Ph.D. Thesis

Realization of Linear Dynamical Systems and Networks

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C E R T I F I C A T E

Certified that the thesis entitled "REALIZATION OF LINEAR DYNAMICAL SYSTEMS AND NETWORKS" which is being submitted by Mr. Satish Chandra Puri in fulfilment of the requirements for the award of the Degree of Doctor of Philosophy in Electronics and Communication Engineering of the University of Roorkee is a record of the student's own work carried out by him under my supervision and guidance. The matter embodied in this thesis has not been submitted for the award of any other degree.

This is further to certify that Mr. Puri has worked for a period of four years from December 1969 to December 1973 for preparing this thesis.

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TABLE OF CONTENTS

Contents (Contd..)

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A B S T R A C T

REALIZATION OF LINEAR DYNAMICAL SYSTEMS AND NETWORKS

With the increasing use of state-space approach in control systems and network theory, considerable interest has been shown in the problem of realization of linear systems. This thesis is concerned with the state-space realization of linear dynamical systems and its application to networks. In particular, both minimal and non-minimal realization techniques have been developed and their application to problems in network theory have been sought with a view to obtain better insight and to improve upon the existing techniques in network and system theory.

The problem of state model realization of a symmetric positive real matrix for passive RLC networks without the use of gyrators has been investigated and a new minimal realization technique based on the moments of impulse response matrix has been proposed. The method is especially preferable for the cases where the data is contaminated with noise.

The algorithms for the realization problem of linear dynamical systems proposed uptil now appear computationally rather cumbersome. A simplified technique for obtaining a non-minimal state-model of a transfer function matrix has been proposed. In order to determine the

dimension of the realization, mode matrices M and M_c are defined for the multiple pole case.

Roveda and Schmid [91] have proposed a procedure for obtaining an upper bound on the dimension of a minimal realization. Their method is applicable under the assumption that no element of the transfer-function matrix H(s) has multiple poles. Here, a generalized algorithm is developed to obtain a non-minimal realization for the case of H(s) having simple as well as multiple poles. The realization results is a still lower dimension, compared with the other methods.

Because of a change from transfer-function description of a dynamical system to a more general state-space characterization, it is quite important to establish a communication link between state-space characterization and frequency domain methods. Some work has already been initiated in this direction. A technique for determining the state-model and the impedance matrix Z(s) of order n from given U(s) = Z(s) + Z'(-s) is presented, which is simpler than the one proposed earlier [61]. It avoids the cumbersome spectral factorization and the determination of a symmetric positive definite matrix P, which gets unwieldy in the case of existing methods especially when the order n of U(s) is large. Z(s) obtained thus is a minimum reactance matrix. An algorithm is also proposed for obtaining state-space realization and the impedance matrix Z(s) when $V(s) = Z(s) - Z'(-s)$ is given. The method is applicable to V(s) of any order n. Further, a state-space interpretation of the Foster synthesis method for driving point immittance functions of LC networks is presented.

A method for determining transfer-function matrix from a knowledge of its moments is presented. It is shown that at the most (n+1) moments of the impulse response matrix are required in the process, where n is the order of the state matrix. Also, a method is given for determining the resolvent matrix $(sI-A)^{-1}$ and its higher powers, where the given matrix A is in Jordan canonical form. Further, when A is in the companion form, an algorithm is proposed to compute A^{-k} , $k = 1,2,\dots$. These results may be employed to find the moments of the impulse-response matrix.

A method is given to construct a transformation N(t) which transforms a time-varying autonomous system to the companion form. In some cases the transformation could be made a constant matrix.

Finally, some suggestions are given for further work in this field.

CHAPTER I

INTRODUCTION TO THE THESIS

1.1 INTRODUCTION

The state-variable approach has emerged as a powerful tool in the study of dynamical systems and networks. State-space techniques adapt easily for computerization, are indispensable for time-varying, and non-linear systems and afford a more general representation of a physical process. A very important advantage of these methods lies in their flexibility in generating "equivalent " canonical representations which are very useful in system analysis. Consequently, there has been a shift in characterizing a dynamical system from impulse-response, or transfer-function matrix to a state-variable vector- differential equation

$$\dot{x} = Ax + Bu$$

$$y = Cx + Du$$

where x is the state-vector, u is the input vector and y is the output vector. While the transfer-function matrix is an input-output description, the state-model gives an internal description of the system. If the state-variable equations are known, the system is said to be realized because these first order differential equations can be easily simulated on an analog computer.

If the state-model (A,B,C,D) of a linear, time-invariant dynamical system is known, it is a simple matter to obtain the corresponding transfer-function matrix H(s) which is given by

$$H(s) = C\ (sI - A)^{-1}\ B\ + D\ .$$

The converse, however, is not true. In general, there are innumerable realizations (A,B,C,D) which will give the same input-output response to a given system characterized by H(s).

The problem of realizing the system (A,B,C,D) from a given H(s) has been actively studied in the recent past. A well-developed theory of realization is now available in the technical literature [90],[100]. Various methods for minimal realizations rely heavily on the dual concepts of observability and controllability. With different degrees of complexity, most of the minimal realization techniques require a lot of computational work. An endeavor has been made to develop nonminimal realization techniques which are easy to apply, so that the realization problem may be solved quickly. If necessary, this sub-optimal realization can be made irreducible, by using standard system reduction techniques.

In the field of network theory, the determination of a realization (A,B,C,D) is the first step for synthesizing a network which corresponds to a specified input-output behavior. If a given (A,B,C,D) satisfies Anderson's positive real lemma [4], a network using only passive elements can be synthesized. Further, the set (A,B,C,D) satisfying reciprocity criterion of Yarlagadda [118] will lead to reciprocal network realizations. System theory techniques have also found applications in the determination of network functions, and in giving state-space interpretation of several well-known properties of network functions and classical synthesis methods. The problem of realization of dynamical systems has thus attracted wide attention because of its manifold applications in studying the problems of control, optimization and network theory.

1.2 STATEMENT OF THE PROBLEM

This thesis is concerned with the problem of a state-model realization of linear dynamical systems and networks from input-output data. The specific problems treated in this thesis can be stated as follows.

(1) New methods of realizations of linear, time-invariant multivariable systems from the given transfer - function matrix H(s), having advantages over the existing methods are sought. In particular, a minimal realization (A,B,C,D) from a symmetric H(s) using moments such that the realization is reciprocal, is obtained. The

use of moments is advantageous in the presence of noise. Further, new methods of obtaining non-minimal realizations are developed, which are easy to construct and have lower upper-bounds on the dimension, compared with other methods.

(2) The link between state-space characterization and frequency domain methods in network theory is investigated. In particular, a state-model realization and the positive real impedance matrix Z(s) are obtained when its Hermitian part Z(s) + Z'(-s), or Z(s) - Z'(-s) is given. A state-space interpretation of the Foster Synthesis method for LC networks, without considering the topology of the network, is presented.

A method is also given for obtaining the transfer - function matrix H(s) from its realization through the intermediation of moments of impulse response.

(3) The companion matrix and Jordan canonical forms, their inverse powers with applications to system analysis are studied. Methods for obtaining inverse powers of these canonical forms are developed. A method to construct transformations, which will reduce a time-varying

autonomous system into the companion form, is also discussed.

1.3 ORGANIZATION OF THE THESIS

The work embodied in this thesis has been arranged in the following manner.

The problem of minimal realization of linear systems is introduced in Chapter II. After giving some preliminaries, an historical review of various realization methods is given. The algorithm of Ho and Kalman is discussed in some details because of its importance in so much of subsequent work in the field. A brief review of the literature on state-space interpretation of classical results of network theory is also included in Chapter II. This Chapter is concluded with the key properties of passive and reciprocal realizations in state-space terms, which are needed in the sequel.

Chapter III presents new and improved methods of obtaining system realizations. A minimal realization technique from a symmetric transfer-function matrix is discussed. The technique results in reciprocal realizations. The problem of developing suboptimal realization methods has also been considered in this chapter. A method is presented which results in many cases in lower order realizations. In order to determine the suitability of the proposed method, mode matrices are defined for the multiple pole case. Another algorithm for realizing a linear time-invariant dynamical system has been given in third Chapter. The dimension of the realization obtained by this algorithm is the lowest compared with other hitherto known methods. All the realization methods developed in third chapter are equally applicable to the multiple pole case. Illustrative examples are also given.

In Chapter IV, an alternative method is proposed for finding a state-model realization and the positive-real impedance matrix Z(s) from its given even part Z(s) + Z'(-s) . A method to determine a state-model realization and the positive-real impedance matrix Z(s) from the given odd part Z(s) - Z'(-s) is also presented. A state-space interpretation of Foster Synthesis method for LC networks is presented. A method is presented for determining the transfer function matrix of a linear time-invariant system represented by the state-variable equations, through the intermediation of the moments of impulse response.

For simplification in system analysis and synthesis, it is desirable to transform the dynamic characterization into a canonical form. Chapter four also deals with these canonical forms. An algorithm for determining the inverse powers of a companion matrix is developed. Another algorithm is proposed to find the inverse powers of a matrix which is in Jordan canonical form. These algorithms can be applied for computing the moments of impulse response. The problem of obtaining transformations which will reduce a linear time-varying autonomous system to companion form has also been discussed. The proposed method is an extension of the method due to Power [76].

The subject matter of this chapter is also illustrated with examples.

A summary of the contributions made in this thesis is given in Chapter V. Some suggestions for further investigations in this field have also been incorporated.

CHAPTER II

REVIEW AND GENERAL CONSIDERATIONS

2.1 INTRODUCTION

One of the most important tasks in the study of dynamical systems is their characterization by a suitable mathematical model. This mathematical representation serves to optimize, control, or predict future behavior of the physical process. Determination of a state-model from input/ output data of a system has attracted the attention of many a researcher in the last decade. A host of literature is available in this field of system theory [1] - [3], [10], [14] - [18], [25] - [26], [38] - [48], [65] - [66], [86] - [91], [95] - [107] [122], which has led to the evolvement of a complete theory of realization. It has provided an understanding between the frequency domain and the state-space descriptions of systems [11] - [12], [30], [60]. State-model realization has assumed great importance because of modern trend of carrying out network synthesis in the state space [6] - [7], [52], [58], [105], [118]. Several well-known classical methods of network synthesis have been given state-space interpretation [4], [50], [78], [119]. Some problems of network analysis have been solved by using system theory concepts [54], [61], [77]. For time-varying, and non-linear systems, state-model representation is most convenient. This review chapter surveys representative literature on minimal state-model realizations of linear dynamical systems, and some results of system theory as they apply to network theory. The realization algorithm of Ho and Kalman [38] has been dealt with in detail because of its importance for much subsequent research in the field.

2.2 DESCRIPTION OF PHYSICAL SYSTEMS

A multivariable finite-dimensional linear dynamical system may be specified in many different ways. However, there are two standard forms in which a precise definition can be given to the system. We may specify the state-variable differential equations

$$\begin{aligned} \dot{x}(t) &= A(t)\ x(t) + B(t)\ u(t) \\ y(t) &= C(t)\ x(t) + D(t)\ u(t) \end{aligned} \qquad (2.1)$$

where $x(t)$, $u(t)$, and $y(t)$ designate the nx1 state vector, the mx1 control or input vector, and the px1 output vector, respectively. The matrices $A(t)$, $B(t)$, $C(t)$, and $D(t)$ have dimensions nxn, nxm, pxn and pxm, respectively. The second basic system description is the pxm impulse-response matrix $H(t,\tau) = \left[h_{ij}(t,\tau)\right]$ which relates the ith output to the jth input. The set of matrices (A,B,C,D) represents the internal description of the system and is termed to be a <u>realization</u>

of $H(t,\tau)$ [41] if for all $t \geqslant \tau$

$$H(t,\tau) = C(t)\,\Phi(t,\tau)\,B(\tau) + D(t) \qquad (2.2)$$

where $\Phi(t,\tau)$ is the transition matrix of(A,B,C,D). For the time-invariant case A,B,C, and D have real and constant elements and the transition matrix becomes exp (At). In many practical cases, the transfer-function matrix $H(s) = \mathcal{L}\left[H(t)\right]$ is given . It is easily shown that

$$H(s) = C\,(sI - A)^{-1}\,B + D \qquad (2.3)$$

where I is nxn unit matrix.

The number n is called the dimension of the system. The realization is said to be <u>minimal</u> or <u>irreducible</u> if there is no system of order less than n which also realizes $H(t,\tau)$.

A transfer-function matrix H(s) is said to be <u>rational</u> if every element of H(s) is a ratio of polynomials in s with real coefficients. H(s) is <u>regular</u> if no element of H(s) has a pole at infinity. H(s) is <u>proper</u> if $H(\infty) = D = 0$.

The <u>realization problem</u> is to pass from an input-output description of a system in the form of an impulse-response matrix, or transfer-function matrix, to a state-space description to the type (2.1) . The term realization comes from the fact that, using the description (2.1) it is possible to build systems, namely analog computers, whose behavior simulates the behavior of the system. In general, there is no unique solution to the realization problem and different realizations of the same input-output response have quite distinct characteristics. It is necessary, therefore, to examine the properties of "equivalent" representations. The following type of equivalence proves to be the most important in the realization problem.

Definition : $[A(t), B(t), C(t)]$ is algebraically equivalent to $[\bar{A}(t), \bar{B}(t), \bar{C}(t)]$ if and only if there exists a continuously differentiable matrix $N(t)$ with $\det N(t) \neq 0$ for all t , such that [41]

$$\begin{aligned} \bar{A}(t) &= N(t)\,A(t)\,N(t)^{-1} + \dot{N}(t)\,N(t)^{-1} \\ \bar{B}(t) &= N(t)\,B(t) \\ \bar{C}(t) &= C(t)\,N(t)^{-1} . \end{aligned} \qquad (2.4)$$

It may be readily verified that

$$\bar{\Phi}(t,\tau) = N(t)\,\Phi(t,\tau)\,N(t)^{-1} . \qquad (2.5)$$

D is not considered since it does not constitute the dynamic part of the system.

After discussing the preceding preliminaries, an historical development of the realization algorithms is given in the next section.

