ANALYZING THE CAPACITY OF GENERALIZED-K FADING CHANNELS AT LOW SNR

A DISSERTATION

Submitted in partial fulfilment of the requirements for the award of the degree

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by

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CANDIDATE'S DECLARATION

I hereby certify that the work which is being presented in the dissertation entitled " **ANALYZING THE CAPACITY OF GENERALIZED-*K* FADING CHANNELS AT LOW SNR**" in partial fulfilment of the requirements for the award of the degree of Master of Technology and submitted in the Department of Electronics and Communication Engineering of the Indian Institute of Technology Roorkee, Roorkee is an authentic record of my own work carried out during a period from July, 2018 to June, 2019 under the supervision of Dr. Debashis Ghosh, Professor, Department of Electronics and Communication Engineering, Indian Institute of Technology Roorkee, Roorkee.

The matter presented in this dissertation has not been submitted by me for the award of any other degree of this or any other Institution.

(PIYUSH GOEL)

This is to certify that the above statement made by the candidate is correct to the best of my knowledge.

(DEBASHIS GHOSH)
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Date:

ABSTRACT

In this work, we have characterize the low SNR capacity of Nakagami-gamma also known as generalized-K fading channel in nats per channel use with perfect channel state information at both transmitter and receiver(CSIT-R). We have focus in Low SNR regime because wireless systems are now operating at Low SNR to have a higher energy efficiency. We have shown that Low SNR capacity of generalized-K fading channel scales as $\frac{\Omega}{4\,m}\,\text{SNR}\,\log^2\left(\frac{1}{\text{SNR}}\right)$, where m is the distribution shaping parameter and Ω is defined in terms of channel mean square as $\Omega = \frac{\mathrm{E}[x^2]}{k}$, where k is also the shaping parameter. Our Asymptotic Low SNR capacity follows the exact results obtained from simulation which justify our analysis. We also provide an on-off scheme that is achieving our Low SNR capacity results which indicates that this scheme can be practically implemented at Low SNR. Further, we also Analyze another important parameter called energy efficiency of generalized-K fading channel at Low SNR in nats per channel use per joules and also characterize the Minimum energy per nat that is required for reliable communication.

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Chapter 1

INTRODUCTION

1.1 Overview

In wireless communication, apart from LOS (line of sight) path, there exist various non LOS paths because of various scatterers such as trees, buildings which causes the signal reflection, diffraction and scattering. Hence, at the receiver multiple signal components combines. This is called as multipath propagation environment. The multiple signal components combine at the receiver antenna to produce a composite received signal. Multipath propagation environment leads to constructive and destructive interference which leads to amplifies and attenuate the received signal amplitude respectively causing the received signal power varies, this is called multipath fading. Channel coefficient depends on attenuation and delays of different multipath components which lead to this variation in received signal power. But in practical life situation, multipath fading and shadowing occurs simultaneously which yields a composite/multipath shadowed fading environment or simply composite fading environment [1]. While evaluating the performance metrics for wireless communication systems, it is necessary to consider the effect of both fading and shadowing. This may be particularly true for the case of slow moving or stationary MSs environment in which receiver is unable to average out the effects of fading.

Depending upon the nature of radio propagation environment, several distributions are used to model them. For multipath fading, we have Rayleigh distribution, Rician distribution, Nakagami distribution, Weibull distribution, shadowing is modelled by lognormal distribution. Depending on the nature of fading and shadowing, we have different multipath fading shadowing environments which are modelled by different distributions which are combinations of fading and shadowing ditributions including widely accepted Rayleigh lognormal, Nakagami lognormal, Weibull lognormal distribution. A problem with these composite distributions in which shadowing is modelled by lognormal distribution is their complicated mathematical expression that make the analysis very difficult. An alternate approach is to approximate lognormal distribution by gamma distribution [2]. Hence using gamma distribution, various composite distributions have been proposed such as Rayleigh-gamma distribution or K distribution [1], Nakagami-gamma distribution or generalized-K distribution [3], all these distributions have simpler mathematical expression, hence analysis of shadowed fading channels becomes simpler.

Energy efficiency has become a important parameter in designing wireless communication systems, so wireless systems are now operating at Low SNR to achieve high energy efficiency. Hence analyzing the performance metric such as capacity at Low SNR is of practical interest in wireless communication systems. Even by operating at low SNR one can still achieve high capacity because today wireless system are operating at huge bandwidth.

1.2 Literature Survey

Performance Analysis of various fading channel has been studied in [4]-[17]. For an independent and identically distributed (i.i.d.) Rayleigh fading channel, the capacity scales linearly $\min(N_r, N_t)$ times that compared to single-input single-output (SISO), where N_t is the number of transmit antennas and N_r is the number of receive antennas with perfect Channel State Information (CSI) at high signal-to-noise ratio (SNR)[4]–[6]. In [7], the capacity of the SISO Rayleigh flat fading channel at low SNR is derived. In [8], Tall have characterized the ergodic capacity of MIMO (Multiple input multiple output) Rayleigh fading channels with perfect channel state information at both transmitter and receiver at asymptotically low SNR. Ergodic capacity of Rician fading channel has been widely investigated in order to derive a closed form expression and/or accurate approximations in [9]–[11]. The low-SNR regime capacity of a Multiple-Input Multiple-Output (MIMO) Rician channel has been looked at in [12], [13], assuming no CSI-T or mean CSI, respectively. In [14], closed form expressions for capacity of an independent identically distributed (i.i.d.) flat Rician fading channel with perfect channel state information at the receiver, and perfect channel state information at the transmitter (CSI-T) is derived at Low SNR and the expressions derived can be seen as a generalization of previous works as they captures the Rayleigh fading channel as a special case, this also characterizes the expression for energy efficiency of Rician fading channels at low-SNR which implies that the energy required to communicate one nat of information reliably is asymptotically very low which is in contrast with no CSI-T case where one cannot achieve a lower energy efficiency than -1.59 dB per information bit.

The capacity of Nakagami fading channel has been investigated at Low SNR in [15] for a multiple antenna channel assuming no CSI-T. In [16], the ergodic capacity of MIMO Nakagami fading channels is analysed with both uniformly an non-uniformly distributed phases where capacity upper bound for the channel is derived and then exact expressions for the low signal-to-noise ratio (SNR) capacity is derived, based on which the impact of fading parameter m on the capacity is examined. In [17], Low SNR capacity of Nakagami-m fading channel has beed studied and closed form expressions for the capacity are derived and the result characterized the capacity of Rayleigh fading channel as a special case, also closed form expressions of energy efficiency of Nakagami-m fading channel is derived.

Performance analysis of generalized-K fading channels is done in [18]-[20]. In [18], the performance metrics of digital communication systems over generalized-K (KG) fading channels are analyzed. Closed form expressions for the SNR statistics, the average Shannon's channel capacity and the bit error rate (BER) are derived. Further, the work in [19] presented the channel capacity under different adaptive transmission policies. In [20] the channel capacity is analysed over generalized-K fading channel with L-branch maximal-ratio combining (MRC). The derived results are obtained in the terms of well known Meijer G function. But no one has provide the closed form expression of capacity of generalized-K fading channel at Low SNR.

1.3 Organization Of Dissertation

In Chapter 2, we introduced modelling of fading and shadowing in which we introduced different fading and shadowing phenomenon. We introduced Log normal shadowing where we study why shadowing from building terrain and trees affect the link quality in wireless communication systems, we also discuss the approximation of lognormal distribution to gamma distribution to describe shadowing effect in satellite and terrestrial systems and also study the advantage of using gamma distribution as an alternate to lognormal distribution. Then we introduced composite multipath fading shadowing environment which consists of multipath fading superimposed in shadowing, in this we discuss the approach for obtaining composite distributions which describes fading and shadowing simultaneously and then we present various composite distributions which are obtained from combinations of multipath fading and shadowing distributions such as Rayleigh-lognormal distribution and then we see approximating lognormal distribution by gamma distribution in Rayleigh-lognormal distribution, we have much simpler composite distribution called K distribution. Then we introduced Rician-lognormal distribution which is used to describe shadowed fading environment where we also compare Rician-lognormal and pure Rician distribution for various levels of shadowing. Then we introduced another composite distribution called Nakagami-lognormal distribution where we also compare Nakagami-lognormal and pure Nakagami distribution for various levels of shadowing then we see approximating lognormal distribution by gamma distribution in Nakagami-lognormal distribution, we have much simpler composite distribution named Nakagami-gamma or generalized-K distribution, we have also plotted generalized-K distribution for different levels of shadowing.

In Chapter 3, we have chosen a shadowed fading channel which is modelled by Nakagami-gamma distribution also known as generalized-K distribution. We discuss how generalized-K distribution is used to describe variety of fading and shadowing model as its special case. Then we will evaluated the capacity of generalized-K fading channel in nats per channel use (ncpu) at asymptotic Low SNR by assuming perfect channel state information at both transmitter and receiver(CSIT-R). We have shown that the capacity of generalized-K fading channel at Low SNR scales as $\frac{\Omega}{4\,m}\,\text{SNR}\,\log^2\left(\frac{1}{\text{SNR}}\right)$. We also provide an on-off scheme that is achieving asymptotic Low SNR capacity results. Further we also characterize the energy efficiency of generalized-K fading channel in nats per channel use per joules at Low SNR, where we concludes as SNR increases energy efficiency decreases or correspondingly energy per nat increases and then we also characterize the Minimum energy per nat that is required for reliable communication.

In Chapter 4, we have compare exact capacity with our asymptotic Low SNR capacity results . The exact capacity curve follows the asymptotic capacity curves which justify our work. We have plotted the erdogic capacity for generalized-K fading channel for different values of shaping parameters m and k. The capacity through on-off scheme is also plotted and we see this rate matches the exact capacity curve for all SNR values. We also plotted another important performance metrics named Energy efficiency that characterize the performance of generalized-K fading channel and we see that asymptotic results are very close to the exact one. Energy efficiency through on-off scheme is also shown which is very close to the exact energy efficiency. Chapter 5 concludes the work.

Chapter 2

MODELLING OF FADING AND SHADOWING

2.1 Log Normal Shadowing

In satellite and terrestrial land mobile wireless communication systems, shadowing from building terrain and trees lead to slow variation of mean signal level which affect the link quality. Emperical measurement shows that this shadowing can be modelled by a lognormal distribution function for various indoor and outdoor environment [1], the lognormal distribuion is given as

$$f(y) = \frac{\xi}{\sqrt{2\pi}\sigma y} \exp\left[-\frac{(\xi \log y - \mu)^2}{2\sigma^2}\right]$$

Where y is the channel amplitude square i.e $y = |h|^2$ where μ and σ are logarithmic mean and logarithmic variance of y and ξ is the constant given as $\xi = \dfrac{10}{log10} = 4.3429$. Also note theat μ and σ are in db.

It is found that log-normal distribution can also be approximated by gamma distribution of [2]. As found in [21], gamma distributions can be used as alternate to lognormal distribution to describe shadowing effect in satellite and terrestrial systems. The fitness of this model was proved in [21] with emperical data. The advantage of using gamma as an substitute to lognormal distribution is that it helps to simplify various composite multipath/shadowing models.

2.2 Composite Multipath Shadowing

A composite multipath/shadowing fading environment consists of multipath fading superimposed over shadowing. While evaluating the performance analysis of wireless communication systems, it is necessary to consider the effect of combination of fading and shadowing. This is particularly true for the case of slow moving MS(mobile station) or pedestrain environment [1] in which receiver is unable to average out envelop fading due to multipath. This type of fading is also observed in land-satellite systems subjected to vegetative and or urban shadowing [1].

