APPROXIMATE CONTROLLABILITY OF INFINITE DIMENSIONAL SEMILINEAR CONTROL SYSTEMS

Ph. D. THESIS

by

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STUDENT'S DECLARATION

I hereby certify that the work presented in the thesis "**APPROXIMATE CONTROLLABILITY OF INFINITE DIMENSIONAL SEMILINEAR CONTROL SYSTEMS**" is my own work carried out during a period from July, 2015 to December, 2020 under the supervision of Dr. N. Sukavanam, Professor, Department of Mathematics, Indian Institute of Technology Roorkee, Roorkee.

The matter presented in this thesis has not been submitted for the award of any other degree of this or any other Institution.

Abdul Haq

Dated: December 16, 2020 **(Abdul Haq)**

SUPERVISOR'S DECLARATION

This is to certify that the above mentioned work is carried out under my supervision.

N. Sukavanam
21/12/2020

Dated: December 16, 2020 **(N. Sukavanam)**

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Dedicated to

My Parents & My Late Sister Shama Parveen

Abstract

The present research work deals with the existence of solutions and approximate controllability of deterministic semilinear integer order systems with control delays and fractional order systems without delay. To derive the existence and controllability results, various techniques have been applied along with the semigroup, cosine and sine families, fractional calculus, fractional cosine family, fractional resolvent, fixed point theory. Some examples are provided for the illustration of the obtained results.

Some introductory matter along with literature survey on controllability of nonlinear and linear control systems of fractional and integer orders are given in Chapter 1. Basic concepts and definitions of control theory, semigroup theory, cosine family, fractional calculus, fractional cosine family and nonlinear functional analysis which are utilized in forthcoming chapters, are given in Chapter 2.

In Chapter 3, the existence of mild solutions of first-order retarded semilinear system with control delay is proved under the locally Lipschitz continuity of nonlinear function and a fixed point theorem. Then the approximate controllability of semilinear system is proved provided that the associated linear system without delay is approximately controllable. Controllability results are obtained by using the method of steps and semigroup theory. The results of this chapter are illustrated with controlled heat equation.

Chapter 4 contains two sections. The first section deals with the approximate and exact controllability of second-order nonlocal retarded semilinear system with control delay. In this section the existence of solutions is derived applying fixed point approach and cosine family. Here the nonlinear function is supposed to be Lipschitz continuous. The controllability of the associated linear delay system is proved by the method of steps and then the controllability of the actual system is

deduced by proving that the reachable set of semilinear system contains the reachable set of associated linear system. The results of this section are illustrated with controlled wave equations. In the second section, the approximate controllability of second-order nonlocal retarded semilinear system with multiple delays in control is discussed in Banach spaces. The existence of solutions is derived applying fixed point approach. For this, nonlinear function is supposed to be locally Lipschitz. Then the approximate controllability of associated linear system and actual system is proved applying the technique similar to Chapter 3. Here, the problem of first section is extended for multiple time delays. The obtained results are illustrated by providing an example.

In Chapter 5, the partial approximate controllability of nonlocal Riemann-Liouville fractional systems with integral initial conditions in Hilbert spaces without assuming the Lipschitz continuity of nonlinear function is investigated. We also exclude the usual assumptions on nonlocal functions such as Lipschitz continuity and compactness. First, the existence results are derived applying Schauder's fixed point theorem and then the partial approximate controllability result is proved. For this, we suppose that the associated linear system is partial approximately controllable for $\varphi = 0$, where φ is nonlocal function. To obtain our result the concept of semigroup is used rather than resolvent operator. Here, we assume that the semigroup generated by linear map is compact. Lastly, an example is given to apply the obtained results.

In Chapter 6, we analyze the approximate controllability of Riemann-Liouville fractional systems with integral initial conditions in Banach space. First we deduce the existence of mild solutions using fractional Riemann- Liouville family and fixed point approach by assuming the Lipschitz continuity of nonlinear term. Then we determine the approximate controllability of the system. We make use of iterative and approximate technique to obtain the controllability result. The obtained results are illustrated by providing an example.

Chapter 7 is concerned with the Riemann-Liouville fractional semilinear integrodifferential systems with damping in Banach spaces. First we prove the existence of solutions by applying fixed point approach. Then the approximate controllability of the system is shown by applying an approximate method. To obtain our results, we use the concept of Riemann-Liouville fractional $(\vartheta, \varphi, \lambda)$ resolvent, where $0 < \varphi < \vartheta \leq 1$ and λ is a real number. Finally, the obtained results are illustrated by providing an example.

The final concluding remarks about the work presented in the thesis, and brief discussion on the future work, are given in Chapter 8.

Keywords: Delay systems; Fractional systems; Integrodifferential systems; Riemann-Liouville derivatives; Damping; Semigroup; Cosine Family; Fractional cosine family; Fractional resolvent; Contraction map; Fixed point; Approximate method; Iterative technique; Mild solutions; Reachable set; Approximate controllability

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Complete List of Publications

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3. *"International conference on recent advances in pure and applied mathematics"*, DTU Delhi, India, October 23-25, 2018.

4. *"International conference on differential equations and control problems: modeling, analysis and computations"*, IIT Mandi, Mandi, India, June 17-19, 2019.

Chapter 1

Introduction

1.1 General introduction

1.1.1 Origin of control theory

Control theory is an interdisciplinary branch of mathematics and engineering that deals with the influence behavior of dynamical systems. Here, a system is defined as an arrangement, collection or set of entities which are related by interactions and produce various outputs in response to different inputs. If a system changes with respect to time or other variable then it is known as a dynamical system. For example, electromechanical machines such as motor car, aircraft or spaceships, biological systems such as human body, economic structures of countries or regions, population growth in a region are dynamical systems. If a dynamical system is controlled by suitable inputs or controls to obtain the desired output or state then it is known as a control system.

In real life, there are many control systems which are in use. For example, our body temperature and blood sugar level needs to be controlled at desired set points, insect and animal populations are controlled by very delicately balanced prey predator relationship. These control systems are provided to us by nature. There are many complex man-made as well as simple control systems which are used in our everyday life. Automatic water heater, washing machine, missiles, etc. are some examples. However, whether a control system is natural or man-made, those all share a common aim, which is to control or regulate a particular variable within certain operating limits.

Controllability is an important area in the study of dynamical systems. It plays a crucial role in control problems such as stabilization of unstable systems by feedback control or in the study of optimal control. For this reason, it has been studied by several authors during the past few decades. Controllability is a mathematical problem, which analyzes the possibility of steering a system from any initial state to any final state utilizing a set of admissible controls. During the last two hundred years, the classical areas of applied mathematics such as thermodynamics, electromagnetic theory, mechanics of fluids, solids and particles etc., have been well developed and generally reflect this emphasis on analysis too.

1.1.2 Controllability

Let $Z = L_2([0,c];V)$ and $U = L_2([0,c];V')$ be the function spaces. Consider the semilinear system

$$\begin{cases} \frac{dz}{dt} = \dot{z}(t) = Az(t) + Bu(t) + F(t, z(t)), \quad t \in (0, c], \\ z(0) = y_0 \in V. \end{cases} \tag{1.1.1}$$

Here, $z(t) \in V$ and $u(t) \in V'$ are the state and control, respectively, of the system where V and V' are Banach spaces. $B : V' \to V$ is a continuous linear map and $F : [0,c] \times V \to V$ is nonlinear. Here, we suppose that $A : D(A) \subseteq V \to V$ is a closed and densely defined linear map with domain $D(A)$, and it generates a C_0-semigroup $\{T(t)\}_{t \in \mathbb{R}_0^+}$ defined on V (see Chapter 2).

The mild solution of the system (1.1.1) is defined as the solution of the integral equation

$$z(t) = T(t)y_0 + \int_0^t T(t-s)\big(Bu(s) + F(s, z(s))\big)\, ds, \quad 0 \le t \le c.$$

Denote by $z(t; y_0, u)$ the mild solution of the system (1.1.1), corresponding to control $u \in U$ and the initial condition y_0.

The system (1.1.1) is called exact controllable in the time interval $[0,c]$, if for any given final state z_c in V, one can find a control $u \in U$ such that the mild solution $z(t; y_0, u)$ of the system (1.1.1) satisfies $z(c; y_0, u) = z_c$.

The system (1.1.1) is called approximate controllable in the time interval $[0,c]$, if for given final state z_c in V and $\varepsilon > 0$, one can find a control $u \in U$ such that the mild solution of (1.1.1) satisfies $\|z(c; y_0, u) - z_c\| < \varepsilon$.

It should be noted that approximate controllability empowers to steer the system to any given neighborhood of any final state but exact controllability means that system can be steered to any given final state. Obviously, the exact controllability is necessarily stronger notion than approximate controllability. Approximate controllability allows to steer the system to states belonging to a dense subset of the state space. Therefore it is interesting to discuss the approximate controllability of a system.

In this thesis, some results on the approximate controllability of semilinear delay control systems of integer order and non-delay control system of fractional order are established.

1.1.3 Motivation of the thesis

Many scientific and engineering problems can be modeled by deterministic and non-deterministic partial differential equations, fractional order differential equations or coupled ordinary and partial differential equations with or without delay in finite or infinite dimensional spaces using semigroup and cosine family. The systems arising in practice are mainly nonlinear to some extent. There are various properties of the system such as existence, uniqueness and regularity of the solutions, stability of equilibrium points, etc. Controllability is also an important area of study in control theory. In many applications, the objective of the control action is to drive the system from one state to another in an optimal fashion. However, before we formulate the question of optimality it is necessary to pose the more fundamental question of whether or not it is possible to reach a desired state from an initial state. So, this gives the motivation to analyze the controllability of dynamical systems of an abstract form.

In many problems, the present rate of change of some unknown function depends upon previous values of the same function. Such problems are modeled by the time delay systems. A system may experience time delay either in control or in state or in both. Some of the examples of physical and biological systems which involves time delays are population growth, a system involving feedback control, mixing of liquids, prey-predator population models etc. Thus the delay of a system constitutes a crucial part of research area in the theory of control.

Retarded systems are the systems involving retarded arguments. In the problems having retarded systems, there has been an expanding interest for several decades. Many natural incidents embrace a significant memory effect. Retarded systems express the mathematical model of such real life problems. For example, many problems occurring from medicine, bio sciences, chemical sciences, physical sciences, economics are affected by their previous results at major scale. Therefore for the study of controllability, it becomes much important to select retarded systems.

It is also seen that in various engineering models, such as semiconductor modeling, heat conduction, nonlocal reactive transport in underground water flows in porous media and biotechnology, mathematical formulation of nonlocal problems arise naturally. The quantity of physical phenomena displayed by partial differential equations with nonlocal conditions which have abstract formulation as a functional differential equation is continually expanding.

The fractional differential systems have drawn the attention of many engineers, physicists and mathematician in last twenty years. Because these systems poured many applications in the areas of economics, engineering and science. The differential systems of fractional order have the capability to describe the memory and hereditary properties of some significant materials and processes. The theory of fractional calculus is the generalization of the theory of integration and differentiation of integer order to arbitrary order (termed as fractional differentiations and integrations). Particularly, this discipline has the concept and methods of solving the differential systems of fractional order. It has been realized that the fractional order differential operator is nonlocal but the integer order differential operator is a local operator.

A detailed review of literature on controllability of various systems is given in the next section.

1.2 Review of literature

1.2.1 Controllability of first-order systems

Theory of controllability was originated from the great work done by Kalman [41] in 1960. In which, he discussed the controllability of linear system of finite dimensional under a rank condition of the controllability matrix (see [10]). In 1967, Tarnove [84] used the fixed point theorem due to Bohnenblust-Karlin to analyze A-controllability for a nonlinear system, where A is a non-empty, closed, bounded convex subset of the set of continuous functions. Subsequently, this idea was utilized by Dauer [20] for the controllability of the systems of the form $\dot{z}(t) = F(t, z) + G(t, u)$ in finite dimensional spaces. Joshi and George [39] analyzed the controllability property of the nonlinear systems (non-autonomous) in finite dimensional spaces with the supposition that its linear part is controllable. For this, they reduced the controllability problem to the solvability of an operator equation. The solvability analysis of the operator system was carried out by applying fixed point and monotone operator theory. In 2009, Klamka [48] extended the results of [44] for the systems with control delays.

A finite dimensional system is usually an approximation of some infinite dimensional system. Therefore motivated by work mentioned above, many authors extended the results to more general cases including infinite dimensional systems.

A more general model for the system was considered by Fattorini [25] in 1966. In this work, the controllability property was investigated for the case when the linear map A is assumed to be closed and densely defined, and it generates a C_0-semigroup $T(t)$. In 1967, he determined the approximate controllability for the case when A is self adjoint, semibounded above and defined on a Hilbert space and the dynamical system has only a finite number of scalar controls [26].

Controllability results were developed by Carmichael and Quinn [14] for the nonlinear control system in an infinite dimensional setting. They formulated the controllability problem as a fixed point problem and used Nussbaum fixed-point theorem to establish conditions under which the nonlinear control system is exact controllable from the origin to some open set contained in an appropriate function space. In 1983, Zhou [90] gave new sufficient conditions for the approximate controllability of the semilinear control systems. The results were obtained for the case when the range of the control operator satisfies an inequality condition. In 1984, he introduced some general conditions for exact reachability and approximate controllability connected with two families of associated quadratic optimal control problems [91].

Making use of Schauder's degree theorem, Naito [71] studied the approximate controllability for semilinear systems of the form $\dot{z}(t) + Az(t) = (Bu)(t) + F(z(t))$ with initial condition $z(0) = 0$ under the uniform boundedness of the nonlinear operator and a range condition on control operator. In [70], he dropped the uniform boundedness condition on nonlinear operator and used an inequality condition to show that the semilinear system is approximate controllable.

In 1998, Bian [11] derived some results on approximate controllability for semilinear systems. In 1999, Jeong et al. [35] extended the results of [11] and discuss the controllability results for semilinear systems with infinite delay. In 1988, Lions [59] investigated the exact controllability of distributed systems. Here, the control was assumed to be a boundary control or a local distributed control. In 1995, George [28] proved the approximate controllability of the non-autonomous semilinear systems under different assumptions on the linear and nonlinear operators. The controllability of impulsive systems was also proved by George et al. [29] in 2000.

In 2002, utilizing fixed point theorem and semigroup, Balachandran and Dauer [3] presented a survey on the controllability of nonlinear systems and functional integrodifferential systems in Banach spaces. In 2002, Dauer and Mahmudov [21] investigated the approximate controllability for the semilinear delay functional differential systems applying Schauder's fixed point theorem. For this, the corresponding semigroup $T(t)$ is assumed to be compact. Sufficient conditions for the exact controllability of the semilinear functional differential systems were also derived when the semigroup is not compact. These conditions were obtained by using the Banach fixed point theorem. By assuming same conditions they also proved the approximate and exact controllability of semilinear systems without delay in 2004 [22].

In 2005, Joshi and Kumar [40] investigated the approximation of exact controllability problem involving parabolic differential equations. In [36], Jeong et al.

analyzed the controllability of nonlinear and linear systems by assuming that the system of generalized eigenspaces of A is complete. In 2007, they investigated the controllability of nonlinear retarded control systems [34]. Further Lipschitz continuity and the uniform boundedness of the nonlinear term have been considerably weakened. Ntouyas and Regan [72] proved some controllability results for semilinear neutral functional differential inclusions with finite and infinite delays in Banach spaces by replacing the compactness of operators with the complete continuity of the nonlinearity.

Utilizing fixed point approach, the controllability property of nonlocal semilinear evolution equation was investigated by Mahmudov [63] in 2008. In 2009, Wang [87] obtained some sufficient conditions for approximate controllability of integrodifferential equations with multiple delays using Schauder's fixed point theorem. In 2012, Liu [60] extended the results of [87] and discussed the controllability of time varying system with multiple delays and impulsive effects.

In 2013, Kang et al. [43] considered the nonlinear evolution equation and studied its controllability property. Here, they assumed that the nonlinear map verifies the monotone condition. Kumar and Sukavanam [55] considered the nonlinear system with delays in control and state and studied its controllability using Lipschitz continuity. In 2015, Shukla et al. [80] determined the approximate controllability of nonlinear systems with state delay by assuming the Lipschitz condition on nonlinearity term. Here, they utilized fundamental solution rather than semigroup.

1.2.2 Controllability of second-order systems

In the beginning, the controllability of second-order systems were analyzed by converting them into first-order systems. Later, it has been observed that the study of a second-order system by converting them into first-order system need not give desired results due to the behaviour of the semigroup generated by the linear part of the transformed system. Therefore, it is more effective to study a second-order system directly.

To discuss the differential systems of second-order as it is rather than converting them into first-order systems, the theory of cosine family is a useful tool which was introduced by Travis and Webb [85] in 1977. They studied the uniform continuity, regularity and some other properties of cosine families. In [86], they obtained the solution of nonlinear second-order differential equations, using the theory of cosine family.

In 1997, Park and Han [73] proved the controllability of second-order nonlinear systems in Banach spaces by assuming the uniform boundedness of nonlinearity term. Utilizing Schauder fixed point theorem, approximate controllability of integrodifferential systems of second-order is investigated in [4] and [74]. In 2003, Mckibben [67] studied approximate controllability for second-order functional evolution equation with the help of properties of cosine family and sequential approach proposed by Zhou [90] in 1983.

Approximate controllability for neutral equations of second-order was studied by Mahmudov et al. [62] in 2006. In [8] Balasubramaniam et al. studied approximate controllability for distributed implicit functional control systems of second-order with unbounded delay. Without assuming the compactness of the cosine family, Sakthivel et al. [76] investigated the complete controllability for nonlinear impulsive control systems of second-order. They established controllability results by utilizing the fixed point approach.

Kowalski and Sadkowski [50] presented some properties of cosine family. Using the cosine family, they established the existence of mild solution of the abstract second-order cauchy problem and gave some equivalent conditions for exact controllability, null-exact controllability and approximate controllability. Finally, they considered the mixed wave problem in the space $L_2[0,1]$ and proved that it is approximately controllable. Utilizing Sadovskii fixed point theorem Kumar et al. [54] obtained controllability results for second-order nonlocal differential system in 2014. Here, they dropped the compactness of nonlinear map and cosine family.

In 2015, a numerical and an analytical estimation for the trajectory controllability of integro-differential systems of second-order was given by Chalishajar D. and Chalishajar H. [16]. In 2016, Mahmudov et al. [66] investigated the controllability results for the evolution differential inclusions of second-order in Hilbert spaces. A survey on controllability of differential systems of second-order was presented by Klamka et al. [49] in 2017. In 2018, Kumar et al. [52] determined the controllability of impulsive semilinear systems of second-order. They also discuss the case of nonlocal initial conditions.

1.2.3 Controllability of fractional order systems

Fractional differential equations are found to be appropriate models in many engineering, biological and physical problems. For this reason, they have attracted much attention in last two decades. In fact, fractional order derivatives confer a better instrument for the illustration of hereditary and memory properties. Therefore in the modeling of systems and processes, they have poured many applications in the areas of electricity mechanics, heat conduction, electrodynamics of complex medium, physics, viscoelasticity, aerodynamics, control theory etc.

Utilizing a fixed point theorem with fractional calculus, Balachandran and Park [5] investigated the controllability of fractional semilinear integrodifferntial system with nonlocal condition. In 2011, Sukavanam et al. [81] studied the approximate controllability of fractional semilinear delay systems utilizing Gronwall's inequality and basics of fractional calculus. A set of sufficient conditions for the controllability of fractional order nonlinear dynamical systems through Schauder's fixed point theorem was established by Balachandran et al. [6].

In 2012, Kumar et al. [53] investigated the approximate controllability of fractional order semilinear systems with bounded delay making use of Schauder's fixed point theorem. In 2012, Tai and Lun [83] proved controllability of fractional impulsive neutral evolution integrodifferential systems. Sufficient conditions for the controllability were established by applying fractional calculus, resolvent operators and Krasnoselskii's fixed point theorem.

In 2013, Kamaljeet and Bahuguna [42] discussed the controllability of impulsive differential equations of fractional order with finite delay and nonlocal conditions. In 2014, Souad and Toufik [24] considered the neutral evolution equations of fractional order with Caputo derivatives and nonlocal conditions, and obtained the approximate controllability results assuming the compactness of the semigroup generated by linear operator.

It is well known that the integrals initial conditions or Riemann-Liouville derivatives play a crucial role in many practical problems. Such initial conditions are more appropriate than other physically interpretable initial conditions. In [31], Heymans and Podlubny have given the physical interpretation of initial conditions expressed in terms of integrals or Riemann-Liouville derivatives of fractional order in the area of viscoelasticity.

In 2015, Liu and Li [61] developed approximate controllability results for a class of Riemann-Liouville fractional equations of the form

$$\begin{cases} D_t^{\vartheta} z(t) = Az(t) + F(t, z(t)) + (Bu)(t), \quad t \in (0, c], \\ \left(I_t^{1-\vartheta} z(t)\right)_{t=0} = y_0 \in V \end{cases}$$

for $0 < \vartheta < 1$ in Banach spaces. The theory of Laplace transform together with probability density function are used to derive the mild solution of the system in terms of semigroup. To obtained the existence of solutions and controllability results, the differentiability of semigroup and Lipschitz condition on F are assumed. Utilizing the ideas of this article, Ibrahim et al. [33] obtained the existence and controllability results for the same system with the initial conditions $\lim_{t\to 0^+} \Gamma(\vartheta) t^{1-\vartheta} z(z) = y_0$. Here, the concept of ϑ-order resolvent is used rather than C_0-semigroup.

Mahmudov and McKibben [64] determined the approximate controllability of fractional systems with generalized Riemann-Liouville derivatives. Here, the nonlinearity term is not Lipschitz but it is measurable with respect to t. In 2017, Zhu et al. [93] considered the same system as in [33] and obtained the optimal controls utilizing the resolvent technique. Here, existence and optimal control were analyzed without Lipschitz condition.

In 2018, Zhu et al. [92] analyzed the system considered in [33] with the condition $1/2 < \vartheta < 1$ and obtained the existence and controllability results with integral contarctor nonlinearity. Utilizing Schauder's fixed point theorem and semigroup, Lian et al. [58] determined the existence of solution and time optimal control for the systems of the form

$$\begin{cases} D_t^{\vartheta} z(t) = Az(t) + F(t, z(t)) + B(t)u(t), \quad t \in (0, c], \\ \left(I_t^{1-\vartheta} z(t)\right)_{t=0} = y_0 \in V \end{cases}$$

for $0 < \vartheta < 1$ in Hilbert spaces without Lipschitz assumption. Making use of compact method, they removed reflexivity of state space.

1.3 Organization of the thesis

In this thesis, existence of mild solution and approximate controllability of semilinear systems of integer order and fractional order have been investigated. Results are obtained using fixed point, semigroup, cosine family and fractional calculus. There are eight chapters in the thesis including the present one containing introduction and literature review. The chapter-wise description is given below.

Chapter 2 contains basic concepts of control theory and nonlinear functional analysis which are used in subsequent chapters.

In Chapter 3, the existence of mild solution of first-order retarded semilinear system with control delay is proved under the locally Lipschitz continuity of nonlinear function and a fixed point theorem. Then the approximate controllability of semilinear system is proved provided that the associated linear system without delay is approximately controllable. Controllability results are obtained by the method of steps and semigroup theory.

The results of this chapter are communicated to **"FILOMAT Journal of Mathematics"**.

Chapter 4 contains two sections. The first section deals with the approximate and exact controllability of second-order nonlocal retarded semilinear system with control delay. In this section the existence of mild solution has been derived using fixed point approach and cosine family. Here the nonlinear function is supposed to be Lipschitz continuous. The controllability of the associated linear system with delay is proved by the method of steps and then the controllability of the actual system is proved. In the second section, the approximate controllability of second-order nonlocal retarded semilinear system with multiple delays in control is discussed in Banach spaces. The existence of mild solution has been derived using fixed point approach. For this, nonlinear function is supposed to be locally Lipschitz. Then the approximate controllability of associated linear system and actual system is proved.

The results of the first section of this chapter are published in **"Applicable Analysis (Taylor & Francis Online)"** and the results of the second section of this chapter are accepted for publication in **"Bulletin of the Iranian Mathematical Society (Springer)"**.

In Chapter 5, the partial approximate controllability of nonlocal fractional systems with integral initial conditions in Hilbert spaces without assuming the Lipschitz continuity of nonlinear function is investigated. We also exclude the conditions of Lipschitz continuity and compactness for the nonlocal function. The existence of solution is derived applying Schauder's fixed point theorem, then the partial approximate controllability result is proved by assuming that the associated linear system is partial approximately controllable for $\varphi = 0$, where φ is nonlocal function.

The results of this chapter are Published in **"Rendiconti del Circolo Matematico di Palermo Series 2 (Springer)"**.

In Chapter 6 we analyzed the approximate controllability of Riemann-Liouville fractional evolution equations with integral initial conditions in Banach spaces. First we deduce the existence of mild solutions using fractional Riemann- Liouville family and fixed point approach by assuming the Lipschitz continuity of nonlinearity term. Then we established new sufficient conditions for the approximate controllability of the system.

The results of this chapter are communicated to **"Numerical Functional Analysis and Optimization (Taylor & Francis Online)"**.

Chapter 7 is concerned with the Riemann-Liouville fractional semilinear integrodifferential systems with damping in Banach spaces. First we proved the existence of mild solutions of the system using fixed point approach. Then we established a set of new sufficient conditions for the approximate controllability of the system by means of iterative and approximate technique. To obtain our results, we use the concept of Riemann-Liouville fractional $(\vartheta, \varphi, \lambda)$ resolvent, where $0 < \varphi < \vartheta \leq 1$ and λ is a real number.

The results of this chapter are published in **"Chaos, Solitons & Fractals (Elsevier)"**.

Chapter 2

Preliminaries

In this chapter, some basic concepts of control theory and nonlinear functional analysis, which are used in subsequent chapters, are presented.

2.1 Basic concepts of control theory

2.1.1 Finite dimensional linear systems

A mathematical formulation of a finite dimensional linear control system can be represented by the differential systems

$$\begin{cases} \frac{dz(t)}{dt} = \dot{z}(t) = A(t)z(t) + B(t)u(t), \quad t \in (t_0, c], \\ z(t_0) = y_0 \in \mathbb{R}^n, \end{cases} \tag{2.1.1}$$

where $z(t) \in \mathbb{R}^n$ is known as the state variable and $u(t) \in \mathbb{R}^m$ is known as the control at time t. $A(t)$ and $B(t)$ are piecewise continuous matrices of order $n \times n$ and $n \times m$ respectively.

Let $L_2([t_0, c]; \mathbb{R}^n)$ and $L_2([t_0, c]; \mathbb{R}^m)$ be the function spaces to which z and u belong, respectively. The solution of the system (2.1.1) is given by the equation

$$z(t) = \Phi(t, t_0)y_0 + \int_{t_0}^{t} \Phi(t, s)B(s)u(s)\, ds, \quad t_0 \leq t \leq c,$$

where $\Phi(t, t_0)$ is an $n \times n$ matrix, known as the state transition matrix and it has the following properties:

(i) $\frac{d}{dt}\Phi(t, t_0) = A(t)\Phi(t, t_0)$;

(ii) $\Phi(t, t) = \mathcal{I}$, the identity matrix of order n;

(iii) $\Phi^{-1}(t, t_0) = \Phi(t_0, t)$;

(iv) $\Phi(t, s)\Phi(s, t_0) = \Phi(t, t_0)$.

The state transition matrix $\Phi(t, t_0)$ can be obtained by the following Peano's series,

$$\Phi(t, s) = \mathcal{I} + \int_s^t A(\tau)\, d\tau + \int_s^t \int_s^{\tau_1} A(\tau_2)A(\tau_1)\, d\tau_2 d\tau_1 + \cdots .$$

If the matrix A does not depend on time t, then from the above series, one has

$$\Phi(t, s) = e^{A(t-s)}.$$

Remark 2.1.1. *If the matrices A and B are constants then the system is known as autonomous system and in this case the solution takes the form*

$$z(t) = e^{A(t-t_0)}y_0 + \int_{t_0}^t e^{A(t-s)}Bu(s)\, ds, \quad t_0 \leq t \leq c.$$

Definition 2.1.1. The system (2.1.1) is said to be controllable on $[t_0, c]$, if for every given vector $z_c \in \mathbb{R}^n$, one can find a control $u \in L_2([t_0, c]; \mathbb{R}^m)$ such that the solution of the system (2.1.1) satisfies $z(c) = z_c$, that is

$$\Phi(c, t_0)y_0 + \int_{t_0}^c \Phi(c, s)B(s)u(s)\, ds = z_c.$$

If above is not the case, we say that the system is uncontrollable.

Remark 2.1.2. *In general, the control u which steers the system from y_0 to the final state z_c, depends on y_0 and z_c and it may not be unique.*

The collection of all points to which the system can be steered in time c is known as the controllable space or reachable set and is denoted by $\mathfrak{R}_c$, that is

$$\mathfrak{R}_c = \left\{ z(c) \in \mathbb{R}^n \,\middle|\, z \text{ is a solution of (2.1.1) associated with } u \in L_2([t_0, c]; \mathbb{R}^m) \right\}.$$

The system (2.1.1) is said to be controllable on $[t_0, c]$ if the reachable set $\mathfrak{R}_c$ equals to the whole space $\mathbb{R}^n$.

Definition 2.1.2. Let B^* and Φ^* be the conjugate transpose of B and Φ respectively. The matrices $G : L_2([t_0, c]; \mathbb{R}^m) \to \mathbb{R}^n$ and $W_{t_0}^c : \mathbb{R}^n \to \mathbb{R}^n$ defined by

$$Gu = \int_{t_0}^{c} \Phi(t, s)B(s)u(s)\, ds \tag{2.1.2}$$

and

$$W_{t_0}^c = \int_{t_0}^{c} \Phi(c, s)B(s)B^*(s)\Phi^*(c, s)\, ds, \tag{2.1.3}$$

are known as the controllability matrix and the controllability Grammian matrix, respectively.

Theorem 2.1.3. *[18] The linear system* (2.1.1) *is controllable if and only if the controllablility Grammian matrix $W_{t_0}^c$ given by* (2.1.3)*, is invertible.*

2.1.2 Infinite dimensional linear systems

In infinite dimensional spaces, the control problems are more difficult and conceptual than the finite dimensional cases. For this reason, it is an important area of research with a rich literature. To study the first-order systems, the theory of semigroup is an important tool. Therefore first we review the theory of semigroup.

As usual, let $\mathcal{B}(V)$ denotes the set of continuous linear maps from the Banach space V to V and $\mathbb{R}_0^+$ be the set of non-negative real numbers.

Definition 2.1.3. [75] A family of operators $\{T(t)\}_{t\in\mathbb{R}_0^+} \subset \mathcal{B}(V)$ is called a strongly continuous semigroup or C_0-semigroup on V, if it satisfies the following properties:

(i) $T(0)y = y$ for $y \in V$;

(ii) $T(s + t) = T(s)T(t)$ for $s,\ t \in \mathbb{R}_0^+$;

(iii) $\lim_{t\downarrow 0} \|T(t)y - y\| = 0\ \forall\, y \in V$.

Definition 2.1.4. [75] The infinitesimal generator A of a strongly continuous semigroup $T(t)$ is defined as

$$Ay = \lim_{t\downarrow 0} \frac{T(t)y - y}{t},$$

if the limit exists. The domain of A is the collection of all points $y \in V$ for which the limit exists.

Theorem 2.1.4. *[18] A C_0-semigroup $T(t)$ generated by A satisfies the following properties:*

(i) *there are constants M and ω such that*

$$\|T(t)\| \leq Me^{\omega t} \quad \forall\, t \in \mathbb{R}_0^+$$

and hence $\|T(t)\|$ is bounded on every bounded interval;

(ii) *for $y \in V$, we have*

$$\lim_{h \to 0} \frac{1}{h} \int_t^{t+h} T(s)y\, ds = T(t)y;$$

(iii) *for $y \in D(A)$, $T(t)y \in D(A)$ and*

$$\frac{d}{dt} T(t)y = AT(t)y = T(t)Ay.$$

An infinite dimensional linear control system can be written as

$$\begin{cases} \dot{z}(t) = Az(t) + Bu(t), \quad t \in (t_0, c], \\ z(t_0) = y_0 \in V, \end{cases} \tag{2.1.4}$$

where for each fixed t, the state $z(t) \in V$ and the control $u(t) \in V'$; V' is another Banach space. Let $Z = L_2([t_0, c]; V)$ and $U = L_2([t_0, c]; V')$ are function spaces of V and V', respectively. The linear map $A : D(A) \subseteq V \to V$ is closed with dense domain $D(A)$ and it generates a semigroup $T(t)$, $B : V' \to V$ is a continuous linear map.

For any $y_0 \in V$, a function $z \in L_2([t_0, c]; V)$ is the **mild solution** of (2.1.4) if it satisfies

$$z(t) = T(t - t_0)y_0 + \int_{t_0}^{t} T(t - s)Bu(s)\, ds, \quad t_0 \leq t \leq c.$$

Definition 2.1.5. [3] The system (2.1.4) is said to be approximate controllable on $[t_0, c]$, if for given $\varepsilon > 0$ and a final state z_c in V, one can find a control u in U steering y_0, along a solution (trajectory) of the system (2.1.4) to an ε-neighborhood of z_c, that is

$$\|z(c) - z_c\| \leq \varepsilon.$$

If $z(c) = z_c$, the system (2.1.4) is said to be an exact controllable system on $[t_0, c]$.

Remark 2.1.5. *In case of finite dimensional systems, the concepts of approximate controllablility and exact controllability are equivalent.*

2.1.3 Infinite dimensional first-order semilinear systems

An abstract form of infinite dimensional semilinear control systems is given as

$$\begin{cases} \dot{z}(t) = Az(t) + Bu(t) + F(t, z(t)), \quad t \in (t_0, c], \\ z(t_0) = y_0 \in V, \end{cases} \tag{2.1.5}$$

where the state space and control space, and the operators $A,\ B$ setting are similar as defined in previous section. The map $F : [t_0, c] \times V \to V$ produces nonlinearity in the system (2.1.5).

A mild solution of (2.1.5) is a function $z(t)$ given by the nonlinear integral equation

$$z(t) = T(t - t_0)y_0 + \int_{t_0}^{t} T(t - s)\big(Bu(s) + F(s, z(s))\big)\, ds, \quad t_0 \le t \le c.$$

Under suitable assumptions on F, the mild solution $z(t)$ is unique for each fixed $u \in U$.

Definition 2.1.6. The reachable set $\mathfrak{R}_c(F)$ of the system (2.1.5) is defined as

$$\mathfrak{R}_c(F) = \big\{z(c) \,\big|\, z(t) \text{ is the mild solution of (2.1.5) associated with } u \in U\big\}.$$

The reachable set of linear system corresponding to semilinear system (2.1.5) is denoted by $\mathfrak{R}_c(0)$.

The controllability in terms of reachable set is defined below:

Definition 2.1.7. A control system is said to be approximate controllable on $[t_0, c]$, if $\mathfrak{R}_c(F)$ is dense in V, that is $\overline{\mathfrak{R}_c(F)} = V$ and it is said to be exact controllable, if $\mathfrak{R}_c(F) = V$.

An important result on the controllability of the system (2.1.5) was given by Naito [71] which is given below:

Theorem 2.1.6. *[71] The semilinear system* (2.1.5) *is approximate controllable if the following conditions are satisfied:*

(*i*) *the semigroup* $T(t)$ *is compact;*

(*ii*) $F(t, y)$ *is Lipschitz in* $y \in V$*;*

(*iii*) $\|F(t, y)\| \leq k_F$*, where* k_F *is a positive constant;*

(*iv*) *for every* $p \in Z = L_2([t_0, c]; V)$*, one can find a* $q \in \overline{Range(B)}$ *satisfying the equation* $\zeta p = \zeta q$*, where* $\zeta : Z \to V$ *is defined as*

$$\zeta z = \int_{t_0}^{c} T(c - s)z(s)\, ds.$$

Condition (*iv*) of the above theorem implies the approximate controllability of corresponding linear control system of (2.1.5) (when $F = 0$ in (2.1.5)) (for proof see [71]).

2.1.4 Infinite dimensional second-order semilinear systems

First we define sine and cosine family, because mild solutions for the systems of second-order are defined in terms of these families (for details see [85; 86]).

Definition 2.1.8. [86] A family of operators $\{\mathcal{C}(t)\}_{t\in\mathbb{R}} \subset \mathcal{B}(V)$ is called strongly continuous cosine family if

(*i*) $\mathcal{C}(0)y = y$ for $y \in V$;

(*ii*) $2\mathcal{C}(s)\mathcal{C}(t) = \mathcal{C}(s - t) + \mathcal{C}(s + t)$ for $s, t \in \mathbb{R}$;

(*iii*) $\mathcal{C}(t)$ is strongly continuous in t.

Definition 2.1.9. [86] The sine family $\{\mathcal{S}(t)\}_{t\in\mathbb{R}} \subset \mathcal{B}(V)$ associated with $\{\mathcal{C}(t)\}_{t\in\mathbb{R}}$ is defined as

$$\mathcal{S}(t)y = \int_0^t \mathcal{C}(s)y\, ds, \quad y \in V, \quad t \in \mathbb{R}.$$

Definition 2.1.10. [86] The infinitesimal generator A of a strongly continuous cosine family $\{\mathcal{C}(t)\}_{t\in\mathbb{R}}$ is defined by

$$Ay = \frac{d^2}{dt^2}\mathcal{C}(t)y\Big|_{t=0},$$

where

$$D(A) = \big\{y \in V \;\big|\; \mathcal{C}(t)y \text{ is twice continuously differentiable function of } t\big\}.$$

Lemma 2.1.7. *[86] If A generates a strongly continuous cosine family $\{\mathcal{C}(t)\}_{t\in\mathbb{R}}$, then*

(i) $\mathcal{S}(t) = -\mathcal{S}(-t)$ *for* $t \in \mathbb{R}$*;*

(ii) $\mathcal{C}(t) = \mathcal{C}(-t)$ *for* $t \in \mathbb{R}$*;*

(iii) $\mathcal{S}(s),\ \mathcal{S}(t),\ \mathcal{C}(s)$ *and* $\mathcal{C}(t)$ *commute for* $s, t \in \mathbb{R}$*;*

(iv) $2\mathcal{S}(s)\mathcal{C}(t) = \mathcal{S}(s-t) + \mathcal{S}(s+t)$ *for* $s, t \in \mathbb{R}$*;*

(v) $\mathcal{C}(s)\mathcal{S}(t) + \mathcal{C}(t)\mathcal{S}(s) = \mathcal{S}(s+t)$ *for* $s, t \in \mathbb{R}$*;*

(vi) $2A\mathcal{S}(s)\mathcal{S}(t) = \mathcal{C}(s+t) - \mathcal{C}(s-t)$ *for* $s, t \in \mathbb{R}$*.*

A second-order abstract semilinear control system can be written as

$$\begin{cases} \frac{d^2}{dt^2} z(t) = \ddot{z}(t) = Az(t) + Bu(t) + F(t, z(t)), \quad t \in (t_0, c], \\ z(t_0) = y_0 \in V, \\ \dot{z}(t_0) = y_1 \in V, \end{cases} \tag{2.1.6}$$

where the operators $A,\ B,\ F$, state space and control space are defined as earlier.

The system represented by the differential equation (2.1.6) is called an infinite dimensional second-order semilinear control system.

The mild solution of (2.1.6) is given by a nonlinear integral equation which can be written as

$$z(t) = \mathcal{C}(t-t_0)y_0 + \mathcal{S}(t-t_0)y_1 + \int_{t_0}^{t} \mathcal{S}(t-s)\big(Bu(s) + F(s, z(s))\big)\, ds, \quad t_0 \le t \le c.$$

2.2 Basic concepts of fractional calculus

First we give some definitions from fractional calculus.

Definition 2.2.1. The Riemann-Lioville fractional integral operator of order $\vartheta > 0$ of a function f is given by

$$I_t^{\vartheta} f(t) = \frac{1}{\Gamma(\vartheta)} \int_0^t (t-s)^{\vartheta-1} f(s)\, ds.$$

Definition 2.2.2. The Mittag-Leffler function $E_{\vartheta,\varphi}(x)$ is defined as

$$E_{\vartheta,\varphi}(x) = \sum_{n=0}^{\infty} \frac{x^n}{\Gamma(\vartheta n + \varphi)}.$$

If $\varphi = 1$ then it is the-one parameter Mittag-Leffler function E_ϑ.

Definition 2.2.3. The two-parameter Mittag-Leffler integral operator is given by

$$\mathcal{E}_t^{\vartheta,\varphi,\lambda} f(t) = \int_0^t (t-s)^{\varphi-1} E_{\vartheta,\varphi}\left(\lambda(t-s)^\vartheta\right) f(s)\, ds, \quad t > 0.$$

Definition 2.2.4. The function defined by

$$\begin{aligned} D_t^\vartheta f(t) = \frac{d^\vartheta}{dt^\vartheta} f(t) &= \frac{d^n}{dt^n}\left(I_t^{n-\vartheta} f(t)\right) \\ &= \frac{1}{\Gamma(n-\vartheta)} \frac{d^\vartheta}{dt^\vartheta} \int_0^t (t-s)^{(n-\vartheta-1)} f(s)\, ds \end{aligned}$$

is called ϑ- order Riemann-Lioville fractional derivative of $f(t)$, where $n-1 \leq \vartheta < n$, $n \in \mathbb{N}$.

Definition 2.2.5. The function defined by

$${}^C D_t^\vartheta f(t) = \frac{1}{\Gamma(n-\vartheta)} \int_0^t (t-s)^{(n-\vartheta-1)} f^n(s)\, ds$$

is called ϑ- order Caputo fractional derivative of $f(t)$, where $n-1 \leq \vartheta < n$, $n \in \mathbb{N}$.

This definition is more restrictive than Riemann-Lioville one because it needs the absolute integrability of the nth-order derivative of the function $f(t)$. Between the two definitions there is the following relation:

$$D_t^\vartheta f(t) = {}^C D_t^\vartheta f(t) + \sum_{k=0}^{m-1} \frac{t^{k-\vartheta}}{\Gamma(k-\vartheta+1)} f^{(k)}(0^+).$$

Lemma 2.2.1. *Let $\vartheta > 0$, $m = [\vartheta] + 1$, and let $z_{m-\vartheta}(t) = I_t^{m-\vartheta} z(t)$. If $z(t) \in L_1([0,c];V)$ and $z_{m-\vartheta}(t) \in AC^m([0,c];V)$, then*

$$I_t^\vartheta D_t^\vartheta z(t) = z(t) - \sum_{j=1}^{m} \frac{z_{m-\vartheta}^{(m-j)}}{\Gamma(\vartheta - j + 1)} t^{\vartheta - j}.$$

2.2.1 Infinite dimensional fractional systems of order $\vartheta \in (0,1)$

Consider the fractional system of the form

$$\begin{cases} {}^C D_t^{\vartheta} z(t) = Az(t), \quad t \in (0,c], \\ z(0) = y_0 \in V, \end{cases} \tag{2.2.1}$$

where ${}^C D_t^{\vartheta}$ is the Caputo fractional derivative of order $\vartheta \in (0,1)$. $A : D(A) \subseteq V \to V$ generates a C_0-semigroup $T(t)$.