2.3 HISTORICAL REVIEW OF REALIZATION PROCEDURES

The problem of realization for linear systems was first stated by Gilbert [31] in 1963, who gave an algorithm for computing state-variable differential equations from a transfer-function matrix. At the same time Kalman [41] proposed an algorithm for the same problem, in which the values of A, B, and C could be found from the coefficients of numerator and denominator polynomials of the elements of H(s). The dual concepts of controllability and observability play an important role in these algorithms. In 1965, Kalman [44] proposed a new algorithm for obtaining the state equations from a given transfer-function matrix having multiple poles. Kalman had employed the classical theory of elementary divisors and the language of modules. This algorithm exhibits the canonical form, under equivalence, of a rectangular polynomial matrix [29]. Based on Kalman's method [44], another realization procedure was suggested by Raju [82]. He employs Kalman's method [44] for finding the order of the system and for finding the state matrix A. The matrices B and C are obtained by drawing a signal flow graph. Another minimum realization algorithm was proposed by Ho and Kalman [38] in 1965 which has been acclaimed to be one of the most useful and computationally simpler one. This method was evolved from a study of the so called Markov parameters [29] ,[106]. The impulse-response data of the system, which is assumed to have zero initial state, can be given in the time- or the s-domain in the form of Markov parameters. Ho and Kalman's algorithm centres on "the generalized Hankel Matrix" built from the Markov parameters. In 1971, Ackermann and Bucy [1] gave a method for constructing a state-variable model in the canonical form of Bucy [15], from the given matrix of impulse-response sequences of a finite-dimensional discrete time, linear, constant dynamical system. The construction is an alternate to the Ho-Kalman algorithm [38] in which two matrices P and Q must be found. Since P and Q in [38] are not unique, the realization obtained by Ho and Kalman is not in any special canonical form ; in general, all n(n+m+p) coefficients of (A,B,C) must be determined. In the canonical form of Bucy, at most n(n+p) parameters have to be evaluated . A procedure was outlined by Alberston and Womack [2] for computing the dimension of and constructing irreducible realization of a given system transfer-function matrix. Their procedure is much simpler and provides more insight into the physical significance of the problem. The resulting realization is in diagonal form. However, H(s) is constrained to have only simple poles. In 1969 , Wolovich and Falb [117] stated and proved a structure theorem for time-invariant multivariable linear systems. The theorem is then applied to obtain an algorithm analogous to that of Mayne [69] for solving the problem of realization. A computer algorithm had been developed for applying the algorithm. A method of realization based on the moments of the impulse-response matrix has been proposed by Bruni et. al [14]. Their procedure utilizes the Ho-Kalman algorithm [38]. The point of difference lies in that the Hankel matrix in [14] is constructed by the moments in place of Markov parameters as in [38.] .In the presence of noise, computation of moments is preferable to that of Markov parameters which are the local time-derivatives of the impulse-response matrix. Kuo [55], and Panda and Chen [73] determine irreducible realizations of a rational matrix in the Jordan form.

The problem of finding a sub-optimal solution to the realization problem has also attracted the attention of several authors. In 1963, Kalman [41] presented a simple method for computing a good upper bound on the dimension of a minimal realization, and provided an algorithm for constructing the corresponding noncanonical realization. Glass [33] proposed a simple procedure for obtaining a non-minimal realization of H(s), in 1968; the resulting realization is always in Jordan form. In 1970, Roveda and Schmid [91] presented a method for computing a new and lower upper-bound, compared with

Kalman [41], on the dimension of minimal realization of linear time-invariant dynamical systems. A simple algorithm was proposed by Roveda and Schmid [91] for constructing realizations with dimension equal to this upper bound. However, H(s) is assumed to have only simple poles. The realizations obtained are the minimal ones having the property of being structurally invariant with respect to the variations of the transfer-function-matrix coefficients. All these methods require significantly less computational work and yield a quick solution to the realization problem. If necessary, a completely controllable and completely observable part may be extracted from the nonminimal realization by the methods suggested by Mayne [69] and Rosenbrock [87], which yields a minimal realization.

Minimal realizations from symmetric impulse-response, or transfer-function matrix have been obtained by Lal and Singh [62] and Lal et.al. [65]- [66]. These methods are modifications of Ho-Kalman algorithm [38]. The importance of such realizations lies in passive network synthesis since they result in reciprocal realizations and further, all reciprocal realizations for RC and RL cases are passive [52], [123].

The minimal realization problem of time-variable linear systems has been considered by Desoer and Varaiya [25], Silverman [96] - [98], [100]. Silverman and Meadows [102], [104], Skoog [107], Youla [122], Lal and Singh [59]. The class of analytic matrices $H(t,\tau)$ have been considered as analyticity simplifies the development of the methods.

It may be mentioned that the discrete case is analogous to the continuous one and the methods of continuous-time solutions are equally applicable to the theory of discrete-time minimum realizations.

The algorithm of Ho and Kalman [38] is given in some details in the following section.

2.4 THE ALGORITHM OF HO AND KALMAN

In this section, minimal realizations are constructed when the system specifications are given in the form of Markov parameters [29]. The problem of realization is then the following :

"Given a sequence of pxm constant matrices Y_k (Markov parameters), k = 0,1,2,... , find a triple (A,B,C) of constant matrices such that

$$Y_k = C A^k B , \quad k = 0,1,2,... \text{"}$$

The sequence Y_k has a finite-dimensional realization if and only if there is an integer r and constants

α_i such that

$$Y_{r+j} = \sum_{i=1}^{r} \alpha_i \; Y_{r+j-i} \qquad \text{for all } j \geqslant 0 \tag{2.6}$$

where the degree r of the annihilating polynomial [29] of A_{min} is assumed to be known. The method to determine r is given later in this section. The algorithm for the construction of a minimal realization is described now.

The algorithm begins by forming the rxr block matrix ("generalized Hankel matrix") composed of the Markov parameters.

$$S_r = \begin{bmatrix} Y_o & Y_1 & \cdots & Y_{r-1} \\ Y_1 & Y_2 & \cdots & Y_r \\ \vdots & & & \vdots \\ Y_{r-1} & Y_r & \cdots & Y_{2r-2} \end{bmatrix} = \begin{bmatrix} Y_{i+j-2} \end{bmatrix} \tag{2.7}$$

If Y_k has a finite-dimensional realization, then the dimension of minimal realization is

$$n = \text{rank } S_r \tag{2.8}$$

The following steps yield a minimal realization.

Step 1) Form the matrix S_r

Step 2) Find non-singular matrices P and Q such that

$$P \, S_r \, Q = \begin{bmatrix} I_z & 0 \\ 0 & 0 \end{bmatrix} = J \tag{2.9}$$

Here, I_z is a zxz unit matrix, $z = \text{rank } S_r$, and J is idempotent.

Step 3) Let E_p be the block matrix $(I_p\ 0_p\ \ldots\ 0_p)$ and let ULH denote the operator which picks out upper left-hand block . Then a minimal realization of Y_k is given by

$$\begin{aligned} A &= \mathrm{ULH}\left[J \ P\, \Upsilon\, S_r\, Q\, J \right] \\ B &= \mathrm{ULH}\left[J \ P\, S_r \ E'_m \right] \\ C &= \mathrm{ULH}\left[E_p \ S_r \ Q \ J \right] \end{aligned} \qquad (2.10)$$

where Υ is a constant and

$$\Upsilon\, S_r = \begin{bmatrix} Y_1 & Y_2 & \cdots & Y_r \\ Y_2 & Y_3 & \cdots & Y_{r+1} \\ \vdots & & & \vdots \\ Y_r & Y_{r+1} & \cdots & Y_{2r-1} \end{bmatrix} \qquad (2.11)$$

The procedure described in the preceding steps makes only one assumption, namely a knowledge of the integer r. In order to determine r , it is given the values 1,2, ... , etc. For each value of r, the rank of S_r is determined. That value of r is chosen when rank S_r = rank S_{r+1} .

After treating the realization problem, some results concerning the state-space interpretation of classical concepts are discussed further in the following section.

2.5 STATE-SPACE INTERPRETATION OF CLASSICAL RESULTS OF NETWORK THEORY

There has been a growing interest in the application of state-space approach in the field of network theory, apart from the development of control and system theory in state-space terms. Many concepts of system theory find their utility in network analysis and synthesis. For example, a state-model of an RLC network which is completely controllable and completely observable can be synthesized using a minimum number of reactive elements . However, frequency domain methods are still extensively being used in the majority of network design problems. This has led to the exploration of communication links between the state-variable characterization and the input-output description of networks. Several authors have put in endeavors in this connection. The expressions for poles and zeros of a system in terms of its matrices have been developed by Brockett [12] . Similar relations were derived by Kuh [52] by signal flow graph representation of the state-space description of linear systems. Sandberg and So [94] developed techniques for evaluating the poles and zeros of a scalar transfer function from the state and output equations of the system. Recently, Lal and Singh [60] have derived some well known properties of LC, RC networks etc. using system theory concepts and have also given state-space interpretation of classical Foster and Cauer methods by considering topological state models of networks. Capacitor voltages and inductor currents are chosen as the state variables.

Consider a single-input single-output system

$$\begin{aligned} \dot{x} &= A x + b u \\ y &= c x + d u \end{aligned} \qquad (2.12)$$

When there is no resistance path between the input and the output, $d = 0$. In that case, the characteristic polynomial of A gives the poles of the transfer function while the zeros of the transfer function are given by the characteristic polynomial of A_o where [52]

$$A_o = \left[I - \frac{bc}{cb} \right] A \qquad (2.13a)$$

whereas for the case $d \neq 0$, A_o is given by [52]

$$A_o = \left[I - \frac{bc}{sd + cb} \right] A \qquad (2.13b)$$

or

$$A_o = A - \frac{bc}{d} \qquad (2.13c)$$

as given in [92]. The expression for the transfer function as given by Sandberg and So [92] is

$$H(s) = \frac{\det \begin{bmatrix} d & -c \\ b & sI - A \end{bmatrix}}{\det \left[sI - A \right]} \qquad (2.13d)$$

Further, the state-space interpretation of multiport Darlington method has been given by Anderson and Brockett [5]. Recently, Khan et. al. [50] have extended the technique of Puri and Takeda [78] for state-space synthesis of LC networks to n-port lossless Foster form. Besides, a state-variable technique has been proposed by Lal and Singh [61] for determining the state-model and the impedance matrix Z(s) from its given Hermitian part Z(s)+ Z'(-s). Youla's factorization of rational matrices [121] and the system theory criterion for positive real matrices, developed by Anderson [4] have been exploited by Lal and Singh [61], alongwith the realization theory, to obtain the minimum reactance matrix Z(s).

The concepts of passivity and reciprocity useful for state-space synthesis are dealt with next.

2.6.1 Passivity Criterion

Anderson's [4] system theory criterion for positive and real matrices is stated here as a lemma.

LEMMA 2.1 Let Z(s) be a matrix of rational transfer functions such that $Z(\infty)$ is finite and Z has poles which lie in Re $s < 0$ or are simple on Re $s = 0$ Let (A,B,C ,D) be a minimal realization of Z. Then Z(s) is positive real if and only if there exist a symmetric positive definite P and matrices W_o and L such that

$$PA + A'P = -L'L ,$$

$$PB = C' - L'W_o \qquad (2.14)$$

$$W_o' W_o = D + D_o' .$$

Here W(s) is found by using a lemma on spectral factorization, due to Youla [121] , such that

$$Z(s) + Z'(-s) = W'(-s) W(s) \cdot \qquad (2.15)$$

W(s) is unique save for multiplication on the left by an arbitrary orthogonal matrix.

2.6.2 Reciprocity Criterion

A theorem on reciprocity, due to Yarlagadda [118] is stated in the following.

THEOREM 2.1 Let Z(s) be an nxn matrix of real rational transfer functions with $Z(\infty)$ finite. Then Z(s) possesses a state model of the form

$$\begin{aligned} \dot{x} &= A x + B u \\ y &= C x + D u , \end{aligned} \qquad (2.16)$$

such that

$$(I \dotplus \Sigma) M \qquad (2.17)$$

is a symmetric matrix, where Σ is a unique ordered diagonal matrix of plus ones and minus ones, $\dotplus$ denotes direct sum and

$$I \dotplus \Sigma = \begin{bmatrix} I & 0 \\ 0 & \Sigma \end{bmatrix} , \text{ and } M = \begin{bmatrix} D & C \\ B & A \end{bmatrix}$$

if and only if $Z(s) = Z'(-s)$.

It may be pointed out that it is rather difficult to satisfy both passivity and reciprocity conditions simultaneously. However, all reciprocal realizations for RL and RC impedance matrices have been shown in [53] to be passive. Thus in these cases a reciprocal realization is automatically passive.

2.7 COMMENTS

It is evident that an abundance of literature can be found in state-model realization techniques. The given data may be in the form of an impulse response which dies out with time, does not disturb the steady state, and if the system is stable, it returns to the initial value after the application of the impulse. The data could also be in the form of Markov parameters, or moments of the impulse response. Of late, there have been fruitful attempts to find quick and computationally simple methods by looking for suboptimal solutions to the realization problem.

While a fairly complete theory of system realization is available, state-space network synthesis methods are still being developed. However, several well-known network properties and classical synthesis methods have been given state- space interpretation.

The following chapters deal with some new methods of realization, state-space interpretation, determination of system functions of positive real network functions, evaluation of inverse powers of canonical matrices, and transformation of time-varying autonomous systems to companion form.

CHAPTER III

REALIZATION OF LINEAR TIME-INVARIANT DYNAMICAL SYSTEMS

3.1 INTRODUCTION

The problem of constructing irreducible (or minimal) realizations of real, finite-dimensional, continuous-time and linear dynamical systems from their external descriptions has been actively studied in recent years. The fundamentals of this problem have been established by Gilbert, and Kalman in 1963. Interest in this fundamental problem of system theory has been generated due to a change from transfer-function description to a more general state-space characterization, for studying problems of control, filtering, identification and those in the field of network theory. Realization theory has served to get a better insight into the relationship between input-output and state-space models of a system. In this chapter, methods have been evolved for obtaining minimal as well as non-minimal realizations. A procedure is presented for constructing minimal reciprocal realizations of a given system transfer-function matrix H(s), for the case when H(s) is symmetric, using moments. Also, the technique proposed by Glass [33] for synthesizing transfer-function matrices having multiple poles is modified which may result in a lower dimension.

In order to determine the dimension of A matrix, mode matrices M and M_c are defined for the multiple pole case. Further, a simple algorithm is proposed for constructing a realization of a dynamical system described by means of its transfer-function matrix with multiple poles. The dimension of the resulting system will be still lower.