There are different approaches that have been suggested in literature for obtaining the composite distribution (distributions describing fading and shadowing simultaneously). In general, the probability density function of combination of fading and shadowing can be expressed as follows

$$f(a) = \int_0^\infty f_{\frac{A}{Z}}\left(\frac{a}{z}\right) f(z)\mathrm{dz} \tag{2.1}$$

Where $f_{\frac{A}{Z}}(\frac{a}{z})$ is the conditional density function of amplitude of fading channel in which average power become random due to the consequence of shadowing and $f(z)$ is the distribution describing shadowing effect.

Now we present various composite distributions which can be obtained from combinations of various multipath fading distributions and shadowing distribution for different composite environment.

2.2.1 Rayleigh-lognormal distribution

Suzuki [22] also proposed a composite Rayleigh/lognormal distribution to model mulipath fading sahdowing environment. Since consequence of shadowing is that the average power of faded signal becomes random, the conditional distribution of channel amplitude for the case of rayleigh fading is given as follows

$$f_{\frac{A}{Z}}\left(\frac{a}{z}\right) = \frac{2a}{z} \exp -\left(\frac{a^2}{z}\right)$$

Now, having this superimposed on shadowing will give rise to Rayleigh-lognormal distribution that describe composite environment and can be obtained from equation (2.1) where $f(z)$ is lognormal distributed.

So the combined distribution for this case can be given as

$$f_{RL}(a) = \int_0^\infty \frac{2a}{z} \exp -\left(\frac{a^2}{z}\right) \frac{\xi}{\sqrt{2\pi}\sigma z} \exp\left[-\frac{(\xi \log z - \mu)^2}{2\sigma^2}\right] \mathrm{dz}$$

.

Where subscript RL indicate that the density function is Rayleigh-lognormal.

For the channel amplitude square y, the distribution can be given as follows

$$f_{RL}(y) = \int_0^\infty \frac{1}{z} \exp -\left(\frac{y}{z}\right) \frac{\xi}{\sqrt{2\pi}\sigma z} \exp\left[-\frac{(\xi \log z - \mu)^2}{2\sigma^2}\right] \mathrm{dz}$$

2.2.2 *K* distribution

The Rayleigh-lognormal density function discussed above is in integral form, no closed form solution exists. Hence performance evaluation of wireless system under such fading shadowed channel is very difficult. Since we have said that the gamma distribution can be used as an alternated to the lognormal distributions. We can have simpler composite distributions if we use gamma distribution to model shadowing effects. So Rayleigh-lognormal distribution can be well approximated by the rayleigh gamma distribution which is so called as a K distribution [1] which can be given as follows

$$f_{RG}(a) = \int_0^\infty \frac{2a}{z} \exp - \left(\frac{a^2}{z}\right) \frac{1}{\Gamma(k)} \left(\frac{k}{P_0}\right)^k z^{k-1} e^{-\left(\frac{k}{P_0}\right)z} \mathrm{dz}$$

$$f_{RG}(a) = \frac{4}{\Gamma(k)} \left(\frac{k}{P_0}\right)^{\frac{k+1}{2}} a^k K_{k-1}\left(\sqrt{\frac{4k}{P_0}}a\right)$$

Where K_{k-1} is the modified Bessel function of the second kind of order $k-1$.
For the channel amplitude square y, the composite distribution can be written as

$$f_{RG}(y) = \frac{2}{\Gamma(k)} \left(\frac{k}{P_0}\right)^{(k+1)/2} y^{(k-1)/2} K_{k-1}\left(\sqrt{\frac{4k}{P_0}}\sqrt{y}\right)$$

2.2.3 Rician-lognormal distribution

In rician fading shadowing, we have Rician lognormal distribution used to describe the coomposite fading shadowing environment. Since consequence of shadowing is average power of fading signal is not deterministic, so the conditional density function of channel amplitude is given as follows

$$f_{\frac{A}{Z}}\left(\frac{a}{z}\right) = \frac{2\,(1+K)\,a}{z} \exp - \left(K + \frac{(K+1)a^2}{z}\right) I_0\left(2a\sqrt{\frac{K(1+K)}{z}}\right)$$

Now, in similar manner incuding combining with lognormal shadowing, then probability density function can be given as

$$f_{R_iL}(a) = \int_0^\infty \frac{2\,(1+K)\,a}{z} \exp - \left(K + \frac{(K+1)a^2}{z}\right) I_0\left(2a\sqrt{\frac{K(1+K)}{z}}\right) \times \frac{\xi}{\sqrt{2\pi}\sigma z} exp\left[-\frac{(\xi \log z - \mu)^2}{2\sigma^2}\right] \mathrm{dz}$$

The density dunction of channel amplitude square y is given by

$$f_{R_iL}(y) = \int_0^\infty \frac{(1+K)}{z} \exp -\left(K + \frac{(K+1)y}{z}\right) I_0 \left(2\sqrt{\frac{K(1+K)y}{z}}\right) \times \frac{\xi}{\sqrt{2\pi}\sigma z} \exp\left[-\frac{(\xi \log z - \mu)^2}{2\sigma^2}\right] \mathrm{dz}$$

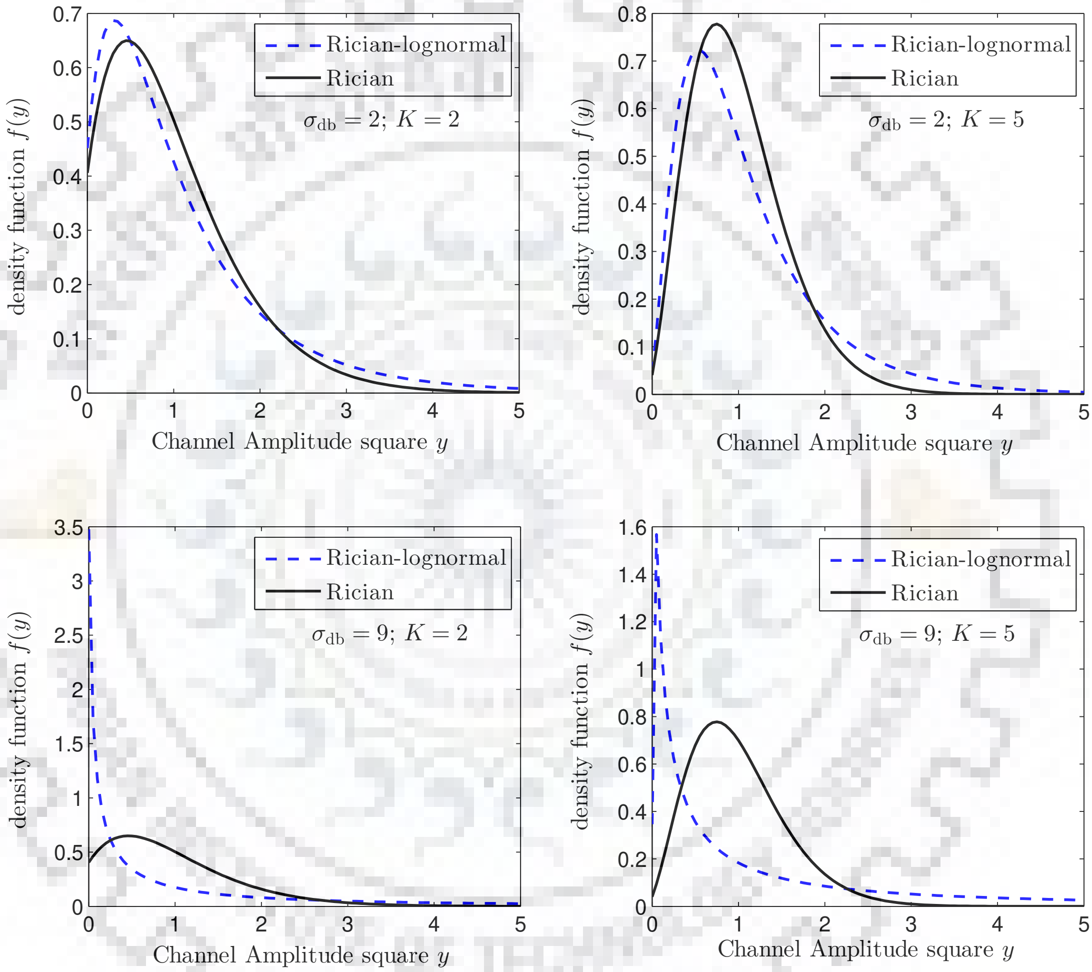

Figure 2.1: (Simulated) Comparision between Rician-lognormal distribution and pure Rician distribution for different levels of shadowing

This equation is plotted in Fig. 2.1 for different levels of shadowing indicated by parameter σ_{db}, for comparison, the density function of channel amplitude square y for pure Rician fading is also plotted. We can inferred from the figure that at low shadowing levels, the density function of pure Rician fading and composite Rician lognormal fading shadowing almost remain same but as severity of shadowing increases indicate by parameter σ_{db}, there is a considerable difference in both the distributions.

2.2.4 Nakagami-lognormal distribution

The composite Nakagami/lognormal probability distribution which arises in Nakagami shadowed environment was introduced by Ho and stuber [23]. Now including the consequence of shadowing for the case of Nakagami fading, the density function of channel amplitude a needs to be rewritten as

$$f_{\frac{A}{Z}}\left(\frac{a}{z}\right) = \frac{2}{\Gamma(m)}\left(\frac{m}{z}\right)^m x^{2m-1}e^{-\left(\frac{m}{z}\right)x^2}$$

Where the average power P_0 has been replaced by the random variable z, now having this fading superimposed on shadowing will give rise to composite distribution known as Nakagami-lognormal fading shadowed distribution that describes the composite multipath fading shadowing environment and can be obtained from (2.1) So, by substituting $f(z)$ to be lognormal distributed the composite distribution is given by

$$f_{NL}(y) = \int_0^\infty \frac{2}{\Gamma(m)}\left(\frac{m}{z}\right)^m x^{2m-1}e^{-\left(\frac{m}{z}\right)x^2} \frac{\xi}{\sqrt{2\pi}\sigma z} \exp\left[-\frac{(\xi \log z - \mu)^2}{2\sigma^2}\right] \text{dz}$$

The density function for the channel amplitude square can be given as

$$f_{NL}(y) = \int_0^\infty \frac{1}{\Gamma(m)}\left(\frac{m}{z}\right)^m y^{m-1}e^{-\left(\frac{m}{z}\right)y} \frac{\xi}{\sqrt{2\pi}\sigma z} \exp\left[-\frac{(\xi \log z - \mu)^2}{2\sigma^2}\right] \text{dz}$$

Above Equation is plotted in Fig. 2.2. For comparison purpose, the density of the channel amplitude square y under pure Nakagami fading conditions is also shown. For low values of the shadowing parameter, the density functions of pure Nakagami fading conditions and Nakagami-lognormal shadowed fading conditions are very close. However, as the extent of shadowing increases (indicated by parameter σ_{db}), the pdf of the channel amplitude square y in shadowed fading channels move to the left indicating the effect of shadowing.

