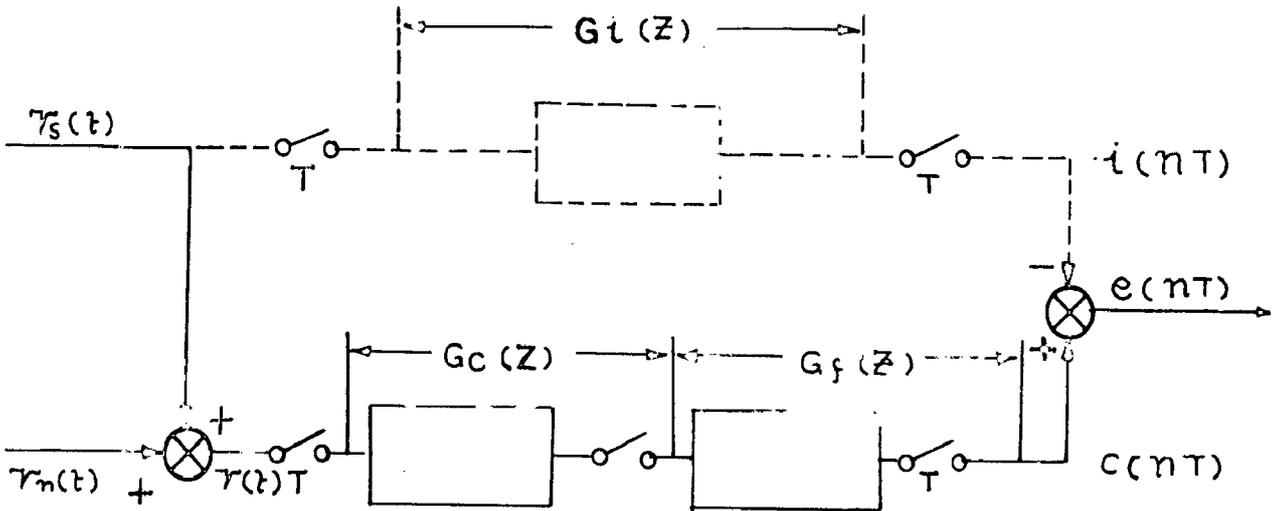


Fig. 10

The block diagram of a sampled-data control system is shown in Fig. 10. The input is a continuous-time signal $r_s(t)$. This signal is sampled at intervals of T to produce $r(nT)$. The sampled signal $r(nT)$ is then processed by a controller $G_c(z)$ and a feedback element $G_f(z)$ to produce a control signal $c(nT)$. The control signal $c(nT)$ is then processed by a plant $G_t(z)$ to produce the output $i(nT)$. The output $i(nT)$ is then sampled to produce $e(nT)$, which is fed back to the controller $G_c(z)$ and the feedback element $G_f(z)$.

For the sampled-data control system, Fig. 10, the symbols used are as follows:

- $r_s(t)$ = sampled input or the control signal
- $r(nT)$ = the sampled signal
- $G_c(z)$ = the controller transfer function of the control signal
- $G_f(z)$ = the feedback transfer function of the control signal

$$\begin{aligned}
 & \sigma^2(z) \\
 &= \frac{1}{2\pi} \int_{\gamma} \eta(z) \underbrace{U_2(z)U_2(z^{-1})}_{\sigma^2(z)} \underbrace{U_0(z^{-1})U_0(z)}_{\sigma^2(z^{-1})} dz \\
 & \quad \underbrace{U_2(z^{-1})U_2(z)}_{\sigma^2(z^{-1})} \underbrace{U_0(z)U_0(z^{-1})}_{\sigma^2(z)} dz + \frac{1}{2\pi} \int_{\gamma} \eta(z^{-1})
 \end{aligned}$$

$$\underbrace{U_2(z^{-1})U_2(z)}_{\sigma^2(z^{-1})} \underbrace{U_0(z)U_0(z^{-1})}_{\sigma^2(z)} dz - \underbrace{U_2(z)U_2(z^{-1})}_{\sigma^2(z)} \underbrace{U_0(z^{-1})U_0(z)}_{\sigma^2(z^{-1})} dz \dots (7)$$

where $\sigma^2(z) = \underbrace{U_2(z)U_2(z^{-1})}_{\sigma^2(z)} \underbrace{U_0(z)U_0(z^{-1})}_{\sigma^2(z)} \dots (8)$

Notation $\sigma^+(z) = \sigma^2(z) \sigma^+(z) \dots (9)$

$$U_2(z)U_2(z^{-1}) = \underbrace{U_2(z)U_2(z^{-1})}_{\sigma^2(z)} \underbrace{U_2(z)U_2(z^{-1})}_{\sigma^2(z)} \dots (10)$$

$\sigma^+(z)$ indicates collecting portion of function $\sigma^2(z)$ that lies within the unit circle and $\sigma^-(z)$ indicates collecting the portion of function $\sigma^2(z)$ that lies outside the unit circle in z -plane.

$$\begin{aligned}
 \sigma^2(z) &= \frac{1}{2\pi} \int_{\gamma} \eta(z) \sigma^+(z) \underbrace{U_2(z)U_2(z^{-1})}_{\sigma^2(z)} \underbrace{U_2(z)U_2(z^{-1})}_{\sigma^2(z)} \\
 & \quad \underbrace{U_0(z^{-1})U_0(z)}_{\sigma^2(z^{-1})} dz + \frac{1}{2\pi} \int_{\gamma} \eta(z^{-1}) \underbrace{U_2(z^{-1})U_2(z)}_{\sigma^2(z^{-1})} \underbrace{U_0(z)U_0(z^{-1})}_{\sigma^2(z)} \underbrace{U_0(z^{-1})U_0(z)}_{\sigma^2(z^{-1})} dz
 \end{aligned}$$

$$\frac{1}{2\pi j} \int_{\Gamma} \eta(\sigma^{-1}) \sigma^{-1}(\sigma) \phi_2(\sigma) \phi_2(\sigma^{-1})^{-1} \phi_2(\sigma) \phi_2(\sigma^{-1})^{-1} \delta(\sigma) \phi_0(\sigma) \dots (11)$$

The second term in the braces contains poles inside the unit circle in the σ -plane, as well as outside the unit circle. The contour integral vanishes if the integral has its poles either all inside the unit circle or all outside the unit circle. Thus,

$$\frac{1}{2\pi j} \int_{\Gamma} \eta(\sigma^{-1}) \sigma^{-1}(\sigma) \phi_2(\sigma) \phi_2(\sigma^{-1})^{-1} \phi_2(\sigma) \phi_2(\sigma^{-1})^{-1} \delta(\sigma) \phi_0(\sigma) \dots (12)$$

$$\frac{1}{2\pi j} \int_{\Gamma} \eta(\sigma) \sigma(\sigma) \phi_2(\sigma) \phi_2(\sigma^{-1})^{-1} \phi_2(\sigma) \phi_2(\sigma^{-1})^{-1} \delta(\sigma) \phi_0(\sigma) \dots (13)$$

where, the symbol \oint implies the operation of picking the part of a function of σ with poles inside the unit circle in σ -plane, and the symbol \oint implies the operation of picking the part of a function of σ with poles outside the unit circle in σ -plane.