3.2 MINIMAL RECIPROCAL REALIZATIONS USING MOMENTS

The problem of minimal realization from a transfer-function matrix has been widely investigated. Various methods are available for the construction of the matrices A,B,C such that (assuming no direct path between input and output)

$$C \exp(At) B = H(t) \qquad (3.1a)$$

$$C (sI - A)^{-1} B = H(s) \qquad (3.1b)$$

where H(t) is the impulse-response matrix of a linear time-invariant finite-dimensional strictly proper system, and H(s) is its Laplace transform. The most relevant from a theoretical and computational point of view is the Ho-Kalman algorithm [38], Their algorithm begins by forming the rxr block matrix S_r (generalized Hankel matrix) built out of the Markov parameters [29], [106] Y_k's, where

$$S_r = \begin{bmatrix} Y_0 & Y_1 & \cdots & Y_{r-1} \\ Y_1 & Y_2 & \cdots & Y_r \\ \vdots & & & \vdots \\ Y_{r-1} & Y_r & \cdots & Y_{2r-2} \end{bmatrix} \qquad (3.2)$$

and Y_k's are coefficients of the negative power series expansion of H(s). Non-singular matrices P and Q are then found using standard methods [23] such that

$$P \; S_r \; Q = \begin{bmatrix} I_z & 0 \\ 0 & 0 \end{bmatrix} = J \qquad (3.3)$$

where I_z is a zxz unit matrix, z ≡ rank S_r and J is idempotent.

If E_p is the block matrix $(I_p, 0_p, \ldots, 0_p)$ and ULH denotes the operator which picks out upper left hand block, then

$$A = \text{ULH}\left[J \; P \; (\tau \; S_r) \; Q \; J\right] \qquad (3.4a)$$

$$B = \text{ULH}\left[J \; P \; S_r \; E_m'\right] \qquad (3.4b)$$

$$C = \text{ULH}\left[E_p \; S_r \; Q \; J\right] \qquad (3.4c)$$

is a minimal realization of H(s) , where

$$\tau \; S_r = \begin{bmatrix} Y_1 & Y_2 & \cdots & Y_r \\ Y_2 & Y_3 & \cdots & Y_{r+1} \\ \vdots & & & \vdots \\ Y_r & Y_{r+1} & \cdots & Y_{2r-1} \end{bmatrix} . \qquad (3.5)$$

Recently, Lal and Singh [62] have suggested a modification of the algorithm of Ho and Kalman [38] for obtaining a minimal realization (A,B,C,D) of a transfer-function matrix H(s), for the case when H(s) is symmetric. The realization obtained in [62] is such that

$$(I \dotplus \Sigma)\, M \qquad (3.6)$$

is symmetric, where

$$M = \begin{bmatrix} D & C \\ B & A \end{bmatrix} \qquad (3.7)$$

$\dotplus$ denotes direct sum

and the diagonal matrix

$$\Sigma = \Sigma_1 \dotplus \Sigma_2$$

where

Σ_1 has only +1's on the diagonal

Σ_2 has only -1's on the diagonal .

Such realizations result in reciprocal networks, as mentioned in [62] . Further, it has been proved in [53] and [123] that all reciprocal realizations for RC and RL cases are passive. In the realization process of [62] , since the given matrix H(s) is symmetric, the Markov parameters and consequently the Hankel matrix S, will also be symmetric . For the symmetric matrix S, a non-singular P can always be found [23] such that

$$P\,S\,P' = (\Sigma \dotplus 0), \qquad (3.8)$$

the order of Σ being equal to the rank of S.

While applying the method of [62] to synthesize H(s), it is also worthwhile to expand H(s) in a positive power series according to

$$H(s) = \sum_{k=0}^{\infty} C_k \, s^k . \qquad (3.9)$$

This series converges in a suitable neighbo-rhood of the origin, and it can be analytically continued on the whole plane except for the singularities of H(s) [14]. Consequently, the sequence C_k, uniquely identifies the H(s). Each C_k is uniquely connected to the corresponding moment M_k of the impulse- response matrix H(t) by the relation [14]

$$M_k = (-1)^k k! \, C_k \qquad (3.10)$$

where

$$M_k = \int_0^{\infty} t^k H(t) \, dt, \quad k = 0,1,2, \ldots \qquad (3.11)$$

Since H(s) is symmetric, its moments are also symmetric as is clear from (3.9) and (3.10). Then the Hankel matrix S constructed from the moments will also be symmetric. Therefore, a non-singular matrix P can again be found such that (3.6) is satisfied. The procedure of [62] can then be applied without any modification to the Hankel matrix constructed from M_k^* where [14]

$$M_k^* = \frac{(-1)^k}{(k-1)!} M_{k-1} , \quad k = 1,2,\ldots \qquad (3.12a)$$

and

$$M_0^* = \left[H(t)\right]_{t=0} = \lim_{s \to \infty} s\,H(s) \cdot \qquad (3.12b)$$

The matrices A, B, C are related with the quantities introduced above according to [14]

$$M_k^* = C\,A^{-k}\,B\ , \quad k = 0,1,2,\ \ldots \qquad (3.13)$$

Then, in the light of the methods given in [14] and [62] we get [79]

$$A^{-1} = ULH\left[J\,P\,(\tau\ S)\,P'(\textstyle\sum \dotplus 0)\,J\right] \qquad (3.14a)$$

$$B = ULH\left[J\,P\,S\ E'_m\right] \qquad (3.14b)$$

$$C = ULH\left[E_p\ S\,P'\ (\textstyle\sum \dotplus 0)\ J\right] \qquad (3.14c)$$

$$D = H(\infty) \cdot \qquad (3.14d)$$

Here $p = m$. It is obvious that this realization will also satisfy (3.6).

At the end of the calculations, it is necessary, of course, to invert the matrix A^{-1} to obtain A . For reciprocal RC and RL networks H(s) will be asymptotically stable, and, from [14] , A is then non-singular.

CONCLUSION

A method of minimal realization based on moments is presented. In the presence of noise, computation of moments is preferable to that of Markov parameters which can be interpreted as time derivatives of the impulse-response matrix calculated in the origin. As pointed out in [14] , when realization is to be constructed from empirically obtained data of H(t) , a method based on moments is advantageous.

The example of [62] is taken for illustration purposes :

Example 3.1

Given a symmetric $H(s) = \dfrac{\frac{1}{8} s^2 + \frac{2}{8} s + \frac{9}{8}}{s^2 + 2s + 1}$

Then

$$D = 1/8$$

and

$$H_1(s) = H(s) - 1/8$$

$$= \frac{1}{s^2 + 2s + 1}$$

$$= 1 - 2s + 3s^2 - 4s^3 + 5s^4 - \ldots .$$

Using (3.12a) and (3.12b), we get

$M_0^* = 0$, $M_1^* = -1$, $M_2^* = 2$

$M_3^* = -3$, $M_4^* = 4$, $M_5^* = -5$, ...

It can be seen that

$$M_k^* = -C_{k-1} , \quad k = 1,2, \ldots$$

The Hankel matrix

$$S = \begin{bmatrix} 0 & -1 \\ -1 & 2 \end{bmatrix}$$

and

$$\tau S = \begin{bmatrix} M_1^* & M_2^* \\ M_2^* & M_3^* \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 2 & -3 \end{bmatrix}$$

As S is symmetric, it can be transformed to the form of Eq. (3.8) with

$$P = \begin{bmatrix} 0 & 1/\sqrt{2} \\ \sqrt{2} & 1/\sqrt{2} \end{bmatrix} ,$$

$$\sum = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} ,$$

$$J = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$E'_m = \begin{bmatrix} 1 \\ 0 \end{bmatrix} , E_p = \begin{bmatrix} 1 & 0 \end{bmatrix} .$$

Using Eq. (3.14) , we get

$$A^{-1} = \begin{bmatrix} -3/2 & -1/2 \\ 1/2 & -1/2 \end{bmatrix}$$

$$B = \begin{bmatrix} -1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix},$$

$$C = \begin{bmatrix} -1/\ 2 & 1/\ 2 \end{bmatrix}, \ D = 1/8 .$$

With

$$A = \begin{bmatrix} -1/2 & 1/2 \\ -1/2 & -3/2 \end{bmatrix}$$

we get

$$M = \begin{bmatrix} D & C \\ B & A \end{bmatrix} = \begin{bmatrix} 1/8 & -1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & -1/2 & 1/2 \\ -1/\ 2 & -1/2 & -3/2 \end{bmatrix}$$

and

$$\left(I \dot{+} \sum\right) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} .$$

It can be seen that Eq. (3.6) is satisfied.

Example 3.2

Let

$$H(s) = \begin{bmatrix} \frac{1}{s+1} & \frac{1}{s+1} \\ \frac{1}{s+1} & \frac{1}{s+1} \end{bmatrix} .$$

From (3.10) we get the moments

$$M_o = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad M_1 = \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix}, \quad M_2 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \ldots$$

Using Eqs (3.12a) and (3.12b) M_k^*'s are obtained as follow .

$$M_o^* = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, M_1^* = \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix}, \quad M_2^* = \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix},$$

The Hankel matrix S is found to have rank = 1

$$S = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$TS = \begin{bmatrix} M_1^* \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix}$$

The congruence transformation matrix is

$$P = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} .$$

Then

$$\sum \dot{+} 0 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad J = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} .$$

With

$$E'_m = E_p = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

and using Eq. (3.14), we get

$$A^{-1} = \begin{bmatrix} -1 \end{bmatrix} = A$$

$$B = \begin{bmatrix} 1 & 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

This realization (A,B,C) satisfies (3.6) and thus would result in reciprocal realizations using passive network elements.

3.3 NON-MINIMAL REALIZATION TECHNIQUES

The realization procedures found in the literature so far appear computationally rather cumbersome, having different degrees of complexity. Thus, it is worthwhile looking for sub-optimal solutions of the realization problem having computational simplicity, by using noncanonical structures. The following two sections deal with this problem of realization.

3.3.1 Synthesis of Transfer-Function Matrices with Multiple Poles

A simplified technique was presented by Glass [33] for obtaining a state-model realization (A,B,C) of a transfer-function matrix H(s), with $H(\infty) = 0$. The constant matrix $H(\infty)$ corresponds to "resistance paths" from system inputs to system outputs, and does not contribute to the dynamical part of the system. The resulting system is not always irreducible and standard system reduction techniques have to be used to obtain an irreducible realization. Therefore, before applying this algorithm, it is perhaps worthwhile to consider whether factoring of H(s) into a slightly different form will lead to a state-model of lower dimension.

It is apparent that the dimension of the resulting A matrix in Glass's technique is equal to the number of rows of Q(s), where Q(s) is obtained by factoring H(s) as the matrix product

$$H(s) = C\,Q(s)\ . \qquad (3.15)$$

The dimension of A matrix can also be easily obtained by constructing mode matrices M , M_c and M_r [2] defined for the multiple-pole case as follows . Consider H(s) of [33].

$$H(s) = \begin{bmatrix} \frac{1}{s+1} + \frac{2}{(s+1)^2} + \frac{3}{s+2} & \frac{2}{s+1} - \frac{3}{s+3} \\ -\frac{2}{s+1} + \frac{1}{s+2} & -\frac{2}{(s+3)^2} + \frac{1}{s+5} \end{bmatrix} . \qquad (3.16)$$

M is defined as a mode matrix whose elements correspond to the modes of H(s), and as such can be written by inspection as

$$M = \begin{bmatrix} (1)\ ,\ (1)^2, (2) & (1),\ (3) \\ (1),\ (2) & (3), (3)^2, (5) \end{bmatrix} . \qquad (3.17)$$

It may be noted that in the multiple-pole case, for a term $\alpha / (s + \sigma)^r$ in an element of H(s), the corresponding entries of M contain all the multiplicities (σ), $(\sigma)^2$ ...., $(\sigma)^r$. However, if this element of H(s) contains $\beta / (s+\sigma)$ in addition, the corresponding entry in the matrix M is nil as indicated in (3.17).

M is reduced to column combinations [2] which are achieved by considering a mode, say $(\sigma)^k$ in a particular column of M, and retaining it in only one entry while cancelling it from others in the same column . The process is repeated for all the modes. Thus M_c becomes

$$M_c = \begin{bmatrix} (1), (1)^2, (2) & (1), (3) \\ - & (3)^2, (5) \end{bmatrix} . \qquad (3.18)$$

M is reduced to M_r by making row combinations, in a similar way. Then we get M_r as,

$$M_r = \begin{bmatrix} (1), (1)^2, (2) & (3) \\ (1), (2) & (3), (3)^2, (5) \end{bmatrix} . \qquad (3.19)$$

Obviously, the sum of the number of modes of M_c (7 in this problem) is equal to the number of rows of Q(s) of (3.15) .

If it is found that the sum of the number of modes in M_c is less than the sum of the number of modes in M_r , a state-model of lower dimension than that given in [33] can be obtained by factoring H(s) as the matrix product

$$H(s) = P(s) B \tag{3.20}$$

where the elements of the kth row of P(s) are the poles appearing in the corresponding row of H(s), with multiple poles listed in order of decreasing multiplicity. (3.20) may be written as

$$H(s) = P(s) T^{-1}(s) T(s) B \tag{3.21}$$

where T(s) is a non-singular matrix. $T^{-1}(s)$ is constructed by a procedure similar to that given in [33] by having linear factors (s+p), which correspond to the poles in the corresponding columns of P(s), as the elements of the principal diagonal. All other elements are zero, except that a minus one is required in the position below the principal diagonal, if the corresponding pole of P(s) is not simple. $T^{-1}(s)$ constructed thus will ensure that C, given by $C = P(s) T^{-1}(s)$, and A, given by $A = sI - T^{-1}(s)$, where I is a unit matrix of appropriate order, are both time-invariant matrices.

The procedure for row or column mode combining provides an upper bound on the minimal dimension of the system. Let α_i be the number of λ_i modes in M_c and β_i the number of λ_i modes in M_r. Then the maximum number of state-variables required to realize H(s) is

$$n_o = \sum_{i=1}^{q} \min(\alpha_i, \beta_i) \tag{3.22}$$

The dimension $n \leq n_o$ of irreducible realizations.

Obviously, in many examples, the realization obtained by the proposed method may be of lower dimension than that obtained by [33], which will require far fewer computations when a completely controllable and completely observable part is extracted from it.