The integral form of the Cauchy problem (2.2.1) is

$$z(t) = y_0 + \frac{1}{\Gamma(\vartheta)} \int_0^t (t-s)^{\vartheta-1} Az(s)\, ds. \tag{2.2.2}$$

By a solution of the Cauchy problem (2.2.1), we mean a function z satisfying the following conditions:

(i) z is continuous on $[0,c]$ and $z(t) \in D(A)$ for each $t \in [0,c]$;

(ii) ${}^C D_t^{\vartheta} z(t)$ is continuous on $[0,c]$, where $0 < \vartheta < 1$;

(iii) z satisfies the equation (2.2.1) on $[0,c]$ and the initial condition $z(0) = y_0$.

Notice that the integral equation (2.2.2) is equivalent to the Cauchy problem (2.2.1).

The solution of (2.2.1) is given by

$$z(t) = \widehat{T}_{\vartheta}(t) y_0,$$

where

$$\widehat{T}_{\vartheta}(t) y = \int_0^{\infty} \phi_{\vartheta}(\varrho) T(t^{\vartheta} \varrho) y \, d\varrho$$

and

$$\phi_{\vartheta}(\varrho) = \frac{1}{\vartheta} \varrho^{-1-1/\vartheta} \psi_{\vartheta}(\varrho^{-1/\vartheta}).$$

($\widehat{T}_{\vartheta}(t)$ is known as ϑ-order semigroup).

Note that $\phi_{\vartheta}(\varrho)$ satisfies the condition of a probability density function. The term $\psi_{\vartheta}(\varrho)$ is defined as

$$\psi_{\vartheta}(\varrho) = \frac{1}{\pi} \sum_{n=1}^{\infty} (-1)^{n-1} \varrho^{-n\vartheta-1} \frac{\Gamma(n\vartheta+1)}{n!} sin(n\pi\vartheta), \ \varrho \in (0,\infty).$$

Now, consider the fractional system

$$\begin{cases} {}^C D_t^{\vartheta} z(t) = Az(t) + f(t), \quad t \in (0, c], \\ z(0) = y_0 \in V, \end{cases} \tag{2.2.3}$$

where the function $f \in L_1([0, c]; V)$.

Definition 2.2.6. [94] A function $z \in C([0, c]; V)$ is said to be a mild solution of (2.2.3) if it satisfies

$$z(t) = \widehat{T}_{\vartheta}(t) y_0 + \int_0^t (t-s)^{\vartheta-1} T_{\vartheta}(t-s) f(s)\, ds, \quad 0 \leq t \leq c,$$

where

$$T_{\vartheta}(t) y = \vartheta \int_0^{\infty} \varrho \phi_{\vartheta}(\varrho) T(t^{\vartheta} \varrho) y \, d\varrho.$$

A mathematical model of an infinite dimensional fractional linear control system is given by

$$\begin{cases} {}^C D_t^{\vartheta} z(t) = Az(t) + Bu(t), \quad t \in (0, c], \\ z(0) = y_0 \in V. \end{cases} \tag{2.2.4}$$

A function $z \in C([0, c]; V)$ given by

$$z(t) = \widehat{T}_{\vartheta}(t) y_0 + \int_0^t (t-s)^{\vartheta-1} T_{\vartheta}(t-s) Bu(s)\, ds,$$

is known as the mild solution of the system (2.2.4).

The controllability of the fractional order control system (2.2.4) is a generalization of the controllability of first-order linear system. Now, we introduce the controllability operator for (2.2.4) as in [77].

$$\Psi_0^c = \int_0^c (c-s)^{\vartheta-1} T_{\vartheta}(T-s) BB^* T_{\vartheta}^*(T-s)\, ds,$$

where B^* and $T_{\vartheta}^*(t)$ are adjoint operators of B and $T_{\vartheta}(t)$, respectively. It is easily seen that Ψ_0^c is continuous linear operator.

Let

$$R(\lambda, \Psi_0^c) = (\lambda I + \Psi_0^c)^{-1} \quad \text{for} \quad \lambda > 0.$$

Lemma 2.2.2. *[77] The fractional order linear control system (2.2.4) is approximate controllable on $[0, c]$ if and only if $\lambda R(\lambda, \Psi_0^c) \to 0$ as $\lambda \to 0^+$ in the strong operator topology.*

Consider the infinite dimensional fractional order semilinear control system in abstract form as follows

$$\begin{cases} {}^C D_t^{\vartheta} z(t) = Az(t) + Bu(t) + F(t, z(t)), & t \in (0, c], \\ z(0) = y_0 \in V, \end{cases} \tag{2.2.5}$$

where $0 < \vartheta \leq 1$ and F is a nonlinear function. The mild solution $z(t)$ of the system (2.2.5) is given by the integral equation

$$z(t) = \widehat{T}_{\vartheta}(t) y_0 + \int_0^t (t-s)^{\vartheta - 1} T_{\vartheta}(t-s) \big(Bu(s) + F(s, z(s)) \big) \, ds, \quad 0 \leq t \leq c.$$

The system (2.2.5) is approximate controllable if the following conditions are satisfied [77]:

(*i*) the C_0-semigroup $T(t)$ generated by A is compact;

(*ii*) $\forall\ t \in [0, c]$, the function $F(t, \cdot) : V \to V$ is continuous and $\forall\ z \in C([0, c]; V)$ the function $F(\cdot, z) : [0, c] \to V$ is strongly measurable;

(*iii*) there is a constant $q_1 \in [0, \vartheta]$ and $m \in L_{\frac{1}{q_1}}([0, c]; \mathbb{R}_0^+)$ such that $|F(t, z)| \leq m(t)$ $\forall\ z \in V$ and almost all $t \in [0, c]$;

(*iv*) the function $F : [0, c] \times V \to V$ is continuous and there is a constant $k_F > 0$ such that $\|F(t, z)\| \leq k_F\ \forall\ (t, z) \in [0, c] \times V$.

2.2.2 Infinite dimensional fractional systems of order $\vartheta \in (1, 2]$

Consider the fractional system of order $\vartheta \in (1, 2]$

$$\begin{cases} {}^C D_t^{\vartheta} z(t) = Az(t) + Bu(t) + F(t, z(t)), & t \in (0, c], \\ z(0) = y_0 \in V, \\ \dot{z}(0) = y_1 \in V, \end{cases} \tag{2.2.6}$$

where $\vartheta \in (1,2]$ and ${}^C D_t^{\vartheta}$ is the Caputo fractional differential operator. A generates a strongly continuous ϑ-order cosine family $\{\mathcal{C}_\vartheta(t)\}_{t\in\mathbb{R}_0^+}$. The other notations are defined as in previous sections.

Now, we define the ϑ-order fractional cosine family. For this consider the system

$$\begin{cases} {}^C D_t^{\vartheta} z(t) = Az(t), \quad t \in (0,c], \\ z(0) = y \in V, \\ \dot{z}(0) = 0 \in V. \end{cases} \tag{2.2.7}$$

Applying Riemann-Liouville fractional integral of order ϑ on both sides of (2.2.7), one can get

$$z(t) = y + \frac{1}{\Gamma(\vartheta)} \int_0^t (t-s)^{\vartheta-1} Az(s)\, ds.$$

Definition 2.2.7. [2] A family of operators $\{\mathcal{C}_\vartheta(t)\}_{t\in\mathbb{R}_0^+} \subset \mathcal{B}(V)$ is called strongly continuous ϑ-order fractional cosine family if

(i) $\mathcal{C}_\vartheta(0)y = y$ for $y \in V$, and $\mathcal{C}_\vartheta(t)$ is strongly continuous in t;

(ii) $\mathcal{C}_\vartheta(t)y \in D(A)$ and $A\mathcal{C}_\vartheta(t)y = \mathcal{C}_\vartheta(t)Ay$ for $y \in D(A)$, $t \in \mathbb{R}_0^+$;

(iii) $\mathcal{C}_\vartheta(t)y$ satisfies $z(t) = y + \frac{1}{\Gamma(\vartheta)}\int_0^t (t-s)^{\vartheta-1} Az(s)\, ds$ for $y \in D(A)$, $t \in \mathbb{R}_0^+$.

A is known as the infinitesimal generator of $C_\vartheta(t)$. The family $C_\vartheta(t)$ is also known as the ϑ-order cosine family.

Definition 2.2.8. The fractional sine family $\{\mathcal{S}_\vartheta(t)\}_{t\in\mathbb{R}_0^+} \subset \mathcal{B}(N)$ associated with $\{\mathcal{C}_\vartheta(t)\}_{t\in\mathbb{R}_0^+}$ is defined by

$$\mathcal{S}_\vartheta(t) = \int_0^t \mathcal{C}_\vartheta(s)\, ds, \quad t \in \mathbb{R}_0^+. \tag{2.2.8}$$

Definition 2.2.9. The fractional Riemann-Liouville family $\{\mathcal{R}_\vartheta(t)\}_{t\in\mathbb{R}_0^+} \subset \mathcal{B}(N)$ is defined by

$$\mathcal{R}_\vartheta(t) = I_t^{\vartheta-1}\mathcal{C}_\vartheta(t), \quad t \in \mathbb{R}_0^+. \tag{2.2.9}$$

Definition 2.2.10. $\mathcal{C}_\vartheta(t)$ is called exponentially bounded if there are constants $\omega \geq 1$ and $\lambda \geq 0$ satisfying

$$\|\mathcal{C}_\vartheta(t)\| \leq \omega e^{\lambda t}, \quad t \in \mathbb{R}_0^+. \tag{2.2.10}$$

If $\check{z}(\rho)$ is the Laplace transform of z, that is

$$L\{z(t)\} = \int_0^\infty e^{-\rho t} z(t)\, dt = \check{z}(\rho),$$

then

$$L\{I_t^\vartheta z(t)\} = \frac{1}{\rho^\vartheta} \check{z}(\rho).$$

Using Laplace transform theory and the following well known relation

$$\int_0^\infty e^{-\rho t} \mathcal{C}_\vartheta(t) y\, dt = \rho^{\vartheta-1}(\rho^\vartheta I - A)^{-1} y, \quad Re(\rho) > \lambda, \quad y \in V, \tag{2.2.11}$$

one can easily obtain

$$\int_0^\infty e^{-\rho t} \mathcal{S}_\vartheta(t) y\, dt = \rho^{\vartheta-2}(\rho^\vartheta I - A)^{-1} y, \quad Re(\rho) > \lambda, \quad y \in V \tag{2.2.12}$$

and

$$\int_0^\infty e^{-\rho t} \mathcal{R}_\vartheta(t) y\, dt = (\rho^\vartheta I - A)^{-1} y, \quad Re(\rho) > \lambda, \quad y \in V. \tag{2.2.13}$$

Definition 2.2.11. A function $z \in C([0, c]; V)$ is called a mild solution of (2.2.6) if it satisfies

$$z(t) = \mathcal{C}_\vartheta(t) y_0 + \mathcal{S}_\vartheta(t) y_1 + \int_0^t \mathcal{R}_\vartheta(t-s)\big(Bu(s) + F(s, z(s))\big)\, ds. \tag{2.2.14}$$

2.2.3 Infinite dimensional fractional systems with damping

Consider the following Riemann-Liouville fractional systems with damping:

$$\begin{cases} D_t^\vartheta z(t) + \lambda D_t^\varphi z(t) = Az(t), \quad t \in (0, c], \\ \big(I_t^{1-\vartheta} z(t)\big)_{t=0} = y_0 \in V, \end{cases} \tag{2.2.15}$$

where $0 < \varphi < \vartheta \le 1$ and λ is a real number.

Let $\mathbb{R}^+$ be the set of positive real numbers.

Definition 2.2.12. [68] A family of operators $\{\mathcal{R}_{\vartheta,\varphi,\lambda}(t)\}_{t\in\mathbb{R}^+} \subset \mathcal{B}(V)$ is called Riemann-Liouville fractional $(\vartheta, \varphi, \lambda)$ resolvent on the Banach space V if

(i) for any $y \in V$, $\mathcal{R}_{\vartheta,\varphi,\lambda} y \in C((0,\infty); V)$, and

$$\lim_{t\to 0^+} \Gamma(\vartheta) t^{1-\vartheta} \mathcal{R}_{\vartheta,\varphi,\lambda}(t) y = y;$$

(ii) $\mathcal{R}_{\vartheta,\varphi,\lambda}(t)$ and $\mathcal{R}_{\vartheta,\varphi,\lambda}(s)$ commute for $t, s > 0$;

(iii) for $t, s > 0$, one has

$$\begin{aligned}
&\mathcal{R}_{\vartheta,\varphi,\lambda}(s)\mathcal{E}_t^{\vartheta-\varphi,\vartheta,-\lambda}\mathcal{R}_{\vartheta,\varphi,\lambda}(t) - \mathcal{E}_s^{\vartheta-\varphi,\vartheta,-\lambda}\mathcal{R}_{\vartheta,\varphi,\lambda}(s)\mathcal{R}_{\vartheta,\varphi,\lambda}(t)\\
&= s^{\vartheta-1}E_{\vartheta-\varphi,\vartheta}(-\lambda s^{\vartheta-\varphi})\mathcal{E}_t^{\vartheta-\varphi,\vartheta,-\lambda}\mathcal{R}_{\vartheta,\varphi,\lambda}(t)\\
&\quad - t^{\vartheta-1}E_{\vartheta-\varphi,\vartheta}(-\lambda t^{\vartheta-\varphi})\mathcal{E}_s^{\vartheta-\varphi,\vartheta,-\lambda}\mathcal{R}_{\vartheta,\varphi,\lambda}(s).
\end{aligned}$$

Definition 2.2.13. The linear operator A defined by

$$Ay = \Gamma(2\vartheta)\lim_{t\to 0^+}\frac{t^{1-\vartheta}\mathcal{R}_{\vartheta,\varphi,\lambda}(t)y - E_{\vartheta-\varphi,\vartheta}\left(-\lambda t^{\vartheta-\varphi}\right)y}{t^{\vartheta}} \quad \text{for} \quad y \in D(A),$$

where

$$D(A) = \left\{ y \in V \,\middle|\, \lim_{t\to 0^+}\frac{t^{1-\vartheta}\mathcal{R}_{\vartheta,\varphi,\lambda}(t)y - E_{\vartheta-\varphi,\vartheta}\left(-\lambda t^{\vartheta-\varphi}\right)y}{t^{\vartheta}} \text{ exists} \right\},$$

is called the generator of Riemann-Liouville fractional $(\vartheta, \varphi, \lambda)$ resolvent $\mathcal{R}_{\vartheta,\varphi,\lambda}(t)$.

Lemma 2.2.3. *[68] If A generates a Riemann-Liouville fractional $(\vartheta, \varphi, \lambda)$ resolvent $\mathcal{R}_{\vartheta,\varphi,\lambda}(t)$, then*

(i) *$\mathcal{R}_{\vartheta,\varphi,\lambda}(t)y \in D(A)$ and $A\mathcal{R}_{\vartheta,\varphi,\lambda}(t)y = \mathcal{R}_{\vartheta,\varphi,\lambda}(t)Ay$ for $y \in D(A)$;*

(ii) *for $y \in V$, $t > 0$,*

$$\mathcal{R}_{\vartheta,\varphi,\lambda}(t)y = t^{\vartheta-1}E_{\vartheta-\varphi,\vartheta}\left(-\lambda t^{\vartheta-\varphi}\right)y + A\mathcal{E}_t^{\vartheta-\varphi,\vartheta,-\lambda}\mathcal{R}_{\vartheta,\varphi,\lambda}(t)y;$$

(iii) *for $y \in D(A)$ and $t > 0$,*

$$\mathcal{R}_{\vartheta,\varphi,\lambda}(t)y = t^{\vartheta-1}E_{\vartheta-\varphi,\vartheta}\left(-\lambda t^{\vartheta-\varphi}\right)y + \mathcal{E}_t^{\vartheta-\varphi,\vartheta,-\lambda}\mathcal{R}_{\vartheta,\varphi,\lambda}(t)Ay.$$

Definition 2.2.14. A function $z \in C_{1-\vartheta}([0, c]; V)$ is said to be a mild solution of (2.2.15) if it satisfies

$$z(t) = \mathcal{R}_{\vartheta,\varphi,\lambda}(t)y_0.$$

2.3 Basics concepts of functional analysis

Now we give basic definitions and theorems from functional analysis.

Definition 2.3.1. Let V and V' be two Banach spaces. Then $F : V \to V'$ is said to be Lipschitz continuous if there exists a constant $k_F > 0$ such that

$$\|F(y) - F(\tilde{y})\|_{V'} \leq k_F \|y - \tilde{y}\|_V \quad \forall\, y, \tilde{y} \in V.$$

Definition 2.3.2. Let V be a Banach space and $F : V \to V'$ be a map then each solution of the equation

$$F(y) = y, \;\; y \in V,$$

is known as a fixed point of the map F.

Theorem 2.3.1. *[51; 69] (Banach contraction fixed point theorem) Let E be a nonempty, closed set in a Banach space V and $F : E \to E$ be a k-contraction map, i.e,*

$$\|F(y) - F(\tilde{y})\| \leq k\|y - \tilde{y}\|, \; \forall\, y, \tilde{y} \in E \text{ and fixed } k, \; 0 \leq k < 1.$$

Then F has a unique fixed point in M.

Definition 2.3.3. (Nemytskii operator) Let $F : [0, c] \times V \to V$ be a function, which satisfies Caratheodory condition, that is $F(t, y)$ is continuous with respect to y for almost all $t \in [0, c]$ and measurable with respect to t for all $y \in V$. Then the operator $\tilde{F} : L_2([0, c]; V) \to L_2([0, c]; V)$ defined by

$$\big(\tilde{F}z\big)(t) = F(t, z(t)), \quad z \in L_2([0, c]; V),$$

is called Nemytskii operator.

Theorem 2.3.2. *[9] (Schauder fixed point theorem) Let E be a closed, convex and bounded subset of a Banach space V. let $T : E \to E$ be a continuous and compact operator that maps E into itself. Then the equation $Ty = y$ has at least one solution in E.*

Theorem 2.3.3. *(Arzela-Ascoli theorem) Suppose E is a compact set in $\mathbb{R}^n$. A set $S \subset C(E)$ is relatively compact in $C(E)$ if and only if the functions in S are uniformly bounded and equicontinuous on E.*

Theorem 2.3.4. *(Dominated convergence theorem) Let $\{\xi_n\}$ be a sequence of measurable functions on a set S satisfying $|\xi_n| \leq g$ a.e. on S for $n = 1, 2, \ldots;$ where g is integrable on S in the Lebesgue sense. If ξ_n converges pointwise to a function ξ a.e. on S. Then ξ is Lebesgue integrable on E and*

$$\lim_{n\to\infty} \int_S \xi_n \, dx = \int_S \xi \, dx.$$

Lemma 2.3.5. *(Fatou's lemma) Let $\{\xi_n\}$ be a sequence of integrable functions on a set S such that $\xi_n \geq g$ a.e. on S for some integrable function g and $\underline{\lim}_{n\to\infty} \int_S \xi_n \, dx < \infty$. Then*

$$\int_S \underline{\lim}_{n\to\infty} \xi_n \, dx \leq \underline{\lim}_{n\to\infty} \int_S \xi_n \, dx.$$

Definition 2.3.4. [18] (Gronwall's inequality) Assume that the continuous functions $\chi,\ \xi : [0, c] \to [0, \infty)$ and $\kappa > 0$ satisfy

$$\chi(t) \leq \kappa + \int_0^t \xi(s)\chi(s) \, ds \quad \forall\, t \in [0, c].$$

Then χ satisfies the following Gronwall's inequality

$$\chi(t) \leq \kappa \exp\left(\int_0^t \xi(s) \, ds \right).$$

Theorem 2.3.6. *[88]. Let $\vartheta > 0$, $\xi(t)$ be nonnegative, nondecreasing and locally integrable on $[0, \tau)$ (some $\tau \leq \infty$) and $\chi(t)$ be a nonnegative, nondecreasing, bounded continuous function defined on $[0, \tau)$, and let $\psi(t)$ be nonnegative and locally integrable on $[0, \tau)$ with*

$$\psi(t) \leq \xi(t) + \chi(t) \int_0^t (t - s)^{\vartheta - 1} \psi(s) \, ds.$$

Then

$$\psi(t) \leq \xi(t) E_\vartheta\big(\chi(t) t^\vartheta \Gamma(\vartheta)\big).$$

Chapter 3

Approximate Controllability of First-Order Retarded Semilinear Systems with Fixed Delay in Control

In this chapter, we prove the controllability of a class of retarded differential equations with fixed delay in control. First we prove the existence of mild solution by applying fixed point theorem and Gronwall's inequality. For this, the nonlinear function is supposed to be locally Lipschitz. Then the approximate controllability of the system is deduced via approximate method. Finally, an illustrative example has been described.

3.1 Introduction and preliminaries

Let V and V' be Hilbert spaces and $Z = L_2([0,c];V)$, $U = L_2([0,c];V')$ be the function spaces. Let $\mathfrak{C}_t = C([-a,t];V)$ be the space of continuous functions from $[-a,t]$ to V with the norm $\|z\|_{\mathfrak{C}_t} = \sup\limits_{-a\leq\varrho\leq t} \|z(\varrho)\|_V$. Consider the retarded differential system

$$\begin{cases} \dot{z}(t) = Az(t) + B_0u(t) + B_1u(t-a) + F\big(t, z_t, u(t)\big), & t \in (0,c], \\ z(t) = \wp(t), \quad u(t) = 0, & t \in [-a,0], \end{cases} \tag{3.1.1}$$

where the state $z \in Z$, the control $u \in U$, A is a densely defined closed linear operator generating a C_0-semigroup $T(t)$, B_0 and B_1 are continuous linear maps from V' to V, $\wp : [-a, 0] \to V$ is continuous and $F : [0, c] \times \mathfrak{C}_0 \times V' \to V$ is a nonlinear map. If $z \in \mathfrak{C}_c$, then $z_t : [-a, 0] \to V$ is defined as $z_t(\theta) = z(t + \theta)\ \forall\, \theta \in [-a, 0]$.

Many natural incidents embrace a significant memory effect. Retarded systems express the mathematical model of such real life problems. For example, many problems occurring from medicine, biosciences, chemical sciences, physical sciences, economics are affected by their previous results at major scale. Therefore for the study of controllability, it becomes much important to select retarded systems. Klamka [46] studied the controllability of linear systems with time-variable delays in control. In [55] Kumar et al. obtained a set of sufficient conditions for exact controllability of semilinear retarded systems.

The system (3.1.1) admits the concept of distributed delay $z_t \in C([-a, 0]; V)$, where delay is incurred as mentioned in above paragraph. The study of such systems covers a wide range of applications. Sukavanam et al. [82] studied the controllability of a semilinear delayed system with growing nonlinear term. In [23], Davies et al. deduced the results for null and exact controllability of linear systems with delay in both state and control. Utilizing sequence method and the concept of fundamental solution Anurag et al. [80] investigated the controllability of semilinear systems with state delay. In [37], Jeong and Roh obtained some results for the approximate controllability of the semilinear retarded control system of first-order under Lipschitz continuity of the nonlinear function. In this chapter, we extend the results for semilinear delay system with control in nonlinearity term.

Definition 3.1.1. A function $z \in \mathfrak{C}_c$ is said to be a mild solution of (3.1.1) corresponding to a control function $u \in U$, if it satisfies

$$z(t) = \begin{cases} T(t)\wp(0) + \displaystyle\int_0^t T(t-s)\big(B_0 u(s) + B_1 u(s-a)\big)\, ds \\ + \displaystyle\int_0^t T(t-s) F\big(s, z_s, u(s)\big)\, ds, & t \in (0, c], \\ \wp(t), & t \in [-a, 0]. \end{cases} \tag{3.1.2}$$

Definition 3.1.2. The system given by (3.1.1) is said to be approximately controllable on $[0, c]$, if for every given $\varepsilon > 0$ and a final state z_c, one can find a control

$u \in U$ such that the mild solution of (3.1.1) corresponding to u satisfies

$$\|z(c) - z_c\| \leq \varepsilon.$$

For the system (3.1.1), the systems

$$\begin{cases} \dot{z}(t) = Az(t) + B_0u(t) + B_1u(t-a), \quad t \in (0,c], \\ z(0) = \wp(0) \end{cases} \tag{3.1.3}$$

and

$$\begin{cases} \dot{z}(t) = Az(t) + B_0u(t), \quad t \in (0,c], \\ z(0) = \wp(0) \end{cases} \tag{3.1.4}$$

are the associated linear systems with delay and without delay, respectively.

Throughout this chapter, we assume that there is a constant $k_T > 0$ satisfying $\|T(t)\| \leq k_T$ for $0 \leq t \leq c$.

3.2 Existence of mild solution

To derive the existence result, we assume the following:

(H_1) F is continuous in t and locally Lipschitz in z that is there is a constant λ_r such that

$$\|F(t, z_1, u) - F(t, z_2, u)\|_V \leq \lambda_r \|z_1 - z_2\|_{\mathfrak{C}_0}$$

holds for all $z_\ell \in \mathfrak{C}_0$ with $\|z_\ell\| \leq r$ $(\ell = 1, 2)$, $u \in V'$ and $t \in [0, c]$;

(H_2) there is a positive constant k_F such that

$$\|F(t, z, u)\|_V \leq k_F\big(1 + \|z\|_{\mathfrak{C}_0} + \|u\|_{V'}\big)$$

holds for all $z \in \mathfrak{C}_0$, $u \in V'$ and $t \in [0, c]$.

Firs we prove the next lemma.

Lemma 3.2.1. *Let $z(t)$ be continuous on $[0, c)$ $(0 < c < \infty)$. If k_1 and k_2 be two positive constants such that*

$$\|z(t)\|_V \leq k_1 + k_2 \int_0^t \|z_s\|_{\mathfrak{C}_0} \, ds \quad \forall\, t \in [0, c).$$

Then

$$\|z(t)\|_V \leq (k_1 + k_\wp)\exp(k_2 c) \quad \forall\, t \in [0, c),$$

where $z(t) = \wp(t)$ *on* $[-a, 0]$ *and* $k_\wp = \|\wp\|_{\mathfrak{C}_0}$.

Proof. Let $t' \in [0, c)$ be arbitrary but fixed. Then there is a $t^* \in [-a, t']$ such that

$$\sup_{\theta \in [-a,0]} \|z(t' + \theta)\|_V = \|z(t^*)\|_V.$$

Now if $t^* \in [-a, 0]$, then

$$\begin{aligned}\sup_{\theta \in [-a,0]} \|z(t' + \theta)\|_V &= \|z(t^*)\|_V \\ &\leq k_\wp \\ &< k_\wp + k_1 + k_2 \int_0^{t'} \|z_s\|_{\mathfrak{C}_0}\, ds.\end{aligned}$$

If $t^* \in (0, t']$, then

$$\begin{aligned}\sup_{\theta \in [-a,0]} \|z(t' + \theta)\|_V &= \|z(t^*)\|_V \\ &\leq k_1 + k_2 \int_0^{t^*} \|z_s\|_{\mathfrak{C}_0}\, ds \\ &\leq k_\wp + k_1 + k_2 \int_0^{t'} \|z_s\|_{\mathfrak{C}_0}\, ds.\end{aligned}$$

Thus

$$\|z(t')\|_V \leq \sup_{\theta \in [-a,0]} \|z(t' + \theta)\|_V \leq k_\wp + k_1 + k_2 \int_0^{t'} \|z_s\|_{\mathfrak{C}_0}\, ds,$$

which gives

$$\|z(t)\|_V \leq \|z_t\|_{\mathfrak{C}_0} \leq k_\wp + k_1 + k_2 \int_0^{t} \|z_s\|_{\mathfrak{C}_0}\, ds \quad \forall\, t \in [0, c).$$

Using Gronwall's inequality, we have

$$\|z(t)\|_V \leq \|z_t\|_{\mathfrak{C}_0} \leq (k_1 + k_\wp)\exp(k_2 c) \quad \forall\, t \in [0, c).$$

This proves the lemma. ■

Theorem 3.2.2. *Under hypotheses* (H_1) *and* (H_2)*, the semilinear system (3.1.1) admits a unique mild solution in* $\mathfrak{C}_c$ *for each control* $u \in U$.

Proof. Let $\max\{\|B_0\|, \|B_1\|\} \leq k_B$. Define a mapping $\mathcal{Q} : \mathfrak{C}_{c_1} \to \mathfrak{C}_{c_1}$ as

$$(\mathcal{Q}z)(t) = \begin{cases} T(t)\wp(0) + \displaystyle\int_0^t T(t-s)\big(B_0u(s) + B_1u(s-a)\big)\, ds \\ + \displaystyle\int_0^t T(t-s)F\big(t, z_s, u(s)\big)\, ds, & t \in (0, c_1], \\ \wp(t), & t \in [-a, 0]. \end{cases}$$

Consider the ball

$$\mathcal{B}_{a_0} = \big\{z \in \mathfrak{C}_{c_1} \,\big|\, \|z\|_{\mathfrak{C}_{c_1}} \leq a_0, z(0) = \wp(0)\big\}.$$

For any $z \in \mathcal{B}_{a_0}$ and $0 \leq s \leq c_1$

$$\|z_s\|_{\mathfrak{C}_0} = \sup_{\theta\in[-a,0]} \|z(s+\theta)\| \leq \sup_{\varrho\in[-a,c_1]} \|z(\varrho)\| \leq a_0.$$

Thus

$$\begin{aligned} \|(\mathcal{Q}z)(t)\| &\leq k_T\|\wp(0)\| + k_Tk_B\bigg(\int_0^t \|u(s)\|\, ds + \int_0^t \|u(s-a)\|\, ds\bigg) \\ &\quad + k_T\int_0^t \big\|F\big(t, z_s, u(s)\big) - F(s, 0, u(s))\big\|\, ds \\ &\quad + k_T\int_0^t \|F(s, 0, u(s))\|\, ds \\ &\leq k_T\|\wp(0)\| + 2k_Tk_B\sqrt{c}\|u\|_U + k_T\lambda_{a_0}\int_0^t \|z_s\|\, ds \\ &\quad + k_Tk_F\int_0^t (1 + \|u\|_U)\, ds \\ &\leq k_T\|\wp(0)\| + 2k_Tk_B\sqrt{c}\|u\|_U + k_T\lambda_{a_0}a_0c_1 + k_Tk_F\left(c_1 + \sqrt{c_1}\|u\|_U\right) \\ &= k_T\big(\|\wp(0)\| + 2k_B\sqrt{c}\|u\|_U + (\lambda_{a_0}a_0\sqrt{c_1} + k_F\|u\|_U + k_F\sqrt{c_1})\sqrt{c_1}\big). \end{aligned}$$

Now choosing $a_0 = 2k_T(\|\wp(0)\| + 2k_B\sqrt{c}\|u\|_U + 1)$ and $0 < c_1 < c$ small enough such that

$$(\lambda_{a_0}a_0\sqrt{c_1} + k_F\|u\|_U + k_F\sqrt{c_1})\sqrt{c_1} \leq \|\wp(0)\| + 2k_B\sqrt{c}\|u\|_U + 1.$$

Then

$$\begin{aligned} \|(\mathcal{Q}z)(t)\| &\leq 2k_T(\|\wp(0)\| + 2k_B\sqrt{c}\|u\|_U + 1) \\ &= a_0 \text{ (say)}. \end{aligned}$$

Therefore $\mathcal{Q}$ maps $\mathcal{B}_{a_0}$ into itself.

Now we show that $\mathcal{Q}^n$ is contraction on $\mathcal{B}_{a_0}$. Take $z, \tilde{z} \in \mathcal{B}_{a_0}$, then

$$\begin{aligned}
\|(\mathcal{Q}z)(t) - (\mathcal{Q}\tilde{z})(t)\| &\leq k_T \int_0^t \big\|F\big(s,(z)_s,u(s)\big) - F\big(s,(\tilde{z})_s,u(s)\big)\big\|\, ds \\
&\leq k_T\lambda_{a_0} \int_0^t \|(z)_s - (\tilde{z})_s\|_{\mathfrak{C}_0}\, ds \\
&\leq k_T\lambda_{a_0} t \|z - \tilde{z}\|_{\mathfrak{C}_{c_1}}.
\end{aligned}$$

Further,

$$\begin{aligned}
\|(\mathcal{Q}^2z)(t) - (\mathcal{Q}^2\tilde{z})(t)\| &\leq k_T \int_0^t \big\|F\big(s,(\mathcal{Q}z)_s,u(s)\big) - F\big(s,(\mathcal{Q}\tilde{z})_s,u(s)\big)\big\|\, ds \\
&\leq k_T\lambda_{a_0} \int_0^t \|(\mathcal{Q}z)_s - (\mathcal{Q}\tilde{z})_s\|_{\mathfrak{C}_0}\, ds \\
&\leq k_T\lambda_{a_0} \int_0^t \sup_{-a\leq\varrho\leq 0} \|(\mathcal{Q}z)(s+\varrho) - (\mathcal{Q}\tilde{z})(s+\varrho)\|\, ds \\
&\leq k_T\lambda_{a_0} \int_0^t \Bigg(\sup_{-a\leq\varrho\leq 0} \|(\mathcal{Q}z)(\varrho) - (\mathcal{Q}\tilde{z})(\varrho)\| \\
&\quad + \sup_{0\leq\varrho\leq s} \|(\mathcal{Q}z)(\varrho) - (\mathcal{Q}\tilde{z})(\varrho)\| \Bigg)\, ds \\
&= k_T\lambda_{a_0} \int_0^t \sup_{0\leq\varrho\leq s} \|(\mathcal{Q}z)(\varrho) - (\mathcal{Q}\tilde{z})(\varrho)\|\, ds \\
&\leq k_T\lambda_{a_0} \int_0^t k_T\lambda_{a_0} s \|z - \tilde{z}\|_{\mathfrak{C}_{c_1}}\, ds \\
&\leq \frac{(k_T\lambda_{a_0} t)^2}{2} \|z - \tilde{z}\|_{\mathfrak{C}_{c_1}}.
\end{aligned}$$

Repeating the above process, we have

$$\begin{aligned}
\|(\mathcal{Q}^nz)(t) - (\mathcal{Q}^n\tilde{z})(t)\| &\leq \frac{(k_T\lambda_{a_0} t)^n}{n!} \|z - \tilde{z}\|_{\mathfrak{C}_{c_1}} \\
&\leq \frac{(k_T\lambda_{a_0} c_1)^n}{n!} \|z - \tilde{z}\|_{\mathfrak{C}_{c_1}}.
\end{aligned}$$

Therefore

$$\|(\mathcal{Q}^nz) - (\mathcal{Q}^n\tilde{z})\|_{\mathfrak{C}_{c_1}} \leq \frac{(k_T\lambda_{a_0} c_1)^n}{n!} \|z - \tilde{z}\|_{\mathfrak{C}_{c_1}},$$

which shows that $\mathcal{Q}^n$ is a contraction map for sufficiently large value of n. By Banach fixed poin theorem, $\mathcal{Q}$ has a fixed point in $\mathcal{B}_{a_0}$. So (3.1.2) is a mild solution on $[-a, c_1]$. In similar way, the existence of mild solution on $[c_1, c_2]$, where $c_1 < c_2$, can be shown. Applying the above technique, one can deduce that (3.1.2) is a mild solution on the maximal existing interval $[-a, c^*)$, $c^* \leq c$.

Next, we show the boundedness of solution. Clearly $z(t)$ is bounded on $[-a, 0]$. For $t \in [0, c^*)$, one has

$$\begin{aligned}\|z(t)\| &\leq k_T\|\wp(0)\| + k_T k_B \int_0^t \|u(s)\|\, ds + k_T k_B \int_0^t \|u(s-a)\|\, ds \\ &\quad + k_T k_F \int_0^t \big(1 + \|z_s\|_{\mathfrak{C}_0} + \|u(s)\|_{V'}\big)\, ds \\ &\leq k_T\big(\|\wp(0)\| + \sqrt{c}\|u\|_U(2k_B + k_F) + k_F c\big) + k_T k_F \int_0^t \|z_s\|_{\mathfrak{C}_0}\, ds.\end{aligned}$$

By Lemma 3.2.1, we have

$$\|z(t)\| \leq \big(k_T\big(\|\wp(0)\| + \sqrt{c}\|u\|_U(2k_B + k_F) + k_F c\big) + k_\wp\big) \exp\big(k_T k_F c\big),$$

which shows that $z(t)$ is bounded on $[-a, c^*)$ and hence it is defined on $[-a, c]$.

For uniqueness, let z_1 and z_2 be any two solutions of (3.1.1). Since $z_1(t) = z_2(t) = \wp(t)$ on $[-a, 0]$, therefore the solution is unique on $[-a, 0]$. For $t \in [0, c]$, set

$$a^* = \max\big\{\|z_1\|_{\mathfrak{C}_c}, \|z_2\|_{\mathfrak{C}_c}\big\}.$$

Then

$$\begin{aligned}\|z_1(t) - z_2(t)\|_V &\leq k_T \int_0^t \big\|F\big(s, (z_1)_s, u(s)\big) - F\big(s, (z_2)_s, u(s)\big)\big\|\, ds \\ &\leq k_T \lambda_{a^*} \int_0^t \big\|(z_1)_s - (z_2)_s\big\|_{\mathfrak{C}_0}\, ds.\end{aligned}$$

Therefore

$$\begin{aligned}\|(z_1)_t - (z_2)_t\|_{\mathfrak{C}_0} &\leq \sup_{-a\leq\varrho\leq 0} \|z_1(\varrho) - z_2(\varrho)\|_V + \sup_{0\leq\varrho\leq t} \|z_1(\varrho) - z_2(\varrho)\|_V \\ &= \sup_{0\leq\varrho\leq t} \|z_1(\varrho) - z_2(\varrho)\|_V \\ &\leq k_T \lambda_{a^*} \int_0^t \big\|(z_1)_s - (z_2)_s\big\|_{\mathfrak{C}_0}\, ds.\end{aligned}$$

By Gronwall's inequality it follows that $(z_1)_t = (z_2)_t$ for all $t \in [0, c]$. Hence $z_1 = z_2$. ■

3.3 Controllability results

For further discussion, we suppose the following conditions:

(H_3) the system (3.1.4) is approximately controllable;

(H_4) there exists a function $q \in L_1[0, c]$ such that

$$\|F(t, z, u)\| \leq q(t)$$

for all $(t, z, u) \in [0, c] \times \mathfrak{C}_0 \times V'$.

First we prove the controllability of the linear delay system (3.1.3).

Theorem 3.3.1. *Under the hypothesis* (H_3), *the system (3.1.3) is approximately controllable.*

Proof. Let $\varepsilon > 0$ be given. Since $0 < c < \infty$ therefore there is a positive integer ℓ such that $c \in ((\ell - 1)a, \ell a]$. Suppose $z_1, z_2, \ldots, z_{\ell-1}$ are given in V. Now consider the system

$$\begin{cases} \dot{\xi}(t) = A\xi(t) + B_0 u(t), & t \in (0, a], \\ \xi(0) = \wp(0). \end{cases} \tag{3.3.1}$$

Set $\xi_1 = z_1$. By assumption (H_3), there is a control u_1 such that the mild solution $\xi(t)$ of (3.3.1) is given by

$$\xi(t) = T(t)\wp(0) + \int_0^t T(t - s)B_0 u_1(s)\, ds, \quad 0 \leq t \leq a$$

and it satisfies $\|\xi(a) - \xi_1\| \leq \varepsilon$.

Define

$$r_1(t) = \begin{cases} 0, & t \in [-a, 0], \\ u_1(t), & t \in (0, a]. \end{cases}$$

Let

$$z(t) = T(t)\wp(0) + \int_0^t T(t - s)B_0 r_1(s)\, ds + \int_0^t T(t - s)B_1 r_1(s - a)\, ds, \quad 0 \leq t \leq a.$$

Then

$$\|z(a) - z_1\| = \|\xi(a) - \xi_1\|$$

$$\leq \varepsilon.$$

Denote $\xi(a)$ by ξ_a and consider the system

$$\begin{cases} \dot{\xi}(t) = A\xi(t) + B_0 u(t), \quad t \in (a, 2a], \\ \xi(a) = \xi_a. \end{cases} \tag{3.3.2}$$

Set $\xi_2 = z_2 - \chi_{2h}$, where $\chi_{2h} = \int_0^{2h} T(2h-s)B_1 r_1(s-a)\,ds$ is known. Again by assumption (H_3), there is a control u_2 such that the mild solution $\xi(t)$ of (3.3.2) is given by

$$\xi(t) = T(t)\wp(0) + \int_0^t T(t-s)B_0 u_2(s)\,ds, \quad a \leq t \leq 2a$$

and it satisfies $\|\xi(2a) - \xi_2\| \leq \varepsilon$.

Define

$$r_2(t) = \begin{cases} r_1(t), \quad t \in [0, a], \\ u_2(t), \quad t \in (a, 2a]. \end{cases}$$

Let

$$z(t) = T(t)\wp(0) + \int_0^t T(t-s)B_0 r_2(s)\,ds + \int_0^t T(t-s)B_1 r_2(s-a)\,ds, \quad a \leq t \leq 2a.$$

Then

$$\begin{aligned} \|z(2a) - z_2\| &= \|\xi(2a) + \chi_{2h} - z_2\| \\ &= \|\xi(2a) - \xi_2\| \\ &\leq \varepsilon. \end{aligned}$$

Continuing in similar way, at the ℓ-th step, we get

$$\begin{cases} \dot{\xi}(t) = A\xi(t) + B_0 u(t), \quad t \in ((\ell-1)a, c], \\ \xi((\ell-1)a) = \xi_{(\ell-1)a}. \end{cases} \tag{3.3.3}$$

Set $\xi_\ell = z_c - \chi_c$, where $\chi_c = \int_0^c T(c-s)B_1 r_{\ell-1}(s-a)\,ds$ is known. Then there is a control u_ℓ such that the mild solution $\xi(t)$ of (3.3.3) is given by

$$\xi(t) = T(t)\wp(0) + \int_0^t T(t-s)B_0 u_\ell(s)\,ds, \quad (\ell-1)a \leq t \leq c$$

and it satisfies $\|\xi(c) - \xi_\ell\| \leq \varepsilon$.

Define

$$r_\ell(t) = \begin{cases} r_{\ell-1}(t), & t \in ((\ell-2)a, (\ell-1)a], \\ u_\ell(t), & t \in ((\ell-1)a, c]. \end{cases}$$

Let

$$z(t) = T(t)\wp(0) + \int_0^t T(t-s)B_0 r_\ell(s)\,ds + \int_0^t T(t-s)B_1 r_\ell(s-a)\,ds, \quad (\ell-1)a \leq t \leq c.$$

Then

$$\begin{aligned} \|z(c) - z_c\| &= \|\xi(c) + \chi_c - z_c\| \\ &= \|\xi(c) - \xi_\ell\| \\ &\leq \varepsilon. \end{aligned}$$

Now if we define the control r on $[-a, c]$ as

$$r(t) = \begin{cases} 0, & t \in [-a, 0], \\ r_i(t), & t \in ((i-1)a, ia], \quad i = 1, 2, \ldots, (\ell-1); \\ u_\ell(t), & t \in ((\ell-1)a, c]. \end{cases}$$

Then we can write the mild solution $z(t)$ of (3.1.3) corresponding to the control $r(t)$ as

$$z(t) = T(t)\wp(0) + \int_0^t T(t-s)B_0 r(s)\,ds + \int_0^t T(t-s)B_1 r(s-a)\,ds, \quad 0 \leq t \leq c$$

and it satisfies $\|z(c) - z_c\| \leq \varepsilon$. ■

Next, we show the controllability of the original system using the above theorem.