Using equation (2.7-11) reduce to :

$$\sigma^2(u\Omega) = \frac{1}{2\pi j} \oint_{\Gamma} \eta(z) \delta^+(z) \left[\begin{array}{c} C_2(z) C_2(z^{-1}) \\ C_2(z^{-1}) C_2(z) \end{array} \right] \left[\begin{array}{c} \delta_{z^{-1}z_0}(z) + \delta_{z_0z^{-1}}(z) \\ \delta^+(z) [C_2(z) C_2(z^{-1})]^{-1} \end{array} \right]^{-1} z^{-1} dz$$

$$+ \frac{1}{2\pi j} \oint_{\Gamma} \eta(z^{-1}) \delta^-(z) \left[\begin{array}{c} C_2(z) C_2(z^{-1}) \\ C_2(z) C_2(z^{-1}) \end{array} \right] \left[\begin{array}{c} \delta_{z^{-1}z_0}(z) + \delta_{z_0z^{-1}}(z) \\ \delta^-(z) [C_2(z) C_2(z^{-1})]^{-1} \end{array} \right]^{-1} z^{-1} dz \quad \dots(14)$$

For the compensation to be optimum, giving minimum mean square error, $\sigma^2(u\Omega)$ should vanish for arbitrary $\eta(z)$. Optimal compensation is given by

$$C_0(z) [C_2(z) C_2(z^{-1})]^{-1} \delta^+(z) = \frac{C_2(z^{-1}) C_2(z) \delta_{z^{-1}z_0}(z) + \delta_{z_0z^{-1}}(z)}{\delta^-(z) [C_2(z) C_2(z^{-1})]^{-1}} = 0$$

$$C_0(z) = \frac{C_2(z^{-1}) C_2(z) [\delta_{z^{-1}z_0}(z) + \delta_{z_0z^{-1}}(z)]}{\delta^-(z) [C_2(z) C_2(z^{-1})]^{-1}} \dots(15)$$

$$\dots(16)$$

Equation (2.7-16), gives the pulse-transfer function of the compensation that will optimize the system in the minimum mean square error sense. This can be obtained in terms of pulse-transfer function of fixed components and pulse spectral density of input signal.

OPTIMUM SYSTEMS

OPTIMUM SYSTEMS ANALYSIS

2.1 INTRODUCTION

In the previous chapter, optimum design for the systems, whose input and output are sampled in synchronism, is carried out in time-domain. The optimum system equation for sampled-data systems with random input signals, is obtained by minimization of the mean-square value of error sequence. The system equation obtained is as follows :

$$\sum_{k=-\infty}^{\infty} c(k) \hat{A}_{xx}(k-n)z^{-n} - \hat{A}_{xy}(k) = 0 \quad \text{for } k \geq 0 \dots 0$$

where

$c(k)$ = weighting sequence of the optimum sampled data system.

$\hat{A}_{xx}(k-n)z^{-n}$ = Autocorrelation sequence of the input signal

$\hat{A}_{xy}(k)$ = Input-ideal output cross-correlation sequence,

which is quite similar to Wiener-Hopf equation for optimum continuous data systems. A solution of the equation is requested and is carried out in z-domain. The equation is modified as

$$\sum_{k=-\infty}^{\infty} c(k) \hat{A}_{xx}^*(k-n)z^{-n} = \hat{A}_{xy}(z)$$

where $\hat{A}_{xy}(z)$ is defined by equations (2.2-4) and (2.2-5)

The transfer function of the system is obtained by taking Laplace transform of both sides of the modified system equation. The transfer function of the system can be written in terms of Laplace transform of system equation, is given by

$$G(s) = \frac{Y(s)}{U(s)} = \frac{G_1(s)}{G_2(s)}$$

where,

$G_1(s)$ Output Laplace transform function

$G_2(s)$ Laplace transform of input signal

$Y(s)$ Laplace transform of output signal.

Optimization of the control system with respect to input signal, minimizing total system error, minimizing the cost and so on. This can be done by using the system with transfer function satisfying minimum error cost criterion can be obtained.

It is further shown that when the control signal is constant and the noise enters the system at different points, identification of noise errors does yields the system transfer function equation.

$$\sum_{k=0}^{\infty} G_k(z^{-1}) f_k(z^{-1}) = G_k(z^{-1}) f_k(z^{-1}) - f_{k+1}(z^{-1}) = 0 \quad \text{for } k \geq 0$$

where,

$G_k(z^{-1})$ Transfer function of the control signal of system.

$U_2(z)$ weighting sequence for random noise component

$U_1(z)$ input control signal component sequence

$U_3(z)$ input noise sequence

Optimization of sampled-data system has been extended to multi-stage configuration systems. For random input elements, it is shown that the minimization of mean square error yields the optimum system equation,

$$\sum_{k_1=-\infty}^{\infty} U_2(z^{k_1}) A_{21} (z^{k_1} + U_1) z^{-k_1} = \sum_{k_2=-\infty}^{\infty} \sum_{k_1=-\infty}^{\infty} \sum_{k_3=-\infty}^{\infty} U_3(z^{k_3})$$

$$U_2(z^{k_2}) U_3(z^{k_3}) A_{22} (z^{k_2} + U_2 - z^{k_1} - U_1) z^{-k_2} = 0 \text{ for } k_1 \geq 0$$

where

$U_2(z)$ = Weight sequence of fixed elements

$U_3(z)$ = Weighting sequence of compensation elements.

Pulse transfer function of the compensation elements in terms of pulse spectral densities of input and ideal output signals and the pulse transfer function of fixed elements is obtained and the expression obtained is as follows:

$$O_2(z) = \frac{O_2(z^{-1}) S_{21}(z)}{O_2(z) O_2(z^{-1}) - S_{22}(z)} = \frac{O_2(z) O_2(z^{-1}) S_{21}(z)}{O_2(z) O_2(z^{-1}) - S_{22}(z)}$$

Next it is shown that when fixed element pulse-transfer function has all its poles and zeroes within the unit circle

In the case, the pulse transfer function of a sampled system is the ratio of pulse-transfer function of a free multi-variable system, with zero performance specifications, and the pulse transfer function of fixed elements.

The expression for output error signal error of the sampled-data system, has been obtained in terms of the optimum system pulse transfer function and spectral densities of the system signals.

Fig. 3. Block diagram of a sampled-data system with feedback.

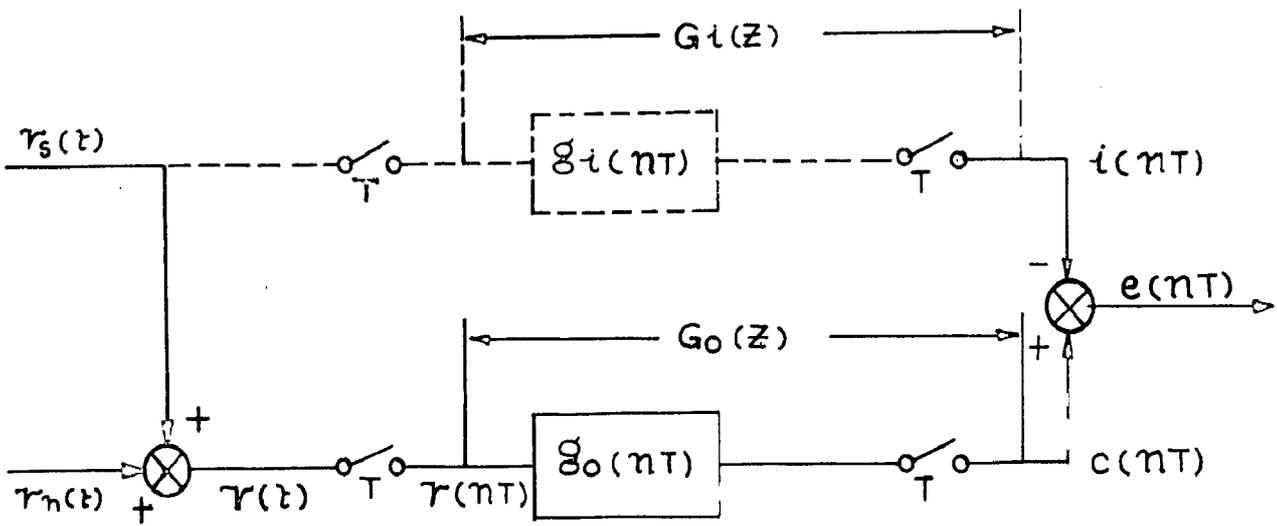


Fig. 3

In the case of a sampled-data system, there are no constraints on the form of the error signal spectral density or of the fixed elements and the designer has to choose both the plant

and elements. In the present section, time-domain equation of the optimum, sampled data system weighting sequence, in the mean square sense has been derived for the zero configuration case.