Suppose a system, specified by the transfer-function matrix H(s) is required to be synthesized, where

$$H(s) = \begin{bmatrix} \frac{1}{s+1} + \frac{2}{(s+1)^2} + \frac{1}{s+2} & \frac{2}{s+1} + \frac{1}{(s+1)^2} \\ \frac{2}{s+3} + \frac{3}{(s+3)^2} + \frac{1}{s+5} & \frac{1}{s+3} + \frac{2}{s+5} \end{bmatrix} .$$

In order to decide about the suitability of the method given by Glass or the one proposed here, mode matrix M is constructed first. Thus

$$M = \begin{bmatrix} (1), (1)^2, (2) & (1), (1)^2 \\ (3), (3)^2, (5) & (3), (5) \end{bmatrix} .$$

M is reduced to M_c by column combinations and to M_r by row combinations, i.e.,

$$M_c = \begin{bmatrix} (1), (1)^2, (2) & (1), (1)^2 \\ (3), (3)^2, (5) & (3), (5) \end{bmatrix}$$

$$M_r = \begin{bmatrix} (1), (1)^2, (2) & - \\ (3), (3)^2, (5) & - \end{bmatrix} .$$

From M_c , the dimension of A is 10 whereas M_r gives the dimension of A as 6. Of course, a realization of dimension 6 is to be prefered . From (3.20), we get

$$H(s) = \begin{bmatrix} \frac{1}{(s+1)^2} & \frac{1}{s+1} & \frac{1}{s+2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{(s+3)^2} & \frac{1}{s+3} & \frac{1}{s+5} \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 2 \\ 1 & 0 \\ 3 & 0 \\ 2 & 1 \\ 1 & 2 \end{bmatrix},$$

which gives

$$T^{-1}(s) = \begin{bmatrix} s+1 & 0 & 0 & 0 & 0 & 0 \\ -1 & s+1 & 0 & 0 & 0 & 0 \\ 0 & 0 & s+2 & 0 & 0 & 0 \\ 0 & 0 & 0 & s+3 & 0 & 0 \\ 0 & 0 & 0 & -1 & s+3 & 0 \\ 0 & 0 & 0 & 0 & 0 & s+5 \end{bmatrix} .$$

Matrix inversion is not required since only $T^{-1}(s)$ is required in the synthesis. The realization (A,B,C,D) is given below.

$$A = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -3 & 0 & 0 \\ 0 & 0 & 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 & -5 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 1 \\ 1 & 2 \\ 1 & 0 \\ 3 & 0 \\ 2 & 1 \\ 1 & 2 \end{bmatrix},$$

$$C = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

3.3.2 An Algorithm for Lower Dimension Realization of Dynamical Systems

The aim of this section is to present a method for computing, a new and lower , upper-bound and to give a simple algorithm for constructing a realization, with dimension equal to this upper bound, of a dynamical system described by means of its transfer-function matrix with multiple poles.

A procedure for computing an upper bound n_o on n was proposed by Kalman [41] , with

$$n_o = \left(\min \sum_{i=1}^{p} \alpha_i, \quad \sum_{j=1}^{m} \beta_j \right), \qquad (3.23)$$

where α_i and β_j are the number of distinct poles (counting each pole with its maximum multiplicity) in the ith row and in the jth column, respectively, of matrix H(s), and he has provided an algorithm for constructing the corresponding non-canonical realization.

Recently, Roveda and Schmid [91] have proposed a procedure for obtaining a good upper bound on the dimension of a minimal realization. They construct realizations with dimension equal to this upper bound. Their method is applicable under the assumption that no element of the transfer-function matrix H(s) has multiple poles. The resulting system is

$$\dot{x} = A\,x + B\,u$$
$$y = C\,x \qquad (3.24)$$

where x,y and u are n,p, and m vectors, and A,B,C are nxn , nxm and pxn constant matrices, respectively. The condition of simple poles is a big drawback. Here, a generalized algorithm is proposed to obtain a non-canonical realization for the case of H(s) having simple as well as multiple poles. The significance of the proposed algorithm is its computational simplicity. The dimension of the resulting system will be smaller than the ones obtained by Kalman [41], Glass [33] or Lal et.al. [64].

The steps of the algorithm are more easily presented by means of an illustrative example given in the following paragraphs. Given a transfer-function matrix

$$H(s) = \begin{bmatrix} \frac{1}{(s+1)^2} + \frac{2}{s+1} + \frac{1}{s+2} & \frac{5}{s+2} + \frac{3}{s+1} & \frac{1}{(s+1)^2} + \frac{4}{s+1} \\ \frac{2}{s+2} + \frac{1}{(s+2)^2} & \frac{2}{(s+1)^2} + \frac{1}{s+2} & \frac{2}{(s+3)^2} + \frac{1}{s+3} \\ \frac{5}{s+3} & \frac{1}{s+3} + \frac{6}{(s+1)^2} + \frac{1}{s+1} & \frac{2}{s+3} \end{bmatrix} \qquad (3.25)$$

Step 1 : Compute the coefficient matrices of H(s) corresponding to the distinct poles and their multiplicities such that

$$R(1^2) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 6 & 0 \end{bmatrix}, \quad R(1) = \begin{bmatrix} 2 & 3 & 4 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},$$

$$R(2) = \begin{bmatrix} 1 & 5 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad R(3^2) = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$R(3) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 5 & 1 & 2 \end{bmatrix}. \qquad (3.26)$$

Step 2 : For each coefficient matrix, find the minimum set of lines (covering set) containing all its non-zero elements, as shown in the preceding step.

Step 3 : In order to construct A, consider a multiple pole. Associate with the coefficient matrix of highest multiplicity, say $R(1^2)$, a real matrix $A(1^2)$ constructed as follows.

1) Scan $R(1^2)$ for columns belonging to the covering set. Form an upper Jordan block having $(s+1)^2$ as its

elementary divisor corresponding to each such column. In the example considered, there is only one such column which would generate a 2x2 upper Jordan block.

Scan $R(1^2)$ for rows belonging to the covering set. Form a lower Jordan block having $(s+1)^2$ as its elementary divisor corresponding to each such row. In the example considered, there is only one such row which gives rise to a 2x2 lower Jordan block. Then the direct sum of the Jordan blocks constructed from $R(1^2)$ by column scanning and row scanning gives $A(1^2)$.

2) Now consider R(1) which is the coefficient matrix of the next lower multiplicity corresponding to the pole of highest multiplicity considered in the preceding. Scan R(1) for columns belonging to the covering set. Ignore those columns of the covering set of R(1) in which the columns in the corresponding position of the covering set of $R(1^2)$ have been considered. Form Jordan blocks with (s+1) as the elementary divisor for each of the remaining columns of the covering set which essentially reduces itself to the same form, as obtained in [91]. Similarly, scan R(1) for rows and form Jordan blocks.

Then the direct sum of the Jordan blocks formed from column scanning and row scanning of R(1) as just described gives A(1).

3) Repeat the preceding procedure for the remaining coefficient matrices. The resulting A matrix for the example considered obviously becomes

$$A = \text{diag}\left[A(1^2), A(2), A(3^2), A(3) \right]$$

$$= \begin{bmatrix} -1 & 1 & & & & & & \\ 0 & -1 & & & & & & \\ & & -1 & 0 & & & & \\ & & 1 & -1 & & & & \\ & & & & -2 & 0 & & \\ & & & & 0 & -2 & & \\ & & & & & & -3 & 0 & \\ & & & & & & 1 & -3 & \\ & & & & & & & & -3 \end{bmatrix} \cdot \quad (3.27)$$

Step 4 : In order to construct B, consider a multiple pole. Associate with the coefficient matrix of the highest multiplicity, say $R(1^2)$, a real matrix $B(1^2)$ as follows

1) Scan $R(1^2)$ for columns belonging to the covering set and let K be the index of the first column in this set [91] . Form the first two rows of $B(1^2)$ by letting

$$b_{1j} = 0 , \quad \text{for } j = 1,\dots, m \qquad (3.28a)$$

$$b_{2j} = \begin{cases} 1 , & \text{for } j = K \\ 0 & \text{for } j \neq K . \end{cases} \qquad (3.28b)$$

Repeat for the remaining columns of $R(1^2)$ belonging to the covering set.

2) Scan $R(1^2)$ for rows belonging to the covering set and let H be the index of the first row in this set. Form the next row of $\Lambda(1^2)$, say, row t, by letting

$$b_{tj}(1^2) = r_{Hj}(1^2) \; , \; j = 1,\ldots, m \qquad (3.29a)$$

if the column does not belong to the covering set,

$$b_{tj}(1^2) = 0 \qquad (3.29b)$$

otherwise.

The (t+1) th row of $\Lambda(1^2)$ is formed as follows. Let the index of the first row of the covering set of R(1) be E. Then,

$$b_{t+1,j}(1^2) = \begin{cases} r_{Ej}(1^2) & \text{for } E = H \; , \; j = 1,\ldots, m \\ 0 & \text{for } E \neq H \; , \; j = 1,\ldots, m \end{cases} \qquad (3.30a)$$

if the column j does not belong to the covering set,

$$b_{t+1,j}(1^2) = 0 \qquad (3.30b)$$

Otherwise. Repeat for the remaining rows of $R(1^2)$ belonging to the covering set.

3) The procedure of [91] for R(1) is now carried out. Those lines of the covering set of R(1) are, however, neglected which have already been taken into account while considering $R(1^2)$.

4) Repeat the preceding procedure for the remaining coefficient matrices. The resulting B matrix for the example considered becomes

$$B = \begin{bmatrix} B(1^2) \\ B(2) \\ B(3^2) \\ B(3) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ \hdashline 1 & 0 & 1 \\ 2 & 3 & 4 \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \\ \hline 1 & 0 & 2 \\ 0 & 0 & 1 \\ \hline 5 & 1 & 2 \end{bmatrix} . \quad (3.31)$$

Step 5 : Again consider a multiple pole in order to construct C. Associate with the coefficient matrix of highest multiplicity, say, $R(1^2)$, a real matrix $C(1^2)$ by letting

$$c_{i1}(1^2) = r_{ik}(1^2) \text{ , } i = 1, \ldots, p. \quad (3.32a)$$

The second column is formed as follows. Let L be the index of the first column of the covering set of $R(1^2)$. Then,

$$c_{i2}(1^2) = \begin{cases} r_{iL}(1^2), & \text{for } L = K \text{ , } i = 1,2,\ldots,p \\ 0 & \text{for } L \neq K \text{ , } i = 1,2,\ldots,p \end{cases} \quad (3.32b)$$

Repeat for the remaining columns of $R(1^2)$ belonging to the covering set.

2) Scan $R(1^2)$ for rows belonging to the covering set and let H be the index of the first row in this set. Form the next two columns of $C(1^2)$, say, columns u and u+1 , by letting

$$c_{i,Hu}(1^2) = 0 \ , \ i = 1, \ldots, p \tag{3.33a}$$

$$c_{i,Hu+1}(1^2) = \begin{cases} 1 & , \text{ for } i = H \\ 0 & , \text{ for } i \neq H \end{cases} \tag{3.33b}$$

Repeat for the remaining rows of $R(1^2)$ belonging to the covering set.

3) The procedure of [91] for R(1) is next carried out. Those lines of the covering set of R(1), are , however, ignored which have already been taken into account while considering $R(1^2)$.

4) Repeat the preceding procedure for the remaining coefficient matrices. The resulting C matrix for the example considered becomes

$$C = \left[C(1^2) \quad C(2) \quad C(3^2) \quad C(3) \right]$$

$$= \left[\begin{array}{cc|cc|cc|cc|c} 0 & 3 & 0 & 1 & 1 & 5 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 2 & 1 & 0 & 1 & 0 \\ 6 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array} \right] \tag{3.34}$$

It will thus be seen that Glass's technique [33] is a special case of the procedure just described when the covering set consists of the columns only; so is the technique proposed in [64] when the covering set consists of the rows only. As the proposed technique considers covering sets from both columns and rows together, the dimension of the resulting realization will obviously be smaller than or at the most equal to the lesser of the one obtained from [33] , [41], or [64]. The proposed algorithm is self-evident in the light of the algorithms already given in [33] and [64]. The upper bound on system dimension can be easily found from step 3 of the proposed algorithm.

For the example considered, the upper bound given by Kalman [41] is

$$n_o = \sum_{i=1}^{3} \alpha_i = 11 .$$

The dimension of the realization obtained by [33] is 13 and of that obtained by [64] and [41] is 11. Applying the algorithm described in this section results in a realization of dimension 9 which gives a lower upper-bound on the dimension of an irreducible realization.

3.4 MISCELLANEOUS COMMENTS

The work is this chapter reveals new characteristics of linear dynamical systems described by transfer-function matrices having multiple poles. A modified procedure of minimal realization utilizing the approach of Ho and Kalman [38] is presented when the given matrix is symmetric. The method is based on moments of impulse - response . In the presence of noise, computation of moments is preferable to that of Markov parameters which can be interpreted as time derivatives of the impulse response matrix calculated in the origin. When realization is to be constructed from empirically obtained data of H(t), a method based on moments is advantageous. The realization obtained satisfies reciprocity constraints. Further, the problem of realization without guaranteeing irreducibility has been investigated. In some applications [77], non-minimal realizations are acceptable if they could be constructed easily. If, however, a minimal realization is required, standard system reduction techniques [69], [87] may be employed to extract a completely controllable and completely observable part which gives the minimal realization. A simplified technique is presented for obtaining a state-model realization of a transfer-function matrix. It has been demonstrated how one can often get lower order realization of a transfer-function matrix.

Continuing with the search for obtaining sub-optimal solutions of system realization problem using noncanonical structures, another algorithm is proposed for constructing a realization of a dynamical system having multiple poles. The proposed algorithm has the advantage that the dimension of the realization is not greater than the dimension of the realization which can be obtained by Kalman [41], Glass [33], or Lal, Singh and Puri [64]. The computational simplicity of the algorithm emphasizes the significance of approaching realization problems through noncanonical structures.

The next chapter is concerned with the application of system theory for the determination of network functions and their state-model realization, and also for a state-space interpretation of classical synthesis methods.

CHAPTER IV

NETWORK FUNCTIONS AND STATE-SPACE INTERPRETATION

4.1 INTRODUCTION

The many advances made in system theory in the past few years have enlarged its scope to a number of established fields. It is the mathematical structure of a system, and not its physical form, that is of interest to a system theorist, for studying the behavior of various types and forms of systems. Consequently, much attention is being paid for network analysis and synthesis using state-space techniques. This chapter discusses the determination of network functions and their state-space realizations utilizing results of system theory. A method for determining transfer-function matrix from a knowledge of its moments is given. Procedures are given for obtaining the higher powers of the inverse of state matrix when it is given in Jordan canonical form or in the companion form. Either of these results may be employed to compute moments of impulse-response matrix. Also, a method is presented to generate transformation matrices which would transform a time-varying autonomous system to companion form. Besides, state-space interpretation of classical network synthesis methods is given.

4.2 DETERMINATION OF IMPEDANCE MATRIX Z(s) FROM GIVEN Z(s) + Z'(-s)

Recently, a state-variable technique for the determination of the impedance matrix Z(s) of order n from given U(s) = Z(s) + Z'(-s) has been given by Lal and Singh [61], where prime denotes matrix transposition. The first step in the procedure of [61] involves the factorization of the nxn matrix Z(s) + Z'(-s) such that

$$Z(s) + Z'(-s) = W'(-s)\ W(s)$$

where W(s) is an rxn matrix and r denotes the normal rank of U(s). It is also necessary in the procedure of [61] to determine a symmetric positive-definite matrix P, and matrices L and W_o such that

$$P\,A + A'\,P = -\,L\,L'$$

$$P\,B = C - L\,W_o$$

$$W_o'\ W_o = D + D'$$

where (A,B,C,D) is a minimal realization of Z(s) and (A, B, L, W_o) is a minimal realization of W(s).