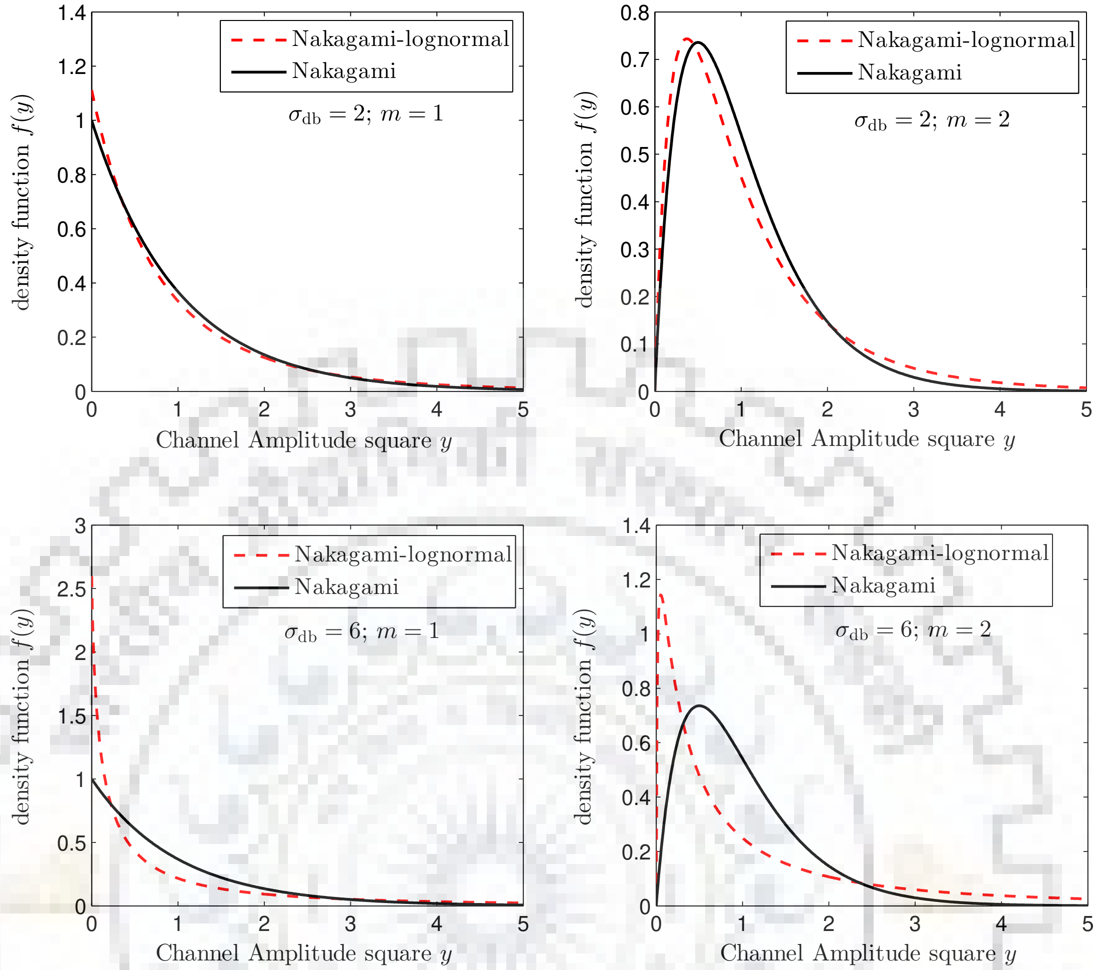

Figure 2.2: (Simulated) Comparision between Nakagami-lognormal distribution and pure Nakagami distribution for different levels of shadowing

2.2.5 Generalized-K distribution

The Nakagami-lognormal density fuction shown discussed is in integral form, no closed form solution exists, hence using the gamma distribution as an alternated to the lognormal distributions, we have simpler composite distributions. Hence, Nakagami-lognormal distribution can be well approximated by the Nakagami gamma distribution which is so called as a generalized-K distribution [3] which can be given as follows

$$f_{NG}(x) = \int_0^\infty \frac{2}{\Gamma(m)} \left(\frac{m}{z}\right)^m x^{2m-1} e^{-\left(\frac{m}{z}\right)x^2} \frac{1}{\Gamma(k)} \left(\frac{k}{P_0}\right)^k z^{k-1} e^{-\left(\frac{k}{P_0}\right)z} dz$$

which can be written as

$$f_{NG}(x) = \frac{4m^{(\beta+1)/2}x^{\beta}}{\Gamma(m)\Gamma(k)\Omega^{(\beta+1)/2}} K_{\alpha}\left[2\sqrt{\frac{m}{\Omega}}\ x\right]$$

where $K_{\alpha}(.)$ is the modified bessel function of second kind of order α and $\Gamma(.)$ is the gamma function, k and m are the shaping parameters and Ω is the given in terms of average power as $\Omega = \frac{E[x]^2}{k} = \frac{P_0}{k}$.

The generalized-K distribution is shown in Fig. 2.3 for $m = 1$ and in Fig. 2.4 for of $m = 2$ for the range of values of shadowing from weak (2 db) to strong shadowing (9db). The effect of shadowed is clear from the density function plots. As the shadowing parameter increase, the peaks of the density function move toward lower values of the channel aplitude square y, indicating that the amount of increase in randomness.

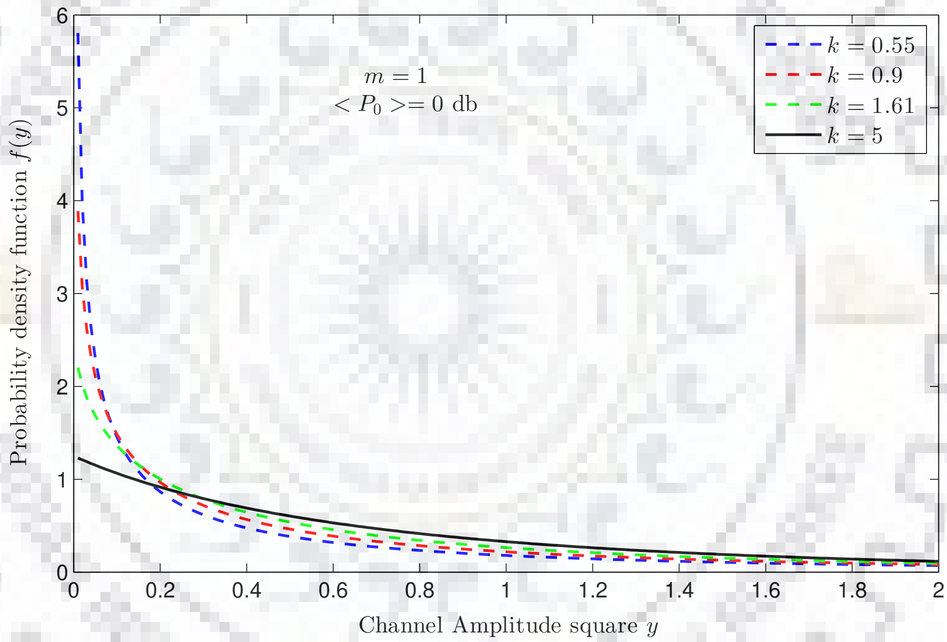

Figure 2.3: (Simulated) Nakagami-gamma or Generalized-K distribution for (m=1) and different levels shadowing

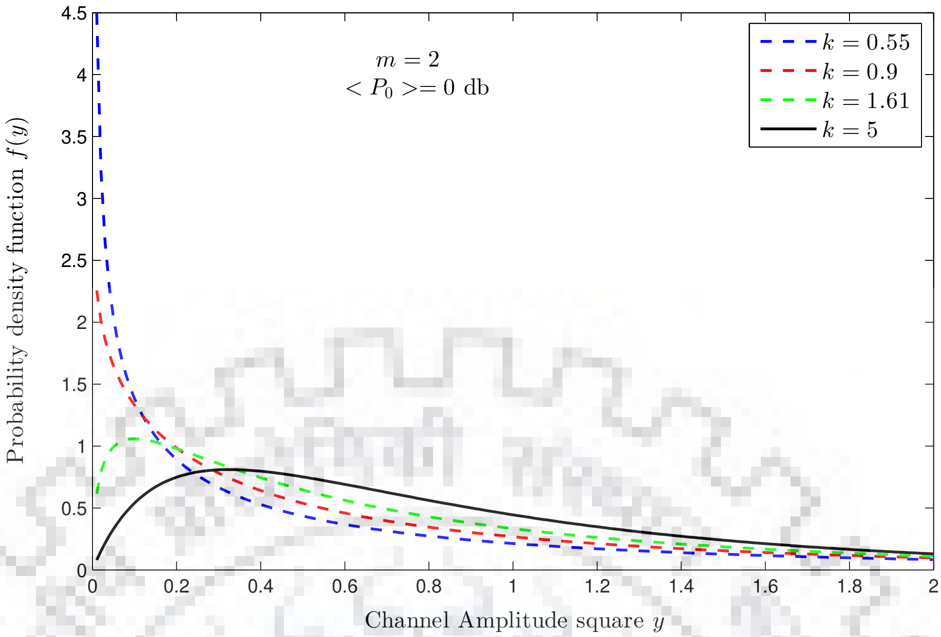

Figure 2.4: (Simulated) Nakagami-gamma or Generalized-K distribution for (m=2) and different levels shadowing

2.3 Summary

In this chapter, we studied modelling of fading and shadowing in which we have studied different fading and shadowing phenomenon. We have introduced Log normal shadowing where we have discussed the approximation of lognormal distribution to gamma distribution to describe shadowing effect. Then we studied composite multipath fading shadowing environment which consists of multipath fading superimposed in shadowing, then we have presented various composite distributions which are obtained from combinations of multipath fading and shadowing distributions such as Rayleigh-lognormal distribution, K distribution. Then we introduced Rician-lognormal distribution which is used to describe shadowed fading environment. We then studied other composite distributions called Nakagami-lognormal distribution and generalized-K distribution.

Chapter 3

PERFORMANCE METRICS FOR GENERALIZED-K FADING CHANNELS

Let x is complex random variable that represent channel input, y is complex random variable which represent channel output, then the system is modelled as

$$\mathbf{y} = \mathbf{h}\,\mathbf{x} + \mathbf{w} \tag{3.1}$$

where $\mathbf{w}$ is zero mean circularly symmetric complex gaussian noise with variance σ^2 written as $\mathbf{w} \sim CN(0, \sigma^2)$ and $\mathbf{h}$ is complex random variable which represent channel coefficient. Let amplitude of channel coefficient follows a mixture of Nakagami fading and gamma shadowing which is so called as generalized-K distribution defined as

$$f(a) = \frac{4\, m^{(\beta+1)/2}\, a^{\beta}}{\Gamma(m)\Gamma(k)\, \Omega^{(\beta+1)/2}}\, K_{\alpha}\left[2\sqrt{\frac{m}{\Omega}}\; a\right] \tag{3.2}$$

where $K_{\alpha}(.)$ is the modified bessel function of second kind of order α and $\Gamma(.)$ is the gamma function. k and m are the shaping parameters, $\beta = k + m - 1$ and $\alpha = k - m$ and Ω is the average power given by $\Omega = \dfrac{E[a]^2}{k}$.

Since the generalized-K distribution depends on two shaping parameters, so can be used to describe variety of fading and shadowing models as its special case, for $m = 1$ it reduces to K distribution, for $m \to \infty$ and $k \to \infty$ it approaches to AWGN channel or no fading and for $k \to \infty$ it approaches to well known Nakagami distribution.