Theorem 3.3.2. *Under hypotheses (H_1)-(H_4), the semilinear system (3.1.1) is approximately controllable.*

Proof. : Since $q \in L_1[0, c]$, we are able to find an increasing sequence $\langle c_n \rangle$ in $[0, c]$ such that $c_n \to c$ and

$$\int_{c_n}^c q(t)\,dt \to 0, \quad as \quad n \to \infty.$$

Now by Theorem 3.3.1, for any given $\varepsilon > 0$ and $z_c \in V$, we can select a control $\tilde{u}_0 \in U$ satisfying

$$\left\| z_c - T(c)\wp(0) - \int_0^c T(c-s)B_0\tilde{u}_0(s)\,ds - \int_0^c T(c-s)B_1\tilde{u}_0(s-a)\,ds \right\| \le \frac{\varepsilon}{2}.$$

Let $z_1 = z(c_1, \wp(0), \tilde{u}_0)$. Again by Theorem 3.3.1, we can select a control $\tilde{u}_1 \in L_2([c_1, c]; V')$ satisfying

$$\left\| z_c - T(c-c_1)z_1 - \int_{c_1}^c T(c-s)B_0\tilde{u}_1(s)\,ds - \int_{c_1}^c T(c-s)B_1\tilde{u}_1(s-a)\,ds \right\| \le \frac{\varepsilon}{2}.$$

Define

$$\tilde{w}_1(t) = \begin{cases} \tilde{u}_0(t), & t \in [0, c_1), \\ \tilde{u}_1(t), & t \in [c_1, c]. \end{cases}$$

Evidently, $\tilde{w}_1 \in U$. Continuing in same manner we obtain three sequences $z_n, \tilde{u}_n$ and $\tilde{w}_n$ such that $\tilde{u}_n \in L_2([c_n, c]; V')$, $\tilde{w}_n \in U$ given by

$$\tilde{w}_n(t) = \begin{cases} \tilde{u}_{n-1}(t), & t \in [0, c_n), \\ \tilde{u}_n(t), & t \in [c_n, c] \end{cases}$$

and $z_n = z(c_n, \wp(0), \tilde{u}_{n-1})$ with

$$\left\| z_c - T(c-c_n)z_n - \int_{c_n}^c T(c-s)B_0\tilde{u}_n(s)\,ds - \int_{c_n}^c T(c-s)B_1\tilde{u}_n(s-a)\,ds \right\| \le \frac{\varepsilon}{2}.$$

Now, if $z(c, \tilde{w}_n)$ be the mild solution of (3.1.1) for the control $\tilde{w}_n$, then

$$\begin{aligned} z(c, \tilde{w}_n) &= T(c-c_n)\Big(T(c_n)\wp(0) + \int_0^{c_n} T(c_n-s)B_0\tilde{w}_n(s)\,ds \\ &\quad + \int_0^{c_n} T(c_n-s)B_1\tilde{w}_n(s-a)\,ds + \int_0^{c_n} T(c_n-s)F\big(s, z_s, \tilde{w}_n(s)\big)\,ds\Big) \\ &\quad + \int_{c_n}^c T(c-s)B_0\tilde{w}_n(s)\,ds + \int_{c_n}^c T(c-s)B_1\tilde{w}_n(s-a)\,ds \\ &\quad + \int_{c_n}^c T(c-s)F\big(s, z_s, \tilde{w}_n(s)\big)\,ds \\ &= T(c-c_n)\Big(T(c_n)\wp(0) + \int_0^{c_n} T(c_n-s)B_0\tilde{u}_{n-1}(s)\,ds \\ &\quad + \int_0^{c_n} T(c_n-s)B_1\tilde{u}_{n-1}(s-a)\,ds + \int_0^{c_n} T(c_n-s)F\big(s, z_s, \tilde{u}_{n-1}(s)\big)\,ds\Big) \end{aligned}$$

$$
\begin{aligned}
&+\int_{c_n}^{c} T(c-s)B_0\tilde{u}_n(s)\,ds + \int_{c_n}^{c} T(c-s)B_1\tilde{u}_n(s-a)\,ds \\
&+\int_{c_n}^{c} T(c-s)F\big(s, z_s, \tilde{u}_n(s)\big)\,ds \\
=\;& T(c-c_n)z_n + \int_{c_n}^{c} T(t-s)B_0\tilde{u}_n(s)\,ds + \int_{c_n}^{c} T(c-s)B_1\tilde{u}_n(s-a)\,ds \\
&+\int_{c_n}^{c} T(c-s)F\big(s, z_s, \tilde{u}_n(s)\big)\,ds.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
&\|z(c,\tilde{w}_n) - z_c\| \\
&\quad\le \left\| T(c-c_n)z_n + \int_{c_n}^{c} T(c-s)B_0\tilde{u}_n(s)\,ds + \int_{c_n}^{c} T(c-s)B_1\tilde{u}_n(s-a)\,ds - z_c \right\| \\
&\qquad + \left\| \int_{c_n}^{c} T(c-s)F\big(s, z_s, \tilde{u}_n(s)\big)\,ds \right\| \\
&\quad\le \frac{\varepsilon}{2} + k_T \int_{c_n}^{c} \big\|F\big(s, z_s, \tilde{u}_n(s)\big)\big\|\,ds \\
&\quad\le \frac{\varepsilon}{2} + k_T \int_{c_n}^{c} q(s)\,ds \\
&\quad\le \frac{\varepsilon}{2} + k_T \frac{\varepsilon}{2k_T} \\
&\quad= \varepsilon
\end{aligned}
$$

for sufficiently large value of n. Hence the systen (3.1.1) is approximately controllable. ∎

Remark 3.3.3. *Under hypotheses* (H_3) *and* (H_4)*, the semilinear system (3.1.1) is approximately controllable if it has a solution for each control* $u \in U$*.*

3.4 Example

Example 1. Consider the semilinear heat equation for $0 < x < \pi$ with control delay

$$
\begin{cases}
\frac{\partial \widehat{z}(t,x)}{\partial t} = \frac{\partial^2 \widehat{z}(t,x)}{\partial x^2} + B_0\widehat{u}(t,x) + \widehat{u}(t-a,x) & \\
\qquad\qquad + F\big(t, \widehat{z}(t+\theta,x), \widehat{u}(t,x)\big), & t \in (0,c], \\
\widehat{z}(t,0) = \widehat{z}(t,\pi) = 0, & t \in (0,c], \\
\widehat{z}(t,x) = \widehat{\wp}(t,x), & t \in [-a,0].
\end{cases}
\tag{3.4.1}
$$

To write it in abstract form, we make the following setting

(i) $V = L_2[0,\pi]$ and define $A : D(A) \subseteq V \to V$ by $Ay = \frac{d^2y}{dx^2}$ with domain

$$D(A) = \left\{ y \in L_2[0,\pi] \,\middle|\, y,\ \frac{\partial y}{\partial x} \quad \text{are absolutely continuous,} \quad \frac{\partial^2 y}{\partial x^2} \in L_2[0,\pi] \right.$$
$$\left. \text{and} \quad y(0) = y(\pi) = 0 \right\}.$$

(ii) Let $\xi_\ell(x) = (\frac{2}{\pi})^{1/2} \sin \ell x$, $0 \le x \le \pi$, $\ell = 1, 2, \ldots$, then $\lambda_\ell = -\ell^2$ is the eigenvalue of A with corresponding eigenfunction ξ_ℓ and the family $\{\xi_\ell\}_{\ell \in \mathbb{N}}$ form a complete orthonormal set for V and $\exp(\lambda_\ell t)$ is the eigenvalue of the C_0−semigroup $T(t)$ generated by A.

(iii) Define

$$V' = \left\{ \upsilon \in L_2[0,\pi] \,\middle|\, \upsilon = \sum_{\ell=2}^{\infty} \alpha_\ell \xi_\ell \quad \text{with} \quad \sum_{\ell=2}^{\infty} \alpha_\ell^2 < \infty \right\}$$

with the norm

$$\|\upsilon\|_{V'} = \sqrt{\sum_{\ell=2}^{\infty} \alpha_\ell^2}.$$

Let $B_0 : V' \to V$ be a continuous linear map defined as

$$B_0 \upsilon = 2\alpha_2 \xi_1 + \sum_{\ell=2}^{\infty} \alpha_\ell \xi_\ell, \quad \sum_{\ell=2}^{\infty} \alpha_\ell \xi_\ell \in V'.$$

The abstract form of (3.4.1) is

$$\begin{cases} \dot{z}(t) = Az(t) + B_0 u(t) + B_1 u(t-a) \\ \qquad\qquad + F\big(t, z(t+\theta), u(t)\big), & t \in (0, c], \\ z(t) = \wp(t), & t \in [-a, 0], \end{cases} \tag{3.4.2}$$

where $B_1 = I$, $z(t) = \widehat{z}(t,\cdot)$, $u(t) = \widehat{u}(t,\cdot)$ and $\wp(t) = \widehat{\wp}(t,\cdot)$.

If we take

$$F\big(t, z_t, u(t)\big) = \left(\frac{t\,\|z_t\|^2_{\mathfrak{C}_0}}{1 + \|z_t\|^2_{\mathfrak{C}_0}} \xi_3(x) + \frac{t^2 \|u(t)\|}{1 + \|u(t)\|} \xi_4(x) \right),$$

then

$$\big\|F\big(t, z_t, u(t)\big)\big\| \le (t + t^2)$$

$$\leq c(1+c)\left(1+\|z_t\|_{\mathfrak{C}_0}+\|u(t)\|\right).$$

Hence (H_2) and (H_4) are satisfied. Also,

$$\begin{aligned}\left\|F\big(t,(z_1)_t,u(t)\big)-F\big(t,(z_2)_t,u(t)\big)\right\| &\leq c\left(\|(z_1)_t\|_{\mathfrak{C}_0}+\|(z_2)_t\|_{\mathfrak{C}_0}\right)\|(z_1)_t-(z_2)_t\|_{\mathfrak{C}_0}\\ &\leq 2cr\,\|(z_1)_t-(z_2)_t\|_{\mathfrak{C}_0}\\ &= \lambda_r\,\|(z_1)_t-(z_2)_t\|_{\mathfrak{C}_0}\end{aligned}$$

for any $(z_1)_t,(z_2)_t\in\mathcal{B}(0,r)\subset\mathfrak{C}_0$ and $u(t)\in V'$. Hence (H_1) is satisfied. Since the hypothesis (H_3) is satisfied [18], therefore approximate controllability of the system (3.4.2) follows from Theorem 3.3.2.

3.5 Concluding remarks

In this chapter, the concept of approximate controllability of first-order retarded semilinear system has been presented. Here, the nonlinear function is assumed to be locally Lipschitz continuous. The result of existence has been deduced by utilizing iterative technique and a fixed point theorem. For this, we proved the Lemma 3.2.1. The controllability results have been deduced by assuming that the associated linear system without delay is approximately controllable.

Chapter 4

Controllability of Second-Order Nonlocal Retarded Semilinear Systems with Fixed Delays in Control

This chapter contains two sections. The first section deals with the approximate and exact controllability of second-order nonlocal retarded semilinear system with control delay. In this section the existence of mild solution is derived applying fixed point approach and cosine family. The controllability of the associated linear system with delay is proved by the method of steps and then the controllability of actual system is shown by proving that the reachable set of semilinear system contains the reachable set of the associated linear system without delay. In the second section, the approximate controllability of second-order nonlocal retarded semilinear system with multiple delays in control is discussed in Banach spaces. The existence of solution is derived under locally Lipschitz continuity of nonlinear function. Then the approximate controllability of associated linear system and actual system is proved by assuming that the associated linear system without delay is approximately controllable. The Banach fixed point theorem combined with the theory of cosine family and iterative technique, are the main tools used in this chapter.

4.1 Controllability of second-order nonlocal retarded semilinear systems with delay in control

4.1.1 Introduction and preliminaries

Let V and V' be Hilbert spaces and $U = L_2([0,c];V')$ be a function space. Let $\mathfrak{C}_t = C([-a,t];V)$ denotes the set of all V-valued continuous functions defined on $[-a,t]$ with the norm $\|z\|_{\mathfrak{C}_t} = \sup_{-a\leq\varrho\leq t} \|z(\varrho)\|$. Consider the following second-order retarded semilinear control system:

$$\begin{cases} \ddot{z}(t) = Az(t) + B_0u(t) + B_1u(t-a) \\ \qquad\qquad + F\big(t, z_{\varsigma(t)}, u(t) + u(t-a)\big), \quad t \in (0,c], \\ \psi(z) = h, \quad u(t) = 0, \qquad\qquad\qquad t \in [-a,0], \\ \dot{z}(0) = y_1, \end{cases} \tag{4.1.1}$$

where the state z takes its values in the space V; the control u takes its value in the Banach space V'; $a > 0$ represents the delay. A generates a strongly continuous cosine family $\{C(t)\}_{t\in\mathbb{R}}$ on V; B_0, B_1 are continuous linear maps from V' to V; the map $\varsigma : [0,c] \to [0,c]$ is nondecreasing and nonexpansive such that $\varsigma(t) \leq t$; the delay function $z_{\varsigma(t)} \in \mathfrak{C}_0$ defined by $z_{\varsigma(t)}(\theta) = z(\varsigma(t)+\theta), \quad \theta \in [-a,0]$, monitors the retarded state; and $F : [0,c] \times \mathfrak{C}_0 \times V' \to V$ is nonlinear. The functions ψ and h together represent the nonlocal delay condition.

The field of nonlocal differential systems has been observed and expeditiously growing after the great work by Chabrowski [15], who introduced the concept of nonlocal condition about three decades ago. The physical significance of nonlocal condition was given by Byszewski [13]. nonlocal initial conditions have a lot of applications in areas such as population dynamics, blood flow problems, thermo-elasticity, underground water flow, chemical engineering, etc. Controllability of nonlocal retarded semilinear stochastic system was studied by Shukla et. al. [79]. Utilizing a fixed point theorem, Urvashi and Sukavanam [1] deduced that nonlocal semilinear stochastic system of second-order is approximately controllable under some conditions. Utilizing a fixed point theorem, Kumar and Sukavanam [54] analyzed the controllability for nonlocal differential systems of second-order. Kumar and Tomar [56] proved the contollability of second-order nonlocal retarded semilinear systems without converting them into first-order systems. However, as for as we know, there is no discussion on controllability of nonlocal semilinear retarded systems of second-order with control delay.

Definition 4.1.1. Let $\wp \in \mathfrak{C}_0$ be such that $\psi(\wp) = h$. A function $z \in \mathfrak{C}_c$ is said to be a mild solution of (4.1.1) if it satisfies

$$z(t) = \begin{cases} \mathcal{C}(t)\wp(0) + \mathcal{S}(t)y_1 + \int_0^t \mathcal{S}(t-s)\big(B_0u(s) + B_1u(s-a)\big)ds \\ + \int_0^t \mathcal{S}(t-s)F\big(s, z_{\varsigma(s)}, u(s) + u(s-a)\big)\, ds, & t \in (0, c], \\ \wp(t), & t \in [-a, 0]. \end{cases}$$

Moreover, if $\wp(0) \in V_1$, z is continuously differentiable on $[0, c]$ and

$$\begin{aligned} \dot{z}(t) =& A\mathcal{S}(t)\wp(0) + \mathcal{C}(t)y_1 + \int_0^t \mathcal{C}(t-s)\left(B_0u(s) + B_1u(s-a)\right)\, ds \\ &+ \int_0^t \mathcal{C}(t-s)F\big(s, z_{\varsigma(s)}, u(s) + u(s-a)\big)\, ds. \end{aligned}$$

Here, the set V_1 is given by

$$V_1 = \big\{y \in V \;\big|\; \mathcal{C}(t)y \quad \text{is once continuously differentiable w. r. to} \quad t\big\}.$$

For the system (4.1.1), the corresponding linear systems with delay and without delay are

$$\begin{cases} \ddot{z}(t) = Az(t) + B_0u(t) + B_1u(t-a), \quad t \in (0, c], \\ z(0) = \wp(0), \\ \dot{z}(0) = y_1 \end{cases} \tag{4.1.2}$$

and

$$\begin{cases} \ddot{z}(t) = Az(t) + B_0u(t), \quad t \in (0, c], \\ z(0) = \wp(0), \\ \dot{z}(0) = y_1, \end{cases} \tag{4.1.3}$$

respectively, provided that $\psi(\wp) = h$.

We define the following sets:

(*i*) The reachable set $\mathfrak{R}_c(F)$ for (4.1.1) is given by

$$\mathfrak{R}_c(F) = \big\{z(c,u) \in V \;\big|\; z \text{ is a mild solution of (4.1.1) associated with } u \in U\big\}.$$

(*ii*) The trajectory reachable set $\mathfrak{R}(F)$ for (4.1.1) is given by

$$\mathfrak{R}(F) = \big\{z(\cdot,u) \in \mathfrak{C}_c \;\big|\; z \text{ is a mild solution of (4.1.1) associated with } u \in U\big\}.$$

(*iii*) The reachable set $\mathfrak{R}_c(0)$ for (4.1.2) is given by

$$\mathfrak{R}_c(0) = \big\{z(c,u) \in V \;\big|\; z \text{ is a mild solution of (4.1.2) associated with } u \in U\big\}.$$

(*iv*) The trajectory reachable set $K(0)$ for (4.1.2) is given by

$$\mathfrak{R}(0) = \big\{z(\cdot,u) \in \mathfrak{C}_c \;\big|\; z \text{ is a mild solution of (4.1.2) associated with } u \in U\big\}.$$

(*v*) $\tilde{\mathfrak{R}}(0) = \big\{\omega(\cdot,u) \in \mathfrak{C}_c \;\big|\; \omega \text{ is a concatenation of } \psi \text{ and } z, \text{ where } z \in \mathfrak{R}(0)\big\}.$

Definition 4.1.2. The system (4.1.1) is approximately (exactly) controllable if $\overline{\mathfrak{R}_c(F)} = V \quad (\mathfrak{R}_c(F) = V)$.

Definition 4.1.3. The system (4.1.2) is approximately (exactly) controllable if $\overline{\mathfrak{R}_c(0)} = V \quad (\mathfrak{R}_c(0) = V)$.

For further discussion we assume that there are constants $k_{\mathcal{C}}$ and $k_{\mathcal{S}}$ satisfying $\|\mathcal{C}(t)\| \le k_{\mathcal{C}}$ and $\|\mathcal{S}(t)\| \le k_{\mathcal{S}}$ for $0 \le t \le c$.

4.1.2 Existence of mild solution

To deduced the existence results we make the following hypotheses:

(H_1) $\psi : C([-a,0];V_1) \to C([-a,0];V_1)$ and for each given h there is a unique function $\wp \in C([-a,0];V_1)$ such that $\psi(\wp) = h$;

(H_2) the nonlinear map $F : [0,c] \times \mathfrak{C}_0 \times V' \to V$ is continuous in t and there is a constant $\lambda > 0$ such that

$$\|F(t,z,u) - F(t,\tilde{z},\tilde{u})\| \le \lambda \left(\|z - \tilde{z}\|_{\mathfrak{C}_0} + \|u - \tilde{u}\|\right)$$

for all $t \in [0,c]$; $z, \tilde{z} \in \mathfrak{C}_0$ and $u, \tilde{u} \in V'$.

Theorem 4.1.1. *If hypotheses (H_1) and (H_2) are true, then the system (4.1.1) admits a unique mild solution for every $u \in U$ and $y_1 \in V$. Moreover, the map $\wp \mapsto z$ from $C([-a,0];V_1)$ to $\mathfrak{C}_c$ is Lipschitz and it induces the uniqueness of the solution.*

Proof. For the existence and uniqueness of mild solution, it is sufficient to show that the map $\mathcal{Q} : \mathfrak{C}_c \to \mathfrak{C}_c$ defined by

$$(\mathcal{Q}z)(t) = \begin{cases} \mathcal{C}(t)\wp(0) + \mathcal{S}(t)y_1 + \displaystyle\int_0^t \mathcal{S}(t-s)\big(B_0u(s) + B_1u(s-a)\big)\, ds \\ + \displaystyle\int_0^t \mathcal{S}(t-s)F\big(s, z_{\varsigma(s)}, u(s) + u(s-a)\big)\, ds, & t \in (0,c], \\ \wp(t), & t \in [-a,0] \end{cases}$$

has a unique fixed point in $\mathfrak{C}_c$. Clearly, $\mathcal{Q}$ has a unique fixed point for $t \in [-a,0]$. Therefore proof is needed only for $t \in [0,c]$. Let $z, \tilde{z} \in C([0,c];V)$, then

$$\begin{aligned} \|(\mathcal{Q}z)(t) - (\mathcal{Q}\tilde{z})(t)\| &\leq k_{\mathcal{S}} \int_0^t \big\|F\big(s, z_{\varsigma(s)}, u(s) + u(s-a)\big) \\ &\qquad - F\big(s, \tilde{z}_{\varsigma(s)}, u(s) + u(s-a)\big)\big\|\, ds \\ &\leq k_{\mathcal{S}}\lambda \int_0^t \big\|z_{\varsigma(s)} - \tilde{z}_{\varsigma(s)}\big\|_{\mathfrak{C}_0}\, ds \\ &\leq k_{\mathcal{S}}\lambda \int_0^t \|z - \tilde{z}\|_{\mathfrak{C}_c}\, ds \\ &= k_{\mathcal{S}}\lambda t \|z - \tilde{z}\|_{\mathfrak{C}_c}. \end{aligned}$$

Since $-a \leq \varsigma(s) + \theta \leq s \leq c$ for $s \in [0,c]$ and $\theta \in [-a,0]$, therefore we have

$$\begin{aligned} \|(\mathcal{Q}^2z)(t) - (\mathcal{Q}^2\tilde{z})(t)\| &\leq k_{\mathcal{S}} \int_0^t \big\|F\big(s, (\mathcal{Q}z)_{\varsigma(s)}, u(s) + u(s-a)\big) \\ &\qquad - F\left(s, (\mathcal{Q}\tilde{z})_{\varsigma(s)}, u(s) + u(s-a)\right)\big\|\, ds \\ &\leq k_{\mathcal{S}}\lambda \int_0^t \big\|(\mathcal{Q}z)_{\varsigma(s)} - (\mathcal{Q}\tilde{z})_{\varsigma(s)}\big\|_{\mathfrak{C}_0}\, ds \\ &\leq k_{\mathcal{S}}\lambda \int_0^t \sup_{-a\leq\varrho\leq 0} \|(\mathcal{Q}z)(\varsigma(s)+\varrho) - (\mathcal{Q}\tilde{z})(\varsigma(s)+\varrho)\|\, ds \\ &\leq k_{\mathcal{S}}\lambda \int_0^t \bigg(\sup_{-a\leq\varrho\leq 0} \|(\mathcal{Q}z)(\varrho) - (\mathcal{Q}\tilde{z})(\varrho)\| \\ &\qquad + \sup_{0\leq\varrho\leq s} \|(\mathcal{Q}z)(\varrho) - (\mathcal{Q}\tilde{z})(\varrho)\|\bigg)\, ds \end{aligned}$$

$$
\begin{aligned}
&= k_{\mathcal{S}}\lambda \int_0^t \sup_{0\le\varrho\le s} \|(\mathcal{Q}z)(\varrho) - (\mathcal{Q}\tilde{z})(\varrho)\|\, ds \\
&\le k_{\mathcal{S}}\lambda \int_0^t k_{\mathcal{S}}\lambda s \|z-\tilde{z}\|_{\mathfrak{C}_c}\, ds \\
&\le \frac{(k_{\mathcal{S}}\lambda t)^2}{2}\|z-\tilde{z}\|_{\mathfrak{C}_c}.
\end{aligned}
$$

Repeating the above process, we obtain

$$
\begin{aligned}
\|(\mathcal{Q}^n z)(t) - (\mathcal{Q}^n \tilde{z})(t)\| &\le \frac{(k_{\mathcal{S}}\lambda t)^n}{n!}\|z-\tilde{z}\|_{\mathfrak{C}_c} \\
&\le \frac{(k_{\mathcal{S}}\lambda c)^n}{n!}\|z-\tilde{z}\|_{\mathfrak{C}_c}.
\end{aligned}
$$

Therefore

$$
\|\mathcal{Q}^n z - \mathcal{Q}^n \tilde{z}\|_{\mathfrak{C}_c} \le \frac{(k_{\mathcal{S}}\lambda c)^n}{n!}\|z-\tilde{z}\|_{\mathfrak{C}_c}.
$$

But $\frac{(k_{\mathcal{S}}\lambda c)^n}{n!} < 1$ for n to be large enough. Therefore by generalized Banach contraction principle $\mathcal{Q}$ has a unique fixed point in $C([0,c];V)$.

Next, let $z_1, z_2 \in \mathfrak{C}_c$ be any two solutions of (4.1.1) associated with the nonlocal delay functions $h_1, h_2 \in \mathfrak{C}_0$, respectively. Then by hypothesis (H_1), one can find $\wp_1, \wp_2 \in \mathfrak{C}_0$ such that $g(\wp_\ell) = h_\ell, \quad \ell = 1,2$. Now, one can write

$$
z_\ell(t) = \begin{cases} \mathcal{C}(t)\wp_\ell(0) + \mathcal{S}(t)y_1 + \int_0^t \mathcal{S}(t-s)\big(B_0u(s) + B_1u(s-a)\big)\, ds & \\ + \int_0^t \mathcal{S}(t-s)F\big(s, (z_\ell)_{\varsigma(s)}, u(s)+u(s-a)\big)\, ds, & t\in(0,c], \\ \wp_\ell(t), & t\in[-a,0]. \end{cases}
$$

For $t\in[-a,0]$, the case is trivial. Therefore we discuss only for $t\in[0,c]$.

$$
\begin{aligned}
\|z_1(t) - z_2(t)\| &\le \|\mathcal{C}(t)\left(\wp_1(0) - \wp_2(0)\right)\| + k_{\mathcal{S}}\int_0^t \lambda \left\|(z_1)_{\varsigma(s)} - (z_2)_{\varsigma(s)}\right\|_{\mathfrak{C}_0} ds \\
&\le k_{\mathcal{C}}\|\wp_1 - \wp_2\|_{\mathfrak{C}_0} + k_{\mathcal{S}}\lambda \int_0^t \left\|(z_1)_{\varsigma(s)} - (z_2)_{\varsigma(s)}\right\|_{\mathfrak{C}_0} ds. \qquad (4.1.4)
\end{aligned}
$$

Now,

$$
\begin{aligned}
\left\|(z_1)_{\varsigma(s)} - (z_2)_{\varsigma(s)}\right\|_{\mathfrak{C}_0} &= \sup_{-a\le\theta\le 0}\|z_1(\varsigma(s)+\theta) - z_2(\varsigma(s)+\theta)\| \\
&\le \sup_{-a\le\varrho\le 0}\|z_1(\varrho) - z_2(\varrho)\| + \sup_{0\le\varrho\le s}\|z_1(\varrho) - z_2(\varrho)\|
\end{aligned}
$$

$$= \sup_{-a\le\varrho\le 0} \|\wp_1(\varrho) - \wp_2(\varrho)\| + \sup_{0\le\varrho\le s} \|z_1(\varrho) - z_2(\varrho)\|$$
$$\le \|\wp_1 - \wp_2\|_{\mathfrak{C}_0} + \sup_{0\le\varrho\le s} \|z_1(\varrho) - z_2(\varrho)\|.$$

and hence from (4.1.4)

$$\sup_{0\le\varrho\le t} \|z_1(\varrho) - z_2(\varrho)\| \le (k_{\mathcal{C}} + k_{\mathcal{S}}\lambda c)\|\wp_1 - \wp_2\|_{\mathfrak{C}_0} + k_{\mathcal{S}}\lambda \int_0^c \sup_{0\le\varrho\le s} \|z_1(\varrho) - z_2(\varrho)\|\, ds, \quad 0 \le t \le c.$$

Applying Gronwall's inequality, we obtain

$$\|z_1 - z_2\|_{\mathfrak{C}_c} \le (k_{\mathcal{C}} + k_{\mathcal{S}}\lambda c)\exp(k_{\mathcal{S}}\lambda c)\|\wp_1 - \wp_2\|_{\mathfrak{C}_0}.$$

This completes the proof. ∎

Remark 4.1.2. *It is easily seen that if the map ψ is not injective, then the system (4.1.1) may have more than one solution for a fixed control $u \in U$.*

4.1.3 Controllability results

First we prove the controllability of linear delay system (4.1.2). For this, we derive a new form of the mild solution of the system

$$\begin{cases} \ddot{\xi}(t) = A\xi(t) + B_0u(t), & t \in (t_0, c], \\ \xi(t_0) = z(t_0) = \xi_{t_0}, \\ \dot{\xi}(t_0) = \dot{z}(t_0) = \dot{\xi}_{t_0}, \end{cases} \tag{4.1.5}$$

where $t_0 \in (0, c)$ is fixed and $z(t)$ is the mild solution of the system

$$\begin{cases} \ddot{z}(t) = Az(t) + B_0u(t), & t \in (0, t_0], \\ z(0) = \wp(0), \\ \dot{z}(0) = y_1. \end{cases} \tag{4.1.6}$$

Lemma 4.1.3. *If $\wp(0) \in V_1$, then the mild solution of (4.1.5) can be expressed as*

$$\xi(t) = \mathcal{C}(t)\wp(0) + \mathcal{S}(t)y_1 + \int_0^t \mathcal{S}(t-s)B_0u(s)\, ds, \quad \le t_0 \le t \le c.$$

Proof. We know that the mild solution of (4.1.5) is

$$\xi(t) = \mathcal{C}(t-t_0)z(t_0) + \mathcal{S}(t-t_0)\dot{z}(t_0) + \int_{t_0}^{t} \mathcal{S}(t-s)B_0u(s)\,ds, \ t_0 \le t \le c. \quad (4.1.7)$$

But $z(t)$ is the mild solution of (4.1.6) therefore we have

$$z(t_0) = \mathcal{C}(t_0)\wp(0) + \mathcal{S}(t_0)y_1 + \int_0^{t_0} \mathcal{S}(t_0-s)B_0u(s)\,ds \quad (4.1.8)$$

and

$$\dot{z}(t_0) = A\mathcal{S}(t_0)\wp(0) + \mathcal{C}(t_0)y_1 + \int_0^{t_0} \mathcal{C}(t_0-s)B_0u(s)\,ds. \quad (4.1.9)$$

Using (4.1.8) and (4.1.9) in (4.1.7), we get

$$\begin{aligned}\xi(t) = &\big(\mathcal{C}(t-t_0)\mathcal{C}(t_0) + A\mathcal{S}(t-t_0)\mathcal{S}(t_0)\big)\wp(0) + \big(\mathcal{C}(t-t_0)\mathcal{S}(t_0) + \mathcal{S}(t-t_0)\mathcal{C}(t_0)\big)y_1 \\ &+ \int_0^{t_0} \big(\mathcal{C}(t-t_0)\mathcal{S}(t_0-s) + \mathcal{S}(t-t_0)\mathcal{C}(t_0-s)\big)B_0u(s)\,ds \\ &+ \int_{t_0}^{t} \mathcal{S}(t-s)B_0u(s)\,ds \\ = &\ \mathcal{C}(t)\wp(0) + \mathcal{S}(t)y_1 + \int_0^t \mathcal{S}(t-s)B_0u(s)\,ds,\end{aligned}$$

which completes the proof. ■

Theorem 4.1.4. *Under the hypothesis* (H_1)*, the associated linear delay system (4.1.2) is approximately (exactly) controllable if the associated linear system (4.1.3) is approximately (exactly) controllable.*

Proof. Let the linear system (4.1.3) be approximately controllable and $\varepsilon > 0$ be given. Since $0 < c < \infty$, there is a positive integer ℓ such that $c \in ((\ell-1)a, \ell a]$. Suppose $\tilde{z}_1, \tilde{z}_2, \dots, \tilde{z}_{\ell-1}$ are given in the Hilbert space V. Now consider the linear system

$$\begin{cases} \ddot{\xi}(t) = A\xi(t) + B_0u(t), \quad t \in (0,a], \\ \xi(0) = \wp(0), \\ \dot{\xi}(0) = \xi_1 = y_1. \end{cases} \quad (4.1.10)$$

Set $\tilde{\xi}_1 = \tilde{z}_1$. By approximate controllability of (4.1.3) there is a control u_1 such that the mild solution $\xi(t)$ of (4.1.10) is given by

$$\xi(t) = \mathcal{C}(t)\wp(0) + \mathcal{S}(t)y_1 + \int_0^t \mathcal{S}(t-s)B_0u_1(s)\,ds, \quad 0 \le t \le a$$

and it satisfies $\|\xi(a) - \tilde{\xi}_1\| \leq \varepsilon$.

Define

$$r_1(t) = \begin{cases} 0, & t \in [-a, 0], \\ u_1(t), & t \in (0, a]. \end{cases}$$

Let

$$z(t) = \mathcal{C}(t)\wp(0) + \mathcal{S}(t)y_1 + \int_0^t \mathcal{S}(t-s)B_0 r_1(s)\, ds + \int_0^t \mathcal{S}(t-s)B_1 r_1(s-a)\, ds,\ 0 \leq t \leq a.$$

Then

$$\begin{aligned} \|z(a) - \tilde{z}_1\| &= \|\xi(a) - \tilde{\xi}_1\| \\ &\leq \varepsilon. \end{aligned}$$

Denote $\xi(a)$ by ξ_a and $\dot{\xi}(a)$ by $\dot{\xi}_a$, and consider the system

$$\begin{cases} \ddot{\xi}(t) = A\xi(t) + B_0 u(t), & t \in (a, 2a], \\ \xi(a) = \xi_a, \\ \dot{\xi}(a) = \dot{\xi}_a. \end{cases} \tag{4.1.11}$$

Set $\tilde{\xi}_2 = \tilde{z}_2 - \chi_{2a}$, where $\chi_{2a} = \int_0^{2a} \mathcal{S}(2a - s)B_1 r_1(s - a)\, ds$ is known. Again by approximate controllability of (4.1.3) there is a control u_2 such that the mild solution $\xi(t)$ of (4.1.11) is given by

$$\xi(t) = \mathcal{C}(t)\wp(0) + \mathcal{S}(t)y_1 + \int_0^t \mathcal{S}(t-s)B_0 u_2(s)\, ds, \quad a \leq t \leq 2a$$

and it satisfies $\|\xi(2a) - \tilde{\xi}_2\| \leq \varepsilon$.

Define

$$r_2(t) = \begin{cases} r_1(t), & t \in [0, a], \\ u_2(t), & t \in (a, 2a]. \end{cases}$$

Let

$$z(t) = \mathcal{C}(t)\wp(0) + \mathcal{S}(t)y_1 + \int_0^t \mathcal{S}(t-s)B_0 r_2(s)\, ds + \int_0^t \mathcal{S}(t-s)B_1 r_2(s-a)\, ds,\ a \leq t \leq 2a.$$

Then

$$\|z(2a) - \tilde{z}_2\| = \|\xi(2a) + \chi_{2a} - \tilde{z}_2\|$$

$$
\begin{aligned}
&= \|\xi(2a) - \tilde{\xi}_2\| \\
&\leq \varepsilon.
\end{aligned}
$$

Continuing in similar way, at the ℓ-th step, we get

$$
\begin{cases}
\ddot{\xi}(t) = A\xi(t) + B_0 u(t), \quad t \in ((\ell-1)a, c], \\
\xi((\ell-1)a) = \xi_{(\ell-1)a}, \\
\dot{\xi}((\ell-1)a) = \dot{\xi}_{(\ell-1)a}.
\end{cases} \tag{4.1.12}
$$

Set $\tilde{\xi}_\ell = z_c - \chi_c$, where $\chi_c = \int_0^c T_0(c-s)\mathcal{B}_1 r_{\ell-1}(s-a)\,ds$ is known. Then, there is a control u_ℓ such that the mild solution $\xi(t)$ of (4.1.12) is given by

$$
\xi(t) = \mathcal{C}(t)\wp(0) + \mathcal{S}(t)y_1 + \int_0^t \mathcal{S}(t-s)B_0 u_\ell(s)\,ds, \quad (\ell-1)a \leq t \leq c
$$

and it satisfies $\|\xi(c) - \tilde{\xi}_\ell\| \leq \varepsilon$.

Define

$$
r_\ell(t) = \begin{cases}
r_{\ell-1}(t), & t \in ((\ell-2)a, (\ell-1)a], \\
u_\ell(t), & t \in ((\ell-1)a, c].
\end{cases}
$$

Let

$$
\begin{aligned}
z(t) = \mathcal{C}(t)\wp(0) + \mathcal{S}(t)y_1 &+ \int_0^t \mathcal{S}(t-s)B_0 r_\ell(s)\,ds \\
&+ \int_0^t \mathcal{S}(t-s)B_1 r_\ell(s-a)\,ds, \quad (\ell-1)a \leq t \leq c.
\end{aligned}
$$

Then

$$
\begin{aligned}
\|z(c) - z_c\| &= \|\xi(c) + \chi_c - z_c\| \\
&= \|\xi(c) - \tilde{\xi}_\ell\| \\
&\leq \varepsilon.
\end{aligned}
$$

Now, if we define the control r on $[-a, c]$ as

$$
r(t) = \begin{cases}
0, & t \in [-a, 0], \\
r_i(t), & t \in ((i-1)a, ia], \quad i = 1, 2, \ldots, (\ell-1); \\
u_\ell(t), & t \in ((\ell-1)a, c],
\end{cases}
$$

then the mild solution $z(t)$ of (4.1.2) associated with $r(t)$ is given by

$$z(t) = \mathcal{C}(t)\wp(0)+\mathcal{S}(t)y_1+\int_0^t \mathcal{S}(t-s)B_0r(s)\,ds+\int_0^t \mathcal{S}(t-s)B_1r(s-a)\,ds,\ 0\le t\le c$$

and it satisfies $\|z(c)-z_c\|\le\varepsilon$. Hence (4.1.2) is approximately controllable. The proof for exact controllability is similar. ■

The forthcoming discussion obeys the following conditions:

(H_3) $\mathcal{R}(B_0)\supseteq\mathcal{R}(B_1)\supseteq\mathcal{R}(F)$;

(H_4) there is a $\delta>0$ such that

$$\|B_0u(t)+B_1u(t-a)\|\ge\delta\|u(t)+u(t-a)\| \quad \forall\ u(t)\in V';$$

(H_5) $\lambda<\delta$.

Following lemma shows that the iterative formula

$$\begin{aligned}&B_0u_n(t)+B_1u_n(t-a)\\ &\quad= B_0u^*(t)+B_1u^*(t-a)-F\big(t,z_{\varsigma(t)},u_{n-1}(t)+u_{n-1}(t-a)\big),\quad n=1,2,\ldots\end{aligned} \tag{4.1.13}$$

makes sense for each given $u^*\in U$ and $z_{\varsigma(t)}\in\mathfrak{C}_0$.

Lemma 4.1.5. *Under the hypothesis (H_3), the iterative formula given by (4.1.13) is well defined for each given $u^*\in U$.*

Proof. Since $0<c<\infty$ therefore $c\in((\ell-1)a,\ell a]$ for some $\ell\in\mathbb{N}$.
Let $u_0\in U$, then for $t\in(0,a]$

$$\begin{aligned}&B_0u^*(t)+B_1u^*(t-a)-F\big(t,z_{\varsigma(t)},u_0(t)+u_0(t-a)\big)-B_1u_1(t-a)\\ &\quad= B_0u^*(t)-F\big(t,z_{\varsigma(t)},u_0(t)+u_0(t-a)\big)\\ &\quad= B_0u_{11}(t)\ (\text{say}).\end{aligned}$$

If we take $u_1(t)=u_{11}(t)$ for $t\in(0,a]$, then for $t\in(a,2a]$

$$\begin{aligned}&B_0u^*(t)+B_1u^*(t-a)-F\big(t,z_{\varsigma(t)},u_0(t)+u_0(t-a)\big)-B_1u_1(t-a)\\ &\quad= B_0u^*(t)+B_1u^*(t-a)-F\big(t,z_{\varsigma(t)},u_0(t)+u_0(t-a)\big)-B_1u_{11}(t-a)\end{aligned}$$

$$= B_0 u_{12}(t) \text{ (say)}.$$

Further, if we take $u_1(t) = u_{12}(t)$ for $t \in (a, 2a]$, then in similar fashion, one can obtain $u_1(t) = u_{13}(t)$ for $t \in (2a, 3a]$. Repeating the above process, at the ℓ-th step, we get

$$u_1(t) = u_{1\ell}(t) \quad \text{for} \quad t \in ((\ell-1)a, c].$$

Clearly, the function u_1 given by

$$u_1(t) = \begin{cases} 0, & t \in [-a, 0], \\ u_{1i}(t), & t \in ((i-1)a, ia], \quad i = 1, 2, \ldots, (\ell-1); \\ u_{1\ell}(t), & t \in ((\ell-1)a, c], \end{cases}$$

satisfies

$$B_0 u_1(t) + B_1 u_1(t-a) = B_0 u^*(t) + B_1 u^*(t-a) - F\big(t, z_{\varsigma(t)}, u_0(t) + u_0(t-a)\big), \quad 0 \le t \le c.$$

This proves the lemma. ■

Remark 4.1.6. *By a similar argument, it is easy to verify that every $u^* \in U$ can be uniquely expressed as $u^* = u + u(\cdot - a)$, where $u \in U$. Moreover, if a sequence $u_n + u_n(\cdot - a) \to u + u(\cdot - a)$ then $u_n \to u$ and $u_n(\cdot - a) \to u(\cdot - a)$.*

In the next lemma, we prove that the operator equation

$$\begin{aligned} B_0 u(t) + B_1 u(t-a) &= B_0 u^*(t) + B_1 u^*(t-a) \\ &\quad - F\big(t, z_{\varsigma(t)}, u(t) + u(t-a)\big), \quad 0 \le t \le c \end{aligned} \tag{4.1.14}$$

admits a solution $u \in U$ for each $u^* \in U$ and $z_{\varsigma(t)} \in \mathfrak{C}_0$.