The aim of this section is to give an alternative approach for determining the state-model and the impedance matrix from given U(s) .The proposed technique is simpler than the one given earlier [61], and is applicable for any n .

For a positive-real impedance matrix Z(s), assuming $Z(\infty) = 0$ for simplicity, and Z(s) possessing no imaginary axis poles, if (A,B,C) is a minimal realization for Z(s), then A will have eigenvalues with negative real parts [4] . Z(s) and Z'(-s) can have no poles in common (those of Z(s) being in Re $s < 0$ and those of Z'(-s) in Re $s > 0$). Therefore .

$$\delta \left[Z(s) + Z'(-s) \right] = 2 \quad \delta \left[Z(s) \right] \tag{4.3}$$

as proved in [4] , where δ denotes the degree of a rational matrix. Hence the dimension of the matrix A is half that of A_H , where (A_H, B_H, C_H) is a realization of U(s) such that

$$U(s) = C_H \, (sI - A_H)^{-1} \, B_H \, . \tag{4.4}$$

Let U(s) satisfy the following conditions [59] .

1) U(s) is a real, rational, para-Hermitian matrix i.e., U(s) = U'(-s)
2) On the $j\omega$ – axis , U(s) is bounded and is non-negative definite.

A para-Hermitian matrix as originally defined by Belevitch, OOno and more precisely by Youla [121] is by definition real, rational. Also , if U(s) is para-Hermitian, then it is Hermitian on $j\omega$ - axis.

The proposed algorithm for determining Z(s) is now given.

Step 1: Obtain a minimal realization (A_H, B_H , C_H) of U(s) in the Jordan form by any of the known methods [53],[71].

Step 2 : Modify the above realization by re-ordering the rows of A_H , B_H , C_H' to get

$$A_H = \begin{bmatrix} A & \vdots \\ \cdots & A^- \end{bmatrix}, \quad B_H = \begin{bmatrix} B \\ \cdots \\ B^- \end{bmatrix}$$

$$C_H = \begin{bmatrix} C & \vdots & C^- \end{bmatrix}, \qquad (4.5)$$

where A consists of Jordan blocks with negative eigen-values and A^- has Jordan blocks with positive eigen-values.

Step 3 : Reject the sub-matrices A^-, B^-, C^- having positive eigen-values which correspond to the right half plane poles, to obtain (A,B,C) realizing Z(s).

Step 4 : Since the inversion of (sI - A) is quite simple as A is in Jordan form, Z(s) can be obtained quite easily by the relation

$$Z(s) = C\,(sI - A)^{-1}\,B \,. \qquad (4.6)$$

The steps enumerated above are, in fact, the extension of Bode's method [113] for determining a positive- real function from even part, for the matrix case. These steps can obviously be followed through successfully keeping in view the properties of a positive-real matrix, conditions on Z(s) + Z'(-s) mentioned above (which are an extension of the corresponding scalar case), and the fact that the eigen-values of A which are to be negative, correspond to the left half-plane poles of $Z(s)$.

It may be noted that the realization (A_H, B_H, C_H) need not be minimal so far as the determination of $Z(s)$ is concerned. In fact, a sub-optimal realization in the Jordan form may be obtained easily from [33] or [62]. Even if a minimal realization is required, a completely controllable and completely observable part may be extracted from the non-minimal (A,B,C) thus obtained, by the method of Mayne [69] or Kalman [41].

$Z(s)$ obtained by this method will have no poles on the imaginary axis. Therefore, $Z(s)$ will be a minimum reactance matrix [71].

The given method is illustrated by an example.

Example

Consider

$$U(s) = Z(s)+Z'(-s)=\begin{bmatrix} \frac{1}{(s+1)(s-1)} & -\frac{2}{(s+1)^2} \\ -\frac{2}{(s-1)^2} & -\frac{4}{(s+1)(s-1)} \end{bmatrix}.$$

The problem is to determine a positive real $Z(s)$ having this $U(s)$.

$U(s)$ is seen to satisfy conditions 1) and 2).

Step 1: $U(s)$ has a Jordan form realization

$$A_H = \left[\begin{array}{c|cc|cc|c} -1 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right], \quad B_H = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix},$$

$$C_H = \begin{bmatrix} 1/2 & 0 & -1/2 & -2 & 0 & 0 \\ 0 & -2 & 0 & 0 & 2 & -2 \end{bmatrix}.$$

Step 2 : The realization is modified to give

$$A_H = \left[\begin{array}{ccc|ccc} -1 & 1 & 0 & & & \\ 0 & -1 & 0 & & & \\ 0 & 0 & -1 & & & \\ \hline & & & 1 & 1 & 0 \\ & & & 0 & 1 & 0 \\ & & & 0 & 0 & 1 \end{array}\right], \quad B_H = \left[\begin{array}{cc} 0 & 0 \\ 0 & 1 \\ 1 & 0 \\ \hline 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{array}\right],$$

$$C_H = \left[\begin{array}{ccc|ccc} -2 & 0 & 1/2 & 0 & -1/2 & 0 \\ 0 & 2 & 0 & -2 & 0 & -2 \end{array}\right].$$

Step 3 : Rejecting the second sub-group of matrices, we get a realization of Z(s) as

$$A = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix},$$

$$C = \begin{bmatrix} -2 & 0 & 1/2 \\ 0 & 2 & 0 \end{bmatrix}.$$

Step 4: Z(s) is found to be

$$Z(s) = \begin{bmatrix} \dfrac{1/2}{(s+1)} & \dfrac{-2}{(s+1)^2} \\ 0 & \dfrac{2}{(s+1)} \end{bmatrix}.$$

It is clear that each entry of a positive-real matrix may not be positive real [71] .

An alternative state-variable method for the determination of Z(s) from the given Z(s) + Z'(-s) is presented. The proposed technique is simpler than the one described in [61] since it does not require the cumbersome spectral factorization, and the determination of a symmetric positive definite matrix P [4] which gets unwieldy especially when the order n of U(s) is large.

4.3 AN ALGORITHM FOR DETERMINING A POSITIVE REAL IMPEDANCE MATRIX Z(s), GIVEN Z(s)-Z'(-s)

The field of network theory is being widely investigated these days in terms of the state space. Several well-known classical analysis and synthesis problems have also been given new solutions in state-space terms . Anderson [4] has given a system theory criterion for positive real matrices. Anderson and Brocket [5] have given a state-space interpretation of the Darlington Synthesis. Recently, Puri and Takeda [78] have described state-space realization of Foster synthesis for LC networks. Another interesting problem in network analysis is to find a relationship between a positive real (PR) impedance matrix Z(s) of dimension nxn and its parts Z(s)+Z'(-s) and Z(s)-Z'(-s), where the prime denotes transpose of a matrix. The problem of determining Z(s) from given Z(s)+Z'(-s), called U(s), has been effectively solved by Lal and Singh [61], where U(s) is a para-Hermitian matrix which is nonnegative definite. An alternative method for solving the same problem has been recently given by Puri et.al. [77]. However the determination of a positive real Z(s) from given Z(s) -Z'(-s) had remained an unsolved problem. This is perhaps partly so because no results are known regarding the factorization of Z(s)-Z'(-s) in a similar form as is possible for Z(s)+Z'(-s) which had been exploited by Lal and Singh [61]. The aim of this section is to determine a state-model realization and the PR impedance matrix Z(s) of order n from given Z(s) - Z'(-s), called herein V(s). The proposed technique is equally valid when multiple poles are present.

For a successful implementation of the proposed algorithm, V(s) must satisfy the following conditions.

1) V(s) is real- rational.

2) V(s) has no multiple pole at the origin.

3) The para-Hermitian part of V(s) is zero.

DISCUSSION OF THE METHOD

Let a partial fraction expansion of V(s) be made, with a subsequent grouping together of terms with poles on the $j\omega$ axis, and poles in the half planes $\operatorname{Re} s < 0$ and $\operatorname{Re} s > 0$. Then V(s) always admits the form

$$V(s) = 2sL + 2\sum_i \frac{F_i s + G_i}{s^2 + \omega_i^2} + 2s^{-1}C_o + V_o(s) \qquad (4.7)$$

where L is real and symmetric, C_o is also real and symmetric, F_i is real and nonnegative definite symmetric, and G_i is skew symmetric. Also,

$$V_o(s) = Z_o(s) - Z_o'(-s) \qquad (4.8)$$

where $Z_o(s)$ is positive real and that all poles of elements of $Z_o(s)$ lie in $\operatorname{Re} s < 0$. The treatment in the preceding holds in the light of the positive real nature of Z(s) [9]. Thus, whenever V(s) has a pole at zero or at infinity, half of it is the share of Z(s). The same is true if there are imaginary poles of V(s). The next step is to construct $Z_o(s)$ from $V_o(s)$.

As in [77], consider a positive real $Z_o(s)$, with $Z_o(\infty) = 0$, and assume that it possesses no imaginary axis poles, i.e., all poles lie in the half-plane $\operatorname{Re} s < 0$.

If (A,B,C) is a minimal realization for $Z_o(s)$, then A will have eigen-values with negative real parts [4]. Because $Z_o(s)$ and $Z_o'(-s)$ can have no poles in common (those of $Z_o(s)$ being in Re $s < 0$ and those of $Z_o'(-s)$ in Re $s > 0$), by lemma 2 of [4]

$$\delta\left[Z_o(s) + Z_o'(-s)\right] = 2\,\delta\left[Z_o(s)\right] \quad (4.9)$$

where δ denotes the degree of a rational matrix and gives the dimension of its minimal realization. If $Z_o'(-s)$ is changed to $-Z'(-s)$, only its residues will become negative. Therefore,

$$\delta\left[Z_o(s) - Z'(-s)\right] = 2\,\delta\left[Z_o(s)\right] \qquad (4.10)$$

Hence, the dimension of the matrix A is half that of A_V. (A_V, B_V, C_V) is a realization of $V_o(s)$ such that

$$V_o(s) = C_V\,(sI - A_V)^{-1}\,B_V\,. \qquad (4.11)$$

$Z_o(s)$ can be found from $V_o(s)$ by the following procedure.

Step 1 : Obtain a minimal realization (A_V, B_V, C_V) of $V_o(s)$ in the Jordan canonical form by any of the known methods [17], [55].

Step 2 : Modify the above realization by re-ordering the rows of A_V, B_V abd C_V' to get

$$A_V = \left[\begin{array}{c|c} A & \\ \hline & A^- \end{array}\right], \quad B_V = \left[\begin{array}{c} B \\ \hline B^- \end{array}\right], \quad C_V = \left[\begin{array}{c|c} C & C^- \end{array}\right]$$

(4.12)

where A consists of Jordan blocks having eigenvalues with negative real parts, and A^- has Jordan blocks of the same dimensions all of whose eigen-values have positive real parts. Such a re-ordering does not affect input/output relations.

Step 3: The set of submatrices (A,B,C) having eigen-values with negative real parts which correspond to the left half-plane poles is picked out. This set is a realization of $Z_o(s)$.

Step 4: $Z_o(s)$ may be computed easily by the relation

$$Z_o(s) = C \ (sI - A)^{-1} B \qquad (4.13)$$

because the evaluation of the resolvent matrix $(sI - A)^{-1}$ is quite simple as A is in Jordan form.

This $Z_o(s)$ is almost the matrix we seek from $V_o(s)$. It passes the direct PR test [71] except possibly for the sign of its Hermitian part at real frequencies. The addition of a nonzero real, constant matrix D to $Z_o(s)$ will take care of this last difficulty if it arises. It should be large enough to ensure that $Z_{OH}(j\omega)$ is nonnegative definite. To this can be added any positive semi-definite constant matrix (resistance), of course. If $V(\infty)$ is a nonzero matrix D_V, then the nxn constant matrix D must satisfy the relation

$$D - D' = D_V \quad . \qquad (4.14)$$

It may be mentioned here that the statement of criteria for positive semi-definiteness of matrices given in [72], [116], among several others, implies that a real symmetric matrix is positive semi-definite iff the <u>leading</u> principal minor determinants are non-negative, where the mth leading principal minor of the nxn matrix Z is the determinant of the matrix formed by deleting the last n-m rows and columns of Z [116]. Application of this criterion, however, leads to erroneous results. This error contained in several engineering texts on system theory regarding Sylvester's criterion for positive semi-definite matrices has been pointed out recently by Swamy [108]. The correct statement given in [29], [71] is as follows.

A Hermitian matrix is positive semi-definite if and only if every principal minor is nonnegative.

Considering nxn matrix $Z_o(s)$, if its Hermitian part is not already singular at some $s_o = j\omega_o$, we must form the minimum matrix Z_m using the result in [71].

$$Z_m(s) = Z_o(s) - \left[r \dotplus 0_{n-1} \right]$$

$$r = \min_{0 \leq \omega \leq \infty} \frac{\Delta(\omega)}{\Delta_{11}(\omega)} \qquad (4.15)$$

Here, $\Delta(\omega)$ and $\Delta_{11}(\omega)$ are the determinant and (1,1) principal minor of $Z_{OH}(' \omega)$, respectively. Z_m is PR with its Hermitian part of rank n-1 at $s_o = j\,\omega_o$, the frequency at which the minimum determining r occurs. Further, $\delta\left[Z_m\right] = \delta\left[Z_o\right]$. The process corresponds to extracting a series resistor from port 1. The result is stated for port 1 but holds by renumbering, for any port. Z_m is called a minimum matrix. By choosing any nonzero positive semi-definite (real, constant) matrix, the above resistance extraction can be made to get an alternate Z_m.

Then the impedance matrix Z(s) is given by

$$Z(s) = sL + s^{-1} C_o + \sum_i \frac{F_i s + G_i}{s^2 + \omega_i^2} + Z_m(s) \,. \quad (4.16)$$

Z(s) thus obtained is real rational, has no poles in $\sigma > 0$; poles of Z(s) on $\sigma = 0$ are simple; for each pole on $\sigma = 0$, the residue is Hermitian and positive semi-definite, and $Z_H(j\omega) \geq 0$ because $Z_H(j\omega) = Z_{mH}(j\omega)$ here. Z(s) is, therefore, a PR matrix.

An illustrative example is given.