3.1 Capacity Analysis at low SNR

Let the channel input is subjected to following constraint as $\mathrm{E}[|\mathbf{x}|^2] \leq P$. According to optimal power algorithm, optimal power can be given as [24]

$$p(h) = \left[\frac{1}{\lambda} - \frac{1}{|h|^2}\right]^{+} \tag{3.3}$$

where,

$$a^+ = \begin{cases} a, & \text{if } a \geq 0. \\ 0, & \text{otherwise.} \end{cases}$$

where λ is the lagrange multiplier which can be obtained through the above condition of power constraint with equality i.e

$$\mathrm{E}\left[\frac{1}{\lambda} - \frac{1}{|h|^2}\right]^+ = P$$

Now assuming unit noise variance power P will become equal to SNR, so above can be written as

$$\mathrm{E}\left[\frac{1}{\lambda} - \frac{1}{|h|^2}\right]^+ = \mathrm{SNR} \tag{3.4}$$

Capacity is then obtained as

$$C = \mathrm{E}\left[\log(1 + p(h)|h|^2)\right]$$

Substituting for $p(h)$ from (3.3), we have $C = \mathrm{E}\left[\log\left(\frac{|h|^2}{\lambda}\right)\right]$ with $|h|^2 \geq \lambda$ which can be written as

$$C = \int_{\lambda}^{\infty} \log\left(\frac{|h|^2}{\lambda}\right) f_{|h|^2}(y)\mathrm{dy} \tag{3.5}$$

Now let us calculate the value of lagrange multiplier λ, for this we need to know the density fuction of magnitude squared of chnanel coefficient.

Now if we know the pdf of $a = |h|$, then the pdf of $y = |h|^2$ can be obtained as follows

$$f(y) = \frac{1}{2\sqrt{y}} f_a(\sqrt{y}) \tag{3.6}$$

so pdf of $y = |h|^2$ using (3.2) is written as

$$f(y) = \frac{1}{2\sqrt{y}} \frac{4\, m^{(\beta+1)/2} \sqrt{y}^{\beta}}{\Gamma(m)\Gamma(k)\, \Omega^{(\beta+1)/2}} K_{\alpha}\left[2\sqrt{\frac{m}{\Omega}}\sqrt{y}\right] \tag{3.7}$$

where $K_{\alpha}(.)$ is the modified bessel function of second kind of order α . So from (3.4) we have

$$\mathrm{SNR} = \int_{\lambda}^{\infty} \left[\frac{1}{\lambda} - \frac{1}{|h|^2}\right] f_{|h|^2}(y)\mathrm{dy}$$

using (3.7), we have

$$\text{SNR} = \frac{2\, m^{(\beta+1)/2}}{\Gamma(m)\Gamma(k)\,\Omega^{(\beta+1)/2}} \int\limits_{\lambda}^{\infty} \left[\frac{1}{\lambda} - \frac{1}{y}\right] \; y^{(\beta-1)/2} \; K_\alpha \left[2\sqrt{\frac{m}{\Omega}}\sqrt{y}\right] \text{dy} \tag{3.8}$$

Let us define a function $G(x) = \text{E}\left[\frac{1}{x} - \frac{1}{|h|^2}\right]^+$, putting $x = \lambda$ and using (3.4), we have

$$G(\lambda) = \text{E}\left[\frac{1}{\lambda} - \frac{1}{|h|^2}\right]^+ = \text{SNR} \tag{3.9}$$

taking limits SNR $\to 0$ both sides, we have

$$\lim_{SNR\to 0} G(\lambda) = 0$$

$$\lim_{SNR\to 0} (\lambda) = G^{-}(0) = \infty \tag{3.10}$$

following this, we need to get series expansion of RHS of (3.8) at infinity, so we need to know the series expansion of modified bessel function of second kind at infinity which can be defined as follows [25].

$$K_\alpha(x) \approx \sqrt{\frac{\pi}{2x}} e^{-x} \left[\sum_{l=0}^{n-1} \frac{\Gamma(\alpha + l - \frac{1}{2})}{(2x)^l\,\Gamma(\alpha - l - \frac{1}{2})\, l!} + o\left(\frac{1}{x^{n-1}}\right)\right]$$

so by using this in (3.8) we have,

$$\text{SNR} = \sqrt{\pi}\frac{m^{(\beta+1)/2}\left(\frac{m}{\Omega}\right)^{-1/4}}{\Gamma(m)\Gamma(k)\,\Omega^{(\beta+1)/2}} \int\limits_{\lambda}^{\infty} \left(\frac{1}{\lambda} - \frac{1}{y}\right) y^{(\beta-1)/2} y^{-1/4} \exp - \left(2\left(\frac{m}{\Omega}\right)^{\frac{1}{2}}\sqrt{y}\right) \text{dy} \tag{3.11}$$

Let $2\left(\frac{m}{\Omega}\right)^{\frac{1}{2}}\sqrt{y} = p$ which implies $y = \left(\frac{p}{2}\right)^2 \frac{\Omega}{m}$, using this (3.11) can be written as

$$\text{SNR} = \frac{\sqrt{\pi}}{\Gamma(m)\Gamma(k)} \left(\frac{1}{2}\right)^{\beta - \frac{1}{2}} \int\limits_{2(\frac{m}{\Omega}\lambda)^{\frac{1}{2}}}^{\infty} \left(\frac{1}{\lambda} - \frac{1}{\left(\frac{p}{2}\right)^2\left(\frac{\Omega}{m}\right)}\right) p^{(\beta - \frac{1}{2})} e^{-p} \text{dp} \tag{3.12}$$

Now Consider integral, $\int\limits_{2(\frac{m}{\Omega}\lambda)^{\frac{1}{2}}}^{\infty} \left(\frac{1}{\lambda} - \frac{1}{\left(\frac{p}{2}\right)^2\left(\frac{\Omega}{m}\right)}\right) p^{(\beta - \frac{1}{2})}\, e^{-p}$ dp in (3.12) which can be written as

$$= \int_{2\left(\frac{m}{\Omega}\lambda\right)^{\frac{1}{2}}}^{\infty} \frac{1}{\lambda} p^{\left(\beta-\frac{1}{2}\right)} e^{-p} dp \; - \; 2^2 \left(\frac{m}{\Omega}\right) \int_{2\left(\frac{m}{\Omega}\lambda\right)^{\frac{1}{2}}}^{\infty} p^{\left(\beta-\frac{5}{2}\right)} e^{-p} \mathrm{dp}$$

$$= \frac{1}{\lambda}\Gamma\left[\beta+\frac{1}{2},\, 2\left(\frac{m}{\Omega}\lambda\right)^{\frac{1}{2}}\right] - 2^2\left(\frac{m}{\Omega}\right)\Gamma\left[\beta-\frac{3}{2},\, 2\left(\frac{m}{\Omega}\lambda\right)^{\frac{1}{2}}\right] \tag{3.13}$$

where, $\Gamma(.)$ is the incomplete gamma function defined in [25] as

$$\Gamma(\beta, \alpha) = (\beta - 1)!\, e^{-\alpha} \sum_{j=0}^{\beta-1} \frac{\alpha^j}{j!} \tag{3.14}$$

so by expanding the $\Gamma(.)$ function in (3.13) by using (3.14) and also using the fact that $\lambda^u >> \lambda^v \quad \forall u$ and $v \in \mathbb{N}$ but $u > v$,which follows from (3.10), equation (3.13) is given as (Appendix A)

$$\approx e^{-2\left(\frac{m}{\Omega}\lambda\right)^{\frac{1}{2}}} 2^{\left(\beta-\frac{1}{2}\right)} \left(\frac{m}{\Omega}\right)^{\frac{\beta}{2}-\frac{3}{4}} \lambda^{\frac{\beta}{2}-\frac{7}{4}} \tag{3.15}$$

so using this result, equation (3.12) can be written as

$$\text{SNR} \approx \frac{\sqrt{\pi}}{\Gamma(m)\Gamma(k)}\, e^{-2\left(\frac{m}{\Omega}\lambda\right)^{\frac{1}{2}}} \left(\frac{m}{\Omega}\right)^{\frac{\beta}{2}-\frac{3}{4}} \lambda^{\frac{\beta}{2}-\frac{7}{4}} \tag{3.16}$$

$$e^{2\left(\frac{m}{\Omega}\lambda\right)^{\frac{1}{2}}} \lambda^{\frac{-\beta}{2}+\frac{7}{4}} \approx \frac{\sqrt{\pi}}{\text{SNR}\ \Gamma(m)\Gamma(k)} \left(\frac{m}{\Omega}\right)^{\frac{\beta}{2}-\frac{3}{4}} = c \tag{3.17}$$

divide both sides of (3.17) by powers of $2\left(\frac{m}{\Omega}\right)^{\frac{1}{2}}$, we have

$$\exp\left(\lambda^{\frac{1}{2}}\right) \left(\lambda^{\frac{1}{2}}\right)^{\frac{\left(\frac{7}{2}-\beta\right)}{2\left(\frac{m}{\Omega}\right)^{\frac{1}{2}}}} \approx \exp\left(\frac{\log(c)}{2\left(\frac{m}{\Omega}\right)^{\frac{1}{2}}}\right) \tag{3.18}$$

This is of the form $e^y\ y^p = e^{c1}$ which can be written as $e^{\frac{y}{p}} \frac{y}{p} = \frac{e^{\frac{c1}{p}}}{p}$. So using this fact equation (3.18) can be written as

$$\frac{\lambda^{\frac{1}{2}}}{\frac{\left(\frac{7}{2}-\beta\right)}{2\left(\frac{m}{\Omega}\right)^{\frac{1}{2}}}} \exp \frac{\lambda^{\frac{1}{2}}}{\frac{\left(\frac{7}{2}-\beta\right)}{2\left(\frac{m}{\Omega}\right)^{\frac{1}{2}}}} \approx \frac{\exp\left(\frac{\log(c)}{\left(\frac{7}{2}-\beta\right)}\right)}{\frac{\left(\frac{7}{2}-\beta\right)}{2\left(\frac{m}{\Omega}\right)^{\frac{1}{2}}}}$$

$$\frac{2\left(\frac{m}{\Omega}\lambda\right)^{\frac{1}{2}}}{\left(\frac{7}{2}-\beta\right)} \exp\left(\frac{2\left(\frac{m}{\Omega}\lambda\right)^{\frac{1}{2}}}{\left(\frac{7}{2}-\beta\right)}\right) \approx \frac{2\left(\frac{m}{\Omega}\right)^{\frac{1}{2}}}{\left(\frac{7}{2}-\beta\right)} \exp\left(\frac{\log(c)}{\left(\frac{7}{2}-\beta\right)}\right) \tag{3.19}$$

This equation is of the form of $xe^x = y$ whose solution is given in terms of the Lambert-W function

so for $\beta < \dfrac{7}{2}$,we get

$$\frac{\lambda^{\frac{1}{2}}}{\frac{\left(\frac{7}{2}-\beta\right)}{2\left(\frac{m}{\Omega}\right)^{\frac{1}{2}}}} \approx W_0\left(\frac{2\left(\frac{m}{\Omega}\right)^{\frac{1}{2}}}{\left(\frac{7}{2}-\beta\right)} \exp\left(\frac{\log(c)}{\left(\frac{7}{2}-\beta\right)}\right)\right)$$

$$\lambda^{\frac{1}{2}} \approx \frac{\left(\frac{7}{2}-\beta\right)}{2\left(\frac{m}{\Omega}\right)^{\frac{1}{2}}} W_0\left(\frac{2\left(\frac{m}{\Omega}\right)^{\frac{1}{2}} c^{\frac{1}{\left(\frac{7}{2}-\beta\right)}}}{\left(\frac{7}{2}-\beta\right)}\right)$$

$$\lambda \approx \frac{\left(\frac{7}{2}-\beta\right)^2}{4\left(\frac{m}{\Omega}\right)} W_0^2\left(\frac{2\left(\frac{m}{\Omega}\right)^{\frac{1}{2}} c^{\frac{1}{\left(\frac{7}{2}-\beta\right)}}}{\left(\frac{7}{2}-\beta\right)}\right) \tag{3.20}$$

