A STUDY ON EXISTENCE OF SOLUTION AND CONTROLLABILITY OF DELAY DIFFERENTIAL SYSTEMS

A THESIS

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by

SANJUKTA DAS

DEPARTMENT OF MATHEMATICS
INDIAN INSTITUTE OF TECHNOLOGY ROORKEE
ROORKEE-247 667 (INDIA)

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INDIAN INSTITUTE OF TECHNOLOGY ROORKEE
ROORKEE

CANDIDATE'S DECLARATION

I hereby certify that the work which is being presented in the thesis entitled "**A STUDY ON EXISTENCE OF SOLUTION AND CONTROLLABILITY OF DELAY DIFFERENTIAL SYSTEMS**" in partial fulfillment of the requirements for the award of the Degree of Doctor of Philosophy and submitted in the Department of Mathematics of the Indian Institute of Technology Roorkee, Roorkee is an authentic record of my own work carried out during the period from July, 2011 to July, 2015 under the supervision of Dr. D. N. Pandey, Assistant Professor and Dr. N. Sukavanam, Professor, Department of Mathematics, Indian Institute of Technology Roorkee, Roorkee, Uttarakhand, India.

The matter presented in this thesis has not been submitted by me for the award of any other degree of this or any other Institute.

Sanjukta Das
(SANJUKTA DAS)

This is to certify that the above statement made by the candidate is correct to the best of our knowledge.

(Dr. D. N. Pandey)
Supervisor

(Dr. N. Sukavanam)
Supervisor

Date: 30 November, 2015

The Ph.D. Viva-Voce Examination of **Mrs. Sanjukta Das,** Research Scholar, has been held on........30.11.2015..........

Chairman, SRC

Signature of External Examiner

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Head of the Department/Chairman, ODC

Abstract

Controllability of distributed parameter systems, essentially of dynamical systems governed by partial differential equations, has evolved into a widely researched topic in less than three decades. Despite generating a distinctive identity and philosophy as a part of the theory of dynamical systems, this research field has played a significant role in the advancement of the extensive theory of partial differential equations.

In last few decades, control theory has contributed enormously to study of realistic problems of elasticity such as thermoelasticity, aeroelasticity, problems depicting interactions between fluids and elastic structures and real world problems of fluid dynamics, to name but a few. Such real world problems present new mathematical challenges. For instance, the mathematical foundations of basic theoretical issues have to be enriched, along with the development of conceptual insights significant to the designers and the practitioners. This poses novel challenges that need to be addressed.

In our present work we focuss on the existence, uniqueness and controllablity of nonlinear functional differential equations. We use theory of semigroup, cosine family, measure of noncompactness and fixed point theorems to obtain the results. The results can be applied to a class of functional differential equations, appearing in the mathematical models of several physical phenomena to which the prototype of partial differential equations modeling the phenomena, belongs.

The layout of the thesis, containing 10 chapters, is as follows.

Chapter 1 is introductory in nature. The delay differential equations and their applications are discussed. The objective of work done, current status of the field and layout of the thesis is also presented in this chapter.

Chapter 2 illustrates some basic properties of semigroup theory, cosine family, measure of noncompactness, controllability, fractional and stochastic differential equations.

In chapter 3 we study a functional differential equation with deviating argument and finite delay to establish that it is approximately controllable.
The results of this chapter are published as 'Approximate Controllability of a Functional Differential Equation with Deviated Argument' in Nonlinear Dynamics and Systems Theory, Infor Math, volume 14, no. 3, (2014), 265-277.

In chapter 4 existence of mild solution of a second order partial neutral differential equation with state dependent delay and non-instantaneous impulses is investigated. We use Hausdorff measure of noncompactness and Darbo Sadovskii fixed point theorem to prove the existence.
The results of this chapter are published as 'Existence of Solution for a Second-Order Neutral Differential Equation with State Dependent Delay and Non-instantaneous Impulses' in International Journal of Nonlinear Science, World Scientific, volume 18, no.2, (2014), 145-155.

Chapter 5 consists of two parts. The first part deals with the existence of mild solution of an instantaneous impulsive second order differential equation with state dependent delay. In second part non-instantaneous impulsive conditions are studied. We introduce new non-instantaneous impulses with fixed delays.
The results of this chapter are in revision as 'Existence of Solution of Impulsive Second-Order Neutral Integro-Differential Equation with State Delay' in Journal of Integral Equations and Applications.

In chapter 6 we establish the existence and uniqueness of mild solution and the approximate controllability of a second order neutral partial differential equation with state dependent delay. The conditions for approximate controllability are investigated for the distributed second order neutral differential system with respect to the approximate controllability of the corresponding linear system in a Hilbert space.
The results of this chapter are published as 'Approximate Controllability of a Second Order Neutral Differential Equation with State Dependent Delay' in Differential Equations and Dynamical Systems, Springer, DOI $10.1007/s12591-014-0218-6, (2014)$.

Chapter 7 is divided in two parts. In the first part we study a second order neutral differential equation with state dependent delay and non-instantaneous impulses. The existence and uniqueness of the mild solution are investigated via Hausdorff measure of non-compactness and Darbo Sadovskii fixed point theorem. In the second part the conditions for approximate controllability are investigated for the neutral second order system under the assumption that the corresponding linear system is approximately controllable. A simple range condition is used to prove

approximate controllability.
The results of this chapter are published as 'Existence of Solution and Approximate Controllability for Neutral Differential Equation with State Dependent Delay' in International Journal of Partial Differential Equations, Hindawi, volume 2014 (2014), Article ID 787092, 12 pages.

In chapter 8 we study a fractional neutral differential equation with deviating argument to establish the existence and uniqueness of mild solution. The approximate controllability of a class of fractional neutral differential equation with deviating argument is discussed by assuming a simple range condition.
The results of this chapter are published as 'Approximate Controllability of a Fractional Neutral System with Deviated Argument in Banach Space' in Differential Equations and Dynamical Systems, Springer, $DOI: 10.1007/s12591-015-0237-y$, (2015).

In chapter 9 the approximate controllability of an impulsive fractional stochastic neutral integro-differential equation with deviating argument and infinite delay is studied. The control parameter is also included inside the nonlinear term. Only Schauder fixed point theorem and a few fundamental hypotheses are used to prove our result.
The results of this chapter are published as 'Approximate controllability of an impulsive neutral fractional stochastic differential equation with deviated argument and infinite delay' in Nonlinear Studies, volume 22, no. 1, 1-16, (2015), CSP - Cambridge, UK; I&S - Florida, USA.

In chapter 10 the existence, uniqueness and convergence of approximate solutions of a stochastic fractional differential equation with deviating argument is established. Analytic semigroup theory is used along with fixed point approach. Then we investigate Faedo-Galerkin approximation of solution and establish some convergence results.
The results of this chapter are accepted for publication as 'Approximations of Solutions of a Fractional Stochastic Differential Equations with Deviated Argument' in Journal of Fractional Calculus and Applications in 2015.

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There is nothing in this world so purifying as knowledge. Here no effort undertaken is lost, no disaster befalls. Even a little of this righteous course delivers one from all evils.

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Sanjukta Das

Sanjukta Das

Table of Contents

Chapter 1

Introduction

Delay differential equations reflect dead-time in hereditary systems or aftereffect in systems with mathematical models containing deviated arguments and differential-difference equations. Delay differential equations are a class of functional differential equations (FDEs). A functional differential equation is a differential equation in which the derivative $y'(t)$ of an unknown function y has a value at time t that is related to y as a function of some other function at time t. A general first-order functional differential equation can be represented as $y'(t) = f(t, y(t), y(h(t)))$, where f and h are some suitable functions. FDEs are infinite-dimensional in contrast to ordinary differential equations (ODEs), which are finite dimensional. The state may be defined as a n-vector $x(t)$ in the Euclidean space $\mathbb{R}^n$ in systems modeled by ODEs. Deviated time-argument attempts to capture hereditary properties. So, in FDEs the state can no longer be represented by a vector $x(t)$ at discrete time t. Then the state may be represented as a history valued function x_t corresponding to a dead time interval. Unlike ODEs, in case of FDEs originally different solutions may coincide after some time, unless the uniqueness of the backward continuation is guaranteed by atomicity property.

In recent years detailed study of parabolic and hyperbolic partial differential equations (PDEs) is done on account of various engineering applications. Such PDEs arise in the study of several dynamical systems like meteorological models, reaction-diffusion or convection-diffusion systems, flame propagation, superconductivity, air pollution etc. As these type of dynamical systems are highly complex, parallel methods play a significant role. The parabolic and hyperbolic (PDEs) can be reformulated as abstract ordinary differential equations. Thereby, semigroup theory is used. We refer [19],[20],[35],[67],[72], [74],[77],[82],[86],[124],[135],[154] for details and applications of semigroup theory .

Neutral differential systems are delay systems, involving highest order derivative of both the unknown quantity and its delayed or deviated part. Generally, initial boundary value problems undergo investigation by reformulation into initial value problems in abstract spaces. Such abstract formulations are generally written as nonlinear functional differential equations. The initial conditions occur as essential conditions. The boundary conditions are included in the domain of the operator and thus appear as natural conditions. Thus certain invariant properties of a prototype of problems can be studied in constrast to study of any particular PDE. Neutral differential equations with unbounded delay appear abundantly as models in problems of mechanical engineering, mathematical biology, electrical systems etc. Hence it is a widely studied topic in several papers and monographs for instance, partial neutral differential equation with infinite delay arise in the study of conduction of heat in substances exhibiting fading memory, (see [83]). For allied applications and more details on neutral differential systems, one may see [55],[62],[64],[78],[81],[92],[95],[96],[134]. Second order neutral differential systems often model variational problems in calculus of variation. Some second order neutral differential systems represent the dynamics of masses exhibiting vibrations, on being connected to an electric bar. For more details related to applications of second order neutral differential equations we refer [54],[105],[118].

Impulsive differential equations appear in systems involving stimulus or in the simulation of any suddenly perturbed process. Discrete impulses are very small compared to the whole time span of the process. We refer [43],[50],[52], [61],[63],[109], [117],[125],[144],[148],[159] regarding discrete impulses.

In contrast to integer-order, fractional-order derivatives efficiently incorporate the hereditary properties of various materials with memory. Practical use of controllers of fractional-order occur in servo systems for controlling hard disk drives, milling of cement, reduction of chaos in electrical circuits, electronic converters for controlling power, composite hydraulic cylinders, irrigation canals, etc. Fractional-order controllers outperforms the traditional controllers by effectively modeling and investigating real-world processes. Use of fractional-order dynamics enhances the precision in modeling the systems. Fractional differential equation occur often in the study of fractals. Some problems of viscoelasticity can also be modeled by fractional differential equation. They also model problems in seismology. Many partial differential systems can be reduced to functional differential equations with deviated arguments, see for instance [73],[88],[115],[145].

Methods based on semigroup theory are quite efficient in the study of infinite-dimensional control theory, population dynamics, quantum mechanics and transport theory. With the advent of new functional-analytic results semigroup theory is increasingly used as an alternative to other validated methods. In the context of complex dynamical systems, limitations to semigroups, in particular, strongly continuous semigroups, are prevalent. This naturally gives rise to cosine families, integrated semigroups, resolvent families etc. The concept of cosine family is quite similar to of semigroup theory just as the fundamental theorem of Sova-DaPrato-Giusti is parallel to the Hille- Yosida-Feller-Phillips theorem. The above two theorems on the generation of cosine families and semigroups find a common origin in the Henning-Neubrander representation theorem. Moreover, the classical form of the Trotter-Kato-Neveu theorem on the convergence of semigroups applies to cosine families with few modifications. Although, despite these similarities, cosine families and semigroups are fundamentally different.

Random noise gives rise to fluctuations in deterministic models. Stochastic problems are more efficient than deterministic ones since they effectively assimilate the randomness of the system. Results of controllability for abstract systems are abundantly available in literature (see for details [91],[111],[129],[140],[142],[166],[169] and references therein), in comparison to fractional stochastic differential systems. We refer [2],[126],[147],[152],[153],[172],[174] for the study of stochastic differential equations.

1.1 Motivation of Thesis

Reformulations to abstract forms are possible for a large number of PDEs. The abstract formulation allows the study of a class of problems rather than just any individual problem. Thus our results can be applied to the whole class consisting of those prototype of problems.

Controllability of nonlinear dynamical systems involving deviating argument had scarcely been studied in literature. Moreover state dependent delay and non-instantaneous impulses are lately introduced in this century to study various real world phenomena. The main objective of this thesis is to provide simple sufficient conditions for the existence, uniqueness, exact or approximate controllability of first, second and fractional order delay differential systems involving deviating argument or impulsive conditions. The state may also depict a required future goal apart from representing any action of the past.

In contrast to ODEs, the controllability of FDEs differ in three fundamental ways:

(1) In the case of functional models, controllability means to attain a function (the vector $x(t)$ from time t_1 to time $t_1 + h$) in contrast to ODEs, where controllability implies reaching a point at a time t_1.

(2) Starting at time t_1, in the case of linear systems with no delays, any point which can be attained at time $t_2 > t_1$ can also be attained at time $t_1+a(t_2-t_1)$, $a > 0$. Whereas, delay differential equations are entitled to the existence of a required, minimum reaching time. Thus, special kind of indices like class of system, delineating the number of units of delays required for attaining the target must be added besides the usual controllability indices corresponding to reachable spaces.

(3) The realization and type of the control law is different. The expression of the state-feedback is $u(t) = \zeta(x_t)$, implying the infinite dimension of the controller. In case of memoryless controls, control law is represented as $u(t) = \zeta(x(t))$. Whereas in point-wise delayed controls, control law is represented as $u(t) = \zeta(x(t); x(t - h_i))$. Here ζ is some appropriate function.

These differences motivate us to study the controllability of delay systems. In [127] the authors studied neutral functional differential equation of the form

$$\begin{aligned} \frac{d}{dt}[x(t)+g(t,x(t))] &= Ax(t)+(Bu)(t)+f(t,x(t)), \\ x(0) &= x_0, \quad t\in[0,T] \end{aligned} \tag{1.1.1}$$

Motivated by [127] we try to extend the problem to second order, fractional order and stochastic case. We also study the effect of non-instantaneous impulses and state dependent delay along with infinite delay. We also study the case where the control term is included inside the nonlinear term.

Hernandez [93] studied new class of non-instantaneous impulses in FDEs. We attempt to study another new class of non-instantaneous impulses in neutral fractional stochastic differential equations.

C.G. Gal [79] investigated an abstract differential equation involving deviating argument. Specifically the local and global solutions were investigated. He established the existence and uniqueness of such solutions.

Pandey et al. [145] investigated a neutral differential equation coupled with a deviating argument. Analytic semigroup theory was used along with fixed point approach to establish the existence and uniqueness of mild solution. The use of compact and fractional operators, analytic semigroups are prevalent in the investigation of such systems. Such strict conditions on the operators restrict their applicability.

Benchohra et al. [52] and Chang [61] discussed the exact controllability of functional systems with impulsive conditions and unbounded delay. However, they assumed that the inverse of a controllability operator exists. Generally due to the compactness of the generated semigroup it is not invertible. Hence their methodology does not work in infinite-dimensional cases. Moreover it is hardly possible to apply and check their condition in real world complex systems.

With a different approach Zhou [176] established approximate controllability of an abstract semilinear differential system. Naito [139] proved that the semilinear problem in [176] is the approximately controllable, if a range condition on the control operator is satisfied. Sakthivel et.al. [152] proposed viable results for both stochastic and deterministic system to be approximate controllable.

Interestingly controllability results for functional differential equation with deviated argument coupled with impulsive conditions are not widely available so far. In an effort to cover this void, we attempt to investigate remote control systems where values of space variable are dependent on some remote space, by using simple functions of deviating argument.

1.2 Review of literature

1.2.1 Existence of solution

The literature related to functional differential equations is very extensive. [86] contains a comprehensive description of such equations. Similarly, for additional material concerning abstract partial functional differential equations and related issues, we refer [1],[21],[27],[30],[90],[102]. For literature related to unbounded delay we refer [92],[94],[95],[101],[108], and for state dependent delay we refer [89],[97],[99],[100]. For details in fixed point theory and inequalities we refer [3],[4],[5],[7]. For related work in second order functional differential equations we refer [25],[28],[34],[38],[161], [162],[163],[165] and for the case of fractional differential equations we refer [11],[51], [112],[113]. For methods in approximation of solutions we refer [22],[24],[25],[46],[87], [125],[128]. Applications in population dynamics, and vibrational problems, and allied fields are available in [12],[18],[23],[26],[29],[47],[48],[58],[103],[104], [130],[131], [132].

Hernández et.al. (1998) [95] investigated the existence solutions of a partial neutral differential equations with unbounded delay. They proved existence of mild and strong solutions by using strongly continuous semigroup. In (1998) Hernández et.al. [94] also proved a result of existence periodic solutions for the same class of quasi-linear neutral differential equations with unbounded delay.

S. Agarwal and Bahuguna (2005) [8] proved the exact and approximate solutions of a delay system coupled various types of nonlocal history conditions. The authors investigated mild, strong, and classical solutions for existence and uniqueness. They used the method of semidiscretization in time. The authors also proved a result about the global existence of solutions.

S. Agarwal and Bahuguna (2006) [9] studied a nonlocal neutral differential equation. The existence of the solutions in a Banach space was proved by using Schauder's fixed point theorem.

Hernández et.al. (2006) [89] proved the existence of mild solutions for a functional differential equation involving state-dependent delay. Also, Hernández et.al. (2006) [99] proved the existence of mild solutions for the impulsive functional differential equations of similar type.

Hernández et.al. (2007) [98] investigated a neutral differential equation of first and second order with impulses, using fixed point approach. The authors established the existence of mild solutions.

Muslim and Bahuguna (2008) [136] proved the existence and uniqueness of solution of a neutral differential equation involving deviating argument using analytic semigroups theory and fixed point method.

P.Balasubramaniam et.al.(2009) [42] proved the existence, uniqueness and approximate solutions of a stochastic integro-differential equation. The convergence of solutions was proved using Faedo-Galerkin approximations.

Tidke et.al. (2010) [160] proved the existence, uniqueness and other properties of solutions of second order Volterra differential equation using strongly continuous cosine family, a modified version of contraction mapping principle and an integral inequality due to B. G. Pachpatte.

Lizhen Chen et. al. (2010) [66] investigated a second-order neutral differential equation using measure of noncompactness and fixed point theory. The authors established the existence of mild solutions. The compactness condition on cosine family was relaxed in deriving the compactness of solution set.

Aihong Li et.al. (2010) [14] established the existence of mild solutions of an impulsive neutral stochastic integro-differential equation with unbounded delays. They assumed that an analytic resolvent operator is generated by the undelayed part. They reformulated it into an integral equation. Sufficient conditions for the existence of solutions were established by using analytic resolvent operators and Sadovskii fixed point theorem.

Fang Li (2011) [120] investigated s fractional neutral differential equation with infinite delay via Kuratowski's measure of noncompactness. Also the existence of mild solution of some integro-differential equation was proved as a part of application.

V. Vijaykumar et.al. (2012) [167] proved the global existence of solutions for a Volterra-Fredholm kind functional integrodifferential equations with impulsive conditions. Assuming the Leray-Schauders Alternative theorem, they established the global existence of solutions.

Zdzislaw Brzeźniak et. al. (2013) [57] investigated a stochastic NavierStokes equations with a multiplicative Gaussian noise. They considered the equation in 2D and 3D possibly unbounded domains. The unknown velocity and its spatial derivatives determined the noise term. The existence of a martingale solution was established. The solution was derived using the classical Faedo-Galerkin approximation, the Jakubowski version of the Skorokhod theorem. Also, some compactness and tightness criteria in nonmetric spaces were established. The compactness results were established using a generalized version of the classical Dubinsky Theorem.

Shengli Xie (2013) [171] investigated a second-order integro-differential system with unbounded delay and impulsive conditions. The author used the Kuratowski measure of noncompactness along with progressive estimation approach to establish the existence of mild solutions

Sakthivel et.al (2013) [153] established the existence of mild solutions of an impulsive fractional stochastic differential equation involving unbounded delay. The authors used fractional calculus, stochastic analysis, fixed point methods and techniques adopted directly from deterministic fractional equations. Sufficient conditions were derived for the existence of mild solutions. Moreover, the existence of solutions for fractional stochastic semilinear differential equations involving nonlocal conditions was established.

Shengli Xie (2014) [170] studied the existence and uniqueness of mild solutions for an impulsive fractional integro-differential evolution equation with unbounded delay. The author generalized the existence theorem for integer order differential equations to the case of fractional order.

Jankowski (2014) [106] considered boundary fractional differential problems with advanced arguments. The existence of initial value problems was established with the initial point defined at the end point of an interval. Moreover, nonhomogeneous linear fractional differential equations were investigated. The existence of solutions for fractional differential equations involving advanced arguments and boundary values was proved with the help of a monotone iterative technique. The corresponding fractional inequalities were also studied.

Zhang et. al. (2014) [173] considered impulsive differential equations with fractional-order $0 < q < 1$. They proved the formula of solutions used in their cited papers to be incorrect. The authors formulated and proved a formula for the general solution of an impulsive Cauchy problem with Caputo fractional derivative of order lying between 0 and 1. Moreover, the authors established an existence result for a type of impulsive fractional differential system with special initial value with the help of fixed point methods.

1.2.2 Controllability

We refer [10],[16],[40],[41],[43],[56],[71],[80],[116] for literature on controllability and related topics

K. Naito (1987) [139] proved the approximate controllability of an abstract semilinear control system The author assumed a relation, that has a simple form. Moreover that can be easily verified in many examples.

Mohan C. Joshi and Raju K. George (1989) [110] established global controllability of a semilinear system with both Lipschitzian and non-Lipschitzian types of monotone nonlinearities.

Nandakumaran et. al. (1995) [143] obtained the partial exact controllability for a nonlinear system. The authors used semigroup formulation along with fixed point approach to investigate the nonlinear system.

Dauer et. al. (2002) [69] investigated the approximate and complete controllability for semilinear functional differential systems. Sufficient conditions were formulated and proved for each of the two types of controllability. The authors removed the limitation that linear systems with compact semigroup cannot be completely controllable in infinite-dimensional spaces. The conditions were derived with aid of the Schauder fixed point theorem in case of compact semigroup and the Banach fixed point theorem in case of noncompactness of semigroup.

Mahmudov et. al. (2003) [127] investigated a semilinear neutral system to check its approximate controllability. The authors used the Schauder fixed point theorem and some fundamental assumptions on the systems operator. The approximate controllability of the semilinear system followed from the approximate controllability of its linear part.

Jin-Mun Jeong et.al. (2007) [107] established the approximate controllability for the semilinear retarded control system. The authors also derived the equivalent relation between controllability and stabilizability of the solution for the corresponding linear control system.

Meili Li et.al. (2007) [122] dealt with the controllability of abstract neutral functional integro-differential systems with infinite delay. The authors used fractional power of operators and Sadovskii fixed point theorem to prove the results.

Sakthivel et.al. (2007) [149] established the approximate controllability of a nonlinear impulsive differential and neutral functional differential equation in Hilbert space. The authors used semigroup theory and fixed point approach. For impulsive differential and neutral functional differential equations, the authors derived sufficient conditions for approximate controllability.

R.K. George, A.K. Nandakumaran, and D.N. Chalishajar (2007) [141] dealt with a nonlinear dispersion system. The authors established exact controllability of the system. The two kinds of nonlinearities, such as Lipschitzian and monotone were used. The exact controllability of the above system with the aid of Integral Contractors, was established. The advantage being the use of Integral Contractors as a weaker condition than the condition of Lipschitz.

Sakthivel et.al. (2009) [150] proved the exact controllability of second order nonlinear impulsive control systems, of certain types. The authors derived sufficient conditions for the exact controllability of those type of systems.

Darwish et. al. (2009) [68] dealt with the existence of controllability available in literature. They established the fact the trivial modification of those available results in literature can lead to the control of infinite dimensional systems. The authors used the complete continuity of the nonlinearity instead of the compactness of operators.

P. Muthukumar and P. Balasubramaniam (2010) [138] formulated and proved the sufficient conditions for the approximate controllability of McKeanVlasov type of second order nonlinear stochastic differential equation. At a given time t the nonlinearities depended on the state of the of the system as well as on the probability distribution at that time.

Yong-Kui Chang et. al. (2010) [65] dealt with the global uniqueness of solutions and controllability of a stochastic integro-differential equation in Fréchet spaces. The authors used the resolvent operators along with a nonlinear alternative of Leray-Schauder type theorem in Fréchet spaces due to Frigon and Granas.

Sukavanam et. al (2010) [158] established some sufficient conditions, for S-controllability of a first order abstract semilinear stochastic control system. The results were derived by the approach of separation principle.

Sukavanam et. al. (2011) [157] established the approximate controllability of a fractional semilinear delay control systems by assuming that the approximate controllability of the linear system. The existence and uniqueness of the mild solution was also investigated.

Surendra Kumar et. al. (2012) [116] proved sufficient conditions of approximate controllability of a fractional semilinear delay control systems. The authors also proved the existence and uniqueness of mild solution of the system. They used contraction principle and the Schauder fixed point theorem. Some examples were provided as well.

Sakthivel et.al (2012) [151] dealt with a type of control systems represented by abstract nonlinear fractional differential neutral equations. The authors established exact controllability for the fractional differential control systems. The authors formulated and proved sufficient conditions of the controllability of the nonlinear fractional systems. The main tool was fixed point analysis. Further, investigation of controllability for systems with nonlocal conditions was done. The authors proved the controllability of nonlinear systems by assuming the exact controllability of the associated linear control systems.

Muslim et. al. (2013) [137] focussed on a control system described by neutral differential equation involving deviating argument. The authors used semigroups of linear operators along with Banach fixed point theorem. The authors established the complete controllability of the deviated system. Further a nonlocal system was investigated by as an extension of the proved results.

K. Balachandran et.al. (2014) [36] considered of nonlinear fractional integrodifferential systems involving implicit fractional derivative. Sufficient conditions for controllability were formulated and proved. The authors used measure of noncompactness together with Darbo's fixed point theorem.

K. Balachandran et.al. (2015) [37] investigated the controllability of nonlinear neutral fractional Volterra integrodifferential systems involving implicit fractional derivatives. These types of systems were based on a number of problems involving complex media. The authors derived sufficient conditions for controllability. The main technique was based on condensing map and measure of noncompactness.

1.3 Organization of Thesis

This thesis contains 10 chapters.

Chapter 1 is introduction.

Chapter 2 gives an introduction to basics of semigroup theory, cosine family, control theory, measure of noncompact-ness, fractional and stochastic differential equations.

In chapter 3 we study the approximate controllability of a functional differential equation with deviating argument and finite delay. Sufficient condition for approximate controllability is provided by assuming that the linear control system is approximately controllable. Schauder fixed point theorem is used and the C_0 semigroup associated with mild solution has been replaced by fundamental solution. The results of this chapter are published as 'Approximate Controllability of a Functional Differential Equation with Deviated Argument' in Nonlinear Dynamics and Systems Theory, Infor Math, volume 14, no. 3, (2014), 265-277.

In chapter 4 the existence of mild solution of a class of second order partial neutral differential equation with state dependent delay and non-instantaneous impulses is investigated. Hausdorff measure of noncompactness is used. Darbo Sadovskii fixed point theorem is applied to prove the existence. Also, some restrictive conditions such as the compactness of the associated cosine or sine operators and the Lipschitz conditions on the nonlinear functions are replaced by simple and natural assumptions. In the last section an example is studied to illustrate the presented result.

The results of this chapter are published as 'Existence of Solution for a Second-Order Neutral Differential Equation with State Dependent Delay and Non-instantaneous Impulses' in International Journal of Nonlinear Science, World Scientific, volume 18, no.2, (2014), 145-155.

Chapter 5 consists of two parts. The first part deals with the existence of mild solution of a class of instantaneous impulsive second order partial neutral differential equation with state dependent delay. The second part studies the non-instantaneous impulsive conditions. We use Kuratowski measure of noncompactness. To establish the existence of mild solution Monch fixed point theorem is applied. We remove the restrictive conditions on the priori estimation available in literature. The compactness of cosine or sine operators, nonlinear terms and associated impulses are also not required in this chapter. The noncompactness measure estimation, the Lipschitz conditions, and compactness of the nonlinear functions are replaced by simple and natural assumptions. We introduce new non-instantaneous impulses with fixed delays. In the last section we study examples to illustrate the presented result.
The results of this chapter are in revision as 'Existence of Solution of Impulsive Second-Order Neutral Integro-Differential Equation with State Delay' in Journal of Integral Equations and Applications.

In chapter 6 we establish the existence and uniqueness of mild solution and approximate controllability of a second order neutral partial differential equation involving state dependent delay. The existence of mild solution is derived with the help of Hausdorff measure of noncompactness and Darbo Sadovskii theorem. Some restrictive conditions such as the compactness of cosine or sine family and the Lipschitz conditions on the nonlinear functions are replaced by simple and natural assumptions. The conditions for approximate controllability are investigated for the distributed second order neutral system by assuming that the corresponding linear system is the approximately controllable.
The results of this chapter are published as 'Approximate Controllability of a Second Order Neutral Differential Equation with State Dependent Delay' in Differential Equations and Dynamical Systems, Springer, DOI $10.1007/s12591-014-0218-6, (2014)$.

Chapter 7 is divided in two parts. In the first part a second order neutral partial differential equation involving state dependent delay and non-instantaneous impulses is studied. The conditions for existence and uniqueness of the mild solution are investigated via Hausdorff measure of non-compactness and Darbo Sadovskii fixed point theorem. Thus, the assumption of compactness of cosine operators is removed. The conditions for approximate controllability are investigated for the neutral second order system with respect to the approximate controllability of the corresponding linear system in a Hilbert space. A simple range condition is used to prove approximate controllability. Thereby, the non-singularity of a controllability operator is not required which was an essential condition in [39]. Since in infinite dimensional spaces, with compact semigroup the controllability operator is not invertible. Our methodology does not require to find the inverse of the controllability Gramian operator. Also the associated limiting condition in [69] are removed. Examples are studied to substantiate the theory.
The results of this chapter are published as 'Existence of Solution and Approximate Controllability for Neutral Differential Equation with State Dependent Delay' in International Journal of Partial Differential Equations, Hindawi, volume 2014 (2014), Article ID 787092, 12 pages.

In chapter 8 a fractional neutral differential equation with deviated argument is investigated. The existence and uniqueness of mild solution is proved by applying Banach contraction principle. We removed additional conditions of compactness of semigroups or nonlinear functions, analyticity, uniform boundedness. We also investigate a fractional neutral differential equation involving deviating argument to establish its the approximate controllability. A simple condition on the range of a operator is used. Therefore assumption of nonsingularity of controllability operator removed. Since in infinite dimensional spaces with compact semigroups, the controllability operator is not invertible. Our methodology does not require to find inverse of the controllability Gramian. We also remove requirement to verify the associated limiting condition. An example is also presented.
The results of this chapter are published as 'Approximate Controllability of a Fractional Neutral System with Deviated Argument in Banach Space' in Differential Equations and Dynamical Systems, Springer, $DOI : 10.1007/s12591-015-0237-$

y, (2015).

In chapter 9 the approximate controllability of a fractional impulsive stochastic neutral integro-differential equation with deviating argument and infinite delay is studied. The control parameter is included in the nonlinear term as well. Only Schauder fixed point theorem and a few fundamental hypotheses are used to prove our result. The assumption of the existence of the inverse of controllability operator is not required. This removes the limitation in infinite-dimensional space of the nonexistence of the inverse incase of compact semigroups. Lipschitz continuity of the nonlinear function is replaced by simple assumptions. An example is also studied.

The results of this chapter are published as 'Approximate controllability of an impulsive neutral fractional stochastic differential equation with deviated argument and infinite delay' in Nonlinear Studies, volume 22, no. 1, (2015), 1-16, CSP - Cambridge, UK; I&S - Florida, USA.

In the chapter 10 the existence, uniqueness and convergence of approximate solutions of a stochastic fractional differential equation with deviating argument is established. Analytic semigroup is used coupled with fixed point approach. Then we consider Faedo-Galerkin approximation of solution and prove some convergence results. We also study an example to illustrate our result.

The results of this chapter are accepted for publication as 'Approximations of Solutions of a Fractional Stochastic Differential Equations with Deviated Argument' in Journal of Fractional Calculus and Applications in 2015.

Chapter 2

Preliminaries

In this chapter, some basic definitions, lemmas and theorems are recalled. This chapter has six sections. In section 2.1 some facts about operators defined on Banach space are given. In section 2.2 introduction to semigroup theory is given briefly. In section 2.3 some facts about control theory are discussed. In section 2.4 some basic facts of fractional calculus and literature, related to the fractional order systems are illustrated. In section 2.5 some basic definitions of measure of noncompactness are given. In section 2.6 some basic facts of stochastic analysis are presented.

2.1 Basic concepts of Banach Space

Suppose X and Y to be the Banach spaces equipped with the norm $\|\cdot\|_X$ and $\|\cdot\|_Y$, respectively. We denote by $\mathcal{L}(X,Y)$ the space of all bounded linear operators from X to Y with the operator norm denoted by $\|\cdot\|_{\mathcal{L}(X,Y)}$ and we may write simply $\mathcal{L}(X)$ and $\|\cdot\|_{\mathcal{L}(X)}$, when $X = Y$. If A is a linear operator in X, then $D(A)$, $N(A)$, and $R(A)$ denote the domain, null space and range space of A, respectively. The notations $\sigma(A)$, $\rho(A)$ stand for the mean spectrum and resolvent set of A, respectively and $R(\lambda, A) := (\lambda I - A)^{-1}$ denotes the resolvent operator of A.

Let $\mathcal{J} = (a,b)$ with $-\infty \leq a < b \leq \infty$. Then, $L^p(\mathcal{J}, X)$ represents the Banach space of all Bochner-measurable functions $F : \mathcal{J} \to X$ with the following norm

$$\begin{aligned} \|F\|_{L^p(\mathcal{J},X)} &:= (\int_{\mathcal{J}} \|F(s)\|_X^p ds)^{1/p}, \ 1 \leq p < \infty, \\ \|F\|_{L^p(\mathcal{J},X)} &:= \sup_{t\in\mathcal{J}} \|F(t)\|_X, \quad p = \infty. \end{aligned}$$

Gronwall's inequality: Let F and G be the non-negative continuous functions and for each $t \geq t_0$ and a constant β. Then the inequality

$$F(t) \leq \beta + \int_{t_0}^{t} G(s)F(s)ds, \quad t \geq t_0, \tag{2.1.1}$$

implies the following inequality

$$F(t) \leq \beta \exp(\int_{t_0}^{t} G(s)ds), \quad t \geq t_0. \tag{2.1.2}$$

Furthermore, the notations $C(\mathcal{J}, X)$ and $C^m(\mathcal{J}, X)$ stand for the space of all continuous and m-times continuously differentiable functions, respectively. The space $C_0^\infty(R, X)$ consists of all infinitely differentiable functions with compact support. Set $J = [0, T]$, $T > 0$. Then, $C(J, X)$ and $C^m(J, X)$ denote the Banach spaces with equipped with the norm denoted by

$$\|F\|_C := \sup_{t \in J} \|F(t)\|_X, \quad \|F\|_{C^m} := \sup_{t \in J} \sum_{k=0}^{m} \|F^k(t)\|_C, \tag{2.1.3}$$

respectively.

Definition 2.1.1. [175] Let $I = (0, T)$, or $I = \mathbb{R}^+$, or $I = \mathbb{R}$, $m \in \mathbb{N}$ and $1 \leq p < \infty$. The Sobolev spaces $W^{m,p}$ is defined as

$$\begin{aligned} W^{m,p}(I, X) : &= \{F : \text{there exists } z \in L^p(I, X) : F(t) = \sum_{k=0}^{m-1} c_k \frac{t^k}{k!} \\ &+ \frac{t^{m-1}}{(m-1)!} * z(t), \ t \in I\}. \end{aligned} \tag{2.1.4}$$

Note that $z(t) = F^m(t)$, $c^k = F^k(0)$. Also

$$W_0^{m,p}(I, X) := \{F \in W^{m,p}(I, X) : \ F^k(0) = 0, \ k = 0, 1, \cdots, m-1\}. \tag{2.1.5}$$

It is clear that $F \in W_0^{m,p}(I, X)$ if and only if $F = \frac{t^{m-1}}{(m-1)!} * z(t)$ for some $z \in L^p(I, X)$.

Now, some basic definitions and theorems which will be used throughout this thesis is provided.

Definition 2.1.2. Let X and $\widetilde{X}$ be the Banach spaces. A mapping $f : X \to \widetilde{X}$ is said to be Lipschitz continuous if there exists a constant $l > 0$ such that

$$\|f(z_1) - f(z_2)\|_{\widetilde{X}} \leq l\|z_1 - z_2\|_X, \ \text{ for all } \ z_1, \ z_2 \in X. \tag{2.1.6}$$

Definition 2.1.3. A function $f : X \to \widetilde{X}$ is said to be a Hölder continuous if there exist nonnegative real constants K and α such that

$$\|f(z_1) - f(z_2)\|_{\widetilde{X}} \leq K\|z_1 - z_2\|_X^{\alpha} \text{ for each } z_1, z_2 \in X. \tag{2.1.7}$$

The number α is known as the exponent of the Hölder condition. The function satisfies a Lipschitz condition, when $\alpha = 1$. If $\alpha = 0$, then the function simply is bounded.

Definition 2.1.4. The family of functions $\mathfrak{F} = \{f \in \mathfrak{F} \textit{ such that } f : X \to \widetilde{X}\}$ is said to be equicontinuous at a point $z_0 \in X$ if for every $\epsilon > 0$, and every $f \in \mathfrak{F}$ there exists a $\delta > 0$ such that

$$\|f(z_0) - f(z)\|_{\widetilde{X}} \leq \epsilon \; \forall \; z \text{ with } \|z_0 - z\|_X \leq \delta. \tag{2.1.8}$$

Definition 2.1.5. Let X be the Banach space and $F : X \to X$ be a nonlinear operator. Then each solution of the equation

$$F(z) = z, \quad z \in X \tag{2.1.9}$$

is called the fixed point of operator F.

Definition 2.1.6. A mapping F from a subset M of a normed space X into X is called a contraction mapping in there exists a positive number $k < 1$ such that

$$\|F(z_1) - F(z_2)\| \leq k\|z_1 - z_2\| \text{ for all } z_1, z_2 \in M. \tag{2.1.10}$$

Theorem 2.1.7. (Banach fixed point theorem) Let N be a closed subset of a Banach space X and let F be a contraction mapping from N into N. Then, there exists a unique $z \in N$ such that $F(z) = z$.

Definition 2.1.8. Let X and $\widetilde{X}$ be normed linear spaces. An operator $T : X \to \widetilde{X}$ is called compact if it maps every bounded subset of X into a relatively compact subset of $\widetilde{X}$.

Theorem 2.1.9. (Arzela-Ascoli theorem:) Assume that K is a compact set in $\mathbb{R}^n$, $n \geq 1$. Then, a set $B \subset C(K)$ is relatively compact in $C(K)$ if and only if the functions in B are uniformly bounded and equicontinuous on K.

Theorem 2.1.10. Let X and $\widetilde{X}$ be normed linear spaces. A linear operator $T : X \to \widetilde{X}$ is compact iff it maps every bounded sequence (z_n) in X onto a sequence $(T(z_n))$ in $\widetilde{X}$ which has a convergent subsequence.

Theorem 2.1.11. (Schauder's fixed point theorem:) Let M be a convex compact set in a Banach space X and mapping $T : M \to M$ is a continuous map. Then T has a fixed point.

Theorem 2.1.12. (Schaefer's fixed point theorem:) Let X be a Banach space and $T : X \to X$ to be a continuous compact mapping. Whenever the set

$$M = \bigcup_{0\leq\lambda\leq 1} \{y \in X : y = \lambda T(y)\} \tag{2.1.11}$$

is bounded, T has a fixed point.

Note that the Schaefer's fixed point theorem is version of Schauder's theorem. Sometimes it is known as the Leray-Schauder principle.

Lemma 2.1.13. [53] Let X be Hilbert space and X_1, X_2 be closed subspaces such that $X = X_1 + X_2$. Then there exists a bounded linear operator $P : X \to X_2$ such that for each $x \in X$, $x_1 = x - Px \in X_1$ and $\|x_1\| = min\{\|y\| : y \in X_1, (1-Q)(y) = (1-Q)(x)\}$ where Q denotes the orthogonal projection on X_2.

2.2 Semigroup Theory and Cosine family

Suppose that X is a complex Banach space. Suppose A to be a closed linear operator dense in X. Assume that $D(A)$ is associated with the graph norm of A that is, $\|y\|_{D(A)} := \|y\|_X + \|Ay\|_X$. Since A is closed, therefore, $D(A)$ is a Banach space, continuously and densely embedded into X.

Definition 2.2.1. [146] The one parameter family $\{\mathcal{S}(t)\}_{t\geq 0}$, of bounded linear operators from Banach X into itself is called a semigroup of bounded linear operators on X if the following conditions hold;

(1) $\mathcal{S}(0) = I$, where I is the identity operator on X.

(2) $\mathcal{S}(t + s) = \mathcal{S}(t)\mathcal{S}(s)$ for every $t, s \geq 0$.

Definition 2.2.2. A semigroup $\{\mathcal{S}(t)\}_{t\geq 0}$ of bounded linear operators on X is said to be a strongly continuous semigroup or C_0 semigroup if

$$\lim_{t\downarrow 0} \mathcal{S}(t)x = x, \quad \text{for every } x \in X. \tag{2.2.12}$$

Definition 2.2.3. The semigroup $\{\mathcal{S}(t)\}_{t\geq 0}$ of bounded linear operators is said to be a uniformly continuous semigroup if $\lim_{t\to 0} \| \mathcal{S}(t) - I\| = 0$.

Definition 2.2.4. The infinitesimal generator of a semigroup of bounded linear operator $\{\mathcal{S}(t)\}_{t\geq 0}$ on Banach space X is a linear operator A on X defined as

$$Az = \lim_{t\to 0} \frac{\| \mathcal{S}(t)z - z\|}{t}, \text{ for } z \in D(A), \tag{2.2.13}$$

whenever this limit exists. The domain of A denoted by $D(A)$ defined as

$$D(A) = \{z \in X : \lim_{t\to 0} \frac{\| \mathcal{S}(t)z - z\|}{t} \text{ exists}\}. \tag{2.2.14}$$

Remark

1 Suppose A to be the infinitesimal generator of a C_0-semigroup $\{\mathcal{S}(t)\}_{t\geq 0}$. Then, $D(A)$ is dense in X and A is a closed bounded linear operator.

2 Let $\mathcal{S}(t)$ be the C_0-semigroup. Then, there exist constants $\delta \in R$ and $M \geq 1$ such that

$$\|\mathcal{S}(t)\| \leq Me^{\delta t}, \text{ for all } t \geq 0. \tag{2.2.15}$$

3 If $\delta = 0$, then, $\mathcal{S}(t)$ is called uniformly bounded semigroup. Moreover, if $M = 1$, then $\mathcal{S}(t)$ is called C_0-semigroup of contractions.

4 The generator of the semigroup $\mathcal{S}(t)$ is unique.

Theorem 2.2.5. Suppose $\mathcal{S}(t)$ to be a uniformly continuous semigroup (C_0−semigroup) of bounded linear operators defined on X which is generated by A. Then,

(1) $\|\mathcal{S}(t)\|$ is bounded on every finite subinterval of $[0, \infty)$,

(2) for each $z \in X$, $\lim_{h\to 0} \frac{1}{h} \int_t^{t+h} \mathcal{S}(s)zds = \mathcal{S}(t)z$,

(3) for all $z \in X$, $\int_0^t \mathcal{S}(s)xds \in D(A)$ and

$$A(\int_0^t \mathcal{S}(s)zds) = \mathcal{S}(t)z - z, \tag{2.2.16}$$

(4) for $z \in D(A)$, $\mathcal{S}(t)z \in D(A)$ and

$$\frac{d}{dt}\mathcal{S}(t)z = A\mathcal{S}(t)z = \mathcal{S}(t)Az, \tag{2.2.17}$$

(5) for all $z \in D(A)$,

$$\mathcal{S}(t)z - \mathcal{S}(s)z = \int_s^t \mathcal{S}(\tau)Azd\tau = \int_s^t A\mathcal{S}(\tau)zd\tau, \tag{2.2.18}$$

(6) if $w_0 = \inf_{t>0}(\frac{1}{t}\log\|\mathcal{S}(t)\|)$, then $w_0 = \lim_{t\to\infty}(\frac{1}{t}\log\|\mathcal{S}(t)\|) < \infty$,

(7) for all $w > w_0$, there is a constant M_w such that $\|\mathcal{S}(t)\| \leq M_w e^{wt}$ for all $t \geq 0$.

The constant w_0 is known as the growth bound of the semigroup.

For a linear operator A (not always bounded) in X, the resolvent set $\rho(A)$ of A consists of all complex numbers λ such that $\lambda I - A$ is invertible. The resolvent of A is a family $R(\lambda, A) = (\lambda I - A)^{-1}$, $\lambda \in \rho(A)$ of bounded linear operators which plays an important role in the application of semigroup. For the resolvent operator $R(\lambda, A)$ of the generator A of a C_0-semigroup, we have the following result which shows that the resolvent operator is just the Laplace transform of the semigroup.

Lemma 2.2.6. Let $\mathcal{S}(t)$ be a C_0-semigroup with infinitesimal generator A and growth bound w_0. If $Re(\lambda) > w > w_0$, then $\lambda \in \rho(A)$, and for all $y \in X$ the following results hold:

(*a*) $R(\lambda, A)y = (\lambda I - A)^{-1}y = \int_0^\infty e^{-\lambda t}\mathcal{S}(t)ydt$ and $\|R(\lambda, A)\| \leq \frac{M}{\mu - w}$; $\mu = Re(\lambda)$;

(*b*) For all $y \in X$, $\lim_{\beta\to\infty}(\beta I - A)^{-1}y = y$, where β is constrained to be real.

In 1948, Hille and Yosida established an result known as Hille-Yosida Theorem which plays an important role in the theory of semigroup of bounded linear operators.

Theorem 2.2.7. [146] (Hille-Yosida Theorem) A necessary and sufficient condition for a closed linear densely defined operator A on a Banach space X to be the infinitesimal generator of a strongly continuous semigroup $\mathcal{S}(t)$, $t \geq 0$ on X is that there exist real numbers M and δ such that every real $\lambda > \delta$ belongs to the resolvent set of A and for such λ and

$$\|R(\lambda, A)^k\| \leq \frac{M}{(\lambda-\delta)^k}, \forall \ k \geq 1, \tag{2.2.19}$$

where $R(\lambda, A) = (\lambda I - A)^{-1}$ denotes the resolvent operator of A.

Theorem 2.2.8. Let U be a bounded linear operator. If $\|U\| \leq v$, then

$$e^{tU} = \frac{1}{2\pi i}\int_{v-i\infty}^{v+i\infty} e^{\lambda t}(\lambda I - U)^{-1} d\lambda. \tag{2.2.20}$$

The convergence in (2.2.20) is in the uniform operator topology and uniformly in t on bounded intervals.

Theorem 2.2.9. Let A be a linear operator dense in X which satisfies the following two conditions:

(1) $\sum_\mu = \{\lambda : |\arg\lambda| < \frac{\pi}{2} + \mu\} \cup \{0\} \subset \rho(A)$, for some $0 < \mu < \pi/2$;

(2) There is a constant M such that

$$(\lambda I - A)^{-1} \leq \frac{M}{|\lambda|}, \text{ for } \lambda \in \sum_\mu \text{ and } \lambda > 0.$$

Then, A generates a C_0-semigroup $\mathcal{S}(t)$ fulfilling $\|\mathcal{S}(t)\| \leq N$, where N is a positive constant and

$$\mathcal{S}(t) = \frac{1}{2\pi i}\int_F e^{\lambda t}(\lambda I - A)^{-1} d\lambda.$$

Here F is a smooth curve in $\sum_\mu$ starting from $\infty e^{-i\theta}$ to $\infty e^{i\theta}$ for some $\theta \in (\pi/2, \pi/2 + \delta)$, with the integral converging in uniform operator topology.

Definition 2.2.10. Let $\sum_\theta = \{\lambda \in C : |\arg\lambda| < \theta\} \cup \{0\}$ for $\theta \in (0, \pi/2]$. The family the bounded linear operators $\mathcal{S}(t)$, $z \in \sum_{\theta \cup \{0\}}$ defined in Banach space X is said to be analytic semigroup if

(*i*) $\mathcal{S}(0) = I$;

(*ii*) $\mathcal{S}(z^* + z^{**}) = \mathcal{S}(z^*)\mathcal{S}(z^{**})$, for each $z^*,\ z^{**} \in \sum_\theta$;

(*iii*) the map $z \mapsto \mathcal{S}(z)$ is analytic in $\sum_\theta$;

(*iv*) $\lim_{z\to 0} \mathcal{S}(z)y = y$ for all $y \in X$ and $z \in \sum_{\theta'}$, $\ 0 < \theta' < \theta$.

Moreover, if

(*v*) $\|\mathcal{S}(z)\|$ is bounded in $\sum_{\theta'}$ for all $0 < \theta' < \theta$.

Then, $\mathcal{S}(z),\ z \in \sum_{\theta\cup\{0\}}$ is called a bounded analytic semigroup.

We also have the following results

Theorem 2.2.11. Let A be a linear operator dense in X. Then, the following hypothesis are equivalent.

(*i*) A is the infinitesimal generator of a bounded analytic semigroup $\mathcal{S}(z),\ z \in \sum_{\theta\cup\{0\}}$ on X.

(*ii*) $\exists\, \vartheta \in (0, \pi/2)$ such that the operator $e^{\pm i\vartheta}$ generates strongly continuous semigroups on X.