EXAMPLE

Consider

$$V(s) = \begin{bmatrix} 4s + \dfrac{2s}{s^2+1} & \dfrac{2}{s^2+1} + \dfrac{s-2}{s+4} \\ \dfrac{-2}{s^2+1} - \dfrac{s+2}{s-4} & \dfrac{2}{s} + \dfrac{2s}{s^2+1} - \dfrac{12s}{s^2-64} \end{bmatrix}$$

which may be re-written as

$$V(s) = 2s \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} + 2s^{-1} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} + \frac{2}{s^2+1} \begin{bmatrix} s & 1 \\ -1 & s \end{bmatrix} + \begin{bmatrix} 0 & \frac{s-2}{s+4} \\ -\frac{s+2}{s-4} & -\frac{12}{s^2-64} \end{bmatrix} .$$

This gives

$$L = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}, \; C_o = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, F_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \geqslant 0$$

$$H_1 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$

and

$$V_o(s) = \begin{bmatrix} 0 & \frac{s-2}{s+4} \\ -\frac{s+2}{s+4} & -\frac{12s}{s^2-64} \end{bmatrix} = \begin{bmatrix} 0 & \frac{-6}{s+4} \\ \frac{-6}{s-4} & -(\frac{6}{s+8} + \frac{6}{s-8}) \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} .$$

$V_o(s)$ has a realization (A_V, B_V, C_V, D_V) given by

$$A_v = \begin{bmatrix} 4 & & & \\ & -4 & & \\ & & -8 & \\ & & & 8 \end{bmatrix}, \quad B_v = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix},$$

$$C_v = \begin{bmatrix} 0 & -6 & 0 & 0 \\ -6 & 0 & -6 & -6 \end{bmatrix}, \quad D_v = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$

which is re-ordered to give

$$A_v = \left[\begin{array}{cc|cc} -4 & & & \\ & -8 & & \\ \hline & & 4 & \\ & & & 8 \end{array}\right], \quad B_v = \left[\begin{array}{cc} 0 & 1 \\ 0 & 1 \\ \hline 1 & 0 \\ 0 & 1 \end{array}\right],$$

$$C_v = \left[\begin{array}{cc|cc} -6 & 0 & 0 & 0 \\ 0 & -6 & -6 & -6 \end{array}\right].$$

D_v is unaffected. Then we get

$$A = \begin{bmatrix} -4 & \\ & -8 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} -6 & 0 \\ 0 & -6 \end{bmatrix},$$

$$Z_o(s) = \begin{bmatrix} 0 & \dfrac{-6}{s+4} \\ 0 & \dfrac{-6}{s+8} \end{bmatrix} + D .$$

The matrix

$$D = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$, satisfying (4.14) when

added to $\begin{bmatrix} 0 & \frac{-6}{s+4} \\ 0 & \frac{-6}{s+8} \end{bmatrix}$

makes it positive definite which otherwise is negative definite.

$$\therefore \quad Z_o(s) = \begin{bmatrix} 1 & \frac{s-2}{s+4} \\ 0 & \frac{s+2}{s+8} \end{bmatrix} .$$

We must make the resistance extraction in order to make $Z_o(s)$ a minimum matrix.

$$Z_{OH}(j\omega) = \begin{bmatrix} 1 & \frac{j\omega - 2}{2(j\omega + 4)} \\ \frac{j\omega + 2}{2(j\omega - 4)} & \frac{\omega^2 + 16}{\omega^2 + 64} \end{bmatrix} .$$

The determinant Δ and (1,1) minor Δ_{11} are

$$\Delta(\omega) = \frac{9\,(\omega^2 - 16)^2}{64(\omega^2+16)(\omega^2+64)} + \frac{39}{64}\;\frac{\omega^2 + 16}{\omega^2 + 64} ,$$

$$\Delta_{11}(\omega) = \frac{\omega^2 + 16}{\omega^2 + 64} .$$

Consequently, the resistance to be extracted from port 1 is

$$r = \min_{\omega} \frac{\Delta(\omega)}{\Delta_{11}(\omega)} = \frac{39}{64},$$

which occurs at $\omega_o = 4$ and gives

$$Z_m(s) = \begin{bmatrix} \frac{25}{64} & \frac{s-2}{s+4} \\ 0 & \frac{s+2}{s+8} \end{bmatrix}.$$

Thus, a PR Z(s) obtained from Z(s) - Z'(-s) is

$$Z(s) = \begin{bmatrix} 2s + \frac{s}{s^2+1} + \frac{25}{64} & \frac{1}{s^2+1} + \frac{s-2}{s+4} \\ -\frac{1}{s^2+1} & \frac{1}{s} + \frac{s}{s^2+1} + \frac{s+2}{s+8} \end{bmatrix}.$$

It has been shown how a positive real impedance matrix Z(s) may be computed from its given part Z(s)-Z'(-s) employing system theory concepts. It may be pointed out that the realization (A_v, B_v, C_v) need not necessarily be minimal so far as the determination of Z(s) is concerned. A sub-optimal realization of $V_o(s)$ may be readily constructed by the technique suggested in [64] which will be in Jordan form . The non-uniqueness of D causes an arbitrariness in Z(s) , except when Z(s) is of order 1. Bode's illuminating method [112] for the scalar case cannot be applied to the matrix case since the off-diagonal entries in a PR matrix need not be pr.

4.4 STATE - SPACE INTERPRETATION OF FOSTER SYNTHESIS METHOD

State-space techniques have generated a lot of interest in network analysis and synthesis in the past few years. It is quite important to establish a communication link between state-space characterization and frequency domain methods. It is of great interest to provide state-space interpretation of the well-known properties of network functions and the comm on synthesis procedures. Some work has already been initiated in this direction [5], [60], [94], [118]. Recently, a procedure has been given by Jain [39] for Foster synthesis of LC networks. The method of [39] uses nonsingular observability matrix as a transformation for a canonical state-model representation of the Foster network which is then compared with a similar canonical state-model written directly in terms of the coefficients of the network function. Thus, the element values are determined, via state-space characterization, in terms of the coefficients of the network function to be synthesized. Here, an altogether different method is presented for Foster synthesis of driving point immittance functions of

LC networks [77] and is much simpler than that in [38].

Consider an LC driving-point immittance function Z(s), which may be written as

$$Z(s) = \alpha s + Z_1(s) \qquad (4.17)$$

where α is a constant and $Z_1(s)$ is regular (no pole at infinity) and proper ($Z_1(\infty) = 0$).

Let $Z_1(s)$ have a state-space representation

$$\begin{aligned} \dot{x} &= A x + b u \\ y &= c x + d u \end{aligned} \qquad (4.18)$$

where x is an n-dimensional column vector, u is the input, y is the output and the dimensions of A, b, c and d are nxn, nx1, 1xn and 1x1, respectively.

Since $Z_1(s)$ is a proper function, d = 0. In a suitable neighborhood of infinity, $Z_1(s)$ can be expanded in a negative power series as [38]

$$Z_1(s) = c\, b\, s^{-1} + c\, A\, b\, s^{-2} + c\, A^2\, b\, s^{-3} + \ldots \qquad (4.19)$$

For lossless networks, there always exists a transformation T [105] which transforms A, b and c to a form such that the new A is skew-symmetric and the new b is equal to the transpose of the new c. Let A_t, b_t and c_t denote the new values after applying the transformation T. Since c_t is a non-zero row vector, the scalar $c_t\, c_t' > 0$. Further, since A_t is a skew symmetric matrix, $c_t\, A_t\, b_t$, which is equal to $c_t\, A_t\, c_t'$, a scalar, will always be zero (A_t is skew-symmetric, and any matrix $c_t A_t c_t'$ is either a skew or a null matrix. In our case, as c_t is a row vector, the product $c_t A_t c_t'$ will be scalar and hence equal to zero) and $c_t A_t^2 c_t' \leq 0$, since

$$c_t A_t c_t' = c_t A_t(-A_t' c_t') = -(c_t A_t)(c_t A_t)' . \quad (4.20)$$

Similarly,

$$c_t A_t^3 c_t' = 0$$

$$c_t A_t^4 c_t' \geq 0$$

$$c_t A_t^5 c_t' = 0$$

$$c_t A_t^6 c_t' \leq 0$$

and so on. Therefore, $Z_1(s)$ becomes

$$Z_1(s) = c_t c_t' s^{-1} + c_t A_t^2 c_t' s^{-3} + c_t A_t^4 c_t' s^{-5} + .. \quad (4.21)$$

$$= c\ b\ s^{-1} + c\ A^2\ b\ s^{-3} + c\ A^4 b\ s^{-5} + ... \quad (4.22)$$

$$= Y_0\ s^{-1} + Y_2\ s^{-3} + Y_4\ s^{-5} + ... \quad (4.23)$$

where the Y_k s are called Markov parameters [29], [106]. It can be seen that Y_0, Y_4, Y_8, ... are positive, while Y_2, Y_6, Y_{10}, ... are negative.

A lossless driving - point function may be written in its partial - fraction expansion as

$$Z(s) = \frac{K_0}{s} + Hs + \sum_{r=1}^{\hat{n}} \frac{2 K_r s}{s^2 + \omega_r^2} \quad (4.24)$$

where K_o, K_r , ω_r^2 and H are positive and real constants ($\omega_i^2 \neq \omega_j^2$) and

$$\hat{n} = \begin{cases} \frac{n}{2} & (n \text{ even}) \\ \frac{n-1}{2} & (n \text{ odd}) \end{cases} .$$

Comparing (4.17) with (4.24), we get $H = \alpha$ and

$$Z_1(s) = \frac{K_o}{s} + \sum_{r=1}^{\hat{n}} \frac{2 K_r s}{s^2 + \omega_r^2} . \quad (4.25)$$

Eq (4.25) can be rewritten as

$$Z_1(s) = K_o s^{-1} + \sum_{r=1}^{\hat{n}} 2 K_r$$

$$\times (s^{-1} - \omega_r^2 s^{-3} + \omega_r^4 s^{-5} - \omega_r^6 s^{-7} + \ldots) \quad (4.26)$$

A comparison of (4.23) and (4.26) gives an infinite set of simultaneous equations

$$\left.\begin{aligned} Y_o &= K_o + 2K_1 + 2 K_2 + \ldots + 2K_{\hat{n}} \\ -Y_2 &= 2K_1\omega_1^2 + 2K_2\omega_2^2 + \ldots + 2K_{\hat{n}} \omega_{\hat{n}}^2 \\ Y_4 &= 2K_1\omega_1^4 + 2K_2 \omega_2^4 + \ldots + 2K_{\hat{n}} \omega_{\hat{n}}^4 \\ &\vdots \end{aligned}\right\} \quad (4.27)$$

Also $Z_1(s)$ can be expressed as [94]

$$Z_1(s) = \frac{n(s)}{q(s)} = \frac{\det\begin{bmatrix} 0 & -c \\ b & sI-A \end{bmatrix}}{\det\begin{bmatrix} sI - A \end{bmatrix}} \quad (4.28)$$

which shows that the poles of $Z_1(s)$ are given by the eigen-values of A, and that they are not affected by the transformation T. Thus ω_1^2, ω_2^2, ..., $\omega_{\hat{n}}^2$ can be evaluated from the matrix A. Now $\hat{n}+1$ unknowns are left in (4.27) . The first $\hat{n}+1$ equations of (4.27) can be written in matrix form as

$$\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & \omega_1^2 & \omega_2^2 & \cdots & \omega_{\hat{n}}^2 \\ 0 & \omega_1^4 & \omega_2^4 & \cdots & \omega_{\hat{n}}^4 \\ \vdots & & & & \vdots \\ 0 & \omega_1^{2\hat{n}-2} & \omega_2^{2\hat{n}-2} & \cdots & \omega_{\hat{n}}^{2\hat{n}-2} \end{bmatrix} \begin{bmatrix} K_o \\ 2K_1 \\ 2K_2 \\ \vdots \\ 2K_{\hat{n}} \end{bmatrix}$$

$$= \begin{bmatrix} Y_o \\ -Y_2 \\ Y_4 \\ \vdots \\ (-1)^{\hat{n}+1}\, Y_{2\hat{n}-2} \end{bmatrix} \quad (4.28)$$

For LC networks, all the poles of $Z_1(s)$ lie on the imaginary axis. Therefore ω_1^2 , ..., $\omega_{\hat{n}}^2$ are all positive. It is well-known that the coefficient matrix of (4.28) is then non-singular. Hence a nontrivial solution of (4.28) exists.

The constants K having been evaluated, a Foster canonical network can be drawn [113] . The procedure is illustrated with an example.

Example :

Given the set (A,b,c) for an LC impedance function .

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1/2 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -3/2 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$b = \begin{bmatrix} 1 \\ 0 \\ 9/2 \\ 0 \\ 4 \end{bmatrix}$$

$$c = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \end{bmatrix} .$$

The Markov parameters, as obtained by (4.22) are

$$Y_0 = 4 \ , Y_2 = -3/2 \ , Y_4 = 5/4 \ , Y_6 = -9/8 \ ,\ldots$$

The order of the matrix A is 5. Therefore ,there will be five network elements, one series capacitor and two parallel LC network configurations. This gives $\hat{n} = 2$. The characteristic equation of the A matrix is

$$s(s^4 + 3/2\, s^2 + 1/2) = 0$$

which gives the eigen-values $s = 0$, $s^2 = -1/2$, and $s^2 = -1$. Then $\omega_1^2 = 1/2$ and $\omega_2^2 = 1$.

From (4.28)

$$\begin{bmatrix} K_o \\ 2K_1 \\ 2K_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} .$$

The procedure given above discusses the Foster realization for 1-port LC networks in state-space terms. Similar steps could be followed for RC and RL networks. It may be noted that there is no necessity to apply the transformation T in the actual procedure, since the products cb, cAb, cA^2b, ... of all realizations corresponding to a given $Z_1(s)$ are the same. Simplicity of the proposed method is self evident. The method presented here has recently been extended to the n-port case by Khan.et.al. [50].

4.5 DETERMINATION OF TRANSFER-FUNCTION MATRIX USING MOMENTS OF IMPULSE RESPONSE

A fast and simple technique for transfer matrix inversion has recently been given in [110]. The formulas stated there for matrix inversion of $(sI-A)^{-1}$ are applicable to system matrices in companion matrix form. Another direct method for the evaluation of transfer-function matrix from the given state equations has been suggested in [63]. The method of [63] is based on the reverse of the approach, discussed by Ho and Kalman [38], for determining (A,B,C) from the given transfer-function matrix. Markov parameters [29] are used in the process of [63.].Various other methods are available for the determination of the resolvent matrix $(sI - A)^{-1}$ [30], [51] which is required for the evaluation of the transfer-function matrix. This section presents a method for determining the transfer-function matrix H(s) of a linear time-invariant system represented by

$$\begin{aligned} \dot{x} &= A x + B u \\ y &= C x \end{aligned} \qquad (4.29)$$

through the intermediation of the moments of the impulse response. This method is especially suitable for processing by a digital computer. It requires the inverse of the real-valued matrix A. Since it is a matrix of numbers, the standard machine routines for inverting matrices can be applied easily. The computation of $H(s)$ is relatively easy even if the system is of a higher order.