Lemma 4.1.7. *Under hypotheses (H_3)-(H_5), the operator equation (4.1.14) is solvable in $u \in U$ for each given $u^* \in U$ and $z_{\varsigma(t)} \in \mathfrak{C}_0$.*

Proof. Let $u_0 \in U$ then by above lemma and assumption (H_4)

$$\begin{aligned} &\|u_{n+1}(t) + u_{n+1}(t-a) - u_n(t) - u_n(t-a)\| \\ &\quad \le \frac{1}{\delta} \|B_0 (u_{n+1}(t) - u_n(t)) + B_1 (u_{n+1}(t-a) - u_n(t-a))\| \\ &\quad = \frac{1}{\delta} \big\|F\big(t, z_{\varsigma(t)}, u_{n-1}(t) + u_{n-1}(t-a)\big) - F\big(t, z_{\varsigma(t)}, u_n(t) + u_n(t-a)\big)\big\| \end{aligned}$$

$$\leq \frac{\lambda}{\delta}\|u_n(t) + u_n(t-a) - u_{n-1}(t) - u_{n-1}(t-a)\|.$$

Hence

$$\begin{aligned}
&\|u_{n+1} + u_{n+1}(\cdot - a) - u_n - u_n(\cdot - a)\|_U \\
&\quad \leq \frac{\lambda}{\delta}\|u_n + u_n(\cdot - a) - u_{n-1} - u_n(\cdot - a)\|_U \\
&\quad \leq \left(\frac{\lambda}{\delta}\right)^2 \|u_{n-1} + u_{n-1}(\cdot - a) - u_{n-2} - u_{n-2}(\cdot - a)\|_U \\
&\quad \vdots \\
&\quad \leq \left(\frac{\lambda}{\delta}\right)^n \|u_1 + u_1(\cdot - a) - u_0(t) - u_0(t-a)\|_U \\
&\qquad \to 0 \quad as \quad n \to \infty,
\end{aligned}$$

which shows that $\langle u_n + u_n(\cdot - a)\rangle$ is a Cauchy sequence in U. It means there is a $u \in U$ satisfying $\lim_{n\to\infty} (u_n + u_n(\cdot - a)) = u + u(\cdot - a)$. Now,

$$\begin{aligned}
&\big\|F\big(t, z_{\varsigma(t)}, u_n(t) + u_n(t-a)\big) - F\big(t, z_{\varsigma(t)}, u(t) + u(t-a)\big)\big\| \\
&\quad \leq \lambda\|u_n(t) + u_n(t-a) - u(t) - u(t-a)\| \\
&\quad \to 0 \quad \text{as} \quad n \to \infty.
\end{aligned}$$

Thus

$$\begin{aligned}
&F\big(t, z_{\varsigma(t)}, u(t) + u(t-a)\big) \\
&\quad = \lim_{n\to\infty} F\big(t, z_{\varsigma(t)}, u_n(t) + u_n(t-a)\big) \\
&\quad = \lim_{n\to\infty} (B_0u^*(t) + B_1u^*(t-a) - B_0u_{n+1}(t) - B_1u_{n+1}(t-a)) \\
&\quad = B_0u^*(t) + B_1u^*(t-a) - B_0u(t) - B_1u(t-a),
\end{aligned}$$

which proves the lemma. ■

Theorem 4.1.8. *Under the hypotheses* (H_1)*-*(H_5)*, the semilinear system (4.1.1) is approximately (exactly) controllable if the associated linear system without delay is approximately (exactly) controllable.*

Proof. It is enough to prove that $\tilde{\mathfrak{R}}(0) \subseteq \mathfrak{R}(F)$. Let $x \in \tilde{\mathfrak{R}}(0)$, then there is a

$u^* \in U$ satisfying

$$x(t) = \begin{cases} \mathcal{C}(t)\wp(0) + \mathcal{S}(t)y_1 + \int_0^t \mathcal{S}(t-s)\big(B_0u^*(s) + B_1u^*(s-a)\big)\,ds, & t \in (0,c], \\ \wp(t), & t \in [-a,0]. \end{cases}$$

By Lemma 4.1.7, one can finda control $u \in U$ such that

$$B_0u(t) + B_1u(t-a) = B_0u^*(t) + B_1u^*(t-a) - F\big(t, x_{\varsigma(t)}, u(t) + u(t-a)\big).$$

Let $z \in \mathfrak{C}_c$ be the mild solution of (4.1.1) corresponding to u. Then we can write

$$\begin{aligned} z(t) = {} & \mathcal{C}(t)\wp(0) + \mathcal{S}(t)y_1 + \int_0^t \mathcal{S}(t-s)B_0u^*(s)\,ds \\ & + \int_0^t \mathcal{S}(t-s)B_1u^*(s-a)\,ds + \int_0^t \mathcal{S}(t-s)F\big(t, z_{\varsigma(s)}, u(s) + u(s-a)\big)\,ds \\ & - \int_0^t \mathcal{S}(t-s)F\big(t, x_{\varsigma(s)}, u(s) + u(s-a)\big)\,ds, \quad 0 < t \le c, \end{aligned}$$

which gives

$$\begin{aligned} & z(t) - x(t) \\ & = \int_0^t \mathcal{S}(t-s)\left(F\big(t, z_{\varsigma(s)}, u(s) + u(s-a)\big) - F\big(t, x_{\varsigma(s)}, u(s) + u(s-a)\big)\right)ds. \end{aligned}$$

Taking norm, we get

$$\begin{aligned} & \|z(t) - x(t)\| \\ & \le k_{\mathcal{S}} \int_0^t \left\|F\big(t, z_{\varsigma(s)}, u(s) + u(s-a)\big) - F\big(t, x_{\varsigma(s)}, u(s) + u(s-a)\big)\right\| ds \\ & \le k_{\mathcal{S}}\lambda \int_0^t \left\|z_{\varsigma(s)} - x_{\varsigma(s)}\right\| ds. \end{aligned} \tag{4.1.15}$$

Now,

$$\begin{aligned} \|z_{\varsigma(s)} - x_{\varsigma(s)}\|_{\mathfrak{C}_0} & = \sup_{-a \le \theta \le 0} \|z(\varsigma(s) + \theta) - x(\varsigma(s) + \theta)\| \\ & \le \sup_{-a \le \varrho \le 0} \|z(\varrho) - x(\varrho)\| + \sup_{0 \le \varrho \le s} \|z(\varrho) - x(\varrho)\|. \end{aligned}$$

Also $z(\varrho) = \psi(\varrho) = x(\varrho)$ for $\varrho \in [-a,0]$. Thus (4.1.15) leads to

$$\sup_{0 \le \varrho \le t} \|z(\varrho) - x(\varrho)\| \le k_{\mathcal{S}}\lambda \int_0^t \sup_{0 \le \varrho \le s} \|z(\varrho) - x(\varrho)\|\,ds.$$

Applying Gronwall's inequality, we have $\|z - x\|_{\mathfrak{C}_c} = 0$. Hence $\tilde{\mathfrak{R}}(0) \subseteq \mathfrak{R}(F)$. ■

4.1.4 Example

Consider the following wave equation on $[0, c]$ for $0 \leq x \leq 1$:

$$\begin{cases} \frac{\partial^2 \widehat{z}(t,x)}{\partial t^2} = \frac{\partial^2 \widehat{z}(t,x)}{\partial x^2} + \widehat{u}(t,x) + u(t-a,x) \\ \qquad\qquad + F\big(t, \widehat{z}(\varsigma(t)+\theta,x), \widehat{u}(t,x) + \widehat{u}(t-a,x)\big), \quad t \in (0,c], \\ \frac{\partial \widehat{z}}{\partial x}(t,0) = \frac{\partial \widehat{z}}{\partial x}(t,1) = 0, \qquad\qquad t \in (0,c], \\ \sum_{i=1}^{n} \alpha_i \widehat{z}(t_i,x) = \widehat{z}_0(x), \\ \frac{\partial \widehat{z}}{\partial t}(0,x) = \widehat{z}_1(x), \end{cases} \tag{4.1.16}$$

where $-a \leq t_1 < t_2 < \cdots < t_n \leq 0$.

The equation (4.1.16) takes the abstract form (4.1.1), if we set
(i) $V = L_2[0,1]$ and define $A : D(A) \subseteq V \to V$ by $Ay = \frac{d^2y}{dx^2}$ with domain

$$D(A) = \left\{ y \in L_2[0,1] \,\middle|\, y,\ \frac{\partial y}{\partial x} \text{ are absolutely continuous, } \frac{\partial^2 y}{\partial x^2} \in L_2[0,1] \right. \\ \left. \text{and } \frac{\partial y}{\partial x}(0) = \frac{\partial y}{\partial x}(1) = 0 \right\}.$$

Let $\xi_\ell(x) = \sqrt{2}\cos \ell\pi x$ and $\lambda_\ell = (\ell\pi)^2$, $\ell = 1, 2, \ldots$, then $0, \lambda_1, \lambda_2, \cdots$ are eigenvalues of A with corresponding eigenfunctions $1, \xi_1, \xi_2, \cdots$ and the family $\{1, \xi_1, \xi_2, \cdots\}$ form a complete orthonormal set for N. Define the control space

$$V' = \left\{ \upsilon \in L_2[0,1] \,\middle|\, \upsilon = \sum_{\ell=2}^{\infty} \tilde{\alpha}_\ell \xi_\ell \quad \text{with} \quad \sum_{\ell=2}^{\infty} \tilde{\alpha}_\ell^2 < \infty \right\},$$

and the linear operators B_0, B_1 from V' to V given by $B_j\upsilon = \upsilon$, $j = 0, 1$. Evidently, B_0 and B_1 are bounded and satisfy assumption (H_5).
(ii) $\varsigma(t) = \frac{t^2}{t+t^2}$, $t \in [0, c]$, which satisfy delay property and $\widehat{z}_{\varsigma(t)}(\theta, x) = \widehat{z}\left(\frac{t^2}{t+t^2} + \theta, x\right)$.
(iii) $\psi(z)(t) = \gamma(z)$ for $z \in \mathfrak{C}_0$, $t \in [-a, 0]$; $h(t) = y_0 = \widehat{z}_0(x)$, where $\gamma : \mathfrak{C}_0 \to V$ is given by

$$\gamma(z) = \sum_{i=1}^{n} \alpha_i z(t_i)$$

and

$$z(t_i) = \widehat{z}(t_i, x).$$

Take $\wp \in \mathfrak{C}_0$ such that $\wp(t_i) = \frac{1}{\alpha_i}\frac{1}{n}y_0 = \wp(t_i)$, then for each $t \in [-a, 0]$,

$$\begin{aligned}\psi(\wp)(t) = \gamma(\wp) &= \sum_{i=1}^{n} \alpha_i \wp(t_i)\\ &= \sum_{i=1}^{n} \alpha_i \frac{1}{\alpha_i}\frac{1}{n} y_0\\ &= y_0\\ &= h(t).\end{aligned}$$

Therefore hypothesis (H_1) is satisfied.

The spectral representation of A is

$$Ay = -\sum_{\ell=1}^{\infty} (\ell\pi)^2 \langle y, \xi_\ell \rangle \xi_\ell, \quad y \in V.$$

A generates a cosine family $\{\mathcal{C}(t)\}_{t\in\mathbb{R}}$ defined by

$$\mathcal{C}(t)y = \sum_{\ell=1}^{\infty} \cos(\ell\pi t) \langle y, \xi_\ell \rangle \xi_\ell, \quad y \in V,$$

with corresponding sine family

$$\mathcal{S}(t)y = \sum_{\ell=1}^{\infty} \frac{1}{\ell\pi} \sin(\ell\pi t) \langle y, \xi_\ell \rangle \xi_\ell, \quad y \in V.$$

If we consider the nonlinear part

$$F\big(t, z_{\varsigma(t)}, u(t) + u(t-a)\big) = \frac{1}{3}\Big(\big\|z_{\varsigma(t)}\big\|_{\mathfrak{C}_0} \xi_3(x) + \|u(t) + u(t-a)\| \, \xi_4(x)\Big).$$

Then by Minkowski's inequality we obtain

$$\begin{aligned}&\big\|F\big(t, z_{\varsigma(t)}, u(t) + u(t-a)\big) - F\big(t, \tilde{z}_{\varsigma(t)}, \tilde{u}(t) + \tilde{u}(t-a)\big)\big\|\\ &\le \frac{1}{3}\Big(\big\|z_{\varsigma(t)} - \tilde{z}_{\varsigma(t)}\big\|_{\mathfrak{C}_0} + \|u(t) + u(t-a) - \tilde{u}(t) - \tilde{u}(t-a)\|\Big)\end{aligned}$$

for any $z_{\varsigma(t)}, \tilde{z}_{\varsigma(t)} \in \mathfrak{C}_0$ and $u(t), \tilde{u}(t) \in V'$.

Above shows that F is Lipschitz with constant $\lambda = \frac{1}{3}$ and hypothesis (H_5) is satisfied. Further, if we take $\alpha_3 = \frac{1}{3}\|z_{\varsigma(t)}\|_{\mathfrak{C}_0}$, $\alpha_4 = \frac{1}{3}$ and rest $\alpha_j = 0$, then the hypothesis (H_3) is satisfied. The linear part of (4.1.16) without delay is controllable [18]. Thus by Theorem 4.1.8, the system (4.1.16) is exactly controllable.

4.2 Approximate controllability of second-order nonlocal retarded semilinear systems with multiple delays in control

In this section, we consider the semilinear system with control delays in Banach spaces. Here the inclusion condition among the range sets of the operators has been dropped and the nonlinear function has been considered in more general form.

4.2.1 Introduction and preliminaries

Let $Z = L_p([0,c];V)$ be a function space, where $p > 1$ and V is a Banach space. Consider the semilinear system

$$\begin{cases} \ddot{z}(t) = Az(t) + \displaystyle\sum_{i=0}^{m} B_i u(t - a_i) \\ \qquad\qquad + F\big(t, z_{\varsigma(t)}, u(t), u(t-\widehat{a}_1), \dots, u(t-\widehat{a}_{\widehat{m}})\big), \quad t \in (0,c], \\ \psi(z) = h, \quad u(t) = 0, \qquad\qquad t \in [-a, 0], \\ \dot{z}(0) = y_1, \end{cases} \tag{4.2.1}$$

where the state $z(t) \in V$ and the control $u \in U = L_p([0,c];V')$, V' is another Banach space; a_i and $\widehat{a}_j$, $j = 1,2,\dots,\widehat{m}$ are fixed delays such that $0 = a_0 < a_1 < a_2 < \cdots < a_m < c$, $0 < \widehat{a}_1 < \widehat{a}_2 < \cdots < \widehat{a}_{\widehat{m}} < c$ and $a = \max\{a_m, \widehat{a}_{\widehat{m}}\}$. $B_0, B_1, \dots, B_m$ are continuous linear maps from V' to V and $F : [0,c] \times \mathfrak{C}_0 \times \underbrace{V' \times V' \times \cdots \times V'}_{(\widehat{m}+1) \text{ times}} \to V$ is nonlinear. The other notations are similar as in previous section.

Controllability results for semilinear and linear systems with delays in state or control in finite and infinite dimensional spaces have been analyzed by many researchers. Among them, Klamka [48] investigated the stochastic controllability of linear systems with multiple delays in control in finite dimesional Hilbert spaces. Controllability of linear systems with multiple delays in state was proved in [47]. Klamka [45] considered a finite dimensional system described by semilinear differential equations with control delays and determined the constrained controllability using rank condition. In [60], Liu et al. analyzed the controllability of time-varying systems of linear equations with impulsive effects and delays in control. Sukavanam et al. [82] studied the controllability of a semilinear delayed system with growing nonlinear term. In [23] Devies et al. deduced the results for null and exact controllability of linear systems with delay in both control and state. Applying sequence method and the concept of fundamental solution Anurag et al. [80] analyzed the controllability of semilinear systems with state delay.

Up to now, there is no result on controllability of semilinear systems of second-order with control delays. To fill this gap, the present section is devoted to analyze the controllability of nonlocal retarded semilinear systems of second-order with multiple delays in control.

Definition 4.2.1. Suppose $\wp \in \mathfrak{C}_0$ satisfies $\psi(\wp) = h$. A function $z \in \mathfrak{C}_c$ is said to be a mild solution of (4.2.1) if it satisfies

$$z(t) = \begin{cases} \mathcal{C}(t)\wp(0) + \mathcal{S}(t)y_1 + \displaystyle\int_0^t \mathcal{S}(t-s)\Big(\sum_{i=0}^{m} B_i u(s-a_i)\Big)\, ds \\ + \displaystyle\int_0^t \mathcal{S}(t-s)F\big(s, z_{\varsigma(s)}, u(s), u(s-\widehat{a}_1), \ldots, u(s-\widehat{a}_{\widehat{m}})\big)\, ds \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{for } \ t \in (0, c], \\ \wp(t) \qquad\qquad\qquad\qquad\qquad\quad \text{for } \ t \in [-a, 0]. \end{cases} \tag{4.2.2}$$

Moreover, if $\wp(0) \in V_1$, then $\dot{z}$ is continuous on $[0, c]$ and is given by

$$\dot{z}(t) = A\mathcal{S}(t)\wp(0) + \mathcal{C}(t)y_1 + \int_0^t \mathcal{C}(t-s)\Big(\sum_{i=0}^{m} B_i u(s-a_i)\Big)\, ds \\ + \int_0^t \mathcal{C}(t-s)F\big(s, z_{\varsigma(s)}, u(s), u(s-\widehat{a}_1), \ldots, u(s-\widehat{a}_{\widehat{m}})\big)\, ds.$$

Here, V_1 is defined as in previous section.

Definition 4.2.2. The system given by (4.2.1) is said to be approximately controllable on $[0, c]$, if for every given $\varepsilon > 0$ and a final state $z_c \in V$ one can find a control $u \in U$ such that the mild solution $z(t)$ of (4.2.1) corresponding to u satisfies

$$\|z(c) - z_c\| \leq \varepsilon.$$

For the system (4.2.1), the systems

$$\begin{cases} \ddot{z}(t) = Az(t) + \displaystyle\sum_{i=0}^{m} B_i u(t-a_i), \quad t \in (0, c], \\ z(0) = \wp(0), \\ \dot{z}(0) = y_1. \end{cases} \tag{4.2.3}$$

and

$$\begin{cases} \ddot{z}(t) = Az(t) + B_0u(t), \quad t \in (0, c], \\ z(0) = \wp(0), \\ \dot{z}(0) = y_1 \end{cases} \tag{4.2.4}$$

are associated linear systems with delays and without delay, respectively, if $\psi(\wp) = h$.

Throughout this section, we again suppose that $\|\mathcal{C}(t)\| \le k_{\mathcal{C}}$ and $\|\mathcal{S}(t)\| \le k_{\mathcal{S}}$, $0 \le t \le c$, where $k_{\mathcal{C}}$ and $k_{\mathcal{S}}$ are constants.

4.2.2 Existence of mild solution

To discuss the existence result we suppose the following:

(H_1) $\psi : C([-a, 0]; V_1) \to C([-a, 0]; V_1)$ and for each given h there is a unique function $\wp \in C([-a, 0]; V_1)$ satisfying $\psi(\wp) = h$;

(H_2) F is continuous in t and locally Lipschitz in z, that is, there exists a constant $\lambda_r > 0$ satisfying

$$\|F(t, z_1, u_0, u_1, \ldots, u_{\widehat{m}}) - F(t, z_2, u_0, u_1, \ldots, u_{\widehat{m}})\| \le \lambda_r \|z_1 - z_2\|_{\mathfrak{C}_0}$$

for all $t \in [0, c]$; $z_\ell \in \mathfrak{C}_0$ with $\|z_\ell\|_{\mathfrak{C}_0} \le r$, $\ell = 1, 2$ and $u_j \in V'$, $j = 0, 1, 2, \ldots, \widehat{m}$;

(H_3) there exists a $k_F > 0$ satisfying

$$\|F(t, z, u_0, u_1, \ldots, u_{\widehat{m}})\| \le k_F(1 + \|z\|_{\mathfrak{C}_0} + \|u_0\| + \|u_1\| + \cdots + \|u_{\widehat{m}}\|)$$

for all $t \in [0, c]$; $z \in \mathfrak{C}_0$ and $u_j \in V'$, $j = 0, 1, 2, \ldots, \widehat{m}$.

Theorem 4.2.1. *Under hypotheses (H_1)-(H_3), the system (4.2.1) has a unique mild solution for each $u \in U$ and $y_1 \in V$.*

Proof. Let $0 < c_1 < c$ and $\max\{\|B_0\|, \|B_1\|, \dots, \|B_m\|\} \le k_B$. Define a mapping $\mathcal{Q} : \mathfrak{C}_{c_1} \to \mathfrak{C}_{c_1}$ by

$$(\mathcal{Q}z)(t) = \begin{cases} \mathcal{C}(t)\wp(0) + \mathcal{S}(t)y_1 + \int_0^t \mathcal{S}(t-s)\Big(\sum_{i=0}^{m} B_i u(s-a_i)\Big)\, ds \\ + \int_0^t \mathcal{S}(t-s)F\big(s, z_{\varsigma(s)}, u(s), u(s-\widehat{a}_1), \dots, u(s-\widehat{a}_{\widehat{m}})\big)\, ds \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{for } \ t \in (0, c_1], \\ \wp(t) \qquad\qquad\qquad\qquad\qquad\quad\; \text{for } \ t \in [-a, 0] \end{cases} \tag{4.2.5}$$

and consider the ball

$$\mathcal{B}_{r_0} = \{z \in \mathfrak{C}_{c_1} \,|\, \|z\|_{\mathfrak{C}_{c_1}} \le r_0, z(0) = \wp(0) \text{ and } \dot{z}(0) = y_1\}.$$

Then, for any $z \in \mathcal{B}_{r_0}$ and $0 \le s \le c_1$

$$\|z_{\varsigma(s)}\|_0 = \max_{\theta\in[-a,0]} \|z(\varsigma(s)+\theta)\| \le \max_{\varrho\in[-a,c_1]} \|z(\varrho)\| \le r_0.$$

Thus

$$\begin{aligned} \|(\mathcal{Q}z)(t)\| \le\ & k_{\mathcal{C}}\|\wp(0)\| + k_{\mathcal{S}}\|y_1\| + k_{\mathcal{S}}k_B\Big(\int_0^t \sum_{i=0}^{m} \|u(s-a_i)\|\, ds\Big) \\ & + k_{\mathcal{S}} \int_0^t \big\|F\big(t, z_{\varsigma(s)}, u(s), u(s-\widehat{a}_1), \dots, u(s-\widehat{a}_{\widehat{m}})\big) \\ & - F\big(t, 0, u(s), u(s-\widehat{a}_1), \dots, u(s-\widehat{a}_{\widehat{m}})\big)\big\|\, ds \\ & + k_{\mathcal{S}} \int_0^t \big\|F\big(t, 0, u(s), u(s-\widehat{a}_1), \dots, u(s-\widehat{a}_{\widehat{m}})\big)\big\|\, ds \\ \le\ & k_{\mathcal{C}}\|\wp(0)\| + k_{\mathcal{S}}\|y_1\| + (m+1)k_{\mathcal{S}}k_B c^{1-\frac{1}{p}}\|u\|_U + k_{\mathcal{S}}\lambda_{r_0}\int_0^t \|z_{\varsigma(s)}\|_{\mathfrak{C}_0}\, ds \\ & + k_{\mathcal{S}}k_F \int_0^t [1 + \|u(s)\| + \|u(s-\widehat{a}_1)\| + \cdots + \|u(s-\widehat{a}_{\widehat{m}})\|]\, ds \\ \le\ & k_{\mathcal{C}}\|\wp(0)\| + k_{\mathcal{S}}\|y_1\| + (m+1)k_{\mathcal{S}}k_B c^{1-\frac{1}{p}}\|u\|_U + k_{\mathcal{S}}\lambda_{r_0}r_0c_1 \\ & + k_{\mathcal{S}}k_F\left(c_1 + (\widehat{m}+1){c_1}^{1-\frac{1}{p}}\|u\|_U\right) \\ =\ & k_{\mathcal{C}}\|\wp(0)\| + k_{\mathcal{S}}\|y_1\| + (m+1)k_{\mathcal{S}}k_B c^{1-\frac{1}{p}}\|u\|_U \\ & + k_{\mathcal{S}}\left(\lambda_{r_0}r_0c_1 + k_F\left(c_1 + (\widehat{m}+1){c_1}^{1-\frac{1}{p}}\|u\|_U\right)\right). \end{aligned}$$

Now choosing $r_0 = 2\left(k_{\mathcal{C}}\|\wp(0)\| + k_{\mathcal{S}}\|y_1\| + (m+1)k_{\mathcal{S}}k_B c^{1-\frac{1}{p}}\|u\|_U\right) + 1$ and c_1 small enough such that

$$k_{\mathcal{S}}\left(\lambda_{r_0}r_0c_1 + k_F\left(c_1 + (\widehat{m}+1){c_1}^{1-\frac{1}{p}}\|u\|_U\right)\right)$$

$$\leq k_{\mathcal{C}}\|\wp(0)\| + k_{\mathcal{S}}\|y_1\| + (m+1)k_{\mathcal{S}}k_B c^{1-\frac{1}{p}}\|u\|_U + 1.$$

Then

$$\begin{aligned}\|(\mathcal{Q}z)(t)\| &\leq 2\left(k_{\mathcal{C}}\|\wp(0)\| + k_{\mathcal{S}}\|y_1\| + (m+1)k_{\mathcal{S}}k_B c^{1-\frac{1}{p}}\|u\|_U\right) + 1\\ &= r_0 \text{ (say)}.\end{aligned}$$

Therefore $\mathcal{Q}$ maps $\mathcal{B}_{r_0}$ into itself.

Now, take $z, \tilde{z} \in \mathcal{B}_{r_0}$, then

$$\begin{aligned}\|(\mathcal{Q}z)(t) - (\mathcal{Q}\tilde{z})(t)\| &\leq k_{\mathcal{S}}\int_0^t \big\|F\big(s, z_{\varsigma(s)}, u(s), u(s-\widehat{a}_1), \ldots, u(s-\widehat{a}_{\widehat{m}})\big)\\ &\quad - F\big(s, \tilde{z}_{\varsigma(s)}, u(s), u(s-\widehat{a}_1), \ldots, u(s-\widehat{a}_{\widehat{m}})\big)\big\|\, ds\\ &\leq k_{\mathcal{S}}\lambda_{r_0}\int_0^t \|z_{\varsigma(s)} - \tilde{z}_{\varsigma(s)}\|_{\mathfrak{C}_0}\, ds\\ &\leq k_{\mathcal{S}}\lambda_{r_0} t\|z - \tilde{z}\|_{\mathfrak{C}_{c_1}}.\end{aligned}$$

Further,

$$\begin{aligned}\|(\mathcal{Q}^2 z)(t) - (\mathcal{Q}^2\tilde{z})(t)\| &\leq k_{\mathcal{S}}\int_0^t \big\|F\big(s, (\mathcal{Q}z)_{\varsigma(s)}, u(s), u(s-\widehat{a}_1), \ldots, u(s-\widehat{a}_{\widehat{m}})\big)\\ &\quad - F\big(s, (\mathcal{Q}\tilde{z})_{\varsigma(s)}, u(s), u(s-\widehat{a}_1), \ldots, u(s-\widehat{a}_{\widehat{m}})\big)\big\|\, ds\\ &\leq k_{\mathcal{S}}\lambda_{r_0}\int_0^t \big\|(\mathcal{Q}z)_{\varsigma(s)} - (\mathcal{Q}\tilde{z})_{\varsigma(s)}\big\|_{\mathfrak{C}_0}\, ds\\ &\leq k_{\mathcal{S}}\lambda_{r_0}\int_0^t \sup_{-a\leq\varrho\leq 0}\|(\mathcal{Q}z)(\varsigma(s)+\varrho) - (\mathcal{Q}\tilde{z})(\varsigma(s)+\varrho)\|\, ds\\ &\leq k_{\mathcal{S}}\lambda_{r_0}\int_0^t \Bigg(\sup_{-a\leq\varrho\leq 0}\|(\mathcal{Q}z)(\varrho) - (\mathcal{Q}\tilde{z})(\varrho)\|\\ &\quad + \sup_{0\leq\varrho\leq s}\|(\mathcal{Q}z)(\varrho) - (\mathcal{Q}\tilde{z})(\varrho)\|\Bigg)\, ds\\ &= k_{\mathcal{S}}\lambda_{r_0}\int_0^t \sup_{0\leq\varrho\leq s}\|(\mathcal{Q}z)(\varrho) - (\mathcal{Q}\tilde{z})(\varrho)\|\, ds\\ &\leq k_{\mathcal{S}}\lambda_{r_0}\int_0^t k_{\mathcal{S}}\lambda_{r_0} s\|z - \tilde{z}\|_{\mathfrak{C}_{c_1}}\, ds\\ &\leq \frac{(k_{\mathcal{S}}\lambda_{r_0} t)^2}{2}\|z - \tilde{z}\|_{\mathfrak{C}_{c_1}}.\end{aligned}$$

Repeating the above process, one can obtain

$$\|(\mathcal{Q}^n z)(t) - (\mathcal{Q}^n\tilde{z})(t)\| \leq \frac{(k_{\mathcal{S}}\lambda_{r_0} t)^n}{n!}\|z - \tilde{z}\|_{\mathfrak{C}_{c_1}}$$

$$\leq \frac{(k_{\mathcal{S}}\lambda_{r_0}c_1)^n}{n!}\|z-\tilde{z}\|_{\mathfrak{C}_{c_1}}.$$

Therefore

$$\|\mathcal{Q}^n z-\mathcal{Q}^n\tilde{z}\|_{\mathfrak{C}_{c_1}} \leq \frac{(k_{\mathcal{S}}\lambda_{r_0}c_1)^n}{n!}\|z-\tilde{z}\|_{\mathfrak{C}_{c_1}}.$$

which shows that the $\mathcal{Q}^n$ is a contraction map for sufficiently large value of n. By Banach fixed point theorem, $\mathcal{Q}$ has a fixed point in $\mathcal{B}_{r_0}$. Hence (4.2.2) is a mild solution on $[-a, c_1]$. In similar way, the existence of mild solution on $[c_1, c_2]$, where $c_1 < c_2$, can be shown. Applying the above technique, one can deduce that (4.2.2) is a mild solution on the maximal existing interval $[-a, c^*)$, $c^* \leq c$.

Next we show the boundedness of solution. Clearly $z(t)$ is bounded on $[-a, 0]$. Now for $t \in [0, c^*)$

$$\begin{aligned}\|z(t)\| &\leq k_{\mathcal{C}}\|\wp(0)\| + k_{\mathcal{S}}\|y_1\| + (m+1)k_{\mathcal{S}}k_B\int_0^t \|u(s)\|\,ds\\ &\quad + k_{\mathcal{S}}k_F\int_0^t \big(1+\|z_{\varsigma(s)}\|_{\mathfrak{C}_0} + (\widehat{m}+1)\|u(s)\|\big)\,ds\\ &\leq k_{\mathcal{C}}\|\wp(0)\| + k_{\mathcal{S}}\Big(\|y_1\| + \big((m+1)k_B + (\widehat{m}+1)k_F\big)c^{1-\frac{1}{p}}\|u\|_U + k_F c\Big)\\ &\quad + k_{\mathcal{S}}k_F\int_0^t \|z_{\varsigma(s)}\|_{\mathfrak{C}_0}\,ds.\end{aligned}$$

Therefore

$$\begin{aligned}\|z(t)\| &\leq \|z_{\varsigma(t)}\|_{\mathfrak{C}_0}\\ &\leq k_{\mathcal{C}}\|\wp(0)\| + k_{\mathcal{S}}\Big(\|y_1\| + \big((m+1)k_B + (\widehat{m}+1)k_F\big)c^{1-\frac{1}{p}}\|u\|_U + k_F c\Big)\\ &\quad + M_{\wp} + k_{\mathcal{S}}k_F\int_0^t \|z_{\varsigma(s)}\|_{\mathfrak{C}_0}\,ds.\end{aligned}$$

By Gronwall's inequality, one has

$$\begin{aligned}\|z(t)\| \leq \bigg(&k_{\mathcal{C}}\|\wp(0)\| + k_{\mathcal{S}}\Big(\|y_1\| + \big((m+1)k_B + (\widehat{m}+1)k_F\big)c^{1-\frac{1}{p}}\|u\|_U + k_F c\Big)\\ &+ M_{\wp}\bigg)\exp(k_{\mathcal{S}}k_F c),\end{aligned}$$

which shows that $z(t)$ is bounded on $[-a, c^*)$ and hence it is defined on $[-a, c]$.

For uniqueness, suppose z_1 and z_2 be two solutions of (4.2.1) for the same control function u. Then $z_1(t) = z_2(t) = \wp(t)$ for $t \in [-a, 0]$. Now for $t \in [0, c]$, set

$$a^* = \max\big\{\|z_1\|_{\mathfrak{C}_c}, \|z_2\|_{\mathfrak{C}_c}\big\}.$$

Then

$$\begin{aligned}\|z_1(t)-z_2(t)\|_V &\le k_{\mathcal{S}}\int_0^t \Big\|F\big(s,(z_1)_{\varsigma(s)},u(s),u(s-\widehat{a}_1),\dots,u(s-\widehat{a}_{\widehat{m}})\big)\\ &\qquad - F\big(s,(z_2)_{\varsigma(s)},u(s),u(s-\widehat{a}_1),\dots,u(s-\widehat{a}_{\widehat{m}})\big)\Big\|\,ds\\ &\le k_{\mathcal{S}}\lambda_{a^*}\int_0^t \big\|(z_1)_{\varsigma(s)}-(z_2)_{\varsigma(s)}\big\|_{\mathfrak{C}_0}\,ds\\ &\le k_{\mathcal{S}}\lambda_{a^*}\int_0^c \big\|(z_1)_{\varsigma(s)}-(z_2)_{\varsigma(s)}\big\|_{\mathfrak{C}_0}\,ds.\end{aligned}$$

Therefore

$$\|(z_1)_{\varsigma(t)}-(z_2)_{\varsigma(t)}\|_{\mathfrak{C}_0} \le k_{\mathcal{S}}\lambda_{a^*}\int_0^c \big\|(z_1)_{\varsigma(s)}-(z_2)_{\varsigma(s)}\big\|_{\mathfrak{C}_0}\,ds.$$

By Gronwall's inequality, one has $(z_1)_{\varsigma(t)}=(z_2)_{\varsigma(t)}$ for all $t\in[0,c]$ and consequently $z_1=z_2$. ■

4.2.3 Controllability results

The subsequent discussion needs the following hypotheses:

(H_4) the system (4.2.4) is approximately controllable;

(H_5) there exists a function $q\in L_1[0,c]$ satisfying

$$\|F(t,z,u_0,u_1,\dots,u_{\widehat{m}})\|\le q(t)$$

for all $(t,z,u_0,u_1,\dots,u_{\widehat{m}})\in[0,c]\times\mathfrak{C}_0\times V'\times V'\times\cdots\times V'$.

Theorem 4.2.2. *Under hypotheses (H_1) and (H_4), the corresponding linear delay system (4.2.3) is approximately controllable.*

Proof. Set $c=a_{m+1}$ and $r=\min\{a_1,a_2-a_1,a_3-a_2,\dots,a_{m+1}-a_m\}$. Since $0=a_0<a_1<a_2<\cdots<a_m<a_{m+1}$. Therefore for each a_{i+1} one can find a positive integer n_i and a constant $\alpha_i\in[0,r)$ satisfying $a_{i+1}=a_i+n_ir+\alpha_i$, $i=1,2,\dots,m$.
Case 1: When $\alpha_1,\alpha_2,\dots,\alpha_m$ are positive.

Let $\tilde{z}_0;\tilde{z}_{11},\tilde{z}_{12}\dots,\tilde{z}_{1n_1},\tilde{z}_{\overline{1n_1+1}};\tilde{z}_{21},\tilde{z}_{22},\dots,\tilde{z}_{2n_2},\tilde{z}_{\overline{2n_2+1}};\cdots;\tilde{z}_{m1},\tilde{z}_{m2},\dots,\tilde{z}_{mn_m}$ be given in V and $z_c\in V$ be the final state. Consider the system

$$\begin{cases}\ddot{\xi}(t)=A\xi(t)+B_0u(t), \quad t\in(0,a_1],\\ \xi(0)=\wp(0),\\ \dot{\xi}(0)=\xi_1=y_1.\end{cases} \tag{4.2.6}$$

Set $\tilde{\xi}_0 = \tilde{z}_0$. By (H_4) one can find a control u_0 such that the mild solution $\xi(t)$ of (4.2.6) is given by

$$\xi(t) = \mathcal{C}(t)\wp(0) + \mathcal{S}(t)y_1 + \int_0^t \mathcal{S}(t-s)B_0u_0(s)\,ds, \quad 0 \le t \le a_1$$

and it satisfies $\|\xi(a_1) - \tilde{\xi}_0\| \le \varepsilon$.

Let

$$w_0(t) = \begin{cases} 0, & t \in [-a, 0], \\ u_0(t), & t \in [0, a_1] \end{cases}$$

and

$$z(t) = \mathcal{C}(t)\wp(0) + \mathcal{S}(t)y_1 + \int_0^t \mathcal{S}(t-s)\bigg(\sum_{i=0}^{m} B_i w_0(s-a_i)\bigg)\,ds, \quad 0 \le t \le a_1.$$

Then

$$\begin{aligned}\|z(a_1) - \tilde{z}_0\| &= \|\xi(a_1) - \tilde{\xi}_0\| \\ &\le \varepsilon.\end{aligned}$$

Denote $\xi(a_1)$ by ξ_{a_1} and $\dot{\xi}(a_1)$ by $\dot{\xi}_{a_1}$, and consider the system

$$\begin{cases} \ddot{\xi}(t) = A\xi(t) + B_0u(t), & t \in (a_1, a_1 + r], \\ \xi(a_1) = \xi_{a_1}, \\ \dot{\xi}(a_1) = \dot{\xi}_{a_1}. \end{cases} \tag{4.2.7}$$

Set $\tilde{\xi}_{11} = \tilde{z}_{11} - \chi_{a_1+r}$, where $\chi_{a_1+r} = \int_0^{a_1+r} \mathcal{S}(a_1+r-s)\big(\sum_{i=1}^{m} B_i w_0(s-a_i)\big)\,ds = \int_0^{a_1+r} \mathcal{S}(a_1+r-s)B_1w_0(s-a_1)\,ds$ is known. Again by (H_4) one can find a control u_{11} such that the mild solution $\xi(t)$ of (4.2.7) is given by

$$\xi(t) = \mathcal{C}(t)\wp(0) + \mathcal{S}(t)y_1 + \int_0^t \mathcal{S}(t-s)B_0u_{11}(s)\,ds, \quad a_1 \le t \le a_1 + r$$

and it satisfies $\|\xi(a_1 + r) - \tilde{\xi}_{11}\| \le \varepsilon$.

Let

$$w_{11}(t) = \begin{cases} w_0(t), & t \in [0, a_1], \\ u_{11}(t), & t \in (a_1, a_1 + r] \end{cases}$$

and

$$z(t) = \mathcal{C}(t)\wp(0) + \mathcal{S}(t)y_1 + \int_0^t \mathcal{S}(t-s)\bigg(\sum_{i=0}^{m} B_i w_{11}(s-a_i)\bigg)\,ds, \quad a_1 \leq t \leq a_1 + r.$$

Then

$$\begin{aligned}\|z(a_1+r) - \tilde{z}_{11}\| &= \|\xi(a_1+r) + \chi_{a_1+r} - \tilde{z}_{11}\| \\ &= \|\xi(a_1+r) - \tilde{\xi}_{11}\| \\ &\leq \varepsilon.\end{aligned}$$

Continuing in similar fashion, at the (n_1+2)-th step (if $\alpha_1 > 0$), we get

$$\begin{cases}\ddot{\xi}(t) = A\xi(t) + B_0 u(t), \quad t \in (a_1 + n_1 r, a_2], \\ \xi(a_1 + n_1 r) = \xi_{a_1+n_1 r}, \\ \dot{\xi}(a_1 + n_1 r) = \dot{\xi}_{a_1+n_1 r}.\end{cases} \tag{4.2.8}$$

Set $\tilde{\xi}_{1\overline{n_1+1}} = \tilde{z}_{1\overline{n_1+1}} - \chi_{a_2}$, where $\chi_{a_2} = \int_0^{a_2} \mathcal{S}(a_2-s)\big(\sum_{i=1}^{m} B_i w_{1n_1}(s-a_i)\big)\,ds = \int_0^{a_2} \mathcal{S}(a_2-s) B_1 w_{1n_1}(s-a_1)\,ds$ is known. Then one can find a control $u_{1\overline{n_1+1}}$ such that the mild solution $\xi(t)$ of (4.2.8) is given by

$$\xi(t) = \mathcal{C}(t)\wp(0) + \mathcal{S}(t)y_1 + \int_0^t \mathcal{S}(t-s)B_0 u_{1\overline{n_1+1}}(s)\,ds, \quad a_1 + n_1 r \leq t \leq a_2$$

and it satisfies $\|\xi(a_2) - \tilde{\xi}_{1\overline{n_1+1}}\| \leq \varepsilon$.

Let

$$w_{1\overline{n_1+1}}(t) = \begin{cases} w_{1n_1}(t), & t \in (a_1 + (n_1-1)r, a_1 + n_1 r], \\ u_{1\overline{n_1+1}}(t), & t \in (a_1 + n_1 r, a_2]\end{cases}$$

and

$$z(t) = \mathcal{C}(t)\wp(0) + \mathcal{S}(t)y_1 + \int_0^t \mathcal{S}(t-s)\bigg(\sum_{i=0}^{m} B_i w_{1\overline{n_1+1}}(s-a_i)\bigg)\,ds, \quad a_1 + n_1 r \leq t \leq a_2.$$

Then

$$\begin{aligned}\|z(a_2) - \tilde{z}_{1\overline{n_1+1}}\| &= \|\xi(a_2) + \chi_{a_2} - \tilde{z}_{1\overline{n_1+1}}\| \\ &= \|\xi(a_2) - \tilde{\xi}_{1\overline{n_1+1}}\| \\ &\leq \varepsilon.\end{aligned}$$

Repeating the above process, at the last step, that is, $(n_1+n_2+\cdots+n_m+m+1)$-th step (if $\alpha_i > 0, \quad \forall\, i = 1, 2, \ldots, m$), we get

$$\begin{cases} \ddot{\xi}(t) = A\xi(t) + B_0u(t), & t \in (a_m + n_m r, c], \\ \xi(a_m + n_m r) = \xi_{a_m+n_m r}, \\ \dot{\xi}(a_m + n_m r) = \dot{\xi}_{a_m+n_m r}. \end{cases} \tag{4.2.9}$$

Set $\tilde{\xi}_c = \tilde{z}_c - \chi_c$, where $\chi_c = \int_0^c \mathcal{S}(c-s)\big(\sum_{i=1}^m B_i w_{mn_m}(s-a_i)\big)\, ds$ is known from previous step. Then one can find a control $u_{m\overline{n_m+1}}$ such that the mild solution $\xi(t)$ of (4.2.9) is given by

$$\xi(t) = \mathcal{C}(t)\wp(0) + \mathcal{S}(t)y_1 + \int_0^t \mathcal{S}(t-s)B_0 u_{m\overline{n_m+1}}(s)\, ds, \quad a_m + n_m r \le t \le c$$

and it satisfies $\|\xi(c) - \tilde{\xi}_c\| \le \varepsilon$.