(*iii*) A is the infinitesimal generator of a strongly continuous semigroup $\mathcal{S}(t),\ t \geq 0$ on X such that $arg(\mathcal{S}(t)) \subset D(A)$, for each $t > 0$ and

$$M := \sup_{t>0} \|tA\mathcal{S}(t)\| < \infty. \tag{2.2.21}$$

(*iv*) A is the infinitesimal generator of a bounded strongly continuous semigroup $\mathcal{S}(t)$, $t \geq 0$ on X and there exists a positive constant C such that

$$\|R(r + is, A)\| \leq \frac{C}{|s|},\ \ \forall\ r > 0 \text{ and } 0 \neq s \in R. \tag{2.2.22}$$

Cosine family

The family $\{C(t) : t \in \mathbb{R}\}$ in $B(X)$, the space of all bounded linear operators, is called a strongly continuous cosine family if the following conditions are satisfied:

(a) $C(0) = I$ (I is the identity operator in X);

(b) $C(t + s) + C(t - s) = 2C(t)C(s)$ for all $t, s \in \mathbb{R}$

(c) The map $t \to C(t)x$ is strongly continuous for each $x \in X$.

$\{S(t) : t \in \mathbb{R}\}$ is the strongly continuous one parameter family of sine operators associated to strongly continuous $\{C(t) : t \in \mathbb{R}\}$. Further $S(t)x = \int_0^t C(s)xds,\ x \in X,\ t \in \mathbb{R}$. We refer books by Goldstein[84] and Fattorini[75] for further study.

The definition of abstract phase space $\mathfrak{B}$ as introduced by Hale and Kato, is given as follows

Definition 2.2.12. [85]: Let $\mathfrak{B}$ be a linear space of maps from $(-\infty, 0]$ into X endowed with the seminorm $\|.\|_{\mathfrak{B}}$ and satisfying the following conditions:

(A) If $x : (-\infty, \sigma+a] \to X, b > 0$, such that $x_t \in \mathfrak{B}$ and $x|_{[\sigma,\sigma+a]} \in C([\sigma, \sigma+a] : X)$, then for all $t \in [\sigma, \sigma + a)$ the following conditions hold
 (i) $x_t \in \mathfrak{B}$,
 (ii) $\|x(t)\| \leq H\|x_t\|_{\mathfrak{B}}$,
 (iii) $\|x_t\|_{\mathfrak{B}} \leq K(t-\sigma)sup\{\|x(s)\| : \sigma \leq s \leq t\} + M(t+\sigma)\|x_\sigma\|_{\mathfrak{B}}$,
 where H is a positive constant $K_a, M_a : [0, \infty) \to [1, \infty)$, K_a is continuous; M_a is locally bounded and H, K_a, M_a are independent of $x(.)$

(B) The space $\mathfrak{B}$ is complete. Then $\mathfrak{B}$ is said to be abstract the phase space.

2.2.1 Fractional Powers of Operators

For the operator A for which $-A$ generates an analytic semigroup $\mathcal{S}(t)$, one can define fractional power of A. In particular, we assume that A is densely defined closed linear operator for which

$$\sum^{+} = \{\lambda : 0 < \upsilon < |\arg\lambda| \leq \pi\} \cup U, \tag{2.2.23}$$

where U denotes a neighborhood of zero, and

$$\|R(\lambda, A)\| \leq \frac{M}{1+|\lambda|}, \text{ for } \lambda \in \sum^{+}. \tag{2.2.24}$$

For $\upsilon = \pi/2$ and $M = 1$, $-A$ generates a C_0-semigroup while for $\upsilon < \pi/2$, $-A$ generates of an analytic semigroup.

For an operator A with condition (2.2.23) and (2.2.24), one can define negative fractional powers $0 < \alpha < \infty$ by the formula

$$A^{-\alpha} := \frac{1}{2\pi i}\int_{\Gamma} \xi^{-\alpha} R(\xi) d\xi = \frac{1}{2\pi i}\int_{\Gamma} \xi^{-\alpha}(A - \xi I)^{-1} d\xi, \tag{2.2.25}$$

where Γ denotes the path starting in the resolvent of A from $\infty e^{-i\theta}$ to $\infty e^{i\theta}$ for $\upsilon < \theta < \pi$ i.e., $\Gamma = \Gamma_1 \cup \Gamma_2$, where $\Gamma_1 = [\varrho \exp(i\theta) : 0 \leq \varrho < \infty]$ and $\Gamma_2 = [\varrho \exp(-i\theta) : 0 \leq \varrho < \infty]$, avoiding the negative real axis and the origin and $\chi^{-\alpha}$ is positive for real positive values of χ. For $\alpha > 0$, the operators $A^{-\alpha}$ are bounded due to convergence of the integral (2.2.25).

By definition of $A^{-\alpha}$, the operators $A^{-\alpha}$ form a semigroup

$$A^{-\beta}A^{-\alpha} = A^{-(\alpha+\beta)} = A^{-\alpha}A^{-\beta}, \quad \text{for} \quad \alpha,\, \beta > 0, \tag{2.2.26}$$

and there is a constant C such that for $\alpha \in [0, 1]$, $\|A^{-\alpha}\| \leq C$. Moreover, let us assume that A satisfies the (2.2.23) and (2.2.24) with $\upsilon < \pi/2$. Then,

$$A^{\alpha} = (A^{-\alpha})^{-1}, \quad \text{for} \quad \alpha > 0.$$

If $\alpha = 0$, we get $A^{\alpha} = I$.

Theorem 2.2.13. [146] Let $-A$ be the infinitesimal generator of an analytic semigroup $\mathcal{S}(t)$ and $0 \in \rho(A)$. Then,

(*i*) for $\alpha \geq 0$, $\mathcal{S}(t) : X \to D(A^{\alpha})$ for every $t > 0$;

(*ii*) $\mathcal{S}(t)A^{\alpha}z = A^{\alpha}\mathcal{S}(t)z$ for each $z \in D(A^{\alpha})$;

(*iii*) for $t > 0$, the operator $A^{\alpha}\mathcal{S}(t)$ is bounded and

$$\|A^{\alpha}\mathcal{S}(t)\| \leq M_{\alpha} t^{-\alpha} e^{-\delta t}, \quad \text{for} \quad \delta > 0,$$

for some constant M_{α} which depends on α;

(*iv*) for $\alpha \in (0, 1]$ and $z \in D(A^{\alpha})$,

$$\|\mathcal{S}(t)z - z\| \leq C_{\alpha} t^{\alpha} \|A^{\alpha} z\|. \tag{2.2.27}$$

2.3 Basic Concepts of Control Theory

Suppose a spaceship is to dock at the international space station. Is there atleast one control strategy to manoeuvre the spaceship to dock? This is the controllability question.

2.3.1 Finite dimensional control systems

A linear nonautonomous linear control system can be represented by

$$\begin{aligned} \frac{dx(t)}{d(t)} &= Ax(t)x(t) + B(t)u(t), \quad t_0 \le t \le \tau \\ x(t_0) &= x_0 \end{aligned} \tag{2.3.1}$$

where t_0, τ are two real numbers and $A(t)$ and $B(t)$ are matrices of order $n \times n$ and $n \times m$ respectively. For all $t \in [t_0, \tau]$ $x(t) \in \mathbb{R}^n$ is known as the state of the system, $u(t) \in \mathbb{R}^m$ is called control. Let $L^2([t_0, \tau]; \mathbb{R}^n)$ and $\mathcal{L}^2([t_0, \tau]; \mathbb{R}^m)$ be function spaces to which x(.) and u(.) belong.

The mild solution of (2.3.1) is given by

$$x(t) = \phi(t, t_0)x_0 + \int_{t_0}^{t} \phi(t, s)Bu(s)ds$$

where $\phi(t, s)$ is called the state transition matrix, since it relates the state at any time t_0 to the state at any other time t.

Remark The control u which steers the initial state x_0 to the final state x_τ need not be unique. If x_0 and x_τ belong to a subset of $\mathbb{R}^n$ then the resulting controllability is called local controllability.

The set of all points to which the initial state x_0 can be steered in time τ is called the reachable set

$$K_\tau = \{x(\tau) \in \mathbb{R}^n : x(.) \;\; is \;\; the \;\; solution \;\; of \;\; (2.3.1)\}$$

The linear system (2.3.1) is said to be controllable over the interval $[t_0, \tau]$ if the reachable set K_τ equals to the whole space $\mathbb{R}^n$.

Definition 2.3.1. The controllability matrix of (2.3.1) $G : L^2([t_0, \tau]; \mathbb{R}^n) \to \mathbb{R}^n$ is defined as

$$Gu = \int_{t_0}^{t} \phi(\tau, s)B(s)u(s)ds.$$

Definition 2.3.2. The controllability Grammian matrix $W_{t_0}^{\tau} : \mathbb{R}^n \to \mathbb{R}^n$ is defined as

$$W_{t_0}^{\tau}u = \int_{t_0}^{t} \phi(\tau, s)B(s)B^*(s)\phi^*(\tau, s)ds.$$

Clearly the controllable Grammian matrix is a symmetric matrix of order $n \times n$. Autonomous System: If the entries of matrices A and B are constants then the system is said to autonomous. The solution of the autonomous system is given by

$$x(t) = e^{A(t-t_0)}x_0 + \int_{t_0}^{t} e^{A(t-s)}Bu(s)ds$$

Theorem 2.3.3. [45] The linear control system (2.3.1) (autonomous or nonautonomous) is controllable iff the controllability Grammian matrix is invertible.

2.3.2 Infinite Dimensional Control Systems

Infinite dimensional control systems is a widely researched field with emphasis on delay control systems. The two basic concepts can be distinguished namely exact controllability and approximate controllability.

The mathematical model of an infinite dimensional linear control system can be written as

$$\begin{aligned} \frac{dx(t)}{d(t)} &= Ax(t)x(t) + B(t)u(t), \quad t_0 \le t \le \tau \\ x(t_0) &= x_0 \end{aligned} \tag{2.3.2}$$

where the state $x(t)$ of the system at time t takes values in a Banach space V. The control function $u(t)$ takes values in another Banach space $\hat{V}$. The operator $A : D(A) \subset V \to V$ is a closed, linear and densely defined operator. $B : \hat{V} \to V$ is a bounded linear operator.

$x(t) = T(t-t_0)x_0 + \int_{t_0}^{t} T(t-s)Bu(s)ds$ is the mild solution of (2.3.2).

Definition 2.3.4. The system (2.3.2) is said to be approximately controllable if for all $\epsilon > 0$ and two initial and final points x_0, x_τ respectively, there exists an admissible control $u(t)$ on $[t_0, \tau]$ steering x_0 along a trajectory (mild solution) x(t) of (2.3.2) to an $\epsilon-$ neighbourhood of x_τ such that

$$\|x(\tau) - x_\tau\| \le \epsilon$$

If $\epsilon = 0$ the above definition gives exact controllability of system (2.3.2).

Definition 2.3.5. For system (2.3.2) the controllability map $G : \mathcal{L}_2([t_0, \tau]; \hat{V}) \to V$ is defined as

$$Gu = \int_{t_0}^{t} T(\tau - s)B(s)u(s)ds.$$

Definition 2.3.6. The controllability Grammian map is defined as

$$W_{t_0}^{\tau}u = \int_{t_0}^{t} T(\tau - s)B(s)B^*(s)T^*(\tau - s)ds.$$

Theorem 2.3.7. [45] The system (2.3.2) is approximately controllable iff $W_{t_0}^{\tau}$ is positive definite.

In (1977) Triggiani [164] proved that if A generated a compact C_0 semigroup $T(t)$, then the linear system can never be exactly controllable in an infinite dimensional space.

2.4 Basic Concepts of Fractional Calculus

There are two main approaches for defining a fractional derivative. One is through Mittag-Leffer functions and the other approach generalizes a convolution type representation of repeated integration. The Riemann-Liouville and Caputo definitions take this approach. Now we consider the few definitions of fractional calculus.

Definition 2.4.1. [114] The Riemann-Liouville fractional integral of order $\alpha > 0$, of the function $F : \mathbb{R}^+ \to X$ is defined by

$$^{RL}J_t^{\alpha}F(t) = \int_0^t \frac{(t-s)^{\alpha-1}F(s)}{\Gamma(\alpha)}ds, \text{ for } t > 0, \tag{2.4.3}$$

where $F \in L^1(\mathbb{R}^+, \mathbb{R})$ and $J_t^0 = I$. We can write

$$J_t^{\alpha}F(t) = g_{\alpha}(t) * F(t), \tag{2.4.4}$$

where g_{α} is defined as

$$g_{\alpha}(\tau) = \begin{cases} \frac{\tau^{\alpha-1}}{\Gamma(\alpha)}, & \tau > 0 \\ 0, & t \leq 0 \end{cases} \tag{2.4.5}$$

and $*$ denotes the the convolution of functions, defined on $\mathbb{R}$ or $\mathbb{R}^+$:

$$\begin{aligned} (g_{\alpha} * F)(t) &= \int_{-\infty}^{+\infty} g_{\alpha}(t-s)F(s)ds, \ t \in \mathbb{R}, \ g_{\alpha} \in L^1(\mathbb{R}), \ F \in L^1(\mathbb{R}, X), \\ (g_{\alpha} * F)(t) &= \int_0^t g_{\alpha}(t-s)F(s)ds, \ t \in \mathbb{R}^+, \ g_{\alpha} \in L^1(\mathbb{R}^+), \ F \in L^1(\mathbb{R}^+, X). \end{aligned}$$

Definition 2.4.2. The Riemann-Liouville derivative of order α, of the function F is defined as

$$\begin{aligned} {}^{RL}D_t^{\alpha}F(t) &= D_t^m J_t^{m-\alpha}F(t) \\ &= \frac{1}{\Gamma(m-\alpha)}\frac{d^m}{dt^m}\int_0^t (t-s)^{m-\alpha-1}F(s)ds,\ t>0, \end{aligned} \tag{2.4.6}$$

for $m-1 \leq \alpha < m \quad m \in \mathbb{R}$. where $D_t^m = \frac{d^m}{dt^m}$, $F \in L^1(\mathbb{R}^+, X)$, $J_t^{m-\alpha}F \in W^{m,1}(\mathbb{R}^+, X)$.

Definition 2.4.3. [177] The Caputo derivative of a function $F(t)$ is defined as

$${}_0^C D_t^{\alpha}F(t) = \frac{1}{\Gamma(m-\alpha)}\int_0^t (t-s)^{m-\alpha-1}F^m(s)ds, \tag{2.4.7}$$

in which $m-1 < \alpha < m,\ \ m \in \mathbb{N}$ and $F \in C^{m-1}((0,T),X) \cap L^1((0,T),X)$.

2.4.1 Basic Concept of Solutions of Fractional Differential Equations

We consider the infinite dimensional fractional order problem illustrated as

$$\begin{aligned} {}^cD_t^q y(t) &= Ay(t),\ t \in [0,T], & (2.4.8) \\ y(0) &= y_0, & (2.4.9) \end{aligned}$$

where ${}^cD_t^q$ denotes the fractional derivative in Caputo sense of order q, $0 < q < 1$, the state $y(\cdot)$ takes its values in a Banach space X, $A : D(A) \subseteq X \to X$ is a closed densely linear operator defined in X. In (2.4.8), A is assumed to be the infinitesimal generator of C_0-semigroup of bounded linear operator $\mathcal{S}(t),\ t \geq 0$.

The equation (2.4.8) is equivalent to the following integral equation

$$y(t) = y_0 + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}Ay(s)ds. \tag{2.4.10}$$

The solution to (2.4.8) is closely associated with a function $y \in C([0,T],X)$ that satisfies the following assumptions

(*i*) y is continuous on $[0,T]$ and $y(t) \in D(A)$ for each $t \in [0,T]$,

(*ii*) ${}^cD_t^q y(t)$ exists and is continuous on $[0,T]$ with $0 < q < 1$,

(*iii*) y satisfies the equation (2.4.8) on $[0,T]$ and the initial condition $y(0) = y_0$.

Taking Laplace transform of equation (2.4.10), we get

$$L[y(t)] = \lambda^{q-1}\int_0^\infty e^{-\lambda^q s}\mathcal{S}(s)y_0 ds, \tag{2.4.11}$$

Consider the one-sided stable probability density

$$\Phi_q(\zeta) = \frac{1}{\pi}\sum_{n=1}^{\infty}(-1)^{n-1}\zeta^{-nq-1}\frac{\Gamma(nq+1)}{n!}\sin(n\pi q),\ \zeta\in(0,\infty). \tag{2.4.12}$$

whose Laplace transform is given by

$$\int_0^\infty e^{-\lambda\zeta}\Phi_q(\zeta)d\zeta = e^{-\lambda^q},\ \ q\in(0,1). \tag{2.4.13}$$

Therefore, we get

$$\begin{aligned}\lambda^{q-1}\int_0^\infty & e^{-\lambda^q t}\mathcal{S}(s)y_0 dt\\ &= \int_0^\infty e^{-\lambda t}\Big[\int_0^\infty \Phi_q(\zeta)\mathcal{S}(t^q/\zeta^q)y_0 d\zeta\Big]dt,\end{aligned} \tag{2.4.14}$$

Then

$$L[y(t)] = \int_0^\infty e^{-\lambda t}\Big[\int_0^\infty \Phi_q(\zeta)\mathcal{S}(t^q/\zeta^q)y_0 d\zeta\Big]dt. \tag{2.4.15}$$

Taking inverse Laplace transformation of above equation

$$\begin{aligned} y(t) &= \int_0^\infty \Phi_q(\zeta)\mathcal{S}(t^q/\zeta^q)y_0 d\zeta,\\ &= \int_0^\infty \Psi_q(\zeta)\mathcal{S}(t^q\zeta)y_0 d\zeta,\\ &= \mathcal{S}_q(t)y_0,\end{aligned} \tag{2.4.16}$$

where $\Psi_q(\zeta) = \frac{1}{q}\zeta^{-1-\frac{1}{q}}\Phi_q(\zeta^{-1/q})$ satisfies the conditions of a probability density function defined on $(0,\infty)$, i.e. $\Psi_q(\zeta)\geq 0$, and $\int_0^\infty \Psi_q(\zeta)d\zeta = 1$. Therefore, the solution of (2.4.8) is given as

$$y(t) = \mathcal{S}_q(t)y_0, \tag{2.4.17}$$

where $\mathcal{S}_q(t),\ t\geq 0$ is defined by

$$\mathcal{S}_q(t)y = \int_0^\infty \Psi_q(\zeta)\mathcal{S}(t^q\zeta)y d\zeta,\ \ y\in D(A). \tag{2.4.18}$$

Next, we consider the following fractional differential equation

$$\begin{aligned} {}^cD_t^q y(t) &= Ay(t) + F(t),\quad t\in[0,T],\ 0\leq T<\infty, && (2.4.19)\\ y(0) &= y_0, && (2.4.20)\end{aligned}$$

where $F \in L^1([0,T],X)$.

Taking Laplace transformation of the equation (2.4.19) on both sides we get

$$\begin{aligned} L[y(t)] &= \int_0^\infty e^{-\lambda t}\big[\int_0^\infty \Phi_q(\zeta)\mathcal{S}(t^q/\zeta^q)y_0 d\zeta\big]dt \\ &\quad + \int_0^\infty e^{-\lambda t}\Big[q\int_0^t\int_0^\infty \Phi_q(\zeta)\mathcal{S}(\frac{(t-s)^{q-1}}{\zeta^q})F(s)\frac{(t-s)^q}{\zeta^q}d\zeta ds\Big]dt. \end{aligned}$$

Taking inverse Laplace transformation of above equation we get

$$y(t) = \mathcal{S}_q(t)y_0 + \int_0^t (t-s)^{q-1}\mathcal{T}_q(t-s)F(s)ds, \tag{2.4.21}$$

where, the operator $\mathcal{T}_q(t)$ is defined by

$$\mathcal{T}_q(t)y = q\int_0^\infty \zeta\Psi_q(\zeta)\mathcal{S}(t^q\zeta)y d\zeta. \tag{2.4.22}$$

Definition 2.4.1. A continuous function $y(\cdot) \in C([0,T],X)$ is said to be the solution problem (2.4.19)-(2.4.20) if the following integral equation

$$y(t) = \mathcal{S}_q(t)y_0 + \int_0^t (t-s)^{q-1}\mathcal{T}_q(t-s)F(s)ds, \tag{2.4.23}$$

is verified.

2.5 Basic Concepts of Measure of Noncompactness

We start with axiomatic definition of measures of noncompactness of bounded sets on a complete metric space.

Suppose (X,d) to be the complete metric space with metric d and $\mathcal{N}_X$ denotes the class of all bounded subsets of X. Now, we present some notations which will be needed. If U is a subset of a metric space (X,d), then $\text{diam}(U) = \sup\{d(y,y') : y,\ y' \in U\}$ is called the diameter of U. A set U in (X,d) is called k-separated if $d(y_1,y_2) \geq k$ for all distinct $y_1, y_2 \in U$ and the set U is said to be a k-separation of X.

Definition 2.5.1. [44] Let X be a complete metric space. A function $\varphi : \mathcal{N}_X \to [0,\infty)$ is said to be a measure of noncompactness on X if it satisfies the following properties:

(*i*) $\varphi(W) = 0$ if and only if $W \in \mathcal{N}_X$ is precompact. (Regularity)

(*ii*) $\varphi(W) = \varphi(\overline{W})$, where $\overline{W}$ denotes the closure of $W \in \mathcal{N}_X$(Invariance under closure)

(*iii*) $\varphi(W_1 \cup W_2) = \max\{\varphi(W_1),\ \varphi(W_2)\}$, $\forall\ W_1, W_2, W_3 \in \mathcal{N}_X$(Semi-additivity).

It is not difficult to see that the following basic results hold for any measure of noncompactness. For any bounded set $W, W_1, W_2 \in \mathcal{N}_X$, any measure of noncompactness φ fulfills the following conditions[44]

(*i*) $\varphi(W_1) \leq \varphi(W_2)$, when $W_1 \subset W_2$, [Monotonicity];

(*ii*) $\varphi(W_1 \cap W_2) \leq \min\{\varphi(W_1), \varphi(W_2)\}$;

(*iii*) $\varphi(W) = 0$ for each finite set W, [Non-singularity];

(*iv*) Let $\{W_n\}$ be a decreasing sequence of nonempty, closed sets in $\mathcal{N}_X$ such that $\lim_{n\to\infty} \varphi(W_n) = 0$. Then $W = \cap_{n=1}^{\infty} W_n \neq \emptyset$ is compact (Cantor's generalized intersection property).

Now, we are going to recall some definitions of the Kuratowski, Hausdorff and separation measures of noncompactness.

Definition 2.5.2. [44] Let (X, d) be a metric space. The Kuratowski measure of noncompactness $\alpha(U)$ of the set $U \subset X$ is the greatest lower bound of those $\kappa > 0$, for which U admits a finite subdivision into sets, whose diameters are less than κ i.e.

$$\alpha(U) := \inf\{\kappa > 0 : U \subset \cup_{k=1}^{n} U_k,\ U_k \subset X,\ \text{diam}(U_k) < \kappa,\ k = 1, 2, \cdots, n \in \mathbb{N}\}.$$

Clearly, the set U is completely bounded if and only if $\alpha(U) = 0$.

Definition 2.5.3. [44]: The Hausdorff's measure of noncompactness χ_Y is denoted by $\chi_Y(D)$ which is infimum of
$\{r > 0, D$ such that D can be covered by finite no. of balls with radius r$\}$
for a bounded set D in any Banach space Y.

Lemma 2.5.4. [44]: Let Y be a Banach space and $D, C \subset Y$ to be bounded and let φ denote both Hausdorff and Kuratowski measure of noncompactness. Then following properties hold:

(1) D is relatively compact if and only if $\varphi_Y(B) = 0$;

(2) $\varphi_Y(D) = \varphi_Y(\overline{D}) = \varphi_Y(convD)$, where $\overline{D}$ and *conv* D are closure and convex hull of D respectively;

(3) $\varphi_Y(D) \leq \varphi_Y(C)$ when $D \subset C$;

(4) $\varphi_Y(D + C) \leq \varphi_Y(D) + \varphi_Y(C)$ where $D + C = \{x + y; x \in D, y \in C\}$;

(5) $\varphi_Y(D \cup C) = max\{\varphi_Y(D), \varphi_Y(C)\}$;

(6) $\varphi_Y(\lambda D) = |\lambda|\varphi_Y(D)$ for any $\lambda \in R$;

(7) If the map $Q : D(Q) \subset Y \to Z$ is Lipschitz continuous with Lipschitz constant k, then $\varphi_Z(Q(D)) \leq k\varphi_Y(D)$ for every bounded subset $D \subset D(Q)$, where Z is a Banach space;

(8) If $\{W_n\}_{n=1}^{+\infty}$ is a decreasing sequence of bounded closed nonempty subset of Y and $lim_{n\to\infty}\varphi_Y(W_n) = 0$, then $\cap_{n=1}^{+\infty} W_n$ is nonempty and compact in Y.

Definition 2.5.5. [44]: The map $Q : W \subset Y \to Y$ is called a $\chi-$ contraction if $\exists\ 0 < k < 1$ such that $\chi_Y Q(C) \leq k\chi_Y(C)$, for any bounded closed subset C of W.

Lemma 2.5.6. (Darbo-Sadovskii) [44]: Let $W \subset Y$ be closed and convex and $0 \in W$, then the continuous map $Q : W \to W$ is $\chi-$contraction, if there exists atleast one fixed point of the map Q.

Lemma 2.5.7. [44]: For $W(t) = \{u(t) : u \in W\} \subset X$
(1) If $W \subset C([a, b]; X)$ is bounded, then for all $t \in [a, b]$, $\chi(W(t)) \leq \chi_C(W)$.
(2) If W is equicontinuous on $[a, b]$, then $\chi(W(t))$ is continuous for all $t \in [a, b]$. Also,

$$\chi_C(W) = sup\{\chi(W(t)), t \in [a, b]\}$$

(3) If $W \subset C([a, b]; X)$ is bounded and equicontinuous , then for all $t \in [a, b]$ $\chi(W(t))$ is continuous. Also,

$$\chi(\int_a^t W(s)ds) \leq \int_a^t \chi(W(s))ds \;\; t \in [a, b]$$

$PC([0,a],X)$ denotes the space of all normalized piecewise continuous function from $[0,a]$ into X. Specifically, it is the space PC comprising of all functions $u : [0,a] \to X$ such that u is continuous at $t \neq t_i$, $u(t_i^-) = u(t_i)$ and $u(t_i^+)$ exists $\forall\ i = 1,2,...,n$. It is clear that PC associated with the norm $\|x\|_{PC} = sup_{t\in J}\|x(t)\|$ is a Banach space. For any $x \in PC$

$$\widetilde{x}_i(t) = \begin{cases} x(t), & t \in (t_i, t_{i+1}]; \\ x(t_i^+), & t = t_i,\ i = 1,2,...,n. \end{cases} \tag{2.5.24}$$

So, $\widetilde{x} \in C([t_i, t_{i+1}], X)$.

$PC([0,a],X)$ denoting the space of all normalized piecewise continuously differentiable function from $[0,a]$ into X endowed with norm $\|x\|_{PC} = sup_{t\in J}\|x(t)\| + sup_{t\in J}\|x'(t)\|$ is a Banach space.

Lemma 2.5.8. [44] : For $W(t) = \{u(t) : u \in W\} \subset X$;
(1) If $W \subset PC([a,b];X)$ is bounded, for all $t \in [a,b]$, $\chi(W(t)) \leq \chi_{PC}(W)$
(2) If W is piecewise equicontinuous on $[a,b]$, then $\chi(W(t))$ is piecewise continuous for every $t \in [a,b]$. Also

$$\chi_{PC}(W) = sup\{\chi(W(t)), t \in [a,b]\}$$

(3) If $W \subset PC([a,b];X)$ is bounded and piecewise equicontinuous, for all $t \in [a,b]$ $\chi(W(t))$ is piecewise continuous. Also

$$\chi(\int_a^t W(s)ds) \leq \int_a^t \chi(W(s))ds \quad t \in [a,b]$$

Lemma 2.5.9. [44]: If the semigroup $S(t)$ is equicontinuous, then for all $t \in [0,a]$ the set $\{\int_0^t S(t-s)u(s)ds : \|u(s)\| \leq \eta(s)\ for\ a.e.\ s \in [0,a]\}$ is equicontinuous. Here $\eta \in L([0,a];R^+)$,

Condensing operator:

Definition 2.5.10. Suppose X_1 and X_2 to be the Banach spaces and φ_1 and φ_2 be the measure of noncompactness in X_1 and X_2, respectively, taking values in $(Q,\leq)$, (a partially ordered set) . A continuous map $F : D(F) \subset X_1 \to x_2$ is said to be (φ_1,φ_2)-condensing whenever $B \subset D(F)$, $\varphi_1(B) \leq \varphi_2(F(B))$ implies that B is precompact.

In other words, F is (φ_1, φ_2)-condensing in the proper sense if for any set $B \subset D(F)$, whose closure is not compact, we have

$$\varphi_2(F(B)) < \varphi_1(B).$$

Condensing operators contain both compact and contracting operators as special cases.

2.6 Basic Concepts of Stochastic Analysis

We first recall some concepts from general probability theory.

Definition 2.6.1. Suppose Ω to be a given set. A σ-algebra $\mathcal{F}$ on Ω is a collection of subsets of Ω with the following properties:
(i) $\emptyset \in \mathcal{F}$;
(ii) $F \in \mathcal{F} \Rightarrow F^C \in \mathcal{F}$, where $F^C = \Omega - F$ is the complement of F in Ω;
(iii) $F_1, F_2, \cdots \in \mathcal{F} \Rightarrow F := \cup_{j=1}^{\infty} F_j \in \mathcal{F}$.
Then $(\Omega, \mathcal{F})$ is said to be a measurable space.

If $\mathcal{F}_1$ and $\mathcal{F}_2$ are two σ-algebras of subsets of Ω, by $\mathcal{F}_1 \vee \mathcal{F}_2$ we denote the smallest σ-algebra of subsets of Ω which contains the σ-algebras $\mathcal{F}_1$ and $\mathcal{F}_2$.

By $\mathcal{B}(\mathbb{R}^n)$, we denote the σ-algebra of Borel subsets of $\mathbb{R}^n$, i.e. the smallest σ-algebra containing all open subsets of $\mathbb{R}^n$.

For a family $\mathcal{C}$ of subsets of Ω, $\sigma(\mathcal{C})$ will denote the smallest σ-algebra of subsets of Ω containing $\mathcal{C}$, $\sigma(\mathcal{C})$ will be termed the σ-algebra generated by $\mathcal{C}$.

Definition 2.6.2. A probability measure P on $(\Omega, \mathcal{F})$ is a map $P : \mathcal{F} \to [0, 1]$ such that
(1) $P(\emptyset) = 0$, $P(\Omega) = 1$.
(2) If $F_1, F_2, \cdots \in \mathcal{F}$ are disjoint, then

$$P(\bigcup_{j=1}^{\infty} F_j) = \sum_{j=1}^{\infty} P(F_j).$$

(3) If $F_1, F_2, \cdots, \in \mathcal{F}$, then

$$P(\bigcup_{j=1}^{\infty} F_j) \leq \sum_{j=1}^{\infty} P(F_j).$$

Definition 2.6.3. Then $(\Omega, \mathcal{F}, P)$ is said to be a probability space.

It is known as a complete probability space whenever $\mathcal{F}$ consists all the subsets B of Ω with P-outer measure zero, i.e. with

$$P^*(B) = \inf\{P(F); F \in \mathcal{F}, B \subset F\} = 0,$$

where P^* represents the outer measure of B.

Definition 2.6.4. Let $(\Omega, \mathcal{F}, P)$ be a probability space. A function $Y : \Omega \to \mathbb{R}^n$ is known as $\mathcal{F}$-measurable whenever

$$Y^{-1}(U) := \{\omega \in \Omega : Y(\omega) \in U\} \in \mathcal{F},$$

$\forall\ U \in \mathbb{R}^n$, where U is any open set.

Definition 2.6.5. Let $(\Omega, \mathcal{F}, P)$ be a probability space. A mapping $\mathbf{X} : \Omega \to \mathbb{R}^n$ is said to be an n-dimensional random variable if for each $F \in \mathcal{B}$, we have

$$\mathbf{X}^{-1}(F) \in \mathcal{F}.$$

The random variable $\mathbf{X}$ is also $\mathcal{F}$-measurable.

Let $F \in \mathcal{F}$. Then the indicator function of F,

$$\chi_F(\omega) := \begin{cases} 1, & \omega \in F \\ 0, & \omega \notin F \end{cases}$$

is a random variable.

Lemma 2.6.6. Let $\mathbf{X} : \Omega \to \mathbb{R}^n$ be a random variable. Then

$$\mathcal{F}(\mathbf{X}) := \{\mathbf{X}^{-1}(F) : F \in \mathcal{B}\}$$

is called the σ-algebra generated by $\mathbf{X}$. This is the smallest sub-σ-algebra of $\mathcal{F}$ with respect to which $\mathbf{X}$ is measurable.

Definition 2.6.7. If $\int_\Omega |\mathbf{X}| dP < \infty$, then the number

$$E[\mathbf{X}] = \int_\Omega \mathbf{X} dP,$$

is called the expectation of $\mathbf{X}$ (w.r.t. P).

Definition 2.6.8. A stochastic process is a parameterized collection of random variables $\{\mathbf{X}(t)|\, t \geq 0\}$ on a probability space $(\Omega, \mathcal{F}, P)$ and taking values in $\mathbb{R}^n$.

For every fixed $t \geq 0$, we get a random variable

$$\omega \to \mathbf{X}(t, \omega); \quad \omega \in \Omega.$$

Again, fixing $\omega \in \Omega$, the function

$$t \to \mathbf{X}(t, \omega); \quad t \geq 0,$$

is called a path of $\mathbf{X}(t)$.

Usually we denote a stochastic process by $\{\mathbf{X}(t),\ t \in J \subset \mathbb{R}\}$, $X = \{\mathbf{X}(t)\}_{t \in J}$ or $\mathbf{X}(t),\ t \in J$, the dependence upon the second argument ω being omitted.

Let $J \subset \mathbb{R}$ be an interval. Now, we state following result which is used to study the stochastic process.

Definition 2.6.9. (*i*) The process $X = \mathbf{X}(t),\ t \in J$ is continuous if for a.a. ω, the functions $\mathbf{X}(\cdot, \omega)$ are continuous on J.
(*ii*) X is called to be right continuous if for a.a. ω, the functions $\mathbf{X}(\cdot, \omega)$ are right continuous on J.
(*iii*) The process $X = \{\mathbf{X}(t) : t \in J\}$ is continuous in probability if $t_n \to t_0$ with $t_n, t_0 \in J$ implies $\mathbf{X}(t_n) \to^P \mathbf{X}(t_0)$.
(*iv*) X is said to be a measurable process if it is measurable on the product space with respect to the σ-algebra $\mathcal{B}(J) \otimes \mathcal{F}$, $\mathcal{B}(J)$ is a σ-algebra of Borel sets in J.

Definition 2.6.10. Suppose $(\Omega, \mathcal{F}, \mathbb{P})$ to be a probability space. A filtration $\{\mathcal{F}_t|\ t \in J\}$ is a weakly increasing collection of σ-algebras on Ω and bounded above by $\mathcal{F}$, i.e. for $s, t \in J$ with $s < t$,

$$\mathcal{F}_s \subseteq \mathcal{F}_t \subseteq \mathcal{F}.$$

A stochastic process X is said to be adapted to the filtration if, for every $t \in J$, $\mathbf{X}(t)$ is $\mathcal{F}_t$-measurable.

Definition 2.6.11. The filtration is said to be normal if
(*i*) $\mathcal{F}_0$ contains all $B \in \mathcal{F}$ such that $P(A) = 0$,
(*ii*) $\mathcal{F}_t = \mathcal{F}_{t+}$, $t \in J$, where $\mathcal{F}_{t+}$ denotes the intersection of all $\mathcal{F}_s$ for $s > t$.

Definition 2.6.12. Suppose $\mathbf{X} = \{\mathbf{X}(t) : t \in J\}$ to be the stochastic process. The natural filtration for process X is the filtration, where $\mathcal{F}_t$ is generated by all values of $\mathbf{X}(s)$ up to time $s = t$, i.e. $\mathcal{F}_t = \sigma(\{\mathbf{X}^{-1}(s)(A) : s \leq t,\ A \in \Sigma\})$. A stochastic process is always adapted to its natural filtration.

Let us consider a family $\mathbf{F} = \{\mathcal{F}_t : t \in J\}$ of σ-algebras $\mathcal{F}_t \subset \mathcal{F}$ with the property that $t_1 < t_2$ gives $\mathcal{F}_{t_1} \subset \mathcal{F}_{t_2}$.

Definition 2.6.13. A continuous stochastic process $W(t),\ t \geq 0$ is called a standard Brownian motion or a standard Wiener process if:
(i) $W(0) = 0$,
(ii) $W(t)$ is a almost surely continuous stochastic process with independent increments,
(iii) $EW(t) = 0,\ t \geq 0$, and $E|W(t) - W(s)|^2 = |t - s|$ for $t \geq s \geq 0$.

Definition 2.6.14. An n-dimensional stochastic process $\mathbf{X}(t) = (\mathbf{X}^1(t), \cdots, \mathbf{X}^n(t))$, $t \geq 0$ is called an n-dimensional standard Wiener process if each process $W^i(t)$ is a standard Brownian motion and the σ-algebras $\sigma(W^i(t) : t \geq 0),\ 1 \leq i \leq n$ are independent.

Definition 2.6.15. Suppose that (X, d) denotes a metric space, and let $G \subseteq \mathbb{R}$. A function $\beta : G \to X$ is said to be a càdlàg function, if $\forall\ t \in G$

- $\beta(t-) := \lim_{s\uparrow t} \beta(s)$ exists; and
- $\beta(t+) : \lim_{s\downarrow t} \beta(s)$ exists and equal to $f(t)$.

i.e., β is right-continuous with left limits.

Definition 2.6.16. A bounded linear operator T over a separable Hilbert space H is called the trace class whenever for every orthonormal bases $\{e_k\}_k$ of H the sum of positive terms

$$\|T\|_1 = Tr|T| = \sum_k < (T^*T)^{1/2} e_k, e_k >,$$

is finite. In this case, the sum

$$Tr(T) = \sum_k < Te_k, e_k >$$

is absolutely convergent. It is independent of the orthonormal basis. It is known as the trace of T.

A linear bounded operator $T : H \to H$ is said to be Hilbert-Schmidt operator if $\sum_{k=1}^{\infty} \|Te_k\|^2 < \infty$.

We consider X and K to be two separable Hilbert spaces. Suppose $\beta_n(t)$ $(n = 1, 2, \cdots)$ be a sequence of real-valued one dimensional standard Brownian motions mutually independent over $(\Omega, \mathcal{F}, P)$.
Let ς_n be a complete orthonormal basis in K. Then $Q \in L(K, K)$ is the operator defined by $Q\varsigma_n = \lambda_n \varsigma_n$ with finite trace $Tr(Q) = \sum_{n=1}^{\infty} \lambda_n \leq \infty$. We define

$$W(t) = \sum_{n=1}^{\infty} \sqrt{\lambda_n}\beta_n(t)\varsigma_n(t), \ t \geq 0.$$

$W(t) \in K$ is the Q-Wiener process. $\mathcal{F}_t = \sigma(W(s) : 0 \leq s \leq t)$ denotes the filtration. We denote $\mathcal{F}_T = \mathcal{F}$.

$\phi \in L(K, X)$. denotes a Q-Hilbert Schmidt function, if

$$\|\phi\|_Q^2 = Tr(\phi Q \phi^*) = \sum_{n=1}^{\infty} \|\sqrt{\lambda_n}\phi\varsigma_n\|^2 \leq \infty.$$

$L_Q(K, X)$ with the norm $\|\phi\|_Q^2 = <\phi, \phi>$ is a Hilbert space.

Chapter 3

Controllability of a Functional Differential System

In this chapter, controllability of functional differential system with bounded delay is studied. We removed the use of analytic semigroup and compactness of the nonlinear function. The limitation of non-existence of the inverse of controllability operator is overcome by assuming a geometric relation between the range of the operator B and a subspace associated with the fundamental solution. An example is studied to substantiate the results.

3.1 Introduction

The controllability of infinite dimensional systems represented by nonlinear evolution equations is widely investigated in various articles such as [39],[70],[156], etc.

Chang and Liu [60] established the existence of mild and strong solutions of a neutral differential equations involing nonlocal conditions. The authors used Sadovskii fixed point theorem combined with compact analytic semigroups of uniformly bounded linear operators.

The use of fractional operators, analyticity and compactness to prove these results, imposed severe restrictions on the semigroup as well as the nonlinear part of the system. Interestingly, the results for controllability of impulsive functional differential equations with deviated argument are not abundantly available. To remove the above limitations, is one of the motivations of this chapter.

The approximate controllability of a class of functional differential equation involving deviating argument and finite delay is discussed. Sufficient conditions for approximate controllability are derived by assuming the approximate controllability of the linear control system. Schauder fixed point theorem is used. We proceed by establishing a connection between the reachable set of linear control problem and of the semilinear delay control problem coupled with deviating argument.

3.2 Functional Differential Equation with Deviated Argument and Bounded delay

In this section we study the approximate controllability of the functional differential equation with finite delay and deviated argument, which is illustrated as follows.

$$\begin{aligned} \frac{dx(t)}{dt} &= Ax(t) + A_1x_t + Bu(t) + f(t, x_t, x(a(x(t), t))), t \in J = [0, \tau] \\ x(t) &= \phi(t), -h \leq t \leq 0 \end{aligned} \tag{3.2.1}$$

where $x(t) \in X$ and $u(t) \in U$, X and U being Hilbert spaces. Let $Z = L^2([0, \tau]; X)$, $Z_h = L^2([-h, \tau]; X)$, $0 < h < \tau$ and $Y = L^2([0, \tau]; U)$ be the corresponding function spaces. $A : D(A) \subset X \to X$ is a closed linear operator which generates a strongly continuous semigroup $T(t)$. A_1 is a bounded linear operator from $C([-h, \tau]; X)$ to $L^2([0, \tau], X)$. $B : Y \to Z$ is a bounded linear operator. When $x : [-h, \tau] \to X$ is a continuous function then $x_t(.)$ is denoted by $x_t(\theta) = x(t + \theta)$, $\theta \in [-h, 0]$ and $\phi \in C([-h, 0]; X)$. $x_t \in C([-h, 0], X)$ a Banach space of all continuous functions from $[-h, 0]$ to X with norm

$$\|x_t\|_C := sup_{\theta \in [-h,0]} \|x_t(\theta)\|_X = sup_{\theta \in [-h,0]} \|x(t + \theta)\|_X \ \ for\ t \in (0, \tau].$$

$C_L(J, X) = \{u \in C(J, X) : \exists l > 0 \ such\ that\ \|u(t) - u(s)\| \leq l|t - s|, \forall t, s \in J\}$. Simple Lipschitz conditions are required to study the differential equation with deviated argument.

3.2.1 Existence and uniqueness of mild solution

Let us state some definitions and lemmas which are used in proving the existence and uniqueness of the mild solution and approximate controllability of (3.2.1). In equation (3.2.1) if we put $f \equiv 0$ the resulting equation without the delay term is called the corresponding linear system (3.2.2)

$$\begin{aligned} \frac{dx(t)}{dt} &= Ax(t) + Bu(t), \ t \in [0, \tau] \\ x(0) &= \phi(0) \ \in [-h, 0] \end{aligned} \tag{3.2.2}$$

Let us consider the linear delayed system

$$\begin{aligned} \frac{dx(t)}{dt} &= Ax(t) + A_1 x_t, \ t \in [0, \tau] \\ x_0 &= \phi \ \in [-h, 0] \end{aligned} \tag{3.2.3}$$

Let $x^{\phi}(t)$ be the unique solution of system (3.2.3). Let $\mathcal{L}(X)$ denote the Banach space of all bounded linear operators on X. Define a map $S : J \to \mathcal{L}(X)$ by

$$S(t)\phi(0) = \begin{cases} x^{\phi}(t), & t \geq 0; \\ 0, & t < 0. \end{cases} \tag{3.2.4}$$

Then $S(t)$ is called the fundamental solution of (3.2.3) satisfying

$$\begin{aligned} &S(t) = T(t)\phi(0) + \int_0^t T(t-s)A_1 S(s+\theta)ds, \ t > 0 \\ &S(0) = I, S(t) = 0, \ -h \leq t < 0 \end{aligned} \tag{3.2.5}$$

It follows from [169] that $S(t)$ is the unique solution of (3.2.3). It can be easily shown that

$$S(t) \leq K_0 \exp(K_0 ||A_1|| \tau) := M$$

where $\{max\|T(t)\|, \ t \in [0, \tau]\} = K_0$.

Therefore the mild solution of semilinear control system (3.2.1) is defined as

Definition 3.2.1. The function $x : (-h, \tau] \to X$ is said to be a mild solution of (3.2.1) if $x(.) \in C_L(J, X)$, $x(t) = \phi(t)$ for $t \in [-h, 0]$ and it satisfies the integral equation.

$$x(t) = S(t)\phi(0) + \int_0^t S(t-s)Bu(s)ds + \int_0^t S(t-s)f(s, x_s, x(a(x(s), s)))ds, \ t \in J \tag{3.2.6}$$

and the mild solution of the corresponding linear system with delay and control term

$$\begin{aligned} \frac{dx(t)}{dt} &= Ax(t) + A_1 x_t + Bu(t), \ t \in [0,\tau] \\ x_0 &= \phi \ \in [-h, 0] \end{aligned} \tag{3.2.7}$$

is defined as

$$\begin{aligned} x(t) =& S(t)\phi(0) + \int_0^t S(t-s)Bu(s)ds, t \in [0,\tau] \\ x(t) =& \phi(t), -h \le t < 0. \end{aligned} \tag{3.2.8}$$

Definition 3.2.2. The set given by $K_\tau(f) = \{x(T) \in X : x \in Z_h\}$ is called reachable set of the system (3.2.1) . $K_\tau(0)$ denotes the reachable set of the associated linear system (3.2.7).

Definition 3.2.3. The system (3.2.1) is called approximately controllable whenever $K_\tau(f)$ is dense in X. The associated linear system is approximately controllable whenever $K_\tau(0)$ is dense in X.

Let us assume that

(H1) The nonlinear function $f : J \times X \times X \to X$ satisfies Lipschitz condition,

$$\|f(t, x_1, z_1) - f(t, x_2, z_2)\| \le P(\|x_1 - x_2\| + \|z_1 - z_2\|)$$

for all $x_1, x_2, z_1, z_2 \in X, \ t \in (0, \tau]$ and $\exists$ a constant $g > 0$,
such that $\|f(s, 0, x(a(x(0), 0)))\| \le g, \ \forall \ s \in J$

(H2) Let $a : X \times R^+ \to R^+$ satisfy the Lipschitz condition $|a(x_1, s) - a(x_2, s)| \le L_a\|x_1 - x_2\|$ and $a(., 0) = 0$

Lemma 1. The fundamental solution $S(t)$ is bounded.

Proof Since

$$\begin{aligned}
||S(t)|| \leq & K_0 + K_0||A_1|| \int_0^t ||S(s+\theta)||ds \\
\leq & K_0 + k_0||A_1|| \int_0^{t+\theta} ||S(\sigma)||d\sigma \\
\leq & K_0 + ||A_1||K_0 \int_{-h}^{t} ||S(\sigma)||d\sigma \\
\leq & K_0 + K_0||A_1|| \int_0^{t+h} ||S(\sigma)||d\sigma
\end{aligned} \tag{3.2.9}$$

$$||S(t)|| \leq K_0 \exp K_0||A_1||(t+h) \leq K_0(1+d)\exp K(\tau+h) = M$$

$$max\{||S(t)|| : t \in [0,\tau]\} = M$$

Hence the fundamental solution is bounded.

Lemma 2. If the C_0–semigroup $T(t)$ is compact then the fundamental solution $S(t)$ is compact.

Proof: Let us define the sequence of operators $S_n(t)$ on $[-h,\tau]$. From the compactness of $T(t)$ and boundedness of $||A_1||$ we conclude that S_n is compact. Let $||A_1|| = K_1$. To prove $S_n(t) \to S(t)$ in $\mathcal{L}(X)$ we first show that $\{S_n(t)\}$ is a Cauchy sequence in $\mathcal{L}(X)$. Let us define

$$\begin{aligned}
S_1(t) &= T(t), t \in [0,\tau] \\
&= 0, t \in [-h,0] \\
S_{n+1}(t) &= T(t) + \int_0^t T(t-s)S_n(s+\theta)ds, t \in (0,\tau], \theta \in [-h,0] \\
&= 0, t \in [-h,0]
\end{aligned} \tag{3.2.10}$$

for $n = 1, 2, ...$

Therefore,

$$\begin{aligned}
||S_2(t) - S_1(t)|| &\leq \int_0^t ||T(t-s)||||A_1||||S(s+\theta)||ds \\
&\leq K_0K_1Mt
\end{aligned} \tag{3.2.11}$$

$$||S_{n+1}(t) - S_n(t)|| \leq \frac{1}{n!}K_0^nK_1^nM_1\tau^n \to 0 \;\; as, \; n \to 0$$

Thus $\{S_n(t)\}$ is a cauchy sequence. As $\mathcal{L}(X)$ is the Banach space of all bounded linear operators on X, $\exists$ an operator $S(t) \in \mathcal{L}(X)$ such that $S_n(t) \to S(t)$ uniformly on $[0,\tau]$ and hence $S(t)$ is compact $\forall t \in [0,\tau]$. It is easy to check that $S(t)$ is unique.

Now, the equation (3.2.6) is checked to be the unique mild solution of (3.2.1).

Theorem 1. The system (3.2.1) has a unique mild solution in $C_L(J,X)$ for every control $u \in L_2([0,T];U)$ whenever assumptions $(H1)$ and $(H2)$ hold.