The output of a system, Fig. 5, input and output of which are sampled at synchronism, will be given by

$$c(nT) = \sum_{k=0}^n g(kT) e(nT-kT) \quad \dots (1)$$

where,

$c(nT)$ = sampled output of the system at n th sampling instant

$e(nT)$ = sampled input of the system at n th sampling instant

$g(kT)$ = weighting sequence of sampled-data system.

T = sampling period

Difference between optimum system output $c(nT)$ and ideal output sequence, i.e., system error sequence $e_e(nT)$ will be given by

$$e_e(nT) = c(nT) - g(nT) \quad \dots (2)$$

$$\sigma_e^2(nT) = e^2(nT) - 2e(nT)g(nT) + g^2(nT) \quad \dots (3)$$

σ_e^2 = Mean square value of error sequence

$$\sigma_e^2(nT) = \lim_{N \rightarrow \infty} \frac{1}{2N+1} \sum_{k=0}^{2N} \sigma_e^2(kT)$$

* Weighting sequence $g(kT)$ is zero for negative values of k .

$$o_{ij}^2(x) = o^2(x) - 2o(x)g_j(x) + g_j^2(x) \quad \dots(4)$$

$$o^2(x) = \lim_{T \rightarrow \infty} \frac{1}{2T+1} \sum_{n=-T}^T o(x) o(x)$$

$$= \lim_{T \rightarrow \infty} \frac{1}{2T+1} \sum_{n=-T}^T \sum_{m=-T}^T o(x)g_j(x-m) \sum_{m=-T}^T o(x)g_j(x-m)$$

Since L, a, b, n are dummy variables interchanging them

$$o^2(x) = \sum_{L=-\infty}^{\infty} \sum_{a=-\infty}^{\infty} o(L)g_j(L) \lim_{T \rightarrow \infty} \frac{1}{2T+1} \sum_{n=-T}^T g_j(x-a)g_j(x-a)$$

From equation (2.3-1)

$$o^2(x) = \sum_{L=-\infty}^{\infty} \sum_{a=-\infty}^{\infty} o(L)g_j(L) \rho_{jj}(L-a) \quad \dots(5)$$

Similarly from equation (2.3-2)

$$o(L)g_j(L) = \sum_{m=-\infty}^{\infty} o(L) \rho_{jj}(L) \quad \dots \quad \dots(6)$$

and

$$g_j^2(x) = \rho_{jj}(0) \quad \dots \quad \dots(7)$$

Thus, from equations (3.2-4), (3.2-5), (3.2-6), (3.2-7)

$$o_j^2(x) = \rho_{jj}(0) = \sum_{L=-\infty}^{\infty} o(L) \rho_{jj}(L) = \sum_{L=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} o(L)g_j(L) \rho_{jj}(L-a) \quad \dots(8)$$

If the original system $U(x)$ is replaced by another system $U(x) \in L(x)$, where $L(x)$ is any realizable

realizable weighting sequence, ϵ is a parameter, that is varied to test the optimality of $U(x)$. Hence with

$U(x) \in \mathcal{R}(x)$ is a function that has the optimal control and will be given by

$$\begin{aligned} \frac{\partial^2}{\partial \epsilon^2} \phi_{opt}^2(x) &= \phi_{opt}^2(x) + 2 \sum_{k=0}^{\infty} (B(k) + D(k)) \phi_{opt}^2(k) \\ &+ \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} C(k) + D(k) + C(l) + D(l) \phi_{opt}^2(k+l) \end{aligned} \quad \dots(9)$$

$$\begin{aligned} \frac{\partial}{\partial \epsilon} \phi_{opt}^2(x) &= 2 \sum_{k=0}^{\infty} D(k) \phi_{opt}^2(k) + 2 \epsilon \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} (B(k)D(l)) \phi_{opt}^2(k+l) \\ &+ \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} C(k) + C(l) + D(k) + D(l) \phi_{opt}^2(k+l) \end{aligned} \quad \dots(10)$$

$$\frac{\partial^2}{\partial \epsilon^2} \phi_{opt}^2(x) = 2 \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} D(k)D(l) \phi_{opt}^2(k+l) \quad \dots(11)$$

$\frac{\partial^2}{\partial \epsilon^2} \phi_{opt}^2(x)$ is a positive quantity.

If $U(x)$ is the optimally regulated of optimal system giving minimum error square error, then

$$\frac{\partial}{\partial \epsilon} \phi_{opt}^2(x) \Big|_{\epsilon=0} = 0 \quad \dots(12)$$

From equation (12) the condition of optimal control will be given by

$$\sum_{k=0}^{\infty} D(k) \left[\sum_{l=0}^{\infty} (B(k) + D(k)) \phi_{opt}^2(k+l) + C(k) \right] = 0 \quad \dots(13)$$

where $u(\lambda)$ is a real function, $u(\lambda) \geq 0$.

$$u(\lambda) = 0 \quad \text{for } \lambda < 0 \quad \text{or } \lambda > 0 \quad \dots (99)$$

where $u(\lambda) \geq 0$ and $u(\lambda) = 0$.

$$\sum_{\lambda=0}^{\infty} u(\lambda) \sum_{\mu=0}^{\infty} u(\mu) A_{\mu}(\lambda) = 0 \quad \text{for } \lambda > 0 \quad \dots (100)$$

Only one of the two expressions can be non-zero for $\lambda > 0$, so for the other one remains the function to be equal to zero for $\lambda > 0$.

$$\sum_{\lambda=0}^{\infty} u(\lambda) A_{\mu}(\lambda) = 0 \quad \text{for } \lambda > 0 \quad \dots (101)$$

where $u(\lambda) \geq 0$ and $u(\lambda) = 0$ for the function $u(\lambda)$ is the same as in (99) by the above expression.

3.3. THEOREM: If the system (97) has a solution $u(\lambda) \geq 0$ for $\lambda > 0$.

Then, the function $u(\lambda)$, of the system (97) that has a solution $u(\lambda) \geq 0$ for $\lambda > 0$, is given by:

$$\sum_{\lambda=0}^{\infty} u(\lambda) A_{\mu}(\lambda) = 0 \quad \text{for } \lambda > 0 \quad (102)$$

The above theorem holds only for $\lambda > 0$, but not for $\lambda < 0$ also, since $A_{\mu}(\lambda)$ and $A_{\mu}(\lambda)$ in general will not be zero for $\lambda < 0$.

Writing $A_{rr}(kT)$ as

$$A_{rr}(kT) = \sum_{l=-\infty}^{\infty} \beta_{rr}^{\dagger}(k-l)T \phi_{rr}^{\dagger}(lT) \quad \dots (2A)$$

$$\text{where } \beta_{rr}^{\dagger}(lT) = 0 \quad \text{for } l < 0 \quad \dots (2B)$$

$$\phi_{rr}^{\dagger}(lT) = 0 \quad \text{for } l > 0 \quad \dots (2C)$$

Multiplying both sides of equation (3.3-2A) by z^{-k} and summing for all values of k from $-\infty$ to $+\infty$,

$$\sum_{k=-\infty}^{\infty} A_{rr}(kT) z^{-k} = \sum_{k=-\infty}^{\infty} z^{-k} \left[\sum_{l=-\infty}^{\infty} \beta_{rr}^{\dagger}(k-l)T \phi_{rr}^{\dagger}(lT) \right]$$

Changing order of summation,

$$D_{rr}(z) = \sum_{l=-\infty}^{\infty} \phi_{rr}^{\dagger}(lT) z^{-lT} \sum_{k=-\infty}^{\infty} \beta_{rr}^{\dagger}(k-l)T z^{-(k-l)T}$$

$$D_{rr}(z) = D_{rr}^{\dagger}(z) D_{rr}^{\dagger}(z) \quad \dots (3)$$

$D_{rr}^{\dagger}(z)$ will have poles and zeroes inside the unit circle in z -plane only, since $A_{rr}(kT)$ is zero for negative values of k . Similarly, $D_{rr}^{\dagger}(z)$ will have poles and zeroes outside the unit circle in z -plane only since $\phi_{rr}^{\dagger}(kT)$ is zero for positive values of k .