Now substituting the value of c from (3.17) in (3.20) we get,

$$\lambda \approx \frac{\left(\frac{7}{2}-\beta\right)^2}{4\left(\frac{m}{\Omega}\right)} W_0^2\left(\frac{2(\frac{m}{\Omega})^{\frac{1}{2}}}{\left(\frac{7}{2}-\beta\right)}\left(\frac{\sqrt{\pi}}{\Gamma(m)\Gamma(k)}\left(\frac{m}{\Omega}\right)^{\frac{\beta}{2}-\frac{3}{4}}\right)^{\frac{1}{\frac{7}{2}-\beta}}\left(\frac{1}{\text{SNR}}\right)^{\frac{1}{\frac{7}{2}-\beta}}\right)$$

$$\lambda \approx \frac{\left(\frac{7}{2}-\beta\right)^2}{4\left(\frac{m}{\Omega}\right)} W_0^2\left(\frac{2}{\left(\frac{7}{2}-\beta\right)}\left(\frac{\sqrt{\pi}}{\Gamma(m)\Gamma(k)}\left(\frac{m}{\Omega}\right)\right)^{\frac{1}{\frac{7}{2}-\beta}}\left(\frac{1}{\text{SNR}}\right)^{\frac{1}{\frac{7}{2}-\beta}}\right) \tag{3.21}$$

Now using the property of lambert-W function, for $A > 0$ $\lim_{x\to\infty} \dfrac{W(Ax)}{W(x)} = 1$, we have

$$\lambda \approx \frac{\left(\frac{7}{2}-\beta\right)^2}{4\left(\frac{m}{\Omega}\right)} W_0^2\left(\left(\frac{1}{\text{SNR}}\right)^{\frac{1}{\frac{7}{2}-\beta}}\right) \tag{3.22}$$

using the expansion of (3.5), it can be easily shown that the Capacity is given by

$$C \approx \text{SNR}\ \lambda \tag{3.23}$$

Hence we get the capacity at full CSIT-R at both the transmitter and receiver as

$$C \approx \text{SNR}\,\frac{\left(\frac{7}{2}-\beta\right)^2}{4\left(\frac{m}{\Omega}\right)} W_0^2\left(\left(\frac{1}{\text{SNR}}\right)^{\frac{1}{\frac{7}{2}-\beta}}\right) \qquad \text{for} \quad \beta < \frac{7}{2} \tag{3.24}$$

Although lambert function now is easily available in many software but it will be interesting to see that above can also be converted into the generally used function which is logarithmic function. The solution to the equation of the form of $xe^x = y$ with $x > 1$ can also be given in terms of an infinite ladder series [26] as

$$x = -\log\left(-\frac{-\log\left(\frac{-\log(..)}{y}\right)}{y}\right)$$

The first three approximations in this ladder are

$$\begin{aligned} x_1(y) &= \log(y) \\ x_2(y) &= \log(y) - \log\log(y) \\ x_3(y) &= \log(y) - \log[\log(y) - \log\log(y)] \\ \text{In general} \quad x_i(y) &= x_1(y) - \log[x_{i-1}(y)] \end{aligned}$$

Now, if we used first approximation the (3.22) can be written as

$$\lambda \approx \frac{\left(\frac{7}{2}-\beta\right)^2}{4\left(\frac{m}{\Omega}\right)} \log^2\left(\left(\frac{1}{\text{SNR}}\right)^{\frac{1}{\frac{7}{2}-\beta}}\right)$$

$$\lambda \approx \frac{1}{4\left(\frac{m}{\Omega}\right)} \log^2\left(\frac{1}{\text{SNR}}\right) \tag{3.25}$$

Hence Capacity is given as

$$C \approx \text{SNR}\frac{1}{4\left(\frac{m}{\Omega}\right)} \log^2\left(\frac{1}{\text{SNR}}\right) \tag{3.26}$$

In the similar way, the solution of (3.19) for $\beta > \frac{7}{2}$ can be given as follows

$$\lambda \approx \frac{\left(\frac{7}{2}-\beta\right)^2}{4\left(\frac{m}{\Omega}\right)} W_{-1}^2\left(\frac{2}{\left(\frac{7}{2}-\beta\right)}\left(\frac{\sqrt{\pi}}{\Gamma(m)\Gamma(k)}\left(\frac{m}{\Omega}\right)\right)^{\frac{1}{\frac{7}{2}-\beta}}\left(\frac{1}{\text{SNR}}\right)^{\frac{1}{\frac{7}{2}-\beta}}\right) \tag{3.27}$$

Now again using the property of lambert-W function for $A < 0$ $\lim_{x\to 0^+} \frac{W(Ax)}{W(-x)} = 1$, above can be given as

$$\lambda \approx \frac{\left(\frac{7}{2}-\beta\right)^2}{4\left(\frac{m}{\Omega}\right)} W_{-1}^2\left(-\left(\frac{1}{\text{SNR}}\right)^{\frac{1}{\frac{7}{2}-\beta}}\right) \tag{3.28}$$

Hence, we get the capacity at full CSIR at both the transmitter and receiver as

$$C \approx SNR\frac{\left(\frac{7}{2}-\beta\right)^2}{4\left(\frac{m}{\Omega}\right)} W_{-1}^2\left(-\left(\frac{1}{\text{SNR}}\right)^{\frac{1}{\frac{7}{2}-\beta}}\right) \qquad \text{for} \quad \beta > \frac{7}{2} \tag{3.29}$$

The solution to the equation of the form of $xe^x = y$ with $x < -1$ can also be given in terms of an infinite ladder series [26] as

$$x = -\log\left(-\frac{\log\left(\frac{-\log(..)}{y}\right)}{y}\right)$$

In this case, first three approximations in the ladder are

$$\begin{aligned}
x_1(y) &= \log(-y) \\
x_2(y) &= \log(-y) - \log(-\log(-y)) \\
x_3(y) &= \log(-y) - \log[\log(-\log(-y)) - \log(-y)] \\
\text{In general} \quad x_i(y) &= x_1(y) - \log[-x_{i-1}(y)] \quad \text{for} \;\; i \geq 2
\end{aligned}$$

Now, if we used first approximation the (3.28) can be written as

$$\lambda \approx \frac{\left(\frac{7}{2}-\beta\right)^2}{4\left(\frac{m}{\Omega}\right)} \log^2\left(\left(\frac{1}{\mathrm{SNR}}\right)^{\frac{1}{\frac{7}{2}-\beta}}\right)$$

$$\lambda \approx \frac{1}{4\left(\frac{m}{\Omega}\right)} \log^2\left(\frac{1}{\mathrm{SNR}}\right) \tag{3.30}$$

Hence, Capacity is given as

$$C \approx \mathrm{SNR}\,\frac{1}{4\left(\frac{m}{\Omega}\right)} \log^2\left(\frac{1}{\mathrm{SNR}}\right) \tag{3.31}$$

Now, for $\beta \approx \frac{7}{2}$ from (3.16), we have

$$e^{2\left(\frac{m}{\Omega}\lambda\right)^{\frac{1}{2}}} \approx \frac{\sqrt{\pi}}{\mathrm{SNR}\,\Gamma(m)\Gamma(k)}\left(\frac{m}{\Omega}\right)$$

$$\lambda \approx \frac{1}{4\left(\frac{m}{\Omega}\right)} \log^2\left(\frac{\sqrt{\pi}}{\mathrm{SNR}\,\Gamma(m)\Gamma(k)}\left(\frac{m}{\Omega}\right)\right) \tag{3.32}$$

Hence we get the capacity in this case given by

$$C \approx \mathrm{SNR}\,\frac{1}{4\left(\frac{m}{\Omega}\right)} \log^2\left(\frac{1}{\mathrm{SNR}}\right) \qquad \text{for} \quad \beta = \frac{7}{2} \tag{3.33}$$

Clearly, from all the above cases, we can say the Low SNR capacity of Generalized-K fading channel in general is given by $C \approx \mathrm{SNR}\,\frac{1}{4\left(\frac{m}{\Omega}\right)} \log^2\left(\frac{1}{\mathrm{SNR}}\right)$

3.2 Low SNR Capacity using ON-OFF Scheme

In this section, we will show that the asymptotic low SNR capacity results can also be achieved by an on-off scheme. Since $p(h)$ is defined as $p(h) = \left[\frac{1}{\lambda} - \frac{1}{|h|^2}\right]^+$, so if $\frac{1}{\lambda} - \frac{1}{|h|^2} \geq 0$ or $|h|^2 \geq \lambda$, then transmission is done otherwise nothing is transmitted. Now transmission is done with power,

$$= \frac{\mathrm{SNR}}{\mathrm{prob}(|\mathrm{h}|^2 \geq \lambda)}$$

so we can write,

$$p(h) = \begin{cases} \frac{\mathrm{SNR}}{\mathrm{prob}(|\mathrm{h}|^2 \geq \lambda)}, & \text{if } |h|^2 > \lambda. \\ 0, & \text{otherwise.} \end{cases} \tag{3.34}$$

capacity in this case is given as

$$C' = \underset{|h|^2}{\mathrm{E}}\left[\log\left(1 + p(h)\,|h|^2\right)\right]$$

$$= \int_0^\infty \left[\log\left(1 + p(h)\,|h|^2\right)\right] p_{\mathrm{y}}(y)\mathrm{d}y$$

$$= \int_\lambda^\infty \left[\log\left(1 + \frac{\mathrm{SNR}\ |h|^2}{\mathrm{prob}(|h|^2 > \lambda)}\right)\right] p_{\mathrm{y}}(y)\mathrm{d}y$$

$$C'_{min} = \int_\lambda^\infty \left[\log\left(1 + \frac{\mathrm{SNR}\ \lambda}{\mathrm{prob}(|h|^2 > \lambda)}\right)\right] p_{\mathrm{y}}(y)\mathrm{d}y$$

$$C' \geq \int_\lambda^\infty \left[\log\left(1 + \frac{\mathrm{SNR}\ \lambda}{\mathrm{prob}(|h|^2 > \lambda)}\right)\right] p_{\mathrm{y}}(y)\mathrm{d}y$$

$$C' \geq \left[\log\left(1 + \frac{\mathrm{SNR}\ \lambda}{\mathrm{prob}(|h|^2 > \lambda)}\right)\right] \int_\lambda^\infty p_{\mathrm{y}}(y)\mathrm{d}y$$

$$C' \geq \left[\log\left(1 + \frac{\mathrm{SNR}\ \lambda}{\mathrm{prob}(|h|^2 > \lambda)}\right)\right] \mathrm{prob}(|h|^2 > \lambda) \tag{3.35}$$

Consider, $\mathrm{prob}(|\mathrm{h}|^2 > \lambda) = \int_\lambda^\infty f_{|\mathrm{h}|}(y)\mathrm{d}y$ which is given by (Appendix B)

$$\approx \frac{\sqrt{\pi}}{\Gamma(m)\Gamma(k)} e^{-2\left(\frac{m}{\Omega}\lambda\right)^{\frac{1}{2}}} \left(\left(\frac{m}{\Omega}\lambda\right)^{\frac{1}{2}}\right)^{\beta-\frac{1}{2}} \tag{3.36}$$

In (3.35) the factor $\dfrac{\text{SNR }\lambda}{\text{prob}(|\mathbf{h}|^2 > \lambda)}$ using (3.36) is given by $\left(\dfrac{\Omega}{m}\right)^{\frac{1}{2}} \dfrac{1}{\lambda^{\frac{1}{2}}}$ which $\to 0$ *as* SNR $\to 0$ and using the fact that $\log(1+x) \approx x$ for $x \to 0$, the capacity in (3.35) is given as $C' \approx$ SNR λ. This proves that the on-off scheme is achieving the asymptotic Low SNR capacity results.