Let

$$w_{m\overline{n_m+1}}(t) = \begin{cases} w_{mn_m}(t), & t \in (a_m + (n_m - 1)r, a_m + n_m r], \\ u_{m\overline{n_m+1}}(t), & t \in (a_m + n_m r, c] \end{cases}$$

and

$$z(t) = \mathcal{C}(t)\wp(0) + \mathcal{S}(t)y_1 + \int_0^t \mathcal{S}(t-s)\left(\sum_{i=0}^m B_i w_{m\overline{n_m+1}}(s-a_i)\right) ds, \; a_m + n_m r \le t \le c.$$

Then

$$\begin{aligned} \|z(c) - z_c\| &= \|\xi(c) + \chi_c - z_c\| \\ &= \|\xi(c) - \tilde{\xi}_c\| \\ &\le \varepsilon. \end{aligned}$$

Now, define the control w on $[-a, c]$ as

$$w(t) = \begin{cases} w_0, & t \in [-a, a_1], \\ w_i(t), & t \in (a_i, a_{i+1}], \quad i = 1, 2, \ldots, m; \end{cases}$$

where

$$w_i(t) = \begin{cases} w_{ij}, & t \in (a_i + (j-1)r, a_i + jr], \quad j = 1, 2, \ldots, n_i; \\ u_{i\overline{n_i+1}}(t), & t \in (a_i + n_i r, a_{i+1}]. \end{cases}$$

Then

$$z(t) = \mathcal{C}(t)\wp(0) + \mathcal{S}(t)y_1 + \int_0^t \mathcal{S}(t-s)\Big(\sum_{i=0}^{m} B_i w(s-a_i)\Big)\, ds, \quad 0 \le t \le c$$

is the mild solution of (4.2.3) for the control function w and it satisfies $\|z(c)-z_c\| \le \varepsilon$. For other cases, the proof is similar. ■

Theorem 4.2.3. *Under hypotheses (H_1)-(H_5), the semilinear system (4.2.1) is approximately controllable.*

Proof. : Since $q \in L_1[0,c]$, one can find an increasing sequence $\langle c_n \rangle$ in $[0,c]$ such that $c_n \to c$ and

$$\int_{c_n}^{c} q(t)\, dt \to 0 \quad \text{as} \quad n \to \infty.$$

Now by approximate controllability of (4.2.3), for any given $\varepsilon > 0$ and $z_c \in V$, one can find a control $\tilde{u}_0 \in U$ satisfying

$$\Big\| z_c - \mathcal{C}(c)\wp(0) - \mathcal{S}(c)y_1 - \int_0^c \mathcal{S}(c-s)\Big(\sum_{i=0}^{m} B_i \tilde{u}_0(s-a_i)\Big)\, ds \Big\| \le \frac{\varepsilon}{2}.$$

Denote $z_1 = z(c_1, \wp, \tilde{u}_0)$ and $\dot{z}_1 = \dot{z}(c_1, \wp, \tilde{u}_0)$, where $z(t, \wp, \tilde{u}_0)$ is the mild solution of (4.2.1) for the control $\tilde{u}_0$. Again by approximate controllability of (4.2.3), one can find a control $\tilde{u}_1 \in L_p([c_1, c]; V')$ satisfying

$$\Big\| z_c - \mathcal{C}(c-c_1)z_1 - \mathcal{S}(c-c_1)\dot{z}_1 - \int_{c_1}^c \mathcal{S}(c-s)\Big(\sum_{i=0}^{m} B_i \tilde{u}_1(s-a_i)\Big)\, ds \Big\| \le \frac{\varepsilon}{2}.$$

Define

$$\tilde{w}_1(t) = \begin{cases} \tilde{u}_0(t), & t \in [0, c_1), \\ \tilde{u}_1(t), & t \in [c_1, c]. \end{cases}$$

Clearly, $\tilde{w}_1 \in U$. Continuing in this manner, one can obtain three sequences $z_n, \tilde{u}_n$ and $\tilde{w}_n$ such that $\tilde{u}_n \in L_p([c_n, c]; V')$, $\tilde{w}_n \in U$ given by

$$\tilde{w}_n(t) = \begin{cases} \tilde{u}_{n-1}(t), & t \in [0, c_n), \\ \tilde{u}_n(t), & t \in [c_n, c] \end{cases}$$

and $z_n = z(c_n, \wp, \tilde{u}_{n-1})$, $\dot{z}_n = \dot{z}(c_n, \wp, \tilde{u}_{n-1})$ with

$$\left\| z_c - \mathcal{C}(c-c_n)z_n - \mathcal{S}(c-c_1)\dot{z}_n - \int_{c_n}^{c} \mathcal{S}(c-s)\left(\sum_{i=0}^{m} B_i \tilde{u}_n(s-a_i)\right) ds \right\| \le \frac{\varepsilon}{2}.$$

Let $z(t, \wp, \tilde{w}_n)$ be the mild solution of (4.2.1) associated with $\tilde{w}_n$.
Denote

$$\mathcal{G}(s) = \sum_{i=0}^{m} B_i \tilde{w}_n(s-a_i) + F\big(s, z_{\varsigma(s)}, \tilde{w}_n(s), \tilde{w}_n(s-\widehat{a}_1), \ldots, \tilde{w}_n(s-\widehat{a}_{\widehat{m}})\big).$$

Then

$$\begin{aligned}
& z(c, \wp, \tilde{w}_n) \\
&\quad = \mathcal{C}(c)\wp(0) + \mathcal{S}(c)y_1 + \int_0^c \mathcal{S}(c-s)\mathcal{G}(s)\, ds \\
&\quad = \mathcal{C}(c-c_n+c_n)\wp(0) + \mathcal{S}(c-c_n+c_n)y_1 \\
&\qquad + \int_0^{c_n} \mathcal{S}(c-c_n+c_n-s)\mathcal{G}(s)\, ds + \int_{c_n}^{c} \mathcal{S}(c-s)\mathcal{G}(s)\, ds \\
&\quad = \big(\mathcal{C}(c-c_n)\mathcal{C}(c_n) + A\mathcal{S}(c-c_n)\mathcal{S}(c_n)\big)\wp(0) \\
&\qquad + \big(\mathcal{S}(c-c_n)\mathcal{C}(c_n) + \mathcal{S}(c_n)\mathcal{C}(c-c_n)\big)y_1 \\
&\qquad + \int_0^{c_n} \big(\mathcal{S}(c-c_n)\mathcal{C}(c_n-s) + \mathcal{S}(c_n-s)\mathcal{C}(c-c_n)\big)\mathcal{G}(s)\, ds \\
&\qquad + \int_{c_n}^{c} \mathcal{S}(c-s)\mathcal{G}(s)\, ds \\
&\quad = \mathcal{C}(c-c_n)\left(\mathcal{C}(c_n)\wp(0) + \mathcal{S}(c_n)y_1 + \int_0^{c_n} \mathcal{S}(c_n-s)\mathcal{G}(s)\, ds\right) \\
&\qquad + \mathcal{S}(c-c_n)\left(A\mathcal{S}(c_n)\wp(0) + \mathcal{C}(c_n)y_1 + \int_0^{c_n} \mathcal{C}(c_n-s)\mathcal{G}(s)\, ds\right) \\
&\qquad + \int_{c_n}^{c} \mathcal{S}(c-s)\mathcal{G}(s)\, ds \\
&\quad = \mathcal{C}(c-c_n)\Bigg(\mathcal{C}(c_n)\wp(0) + \mathcal{S}(c_n)y_1 + \int_0^{c_n} \mathcal{S}(c_n-s)\Bigg(\sum_{i=0}^{m} B_i \tilde{w}_n(s-a_i) \\
&\qquad + F\big(s, z_{\varsigma(s)}, \tilde{w}_n(s), \tilde{w}_n(s-\widehat{a}_1), \ldots, \tilde{w}_n(s-\widehat{a}_{\widehat{m}})\big)\Bigg) ds\Bigg) \\
&\qquad + \mathcal{S}(c-c_n)\Bigg(A\mathcal{S}(c_n)\wp(0) + \mathcal{C}(c_n)y_1 + \int_0^{c_n} \mathcal{C}(c_n-s)\Bigg(\sum_{i=0}^{m} B_i \tilde{w}_n(s-a_i) \\
&\qquad + F\big(s, z_{\varsigma(s)}, \tilde{w}_n(s), \tilde{w}_n(s-\widehat{a}_1), \ldots, \tilde{w}_n(s-\widehat{a}_{\widehat{m}})\big)\Bigg) ds\Bigg)
\end{aligned}$$

$$
\begin{aligned}
&+\int_{c_n}^{c}\mathcal{S}(c-s)\bigg(\sum_{i=0}^{m}B_i\tilde{w}_n(s-a_i)\\
&+F\big(s,z_{\varsigma(s)},\tilde{w}_n(s),\tilde{w}_n(s-\widehat{a}_1),\ldots,\tilde{w}_n(s-\widehat{a}_{\widehat{m}})\big)\bigg)\,ds\\
=\,&\mathcal{C}(c-c_n)\bigg(\mathcal{C}(c_n)\wp(0)+\mathcal{S}(c_n)y_1+\int_0^{c_n}\mathcal{S}(c_n-s)\bigg(\sum_{i=0}^{m}B_i\tilde{u}_{n-1}(s-a_i)\\
&+F\big(s,z_{\varsigma(s)},\tilde{u}_{n-1}(s),\tilde{u}_{n-1}(s-\widehat{a}_1),\ldots,\tilde{u}_{n-1}(s-\widehat{a}_{\widehat{m}})\big)\bigg)\,ds\bigg)\\
&+\mathcal{S}(c-c_n)\bigg(A\mathcal{S}(c_n)\wp(0)+\mathcal{C}(c_n)y_1+\int_0^{c_n}\mathcal{C}(c_n-s)\bigg(\sum_{i=0}^{m}B_i\tilde{u}_{n-1}(s-a_i)\\
&+F\big(s,z_{\varsigma(s)},\tilde{u}_{n-1}(s),\tilde{u}_{n-1}(s-\widehat{a}_1),\ldots,\tilde{u}_{n-1}(s-\widehat{a}_{\widehat{m}})\big)\bigg)\,ds\bigg)\\
&+\int_{c_n}^{c}\mathcal{S}(c-s)\bigg(\sum_{i=0}^{m}B_i\tilde{u}_n(s-a_i)\\
&+F\big(s,z_{\varsigma(s)},\tilde{u}_n(s),\tilde{u}_n(s-\widehat{a}_1),\ldots,\tilde{u}_n(s-\widehat{a}_{\widehat{m}})\big)\bigg)\,ds\\
=\,&\mathcal{C}(c-c_n)z_n+\mathcal{S}(c-c_n)\dot{z}_n+\int_{c_n}^{c}\mathcal{S}(c-s)\bigg(\sum_{i=0}^{m}B_i\tilde{u}_n(s-a_i)\bigg)\,ds\\
&+\int_{c_n}^{c}\mathcal{S}(c-s)F\big(s,z_{\varsigma(s)},\tilde{u}_n(s),\tilde{u}_n(s-\widehat{a}_1),\ldots,\tilde{u}_n(s-\widehat{a}_{\widehat{m}})\big)\,ds.
\end{aligned}
$$

Now,

$$
\begin{aligned}
&\|z(c,\wp,\tilde{w}_n)-z_c\|\\
&\le\bigg\|z_c-\mathcal{C}(c-c_n)z_n-\mathcal{S}(c-c_n)\dot{z}_n-\int_{c_n}^{c}\mathcal{S}(c-s)\bigg(\sum_{i=0}^{m}B_i\tilde{u}_n(s-a_i)\bigg)\,ds\bigg\|\\
&\quad+\bigg\|\int_{c_n}^{c}\mathcal{S}(c-s)F\big(s,z_{\varsigma(s)},\tilde{u}_n(s),\tilde{u}_n(s-\widehat{a}_1),\ldots,\tilde{u}_n(s-\widehat{a}_{\widehat{m}})\big)\,ds\bigg\|\\
&\le\frac{\varepsilon}{2}+k_{\mathcal{S}}\int_{c_n}^{c}\big\|F\big(s,z_{\varsigma(s)},\tilde{u}_n(s),\tilde{u}_n(s-\widehat{a}_1),\ldots,\tilde{u}_n(s-\widehat{a}_{\widehat{m}})\big)\big\|\,ds\\
&\le\frac{\varepsilon}{2}+k_{\mathcal{S}}\int_{c_n}^{c}q(s)\,ds\\
&\le\frac{\varepsilon}{2}+k_{\mathcal{S}}\frac{\varepsilon}{2k_{\mathcal{S}}}\\
&=\varepsilon
\end{aligned}
$$

for sufficiently large value of n. Hence the systen (4.2.1) is approximately controllable. ■

Remark 4.2.4. *Under assumptions* (H_1)*,* (H_4) *and* (H_5)*, the system (4.2.1) is approximately controllable if it has a solution for each given* $u \in U$ *and* $y_1 \in V$*.*

4.2.4 Example

Consider the semilinear wave equation for $x \in [0,1]$

$$\begin{cases} \frac{\partial^2 \widehat{z}(t,x)}{\partial t^2} = \frac{\partial^2 \widehat{z}(t,x)}{\partial x^2} + \sum_{i=0}^{m} B_i \widehat{u}(t-a_i,x) \\ \qquad + F\big(t, \widehat{z}(\varsigma(t)+\theta,x), \widehat{u}(t,x), \widehat{u}(t-\widehat{a}_1,x), \ldots, \widehat{u}(t-\widehat{a}_{\widehat{m}})\big) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for } \ t \in (0,c], \\ \frac{\partial \widehat{z}}{\partial x}(t,0) = \frac{\partial \widehat{z}}{\partial x}(t,1) = 0 \qquad\qquad\qquad\qquad \text{for } \ t \in (0,c], \\ \widehat{z}_0(x) = \sum_{j=1}^{n} \beta_j \widehat{z}(t_j,x), \\ \frac{\partial \widehat{z}}{\partial t}(0,x) = \widehat{y}_1(x), \end{cases} \tag{4.2.10}$$

where $-a \le t_1 < t_2 < \cdots < t_n \le 0$ and $\varsigma(t) = \frac{t^3}{1+c^3}$, $0 \le t \le c$.

Clearly, ς satisfies $\varsigma(t) \le t$ and

$$\tilde{z}_{\varsigma(t)}(\theta,x) = \tilde{z}\left(\frac{t^3}{1+c^3} + \theta, x\right).$$

The equation (4.2.10) can be converted in the abstract form (4.2.1), if we make the setting similar to the previous example.

If we take

$$\begin{aligned} &F\big(t, z_{\varsigma(t)}, u(t), u(t-\widehat{a}_1), \ldots, u(t-\widehat{a}_{\widehat{m}})\big) \\ &\quad = \left(\frac{t\left\|z_{\varsigma(t)}\right\|^2_{\mathfrak{C}_0}}{1+\left\|z_{\varsigma(t)}\right\|^2_{\mathfrak{C}_0}} \xi_3(x) + \frac{t^2(\|u(t)\| + \|u(t-\widehat{a}_1)\| + \cdots + \|u(t-\widehat{a}_{\widehat{m}})\|)}{1+\|u(t)\| + \|u(t-\widehat{a}_1)\| + \cdots + \|u(t-\widehat{a}_{\widehat{m}})\|} \xi_4(x) \right), \end{aligned}$$

then

$$\begin{aligned} &\left\|F\big(t, z_{\varsigma(t)}, u(t), u(t-\widehat{a}_1), \ldots, u(t-\widehat{a}_{\widehat{m}})\big)\right\| \\ &\quad \le (t\|\xi_3\|_V + t^2\|\xi_4\|_V) \\ &\quad = (t+t^2) \\ &\quad \le c(1+c)\left(1 + \|z_{\varsigma(t)}\|_{\mathfrak{C}_0} + \|u(t)\| + \|u(t-\widehat{a}_1)\| + \cdots + \|u(t-\widehat{a}_{\widehat{m}})\|\right). \end{aligned}$$

Hence (H_3) and (H_5) are satisfied.

Also,

$$\begin{aligned}&\big\|F\big(t,(z_1)_{\varsigma(t)},u(t),u(t-\widehat{a}_1),\ldots,u(t-\widehat{a}_{\widehat{m}})\big)\\&\quad -F\big(t,(z_2)_{\varsigma(t)},u(t),u(t-\widehat{a}_1),\ldots,u(t-\widehat{a}_{\widehat{m}})\big)\big\|\\&\qquad\le c\left(\left\|(z_1)_{\varsigma(t)}\right\|_{\mathfrak{C}_0}+\left\|(z_2)_{\varsigma(t)}\right\|_{\mathfrak{C}_0}\right)\left\|(z_1)_{\varsigma(t)}-(z_2)_{\varsigma(t)}\right\|_{\mathfrak{C}_0}\\&\qquad\le 2cr\left\|(z_1)_{\varsigma(t)}-(z_2)_{\varsigma(t)}\right\|_{\mathfrak{C}_0}\\&\qquad=\lambda_r\left\|(z_1)_{\varsigma(t)}-(z_2)_{\varsigma(t)}\right\|_{\mathfrak{C}_0}\end{aligned}$$

for any $(z_1)_{\varsigma(t)},(z_2)_{\varsigma(t)}\in\mathcal{B}(0,r)\subset\mathfrak{C}_0$ and $u(t)\in V'$. Hence F is locally Lipschitz, that is, (H_2) is satisfied. The linear part of (4.2.10) is approximately controllable (in fact, it is exactly controllable) [18]. Thus by previous theorem, the system (4.2.10) is controllable.

4.3 Concluding remarks

In this chapter, the existence and approximate controllability of second-order nonlocal retarded systems have been analyzed. In first section, we determined the approximate and exact controllability of second-order nonlocal retarded semilinear systems with fixed delay in control under Lipschitz assumption. Utilizing a fixed point theorem, the result of existence and uniqueness has been deduced. The controllability of associated linear delay system has been proved by the method of steps and then the controllability of actual system is shown by proving that the reachable set of semilinear system contains the reachable set of the associated linear system without delay. In second section, the approximate controllability for retarded systems of second-order with control delays and nonlocal conditions has been discussed by assuming that the nonlinear term is locally Lipschitz which is a weaker condition than Lipschitz continuity. Using fixed point approach, the existence and uniqueness results have been derived. Then, under some assumptions, we proved that the controllability of the corresponding linear system without delay implies the controllability of the corresponding linear delay system and the actual system by applying an iterative technique. Here, the results have been proved without assuming the inclusion condition among the range sets of the operators. But conditions (H_3) and

(H_5) are very strong and may not be easily satisfied in many practical problems. For this reason, an study on approximate controllability of the same system without assuming the conditions (H_3) and (H_4) is a matter of next investigation.

Chapter 5

Partial Approximate Controllability of Nonlocal Riemann-Liouville Fractional Semilinear Systems

In this chapter, we investigate the partial approximate controllability of nonlocal Riemann-Liouville fractional systems with integral initial conditions in Hilbert spaces without Lipschitz condition on nonlinear function. We also exclude the conditions of Lipschitz continuity and compactness for the nonlocal function. The existence results are derived by applying Schauder's fixed point theorem, then the partial controllability result is proved by assuming that the associated linear system with local initial condition is partial approximately controllable. Lastly, an example is provided to apply our results.

5.1 Introduction and preliminaries

Let V and V' be Hilbert spaces with the corresponding function spaces $Z = L_2([0, c]; V)$ and $U = L_2([0, c]; V')$ respectively. Consider the semilinear system

$$\begin{cases} D_t^{\vartheta} z(t) = Az(t) + Bu(t) + F(t, z(t)), \quad t \in (0, c], \\ \left(I_t^{1-\vartheta} z(t)\right)_{t=0} = y_0 - \wp(z), \end{cases} \tag{5.1.1}$$

where $0 < \vartheta \leq 1$ and D_t^{ϑ} denotes the Riemann-Liouville fractional derivative of order ϑ. The state $z \in Z$, the control $u \in U$ and $y_0 \in V$ is given. The linear map $A : D(A) \subseteq V \to V$ generates a C_0- semigroup $T(t)$. B is continuous linear map from V' to V. $F : [0, c] \times V \to V$ is nonlinear and $\wp$ is a function to be specified later.

In last few decades, controllability results for various types of semilinear and linear differential systems of fractional order have been analyzed in many articles. Among them, Kumar et al. [53] investigated the approximate controllability for nonlinear systems of fractional order with bounded delay by applying fixed point theorem. Tai and Lun [83] proved controllability of impulsive fractional neutral integrodifferential equations by applying fractional calculus and resolvent operators. Liu and Li [61] determined the controllability of Riemann-Liouville fractional systems in infinite dimensional Banach spaces by using C_0-semigroup and Lipschitz nonlinearity. In [92], Zhu et al. analyzed the controllability of Riemann-Liouville fractional nonlinear systems using itegral contractor. Mahmudov [65] is the one who analyzed the partial controllability for semilinear equations of fractional order with Caputo derivatives. However, the partial controllability of fractional systems with Riemann-Liouville derivatives is still untreated topic in the literature so for.

Our aim is to analyze the partial approximate controllability of the fractional system (5.1.1) without Lipschitz condition or compactness of nonlocal function. For this, we make an approximate problem of the fractional system (5.1.1) and prove the compactness of the set of its solutions. Then we prove that it is possible to steer the system to any open set containing any given final state in a closed subspace.

To define the mild solution of (5.1.1), consider the Banach space $C_{1-\vartheta}([0, c]; V) = \{z \,|\, t^{1-\vartheta} z(t) \in C([0, c]; V)\}$ with the norm $\|z\|_{C_{1-\vartheta}} = \sup_{t\in[0,c]}\{t^{1-\vartheta}\|z(t)\|_V\}$. For C_0-semigroup $T(t)$, we set $\sup_{t\in[0,c]} \|T(t)\| \leq k_T < \infty$ and $\|B\| \leq k_B$. Let V_0 be the closed subspace of V, P be the projection from V onto V_0.

Definition 5.1.1. ([61]) A function $z \in C_{1-\vartheta}([0, c]; V)$ is said to be a mild solution of (5.1.1) if it satisfies

$$\begin{aligned} z(t) &= t^{\vartheta-1} T_\vartheta(t) \left(y_0 - \wp(z)\right) \\ &\quad + \int_0^t (t-s)^{\vartheta-1} T_\vartheta(t-s)\big(Bu(s) + F(s, z(s))\big)\, ds, \end{aligned} \tag{5.1.2}$$

where

$$T_\vartheta(t) = \vartheta \int_0^\infty \alpha \xi_\vartheta(\alpha) T(t^\vartheta \alpha)\, d\alpha,$$
$$\xi_\vartheta(\alpha) = \frac{1}{\vartheta}\alpha^{-1-\frac{1}{\vartheta}}\omega_\vartheta\left(\alpha^{-\frac{1}{\vartheta}}\right),$$
$$\omega_\vartheta(\alpha) = \frac{1}{\pi}\sum_{\ell=1}^{\infty}(-1)^{\ell-1}\alpha^{-\ell\vartheta-1}\frac{\Gamma(\ell\vartheta+1)}{\ell!}\sin(\ell\pi\vartheta), \quad \alpha \in (0,\infty)$$

and ξ_ϑ is a probability density function defined on $(0,\infty)$.

Definition 5.1.2. The system (5.1.1) is said to be partial approximately controllable on $[0,c]$ if for any given $\varepsilon > 0$ and $z_c \in V_0$, one can find a control $u \in U$ such that the mild solution $z(t,u)$ satisfies

$$\|Pz(c,u) - z_c\| \leq \varepsilon.$$

Lemma 5.1.1. *([61]) For each fixed $t \geq 0$, $T_\vartheta(t)$ is continuous linear map satisfying*

$$\|T_\vartheta(t)y\| \leq \frac{k_T}{\Gamma(\vartheta)}\|y\| \ \forall\, z \in V.$$

Remark 5.1.2. *It is notable that $T_\vartheta(t)$ is point wise bounded. Therefore by uniform boundedness theorem, there is a constant $\widehat{k}_T > 0$ such that $\sup_{t\in[0,c]}\|T_\vartheta(t)\| \leq \widehat{k}_T$.*

Throughout this chapter, we suppose the following conditions:

(H_1) the semigroup $T(t)$ generated by A is compact;

(H_2) the function $F : [0,c] \times V \to V$ is jointly continuous and there is a $g \in C([0,c];\mathbb{R}_0^+)$ satisfying

$$\|F(t,y)\| \leq g(t), \ \forall\, (t,y) \in [0,c] \times V;$$

(H_3) the function $\wp : C_{1-\vartheta}([0,c];V) \to V$ is continuous and there is a constant $k_\wp$ satisfying

$$\|\wp(z)\| \leq k_\wp;$$

(H_4) there exists a $b \in (0,c)$ such that for any $z, \tilde{z} \in C_{1-\vartheta}([0,c];V)$ satisfying $z(t) = \tilde{z}(t),\ t \in [b,c],$ we have $\wp(z) = \wp(\tilde{z})$;

(H_5) the linear system

$$z(t) = t^{\vartheta-1}T_\vartheta(t)y_0 + \int_0^t (t-s)^{\vartheta-1}T_\vartheta(t-s)Bu(s)\,ds \tag{5.1.3}$$

is partial approximately controllable.

Remark 5.1.3. *The system (5.1.3) is partial approximately controllable iff* $B^*T_\vartheta^*(c-s)P^*y = 0,\ 0 < s < c$ *implies that* $y = 0$.

For $\varepsilon > 0$ and $n \geq 1$, define the functional

$$\zeta_{\varepsilon,n}(y,z) = \frac{1}{2}\int_0^c (c-s)^{\vartheta-1}\|B^*T_\vartheta^*(c-s)P^*y\|^2\,ds + \varepsilon\|y\| - \langle y, \delta_n(z)\rangle, \tag{5.1.4}$$

where

$$\delta_n(z) = c^{\vartheta-1}PT_\vartheta(c)\left(y_0 - T\left(\frac{1}{n}\right)\wp(z)\right) + \int_0^c (c-s)^{\vartheta-1}PT_\vartheta(c-s)F(s,z(s))\,ds - z_c,$$

and the operator $G : C_{1-\vartheta}([0,c];V) \to V_0$ given by

$$G(z) = c^{\vartheta-1}PT_\vartheta(c)\left(y_0 - T\left(\frac{1}{n}\right)\wp(z)\right) + \int_0^c (c-s)^{\vartheta-1}PT_\vartheta(c-s)F(s,z(s))\,ds.$$

Consider the ball

$$\mathcal{B}(0,\lambda) = \left\{z \in C_{1-\vartheta}([0,c];V) \,\middle|\, \|z\|_{C_{1-\vartheta}} \leq \lambda\right\}.$$

Then, it can be easily prove that the set

$$\left\{c^{\vartheta-1}PT_\vartheta(c)\left(y_0 - T\left(\frac{1}{n}\right)\wp(z)\right) \,\middle|\, z \in \mathcal{B}(0,\lambda)\right\}$$

is relatively compact in V and hence the map $G : \mathcal{B}(0,\lambda) \to V_0$ is compact [65].

5.2 Existence of mild solution

Lemma 5.2.1. *For any* $\mathcal{B}(0,\lambda)$*,*

$$\underline{\lim}_{\|y\|\to\infty} \inf_{z\in\mathcal{B}(0,\lambda)} \frac{\zeta_{\varepsilon,n}(y;z)}{\|y\|} \geq \varepsilon. \tag{5.2.1}$$

Proof. First we prove $\delta_n : \mathcal{B}(0,\lambda) \to V_0$ is continuous for any $n \geq 1$. For this take $z_\nu, z \in \mathcal{B}(0,\lambda)$ satisfying $\lim_{\nu\to\infty} \|z_\nu - z\|_{C_{1-\vartheta}} = 0$. In view of (H_2) and continuity of norm, one can get $\lim_{\nu\to\infty} \|\tilde{F}(z_\nu) - \tilde{F}(z)\|_{C_{1-\vartheta}} = 0$ where the map $\tilde{F} : C_{1-\vartheta}([0,c];V) \to C_{1-\vartheta}([0,c];V)$ is defined by $(\tilde{F}(z))(t) = F(t, z(t))$. Now,

$$\begin{aligned}\|\delta_n(z_\nu) - \delta_n(z)\|_{V_0} \leq & \left\| Pc^{\vartheta-1}T_\vartheta(c)T\left(\frac{1}{n}\right)(\wp(z_\nu) - \wp(z))\right\| \\ & + \left\| \int_0^c (c-s)^{\vartheta-1} PT_\vartheta(c-s)\big(F(s,z_\nu(s)) - F(s,z(s))\big)\, ds \right\| \\ = & \widehat{k}_T k_T \|P\| \|\wp(z_\nu) - \wp(z)\| \\ & + \frac{\widehat{k}_T c^{2\vartheta-1}(\Gamma(\vartheta))^2}{\Gamma(2\vartheta)} \|P\| \|\tilde{F}(z_\nu) - \tilde{F}(z)\|_{C_{1-\vartheta}} \\ & \to 0 \text{ as } \nu \to \infty.\end{aligned}$$

Now, suppose (5.2.1) does not hold. Then one can select sequences y_ν in V and z_ν in $\mathcal{B}(0,\lambda)$, with $\|y_\nu\| \to \infty$ satisfying

$$\underline{\lim}_{\nu\to\infty} \frac{\zeta_{\varepsilon,n}(y_\nu; z_\nu)}{\|y_\nu\|} < \varepsilon. \tag{5.2.2}$$

Since $\{\delta_n(z_\nu) \,|\, \nu \geq 1\} \subset Range(G)$ is relatively compact. Therefore by taking a subsequence, one can assume that

$$\delta_n(z_\nu) \to \delta_n \in V. \tag{5.2.3}$$

Denote $\hat{y}_\nu = \frac{y_\nu}{\|y_\nu\|}$. Boundedness of $\hat{y}_\nu$ enables to select a subsequence (still denoted by $\hat{y}_\nu$) such that $\hat{y}_\nu \rightharpoonup \hat{y} \in V$. Since $T(t)$ is compact, one can see that

$$B^*T^*(c-\cdot)P^*\hat{y}_\nu \to B^*T^*(c-\cdot)P^*\hat{y} \text{ in } C_{1-\vartheta}([0,c];V). \tag{5.2.4}$$

From (5.2.1), one can obtain

$$\frac{\zeta_{\varepsilon,n}(y_\nu; z_\nu)}{\|y_\nu\|} = \frac{\|y_\nu\|}{2} \int_0^c (c-s)^{\vartheta-1} \|B^*T_\vartheta^*(c-s)P^*\hat{y}_\nu\|^2\, ds + \varepsilon\|\hat{y}_\nu\| - \langle \hat{y}_\nu, \delta_n(z_\nu)\rangle.$$

Since $\|y_\nu\| \to \infty$, using Fatou lemma and (5.2.2)-(5.2.4), one can obtain

$$\begin{aligned}\int_0^c (c-s)^{\vartheta-1} \|B^*T_\vartheta^*(c-s)P^*\hat{y}\|^2\, ds \leq & \underline{\lim}_{\nu\to\infty} \int_0^c (c-s)^{\vartheta-1} \|B^*T_\vartheta^*(c-s)P^*\hat{y}_\nu\|^2\, ds \\ = & 0.\end{aligned}$$

By (H_5), $\hat{y} = 0$ and hence $\hat{y}_\nu \rightharpoonup 0 \in V$.

Thus

$$\begin{aligned}\underline{\lim}_{\nu\to\infty}\frac{\zeta_{\varepsilon,n}(y_\nu; z_\nu)}{\|y_\nu\|} &\geq \underline{\lim}_{\nu\to\infty}(\varepsilon\|\hat{y}_\nu\| - \langle \hat{y}_\nu, \delta_n(z_\nu)\rangle) \\ &= \varepsilon,\end{aligned}$$

which is a contradiction. ■

Remark 5.2.2. *It is easy to verify that for any fixed* $z \in \mathcal{B}(0,\lambda)$*, the map* $y \mapsto \zeta_{\varepsilon,n}(y; z)$ *is strictly convex and continuous.*

Remark 5.2.3. *For every fixed* $z \in C_{1-\vartheta}([0,c]; V)$*, the functional* $\zeta_{\varepsilon,n}(\cdot; z)$ *has a unique minimizer* $\tilde{y}_{\varepsilon,n}$ *that defines a map* $\chi_{\varepsilon,n} : C_{1-\vartheta}([0,c]; V) \to V$ *given by* $\chi_{\varepsilon,n}(z) = \tilde{y}_{\varepsilon,n}$*, where* $\zeta_{\varepsilon,n}(\tilde{y}_{\varepsilon,n}; z) = \min_{y\in V}\zeta_{\varepsilon,n}(y; z)$*; which is bounded on* $\mathcal{B}(0,\lambda)$ *for each fixed* $\varepsilon > 0$*, that is there is a constant* κ_ε *such that* $\|\chi_{\varepsilon,n}(z)\| < \kappa_\varepsilon$ *for any* $z \in \mathcal{B}(0,\lambda)$*,* $n \geq 1$*.*

Lemma 5.2.4. *If* $z_\nu, z \in \mathcal{B}(0,\lambda)$ *such that*

$$\lim_{\nu\to\infty}\|z_\nu - z\|_{C_{1-\vartheta}} = 0,$$

then

$$\lim_{\nu\to\infty}\|\chi_{\varepsilon,n}(z_\nu) - \chi_{\varepsilon,n}(z)\|_V = 0.$$

Proof. Boundedness of $\tilde{y}_{\varepsilon,n,\nu} = \chi_{\varepsilon,n}(z_\nu)$ enables to assume that $\tilde{y}_{\varepsilon,n,\nu} \rightharpoonup \hat{y}_{\varepsilon,n}$. By the definition of $\zeta_{\varepsilon,n}$ and the optimality of $\tilde{y}_{\varepsilon,n,\nu} = \chi_{\varepsilon,n}(z_\nu)$ and $\tilde{y}_{\varepsilon,n} = \chi_{\varepsilon,n}(z)$, one obtains

$$\begin{aligned}\zeta_{\varepsilon,n}(\tilde{y}_{\varepsilon,n}; z) &\leq \zeta_{\varepsilon,n}(\hat{y}_{\varepsilon,n}; z) \\ &\leq \underline{\lim}_{\nu\to\infty}\zeta_{\varepsilon,n}(\tilde{y}_{\varepsilon,n,\nu}; z_\nu) \\ &\leq \overline{\lim}_{\nu\to\infty}\zeta_{\varepsilon,n}(\tilde{y}_{\varepsilon,n,\nu}; z_\nu) \\ &\leq \lim_{\nu\to\infty}\zeta_{\varepsilon,n}(\tilde{y}_{\varepsilon,n}; z_\nu) \\ &= \zeta_{\varepsilon,n}(\tilde{y}_{\varepsilon,n}; z).\end{aligned}$$

Above shows that $\hat{y}_{\varepsilon,n}$ also minimize $\zeta_{\varepsilon,n}(\cdot; z)$, which means $\hat{y}_{\varepsilon,n} = \tilde{y}_{\varepsilon,n}$. Therefore

$$\lim_{\nu\to\infty}\zeta_{\varepsilon,n}(\tilde{y}_{\varepsilon,n,\nu}; z_\nu) = \zeta_{\varepsilon,n}(\tilde{y}_{\varepsilon,n}; z),$$

$$\lim_{\nu\to\infty}\int_0^c (c-s)^{\vartheta-1}\|B^*T_\vartheta^*(c-s)P^*\tilde{y}_{\varepsilon,n,\nu}\|^2\,ds$$
$$=\int_0^c (c-s)^{\vartheta-1}\|B^*T_\vartheta^*(c-s)P^*\tilde{y}_{\varepsilon,n}\|^2\,ds,$$
$$\lim_{\nu\to\infty}\langle \tilde{y}_{\varepsilon,n,\nu},\delta_n(z_\nu)\rangle=\langle \tilde{y}_{\varepsilon,n},\delta_n(z)\rangle$$

and

$$\|\tilde{y}_{\varepsilon,n}\|\le\lim_{\nu\to\infty}\|\tilde{y}_{\varepsilon,n,\nu}\|.$$

From above relations, one has

$$\|\tilde{y}_{\varepsilon,n}\|=\lim_{\nu\to\infty}\|\tilde{y}_{\varepsilon,n,\nu}\|. \tag{5.2.5}$$

Since V is a Hilbert space and $\tilde{y}_{\varepsilon,n,\nu}\rightharpoonup\tilde{y}_{\varepsilon,n}$, therefore the result follows. ■

In the next theorem, we show that the map $\Psi_{\varepsilon,n}: C_{1-\vartheta}([0,c];V)\to C_{1-\vartheta}([0,c];V)$ defined by

$$(\Psi_{\varepsilon,n}z)(t)=t^{\vartheta-1}T_\vartheta(t)\left(y_0-T\left(\frac{1}{n}\right)\wp(z)\right)$$
$$+\int_0^t (t-s)^{\vartheta-1}T_\vartheta(t-s)\big(Bu_{\varepsilon,n}(s,z)+F(s,z(s))\big)\,ds \tag{5.2.6}$$

with

$$u_{\varepsilon,n}(s,z)=B^*T_\vartheta^*(c-s)P^*\chi_{\varepsilon,n}(z)=B^*T_\vartheta^*(c-s)P^*\tilde{y}_{\varepsilon,n}, \tag{5.2.7}$$

has a fixed point.

Theorem 5.2.5. *The operator $\Psi_{\varepsilon,n}$ has a fixed point in $C_{1-\vartheta}([0,c];V)$, for $n\ge 1$.*

Proof. Claim 1: $\Psi_{\varepsilon,n}$ is continuous.

Let $z_j\in C_{1-\vartheta}([0,c];V)$ with $z_j\to z\in C_{1-\vartheta}([0,c];V)$ as $j\to\infty$. Since $\tilde{F}$ and $u_{\varepsilon,n}$ are continuous, therefore

$$t^{1-\vartheta}\|(\Psi_{\varepsilon,n}z_j)(t)-(\Psi_{\varepsilon,n}z)(t)\|$$
$$\le k_T\widehat{k}_T\|\wp(z_j)-\wp(z)\|+\frac{\widehat{k}_Tc^\vartheta(\Gamma(\vartheta))^2}{\Gamma(2\vartheta)}\|\tilde{F}(z_j)-\tilde{F}(z)\|_{C_{1-\vartheta}}$$
$$+\frac{\widehat{k}_Tk_Bc^\vartheta(\Gamma(\vartheta))^2}{\Gamma(2\vartheta)}\|u_{\varepsilon,n}(\cdot,z_j)-u_{\varepsilon,n}(\cdot,z)\|_{C_{1-\vartheta}}.$$

Hence $\|\Psi_{\varepsilon,n}z_j - \Psi_{\varepsilon,n}z\|_{C_{1-\vartheta}} \to 0$ as $j \to \infty$.

Claim 2: There exists a positive number $\lambda(\varepsilon) > 0$ such that $\Psi_{\varepsilon,n}(\mathcal{B}(0,\lambda(\varepsilon))) \subset \mathcal{B}(0,\lambda(\varepsilon))$.

$$\begin{aligned} t^{1-\vartheta}\|(\Psi_{\varepsilon,n}z)(t)\| &\leq \widehat{k}_T\|y_0\| + k_T\widehat{k}_T k_\wp + \frac{c\widehat{k}_T}{\vartheta}\left(\|g\| + \widehat{k}_T k_B^2\|P\|\kappa_\varepsilon\right) \\ &= \lambda(\varepsilon). \end{aligned}$$

Claim 3: $\Psi_{\varepsilon,n}$ is compact for $n \geq 1$.

Hence by Schauder fixed point theorem, $\Psi_{\varepsilon,n}$ has a fixed point. ■

Suppose that $z_{\varepsilon,n} \in \mathcal{B}(0,\lambda(\varepsilon))$ be a fixed point of $\Psi_{\varepsilon,n}$ and $\chi_{\varepsilon,n}(z_{\varepsilon,n})$ minimizes $\zeta_{\varepsilon,n}(y; z_\varepsilon)$, and

$$u_{\varepsilon,n}(s, z_{\varepsilon,n}) = B^*T_\vartheta^*(c-s)P^*\chi_{\varepsilon,n}(z_{\varepsilon,n})$$

is the associated control. Further, suppose that

$$z_{\varepsilon,n} \to z_\varepsilon \text{ in } C_{1-\vartheta}([0,c];V) \text{ as } n \to \infty$$

and $\chi_\varepsilon(z_\varepsilon)$ minimizes $\zeta_\varepsilon(y; z_\varepsilon)$, and

$$u_\varepsilon(s, z_\varepsilon) = B^*T_\vartheta^*(c-s)P^*\chi_\varepsilon(z_\varepsilon)$$

is the associated control.

Lemma 5.2.6. *If*

$$\lim_{n\to\infty} \|z_{\varepsilon,n} - z_\varepsilon\|_{C_{1-\vartheta}} = 0,$$

then

$$\lim_{n\to\infty} \|\chi_{\varepsilon,n}(z_{\varepsilon,n}) - \chi_\varepsilon(z_\varepsilon)\| = 0$$

and

$$\lim_{n\to\infty} \|u_{\varepsilon,n}(s, z_{\varepsilon,n}) - u_\varepsilon(s, z_\varepsilon)\| = 0$$

Proof. By definition $\chi_{\varepsilon,n}(z_{\varepsilon,n})$ and $\chi_\varepsilon(z_\varepsilon)$ minimize

$$\zeta_{\varepsilon,n}(y, z_{\varepsilon,n}) = \int_0^c (c-s)^{\vartheta-1}\|B^*T_\vartheta^*(c-s)P^*y\|^2\,ds + \varepsilon\|y\| - \langle y, \delta_n(z_{\varepsilon,n})\rangle$$

and

$$\zeta_\varepsilon(y, z_\varepsilon) = \int_0^c (c-s)^{\vartheta-1}\|B^*T_\vartheta^*(c-s)P^*y\|^2\,ds + \varepsilon\|y\| - \langle y, \delta_n(z_\varepsilon)\rangle,$$

respectively. The boundedness of $\chi_{\varepsilon,n}(z_{\varepsilon,n})$ enables to suppose that $\chi_{\varepsilon,n}(z_{\varepsilon,n}) \rightharpoonup \hat{\chi}_\varepsilon \in V$. Therefore the optimality of $\chi_{\varepsilon,n}(z_{\varepsilon,n})$ and $\chi_\varepsilon(z_\varepsilon)$ give

$$\zeta_\varepsilon(\chi_\varepsilon(z_\varepsilon); z_\varepsilon) \leq \zeta_\varepsilon(\hat{\chi}_\varepsilon; z_\varepsilon) \leq \underline{\lim}_{n\to\infty}\zeta_\varepsilon(\chi_{\varepsilon,n}(z_{\varepsilon,n}); z_\varepsilon) \tag{5.2.8}$$

and

$$\begin{aligned}\underline{\lim}_{n\to\infty}\zeta_{\varepsilon,n}(\chi_{\varepsilon,n}(z_{\varepsilon,n}); z_{\varepsilon,n}) &\leq \overline{\lim}_{n\to\infty}\zeta_{\varepsilon,n}(\chi_{\varepsilon,n}(z_{\varepsilon,n}); z_{\varepsilon,n})\\ &\leq \lim_{n\to\infty}\zeta_{\varepsilon,n}(\chi_\varepsilon(z_\varepsilon); z_{\varepsilon,n})\\ &= \zeta_\varepsilon(\chi_\varepsilon(z_\varepsilon); z_\varepsilon).\end{aligned} \tag{5.2.9}$$

From (5.2.8) and (5.2.9), one has

$$\zeta_\varepsilon(\chi_\varepsilon(z_\varepsilon); z_\varepsilon) = \zeta_\varepsilon(\hat{\chi}_\varepsilon; z_\varepsilon)$$

and

$$\underline{\lim}_{n\to\infty}\zeta_{\varepsilon,n}(\chi_{\varepsilon,n}(z_{\varepsilon,n}); z_{\varepsilon,n}) = \overline{\lim}_{n\to\infty}\zeta_{\varepsilon,n}(\chi_{\varepsilon,n}(z_{\varepsilon,n}); z_{\varepsilon,n}) = \zeta_\varepsilon(\chi_\varepsilon(z_\varepsilon); z_\varepsilon),$$

which shows that $\hat{\chi}_\varepsilon$ is also a minimizer of $\zeta_\varepsilon(\cdot; z_\varepsilon)$. From the uniqueness of the minimizer $\hat{\chi}_\varepsilon = \chi_\varepsilon(z_\varepsilon)$. Therefore

$$\begin{aligned}&\lim_{n\to\infty}\zeta_{\varepsilon,n}(\chi_{\varepsilon,n}(z_{\varepsilon,n}); z_{\varepsilon,n}) = \zeta_\varepsilon(\chi_\varepsilon(z_\varepsilon); z_\varepsilon),\\ &\lim_{n\to\infty}\int_0^c (c-s)^{\vartheta-1}\|B^*T_\vartheta^*(c-s)P^*\chi_{\varepsilon,n}(z_{\varepsilon,n})\|^2\,ds\\ &\quad= \int_0^c (c-s)^{\vartheta-1}\|B^*T_\vartheta^*(c-s)P^*\chi_\varepsilon(z_\varepsilon)\|^2\,ds,\\ &\lim_{n\to\infty}\langle\chi_{\varepsilon,n}(z_{\varepsilon,n}), \delta_n(z_{\varepsilon,n})\rangle = \langle\chi_\varepsilon(z_\varepsilon), \delta(z_\varepsilon)\rangle\end{aligned}$$

and

$$\|\chi_\varepsilon(z_\varepsilon)\| \leq \lim_{n\to\infty}\|\chi_{\varepsilon,n}(z_{\varepsilon,n})\|.$$

From above relations, one has

$$\|\chi_\varepsilon(z_\varepsilon)\| = \lim_{n\to\infty}\|\chi_{\varepsilon,n}(z_{\varepsilon,n})\|. \tag{5.2.10}$$

Since V is a Hilbert space and $\chi_{\varepsilon,n}(z_{\varepsilon,n}) \rightharpoonup \chi_\varepsilon(z_\varepsilon)$, therefore we obtain the strong convergence. ■

In the next theorem, we show that the map $\Psi_\varepsilon : C_{1-\vartheta}([0,c];V) \to C_{1-\vartheta}([0,c];V)$ defined by

$$(\Psi_\varepsilon z)(t) = t^{\vartheta-1}T_\vartheta(t)\left(y_0 - \wp(z)\right) + \int_0^t (t-s)^{\vartheta-1}T_\vartheta(t-s)\big(Bu_\varepsilon(s,z) + F(s,z(s))\big)ds$$

with

$$u_\varepsilon(s,z) = B^*T_\vartheta^*(c-s)P^*\chi_\varepsilon(z) = B^*T_\vartheta^*(c-s)P^*\tilde{y}_\varepsilon,$$

has a fixed point.