By equation (3.3-3), spectral density $D_{rr}(z)$ can be expressed as product of two parts, one containing poles and zeroes inside the unit circle in z -plane, while the other has poles and zeroes outside the unit circle only.

Given correlation sequence $\beta_{rr}(kT)$ can be written as

$$P_{11}(z) = \sum_{k=0}^{\infty} P(z, k) P_{11}^*(z) \quad \dots (4)$$

where, $P(z, k)$, $P(z, k)$ in (3.3-1), can be

$$\sum_{k=0}^{\infty} P(z, k) \sum_{l=0}^{\infty} P_{11}^*(z) (z^{-k})^l = \sum_{k=0}^{\infty} P(z, k) P_{11}^*(z) = 0 \quad \text{for } z > 0 \quad \dots (5)$$

where, $P(z, k)$, $P_{11}^*(z)$

$$\sum_{k=0}^{\infty} P_{11}^*(z) \sum_{l=0}^{\infty} P(z, k) P_{11}^*(z) (z^{-k})^l = \beta (z^{-k})^l = 0 \quad \text{for } z > 0 \quad \dots (6)$$

and equation (3.3-5)

$$P_{11}^*(z) = 0 \quad \text{for } z > 0$$

Thus, the equation will hold for $z > 0$, if the equation holds for $z < 0$ also,

$$\sum_{k=0}^{\infty} P(z, k) P_{11}^*(z) (z^{-k})^l = \beta (z^{-k})^l = 0 \quad \text{for } z > 0 \quad \dots (7)$$

0 $z < 0$

The above equation holds for $z > 0$, hence, it will hold for $z > 0$ also, where, z is a negative number. The equation (3.3-7) may be written as, with a change of variable

$$z = z^{-1}$$

$$\sum_{k=0}^{\infty} P(z, k) P_{11}^*(z) (z^{-k})^l = (z^{-k})^l = 0 \quad \text{for } z > 0 \quad \dots (8)$$

So a suitable variable sequence,

$$z^{-k} = 0 \quad \text{for } z < 0$$

$$\text{also } P_{11}^*(z) = 0 \quad \text{for } z < 0$$

where $\beta_0(x)$ is the function $\beta_0(x)$ for $x < 0$ and $\beta_1(x)$ is the function $\beta_1(x)$ for $x > 0$. Similarly, we define $\beta_2(x)$ as the function

$$\beta_2(x) = \beta_0(x) + \beta_1(x) \quad \dots(91)$$

and so

$$\beta_0(x) = 0 \quad \text{for } x < 0 \quad \dots(92)$$

$$\beta_1(x) = 0 \quad \text{for } x > 0 \quad \dots(93)$$

where $\beta_0(x)$ is the function

$$\sum_{n=0}^{\infty} \beta_n(x) \beta_n^*(x) \quad \text{for } x < 0 \quad \dots(94)$$

where $\beta_n(x)$ is the function $\beta_n(x)$ for all values of x and $\beta_n^*(x)$ is the function $\beta_n^*(x)$ for all values of x . This function $\beta_n(x)$ is the function $\beta_n(x)$ for all values of x and $\beta_n^*(x)$ is the function $\beta_n^*(x)$ for all values of x .

where $\beta_n(x)$ is the function $\beta_n(x)$ for all values of x and $\beta_n^*(x)$ is the function $\beta_n^*(x)$ for all values of x .

$$\sum_{n=0}^{\infty} \beta_n(x) \beta_n^*(x) = \sum_{n=0}^{\infty} \beta_n(x) \beta_n^*(x)$$

$$\sum_{n=0}^{\infty} \beta_n(x) \beta_n^*(x) = \sum_{n=0}^{\infty} \beta_n(x) \beta_n^*(x)$$

$$\beta_n(x) \beta_n^*(x) = \beta_n(x) \quad \dots(95)$$

where $\beta_n(x)$ is the function $\beta_n(x)$ for all values of x and $\beta_n^*(x)$ is the function $\beta_n^*(x)$ for all values of x .

$$\beta_n(x) = \beta_n(x) \quad \dots(96)$$

where $\beta_n(x)$ is the function $\beta_n(x)$ for all values of x and $\beta_n^*(x)$ is the function $\beta_n^*(x)$ for all values of x .

$$Q_{20}(s) = P(s) Q_{20}(s)$$

$$\dots(11)$$

Let $(P(s))$ and $(Q(s))$ contain only real zeros. Let $(P(s))$ be written as

$$P(s) = \frac{Q_1(s)}{Q_2(s)} \dots(12)$$

where $Q_1(s)$ and $Q_2(s)$ are polynomials with real coefficients. Let $(Q(s))$ be written as $Q(s) = \frac{R(s)}{S(s)}$ where $R(s)$ and $S(s)$ are polynomials with real coefficients. Let $(Q_1(s))$ and $(Q_2(s))$ be written as $Q_1(s) = \frac{U(s)}{V(s)}$ and $Q_2(s) = \frac{W(s)}{X(s)}$ where $U(s)$, $V(s)$, $W(s)$ and $X(s)$ are polynomials with real coefficients. Let $(R(s))$ and $(S(s))$ be written as $R(s) = \frac{Y(s)}{Z(s)}$ and $S(s) = \frac{T(s)}{U(s)}$ where $Y(s)$, $Z(s)$, $T(s)$ and $U(s)$ are polynomials with real coefficients.

$$Q_{20}(s) = \frac{Q_1(s)}{Q_2(s)} \dots(13)$$

$$Q_{20}(s) = \frac{U(s)}{V(s)} \dots(14)$$

$$Q(s) = \frac{Y(s)}{Z(s)} \dots(15)$$

A relatively direct proof can also be given as follows:

It is obvious that the system equation 3.2-16 is

$$\sum_{k=-\infty}^{\infty} u(k) \Delta_{TT}(k-n)T - \beta_{TT}(k) = 0 \quad \text{for } k > 0$$

$$\text{For } \sum_{k=-\infty}^{\infty} u(k) \Delta_{TT}(k-n)T - \beta_{TT}(k) = q(k) \quad \text{for } k < 0 \quad \dots(17)$$

$q(k)$ is an general nonzero for $k < 0$, but vanishes for $k > 0$. The original sampled data system equation can be written as

$$\sum_{k=-\infty}^{\infty} u(k) \Delta_{TT}(k-n)T - \beta_{TT}(k) = q(k) \quad \dots(18)$$

Taking z-transform of $q(k)$

$$Q(z) = \sum_{k=-\infty}^{\infty} q(k) z^{-k}$$

$$Q(z) = \sum_{k=-\infty}^{\infty} q(k) z^{-k} \quad \dots(19)$$

$Q(z)$ has poles outside the unit circle in a plane z .

Using Laplace called a transform of equation 3.2-19,

$$Q_L(s) D_{TT}(s) - D_{TT}(s) = Q(s) \quad \dots(20)$$

$D_{TT}(s)$ is an even function and can be expressed by equation (3.2-9) as the product of $D_{TT}^+(s)$ and $D_{TT}^-(s)$. Equation (3.2-20) can be written as

$$Q_L(s) D_{TT}^+(s) D_{TT}^-(s) - D_{TT}^-(s) = Q(s)$$

$$Q_L(s) D_{TT}^+(s) = \frac{D_{TT}^-(s)}{D_{TT}^-(s)} = \frac{Q(s)}{D_{TT}^-(s)} \quad \dots(21)$$

(f) The first two terms in the series are $1 + 2x$ and $1 + 2x + 2^2x^2$ and so on. The general term is $1 + 2x + 2^2x^2 + \dots + 2^{n-1}x^{n-1}$. This is a geometric series with first term 1 and common ratio $1 + 2x$. The sum of the first n terms is $\frac{1 - (1 + 2x)^n}{1 - (1 + 2x)}$. The series converges for $|1 + 2x| < 1$, i.e. $-1 < 1 + 2x < 1$, which gives $-1 < 2x < 0$, or $-\frac{1}{2} < x < 0$.