3.3 Energy Efficiency at Low SNR

Now, we consider the energy efficiency of generalized-K fading channel. Energy efficiency can be defined as capacity per unit SNR or the number of nats per unit of transmitted energy that are reliably transmitted from transmitter to receiver i.e [27]

$$\eta_{EE} = \frac{C}{\text{SNR}} \tag{3.37}$$

$$\approx \frac{1}{4\left(\frac{m}{\Omega}\right)} \log^2\left(\frac{1}{\text{SNR}}\right) \tag{3.38}$$

As SNR increases, we can find that energy efficiency decreases or correspondingly energy per nat increases. Also, it can be noted from above expression that energy efficiency also depends on parameter m and Ω, as m increases energy efficiency decreases.

3.4 Minimum energy per nat

Now we will calculate $\frac{E_b}{N_{o\,min}}$, that is minimum Energy per nat required for reliable communication which can be defined as follows [28]

$$\frac{E_b}{N_{o\,min}} = \lim_{\text{SNR}\to 0} \frac{\text{SNR}}{C(\text{SNR})} = \frac{1}{\dot{C}(0)} \tag{3.39}$$

Since, Capacity is given by

$$C(\text{SNR}) = \text{SNR}\,\frac{1}{4\left(\frac{m}{\Omega}\right)} \log^2\left(\frac{1}{\text{SNR}}\right)$$

Differentiating this respect to SNR we have

$$\frac{d\,C(SNR)}{d\,SNR} = \frac{\Omega}{4\,m}\left[2\,\text{SNR}\,\log\left(\frac{1}{\text{SNR}}\right)\left(\frac{1}{\text{SNR}^2}\right) + \log^2\left(\frac{1}{\text{SNR}}\right)\right]$$

$$= \frac{\Omega}{4\,m}\log^2\left(\frac{1}{\text{SNR}}\right)\left[1 - 2\frac{\log^{-1}\left(\frac{1}{\text{SNR}}\right)}{\text{SNR}}\right]$$

Taking limit SNR $\rightarrow 0$ on both sides, we have we have

$$\dot{C}(0) = \frac{\Omega}{4\,m}\log^2\left(\frac{1}{\text{SNR}}\right)\lim_{SNR\rightarrow 0}\left[1 - 2(-1)\,\log^{-2}\left(\frac{1}{\text{SNR}}\right)\right]$$

$$\dot{C}(0) \approx \frac{\Omega}{4\,m}\log^2\left(\frac{1}{SNR}\right) \tag{3.40}$$

Hence using (3.40), minimum energy per nat in (3.39) is given as

$$\frac{E_b}{N_{o\,min}} \approx \frac{1}{\frac{\Omega}{4\,m}\log^2\left(\frac{1}{\text{SNR}}\right)} \tag{3.41}$$

This relation shows how minimum energy per nat varies with SNR, as SNR increases minimum energy per nat increases.

3.5 Summary

In this chapter, we evaluated performance metrics for generalized-K fading channel. We have evaluated the capacity of generalized-K fading channel in nats per channel use (ncpu) at asymptotic Low SNR by assuming perfect channel state information at both transmitter and receiver (CSIT-R). We have shown that the capacity of generalized-K fading channel at Low SNR scales as $\frac{\Omega}{4\,m}\,\text{SNR}\,\log^2\left(\frac{1}{\text{SNR}}\right)$. We also provide an on-off scheme that is achieving asymptotic Low SNR capacity results. Further we have also characterize the energy efficiency of generalized-K fading channel at Low SNR, where we concludes as SNR increases energy efficiency decreases and then we also characterize the Minimum energy per nat that is required for reliable communication.

Chapter 4

RESULTS AND DISCUSSIONS

In this section we present some results, where we have compared exact capacity with our asymptotic Low SNR results which justify our work. we have plotted the erdogic capacity for generalized-K fading channel for different values of shaping parameters m and k. We have set $\Omega = \frac{1}{k}$ in all our results presented so as to have a unity channel mean square value.

In fig. 4.1, we have plotted erdogic capacity of generalized-K fading channel for $m = 1$ and $k = 1$, where the exact capacity is plotted using standard optimization tools. To obtained exact curve, we first calculate the lagrange multiplier λ that satisfy the power constraint with equality and then we calculate the capacity using (3.5). Also low SNR capacity results using Lambert and log function are also plotted. We can see from the figure that our asymptotic capacity curves are following the exact capacity curve. For the case of $k = 1$ and $m = 1$, we can also note that the lambert function is lower bound on log function and we see that capacity given by lambert function is close to exact curve than log function for this considered case. The capacity through on-off scheme is also shown and we can see from figure that this rate matches with exact capacity curve for all SNR values.

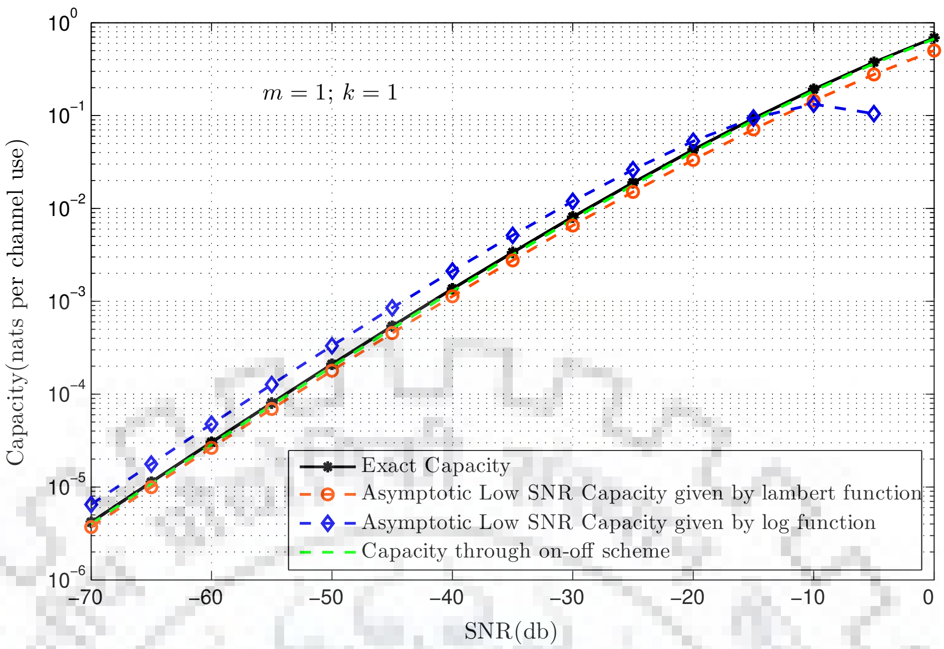

Figure 4.1: (Simulated) Low SNR capacity in nats per channel use with respect to SNR(db) for $m = 1$ and $k = 1$

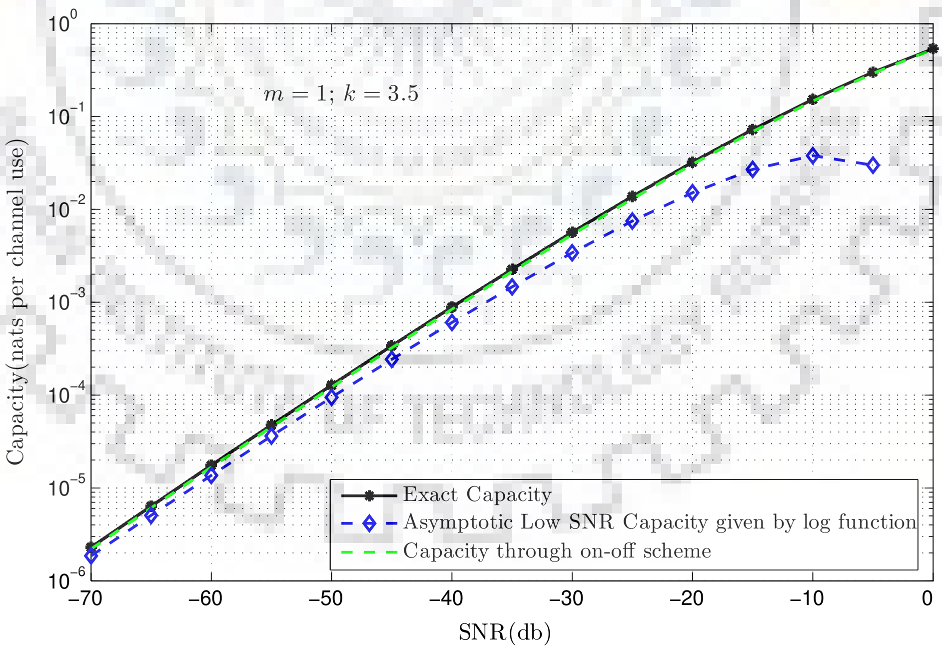

Figure 4.2: (Simulated) Low SNR capacity in nats per channel use with respect to SNR(db) for $m = 1$ and $k = 3.5$

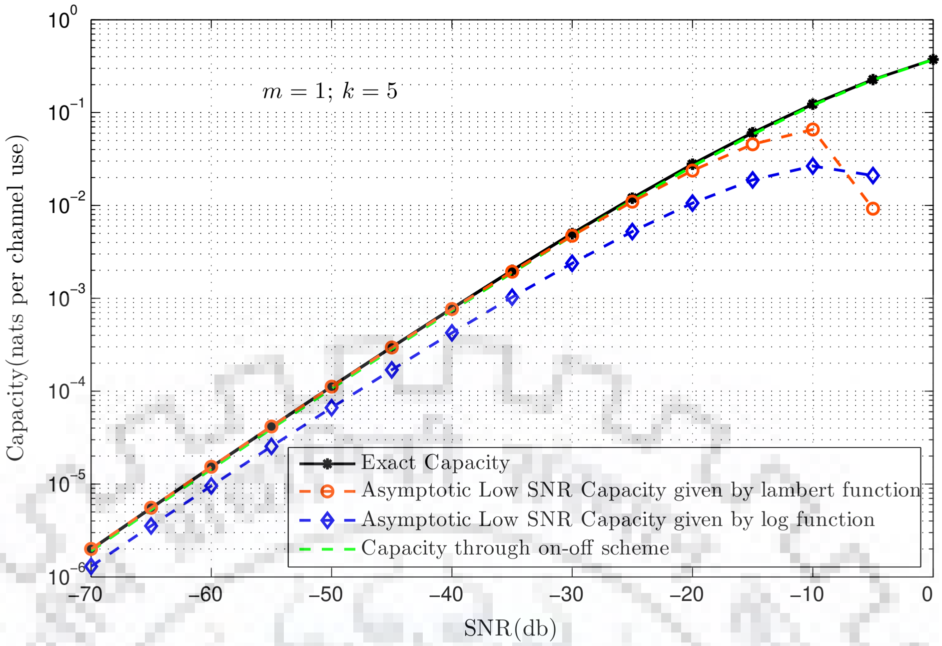

Figure 4.3: (Simulated) Low SNR capacity in nats per channel use with respect to SNR(db) for $m = 1$ and $k = 5$

Capacity results for different values of m and k are shown in Fig. 4.2 and Fig. 4.3. We can see from figures that all our asymptotic results for the capacity matches the exact capacity curves. For the case of $m = 1$ and $k = 3.5$ we can see that the capacity is only given by log function. We can see that for the case of $m = 1$ and $k = 5$ log function is lower bound on lambert function. We also note that capacity results given by lambert function is very close as compared to the exact capacity curve. We also note that in both figures, capacity through on-off scheme coincides with the exact capacity curve which indicate that on-off scheme can be practically implemented. We can also note that capacity decreases as k increases for same value of m.