Proof: Suppose we define the space $C_{L_0}([-h,\tau],X) = \{x \in C([-h,\tau],X) : x \in C_L([0,\tau],X)\}$. Fix $0 < t_1 < T$ such that

$$PMt_1(l + 2lL_a)R < M\|\phi\| + MM_BT\|u\| + MTg + 1$$

Define the mapping $\Phi : C_{L_0}([-h,t_1],X) \to C_{L_0}([-h,t_1],X)$ as

$$\begin{aligned}(\Phi x)(t) &= S(t)\phi(0) + \int_0^t S(t-s)[Bu(s) + f(s,x_s,x(a(x(s),s)))]ds,\ t \in (0,t_1] \\ &= \phi(\theta),\ \ \theta \in [-h,0] \qquad (3.2.12)\end{aligned}$$

Suppose we take the space $B_R = \{x(.) \in C_{L_0}([-h,t_1],X) : \|x\|_{C([-h,t_1],X)} \leq R,\ \ x(0) = \phi(0)\}$ endowed with the norm of uniform convergence. For any $x \in B_R$ and $0 \leq t \leq t_1$,

$$\|x_t\|_C = sup_{-h\leq\theta\leq 0}\|x_t(\theta)\|_X \leq sup_{-h\leq\zeta\leq t_1}\|x(\zeta)\|_X \leq R.$$

Then

$$\begin{aligned}\|(\Phi x)(t)\| &\leq M\|\phi(0)\| + MM_BT\|u\| \\ &+ \int_0^t M[\|f(s,x_s,x(a(x(s),s))) - f(s,0,x(a(x(0),0)))\| \\ &+ \|f(s,0,x(a(x(0),0)))\|]ds \\ &\leq M\|\phi\| + MM_BT\|u\| \\ &+ \int_0^t M[P(\|x(s+\theta) - 0\| + lL_a\|x(s) - x(0)\|) + g]ds \\ &\leq M\|\phi(0)\| + MM_Bt_1\|u\| \\ &+ \int_{-h}^{t_1} MP(\|x(\sigma)\|d(\sigma) + \int_0^{t_1}[MlL_a\|x(s) - x(0)\| + g]ds \\ &\leq M\|\phi(0)\| + MM_Bt_1\|u\| + M(t_1+h)P\|x\| + 2Mt_1PlL_a\|x\| + gt_1 \\ &\leq M\|\phi(0)\| + MM_Bt_1\|u\| + M(t_1+h)PR + 2Mt_1PlL_aR + gt_1\end{aligned}$$

Let

$$M\|\phi\| + MM_Bt_1\|u\| + M(t_1+h)PR + 2Mt_1PlL_aR + gt_1 < R$$

Then

$$M\|\phi\| + MM_Bt_1\|u\| + gt_1 < R(1 - M(t_1+h)P - 2Mt_1PlL_a).$$

RHS is positive if

$$\begin{aligned} t_1(PM + 2MPlL_a) &< M(t_1+h)P + 2Mt_1PlL_a < 1 \\ t_1 &< \frac{1}{(PM + 2MPlL_a)} \end{aligned} \tag{3.2.13}$$

Hence Φ maps B_R into itself when t_1 satisfies (3.2.13). Then we prove that Φ is a contraction. Let $x_1,\ x_2 \in B_R$

$$\begin{aligned} \|(\Phi x_1)(t) - (\Phi x_2)(t)\| &\leq \int_0^t M\|f(s,(x_1)_s, x_1(a(x_1(s),s))) \\ &- f(s,(x_1)_s, x_1(a(x_2(s),s))) - f(s,(x_2)_s, x_2(a(x_2(s),s))) \\ &+ f(s,(x_1)_s, x_1(a(x_2(s),s)))\|ds \\ &\leq tMP[\|x_1(a(x_1(s),s)) - x_1(a(x_2(s),s))\| \\ &+ (\|(x_2)_s - (x_1)_s\| \\ &+ \|x_2(a(x_2(s),s) - x_1(a(x_2(s),s)))\|)] \\ &\leq tMP[l|a(x_1(s),s) - a(x_2(s),s)| \\ &+ \|x_2(s+\theta) - x_1(s+\theta)\| + (\|x_2 - x_1\|_{C([-h,t_1];X)})] \\ &\leq tM(lPL_a\|x_1(s) - x_2(s)\|_{C([-h,t_1],X)} \\ &+ P\|x_2(t_1) - x_1(t_1)\| + P\|x_2 - x_1\|_{C([-h,t_1],X)}) \\ &\leq Mt(lPL_a + 2P)\|x_2 - x_1\|_{C([-h,t_1],X)} \end{aligned} \tag{3.2.14}$$

So, $\|\Phi x_1 - \Phi x_2\|_{C([-h,t_1],X)} \leq Mt(lPL_a + 2P)\|x_1 - x_2\|_{C([-h,t_1],X)}$. Thus Φ is a contraction mapping. Therefore, Φ has a fixed point in B_R. Hence (3.2.6) is the mild solution on $[-h, t_1]$. Similarly it can be proved that (3.2.6) is the mild solution on the interval $[t_1, t_2],\ t_1 < t_2$ Repeating the above process we get that

$$\|\Phi^n x_1 - \Phi^n x_2\|_{C([-h,t_1],X)} \leq \frac{Mt^n}{n!}(lPL_a + 2P)\|x_1 - x_2\|_{C([-h,t_1];X)}.$$

Thus (3.2.6) is the mild solution on the maximal existence interval $[-h, t^*],\ t^* < \infty$.

Then we prove that x is well defined in $[-h, \tau]$. For that when $t \in [-h, 0]$, then

$x(t) = \phi(t)$. Therefore it is bounded. When $t \in [0, t^*)$ then

$$\begin{aligned}
\|x(t)\| &\leq M\|\phi\| + M\int_0^t [M_B\|u(s)\| + P\|x_s - 0\| \\
&+ P|x(a(x(s), s) - x(a(x(0), 0)\| + g]ds \\
&\leq M\|\phi\| + MM_B\tau\|u(s)\| \\
&+ M\int_0^t P[\|x_s\| + lL_a\|x(s) - x(0)\| + g] \\
&\leq M\|\phi\| + MM_B\tau\|u(s)\| \\
&+ M\tau P(\|x(0)\| + g) + M\int_0^t l\|x(s)\|ds \qquad (3.2.15)
\end{aligned}$$

By Gronwall's inequality $\|x(t)\| \leq \|x_t\|_C \leq [M\|\phi\| + MM_B\tau\|u(s)\| + MTP(\|x(0)\| + g)] \exp(M\tau P)$. Thus $\|x(t)\|$ is bounded. Hence x is well defined on $[-h, \tau]$. To prove the uniqueness of solution let x_1 and x_2 be any two mild solutions of (3.2.6) such that for $t \in [-h, 0]$, $x_1(t) = x_2(t) = \phi$. For $t \in [0, t^*)$

$$\begin{aligned}
\|x_1(t) - x_2(t)\| &\leq M\int_0^t \|f(s, (x_1)_s, x_1(a(x_1(s), s))) \\
&- f(s, (x_2)_s, x_2(a(x_1(s), s)))\|ds + f(s, (x_2)_s, x_2(a(x_1(s), s))) \\
&- f(s, (x_2)_s, x_2(a(x_2(s), s)))\| \\
&\leq M\int_0^t P\{\|(x_1)_s - (x_2)_s\| + \|x_1(s) - x_2(s)\| \\
&+ lL_a\|x_1(s) - x_2(s)\|\}ds \\
&\leq M\int_{-h}^t P\|x_1(\eta) - x_2(\eta)\|d\eta + M\int_0^t P\|x_1(s) - x_2(s)\|ds \\
&+ M\int_0^t PlL_a\|x_1(s) - x_2(s)\|ds \\
&\leq M\int_{-h}^0 P\|x_1(\eta) - x_2(\eta)\|d\eta + M\int_0^t P(2 + lL_a)\|x_1(s) - x_2(s)\|ds
\end{aligned}$$

Since uniqueness of the mild solution is proved on $[-h, 0]$ we get

$$\|x_1(t) - x_2(t)\| \leq MP(2 + lL_a)\int_0^t \|x_1(s) - x_2(s)\|ds$$

Hence by Gronwall's inequality $x_1(t) = x_2(t)$ for all $t \in [-h, \tau]$.

3.2.2 Approximate Controllability

Define a linear operator L from Z to $C_L([0, \tau], X)$ by $Lx = \int_0^\tau S(t - s)x(s)ds, t \in [0, \tau]$.

Let $Kx(t) = \int_0^t S(t-s)x(s)ds, t \in [0,\tau]$.
Z can be decomposed uniquely as $Z = N_0(L) \oplus N_0^{\perp}(L)$ where $N_0(L)$ is the null space of the operator L and $N_0(L)$ is its orthogonal space.
Let us assume

(H3) $\forall\, p \in Z$, $\exists$ a function $q \in \overline{R(B)}$ such that $Lp = Lq$.

The approximate controllability of the corresponding linear system (3.2.2) follows from the hypothesis $(H3)$. Then it is to be proved that the linear system (3.2.7) with finite delay is approximately controllable. Next by assuming that the linear system with delay (3.2.7) is approximately controllable, the system (3.2.1) is to be proved to be approximately controllable using Schauder fixed point theorem. Define the operator $F : C_{L_0}([0,\tau], X) \to L^2([0,\tau], X)$ as

$$F(x)(t) = f(t, x_t, x(a(x(t),t))); 0 < t \leq \tau.$$

Hypotheses $(H1)$, $(H2)$ imply that F is a continuous map. Hypothesis $(H3)$ implies that for any $p \in Z$, $\exists$ a $q \in R(B)$ such that $L(p-q) = 0$. Therefore $p - q = n \in N_0(L)$ which implies that $Z = N_0(L) \oplus \overline{R(B)}$. Thus, it follows that the existence of a linear and continuous mapping Q from $N_0^{\perp}(L)$ into $\overline{R(B)}$. It is defined as $Qu^* = v$ where v is the unique minimum norm element $v \in (u^* + N_0(L)) \bigcap \overline{R(B)}$, i.e. $\|Qu^*\| = \|v\| = min\{\|v\| : v \in \{(u^* + N_0(L)) \bigcap \overline{R(B)}\}$. $(H3)$, implies that for every $v \in \{u^* + N_0^{\perp}\} \cap \overline{R(B)}$ is not empty and every $z \in Z$ has a unique decomposition $z = n + q$. Thus, the operator Q is well defined. Moreover, $\|Q\| = c$ for some constant c.

Let us consider the subspace M_0 of $C_{L_0}([0,\tau], X)$ which is defined as

$$M_0 = \begin{cases} m \in C_{L_0}([0,\tau], X) : m(t) = Kn(t), & n \in N_0(L); 0 \leq t \leq \tau \\ m(t) = 0, & -h \leq t \leq 0; \end{cases} \tag{3.2.16}$$

Let

$$f_x : \overline{M_0} \to \overline{M_0}$$

defined by

$$f_x = \begin{cases} Kn, & 0 < t \leq \tau; \\ 0, & -h \leq t \leq 0; \end{cases} \tag{3.2.17}$$

where n is given by the unique decomposition of $F(x+m)(t) = n(t)+q(t)$, $n \in N_0(L)$ and $q \in \overline{R(B)}$.

The following assumption is made .

$(A1)$ $\overline{R(A_1)} \subset \overline{R(B)}$

Theorem 2. The operator f_x has a fixed point in M_0 if $M(1+c)P\tau < 1$.

Proof: Since $S(t)$ is compact so K is compact and f_x is compact. Let $z \in Z$ then $z = q + n$, $n \in N_0(L)$, $q \in \overline{R(B)}$. Also $\|n\|_Z \leq (1+c)\|z\|_Z$ (see [158]). Let

$$B_r = \{v \in \overline{M_0} : ||v|| \leq r\}.$$

Let $m \in B_r$. Let $\|f(0,0,(x+m)(a(m(s),0)\| \leq l_f$ Suppose on the other hand

$$\begin{aligned}
r < \|f_x(m)\| =& \|Kn\| \leq \int_0^t \|S(t-s)n(s)\|ds \\
&\leq \int_0^t M(1+c)\|F(x+m)\|_Z ds \\
&\leq \int_0^t M(1+c)[\|f(s,(x+m)_s,(x+m)(a((x+m)(s),s)))\| \\
&-\|f(0,0,(x+m)(a(m(s),0))))\| + \|f(0,0,(x+m)(a(m(s),0))))\|] \\
&\leq M(1+c)\int_0^t P[\|(x+m)(s+\theta)-0\| \\
&+\|(x+m)(a((x+m)(s),s)) - (x+m)(a((m)(s),0))\| + l_f]ds \\
&\leq M(1+c)\int_0^t P[\|x\| + \|m\| + l|a((x+m)(s),s) - a(m(s),0)| + l_f]ds \\
&\leq M(1+c)\int_0^t P[\|x\| + r + lL_a\|(x+m)(s) - m(s)\| + l_f]ds \\
&\leq M(1+c)\int_0^t P[\|x\| + r + lL_a\|x\| + l_f]ds \\
&\leq M(1+c)P(\|x\|T + r\tau + lL_a\|x\|T + l_fT)
\end{aligned} \tag{3.2.18}$$

Dividing by r and taking limit as r tends to ∞ we get a contradiction. So f_x maps B_r into itself. Therefore by Schauder fixed point theorem it has a fixed point.

Theorem 3. Suppose the linear control system (3.2.2)

$$\begin{aligned}
\frac{dx(t)}{dt} &= Ax(t) + Bu(t) \\
x(0) &= \phi(0)
\end{aligned} \tag{3.2.19}$$

is approximately controllable then the linear delay control system (3.2.7)

$$\begin{aligned} \frac{dx(t)}{dt} &= Ax(t) + A_1x_t + Bu(t) \\ x(t) &= \phi(t), \ -h \le t \le 0 \end{aligned}$$

is controllable if assumptions $(A1)$ hold.

Proof: Consider

$$\begin{aligned} y'(t) = Ay(t) + Bu(t), t \ \in [0, \tau] \\ y(t) = \phi(t), \ t \in [-h, 0] \end{aligned} \tag{3.2.20}$$

The mild solution of equation (3.2.20) is as follows

$$\begin{aligned} y(t) = T(t)\phi(0) + \int_0^t T(t-s)Bu(s)ds, \ t > 0 \\ y(t) = \phi(t), \ t \in [-h, 0] \end{aligned} \tag{3.2.21}$$

Since $\overline{R(A_1)} \subset \overline{R(B)}$. So $\forall \ \epsilon > 0, \exists \ w \in U$ such that

$$\|A_1 y_s - Bw\|_Z \le \epsilon$$

Let $x(t)$ be a solution of linear delay control system corresponding to control $(u-w)$ satisfying

$$\begin{aligned} x(t) = T(t)\phi(0) + \int_0^t T(t-s)\{B(u-w) + A_1x_s\}ds, t > 0 \\ x(t) = \phi(t), t \in [-h, 0] \end{aligned} \tag{3.2.22}$$

If $t \in [-h, 0]$, then

$$x_0(t) - y_0(t) = 0$$

and if $t \in (0, \tau]$ then we get

$$\begin{aligned} x(t) - y(t) &= \int_0^t T(t-s)[-Bw(s) + A_1x_s] \\ &= \int_0^t T(t-s)[-Bw(s) + A_1y_s]ds \\ &+ \int_0^t T(t-s)[A_1x_s - A_1y_s]ds \end{aligned}$$

Taking norm on both sides

$$\begin{aligned}\|x(t)-y(t)\| &\le K_0\int_0^t \|Bw(s)-A_1x_s\|ds\\ &+K_0\int_0^t\|A_1x_s-A_1y_s\|ds\\ &\le K_0\tau\|Bw(s)-A_1x_s\|_Z+K_0\int_0^t K_1\|x_s-y_s\|ds\\ &\le K_0\epsilon\tau+K_0\int_0^t K_1\|x_s-y_s\|ds\\ &\le K_0\epsilon\tau+K_0\int_{-h}^t K_1\|x(\eta)-y(\eta)\|d\eta\end{aligned}$$

where $\|A_1\| \le K_1$ since A_1 is bounded linear operator from $C_{L_0}([-h,\tau],X)$ to $L^2([0,\tau],X)$. This implies

$$||x(t)-y(t)|| \le K_0\epsilon\tau+K_0K_1\int_{-h}^t\|x(\eta)-y(\eta)\|d\eta \tag{3.2.23}$$

Using Gronwall's inequality

$$||x(t)-y(t)|| \le K_0\epsilon\tau\exp(K_0K_1\{\tau+h\})$$

Since RHS depends on ϵ so it can be made as small as possible. This implies that the reachable set of linear delay control system is dense in the reachable set of the linear control system (3.2.2) which in turn is dense in X as (3.2.7) is apprroximately controllable. Hence the linear delay control system is controllable.

Theorem 4. The semilinear control system (3.2.1) is approximately controllable if the linear delay control system (3.2.7)

$$\begin{aligned}\frac{dx(t)}{dt} &= Ax(t)+A_1x_t+Bu(t)\\ x(t) &= \phi(t),\ -h\le t\le 0\end{aligned}$$

is approximately controllable .

Proof: Let $x(.)$ be the mild solution of the linear delay control system (3.2.7) is given by

$$x(t)=S(t)\phi(0)+KBu(t), t\in(0,\tau]$$

$$x(t)=\phi(t), t\in[-h,0]$$

We prove

$$y(t) = x(t) + m_0(t)$$

to be mild solution semilinear problem (3.2.1). Since

$$KF_h(x + m_0)(t) = Kn(t) + Kq(t)$$

operating K on both sides at $m = m_0$, fixed point of f_x.

$$\begin{aligned} KF_h(x + m_0)(t) &= Kn(t) + Kq(t) \\ &= m_0(t) + Kq(t) \end{aligned} \tag{3.2.24}$$

Add $x(.)$ to both sides and using $y(t) = x(t) + m_0(t)$

$$\begin{aligned} x(t) + KF_h(x + m_0)(t) &= x(t) + m_0(t) + Kq(t) \\ x(t) + KF_h(y)(t) &= y(t) + Kq(t) \\ \Rightarrow y(t) &= x(t) + KF_h(y)(t) - Kq(t) \\ \Rightarrow y(t) &= S(t)\phi(0) + K(Bu - q)(t) + KF_h(y)(t) \end{aligned} \tag{3.2.25}$$

This is the mild solution of semilinear problem with control $(Bu - q)$. By following the same proof in [155] we get the following conclusion that since $q \in \overline{R(B)}$ there exists a $v \in U$ such that $\|Bv - q\| < \epsilon$ for any given $\epsilon > 0$. Let x_v be a solution of the given semilinear delay control system (3.2.1) corresponding to the control v. Then as shown by [139] we have $||y(\tau) - x_v(\tau)|| = ||x(\tau) - x_v(\tau)|| \leq \epsilon$. This implies that $x(\tau) \in \overline{K_\tau(f)}$. Then it follows that $\overline{K_\tau(0)} \subset \overline{K_\tau(f)}$. Thus (3.2.1) is approximately controllable since the corresponding linear system (3.2.7) is approximately controllable.

3.2.3 Example

Let us consider the heat control system with finite delay

$$\begin{aligned} \frac{\partial y(t,x)}{\partial t} &= \frac{\partial^2 y(t,x)}{\partial x^2} + y(t+\theta, x) + Bu(t,x) + f(t, x(t+\theta), x(a(x(s), s)))ds \\ &\quad 0 < t < T,\ -h < \theta < 0,\ 0 < x < \pi \\ y(t,0) &= y(t,\pi) = 0,\ 0 \leq t \leq T \\ y(t,x) &= \xi(x),\ \ -h \leq t \leq 0,\ \ 0 \leq x \leq \pi. \end{aligned} \tag{3.2.26}$$

Let $X = L^2(0, \pi)$ and $A = -\frac{d^2}{dx^2}$. Define

$$D(A) = \{y \in X : y, \frac{dy}{dx} \text{ are absolutely continuous}, \frac{d^2y}{dx^2} \in X \text{ and } y(0) = y(\pi) = 0\}.$$

For $y \in D(A)$, $y = \sum_{n=1}^{\infty} < y, \phi_n > \phi_n$ and $Ay = -\sum_{n=1}^{\infty} n^2 < y, n > \phi_n$. where $\phi_n(x) = \frac{2}{\pi}^{\frac{1}{2}} \sin nx, \ 0 \leq x \leq \pi, \ n = 1, 2, 3...$ is the eigenfunction corresponding to the eigenvalue $\lambda_n = -n^2$ of the operator A. ϕ_n is an orthonormal base. A will generate a compact semigroup $T(t)$. such that $T(t)y = \sum_{n=1}^{\infty} e^{-n^2 t} < y, \phi_n > \phi_n, \ n = 1, 2, ... \ \forall \ y \in X$. Let the infinite dimensional control space be defined as $U = \{u : u = \sum_{n=2}^{\infty} u_n \phi_n, \ \sum_{n=2}^{\infty} u_n^2 < \infty\}$ with norm $\|u\|_U = (\sum_{n=2}^{\infty} u_n^2)^{\frac{1}{2}}$. Thus U is a Hilbert space.

Let $\widetilde{B} : U \to X : \ \widetilde{B}u = 2u_2\phi_1 + \sum_{n=2}^{\infty} u_n \phi_n \ \ for \ u = \sum_{n=2}^{\infty} u_n \phi_n \in U$. The bounded linear operator $B : L^2(0, T : U) \to L^2(0, T; X)$ is defined by $(Bu)(t) = \widetilde{B}u(t)$. Then this problem (3.2.26) can be reformulated into an abstract semilinear differential equation with deviated argument and finite delay by substituting $I = A_1$. If the hypotheses $(H1) - (H3)$ and assumption $(A1)$ are satisfied then it can be shown that this system (3.2.26) is approximately controllable.

3.3 Conclusion

We proved the existence and uniqueness and approximate controllability of the functional differential equation (3.2.1) with deviated argument and finite delay by using Schuader fixed point theorem, fundamental solution instead of C_0 semigroup and by establishing a geometric relation between the range of the operator B and a subspace related with the fundamental solution.

Chapter 4

Existence of Solution for a Second-order Neutral Differential Equation with State Dependent Delay and Non-instantaneous Impulses

In this chapter the existence of mild solution of a class of second order neutral differential equation invoving state dependent delay and non-instantaneous impulses is investigated. Hausdorff measure of noncompactness is used. Darbo Sadovskii fixed point theorem is applied to prove the existence. Also, some restrictive conditions such as the compactness assumption on the associated cosine or sine family of operators and the Lipschitz conditions on the nonlinear functions are replaced by simple and natural assumptions. In the last section we also study an example to illustrate the presented result.

4.1 Introduction

Non-instantaneous impulses occur abruptly at certain time points and continue their action for a specified duration of time. The study of non-instantaneous impulsive differential equations is significant to varied fields of applications like in the modeling of stage by stage rocket combustion, maintaining hemodynamical equilibrium, etc. A particular application is the abrupt injection of insulin in the bloodstream with consequent gradual absorption since it acts for a finite interval of time. Differential equation with non-instantaneous impulses are recently studied by Hernandez et.al [93].

We study the second order partial neutral differential equation with state dependent delay modeled in the form

$$\begin{aligned} \frac{d^2}{dt^2}(x(t) - g(t, x_t)) &= Ax(t) + f(t, x_{\rho(t,x_t)}, x'(t)),\ t \in (s_i, t_{i+1}],\ i = 0, ..., n \\ x_0 &= \phi \in \mathfrak{B}, \\ x'(0) &= \xi \in X, \\ x(t) &= J_i^1(t, x_t),\ t \in (t_i, s_i],\ i = 1, 2, ..., n \\ x'(t) &= J_i^2(t, x_t),\ t \in (t_i, s_i],\ i = 1, 2, ..., n \end{aligned} \tag{4.1.1}$$

where A denotes the infinitesimal generator of a strongly continuous cosine family $\{C(t) : t \in \mathsf{R}\}$ and $S(t)$ denotes the associated sine function. Here X is a Banach space. The history valued function $x_t : (-\infty, 0] \to X$, $x_t(\theta) = x(t + \theta)$ takes values in some abstract phase space $\mathfrak{B}$ defined in chapter 2 as Definition 2.2.12; $g, f, J_i^1, J_i^2, i = 1, .., n$ are defined in the following section. $0 = t_0 = s_0 < t_1 \leq s_1 \leq t_2, ..., < t_n \leq s_n \leq t_{n+1} = a$ are prefixed numbers. Let $J = [0, a]$.

Let $N, \widetilde{N}$ be certain constants such that $\|C(t)\| \leq N$ and $\|S(t)\| \leq \widetilde{N}$ for every $t \in J = [0, a]$. For more details refer books by Goldstein[84] and Fattorini[75].

$PC([0, a], X)$ denotes the space of normalized piecewise continuous function from $[0, a]$ into X. PC consists by all functions $u : [0, a] \to X$ such that u is continuous at $t \neq t_i$, $u(t_i^-) = u(t_i)$ and $u(t_i^+)$ exists for all $i = 1, 2, ..., n$. PC associated with the norm $\|x\|_{PC} = sup_{t \in J}\|x(t)\|$ is a Banach space. For any $x \in PC$

$$\widetilde{x}_i(t) = \begin{cases} x(t), & t \in (t_i, t_{i+1}]; \\ x(t_i^+), & t = t_i,\ i = 1, 2, ..., n. \end{cases} \tag{4.1.2}$$

So, $\widetilde{x} \in C([t_i, t_{i+1}], X)$.

Lemma 4.1.1. [44] If $W \subset PC^1(J,X)$ is bounded and the elements of W' are equicontinuous, then

$$\chi_{PC^1}(W) = max\{sup_{t\in J}\chi W(t), sup_{t\in J}\chi(W'(t))\}$$

where χ_{PC^1} denotes the Hausdorff measure of noncompactness in the space $PC^1(J,X)$.

4.2 Main Result

The mild solution of the problem (4.1.1) is as follows.

Definition 4.2.1. A function $x : (-\infty, a] \to X$ is a mild solution of the problem (4.1.1) if $x_0 = \phi$, $x'(0) = \xi$, $x(.)|_{[0,a]} \in PC^1(X)$, $x(t) = J_i^1(t, x_t)$, $\forall\ t \in (t_i, s_i]$, $i = 1, ..., n$, $x'(t) = J_i^2(t, x_t)$, $t \in (t_i, s_i]$, $i = 1, 2, ..., n$ and

$$\begin{aligned} x(t) &= C(t)(\phi(0) - g(0,\phi)) + S(t)(\xi - \eta) + g(t, x_t) + \int_0^t AS(t-s)g(s, x_s)ds \\ &+ \int_0^t S(t-s)f(s, x_{\rho(s,x_s)}, x'(s))ds,\ t \in [0, t_1] \\ x(t) &= C(t-s_i)(J_i^1(s_i, x_{s_i}) - g(s_i, x_{s_i})) + S(t-s_i)(J_i^2(s_i, x_{s_i}) - g'(s_i, x_{s_i})) \\ &+ g(t, x_t) + \int_{s_i}^t AS(t-s)g(s, x_s)ds + \int_{s_i}^t S(t-s)f(s, x_{\rho(s,x_s)}, x'(s))ds, \\ &\quad for\ \ t \in [s_i, t_{i+1}],\ i = 1, ..., n \end{aligned} \tag{4.2.1}$$

where $\frac{d}{dt}g(t, x_t)|_{t=0} = \eta$, where η is independent of x. To prove our result we always assume $\rho : J \times \mathfrak{B} \to (-\infty, a]$ is a continuous function. In this section $y : (-\infty, a] \to X$ is the function defined by $y_0 = \phi$ *and* $y(t) = C(t)(\phi(0) - g(0,\phi)) + S(t)(\xi - \eta)$ on $[0, t_1]$. Clearly $\|y_t\|_{\mathfrak{B}} \le K_a\|y\|_a + M_a\|\phi\|_{\mathfrak{B}}$ where $\|y\|_a = sup_{0\le t\le a}\|y(t)\|$

Let $S(a)$ be the space $S(a) = \{x : (-\infty, a] \to X :\ x_0 = 0,\ x'(0) = 0,\ x|_J \in PC^1\}$ endowed with norm $\|u\|_1 = \|u\|_\infty + \|u'\|_\infty$. The following hypotheses are used.

(H_ϕ) The function $t \to \phi_t$ is continuous from $\mathbb{R}(\rho^-) = \{\rho(s,\psi) : \rho(s,\psi) \le 0\}$ into $\mathfrak{B}$ and $\exists$ a continuous bounded function $J^\phi : \mathbb{R}(\rho^-) \to (0, \infty)$ such that $\|\phi_t\|_{\mathfrak{B}} \le J^\phi(t)\|\phi\|_{\mathfrak{B}}$ for every $t \in \mathbb{R}(\rho^-)$.

(Hf) The function $f : J \times \mathfrak{B} \to X$ satisfies the following:

(1) For every $x : (-\infty, a] \to X, x_0 \in \mathfrak{B}$ *and* $x|_J \in PC$, the function $f(., \psi, x) : J \to X$ is strongly measurable for every $\psi \in \mathfrak{B}$, $x \in X$ and $f(t, ., .)$ is continuous for a.e. $t \in J$.

(2) $\exists$ an integrable function $\alpha : J \to [0, +\infty)$ and a monotone continuous nondecreasing function $\Omega : [0, +\infty) \to (0, +\infty)$ such that $\|f(t, v, x)\| \leq \alpha(t)\Omega(\|v\|_{\mathfrak{B}} + \|x\|)$ $\forall t \in J$ *and* $v \in \mathfrak{B}$.

(3) Suppose $D_1(\theta) = \{v(\theta) : v \in D_1\}$. For a.e. $s, t, \in J$ $\exists$ an integrable function $\eta : J \to [0, \infty)$ such that
$\chi(S(s)f(t, D_1, D_2)) \leq \eta_1(t) \sup_{-\infty<\theta\leq 0} \chi(D_1(\theta))$. For $D_2(\theta) = \{v(\theta) : v \in D_2\}$, for a.e. $s, t, \in J$, $\chi(C(s)f(t, D_1, D_2)) \leq \eta_2(t) \sup_{-\infty<\theta\leq 0} \chi(D_2(\theta))$

(Hg) The function $g : J \times \mathfrak{B}$ satisfies the following.

(1) $g(t, .) : \mathfrak{B} \to X$ is continuous $\forall t \in J$.

(2) If $x : (-\infty, a] \to X$ be such that $x_0 = \phi$ and $x|_J \in PC$ then the function $t \to g(t, x_t)$ belongs to PC and $t \to g(t, x_t)$ is strongly measurable function.

(3) There exists a function a non decreasing function ω_g such that
$\|g(t, \psi)\|_Y \leq m_g(t)\Omega_g(\|\psi\|_{\mathfrak{B}})$, for all $(t, \psi) \in J \times \mathfrak{B}$

(4) The set $V(r) = \{AS(\theta)g(s, \psi) : \theta, s \in J, \ \psi \in B_r(0, \mathfrak{B})\}$ is precompact in X for all $r > 0$.

(5) The set $\{\widetilde{v}_i : v \in V(r, g)\}$ is equicontinuous subset of $C([t_i, t_{i+1}], X)$ for all $i = 1, \cdots, n$

(6) $t \to g(t, x_t)$ is C^1 on J and $\frac{d}{dt}g(t, x_t)|_{t=0} = \eta$ where η is independent of x.

(7) The operator $P : S(a) \to C(J, X)$, is a completely continuous operator defined as $P(x)(t) = \frac{d}{dt}g(t, x_t + y_t)$ is such that $\|Px\| \leq c_p\|x\| + d_p$. Thus, the set $\{Px(t) : x \in S_a, t \in J\}$ is precompact in X.

(HJ) (1) For the maps $J_i^1(t, \phi) : J \times \mathfrak{B} \to X$ there exist positive constants $c_i^1, c_i^2, d_i^1, d_i^2$ such that
$\|J_i^j(t, v)\| \leq c_i^j\|v\|_{\mathfrak{B}} + d_i^j, \ \forall \ j = 1, 2,$

(2) The maps $J_i^1(.,\psi), J_i^2(.,\psi)$ are completely continuous $\forall\ (.,\psi) \in (t_i, s_i] \times \mathfrak{B}\ \ i = 1, ..., n$,

(HI) $c_p(a+1) + ((N+\widetilde{N})c_i^1 + (\widetilde{N}+N)c_i^2)K_a + (c_i^1 + c_i^2)K_a + limsup_{\tau\to\infty}\frac{\Omega(\tau)}{\tau}\int_{s_j}^{t_k}((\widetilde{N_1}+\widetilde{N_2})m_g(s) + (\widetilde{N}+N)m_f(s))ds + max\{\int_0^a \eta_1(s)ds, \int_0^t \eta_2(s)ds\} < 1$

(H1) There exists a Banach space $(Y, \|.\|_Y)$, continuously included in X with $AS(t) \in \mathcal{L}(Y,X)$, $\forall t \in J$ and $AS(.)x \in C(J;X)$ for every $x \in Y$. $\exists$ constants $N_Y, \widetilde{N_1}$ such that $\|y\| \leq N_Y\|y\|_Y$, $\forall y \in Y$ and $\|AS(t)\|_{\mathcal{L}(Y,X)} \leq \widetilde{N_1}$, $\forall t \in J$

(H2) $\mathcal{R}(C(t)-I)$ is closed and dim Ker$(C(t)-I) < \infty$, $\forall\ 0 < t \leq a$

Lemma 4.2.2. [96]: If $y : (-\infty, a] \to X$ is a function such that $y_0 = \phi$ and $y|_J \in PC(X)$ then

$$\|y_{\rho(s,y_s)}\|_{\mathfrak{B}} \leq (M_a + \widetilde{J^\phi})\|\phi\|_{\mathfrak{B}} + K_a sup\{\|y(\theta)\|; \theta \in [0, max\{0,s\}]\},$$
$$s \in \mathbb{R}(\rho^-) \cup [0,a]$$

where $\widetilde{J^\phi} = sup_{t\in\mathbb{R}(\rho^-)} J^\phi(t)$, $M_a = sup_{t\in J} M(t)$ *and* $K_a = max_{t\in J} K(t)$.

Lemma 4.2.3. [96]: Let condition $(H2)$ be satisfied and $B \subset Y$. If B is bounded in X and the set $\{AS(t)y : t \in [0,a], y \in B\}$ is precompact in X, then B is precompact in X.

Proof: Since for $y \in B$, $C(t)y - y = A\int_0^t S(s)ydy = \int_0^t AS(s)ydy$. The mean value theorem for Bochner integral implies that $C(t)y - y \in t \times \overline{conv(AS(s)y : 0 \leq s \leq t, y \in B)}$, where $conv$ is the convex hull. Then by hypothesis $(H2)$ the result follows.

Lemma 4.2.4. [98]: A set $B \subset PC^1$ is precompact in PC^1 if and only if each set $\widetilde{B_i}, i = 1, ..., n$ is precompact in $C^1([t_i, t_{i+1}], X)$.

Theorem 4.2.5. If the hypotheses $(H_\phi), (Hf), (Hg), (HI), (H1)$ are satisfied, then the initial value problem (4.1.1) has atleast one mild solution.

Proof: Let $\Gamma = \Gamma_i^1 + \Gamma_j^2$, $\forall i = 1, \cdots, n$ *and* $j = 0, \cdots, n$

$$(\Gamma_i^1 x)(t) = \begin{cases} J_i^1(t, \overline{x}_t), & t \in (t_i, s_i]; i = 1, \cdots, n \\ C(t-s_i)[J_i^1(t, x_{s_i}) - g(s_i, x_{s_i})] & \\ +S(t-s_i)[J_i^2(t, x_{s_i}) - g'(s_i, x_{s_i})], & t \in (s_i, t_{i+1}]; i = 1, \cdots, n \end{cases} \quad (4.2.2)$$

$$(\Gamma_j^2 x)(t) = \begin{cases} g(t,\overline{x}_t) + \int_{s_j}^t AS(t-s)g(s,\overline{x}_s)ds \\ + \int_{s_j}^t S(t-s)f(s,\overline{x}_{\rho(s,\overline{x_s})}, x_s')ds, & t \in (s_j, t_{j+1}]; j = 0,\cdots,n \\ 0, & t \notin (s_j, t_{j+1}], j = 0,\cdots,n. \end{cases} \tag{4.2.3}$$

where $\overline{x}_0 = \phi$ and $\overline{x} = x + y$ on J.

$$(\Gamma_i^1 x)'(t) = \begin{cases} J_i^2(t,\overline{x}_t), & t \in (t_i, s_i]; i = 1,\cdots,n \\ AS(t-s_i)[J_i^1(t,x_{s_i}) - g(s_i,x_{s_i})] \\ +C(t-s_i)[J_i^2(t,x_{s_i}) - g'(s_i,x_{s_i})], & t \in (s_i, t_{i+1}]; i = 1,\cdots,n \end{cases} \tag{4.2.4}$$

$$(\Gamma_j^2 x)'(t) = \begin{cases} Px(t) + \int_{s_j}^t AC(t-s)g(s,\overline{x}_s)ds \\ + \int_{s_j}^t C(t-s)f(s,\overline{x}_{\rho(s,\overline{x_s})})ds, & t \in (s_j, t_{j+1}]; j = 0,\cdots,n \\ 0, & t \notin (s_j, t_{j+1}], j = 0,\cdots,n. \end{cases} \tag{4.2.5}$$

It is easy to check that

$$\|\overline{x}_t\|_{\mathfrak{B}} \leq K_a\|y\|_a + M_a\|\phi\|_{\mathfrak{B}} + K_a\|x\|_t,$$

where $\|x\|_t = sup_{0\leq s\leq t}\|x(s)\|$.

$$\|\overline{x}_{\rho(s,\overline{x}_s)}\|_{\mathfrak{B}} \leq k^* := (M_a + \widetilde{J^\phi})\|\phi\|_{\mathfrak{B}} + K_a\|y\|_a + K_a\|x\|_a.$$

Thus Γ is well defined and has values in $S(a)$. Applying Lebesgue dominated convergence theorem, axioms of phase space and the hypotheses $(Hf), (Hg)$ and it can be easily proved that Γ is continuous.

Step1 : We assert that there exist $k > 0$ such that $\Gamma(B_k) \subset B_k$, where $B_k = \{x \in S(a) : \|x\|_a \leq k\}$. In the following we define $\hat{k} := K_a k + \|y_s\|_{\mathfrak{B}} = K_a k + K_a\|y\|_a + M_a\|\phi\|_{\mathfrak{B}}$. When we assume that the assertion is false, then $\forall\ k > 0$, there exists

$x_k \in B_k$. and $t_k \in (s_j, t_{j+1}]$ for some $j \in \{0, \cdots, n\}$ such that $k < \|\Gamma x_k(t_k)\|_1$. Then,

$$\begin{aligned}
k &\leq \|\Gamma_j^2 x(t))\| + \|\Gamma_i^1 x(t))\| + \|(\Gamma_j^2 x)'(t)\| + \|(\Gamma_i^1 x)'(t)\| \\
&\leq c_p a\|x\|_1 + c + \widetilde{N_1}\int_{s_j}^{t_k} m_g(s)\Omega\|\overline{x_{k}}_s\|_{\mathfrak{B}} ds \\
&+ \widetilde{N}\int_{s_j}^{t_k} \alpha(s)\Omega(\|\overline{x_k}_{\rho(s,\overline{x_k}_s)}\|_{\mathfrak{B}} + \|\overline{x_k'}(s)\|)ds + (Nc_i^1 + \widetilde{N}c_i^2)\|x_{s_i}\| \\
&+ c_p\|x\|_1 + c + \widetilde{N_2}\int_{s_j}^{t_k} m_g(s)\Omega\|\overline{x_{k}}_s\|_{\mathfrak{B}} ds \\
&+ N\int_{s_j}^{t_k} \alpha(s)\Omega(\|\overline{x_k}_{\rho(s,\overline{x_k}_s)}\|_{\mathfrak{B}} + \|\overline{x_k'}(s)\|) + (\widetilde{N}c_i^1 + Nc_i^2)\|x_{s_i}\| \\
&\leq c_p(a+1)k + c + ((N+\widetilde{N})c_i^1 + (\widetilde{N}+N)c_i^2)K_a k \\
&+ (\widetilde{N_1}+\widetilde{N_2})\int_{s_j}^{t_k} m_g(s)\Omega(K_a\|y\|_a + M_a\|\phi\|_{\mathfrak{B}} + K_a k)ds \\
&+ (\widetilde{N}+N)\int_{s_j}^{t_k} \alpha(s)ds\Omega(K_a\|y\|_a + (M_a + \widetilde{J^\phi})\|\phi\|_{\mathfrak{B}} + K_a k + k)ds \\
&\leq c_p(a+1)k + c + ((N+\widetilde{N})c_i^1 + (\widetilde{N}+N)c_i^2)K_a k \\
&+ \int_{s_j}^{t_k} ((\widetilde{N_1}+\widetilde{N_2})m_g(s) + (\widetilde{N}+N)m_f(s)) \\
&\times \Omega(K_a\|y\|_a + (M_a + \widetilde{J^\phi})\|\phi\|_{\mathfrak{B}} + K_a k + k)ds
\end{aligned} \tag{4.2.6}$$

Hence

$$\begin{aligned}
1 &< c_p(a+1) + ((N+\widetilde{N})c_i^1 + (\widetilde{N}+N)c_i^2)K_a \\
&+ limsup_{\tau\to\infty}\frac{\Omega(K_a\|y\|_a + (M_a + \widetilde{J^\phi})\|\phi\|_{\mathfrak{B}} + K_a k + k)}{k} \\
&\times \int_{s_j}^{t_k}((\widetilde{N_1}+\widetilde{N_2})m_g(s) + (\widetilde{N}+N)m_f(s))ds \\
&\leq c_p(a+1) + ((N+\widetilde{N})c_i^1 + (\widetilde{N}+N)c_i^2)K_a \\
&+ limsup_{\tau\to\infty}\frac{\Omega(\tau)}{\tau}\int_{s_j}^{t_k}((\widetilde{N_1}+\widetilde{N_2})m_g(s) + (\widetilde{N}+N)m_f(s))ds
\end{aligned} \tag{4.2.7}$$

which is a contradiction to the hypothesis $(H1)$. Similarly, suppose there exists

$x_k \in B_k$ and $t_k \in (t_i, s_i]$ for some $i \in \{1, \cdots, n\}$ such that $(\Gamma x_k)(t_k) > k$. Then,

$$\begin{aligned} k < \|(\Gamma_i^1 x_k)(t_k)\|_1 &= \|J_i^1(t_k, \overline{x_{kt_k}})\| + \|J_i^2(t_k, \overline{x_{kt_k}})\| \\ &\leq \{c_i^1\|\overline{x_{kt_k}}\|_{\mathfrak{B}} + d_i^1\} + \{c_i^2\|\overline{x_{kt_k}}\|_{\mathfrak{B}} + d_i^2\} \\ &\leq \{c_i^1(K_a\|y\|_a + M_a\|\phi\|_{\mathfrak{B}} + K_a k) + d_i^1\} \\ &+ \{c_i^2(K_a\|y\|_a + M_a\|\phi\|_{\mathfrak{B}} + K_a k) + d_i^2\} \end{aligned} \quad (4.2.8)$$

Hence,

$$1 < c_i^1 K_a \quad (4.2.9)$$

which is a contradiction. Hence $\Gamma(B_k) \subset B_k$.

Step 2 : To prove that Γ is a $\chi-$contraction. Let $\Gamma = \Gamma_i^1 + \Gamma_j^2\ \forall\ i = 1, \cdots, n; j = 0, \cdots, n$ be split into $\Gamma = \Gamma_i^{1a} + \Gamma_i^{1b} + \Gamma_i^{1c} + \{\Gamma_j^{2a} + \Gamma_j^{2b} + \Gamma_j^{2c}\}$, $\forall\ i = 1, \cdots, n; j = 0, \cdots, n$

$$\Gamma_i^{1a} = C(t - s_i)(-g(s, x_{s_i}) + J_i^1(s_i, x_{s_i}))$$
$$\Gamma_i^{1b} = S(t - s_i)(J_i^2(s_i, x_{s_i}) - g'(s_i, x_{s_i}))$$
$$\Gamma_i^{1c} = J_i^1(t, x_t)$$
$$\Gamma_j^{2a}x(t) = g(s, \overline{x_t})$$
$$\Gamma_j^{2b}x(t) = \int_{s_j}^t AS(t - s)g(s, \overline{x_s})ds$$
$$\Gamma_j^{2c}x(t) = \int_{s_j}^t S(t - s)f(s, \overline{x}_{\rho(s,\overline{x}_s)})ds$$

The properties of the function g in (Hg) , lemmas 4.2.3 and lemma 4.2.4 imply that for all $j = 0, ..., n$, the set of functions $V(k, g)_j = \{t \to [\widetilde{g}(t, x_t + y_t)]_j : x \in B_k\, j = 0, \cdots, n\}$ is precompact in $C([s_j, t_{j+1}], X)$. By lemma 2.5.8(2) $\chi_{PC}(W) = sup\{\chi(W(t)), t \in J\}$. By lemma 2.5.4 (1) for any $W \subset \Gamma_j^{2a}(B_k)$

$$\begin{aligned} \chi_{PC^1}(\Gamma_j^{2a}W(t)) &= \chi_{PC^1}(g(t, W_t + y_t)) \\ &= max\{sup_{t\in J}\chi_{PC}g(t, W_t + y_t)), sup_{t\in J}\chi_{PC}g'(t, W_t + y_t)\} \\ &= 0 \end{aligned} \quad (4.2.10)$$

By mean value theorem for Bochner integral, we derive

$$\{\Gamma_j^{2b}x(t) : x \in B_k\} \subset t \times \overline{conv(\{AS(h)g(s, \psi) : 0 \leq h, s \leq t, \|\psi\|_{\mathfrak{B}} \leq \hat{k}\})}$$

$$\{(\Gamma_j^{2b}x(t))' : x \in B_k\} \subset t \times \overline{conv(\{AC(h)g(s,\psi) : 0 \leq h, s \leq t, \|\psi\|_{\mathfrak{B}} \leq \hat{k}\})}$$

This implies $\{\Gamma_j^{2b}x(t) : x \in B_k\}$ and $\{(\Gamma_j^{2b}x(t))' : x \in B_k\}$ is precompact in X for all $t \in J$. Hence by Lemma 2.5.4(1),

$$\begin{aligned}\chi_{PC^1}(\Gamma_j^{2b}W(t)) &= max\{sup_{t\in J}\chi_{PC}(\int_{s_j}^t AS(t-s)g(s,W_s+y_s)ds),\\ &sup_{t\in J}\chi_{PC}(\int_{s_j}^t AC(t-s)g(s,W_s+y_s)ds)\} = 0 \qquad (4.2.11)\end{aligned}$$

By lemma 2.5.9 for any $W \subset \Gamma_j^{2c}(B_k)$, since $S(t)$ is equicontinuous so, W is piecewise equicontinuous. Hence from the fact that $\rho(s,\overline{x_s}) \leq s, s \in [0,a]$ and lemma 2.5.8(2) and $\chi_{PC}(W) = \sup\{\chi(W(t)), t \in [s_j, t_{j+1}],\ j = 0, \cdots, n\}$ such that for all $j = 0, \cdots, n$.

$$\begin{aligned}&\chi_{PC^1}(\Gamma_j^{2c}W(t)) = \chi_{PC^1}(\int_{s_j}^t S(t-s)f(s, W_{\rho(s,\overline{x}_s)} + y_s, W'(s) + y'(s))ds\\ &= max\{sup_{t\in J}\chi_{PC}(\int_{s_j}^t S(t-s)f(s, W_{\rho(s,\overline{x}_s)} + y_s, W'(s) + y'(s))ds,\\ &sup_{t\in J}\chi_{PC}(\int_{s_j}^t C(t-s)f(s, W_{\rho(s,\overline{x}_s)} + y_s, W'(s) + y'(s))\}ds\\ &\leq max\{sup_{t\in J}\int_{s_j}^t \eta_1(s)sup_{-\infty<\theta\leq 0}\chi(W(\rho(s,\overline{x}_s)+\theta) + y(s+\theta))ds,\\ &sup_{t\in J}\int_{s_j}^t \eta_2(s)sup_{-\infty<\theta\leq 0}\chi(W'(s+\theta) + y'(s+\theta))ds\}\\ &\leq max\{sup_{t\in J}\int_{s_i}^t \eta_1(s)sup_{-\infty<\theta\leq 0}\chi(W(s+\theta) + y(s+\theta))ds,\\ &sup_{t\in J}\int_{s_i}^t \eta_2(s)sup_{-\infty<\theta\leq 0}\chi(W'(s+\theta) + y'(s+\theta))ds\}\\ &\leq max\{sup_{t\in J}\int_{s_i}^t \eta_1(s)sup_{0<\tau\leq s}\chi W(\tau)ds,\\ &sup_{t\in J}\int_{s_i}^t \eta_2(s)sup_{0<\tau\leq s}\chi W(\tau)ds\}\\ &\leq max\{\int_0^a \eta_1(s)ds, \int_0^a \eta_2(s)ds\}\chi_{PC^1}(W) \qquad (4.2.12)\end{aligned}$$

Hence

$$\begin{aligned}\chi_{PC^1}(\Gamma_j^{2c}W) &= \sup_{t\in J}\{\chi_{PC^1}(\Gamma_j^{2c}W(t)),\ t \in [s_j, t_{j+1}],\ j = 0, \cdots, n\}\\ &\leq \chi_{PC^1}(W)max\{\int_0^a \eta_1(s)ds, \int_0^a \eta_2(s)ds\} \qquad (4.2.13)\end{aligned}$$

For arbitrary $x_1, x_2 \in B_k$ and $t \in (s_i, t_{i+1}]\ \forall\ i = 1, \cdots, n$

$$\begin{aligned}
&\chi_{PC^1}(\Gamma_i^{1a}x)(t) = max\{sup\chi_{PC}(\Gamma_i^{1a}x)(t), sup\chi_{PC}(\Gamma_i^{1b}x)'(t)\} \\
&\leq max\{sup\chi_{PC}\overline{conv(\{C(\theta)[J_i^1(s,\psi) - g(s,\psi)] : 0 \leq \theta, s \leq t, \|\psi\| \leq k\})}, \\
&sup\chi_{PC}\overline{conv(\{S(\theta)(J_i^1(s,\psi) - g(s,\psi)) : 0 \leq \theta, s \leq t, \|\psi\| \leq k\})}\} \\
&= 0 \qquad (4.2.14)
\end{aligned}$$

Since

$$\begin{aligned}
C(t-s_i)[J_i^1(s_i, x_{s_i}) &- g(s_i, x_{s_i})] \\
&\in \overline{conv(\{C(\theta)[J_i^1(s,\psi) - g(s,\psi)] : 0 \leq \theta, s \leq t, \|\psi\| \leq k\})}.
\end{aligned}$$

and

$$\begin{aligned}
S(t-s_i)[J_i^1(s_i, x_{s_i}) &- g(s_i, x_{s_i})] \\
&\in \overline{conv(\{C(\theta)[J_i^1(s,\psi) - g(s,\psi)] : 0 \leq \theta, s \leq t, \|\psi\| \leq k\})}.
\end{aligned}$$

is precompact.