$\frac{1}{1 - (1 + 2x)}$ may be written as the sum of two series, $\frac{1}{1 - 2x} - \frac{1}{1 - 2x}$. The function $\frac{1}{1 - 2x}$ is a geometric series with first term 1 and common ratio $2x$. The function $\frac{1}{1 - 2x}$ is a geometric series with first term 1 and common ratio $2x$. The function $\frac{1}{1 - 2x}$ is a geometric series with first term 1 and common ratio $2x$. The function $\frac{1}{1 - 2x}$ is a geometric series with first term 1 and common ratio $2x$.

$$\frac{1}{1 - (1 + 2x)} = \frac{1}{1 - 2x} - \frac{1}{1 - 2x} = \frac{1}{1 - 2x} - \frac{1}{1 - 2x} = \dots (1)$$

$\frac{1}{1 - (1 + 2x)}$ may be written as the sum of two series.

The first series is a geometric series with first term 1 and common ratio $2x$. The second series is a geometric series with first term 1 and common ratio $2x$. The series converges for $|2x| < 1$, i.e. $|x| < \frac{1}{2}$.

$$\frac{1}{1 - (1 + 2x)} = \frac{1}{1 - 2x} - \frac{1}{1 - 2x} = \frac{1}{1 - 2x} - \frac{1}{1 - 2x} = \dots (2)$$

Similarly, the series $1 + 2x + 2^2x^2 + \dots$ converges for $|2x| < 1$, i.e. $|x| < \frac{1}{2}$.

$$Q_D(s) D_{TT}^{-1}(s) = \frac{D_{T1}(s)}{D_{T2}^{-1}(s)} = 0 \quad \dots(25)$$

Then the optimum sampled data system pulse transfer function is given by

$$Q_D(s) = \frac{1}{D_{TT}^{-1}(s)} \cdot \begin{bmatrix} D_{T1}(s) \\ D_{T2}^{-1}(s) \end{bmatrix} \quad \dots(26)$$

When $D_{T1}(s)/D_{T2}^{-1}(s)$ is not a rational function of s

$$Q_D(s) = \frac{1}{D_{TT}^{-1}(s)} \sum_{n=0}^{\infty} s^{-n} = \frac{1}{2\pi j} \int_{\Gamma} \frac{D_{T1}(s)}{D_{T2}^{-1}(s)} s^{-1} ds \quad \dots(27)$$

Fig. 6. Block diagram of a sampled data system for deterministic signals.

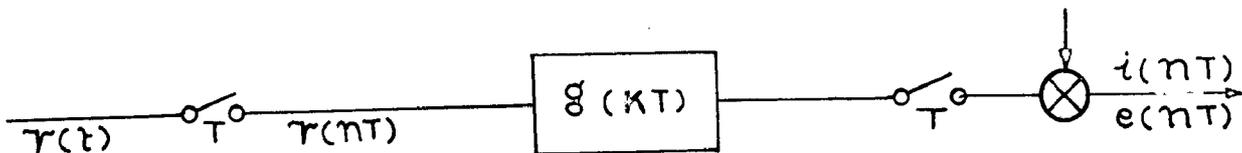


Fig. 6.

System design of a sampled data system for deterministic signals can be achieved on the basis of "total square error" criterion in a way analogous to the design of continuous data control systems with integral square error criterion.

Error sequence for the sampled data system shown in Fig. 6 is given by

$$e(nT) = c(nT) - i(nT) \quad \dots(1)$$

$$y(n) = \sum_{k=0}^n x(k) v(n-k) \quad \dots(8)$$

= (input sequence)

$v(n)$ = Impulse response

$h(n)$ = overall output response

$x(n)$ = arbitrary; sequence of arbitrary sampled data
signal.

Note: square wave, if try to obtain an

$$y = \sum_{k=0}^n x^2(k)$$

$$y = \sum_{k=0}^n x^2(k) \cos(\omega k) \cos(\omega k) \quad \dots(9)$$

(convolution) = sequence for the discrete time system to obtain an

$$h_1(n) = \sum_{k=0}^n g(k) g(n-k) \quad \dots(10)$$

$$h_2(n) = \sum_{k=0}^n g(k) g(n-k) \quad \dots(11)$$

Note: square wave first, to represent in terms of convolution
operation and resulting sequence by equation (10), (11),
(12), etc.

$$y = \sum_{k=0}^n \sum_{l=0}^n x(k) x(l) h_1(n-k-l) = \sum_{k=0}^n x(k) h_2(n-k) \quad \dots(12)$$

If the overall system $y(n)$ is defined by equation (12) $y(n) = \sum_{k=0}^n x(k) h_2(n-k)$
then $h_2(n)$ is overall impulse response which
is related to the convolution of $x(n)$ the input signal to
the system. This can be used to obtain by

$$\delta S = 2 \sum_{k=0}^{\infty} b(k) \lambda(k) = 2 \sum_{k=0}^{\infty} c(k)$$

$$\begin{aligned} \delta S &= \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} 2 \epsilon c(k) \lambda(l) \cdot \epsilon \lambda(l) \lambda(k) \rho_{kl}^{(k-l)/\epsilon} \\ &= 2 \epsilon \sum_{k=0}^{\infty} \lambda(k) \rho_{k0}^{(k-0)/\epsilon} \end{aligned} \quad \dots(7)$$

$$\begin{aligned} \therefore \frac{\partial}{\partial \epsilon} (\delta S) &= 2 \sum_{k=0}^{\infty} \lambda(k) \left[\sum_{l=0}^{\infty} c(l) \rho_{kl}^{(k-l)/\epsilon} - \rho_{k0}^{(k-0)/\epsilon} \right] \\ &= 2 \epsilon \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \lambda(k) \lambda(l) \rho_{kl}^{(k-l)/\epsilon} \quad \dots(8) \end{aligned}$$

$$\frac{\partial^2}{\partial \epsilon^2} (\delta S) = 2 \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \lambda(k) \lambda(l) \rho_{kl}^{(k-l)/\epsilon} \quad \dots(9)$$

Now on differentiating (7) we get

$$\left. \frac{\partial}{\partial \epsilon} (\delta S) \right|_{\epsilon=0} = 0$$

$$\sum_{k=0}^{\infty} \lambda(k) \left[\sum_{l=0}^{\infty} c(l) \rho_{kl}^{(k-l)/\epsilon} - \rho_{k0}^{(k-0)/\epsilon} \right] = 0 \quad \dots(10)$$

Since $\lambda(k)$ is a realizable sequence,

$$\lambda(k) = 0 \quad \text{for } k < 0$$

Equation 9, 4-10 will hold good for $k > 0$, else, if, the expression within brackets is equal to zero for $k > 0$

$$\sum_{l=0}^{\infty} c(l) \rho_{kl}^{(k-l)/\epsilon} - \rho_{k0}^{(k-0)/\epsilon} = 0 \quad \text{for } k > 0 \quad \dots(11)$$

Equation 9.4-11 gives the optimum system, with minimum total-square error criterion, since as seen from

$$(9.4-9) \quad \frac{\delta^2}{\delta \epsilon^2} (\delta \epsilon) \text{ is a positive quantity.}$$