Theorem 5.2.7. *The operator Ψ_ε has a fixed point in $C_{1-\vartheta}([0,c];V)$.*

Proof. Define the following sets

$$E = \left\{z_{\varepsilon,n} \in C_{1-\vartheta}([0,c];V) \,\middle|\, \Psi_{\varepsilon,n}z_{\varepsilon,n} = z_{\varepsilon,n},\ n \geq 1\right\},$$
$$E_{1-\vartheta} = \left\{w_{\varepsilon,n} \,\middle|\, w_{\varepsilon,n}(t) = t^{1-\vartheta}z_{\varepsilon,n}(t),\ z_{\varepsilon,n} \in E\right\}$$

and

$$E_{1-\vartheta}(0) = \left\{w_{\varepsilon,n}(0) = T_\vartheta(0)\left(y_0 - T\left(\frac{1}{n}\right)\wp(z_{\varepsilon,n})\right)\right\}.$$

Claim 1: $E_{1-\vartheta}(0)$ is relatively compact in V.

For $w_{\varepsilon,n} \in E_{1-\vartheta},\ n \geq 1$, define

$$\hat{w}_{\varepsilon,n}(t) = t^{1-\vartheta}\hat{z}_{\varepsilon,n}(t) = \begin{cases} w_{\varepsilon,n}(t), & t \in [b,c], \\ w_{\varepsilon,n}(b), & t \in [0,b). \end{cases}$$

Clearly, $\left\{\hat{w}_{\varepsilon,n} \,\middle|\, n \geq 1\right\}$ is uniformly bounded and equicontinuous on $[0,c]$ therefore by Arzela theorem it has a subsequence which converges in $C([0,c];V)$. Without loss of generality, one can suppose that $\hat{w}_{\varepsilon,n} \to \hat{w}_\varepsilon \in C([0,c];V)$. Then $\hat{z}_{\varepsilon,n} \to \hat{z}_\varepsilon \in C_{1-\vartheta}([0,c];V)$, where $\hat{z}_\varepsilon(t) = t^{\vartheta-1}\hat{w}_\varepsilon(t)$. By hypothesis (H_4), one has $\wp(z_{\varepsilon,n}) = \wp(\hat{z}_{\varepsilon,n}) \to \wp(\hat{z}_\varepsilon)$.

Now,

$$\begin{aligned}
&\|w_{\varepsilon,n}(0) - T_\vartheta(0)(y_0 - \wp(\hat{z}_\varepsilon))\| \\
&\quad = \left\|T_\vartheta(0)\left(y_0 - T\left(\frac{1}{n}\right)\wp(z_{\varepsilon,n})\right) - T_\vartheta(0)(y_0 - \wp(\hat{z}_\varepsilon))\right\| \\
&\quad \leq \widehat{k}_T \left\|T\left(\frac{1}{n}\right)\wp(z_{\varepsilon,n}) - \wp(\hat{z}_\varepsilon)\right\|
\end{aligned}$$

$$\leq \widehat{k}_T \left(\left\| T\left(\frac{1}{n}\right)\wp(z_{\varepsilon,n}) - T\left(\frac{1}{n}\right)\wp(\hat{z}_\varepsilon) \right\| + \left\| T\left(\frac{1}{n}\right)\wp(\hat{z}_\varepsilon) - \wp(\hat{z}_\varepsilon) \right\| \right)$$
$$\leq \widehat{k}_T \left(k_T \left\|\wp(z_{\varepsilon,n}) - \wp(\hat{z}_\varepsilon)\right\| + \left\| T\left(\frac{1}{n}\right)\wp(\hat{z}_\varepsilon) - \wp(\hat{z}_\varepsilon) \right\| \right).$$

From above inequality, one has $\|w_{\varepsilon,n}(0) - T_\vartheta(0)(y_0 - \wp(\hat{z}_\varepsilon))\| \to 0$ as $n \to \infty$. Hence $E_{1-\vartheta}(0)$ is precompact in V.

Claim 2: For fixed $t \in (0, c]$, the set $E_{1-\vartheta}(t) = \{w_{\varepsilon,n}(t) = t^{1-\vartheta} z_{\varepsilon,n}(t),\ z_{\varepsilon,n} \in E\}$ is precompact in V.

Claim 3: At $t = 0$, $E_{1-\vartheta}$ is equicontinuous.

For $t \in (0, c)$, one has

$$\begin{aligned}
&\|w_{\varepsilon,n}(t) - w_{\varepsilon,n}(0)\| \\
&\quad \leq \left\| T_\vartheta(t)\left(y_0 - T\left(\frac{1}{n}\right)\wp(z_{\varepsilon,n})\right) - T_\vartheta(0)\left(y_0 - T\left(\frac{1}{n}\right)\wp(z_{\varepsilon,n})\right) \right\| \\
&\qquad + \widehat{k}_T t^{1-\vartheta} \sup \|Bu_{\varepsilon,n}(s, z_{\varepsilon,n}) + F(s, z_{\varepsilon,n}(s))\| \int_0^t (t-s)^{\vartheta-1}\, ds \\
&\quad \leq \|\left(T_\vartheta(t) - T_\vartheta(0)\right) y_0\| + \left\| \left(T_\vartheta(t) - T_\vartheta(0)\right) T\left(\frac{1}{n}\right)\wp(z_{\varepsilon,n}) \right\| \\
&\qquad + \frac{\widehat{k}_T t}{\vartheta}\left(\widehat{k}_T k_B^2 \|P\| \kappa_\varepsilon + \|g\|_C\right).
\end{aligned}$$

From above inequality, one has $\|w_{\varepsilon,n}(t) - w_{\varepsilon,n}(0)\| \to 0$ as $t \to 0$. Hence $E_{1-\vartheta}$ is equicontinuous at $t = 0$.

Claim 4: $E_{1-\vartheta}$ is equicontinuous on $(0, c]$.

Take $\tau_1, \tau_2 \in (0, c]$ with $\tau_2 > \tau_1$ and $h \in (0, \tau_1)$. Then for $w_{\varepsilon,n} \in E_{1-\vartheta}$, one has

$$\begin{aligned}
&\|w_{\varepsilon,n}(\tau_2) - w_{\varepsilon,n}(\tau_1)\| \\
&\quad \leq \left\| \left(T_\vartheta(\tau_2) - T_\vartheta(\tau_1)\right)\left(y_0 - T\left(\frac{1}{n}\right)\wp(z_{\varepsilon,n})\right) \right\| \\
&\qquad + \left\| \int_{\tau_1}^{\tau_2} \tau_2^{1-\vartheta}(\tau_2 - s)^{\vartheta-1} T_\vartheta(\tau_2 - s)\big(Bu_{\varepsilon,n}(s, z_{\varepsilon,n}) + F(s, z_{\varepsilon,n}(s))\big)\, ds \right\| \\
&\qquad + \left\| \int_0^{\tau_1} \tau_2^{1-\vartheta}(\tau_2 - s)^{\vartheta-1} T_\vartheta(\tau_2 - s)\big(Bu_{\varepsilon,n}(s, z_{\varepsilon,n}) + F(s, z_{\varepsilon,n}(s))\big)\, ds \right. \\
&\qquad \left. - \int_0^{\tau_1} \tau_1^{1-\vartheta}(\tau_1 - s)^{\vartheta-1} T_\vartheta(\tau_1 - s)\big(Bu_{\varepsilon,n}(s, z_{\varepsilon,n}) + F(s, z_{\varepsilon,n}(s))\big)\, ds \right\|
\end{aligned}$$

which gives

$$\|w_{\varepsilon,n}(\tau_2) - w_{\varepsilon,n}(\tau_1)\|$$

$$
\begin{aligned}
&\leq \left\| (T_\vartheta(\tau_2) - T_\vartheta(\tau_1)) \left(y_0 - T\left(\frac{1}{n}\right) \wp(z_{\varepsilon,n}) \right) \right\| \\
&\quad + \left\| \int_{\tau_1}^{\tau_2} \tau_2^{1-\vartheta} (\tau_2 - s)^{\vartheta-1} T_\vartheta(\tau_2 - s) \big(B u_{\varepsilon,n}(s, z_{\varepsilon,n}) + F(s, z_{\varepsilon,n}(s)) \big) \, ds \right\| \\
&\quad + \left\| \int_0^{\tau_1} \left(\tau_2^{1-\vartheta} (\tau_2 - s)^{\vartheta-1} - \tau_1^{1-\vartheta} (\tau_1 - s)^{\vartheta-1} \right) T_\vartheta(\tau_2 - s) \big(B u_{\varepsilon,n}(s, z_{\varepsilon,n}) \right. \\
&\quad \left. + F(s, z_{\varepsilon,n}(s)) \big) \, ds \right\| + \left\| \int_0^{\tau_1} \tau_1^{1-\vartheta} (\tau_1 - s)^{\vartheta-1} (T_\vartheta(\tau_2 - s) - T_\vartheta(\tau_1 - s)) \cdot \right. \\
&\qquad \left. \big(B u_{\varepsilon,n}(s, z_{\varepsilon,n}) + F(s, z_{\varepsilon,n}(s)) \big) \, ds \right\|.
\end{aligned}
$$

By assumption (H_3) and continuity of $T_\vartheta(t)$, one has

$$
\left\| (T_\vartheta(\tau_2) - T_\vartheta(\tau_1)) \left(y_0 - T\left(\frac{1}{n}\right) \wp(z_{\varepsilon,n}) \right) \right\| \to 0 \tag{5.2.11}
$$

as $\tau_2 - \tau_1 \to 0$ for all $w_{\varepsilon,n} \in E_{1-\vartheta}$.

Now,

$$
\begin{aligned}
&\left\| \int_{\tau_1}^{\tau_2} \tau_2^{1-\vartheta} (\tau_2 - s)^{\vartheta-1} T_\vartheta(\tau_2 - s) \big(B u_{\varepsilon,n}(s, z_{\varepsilon,n}) + F(s, z_{\varepsilon,n}(s)) \big) \, ds \right\| \\
&\quad \leq \frac{\widehat{k}_T}{\vartheta} \tau_2^{1-\vartheta} (\tau_2 - \tau_1)^\vartheta \left(\widehat{k}_T k_B^2 \|P\| \kappa_\varepsilon + \|g\|_C \right)
\end{aligned} \tag{5.2.12}
$$

and

$$
\begin{aligned}
&\left\| \int_0^{\tau_1} \left(\tau_2^{1-\vartheta} (\tau_2 - s)^{\vartheta-1} - \tau_1^{1-\vartheta} (\tau_1 - s)^{\vartheta-1} \right) T_\vartheta(\tau_2 - s) \big(B u_{\varepsilon,n}(s, z_{\varepsilon,n}) \right. \\
&\quad \left. + F(s, z_{\varepsilon,n}(s)) \big) \, ds \right\| \\
&\qquad \leq \frac{\widehat{k}_T}{\vartheta} \left| (\tau_2 - \tau_1) - \tau_2^{1-\vartheta} (\tau_2 - \tau_1)^\vartheta \right| \left(\widehat{k}_T k_B^2 \|P\| \kappa_\varepsilon + \|g\|_C \right).
\end{aligned} \tag{5.2.13}
$$

For $\tau_1 - h > 0$, one has

$$
\begin{aligned}
&\left\| \int_0^{\tau_1} \tau_1^{1-\vartheta} (\tau_1 - s)^{\vartheta-1} (T_\vartheta(\tau_2 - s) - T_\vartheta(\tau_1 - s)) \big(B u_{\varepsilon,n}(s, z_{\varepsilon,n}) + F(s, z_{\varepsilon,n}(s)) \big) \, ds \right\| \\
&\quad = \left\| \left(\int_0^{\tau_1 - h} + \int_{\tau_1 - h}^{\tau_1} \right) \tau_1^{1-\vartheta} (\tau_1 - s)^{\vartheta-1} (T_\vartheta(\tau_2 - s) - T_\vartheta(\tau_1 - s)) \big(B u_{\varepsilon,n}(s, z_{\varepsilon,n}) \right. \\
&\qquad \left. + F(s, z_{\varepsilon,n}(s)) \big) \, ds \right\| \\
&\quad \leq \frac{\tau_1^\vartheta - h^\vartheta}{\vartheta} \tau_1^{1-\vartheta} \left(\widehat{k}_T k_B^2 \|P\| \kappa_\varepsilon + \|g\|_C \right) \sup_{s \in [0, \tau_1 - h]} \|T_\vartheta(\tau_2 - s) - T_\vartheta(\tau_1 - s)\|
\end{aligned}
$$

$$+\frac{2\widehat{k}_T}{\vartheta}\tau_1^{1-\vartheta}h^{\vartheta}\left(\widehat{k}_T k_B^2\|P\|\kappa_\varepsilon+\|g\|_C\right). \tag{5.2.14}$$

Since $T_\vartheta(t)$ $(t > 0)$ is continuous in operator norm therefore from inequalities (5.2.11)-(5.2.14) with $h \to 0^+$, one can conclude that $E_{1-\vartheta}$ is equicontinuous in $(0, c]$. Hence $E_{1-\vartheta}$ is relatively compact in $C([0, c]; V)$ and one can suppose that $w_{\varepsilon,n} \to w_\varepsilon \in C([0, c]; V)$ as $n \to \infty$. It means that $z_{\varepsilon,n} \to z_\varepsilon \in C_{1-\vartheta}([0, c]; V)$ as $n \to \infty$, where $z_\varepsilon(t) = t^{\vartheta-1}w_\varepsilon(t)$. Letting $n \to \infty$ in $\Psi_{\varepsilon,n}z_{\varepsilon,n} = z_{\varepsilon,n}$ and applying Lebesgue dominated convergence theorem, one can get

$$w_\varepsilon(t) = T_\vartheta(y_0 - \wp(z_\varepsilon)) + t^{1-\vartheta}\int_0^t (t-s)^{\vartheta-1}T_\vartheta(t-s)\big(Bu_\varepsilon(s, z_\varepsilon) + F(s, z_\varepsilon(s))\big)\, ds$$

for $0 \le t \le c$.

Thus

$$z_\varepsilon(t) = t^{\vartheta-1}T_\vartheta(y_0 - \wp(z_\varepsilon)) + \int_0^t (t-s)^{\vartheta-1}T_\vartheta(t-s)\big(Bu_\varepsilon(s, z_\varepsilon) + F(s, z_\varepsilon(s))\big)\, ds$$

which is a mild solution of the original system (5.1.1). ∎

5.3 Controllability result

By previous theorem for given $\varepsilon > 0$ there is a $z_\varepsilon \in C_{1-\vartheta}([0, c]; V)$ satisfying

$$z_\varepsilon(t) = t^{\vartheta-1}T_\vartheta(y_0 - \wp(z_\varepsilon)) + \int_0^t (t-s)^{\vartheta-1}T_\vartheta(t-s)\big(Bu_\varepsilon(s, z_\varepsilon) + F(s, z_\varepsilon(s))\big)\, ds,$$

with $u_\varepsilon(s, z_\varepsilon) = B^*T_\vartheta^*(c-s)\chi_\varepsilon(z_\varepsilon)$.

Next, we show the partial controllability of the original system.

Theorem 5.3.1. *The semilinear system (5.1.1) is partial approximately controllable.*

Proof. Since ζ_ε is strictly convex therefore $\zeta_\varepsilon(y, z_\varepsilon)$ has a unique minimizer $\tilde{y}_\varepsilon \in V$ satisfying

$$\zeta_\varepsilon(\tilde{y}_\varepsilon; z_\varepsilon) = \min_{y\in V}\zeta_\varepsilon(y; z_\varepsilon).$$

Now, for any $y \in V$ and $r \in \mathbb{R}$ one can write

$$\zeta_\varepsilon(\tilde{y}_\varepsilon; z_\varepsilon) \le \zeta_\varepsilon(\tilde{y}_\varepsilon + ry; z_\varepsilon),$$

which means

$$\begin{aligned}\varepsilon\|\tilde{y}_\varepsilon\| \leq \frac{r^2}{2}\int_0^c (c-s)^{\vartheta-1}\|B^*T_\vartheta^*(c-s)P^*y\|^2\,ds \\ + r\int_0^c (c-s)^{\vartheta-1}\langle B^*T_\vartheta^*(c-s)P^*\tilde{y}_\varepsilon, B^*T_\vartheta^*(c-s)P^*y\rangle ds \\ + \varepsilon\|\tilde{y}_\varepsilon + ry\| - r\langle y, \delta(z_\varepsilon)\rangle.\end{aligned}$$

Dividing by $r > 0$ and taking $r \to 0^+$, one can get

$$\begin{aligned}\langle y, \delta(z_\varepsilon)\rangle &\leq \int_0^c (c-s)^{\vartheta-1}\langle B^*T_\vartheta^*(c-s)P^*\tilde{y}_\varepsilon, B^*T_\vartheta^*(c-s)P^*y\rangle ds \\ &\quad + \varepsilon\underline{\lim}_{r\to 0^+}\frac{\|\tilde{y}_\varepsilon + ry\| - \|\tilde{y}_\varepsilon\|}{r} \\ &\leq \int_0^c (c-s)^{\vartheta-1}\langle B^*T_\vartheta^*(c-s)P^*\tilde{y}_\varepsilon, B^*T_\vartheta^*(c-s)P^*y\rangle ds + \varepsilon\|y\|.\end{aligned}$$

In similar fashion with $r < 0$, one can get

$$\langle y, \delta(z_\varepsilon)\rangle \geq \int_0^c (c-s)^{\vartheta-1}\langle B^*T_\vartheta^*(c-s)P^*\tilde{y}_\varepsilon, B^*T_\vartheta^*(c-s)P^*y\rangle ds - \varepsilon\|y\|.$$

Thus

$$\left|\int_0^c (c-s)^{\vartheta-1}\langle B^*T_\vartheta^*(c-s)P^*\tilde{y}_\varepsilon, B^*T_\vartheta^*(c-s)P^*y\rangle ds - \langle y, \delta(z_\varepsilon)\rangle\right| \leq \varepsilon\|y\|. \tag{5.3.1}$$

But for $u_\varepsilon = B^*T_\vartheta^*(c-s)P^*\tilde{y}_\varepsilon$, one can get

$$\begin{aligned}&\int_0^c (c-s)^{\vartheta-1}\langle B^*T_\vartheta^*(c-s)P^*\tilde{y}_\varepsilon, B^*T_\vartheta^*(c-s)P^*y\rangle ds - \langle y, \delta(z_\varepsilon)\rangle \\ &= \left\langle \int_0^c (c-s)^{\vartheta-1}PT_\vartheta(c-s)BB^*T_\vartheta^*(c-s)P^*\tilde{y}_\varepsilon ds - \delta(z_\varepsilon), y\right\rangle \\ &= \langle Pz_\varepsilon(c) - z_c, y\rangle,\end{aligned} \tag{5.3.2}$$

where

$$\begin{aligned}\delta(z_\varepsilon) &= z_c - c^{\vartheta-1}T_\vartheta(c)(y_0 - \wp(z_\varepsilon)) \\ &\quad - \int_0^c (c-s)^{\vartheta-1}PT_\vartheta(c-s)F(s, z_\varepsilon(s))\,ds.\end{aligned} \tag{5.3.3}$$

From (5.3.1) and (5.3.3), one has

$$|\langle Pz_\varepsilon(c) - z_c, y\rangle| \leq \varepsilon\|y\| \quad \text{for any } y \in V.$$

Hence

$$\|Pz_\varepsilon(c) - z_c\| \leq \varepsilon.$$

This proves the theorem. ■

5.4 Example

Consider the following initial-boundary value problem with Riemann-Liouville derivative for $x \in [0,\pi]$:

$$\begin{cases} D_t^{2/3}\widehat{z}(t,x) = \frac{\partial^2}{\partial x^2}\widehat{z}(t,x) + B\widehat{u}(t,x) + F(t,\widehat{z}(t,x)), & t \in (0,c], \\ \widehat{z}(t,0) = \widehat{z}(t,\pi) = 0, & t \in (0,c], \\ \left(I_t^{1/3}\widehat{z}(t,x)\right)_{t=0} = \widehat{y}_0(x) - \sum_{j=0}^{m} \int_0^{\pi} \omega(x,s)\widehat{z}(\tau_j,s)\,ds, & \tau_j \in (0,c), \end{cases} \tag{5.4.1}$$

where $m \in \mathbb{N},\ 0 < \tau_0 < \tau_1 < \cdots < \tau_m < c$ and $\omega(\cdot,\cdot) \in L_2([0,\pi]\times[0,\pi];\mathbb{R}_0^+)$.

Take $V = L_2[0,\pi]$ and the operator $A : D(A) \subset V \to V$ is defined as

$$Ay = y''$$

with the domain

$$D(A) = \{y \in V \,|\, y \text{ and } y' \text{ are absolutely continuous},\ y'' \in V,\ y(0) = y(\pi) = 0\}.$$

Then A can be expressed as

$$Ay = \sum_{\ell=1}^{\infty}(-\ell^2)\langle y,\xi_\ell\rangle\xi_\ell, \quad y \in D(A)$$

and it generates a compact semigroup $T(t)$ given by

$$T(t)y = \sum_{\ell=1}^{\infty} e^{-\ell^2 t}\langle y,\xi_\ell\rangle\xi_\ell, \quad y \in V \quad \text{with} \quad \|T(t)\| \le e^{-1},\ \ k_T = 1;$$

where $\xi_\ell(x) = \sqrt{\frac{2}{\pi}}\sin \ell x$ are eigen functions of A corresponding to the eigenvalues $\lambda_\ell = -\ell^2$, $\ell = 1,2,\ldots$ and the set $\{\xi_\ell \,|\, \ell = 1,2,\ldots\}$ form an orthonormal basis for V. Now, define the space

$$V' = \left\{ v = \sum_{\ell=2}^{\infty} a_\ell \xi_\ell(x) \,\middle|\, \sum_{\ell=2}^{\infty} a_\ell^2 < \infty \right\}$$

with

$$\|v\| = \left(\sum_{\ell=2}^{\infty} a_\ell^2\right)^{\frac{1}{2}},$$

and the operator $B : V' \to V$ as

$$Bv = 2a_2\xi_1(x) + \sum_{\ell=2}^{\infty} a_\ell \xi_\ell(x).$$

Then the system (5.4.1) takes the form

$$\begin{cases} D_t^{2/3} z(t) = Az(t) + Bu(t) + F(t, z(t)), \quad t \in (0, c], \\ \left(I_t^{1/3} z(t)\right)_{t=0} = y_0 - \wp(z), \end{cases} \tag{5.4.2}$$

where $z(t) = \widehat{z}(t, \cdot)$, $u(t) = \widehat{u}(t, \cdot)$, $y_0 = \widehat{y}_0(\cdot)$ and $\wp : C_{1/3}([0, c]; V) \to V$ is defined as

$$\wp(z) = \sum_{j=0}^{m} \int_0^{\pi} \omega(\cdot, s) z(\tau_j, s)\, ds.$$

Clearly, (H_5) is satisfied [18]. If (H_2)-(H_4) are satisfied, then the partial approximate controllability of the system (5.4.2) follows by previous theorem.

5.5 Concluding remarks

In this chapter, the partial approximate controllability of nonlocal Riemann-Liouville fractional systems with integral initial conditions in Hilbert spaces has been studied. Here, we used the continuity of nonlinear function. For the nonlocal function $\wp$, the conditions of Lipschitz continuity and compactness have also been dropped. For the function $\wp$, the assumption (H_4) means that $\wp$ does not depend on the value of z for $t \in [0, b)$. Applying these ideas and techniques, one can analyze the partial controllability of fractional systems with impulses or control delay.

Chapter 6

Approximate Controllability of Riemann-Liouville Fractional Semilinear Systems of Higher-Order

The objective of this chapter is to analyze the approximate controllability of Riemann-Liouville fractional evolution equations of order $\vartheta \in (1, 2)$. First we deduce the existence of solutions using fractional Riemann-Liouville family and fixed point approach. We make use of iterative and approximate technique to prove the controllability of the system. Finally, an illustrative example has been provided.

6.1 Introduction and preliminaries

Let V be a Banach space and $Z_{\vartheta-1} = \{z \,|\, (c-t)^{\vartheta-1}z(t) \in L_p([0, c]; V)\}$ be a function space with the norm $\|z\|_{Z_{\vartheta-1}} = \left(\int_0^c \|(c-t)^{\vartheta-1}z(t)\|_V^p \, dt\right)^{\frac{1}{p}}$. Consider the fractional order system

$$\begin{cases} D_t^{\vartheta} z(t) = Az(t) + Bu(t) + F(t, z(t)), \quad t \in (0, c], \\ \left(I_t^{2-\vartheta} z(t)\right)_{t=0} = y_0 \in D(A), \\ \left(D_t^{\vartheta-1} z(t)\right)_{t=0} = y_1 \in V, \end{cases} \tag{6.1.1}$$

where $p > \frac{1}{2-\vartheta}$, $1 < \vartheta < 2$ and D_t^ϑ stands for Riemann-Liouville derivative of order ϑ. The state $z \in Z_{\vartheta-1}$, the control $u \in U$, where $U = L_p([0,c];V')$ with the norm $\|u\|_U = \left(\int_0^c \|u(t)\|_{V'}^p \, dt\right)^{\frac{1}{p}}$ and V' is another Banach space. $A : D(A) \subseteq V \to V$ is densely defined and it generates a fractional cosine family $\mathcal{C}_\vartheta(t)$. B is the continuous linear map from U to $Z_{\vartheta-1}$. $F : [0,c] \times V \to V$ is nonlinear.

Liu and Li [61] developed approximate controllability results for Riemann-Liouville fractional equations of the form $D_t^\vartheta z(t) = Az(t) + F(t, z(t)) + (Bu)(t), \quad 0 < \vartheta < 1$, with integral initial condition $\left(I_t^{1-\vartheta} z(t)\right)_{t=0} = y_0$ in Banach spaces. Here, the theory of Laplace transform together with probability density function were used to analyze the existence of solution and controllability for the system. Ibrahim et al. [33] determined the existence and controllability results for the same system with the initial condition $\lim_{t\to 0^+} \Gamma(\vartheta) t^{1-\vartheta} z(z) = y_0$ using the concept of ϑ-order resolvent rather than C_0-semigroup. Mahmudov and McKibben [64] determined the approximate controllability of fractional systems with generalized Riemann-Liouville derivatives. In this chapter, we extend the existence and controllability results for fractional systems of order $\vartheta \in (1,2)$ with Riemann-Liouville derivatives.

To define the mild solution of (6.1.1), we derive the next lemma.

Lemma 6.1.1. *Let $\vartheta \in (1,2)$ and $g \in L_p([0,c];V)$. If $z(t) \in L_1([0,c];V)$, $z_{2-\vartheta}(t) \in AC^2([0,c];V)$ and z is a solution of the system*

$$\begin{cases} D_t^\vartheta z(t) = Az(t) + g(t), & t \in (0,c], \\ \left(I_t^{2-\vartheta} z(t)\right)_{t=0} = y_0 \in D(A), \\ \left(D_t^{\vartheta-1} z(t)\right)_{t=0} = y_1 \in V. \end{cases} \tag{6.1.2}$$

Then

$$z(t) = \dot{\mathcal{R}}_\vartheta(t) y_0 + \mathcal{R}_\vartheta(t) y_1 + \int_0^t \mathcal{R}_\vartheta(t-s) g(s)\, ds, \quad 0 < t \leq c.$$

Proof. From Lemma 2.2.1 of Chapter 2, one can obtain

$$\begin{aligned} z(t) &= \frac{\left(D_t^{\vartheta-1} z(t)\right)_{t=0}}{\Gamma(\vartheta)} t^{\vartheta-1} + \frac{\left(I_t^{2-\vartheta} z(t)\right)_{t=0}}{\Gamma(\vartheta-1)} t^{\vartheta-2} + I_t^\vartheta A z(t) + I_t^\vartheta g(t) \\ &= \frac{t^{\vartheta-1}}{\Gamma(\vartheta)} y_1 + \frac{t^{\vartheta-2}}{\Gamma(\vartheta-1)} y_0 + I_t^\vartheta A z(t) + I_t^\vartheta g(t). \end{aligned}$$

Taking Laplace-transforms, one has

$$\check{z}(\rho) = \frac{1}{\rho^\vartheta} y_1 + \frac{1}{\rho^{\vartheta-1}} y_0 + \frac{1}{\rho^\vartheta}\left(A\check{z}(\rho) + \check{g}(\rho)\right)$$

$$= (\rho^\vartheta I - A)^{-1}y_1 + \rho(\rho^\vartheta I - A)^{-1}y_0 + (\rho^\vartheta I - A)^{-1}\breve{g}(\rho).$$

If δ is the unit impulse function, then by inverting the Laplace transform on both sides and using (2.2.13), one can obtain

$$\begin{aligned} z(t) &= \mathcal{R}_\vartheta(t)y_1 + \dot{\mathcal{R}}_\vartheta(t)y_0 + \mathcal{R}_\vartheta(0)y_0\delta(t) + \int_0^t \mathcal{R}_\vartheta(t-s)g(s)\,ds \\ &= \dot{\mathcal{R}}_\vartheta(t)y_0 + \mathcal{R}_\vartheta(t)y_1 + \int_0^t \mathcal{R}_\vartheta(t-s)g(s)\,ds. \end{aligned}$$

This completes the proof. ■

Since $\mathcal{R}_\vartheta(t)y$ is continuously differentiable on $(0,\infty)$ for all $y \in D(A)$ and

$$\lim_{t\to 0^+} \frac{\mathcal{R}_\vartheta(t)}{t^{\vartheta-1}}y = \frac{y}{\Gamma(\vartheta)} \quad \text{for} \quad y \in V \text{ (see [2])}.$$

Therefore

$$\lim_{t\to 0^+} \frac{\dot{\mathcal{R}}_\vartheta(t)}{t^{\vartheta-2}}y = \frac{y}{\Gamma(\vartheta-1)} \quad \text{for} \quad y \in D(A),$$

which shows that $\dot{\mathcal{R}}_\vartheta(t)y$ has singularity at $t=0$ and there is a constant $k'_{\mathcal{R}} > 0$ such that $t^{2-\vartheta}\|\dot{\mathcal{R}}_\vartheta(t)y\| \le k'_{\mathcal{R}}\|y\|$ for $y \in D(A)$. For this reason, to define the mild solution of (6.1.1), we consider the Banach space $C_{2-\vartheta}([0,c];V) = \big\{z \,\big|\, t^{2-\vartheta}z(t) \in C([0,c];V)\big\}$ with the norm $\|z\|_{C_{2-\vartheta}} = \sup_{t\in[0,c]}\{t^{2-\vartheta}\|z(t)\|_V\}$.

Definition 6.1.1. A function $z \in C_{2-\vartheta}([0,c];V)$ is said to be a mild solution of (6.1.1) if it satisfies

$$z(t) = \dot{\mathcal{R}}_\vartheta(t)y_0 + \mathcal{R}_\vartheta(t)y_1 + \int_0^t \mathcal{R}_\vartheta(t-s)\big(Bu(s) + F(s,z(s))\big)\,ds.$$

Definition 6.1.2. Let $z(t,u)$ be a mild solution of (6.1.1) associated with a control $u \in U$. The set given by

$$\mathfrak{R}_c(F) = \big\{z(c,u) \in V \,\big|\, u \in U\big\},$$

is known as the reachable set of (6.1.1). Moreover, if $\mathfrak{R}_c(F)$ is dense in V, then we say that the system (6.1.1) is approximately controllable on $[0,c]$.

6.2 Existence of mild solution

To derive the existence result we suppose the following conditions:

(H_1) $\mathcal{R}_\vartheta(t)$ is the fractional Riemann-Liouville family associated with the fractional cosine family $\mathcal{C}_\vartheta(t)$ and there exists a constant $k_{\mathcal{R}} > 0$ satisfying

$$\|\mathcal{R}_\vartheta(t)\| \leq k_{\mathcal{R}}, \quad 0 \leq t \leq c;$$

(H_2) there exists a constant $k_F > 0$ satisfying

$$\|F(t,y) - F(t,\tilde{y})\| \leq k_F \|y - \tilde{y}\| \quad \forall\, y, \tilde{y} \in V;$$

(H_3) there is a function $h \in L_p([0,c];\mathbb{R}_0^+)$ and a constant $k'_F > 0$ such that

$$\|F(t,y)\| \leq h(t) + k'_F t^{2-\vartheta}\|y\|$$

for a.e. $t \in [0,c]$ and all $y \in V$.

Theorem 6.2.1. *Under hypotheses (H_1)-(H_3), the semilinear system (6.1.1) has a unique mild solution in $C_{2-\vartheta}([0,c];V)$ for each control $u \in U$.*

Proof. Theorem will be proved if we show that the map $\mathcal{Q} : C_{2-\vartheta}([0,c];V) \to C_{2-\vartheta}([0,c];V)$ defined by

$$(\mathcal{Q}z)(t) = \dot{\mathcal{R}}_\vartheta(t)y_0 + \mathcal{R}_\vartheta(t)y_1 + \int_0^t \mathcal{R}_\vartheta(t-s)\big(Bu(s) + F(s,z(s))\big)\,ds,$$

has a unique fixed point in $C_{2-\vartheta}([0,c];V)$.

By above assumptions it is easily seen that the map $\mathcal{Q}$ is well defined.

For any $z, \tilde{z} \in C_{2-\vartheta}([0,c];V)$, one can obtain

$$\begin{aligned} t^{2-\vartheta}\|(\mathcal{Q}z)(t) - (\mathcal{Q}\tilde{z})(t)\| &\leq t^{2-\vartheta}\int_0^t \big\|\mathcal{R}_\vartheta(t-s)\big(F(s,z(s)) - F(s,\tilde{z}(s))\big)\big\|\,ds \\ &\leq k_F k_{\mathcal{R}} t^{2-\vartheta}\int_0^t s^{\vartheta-2}s^{2-\vartheta}\|z(s) - \tilde{z}(s)\|\,ds \\ &\leq \frac{k_F k_{\mathcal{R}} t}{(\vartheta-1)}\|z - \tilde{z}\|_{C_{2-\vartheta}}. \end{aligned}$$

Further,

$$t^{2-\vartheta}\|(\mathcal{Q}^2 z)(t) - (\mathcal{Q}^2\tilde{z})(t)\| \leq k_F k_{\mathcal{R}} t^{2-\vartheta}\int_0^t s^{\vartheta-2}s^{2-\vartheta}\|(\mathcal{Q}z)(s) - (\mathcal{Q}\tilde{z})(s)\|\,ds$$

$$\leq k_F k_{\mathcal{R}} t^{2-\vartheta} \int_0^t s^{\vartheta-2} \frac{k_F k_{\mathcal{R}} s}{(\vartheta-1)} \|z - \tilde{z}\|_{C_{2-\vartheta}} \, ds$$
$$\leq \frac{(k_F k_{\mathcal{R}} t)^2}{(\vartheta-1)\vartheta} \|z - \tilde{z}\|_{C_{2-\vartheta}}.$$

By inductions, one can obtain

$$t^{2-\vartheta} \|(\mathcal{Q}^n z)(t) - (\mathcal{Q}^n \tilde{z})(t)\| \leq \frac{(k_F k_{\mathcal{R}} t)^n}{(\vartheta-1)(\vartheta)\cdots(\vartheta+n-2)} \|z - \tilde{z}\|_{C_{2-\vartheta}}$$
$$\leq \frac{(k_F k_{\mathcal{R}} c)^n}{(\vartheta-1)((n-1)!)} \|z - \tilde{z}\|_{C_{2-\vartheta}},$$

which gives

$$\|\mathcal{Q}^n z - \mathcal{Q}^n \tilde{z}\|_{C_{2-\vartheta}} \leq \frac{(k_F k_{\mathcal{R}} c)^n}{(\vartheta-1)((n-1)!)} \|z - \tilde{z}\|_{C_{2-\vartheta}}.$$

But the exponential series $\exp(k_F k_{\mathcal{R}} c) = \sum_{\ell=1}^{\infty} \frac{(k_F k_{\mathcal{R}} c)^{\ell-1}}{(\ell-1)!}$ is convergent. Therefore $\frac{(k_F k_{\mathcal{R}} c)^n}{(n)!} < \frac{(\vartheta-1)}{k_F k_{\mathcal{R}} c}$ for some positive integer n. Hence by generalized Banach contraction theorem, $\mathcal{Q}$ has a unique fixed point in $C_{2-\vartheta}([0,c];V)$. ∎

6.3 Controllability results

In the next lemma, we prove some properties of the space $Z_{\vartheta-1}$.

Lemma 6.3.1. *The space $Z_{\vartheta-1}$ has the following properties:*

(i) $Z_{\vartheta-1}$ is a Banach space;

(ii) $C([0,c];V)$ is dense in $Z_{\vartheta-1}$.

Proof. (i) Let $\{z_n\}$ be a Cauchy sequence in $Z_{\vartheta-1}$. Then it follows that $\{y_n\}$ is a Cauchy sequence in $L_p([0,c];V)$, where $y_n(t) = (c-t)^{\vartheta-1} z(t)$. But $L_p([0,c];V)$ is complete, therefore $y_n \to y \in L_p([0,c];V)$. If we take $z(t) = \frac{y(t)}{(c-t)^{\vartheta-1}}$, then $z \in Z_{\vartheta-1}$ and $z_n \to z$.

(ii) First we show that $L_p([0,c];V)$ is dense in $Z_{\vartheta-1}$. For this, take any $z \in Z_{\vartheta-1}$. Then for any $c' \in (0,c)$, $z \in L_p([0,c'];V)$. Define the following sequence:

$$z_n(t) = \begin{cases} z(t), & 0 \leq t < \frac{cn}{n+1}, \\ (c-t)^{\vartheta-1} z(t), & \frac{cn}{n+1} \leq t \leq c; \quad n = 1, 2, \ldots. \end{cases}$$

Clearly, $z_n \in L_p([0,c];V)$.

Now,

$$\|z - z_n\|^p_{Z_{\vartheta-1}} = \int_{\frac{cn}{n+1}}^{c} \left\|(c-t)^{\vartheta-1}\left(z(t) - (c-t)^{\vartheta-1}z(t)\right)\right\|^p_V \, dt$$
$$\to 0 \quad \text{as} \quad n \to \infty,$$

which shows that the space $L_p([0,c];V)$ is dense in $Z_{\vartheta-1}$. Since $C([0,c];V)$ is dense in $L_p([0,c];V)$, therefore for any given $\varepsilon > 0$ and a $f \in L_p([0,c];V)$, there is a $g \in C([0,c];V)$ satisfying

$$\|f - g\|_{L_p} < c^{1-\vartheta}\varepsilon.$$

Therefore

$$\begin{aligned}\|f - g\|_{Z_{\vartheta-1}} &= \left(\int_0^c \left\|(c-t)^{\vartheta-1}[f(t) - g(t)]\right\|^p_V \, dt\right)^{\frac{1}{p}} \\ &\le c^{\vartheta-1}\|f - g\|_{L_p} \\ &< \varepsilon.\end{aligned}$$

Hence $C([0,c];V)$ is dense in $Z_{\vartheta-1}$. ■

Remark 6.3.2. *It is notable that, inclusion maps $\mathcal{I}_1 : C([0,c];V) \to Z_{\vartheta-1}$ and $\mathcal{I}_2 : L_p([0,c];V) \to Z_{\vartheta-1}$ are continuous. But the density result makes it clear that the norm $\|\cdot\|_{Z_{\vartheta-1}}$ is not equivalent to any of norms $\|\cdot\|_C$ and $\|\cdot\|_{L_p}$.*

We define the following operators:

The Nemytskii type operator $\tilde{F} : C_{2-\vartheta}([0,c];V) \to Z_{\vartheta-1}$ is defined as

$$(\tilde{F}z)(t) = F(t, z(t)), \; z \in C_{2-\vartheta}([0,c];V)$$

and the linear operator $\zeta : Z_{\vartheta-1} \to N$ is defined as

$$\zeta z = \int_0^c \mathcal{R}_\vartheta(c-s)z(s) \, ds, \; z \in Z_{\vartheta-1}.$$

We observe that

$$\begin{aligned}\|\zeta z\|_V &\le k_{\mathcal{R}} \int_0^c (c-t)^{1-\vartheta}(c-t)^{\vartheta-1}\|z(s)\|_V \, ds \\ &\le k_{\mathcal{R}} c^{\frac{2p-p\vartheta-1}{p}} \left(\frac{p-1}{2p-p\vartheta-1}\right)^{\frac{p-1}{p}} \|z\|_{Z_{\vartheta-1}}, \qquad p > \frac{1}{2-\vartheta}.\end{aligned}$$

Hence the operator ζ is bounded.