Similarly $\chi(\Gamma_i^{1b}) = 0,\ \forall i = 1, \cdots, n$ and $\chi(\Gamma_i^{1c}) = 0\ \forall i = 1, \cdots, n$

For each bounded set $W \in PC^1(J; X)$ we have,

$$\begin{aligned}
\chi_{PC^1}(\Gamma W) &\leq \chi_{PC^1}(\Gamma_i^{1a}W + \Gamma_i^{1b}W + \Gamma_i^{1c}W) + \chi_{PC^1}(\Gamma_j^{2a}W + \Gamma_j^{2b}W + \Gamma_j^{2c}W) \\
&\leq 0 + 0 + 0 + max\{\int_0^a \eta_1(s)ds, \int_0^t \eta_2(s)ds\}\chi_{PC^1}(W)
\end{aligned}$$

Therefore, Γ is a $\chi-$contraction. Applying Darbo-Sadovskii fixed point theorem it is established that there exists a fixed point of Γ in $S(a)$. So, $z = x + y$ is a mild solution of (4.1.1).

Remark: Our abstract approach permits application to partial differential equations with instantaneous impulsive term involving nonlinear expression also.

4.3 Example

In this section we discuss a partial differential equation applying the abstract results of this paper. In this application, $\mathfrak{B}$ is the phase space $C_0 \times L^2(h, X)$ see ([98]).

Consider the second order neutral differential equation

$$\begin{aligned}
\frac{\partial^2}{\partial t^2}(x(t,\sigma) &- \int_{-\infty}^{t}\int_{0}^{\pi} n(t-s,v,\sigma)x(s,v)dvds) \\
= \frac{\partial^2 x(t,\sigma)}{\partial \sigma^2} &+ \int_{-\infty}^{t} m(t-s)x(s-\rho_1(t)\rho_2(\|x(t)\|,\sigma,\zeta)ds \;\; t\in[0,a], \sigma\in[0,\pi], \\
x(t,0) &= x(t,\pi)=0, \; t\in[0,a], \\
x(s,\sigma) &= \phi(s,\sigma) \; -\infty \le s \le 0, 0\le\sigma\le\pi, \\
\frac{\partial}{\partial t}x(0,\sigma) &= \xi(\sigma), \; 0\le\sigma\le\pi, \\
x(t)(\sigma) &= \int_{\infty}^{t_i} n_i^1(t_i-s)x(s,\sigma)ds, \; t\in(t_i,s_i], \; i=1,...,n \qquad (4.3.1)
\end{aligned}$$

where $\phi \in H^1([0,\pi])$, $\xi \in X$, $0 = t_0 = s_0 < t_1 \le s_1 \le t_2, ..., t_n \le s_n \le t_{n+1} = a$ Here, $X = L^2([0,\pi])$, $\mathfrak{B} = PC_0 \times L^2(\rho, X)$, $A \subset D(A) \subset X \to X$ is the map defined by $Af = f''$ with domain $D(A) = \{f \in X : f'' \in X, \; f(0) = f(\pi) = 0\}$. A denotes the infinitesimal generator $(C(t))_{t\in R}$ on X. A has a discrete spectrum, the eigenvalues are $-n^2$, $n \in \mathbb{N}$; with corresponding eigenvectors $z_n(\theta) = (\frac{2}{\pi})^{\frac{1}{2}} sin(n\theta)$ and the following properties hold

(C1) $A\phi = -\sum_{n=1}^{\infty} n^2 < \phi, z_n > z_n$ where $\phi \in D(A)$

(C2) $C(t)\phi = \sum_{n=1}^{\infty} cos(nt) < \phi, z_n > z_n$ and $S(t)\phi = \sum_{n=1}^{\infty} \frac{sin(nt)}{n} < \phi, z_n > z_n$, for $\phi \in X$.

By defining maps $\rho, g, f : [0,a] \times \mathfrak{B} \times X \to X$ by

$$\rho(t,\psi) := \rho_1(t)\rho_2(\|\psi(0)\|),$$

$$g(\psi)(\sigma) := \int_{-\infty}^{0}\int_{0}^{\pi} n(s,v,\sigma)\psi(s,v)dvds,$$

$$f(\psi)(\sigma) := \int_{-\infty}^{0} m(s)\psi(s,\sigma,\zeta)ds$$

the system (4.3.1) can be transformed into system (4.1.1) Assume that the functions $\rho_i : \mathbb{R} \to [0,\infty)$, $m : \mathbb{R} \to \mathbb{R}$ are piecewise continuous.

(1) The functions $n(s,v,\sigma)$, $\frac{\partial n(s,v,\sigma)}{\partial \sigma}$ are measurable, $n(s,v,\pi) = n(s,\eta,0) = 0$ and

$$L_g := max\{(\int_0^{\pi}\int_{-\infty}^{0}\int_0^{\pi} \frac{1}{h(s)}(\frac{\partial^i n(s,\eta,\sigma)}{\partial \sigma^i})^2 d\eta ds d\sigma)^{1/2} : \; i = 0,1\} < \infty$$

$$\widetilde{L}_g := (\int_0^\pi \int_{-\infty}^0 \int_0^\pi \frac{1}{h(s)}(\frac{\partial^i n(s,\eta,\sigma)}{\partial\sigma^i})^2 d\eta ds d\sigma) < \infty$$

(2) The function $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous and there is continuous function $\int_{-\infty}^0 \frac{\mu(s)^2}{q(s)} ds < \infty$. and $\|f(t,\sigma)\| \leq \mu(s)(\|\sigma\| + \|\zeta\|)$

(3) The functions $n_i^j \in C([0,\infty);\mathbb{R})$ and $L_i^j := (\int_{-\infty}^0 \frac{(n_i^j(s))^2}{q(s)} ds)^{1/2} < \infty, \ \forall \ i = 1,2,...,n, \ j = 1,2$

So, $g(t,.), J_i, (i = 1,...,n), f$ are bounded linear operators. We take $Y = D(A)$. Therefore if $\iota : Y \to X$ is the inclusion then $t \to AS(t)$ is uniformly continuous into $L(Y,X)$ and $\|AS(t)\|_{L(Y,X)} \leq 1$ for $t \in [0,a]$ Suppose $u(t)(\sigma) = x(t,\sigma)$ such that $x_0 = \phi$ and continuous on $[0,t_1)$ then the right derivative

$$\begin{aligned} \frac{d}{dt}g(u_t)|t = 0(\sigma) &= \int_{-\infty}^0 \int_0^\pi \frac{\partial}{\partial s} n(s,v,\sigma)\phi(s,v) dv ds + \int_0^\pi n(0,v,\sigma)\psi(0,v) dv \\ &= \eta(\sigma) \end{aligned} \tag{4.3.2}$$

exists and is independent of x. Hence by assumptions $(a)-(c)$ and theorem 4.2.5 it is ensured that mild solution to the problem (4.3.1) exists.

4.4 Conclusion

The existence of mild solution of a class of second order partial neutral differential equation involving state dependent delay and non-instantaneous impulses is proved. Hausdorff measure of noncompactness and Darbo Sadovskii fixed point theorem were used to replace some restrictive conditions such as the compactness of cosine or sine family of operators and the Lipschitz conditions on the nonlinear functions.

Chapter 5

Existence of Solution of Impulsive Second-Order Integro-Differential Equation with State Delay

This chapter consists of two parts. The first part deals with the existence of mild solution of a class of instantaneous impulsive second order partial differential equation involving state dependent delay. The second part studies the non-instantaneous impulsive conditions on the same problem. Kuratowski measure of noncompactness and Mónch fixed point theorem are required to establish the existence of mild solution. We remove the restrictive conditions on the priori estimation available in literature. The compactness of cosine or sine operators, nonlinear terms and associated impulses is removed. The noncompactness measure estimation, the Lipschitz conditions, and compactness on the nonlinear functions are replaced by simple and natural assumptions. We introduce new non-instantaneous impulses with fixed delays. In the last section we study examples to illustrate the presented result.

5.1 Introduction

In recent times, much attention is paid to functional differential equation with state dependent delay. We refer [13],[15],[59],[76] for details. For work in impulsive differential equations, we refer [63],[117],[144] regarding discrete impulses. However, in general the compactness of the impulsive terms, boundedness of estimates of measure of noncompactness and a priori estimates are used to establish existence results.

In this chapter we study the second order partial neutral differential equation with state dependent delay represented in the form

$$\begin{aligned}
\frac{d^2}{dt^2}x(t) &= A(x(t)-g(t,x_t)) + \int_0^t f(t, x_{\rho(t,x_t)}, x'_t)dt,\ t \in [0,b],\ t \neq t_i, \\
&\qquad i = 1,...,n \\
x_0 &= \phi \in \mathfrak{B}, \\
x'(0) &= \xi \in X, \\
\Delta x(t_i) &= I_i^1(x_{t_i}, x'_{t_i}),\ i = 1,2,...,n \\
\Delta x'(t) &= I_i^2(x_{t_i}, x'_{t_i}),\ i = 1,2,...,n \qquad (5.1.1)
\end{aligned}$$

Here $0 = t_0 < t_1 \leq t_2, ..., < t_n \leq t_{n+1} = b$ are prefixed numbers.

We also study the second order neutral differential equation

$$\begin{aligned}
\frac{d^2}{dt^2}x(t) &= A(x(t)-g(t,x_t)) + \int_0^t f(t, x_{\rho(t,x_t)}, x'(t))dt,\ t \in (s_i, t_{i+1}], \\
&\qquad i = 0,...,n \\
x_0 &= \phi \in \mathfrak{B}, \\
x'(0) &= \xi \in X, \\
x(t) &= J_i^1(t, x(t-t_1)),\ t \in (t_i, s_i],\ i = 1,2,...,n \\
x'(t) &= J_i^2(t, x(t-t_1)),\ t \in (t_i, s_i],\ i = 1,2,...,n \qquad (5.1.2)
\end{aligned}$$

Here $0 = t_0 = s_0 < t_1 \leq s_1 \leq t_2, ..., < t_n \leq s_n \leq t_{n+1} = b$ are prefixed numbers.

In (5.1.1),(5.1.2) A is the infinitesimal generator of a strongly continuous cosine family $\{C(t) : t \in \mathsf{R}\}$ of bounded linear operators on a Banach space X and $t \in [0,b] = J$. $S(t)$ is the associated sine function. The history valued function $x_t : (-\infty, 0] \to X$, $x_t(\theta) = x(t+\theta)$ belongs to some abstract phase space $\mathfrak{B}$ defined axiomatically in chapter 2 as Definition 2.2.12 and $g, f, I_i^1, I_i^2, J_i^1, J_i^2, i = 1,..,n$ are appropriate functions which are defined in the following section, in the hypotheses $(Hf), (Hg), (HI)$ and (HJ) respectively.

The second order abstract partial neutral differential equation similar to (5.1.1) is extensively studied in [34],[38]. Actually, in these articles strict assumptions on semigroup or cosine family are assumed. This resulted in the finite dimensionality of the abstract space. Thus the equations studied in those articles are actually ordinary instead of being partial differential equations. Hence motivated by the need to redress this issue and by the results in [98] and their various applications we study partial neutral differential equation of second order involving state dependent delay, instantaneous and non-instantaneous impulses.

The main contribution of this work lies in the removal of compactness assumption on the associated cosine or sine family of operators and associated impulsive term. The noncompactness measure estimation and the Lipschitz conditions on the nonlinear functions are replaced by simple and natural assumptions.

Suppose $N, \widetilde{N}, \widetilde{N_1}, \widetilde{N_2}$ be certain constants such that $\|C(t)\| \leq N$, $\|S(t)\| \leq \widetilde{N}$, $\|AS\| \leq \widetilde{N_1}$ $\|AC\| \leq \widetilde{N_2}$ for every $t \in J = [0, b]$. For more details see books by Goldstein[84] and Fattorini[75]. Let E denote the Banach space of all vectors $x \in X$ for which $C(.)x$ is a continuously differential function on $\mathbb{R}$, endowed with the norm $\|x(t)\|_E = \|x\| + sup_{0\leq t\leq b}\|AS(t)x\|, \ \ x \in E$.

$PC([0, b], X)$ denotes the space of normalized piecewise continuous function from $[0, b]$ into X. For any $x \in PC$

$$\widetilde{x}_i(t) = \begin{cases} x(t), & t \in (t_i, t_{i+1}]; \\ x(t_i^+), & t = t_i, \ i = 1, 2, ..., n. \end{cases}$$

So, $\widetilde{x} \in C([t_i, t_{i+1}], X)$.

Definition 5.1.1. [44]:For a bounded set B in any Banach space Y the Kuratowski measure of noncompactness α_Y is defined by
$\alpha_Y(B) = inf\{r > 0, B$ can be covered by finite no. of balls with diameter r$\}$

Lemma 5.1.1. [44] Let $h : [0, b] \to E$ be an integrable function such that $h \in PC$. Then the function $v(t) = \int_0^t C(t - s)h(s)ds$ belongs to PC^1, the function $s \to AS(t - s)h(s)$ is integrable on $[0, t]$ for $t \in [0, b]$ and

$$v'(t) = h(t) + A\int_0^t S(t - s)h(s)ds = h(t) + \int_0^t AS(t - s)h(s)ds, \ t \in [0, b]$$

Lemma 5.1.2. [44] Let $h_n \in H \subset L^1([0, b], X)$.If there exists $\varrho \in L^1([0, b], [0, +\infty))$ such that $\|h_n(t)\| \leq \varrho(t)$ for $h_n \in H$ and *a.e.* $t \in [0, b]$, then $\alpha(H(t)) \in L^1([0, b], [0, +\infty))$

and

$$\alpha(\{\int_0^t h_n(s)ds : n \in \mathbb{N}\}) \leq 2\int_0^t \alpha(H(s))ds,\ t \in [0,b]$$

Lemma 5.1.3. [44](Mónch): Let X be a Banach space, Ω be a bounded open subset in X and $0 \in \Omega$. Assume that the operator $F : \Omega \to X$ is continuous and satisfies the following conditions:

(1) $x \neq \lambda F x,\ \forall \lambda \in (0,1),\ x \in \partial\Omega$

(2) D is relatively compact if $D \subset \overline{co}(0 \cup F(D))$ for any countable set $D \subset \overline{\Omega}$. Then F has a fixed point in $\overline{\Omega}$.

5.2 Instantaneous Impulsive Second-order Differential Equation

We define the mild solution of the problem (5.1.1) as follows.

Definition 5.2.1. A function $x : (-\infty, b] \to X$ is a mild solution of the problem (5.1.1) if $x_0 = \phi,\ x'(0) = \xi,\ x(.)|_{[0,b]} \in PC^1(X)$, and

$$\begin{aligned} x(t) &= C(t)\phi(0) + S(t)\xi + g(t,x_t) - \int_0^t AS(t-s)g(s,x_s)ds \\ &+ \int_0^t S(t-s)\int_0^s f(r, x_{\rho(r,x_r)}, x_r')drds + \sum_{0<t_i<t} C(t-t_i)I_i^1(x_{t_i}, x_{t_i}') \\ &+ \sum_{0<t_i<t} S(t-t_i)I_i^2(x_{t_i}, x_{t_i}') \end{aligned} \quad (5.2.3)$$

To prove our result we always assume $\rho : J \times \mathfrak{B} \to (-\infty, b]$ is a continuous function. Let $y : (-\infty, b] \to X$ is the function defined by $y_0 = \phi$ *and* $y(t) = C(t)(\phi(0)) + S(t)(\xi)$ on $[0, t_1]$. From the definition of abstract phase space $\mathfrak{B}$ introduced by Hale and Kato and given in chapter 2, it clearly follows that $\|y_t\|_{\mathfrak{B}} \leq M_1 := K_b\|y\|_b + M_b\|\phi\|_{\mathfrak{B}}$ where $\|y\|_b = sup_{0\leq t\leq b}\|y(t)\|$. Let $\overline{x} = x + y$

$$\|\overline{x}_{\rho(s,\overline{x}_s)}\|_{\mathfrak{B}} \leq M_2^* := (M_b + \widetilde{J^\phi})\|\phi\|_{\mathfrak{B}} + K_b\|y\|_b + K_b\|x\|_b.$$

Taking supremum of M_1, M_2 as $\overline{M}$ and supremum of y' as M' we define the space $S(b)$ as $S(b) = \{x : (-\infty, b] \to X :\ x_0 = 0,\ x'(0) = 0,\ x|_J \in PC^1\}$ endowed with norm $\|u\|_1 = \|u\|_\infty + \|u'\|_\infty$.

The following hypotheses are required to prove our result.

(H_ϕ) The function $t \to \phi_t$ is continuous from $\mathbb{R}(\rho^-) = \{\rho(s,\psi) : \rho(s,\psi) \leq 0\}$ into $\mathfrak{B}$ and there exists a continuous bounded function $J^\phi : \mathbb{R}(\rho^-) \to (0,\infty)$ such that $\|\phi_t\|_{\mathfrak{B}} \leq J^\phi(t)\|\phi\|_{\mathfrak{B}}$ for every $t \in \mathbb{R}(\rho^-)$.

(Hf) The function $f : J \times \mathfrak{B} \times \mathfrak{B} \to X$ satisfies the following:

(1) For every $x : (-\infty, a] \to X, x_0 = 0,\ x'(0) = 0,\ x|_J \in PC^1$ the function $f(., x_t, x'_t) : J \to X$ is strongly measurable and $f(t,.,.)$ is continuous for a.e. $t \in J$.

(2) There exists an integrable function $p : J \to [0, +\infty)$ such that $\|f(t,u,v)\| \leq p(t)(\|u\|_{\mathfrak{B}} + \|v\|_{\mathfrak{B}})\ \forall t \in J$ *and* $u, v \in \mathfrak{B}$.

(3) There exists an integrable function $\mu : J \to [0,\infty)$ such that $\alpha(f(t, D_t, D'_t)) \leq \mu(t)(\alpha(D_t) + \alpha(D'_t))$ for a.e. $t \in J$, where $D_t = \{v_t : v \in D\}$. $D'_t = \{v'_t : v' \in D'\} \subset \mathfrak{B}(t \in J)$, $V' \subset PC^1$

(Hg) The function $g : J \times \mathfrak{B}$ satisfies the following.

(1) $g(t,.)$ is continuous $\forall\ t \in J$.

(2) For every bounded $V \subset S(b)$ the set $\{\widetilde{(v_x)}_i(t) : x \in V\}$ is uniformly equicontinuous on $[t_i, t_{i+1}]$ for all $i = 0, \cdots, n$ where $v_x(t) = g(t, x_t)$

(3) For any bounded set $Q \subset PC^1$, $\alpha(g(t, Q_t)) < c\alpha(Q_t)$, $t \in J$ where c is a positive constant.

(HI) For the maps $I_i^1 : \mathfrak{B} \times \mathfrak{B} \to E$, $I_i^2 : \mathfrak{B} \times \mathfrak{B} \to E$ there exist positive constants $c_i^1, c_i^2, d_i^1, d_i^2$ such that $\|I_i^j(t,v)\| \leq c_i^j\|v\|_{\mathfrak{B}} + d_i^j,\ \forall\ j = 1, 2,$

(H1) There exists a Banach space $(Y, \|.\|_Y)$ continuously included in X such that $AS(t) \in \mathcal{L}(Y, X)$, for all $t \in J$ and $AS(.)x \in C(J; X)$ for every $x \in Y$. $\exists$ constants $N_Y, \widetilde{N_1}$ such that $\|y\| \leq N_Y\|y\|_Y,\ \forall y \in Y$ and $\|AS(t)\|_{\mathcal{L}(Y,X)} \leq \widetilde{N_1},\ \forall t \in J$

(H2) $\mathcal{R}(C(t) - I)$ is closed and dim $\mathrm{Ker}(C(t) - I) < \infty,\ \forall\ 0 < t \leq b$

(HJ) (1) For the maps $J_i^1(t,\phi) : J \times \mathfrak{B} \to X$ there exist positive constants $c_i^1, c_i^2, d_i^1, d_i^2$ such that
$\|J_i^j(t,v)\| \leq c_i^j\|v\|_{\mathfrak{B}} + d_i^j, \ \forall\, j = 1, 2,$

(2) The maps $J_i^1(.,\psi), J_i^2(.,\psi)$ are continuous $\forall\ (.,\psi) \in (t_i, s_i] \times \mathfrak{B} \quad i = 1, ..., n,$

Lemma 5.2.1. [96]: If $y : (-\infty, b] \to X$ is a function such that $y_0 = \phi$ and $y|_J \in PC(X)$ then
$\|y_{\rho(s,y_s)}\|_{\mathfrak{B}} \leq (M_b + \widetilde{J^\phi})\|\phi\|_{\mathfrak{B}} + K_b sup\{\|y(\theta)\|; \theta \in [0, \max\{0, s\}]\},$

$$s \in \mathbb{R}(\rho^-) \cup [0, b]$$

where $\widetilde{J^\phi} = \sup_{t\in\mathbb{R}(\rho^-)} J^\phi(t),\ M_b = \sup_{t\in J} M(t)\ and\ K_b = \max_{t\in J} K(t)$.

Lemma 5.2.2. [84]: Let condition $(H2)$ be satisfied and $B \subset Y$. If B is bounded in X and the set $\{AS(t)y : t \in [0,b], y \in B\}$ is relatively compact in X, then B is relatively compact in X.

Proof: Since for $y \in B$, $C(t)y - y = A\int_0^t S(s)ydy = \int_0^t AS(s)ydy$, it follows from mean value theorem for Bochner integral that
$C(t)y - y \in t \times \overline{co(AS(s)y : 0 \leq s \leq t, y \in B)}$, where co is the convex hull. Then by hypothesis $(H2)$ the result follows.

Lemma 5.2.3. [98]: A set $B \subset PC^1$ is relatively compact in PC^1 if and only if each set $\widetilde{B_i}, i = 1, ..., n$ is relatively compact in $C^1([t_i, t_{i+1}], X)$.

Theorem 5.2.4. If the hypothesis $(H\phi), (Hf), (Hg), (HI), (H1)$ and $(H2)$ holds and the cosine family is equicontinuous then there exists a mild solution of the problem (5.1.1)

Proof: Let us define the function $z : (-\infty, 0] \to X$ as $z_0 = x'_0,\ z(t) = x'(t),\ t \in J$ $S(b) = \{x : (-\infty, b] \to X : x_0 = 0, x'(0) = 0,\ x(.)|_J \in PC^1\}$ Let $\Gamma = (\Gamma_1, \Gamma_2) : S(b) \times S(b) \to S(b)$ be defined as

$$\Gamma_1(x,z)(t) = \begin{cases} 0, & t \leq 0; \\ +\int_0^t AS(t-s)g(s, x_s + y_s)ds & \\ +\int_0^t S(t-s)\int_0^s f(r, \overline{x}_{rho(r,x_r)}, x'_r + y'_r)dr & \\ +\sum_{0<t_i<t} C(t-t_i)I_i^1(x_{t_i} + y_{t_i}, z_{t_i} + y'_{t_i}) & \\ +\sum_{0<t_i<t} S(t-t_i)I_i^2(x_{t_i} + y_{t_i}, z_{t_i} + y'_{t_i}), & t \in J. \end{cases} \tag{5.2.4}$$

and $\Gamma_2(x,z)(t) = \Gamma_1(x,z)'(t)$ Therefore,

$$\Gamma_2(x,z)(t) = \begin{cases} 0, & t \leq 0; \\ + \int_0^t AC(t-s)g(s, x_s + y_s)ds & \\ + \int_0^t C(t-s) \int_0^s f(r, \overline{x}_{\rho(r,x_r)} + y_r, x'_r + y'_r)dr & \\ + \sum_{0<t_i<t} AS(t-t_i) I_i^1(x_{t_i} + y_{t_i}, z_{t_i} + y'_{t_i}) & \\ + \sum_{0<t_i<t} C(t-t_i) I_i^2(x_{t_i} + y_{t_i}, z_{t_i} + y'_{t_i}), & t \in J. \end{cases} \tag{5.2.5}$$

Γ is seen to be continuous by Lebesgue dominated convergence theorem, axioms of phase space and the hypotheses $(H\phi), (Hf), (Hg), (HI)$.
Step 1 : It is shown that $\Omega_0 = \{(x,z) \in S(b) \times S(b) : (x,z) = \lambda\Gamma(x,z) \; for \; some \; \lambda \in (0,1)\}$ is bounded. If $t \in J_0 = [0, t_1]$ then

$$\begin{aligned} \|x(t)\| &= \|\Gamma_1(x,z)(t)\| \leq \widetilde{N_1} \int_0^t [c(\|x\|_{\mathfrak{B}} + \overline{M}) + d]ds \\ &+ \widetilde{N} \int_0^t \int_0^s p(r)(\|x_r\|_{\mathfrak{B}} + \|z_r\|_{\mathfrak{B}} + M' + \overline{M})drds \\ &\leq \overline{M} \int_0^t (\widetilde{N_1}c + \widetilde{N} \int_0^s p(r)dr)ds + \widetilde{N_1}bd \\ &+ K_b \int_0^t (\widetilde{N_1}c + \widetilde{N} \int_0^s p(r)dr)(\|x\|_s + \|z\|_s)ds \\ &+ M'\widetilde{N} \int_0^t \int_0^s p(r)drds, \end{aligned} \tag{5.2.6}$$

$$\begin{aligned} \|z(t)\| &\leq \|\Gamma_2(x,z)(t)\| \leq \widetilde{N_2} \int_0^t [c(\|x\|_{\mathfrak{B}} + \overline{M}) + d]\|ds \\ &+ N \int_0^t \int_0^s p(r)(\|x_r\|_{\mathfrak{B}} + \|z_r\|_{\mathfrak{B}} + M' + \overline{M})drds \\ &\leq \overline{M} \int_0^t (\widetilde{N_2}c + N \int_0^s p(r)dr)ds + \widetilde{N_2}bd \\ &+ K_b \int_0^t (\widetilde{N_2}c + N \int_0^s p(r)dr)(\|x\|_s + \|z\|_s)ds \\ &+ M'\widetilde{N} \int_0^t \int_0^s p(r)drds. \end{aligned} \tag{5.2.7}$$

Therefore,

$$\begin{aligned}
\|x\|_t + \|z\|_t &\leq (\widetilde{N_1} + \widetilde{N_2})bd \\
&+ \overline{M}[\int_0^t [c(\widetilde{N_1} + \widetilde{N_2}) + (N + \widetilde{N}) \int_0^s p(r)dr]ds \\
&+ M'(\widetilde{N} + N) \int_0^t (\int_0^s p(r)dr)ds + \int_0^t [(\widetilde{N_1}c + \widetilde{N_2}c)K_b \\
&+ (N + \widetilde{N})K_b \int_0^s p(r)dr](\|x\|_s + \|z\|_s)]ds. \qquad (5.2.8)
\end{aligned}$$

Since $\|x\|_t + \|z\|_t \in C(J_0, X)$ by Gronwall's lemma there is a constant $G_0 > 0$ such that $\|x\|_t + \|z\|_t \leq G_0,\ t \in J$ and $\|x_t\|_{\mathfrak{B}} \leq K_b G_0$ and $\|z_t\|_{\mathfrak{B}} \leq K_b G_0,\ t \in J_0$. By condition (HI) it is observed that

$$\begin{aligned}
\|I_1^j(x_{t_1} + y_{t_1}, z_{t_1} + y'_{t_1})\| &\leq c_1^j(2K_bG_0 + \overline{M} + M') + d_1^j := \eta_j \\
\|x(t_1^+)\| &= \|x(t_1) + I_1^1(x_{t_1} + y_{t_1}, z_{t_1} + y'_{t_1})\| \leq G_0 + \eta_1 \\
\|z(t_1^+)\| &= \|z(t_1) + I_1^2(x_{t_1} + y_{t_1}, z_{t_1} + y'_{t_1})\| \leq G_0 + \eta_2
\end{aligned} \qquad (5.2.9)$$

When $t \in J_1 = (t_1, t_2]$, let $u(t) = \begin{cases} x(t), & t \in (t_1, t_2]; \\ x(t_1^+), & t = t_1. \end{cases}$

$v(t) = \begin{cases} z(t), & t \in (t_1, t_2]; \\ z(t_1^+), & t = t_1. \end{cases}$ Then $u, v \in C([t_1, t_2], X)$

$$\begin{aligned}
\|u(t)\| &\leq \int_0^t (\widetilde{N_1}cK_b + \widetilde{N}K_b \int_0^s p(r)dr)(\|x\|_s + \|z\|_s)ds \\
&+ \int_0^t [\widetilde{N_1}cM + \widetilde{N} \int_0^s p(r)dr(\overline{M} + M')]ds + \widetilde{N_1}bd \\
&+ N\|I_1^1(x_{t_1} + y_{t_1}, z_{t_1} + y'_{t_1})\| + \widetilde{N}\|I_1^2(x_{t_1} + y_{t_1}, z_{t_1} + y'_{t_1})\| \\
&\leq \int_0^{t_1} (2\widetilde{N_1}cK_bG_0 + \widetilde{N} \int_0^s 2K_bG_0 p(r)dr)ds \\
&+ \int_0^t [N_1c\overline{M} + \widetilde{N} \int_0^s p(r)dr(\overline{M} + M')ds + \int_{t_1}^t (\widetilde{N_1}cK_b \\
&+ \widetilde{N}K_b \int_0^s p(r)dr)(\sup_{t_1 \leq \tau \leq s} \|u(\tau)\| + \sup_{t_1 \leq \tau \leq s} \|v(\tau)\|)ds \qquad (5.2.10)
\end{aligned}$$

$$\begin{aligned}\|v(t)\| &\leq \int_0^{t_1}(2\widetilde{N_2}cK_bG_0 + N\int_0^s 2K_bG_0p(r)dr)(\overline{M}+M')ds + \widetilde{N_2}bd \\ &+ \int_0^t [\widetilde{N}_2c\overline{M} + N\int_0^s p(r)dr(\overline{M}+M')]ds \\ &+ \int_{t_1}^{t_2}(\widetilde{N_2}cK_b + NK_b\int_0^s p(r)dr)(\sup_{t_1\leq t\leq s}\|u(\tau)\| + \sup_{t_1\leq t\leq s}\|v(\tau)\|)ds \end{aligned} \tag{5.2.11}$$

Therefore, from equation (9.2.1), (9.2.2)

$$\begin{aligned}\sup_{t_1\leq s\leq t}\|u(s)\| + \sup_{t_1\leq s\leq t}\|v(s)\| &\leq e_1 + e_2 \\ &+ \int_{t_1}^t [\widetilde{N_1}c + \widetilde{N_2}c + (N+\widetilde{N})\int_0^s p(r)dr]K_b \\ &\times (\sup_{t_1\leq \tau\leq s}\|u(\tau)\| + \sup_{t_1\leq \tau\leq s}\|v(\tau)\|)ds \end{aligned} \tag{5.2.12}$$

where e_1, e_2 are appropriate constants.

Using Gronwall's lemma there exists constants $G_1 > 0$ such that $\|u(t)\| + \|v(t)\| \leq G_1$ for $t \in [t_1, t_2]$. So $\|x(t)\| + \|z(t)\| \leq G_1$, for $t \in J_1$.

Similarly let $G = \max\{G_0, G_1, \cdots, G_n\}$, then $\|(x,z)\| \leq G$ and Ω_0 is bounded.

Let $R > G$ and $\Omega_R = \{(x,z) \in S(b) \times S(b) : \|(x,z)\|_b < R\}$.

Since $R > G$, so

$$(x,z) \neq \lambda\Gamma(x,z), \;\; \forall (x,z) \in \partial\Omega_R \tag{5.2.13}$$

Step2 : Suppose $V \subset \overline{\Omega_R}$ be countable set and $V \subset \overline{co}(\{0,0\} \subset \Gamma(V))$. Let

$$V_1 = \{x \in S(b) : \exists z \in S(b), (x,z) \in V\},$$

$$V_2 = \{z \in S(b) : \exists x \in S(b), (x,z) \in V\}$$

$$V \subset V_1 \times V_2 \subset \overline{co}(\{0\} \cup \Gamma_1(V_1 \times V_2)) \times \overline{co}(\{0\} \cup \Gamma_2(V_1 \times V_2)) \tag{5.2.14}$$

From equations (5.2.4),(5.2.5), lemma 2.5.9 and $(Hg)(2)$ we get that $\Gamma_j((\widetilde{V_1})_i \times (\widetilde{V_2})_i)$, $(j = 1,2)$ are equicontinuous on $J_i (i = 0, 1, \cdots, n)$. From (5.2.14) it is implied that $(\widetilde{V_k})_i (k = 1,2)$ are equicontinuous.

Step3 : Now we prove that V_1 and V_2 are relatively compact. We identify $V_k|_{J_i}$, $(k = 1,2)$ with $\widetilde{V_i}$ where $V_k|_{J_i}$ is the restriction of V_k on $J_i = (t_i, t_{i+1}]$. When

$t \in J_0 = [0, t_1]$, from hypotheses $(Hf)(3)$, $(Hg)(5)$ and Lemma 5.1.2 we get that

$$\begin{aligned}
\alpha(V_1(t)) &\leq \alpha(\Gamma_1(V_1 \times V_2)(t)) \\
&\leq 2\widetilde{N_1}\int_0^t \alpha(g(s, V_{1s} + y_s))ds \\
&+ 2\widetilde{N}\int_0^t \alpha \int_0^s f(r, V_{1\rho(r,x_r)} + y_{\rho(r,x_r)}, V_{2r} + y_r')drds \\
&\leq 2\int_0^t \widetilde{N_1}c\alpha(V_{1s} + y_s)ds + 2\int_0^t 2\widetilde{N}\int_0^s \mu(r)dr(\alpha(V_{1s} + y_s) \\
&+ \alpha(V_{2s} + y_s'))ds \\
&\leq 2\int_0^t (\widetilde{N_1}c + 2\widetilde{N}\int_0^s \mu(r)dr)(\alpha(V_{1s} + y_s) + \alpha(V_{2s} + y_s'))ds \\
&\leq 2\int_0^t [\widetilde{N_1}cK_b + 2K_b\widetilde{N}\int_0^s \mu(r)dr(\sup_{0\leq\tau\leq s} \alpha(V_1(\tau)) \\
&+ \sup_{0\leq\tau\leq s} \alpha(V_1(\tau)))]ds \qquad (5.2.15)
\end{aligned}$$

$$\begin{aligned}
\alpha(V_2(t)) &\leq \alpha(\Gamma_2(V_1 \times V_2)(t)) \\
&\leq 2\widetilde{N_2}\int_0^t \alpha(g(s, V_{1s} + y_s))ds \\
&+ 2N\int_0^t \alpha \int_0^s f(r, V_{1\rho(r,x_r)} + y_{\rho(r,x_r)}, V_{2r} + y_r')drds \\
&\leq 2\int_0^t \widetilde{N_2}c\alpha(V_{1s} + y_s)ds \\
&+ 2\int_0^t 2N\int_0^s \mu(r)dr(\alpha(V_{1s} + y_s) + \alpha(V_{2s} + y_s'))ds \\
&\leq 2\int_0^t (\widetilde{N_2}c + 2N\int_0^s \mu(r)dr)(\alpha(V_{1s} + y_s) + \alpha(V_{2s} + y_s'))ds \\
&\leq 2\int_0^t [(\widetilde{N_1}cK_b + 2K_b\widetilde{N}\int_0^s \mu(r)dr)(\sup_{0\leq\tau\leq s} \alpha(V_1(\tau)) \\
&+ \sup_{0\leq\tau\leq s} \alpha(V_1(\tau)))]ds \qquad (5.2.16)
\end{aligned}$$

Since $m_j(t) := \sup_{0\leq s\leq t} \alpha(V_j(s))$ $(j = 1, 2)$ are continuous and nondecreasing functions on J_0. From equations (5.2.15),(5.2.16) we get that

$$m_1(t) + m_2(t) \leq \int_0^t K(c + \int_0^s \mu(r)dr)(m_1(s) + m_2(s))ds \qquad (5.2.17)$$

where K is an appropriate constant. So, by Gronwall's Lemma and (5.2.17) we see that $\alpha(V_k(t)) = 0$, $(k = 1, 2)$ $t \in J_0$. By lemma 2.5.4(1) we prove that

$V_k, (k = 1, 2)$ is relatively compact in $C(J_0, X)$. Since $\alpha(V_{jt_1} + y_{t_1}) \leq \alpha(V_{jt_1}) \leq K_b \sup_{0\leq s\leq t_1} \alpha(V_j(s)) = 0$ also $I_1^j(.,.)$ $(j = 1, 2)$ is continuous, we can show that

$$\alpha(I_1^1(V_{1t_1} + y_{t_1}, V_{2t_1} + y'_{t_1})) = \alpha(I_1^2(V_{1t_1} + y_{t_1}, V_{2t_1} + y'_{t_1})) = 0$$

Similarly, when $t \in J_1 = [t_1, t_1]$,

$$\begin{aligned} \alpha(V_1(t)) &\leq \alpha(\Gamma_1(V_1 \times V_2)(t)) \\ &\leq 2\int_{t_1}^{t} [\widetilde{N}_1 cK_b + 2K_b\widetilde{N} \int_0^s \mu(r)dr(\sup_{0\leq\tau\leq s} \alpha(V_1(\tau)) \\ &+ \sup_{0\leq\tau\leq s} \alpha(V_1(\tau)))]ds \end{aligned} \tag{5.2.18}$$

$$\begin{aligned} \alpha(V_2(t)) &\leq 2\int_{t_1}^{t} [(\widetilde{N}_1 cK_b + 2K_b\widetilde{N} \int_0^s \mu(r)dr)(\sup_{t_1\leq\tau\leq s} \alpha(V_1(\tau)) \\ &+ \sup_{t_1\leq\tau\leq s} \alpha(V_1(\tau)))]ds \end{aligned} \tag{5.2.19}$$

From equations (5.2.18),(5.2.19) we get that

$$\begin{aligned} &\sup_{t_1\leq s\leq t} \alpha(V_1(s)) + \sup_{t_1\leq s\leq t} \alpha(V_2(s)) \leq \\ &\int_{t_1}^{t} K(\{c + \int_0^s \mu(r)dr\})(\sup_{t_1\leq s\leq t} V_1(s) + \sup_{t_1\leq s\leq t} V_2(s))ds \end{aligned} \tag{5.2.20}$$

where K is an appropriate constant. So, by Gronwall's Lemma and (5.2.20) we see that $\alpha(V_k(t)) = 0$, $(k = 1, 2)$ $t \in J_1$. By lemma 2.5.4(1) we prove that $V_k, (k = 1, 2)$ is relatively compact in $C(J_1, X)$. Since $\alpha(V_{jt_1} + y_{t_1}) \leq \alpha(V_{jt_1}) \leq K_b \sup_{0\leq s\leq t_1} \alpha(V_j(s)) = 0$ also $I_2^j(.,.)$ $(j = 1, 2)$ is continuous, we can show that

$$\alpha(I_2^1(V_{1t_1} + y_{t_1}, V_{2t_1} + y'_{t_1})) = \alpha(I_2^2(V_{1t_1} + y_{t_1}, V_{2t_1} + y'_{t_1})) = 0.$$

Similarly V_k $(k = 1, 2)$ are relatively compact in $C(J_i, X)$, $(i = 2, 3, \cdots, n)$. Thus V_k $(k = 1, 2)$ are relatively compact in $S(b)$. Now by lemma 5.1.3 we can prove that Γ has fixed point in $\overline{\Omega_R}$. If (x, z) is a fixed point of Γ on $S(b)$ then $(x + y)$ is a mild solution of problem (5.1.1).

5.3 Non-instantaneous impulsive second order neutral differential equation

In this section we will find the conditions for the existence of mild solution of the problem (5.1.2). Let us define the mild solution as follows.

Definition 5.3.1. A function $x : (-\infty, a] \to X$ is a mild solution of the problem (5.1.2) if $x_0 = \phi$, $x'(0) = \xi$, $x(.)|_{[0,b]} \in PC^1(X)$, $x(t) = J_i^1(t, x(t - t_1))$, $\forall\ t \in (t_i, s_i]$, $i = 1, ..., n$, $x'(t) = J_i^2(t, x(t - t_1))$, $t \in (t_i, s_i]$, $i = 1, 2, ..., n$ and

$$\begin{aligned} x(t) &= C(t)\phi(0) + S(t)\xi - \int_0^t AS(t-s)g(s, x_s)ds \\ &+ \int_0^t S(t-s)\int_0^s f(r, x_{\rho(s,x_r)}, x'(r))drds,\ t \in [0, t_1] \\ x(t) &= C(t - s_i)J_i^1(s_i, x(t - t_1)) \\ &+ S(t - s_i)J_i^2(s_i, x(t - t_1)) \\ &- \int_{s_i}^t AS(t-s)g(s, x_s)ds \\ &+ \int_{s_i}^t S(t-s)\int_0^s f(s, x_{\rho(r,x_r)}, x'(r))drds, \\ &\quad for\ \ t \in [s_i, t_{i+1}],\ i = 1, ..., n \end{aligned} \tag{5.3.21}$$

Let $y : (-\infty, b] \to X$ is the function defined by $y_0 = \phi\ and\ y(t) = C(t)(\phi(0)) + S(t)(\xi)$ on $[0, t_1]$. Clearly $\|y_t\|_{\mathfrak{B}} \le K_b\|y\|_b + M_b\|\phi\|_{\mathfrak{B}}$ where $\|y\|_b = sup_{0\le t\le b}\|y(t)\|$. Since $S(b) = \{x : (-\infty, b] \to X : x_0 = 0, x'(0) = 0,\ x(.)|_J \in PC^1\}$. Therefore $\overline{x} = x + y$ is a mild solution of (5.1.2).

Theorem 5.3.1. If the hypothesis $(H\phi), (Hf), (Hg), (HJ), (H1)$ and $(H2)$ holds and the cosine family is equicontinuous then there exists a mild solution of the problem (5.1.1)

Proof: Let us define the function $z : (-\infty, 0] \to X$ as $z_0 = x_0'$, $z(t) = x'(t)$, $t \in J$ Let $\Gamma = (\Gamma_1, \Gamma_2) : S(b) \times S(b) \to S(b)$ be defined as

$$\Gamma_1(x, z)(t) = \begin{cases} 0, & t \le 0; \\ -\int_0^t AS(t-s)g(s, x_s + y_s)ds & \\ +\int_0^t S(t-s)\int_0^s f(r, \overline{x}_{\rho(r,x_r)}, x_r' + y_r')dr, & t \in J_1 = [0, t_1]. \end{cases} \tag{5.3.22}$$

and $\Gamma_2(x, z)(t) = \Gamma_1(x, z)'(t)$ Therefore,

$$\Gamma_2(x, z)(t) = \begin{cases} 0, & t \le 0 \\ -\int_0^t AC(t-s)g(s, x_s + y_s)ds & \\ +\int_0^t C(t-s)\int_0^s f(r, \overline{x}_{\rho(r,x_r)}, x_r' + y_r')drds, & \\ & t \in J_1 = [0, t_1]. \end{cases} \tag{5.3.23}$$

$$\Gamma_1(x,z)(t) = \begin{cases} J_i^1(t, x(t-t_1)), \quad t \in (t_i, s_i], \\ C(t-s_i)J_i^1(s_i, x(t-t_1)) \\ -S(t-s_i)J_i^2(s_i, x(t-t_1)) \\ -\int_{s_i}^t AS(t-s)g(s, x_s+y_s)ds \\ +\int_{s_i}^t S(t-s)\int_0^s f(r, \overline{x}_{\rho(r,x_r)}, x'_r+y'_r)drds, \ t \in J_i = (s_i, t_{i+1}]. \end{cases} \tag{5.3.24}$$

and $\Gamma_2(x,z)(t) = \Gamma_1(x,z)'(t)$ Therefore,

$$\Gamma_2(x,z)(t) = \begin{cases} J_i^2(t, x(t-t_1)), & t \in (t_i, s_i] \\ AS(t-s_i)J_i^1(s_i, x(t-t_1)) \\ -C(t-s_i)J_i^2(s_i, x(t-t_1)) \\ -\int_{s_i}^t AC(t-s)g(s, x_s+y_s)ds \\ +\int_{s_i}^t C(t-s)\int_0^s f(r, \overline{x}_{\rho(r,x_r)}, x'_r+y'_r)drds, \\ & t \in J_i = (s_i, t_{i+1}]. \end{cases} \tag{5.3.25}$$

It can be easily proved that Γ is continuous by Lebesgue Dominated Convergence theorem, axioms of phase space and the hypotheses $(H\phi), (Hf), (Hg), (HJ)$.

Step 1 : We show that $\Omega_0 = \{(x,z) \in S(b) \times S(b) : (x,z) = \lambda\Gamma(x,z) \ for \ some \ \lambda \in (0,1)\}$ is bounded. When $t \in J_0 = [0, t_1]$

$$\begin{aligned} \|x(t)\| &\leq \|\Gamma_1(x,z)(t)\| \leq \widetilde{N_1}\int_0^t [c(\|x\|_{\mathfrak{B}} + \overline{M}) + d]ds \\ &+ \widetilde{N}\int_0^t\int_0^s p(r)(\|x_r\|_{\mathfrak{B}} + \|z_r\|_{\mathfrak{B}} + M' + \overline{M})drds \\ &\leq \overline{M}\int_0^t (\widetilde{N_1}c + \widetilde{N}\int_0^s p(r)dr)ds + \widetilde{N_1}bd \\ &+ K_b\int_0^t (\widetilde{N_1}c + \widetilde{N}\int_0^s p(r)dr)(\|x\|_s + \|z\|_s)ds \\ &+ M'\widetilde{N}\int_0^t\int_0^s p(r)drds \end{aligned} \tag{5.3.26}$$

$$\begin{aligned}
\|z(t)\| &\leq \|\Gamma_2(x,z)(t)\| \leq \widetilde{N_2}\int_0^t [c(\|x\|_{\mathfrak{B}}+\overline{M})+d]ds \\
&+ N\int_0^t\int_0^s p(r)(\|x_r\|_{\mathfrak{B}}+\|z_r\|_{\mathfrak{B}}+M'+\overline{M})drds \\
&\leq \overline{M}\int_0^t(\widetilde{N_2}c+N\int_0^s p(r)dr)ds+\widetilde{N_2}bd \\
&+ K_b\int_0^t(\widetilde{N_2}c+N\int_0^s p(r)dr)(\|x\|_s+\|z\|_s)ds \\
&+ M'\widetilde{N}\int_0^t\int_0^s p(r)drds \qquad (5.3.27)
\end{aligned}$$

Therefore,

$$\begin{aligned}
\|x\|_t+\|z\|_t &\leq [(\widetilde{N_1}+\widetilde{N_2})bd \\
&+ \overline{M}\int_0^t[c(\widetilde{N_1}+\widetilde{N_2})+(N+\widetilde{N})\int_0^s p(r)dr]ds \\
&+ M'(\widetilde{N}+N)\int_0^t(\int_0^s p(r)dr)ds+\int_0^t[(\widetilde{N_1}c+\widetilde{N_2}c)K_b \\
&+ (N+\widetilde{N})K_b\int_0^s p(r)dr](\|x\|_s+\|z\|_s)]ds \qquad (5.3.28)
\end{aligned}$$

Since $\|x\|_t+\|z\|_t \in C(J_0,X)$ by Gronwall's lemma there is a constant $G_0>0$ such that $\|x\|_t+\|z\|_t \leq G_0,\ t\in J$ and $\|x_t\|_{\mathfrak{B}} \leq K_bG_0$ and $\|z_t\|_{\mathfrak{B}} \leq K_bG_0,\ t\in J_0$. By condition (HJ) it is observed that for $t\in[t_1,s_1)$

$$\|J_1^j(t,x(t-t_1))\|_E \leq c_1^j(2K_bG_0+\overline{M})+d_1^j := \eta_1^j \qquad (5.3.29)$$

When $t\in J_2=[s_1,t_2]$

$$\begin{aligned}
\|x(t)\| &\leq \|\Gamma_1(x,z)(t)\| \leq N[c_i^1(\|x_{s_i}\|_{\mathfrak{B}})+d_i^1] \\
&+ \widetilde{N}[c_i^2\|x_{s_i}\|_{\mathfrak{B}}+d_i^2] \\
&+ \widetilde{N_1}\int_{s_i}^t[c(\|x(s)\|_{\mathfrak{B}}+\overline{M})+d]ds \\
&+ \widetilde{N}\int_{s_i}^t\int_0^s p(r)(\|x_r\|_{\mathfrak{B}}+\|z_r\|_{\mathfrak{B}}+M'+\overline{M})drds \\
&\leq \overline{M}\int_{s_i}^t(\widetilde{N_1}c+\widetilde{N}\int_0^s p(r)dr)ds+M'\widetilde{N}\int_{s_i}^t\int_0^s p(r)drds \\
&+ K_b\int_{s_i}^t(\widetilde{N_1}c+\widetilde{N}\int_0^s p(r)dr)(\|x\|_s+\|z\|_s)ds \\
&+ [N(c_i^1K_b)+\widetilde{N}(c_i^2K_b)](\|x\|+\|z\|) \\
&+ \widetilde{N_1}bd+N(d_i^1)+\widetilde{N}(d_i^2) \qquad (5.3.30)
\end{aligned}$$

$$\begin{aligned}
\|z(t)\| &\leq \|\Gamma_2(x,z)(t)\| \leq \widetilde{N_1}\|[c_i^1\|x_{s_i}\|_{\mathfrak{B}} + d_i^1\| \\
&+ N[c_i^2\|x_{s_i}\|_{\mathfrak{B}} + d_i^2 \\
&+ \widetilde{N_2}\int_{s_i}^t \|[c(\|x(s)\|_{\mathfrak{B}} + \overline{M}) + d]ds \\
&+ N\int_{s_i}^t\int_0^s p(r)(\|x_r\|_{\mathfrak{B}} + \|z_r\|_{\mathfrak{B}} + M' + \overline{M})drds \\
&\leq \overline{M}\int_0^t(\widetilde{N_2}c + N\int_0^s p(r)dr)ds + M'\widetilde{N}\int_0^t\int_0^s p(r)drds \\
&+ K_b\int_0^t(\widetilde{N_2}c + N\int_0^s p(r)dr)(\|x\|_s + \|z\|_s)ds \\
&+ [\widetilde{N_2}(c_i^1K_b) + N(c_i^2K_b)](\|x\| + \|z\|) \\
&+ \widetilde{N_2}bd + N(d_i^1) + N(d_i^2) + \widetilde{N_2}bd \qquad (5.3.31)
\end{aligned}$$

Therefore,

$$\begin{aligned}
\|x\|_t + \|z\|_t &\leq \{K + \overline{M}\int_0^t[c(\widetilde{N_1} + \widetilde{N_2}) + (N + \widetilde{N})\int_0^s p(r)dr]ds \\
&+ M'(\widetilde{N} + N)\int_0^t(\int_0^s p(r)dr)ds + \int_0^t[(\widetilde{N_1}c + \widetilde{N_2}c)K_b \\
&+ (N + \widetilde{N})K_b\int_0^s p(r)dr](\|x\|_s + \|z\|_s)]ds\} \qquad (5.3.32)
\end{aligned}$$

where K is an appropriate constant. Since $\|x\|_t + \|z\|_t \in C(J_1, X)$ by Gronwall's lemma there is a constant $G_1 > 0$ such that $\|x\|_t + \|z\|_t \leq G_1,\ t \in J$ and $\|x_t\|_{\mathfrak{B}} \leq K_bG_0$ and $\|z_t\|_{\mathfrak{B}} \leq K_bG_1,\ t \in J_0$. By condition (HJ) it is observed that for $t \in [t_2, s_2)$

$$\|J_2^j(t, x(t-t_1)\| \leq c_2^j(2K_bG_1 + \overline{M}) + d_2^j := \eta_2^j\ j = 1,2 \qquad (5.3.33)$$

Similarly let $G = \max\{G_0, \eta_1, G_1, \eta_2 \cdots, G_n\}$, then $\|(x,z)\|_b \leq G$ and Ω_0 is bounded.