DETERMINATION OF H(s)

For the determination of $H(s)$, it is necessary to compute the characteristic polynomial $\det(sI-A)$ which gives the denominator of each entry of $H(s)$, and the moments M_o, M_1, ..., M_n (suffix denotes the order of the moment) which are used to determine the numerator of each entry.

Before giving the explanation of the determination of $H(s)$, the following lemma is presented.

LEMMA [14]

If a minimal realization (A,B,C) corresponds to an impulse-response matrix $H(t)$ which is asymptotically stable, the matrix A is always non-singular.

The transfer-function $H(s)$ can be described by an infinite series [14] as

$$H(s) = \sum_{k=0}^{\infty} \frac{(-1)^k M_k}{k!} s^k \qquad (4.30)$$

where the coefficients of this series are determined from moments. The kth moment M_k is given by

$$M_k = (-1)^{k+1} k! \, C A^{-k-1} B \, , \; k = 0,1,2,.. \qquad (4.31)$$

Substituting (4.31) into (4.30), we get

$$H(s) = -\sum_{k=0}^{\infty} C A^{-k-1} B s^k \tag{4.32}$$

Both (4.30) and (4.32) represent an infinite series in s. Therefore, H(s) may be written as

$$H(s) = \sum_{k=0}^{\infty} H_k s^k \tag{4.33}$$

where H_k is a pxm constant matrix. Then even if the system is asymptotically stable and A^{-k-1},(k=0,1,2,...)exists, H(s) cannot be found in a closed form, from (4.33). If, however, the characteristic polynomial is replaced as

$$q(s) = \det(sI-A) = \sum_{k=0}^{n} q_k s^k \tag{4.34}$$

H(s) can be represented by a polynomial

$$H(s) = \frac{N(s)}{q(s)} = \sum_{k=0}^{w} \frac{N_k s^k}{q(s)}, \quad (w \leq n) \tag{4.35}$$

where N(s), N_k are pxm matrices. From (4.35) it is evident that only (w+1) coefficient matrices $N_0, N_1, \ldots, N_w$ are necessary to determine H(s). From (4.33) and (4.35) we get

$$N(s) = \sum_{k=0}^{\infty} N_k s^k$$

$$= q(s) H(s)$$

$$= \sum_{k=0}^{n} q_k s^k \sum_{i=0}^{\infty} H_i s^i . \tag{4.36}$$

From (4.36), therefore,

$$N_k = \begin{cases} \sum_{j=0}^{k} q_j \; H_{k-j} & , k \leq w \leq n \\ 0 \; , & k > w+1 \end{cases} \tag{4.37}$$

From (4.37) it is evident that the coefficient matrices N_k (k= 0,1,...., w) are determined from q_k(k=0,1,...,n) and H_k(k= 0,1,....,w) , i.e., in the determination of N_k , it is enough to determine at the most upto the nth moment. So, a maximum of (n+1) moments are required in the above procedure, to determine H(s). In other words, first the coefficient matrices N_k are determined from the moments M_o, M_1, ..., M_n and then H(s) is computed from N(s) and q(s) .

Example

The above procedure is illustrated by an example. Let (A,B,C) be given by

$$A = \left[\begin{array}{ccc|ccc} -1 & 0 & 0 & & & \\ 0 & -2 & 1 & & & \\ 0 & 0 & -2 & & & \\ \hline & & & -2 & 0 & 0 \\ & & & 0 & -1 & 1 \\ & & & 0 & 0 & -1 \end{array}\right], \; B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 2 & 0 & 0 \\ 0 & 4 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

The characteristic polynomial of A is

$$q(s) = s^6 + 9s^5 + 33s^4 + 63s^3 + 66s^2 + 36s + 8.$$

The moments obtained using (4.31) are

$$M_o = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad M_1 = \begin{bmatrix} 1 & 1/2 \\ 1 & 2 \end{bmatrix}, \quad M_2 = \begin{bmatrix} 2 & 1/2 \\ 3/2 & 6 \end{bmatrix},$$

$$M_3 = \begin{bmatrix} 6 & 3/6 \\ 3 & 34 \end{bmatrix}, \quad M_4 = \begin{bmatrix} 24 & 3/2 \\ 15/2 & 120 \end{bmatrix}, \quad M_5 = \begin{bmatrix} 120 & 15/4 \\ 45/2 & 720 \end{bmatrix},$$

$$M_6 = \begin{bmatrix} 720 & 45/4 \\ 315/4 & 5040 \end{bmatrix}.$$

Thus, H(s) becomes

$$H(s) = \frac{s^6 + 9s^5 + 33s^4 + 63s^3 + 66s^2 + 36s + 8}{s^6 + 9s^5 + 33s^4 + 63s^3 + 66s^2 + 36s + 8} \begin{bmatrix} 1 - s + s^2 - s^3 + s^4 - s^5 + s^6 - \ldots, & 1 - 1/2\, s + 1/4\, s^2 - 1/8\, s^3 + 1/16\, s^4 - 1/32\, s^5 + 1/64\, s^6 - \ldots, \\ 1 - s + 3/4\, s^2 - 1/2\, s^3 + 5/16\, s^4 - 3/16\, s^5 + 7/64\, s^6 \ldots, & 1 - 2s + 3s^2 - 4s^3 + 5s^4 - 6s^5 + 7s^6 - \ldots, \end{bmatrix}$$

which gives

$$H(s) = \begin{bmatrix} \frac{1}{s+1} & \frac{2}{s+2} \\ \frac{4}{(s+2)^2} & \frac{1}{(s+1)^2} \end{bmatrix}.$$

4.6 ALGORITHM FOR OBTAINING INVERSE POWERS OF MATRICES

A linear dynamical system may be characterized by different state-model representations. However, for simplification in system analysis and synthesis, it is desirable to transform the dynamic characterization into a canonical form. Two most convenient canonical forms are the companion form and Jordan canonical form. Recently, an algorithm has been proposed [92] for determining the power of the companion matrix. This section presents an algorithm for determining the inverse powers of a companion matrix. Another algorithm is given to find the powers of resolvent matrix $(sI - J)^{-1}$ when the state matrix is in Jordan canonical form, from which the inverse powers of J can also be found.

INVERSE POWERS OF COMPANION MATRIX

Let the (nxn) companion matrix $\mathcal{C}$ be represented as

$$\mathcal{C} = \begin{bmatrix} 0_{n-1,1} & I_{n-1} \\ \hline \multicolumn{2}{|c|}{\alpha} \end{bmatrix} \qquad (4.38)$$

where 0 is $(n-1)x1$ zero matrix, I is $(n-1)x(n-1)$ unit matrix, and α is 1xn row vector whose elements α_j , $j = 1,2,\ldots, n$ are the coefficients of the characteristic polynomial of $\mathcal{C}$. It may be seen immediately that the inverse of $\mathcal{C}$ is

$$\mathcal{C}^{-1} = \begin{bmatrix} & \boxed{\quad\beta\quad} & \\ I & & 0 \end{bmatrix} , \qquad (4.39)$$

where

$$\beta_j = \begin{cases} \dfrac{\alpha_{j+1}}{\alpha_1} & , \; j = 1,2,\ldots,(n-1) \\ \dfrac{1}{\alpha_1} & , \; j = n \end{cases} \qquad (4.40)$$

The powers of $\mathcal{C}^{-1}$ are obtained by successive multiplications :

$$[\mathcal{C}]^{-(r+1)} = [\mathcal{C}]^{-r} \, [\mathcal{C}]^{-1} . \qquad (4.41)$$

First partition the matrices $[\mathcal{C}]^{-r}$ and $[\mathcal{C}]^{-1}$ into submatrices A_1, B_1, D_1, E_1 and A_2, B_2, D_2, E_2, where the dimensions of A_1, B_1, D_1, E_1 and A_2, B_2, D_2, E_2 are 1x(n-1), 1x1 , (n-1)x(n-1) , (n-1)x1 and (n-1)x(n-1), (n-1)x1, 1x(n-1), 1x1, respectively. Substituting the partitioned form of $[\mathcal{C}]^{-r}$ and $[\mathcal{C}]^{-1}$ and carrying out the multiplication of (4.41) utilizing (4.40), and after simplification, it is discovered that the element of ith row and kth column of the -(r+1)th power of $\mathcal{C}$ is identical with the (i-1)th row of the corresponding column of the -rth power of $\mathcal{C}$, for i= 2, ..., n .

Thus

$$\mathcal{C}_{ik}^{-(r+1)} = \mathcal{C}_{i-1,k}^{-(r)} \quad , \; i = 2,3,\ldots,n \qquad (4.42)$$
$$k = 1,2,\ldots,n$$

and the elements of the first row of the -(r+1) th power of $\mathcal{C}$ are given by the recurrence relation

$$\mathcal{C}_{1k}^{-(r+1)} = \mathcal{C}_{11}^{-(r)} \beta_k + \mathcal{C}_{1,k+1}^{-(r)} \; , \; r = 1,2,\ldots (4.43)$$

Following the preceding development, the inverse powers of the companion matrix are obtained by the algorithm given below.

For the given companion matrix $\mathcal{C}$, find $\mathcal{C}^{-1}$ using (4.40). This gives the row vector β. Let the element β_k of β be denoted by $\beta_k = \mathcal{C}_{1,k}$. Write the rows of $\mathcal{C}^{-1}$ in the reverse order. Starting from the last row, successive rows may be generated by the relation

$$\mathcal{C}_{q+1,k} = \mathcal{C}_{q,1} \, \mathcal{C}_{1,k} + \mathcal{C}_{q,k+1} \qquad (4.44)$$

where q is a positive integer. Thus, we have

$$\begin{matrix} 0 & 0 & \ldots. & 1 & 0 \\ 0 & 0 & \ldots. & 0 & 0 \\ .. & .. & \ldots. & .. & .. \\ 1 & 0 & \ldots & 0 & 0 \\ \mathcal{C}_{11} & \mathcal{C}_{12} & \ldots & \mathcal{C}_{1,n-1} & \mathcal{C}_{1n} \\ \mathcal{C}_{21} & \mathcal{C}_{2,2} & \ldots. & \mathcal{C}_{2,n-1} & \mathcal{C}_{2n} \\ \ldots & \ldots & \ldots. & .. & \ldots \end{matrix} \qquad (4.45)$$

In order to obtain the -rth power of $\mathfrak{C}$, construction of $n+r-1$ rows in (4.45) is required. Of these, the first $n-1$ rows are already known. Thus only r rows have to be formed for determining $\mathfrak{C}^{-r}$. Finally $\mathfrak{C}^{-r}$ is obtained by selecting the last n rows of (4.45), and writing these rows in the reverse order starting from the last row. The evaluation of the next higher power $\mathfrak{C}^{-(r+1)}$, of course, will require the computation of only one more row, as per (4.43).

Example :

Consider

$$\mathfrak{C} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & -1 & -2 & 1 \end{bmatrix}.$$

Let it be required to find $\mathfrak{C}^{-3}$. The set of $4+3-1 = 6$ rows of (4.45) is formed, utilizing (4.40) and (4.44), as

$$\begin{matrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ -1 & -2 & 1 & 1 \\ -1 & 3 & 0 & -1 \\ 4 & 2 & -2 & -1 \\ \cdots & \cdots & \cdots & \cdots \end{matrix}$$

Choosing the last four rows of the above array and writing them in the reverse order gives

$$[\mathcal{C}]^{-3} = \begin{bmatrix} 4 & 2 & -2 & -1 \\ -1 & 3 & 0 & -1 \\ -1 & -2 & 1 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

and so on.

POWERS OF RESOLVENT MATRIX

If the state matrix is in Jordan canonical form, an algorithm is developed for evaluating powers of the resolvent matrix. Since the Jordan canonical form is the direct sum of several elementary Jordan matrices, it is convenient to consider only one elementary matrix. Let J be such a matrix having the eigen value λ of multiplicity n. Then the resolvent matrix R can be found from

$$R = (sI - J)^{-1}. \tag{4.46}$$

As in the case of companion matrix, the powers of R are obtained by successive multiplications :

$$[R]^{r+1} = [R]\,[R]^{r} \tag{4.47}$$

Carrying out the operations of (4.47), it is seen that each power of the resolvent matrix is an upper-triangular matrix. Successive rows of $[R]^r$ are generated by the relation

$$R^{(r)}_{q+1,k} = \begin{cases} R^{(r)}_{q,k-1} & , k \geq q+1 \\ 0 & , k < q \end{cases} \tag{4.48}$$

q being any positive integer. The elements of $[R]^r$ on the main diagonal are all identical. The elements, of $[R]^r$ on the first principal diagonal, are also identical to one another, and so on. Thus the matrix $[R]^r$ can be obtained from a knowledge of its first row only. The first row of the pth power of R can be generated by the relation

$$R^{(p)}_{1,k} = \frac{\binom{p+k-2}{k-1}}{(s-\lambda)^{p+k-1}} \qquad (4.49)$$

where

$$\binom{N}{K} = \frac{N!}{(N-K)!\ K!} . \qquad (4.50)$$

It is convenient to remember that several sets of coefficients $\hbar_{i,j}$ of $\frac{1}{(s-\lambda)^{p+k-1}}$ may be quickly reproduced with the help of

$$\hbar_{i,j} = \hbar_{i,j-1} + \hbar_{i-1,j} \qquad (4.51)$$

in the following scheme .

r = 1	1	1	1	1	1	...	
r = 2	1	2	3	4	5	...	
r = 3	1	3	6	10	15	...	(4.52)
r = 4	1	4	10	20	35	...	
r = 5	1	5	15	35	70	...	
...	.....		...			...	

each number being formed at once as the sum of the one immediately above it and the one preceding the number. Thus in forming the 5th row, we have

0+1 = 1 , 1+4 = 5 , 5 + 10 = 15 , 15+20 = 35 , etc.

It is clear that $[J]^{-r}$ is obtained from $[sI - J]^{-r}$ by putting s = 0 and then multiplying it by $(-1)^r I$.