The subsequent discussion needs the following hypotheses:

(H_4) there exists a constant $\widehat{k}_F > 0$ satisfying

$$\|F(t,y) - F(t,\tilde{y})\| \leq \widehat{k}_F t^{2-\vartheta} \|y - \tilde{y}\| \quad \forall\, y, \tilde{y} \in V;$$

(H_5) for each $\varepsilon > 0$ and $\phi \in Z_{\vartheta-1}$, there is a $u \in U$ such that

$$\|\zeta\phi - \zeta(Bu)\|_V \leq \varepsilon$$

and

$$\|Bu\|_{Z_{\vartheta-1}} \leq b\|\phi\|_{Z_{\vartheta-1}},$$

where the constant b is independent of ϕ;

(H_6) $bk_{\mathcal{R}}\widehat{k}_F c^{3-\vartheta} \left(\frac{p-1}{2p-p\vartheta-1}\right)^{\frac{p-1}{p}} \exp\left(k_{\mathcal{R}}\widehat{k}_F c^{3-\vartheta}\right) < 1.$

Remark 6.3.3. *It is easily seen that (H_4) is stronger condition than (H_2). Therefore by previous theorem, the system (6.1.1) has a unique mild solution in $C_{2-\vartheta}([0,c];V)$ for each given $u \in U$ if conditions (H_1), (H_3) and (H_4) are satisfied.*

Lemma 6.3.4. *Under hypotheses (H_1), (H_3) and (H_4), any mild solutions of the system (6.1.1) satisfy the following:*

(i) $\|z(\cdot,u)\|_{C_{2-\vartheta}} \leq \sigma_1 \exp\left(k_{\mathcal{R}}k'_F c^{3-\vartheta}\right), \quad u \in U;$

(ii) $\|z_1(\cdot,u_1) - z_2(\cdot,u_2)\|_{C_{2-\vartheta}} \leq \sigma_2 \exp\left(k_{\mathcal{R}}\widehat{k}_F c^{3-\vartheta}\right)\|Bu_1 - Bu_2\|_{Z_{\vartheta-1}}, \quad u_1, u_2 \in U;$

where

$$\sigma_1 = k'_{\mathcal{R}}\|y_0\| + k_{\mathcal{R}}\left(c^{2-\vartheta}\|y_1\| + c^{\frac{4p-2p\vartheta-1}{p}}\left(\frac{p-1}{2p-p\vartheta-1}\right)^{\frac{p-1}{p}}\left(\|Bu\|_{Z_{\vartheta-1}} + \|h\|_{Z_{\vartheta-1}}\right)\right)$$

and

$$\sigma_2 = k_{\mathcal{R}} c^{\frac{4p-2p\vartheta-1}{p}}\left(\frac{p-1}{2p-p\vartheta-1}\right)^{\frac{p-1}{p}}.$$

Proof. (i) Let $z \in C_{1-\vartheta}([0,c];V)$ be a mild solution of (6.1.1) associated with $u \in U$, then

$$z(t) = \dot{\mathcal{R}}_\vartheta(t)y_0 + \mathcal{R}_\vartheta(t)y_1 + \int_0^t \mathcal{R}_\vartheta(t-s)\big(Bu(s) + F(s,z(s))\big)\,ds.$$

We have

$$t^{2-\vartheta}\|z(t)\|_V \leq t^{2-\vartheta}\|\dot{\mathcal{R}}_\vartheta(t)y_0\| + t^{2-\vartheta}\|\mathcal{R}_\vartheta(t)y_1\|$$

$$
\begin{aligned}
&+ t^{2-\vartheta}\int_0^t \big\|\mathcal{R}_\vartheta(t-s)\big(Bu(s)+F(s,z(s))\big)\big\|\ ds\\
&\le k'_{\mathcal{R}}\|y_0\| + k_{\mathcal{R}}\bigg(t^{2-\vartheta}\|y_1\| + t^{2-\vartheta}\int_0^t (t-s)^{1-\vartheta}(t-s)^{\vartheta-1}\|(Bu(s)\|\,ds\\
&+ t^{2-\vartheta}\int_0^t \big(h(s)+k'_F s^{2-\vartheta}\|z(s)\|_V\big)\,ds\bigg)\\
&\le k'_{\mathcal{R}}\|y_0\| + k_{\mathcal{R}}\bigg(c^{2-\vartheta}\|y_1\| + c^{\frac{4p-2p\vartheta-1}{p}}\left(\frac{p-1}{2p-p\vartheta-1}\right)^{\frac{p-1}{p}}\\
&\times\big(\|Bu\|_{Z_{\vartheta-1}}+\|h\|_{Z_{\vartheta-1}}\big)\bigg) + k_{\mathcal{R}}k'_F c^{2-\vartheta}\int_0^t s^{2-\vartheta}\|z(s)\|_V\,ds\\
&= \sigma_1 + k_{\mathcal{R}}k'_F c^{2-\vartheta}\int_0^t s^{2-\vartheta}\|z(s)\|_V\,ds.
\end{aligned}
$$

In view of Gronwall's inequality, one can obtain

$$
t^{2-\vartheta}\|z(t)\|_V \le \sigma_1 \exp\big(k_{\mathcal{R}}k'_F c^{3-\vartheta}\big).
$$

Therefore

$$
\|z(\cdot,u)\|_{C_{2-\vartheta}} \le \sigma_1 \exp\big(k_{\mathcal{R}}k'_F c^{3-\vartheta}\big).
$$

(ii) Let $z_\ell \in C_{2-\vartheta}([0,c];V)$ be the mild solution of (6.1.1) associated with $u_\ell \in U,\ \ \ell = 1,2$. Then

$$
z_\ell(t) = \dot{\mathcal{R}}_\vartheta(t)y_0 + \mathcal{R}_\vartheta(t)y_1 + \int_0^t \mathcal{R}_\vartheta(t-s)\big(Bu_\ell(s)+F(s,z_\ell(s))\big)\,ds.
$$

We have

$$
\begin{aligned}
t^{2-\vartheta}\|z_1(t)-z_2(t)\|_V &\le k_{\mathcal{R}}t^{2-\vartheta}\bigg(\int_0^t \|Bu_1(s)-Bu_2(s)\|\,ds\\
&+ \int_0^t \|F(s,z_1(s))-F(s,z_2(s))\|\,ds\bigg)\\
&\le k_{\mathcal{R}}t^{2-\vartheta}\bigg(\int_0^t (c-s)^{1-\vartheta}(c-s)^{\vartheta-1}\|Bu_1(s)-Bu_2(s)\|\,ds\\
&+ \widehat{k}_F\int_0^t s^{2-\vartheta}\|z_1(s)-z_2(s)\|\,ds\bigg)\\
&\le k_{\mathcal{R}}c^{\frac{4p-2p\vartheta-1}{p}}\left(\frac{p-1}{2p-p\vartheta-1}\right)^{\frac{p-1}{p}}\|Bu_1-Bu_2\|_{Z_{\vartheta-1}}\\
&+ k_{\mathcal{R}}\widehat{k}_F c^{2-\vartheta}\int_0^t s^{2-\vartheta}\|z_1(s)-z_2(s)\|\,ds
\end{aligned}
$$

$$= \sigma_2 \|Bu_1 - Bu_2\|_{Z_{\vartheta-1}} + k_{\mathcal{R}} \widehat{k}_F c^{2-\vartheta} \int_0^t s^{2-\vartheta} \|z_1(s) - z_2(s)\| \, ds.$$

In view of Gronwall's inequality, one can obtain

$$t^{2-\vartheta} \|z_1(t) - z_2(t)\|_V \leq \sigma_2 \exp\left(k_{\mathcal{R}} \widehat{k}_F c^{3-\vartheta}\right) \|Bu_1 - Bu_2\|_{Z_{\vartheta-1}}.$$

Hence

$$\|z_1(\cdot, u_1) - z_2(\cdot, u_2)\|_{C_{2-\vartheta}} \leq \sigma_2 \exp\left(k_{\mathcal{R}} \widehat{k}_F c^{3-\vartheta}\right) \|Bu_1 - Bu_2\|_{Z_{\vartheta-1}}.$$

This proves the lemma. ■

Lemma 6.3.5. *If $\mathcal{R}_\vartheta(t)$ is the fractional Riemann-Liouville family associated with the fractional cosine family $\mathcal{C}_\vartheta(t)$ generated by A, then for any $y \in D(A)$, there is a $\wp \in Z_{\vartheta-1}$ such that $\zeta \wp = y$.*

Proof. Since for any $y \in V$, $\lim_{t \to 0^+} \frac{\mathcal{R}_\vartheta(t)}{t^{\vartheta-1}} y = \frac{y}{\Gamma(\vartheta)}$. Therefore $(c-t)^{1-\vartheta} \mathcal{R}_\vartheta(c-t) y \in L_p([0,c]; V)$ and $(c-t)^{2(1-\vartheta)} \mathcal{R}_\vartheta(c-t) y \in Z_{\vartheta-1}$. Now, if we take

$$\wp_1(t) = \frac{[\Gamma(\vartheta)]^2}{c} (c-t)^{2(1-\vartheta)} \mathcal{R}_\vartheta(c-t) y,$$

then

$$\begin{aligned}
\zeta \wp_1 &= \frac{(\Gamma(\vartheta))^2}{c} \int_0^c \mathcal{R}_\vartheta(c-s) \left((c-s)^{2(1-\vartheta)} \mathcal{R}_\vartheta(c-s) y\right) ds \\
&= \frac{(\Gamma(\vartheta))^2}{c} \left(s(c-s)^{2(1-\vartheta)} \mathcal{R}_\vartheta^2(c-s) y\right)_0^c \\
&\quad - \frac{(\Gamma(\vartheta))^2}{c} \int_0^c s \frac{d}{ds} \left((c-s)^{2(1-\vartheta)} \mathcal{R}_\vartheta^2(c-s) y\right) ds \\
&= \frac{(\Gamma(\vartheta))^2}{c} c \frac{y}{(\Gamma(\vartheta))^2} - \frac{(\Gamma(\vartheta))^2}{c} \int_0^c 2s(c-s)^{1-\vartheta} \mathcal{R}_\vartheta(c-s) \\
&\quad \cdot \frac{d}{ds} \left((c-s)^{(1-\vartheta)} \mathcal{R}_\vartheta(c-s) y\right) ds \\
&= y - \zeta \wp_2 \\
\Longrightarrow \zeta \wp &= y,
\end{aligned}$$

where $\wp_2(s) = \frac{2[\Gamma(\vartheta)]^2}{c} s(c-s)^{(1-\vartheta)} \frac{d}{ds} \left((c-s)^{1-\vartheta} \mathcal{R}_\vartheta(c-s) y\right)$ and $\wp = \wp_1 + \wp_2$. ■

Theorem 6.3.6. *Under hypotheses (H_1) and (H_3)-(H_6), the semilinear system (6.1.1) is approximately controllable.*

Proof. It is sufficient to show that $D(A) \subseteq \overline{\mathfrak{R}_c(F)}$, i.e, for any given $\varepsilon > 0$ and $\hat{y} \in D(A)$, one can find a control $u_\varepsilon \in U$ satisfying

$$\left\|y^* - \zeta(\tilde{F}(z_\varepsilon)) - \zeta(Bu_\varepsilon)\right\|_V \leq \varepsilon,$$

where $y^* = \hat{y} - \dot{\mathcal{R}}_\vartheta(c)y_0 - \mathcal{R}_\vartheta(c)y_1$ and $z_\varepsilon(t) = z(t, u_\varepsilon)$. Since $y_0 \in D(A)$, it can be seen that $\dot{\mathcal{R}}_\vartheta(c)y_0 + \mathcal{R}_\vartheta(c)y_1 \in D(A)$. By previous lemma, there is a $\wp \in Z_{\vartheta-1}$ such that $\zeta\wp = y^*$.

Let $\varepsilon > 0$ be given and $u_1 \in U$. Then by hypothesis (H_5) , there is a $u_2 \in U$ satisfying

$$\left\|y^* - \zeta(\tilde{F}(z_1)) - \zeta(Bu_2)\right\|_V \leq \frac{\varepsilon}{3^2},$$

where $z_1(t) = z(t, u_1)$. Denote $z_2(t) = z(t, u_2)$, again by hypothesis (H_5) there is a $\omega_2 \in U$ satisfying

$$\left\|\zeta(\tilde{F}(z_2) - \tilde{F}(z_1)) - \zeta(B\omega_2)\right\|_V \leq \frac{\varepsilon}{3^3}$$

and

$$\begin{aligned}
\|B\omega_2\|_{Z_{\vartheta-1}} &\leq b\left\|\tilde{F}(z_2) - \tilde{F}(z_1)\right\|_{Z_{\vartheta-1}} \\
&= b\left(\int_0^c \left((c-t)^{\vartheta-1}\|F(t, z_2(t)) - F(t, z_1(t))\|_V\right)^p dt\right)^{\frac{1}{p}} \\
&\leq b\widehat{k}_F c^{\vartheta-1}\left(\int_0^c \left(t^{2-\vartheta}\|z_2(t) - z_1(t)\|_V\right)^p dt\right)^{\frac{1}{p}} \\
&\leq b\widehat{k}_F c^{\frac{p\vartheta-p+1}{p}}\|z_2 - z_1\|_{C_{2-\vartheta}} \\
&\leq b\widehat{k}_F c^{\frac{p\vartheta-p+1}{p}}\sigma_2 \exp\left(k_{\mathcal{R}}\widehat{k}_F c^{3-\vartheta}\right)\|Bu_1 - Bu_2\|_{Z_{\vartheta-1}} \\
&= bk_{\mathcal{R}}\widehat{k}_F c^{3-\vartheta}\left(\frac{p-1}{2p-p\vartheta-1}\right)^{\frac{p-1}{p}} \exp\left(k_{\mathcal{R}}\widehat{k}_F c^{3-\vartheta}\right)\|Bu_1 - Bu_2\|_{Z_{\vartheta-1}}.
\end{aligned}$$

Now, if we define

$$u_3(t) = u_2(t) - \omega_2(t), \quad u_3 \in U,$$

then

$$\begin{aligned}
\left\|y^* - \zeta(\tilde{F}(z_2)) - \zeta(Bu_3)\right\|_V &\leq \left\|y^* - \zeta(\tilde{F}(z_1)) - \zeta(Bu_2)\right\|_V \\
&\quad + \left\|\zeta(\tilde{F}(z_2) - \tilde{F}(z_1)) - \zeta(B\omega_2)\right\|_V \\
&\leq \left(\frac{1}{3^2} + \frac{1}{3^3}\right)\varepsilon.
\end{aligned}$$

Applying inductions, one can obtain a sequence $\{u_n\}$ in U such that

$$\left\|y^* - \zeta(\tilde{F}(z_n)) - \zeta(Bu_{n+1})\right\|_V \leq \left(\frac{1}{3^2} + \frac{1}{3^3} + \cdots + \frac{1}{3^{n+1}}\right)\varepsilon,$$

where $z_n(t) = z(t, u_n)$, and

$$\begin{aligned}&\|Bu_{n+1} - Bu_n\|_{Z_{\vartheta-1}}\\ &\quad\leq bk_{\mathcal{R}}\widehat{k}_F c^{3-\vartheta}\left(\frac{p-1}{2p-p\vartheta-1}\right)^{\frac{p-1}{p}} \exp\left(k_{\mathcal{R}}\widehat{k}_F c^{3-\vartheta}\right)\|Bu_n - Bu_{n-1}\|_{Z_{\vartheta-1}}.\end{aligned}$$

Above shows that $\left\{Bu_n\right\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $Z_{\vartheta-1}$. Since $Z_{\vartheta-1}$ is a Banach space and ζ is continuous therefore the sequence $\left\{\zeta(Bu_n)\right\}_{n\in\mathbb{N}}$ is Cauchy in V. Thus one can find a positive integer n_0 satisfying

$$\left\|\zeta(Bu_{n_0+1}) - \zeta(Bu_{n_0})\right\|_V \leq \frac{\varepsilon}{3}.$$

Now,

$$\begin{aligned}\left\|y^* - \zeta(\tilde{F}(z_{n_0})) - \zeta(Bu_{n_0})\right\|_V &\leq \left\|y^* - \zeta(\tilde{F}(z_{n_0})) - \zeta(Bu_{n_0+1})\right\|_V\\ &\quad+\left\|\zeta(Bu_{n_0+1}) - \zeta(Bu_{n_0})\right\|_V\\ &\leq \left(\frac{1}{3^2} + \frac{1}{3^3} + \cdots + \frac{1}{3^{n_0+1}}\right)\varepsilon + \frac{\varepsilon}{3}\\ &< \varepsilon.\end{aligned}$$

This proves the theorem. ■

Corollary 6.3.7. *Under hypotheses (H_1), (H_3) and (H_4), the system (6.1.1) is approximately controllable if Range(B) is dense in $Z_{\vartheta-1}$.*

Proof. Let $\varepsilon > 0$ be given. Since *Range*(B) is dense in $Z_{\vartheta-1}$ therefore for any $\varepsilon' > 0$ and a nonzero function $g \in Z_{\vartheta-1}$, one can find a control $u \in U$ satisfying

$$\|g - Bu\|_{Z_{\vartheta-1}} \leq \varepsilon'\|g\|_{Z_{\vartheta-1}}.$$

Now,

$$\begin{aligned}\|\zeta g - \zeta(Bu)\|_V &\leq k_{\mathcal{R}}\int_0^c (c-s)^{1-\vartheta}(c-s)^{\vartheta-1}\|g(s) - Bu(s)\|_V\, ds\\ &\leq k_{\mathcal{R}} c^{\frac{2p-p\vartheta-1}{p}}\left(\frac{p-1}{2p-p\vartheta-1}\right)^{\frac{p-1}{p}}\|g - Bu\|_{Z_{\vartheta-1}}\end{aligned}$$

$$\leq k_{\mathcal{R}} c^{\frac{2p-p\vartheta-1}{p}} \left(\frac{p-1}{2p-p\vartheta-1}\right)^{\frac{p-1}{p}} \varepsilon' \|g\|_{Z_{\vartheta-1}}$$
$$\leq \varepsilon.$$

Thus

$$\begin{aligned}\|Bu\|_{Z_{\vartheta-1}} &\leq \|Bu - g\|_{Z_{\vartheta-1}} + \|g\|_{Z_{\vartheta-1}} \\ &\leq \varepsilon' \|g\|_{Z_{\vartheta-1}} + \|g\|_{Z_{\vartheta-1}} \\ &= (\varepsilon' + 1)\|g\|_{Z_{\vartheta-1}}.\end{aligned}$$

Hence the condition (H_5) is satisfied. If we choose ε' in such a way so that (H_6) is satisfied. Then by previous theorem, the system (6.1.1) is approximately controllable. ■

6.4 Example

Consider the following initial-boundary value problem with Riemann-Liouville derivative for $x \in [0, \pi]$:

$$\begin{cases} D_t^{4/3}\widehat{z}(t,x) = \frac{\partial^2}{\partial x^2}\widehat{z}(t,x) + \widehat{u}(t,x) + F(t,\widehat{z}(t,x)), & t \in (0,1], \\ \left(I_t^{2/3}\widehat{z}(t,x)\right)_{t=0} = \widehat{y}_0(x), \\ \left(D_t^{1/3}\widehat{z}(t,x)\right)_{t=0} = \widehat{y}_1(x), \\ \widehat{z}(t,0) = 0 = \widehat{z}(t,\pi), & t \in (0,1]. \end{cases} \tag{6.4.1}$$

Take $V = V' = L_2[0,\pi]$ and $A : D(A) \subset V \to V$ is defined as

$$Ay = y''$$

where

$$D(A) = \left\{ y \in V \,\middle|\, y,\ \frac{\partial y}{\partial x} \text{ are absolutely continuous}, \quad \frac{\partial^2 y}{\partial x^2} \in V \right.$$
$$\left. \text{and} \quad y(0) = 0 = y(\pi) \right\}.$$

Then, A can be expressed as

$$Ay = \sum_{\ell=1}^{\infty} (-\ell^2)\langle y, \xi_\ell \rangle \xi_\ell, \quad y \in D(A)$$

and it generates a cosine family $\{\mathcal{C}(t)\}_{t\in\mathbb{R}} \subset \mathcal{B}(V)$ defined by

$$\mathcal{C}(t)y = \sum_{\ell=1}^{\infty} \cos(\ell t)\langle y, \xi_\ell\rangle \xi_\ell, \quad y \in V,$$

where $\xi_\ell(x) = \sqrt{\frac{2}{\pi}} \sin \ell x$ are eigen functions of A for the eigenvalues $\lambda_\ell = -\ell^2$, $\ell = 1, 2, \ldots$; respectively and the orthonormal set $\{\xi_1, \xi_2, \ldots\}$ is a basis for V. The sine family $\{\mathcal{S}(t)\}_{t\in\mathbb{R}}$ associated with $\{\mathcal{C}(t)\}_{t\in\mathbb{R}}$ is defined as

$$\mathcal{S}(t)y = \sum_{\ell=1}^{\infty} \frac{1}{\ell} \sin(\ell t)\langle y, \xi_\ell\rangle \xi_\ell, \quad y \in V.$$

As $\vartheta = \frac{4}{3} \in (1, 2)$, in view of subordinate principle A also generates an exponentially bounded strongly continuous fractional cosine family $\mathcal{C}_{4/3}(t)$ satisfying $\mathcal{C}_{4/3}(0) = \mathcal{I}$, and

$$\mathcal{C}_{4/3}(t) = \int_0^\infty \mu_{t,2/3}(s)\mathcal{C}(s)\, ds, \quad t > 0,$$

where

$$\mu_{t,2/3}(s) = t^{-2/3}\psi_{2/3}(t^{-2/3}s)$$

and

$$\psi_\alpha(y) = \sum_{\ell=0}^{\infty} (-1)^\ell \frac{y^\ell}{\ell!\Gamma(-\alpha\ell + 1 - \alpha)}, \quad 0 < \alpha < 1.$$

Clearly, the corresponding Riemann-Liouville family $\mathcal{R}_{4/3}(t)$ satisfies (H_1).

The abstract form of (6.4.1) is

$$\begin{cases} D_t^{4/3} z(t) = Az(t) + Bu(t) + F(t, z(t)), \quad t \in (0, 1], \\ \left(I_t^{2/3} z(t)\right)_{t=0} = y_0, \\ \left(D_t^{1/3} z(t)\right)_{t=0} = y_1, \end{cases} \tag{6.4.2}$$

where $z(t) = \widehat{z}(t, \cdot)$, $u(t) = \widehat{u}(t, \cdot)$, $y_0 = \widehat{y}_0(\cdot)$, $y_1 = \widehat{y}_1(\cdot)$ and B is the identity map.

If we take

$$F(t, \widehat{z}(t, x)) = (1 + t^2) + k_0 t^{2/3} \|\widehat{z}(t, \cdot)\|_V \xi_3(x),$$

then

$$\begin{aligned} \|F(t, z(t)) - F(t, \tilde{z}(t))\|_V &\leq |k_0| t^{2/3} \|z(t) - \tilde{z}(t)\|_V \\ &\leq |k_0| \|z(t) - \tilde{z}(t)\|_V. \end{aligned}$$

Thus the hypothesis (H_2), (H_3) and (H_4) are satisfied with $k_F = \widehat{k}_F = k'_F = |k_0|$. Since $p = 2$, therefore $p > \frac{1}{2-4/3}$. By (ii) of Lemma 6.3.1 and Theorem 6.3.7, we see that (H_5) is satisfied. Now, if we take k_0 satisfying

$$|k_0| \leq \frac{1}{1 + \sqrt{3} b k_{\mathcal{R}} \exp(k_{\mathcal{R}})},$$

then

$$\begin{aligned} b k_{\mathcal{R}} \widehat{k}_F c^{3-\vartheta} \left(\frac{p-1}{2p - p\vartheta - 1} \right)^{\frac{p-1}{p}} \exp\left(k_{\mathcal{R}} \widehat{k}_F c^{3-\vartheta} \right) &= \sqrt{3} b k_F k_{\mathcal{R}} \exp(k_F k_{\mathcal{R}}) \\ &\leq \frac{\sqrt{3} b k_{\mathcal{R}} \exp\left(\frac{k_{\mathcal{R}}}{1+\sqrt{3} b k_{\mathcal{R}} \exp(k_{\mathcal{R}})} \right)}{1 + \sqrt{3} b k_{\mathcal{R}} \exp(k_{\mathcal{R}})} \\ &< \frac{\sqrt{3} b k_{\mathcal{R}} \exp(k_{\mathcal{R}})}{1 + \sqrt{3} b k_{\mathcal{R}} \exp(k_{\mathcal{R}})} \\ &< 1. \end{aligned}$$

Above shows that (H_6) is satisfied. Hence the approximately controllability of the system (6.4.2) follows from Theorem 6.3.6 if $y_0 \in D(A)$.

6.5 Concluding remarks

In this chapter, approximate controllability result for semilinear fractional systems of order $\vartheta \in (1, 2)$ with integral initial conditions has been presented by assuming that the nonlinear term is Lipschitz continuous. Using fixed point approach, the results of existence and uniqueness have been derived. Here, we introduced a bigger state space $Z_{\vartheta-1}$ containing $L_p([0, c]; V)$ as a dense subspace. Controllability of the system is shown using sequence method.

Chapter 7

Approximate Controllability of Riemann-Liouville Fractional Semilinear Integrodifferential Systems with Damping

This chapter is concerned with Riemann-Liouville fractional semilinear integrodifferential systems with damping in Banach spaces. First we prove the existence of mild solutions of the system using fixed point principle. Then we establish new sufficient conditions for the approximate controllability of the system by means of iterative and approximate technique. Finally, an example is provided for the illustration of the obtained results.

7.1 Introduction and preliminaries

Let V be a Banach space and $Z = L_p([0,c];V)$ be the function space. Consider the fractional order system

$$\begin{cases} D_t^{\vartheta} z(t) + \lambda D_t^{\varphi} z(t) \\ \quad = Az(t) + Bu(t) + F\left(t, z(t), \int_0^t \Theta(t,s,z(s))\, ds\right), \quad t \in (0,c], \\ \left(I_t^{1-\vartheta} z(t)\right)_{t=0} = y_0 \in V, \end{cases} \tag{7.1.1}$$

where $p > \frac{1}{\vartheta}$, $0 < \varphi < \vartheta \leq 1$ and λ is a real number. D_t^{ϑ} and D_t^{φ} stand for Riemann-Liouville derivative of order ϑ and φ respectively. The state $z \in Z$ and the control $u \in U$, where $U = L_p([0,c];V)$. $A : D(A) \subseteq V \to V$ is densely defined and it generates a Riemann-Liouville fractional $(\vartheta, \varphi, \lambda)$ resolvent $\mathcal{R}_{\vartheta,\varphi,\lambda}(t)$. B is the linear map from U to Z. $F : [0,c] \times V \times V \to V$ and $\Theta : \Omega \times V \to V$ are nonlinear, where $\Omega = \big\{(t,s) \,\big|\, 0 \leq s \leq t \leq c\big\}$.

The existence of damping is inevitable in real material. For this reason, in the field of applications, anomalous diffusion equations with damping became an active area of research. The tuned mass dampers provide an effective and relatively simple way of reducing excessive vibrations of chimneys, towers and buildings. For example the damped differential equation of integer order corresponding to a simple linear oscillator system,

$$\ddot{z}(t) + 2\alpha\kappa\dot{z}(t) + \kappa^2 z(t) = \psi(t), \tag{7.1.2}$$

where $z(t)$ is the displacement of structure, $\psi(t)$ is the external force which is supposed to be white noise, α is the damping ratio and κ is the natural frequency of the structure. However, to describe a damped system with a viscoelastic damping elements, fractional order damping gives a better model. Therefore, it is reasonable to introduce the fractional derivatives of orders ϑ and φ to the displacement. Thus, the equation (7.1.2) can be converted into the form

$$D_t^{\vartheta} z(t) + 2\alpha\kappa D_t^{\varphi} z(t) + \kappa^2 z(t) = \psi(t). \tag{7.1.3}$$

If $\vartheta = \varphi = 1$, (7.1.3) is a linear restoring model. In last few years, the dynamics and vibration analysis of damped systems of fractional order have been of great interest for researchers [12; 17; 19; 27]. In [89], Zarraga et al. analyzed the dynamical behavior of fractional damped systems for mechanical engineering applications. In [68], Mei and Peng obtained the existence results for the abstract fractional Cauchy problem with damping using fractional $(\vartheta, \varphi, \lambda)$ resolvent $\mathcal{R}_{\vartheta,\varphi,\lambda}(t)$. In [78], Sheng and Jiang derived the existence of solution for semilinear fractional systems with damping.

In many fields such as thermoelasticity and nuclear reactor dynamics, we need to reflect the memory effect of the systems in model. If differential equations are used in the modeling of such systems which embraces functions at any given space and time, the impact of previous results is neglected. Therefore, in order to incorporate the effect of memory in such systems, an integral part is introduced in the basic differential equation, which leads to integrodifferential equation. To model dynamical systems, integrodifferential equations are utilized in many problems of applied sciences. The integrodifferential equations have poured many applications in aerospace systems, chemical kinetics, biological models, financial mathematics, industrial mathematics, heat conduction, control theory, thermo elastic contact, viscoelastic mechanics and fluid dynamics etc. (see [32; 38] and references therein). In integrodifferential models of many real life problems, the integral part may not appear linearly. Therefore, it is important to consider an integrodifferential system in which the integral term is introduced in nonlinear function.

The existence of mild solutions and controllability for different types of nonlinear and linear systems by applying various techniques have been discussed by many researchers. Among them, Liu and Li [61] proved the approximate controllability of Riemann-Liouville fractional semilinear systems in infinite dimensional Banach spaces by using C_0-semigroup and Lipschitz nonlinearity. In [92], Zhu et al., proved the approximate controllability of Riemann-Liouville fractional semilinear systems using itegral contractor. Using fractional resovent, Ji and Yang [38] obtained the solution to fractional semilinear integrodifferential systems with Riemann-Liouville derivative without Lipschitz nonlinearity. In finite dimensional spaces, Balachandran et al. [7] analyzed the controllability of fractional damped dynamical systems. He et al. [30] obtained necessary and sufficient conditions for the controllability of dynamical systems of fractional order with damping and control delay. However, to the best of our knowledge, there is no result on the controllability of Riemann-Liouville fractional integrodifferential systems with damping in infinite dimensional spaces and this fact is the motivation of this chapter.

To define the mild solution of (7.1.1) in terms of Riemann-Liouville fractional $(\vartheta,\varphi,\lambda)$ resolvent $\mathcal{R}_{\vartheta,\varphi,\lambda}$ (see Section 2.2), we consider the damped system

$$\begin{cases} D_t^{\vartheta} z(t) + \lambda D_t^{\varphi} z(t) = Az(t) + \psi(t), \quad t \in (0,c], \\ \left(I_t^{1-\vartheta} z(t)\right)_{t=0} = y_0 \in V, \end{cases} \tag{7.1.4}$$

where $0 < \varphi < \vartheta \leq 1$ and $\psi \in L_p([0,c];V)$.

Definition 7.1.1. A function $z \in C_{1-\vartheta}([0,c];V)$ is said to be a mild solution of (7.1.4) if it satisfies

$$z(t) = t^{\vartheta-1}E_{\vartheta-\varphi,\vartheta}\left(-\lambda t^{\vartheta-\varphi}\right)y_0 + A\mathcal{E}_t^{\vartheta-\varphi,\vartheta,-\lambda}z(t) + \mathcal{E}_t^{\vartheta-\varphi,\vartheta,-\lambda}\psi(t),$$

where

$$E_{\vartheta,\varphi}(x) = \sum_{n=0}^{\infty}\frac{x^n}{\Gamma(\vartheta n+\varphi)}$$

and

$$\mathcal{E}_t^{\vartheta,\varphi,\lambda}f(t) = \int_0^t (t-s)^{\varphi-1}E_{\vartheta,\varphi}\left(\lambda(t-s)^{\vartheta}\right)f(s)\,ds, \quad t>0.$$

Theorem 7.1.1. *A function $z \in C_{1-\vartheta}([0,c];V)$ is a mild solution of (7.1.4) if and only if it satisfies*

$$z(t) = \mathcal{R}_{\vartheta,\varphi,\lambda}(t)y_0 + \int_0^t \mathcal{R}_{\vartheta,\varphi,\lambda}(t-s)\psi(s)\,ds. \tag{7.1.5}$$

Proof. Let $\xi_{\vartheta,\varphi,\lambda}(t) = t^{\vartheta-1}E_{\vartheta-\varphi,\vartheta}\left(-\lambda t^{\vartheta-\varphi}\right)$. By Lemma 2.2.3, one has

$$\xi_{\vartheta,\varphi,\lambda}(t) = \mathcal{R}_{\vartheta,\varphi,\lambda}(t) - \left(A\xi_{\vartheta,\varphi,\lambda} * \mathcal{R}_{\vartheta,\varphi,\lambda}\right)(t)$$

Now,

$$\begin{aligned}
\xi_{\vartheta,\varphi,\lambda} * z &= \left(\mathcal{R}_{\vartheta,\varphi,\lambda} - A\xi_{\vartheta,\varphi,\lambda} * \mathcal{R}_{\vartheta,\varphi,\lambda}\right) * z \\
&= \mathcal{R}_{\vartheta,\varphi,\lambda} * z - \mathcal{R}_{\vartheta,\varphi,\lambda} * \left(A\xi_{\vartheta,\varphi,\lambda} * z\right) \\
&= \mathcal{R}_{\vartheta,\varphi,\lambda} * \left(z - A\xi_{\vartheta,\varphi,\lambda} * z\right) \\
&= \mathcal{R}_{\vartheta,\varphi,\lambda} * \left(\xi_{\vartheta,\varphi,\lambda}y_0 + \xi_{\vartheta,\varphi,\lambda} * \psi\right) \\
&= \xi_{\vartheta,\varphi,\lambda} * \left(\mathcal{R}_{\vartheta,\varphi,\lambda}y_0 + \mathcal{R}_{\vartheta,\varphi,\lambda} * \psi\right),
\end{aligned}$$

which implies

$$z(t) = \mathcal{R}_{\vartheta,\varphi,\lambda}(t)y_0 + \int_0^t \mathcal{R}_{\vartheta,\varphi,\lambda}(t-s)\psi(s)\,ds.$$

Conversely, suppose z satisfies (7.1.5). By Lemma 2.2.3 z is well defined on $(0,c]$. One can write

$$\left(s^{1-\vartheta}\mathcal{R}_{\vartheta,\varphi,\lambda}(s) - E_{\vartheta-\varphi,\vartheta}\left(-\lambda s^{\vartheta-\varphi}\right)\right)\mathcal{E}_t^{\vartheta-\varphi,\vartheta,-\lambda}z(t)$$

$$
\begin{aligned}
&= \left(s^{1-\vartheta}\mathcal{R}_{\vartheta,\varphi,\lambda}(s) - E_{\vartheta-\varphi,\vartheta}\left(-\lambda s^{\vartheta-\varphi}\right)\right) \\
&\quad \times \mathcal{E}_t^{\vartheta-\varphi,\vartheta,-\lambda}\left(\mathcal{R}_{\vartheta,\varphi,\lambda}(t)y_0 + \int_0^t \mathcal{R}_{\vartheta,\varphi,\lambda}(t-s)\psi(s)\,ds\right) \\
&= \left(s^{1-\vartheta}\mathcal{R}_{\vartheta,\varphi,\lambda}(s) - E_{\vartheta-\varphi,\vartheta}\left(-\lambda s^{\vartheta-\varphi}\right)\right) \\
&\quad \times \left(\mathcal{E}_t^{\vartheta-\varphi,\vartheta,-\lambda}\mathcal{R}_{\vartheta,\varphi,\lambda}(t)y_0 + \mathcal{E}_t^{\vartheta-\varphi,\vartheta,-\lambda}\left(\left(\mathcal{R}_{\vartheta,\varphi,\lambda} * \psi\right)(t)\right)\right) \\
&= s^{1-\vartheta}\mathcal{R}_{\vartheta,\varphi,\lambda}(s)\mathcal{E}_t^{\vartheta-\varphi,\vartheta,-\lambda}\mathcal{R}_{\vartheta,\varphi,\lambda}(t)y_0 - E_{\vartheta-\varphi,\vartheta}\left(-\lambda s^{\vartheta-\varphi}\right)\mathcal{E}_t^{\vartheta-\varphi,\vartheta,-\lambda}\mathcal{R}_{\vartheta,\varphi,\lambda}(t)y_0 \\
&\quad + s^{1-\vartheta}\mathcal{R}_{\vartheta,\varphi,\lambda}(s)\left(\mathcal{E}_t^{\vartheta-\varphi,\vartheta,-\lambda}\mathcal{R}_{\vartheta,\varphi,\lambda} * \psi\right)(t) \\
&\quad - E_{\vartheta-\varphi,\vartheta}\left(-\lambda s^{\vartheta-\varphi}\right)\left(\mathcal{E}_t^{\vartheta-\varphi,\vartheta,-\lambda}\mathcal{R}_{\vartheta,\varphi,\lambda} * \psi\right)(t) \\
&= s^{1-\vartheta}\Big(\mathcal{R}_{\vartheta,\varphi,\lambda}(s)\mathcal{E}_t^{\vartheta-\varphi,\vartheta,-\lambda}\mathcal{R}_{\vartheta,\varphi,\lambda}(t)y_0 - s^{\vartheta-1}E_{\vartheta-\varphi,\vartheta}\left(-\lambda s^{\vartheta-\varphi}\right) \\
&\quad \times \mathcal{E}_t^{\vartheta-\varphi,\vartheta,-\lambda}\mathcal{R}_{\vartheta,\varphi,\lambda}(t)y_0\Big) + s^{1-\vartheta}\Big(\mathcal{R}_{\vartheta,\varphi,\lambda}(s)\mathcal{E}_t^{\vartheta-\varphi,\vartheta,-\lambda}\mathcal{R}_{\vartheta,\varphi,\lambda}(t) \\
&\quad - s^{\vartheta-1}E_{\vartheta-\varphi,\vartheta}\left(-\lambda s^{\vartheta-\varphi}\right)\mathcal{E}_t^{\vartheta-\varphi,\vartheta,-\lambda}\mathcal{R}_{\vartheta,\varphi,\lambda}(t)\Big) * \psi(t) \\
&= s^{1-\vartheta}\Big(\mathcal{E}_s^{\vartheta-\varphi,\vartheta,-\lambda}\mathcal{R}_{\vartheta,\varphi,\lambda}(s)\mathcal{R}_{\vartheta,\varphi,\lambda}(t)y_0 - t^{\vartheta-1}E_{\vartheta-\varphi,\vartheta}\left(-\lambda t^{\vartheta-\varphi}\right) \\
&\quad \times \mathcal{E}_s^{\vartheta-\varphi,\vartheta,-\lambda}\mathcal{R}_{\vartheta,\varphi,\lambda}(s)y_0\Big) + s^{1-\vartheta}\Big(\mathcal{E}_s^{\vartheta-\varphi,\vartheta,-\lambda}\mathcal{R}_{\vartheta,\varphi,\lambda}(s)\mathcal{R}_{\vartheta,\varphi,\lambda}(t) \\
&\quad - t^{\vartheta-1}E_{\vartheta-\varphi,\vartheta}\left(-\lambda t^{\vartheta-\varphi}\right)\mathcal{E}_s^{\vartheta-\varphi,\vartheta,-\lambda}\mathcal{R}_{\vartheta,\varphi,\lambda}(s)\Big) * \psi(t) \\
&= s^{1-\vartheta}\mathcal{E}_s^{\vartheta-\varphi,\vartheta,-\lambda}\mathcal{R}_{\vartheta,\varphi,\lambda}(s)\Big(\mathcal{R}_{\vartheta,\varphi,\lambda}(t)y_0 - t^{\vartheta-1}E_{\vartheta-\varphi,\vartheta}\left(-\lambda t^{\vartheta-\varphi}\right)y_0 \\
&\quad + \left(\mathcal{R}_{\vartheta,\varphi,\lambda} * \psi\right)(t) - \mathcal{E}_t^{\vartheta-\varphi,\vartheta,-\lambda}\psi(t)\Big) \\
&= s^{1-\vartheta}\mathcal{E}_s^{\vartheta-\varphi,\vartheta,-\lambda}\mathcal{R}_{\vartheta,\varphi,\lambda}(s)\Big(z(t) - t^{\vartheta-1}E_{\vartheta-\varphi,\vartheta}\left(-\lambda t^{\vartheta-\varphi}\right)y_0 - \mathcal{E}_t^{\vartheta-\varphi,\vartheta,-\lambda}\psi(t)\Big).
\end{aligned}
$$

Therefore

$$
\begin{aligned}
&A\mathcal{E}_t^{\vartheta-\varphi,\vartheta,-\lambda}z(t) \\
&\quad = \Gamma(2\vartheta)\lim_{s\to0^+}\frac{\left(s^{1-\vartheta}\mathcal{R}_{\vartheta,\varphi,\lambda}(s) - E_{\vartheta-\varphi,\vartheta}\left(-\lambda s^{\vartheta-\varphi}\right)\right)\mathcal{E}_t^{\vartheta-\varphi,\vartheta,-\lambda}z(t)}{s^{\vartheta}} \\
&\quad = \Gamma(2\vartheta)\lim_{s\to0^+}s^{1-2\vartheta}\mathcal{E}_s^{\vartheta-\varphi,\vartheta,-\lambda}\mathcal{R}_{\vartheta,\varphi,\lambda}(s)\Big(z(t) - t^{\vartheta-1}E_{\vartheta-\varphi,\vartheta}\left(-\lambda t^{\vartheta-\varphi}\right)y_0 \\
&\qquad - \mathcal{E}_t^{\vartheta-\varphi,\vartheta,-\lambda}\psi(t)\Big).
\end{aligned}
\tag{7.1.6}
$$

Now, for any $y \in N$

$$
\begin{aligned}
&\left\|\Gamma(2\vartheta)s^{1-2\vartheta}\mathcal{E}_s^{\vartheta-\varphi,\vartheta,-\lambda}\mathcal{R}_{\vartheta,\varphi,\lambda}(s)y - y\right\| \\
&\quad = \left\|\Gamma(2\vartheta)\int_0^s s^{1-2\vartheta}(s-\varrho)^{\vartheta-1}E_{\vartheta-\varphi,\vartheta}(-\lambda(s-\varrho)^{\vartheta-\varphi})\mathcal{R}_{\vartheta,\varphi,\lambda}(\varrho)y\,d\varrho - y\right\|
\end{aligned}
$$

$$
\begin{aligned}
&= \left\| \Gamma(2\vartheta) \int_0^1 s^{1-\vartheta}(1-\varrho)^{\vartheta-1} E_{\vartheta-\varphi,\vartheta}(-\lambda(s-s\varrho)^{\vartheta-\varphi}) \mathcal{R}_{\vartheta,\varphi,\lambda}(s\varrho) y \, d\varrho - y \right\| \\
&= \left\| \frac{\Gamma(2\vartheta)}{\Gamma(\vartheta)} \int_0^1 \varrho^{\vartheta-1}(1-\varrho)^{\vartheta-1} E_{\vartheta-\varphi,\vartheta}(-\lambda(s-s\varrho)^{\vartheta-\varphi}) \right. \\
&\quad \left. \times \Gamma(\vartheta)(s\varrho)^{1-\vartheta} \mathcal{R}_{\vartheta,\varphi,\lambda}(s\varrho) y \, d\varrho - \frac{\Gamma(2\vartheta)}{(\Gamma(\vartheta))^2} \int_0^1 \varrho^{\vartheta-1}(1-\varrho)^{\vartheta-1} y \, d\varrho \right\| \\
&\leq \sup_{\varrho \in (0,1]} \left\| \Gamma(\vartheta) E_{\vartheta-\varphi,\vartheta}(-\lambda(s-s\varrho)^{\vartheta-\varphi}) \Gamma(\vartheta)(s\varrho)^{1-\vartheta} \mathcal{R}_{\vartheta,\varphi,\lambda}(s\varrho) y - y \right\| \qquad (7.1.7)
\end{aligned}
$$

Since $E_{\vartheta-\varphi,\vartheta}(0) = 1/\Gamma(\vartheta)$ therefore from (7.1.6) and (7.1.7), we have

$$A\mathcal{E}_t^{\vartheta-\varphi,\vartheta,-\lambda} z(t) = z(t) - t^{\vartheta-1} E_{\vartheta-\varphi,\vartheta}\left(-\lambda t^{\vartheta-\varphi}\right) y_0 - \mathcal{E}_t^{\vartheta-\varphi,\vartheta,-\lambda} \psi(t).$$

This implies

$$z(t) = t^{\vartheta-1} E_{\vartheta-\varphi,\vartheta}\left(-\lambda t^{\vartheta-\varphi}\right) y_0 + A\mathcal{E}_t^{\vartheta-\varphi,\vartheta,-\lambda} z(t) + \mathcal{E}_t^{\vartheta-\varphi,\vartheta,-\lambda} \psi(t).$$

This proves the theorem. ∎

In view of above theorem, we give the following definition:

Definition 7.1.2. A function $z \in C_{1-\vartheta}([0,c];V)$ is said to be a mild solution of (7.1.1) if it satisfies

$$z(t) = \mathcal{R}_{\vartheta,\varphi,\lambda}(t) y_0 + \int_0^t \mathcal{R}_{\vartheta,\varphi,\lambda}(t-s) \left(Bu(s) + F\left(s, z(s), \int_0^s \Theta(s,\varrho,z(\varrho))\, d\varrho \right)\right) ds.$$

Definition 7.1.3. Let $z(t,u)$ be a mild solution of (7.1.1) associated with a control $u \in U$. The set given by

$$\mathfrak{R}_c(F) = \left\{ z(c,u) \in V \,\middle|\, u \in U \right\}$$

is known as the reachable set of (7.1.1). Moreover, if $\mathfrak{R}_c(F)$ is dense in V, then we say that the system (7.1.1) is approximately controllable on $[0,c]$.