Let $R > G$ and $\Omega_R = \{(x,z) \in S(b) \times S(b) : \|(x,z)\|_b < R\}$,

Since $R > G$,

$$(x,z) \neq \lambda\Gamma(x,z)\ \ \forall(x,z) \in \partial\Omega_R \qquad (5.3.34)$$

Step2 : Suppose $V \subset \overline{\Omega_R}$ be countable set and $V \subset \overline{co}(\{0,0\} \subset \Gamma(V))$. Let

$$V_1 = \{x \in S(b) : \exists z \in S(b), (x,z) \in V\},$$

$$V_2 = \{z \in S(b) : \exists x \in S(b), (x, z) \in V\}$$

$$V \subset V_1 \times V_2 \subset \overline{co}(\{0\} \cup \Gamma_1(V_1 \times V_2)) \times \overline{co}(\{0\} \cup \Gamma_2(V_1 \times V_2)) \tag{5.3.35}$$

From equations (5.3.24),(5.3.25), lemma 2.5.9 and $(Hg)(2)$ we get that $\Gamma_j((\widetilde{V_1})_i \times (\widetilde{V_2})_i)$, $(j = 1, 2)$ are equicontinuous on $J_i(i = 0, 1, \cdots, n)$. From (5.3.35) it is seen that $(\widetilde{V_k})_i(k = 1, 2)$ are equicontinuous. Next we prove that V_1 and V_2 are relatively compact. We identify $V_k|_{J_i}$, $(k = 1, 2)$ with $\widetilde{V_i}$ where $V_k|_{J_i}$ is the restriction of V_k on $J_i = (s_i, t_{i+1}]$. When $t \in J_0 = [0, t_1]$, from hypotheses $(Hf)(3)$, $(Hg)(5)$ and Lemma 5.1.2 we get that

$$\begin{aligned}
\alpha(V_1(t)) &\leq \alpha(\Gamma_1(V_1 \times V_2)(t)) \\
&\leq 2\widetilde{N_1} \int_0^t \alpha(g(s, V_{1s} + y_s))ds \\
&+ 2\widetilde{N} \int_0^t \alpha \int_0^s f(r, V_{1\rho(r,x_r)} + y_{\rho(r,x_r)}, V_{2r} + y'_r)drds \\
&\leq 2\int_0^t \widetilde{N_1} c\alpha(V_{1s} + y_s)ds + 2\int_0^t 2\widetilde{N} \int_0^s \mu(r)dr(\alpha(V_{1s} + y_s) \\
&+ \alpha(V_{2s} + y'_s))ds \\
&\leq 2\int_0^t (\widetilde{N_1} c + 2\widetilde{N} \int_0^s \mu(r)dr)(\alpha(V_{1s} + y_s) + \alpha(V_{2s} + y'_s)) \\
&\leq 2\int_0^t [\widetilde{N_1} cK_b + 2K_b\widetilde{N} \int_0^s \mu(r)dr(\sup_{0\leq\tau\leq s} \alpha(V_1(\tau)) \\
&+ \sup_{0\leq\tau\leq s} \alpha(V_1(\tau)))]ds
\end{aligned} \tag{5.3.36}$$

$$\begin{aligned}
\alpha(V_2(t)) &\leq \alpha(\Gamma_2(V_1 \times V_2)(t)) \\
&\leq 2\widetilde{N_2} \int_0^t \alpha(g(s, V_{1s} + y_s))ds \\
&+ 2N \int_0^t \alpha \int_0^s f(r, V_{1\rho(r,x_r)} + y_{\rho(r,x_r)}, V_{2r} + y'_r)drds \\
&\leq 2\int_0^t \widetilde{N_2} c\alpha(V_{1s} + y_s)ds \\
&+ 2\int_0^t 2N \int_0^s \mu(r)dr(\alpha(V_{1s} + y_s) + \alpha(V_{2s} + y'_s))ds \\
&\leq 2\int_0^t (\widetilde{N_2} c + 2N \int_0^s \mu(r)dr)(\alpha(V_{1s} + y_s) + \alpha(V_{2s} + y'_s))ds
\end{aligned}$$

$$\leq 2\int_0^t [(\widetilde{N_1}cK_b + 2K_b\widetilde{N}\int_0^s \mu(r)dr)(\sup_{0\leq\tau\leq s}\alpha(V_1(\tau)) + \sup_{0\leq\tau\leq s}\alpha(V_1(\tau)))]ds \quad (5.3.37)$$

Since $m_j(t) := \sup_{0\leq s\leq t}\alpha(V_j(s))$ $(j=1,2)$ are continuous and nondecreasing functions on J_0. From equations (5.3.36),(5.3.37) we get that

$$m_1(t) + m_2(t) \leq \int_0^t K(c + \int_0^s \mu(r)dr)(m_1(s) + m_2(s))ds \quad (5.3.38)$$

where K is an appropriate constant. So, by Gronwall's Lemma and (5.3.38) we see that $\alpha(V_k(t)) = 0$, $(k = 1,2)$ $t \in J_0$. By lemma 2.5.4(1) we prove that $V_k, (k = 1,2)$ is relatively compact in $C(J_0, X)$. Since $\alpha(V_{jt_1} + y_{t_1}) \leq \alpha(V_{jt_1}) \leq K_b \sup_{0\leq s\leq t_1}\alpha(V_j(s)) = 0$ also $J_1^j(.,.)$ $(j=1,2)$ is continuous, we can show that

$$\alpha(J_1^1(V_{1t_1} + y_{t_1})) = \alpha(J_1^2(V_{1t_1} + y_{t_1})) = 0$$

Similarly when $t \in J_1 = [t_1, s_1]$,

$$\begin{aligned}\alpha(V_1(t)) &\leq \alpha(\Gamma_1(V_1 \times V_2)(t)) \\ &\leq 2\int_{t_1}^t [\widetilde{N_1}cK_b + 2K_b\widetilde{N}\int_0^s \mu(r)dr(\sup_{0\leq\tau\leq s}\alpha(V_1(\tau)) \\ &\quad + \sup_{0\leq\tau\leq s}\alpha(V_1(\tau)))]ds + \int_0^t cK_b \sup_{t_1\leq s\leq t}\alpha(V_1(s))ds\end{aligned} \quad (5.3.39)$$

$$\begin{aligned}\alpha(V_2(t)) &\leq 2\int_{t_1}^t [(\widetilde{N_1}cK_b + 2K_b\widetilde{N}\int_0^s \mu(r)dr)(\sup_{t_1\leq\tau\leq s}\alpha(V_1(\tau)) \\ &\quad + \sup_{t_1\leq\tau\leq s}\alpha(V_1(\tau)))]ds + cK_b \sup_{t_1\leq s\leq t}\alpha(V_2(s))ds\end{aligned} \quad (5.3.40)$$

From equations (5.3.39),(5.3.40) we get that

$$\sup_{t_1\leq s\leq t}\alpha(V_1(s)) + \sup_{t_1\leq s\leq t}\alpha(V_2(s)) \leq$$

$$\int_{t_1}^t (K\{c + \int_0^s \mu(r)dr\} + cK_b)(\sup_{t_1\leq s\leq t}V_1(s) + \sup_{t_1\leq s\leq t}V_2(s))ds \quad (5.3.41)$$

where K is the appropriate constant. So, by Gronwall's Lemma and (5.3.41) we see that $\alpha(V_k(t)) = 0$, $(k = 1,2)$ $t \in J_1$. By lemma 2.5.4(1) we prove that

$V_k, (k = 1, 2)$ is relatively compact in $C(J_1, X)$. Since $\alpha(V_{jt_1} + y_{t_1}) \leq \alpha(V_{jt_1}) \leq K_b \sup_{0\leq s\leq t_1} \alpha(V_j(s)) = 0$ also $J_2^j(.,.)$ $(j = 1, 2)$ is continuous, we can show that

$$\alpha(J_2^1(V_{1t_1} + y_{t_1})) = \alpha(J_2^2(V_{1t_1} + y_{t_1})) = 0$$

Similarly V_k $(k = 1, 2)$ are relatively compact in $C(J_i, X)$, $(i = 2, 3, \cdots, n)$. Thus V_k $(k = 1, 2)$ are relatively compact in $S(b)$. Now by lemma 5.1.3 we can prove that Γ has fixed point in $\overline{\Omega_R}$. If (x, z) is a fixed point of Γ on $S(b)$ then $(x + y)$ is a mild solution of problem (5.1.2).

Remark : We can also apply the above methodology to the following:

$$\begin{aligned}
\frac{d^2}{dt^2}x(t) &= A(x(t) - \int_0^t g(\tau, x_\tau)d\tau) + \int_0^t f(t, x_{\rho(t,x_t)})dt,\ t \in [0, b],\ t \neq t_i, \\
&\qquad i = 1, ..., n \\
x_0 &= \phi \in \mathfrak{B}, \\
x'(0) &= \xi \in X, \\
\Delta x(t_i) &= I_i^1(x_{t_i}),\ i = 1, 2, ..., n \\
\Delta x'(t) &= I_i^2(x_{t_i}),\ i = 1, 2, ..., n
\end{aligned} \tag{5.3.42}$$

Here $0 = t_0 < t_1 \leq t_2, ..., < t_n \leq t_{n+1} = b$ are prefixed numbers.

$$\begin{aligned}
\frac{d^2}{dt^2}x(t) &= A(x(t) - \int_0^t g(\tau, x_\tau)d\tau) + \int_0^t f(t, x_{\rho(t,x_t)})dt,\ t \in (s_i, t_{i+1}], \\
&\qquad i = 0, ..., n \\
x_0 &= \phi \in \mathfrak{B}, \\
x'(0) &= \xi \in X, \\
x(t) &= J_i^1(t, x(t - t_1)),\ t \in (t_i, s_i],\ i = 1, 2, ..., n \\
x'(t) &= J_i^2(t, x(t - t_1)),\ t \in (t_i, s_i],\ i = 1, 2, ..., n
\end{aligned} \tag{5.3.43}$$

Here $0 = t_0 = s_0 < t_1 \leq s_1 \leq t_2, ..., < t_n \leq s_n \leq t_{n+1} = b$ are prefixed numbers. The mild solution of (5.3.42) is defined as

Definition 5.3.2. A function $x : (-\infty, b] \to X$ is a mild solution of the problem

(5.3.42) if $x_0 = \phi,\ x'(0) = \xi,\ x(.)|_{[0,b]} \in PC^1(X)$, and

$$\begin{aligned} x(t) &= C(t)\phi(0) + S(t)\xi + \int_0^t g(s, x_s)ds - \int_0^t C(t-s)g(s, x_s)ds \\ &+ \int_0^t S(t-s)\int_0^s f(r, x_{\rho(r,x_r)})drds + \sum_{0<t_i<t} C(t-t_i)I_i^1(x_{t_i}) \\ &+ \sum_{0<t_i<t} S(t-t_i)I_i^2(x_{t_i}) \end{aligned} \quad (5.3.44)$$

We define $S(b) = \{x : (-\infty, b] \to X : x_0 = 0, x'(0) = 0,\ x(.)|_J \in PC^1\}$. We define $\Gamma =: S(b) \times S(b) \to S(b)$

$$\Gamma(x)(t) = \begin{cases} 0, & t \le 0; \\ +\int_0^t C(t-s)g(s, x_s + y_s)ds + \int_0^t g(s, x_s + y_s)ds & \\ +\int_0^t S(t-s)\int_0^s f(r, \overline{x}_{\rho(r,x_r)})dr & \\ +\sum_{0<t_i<t} C(t-t_i)I_i^1(x_{t_i} + y_{t_i}) & \\ +\sum_{0<t_i<t} S(t-t_i)I_i^2(x_{t_i} + y_{t_i}), & t \in J \end{cases} \quad (5.3.45)$$

and proceed as in the first case of theorem 5.2.4.

Definition 5.3.3. A function $x : (-\infty, b] \to X$ is a mild solution of the problem (5.3.43) if $x_0 = \phi,\ x'(0) = \xi,\ x(.)|_{[0,b]} \in PC^1(X),\ x(t) = J_i^1(t, x(t-t_1)),\ \forall\ t \in (t_i, s_i],\ i = 1, ..., n,\ x'(t) = J_i^2(t, x(t-t_1)),\ t \in (t_i, s_i],\ i = 1, 2, ..., n$ and

$$\begin{aligned} x(t) &= C(t)\phi(0) + S(t)\xi - \int_0^t C(t-s)g(s, x_s)ds \\ &+ \int_0^t g(s, x_s)ds + \int_0^t S(t-s)\int_0^s f(r, x_{\rho(s,x_r)})drds,\ t \in [0, t_1] \\ x(t) &= C(t-s_i)J_i^1(s_i, x(t-t_1)) \\ &+ S(t-s_i)J_i^2(s_i, x(t-t_1)) \\ &- \int_{s_i}^t C(t-s)g(s, x_s)ds \\ &+ \int_0^t g(s, x_s)ds + \int_{s_i}^t S(t-s)\int_0^s f(s, x_{\rho(r,x_r)})drds, \\ &\qquad for\ \ t \in [s_i, t_{i+1}],\ i = 1, ..., n \end{aligned} \quad (5.3.46)$$

We define $\Gamma : S(b) \times S(b) \to S(b)$ be defined as

$$\Gamma(x)(t) = \begin{cases} 0, & t \leq 0; \\ -\int_0^t C(t-s)g(s, x_s + y_s)ds + \int_0^t g(s, x_s)ds \\ +\int_0^t S(t-s)\int_0^s f(r, \overline{x}_{\rho(r,x_r)})dr, & t \in J_1 = [0, t_1]. \end{cases} \tag{5.3.47}$$

$$\Gamma(x)(t) = \begin{cases} J_i^1(t, x(t-t_1)), & t \in (t_i, s_i], \\ C(t-s_i)J_i^1(s_i, x(t-t_1)) \\ -S(t-s_i)J_i^2(s_i, x(t-t_1)) + \int_{s_i}^{t_i} g(s, x_s)ds \\ -\int_{s_i}^{t_i} C(t-s)g(s, x_s + y_s)ds \\ +\int_0^t C(t-s)\int_0^s f(r, \overline{x}_{\rho(r,x_r)}, x_r' + y_r')drds, \\ & t \in J_1 = [0, t_1]. \end{cases} \tag{5.3.48}$$

and proceed as in Theorem 5.3.1.

5.4 Examples

In this section we discuss a partial differential equation applying the abstract results of this paper. We discuss the partial differential equation in two examples. In Example 1 instantaneous impulsive differential system is studied while in Example 2 non-instantaneous impulsive differential system is studied. As a result the dynamics and solutions of these two examples will be different as we can perceive from equations (5.2.3) and (5.3.21). In this application, $\mathfrak{B}$ is the phase space $PC_0 \times L^2(h, X)$ see ([98]).

Example 1 : We study following system with instantaneous impulses

$$\begin{aligned} \frac{\partial^2}{\partial t^2}x(t,\sigma) &= (i\triangle - iV(\sigma))(x(t,\sigma) - \int_{-\infty}^{t}\int_0^{\pi} x(s, \sigma - v)d\sigma ds) \\ &+ \int_{-\infty}^{t}(a(x) + B(x(s, \sigma - h(x(s,\sigma))))sin(\frac{t}{\epsilon}))ds, \ t \in [0,b], \sigma \in [0,\pi], \\ x(t,0) &= x(t,\pi) = 0, \ t \in [0,b], \\ x(s,\sigma) &= \phi(s,\sigma), \ -\infty \leq s \leq 0, 0 \leq \sigma \leq \pi, \\ \frac{\partial}{\partial t}x(0,\sigma) &= \xi(\sigma), \ 0 \leq \sigma \leq \pi, \\ \Delta x(t_i)(\sigma) &= \int_{\infty}^{t_i} n_i^1(t_i - s)x(s,\sigma)ds, \ i = 1, ..., n \\ \Delta x'(t_i)(\sigma) &= \int_{\infty}^{t_i} n_i^1(t_i - s)x(s,\sigma)ds, \ i = 1, ..., n \end{aligned} \tag{5.4.49}$$

where $\phi \in H^1([0,\pi]),\ \xi \in X,\ 0 = t_0 = s_0 < t_1 \leq s_1 \leq t_2, ..., t_n \leq s_n \leq t_{n+1} = b$ Here, $X = L^2([0,\pi])$, $\mathfrak{B} = PC_0 \times L^2(\rho, X)$, $A \subset D(A) \subset X \to X$ is the map defined by $A = (i\Delta - iV))$ with domain $D(A) = H^2 \cap H_0^1$. A denotes the infinitesimal generator $(C(t))_{t\in R}$ on X. A has a discrete spectrum, and the following properties hold

(C1) $A\phi = -\sum_{n=1}^{\infty} \lambda_n^2 < \phi, z_n > z_n$ where $\phi \in D(A)$, $\lambda_n,\ z_n,\ n \in \mathfrak{N}$ are eigenvalues and eigenvectors of A.

(C2) $C(t)\phi = \sum_{n=1}^{\infty} cos(\lambda_n t) < \phi, z_n > z_n$ and $S(t)\phi = \sum_{n=1}^{\infty} \frac{sin(\lambda_n t)}{n} < \phi, z_n > z_n$, for $\phi \in X$.

By defining maps $\rho, g, f : [0,b] \times \mathfrak{B} \times X \to X$ by

$$\rho(t,\sigma) := \sigma - h(x(s,\sigma))$$

$$g(\psi)(\sigma) := \int_{-\infty}^{t} \int_{0}^{\pi} x(s, \sigma - v) d\sigma ds),$$

$$f(\psi)(\sigma) := \int_{-\infty}^{t} (a(x) + B(x(s, \sigma - h(x(s,\sigma)))) sin(\frac{t}{\epsilon}))$$

the system (5.4.50) can be transformed into system (5.1.1) Assume that the functions $\rho_i : \mathbb{R} \to [0,\infty)$, $m : \mathbb{R} \to \mathbb{R}$ are piecewise continuous.
$g(t,.), I_i, (i = 1, ..., n), f$ are bounded linear operators. We take $Y = D(A)$. Therefore if $\iota : Y \to X$ is the inclusion then $t \to AS(t)$ is uniformly continuous into $L(Y,X)$ and $\|AS(t)\|_{L(Y,X)} \leq 1$ for $t \in [0,a]$ Hence by assumptions (Hϕ),(Hf),(Hg), (HI), (H1),(H2) and theorem 5.3.1 it is ensured that mild solution to the problem (5.4.50) exists.

Example 2 : We study the following system with non-instantaneous impulses

$$
\begin{aligned}
\frac{\partial^2}{\partial t^2}(x(t,\sigma) &- \int_{-\infty}^{t}\int_{0}^{\pi} x(s,\sigma-v)d\sigma ds) = (i\Delta - iV(\sigma))x(t,\sigma) \\
&+ \int_{-\infty}^{t}(a(x)+B(x(s,\sigma-h(x(s,\sigma)))) sin(\frac{t}{\epsilon}))ds,\ t\in[0,b],\sigma\in[0,\pi], \\
x(t,0) &= x(t,\pi)=0,\ t\in[0,b], \\
x(s,\sigma) &= \phi(s,\sigma),\ -\infty\le s\le 0, 0\le\sigma\le\pi, \\
\frac{\partial}{\partial t}x(0,\sigma) &= \xi(\sigma),\ 0\le\sigma\le\pi, \\
x(t)(\sigma) &= \int_{t_i}^{s_i} n_i^1(t-t_1)x(s,\sigma)ds,\ t\in[s_i,t_i],\ i=1,...,n \\
x'(t)(\sigma) &= \int_{t_i}^{s_i} n_i^1(t-t_1)x(s,\sigma)ds,\ t\in[s_i,t_i],\ i=1,...,n
\end{aligned} \tag{5.4.50}
$$

where $\phi \in H^1([0,\pi])$, $\xi \in X$, $0 = t_0 = s_0 < t_1 \le s_1 \le t_2, ..., t_n \le s_n \le t_{n+1} = b$ Here, $X = L^2([0,\pi])$, $\mathfrak{B} = PC_0 \times L^2(\rho, X)$, $A \subset D(A) \subset X \to X$ is the map defined by $A = (i\Delta - iV))$ with domain $D(A) = H^2 \cap H_0^1$. A denotes infinitesimal generator of $(C(t))_{t\in R}$ on X. Also, A has a discrete spectrum, A has a discrete spectrum, and the following properties hold

(C1) $A\phi = -\sum_{n=1}^{\infty} \lambda_n^2 < \phi, z_n > z_n$ where $\phi \in D(A)$, λ_n, z_n, $n \in \mathfrak{N}$ are eigenvalues and eigenvectors of A.

(C2) $C(t)\phi = \sum_{n=1}^{\infty} cos(\lambda_n t) < \phi, z_n > z_n$ and $S(t)\phi = \sum_{n=1}^{\infty} \frac{sin(\lambda_n t)}{n} < \phi, z_n > z_n$, for $\phi \in X$.

By defining maps $\rho, g, f : [0,b] \times \mathfrak{B} \times X \to X$ as in example 1 the system (5.4.50) can be transformed into system (5.1.2) Assume that the functions $\rho_i : \mathbb{R} \to [0,\infty)$, $m : \mathbb{R} \to \mathbb{R}$ are piecewise continuous. Hence by assumptions $(H\phi), (Hf), (Hg), (HJ)$, $(H1), (H2)$ and theorem 5.3.1 it is ensured that mild solution to the problem (5.4.50) exists.

5.5 Conclusion

Thus we establish the existence of mild solution of the non-instantaneous impulsive partial second order functional differential equations (5.1.1) and (5.1.2) , using

Kuratowski measure of noncompactness and Mónch fixed point theorem. The compactness Lipschitz condition and other restrictive conditions have been removed.

Chapter 6

Approximate Controllability of a Second Order Neutral Differential Equation with State Dependent Delay

This chapter investigates the existence of mild solution and approximate controllability of a second order neutral partial differential equation involving state dependent delay. The Hausdorff measure of noncompactness combined with Darbo Sadovskii theorem are used to establish the existence of mild solution of the system. The strict assumption such as the compactness of the associated cosine or sine family of operators is removed. Some fundamental and natural assumptions are used instead. The conditions for approximate controllability are proposed for the distributed second order neutral system by assuming the approximate controllability of the corresponding linear system in a Hilbert space.

6.1 Introduction

Of late, much attention is paid to functional differential equations with state dependent delay. We refer [15],[17],[76],[123], for related information. Generally the literature related delay differential equations dealt with functional differential equations in which the state actually belonged to a finite dimensional space. As a result, partial functional differential equations involving state dependent delay were mostly abandoned. This is one of the motivations of our work.

In this paper, we study a second order neutral differential equation modeled in the form

$$\begin{aligned} \frac{d^2}{dt^2}(x(t) &- g(t,x_t)) = Ax(t) + f(t, x_{\rho(t,x_t)}) + Bu(t),\ t \in J = [0,a] \\ x_0 &= \phi \in \mathfrak{B},\ \frac{d}{dt}[x(t) - g(t,x_t)]|_{t=0} = z,\ z \in X \end{aligned} \tag{6.1.1}$$

where A denotes the infinitesimal generator of a strongly continuous cosine family $\{C(t) : t \in \mathsf{R}\}$ of bounded linear operators on a Hilbert space X and $S(t)$ is the associated sine function.. The history valued function $x_t : (-\infty, 0] \to X$, $x_t(\theta) = x(t+\theta)$ takes values in some abstract phase space $\mathfrak{B}$ defined in chapter 2 as Definition 2.2.12; g, f are appropriate functions. Let U be another Hilbert space. $u \in U$ is a control parameter. B is a bounded linear operator defined from a Hilbert space U to X.

The existence and uniqueness of mild solutions of a second order abstract partial neutral differential equation related to (6.1.2) is discussed in [34],[38]. The authors assumed strict conditions on the cosine family generated by A, that limited the underlying space X to finite dimension. Consequently, the equations discussed in these works are actually ordinary instead of being partial differential equations.

The approximate controllability of infinite dimensional systems has been extensively discussed, see for instance [33],[39],[61],[121],[159] and the references therein. However, in these papers the invertibility of a controllability operator is assumed. As a consequence, their approach is unsuccessful in infinite dimensional spaces if the generated semigroup is compact. Moreover it is practically troublesome to verify their conditions directly. This is one of the motivations of our paper.

[139],[156] proposed conditions on the systems operators together with the assumption of approximate controllability of the corresponding linear system. To the best of our knowledge only a few papers are available in literature regarding approximate controllability of neutral partial differential equation with state dependent delay.

In the section 6.2 the existence of mild solution of the following second order equation

$$\begin{aligned} \frac{d^2}{dt^2}(x(t) &- g(t,x_t)) = Ax(t) + f(t, x_{\rho(t,x_t)}),\ t \in J = [0,a] \\ x_0 &= \phi \in \mathfrak{B},\ \frac{d}{dt}[x(t) - g(t,x_t)]|_{t=0} = z,\ z \in X \end{aligned} \quad (6.1.2)$$

is discussed. Then in the next section the approximate controllability of the problem (6.1.1) is proved. The last section illustrates the result with an example.

6.2 Existence of mild solution

Let $N, \widetilde{N}$ be certain constants such that $\|C(t)\| \leq N$ and $\|S(t)\| \leq \widetilde{N}$ for every $t \in J = [0,a]$.

Definition 6.2.1. The set given by $\mathcal{R}(f) = \{x(T) \in X : x \in X$ *is a mild solution of*(6.1.2)$\}$ is called reachable set of the system (6.1.2) . $\mathcal{R}_0$ is the reachable set of the corresponding linear control system (6.2.3).

The system (6.1.2) will be approximately controllable if $\mathcal{R}(f)$ is dense in X. Also the corresponding linear system is approximately controllable if $\mathcal{R}_0$ is dense in X.

The approximate controllability of the following linear control system

$$\begin{aligned} x''(t) &= Ax(t) + Bu(t), t \in J \\ x(0) &= x^0, \\ x'(0) &= x^1 \end{aligned} \quad (6.2.3)$$

has been studied by several authors. The existence of solutions of the second order abstract Cauchy problem

$$\begin{aligned} x''(t) &= Ax(t) + h(t), t \in J \\ x(0) &= x^0, \\ x'(0) &= x^1 \end{aligned} \quad (6.2.4)$$

where $h : [0,a] \to X$ is an integrable function has been discussed in [162]

Lemma 6.2.2. [75] Under the preceding assumptions, if h is a continuously differentiable function, then

$$\int_0^t C(t-s)h(s)ds = S(t)h(0) + \int_0^t S(t-s)h'(s)ds$$

We assume that the following conditions hold

(H1) There exists $(Y, \|.\|_Y)$ (a Banach space) continuously included in X such that $AS(t) \in L(Y,X)$, for all $t \in J$ and $AS(.)x \in C(J;X)$, for every $x \in Y$. There exists constants $N_Y, \widetilde{N_1}$, such that $\|y\| \leq N_Y\|y\|_Y$, $\forall y \in Y$ and $\|AS(t)\|_{L(Y,X)} \leq \widetilde{N_1}$, $\forall t \in J$

(H2) $\mathfrak{R}(C(t) - I)$ is closed and dim $\mathrm{Ker}(C(t) - I) < \infty$, $\forall\, 0 < t \leq a$

Lemma 6.2.3. [96] Suppose that the condition $(H2)$ be satisfied and $D \subset Y$. If D is bounded in X and the set $\{AS(t)y : t \in [0, a], y \in D\}$ is relatively compact in X, then D is relatively compact in X.

Lemma 6.2.4. Suppose that h' be continuously differentiable function, then

$$\int_0^t AS(t-s)h(s)ds = -h(t) + C(t)h(0) + \int_0^t S(t-s)h''(s) + S(t)h'(0)$$

Proof : By integration of parts formula, we get

$$\int_0^t AS(t-s)ds = -\int_t^0 AS(p)dp = \int_0^t AS(p)dp = [C(p)]_0^t = C(t)h(0) - I$$

and by applying lemma 6.2.2, we get

$$\begin{aligned}\int_0^t AS(t-s)h(s)ds &= C(t)h(0) - h(t) + \int_0^t C(t-s)h'(s)ds \\ &= C(t)h(0) - h(t) + \int_0^t S(t-s)h''(s)ds + S(t)h'(0)\end{aligned}$$

We define mild solution of problem (6.1.2) as follows.

Definition 6.2.5. A function $x : (-\infty, a] \to X$ is a mild solution of the problem (6.1.2) if $x_0 = \phi$; $x(.)|_{[0,a]} \in C(J,X)$, the functions $f(s, x_{\rho(s,x_s)})$ and $g(s, x_s)$ are integrable and the integral equation is satisfied

$$\begin{aligned}x(t) &= C(t)(\phi(0) - g(0, \phi(0)) + S(t)z + \int_0^t AS(t-s)g(s, x_s)ds \\ &+ g(t, x_t) + \int_0^t S(t-s)f(s, x_{\rho(s,x_s)})ds,\ t \in [0, a]\end{aligned} \tag{6.2.5}$$

To prove our result we always assume $\rho : J \times \mathfrak{B} \to (-\infty, a]$ is a continuous function .

Lemma 6.2.6. [96] If $y : (-\infty, a] \to X$ is a function such that $y_0 = \phi$ and $y|_J \in C(X)$ then

$\|y_{\rho(s,y_s)}\|_{\mathfrak{B}} \leq (M_a + \widetilde{J^\phi})\|\phi\|_{\mathfrak{B}} + K_a sup\{\|y(\theta)\|; \theta \in [0, max\{0, s\}]\},$

$$s \in \mathbb{R}(\rho^-) \cup [0, a]$$

where $\widetilde{J^\phi} = \sup_{t \in \mathbb{R}(\rho^-)} J^\phi(t),\ M_a = \sup_{t \in J} M(t)\ and\ K_a = \sup_{t \in J} K(t).$

The function J^ϕ is defined as follows. The following hypotheses are used.

(H_ϕ) The function $t \to \phi_t$ is continuous from $\mathbb{R}(\rho^-) = \{\rho(s, \psi) : \rho(s, \psi) \leq 0\}$ into $\mathfrak{B}$ and there exists a continuous bounded function $J^\phi : \mathbb{R}(\rho^-) \to (0, \infty)$ such that $\|\phi_t\|_{\mathfrak{B}} \leq J^\phi(t)\|\phi\|_{\mathfrak{B}}$ for every $t \in \mathbb{R}(\rho^-)$.

(Hf) The function $f : J \times \mathfrak{B} \to X$ satisfies the following conditions:

(1) For each $x : (-\infty, a] \to X, x_0 \in \mathfrak{B}\ and\ x|_J \in C([0, a], X),\ f(., \psi) : J \to X$ is strongly measurable for every $\psi \in \mathfrak{B}$ and $f(t, .)$ is continuous for a.e. $t \in J$.

(2) $\exists$ an integrable function $\alpha_f : J \to [0, +\infty)$ and a monotone continuous nondecreasing function $\Omega_f : [0, +\infty) \to (0, +\infty)$ with the property that $\|f(t, v)\| \leq \alpha_f(t)\Omega_f(\|v\|_{\mathfrak{B}})\ \forall t \in J\ and\ v \in \mathfrak{B}$.

(3) Let $D(\theta) = \{v(\theta) : v \in D\}$. For a.e. $s, t, \in J$, $\exists$ an integrable function $\eta : J \to [0, \infty)$ such that

$$\chi(S(s)f(t, D)) \leq \eta(t) \sup_{-\infty<\theta<0} \chi(D(\theta))$$

(Hg) The function $g : J \times \mathfrak{B} \to Y$ satisfies the following

(i) $g(t, .) : \mathfrak{B} \to X$ is continuous $\forall t \in J$. Let us define $V(k, g)$ as the set of function $V(k, g) = \{t \to g(t, x_t) : x \in B_k(0, S(a))\}$, where $S(a) = \{x : (-\infty, a] \to X\ such\ that\ x_0 = 0,\ x|_J \in C\}$. The set $V(k) = \{AS(\theta)g(s, \psi) : \theta, s \in J,\ \psi \in B_k(0, \mathfrak{B})\}$ be relatively compact in X. The set $\{v : v \in V(k, g)\}$ is an equicontinuous subset of $C([0, a], X)$.

(ii) If $x : (-\infty, a] \to X$ be such that $x_0 = \phi$ and $x|_J \in C$ then the function $t \to g(t, x_t)$ belongs to $C([0, a], X)$ and is strongly measurable from J into X.

(iii) There exists an integrable function $\alpha_g : J \to [0, +\infty)$ and a monotone continuous nondecreasing function $\Omega_g : [0, +\infty) \to (0, +\infty)$ such that $\|g(t, v)\| \leq \alpha_g(t)\Omega_g(\|v\|_{\mathfrak{B}})\ \forall t \in J\ and\ v \in \mathfrak{B}$.

(iv) $g(a, x_a) = 0,\ \forall\, x \in X$ and $\|g(t, \psi)\|_Y \leq c_1\|\psi\|_{\mathfrak{B}} + c_2$.

(HI) $K_a(N_Y c_1 + \int_0^a [\widetilde{N} + \widetilde{N_1}]\alpha(s)ds\ lim_{\tau\to\infty} \sup \frac{\Omega(\tau)}{\tau}) < 1$

Let $\Omega = \max\{\Omega_g, \Omega_f\}$ and $\alpha = \max\{\alpha_g, \alpha_f\}$.

In this section $y : (-\infty, a] \to X$ is the function defined by $y_0 = \phi\ on\ (-\infty, 0]$ and $y(t) = C(t)(\phi(0) - g(0, \phi)) + S(t)z$ on $[0, a]$. Clearly $\|y_t\|_{\mathfrak{B}} \leq K_a\|y\|_a + M_a\|\phi\|_{\mathfrak{B}}$ where $\|y\|_a = sup_{0\leq t\leq a}\|y(t)\|$. This follows from the definition of abstract phase space $\mathfrak{B}$ introduced by Hale and Kato and given in chapter 2.

Theorem 6.2.7. Whenever the hypotheses $(H_\phi), (Hf), (Hg), (HI)$ hold, then (6.1.2) has atleast one mild solution.

Proof: Suppose that $S(a)$ denote the space $S(a) = \{x : (-\infty, a] \to X\ such\ that\ x_0 = 0,\ x|_J \in C\}$ associated with supremum norm $\|.\|_a$.
Suppose that $\Gamma : S(a) \to S(a)$ be the map denoted by $(\Gamma x)_0 = 0$ and

$$\begin{aligned}(\Gamma x)(t) &= g(t, x_t) + \int_0^t AS(t-s)g(s, \overline{x_s})ds \\ &+ \int_0^t S(t-s)f(s, \overline{x}_{\rho(s,\overline{x_s})})ds \end{aligned} \qquad (6.2.6)$$

where $\overline{x}_0 = \phi$ and $\overline{x} = x + y$ on J. It is easy to check that

$$\|\overline{x}_t\|_{\mathfrak{B}} \leq K_a\|y\|_a + M_a\|\phi\|_{\mathfrak{B}} + K_a\|x\|_t,$$

where $\|x\|_t = sup_{0\leq s\leq t}\|x(s)\|$.

$$\|\overline{x}_{\rho(s,\overline{x}_s)}\|_{\mathfrak{B}} \leq k^* := (M_a + \widetilde{J^\phi})\|\phi\|_{\mathfrak{B}} + K_a\|y\|_a + K_a\|x\|_a.$$

Therefore Γ is well defined. Γ takes values in $S(a)$. Also by axioms of phase space, the Lebesgue dominated convergence theorem, and the conditions $(Hf), (Hg), (H\phi)$

it can be shown that Γ is continuous.

Step1 : There exists $k > 0$ such that $\Gamma(B_k) \subset B_k$, where $B_k = \{x \in S(a) : \|x\|_a \leq k\}$. In the following $\hat{k} = K_a k + \|y_s\|_{\mathfrak{B}} = K_a k + K_a\|y\|_a + M_a\|\phi\|_{\mathfrak{B}}$. Now if we assume the assertion to be false, then $\forall\ k > 0\ \exists\ x_k \in B_k$ and $t_k \in [0, a]$ such that $k < \|\Gamma x_k(t_k)\|$. Then,

$$\begin{aligned}
k &\leq \|\Gamma x_k(t_k)\| \\
&\leq N_Y c_1\|\overline{x_t}\|_{\mathfrak{B}} + N_Y c_2 + \widetilde{N_1}\int_s^{t_k} m_g(s)\Omega\|\overline{x_{k s}}\|_{\mathfrak{B}} ds \\
&+ \widetilde{N}\int_s^{t_k} \alpha(s)\Omega(\|\overline{x_k}_{\rho(s,\overline{x_{k_s}})}\|_{\mathfrak{B}}) \\
&\leq N_Y c_1(K_a\|y\|_a + M_a\|\phi\|_{\mathfrak{B}} + K_a k) + N_Y c_2 \\
&+ \widetilde{N_1}\int_s^{t_k} \alpha(s)\Omega(K_a\|y\|_a + M_a\|\phi\|_{\mathfrak{B}} + K_a k) ds \\
&+ \widetilde{N}\int_s^{t_k} \alpha(s) ds \Omega(K_a\|y\|_a + (M_a + \widetilde{J^{\phi}})\|\phi\|_{\mathfrak{B}} + K_a k)
\end{aligned}$$

Hence

$$\begin{aligned}
1 &< (N_Y c_1 K_a + \widetilde{N_1}\int_0^a m_g(s) ds lim_{k\to\infty}\sup\frac{\Omega(K_a\|y\|_a + M_a\|\phi\|_{\mathfrak{B}} + K_a k)}{k} \\
&+ \widetilde{N}(\int_0^a \alpha(s) ds\ lim_{k\to\infty} sup\frac{\Omega(K_a\|y\|_a + (M_a + \widetilde{J^{\phi}})\|\phi\|_{\mathfrak{B}} + K_a k)}{k})) \\
&\leq K_a(N_Y c_1 + \int_0^a [\widetilde{N} + \widetilde{N_1}]\alpha(s) ds\ lim_{\tau\to\infty} sup\frac{\Omega(\tau)}{\tau}) \qquad (6.2.7)
\end{aligned}$$

which is a contradiction to the hypothesis (HI). Hence $\Gamma(B_k) \subset B_k$.

Step 2 : To prove that Γ is a $\chi-$contraction. Let Γ be split into $\Gamma = \{\Gamma^a + \Gamma^b + \Gamma^c\}$,

$$\Gamma^a x(t) = g(s, \overline{x_t}),$$

$$\Gamma^b x(t) = \int_s^t AS(t-s) g(s, \overline{x_s}) ds,$$

$$\Gamma^c x(t) = \int_s^t S(t-s) f(s, \overline{x}_{\rho(s,\overline{x}_s)}) ds,$$

The properties of the function g in (Hg), lemma 6.2.3 imply that the set of function $V(k, g) = \{t \to g(t, x_t + y_t) : x \in B_k\}$ is relatively compact in $C([0, a], X)$. By lemma 2.5.7(2) $\chi(W) = sup\{\chi(W(t)), t \in J\}$. By lemma 2.5.4(1), for any $W \subset \Gamma^a(B_k)$

$$\chi(\Gamma^a W(t)) = \chi(g(t, W(t))) = 0 \qquad (6.2.8)$$

By mean value theorem for Bochner integral, we derive

$$\{\Gamma^b x(t) : x \in B_k\} \subset t \times \overline{conv(\{AS(h)g(s,\psi) : 0 \le h, s \le t, \|\psi\|_{\mathfrak{B}} \le \hat{k}\})}$$

This implies $\{\Gamma^b x(t) : x \in B_k\}$ is relatively compact in X for all $t \in J$. Hence by lemma 2.5.4(1),

$$\chi(\Gamma^b W(t)) = 0. \tag{6.2.9}$$

By lemma 2.5.9 for any $W \subset \Gamma^c(B_k)$, since $S(t)$ is equicontinuous so, W is equicontinuous. Hence from the fact that $\rho(s, \overline{x_s}) \le s, s \in [0,a]$ and lemma 2.5.7(3) and $\chi(W) = \sup\{\chi(W(t)), t \in [0,a],\}$ it implies that

$$\begin{aligned}
\chi(\Gamma^c W(t)) &= \chi\left(\int_s^t S(t-s)f(s, W_{\rho(s,\overline{x}_s)} + y_s)ds\right. \\
&\le \int_s^t \eta(s) sup_{-\infty<\theta\le 0}\chi(W(\rho(s,\overline{x}_s)+\theta) + y(s+\theta))ds \\
&\le \int_s^t \eta(s) sup_{-\infty<\theta\le 0}\chi(W(s+\theta) + y(s+\theta))ds \\
&\le \int_s^t \eta(s) sup_{0<\tau\le s}\chi W(\tau)ds \\
&\le \chi(W)\int_s^t \eta(s)ds \\
&\le \chi(W)\int_0^a \eta(s)ds
\end{aligned}$$

Hence

$$\chi(\Gamma^c W) = \sup\{\chi(\Gamma W(t)), t \in [0,a],\} \le \chi(W)\int_0^a \eta(s)ds$$

For each bounded set $W \in C(J;X)$ we have,

$$\begin{aligned}
\chi_C(\Gamma W) &\le \chi_C(\Gamma^a W + \Gamma^b W + \Gamma^c W) \\
&\le (0 + 0 + \int_0^a \eta(s)ds)\chi_{PC}(W)
\end{aligned}$$

Therefore, Γ is a $\chi-$contraction. So, by applying Darbo-Sadovskii fixed point theorem it is proved that there exists a fixed point of Γ in $S(a)$. Thence, $\overline{x} = x + y$ is a mild solution of (6.1.2).

Remark : If the Lipschitz conditions on the nonlinear functions f, g are assumed then it is easy to see that the mild solution is unique.

6.3 Approximate controllability

In this section the approximate controllability of the control system (6.1.1) is studied. Assume that f, g satisfy following condition

(C1) There exists positive constants L_g, L_f such that f, g are Lipschitz continuous in second variable.

Also, $y : (-\infty, a] \to X$ is the function defined by $y_0 = \phi$ *and* $y(t) = C(t)\phi(0) + S(t)(z+g(0,\phi)$ on J. Clearly $\|y_t\|_{\mathfrak{B}} \leq K_a\|y\|_a + M_a\|\phi\|_{\mathfrak{B}}$ where $\|y\|_a = sup_{0\leq t\leq a}\|y(t)\|$

The operators $\Lambda_i : L^2(J, X) \to X$ $i = 1, 2$ are defined as

$$\Lambda_1 x(t) = \int_0^a S(t-s)x(s)ds,$$

$$\Lambda_2 x(t) = \int_0^a AS(t-s)x(s)ds.$$

Clearly Λ_i are bounded linear operators. We set $\mathfrak{N}_i = ker(\Lambda_i)$, $\Lambda = (\Lambda_1, \Lambda_2)$ and $\mathfrak{N} = ker(\Lambda)$ Let $C_0(J, X)$ denote the space consisting of continuous functions $x : J \to X$ such that $x(0) = 0$, endowed with the norm of uniform convergence. Let $J_i : L^2(J, X) \to C_0(J, X)$, $i = 1, 2$ be maps defined as follows

$$J_1 x(t) = \int_0^t S(t-s)x(s)ds,$$

$$J_2 x(t) = \int_0^t AS(t-s)x(s)ds.$$

So, $J_i x(a) = \Lambda_i(x)$, $i = 1, 2$. For a fixed $\phi \in \mathfrak{B}$ and $x \in C(J, X)$ such that $x(0) = \phi(0)$, we define maps $F, G : C_0(J, X) \to L^2(J, X)$ by $F(m)(t) = f(t, m_t + x_t)$ and $G(m)(t) = g(t, m_t + x_t)$. Here $x_t(\theta) = x(t+\theta)$, *for* $t + \theta \geq 0$ and $x_t(\theta) = \phi(t+\theta)$ for $t + \theta \leq 0$ and $m_t(\theta) = m(t+\theta)$ for $t + \theta \geq 0$ and $m_t(\theta) = 0$ for $t + \theta \leq 0$. Clearly, F, G are continuous maps. We also assume that $L^2(J, X) = \mathfrak{N}_i + \overline{R(B)}$, $i = 1, 2$. Referring Lemma 2.1.13 we denote P_i the map associated to this decomposition and construct $X_2 = \mathfrak{N}_i$ and $X_1 = \overline{R(B)}$. Also set $c_i = \|P_i\|$. We introduce the space

$$Z = \{m \in C_0(J, X) : m = J_1(n_1) + J_2(n_2) + P_2(g(t, x_t + m_t)), n_i \in \mathfrak{N}_i, \ i = 1, 2\}$$

and we define the map $\Gamma : \overline{Z} \to C_0(J, X)$ by

$$\Gamma = J_1 \circ P_1 \circ F + J_2 \circ P_2 \circ G + P_2 \circ G.$$

Lemma 6.3.1. If the hypothesis $(H_\phi)-(H_g)$ and conditions $(C1)$ hold for f,g and $aK_a(c_1\widetilde{N}L_f+c_2NL_g)<\sqrt{2}$ then Γ has a fixed point.

Proof: For $z^1,z^2\in\overline{Z}$ let $\Delta f(s)=f(s,z^2_{\rho(s,z^2(s))}+x_{\rho(s,x(s))})-f(s,z^1_{\rho(s,z^1(s))}+x_{\rho(s,x(s))})$ and $\Delta g(s)=g(s,z^2_s+x_s)-g(s,z^1_s+x_s)$. $\forall\, 0\le t\le a$.

$$\begin{aligned}\|(\Gamma z^2-\Gamma z^2)(t)\| &\le \|\int_0^t S(t-s)[P_1(\Delta f)](s)ds\|\\ &+ \|\int_0^t AS(t-s)[P_2(\Delta g)](s)ds\|+\|P_2(\Delta g(t))\|\\ &\le \widetilde{N}\int_0^t \|[P_1(\Delta f)](s)\|ds+\widetilde{N_1}\int_0^t\|[P_2(\Delta g)](s)\|ds\\ &+ \|P_2(\Delta g(s))\|\\ &\le \widetilde{N}t^{1/2}c_1\|\Delta f\|_2+\widetilde{N_1}t^{1/2}c_2\|\Delta g\|_2+c_2\|\Delta g(s)\|.\end{aligned}$$

Now

$$\begin{aligned}\|\Delta f\|_2^2 &= \int_0^a \|f(s,z^2_{\rho(s,z^2(s))}+x_{\rho(s,x(s))})-f(s,z^1_{\rho(s,z^1(s))}+x_{\rho(s,x(s))})\|^2ds\\ &\le L_f^2\int_0^a\|z^2_{\rho(s,z^2(s))}-z^1_{\rho(s,z^1(s))}\|^2_{\mathfrak{B}}ds\\ &\le L_f^2\int_0^a\|z^2_s-z^1_s\|^2_{\mathfrak{B}}ds\\ &\le aL_f^2K_a^2\|z^2-z^1\|^2_\infty ds.\end{aligned}$$

Similarly we find for g. So,

$$\|(\Gamma z^2-\Gamma z^1)(t)\|\le bt^{1/2}\|z^2-z^1\|_\infty$$

where $b=a^{1/2}K_a(c_1\widetilde{N}L_f+c_2\widetilde{N_1}L_g)$ Repeating this get

$$\|(\Gamma^n z^2-\Gamma^n z^1)(t)\|_\infty\le\frac{(bt^{1/2})^n}{2^{(n-1)/(2n)}}\|z^2-z^1\|_\infty$$

As $b=aK_a(c_1\widetilde{N}L_f+c_2\widetilde{N_1}L_g+c_2L_g)<\sqrt{2}$ and $2^{\frac{n-1}{2n}}\to\sqrt{2}$ *as* $n\to\infty$, the map Γ^n is a contraction for n sufficiently large and therefore Γ has a fixed point.

Theorem 6.3.2. If the associated linear control system (6.2.3) is approximately controllable on J, the space $L^2([0,a],X)=\mathfrak{N}_i+\overline{R(B)}, i=1,2$ and condition of the preceding lemma hold then the neutral second order differential control system (6.1.1) with state dependent delay is approximately controllable on J.