Example :

Consider

$$(sI - J) = \begin{bmatrix} s+2 & -1 & 0 & 0 \\ 0 & s+2 & -1 & 0 \\ 0 & 0 & s+2 & -1 \\ 0 & 0 & 0 & s+2 \end{bmatrix}.$$

Utilizing (4.48) , (4.49) and (4.52), -5th power of (sI-J) is

$$(sI-J)^{-5} = \begin{bmatrix} \frac{1}{(s+2)^5} & \frac{5}{(s+2)^6} & \frac{15}{(s+2)^7} & \frac{35}{(s+2)^8} \\ 0 & \frac{1}{(s+2)^5} & \frac{5}{(s+2)^6} & \frac{15}{(s+2)^7} \\ 0 & 0 & \frac{1}{(s+2)^5} & \frac{5}{(s+2)^6} \\ 0 & 0 & 0 & \frac{1}{(s+2)^5} \end{bmatrix}.$$

One possible application of these algorithms is in computing the moments of impulse-response matrix. By a suitable similarity transformation, the state matrix A can be transformed to the companion form $\mathcal{C}$ or Jordan form J . Hence we can write

$$[A]^{-r} = N^{-1} [\mathcal{C}]^{-r} N \qquad (4.53)$$

$$= N^{-1} [J]^{-r} N . \qquad (4.54)$$

Equations (4.53) and (4.54) may be employed to compute moments M_k of impulse response [81] by the relation

$$M_k = (-1)^{k+1} k! \; C A^{-k-1} B \quad , k = 0,1,2,.. \qquad (4.55)$$

where (A,B,C) is a realization of the impulse response matrix. Different inverse powers of A can be evaluated using either of the algorithms developed earlier.

4.7 TRANSFORMATION OF TIME-VARYING AUTONOMOUS SYSTEMS TO COMPANION FORM

The problem of obtaining a phase-variable canonical form for a linear dynamical system characterized by state-variable equations has been considered by several authors . Silverman [95] gave a method for determining the transformation matrix for reducing a single-input single-output system to the phase-variable form. Ramaswami and Ramar [83],[84] later presented simpler methods of finding the transformation matrix. An algorithm developed by Power [76] generates a class of matrices N which transforms a linear time-invariant multivariable autonomous system into companion form. The aim of this section is to extend the method of Power [76] to the time-varying case.

Let the dynamic behavior of the force-free time-varying system be represented by the vector-matrix differential equation

$$\dot{x}(t) = A(t) \quad x(t) \tag{4.56}$$

where $A(t)$ is nxn matrix and $x(t)$ is nx1 column vector. Let $N(t)$ be the transformation matrix which transforms the system of (4.56) into the system

$$\dot{z}(t) = \mathcal{C}(t) \quad z(t) \tag{4.57}$$

where

$$\mathcal{C}(t) = N(t)\, A(t)\, N(t)^{-1} + \dot{N}(t)\, N(t)^{-1} \tag{4.58}$$

is in companion form.

Let us consider fundamental transformation matrices $N_k(t)$ of the coordinate transformation

$$z(t) = N_k(t)\, x(t) \tag{4.59}$$

with

$$z(t) = \text{column}\,(z_1, \dot{z}_1, \ldots, \overset{(n-1)}{z_1})$$

when $z_1(t) = x_k(t)$. The first two rows of $N_k(t)$ are immediately obtained as

$$0 \quad 0 \quad \ldots\ldots \quad 1 \quad \ldots \quad 0 \tag{4.60}$$

$$a_{k1}(t) \quad a_{k2}(t) \ \ldots \ a_{kk}(t) \ \ldots \ a_{kn}(t)$$

The (p+2)th row of $N_k(t)$, denoted by $\langle_{p+2} N_k(t)$, for $p = 1,2,\ldots,(n-2)$ is obtained by successive differentiations. The jth row of $N_k(t)$ represents the equation

$$^{(j-1)}x_k(t) = N_{kj1}(t)\, x_1(t) + N_{kj2}(t) x_2(t) + \ldots. + N_{kjn}(t)\, x_n(t) \quad . \tag{4.61}$$

Differentiating (4.61) and after simplification, we get

$$^{(j)}x_k(t) = \left(\langle_j N_k(t), A_1(t)\rangle + \dot{N}_{kj1}(t)\right) x_1(t) + \left(\langle_j N_k(t), A_2(t)\rangle + \dot{N}_{kj2}(t)\right) x_2(t) + \ldots + \left(\langle_j N_k(t), A_n(t)\rangle + \dot{N}_{kjn}(t)\right). x_n(t) \tag{4.62}$$

where $_jN_k(t), A_i(t)$ is the scalar product of the jth row of $N_k(t)$ with the ith column of $A(t)$.

From (4.62), the (j+1)th row of $N_k(t)$ becomes

$$\langle_j N_k(t) = \langle_j N_k(t), A_1(t)\rangle + \dot{N}_{kj1}(t) \ , \ \langle_j N_k(t), A_2(t)\rangle + \dot{N}_{kj2}(t), \ , \ldots, \langle_j N_k(t), A_n(t)\rangle + \dot{N}_{kjn}(t) \quad . \tag{4.63}$$

Thus the elements of each row are obtained by taking the scalar product of the previous row with each of the columns of A(t) in turn, and to the row thus obtained, adding the derivatives of the previous row.

Example :

Let

$$A(t) = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & -1 \\ -e^{-t} & e^{-t} & -2 \end{bmatrix} .$$

Using (4.60) and (4.63) gives $N_1(t)$:

$$N_1(t) = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -2 & -1 \end{bmatrix}, \quad |N_1(t)| = -1 .$$

Also,

$$N_2(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -1 & -1 \\ e^{-t} & 1-e^{-t} & 3 \end{bmatrix}, \quad |N_2(t)| = e^{-t} .$$

Similarly, another transformation $N_3(t)$ is

$$N_3(t) = \begin{bmatrix} 0 & 0 & 1 \\ -e^{-t} & e^{-t} & -2 \\ 4e^{-t} & 3e^{-t} & -e^{-t}+4 \end{bmatrix}, \quad |N_3(t)| = 7\, e^{-2t} .$$

Any of these transformations will reduce $A(t)$ to $\mathcal{C}(t)$. Thus

$$\mathcal{C}(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -5-e^{-t} & -4 \end{bmatrix} .$$

The example shows that by a judicious choice of $z_1 = x_k$, one of the state variables of (4.56), one may in some problems obtain a time-invariant transformation matrix (N_1 in this case) so that subsequent computations are simplified.

CHAPTER V

SUMMARY AND SUGGESTIONS FOR FURTHER WORK

5.1 INTRODUCTION

The problem of realization for linear systems first stated by Gilbert in 1963, and subsequently investigated by several researchers, has been discussed in this thesis. Various results obtained in the preceding chapters have been summarized in the present chapter. Some suggestions for further investigations in this field are also made.

5.2 SUMMARY

In this thesis, a mathematical description of linear dynamical systems in the input-output form, and in the state-variable vector differential equation form is reviewed, first. The realization problem of linear systems is introduced next, giving some mathematical preliminaries. An historical survey of the technical literature on realization theory, scattered in different research journals, is provided. The review work also signifies the importance of this fundamental problem of system theory. The application of system theory concepts in the field of network theory is considered and a review of the literature giving an interpretation of some well-known properties of network functions, is given in state-space terms. State-space interpretation of classical synthesis methods has also been discussed. System theory criteria for synthesizing a network by using passive elements, or realizing a network which satisfies reciprocity constraints have also been included.

New methods of finding state-model realizations of a linear dynamical system, from the specified input-output data, have been evolved. The data could be in the form of moments of impulse-response, or transfer-function matrix. In particular, a method of minimal realization of a symmetric transfer-function matrix based on moments is given. The method is a modification of the Ho-Kalman algorithm [38] in which a Hankel matrix is constructed from Markov parameters. The realizations obtained by the proposed method result in reciprocal networks. Further, for RC and RL cases, both reciprocity and passivity constraints are satisfied. In the realization process, the given symmetric transfer-function matrix is expanded in a positive power series of s. The moments M_k are obtained uniquely from the coefficient matrices of s . The moments will be symmetric. The Hankel matrix built from the moments will also be symmetric. Then a congruence transformation can be applied, which will result in reciprocal realizations. In the presence of noise, computation of moments is preferable to that of Markov parameters. As such, a method based on moments is advantageous when a realization is to be obtained from a data contaminated with noise.

The problem of finding quick and computationally simple realization procedures by using noncanonical structures has also been considered. In some applications, nonminimal realizations are acceptable. In case a minimal realization is required, standard system reduction techniques may be employed to extract a completely controllable and completely observable part which gives a minimal realization. A technique has been proposed for obtaining a state-model realization, without guaranteeing irreducibility. It is valid for multiple poles. The dimension of the system can be easily found by constructing mode matrices M, M_c and M_r which have been defined for the multiple pole case. The sum of the number of modes in M_c gives the dimension of the realization. In many examples, the realization obtained by the proposed method may be of lower dimension than that obtained by existing methods.

Another algorithm for obtaining a lower-dimension realization of dynamical systems has been proposed. The method is equally applicable for multiple poles. In the realization procedure, coefficient matrices of the transfer-function matrix H(s), corresponding to the distinct poles and their multiplicities, are computed. For each coefficient matrix, the minimum set of lines (covering set) containing all its non-zero elements are found. Then, submatrices of the realization (A, B, C) are constructed, starting from the coefficient matrix of highest multiplicity and scanning columns and rows of the covering set. The direct sum of the submatrices thus obtained gives the desired realization. It is pointed out that Glass's technique [33] is a special case of the proposed method when the covering set consists of columns only; so is the tèchnique proposed in [64] when the covering set consists of rows only. The dimension of the realization obtained by the proposed algorithm is lower compared with the methods of Kalman [41], Glass [33], Lal et.al. [64].

A new method has been presented to find a state-model and the positive-real impedance matrix Z(s) from its given even part Z(s) + Z'(-s). This is an alternative to the method of Lal and Singh [61]. The proposed technique utilizes a result from Anderson [4] that if (A,B,C) is a minimal realization of Z(s) (Z(s) possessing no poles on the imaginary axis), then A will have eigenvalues with negative real parts. A state-model (A_H, B_H,C_H) is obtained in Jordan form, realizing U(s) = Z(s) + Z'(-s), from which (A,B,C) is picked out which corresponds to a positive real Z(s). The proposed technique is simpler than the one described in [61]. It does not require the cumbersome spectral factorization, and the determination of a symmetric positive definite matrix P [4] which gets unwieldy, especially when the order n of U(s) is large.

Another algorithm has been proposed to determine a state-model realization and the positive real impedance matrix Z(s) of order n when its odd part V(s) = Z(s)-Z'(-s) is given. Conditions on V(s) are given for a successful implementation of the procedure. The positive real nature of Z(s) and Anderson's results [4] have been utilized in the proposed method. The resulting Z(s) is a minimum matrix.

An attempt has been made to establish yet another link between state-space and frequency domain methods. A state-space interpretation of the classical Foster synthesis method for driving-point immittance functions of LC networks has been presented. The poles of the network function Z(s) are given by the eigen-values of the state matrix A of Z(s). The residues of a partial-fraction expansion of Z(s) are obtained in terms of Markov parameters which are related to (A,b,c) realizing Z(s). A transformation T is used in arriving at the results. However, there is no necessity to apply the transformation in the actual synthesis procedure.

A method for determining transfer-function matrix from a knowledge of its moments is proposed. It is shown that at the most (n+1) moments of the impulse-response matrix are required in the process, where n is the order of the state matrix A. In the determination of the transfer-function matrix H(s), it is necessary to compute the characteristic polynomial det (sI-A) which gives the denominator of each entry of H(s), and the moments M_0 , M_1 , ..., M_n which are used to determine the numerator of each entry. The computation of H(s) is relatively easy even if the system is of a higher order.

A given dynamical system may be described by different state-model representations(A,B,C). However, from the stand point of system analysis, it is convenient to deal with canonical representations of the system, like the companion form and the Jordan canonical form. Algorithms have been evolved for determining inverse powers of matrices given in companion or Jordan forms. In order to obtain the -rth power of the companion matrix $\mathcal{C}$ of order n, construction of n-r+1 rows of an array is required. In order to compute the next higher power -(r+1) , the formation of only one more rows is necessary. As in the case of companion matrix, the inverse powers of (sI-J) where J is in Jordan form, are obtained by successive multiplications. It is seen that each inverse power of (sI-J) is an upper triangular matrix. These algorithms find application in the computation of moments of an impulse-response matrix.

The problem of finding a canonical form representation of a linear time-varying system has been considered

A method of generating transformation matrices which will transform a time-varying autonomous system to companion form has been given. In some problems, it could be possible to obtain a time-invariant transformation matrix. In such a case, subsequent computations become simple .

5.3 SUGGESTIONS FOR FURTHER INVESTIGATIONS

The problem of giving a mathematical description to dynamical systems has been investigated thoroughly in the past decade. With different degrees of complexity, a large number of methods for finding a state-model realization from specified input-output data are available . While a fairly complete theory of realization for linear time-invariant systems exists, there is a scope for further work for time-varying systems. State-variable approach being more general in nature, the field of network theory is also being investigated in state-space terms. Several concepts and results of system theory have been applied in network problems and a lot more could be done in this direction. In the following paragraphs, some suggestions are given along which further investigations could be carried on.

If a non-minimal realization is given, there exist methods by which the realization could be made minimal for both time-invariant and time-varying systems.

There are several quick methods of obtaining a suboptimal realization from a given transfer-function matrix of a time-invariant system. However, no similar attempt seems to have been made for the time-varying case. It is worth developing simpler methods of realization for time-varying systems.

A method of state-model realization satisfying reciprocity constraints has been given . With such a realization as the starting point, procedures for synthesizing networks without using gyrators are worth investigating.

There is a wide scope for strengthening the link between frequency domain and state-space characterizations with particular application to networks. State-space interpretation of some of the one-port synthesis methods has been given recently. The interpretation of some of the remaining one-port and two-port methods is worth investigating, e.g., Bott-Duffin procedure etc. Besides, the synthesis of Foster, Cauer , Brune and other networks seems possible in state-space terms.

State-space interpretation of poles, zeros, residues, positive-real matrices, reciprocity has been done. It will be worthwhile to give similar meaning to some other comm on concepts in network synthesis , e.g., removing a pole, shifting a zero etc.

Algorithms have been given in this thesis for determining a state-model realization and the positive-real impedance matrix Z(s) from the given even part Z (s) + Z'(-s) , or the odd part Z(s)-Z'(-s). It will be useful to find a realization and the network function when the magnitude function is given.

A method has been given in this thesis to find transformation matrices which will reduce a time-varying autonomous system to the companion form. Its extension to non-autonomous systems and the possibility of developing other simple transformations is worth investigating.

Algorithms for finding inverse powers of companion and Jordan form matrices, evaluation of moments and their application in system realization has been given. Further work along these lines and its possible application for time-varying systems and system identification may lead to some interesting results.

It is hoped that the investigations carried out in this thesis and further work on suggestions contained herein will make some more contributions to systems science.

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