7.2 Existence of mild solution

To derive the existence result we suppose the following conditions:

(H_1) $\mathcal{R}_{\vartheta,\varphi,\lambda}(t)$ is differentiable and there exists a constant $k_{\mathcal{R}} > 0$ satisfying

$$\|t^{1-\vartheta} \mathcal{R}_{\vartheta,\varphi,\lambda}(t)\| \leq k_{\mathcal{R}}, \quad 0 < t \leq c;$$

(H_2) there is a constant $k_F > 0$ satisfying

$$\|F(t, y_1, \tilde{y}_1) - F(t, y_2, \tilde{y}_2)\| \leq k_F \left(\|y_1 - y_2\| + \|\tilde{y}_1 - \tilde{y}_2\|\right)$$

for all $y_\ell, \tilde{y}_\ell \in V,\ \ell = 1, 2$;

(H_3) there is a function $h \in L_p([0, c]; \mathbb{R}_0^+)$, and a constant $k'_F > 0$ satisfying

$$\|F(t, y, \tilde{y})\| \leq h(t) + k'_F t^{1-\vartheta} \left(\|y\| + \|\tilde{y}\|\right)$$

for a.e. $t \in [0, c]$ and all $y, \tilde{y} \in V$;

(H_4) there is a constant $k_\Theta > 0$ satisfying

$$\|\Theta(t, s, y_1) - \Theta(t, s, y_2)\| \leq k_\Theta \|y_1 - y_2\| \ \forall\ y_\ell \in V,\ \ell = 1, 2;$$

(H_5) there is a function $g \in L_p([0, c]; \mathbb{R}_0^+)$ satisfying

$$\|\Theta(t, s, y)\| \leq g(s)$$

for a.e. $(t, s) \in \Omega$ and all $y \in V$.

Theorem 7.2.1. *Under hypotheses (H_1)-(H_5), the semilinear system (7.1.1) has a unique mild solution in $C_{1-\vartheta}([0, c]; V)$ for each control $u \in U$.*

Proof. Theorem will be proved if we show that the map $\mathcal{Q} : C_{1-\vartheta}([0, c]; V) \to C_{1-\vartheta}([0, c]; V)$ defined by

$$(\mathcal{Q}z)(t) = \mathcal{R}_{\vartheta,\varphi,\lambda}(t) y_0 + \int_0^t \mathcal{R}_{\vartheta,\varphi,\lambda}(t - s) B u(s)\, ds + \int_0^t \mathcal{R}_{\vartheta,\varphi,\lambda}(t - s) F\left(s, z(s), \int_0^s \Theta(s, \varrho, z(\varrho))\, d\varrho\right) ds,$$

has a unique fixed point in $C_{1-\vartheta}([0, c]; V)$.

By above assumptions, it is easily seen that the map $\mathcal{Q}$ is well defined.

Now, for any $z, \tilde{z} \in C_{1-\vartheta}([0, c]; V)$, one can obtain

$$t^{1-\vartheta} \|(\mathcal{Q}z)(t) - (\mathcal{Q}\tilde{z})(t)\| \leq t^{1-\vartheta} \int_0^t \left\| \mathcal{R}_{\vartheta,\varphi,\lambda}(t - s) \left(F\left(s, z(s), \int_0^s \Theta(s, \varrho, z(\varrho))\, d\varrho\right)\right.\right.$$

$$
\begin{aligned}
&- F\left(s, \tilde{z}(s), \int_0^s \Theta(s, \varrho, \tilde{z}(\varrho))\, d\varrho\right)\right)\Bigg\| ds \\
&\leq k_{\mathcal{R}} k_F t^{1-\vartheta} \int_0^t (t-s)^{\vartheta-1}\Bigg(\|z(s) - \tilde{z}(s)\| \\
&\quad + \int_0^s \|\Theta(s, \varrho, z(\varrho)) - \Theta(s, \varrho, \tilde{z}(\varrho))\|\, d\varrho \Bigg) ds \\
&\leq k_{\mathcal{R}} k_F t^{1-\vartheta} \int_0^t (t-s)^{\vartheta-1}\Bigg(s^{\vartheta-1} s^{1-\vartheta}\|z(s) - \tilde{z}(s)\| \\
&\quad + k_\Theta \int_0^s \varrho^{\vartheta-1}\varrho^{1-\vartheta}\|z(\varrho) - \tilde{z}(\varrho)\|\, d\varrho \Bigg) ds \\
&\leq k_{\mathcal{R}} k_F t^{1-\vartheta} \int_0^t (t-s)^{\vartheta-1}\left(s^{\vartheta-1} + k_\Theta \frac{s^\vartheta}{\vartheta}\right) ds\, \|z - \tilde{z}\|_{C_{1-\vartheta}} \\
&= k_{\mathcal{R}} k_F t^{\vartheta}\left(\frac{(\Gamma(\vartheta))^2}{\Gamma(2\vartheta)} + \frac{k_\Theta \Gamma(\vartheta)\Gamma(\vartheta+1)t}{\vartheta\Gamma(2\vartheta+1)}\right) \|z - \tilde{z}\|_{C_{1-\vartheta}} \\
&\leq k_{\mathcal{R}} k_F t^{\vartheta} \frac{(\Gamma(\vartheta))^2}{\Gamma(2\vartheta)}\left(1 + \frac{k_\Theta c}{2\vartheta}\right) \|z - \tilde{z}\|_{C_{1-\vartheta}}.
\end{aligned}
$$

Further,

$$
\begin{aligned}
&t^{1-\vartheta}\|(\mathcal{Q}^2 z)(t) - (\mathcal{Q}^2 \tilde{z})(t)\| \\
&\quad \leq k_{\mathcal{R}} k_F t^{1-\vartheta} \int_0^t (t-s)^{\vartheta-1}\Bigg(s^{\vartheta-1}s^{1-\vartheta}\|(\mathcal{Q}z)(s) - (\mathcal{Q}\tilde{z})(s)\| \\
&\quad\quad + k_\Theta \int_0^s \varrho^{\vartheta-1}\varrho^{1-\vartheta}\|(\mathcal{Q}z)(\varrho) - (\mathcal{Q}\tilde{z})(\varrho)\|\, d\varrho\Bigg) ds \\
&\quad \leq (k_{\mathcal{R}} k_F)^2 \frac{(\Gamma(\vartheta))^2}{\Gamma(2\vartheta)}\left(1 + \frac{k_\Theta c}{2\vartheta}\right) t^{1-\vartheta} \\
&\quad\quad \times \int_0^t (t-s)^{\vartheta-1}\left(s^{2\vartheta-1} + k_\Theta \frac{s^{2\vartheta}}{2\vartheta}\right) ds \cdot \|z - \tilde{z}\|_{C_{1-\vartheta}} \\
&\quad = (k_{\mathcal{R}} k_F)^2 \frac{(\Gamma(\vartheta))^2}{\Gamma(2\vartheta)}\left(1 + \frac{k_\Theta c}{2\vartheta}\right) t^{2\vartheta} \\
&\quad\quad \times \left(\frac{\Gamma(\vartheta)\Gamma(2\vartheta)}{\Gamma(3\vartheta)} + \frac{k_\Theta \Gamma(\vartheta)\Gamma(2\vartheta+1)t}{2\vartheta\Gamma(3\vartheta+1)}\right) \|z - \tilde{z}\|_{C_{1-\vartheta}} \\
&\quad \leq (k_{\mathcal{R}} k_F t^\vartheta)^2 \frac{(\Gamma(\vartheta))^3}{\Gamma(3\vartheta)}\left(1 + \frac{k_\Theta c}{2\vartheta}\right)\left(1 + \frac{k_\Theta c}{3\vartheta}\right) \|z - \tilde{z}\|_{C_{1-\vartheta}}.
\end{aligned}
$$

By induction, one can obtain

$$
\begin{aligned}
&t^{1-\vartheta}\|(\mathcal{Q}^n z)(t) - (\mathcal{Q}^n \tilde{z})(t)\| \\
&\quad \leq (k_{\mathcal{R}} k_F t^\vartheta)^n \frac{(\Gamma(\vartheta))^{n+1}}{\Gamma((n+1)\vartheta)}\left(\prod_{\ell=1}^{n}\left(1 + \frac{k_\Theta c}{(\ell+1)\vartheta}\right)\right) \|z - \tilde{z}\|_{C_{1-\vartheta}}
\end{aligned}
$$

$$\leq \frac{\Gamma(\vartheta)\left(k_{\mathcal{R}}k_F t^{\vartheta}\Gamma(\vartheta)\left(1+\frac{k_\Theta c}{2\vartheta}\right)\right)^n}{\Gamma((n+1)\vartheta)}\|z-\tilde{z}\|_{C_{1-\vartheta}}$$
$$\leq \frac{\Gamma(\vartheta)\left(k_{\mathcal{R}}k_F c^{\vartheta}\Gamma(\vartheta)\left(1+\frac{k_\Theta c}{2\vartheta}\right)\right)^n}{\Gamma((n+1)\vartheta)}\|z-\tilde{z}\|_{C_{1-\vartheta}}.$$

Therefore

$$\|\mathcal{Q}^n z-\mathcal{Q}^n\tilde{z}\|_{C_{1-\vartheta}} \leq \frac{\Gamma(\vartheta)\left(k_{\mathcal{R}}k_F c^{\vartheta}\Gamma(\vartheta)\left(1+\frac{k_\Theta c}{2\vartheta}\right)\right)^n}{\Gamma((n+1)\vartheta)}\|z-\tilde{z}\|_{C_{1-\vartheta}}.$$

But the Mittag-Leffler series

$$E_{\vartheta,\vartheta}\left(k_{\mathcal{R}}k_F c^{\vartheta}\Gamma(\vartheta)\left(1+\frac{k_\Theta c}{2\vartheta}\right)\right)=\sum_{\ell=0}^{\infty}\frac{\left(k_{\mathcal{R}}k_F c^{\vartheta}\Gamma(\vartheta)\left(1+\frac{k_\Theta c}{2\vartheta}\right)\right)^{\ell}}{\Gamma((\ell+1)\vartheta)}$$

is convergent. Therefore $\frac{\left(k_{\mathcal{R}}k_F c^{\vartheta}\Gamma(\vartheta)\left(1+\frac{k_\Theta c}{2\vartheta}\right)\right)^n}{\Gamma((n+1)\vartheta)} < \frac{1}{\Gamma(\vartheta)}$ for n large enough. Hence by generalized Banach contraction theorem, $\mathcal{Q}$ has a unique fixed point in the space $C_{1-\vartheta}([0,c];V)$. ■

Remark 7.2.2. *Here, we assumed the Lipschitz continuity of both the nonlinear functions* Θ *and* F *. To prove the existence results for semilinear systems, the Lipschitz continuity of nonlinear functions is broadly used by researchers. For example, Liu and Li [61] proved the existence of solutions for Riemann-Liouville fractional systems using Lipschitz continuity of nonlinear function. Recently, Li et al. [57] proved the existence of solutions for Caputo fractional systems with damping by assuming Lipschitz continuity.*

The conditions (H_3) *and* (H_5) *guarantee that the map* $\mathcal{Q}$ *is well defined, that is* $\mathcal{Q}z \in C_{1-\vartheta}([0,c];V)$ *whenever* $z \in C_{1-\vartheta}([0,c];V)$.

7.3 Controllability results

We define the following operators:

The Nemytskii type operator $\tilde{F}: C_{1-\vartheta}([0,c];V)\to Z$ is defined as

$$(\tilde{F}z)(t)=F\left(t,z(t),\int_0^t \Theta(t,s,z(s))\,ds\right),\ z\in C_{1-\vartheta}([0,c];V)$$

and the linear operator $\zeta: Z\to V$ is defined as

$$\zeta z=\int_0^c \mathcal{R}_{\vartheta,\varphi,\lambda}(c-s)z(s)\,ds,\ z\in Z.$$

We observe that the operator ζ is continuous.

Remark 7.3.1. *In view of Definition 7.1.3, the reachable set $\mathfrak{R}_c(F)$ is dense in V if and only if for each $\varepsilon > 0$ and a $\widehat{y} \in V$, one can find a control $u_\varepsilon \in U$ such that the mild solution z_ε associated with the control u_ε satisfies*

$$\|\widehat{y} - \mathcal{R}_{\vartheta,\varphi,\lambda}(c)y_0 - \zeta(\tilde{F}(z_\varepsilon)) - \zeta(Bu_\varepsilon)\| \leq \varepsilon.$$

The subsequent discussion needs the following hypotheses:

(H_6) there exists a constant $\widehat{k}_F > 0$ satisfying

$$\|F(t, y_1, \tilde{y}_1) - F(t, y_2, \tilde{y}_2)\| \leq \widehat{k}_F t^{1-\vartheta} \left(\|y_1 - y_2\| + \|\tilde{y}_1 - \tilde{y}_2\|\right)$$

for all $y_\ell, \tilde{y}_\ell \in V,\ \ell = 1, 2$;

(H_7) there exists a constant $\widehat{k}_\Theta > 0$ satisfying

$$\|\Theta(t, s, y_1) - \Theta(t, s, y_2)\| \leq \widehat{k}_\Theta s^{1-\vartheta} \|y_1 - y_2\| \ \ \forall\, y_1, y_2 \in V;$$

(H_8) $k_{\mathcal{R}} \widehat{k}_F \widehat{k}_\Theta c^{3-\vartheta} \vartheta^{-1} E_\vartheta\big(k_{\mathcal{R}} \widehat{k}_F c \Gamma(\vartheta)\big) < 1$;

(H_9) for each $\varepsilon > 0$ and $\phi \in Z$, there is a $u \in U$ such that

$$\|\zeta\phi - \zeta(Bu)\|_V \leq \varepsilon$$

and

$$\|Bu\|_Z \leq b\|\phi\|_Z,$$

where the constant b is independent of ϕ;

(H_{10}) $\dfrac{k_{\mathcal{R}} \widehat{k}_F bc\big(1+\widehat{k}_\Theta c^{2-\vartheta}\big)\big(\frac{p-1}{p\vartheta-1}\big)^{\frac{p-1}{p}} E_\vartheta\big(k_{\mathcal{R}} \widehat{k}_F c\Gamma(\vartheta)\big)}{1-k_{\mathcal{R}} \widehat{k}_F \widehat{k}_\Theta c^{3-\vartheta} \vartheta^{-1} E_\vartheta\big(k_{\mathcal{R}} \widehat{k}_F c\Gamma(\vartheta)\big)} < 1.$

Remark 7.3.2. *It is easily seen that (H_6) and (H_7) are stronger conditions than (H_2) and (H_4), respectively. Therefore by previous theorem, the system (7.1.1) admits a unique mild solution in $C_{1-\vartheta}([0, c]; V)$ for each given $u \in U$ if conditions (H_1), (H_3) and (H_5)-(H_7) are satisfied.*

Lemma 7.3.3. *Under hypotheses $(H_1), (H_3)$ and (H_5)-(H_8), any mild solutions of the system (7.1.1) satisfy the following:*

(i) $\|z(\cdot, u)\|_{C_{1-\vartheta}} \leq \rho_1 E_\vartheta\big(k_{\mathcal{R}} k'_F c\Gamma(\vartheta)\big), \quad u \in U;$

$$(ii)\ \|z_1(\cdot,u_1)-z_2(\cdot,u_2)\|_{C_{1-\vartheta}} \le \rho_2 E_\vartheta\big(k_{\mathcal{R}}\widehat{k}_F c\Gamma(\vartheta)\big)\|Bu_1-Bu_2\|_Z, \quad u_1,u_2\in U; \tag{7.3.1}$$

where

$$\rho_1 = k_{\mathcal{R}}\left(\|z_0\| + \left(\frac{p-1}{p\vartheta-1}\right)^{\frac{p-1}{p}}(\|Bu\|_Z+\|h\|_{L_p})c^{\frac{p-1}{p}} + \frac{k'_F c^{3-\vartheta-\frac{1}{p}}}{\vartheta}\|g\|_{L_p}\right)$$

and

$$\rho_2 = \frac{k_{\mathcal{R}} c^{\frac{p-1}{p}}\left(\frac{p-1}{p\vartheta-1}\right)^{\frac{p-1}{p}}}{1-k_{\mathcal{R}}\widehat{k}_F\widehat{k}_\Theta c^{3-\vartheta}\vartheta^{-1}E_\vartheta\big(k_{\mathcal{R}}\widehat{k}_F c\Gamma(\vartheta)\big)}.$$

Proof. (i) Let $z\in C_{1-\vartheta}([0,c];V)$ be a mild solution of (7.1.1) associated with $u\in U$, then

$$z(t)=\mathcal{R}_{\vartheta,\varphi,\lambda}(t)y_0+\int_0^t \mathcal{R}_{\vartheta,\varphi,\lambda}(t-s)\left(Bu(s)+F\left(s,z(s),\int_0^s\Theta(s,\varrho,z(\varrho))\,d\varrho\right)\right)ds.$$

We have

$$\begin{aligned}
t^{1-\vartheta}\|z(t)\|_V &\le t^{1-\vartheta}\|\mathcal{R}_{\vartheta,\varphi,\lambda}(t)y_0\| + t^{1-\vartheta}\int_0^t\|\mathcal{R}_{\vartheta,\varphi,\lambda}(t-s)\| \\
&\quad\cdot\left\|Bu(s)+F\left(s,z(s),\int_0^s\Theta(s,\varrho,z(\varrho))\,d\varrho\right)\right\|ds \\
&\le k_{\mathcal{R}}\Bigg(\|y_0\| + t^{1-\vartheta}\int_0^t(t-s)^{\vartheta-1}\|Bu(s)\|\,ds + t^{1-\vartheta}\int_0^t(t-s)^{\vartheta-1} \\
&\quad\cdot\left(h(s)+k'_F s^{1-\vartheta}\|z(s)\|_V + k'_F s^{1-\vartheta}\int_0^s g(\varrho)\,d\varrho\right)ds\Bigg) \\
&\le k_{\mathcal{R}}\Bigg(\|y_0\| + \left(\frac{p-1}{p\vartheta-1}\right)^{\frac{p-1}{p}}(\|Bu\|_Z+\|h\|_{L_p})c^{\frac{p-1}{p}} \\
&\quad + k'_F c^{3-2\vartheta-\frac{1}{p}}\int_0^t(t-s)^{\vartheta-1}\,ds\|g\|_{L_p} \\
&\quad + k'_F c^{1-\vartheta}\int_0^t(t-s)^{\vartheta-1}s^{1-\vartheta}\|z(s)\|_V\,ds\Bigg) \\
&\le \rho_1 + k_{\mathcal{R}}k'_F c^{1-\vartheta}\int_0^t(t-s)^{\vartheta-1}s^{1-\vartheta}\|z(s)\|_V\,ds.
\end{aligned}$$

From Theorem 2.3.6, we obtain

$$t^{1-\vartheta}\|z(t)\|_V \le \rho_1 E_\vartheta\big(k_{\mathcal{R}}k'_F c\Gamma(\vartheta)\big)$$

Therefore

$$\|z(\cdot,u)\|_{C_{1-\vartheta}} \le \rho_1 E_\vartheta\big(k_{\mathcal{R}}k'_F c\Gamma(\vartheta)\big).$$

(*ii*) Let $z_\ell \in C_{1-\vartheta}([0,c];V)$ be the mild solution of (7.1.1) associated with $u_\ell \in U, \quad \ell = 1,2$. Then We have

$$
\begin{aligned}
& t^{1-\vartheta}\|z_1(t)-z_2(t)\|_V \\
& \leq k_{\mathcal{R}} t^{1-\vartheta}\bigg(\int_0^t (t-s)^{\vartheta-1}\|Bu_1(s)-Bu_2(s)\|\,ds + \int_0^t (t-s)^{\vartheta-1} \\
& \quad \cdot \bigg\|F\bigg(s, z_1(s), \int_0^s \Theta(s,\varrho,z_1(\varrho))\,d\varrho\bigg) - F\bigg(s, z_2(s), \int_0^s \Theta(s,\varrho,z_2(\varrho))\,d\varrho\bigg)\bigg\|\,ds\bigg) \\
& \leq k_{\mathcal{R}} c^{\frac{p-1}{p}}\left(\frac{p-1}{p\vartheta-1}\right)^{\frac{p-1}{p}}\|Bu_1-Bu_2\|_Z + k_{\mathcal{R}}\widehat{k}_F c^{1-\vartheta}\int_0^t (t-s)^{\vartheta-1}s^{1-\vartheta} \\
& \quad \cdot \bigg(\|z_1(s)-z_2(s)\| + \widehat{k}_\Theta \int_0^s \varrho^{1-\vartheta}\|z_1(\varrho)-z_2(\varrho)\|\,d\varrho\bigg)\,ds \\
& \leq k_{\mathcal{R}} c^{\frac{p-1}{p}}\left(\frac{p-1}{p\vartheta-1}\right)^{\frac{p-1}{p}}\|Bu_1-Bu_2\|_Z + k_{\mathcal{R}}\widehat{k}_F c^{1-\vartheta}\bigg(\int_0^t (t-s)^{\vartheta-1}s^{1-\vartheta} \\
& \quad \cdot \|z_1(s)-z_2(s)\|\,ds + \widehat{k}_\Theta \int_0^t (t-s)^{\vartheta-1}s^{1-\vartheta}\int_0^s d\varrho\,ds\|z_1-z_2\|_{C_{1-\vartheta}}\bigg) \\
& \leq k_{\mathcal{R}} c^{\frac{p-1}{p}}\left(\frac{p-1}{p\vartheta-1}\right)^{\frac{p-1}{p}}\|Bu_1-Bu_2\|_Z + k_{\mathcal{R}}\widehat{k}_F c^{1-\vartheta}\bigg(\int_0^t (t-s)^{\vartheta-1}s^{1-\vartheta} \\
& \quad \cdot \|z_1(s)-z_2(s)\|\,ds + \widehat{k}_\Theta \int_0^t (t-s)^{\vartheta-1}c^{2-\vartheta}\,ds\|z_1-z_2\|_{C_{1-\vartheta}}\bigg) \\
& \leq k_{\mathcal{R}} c^{\frac{p-1}{p}}\left(\frac{p-1}{p\vartheta-1}\right)^{\frac{p-1}{p}}\|Bu_1-Bu_2\|_Z + k_{\mathcal{R}}\widehat{k}_F\widehat{k}_\Theta c^{3-\vartheta}\vartheta^{-1}\|z_1-z_2\|_{C_{1-\vartheta}} \\
& \quad + k_{\mathcal{R}}\widehat{k}_F c^{1-\vartheta}\int_0^t (t-s)^{\vartheta-1}s^{1-\vartheta}\|z_1(s)-z_2(s)\|\,ds \\
& \leq k_{\mathcal{R}} c^{\frac{p-1}{p}}\left(\frac{p-1}{p\vartheta-1}\right)^{\frac{p-1}{p}}\|Bu_1-Bu_2\|_Z + k_{\mathcal{R}}\widehat{k}_F\widehat{k}_\Theta c^{3-\vartheta}\vartheta^{-1}\|z_1-z_2\|_{C_{1-\vartheta}} \\
& \quad + k_{\mathcal{R}}\widehat{k}_F c^{1-\vartheta}\int_0^t (t-s)^{\vartheta-1}s^{1-\vartheta}\|z_1(s)-z_2(s)\|\,ds.
\end{aligned}
$$

From Theorem 2.3.6, we obtain

$$
\begin{aligned}
t^{1-\vartheta}\|z_1(t)-z_2(t)\|_N \leq \Bigg(& k_{\mathcal{R}} c^{\frac{p-1}{p}}\left(\frac{p-1}{p\vartheta-1}\right)^{\frac{p-1}{p}}\|Bu_1-Bu_2\|_Z \\
& + k_{\mathcal{R}}\widehat{k}_F\widehat{k}_\Theta c^{3-\vartheta}\vartheta^{-1}\|z_1-z_2\|_{C_{1-\vartheta}}\Bigg) E_\vartheta\big(k_{\mathcal{R}}\widehat{k}_F c\Gamma(\vartheta)\big).
\end{aligned}
$$

Therefore

$$
\|z_1-z_2\|_{C_{1-\vartheta}} \leq \Bigg(k_{\mathcal{R}} c^{\frac{p-1}{p}}\left(\frac{p-1}{p\vartheta-1}\right)^{\frac{p-1}{p}}\|Bu_1-Bu_2\|_Z
$$

$$+ k_{\mathcal{R}}\widehat{k}_F\widehat{k}_\Theta c^{3-\vartheta}\vartheta^{-1}\|z_1 - z_2\|_{C_{1-\vartheta}}\Bigg) E_\vartheta\big(k_{\mathcal{R}}\widehat{k}_F c\Gamma(\vartheta)\big),$$

which gives

$$\begin{aligned}\|z_1(\cdot, u_1) - z_2(\cdot, u_2)\|_{C_{1-\vartheta}} &\leq \frac{k_{\mathcal{R}} c^{\frac{p-1}{p}} \left(\frac{p-1}{p\vartheta-1}\right)^{\frac{p-1}{p}} E_\vartheta\big(k_{\mathcal{R}}\widehat{k}_F c\Gamma(\vartheta)\big)}{1 - k_{\mathcal{R}}\widehat{k}_F\widehat{k}_\Theta c^{3-\vartheta}\vartheta^{-1} E_\vartheta\big(k_{\mathcal{R}}\widehat{k}_F c\Gamma(\vartheta)\big)}\|Bu_1 - Bu_2\|_Z \\ &= \rho_2 E_\vartheta\big(k_{\mathcal{R}}\widehat{k}_F c\Gamma(\vartheta)\big)\|Bu_1 - Bu_2\|_Z.\end{aligned}$$

This completes the proof. ∎

Theorem 7.3.4. *Under hypotheses (H_1), (H_3) and (H_5)-(H_{10}), the semilinear system (7.1.1) is approximately controllable.*

Proof. It is sufficient to show that $D(A) \subseteq \overline{\mathfrak{R}_c(F)}$, i.e, for any given $\varepsilon > 0$ and $\widehat{y} \in D(A)$, one can find a control $u_\varepsilon \in U$ satisfying

$$\big\|y^* - \zeta(\tilde{F}(z_\varepsilon)) - \zeta(Bu_\varepsilon)\big\|_V \leq \varepsilon,$$

where $y^* = \widehat{y} - \mathcal{R}_{\vartheta,\varphi,\lambda}(c)y_0$ and $z_\varepsilon(t) = z(t, u_\varepsilon)$.
It can be seen that there is a $\wp \in Z$ such that $\zeta\wp = y^*$, if we take

$$\begin{aligned}\wp(t) = \frac{(\Gamma(\vartheta))^2 (c-t)^{1-\vartheta}}{c}\Bigg(&(c-t)^{1-\vartheta}\mathcal{R}_{\vartheta,\varphi,\lambda}(c-t)y^* \\ &+ 2t\frac{d}{dt}\left((c-t)^{1-\vartheta}\mathcal{R}_{\vartheta,\varphi,\lambda}(c-t)y^*\right)\Bigg).\end{aligned}$$

Let $\varepsilon > 0$ be given and $u_1 \in U$. Then by (H_9), there is a $u_2 \in U$ satisfying

$$\big\|y^* - \zeta(\tilde{F}(z_1)) - \zeta(Bu_2)\big\|_V \leq \frac{\varepsilon}{3^2},$$

where $z_1(t) = z(t, u_1)$. Denote $z_2(t) = z(t, u_2)$, again by (H_9) there is a $\omega_2 \in U$ satisfying

$$\big\|\zeta(\tilde{F}(z_2) - \tilde{F}(z_1)) - \zeta(B\omega_2)\big\|_V \leq \frac{\varepsilon}{3^3}$$

and

$$\begin{aligned}\|B\omega_2\|_Z &\leq b\big\|\tilde{F}(z_2) - \tilde{F}(z_1)\big\|_Z \\ &= b\left(\int_0^c \big\|\tilde{F}(z_2)(t) - \tilde{F}(z_1)(t)\big\|_V^p\, dt\right)^{\frac{1}{p}}\end{aligned}$$

$$= b\Bigg(\int_0^c \bigg\| F\left(t, z_2(t), \int_0^t \Theta(t,\varrho,z_2(\varrho))\,d\varrho\right) - F\left(t, z_1(t), \int_0^t \Theta(t,\varrho,z_1(\varrho))\,d\varrho\right)\bigg\|_V^p dt\Bigg)^{\frac{1}{p}}$$

$$\leq b\widehat{k}_F\Bigg(\int_0^c \bigg(t^{1-\vartheta}\|z_2(t)-z_1(t)\| + \widehat{k}_\Theta t^{1-\vartheta}\int_0^t \varrho^{1-\vartheta}\|z_2(\varrho)-z_1(\varrho)\|\,d\varrho\bigg)^p dt\Bigg)^{\frac{1}{p}}$$

$$\leq b\widehat{k}_F\left(\int_0^c \left(1+\widehat{k}_\Theta c^{2-\vartheta}\right)^p dt\right)^{\frac{1}{p}}\|z_2-z_1\|_{C_{1-\vartheta}}$$

$$= b\widehat{k}_F c^{\frac{1}{p}}\left(1+\widehat{k}_\Theta c^{2-\vartheta}\right)\|z_2-z_1\|_{C_{1-\vartheta}}$$

$$\leq b\widehat{k}_F c^{\frac{1}{p}}\left(1+\widehat{k}_\Theta c^{2-\vartheta}\right)\rho_2 E_\vartheta\big(k_{\mathcal{R}}\widehat{k}_F c\Gamma(\vartheta)\big)\|Bu_1-Bu_2\|_Z$$

$$= \frac{k_{\mathcal{R}}\widehat{k}_F bc\left(1+\widehat{k}_\Theta c^{2-\vartheta}\right)\left(\frac{p-1}{p\vartheta-1}\right)^{\frac{p-1}{p}} E_\vartheta\big(k_{\mathcal{R}}\widehat{k}_F c\Gamma(\vartheta)\big)}{1-k_{\mathcal{R}}\widehat{k}_F\widehat{k}_\Theta c^{3-\vartheta}\vartheta^{-1}E_\vartheta\big(k_{\mathcal{R}}\widehat{k}_F c\Gamma(\vartheta)\big)}\|Bu_1-Bu_2\|_Z.$$

Now, if we define

$$u_3(t) = u_2(t) - \omega_2(t), \quad u_3 \in U,$$

then

$$\begin{aligned}\left\|y^* - \zeta(\tilde{F}(z_2)) - \zeta(Bu_3)\right\|_V &\leq \left\|y^* - \zeta(\tilde{F}(z_1)) - \zeta(Bu_2)\right\|_V \\ &\quad + \left\|\zeta(\tilde{F}(z_2) - \tilde{F}(z_1)) - \zeta(B\omega_2)\right\|_V \\ &\leq \left(\frac{1}{3^2} + \frac{1}{3^3}\right)\varepsilon.\end{aligned}$$

Applying induction, one can obtain a sequence $\{u_n\}$ in U such that

$$\left\|y^* - \zeta(\tilde{F}(z_n)) - \zeta(Bu_{n+1})\right\|_V \leq \left(\frac{1}{3^2} + \frac{1}{3^3} + \cdots + \frac{1}{3^{n+1}}\right)\varepsilon,$$

where $z_n(t) = z(t, u_n)$, and

$$\|Bu_{n+1} - Bu_n\|_Z \leq \frac{k_{\mathcal{R}}\widehat{k}_F bc\left(1+\widehat{k}_\Theta c^{2-\vartheta}\right)\left(\frac{p-1}{p\vartheta-1}\right)^{\frac{p-1}{p}} E_\vartheta\big(k_{\mathcal{R}}\widehat{k}_F c\Gamma(\vartheta)\big)}{1-k_{\mathcal{R}}\widehat{k}_F\widehat{k}_\Theta c^{3-\vartheta}\vartheta^{-1}E_\vartheta\big(k_{\mathcal{R}}\widehat{k}_F c\Gamma(\vartheta)\big)}\|Bu_n - Bu_{n-1}\|_Z,$$

which shows that $\{Bu_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in Z. Since Z is a Banach space and ζ is continuous therefore the sequence $\{\zeta(Bu_n)\}_{n\in\mathbb{N}}$ is Cauchy in V. Thus, one can find a positive integer n_0 satisfying

$$\left\|\zeta(Bu_{n_0+1}) - \zeta(Bu_{n_0})\right\|_V \leq \frac{\varepsilon}{3}.$$

Now,

$$\begin{aligned}
\left\|y^* - \zeta(\tilde{F}(z_{n_0})) - \zeta(Bu_{n_0})\right\|_V &\leq \left\|y^* - \zeta(\tilde{F}(z_{n_0})) - \zeta(Bu_{n_0+1})\right\|_V \\
&\quad + \left\|\zeta(Bu_{n_0+1}) - \zeta(Bu_{n_0})\right\|_V \\
&\leq \left(\frac{1}{3^2} + \frac{1}{3^3} + \cdots + \frac{1}{3^{n_0+1}}\right)\varepsilon + \frac{\varepsilon}{3} \\
&< \varepsilon.
\end{aligned}$$

This proves the theorem. ■

Remark 7.3.5. *If $k_{\mathcal{R}}\widehat{k}_F\widehat{k}_\Theta c^{3-\vartheta}\vartheta^{-1}E_\vartheta\big(k_{\mathcal{R}}\widehat{k}_F c\Gamma(\vartheta)\big) \geq 1$, then the inequality (7.3.1) of Lemma 7.3.3, which is used in the proof of above theorem, doesn't make sense.*

Corollary 7.3.6. *Under hypotheses (H_1) and (H_3)-(H_7), the system (7.1.1) is approximately controllable if Range(B) is dense in Z.*

Proof. Let $\varepsilon > 0$ be given. Since *Range*(B) is dense in Z therefore for any $\varepsilon' > 0$ and a nonzero function $g \in Z$, one can find a control $u \in U$ satisfying

$$\|g - Bu\|_Z \leq \varepsilon'\|g\|_Z.$$

Now,

$$\begin{aligned}
\|\zeta g - \zeta(Bu)\|_V &\leq k_{\mathcal{R}}\int_0^c (c-s)^{\vartheta-1}\|g(s) - Bu(s)\|_V\, ds \\
&\leq k_{\mathcal{R}}c^{\vartheta-\frac{1}{p}}\left(\frac{p-1}{p\vartheta-1}\right)^{\frac{p-1}{p}}\|g - Bu\|_Z \\
&\leq k_{\mathcal{R}}c^{\vartheta-\frac{1}{p}}\left(\frac{p-1}{p\vartheta-1}\right)^{\frac{p-1}{p}}\varepsilon'\|g\|_Z \\
&\leq \varepsilon.
\end{aligned}$$

Thus

$$\|Bu\|_Z \leq \|Bu - g\|_Z + \|g\|_Z$$

$$
\begin{aligned}
&\leq \varepsilon' \|g\|_Z + \|g\|_Z \\
&= (\varepsilon' + 1)\|g\|_Z.
\end{aligned}
$$

Hence the condition (H_9) is satisfied, if we choose ε' in such a way that (H_8) and (H_{10}) are satisfied. Then by previous theorem, the system (7.1.1) is approximately controllable. ■

7.4 Example

Consider the following initial-boundary value problem with Riemann-Liouville derivative for $x \in [0, \pi]$:

$$
\begin{cases}
D_t^{\vartheta} z(t,x) + \lambda D_t^{\varphi} z(t,x) \\
\quad = \kappa^2 \frac{\partial^2}{\partial x^2} z(t,x) + u(t,x) + F\left(t, z(t,x), \int_0^t \Theta(t,s,z(s,x))\,ds\right) \\
\qquad\qquad\qquad\qquad\qquad\qquad \text{for} \quad t \in (0,1], \\
\left(I_t^{1-\vartheta} z(t,x)\right)_{t=0} = y_0(x), \\
z(t,0) = 0 = z(t,1) \qquad\qquad\qquad \text{for} \quad t \in (0,1].
\end{cases}
\tag{7.4.1}
$$

Take $V = V' = L_2[0,1]$, $B = \mathcal{I}$, the identity map and $A : D(A) \subset V \to V$ is defined as

$$
Ay = \kappa^2 y'',
$$

where

$$
D(A) = \left\{ y \in W^{2,2}(0,1) \,\middle|\, y(0) = 0 = y(1) \right\}.
$$

Then, A generates a Riemann-Liouville fractional $(\vartheta, \varphi, \lambda)$ resolvent $\mathcal{R}_{\vartheta,\varphi,\lambda}(t)$ given by

$$
(\mathcal{R}_{\vartheta,\varphi,\lambda}(t)y)(x) = \sum_{n=1}^{\infty} \left(\sum_{\ell=0}^{\infty} \left(-\kappa^2 n^2 \pi^2\right)^{\ell} t^{\vartheta(\ell+1)-1} E_{\vartheta-\varphi,\vartheta(\ell+1)} \left(-\lambda t^{\vartheta-\varphi}\right) \right) \kappa_n \sin(n\pi x),
$$

where $\xi_n(x) = \sin(n\pi x)$ are eigen functions of A for the eigenvalues $\lambda_n = -\kappa^2 n^2 \pi^2$, $n = 1, 2, \ldots$ and $y(x) = \sum_{n=1}^{\infty} \kappa_n \sin(n\pi x)$ (see [68]). It is easily seen that (H_1) is satisfied.

If we take

$$
\tilde{z}(t,x) = \int_0^t \Theta(t,s,z(s,x))\,ds
$$

and

$$F\left(t, z(t,x), \tilde{z}(t,x)\right) = F\left(t, z(t,x), \int_0^t \Theta(t,s,z(s,x))\,ds\right)$$
$$= (1+t^2) + k_0 t^{a_0}\left(z(t,x) + \int_0^t \Theta(t,s,z(s,x))\,ds\right),$$

where

$$\Theta(t,s,z(s,x)) = k_1\left(t^2+s^2\right)s^{a_1}\cos(ts)\sin(z(s,x)).$$

and $a_\ell \geq 1-\vartheta,\ \ell = 0,1$. Then (H_2), (H_3) and (H_6) are satisfied with $k_F = k'_F = \widehat{k}_F = |k_0|$, because

$$\begin{aligned}\|F(t,y_1,\tilde{y}_1) - F(t,y_2,\tilde{y}_2)\| &\leq |k_0|t^{a_0}(\|y_1-y_2\| + \|\tilde{y}_1-\tilde{y}_2\|)\\ &\leq |k_0|t^{a_0+\vartheta-1}t^{1-\vartheta}(\|y_1-y_2\| + \|\tilde{y}_1-\tilde{y}_2\|)\\ &\leq |k_0|t^{1-\vartheta}(\|y_1-y_2\| + \|\tilde{y}_1-\tilde{y}_2\|)\\ &\leq |k_0|(\|y_1-y_2\| + \|\tilde{y}_1-\tilde{y}_2\|)\end{aligned}$$

and

$$\begin{aligned}\|F(t,y,\tilde{y})\| &\leq (1+t^2) + |k_0|t^{a_0+\vartheta-1}t^{1-\vartheta}(\|y\| + \|\tilde{y}\|)\\ &\leq (1+t^2) + |k_0|t^{1-\vartheta}(\|y\| + \|\tilde{y}\|).\end{aligned}$$

Also, the conditions (H_4) and (H_7) are satisfied with $k_\Theta = \widehat{k}_\Theta = 2|k_1|$, because

$$\begin{aligned}\|\Theta(t,s,y_1) - \Theta(t,s,y_2)\| &\leq |k_1|(t^2+s^2)s^{a_1}|\cos(ts)|\|\sin(y_1) - \sin(y_2)\|\\ &\leq |k_1|(t^2+s^2)s^{1-\vartheta}\|y_1-y_2\|\\ &\leq 2|k_1|s^{1-\vartheta}(\|y_1-y_2\|)\\ &\leq 2|k_1|\|y_1-y_2\|.\end{aligned}$$

Now,

$$\begin{aligned}\|\Theta(t,s,z(s,x))\| &\leq |k_1|\left(1+s^2\right)s^{a_1}\\ &= g(s).\end{aligned}$$

Hence (H_5) is satisfied. If we choose the constants k_0 and k_1 sufficiently closed to zero so that (H_8) and (H_{10}) are satisfied, then approximate controllability of the system (7.4.1) follows from Theorem 7.3.4.

7.5 Concluding remarks

In this chapter, existence and approximate controllability results for semilinear fractional integrodifferential systems with damping have been presented. To prove our results, the concept of Riemann-Liouville fractional $(\vartheta, \varphi, \lambda)$ resolvent has been used. Using fixed point approach, the results of existence and uniqueness have been derived. For this, we derived the definition of mild solution in terms of Riemann-Liouville fractional $(\vartheta, \varphi, \lambda)$ resolvent $\mathcal{R}_{\vartheta,\varphi,\lambda}$. Controllability result has been derived using sequence method. For this, the Lamma 7.3.3 has been proved. The study of such systems covers a broad area of applications. However, from physical viewpoint, it is more appropriate to study the higher order fractional damped systems. For this reason, we are committed to analyzing the existence and controllability for Riemann-Liouville fractional integrodifferential systems of the form

$$\begin{cases} D_t^{\vartheta} z(t) + \lambda D_t^{\varphi} z(t) = Az(t) + Bu(t) + F\left(t, z(t), \int_0^t \Theta(t, s, z(s))\, ds\right), \quad t \in (0, c], \\ \left(I_t^{2-\vartheta} z(t)\right)_{t=0} = y_0 \in V, \\ \left(D_t^{\vartheta-1} z(t)\right)_{t=0} = y_1 \in V, \end{cases}$$

with $0 \leq \varphi \leq 1 < \vartheta \leq 2,$ in future.

Chapter 8

Conclusion and Future Directions

In this thesis, the approximate controllability of semilinear control systems of various types with local and nonlocal conditions have been investigated. Particularly, the retarded systems of integer order with fixed point delays in control, and the fractional systems of order $\vartheta \in (0, 1]$ and order $\vartheta \in (1, 2]$ have been considered. Main assumptions made on nonlinear operator are continuity, locally Lipschitz continuity and Lipschitz continuity. Various inequality conditions have been obtained on the systems constants. For obtaining the main results, we have used fixed point theory, sequential approach along with Gronwall's inequality. Moreover, a weaker notion of approximate controllability, namely partial approximate controllability for fractional systems has been discussed.

Some possible directions, in which the obtained results can be extended, are described below:

- Utilizing the techniques and ideas of Chapter 3 and Chapter 4, one can determine the controllability of nonautonomous systems or systems with non instantaneous impulses assuming the continuity of the nonlinear operator rather than assuming the locally Lipschitz continuity.

- The results of Chapter 5 can be extended for the partial controllability of fractional systems of order $\vartheta \in (1, 2]$ with Riemann-Liouville or Caputo derivatives. Also, these results can be extended for the retarded systems or systems with control delays.

- The results of Chapter 6 and Chapter 7 can be extended for the controllability of nonlocal fractional systems. Also, one can drop the assumption of Lipschitz continuity and investigate the controllability of the same systems considered in these chapters. If nonlinear operator is not Lipschitz, then even existence of mild solution is the matter of investigation. Further, one can investigate the partial controllabiliy of the semilinear damped systems.

- There is not much research work on trajectory controllability, null controllability, constrained controllability and complete controllability for Riemann-Liouville fractional systems. Thus, there is a lot of scope in this area of research. Also, this research work can be extended for higher order (bigger than two) systems.

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