Proof: Assume that $x(.)$ to be the mild solution and $u(.)$ to be an admissible control function of system (6.2.3) with initial conditions $x(0) = (\phi(0) - g(0, \phi(0))$ and $x'(0) = z$. Let m be the fixed point of Γ. So, $m(0) = 0$ and $m(a) = \Lambda_1(P_1(F(m))) + \Lambda_2(P_2(G(m))) + P_2(G(m(a))) = 0$. By lemma 2.1.13 we can split the functions $F(m), G(m)$ with respect to the decomposition $L^2(J, X) = \mathfrak{N}_i + \overline{R(B)}$ $i = 1, 2$ respectively by setting $q_1 = F(m) - P_1(F(m))$ and $q_2 = G(m) - P_2(G(m))$. We define the function $y(t) = m(t) + x(t)$ for $t \in J$ and $y_0 = \phi$. So, $x(a) = y(a)$. We claim $y = x + m$ is the mild solution of the system (6.1.2). By applying lemma 6.3.1 we get

$$\begin{aligned} x(t) + m(t) &= C(t)x(0) + S(t)x'(0) + \int_0^t S(t-s)Bu(s)ds + P_2(G(m)) \\ &+ \int_0^t AS(t-s)P_2G(m)ds + \int_0^t S(t-s)P_1(F(m))ds. \end{aligned} \tag{6.3.10}$$

So,

$$\begin{aligned} y(t) &= C(t)x(0) + S(t)z + P_2(g(t, x_t + m_t)) + \int_0^t S(t-s)[F(m) - q_1 \\ &+ Bu(s)]ds + \int_0^t AS(t-s)[G(m) - q_2]ds \\ &= C(t)x(0) + S(t)z + P_2(G(m)) + \int_0^t S(t-s)[f(s, y_{\rho(s,y(s))}) - q_1 \\ &+ Bu(s)]ds + \int_0^t AS(t-s)[g(s, y_s) - q_2]ds \end{aligned} \tag{6.3.11}$$

As $C_0^1(J, U)$ and $C_0^2(J, U)$ are dense in $L^2(J, U)$ we can choose a sequence $v_n^1 \in L^2(J, U)$ and a sequence $v_n^2 \in L^2(J, X)$ such that $Bv_n^1 \to q_1$ and $Bv_n^2 \to q_2$ as $n \to \infty$. Let y^n denote the mild solution of the integral equation (6.3.11) when q_1 is substituted by Bv_n^1 and q_2 by Bv_n^2. Using lemma 6.2.4 we get

$$\begin{aligned} y^n(t) &= C(t)x(0) + S(t)z + P_2(G(m)) + \int_0^t S(t-s)[f(s, y^n_{\rho(s,y^n(s))}) \\ &- Bv_n^1(s) + Bu(s)]ds + \int_0^t AS(t-s)[g(s, y_s^n) - Bv_n^2(s)]ds \\ &= C(t)x(0) + S(t)z + P_2(g(t, y_t^n)) + Bv_n^2 \\ &+ \int_0^t S(t-s)[f(s, y^n_{\rho(s,y(s))}) - Bv_n^1(s) + B\frac{d^2}{ds^2}v_n^2(s) + Bu(s)]ds \\ &+ \int_0^t AS(t-s)g(s, y_s^n)ds \end{aligned} \tag{6.3.12}$$

As $n \to \infty$

$$
\begin{aligned}
y(t) &= C(t)x(0) + S(t)z + P_2G(m) + q_2 \\
&+ \int_0^t S(t-s)[f(s, y_{\rho(s,y(s))}) + B(-v_n^1 - \frac{d^2}{ds^2}v_n^2(s) + u(s))]ds \\
&+ \int_0^t AS(t-s)g(s, y_s)ds \\
&= C(t)x(0) + S(t)z + g(t, y_t) + \int_0^t S(t-s)[f(s, y_{\rho(s,y(s))}) \\
&+ B(-v_n^1 - \frac{d^2}{ds^2}v_n^2(s) + u(s))]ds + \int_0^t AS(t-s)g(s, y_s)ds \qquad (6.3.13)
\end{aligned}
$$

Hence by definition (6.2.5) and equation (6.3.13) we conclude that y is the mild solution of the following equation

$$\frac{d^2}{dt^2}(y(t) - g(t, x_t)) = Ay(t) + f(t, y_{\rho(t,y(t))}) + B(-v_n^1(t) + \frac{d^2}{dt^2}v_n^2(t) + u(t))$$

$$x(0) = \phi \in \mathfrak{B} \quad \frac{d}{dt}[x(t) - g(t, x_t)]|_{t=0} = z$$

Hence $y^n(a) \in \mathcal{R}(a, f, g, \phi, z)$. Since the solution map is generally continuous, $y^n \to y$ as $n \to \infty$. Thus $y(a) \in \mathcal{R}(a, f, g, \phi, z)$. Therefore $\mathcal{R}_0(a, (\phi(0) - g(0, \phi(0)), z) \subset \overline{\mathcal{R}(a, f, g, \phi, z)}$, which means $\mathcal{R}(a, f, g, \phi, z)$ is dense in X. Thus the system (6.1.1) is controllable.

6.4 Example

In this section we discuss a partial differential equation applying the abstract results of this paper. In this application, $\mathfrak{B}$ is the phase space $C_0 \times L^2(h, X)$ (see[96]).
We study the following system

$$
\begin{aligned}
\frac{\partial}{\partial t}(\frac{\partial u(t,\xi)}{\partial t} &+ \int_{-\infty}^t \int_0^\pi b(t-s, \eta, \xi)u(s, \eta)d\eta ds) \\
= \frac{\partial^2 u(t,\xi)}{\partial \xi^2} &+ \int_{-\infty}^t a(t-s)u(s - \rho_1(t)\rho_2(\|u(t)\|, \xi)ds + Bu(t), \\
& \quad t \in [0, a], \xi \in [0, \pi], \\
u(t, 0) &= u(t, \pi) = 0, \ t \in [0, a], \\
u(t, \xi) &= \phi(t, \xi) \ \tau \le 0, 0 \le \xi \le \pi,
\end{aligned}
$$

(6.4.1)

where $\phi \in C_0 \times L^2(h, X),\ 0 < t_1 <, ..., t_n < a$ By defining maps $\rho, G, F : [0, a] \times \mathfrak{B} \to X$ by

$$\rho(t, \psi) := \rho_1(t)\rho_2(\|\psi(0)\|),$$

$$G(\psi)(\xi) := \int_{-\infty}^{0}\int_{0}^{\pi} b(s, v, \xi)\psi(s, v)dvds,$$

$$F(\psi)(\xi) := \int_{-\infty}^{0} a(s)\psi(s, \xi)ds$$

the system (6.4.1) can be transformed into system (6.1.1) Assume that the functions $\rho_i : \mathbb{R} \to [0, \infty),\ a : \mathbb{R} \to \mathbb{R}$ are piecewise continuous.

(a) The functions $b(s, \eta, \xi), \frac{\partial b(s,\eta,\xi)}{\partial \xi}$ are measurable, $b(s, \eta, \pi) = b(s, \eta, 0) = 0$ and

$$L_g := max\{(\int_0^{\pi}\int_{-\infty}^{0}\int_0^{\pi} \frac{1}{h(s)}(\frac{\partial^i b(s, \eta, \xi)}{\partial \xi^i})^2 d\eta ds d\xi)^{1/2} : \ i = 0, 1\} < \infty$$

(b) The function $F : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous and there is continuous function $\int_{-\infty}^{0} \frac{\mu(s)^2}{h(s)} ds < \infty$. and $\|F(t, \xi)\| \leq \mu(s)\|\xi\|$

(c) The functions $a_i^j \in C([0, \infty); \mathbb{R})$ and $L_i^j := (\int_{-\infty}^{0} \frac{(a_i^j(s))^2}{h(s)} ds)^{1/2} < \infty$ for all $i = 1, 2, ..., n\ j = 1, 2$

Moreover $g(t, .), I_i, i = 1, ..., n$ are bounded linear operators .
Hence by assumptions $(a) - (c)$ and theorem 6.3.2 it is ensured that mild solution to the problem (6.4.1) exists.

6.5 Conclusion

Thus, we establish the existence of mild solution and approximate controllability of a second order neutral partial differential equation involving state dependent delay. The conditions for approximate controllability were derived for the distributed second order neutral system by assuming the approximate controllability of the corresponding linear system in a Hilbert space. The strict assumptions such as the compactness of the associated cosine or sine family of operators were removed. We also removed the limitation of the non-existence of the inverse of the controllability operator due to the compactness of the semigroup in infinite-dimensional spaces.

Chapter 7

Existence of Solution and Approximate Controllability of a Neutral Differential Equation with State Dependent Delay

This chapter is divided in two parts. In first part we study a second order neutral partial differential equation with state dependent delay and non-instantaneous impulses. The conditions for existence of the mild solution are investigated via Hausdorff measure of noncompactness. Darbo Sadovskii fixed point theorem is applied. Thus we remove the need to assume the compactness assumption on the associated family of operators. The conditions for approximate controllability are investigated for the neutral second order system with respect to the approximate controllability of the corresponding linear system in a Hilbert space. A simple condition on the range of an operator is used to prove approximate controllability. Thereby, the non-singularity of a controllability operator is not required which was an essential condition in [39]. Since in infinite dimensional spaces, with compact semigroup the controllability operator is not invertible. Our methodology does not require to find the inverse of the controllability Gramian operator. Also the associated limiting condition in [69] is removed. Examples are studied to substantiate the theory.

7.1 Introduction

On account of the extensive use of non-instantaneous impulsive differential equations in electrical and mechanical engineering and other fields, they are recently investigated by Hernandez [84] and many others.

The literature related to state dependent delay mostly deals with functional differential equations in which the state belongs to a finite dimensional space. As a consequence, the study of partial functional differential equations involving state dependent delay is neglected. This is one of the motivations of our paper.

The paper [109] studies existence of differential equation via measure of non-compactness. Measure of non-compactness significantly removes the need to assume Lipschitz continuity of nonlinear functions and operators.

Infinite dimensional systems has been extensively investigated to establish their controllability on account of their applicability in various processes. In the papers [39; 159] the authors established the exact controllability by using compact semi-group. As we know that compactness of the controllability operator follows from compactnes of the operator B C_0-semigroup. Therefore in infinite dimensional, due to a result of Triggiani [164], the controllability operator is no longer invertible.

First we study the existence and uniqueness of mild solution of the second order equation modeled in the form

$$\begin{aligned} \frac{d}{dt}(x'(t)+g(t,x_t)) &= Ax(t)+f(t,x_{\rho(t,x_t)}),\ t\in(s_i,t_{i+1}],\ i=0,...,n \\ x_0=\phi\in\mathfrak{B},\ \ x'(0) &= z\in X \\ x(t)=J_i^1(t,x_t),\ t &\in (t_i,s_i],\ i=1,2,...,n \\ x'(t)=J_i^2(t,x_t),\ t &\in (t_i,s_i],\ i=1,2,...,n \end{aligned} \tag{7.1.1}$$

where A denotes the infinitesimal generator of a strongly continuous cosine family $\{C(t):t\in\mathsf{R}\}$ in the Hilbert space X. The history valued function $x_t:(-\infty,0]\to X$, $x_t(\theta)=x(t+\theta)$ takes values in the abstract phase space $\mathfrak{B}$ defined defined in chapter 2 as Definition 2.2.12 ; $g,f,J_i^1,J_i^2,i=1,..,n$ are appropriately defined functions. $0=t_0=s_0<t_1\le s_1\le t_2,...,<t_n\le s_n\le t_{n+1}=a$ are prefixed numbers.

Specifically, we study the approximate controllability of

$$\begin{aligned} \frac{d}{dt}(x'(t) + g(t, x_t)) &= Ax(t) + f(t, x_{\rho(t,x_t)}) + Bu(t), \ t \in J = [0, a] \\ x_0 = \phi \in \mathfrak{B}, \ x'(0) &= w \in X \end{aligned} \tag{7.1.2}$$

B is a bounded linear operator on a Hilbert space U.

Let $N, \widetilde{N}$ be certain constants such that $\|C(t)\| \leq N$ and $\|S(t)\| \leq \widetilde{N}$ for every $t \in J = [0, a]$. For more details see book by Fattorini [75] and articles [161],[162],[163].

7.2 Existence of Mild solution

We define mild solution of problem (7.1.1) as follows.

Definition 7.2.1. A function $x : (-\infty, a] \to X$ is said to be a mild solution of the problem (5.1.1) if $x_0 = \phi$; $x(.)|_{[0,a]} \in PC(X)$, $x(t) = J_i^1(t, x_t) \ \forall \ t \in (t_i, s_i]$ $i = 1, ..., n$ and

$$\begin{aligned} x(t) &= C(t)\phi(0) + S(t)[z + g(0, \phi)] - \int_0^t C(t-s)g(s, x_s)ds \\ &+ \int_0^t S(t-s)f(s, x_{\rho(s,x_s)})ds, \ t \in [0, t_1] \\ x(t) &= C(t-s_i)J_i^1(s_i, x_{s_i}) + S(t-s_i)(J_i^2(s_i, x_{s_i}) + g(s_i, x_{s_i})) \\ &- \int_{s_i}^t C(t-s)g(s, x_s)ds \\ &+ \int_{s_i}^t S(t-s)f(s, x_{\rho(s,x_s)})ds \ \ for \ \ t \in [s_i, t_{i+1}] \ i = 1, ..., n \end{aligned} \tag{7.2.1}$$

To prove our result we always assume $\rho : J \times \mathfrak{B} \to (-\infty, a]$ is a continuous function. The following hypotheses are used.

(H_ϕ) The function $t \to \phi_t$ is continuous from $\mathbb{R}(\rho^-) = \{\rho(s, \psi) : \rho(s, \psi) \leq 0\}$ into $\mathfrak{B}$ and there exists a continuous bounded function $J^\phi : \mathbb{R}(\rho^-) \to (0, \infty)$ such that $\|\phi_t\|_{\mathfrak{B}} \leq J^\phi(t)\|\phi\|_{\mathfrak{B}}$ for every $t \in \mathbb{R}(\rho^-)$.

(Hf) The function $f : J \times \mathfrak{B} \to X$ satisfies the following properties:

(1) For each $x : (-\infty, a] \to X, x_0 \in \mathfrak{B}$ *and* $x|_J \in PC$, the function $f(., \psi) : J \to X$ is strongly measurable for each $\psi \in \mathfrak{B}$ and $f(t, .)$ is continuous for a.e. $t \in J$.

(2) There exists an integrable function $\alpha : J \to [0, +\infty)$ and a monotone continuous nondecreasing function $\Omega : [0, +\infty) \to (0, +\infty)$ such that $\|f(t, v)\| \leq \alpha(t)\Omega(\|v\|_{\mathfrak{B}})\ \forall t \in J\ and\ v \in \mathfrak{B}$.

(3) Suppose $D(\theta) = \{v(\theta) : v \in D\}$. For a.e. $s, t, \in J$, there exists an integrable function $\eta : J \to [0, \infty)$ such that
$\chi(S(s)f(t, D)) \leq \eta(t) \sup_{-\infty<\theta<0} \chi(D(\theta))$

(Hg) The function $g(.)$ is continuous $\forall t, v \in J \times \mathfrak{B}$ and $g(t, .)$ is Lipschitz continuous such that there exists positive constant L_g such that

$$\|g(t, v_1) - g(t, v_2)\| \leq L_g\|v_1 - v_2\|_{\mathfrak{B}},\ \ (t, v_i) \in J \times \mathfrak{B},\ i = 1, 2.$$

(HJ) (1) There exist positive constants $c_i^1, c_i^2, d_i^1, d_i^2$ such that
$\|J_i^1(t, v)\| \leq c_i^1\|v\|_{\mathfrak{B}} + c_i^2$ and $\|J_i^2(t, v)\| \leq d_i^1\|v\|_{\mathfrak{B}} + d_i^2$

(2) $\|J_i^j(t, u) - J_i^j(t, v)\| \leq L_{J_i^j}\|u - v\|_{\mathfrak{B}}$ for all $u, v \in \mathfrak{B}$ $i = 1, ..., n,\ j = 1, 2$

(H1) (1) $K_a(NaL_g + \widetilde{N}\int_0^a \alpha(s)ds\ \lim_{\tau\to\infty}\ \sup\frac{\Omega(\tau)}{\tau} + \sum_{i=1}^n(Nc_i^1 + \widetilde{N}(d_i^1 + L_g)))$
$+ \sum_{i=1}^n c_i^1 K_a < 1$

(2) $(K_aNL_ga + \sum_{i=1}^n\{NL_{J_i^1} + \widetilde{N}(L_{J_i^2} + L_g)\}K_a + \sum_{i=0}^n \int_{s_i}^t \eta(s)ds)$
$+ \sum_{i=1}^n\{L_{J_i^1}\}K_a < 1$

Lemma 7.2.2. ([96]) If $y : (-\infty, a] \to X$ is a function such that $y_0 = \phi$ and $y|_J \in PC(X)$ then
$\|y_{\rho(s,y_s)}\|_{\mathfrak{B}} \leq (M_a + \widetilde{J^\phi})\|\phi\|_{\mathfrak{B}} + K_a sup\{\|y(\theta)\|; \theta \in [0, max\{0, s\}]\},$

$$s \in \mathbb{R}(\rho^-) \cup [0, a]$$

where $\widetilde{J^\phi} = sup_{t\in\mathbb{R}(\rho^-)}J^\phi(t),\ M_a = sup_{t\in J}M(t)\ and\ K_a = max_{t\in J}K(t)$.

In this section $y : (-\infty, a] \to X$ is the function defined by $y_0 = \phi\ and\ y(t) = C(t)\phi(0) + S(t)(z + g(0, \phi))$ on $J_1 = [0, t_1]$. Clearly $\|y_t\|_{\mathfrak{B}} \leq K_a\|y\|_a + M_a\|\phi\|_{\mathfrak{B}}$ where $\|y\|_a = sup_{0\leq t\leq a}\|y(t)\|$

Theorem 7.2.3. Whenever the hypotheses $(Hf), (Hg), (HI), (H1)$ hold, the system (7.1.1) will have atleast one mild solution.

Proof: Suppose that $S(a)$ denotes the space $S(a) = \{x : (-\infty, a] \to X|\ x_0 = 0,\ x|_J \in PC\}$ associated with supremum norm $\|.\|_a$.
Suppose $\Gamma : S(a) \to S(a)$ denote the map $(\Gamma x)_0 = 0$ and $\Gamma = \sum_{i=1}^{n} \Gamma_i^1 + \sum_{i=1}^{n} \Gamma_i^2$

$$(\Gamma_i^1 x)(t) = \begin{cases} J_i^1(t, \overline{x}_t), & t \in (t_i, s_i]; i = 1, \cdots, n \\ C(t - s_i) J_i^1(s_i, \overline{x}_{s_i}) & \\ +S(t - s_i)(J_i^2(s_i, \overline{x}_{s_i}) + g(s_i, x_{s_i})), & t \in (s_i, t_{i+1}]; i = 1, \cdots, n; \end{cases} \tag{7.2.2}$$

$$(\Gamma_i^2 x)(t) = \begin{cases} \int_{s_i}^{t} C(t - s) g(s, \overline{x_s}) ds & \\ + \int_{s_i}^{t} S(t - s) f(s, \overline{x}_{\rho(s, \overline{x_s})}) ds, & t \in (s_i, t_{i+1}]; i = 0, \cdots, n \\ 0, & t \notin (s_i, t_{i+1}], i = 0, ..., n. \end{cases} \tag{7.2.3}$$

where $\overline{x}_0 = \phi$ and $\overline{x} = x + y$ on J. It is easy to check that

$$\|\overline{x}_t\|_{\mathfrak{B}} \leq K_a \|y\|_a + M_a \|\phi\|_{\mathfrak{B}} + K_a \|x\|_t,$$

where $\|x\|_t = sup_{0 \leq s \leq t} \|x(s)\|$.

$$\|\overline{x}_{\rho(s, \overline{x}_s)}\|_{\mathfrak{B}} \leq k^* := (M_a + \widetilde{J^{\phi}}) \|\phi\|_{\mathfrak{B}} + K_a \|y\|_a + K_a \|x\|_a.$$

So, Γ is well defined. Moreover Γ takes values in $S(a)$. By applying the Lebesgue dominated convergence theorem, and the hypotheses $(Hf), (Hg)$ coupled with the axioms of phase space, we can prove continuity of Γ.
Step1 : There exists $k > 0$ such that $\Gamma(B_k) \subset B_k$, where $B_k = \{x \in S(a) : \|x\|_a \leq k\}$. When we assume the assertion to be false, then $\forall\ k > 0$ there exists $x_k \in B_k$

and $t_k \in (s_i, t_{i+1}]$ such that $k < \|\Gamma x_k(t_k)\|$.

$$\begin{aligned}
k &\leq \sum_{i=0}^{n} \|\Gamma_i^2 x_k(t_k)\| + \sum_{i=1}^{n} \|\Gamma_i^1 x_k(t_k)\| \\
&\leq \sum_{i=0}^{n} N \int_{s_i}^{t_k} L_g(\|\overline{x_k}_s\|_{\mathfrak{B}} + \|g(s,0)\|)ds + \sum_{i=1}^{n} \widetilde{N} \int_{s_i}^{t_k} \alpha(s)\Omega(\|\overline{x_k}_{\rho(s,\overline{x_k}_s)}\|_{\mathfrak{B}}) \\
&+ \sum_{i=1}^{n} N(c_i^1\|\overline{x_k}_s\| + c_i^2) + \sum_{i=1}^{n} \widetilde{N}(d_i^1\|\overline{x_k}_s\| + d_i^2 + L_g\|\overline{x_k}_{s_i} - 0\| + \|g(s,0)\|) \\
&\leq \sum_{i=0}^{n} N \int_{s_i}^{t_k} L_g(K_a\|y\|_a + M_a\|\phi\|_{\mathfrak{B}} + K_a k + \|g(s,0)\|)ds \\
&+ \sum_{i=0}^{n} \widetilde{N} \int_{s_i}^{t_k} \alpha(s)ds\Omega(K_a\|y\|_a + (M_a + \widetilde{J^\phi})\|\phi\|_{\mathfrak{B}} + K_a k) \\
&+ \sum_{i=1}^{n} N(c_i^1(K_a\|y\|_a + M_a\|\phi\|_{\mathfrak{B}} + K_a k) + c_i^2) \\
&+ \widetilde{N}(d_i^1(K_a\|y\|_a + M_a\|\phi\|_{\mathfrak{B}} + K_a k) + d_i^2 \\
&+ L_g(K_a\|y\|_a + M_a\|\phi\|_{\mathfrak{B}} + K_a k) + \|g(s,0)\|)
\end{aligned} \tag{7.2.4}$$

Hence

$$\begin{aligned}
1 &< (\widetilde{N} \int_0^a \alpha(s)ds \lim_{k\to\infty} sup \frac{\Omega(K_a\|y\|_a + (M_a + \widetilde{J^\phi})\|\phi\|_{\mathfrak{B}} + K_a k)}{k} \\
&+ NaK_aL_g) + K_a \sum_{i=1}^{n}(Nc_i^1 + \widetilde{N}(d_i^1 + L_g)) \\
&\leq K_a(NaL_g + \widetilde{N} \int_0^a \alpha(s)ds \lim_{\tau\to\infty} sup \frac{\Omega(\tau)}{\tau} + \sum_{i=1}^{n}(Nc_i^1 + \widetilde{N}(d_i^1 + L_g)))
\end{aligned} \tag{7.2.5}$$

which is a contradiction to the hypothesis $(H1)$. Similarly $(\Gamma x)(t) < k$. for $t_k \in (t_i, s_i]\ \forall\ i = 1, 2, ..., n$. Suppose on the contrary,

$$\begin{aligned}
k < \sum_{i=1}^{n}(\Gamma_i^1 x_k)(t_k) &= \sum_{i=1}^{n} \|J_i^1(t_k, \overline{x_k}_{t_k})\| \\
&\leq \sum_{i=1}^{n}\{c_i^1\|\overline{x_k}_{t_k}\|_{\mathfrak{B}} + c_i^2\} \\
&\leq \sum_{i=1}^{n}\{c_i^1(K_a\|y\|_a + M_a\|\phi\|_{\mathfrak{B}} + K_a k) + c_i^2\}
\end{aligned} \tag{7.2.6}$$

Hence,

$$1 < \sum_{i=1}^{n} c_i^1 K_a \tag{7.2.7}$$

which is a contradiction.

Step 2 : To prove that Γ is a $\chi-$contraction. Let $\Gamma = \sum_{i=1}^{n}\Gamma_i^1 + \sum_{i=0}^{n}\Gamma_i^2$ be split into $\Gamma = \sum_{i=1}^{n}\Gamma_i^1 + \sum_{i=0}^{n}\{\Gamma_{i_1}^2 + \Gamma_{i_2}^2\}$ for $t > 0$

$$\Gamma_{i_1}^2 x(t) = \int_{s_i}^{t} C(t-s)g(s,\overline{x_s})ds$$

$$\Gamma_{i_2}^2 x(t) = \int_{s_i}^{t} S(t-s)f(s,\overline{x}_{\rho(s,\overline{x}_s)})ds$$

For arbitrary $x_1, x_2 \in B_k$, and $t \in (s_i, t_{i+1}]$

$$\begin{aligned}
\sum_{i=0}^{n}\|\Gamma_{i_1}^2 x_1(t) - \sum_{i=0}^{n}\Gamma_{i_2}^1 x_2(t)\| &\leq \sum_{i=0}^{n}\|\int_{s_i}^{t} C(t-s)(g(s,x_{1_s}+y_s) \\
&\quad - g(s,x_{2_s}+y_s))ds\| \\
&\leq \sum_{i=0}^{n} NL_g a\|x_{1_t} - x_{2_t}\|_{\mathfrak{B}} \\
&\leq K_a NL_g a\|x_1 - x_2\|_a
\end{aligned} \tag{7.2.8}$$

So, $\Gamma_{i_1}^2\ \forall i = 0,...,n$ is Lipschitz continuous with Lipschitz constant NL_gaK_a.

For any $W \subset \Gamma_{i_1}^2(B_k)$, W is piecewise equicontinuous since $S(t)$ is equicontinuous. Hence from the fact that $\rho(s,\overline{x_s}) \leq s, s \in [0,a]$ and Lemma 2.5.9 and $\chi_{PC}(W) = sup\{\chi(W(t)), t \in J\}$ we have

$$\begin{aligned}
\chi(\sum_{i=0}^{n}\Gamma_{i_1}^2 W(t)) &= \sum_{i=0}^{n}\chi(\int_{s_i}^{t} S(t-s)f(s,W_{\rho(s,\overline{x}_s)}+y_s)ds \\
&\leq \sum_{i=0}^{n}\int_{s_i}^{t}\eta(s)sup_{-\infty<\theta\leq 0}\chi(W(\rho(s,\overline{x}_s)+\theta)+y(s+\theta))ds \\
&\leq \sum_{i=0}^{n}\int_{s_i}^{t}\eta(s)sup_{-\infty<\theta\leq 0}\chi(W(s+\theta)+y(s+\theta))ds \\
&\leq \sum_{i=0}^{n}\int_{s_i}^{t}\eta(s)sup_{-\infty<\tau\leq 0}\chi W(\tau)ds \\
&\leq \chi_{PC}(W)\sum_{i=0}^{n}\int_{s_i}^{t}\eta(s)ds
\end{aligned}$$

For arbitrary $x_1, x_2 \in B_k$ and $t \in (s_i, t_{i+1}]$

$$\begin{aligned}\sum_{i=1}^{n} \|(\Gamma_i^1 x_1)(t) - \sum_{i=1}^{n}(\Gamma_i^1 x_2)(t)\| &\leq \sum_{i=1}^{n}\{NL_{J_i^1}\|\overline{x_1}_{s_i} - \overline{x_2}_{s_i}\| \\ &+ \widetilde{N}(L_{J_i^2}\|\overline{x_1}_{s_i} - \overline{x_2}_{s_i}\| + L_g\|x_{2s_i} - x_{1s_i}\|)\} \\ &\leq \sum\{NL_{J_i^1} + \widetilde{N}(L_{J_i^2} + L_g)\}\|x_{1s_i} + y_s \\ &- x_{2s_i} - y_s\| \\ &\leq \sum\{NL_{J_i^1} + \widetilde{N}(L_{J_i^2} + L_g)\}\|x_{1s_i} - x_{2s_i}\|_{\mathfrak{B}} \\ &\leq \sum\{NL_{J_i^1} + \widetilde{N}(L_{J_i^2} + L_g)\}K_a\|x_1 - x_2\|_a \end{aligned} \tag{7.2.9}$$

So, Γ_i^1 $\forall i = 1, ..., n$ is Lipschitz continuous with Lipschitz constant $(NL_{J_i^1} + \widetilde{N}L_{J_i^2})K_a$.
For arbitrary $x_1, x_2 \in B_k$ and $t \in (t_i, s_i]$,

$$\begin{aligned}\sum_{i=1}^{n} \|(\Gamma_i^1 x_1)(t) - \sum_{i=1}^{n}(\Gamma_i^1 x_2)(t)\| &\leq \sum_{i=1}^{n} L_{J_i^1}\|x_{1_t} - x_{2_t}\|_{\mathfrak{B}} \\ &\leq \sum_{i=1}^{n} K_a L_{J_i^1}\|x_1 - x_2\|_a \end{aligned} \tag{7.2.10}$$

For each bounded set $W \in PC(J; X)$ and $t \in (s_i, t_{i+1}]$, $\forall\, i = 0, ..., n$ we have,

$$\begin{aligned}\chi_{PC}(\Gamma W) &\leq \sum_{i=1}^{n} \chi_{PC}(\Gamma_i^1 W) + \sum_{i=0}^{n} \chi_{PC}(\Gamma_{i_1}^2 W + \Gamma_{i_2}^2 W) \\ &\leq (K_a N L_g a + \sum\{NL_{J_i^1} + \widetilde{N}(L_{J_i^2} + L_g)\}K_a + \sum_{i=0}^{n}\int_{s_i}^{t} \eta(s)ds)\chi_{PC}(W)\end{aligned}$$

For each bounded set $W \in PC(J; X)$ and $t \in (t_i, s_i]$ $\forall\, i = 1, 2, ..., n$ we have,

$$\begin{aligned}\chi_{PC}(\Gamma W) &\leq \sum_{i=1}^{n} \chi_{PC}(\Gamma_i^1 W) + \sum_{i=0}^{n} \chi_{PC}(\Gamma_{i_1}^2 W + \Gamma_{i_2}^2 W) \\ &\leq (\sum_{i=1}^{n}\{L_{J_i^1}\}K_a + 0 + 0)\chi_{PC}(W)\end{aligned}$$

Therefore, Γ is a $\chi-$contraction. Thus Γ has a fixed point in $S(a)$. This follows from Darbo-Sadovskii fixed point theorem. So, $z = x + y$ is the mild solution of (7.1.1).

7.3 Approximate controllability

In this section the approximate controllability of the control corresponding to (7.1.1) without the impulsive conditions is studied. We consider

$$\begin{aligned} \frac{d}{dt}(x'(t) + g(t, x_t)) &= Ax(t) + f(t, x_{\rho(t,x_t)}) + Bu(t),\ t \in J = [0, a] \\ x_0 = \phi \in \mathfrak{B},\ x'(0) &= w \in X \end{aligned} \tag{7.3.1}$$

where A denotes the infinitesimal generator of a strongly continuous cosine family $\{C(t) : t \in \mathsf{R}\}$. We define mild solution of problem (7.3.1) as follows.

Definition 7.3.1. A function $x : (-\infty, a] \to X$ is said to be a mild solution of the problem (7.3.1) if $x_0 = \phi$; $x(.)|_{[0,a]} \in C(J, X)$, the functions $f(s, x_{\rho(s,x_s)})$ and $g(s, x_s)$ are integrable and the integral equation is satisfied

$$\begin{aligned} x(t) &= C(t)\phi(0) + S(t)[w + g(0, \phi)] - \int_0^t C(t-s)g(s, x_s)ds \\ &+ \int_0^t S(t-s)[f(s, x_\rho(s, x_s)) + Bu(s)]ds,\ t \in [0, a] \end{aligned} \tag{7.3.2}$$

Lemma 7.3.2. [162] Under the assumption that $h : [0, a] \to X$ is an integrable function , such that

$$\begin{aligned} x''(t) &= Ax(t) + h(t), t \in J \\ x(0) &= x^0, \\ x'(0) &= x^1 \end{aligned} \tag{7.3.3}$$

and h is a function continuously differentiable, then

$$\int_0^t C(t-s)h(s)ds = S(t)h(0) + \int_0^t S(t-s)h'(s)ds$$

Let $a = T$

Definition 7.3.3. The set given by $\mathcal{R}_T(f) = \{x(T) \in X : x$ is the mild solution of (7.3.1) $\}$ is called reachable set of the system (7.3.1) . $\mathcal{R}_T(0)$ is the reachable set of the corresponding linear control system (7.3.5).

Definition 7.3.4. The system (7.3.1) is called approximately controllable on $[0, T]$ if $\mathcal{R}_T(f)$ is dense in X. The corresponding linear system is approximately controllable if $\mathcal{R}(0)$ is dense in X.

A continuous linear operator $\mathfrak{L} : L^2([0,T];X) \to C([0,T];X)$ is defined as

$$\mathfrak{L}p = \int_0^T S(T-s)p(s)ds, \quad p \in \mathcal{L}_2([0,T];X).$$

The kernel of the operator $\mathfrak{L}$ is indicated by N. This is a closed subspace of $L^2([0,T];X)$. Suppose $N_0^{\perp}$ be the corresponding orthogonal subspace of $L^2([0,T];X)$. P denotes the projection on $L^2([0,T];X)$ with range $N_0^{\perp}$. Here $\overline{R(B)}$ is the closure of the range of operator B. The following hypothesis are used

(HR) $\forall \epsilon > 0$ and $p(.) \in L^2([0,T];X)$, $\exists u(.) \in U$ such that $\|\mathfrak{L}p - \mathfrak{L}Bu\|_X < \epsilon$

The hypothesis (HR) is equivalent to the $L^2([0,T];X) = \overline{R(B)} + N_0$ or $P\overline{R(B)} = N_0^{\perp}$. Theorem (7.3.5) proves that from hypothesis (HR) the approximate controllability of the system (7.3.4) follows. We know that $L^2([0,T];X) = \overline{R(B) + N_0}$ follows from the approximate controllability of (7.3.5). Thus from the closedness of the product space it follows that (HR) is equivalent to the approximate controllability of (7.3.4).

Theorem 7.3.5. Whenever the assumptions (Hg) and (HR) hold then the associated neutral system

$$\begin{aligned} \frac{d(x'(t)+g(t,x_t))}{dt} &= Ax(t) + Bu(t), t \in J \\ x(0) &= \phi(0), \\ x'(0) &= w \end{aligned} \tag{7.3.4}$$

with $f \equiv 0$ is approximately controllable.

Proof: It is sufficient to prove that $D(A) \subset \overline{\mathcal{R}_T(0)}$ since $D(A)$ is dense in X. Let $h(T,\phi) = C(t)\phi(0) + S(t)[w + g(0,\phi(0))] - \int_0^T C(t-s)g(s,x_s)ds$. For any chosen $\xi \in D(A)$, then $\xi - h(T,\phi) \in D(A)$. It can be easily seen from Lemma (7.3.2) and [139] that there exists some $p \in C^1([0,T];X)$ such that

$$\eta = \xi - h(T,\phi) = \int_0^T S(T-s)p(s)ds.$$

By hypothesis (HR) there exists a control function $u(.) \in L^2([0,T];U)$ such that $\|\eta - \mathfrak{L}Bu\| < \epsilon$. As ϵ is arbitrary it implies that $K_T(0) \subset D(A)$. Since the $D(A)$ is dense in X, $K_T(0)$ is dense in X. Hence the neutral system with $f \equiv 0$ is approximately controllable.

We state the corresponding linear control system

$$\begin{aligned} x''(t) &= Ax(t) + Bu(t), t \in J \\ x(0) &= x^0, \\ x'(0) &= x^1 \end{aligned} \tag{7.3.5}$$

Both exact and approximate controllability of the above system is studied extensively in literature.

Assume that f, g satisfy following conditions. For a fixed $\phi \in \mathfrak{B}$ and $x \in C(J, X)$ such that $x(0) = \phi(0)$, we define maps $F, G : C_0(J, X) \to L^2(J, X)$ by $F(z)(t) = f(t, z_t + x_t)$ and $G(z)(t) = g(t, z_t + x_t)$. Here $x_t(\theta) = x(t + \theta),\ for\ t + \theta \geq 0$ and $x_t(\theta) = \phi(t + \theta)$ for $t + \theta \leq 0$ and $z_t(\theta) = z(t + \theta)$ for $t + \theta \geq 0$ and $z_t(\theta) = 0$ for $t + \theta \leq 0$. Clearly, F, G are continuous maps.

(C1) The function $f(.)$ is continuous $\forall t, v \in J \times \mathfrak{B}$ and $f(t, .)$ is Lipschitz continuous such that there exists positive constant L_f such that

$$\|f(t, v_1) - f(t, v_2)\| \leq L_f \|v_1 - v_2\|_{\mathfrak{B}},\ \ (t, v_i) \in J \times \mathfrak{B},\ i = 1, 2.$$

The above same condition also hold for G.
Also, $y : (-\infty, a] \to X$ is the function defined by $y_0 = \phi\ and\ y(t) = C(t)\phi(0) + S(t)(z+g(0, \phi))$ on J. Clearly $\|y_t\|_{\mathfrak{B}} \leq K_a\|y\|_a + M_a\|\phi\|_{\mathfrak{B}}$ where $\|y\|_a = sup_{0\leq t\leq a}\|y(t)\|$.

The operators $\Lambda_i : L^2(J, X) \to X\ i = 1, 2$ are defined as

$$\Lambda_1 x(t) = \int_0^a S(t - s)x(s)ds$$

$$\Lambda_2 x(t) = \int_0^a C(t - s)x(s)ds$$

Clearly Λ_i are bounded linear operators. We set $\mathfrak{N}_i = ker(\Lambda_i),\ \Lambda = (\Lambda_1, \Lambda_2)$ and $\mathfrak{N} = ker(\Lambda)$. Let $C_0(J, X)$ denote the space consisting of continuous functions $x : J \to X$ such that $x(0) = 0$, endowed with the norm of uniform convergence. Let $J_i : L^2(J, X) \to C_0(J, X),\ i = 1, 2$ be maps defined as follows

$$J_1 x(t) = \int_0^t S(t - s)x(s)ds$$

$$J_2 x(t) = \int_0^t C(t - s)x(s)ds$$

So, $J_i x(a) = \Lambda_i(x),\ i = 1, 2$.

As a continuation of co-author N. Sukavanam's work [155] and from hypothesis $(B1)$ in [139] we assume that $L^2(J, X) = \mathfrak{N}_i + \overline{R(B)},\ i = 1, 2$.

By using lemma (2.1.13) we denote P_i the map associated to this decomposition and construct $X_2 = \mathfrak{N}_i$ and $X_1 = \overline{R(B)}$. Also set $c_i = \|P_i\|$.

We introduce the space

$$Z = \{z \in C_0(J, X) : z = J_1(n_1) + J_2(n_2), n_i \in \mathfrak{N}_i,\ i = 1, 2\}$$

and we define the map $\Gamma : \overline{Z} \to C_0(J, X)$ by

$$\Gamma = J_1 \circ P_1 \circ F - J_2 \circ P_2 \circ G$$

Lemma 7.3.6. If the hypothesis $(H_\phi) - (H_g)$ and conditions $(C1)$ hold for f, g and $aK_a(c_1 \widetilde{N} L_f + c_2 N L_g) < \sqrt{2}$ then Γ has a fixed point.

Proof: For $z^1, z^2 \in \overline{Z}$ let $\Delta f(s) = f(s, z^2_{\rho(s,z^2(s))} + x_{\rho(s,x(s))}) - f(s, z^1_{\rho(s,z^1(s))} + x_{\rho(s,x(s))})$ and $\Delta g(s) = g(s, z^2_s + x_s) - f(s, z^1_s + x_s)$. $\forall\, 0 \le t \le a$

$$\begin{aligned}
\|(\Gamma z^2 - \Gamma z^2)(t)\| &\le \|\int_0^t S(t-s)[P_1(\Delta f)](s)ds\| + \|\int_0^t C(t-s)[P_2(\Delta g)](s)ds\| \\
&\le \widetilde{N} \int_0^t \|[P_1(\Delta f)](s)\|ds + N \int_0^t \|[P_2(\Delta g)](s)\|ds \\
&\le \widetilde{N} t^{1/2} c_1 \|\Delta f\|_2 + N t^{1/2} c_2 \|\Delta g\|_2
\end{aligned}$$

Now

$$\begin{aligned}
\|\Delta f\|_2^2 &= \int_0^a \|f(s, z^2_{\rho(s,z^2(s))} + x_{\rho(s,x(s))}) - f(s, z^1_{\rho(s,z^1(s))} + x_{\rho(s,x(s))})\|^2 ds \\
&\le L_f^2 \int_0^a \|z^2_{\rho(s,z^2(s))} - z^1_{\rho(s,z^1(s))}\|^2_{\mathfrak{B}} ds \\
&\le L_f^2 \int_0^a \|z^2_s - z^1_s\|^2_{\mathfrak{B}} ds \\
&\le a L_f^2 K_a^2 \|z^2 - z^1\|^2_\infty ds
\end{aligned}$$

Similarly we find for g. So,

$$\|(\Gamma z^2 - \Gamma z^1)(t)\| \le b t^{1/2} \|z^2 - z^1\|_\infty$$

where $b = a^{1/2} K_a (c_1 \widetilde{N} L_f + c_2 N L_g)$. Repeating this get

$$\|(\Gamma^n z^2 - \Gamma^n z^1)(t)\|_\infty \le \frac{(bt^{1/2})^n}{2^{(n-1)/(2n)}} \|z^2 - z^1\|_\infty$$

As $b = aK_a(c_1\widetilde{N}L_f + c_2NL_g) < \sqrt{2}$ and $2^{\frac{n-1}{2n}} \to \sqrt{2}$ *as* $n \to \infty$, the map Γ^n is a contraction for n sufficiently large and therefore Γ has a fixed point.

Theorem 7.3.7. If the associated linear control system (7.3.4) is approximately controllable on J, the space $L^2([0,a],X) = \mathfrak{N}_i + \overline{R(B)}, i = 1,2$ and condition of the preceding lemma (7.3.6) hold then the semilinear control system (7.3.1) with state dependent delay is approximately controllable on J.

Proof: Assume $x(.)$ to be the mild solution and $u(.)$ to be an admissible control function of system (7.3.4) with initial conditions $x(0) = \phi(0)$ and $x'(0) = w + g(0,\phi)$. Let z be the fixed point of Γ. So, $z(0) = 0$ and $z(a) = \Lambda_1(P_1(F(z))) - \Lambda_2(P_2(G(z))) = 0$. By Lemma 2.1.13 we can split the functions $F(z), G(z)$ with respect to the decomposition $L^2(J,X) = \mathfrak{N}_i + \overline{R(B)}$ $i = 1,2$ respectively by setting $q_1 = F(z) - P_1(F(z))$ and $q_2 = G(z) - P_2(G(z))$. We define the function $y(t) = z(t) + x(t)$ for $t \in J$ and $y_0 = \phi$. So, $x(a) = y(a)$. Thus by the properties of x and z

$$\begin{aligned} y(t) &= \int_0^t S(t-s)(f(s, y_{\rho(s,y(s))}) - q_1(s) + Bu(s))ds \\ &- \int_0^t C(t-s)(g(s,y_s) - q_2(s))ds + C(t)x(0) + S(t)x'(0) \end{aligned} \tag{7.3.6}$$

As $C_0^1(J,U)$ is dense in $L^2(J,U)$ we can choose a sequence $v_n^1 \in L^2(J,U)$ and a sequence $v_n^2 \in L^2(J,X)$ such that $Bv_n^1 \to q_1$ and $Bv_n^2 \to q_2$ as $n \to \infty$. By Lemma 7.3.2 we get

$$\begin{aligned} y^n(t) &= \int_0^t S(t-s)(f(s, y^n_{\rho(s,y(s))}) - Bv_n^1(s) + Bu(s))ds \\ &- \int_0^t C(t-s)(g(s,y^n_s) - Bv_n^2(s))ds + C(t)\phi(0) + S(t)(w - g(0,\phi)) \\ &= \int_0^t S(t-s)(f(s, y^n_{\rho(s,y(s))}) - Bv_n^1(s) + B\frac{d}{ds}v_n^2(s) + Bu(s))ds \\ &- \int_0^t C(t-s)g(s,y^n_s)ds + C(t)\phi(0) + S(t)(w + g(0,\phi)) \end{aligned}$$

Hence by definition (7.3.1) and the last expression we conclude that y^n is the mild solution of the following equation

$$\frac{d}{dt}(y'(t) + g(t,x_t)) = Ay(t) + f(t, y_{\rho(t,y(t))}) + B(-v_n^1(t) + \frac{d}{dt}v_n^2(t) + u(t))$$

$$x(0) = \phi \in \mathfrak{B} \;\; x'(0) = w$$

Hence $y^n(a) \in \mathcal{R}_T(a, f, g, \phi, w)$. Since the solution map is generally continuous, $y^n \to y$ as $n \to \infty$. Thus $y(a) \in \mathcal{R}_T(a, f, g, \phi, w)$. Therefore $\mathcal{R}_T(0)(a, \phi(0), w + g(0, \phi)) \subset \overline{\mathcal{R}_T(a, f, g, \phi, w)}$, which means $\mathcal{R}_T(a, f, g, \phi, w)$ is dense in X . Thus the system (7.1.1) is controllable.

7.4 Examples

Example 1:

In this section we discuss a concrete partial differential equation applying the abstract results of this paper. In this application, $\mathfrak{B}$ is the phase space $C_0 \times L^2(h, X)$ (see [97]).

Consider the second order neutral differential equation

$$\begin{aligned}
\frac{\partial}{\partial t}(\frac{\partial u(t,\xi)}{\partial t} &+ \int_{-\infty}^{t}\int_{0}^{\pi} b(t-s,\eta,\xi)u(s,\eta)d\eta ds) \\
= \frac{\partial^2 u(t,\xi)}{\partial \xi^2} &+ \int_{-\infty}^{t} a(t-s)u(s-\rho_1(t)\rho_2(\|u(t)\|),\xi)ds, \\
&\qquad t \in (s_i, t_{i+1}],\ i = 0,\cdots,n,\ \xi \in [0,\pi], \\
u(t,0) &= u(t,\pi) = 0,\ t \in [0,a], \\
u(\tau,\xi) &= \phi(\tau,\xi)\ \tau \le 0, 0 \le \xi \le \pi, \\
u'(\tau,\xi) &= \omega(t,\xi)\ \tau \le 0, 0 \le \xi \le \pi, \\
u(t)(\xi) &= \int_{\infty}^{t_i} a_i^1(t_i-s)u(s,\xi)ds\ \ t \in (t_i, s_i],\ i = 1,2,\cdots,n \\
u'(t)(\xi) &= \int_{\infty}^{t_i} a_i^2(t_i-s)u(s,\xi)ds\ \ t \in (t_i, s_i],\ i = 1,2,\cdots,n
\end{aligned} \tag{7.4.1}$$

where $\phi \in C_0 \times L^2(h, X)$, $0 < t_1 <, ..., t_n < a$. For $y \in D(A)$, $y = \sum_{n=1}^{\infty} < y, \phi_n > \phi_n$ and $Ay = -\sum_{n=1}^{\infty} n^2 < y, n > \phi_n$. where $\phi_n(x) = \sqrt{\frac{2}{\pi}} \sin nx,\ 0 \le x \le \pi,\ n = 1, 2, 3...$ is the eigenfunction corresponding to the eigenvalue $\lambda_n = -n^2$ of the operator A. ϕ_n is an orthonormal base. A will generate the operators $S(t)$, $C(t)$ such that $S(t)y = \sum_{n=1}^{\infty} \frac{\sin(nt)}{n} < y, \phi_n > \phi_n,\ n = 1, 2, ... \ \forall\, y \in X$, and the operator $C(t)y = \sum_{n=1}^{\infty} \cos(nt) < y, \phi_n > \phi_n,\ n = 1, 2, ... \ \forall\, y \in X, .$

Let us suppose that the functions $\rho_i : \mathbb{R} \to [0, \infty)$, $a : \mathbb{R} \to \mathbb{R}$ are piecewise continuous. By defining maps $\rho, G, F : [0,a] \times \mathfrak{B} \to X$ by

$$\rho(t,\psi) := \rho_1(t)\rho_2(\|\psi(0)\|),$$

$$g(t,\psi)(\xi) := \int_{-\infty}^{0}\int_{0}^{\pi} b(s,v,\xi)\psi(s,v)dvds,$$

$$f(t,\psi)(\xi) := \int_{-\infty}^{0} a(s)\psi(s,\xi)ds$$

$$J_i^j(\psi)(\xi) := \int_{-\infty}^{0} a_i^j(s)\psi(s,\xi)ds \quad i = 1,\cdots,n \;\; j = 1,2$$

the system (7.4.1) can be transformed into system (7.1.1) Assume that the following conditions hold

(a) The functions $b(s,\eta,\xi), \frac{\partial b(s,\eta,\xi)}{\partial \xi}$ are measurable, $b(s,\eta,\pi) = b(s,\eta,0) = 0$ and

$$L_g := max\{(\int_0^{\pi}\int_{-\infty}^{0}\int_0^{\pi} \frac{1}{h(s)}(\frac{\partial^i b(s,\eta,\xi)}{\partial \xi^i})^2 d\eta ds d\xi)^{1/2} : \; i = 0,1\} < \infty$$

such that $\|g\|_{\mathcal{L}(X)} \leq L_g$.

(b) The function $F : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ is continuous and there is continuous function $L_f = \int_{-\infty}^{0} \frac{a(s)^2}{h(s)}ds < \infty$ and $\|F\|_{\mathcal{L}(X)} \leq L_f$.

(c) The functions $a_i^j \in C([0,\infty);\mathbb{R})$ and $L_i^j := (\int_{-\infty}^{0} \frac{(a_i^j(s))^2}{h(s)}ds)^{1/2} < \infty$ for all $i = 1,2,...,n \; j = 1,2$

Moreover $g(t,.), J_i^j, i = 1,...,n, j = 1,2$ are bounded linear operators .
Hence by assumptions $(a)-(c)$ and Theorem (7.2.3) it is ensured that mild solution to the problem (7.4.1) exists.