The capacity for some other values of m and k are also shown in Fig. 4.4 and Fig. 4.5. In Fig. 4.4, it can be noted that log function is upper bounding the lambert function and capacity through log function is close to the exact capacity, whereas in Fig. 4.5 lambert function is upper bounding the log function and capacity through lambert function is close to the exact capacity. The capacity through on-off scheme is also shown and it almost matches the exact capacity curves for all SNR values.

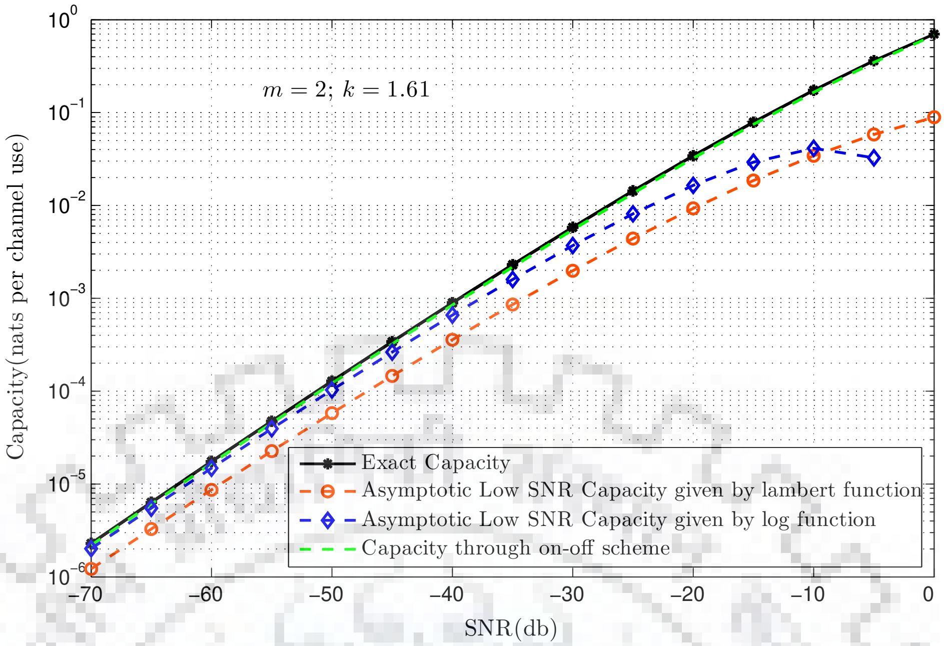

Figure 4.4: (Simulated) Low SNR capacity in nats per channel use with respect to SNR(db) for $m = 2$ and $k = 1.61$

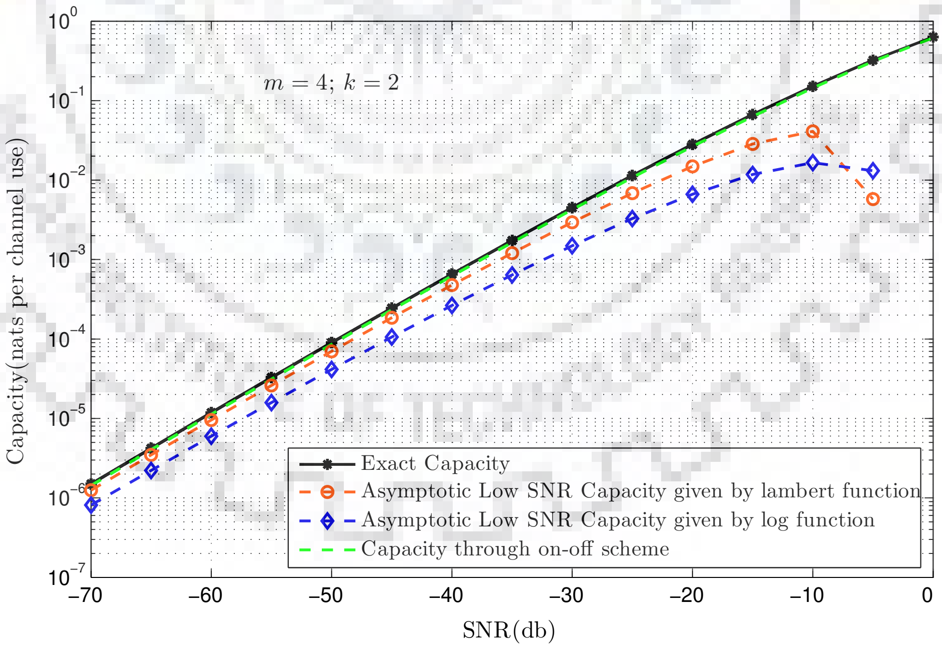

Figure 4.5: (Simulated) Low SNR capacity in nats per channel use with respect to SNR(db) for $m = 4$ and $k = 2$

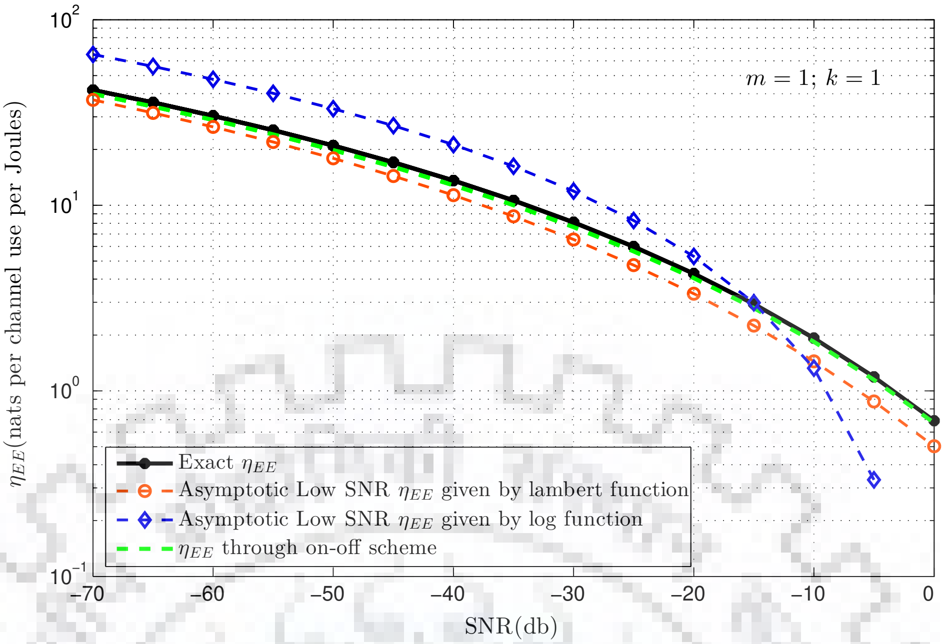

Figure 4.6: (Simulated) Low SNR Energy efficiency in nats per channel use per joules with respect to SNR(db) for $m = 1$ and $k = 1$

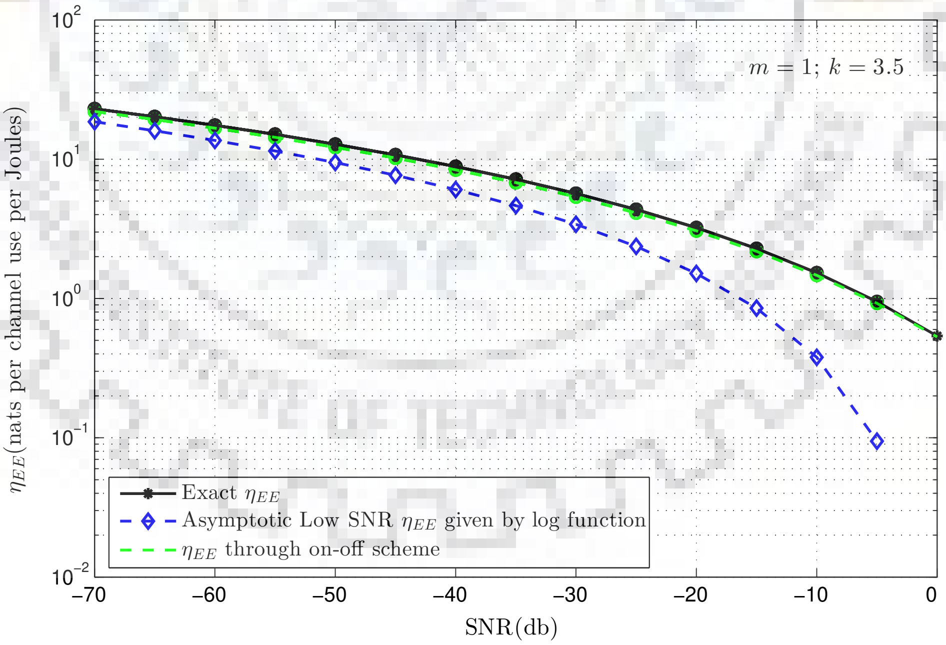

Figure 4.7: (Simulated) Low SNR Energy efficiency in nats per channel use per joules with respect to SNR(db) for $m = 1$ and $k = 3.5$

Now, we shown another important parameter Energy efficiency that characterize the performance of generalized-K fading channel. Energy efficiency in nats per channel use per joules with respect to SNR in db is shown in Fig. 4.6 and Fig. 4.7 for $m = 1$and $k = 1$ and for $m = 1$ and $k = 3.5$ respectively. The exact energy efficiency is shown which is calculated using (3.37) where we substitute exact capacity obtained from numerical integration. We have also plotted asymptotic energy eficiency given by lambert and log function. Clearly we can see that asymptotic results are very closed to the exact one. Efficiency through on-off scheme is also shown which is very close to the exact energy efficiency. We can also notes that energy efficiency of generalized-K fading channel at low SNR decreases with increase in k for constant m.

Chapter 5

CONCLUSION AND FUTURE WORK

We have studied modelling of fading and shadowing in which we studied different fading and shadowing phenomenon. We then introduced Log normal shadowing, composite multipath fading shadowing environment which consists of multipath fading superimposed in shadowing, then we presented various composite distributions which are obtained from combinations of multipath fading and shadowing distributions such as Rayleigh-lognormal distribution, K distribution, Rician-lognormal distribution, Nakagami-lognormal distribution, generalized-K distribution.

We have also characterized the erdogic capacity of shadowing fading channel particularly generalized-K fading channel in nats per channel use with perfect channel state information at both transmitter and receiver. We have provided Asymptotic Low SNR capacity expression Our Asymptotic Low SNR capacity follows the exact result obtained by simulation. We have also seen that the Low SNR capacity results are also obtained through an on-off scheme. We have also characterized two performance metrics, energy efficiency and minimum energy per nat respectively.

We have compared exact capacity with our asymptotic Low SNR capacity results. The exact capacity curve follows the asymptotic capacity curves. We have plotted the erdogic capacity for generalized-K fading channel for different values of shaping parameters m and k. The capacity through on-off scheme is also plotted and we found that this rate matches the exact capacity curve for all SNR values. We have also plotted another important performance metrics named Energy efficiency that characterize the performance of generalized-K fading channel where we have see that asymptotic capacity results are very close to the exact one. Energy efficiency through on-off scheme had also been shown which is very close to the exact energy efficiency.

Analysis of capacity of Generalized-K fading channel with different estimated channels at transmitter and the receiver might be the object of further research work.