9.5 ERROR CRITERIA WHEN SIGNAL & NOISE ARE SEPARATE POINTS

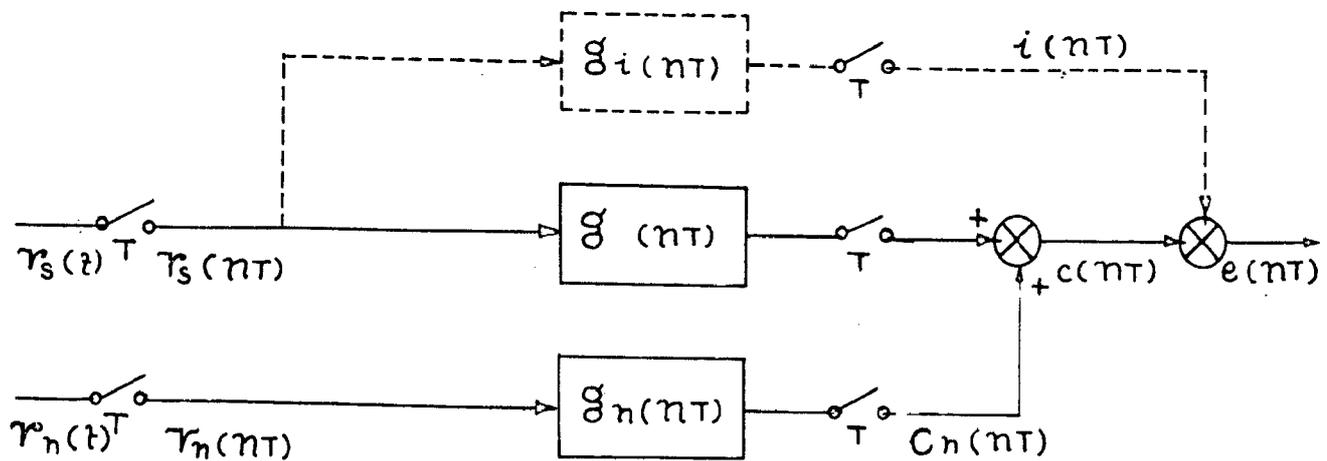


Fig. 9

In the present section, optimization of the sampled data system, Fig. 7, is considered, when noise and signal components enter the system at different points. In the block diagram shown in Fig. 7 $G_i(z)$ is the weighting response for the stochastic control signal components and $G_n(z)$ the weighting response for random noise signals. Both the signal and noise components are assumed to be stationary random functions.

Error response of the system

$$o(n) = q_1(n) + q_2(n) = z(n) \quad \dots(1)$$

where

$q_1(n)$ = output response due to signal component

$q_2(n)$ = output response due to noise

$o(n)$ = output response of the system

$z(n)$ = ideal output response.

$q_1(n)$ and $q_2(n)$ can be written in terms of system input and weighting responses,

$$q_1(n) = \sum_{k=-\infty}^n q_1(k) r_1(n-k) \quad \dots(2)$$

$$q_2(n) = \sum_{k=-\infty}^n q_2(k) r_2(n-k) \quad \dots(3)$$

These equations in terms of input and output, from the conditions $r_1(n) = r_2(n) = \delta(n)$, $r_1(n) = \delta(n)$, $r_2(n) = \delta(n)$ will be given by

$$r_1(n) = \sum_{k=-\infty}^n \delta(k) r_1(n-k)$$

$$r_2(n) = \sum_{k=-\infty}^n \delta(k) r_2(n-k)$$

$$\sum_{k=-\infty}^n \sum_{l=-\infty}^n q_1(k) r_1(l) r_1(n-k-l) = \sum_{k=-\infty}^n q_1(k) r_1(n-k) \quad \dots(4)$$

Following procedure, analogous to that followed by above - configuration also, this method requires the choice of weights which the system at the end point, the output

system equations obtained are

$$\sum_{k=0}^{\infty} \beta_{co}(k) \beta_{cn}^T(k-n) \beta_{co}(k) \beta_{cn}^T(k-n) \beta_{cn}^T(k-n) = \beta_{cn}^T(k) = 0 \quad \text{for } k > 0 \quad \dots(9)$$

Using the noise and control signal components are statistically independent, cross-correlation equation for noise and control system are then reduced to

$$\sum_{k=0}^{\infty} \beta_{co}(k) \beta_{cn}^T(k-n) = \beta_{cn}^T(k) = 0 \quad \text{for } k > 0 \quad \dots(10)$$

2.6. BLOCK DIAGRAM OF A DISCRETE-TIME CONTROL SYSTEM WITH FEEDBACK

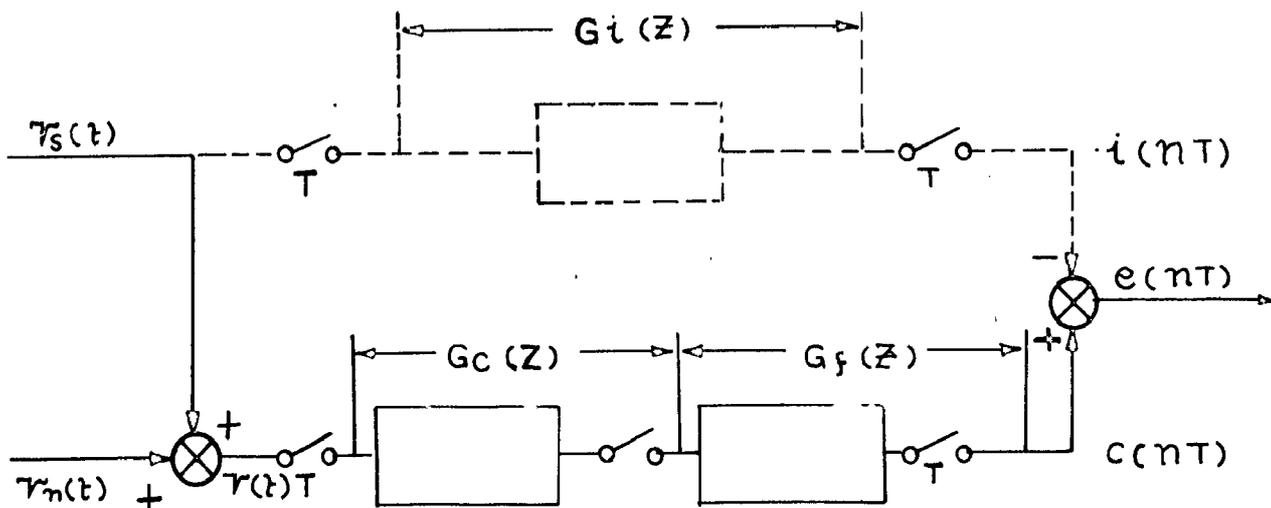


Figure 2.6

In this system, the controller, the error signal is measured, the reference input to the system is assumed to be stationary and ergodic random signal. For the analysis this system block diagram of Figure 2.6, the noise and control signal of the plant elements of the system, and $\beta_{co}(k)$ and $\beta_{cn}(k)$ are to be obtained.