Now let us consider a particular example from the point of view of concrete application

$$
\begin{aligned}
\frac{\partial}{\partial t}(\frac{\partial u(t,\xi)}{\partial t} &+ \int_{-\infty}^{t}\int_{0}^{\pi} b(t-s,\eta,\xi)u(s,\eta)d\eta ds) \\
= \frac{\partial^2 u(t,\xi)}{\partial \xi^2} &+ a(t)b(u(t-\mu(u(t,0)),\xi)), \\
& \qquad t \in (s_i, t_{i+1}],\ i = 0,\cdots,n,\ \xi \in [0,\pi], \\
u(t,0) &= u(t,\pi) = 0,\ t \in [0,a], \\
u(\tau,\xi) &= \phi(\tau,\xi)\ \tau \le 0, 0 \le \xi \le \pi, \\
u'(\tau,\xi) &= \omega(\tau,\xi)\ \tau \le 0, 0 \le \xi \le \pi, \\
u(t)(\xi) &= d_i^1 \sin|u(t,\xi)|,\quad t \in (t_i, s_i],\ i = 1,2,\cdots,n \\
u'(t)(\xi) &= d_i^2 \cos|u(t,\xi)|,\quad t \in (t_i, s_i],\ i = 1,2,\cdots,n. \qquad (7.4.2)
\end{aligned}
$$

where $\phi \in \mathfrak{B} = C_h^0(X)$. The functions $a : J \to \mathbb{R}$, $b : \mathbb{R} \times J \to \mathbb{R}$, $\mu : \mathbb{R} \to \mathbb{R}^+$ are piecewise continuous. We assume the existence of positive constants b_1, b_2 such that

$$|b(t)| \le b_1|t| + b_2, \quad \forall\ t \in \mathbb{R}.$$

If we define maps

$$f(t,\psi)(\xi) = a(t)b(\psi(0,\xi)),$$

$$\rho(t,\psi) = t - \mu(\psi(0,0)),$$

and $g(t,\psi)(\xi)$ as in the problem (7.4.1) we can transform (7.4.2) into (7.1.1). Also a simple estimate shows that $\|f(t,\psi)\| \le a(t)[b_1\|\psi\|_{\mathfrak{B}} + b_2\pi^{1/2}]\ \forall\ (t,\psi) \in J \times \mathfrak{B}$.

Also if we define $J_i^1(t,u(t)) = d_i^1 \sin|u(t)|$ *and* $J_i^2 = d_i^2 \cos|u(t)|$ for all $i = 1,\cdots,n$ then the hypotheses (HJ) can be easily proved. For instance,

$$\|J_i^1(t,u(t))\| = \|d_i^1 \sin|u(t)|\| \le d_i^1\|u(t)\|$$

and

$$
\begin{aligned}
\|J_i^1(t,u_1(t)) - J_i^1(t,u_2(t))\| &= \|d_i^1 \sin|u_1(t)| - d_i^1 \sin|u_2(t)|\| \\
&\le \|d_i^1[|u_1(t)| - |u_2(t)|]\|. \qquad (7.4.3)
\end{aligned}
$$

Similarly it is easily seen for J_i^2. Now, if ϕ satisfies the hypothesis H_ϕ then $\exists$ a mild solution of (7.4.2).

Example 2:

Consider the second order neutral differential equation

$$\begin{aligned}
\frac{\partial}{\partial t}(\frac{\partial u(t,\xi)}{\partial t} &+ \int_{-\infty}^{t}\int_{0}^{\pi} b(t-s,\eta,\xi)u(s,\eta)d\eta ds) \\
= \frac{\partial^2 u(t,\xi)}{\partial \xi^2} &+ \int_{-\infty}^{t} a(t-s)u(s-\rho_1(t)\rho_2(\|u(t)\|),\xi)ds + Bv(t) \\
&\qquad t\in[0,a], \xi\in[0,\pi], \\
u(t,0) &= u(t,\pi)=0,\ t\in[0,a], \\
u(t,\xi) &= \phi(t,\xi)\ \tau\le 0, 0\le\xi\le\pi,
\end{aligned} \tag{7.4.4}$$

where $\phi \in C_0 \times L^2(h,X)$, $0 < t_1 <, ..., t_n < a$ For $y \in D(A)$, $y = \sum_{n=1}^{\infty} < y, \phi_n > \phi_n$ and $Ay = -\sum_{n=1}^{\infty} n^2 < y, n > \phi_n$. where $\phi_n(x) = \sqrt{\frac{2}{\pi}}\sin nx,\ 0 \le x \le \pi,\ n = 1,2,3...$ is the eigenfunction corresponding to the eigenvalue $\lambda_n = -n^2$ of the operator A. ϕ_n is an orthonormal base. A will generate the operators $S(t),\ C(t)$ such that $S(t)y = \sum_{n=1}^{\infty} \frac{\sin(nt)}{n} < y, \phi_n > \phi_n,\ n = 1,2,...\ \forall\ y \in X$, and the operator $C(t)y = \sum_{n=1}^{\infty} \cos(nt) < y, \phi_n > \phi_n,\ n = 1,2,...\ \forall\, y \in X$,. Let the infinite dimensional control space be defined as $U = \{u : u = \sum_{n=2}^{\infty} u_n\phi_n,\ \sum_{n=2}^{\infty} u_n^2 < \infty\}$ with norm $\|u\|_U = (\sum_{n=2}^{\infty} u_n^2)^{\frac{1}{2}}$. Thus U is a Hilbert space. By defining maps $\rho, G, F : [0,a] \times \mathfrak{B} \to X$ by

$$\rho(t,\psi) := \rho_1(t)\rho_2(\|\psi(0)\|),$$

$$G(\psi)(\xi) := \int_{-\infty}^{0}\int_{0}^{\pi} b(s,v,\xi)\psi(s,v)dvds,$$

$$F(\psi)(\xi) := \int_{-\infty}^{0} a(s)\psi(s,\xi)ds$$

the system (7.4.4) can be transformed into system (7.1.1) Assume that the functions $\rho_i : \mathbb{R} \to [0,\infty),\ a : \mathbb{R} \to \mathbb{R}$ are continuous and satisfies the following conditions.

(a) The functions $b(s,\eta,\xi), \frac{\partial b(s,\eta,\xi)}{\partial\xi}$ are measurable, $b(s,\eta,\pi) = b(s,\eta,0) = 0$ and

$$L_g := max\{(\int_0^{\pi}\int_{-\infty}^{0}\int_0^{\pi} \frac{1}{h(s)}(\frac{\partial^i b(s,\eta,\xi)}{\partial\xi^i})^2 d\eta ds d\xi)^{1/2} :\ i = 0,1\} < \infty$$

such that $\|g\|_{\mathcal{L}(X)} \le L_g$.

(b) The function $F : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous and there is continuous function $L_f = \int_{-\infty}^{0} \frac{a(s)^2}{h(s)} ds < \infty$ and $\|F\|_{\mathcal{L}(X)} \leq L_f$.

(c) The functions $a_i^j \in C([0,\infty);\mathbb{R})$ and $L_i^j := (\int_{-\infty}^{0} \frac{(a_i^j(s))^2}{h(s)} ds)^{1/2} < \infty$ for all $i = 1, 2, ..., n$ $j = 1, 2$

Moreover $g(t, .)$ is bounded linear operators.

Here we examine the conditions (HR) for this control system. Then by using theorem (7.3.7) we show its approximate controllability. Let $\widetilde{B} : U \to X :\ \widetilde{B}u = 2u_2\phi_1 + \sum_{n=2}^{\infty} u_n\phi_n \quad for\ u = \sum_{n=2}^{\infty} u_n\phi_n \in U$. The bounded linear operator $B : L_2([0,T];U) \to L_2([0,T];X)$ is defined by $(Bu)(t) = \widetilde{B}u(t)$.
Let $\alpha \in N \subset L_2(0,T : X)$, N is the null space of Γ. Also $\alpha = \sum_1^{\infty} \alpha_n(s)\phi_n$. Therefore

$$\int_0^T S(T-s)\alpha(s)ds = 0. \tag{7.4.5}$$

This implies that

$$\int_0^T \frac{\sin n(T-s)}{n} \alpha_n(s)ds = 0,\ n \in \mathcal{N}$$

The Hilbert space $L_2(0,T)$ can be written as

$$L_2(0,T) = Sp\{\sin s\}^{\perp} + Sp\{\sin 4s\}^{\perp}.$$

Thus for $h_1, h_2 \in L_2(0,T)$ there exists $\alpha_1 \in \{\sin s\} \perp,\ \alpha_2 \in \{\sin 4s\}^{\perp}$ such that $h_1 - 2h_2 = \alpha_1 - 2\alpha_2$. So let $u_2 = h_2 - \alpha_2$. Then $h_1 = \alpha_1 + 2u_2,\ \ h_2 = \alpha_2 + u_2$ also let $u_n = h_n,\ n = 3, 4, \cdots$ and $\alpha_n = 0,\ n = 3, 4, \cdots$. Thus we see that hypothesis (HR) is satisfied as $U = \{u : u = \sum_{n=2}^{\infty} u_n\phi_n, \sum_{n=2}^{\infty} u_n^2 < \infty\}$ and $\widetilde{B} : U \to X : \widetilde{B}u = 2u_2\phi_1 + \sum_{n=2}^{\infty} u_n\phi_n$.

Hence by assumptions $(a)-(c)$ and Theorem (7.3.7) it is ensured that the problem (7.4.4) is approximately controllable.

7.5 Conclusion

The conditions for existence and uniqueness of the mild solution are derived via Hausdorff measure of non-compactness and Darbo Sadovskii fixed point theorem. The conditions of approximate controllability are established for the neutral second order system. A simple condition on the range of an operator is used to prove approximate controllability of the system.

Chapter 8

Approximate Controllability of a Fractional Neutral System with Deviated Argument in a Banach Space

In this chapter we proved the approximate controllability of an impulsive fractional stochastic neutral integro-differential equation with deviating argument and infinite delay. We use Schauder fixed point theorem and fundamental assumptions on system operators. In infinite dimensional space, the assumption of invertibility of controllability operator is removed as it not invertible in case of compact semigroup. Specifically, we studied a remote control dynamical system represented by a neutral fractional differential equation with deviated argument which may take values in a remote space.

8.1 Introduction

Several papers studied the approximate controllability of semilinear control systems, see for instance [69], [139], [156] and references therein. Generally these papers proposed conditions on the systems operators with assumption of approximate controllability of the corresponding linear system. For instance, Naito [139] proved that a semilinear system is approximately controllable under a range condition on the control operator and uniform boundedness of the nonlinear operator. Sukavanam [156] proved sufficient conditions for approximate controllability where the nonlinear function satisfies growth conditions.

Motivated by results in [139] and [156] the purpose of this chapter is to study the existence and uniqueness of mild solution and approximate controllability of a functional differential equation with deviated argument and finite delay using Schuader fixed point theorem. We proceed by establishing a relation between the reachable set of linear control problem and that of the semilinear delay control problem.

In this chapter we studied the existence, uniqueness and approximate controllability of the following fractional order neutral differential equation

$$\begin{aligned} {}^{C}D_t^{\alpha}[x(t) + g(t, x_t)] &= Ax(t) + f(t, x_t, x(a(x(t), t))) + Bu(t), \ t \in J = [0, T] \\ x(t) &= x_0 = \phi \in \mathfrak{B}, \quad t \in (-\infty, 0] \end{aligned} \tag{8.1.1}$$

where $\alpha \in (1/2, 1)$, $0 < T < \infty$ and ${}^{C}D_t^{\alpha}$ denotes the fractional derivative in Caputo sense. The state function $x(.)$ belong to the Banach Space X. The control function $u(.) \in L^2(J, U)$ where U is a Banach space. $B : U \to X$ is a bounded linear operator. $A : D(A) \subset X \to X$ is a the infinitesimal generator of an strongly continuous semigroup of bounded linear operators $S(t)$, $t > 0$ on X. The history valued function $x_t : (-\infty, 0] \to X$, $x_t(\theta) = x(t + \theta)$ belongs to some abstract phase space $\mathfrak{B}$ defined axiomatically in chapter 2 as Definition 2.2.12. f, g and a are suitably defined functions satisfying certain conditions to be specified in the following hypotheses.

Let W be the closed subspace of all continuous functions $x : (-\infty, 0] \to L^2((-\infty, T]; \mathfrak{B})$ such that the restriction $x : [0, T] \to L_2((-\infty, T]; \mathfrak{B})$ is continuous. Let $\|.\|_W$ be a seminorm defined by $\|x\|_W = \sup_{t \in [0,T]} \|x_t\|_{\mathfrak{B}}$. Let $D = C_L(J, X) = \{u \in C(J, X) : \|u(t) - u(s)\| \leq L|t - s|, \forall t, s \in J\}$.

Definition 8.1.1. The function $x(t) \in C((-\infty, T]; X)$ is said to be a mild solution of (8.1.1) if $x(.) \in C_L(J, X)$, $x(t) = \phi(t)$ for $t \in [-\infty, 0]$ and it satisfies the integral equation.

$$\begin{aligned} x(t) &= S_\alpha(t)[\phi(0)+g(0,\phi(0))]-g(t,x_t) \\ &- \int_0^t (t-s)^{\alpha-1} A T_\alpha(t-s) g(s,x_s) ds \\ &+ \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s)[Bu(s)+f(s,x_s,x(a(x(s),s)))]ds,\ t\in J \end{aligned} \tag{8.1.2}$$

where $S_\alpha(t)x = \int_0^\infty \varphi_\alpha(\theta)T(t^\alpha\theta)xd\theta$ and $T_\alpha(t)x = \alpha\int_0^\infty \theta\varphi_\alpha(\theta)T(t^\alpha\theta)xd\theta$. Here $\varphi_\alpha(\theta) = \frac{1}{\alpha}\theta^{-1-1/\alpha}\psi_\alpha(\theta^{-1/\alpha})$ is the probability density function defined on $(0,\infty)$, that is $\varphi_\alpha(\theta)\geq 0$, and $\int_0^\infty \varphi_\alpha(\theta)d\theta = 1$ and

$$\psi_\alpha(\theta) = \frac{1}{\pi}\sum_{n=1}^{\infty}(-1)^{n-1}\theta^{-n\alpha-1}\frac{\Gamma(n\alpha+1)}{n!}sin(n\pi\alpha),\ \theta\in(0,\pi)$$

Lemma 8.1.2. [177] $S_\alpha(t)$ and $T_\alpha(t)$ are linear bounded operators for any chosen $t>0$ such that for any $x\in X$, $\|S_\alpha(t)x\|\leq M_1\|x\|$ and $\|T_\alpha(t)x\|\leq \frac{M_1\alpha}{\Gamma(1+\alpha)}\|x\|$.

Lemma 8.1.3. [177] For any $x\in X$, $0<\beta<1$ and $\eta\in(0,1]$, we have $AT_\alpha(t)x = A^{1-\beta}T_\alpha(t)A^\beta x$ and $\|A^\eta T_\alpha(t)\|\leq \frac{\alpha C_\eta}{t^{\alpha\eta}}\frac{\Gamma(2-\eta)}{\Gamma(1+\alpha(1-\eta))}$, $t\in[0,T]$.

Definition 8.1.4. The set given by $K_\tau(f) = \{x(T)\in X : x\in X\}$ is called reachable set of the system (8.1.1) . $K_\tau(0)$ is the reachable set of the corresponding linear control system (8.3.1).

Definition 8.1.5. The system (8.1.1) is called approximately controllable if $K_\tau(f)$ is dense in X. The corresponding linear system is approximately controllable if $K_\tau(0)$ is dense in X.

The following assumptions are required to prove our results

(H1) $\forall x_1,x_2,z_1,z_2\in X,\ t\in(0,\tau]$ the nonlinear function $f: J\times X\times X\to X$ satisfies Lipschitz condition

$$\|f(t,x_1,z_1)-f(t,x_2,z_2)\|\leq P(\|x_1-x_2\|+\|z_1-z_2\|)$$

with Lipschitz constant $P>0$. and $\exists$ a constant $f_0>0$,
such that $\|f(s,0,x(a(x(0),0)))\|\leq f_0,\ \forall\, s\in J$

(H2) Let $a : X \times R^+ \to R^+$ satisfy the Lipschitz condition $|a(x_1, s) - a(x_2, s)| \leq L_a\|x_1 - x_2\|$ and $a(.,0) = 0$

(H3) The nonlinear function $g : [0,T] \times \mathfrak{B} \to X$ is continuous and there exists constant $0 < \beta < 1$ such that $g \in D(A^\beta)$.
$A^\beta g(.,x)$ is strongly measurable.
$\forall t \in [0,T]$, and $x_1, x_2 \in \mathfrak{B}$, $A^\beta g(t,.)$ satisfies the Lipschitz condition $\|A^\beta g(t,x) - A^\beta g(t,y)\|_X \leq L_g\|x-y\|_{\mathfrak{B}}$ with Lipschitz constant $L_g > 0$

8.2 Existence and uniqueness of mild solution

In this subsection the equation (8.1.2) is verified to be the unique mild solution of the semilinear delay control system (8.1.1).

Theorem 8.2.1. The system (8.1.1) has a unique mild solution in $C_L(J, X)$ for each control $u \in L_2([0,T]; U)$ if assumptions $(H1)$, $(H2)$ and $(H3)$ are satisfied.

Proof: Define the space

$$C_{L_0}([-\infty, T], X) = \{x : x \in C([-\infty, T], X) \cap C_L([0,T], X)\}.$$

Let

$$\begin{aligned} R &= 2[M\|\phi(0) + g(0, \phi(0))\| + \frac{M\alpha}{\Gamma(1+\alpha)}\sqrt{\frac{t_1^{2\alpha-1}}{2\alpha-1}}\|Bu(s)\|_{C((-\infty,T];X)} \\ &+ \frac{M}{\Gamma(1+\alpha)}(PlL_a\|x(0)\| + f_0)t_1^\alpha + L_g\frac{\Gamma(1+\beta)C_{1-\beta}}{\beta\Gamma(1+\alpha\beta)}t_1^{\alpha\beta}(l_g) \\ &+ \frac{\Gamma(1+\beta)C_{1-\beta}}{\Gamma(1+\alpha\beta)\beta}t_1^{\alpha\beta}\|g(t,0)\|_{C((-\infty,T];X)}] + 1 \end{aligned} \quad (8.2.1)$$

Fix $0 < t_1 < T$ such that

$$\begin{aligned} &\frac{MP}{\Gamma(1+\alpha)}P(R + lL_aHR)t_1^\alpha + \|A^{-\beta}\|L_g + L_g t_1^\alpha \beta\frac{\Gamma(1+\beta)C_{1-\beta}}{\beta\Gamma(1+\alpha\beta)} \\ \leq\ & [M\|\phi(0) + g(0,\phi(0))\| + \frac{M\alpha}{\Gamma(1+\alpha)}\sqrt{\frac{t_1^{2\alpha-1}}{2\alpha-1}}\|Bu(s)\|_{C((-\infty,T];X)} \\ +\ & \frac{M}{\Gamma(1+\alpha)}(PlL_a\|x(0)\| + f_0)t_1^\alpha + L_g\frac{\Gamma(1+\beta)C_{1-\beta}}{\beta\Gamma(1+\alpha\beta)}t_1^{\alpha\beta}(l_g) \\ +\ & \frac{\Gamma(1+\beta)C_{1-\beta}}{\Gamma(1+\alpha\beta)}t_1^\alpha\|g(t,0)\|_{C((-\infty,T];X)}] + 1 \end{aligned} \quad (8.2.2)$$

Define the mapping $\Phi : C_{L_0}([-\infty, t_1], X) \to C_{L_0}([-\infty, t_1], X)$ as

$$(\Phi x)(t) = \begin{cases} S_\alpha(t)(\phi(0) + g(0, \phi(0))) - g(t, x_t) \\ - \int_0^t (t-s)^{\alpha-1} A T_\alpha(t-s) g(s, x_s) ds \\ + \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s) B u(s) ds \\ + \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s) f(s, x_s, x(a(x(s), s))) ds, \ t \in (0, t_1] \\ \phi(\theta), \qquad \theta \in [-\infty, 0] \end{cases}$$

Let us consider the space $B_R = \{x(.) \in C_{L_0}([-\infty, t_1], X) : \|x\|_{C([-\infty,t_1],X)} \leq R, \ x(0) = \phi(0)\}$ endowed with the norm of uniform convergence. For any $x \in B_R$ and $0 \leq t \leq t_1$,

$$\|x_t\|_C \leq K \sup\{\|x(s)\| : 0 \leq s \leq T\} + M\|\phi\|_{\mathfrak{B}}.$$

where $K = \sup_{t\in[0,T]}\{K(t)\}$ and $M = \sup_{t\in[0,T]}\{M(t)\}$. Now $(\Phi x)(t) = x(t)$ is given by

$$\begin{aligned} x(t) = x_t(0) &= S_\alpha(t)(\phi(0) + g(0, \phi(0))) \\ &+ \int_0^t (t-s)^{\alpha-1} A T_\alpha(t-s) g(s, x_s) ds \\ &+ \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s)[Bu(s) + f(s, x_s, x(a(x(s), s)))] ds \end{aligned} \tag{8.2.3}$$

Then

$$\begin{aligned} \|x_t(0)\|_X &\leq \|S_\alpha(t)\| \|\phi(0) + g(0, \phi(0))\| + \|A^{-\beta}\| \|A^\beta g(t, x_t) - A^\beta g(t, 0)\| \\ &+ \int_0^t (t-s)^{\alpha-1} \|T_\alpha(t-s)\| \|Bu(s) + f(s, x_s, x(a(x(s), s)))\| ds \\ &+ \int_0^t (t-s)^{\alpha-1} \|A^{1-\beta} t_\alpha(t-s)\| \|A^\beta g(s, x_s) - A^\beta g(s, 0)\| ds \\ &+ \|g(t, 0)\| + \frac{\Gamma(1+\beta) C_{1-\beta}}{\Gamma(1+\alpha\beta)\beta} t^{\alpha\beta} \|g(t, 0)\|_{C((-\infty,T];X)} \\ &\leq M\|\phi(0) + g(0, \phi(0))\| + \frac{M\alpha}{\Gamma(1+\alpha)} \int_0^t (t-s)^{\alpha-1} \|Bu(s)\| ds \\ &+ \int_0^t \frac{M\alpha}{\Gamma(1+\alpha)} (t-s)^{\alpha-1} [\|f(s, x_s, x(a(x(s), s))) \\ &- f(s, 0, x(a(x(0), 0)))\| + \|f(s, 0, x(a(x(0), 0)))\|] ds \end{aligned} \tag{8.2.4}$$

$$
\begin{aligned}
&+ \quad \|A^{-\beta}\|L_g\|x_s\| + L_g\frac{\alpha\Gamma(1+\beta)C_{1-\beta}}{\Gamma(1+\alpha\beta)}\int_0^t (t-s)^{\alpha\beta-1}\|x_s\| + l_g \\
&+ \quad \frac{\Gamma(1+\beta)C_{1-\beta}}{\Gamma(1+\alpha\beta)\beta}t^{\alpha\beta}\|g(t,0)\|_{C((-\infty,T];X)} \\
&\leq \quad M\|\phi(0)+g(0,\phi(0))\| + \frac{M\alpha}{\Gamma(1+\alpha)}\sqrt{\frac{t^{2\alpha-1}}{2\alpha-1}}\|Bu(s)\|_{C((-\infty,T];X)} \\
&+ \quad \frac{M\alpha}{\Gamma(1+\alpha)}\int_0^t (t-s)^{\alpha-1}[P(\|x_s-0\| + lL_a\|x(s)-x(0)\|) + f_0]ds \\
&+ \quad \|A^{-\beta}\|L_g\|x_s\| + L_g\frac{\alpha\Gamma(1+\beta)C_{1-\beta}}{\Gamma(1+\alpha\beta)}\int_0^t (t-s)^{\alpha\beta-1}\|x_s\| + l_g \\
&+ \quad \frac{\Gamma(1+\beta)C_{1-\beta}}{\Gamma(1+\alpha\beta)\beta}t^{\alpha\beta}\|g(t,0)\|_{C((-\infty,T];X)} \\
&\leq \quad M\|\phi(0)+g(0,\phi(0))\| + \frac{M\alpha}{\Gamma(1+\alpha)}\sqrt{\frac{t_1^{2\alpha-1}}{2\alpha-1}}\|Bu(s)\|_{C((-\infty,T];X)} \\
&+ \quad \frac{M\alpha}{\Gamma(1+\alpha)}\int_0^t (t-s)^{\alpha-1}[P(\|x_s\| + lL_aH\|x_s\|)]ds \\
&+ \quad \frac{M}{\Gamma(1+\alpha)}[P(lL_a\|x(0)\|) + g]t_1^{\alpha} \\
&+ \quad \|A^{-\beta}\|L_g\|x_s\| + L_g\frac{\alpha\Gamma(1+\beta)C_{1-\beta}}{\Gamma(1+\alpha\beta)}\int_0^t (t-s)^{\alpha\beta-1}(\|x_s\| + l_g)ds \\
&+ \quad \frac{\Gamma(1+\beta)C_{1-\beta}}{\Gamma(1+\alpha\beta)\beta}t^{\alpha\beta}\|g(t,0)\|_{C((-\infty,T];X)} \\
&\leq \quad M\|\phi(0)+g(0,\phi(0))\| + \frac{M\alpha}{\Gamma(1+\alpha)}\sqrt{\frac{t_1^{2\alpha-1}}{2\alpha-1}}\|Bu(s)\|_{C((-\infty,T];X)} \\
&+ \quad \frac{M}{\Gamma(1+\alpha)}t_1^{\alpha}[P(R+lL_aHR)] + \frac{M}{\Gamma(1+\alpha)}[P(lL_a\|x(0)\|) + f_0]t_1^{\alpha} \\
&+ \quad \|A^{-\beta}\|L_gR + L_g\frac{\Gamma(1+\beta)C_{1-\beta}}{\beta\Gamma(1+\alpha\beta)}t_1^{\alpha\beta}(R+l_g) \\
&+ \quad \frac{\Gamma(1+\beta)C_{1-\beta}}{\Gamma(1+\alpha\beta)\beta}t^{\alpha\beta}\|g(t,0)\|_{C((-\infty,T];X)} < R
\end{aligned} \tag{8.2.5}
$$

Hence Φ maps B_R into itself when t_1 satisfies (8.2.2). Next it is shown that Φ is a contraction.

Let us define $x(t) = z(t) + y(t)$ where $y(t) = \begin{cases} \phi(t), & t \in (-\infty, 0]; \\ S_\alpha(t)\phi(0), & t \in [0, T] \end{cases}$ Let

$x_1,\ x_2 \in B_R$

$$\begin{aligned}
\|\Phi x_1(t) - \Phi x_2(t)\| &\leq \int_0^t \frac{M\alpha}{\Gamma(1+\alpha)}(t-s)^{\alpha-1} \\
&\times \|f(s,(z_1)_s + y_s,(z_1+y)(a((z_1+y)(s),s))) \\
&- f(s,(z_1)_s + y_s,(z_1+y)(a(z_2(s)+y(s),s))) \\
&- f(s,(z_2)_s + y_s,(z_2+y)(a(z_2(s)+y(s),s))) \\
&+ f(s,(z_1)_s + y_s,(z_1+y)(a(z_2(s)+y(s),s)))\|ds \\
&+ \|A^{-\beta}\|\|A^{\beta}[g(t,(z_1)_t + y_t) - g(t,(z_2)_t + y_t)]\| \\
&+ \int_0^t (t-s)^{\alpha-1}\|A^{1-\beta}T_\alpha(t-s)\|\|A^{\beta}[g(t,(z_1)_t + y_t) \\
&- g(t,(z_2)_t + y_t)]\| \\
&\leq \int_0^t \frac{PM\alpha}{\Gamma(1+\alpha)}(t-s)^{\alpha-1}(\|(z_1+y)(a(z_1(s)+y(s),s)) \\
&- (z_1+y)(a(z_2(s)+y(s),s))\| + \|(z_2)_s - (z_1)_s\| \\
&+ \|(z_2+y)(a(z_2(s)+y(s),s)) \\
&- (z_1+y)(a(z_2(s)+y(s),s))\|) \\
&+ \|A^{-\beta}\|L_g\|(z_1)_s - (z_2)_s\| \\
&+ \frac{\alpha L_g\Gamma(1+\beta)C_{1-\beta}}{\Gamma(1+\alpha\beta)}\int_0^t (t-s)^{\alpha\beta-1}\|(z_2)_s - (z_1)_s\|_{\mathfrak{B}}ds \\
&\leq \frac{PM}{\Gamma(1+\alpha)}t_1^{\alpha}(lL_a\|z_2 - z_1\|_{C((-\infty,t_1];X)} \\
&+ K\|z_2 - z_1\|_{C((-\infty,t_1];X)} + \|z_2 - z_1\|_{C((-\infty,t_1];X)}) \\
&+ \|A^{-\beta}\|L_gK\|z_2 - z_1\|_{C((-\infty,t_1];X)} \\
&+ \frac{L_g\Gamma(1+\beta)C_{1-\beta}}{\Gamma(1+\alpha\beta)\beta}t_1^{\alpha\beta}\|z_2 - z_1\|_{C((-\infty,t_1];X)} \\
&\leq \{\frac{PMt_1^{\alpha}}{\Gamma(1+\alpha)}(lL_a + K + 1) + \frac{L_g\Gamma(1+\beta)C_{1-\beta}t^{\alpha\beta}}{\Gamma(1+\alpha\beta)\beta} \\
&+ \|A^{-\beta}\|L_gK\}\|z_2 - z_1\|_{C((-\infty,t_1];X)} \qquad (8.2.6)
\end{aligned}$$

So,

$$\begin{aligned}
\|\Phi x_1 - \Phi x_2\|_{C((-\infty,t_1],X)} &\leq \{\frac{PMt_1^{\alpha}}{\Gamma(1+\alpha)}(lL_a + K + 1) + \|A^{-\beta}\|L_gK \\
&+ \frac{L_g\Gamma(1+\beta)C_{1-\beta}t^{\alpha\beta}}{\Gamma(1+\alpha\beta)\beta}\}\|x_1 - x_2\|_{C((-\infty,t_1],X)}.
\end{aligned}$$

Repeating the above process we get that

$$\begin{aligned} \|\Phi^n x_1 - \Phi^n x_2\|_{C((-\infty,t_1],X)} &\leq \{\frac{PMt_1^\alpha}{\Gamma(1+\alpha)n!}(lL_a + K + 1) + \frac{\|A^{-\beta}\|L_g K}{n!} \\ &+ \frac{L_g\Gamma(1+\beta)C_{1-\beta}t^{\alpha\beta}}{\Gamma(1+\alpha\beta)n!}\}^n \|x_1 - x_2\|_{C((-\infty,t_1];X)}. \end{aligned} \tag{8.2.7}$$

Thus Φ^n is a contraction mapping for large integer n. Therefore, Φ has a fixed point in B_R. Hence (8.1.2) is the mild solution on $(-\infty, t_1]$. Similarly it can be shown that (8.1.2) is the mild solution on the interval $[t_1, t_2]$, $t_1 < t_2$. Thus (8.1.2) is the mild solution on the maximal existence interval $(-\infty, t^*]$, $t^* < \infty$.

Now it is shown that x is well defined in $(-\infty, T]$. If $t \in (-\infty, t^*]$

$$\begin{aligned} \|x(t)\| &\leq M\|\phi\| + \frac{M\alpha(M_B)}{\Gamma(1+\alpha)}\int_0^t M_B\|u(s)\|ds \\ &+ \frac{M\alpha}{\Gamma(1+\alpha)}\int_0^t [P\|x_s - 0\| + P\|x(a(x(s),s) - x(a(x(0),0)\| + f_0]ds \\ &+ \|A^{-\beta}\|L_g K(\|x\| + l_g)_{C((-\infty,t_1];X)} \\ &+ \frac{\alpha L_g\Gamma(1+\beta)C_{1-\beta}}{\Gamma(1+\alpha\beta)}\int_0^t (t-s)^{\alpha-1}\|A^{1-\beta}T_\alpha(t-s)\|(\|x\| + l_g)_{C((-\infty,t_1];X)} \\ &\leq M\|\phi\| + \frac{M\alpha(M_B)}{\Gamma(1+\alpha)}\sqrt{\frac{t_1^{2\alpha-1}}{2\alpha-1}}\|u(s)\| \\ &+ \int_0^t \frac{PM\alpha}{\Gamma(1+\alpha)}(t-s)^{\alpha-1}[\|x_s\| + lL_a\|x(s) - x(0)\| + f_0] \\ &\leq \frac{[M\|\phi\| + \frac{M\alpha(M_B)}{\Gamma(1+\alpha)}\sqrt{\frac{t_1^{2\alpha-1}}{2\alpha-1}}\|u(s)\|]}{1 - \|A^{-\beta}\|L_g K} \\ &+ \frac{[\frac{PM\alpha}{\Gamma(1+\alpha)}\sqrt{\frac{t_1^{2\alpha-1}}{2\alpha-1}}(\|x(0)\| + f_0) + \|A^{-\beta}\|L_g K l_g]}{1 - \|A^{-\beta}\|L_g K} \\ &+ \frac{[\frac{MP\alpha}{\Gamma(1+\alpha)}\int_0^t (K + lL_a)(t-s)^{\alpha-1}\|x(s)\|ds]}{1 - \|A^{-\beta}\|L_g K} \end{aligned}$$

By Gronwall's inequality

$$
\begin{aligned}
\|x(t)\| &\leq \|x_t\|_C \leq \{\frac{[M\|\phi\| + \frac{M\alpha(M_B)}{\Gamma(1+\alpha)}\sqrt{\frac{t_1^{2\alpha-1}}{2\alpha-1}}\|u(s)\|]}{1-\|A^{-\beta}\|L_gK} \\
&+ \frac{[\frac{PM}{\Gamma(1+\alpha)}t_1^{\alpha}(\|x(0)\|+f_0) + \|A^{-\beta}\|L_gKl_g]}{1-\|A^{-\beta}\|L_gK}\} \\
&\times \exp(\frac{MPT^{\alpha}}{\Gamma(1+\alpha)1-\|A^{-\beta}\|L_gK}). \qquad (8.2.8)
\end{aligned}
$$

So $\|x(t)\|$ is bounded on $[-\infty, t^*]$.
The uniqueness of mild solution is proved as follows. Let x_1 and x_2 be any two solutions then since $x(t) = z(t) + y(t)$ where $y(t) = \begin{cases} \phi(t), & t \in (-\infty, 0]; \\ S_\alpha(t)\phi(0), & t \in [0,T] \end{cases}$

$$
\begin{aligned}
\|z_1(t) - z_2(t)\| &\leq \int_0^t \frac{M\alpha}{\Gamma(1+\alpha)}(t-s)^{\alpha-1} \\
&\times \|f(s,(z_1)_s + y_s, (z_1+y)(a((z_1+y)(s),s))) \\
&- f(s,(z_1)_s + y_s, (z_1+y)(a(z_2(s)+y(s),s))) \\
&- f(s,(z_2)_s + y_s, (z_2+y)(a(z_2(s)+y(s),s))) \\
&+ f(s,(z_1)_s + y_s, (z_1+y)(a(z_2(s)+y(s),s)))\|ds \\
&+ \|A^{-\beta}\|\|A^{\beta}[g(t,(z_1)_t+y_t) - g(t,(z_2)_t+y_t)]\| \\
&+ \int_0^t (t-s)^{\alpha-1}\|A^{1-\beta}T_\alpha(t-s)\|\|A^{\beta}[g(t,(z_1)_t+y_t) \\
&- g(t,(z_2)_t+y_t)]\| \\
&\leq \int_0^t \frac{PM\alpha}{\Gamma(1+\alpha)}(t-s)^{\alpha-1}(\|(z_1+y)(a(z_1(s)+y(s),s)) \\
&- (z_1+y)(a(z_2(s)+y(s),s))\| + \|(z_2)_s - (z_1)_s\| \\
&+ \|(z_2+y)(a(z_2(s)+y(s),s)) \\
&- (z_1+y)(a(z_2(s)+y(s),s))\|) \\
&+ \|A^{-\beta}\|L_g\|(z_1)_s - (z_2)_s\| \\
&+ \frac{\alpha L_g\Gamma(1+\beta)C_{1-\beta}}{\Gamma(1+\alpha\beta)}\int_0^t (t-s)^{\alpha\beta-1}\|(z_2)_s - (z_1)_s\|_{\mathfrak{B}}ds
\end{aligned}
$$

(8.2.9)

$$\leq \frac{PM}{\Gamma(1+\alpha)}\int_0^t (t-s)^{\alpha-1}ds(lL_a\|z_2-z_1\|_{C(-\infty,t_1];X)}$$
$$+ \quad K\|z_2-z_1\|_{C((-\infty,t_1];X)} + \|z_2-z_1\|_{C((-\infty,t_1];X)})$$
$$+ \quad \|A^{-\beta}\|L_gK\|z_2-z_1\|_{C((-\infty,t_1];X)}$$
$$+ \quad \frac{L_g\Gamma(1+\beta)C_{1-\beta}}{\Gamma(1+\alpha\beta)\beta}t_1^{\alpha\beta}\|z_2-z_1\|_{C((-\infty,t_1];X)}$$
$$\leq \quad \{\frac{PM}{\Gamma(1+\alpha)}\int_0^t (t-s)^{\alpha-1}(lL_a+K+1)ds$$
$$+ \quad \|A^{-\beta}\|L_gK\}\|z_2-z_1\|_{C((-\infty,t_1];X)} + \frac{L_g\Gamma(1+\beta)C_{1-\beta}t^{\alpha\beta}}{\Gamma(1+\alpha\beta)\beta}$$
$$\leq \quad \frac{PM}{\Gamma(1+\alpha)\mu}\int_0^t (t-s)^{\alpha-1}(lL_a+K+1)ds \tag{8.2.10}$$

where $\mu = 1 - \frac{L_g\Gamma(1+\beta)C_{1-\beta}t^{\alpha\beta}}{\Gamma(1+\alpha\beta)\beta} + \|A^{-\beta}\|L_gK$. So, by Gronwall's inequality we see that $z_1 = z_2$ which implies $x_1 = x_2$ on $(-\infty, T]$. Thus x is well defined on $(-\infty, T]$.

8.3 Approximate Controllability

Let us define a continuous linear operator $\mathfrak{L} : L^2([0,T];X) \to C([0,T];X)$ as

$$\mathfrak{L}p = \int_0^T (T-s)^{\alpha-1}T_\alpha(T-s)p(s)ds, \quad p \in \mathcal{L}_2([0,T];X)$$

The following hypothesis is required to prove the approximate controllability

(HR) $\forall\epsilon > 0$ and $p(.) \in L^2([0,\tau];V)$, $\exists u(.) \in U$ such that $\|\mathfrak{L}p - \mathfrak{L}Bu\|_X < \epsilon$

Theorem 8.3.1. If the assumptions $H3$ and HR hold then the corresponding neutral system

$$\frac{d(x(t)+g(t,x_t))}{dt} = Ax(t) + Bu(t)$$
$$x(t) = \phi(t), \ -\infty \leq t \leq 0 \tag{8.3.1}$$

with $f \equiv 0$ is approximately controllable.

Proof: It is sufficient to prove that $D(A) \subset \overline{K_T(0)}$ since $D(A)$ is dense in X. Let $d(T,\phi) = S_\alpha(T)[\phi(0) + g(0,\phi(0))] - g(T,x_T) - \int_0^T (T-s)^{\alpha-1}AT_\alpha(T-s)g(s,x_s)ds$ For any chosen $\xi \in D(A)$, then $\xi - d(T,\phi) \in D(A)$. It can be seen that there exists some $p \in C^1([0,T];X)$ such that

$$\eta = \xi - d(T,\phi) = \int_0^T (T-s)^{\alpha-1}T_\alpha(T-s)p(s)ds.$$

By hypothesis (HR) there exists a control function $u(.) \in L^2([0,T];U)$ such that $\|\eta - \mathfrak{L}Bu\| < \epsilon$. As ϵ is arbitrary it implies that $K_T(0) \subset D(A)$. Since the $D(A)$ is dense in X, $K_T(0)$ is dense in X. Hence the neutral system with $f \equiv 0$ is approximately controllable.

Let us define the operator $K : Z = L^2([0,T];X) \to C((-\infty,T];X)$, $Kx(t) = \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s)x(s)ds, t \in [0,\tau]$.
Z can be decomposed uniquely as $Z = N_0(L) \oplus N_0^{\perp}(L)$ where $N_0(L)$ is the null space of the operator L and $N_0(L)$ is its orthogonal space.
Define the operator $F : C_{L_0}([0,T],X) \to L^2([0,T],X)$ as

$$F(x)(t) = f(t, x_t, x(a(x(t),t))); 0 < t \leq \tau.$$

Hypotheses $(H1), (H2)$ imply that F is a continuous map. Hypothesis (HR) implies that for any $p \in Z$, there exists a $q \in R(B)$ such that $L(p-q) = 0$. So, $p - q = n \in N_0(L)$ from which it follows that $Z = N_0(L) \oplus \overline{R(B)}$. Thus, it implies the existence of a linear and continuous mapping Q from $N_0^{\perp}(L)$ into $\overline{R(B)}$ which is defined as $Qu^* = v$ where v is the unique minimum norm element $v \in (u^* + N_0(L)) \bigcap \overline{R(B)}$, i.e. $\|Qu^*\| = \|v\| = min\{\|v\| : v \in \{(u^* + N_0(L)) \bigcap \overline{R(B)}\}$. $(H3)$, implies that $\forall v \in \{u^* + N_0^{\perp}\} \cap \overline{R(B)}$ is not empty and $\forall z \in Z$ has a unique decomposition $z = n + q$. Hence the operator Q is well defined. Moreover, $\|Q\| = c$ for some constant c.

Let us consider the subspace M_0 of $C_{L_0}([0,T];X)$ which is defined as

$$M_0 = \begin{cases} m \in C_{L_0}([0,T],X) : m(t) = Kn(t), & n \in N_0(L); 0 \leq t \leq \tau \\ m(t) = 0, & -\infty \leq t \leq 0; \end{cases} \tag{8.3.2}$$

Let

$$f_x : \overline{M_0} \to \overline{M_0}$$

defined by

$$f_x = \begin{cases} Kn, & 0 < t \leq \tau; \\ 0, & -h \leq t \leq 0; \end{cases} \tag{8.3.3}$$

where n is given by the unique decomposition of $F(x+m)(t) = n(t) + q(t)$, $n \in N_0(L)$ and $q \in \overline{R(B)}$.

Theorem 8.3.2. The operator f_x has a fixed point in M_0 if the hypotheses $(H1)-(H2)$ hold and $\frac{M(1+c)PT^\alpha}{\Gamma(1+\alpha)} < 1$.

Proof: Since the semigroup $T(t)$ is compact by hypothesis (HS) so $T_\alpha(t)$ is compact and hence f_x is compact. Let $z \in Z$ then $z = q + n$, $n \in N_0(L)$, $q \in \overline{R(B)}$. Also $\|n\|_Z \leq (1+c)\|z\|_Z$ for some constant c. Let

$$B_r = \{v \in \overline{M_0} : ||v|| \leq r\}.$$

Let $m \in B_r$. Let $\|f(0,0,(x+m)(a(m(s),0)\| \leq l_f$ Suppose on the other hand

$$\begin{aligned}
r < \|f_x(m)\| &= \|Kn\| \leq \int_0^t (t-s)^{\alpha-1}\|T_\alpha(t-s)\|\|n(s)\|ds \\
&\leq \frac{M\alpha}{\Gamma(1+\alpha)}\int_0^t (t-s)^{\alpha-1}(1+c)\|F(x+m)\|_Z ds \\
&\leq \frac{M\alpha(1+c)}{\Gamma(1+\alpha)}\int_0^t [\|f(s,(x+m)_s,(x+m)(a((x+m)(s),s)))\| \\
&- \|f(0,0,(x+m)(a(m(s),0)))\| \\
&+ \|f(0,0,(x+m)(a(m(s),0)))\|]ds \\
&\leq \frac{M\alpha(1+c)}{\Gamma(1+\alpha)}\int_0^t P[\|(x+m)(s+\theta)-0\| \\
&+ \|(x+m)(a((x+m)(s),s)) - (x+m)(a((m)(s),0))\| + l_f]ds \\
&\leq \frac{M(1+c)\alpha}{\Gamma(1+\alpha)}\int_0^t P[\|x\| + \|m\| + l|a((x+m)(s),s) - a(m(s),0)| \\
&+ l_f]ds \\
&\leq \frac{M(1+c)\alpha}{\Gamma(1+\alpha)}\int_0^t P[\|x\| + r + lL_a\|(x+m)(s) - m(s)\| + l_f]ds \\
&\leq \frac{M(1+c)\alpha}{\Gamma(1+\alpha)}\int_0^t P[\|x\| + r + lL_a\|x\| + l_f]ds \\
&\leq \frac{M(1+c)}{\Gamma(1+\alpha)}P(\alpha\|x\|\sqrt{\frac{T^{2\alpha-1}}{2\alpha-1}} + rT^\alpha + lL_a\|x\|\sqrt{\frac{T^{2\alpha-1}}{2\alpha-1}} + l_fT^\alpha)
\end{aligned} \tag{8.3.4}$$

Dividing by r and taking limit as r tends to ∞ we get a contradiction. So f_x maps B_r into itself. Therefore by Schauder fixed point theorem it has a fixed point.

Theorem 8.3.3. The semilinear control system (8.1.1) is approximately controllable if the linear delay control system (8.3.1) is approximately controllable.

Proof: Let $x(.)$ be the mild solution of the linear delay control system (8.3.1) is given by

$$\begin{aligned} x(t) &= S_\alpha(t)(\phi(0) + g(0,\phi(0))) + g(t,x_t) \\ &+ \int_0^t (t-s)^{\alpha-1} A T_\alpha(t-s) g(s,x_s) ds + KBu(t), \quad t \in (0,T] \\ x(t) &= \phi(t), t \in [-\infty, 0] \end{aligned} \tag{8.3.5}$$

We prove

$$y(t) = x(t) + m_0(t)$$

to be mild solution semilinear problem (8.1.1). Since

$$KF_h(x+m_0)(t) = Kn(t) + Kq(t)$$

operating K on both sides at $m = m_0$, fixed point of f_x.

$$\begin{aligned} KF_h(x+m_0)(t) &= Kn(t) + Kq(t) \\ &= m_0(t) + Kq(t) \end{aligned} \tag{8.3.6}$$

Add $x(.)$ to both sides and using $y(t) = x(t) + m_0(t)$

$$\begin{aligned} x(t) + KF_h(x+m_0)(t) &= x(t) + m_0(t) + Kq(t) \\ x(t) + KF_h(y)(t) &= y(t) + Kq(t) \\ \Rightarrow y(t) &= x(t) + KF_h(y)(t) - Kq(t) \\ \Rightarrow y(t) &= S_\alpha(t)(\phi(0) + g(0,\phi(0))) + g(t,x_t) \\ &+ \int_0^t (t-s)^{\alpha-1} A T_\alpha(t-s) g(s,x_s) ds \\ &+ K(Bu-q)(t) + KF_h(y)(t) \end{aligned} \tag{8.3.7}$$

This is the mild solution of semilinear problem with control $(Bu - q)$. By following the same proof in [155] we get the following conclusion that since $q \in \overline{R(B)}$ there exists a $v \in U$ such that $\|Bv - q\| < \epsilon$ for any given $\epsilon > 0$. Let x_v be a solution of the given semilinear delay control system (8.3.1) corresponding to the control v. Then as shown by [139] we have $||y(T) - x_v(T)|| = ||x(T) - x_v(T)|| \leq \epsilon$. This implies that $x(\tau) \in \overline{K_T(f)}$. Then it follows that $\overline{K_T(0)} \subset \overline{K_T(f)}$. Thus (8.1.1) is approximately controllable since the corresponding linear system (8.3.1) is approximately controllable.

8.4 Example

Let us consider the heat control system with finite delay

$$\begin{aligned} {}^{C}D_t^{\alpha}[y(t,x) &+ \int_{-\infty}^{t}\int_{0}^{\pi} a_1(s-t,\eta,x)y(s,\eta)d\eta ds] + \frac{\partial^2 y(t,x)}{\partial x^2} \\ &= Bu(t,x) + \int_{-\infty}^{t} a_2(s-t)y(s,a(y(s,x),s))ds, \\ & \quad 0<t<T,\ -\infty<\theta<0,\ 0<x<\pi \\ y(t,0) &= y(t,\pi)=0,\ 0\le t\le T \\ y(t,x) &= \xi(x),\ -\infty\le t\le 0,\ 0\le x\le\pi. \end{aligned} \tag{8.4.1}$$

Let $X = L_2(0,\pi)$ and $A = \frac{d^2}{dx^2}$. Define

$$\begin{aligned} D(A) = \{y &\in X : y, \frac{dy}{dx} \text{ are absolutely continuous}, \\ \frac{d^2y}{dx^2} &\in X \text{ and } y(0)=y(\pi)=0\}. \end{aligned}$$

For $y \in D(A)$, $y = \sum_{n=1}^{\infty} < y, \phi_n > \phi_n$ and $Ay = -\sum_{n=1}^{\infty} n^2 < y, n > \phi_n$. where $\phi_n(x) = \frac{2}{\pi}^{\frac{1}{2}} \sin nx,\ 0 \le x \le \pi,\ n = 1,2,3...$ is the eigenfunction corresponding to the eigenvalue $\lambda_n = -n^2$ of the operator A. ϕ_n is an orthonormal base. A will generate a compact semigroup $T(t)$. such that $T(t)y = \sum_{n=1}^{\infty} e^{-n^2 t} < y, \phi_n > \phi_n,\ n = 1,2,...\ \forall\ y \in X$. Let the infinite dimensional control space be defined as $U = \{u : u = \sum_{n=2}^{\infty} u_n\phi_n,\ \sum_{n=2}^{\infty} u_n^2 < \infty\}$ with norm $\|u\|_U = (\sum_{n=2}^{\infty} u_n^2)^{\frac{1}{2}}$. Thus U is a Hilbert space.
Let $\widetilde{B} : U \to X :\ \widetilde{B}u = 2u_2\phi_1 + \sum_{n=2}^{\infty} u_n\phi_n\ \ for\ u = \sum_{n=2}^{\infty} u_n\phi_n \in U$. The bounded linear operator $B : L_2(0,T : U) \to L_2(0,T;X)$ is defined by $(Bu)(t) = \widetilde{B}u(t)$. Then this problem (8.4.1) can be transformed into an abstract semilinear differential equation with deviating argument and infinite delay. Following the hypotheses $(H1)-(H3)$ and (HR) the approximate controllability of the system (8.4.1) is proved by help of Theorem 8.3.3.