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APPENDIX A

Proof of equation (3.15)

Expanding incomplete gamma functions in (3.13) using the definition in (3.14), we have

$$= \frac{1}{\lambda}\left(\beta-\frac{1}{2}\right)!\; e^{-2\left(\frac{m}{\Omega}\lambda\right)^{\frac{1}{2}}}\left[1+\frac{2\left(\frac{m}{\Omega}\lambda\right)^{\frac{1}{2}}}{1!}+\frac{\left(2\left(\frac{m}{\Omega}\lambda\right)^{\frac{1}{2}}\right)^{2}}{2!}+ \cdots+\frac{\left(2\left(\frac{m}{\Omega}\lambda\right)^{\frac{1}{2}}\right)^{(\beta-\frac{3}{2})}}{(\beta-\frac{3}{2})!}+\frac{\left(2\left(\frac{m}{\Omega}\lambda\right)^{\frac{1}{2}}\right)^{(\beta-\frac{1}{2})}}{(\beta-\frac{1}{2})!}\right]$$

$$-\,2^2\left(\frac{m}{\Omega}\right)\left(\beta-\frac{5}{2}\right)!\; e^{-2\left(\frac{m}{\Omega}\lambda\right)^{\frac{1}{2}}}\left[1+\frac{2\left(\frac{m}{\Omega}\lambda\right)^{\frac{1}{2}}}{1!}+\frac{\left(2\left(\frac{m}{\Omega}\lambda\right)^{\frac{1}{2}}\right)^{2}}{2!}+ \cdots+\frac{\left(2\left(\frac{m}{\Omega}\lambda\right)^{\frac{1}{2}}\right)^{(\beta-\frac{7}{2})}}{(\beta-\frac{7}{2})!}+\frac{\left(2\left(\frac{m}{\Omega}\lambda\right)^{\frac{1}{2}}\right)^{(\beta-\frac{5}{2})}}{(\beta-\frac{5}{2})!}\right]$$

$$= \left(\beta-\frac{1}{2}\right)!\; e^{-2\left(\frac{m}{\Omega}\lambda\right)^{\frac{1}{2}}}\left[\lambda^{-1}+\frac{2\left(\frac{m}{\Omega}\lambda\right)^{\frac{1}{2}}\lambda^{-1}}{1!}+\frac{\left(2\left(\frac{m}{\Omega}\lambda\right)^{\frac{1}{2}}\right)^{2}\lambda^{-1}}{2!}+ \cdots+\frac{\left(2\left(\frac{m}{\Omega}\lambda\right)^{\frac{1}{2}}\right)^{(\beta-\frac{3}{2})}\lambda^{-1}}{(\beta-\frac{3}{2})!}+\frac{\left(2\left(\frac{m}{\Omega}\lambda\right)^{\frac{1}{2}}\right)^{(\beta-\frac{1}{2})}\lambda^{-1}}{(\beta-\frac{1}{2})!}\right]$$

$$-\left(\beta-\frac{5}{2}\right)!\; e^{-2\left(\frac{m}{\Omega}\lambda\right)^{\frac{1}{2}}}\left[2^2\left(\frac{m}{\Omega}\right)+\frac{2\left(\frac{m}{\Omega}\lambda\right)^{\frac{1}{2}}2^2\left(\frac{m}{\Omega}\right)}{1!}+\frac{\left(2\left(\frac{m}{\Omega}\lambda\right)^{\frac{1}{2}}\right)^{2}2^2\left(\frac{m}{\Omega}\right)}{2!}+ \cdots+\frac{\left(2\left(\frac{m}{\Omega}\lambda\right)^{\frac{1}{2}}\right)^{(\beta-\frac{7}{2})}2^2\left(\frac{m}{\Omega}\right)}{(\beta-\frac{7}{2})!}+\frac{\left(2\left(\frac{m}{\Omega}\lambda\right)^{\frac{1}{2}}\right)^{(\beta-\frac{5}{2})}2^2\left(\frac{m}{\Omega}\right)}{(\beta-\frac{5}{2})!}\right]$$

$$= \left(\beta - \frac{1}{2}\right)\left(\beta - \frac{3}{2}\right)\left(\beta - \frac{5}{2}\right)!\, e^{-2\left(\frac{m}{\Omega}\lambda\right)^{\frac{1}{2}}} \left[\lambda^{-1} + \frac{2\left(\frac{m}{\Omega}\lambda\right)^{\frac{1}{2}}\lambda^{-1}}{1!} + \frac{\left(2\left(\frac{m}{\Omega}\lambda\right)^{\frac{1}{2}}\right)^{2}\lambda^{-1}}{2!} + \right.$$

$$\left. \cdots + \frac{\left(2\left(\frac{m}{\Omega}\lambda\right)^{\frac{1}{2}}\right)^{(\beta-\frac{3}{2})}\lambda^{-1}}{(\beta-\frac{3}{2})!} - \frac{\left(2\left(\frac{m}{\Omega}\lambda\right)^{\frac{1}{2}}\right)^{(\beta-\frac{1}{2})}\lambda^{-1}}{(\beta-\frac{1}{2})!}\right]$$

$$- \left(\beta - \frac{5}{2}\right)!\, e^{-2\left(\frac{m}{\Omega}\lambda\right)^{\frac{1}{2}}} \left[2^2\left(\frac{m}{\Omega}\right) + \frac{2\left(\frac{m}{\Omega}\lambda\right)^{\frac{1}{2}} 2^2\left(\frac{m}{\Omega}\right)}{1!} + \frac{\left(2\left(\frac{m}{\Omega}\lambda\right)^{\frac{1}{2}}\right)^{2} 2^2\left(\frac{m}{\Omega}\right)}{2!} + \right.$$

$$\left. \cdots + \frac{\left(2\left(\frac{m}{\Omega}\lambda\right)^{\frac{1}{2}}\right)^{(\beta-\frac{7}{2})} 2^2\left(\frac{m}{\Omega}\right)}{(\beta-\frac{7}{2})!} \frac{\left(2\left(\frac{m}{\Omega}\lambda\right)^{\frac{1}{2}}\right)^{(\beta-\frac{5}{2})} 2^2\left(\frac{m}{\Omega}\right)}{(\beta-\frac{5}{2})!}\right]$$

$$= \left(\beta - \frac{5}{2}\right)! e^{-2\left(\frac{m}{\Omega}\lambda\right)^{\frac{1}{2}}} \left[\left(\beta - \frac{1}{2}\right)\left(\beta - \frac{3}{2}\right)\lambda^{-1} + \frac{\left(\beta-\frac{1}{2}\right)\left(\beta-\frac{3}{2}\right)2\left(\frac{m}{\Omega}\right)^{\frac{1}{2}}\lambda^{-\frac{1}{2}}}{2!} + \right.$$

$$\left. \cdots + \left(\beta - \frac{1}{2}\right)\left(\beta - \frac{3}{2}\right)\frac{\left(2\left(\frac{m}{\Omega}\lambda\right)^{\frac{1}{2}}\right)^{(\beta-\frac{3}{2})}\lambda^{-1}}{(\beta-\frac{3}{2})!} - \frac{\left(2\left(\frac{m}{\Omega}\lambda\right)^{\frac{1}{2}}\right)^{(\beta-\frac{7}{2})} 2^2\left(\frac{m}{\Omega}\right)}{(\beta-\frac{7}{2})!}\right]$$

$$= \left(\beta - \frac{5}{2}\right)! e^{-2\left(\frac{m}{\Omega}\lambda\right)^{\frac{1}{2}}} \left[\left(\beta - \frac{1}{2}\right)\left(\beta - \frac{3}{2}\right)\lambda^{-1} + \frac{\left(\beta-\frac{1}{2}\right)\left(\beta-\frac{3}{2}\right)2\left(\frac{m}{\Omega}\right)^{\frac{1}{2}}\lambda^{-\frac{1}{2}}}{2!} + \right.$$

$$\left. \cdots + \frac{\left(2\left(\frac{m}{\Omega}\lambda\right)^{\frac{1}{2}}\right)^{(\beta-\frac{3}{2})}\lambda^{-1}}{(\beta-\frac{7}{2})!}\left(\frac{2}{(\beta-\frac{5}{2})}\right)\right]$$

Now, using the fact that $\lambda^u >> \lambda^v \quad \forall u$ and $v \in \mathbb{N}$ but $u > v$, which follows from (3.10), above becomes

$$\approx \left(\beta - \frac{5}{2}\right)!\, e^{-2\left(\frac{m}{\Omega}\lambda\right)^{\frac{1}{2}}} \frac{\left(2\left(\frac{m}{\Omega}\lambda\right)^{\frac{1}{2}}\right)^{(\beta-\frac{3}{2})}\lambda^{-1}}{(\beta-\frac{7}{2})!}\left[\frac{2}{(\beta-\frac{5}{2})}\right]$$

$$\approx e^{-2\left(\frac{m}{\Omega}\lambda\right)^{\frac{1}{2}}} 2^{\beta-\frac{1}{2}} \left(\frac{m}{\Omega}\right)^{\frac{\beta}{2}-\frac{3}{4}} \lambda^{\frac{\beta}{2}-\frac{7}{4}}$$

APPENDIX B

Proof of equation (3.36)

Consider, $\text{prob}(|h|^2 \geq \lambda) \ = \int\limits_{\lambda}^{\infty} f_{|h|}(y)\text{dy}$

substituting the density function of y given in (3.7) , probability becomes

$$= \frac{\sqrt{\pi}}{\Gamma(m)\Gamma(k)} \left(\frac{1}{2}\right)^{\beta-\frac{1}{2}} \Gamma\left[\frac{\beta+1}{2}, 2\left(\frac{m}{\Omega}\lambda\right)^{\frac{1}{2}}\right]$$

now using the definition of incomplete gamma function in (3.14)

$$\frac{\sqrt{\pi}}{\Gamma(m)\Gamma(k)} \left(\frac{1}{2}\right)^{\beta-\frac{1}{2}} \left(\beta-\frac{1}{2}\right)! \, e^{-2\left(\frac{m}{\Omega}\lambda\right)^{\frac{1}{2}}} \left[1+\frac{2\left(\frac{m}{\Omega}\lambda\right)^{\frac{1}{2}}}{1!}+\frac{\left(2\left(\frac{m}{\Omega}\lambda\right)^{\frac{1}{2}}\right)^{2}}{2!}+ \cdots + \frac{\left(2\left(\frac{m}{\Omega}\lambda\right)^{\frac{1}{2}}\right)^{(\beta-\frac{3}{2})}}{(\beta-\frac{3}{2})!}+\frac{\left(2\left(\frac{m}{\Omega}\lambda\right)^{\frac{1}{2}}\right)^{(\beta-\frac{1}{2})}}{(\beta-\frac{1}{2})!}\right]$$

Now, using the fact that $\lambda^u >> \lambda^v \quad \forall u$ and $v \in \mathbb{N}$ but $u > v$, which follows from (3.10), above becomes

$$\approx \frac{\sqrt{\pi}}{\Gamma(m)\Gamma(k)} \left(\frac{1}{2}\right)^{\beta-\frac{1}{2}} e^{-2\left(\frac{m}{\Omega}\lambda\right)^{\frac{1}{2}}} \left(2\left(\frac{m}{\Omega}\lambda\right)^{\frac{1}{2}}\right)^{\beta-\frac{1}{2}}$$

$$\approx \frac{\sqrt{\pi}}{\Gamma(m)\Gamma(k)} e^{-2\left(\frac{m}{\Omega}\lambda\right)^{\frac{1}{2}}} \left(\left(\frac{m}{\Omega}\lambda\right)^{\frac{1}{2}}\right)^{\beta-\frac{1}{2}}$$