The output response of the system is given by

$$y(t) = \sum_{k=0}^{n-1} q_k(t) v_k(t) \quad \dots (1)$$

$$y(t) = \sum_{k=0}^{n-1} \sum_{l=0}^{n-1} q_k(t) v_l(t) v_l(t) \quad \dots (2)$$

where

$v(t)$ = input response to the system

$q_k(t)$ = weighting response of direct elements

$v_l(t)$ = weighting response of complementary elements

The output of the system is given by

$$y(t) = v(t) \cdot S(t) \quad \dots (3)$$

The output value of the system is given by

$$y^2(t) = \lim_{\Delta t \rightarrow 0} \frac{1}{\Delta t} \sum_{k=0}^{n-1} v^2(t)$$

$$y^2(t) = v(t) \cdot S^2(t) \quad \dots (4)$$

$y^2(t)$, which is the value of the output of the complementary elements, is given by $v(t)$, and the complementary value of the output $S^2(t)$, $S^2(t)$ is

$$S^2(t) = S(t) = 2 \sum_{k=0}^{n-1} \sum_{l=0}^{n-1} q_k(t) v_l(t) v_l(t)$$

$$\sum_{k=0}^{n-1} \sum_{l=0}^{n-1} \sum_{m=0}^{n-1} \sum_{p=0}^{n-1} [q_k(t) v_l(t) v_m(t) v_p(t)] \quad \dots (5)$$

Denoting the position, relative to the origin
 for system of two particles, regarded for free configurations
 system, system coordinates relative, regarded as given by

$$\sum_{j=1}^N \sum_{k=1}^N \sum_{l=1}^N \psi_j(\mathbf{r}_1) \psi_k(\mathbf{r}_2) \psi_l(\mathbf{r}_3) \psi_m(\mathbf{r}_4) \dots \psi_n(\mathbf{r}_N) \dots$$

$$\sum_{j=1}^N \psi_j(\mathbf{r}_1) \psi_m(\mathbf{r}_2) = 0 \quad \text{for } m \neq j \dots (3)$$

By using the relation (condition 3.5), the value of
 another function of the coordinates, system can be given by

$$\psi_0(\mathbf{r}) = \frac{1}{\sqrt{N}} \sum_{j=1}^N \psi_j(\mathbf{r}_1) \psi_m(\mathbf{r}_2) \dots \psi_n(\mathbf{r}_N) \dots \dots (4)$$

Then, taking into the account, system consists of identical
 particles $\psi_j(\mathbf{r}_1)$ and $\psi_m(\mathbf{r}_2)$, both being real, odd,
 stationary and orthonormal functions. The first value corresponds
 to the simple case $\psi_j(\mathbf{r}_1)$. From equations (3.5) to (3.8)

$$\psi_{j,m}(\mathbf{r}) = \psi_{j_0 j_0}(\mathbf{r}) \psi_{j_0 j_0}(\mathbf{r}) \psi_{j_0 j_0}(\mathbf{r}) \psi_{j_0 j_0}(\mathbf{r}) \dots \dots (5)$$

$$\psi_{j,m}(\mathbf{r}) = 0 \quad \text{for } \psi_{j_0 j_0}(\mathbf{r}) \neq \psi_{j_0 j_0}(\mathbf{r}) \dots \dots (6)$$

Other similar functions of coordinates, system is given by

$$C_2(z^{-1}) C_2(z) = \frac{D_{2,2}(z) \cdot D_{2,2}(z)}{D_{2,1}(z) \cdot D_{2,1}(z)}$$

$$C_2(z) = \frac{D_{2,2}(z)}{D_{2,1}(z)}$$

... (10)

If $C_2(z)$, the pulse-transfer-function of the fixed elements has no poles or zeroes outside the unit circle in z -plane, then

$$C_2(z) C_2(z^{-1}) = C_2(z) \quad \dots (11A)$$

$$C_2(z) C_2(z^{-1}) = C_2(z^{-1}) \quad \dots (11B)$$

Pulse transfer function of optimum compensation is given by

$$C_c(z) = \frac{C_2(z) \cdot \frac{D_{2,2}(z) \cdot D_{2,2}(z)}{D_{2,1}(z) \cdot D_{2,1}(z)}}{D_{2,1}(z)} \quad \dots (12)$$

Pulse transfer function of optimum system when no fixed elements are present.

Pulse-transfer function of fixed elements

3.7 RMSE SQUARE ERROR

Mean square error of sampled data system is given by

$$e^2(z) = |o(z) - d(z)|^2 \quad \dots (1)$$

$$e^2(z) = \beta_{00}(z) \cdot \beta_{11}(z) - \beta_{01}(z) - \beta_{10}(z) \quad \dots (2)$$

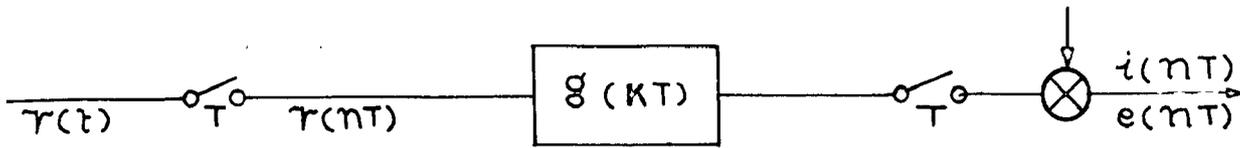


Fig. 9

The output $e(nT)$ can be given in terms of initial conditions

$$e^2(nT) = \frac{1}{T} \int_T D_{11}(s) D_{22}(s) - D_{12}(s) D_{21}(s) u^{-s} ds$$

The output $e(nT)$ of the system excited with system can be given by

$$e^2(nT) = \sum_{k=0}^{n-1} g_1(kT) r(n-kT) + \dots \quad \dots (1)$$

where $g_1(kT)$ = coefficient sequence of the system $r(nT)$.

$$\begin{aligned} \overline{e^2(nT)} &= \sum_{k=0}^{n-1} \sum_{l=0}^{n-k} r(n-kT) r(n-lT) A_{11}(l-k) + \dots \\ &\quad + A_{22}(0) \dots (2) \end{aligned}$$

For the system $r(nT) = \delta(nT)$,

$$\sum_{k=0}^{n-1} r(n-kT) A_{11}(l-k) = A_{11}(l) = 0 \quad \text{for } l > 0$$

$$e^2(nT) = A_{11}(0) = \sum_{k=0}^{n-1} g_1(kT) A_{11}(kT) \quad \dots (3)$$

where $\psi(z)$ is a function of the complex variable z defined by

$$Q_2(z) = \frac{Q_2'(z)}{Q_2(z)} = \frac{Q_2'(z)}{Q_2(z)} \dots (1)$$

and can be written as

$$Q_2(z) = \frac{1}{Q_2(z)} \sum_{n=0}^{\infty} \psi(n) z^{-n}$$

$$\psi(z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{Q_2'(z)}{Q_2(z)} z^{-1} dz \dots (2)$$

$$\text{and } Q_2'(z) = Q_2(z) = \sum_{n=0}^{\infty} A_n(z) \frac{1}{2\pi i} \int_{\Gamma} Q_2(z) z^{-n} dz \dots (3)$$

$$Q_2'(z) = \sum_{n=0}^{\infty} A_n(z) \frac{1}{2\pi i} \int_{\Gamma} Q_2(z) z^{-n} dz = \frac{1}{Q_2(z)} \dots (4)$$

Therefore we can write as

$$Q_2'(z) = Q_2(z) = \sum_{n=0}^{\infty} \psi(n) \frac{1}{2\pi i} \int_{\Gamma} \frac{Q_2'(z)}{Q_2(z)} z^{-n} dz = \sum_{n=0}^{\infty} \psi(n) z^{-n} \dots (5)$$

$$Q_2'(z) = \sum_{n=0}^{\infty} \psi(n) \frac{1}{2\pi i} \int_{\Gamma} z^{-n} dz = \frac{Q_2'(z)}{Q_2(z)} \dots (6)$$

$$\text{and } Q_2'(z) = Q_2(z) = \sum_{n=0}^{\infty} \psi(n) z^{-n} \dots (7)$$

Thus equation (5.7-6), gives the relation between $Q_2(z)$ and $Q_2'(z)$ and the function $\psi(z)$ defined by equation (5.7-1).

3.0

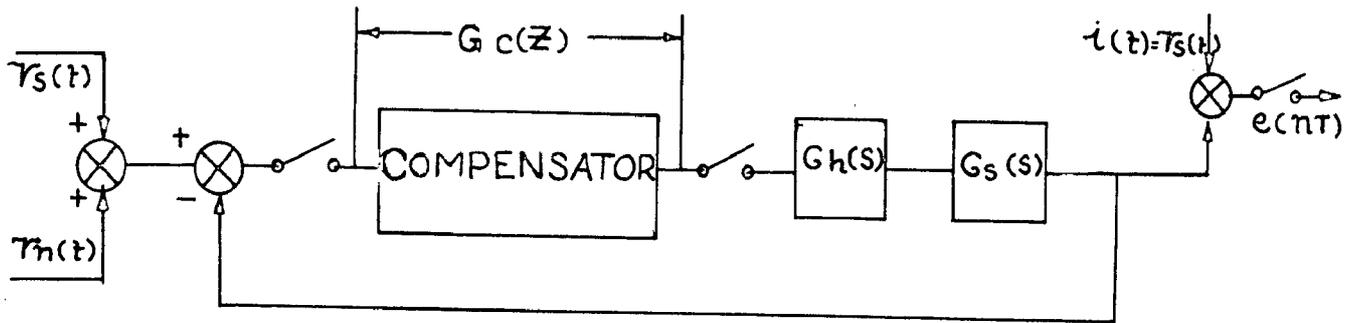


Fig. 10

A sampled data system has the configuration, as shown in Fig. 10. The transfer functions and input signals are described by

$$O_0(z) = 1 + 0.003z^{-1} \quad O_1(z) = \frac{1 - e^{-sT}}{s}$$

$$O_2(z) = \left[1 + \left(\frac{0.1}{T}\right)z^{-1} + \left(\frac{0.1}{T}\right)^2 z^{-2} \right]$$

$$P_{00}(z) = \frac{4.5}{(0.25 - z^{-1})} \quad P_{11}(z) = \frac{0.1}{K}$$

$$P_{22}(z) = 0$$

Sampling period is taken as 0.1 second. The design of sampled data compensator is to be carried out, that will minimize the mean square sampled error $\bar{e}^2(nT)$.

The block diagram of sampled data system Fig. 10, can be rephrased as that of Fig. 4. Here $O_1(z)$ and $O_0(z)$ are the ideal and optimum systems respectively.

Use the relation of a linear, following formula
 and obtain.

$$E_{2,2}^{(1)}(z) = \frac{0.433 z}{(z-0.9512)(z-0.9512)} + E_{2,2}^{(0)}(z) = 0.03 z$$

$$E_{2,2}^{(2)}(z) = E_{2,2}^{(1)}(z) = 0$$

$$E_{2,0}^{(1)}(z) = \frac{0.433 z (z^2 - 0.97 z + 0.92)}{(z-1)(z^2 - 0.97 z + 0.92)}$$

$$\therefore E_{2,0}^{(1)}(z) = E_{2,0}^{(1)}(z) + E_{2,0}^{(0)}(z) + E_{2,0}^{(-1)}(z) + E_{2,0}^{(-2)}(z)$$

$$= \frac{0.03 z (z-0.97) (z-0.92)}{(z-0.9512) (z-1.0318)}$$

Use the relation of a linear, following formula
 and obtain.

$$\therefore E_{2,0}^{(1)}(z) = \frac{0.17 z (z-0.92)}{(z-0.9512)}$$

$$E_{2,0}^{(2)}(z) = \frac{0.17 z (z-0.97)}{(z-1.0318)}$$

Use the relation under consideration, $O_2(z) = 1$.

then,

$$O_2(z) = \frac{E_{2,0}^{(1)}(z) + E_{2,0}^{(2)}(z)}{E_{2,0}^{(1)}(z)}$$

$$= \frac{0.03 z}{\frac{0.17 z (z-0.92)}{(z-0.9512)(z-0.97)} + \frac{0.17 z (z-0.97)}{(z-1.0318)}}$$

$$O_2(z) = \frac{E_{2,0}^{(1)}(z) + E_{2,0}^{(2)}(z)}{E_{2,0}^{(1)}(z)}$$

and the poles under the consideration are $z = 0.9512$.

$$E_{22}(s) = E_{21}(s) \cdot C_2(s)$$

$$E_{22}(s) = \frac{0.162}{s - 0.9912}$$

Using the partial fraction expansion of certain system is given by

$$C_0(s) = \frac{E_{22}(s)}{E_{21}(s)} = \frac{E_{22}(s) \cdot E_{11}(s)}{E_{21}(s) \cdot E_{11}(s)}$$

$$= \frac{0.162}{s - 0.9912} \cdot \frac{0.170}{s - 0.9912}$$

$$= \frac{0.267}{s - 0.9912}$$

Using the partial fraction expansion of certain system is given

$$C_0(s) = \frac{E_{22}(s)}{E_{21}(s)} = \frac{E_{22}(s) \cdot E_{11}(s)}{E_{21}(s) \cdot E_{11}(s)}$$

$$= \frac{0.162 (s-1)(s^2 - 0.9903s + 0.9903)}{(s-1)(s^2 - 0.9903s + 0.9903)}$$

$$= \frac{0.162}{s - 0.9912}$$

CHAPTER IV

STATISTICAL AND ANALYTICAL DESIGN

4.1 INTRODUCTION

In the present dissertation, statistical design of analog-digital control systems in time domain has been considered.

In the first chapter, illustrations of classical approach to the design of control systems have been pointed out. Usually, this is a trial and error approach for a system with identified signals only, disregarding random nature and the existence of noise. The drawbacks of classical design techniques are overcome by analytical design approach. The analytical design proceeds directly from the system specifications. It considers both control system component and noise in the system, as well as the random nature of input signals to an actual system.

The control signal component and noise component can be represented only statistically. For random signal input to a control system, mean-square value of error output is the most convenient performance index, because of its mathematical amenability, besides being collective. For deterministic signal inputs, integral square error is a convenient performance index.

In the second chapter, the fundamental operations, such as addition, subtraction, multiplication, and division, are discussed in detail. The text explains how these operations are performed and the properties that govern them. It also introduces the concept of the number line and how numbers are represented on it.

Chapter three discusses the properties of numbers, including the commutative, associative, and distributive laws. It also covers the concept of prime and composite numbers, and the least common multiple (LCM) and highest common factor (HCF) of two or more numbers. The text provides examples and exercises to illustrate these concepts.

In the third chapter of the book, the concept of fractions is introduced. It explains how to add, subtract, multiply, and divide fractions. The text also discusses the conversion of fractions to decimal form and vice versa. It covers the concept of equivalent fractions and how to compare and order fractions. The chapter includes numerous examples and exercises to help students understand and apply these concepts.

Optimization equation for compensator has also been obtained for the anti-free configuration. Pulse transfer function of the compensator has been obtained in terms of the pulse spectral density of input signal, pulse-transfer functions of ideal system and fixed elements. If the fixed-element pulse transfer function has no poles or zeroes outside the unit circle in z -plane pulse-transfer function of the compensator, is the ratio of pulse-transfer function of the optimum system without fixed elements, and the fixed element pulse transfer function.

Optimum system equation has also been obtained for the sampled data systems, where the control signal and noise enter the system at different points.

4.2 STATISTICAL DESIGN OF LTR SYSTEMS

The study of statistical design is restricted to the sampled-data control systems with infinitesimal sampling duration. Not all the physical sampled data system have infinite sampling duration. The design will be more realistic especially when the time constant of the system is not large enough compared to the sampling duration, if the finite width of sampled pulses is also considered.

Optimum sampled data system design is restricted to the cases, where the input and output signals are sampled in synchronism. It should be possible to extend the statistical

design approach to the system with input and output being sampled at different rates. The technique may also be extended to the system where sampling itself is random.

Statistical design procedure for sampled data systems is restricted to the system, where control signal component and noise enter the system at the same point. It should be possible to extend the concept to the systems, where control signal component and the noise enter the system at different points, as well as to the multiple input systems.

With the advancements in mathematical techniques, it should be possible to extend the statistical design approach to the nonlinear sampled-data control systems, and also the time varying systems.

In conclusion it can be said that the problem of statistical design of sampled data control systems in time domain, has been discussed briefly, in this dissertation. It is hoped that further investigations in this field will lead to more worthwhile results.