8.5 Conclusion

Thus, we proved the existence and uniqueness and approximate controllability of the functional differential equation (8.1.1) with deviating argument and finite delay by using Schuader fixed point theorem and fundamental solution instead of C_0 semigroup. We also removed the limitation of the non-existence of the inverse of the controllability operator due to the compactness of the semigroup in infinite-dimensional spaces. We achieved this by establishing a relation between the reachable set of linear control problem and that of the semilinear delay control problem.

Chapter 9

Approximate Controllability of an Impulsive Fractional Stochastic Differential Equation

The approximate controllability of a fractional impulsive stochastic neutral integro-differential equation with deviating argument and infinite delay is studied. The control parameter is also included inside the nonlinear term. Schauder fixed point theorem is used to prove our result. The assumption of invertibility of controllability operator is removed as the inverse fails to exist in infinite-dimensional space, in case of compactness of the semigroup. Lipschitz continuity of the nonlinear function is replaced by fundamental assumptions on the system operators. We also give an example to illustrate our result.

9.1 Introduction

Results of controllability for abstract systems are abundant (see for details [61; 174] and references therein) rather than for fractional stochastic neutral differential equation with deviated argument and control parameter included inside the nonlinear term.

Benchohra et al. [52] and Chang [61] discussed the exact controllability of functional differential systems with impulsive conditions and unbounded delay. However, they assumed that the inverse of a controllability operator exists. Generally due to the compactness of the generated semigroup it is not invertible in infinite-dimensional space. Hence their methodology does not work in infinite-dimensional cases. Moreover it is not always possible to apply their results.

Although with a different approach Zhou [176] established approximate controllability of an abstract semilinear control system. Mahmudov [69] established that approximate controllability of semilinear system follows from the approximate controllability of its associated linear part.

In this chapter we studied the control system containing deviating argument.

$$\begin{aligned} {}^cD_t^q[x(t)+g(t,x_t)] &= A[x(t)+g(t,x_t)]+Bu(t)+f(t,x(a(x(t),t)),u(t)) \\ &+ \int_{-\infty}^{t} G(t,s,x_s)dW(s), t\in J=[0,T], t\neq t_k, k=1,...,m \\ x_0(t) &= \phi(t), \qquad t\in J_1=(-\infty,0] \\ x(t_k^+)-x(t_k^-) &= I_k(x(t_k)), \qquad k=1,...,m, \end{aligned} \tag{9.1.1}$$

where A is the infinitesimal generator compact semigroup of uniformly bounded linear operators $\{S(t): t\in R^+\}$ on a Hilbert space X. ${}^cD_t^q$ denotes the Caputo fractional derivative of order $0<q<1$. X and U are two separable Hilbert spaces. There are three separable spaces X, K, U. The state space is denoted by X. Suppose $(\Omega,\mathcal{F},P)$ be a probability space together with a normal filtration $\mathcal{F}_t,\ t\in J=[0,T]$. $J_0=J_1+J=(-\infty,T]$.

$$\mathcal{F}_t=\sigma(W(s):0\le s\le t)\ and\ \mathcal{F}_T=\mathcal{F}.$$

Suppose $L^2(\Omega,\mathcal{F},P;X)\equiv L^2(\Omega;X)$ be the Banach space of all strongly measurable, square integrable, X-valued random variables equipped with the norm $\|x(.)\|^2_{L^2}=E\|x(.;w)\|^2_X$. The stochastic process is a collection of random variables $S=\{x(t,w):\Omega\to X: t\in J\}$. We usually suppress w and write $x(t)$ instead of $x(t,w)$. $W(t)\in K$ is the Q-Wiener process. The control parameter $u(t)\in L^2([0,T],\mathcal{F},U)$. The history valued function $x_t:(-\infty,0]\to X$, $x_t(\theta)=x(t+\theta)$ lies in some abstract phase space $\mathfrak{B}$ defined below. B is a bounded linear operator on a Hilbert space U.

Let $h_0 : J_1 \to \mathbb{R}$ be a continuous function such that $l = \int_{-\infty}^{0} h_0(t)dt \leq \infty$. Then $\mathfrak{B} = \{\phi : (-\infty, 0] \to X$ is such that $\forall\ a > 0$, $(E\|\phi(\theta)\|^2)^{1/2}$ is a bounded measurable function on $[-a, 0]$, $\|\phi\|_{[-a,0]} = \sup_{-a\leq\theta\leq 0}\|\phi(\theta)\|$ and

$$\int_{-\infty}^{0} h(s) \sup_{s\leq\theta\leq 0}(E\|\phi(\theta)\|^2)^{1/2}ds < \infty\}.$$

$\mathfrak{B}$ is a Banach space with respect to the norm

$$\int_{-\infty}^{0} h(s) \sup_{s\leq\theta\leq 0}(E\|\phi(\theta)\|^2)^{1/2}ds < \infty, \quad \phi \in \mathfrak{B}$$

Let $f, g : J \times \mathfrak{B} \to X$ be measurable in X and $G : J \times J \times \mathfrak{B} \to L_0^2(K, X)$ is measurable in $L_Q(J, X)$ norm. The space containing all Q-Hilbert Schmidt operators with domain K and range X is denoted by $L_Q(J, X)$. B is a bounded linear operator from U into X. $\phi(t) \in \mathfrak{B}-$ is a random variable independent of $W(t)$. It has finite second moment. Also $\psi(t) \in X$ is a $\mathcal{F}_t$ measurable function.

Let $D = t_1, t_2, \cdots, t_m \subset J = [0, T]$, $0 = t_0 < t_l < \cdots < t_m < t_{m+1} = T$. $C(J_0, L^2(\Omega; X))$ denotes the Banach space of all continuous maps from $J_0 = (-\infty, T]$ into $L^2(\Omega; X)$ which satisfy $sup_{t\in J_0}E\|x(t)\|^2 < \infty$. $L_0^2(\Omega, X) = \{f \in L_2(\Omega, X) : f\ is\ \mathcal{F}_0-measurable\}$ denotes an important subspace. $PC((-\infty, T], L^2(\Omega, \mathcal{F}_t, X)) = \{x(t) : J_0 = (-\infty, T] \to L^2(\Omega, \mathcal{F}_t, X)$ is continuous everywhere except at t_k at which $x(t_k^+), x(t_k^-)$ exists and $x(t_k^-) = x(t_k)$ satisfying $\sup_{t\in J_0} E\|x(t)\|^2 \leq \infty\}$.

$I_k(k = 1, 2, \cdots, m) : X \to X$ is a nonlinear map and $\Delta x(t_k) = x(t_k^+) - x(t_k^-)$ is the change in the state x at time t_k. I_k indicates the amount of the change. Suppose H be a closed subspace of $PC(J_0, L^2(\Omega, \mathcal{F}_t, \mathfrak{B}))$ consisting of measurable and $\mathcal{F}_t-$ adapted $X-$valued process and $\mathcal{F}_0-$ adapted processes $x \in L^2(\Omega, \mathcal{F}_0, \mathfrak{B})$ endowed with norm $\|x\|_H = (\sup_{t\in J} E\|x_t\|_{\mathfrak{B}}^2)^{1/2}$.

Now we define few notations that are used in the following sections

$M = \sup\{\|S(t)\| : 0 \leq t \leq T\}$, $M_1 = \|B\|$,

$\|\lambda\| = \int_0^b |\lambda_i(s)|ds$.

$\lim_{r\to\infty} \frac{\mu_i(r)}{r} = \Xi, \quad \lim_{r\to\infty} \frac{\psi(r)}{r} = \Theta$

Let us define the following operators:

Let $\Gamma_0^T = \int_0^T (T-s)^{(q-1)} T_q(T-s) B B^* T_q^*(T-s) ds$ be the controllability Gramian

Let $R(\alpha, \Gamma_0^T) = (\alpha I + \Gamma_0^T)^{-1}$

We assume the following hypotheses:

(H1) $S(t)$, $t > 0$ is the compact semigroup.

(H2) $f : J \times X \times U \to X$ is continuous and $\exists$ function $\lambda(.) \in L_1(J, R^+)$ and a non decreasing function $g_i \in L_1(C \times U, R^+), i = 1, 2, ..., q, :$

$$E\|f(t,x,u)\|^2 \leq \sum_{i=1}^{q} \lambda_i(t) g_i(x,u)$$

$\forall (t,x,u) \in I \times X \times U$.

(H3) For each $\alpha > 0$

$$\lim_{r\to\infty} \sup \left(r - \sum_{i=1}^{q} \frac{c_i}{\alpha} sup\{g_i(x,u) : \|(x,u)\| \leq r\}\right) = \infty$$

(H4) I_k is continuous and $\exists d_k$:

$$E\|I_k(x)\|^2 \leq d_k$$

$\forall x \in X (k = 1, 2, ..., m)$.

(H5) $g : J \times \mathfrak{B}$ is completely continuous and uniformly bounded $E\|g(.,\phi)\|^2 \leq M_g(1 + \|\phi\|^2_{\mathfrak{B}})$.

(H6) $\alpha R(\alpha, \Gamma_0^T) \to 0$ in strong operator topology as $\alpha \to 0^+$

(H7) $\|I_k(x(t_k)) - I_k(y(t_k))\| \leq L_I(\|x(t_k) - y(t_k)\|)$, $\forall\ x(t_k), y(t_k),\ k = 1, \cdots, m$.

(H8) $a : X \times J \to J$ is a continuous function such that $|a(x(s), s)| \leq s$.

(H9) The function $G : J \times J \times \mathfrak{B} \to L(K, X)$ satisfies the following conditions:

(i) $\forall\ (t,s) \in J \times J$, $G(t,s,.) : \mathfrak{B} \to L(K, X)$ is continuous.

(ii) For all $x \in \mathfrak{B}$, $G(.,.,x) : J \times J \to L(K, X)$ is strongly measurable.

(iii) There exists a positive integrable function $n \in L^1([0,T])$ and there is a nondecreasing function $\psi : [0,\infty) \to (0,\infty)$ such that $\forall\ (t,s,x) \in J \times J \times \mathfrak{B}$ we have

$$\int_0^t E\|G(t,s,x)\|^2_{L_0^2} ds \leq n(t)\psi(\|x\|^2_{\mathfrak{B}})$$

(iv) $\forall\ \phi \in \mathfrak{B}, \quad k(t) = \lim_{a\to\infty} \int_{-a}^0 G(t,s,\phi)dw(s)$ exists and is continuous. $\exists\ M_k > 0$ such that $E\|k(t)\|^2 \leq M_k$.

Remark : The assumption $(H6)$ holds iff the following linear fractional control problem is approximately controllable.

$${}^cD_t^q(t) = Ax(t) + (Bu)(t), t \in [0,T]$$

$$x(0) = x_0.$$

It is an extension of approximate controllability of linear first order problem in [69]. We define mild solution of problem (9.1.1) as follows.

Definition 9.1.1. $x \in H$ is a mild solution of the problem (9.1.1) if $x(t)$ is a $\mathcal{F}_t$−adapted process such that $x(t) = \phi(t)$ on $(-\infty, 0)$, and the following integral equation is satisfied

$$\begin{aligned} x(t) &= S_q(t)[\phi(0) + g(0,\phi)] - g(t,x_t) \\ &+ \int_0^t (t-s)^{q-1}T_q(t-s)[f(t,x(a(x(t),t)),u(t)) + Bu(s)]ds \\ &+ \int_0^t (t-s)^{q-1}T_q(t-s)[\int_{-\infty}^s G(s,\tau,x_\tau)dW(\tau)]ds \\ &+ \sum_{0<t_k<t} (t-t_k)^{q-1}T_q(t-t_k)I_k(x(t_k^-)),\ t \in [0,T] \end{aligned} \tag{9.1.2}$$

where $S_q(t) = \int_0^\infty \zeta_q(\theta)S(t^q\theta)d\theta$; and $T_q(t) = q\int_0^\infty \theta\zeta_q(\theta)S(t^q\theta)d\theta$; ζ_q is a probability density function defined on $(0,\infty)$, i.e. $\zeta_q(\theta) \geq 0,\ \theta \in (0,\infty)$ and

$$\int_0^\infty \zeta_q(\theta)d\theta = 1.$$

Lemma 9.1.2. [152] Let $G : J \times J \times \Omega \to L_0^2$ denote a strongly measurable function with $\int_0^T E\|G(t)\|_{L_0^2}^p dt < \infty$. We have

$$E\| \int_0^t G(s)dw(s)\|^p \leq L_G \int_0^t E\|G(s)\|_{L_0^2}^p ds$$

$\forall t \in [0,T]$ and $p \geq 2$. Here L_G is a constant containing p and T.

Lemma 9.1.3. [168] $S_q(t), T_q(t)$ are strongly continuous and compact. For all chosen $t \geq 0$ $S_q(t), T_q(t)$ are bounded linear operators i.e.

$$\forall\ t \geq 0, \quad \|S_q(t)x\| \leq M\|x\|, \qquad \|T_q(t)x\| \leq \frac{Mq}{\Gamma(q+1)}\|x\|$$

Lemma 9.1.4. [152] Let $x(t) \in PC(J_0, L^2(\Omega, \mathcal{F}_t, X))$ then for any $t \in J$, $x_t \in \mathfrak{B}$

$$l\|x(t)\| \leq \|x_t\|_{\mathfrak{B}} \leq \|\phi\|_{\mathfrak{B}} + l \sup_{s\in[0,t]} \|x(s)\|, \quad where \quad l = \int_\infty^0 h_0(t)dt$$

Lemma 9.1.5. [152] $\forall\ x_T \in L^2(\Omega, \mathcal{F}_T, X)$, $\exists w \in L^2_{\mathcal{F}_t}(\Omega; (J, L_0^2))$ such that $x_T = Ex_T + \int_0^T w(s)dW(s)$

Definition 9.1.6. The reachable set is $R(T;\phi,u) = \{x_T(\phi,u)(0), \quad such\ that \quad u \in L^2_{\mathcal{F}_t}(J,U)\}$. If $\overline{R(T;\phi,u)} = L^2(\Omega, \mathcal{F}_t, X)$, then the problem (9.1.1) is approximately controllable on $[0,T]$

In the next section we prove that (9.1.1) is approximately controllable if $\forall \alpha > 0$, $\exists\ (x,u) \in PC(J_0, L^2(\Omega, \mathcal{F}_t, X)) \times C(J,U)$ such that

$$u(t) = (T-t)^{q-1}B^*T_q^*(T-t)R(\alpha, \Gamma_0^T)p(x,u) \tag{9.1.3}$$

$$\begin{aligned} z(t) &= S_q(t)(x_0 + g(0,\phi)) - g(t,x_t) + \sum_{0<t_k<t} (t-t_k)^{q-1}T_q(t-t_k)I_k(x(t_k)) \\ &+ \int_0^t (t-s)^{q-1}T_q(t-s)[f(s, x(a(x(s),s)), u) + Bv(s)]ds \\ &+ \int_0^t (t-s)^{q-1}T_q(t-s)[\int_{-\infty}^s G(s,\tau,x_\tau)dW(\tau)]ds \end{aligned} \tag{9.1.4}$$

where

$$\begin{aligned} p(x,u) &= Ex_T + \int_0^T w(s)dW(s) - S_q(T)(x_0 + g(0,\phi)) + g(T,x_T) \\ &- \int_0^T (T-s)^{q-1}T_q(T-s)f(s, x(h(x(s),s)), u)ds \\ &- \int_0^T (T-s)^{q-1}T_q(T-s)[\int_{-\infty}^s G(s,\tau,x_\tau)dW(\tau)]ds \\ &- \sum_{k=1}^m (t-t_k)^{q-1}T_q(T-t_k)I_k(x(t_k)) \end{aligned}$$

9.2 Approximate Controllability

Now $\forall\ \alpha > 0$, we define an operator $F_\alpha(x,u) = (z,v)$ on $PC(J_0, L^2(\Omega, \mathcal{F}_t, X)) \times C(J,U)$ where

$$\begin{aligned} v(t) &= (T-t)^{q-1}B^*T_q^*(T-t)R(\alpha,\Gamma_0^T)p(x,u) \qquad (9.2.1)\\ z(t) &= S_q(t)(x_0+g(0,\phi)) - g(t,x_t) + \sum_{0<t_k<t}(t-t_k)^{q-1}T_q(t-t_k)I_k(x(t_k))\\ &+ \int_0^t (t-s)^{q-1}T_q(t-s)[f(s,x(a(x(s),s)),u)+Bv(s)]ds\\ &+ \int_0^t (t-s)^{q-1}T_q(t-s)[\int_{-\infty}^s G(s,\tau,x_\tau)dW(\tau)]ds \qquad (9.2.2)\end{aligned}$$

$$\begin{aligned} p(x,u) &= Ex_T + \int_0^T w(s)dW(s) - S_q(T)(x_0+g(0,\phi)) + g(T,x_T)\\ &- \int_0^T (T-s)^{q-1}T_q(T-s)f(s,x(h(x(s),s)),u)ds\\ &- \int_0^T (T-s)^{q-1}T_q(T-s)[\int_{-\infty}^s G(s,\tau,x_\tau)dW(\tau)]ds\\ &- \sum_{k=1}^m (T-t_k)^{q-1}T_q(T-t_k)I_k(x(t_k)) \qquad (9.2.3)\end{aligned}$$

It will be shown that (9.1.1) is approximately controllable if for all $\alpha > 0$ there exists a fixed point of the operator F_α, which is the mild solution of (9.1.1).

Theorem 9.2.1. There exists a fixed point of the operator F_α i.e. $\exists$ a mild solution of (9.1.1) on J, if the hypotheses $(H1)-(H9)$ are satisfied and

$$\begin{aligned} (\frac{1}{\alpha^2}(\frac{qM^2}{\Gamma(q+1)})^2M_1^2 &+ 1)6[4M_gl^2 + \frac{MT^q}{\Gamma(q+1)}\{\sum_{i=1}^m \|\lambda_i\|_1\Xi\}\\ &+ 2(\frac{MT^q}{\Gamma(q+1)})^2 2L_G \sup_{s\in J} n(s)\Theta] < 1 \qquad (9.2.4)\end{aligned}$$

Proof: Let $Y_{r_0} = \{\{x(.),u(.)\} \in PC(J_0, L^2(\Omega,\mathcal{F}_t,X)) \times C(J\times U) : E\|x\|^2 + E\|u(t)\|^2 \le r_0\}$ and r_0 is a positive constant. Thus, Y_{r_0} is a closed convex subset of a Banach space $PC(J_1, L^2(\Omega,\mathcal{F}_t,X)) \times C(J\times U)$.

Step1: For $0 < \alpha \le 1$, there is a positive constant $r_0 = r_0(\alpha)$ such that $F_\alpha : Y_{r_0} \to Y_{r_0}$.

$$(F_\alpha(x,u))(t) = (z,v) \qquad (9.2.5)$$

$$\begin{aligned}
v(t) &= (T-t)^{q-1}B^*T_q^*(T-t)R(\alpha,\Gamma_0^T)\\
&\times [Ex_T + \int_0^T w(s)dW(s) - S_q(T)(x_0+g(0,\phi)) + g(T,x_T)\\
&- \int_0^T (T-s)^{q-1}T_q(T-s)f(s,x(h(x(s),s)),u)ds\\
&- \int_0^T (T-s)^{q-1}T_q(T-s)[\int_{-\infty}^s G(s,\tau,x_\tau)dW(\tau)]ds\\
&- \sum_{k=1}^m (T-t_k)^{q-1}T_q(T-t_k)I_k(x(t_k))] \qquad (9.2.6)
\end{aligned}$$

$$z(t) = \begin{cases} \phi, \quad t\in(-\infty,0];\\ S_q(t)(x_0+g(0,\phi)) - g(t,x_t)\\ + \sum_{0<t_k<t}(t-t_k)^{q-1}T_q(t-t_k)I_k(x(t_k))\\ + \int_0^t (t-s)^{q-1}T_q(t-s)[f(s,x(a(x(s),s)),u)+Bv(s)]ds\\ + \int_0^t (t-s)^{q-1}T_q(t-s)[\int_{-\infty}^s G(s,\tau,x_\tau)dW(\tau)]ds, \quad t\in J. \end{cases}$$

Let $\Phi = \begin{cases} \phi, & t\le 0;\\ S(t)\phi(0), & t\ge 0. \end{cases}$

Therefore $x(t) = \widetilde{x}(t) + \Phi(t), \quad t\in(-\infty,T]$. where $\widetilde{x}=0, \quad t\le 0$ and for $t>0$,

$$\begin{aligned}
\widetilde{x}(t) &= S_q(t)(g(0,\phi)) - g(t,\widetilde{x}_t+\Phi_t)\\
&+ \sum_{0<t_k<t}(t-t_k)^{q-1}T_q(t-t_k)I_k((\widetilde{x}+\Phi)(t_k))\\
&+ \int_0^t (t-s)^{q-1}T_q(t-s)[f(s,(\widetilde{x}+\Phi)(a(\widetilde{x}+\Phi(s),s)),u)+Bv(s)]ds\\
&+ \int_0^t (t-s)^{q-1}T_q(t-s)[\int_{-\infty}^s G(s,\tau,\widetilde{x}_\tau+\Phi_\tau)dW(\tau)]ds \qquad (9.2.7)
\end{aligned}$$

So let $Y_{r_0}^0 = \{\{\widetilde{x}(.),u(.)\}\in PC(J_0,L^2(\Omega,\mathcal{F}_t,X))\times C(J\times U): E\|\widetilde{x}\|^2+E\|u(t)\|^2\le r_0 \;\; and \;\; \widetilde{x}_0=0\}$. Thus $Y_{r_0}^0$ is a bounded closed convex set.

$$\begin{aligned}
\|x_t\|_{\mathfrak{B}}^2 &= \|\widetilde{x}_t+\Phi_t\|_{\mathfrak{B}}^2\\
&\le 2(\|\widetilde{x}_t\|_{\mathfrak{B}}^2+\|\Phi_t\|_{\mathfrak{B}}^2)\\
&\le 4(l^2\sup_{s\in[0,t]}E\|\widetilde{x}(s)\|^2+\|\widetilde{x}_0\|_{\mathfrak{B}}^2+l^2\sup_{s\in[0,t)}E\|\Phi(s)\|^2+\|\Phi_0\|_{\mathfrak{B}}^2)\\
&\le 4(l^2r_0+l^2M^2E\|\Phi(0)\|^2+\|\Phi\|_{\mathfrak{B}}^2) \qquad (9.2.8)
\end{aligned}$$

$\mu_i(r) = sup\{g_i(x(a(x(s),s)),v): E\|x\|^2+E\|v\|^2\le r,$
$\forall(x,v)\in PC(J_0,L^2(\Omega,\mathcal{F}_t,X))\times U\}$.

If $(x,u) \in Y_{r_0}$ then

$$\begin{aligned}
E\|v(t)\|^2 &\leq \frac{6}{\alpha^2}(\frac{qM^2}{\Gamma(q+1)})^2 M_1^2[2E\|x_T\|^2 + 2\int_0^T E\|w(s)ds\|^2 \\
&+ M^2\|\phi\|_{\mathfrak{B}}^2 + M_g M^2(1+\|\phi\|^2) \\
&+ M_g(1+4(l^2 r_0 + l^2 M^2 E\|\phi(0)\|^2 + \|\phi\|_{\mathfrak{B}}^2) \\
&+ \frac{MT^q}{\Gamma(q+1)}\{\sum_{i=1}^m \lambda_i(s) g_i(x(a(x(s),s)),u)\} + \sum_{0\leq t\leq T}\frac{MT^{q-1}}{\Gamma(q+1)}d_k \\
&+ 2(\frac{MT^q}{\Gamma(q+1)})^2\{2M_k \\
&+ 2L_G \sup_{s\in J} n(s)\psi(l^2 r_0 + l^2 M^2 E\|\phi(0)\|^2 + \|\phi\|_{\mathfrak{B}}^2)\}] \\
&\leq \frac{6}{\alpha^2}(\frac{qM^2}{\Gamma(q+1)})^2 M_1^2[2E\|x_T\|^2 + 2\int_0^T E\|w(s)ds\|^2 \\
&+ M^2\|\phi\|_{\mathfrak{B}}^2 + M_g M^2(1+\|\phi\|^2) \\
&+ M_g(1+4(l^2 r_0 + l^2 M^2 E\|\phi(0)\|^2 + \|\phi\|_{\mathfrak{B}}^2)) \\
&+ \frac{MT^q}{\Gamma(q+1)}\{\sum_{i=1}^m \|\lambda_i\|_1 \mu_i(r_0)\} + \sum_{0\leq t\leq T}\frac{MT^{q-1}}{\Gamma(q+1)}d_k \\
&+ 2(\frac{MT^q}{\Gamma(q+1)})^2\{2M_k \\
&+ 2L_G \sup_{s\in J} n(s)\psi(l^2 r_0 + l^2 M^2 E\|\phi(0)\|^2 + \|\phi\|_{\mathfrak{B}}^2)\}] \\
&= M_c \qquad (9.2.9)
\end{aligned}$$

and

$$\begin{aligned}
E\|z(t)\|^2 &= 6[M^2 M_g(1+\|\phi\|_{\mathfrak{B}}^2) + M_g(1+4(l^2 r_0 + l^2 M^2 E\|\phi(0)\|^2 + \|\phi\|_{\mathfrak{B}}^2)) \\
&+ \frac{MT^q}{\Gamma(q+1)} M_1^2(M_c + \{\sum_{i=1}^m \|\lambda_i\|_1 \mu_i(r_0)\}) \\
&+ \sum_{0\leq t\leq T}\frac{MT^{q-1}}{\Gamma(q+1)}d_k \\
&+ 2(\frac{MT^q}{\Gamma(q+1)})^2\{2M_k \\
&+ 2L_G \sup_{s\in J} n(s)\psi(l^2 r_0 + l^2 M^2 E\|\phi(0)\|^2 + \|\phi\|_{\mathfrak{B}}^2)\}] \\
&= L_1 \qquad (9.2.10)
\end{aligned}$$

Dividing $M_c + L_1$ by r_0 and letting $r_0 \to \infty$, and by assumption (9.2.4) we get that

$$\begin{aligned}
\lim_{r_0\to\infty} \quad & (\frac{1}{\alpha^2}(\frac{qM^2}{\Gamma(q+1)})^2 M_1^2 + 1)6[4M_g l^2 + \frac{MT^q}{\Gamma(q+1)}\{\sum_{i=1}^{m}\|\lambda_i\|_1 \frac{\mu_i(r_0)}{r_0}\} \\
+ & \; 2(\frac{MT^q}{\Gamma(q+1)})^2 2L_G \sup_{s\in J} n(s)\frac{\psi(l^2 r_0)}{r_0}] \\
\leq & \; (\frac{1}{\alpha^2}(\frac{qM^2}{\Gamma(q+1)})^2 M_1^2 + 1)6[4M_g l^2 + \frac{MT^q}{\Gamma(q+1)}\{\sum_{i=1}^{m}\|\lambda_i\|_1 \Xi\} \\
+ & \; 2(\frac{MT^q}{\Gamma(q+1)})^2 2L_G \sup_{s\in J} n(s)\Theta] < 1 \qquad (9.2.11)
\end{aligned}$$

Hence

$$\|F^\alpha(x,u)(t)\| = E\|z(t)\|^2 + E\|v(t)\|^2 \leq L_1 + M_c \leq r_0 \leq \infty.$$

Therefore, F^α maps bounded sets of Y_{r_0} into bounded sets of Y_{r_0}.
Step 2 : As per Arzela-Ascoli theorem and step1 there is a need to prove :
(i) $\forall t \in J$ $V(t) = \{F^\alpha(x,u)(t) : (x,u) \in Y_{r_0}\}$ is pre-compact,
(ii) $\forall \epsilon > 0$ $\exists \delta > 0$: $\|F^\alpha(x,u)(t_1) - F^\alpha(x,u)(t_2)\|$
$< \epsilon$ if $(x,u) \in Y_{r_0}$, $|t_1 - t_2| \leq \delta$, for all $t_1, t_2 \in J$.
For $t = 0$ it is trivial, as $V(0) = \phi(0)$. Therefore fix a real number $0 < t \leq T$ and suppose $\tau \in \mathbb{R}$ is such that $0 < \tau < t$.
We explicity state the operator

$$\begin{aligned}
F_\tau^\alpha(x,u)(t) &= [F^{\epsilon,\delta}z, (T-t)^{q-1}B^* T_q^*(T-s)R(\alpha, \Gamma_0^T)p(x,u)] \\
(F^{\epsilon,\delta}z)t &= q\int_0^{t-\epsilon}\int_\delta^\infty \theta(t-s)^{q-1}\xi(\theta)S((t-s)^q\theta)Bu^\alpha(s)d\theta ds \\
&+ q\int_0^{t-\epsilon}\int_\delta^\infty \theta(t-s)^{q-1}\xi(\theta)S((t-s)^q\theta)f(s, x(a(x(s),s)), u)d\theta ds \\
&+ q\int_0^{t-\epsilon}\int_\delta^\infty \theta(t-s)^{q-1}\xi(\theta)S((t-s)^q\theta) \\
&\times \int_{-\infty}^{s}[G(s,\tau,x_\tau)dW(\tau)]d\theta ds \\
&+ \sum_{0<t_k<t}(t-t_k)^{q-1}q\int_\delta^\infty \theta\xi S((t-t_k)^q\delta)I_k(x(t_k^-))d\theta
\end{aligned}$$

$$
\begin{aligned}
&= qS(\epsilon^q\delta)\int_0^{t-\epsilon}\int_\delta^\infty \theta(t-s)^{q-1}\xi(\theta)S((t-s)^q\theta-\epsilon^q\delta)Bu^\alpha(s)d\theta ds \\
&+ qS(\epsilon^q\delta)\int_0^{t-\epsilon}\int_\delta^\infty \theta(t-s)^{q-1}\xi(\theta)S((t-s)^q\theta-\epsilon^q\delta) \\
&\times f(s,x(a(x(s),s)),u)d\theta ds \\
&+ qS(\epsilon^q\delta)\int_0^{t-\epsilon}\int_\delta^\infty \theta(t-s)^{q-1}\xi(\theta)S((t-s)^q\theta-\epsilon^q\delta) \\
&\times \int_{-\infty}^s [G(s,\tau,x_\tau)dW(\tau)]d\theta ds \\
&+ \sum_{0<t_k<t}(t-t_k)^{q-1}qS(\epsilon^q\delta)\int_\delta^\infty \theta\xi S((t-t_k)^q\theta-\epsilon^q\delta)I_k(x(t_k^-))d\theta \\
&+ \sum_{t-\epsilon<t_k<t}(t-t_k)^{q-1}qS(\epsilon^q\delta)\int_\delta^\infty \theta\xi S((t-t_k)^q\theta-\epsilon^q\delta)I_k(x(t_k^-))d\theta
\end{aligned} \tag{9.2.12}
$$

Since $S(\epsilon^q\delta)$ is compact and $z(t)$ is bounded on Y_{r_0} the set $(F^{\epsilon,\delta}z)(t)$ is pre-compact in X for all $\epsilon,\delta>0$. Also $T_q(t), S_q(t)$ is compact if $S(t)$ is compact, therefore the set $\{F^{\epsilon,\delta}z(t)+g(t,x_t)+S_q(t)g(0,\phi)=F^2(x,u)(t)\}$ is relatively compact. Let

$$\mathbf{F}_\tau^\alpha(x,u)(t)=[F^2(x,u)(t),(T-t)^{q-1}B^*T_q^*(T-s)R(\alpha,\Gamma_0^T)p(x,u)].$$

So

$$V_\tau(t)=\{\mathbf{F}_\tau^\alpha(x,u)(t):(x,u)\in Y_{r_0}\}$$

is pre-compact in X. i.e. $\exists\ \{y_i, 1\le i\le n\}$ in $PC\times U$ s.t.

$$V_\tau(t)\subset\cup_{i=1}^m B(y_i,\epsilon/2).$$

Here $B(y_i,\epsilon/2)$ is an open ball in $PC\times U$ having y_i as center and $\epsilon/2$ as radius. Also, $\forall\ (x,u)\in Y_{r_0}$ we have

$$
\begin{aligned}
&\| \quad (F^\alpha(x,u))(t)-(\mathbf{F}_\tau^\alpha(x,u))(t)\| \\
&\le 7q^2E\|\int_0^t\int_0^\delta \theta(t-s)^{q-1}\xi_q(\theta)S((t-s)^q\theta)Bu^\alpha(s)d\theta ds\|^2 \\
&+ 7q^2E\|\int_{t-\epsilon}^t\int_\delta^\infty \theta(t-s)^{q-1}\xi_q(\theta)S((t-s)^q\theta)Bu^\alpha(s)d\theta ds\|^2 \\
&+ 7q^2E\|\int_0^t\int_0^\delta \theta(t-s)^{q-1}\xi_q(\theta)S((t-s)^q\theta)f(s,x(a(x(s),s)),u(s))d\theta ds\|^2 \\
&+ 7q^2E\|\int_{t-\epsilon}^t\int_\delta^\infty \theta(t-s)^{q-1}\xi_q(\theta)S((t-s)^q\theta)f(s,x(a(x(s),s)),u(s))d\theta ds\|^2
\end{aligned}
$$

$$
\begin{aligned}
&+ \; 7q^2E\|\int_0^t\int_0^\delta \theta(t-s)^{q-1}\xi_q(\theta)S((t-s)^q\theta)\int_{-\infty}^s G(s,\tau,x_\tau)dW(\tau)d\theta ds\|^2 \\
&+ \; 7q^2E\|\int_{t-\epsilon}^t\int_\delta^\infty \theta(t-s)^{q-1}\xi_q(\theta)S((t-s)^q\theta)\int_{-\infty}^s G(s,\tau,x_\tau)dW(\tau)d\theta ds\|^2 \\
&+ \; 7q^2E\|\sum_{0\le t_k\le t}\int_0^\delta \theta(t-s)^{q-1}\xi_q(\theta)S((t-s)^q\theta)I(x_k(t_k))d\theta ds\|^2 \\
&+ \; 7q^2E\|\sum_{t-\epsilon\le t_k\le t}\int_\delta^\infty \theta(t-s)^{q-1}\xi_q(\theta)S((t-s)^q\theta)I(x_k(t_k))d\theta ds\|^2 \\
&\le \; \{7q^2M^2T^q\int_0^t (t-s)^{q-1}[M_B^2M_C+\sum_{i=1}^m\|\lambda_i\|\mu(r_0)+2M_k \\
&+ \; 2L_G\sup_{s\in J} n(s)\psi(l^2r_0+l^2M^2E\|\phi(0)\|^2+\|\phi\|_{\mathfrak{B}}^2)]ds \\
&+ \; 7q^2M^2md_k\}(\int_0^\delta \theta\xi(\theta)d\theta)^2 \\
&\le \; \frac{7qM^2\epsilon^2}{\Gamma^2(q+1)}\int_{t-\epsilon}^t (t-s)^{q-1}[M_B^2M_C+\sum_{i=1}^m\|\lambda_i\|\mu(r_0)+2M_k \\
&+ \; 2L_G\sup_{s\in J} n(s)\psi(l^2r_0+l^2M^2E\|\phi(0)\|^2+\|\phi\|_{\mathfrak{B}}^2)]ds+7q62M^2nd_k\to 0 \\
&\quad as \;\; \epsilon,\delta\to 0^+
\end{aligned}
\tag{9.2.13}
$$

Thus there exists relatively compact sets arbitrarily close to the set $V(t)$, $\forall\, t\in(0,T]$, Hence for each $t\in[0,T]$, $V(t)$ is relatively compact in $C\times U$.

Step 3: We prove the equicontinuity of $V=\{F^\alpha(x,u)(.)|(x,u)\in Y_{r_0}\}$ on $[0,T]$. When $0<t_a<t_b\le T$

$$
\begin{aligned}
E\|v(t_a)-v(t_b)\|^2 \le\; & \|(T-t_a)^{q-1}B^*T_q^*(T-t_a)-(T-t_b)^{q-1}B^*T_q^*(T-t_b)\| \\
& \times\frac{1}{\alpha^2}[2E\|x_T\|^2+2\int_0^t E\|w(s)ds\|^2+M^2\|\phi\|_{\mathfrak{B}} \\
& +M_gM^2(1+\|\phi\|^2) \\
& +M_g(1+4(l^2r_0+l^2M^2E\|\phi(0)\|^2+\|\phi\|_{\mathfrak{B}}^2)) \\
& +\frac{MT^q}{\Gamma(q+1)}\{\sum_{i=1}^m\|\lambda_i\|_1\mu_i(r_0)\}+\sum_{0\le t\le T}\frac{MT^{q-1}}{\Gamma(q+1)}d_k \\
& +2(\frac{MT^q}{\Gamma(q+1)})^2\{2M_k \\
& +2L_G\sup_{s\in J} n(s)\psi(l^2r_0+l^2M^2E\|\phi(0)\|^2+\|\phi\|_{\mathfrak{B}}^2)\}]
\end{aligned}
\tag{9.2.14}
$$

and

$$E \quad \|z(t_b) - z(t_a)\|^2 \leq 13E\| \int_0^{t_a} (t_a - s)^{q-1}[T_q(t_b - s) - T_q(t_a - s)]Bu(s)ds\|^2$$
$$+ \quad 13E\| \int_0^{t_a} [(t_b - s)^{q-1} - (t_a - s)^{q-1}]T_q(t_b - s)Bu(s)ds\|^2$$
$$+ \quad 13E\| \int_{t_a}^{t_b} (t_b - s)^{q-1}T_q(t_b - s)Bu(s)ds\|^2$$
$$+ \quad 13E\| \int_0^{t_a} (t_a - s)^{q-1}[T_q(t_b - s) - T_q(t_a - s)]f(s, x(a(x(s), s)), u)ds\|^2$$
$$+ \quad 13E\| \int_0^{t_a} [(t_b - s)^{q-1} - (t_a - s)^{q-1}]T_q(t_b - s)f(s, x(a(x(s), s)), u)ds\|^2$$
$$+ \quad 13E\| \int_{t_a}^{t_b} (t_b - s)^{q-1}T_q(t_b - s)f(s, x(a(x(s), s)), u)ds\|^2$$
$$+ \quad 13E\| \int_0^{t_a} (t_a - s)^{q-1}[T_q(t_b - s) - T_q(t_a - s)][\int_{\infty}^{s} G(s, \tau, x_\tau)dw(\tau)]ds\|^2$$
$$+ \quad 13E\| \int_0^{t_a} [(t_b - s)^{q-1} - (t_a - s)^{q-1}]T_q(t_b - s)[\int_{\infty}^{s} G(s, \tau, x_\tau)dw(\tau)]ds\|^2$$
$$+ \quad 13E\| \int_{t_a}^{t_b} (t_b - s)^{q-1}T_q(t_b - s)[\int_{\infty}^{s} G(s, \tau, x_\tau)dw(\tau)]ds\|^2$$
$$+ \quad 13E\| \sum_{0<t_k<t_a} (t_a - t_k)^{q-1}(T_q(t_b - t_k) - T_q(t_a - t_k))I(x(t_k))\|^2$$
$$+ \quad 13E\| \sum_{t_a<t_k<t_b} (t_b - t_k)^{q-1}T_q(t_b - t_k)I(x(t_k))\|^2$$
$$+ \quad 13E\| \sum_{0<t_k<t_a} ((t_b - t_k)^{q-1} - (t_a - t_k)^{q-1})T_q(t_b - t_k)I(x(t_k))\|^2$$
$$+ \quad 13E\|g(t_a, x_{t_a}) - g(t_b, x_{t_b})\|^2 \tag{9.2.15}$$

Therefore,

$$E \quad \|z(t_b) - z(t_a)\|^2 \leq 13\epsilon^2 M_B^2 \frac{t_1^q}{q} \int_0^{t_a} (t_a - s)q - 1E\|u(s)\|^2 ds$$
$$+ \quad 13(\frac{qMM_B}{\Gamma(q+1)})^2 \int_0^{t_a} [(t_b - s)^{q-1} - (t_a - s)^{q-1}]ds$$
$$\times \quad \int_0^{t_a} [(t_b - s)^{q-1} - (t_a - s)^{q-1}]E\|u(s)\|^2 ds$$
$$+ \quad 13(\frac{qMM_B}{\Gamma(q+1)})^2 \frac{(t_b - t_a)^q}{q} \int_{t_a}^{t_b} (t_b - s)^{q-1}E\|u(s)\|^2 ds$$
$$+ \quad 13\epsilon^2 \frac{t_1^q}{q} \int_0^{t_a} (t_a - s)^{q-1}\{\sum_{i=1}^{m} \|\lambda_i\|_1 \mu_i(r_0)\}ds$$

$$
\begin{aligned}
&+ \quad 13(\frac{qM}{\Gamma(q+1)})^2 \int_0^{t_a} [(t_b - s)^{q-1} - (t_a - s)^{q-1}] ds \\
&\times \quad \int_0^{t_a} [(t_b - s)^{q-1} - (t_a - s)^{q-1}] \{\sum_{i=1}^{m} \|\lambda_i\|_1 \mu_i(r_0)\} ds \\
&+ \quad 13(\frac{qM}{\Gamma(q+1)})^2 \frac{(t_b - t_a)^q}{q} \int_{t_a}^{t_b} (t_b - s)^{q-1} \{\sum_{i=1}^{m} \|\lambda_i\|_1 \mu_i(r_0)\} ds \\
&+ \quad 13\epsilon^2 \frac{t_1^q}{q} \int_0^{t_a} (t_a - s)^{q-1} 2\{2M_k \\
&+ \quad 2L_G \sup_{s \in J} n(s) \psi(l^2 r_0 + l^2 M^2 E\|\phi(0)\|^2 + \|\phi\|_{\mathfrak{B}}^2)\} ds \\
&+ \quad 13(\frac{qM}{\Gamma(q+1)})^2 \int_0^{t_a} [(t_b - s)^{q-1} - (t_a - s)^{q-1}] ds \\
&\times \quad \int_0^{t_a} [(t_b - s)^{q-1} - (t_a - s)^{q-1}] 2\{2M_k \\
&+ \quad 2L_G \sup_{s \in J} n(s) \psi(l^2 r_0 + l^2 M^2 E\|\phi(0)\|^2 + \|\phi\|_{\mathfrak{B}}^2)\} ds \\
&+ \quad 13(\frac{qM}{\Gamma(q+1)})^2 \frac{(t_b - t_a)^q}{q} \int_{t_a}^{t_b} (t_b - s)^{q-1} 2\{2M_k \\
&+ \quad 2L_G \sup_{s \in J} n(s) \psi(l^2 r_0 + l^2 M^2 E\|\phi(0)\|^2 + \|\phi\|_{\mathfrak{B}}^2)\} ds \\
&+ \quad 13 \frac{t_a^{q-1}}{q-1} \sum_{0 < t_k < t_a} E\|(T_q^*(t_b - t_k) - T_q^*(t_a - t_k) I_(x(t_k))\|^2 \\
&+ \quad 13(\sum_{0 < t_k < t_a} E\|T_q(t_b - t_a) I(x(t_k))\|^2) \\
&\times \quad (\sum_{0 < t_k < t_a} (t_b - t_k)^{q-1} - (t_a - t_k)^{q-1}) \\
&+ \quad 13 \frac{(t_b - t_a)^q}{q+1} (\frac{Mq}{\Gamma(1+q)})^2 \sum_{t_a < t_k < t_b} E\|T_q(t_b - t_k) I(x(t_k))\|^2 \\
&+ \quad 13 L_g \|x_{t_a} - x_{t_b}\|^2 \qquad (9.2.16)
\end{aligned}
$$

Thus RHS is independent of choice of (x, u). This follows from strong continuity and compactness of T_q and by Lebesgue Dominated Convergence theorem $F^\alpha \to 0$ as $t_a - t_b \to 0$. So $F^\alpha[Y_{r_0}]$ is equicontinuous and bounded. So equicontinuity of V is shown. By Arzela-Ascoli, $F^\alpha[Y_{r_0}]$ is relatively compact.

Theorem 9.2.2. Suppose that the hypotheses of the previous theorem (9.2.1) are satisfied and f, G, g are uniformly bounded then (9.1.1) is approximately controllable on $[0, T]$.

Proof: Suppose x^α be a fixed point of F^α in Y_r. Applying Stochastic Fubini's theorem, we get

$$x^\alpha(T) = x_T - \alpha R(\alpha, \Gamma_0^T)p(x, u).$$

By using the property that f, g, G and a are uniformly bounded we get that there exists $L_{fG}, M_{gu} > 0$ such that

$$\|\{f(s, x^\alpha(a(x^\alpha(s), s)), u^\alpha(s)) + \int_{-\infty}^{s} G(s, \tau, x_\tau^\alpha)dW(\tau)\}\|_{L_2^0}^2 \leq L_{fG}^2$$

and $\|g(s, x_s^\alpha)\| \leq M_{gu}$.

Therefore there exists subsequences,

$$\{f(s, x^\alpha(a(x^\alpha(s), s)), u^\alpha(s)), g(s, x_s^\alpha), G(s, \tau, x_\tau^\alpha)\}$$

which converge weakly to

$$\{f(s, x(a(x(s), s)), u(s)), g(s, x_s), G(s, \tau, x_\tau)\}.$$

Since a is continuous so, $a(x^\alpha(s), s) \to a(x(s), s)$ as $x^\alpha(s) \to x(s)$.

From the above equation we get

$$\begin{aligned}
E\|x^\alpha(T) &- x_T\|^2 \leq 8\|\alpha(\alpha I + \Gamma_0^T)^{-1}[Ex_T - S_q(T)(\phi(0) + g(0, \phi(0)))]\|^2 \\
&+ 8E(\int_0^T \|\alpha(\alpha I + \Gamma_0^T)^{-1}w(s)\|_{L_2^0}^2 ds) + 8E\|\alpha(\alpha I + \Gamma_0^T)^{-1}g(T, x_T^\alpha)\|^2 \\
&+ 8E(\int_0^T (T-s)^{q-1}\|\alpha(\alpha I + \Gamma_0^T)^{-1}T_q(T-s)[f(s, x_s^\alpha) - f(s)]\|ds)^2 \\
&+ 8E(\int_0^T (T-s)^{q-1}\|\alpha(\alpha I + \Gamma_0^T)^{-1}T_q(T-s)f(s)\|ds)^2 \\
&+ 8E(\int_0^T (T-s)^{q-1}\|\alpha(\alpha I + \Gamma_0^T)^{-1}T_q(T-s) \\
&\times [\int_{-\infty}^{s} [[G(s, \tau, x_\tau^\alpha) - G(s, \tau, x_\tau)]dW(\tau)]\|ds)^2 \\
&+ 8E(\int_0^T (T-s)^{q-1}\|\alpha(\alpha I + \Gamma_0^T)^{-1}T_q(T-s) \\
&\times \int_{-\infty}^{s} G(s, \tau, x_\tau)dW(\tau))\|ds)^2 \\
&+ 8E\sum_{0<t_k<t}(t-t_k)^{q-1}T_q(t-t_k)I_k(x(t_k)) \to 0 \qquad (9.2.17)
\end{aligned}$$

as $\alpha^+ \to 0$. This is due to the fact that $T_q(t)$ is compact and also due to the theorem of Lebesgue Dominated Convergence. Hence the approximate controllability of (9.1.1) is proved.

9.3 Example

Let us consider the following controlled neutral system with impulses

$$\begin{aligned}
{}^cD^q[x(t,\xi)-\zeta(t,x(t-h,\xi))] &= \frac{\partial^2}{\partial\xi^2}[x(t,\xi)-\zeta(t,x(t-h,\xi))]+u(t,\xi)\\
&+ f(t,x(a(x(t,\xi),t),\xi),u(t,\xi))\\
&+ \int_{-\infty}^{t} G(t,s,x_s)dw(s)\ 0<y<1\\
x(t_k^+,\xi)-x(t_k^-,\xi) &= \overline{I}_k(x(t_k^-,\xi)),\ k=1,\cdots,m.\\
x(t,0) &= x(t,1)=0,\ t>0\\
x(t,\xi) &= \phi(t,\xi),\ -h\le t\le 0; \qquad (9.3.1)
\end{aligned}$$

Here ϕ is continuous. Also $I_k \in C(\mathbb{R},\mathbb{R})$.
Let $g(t,x_t)(\xi)=\zeta(t,x(t-h,\xi))$,

$$F(t,x(a(x(t),t)),u(t))(\xi)=f(t,x(a(x(t,\xi),t),\xi),u(t,\xi))$$

and $(Bu)(t)(\xi)=u(t,\xi)$. Taking $X=L^2(0,1)$ and we define $A:X\to X$ by $Ax=\frac{d^2x}{d\xi^2}$ where domain of A is

$$\begin{aligned}
D(A)=\{x\ \in\ X,\ x,\frac{dx}{d\xi}\ &\textit{are absolutely continuous},\\
&\frac{d^2x}{d\xi^2}\in X,\frac{dx}{dy}(0)=\frac{dx}{dy}(1)=0\} \qquad (9.3.2)
\end{aligned}$$

Then $Ax=\sum_{n=1}^{\infty}(-n^2\pi^2)<x,e_n>e_n,\ x\in D(A)$.
where $e_n(\theta)=\sqrt{2}\cos(n\pi\theta)\ 0<x<1,\ n=1,2,...$
The operator A generates a compact semigroup

$$\begin{aligned}
S(t)x &= \sum_{n=1}^{\infty}2e^{-n^2\pi^2t}\cos(n\pi\xi)\int_0^1\cos(n\pi\xi)x(\psi)d\psi\\
&+ \int_0^1 x(\psi)d\psi,\ \ x\in X \qquad (9.3.3)
\end{aligned}$$

Further, the functions f,ζ are continuous and there exists constants k_1,k_2 such that $f(t,x(a(x(t,\xi),t),\xi),u(t,\xi))\le k_1$, $\zeta(t,x(t-h,\xi))\le k_2$ and there exists constants d_k such that $\|I_k(x)\|\le d_k$.
Hence (9.3.1) can be expressed as (9.1.1). Since the associated linear system of the (9.3.1) is approximately controllable, and theorem 9.2.2, the approximate controllability of (9.3.1) is guaranteed.

9.4 Conclusion

We proved the approximate controllability of an impulsive stochastic fractional differential equation. We substituted the use of Lipschitz continuity of the nonlinear function and the inverse of the controllability operator with simple assumptions on systems operator. Thereby we removed the problem of nonexistence of the inverse of the controllability operator in case of compactness of the generated semigroup.

Chapter 10

Approximation of Solutions of a Stochastic Fractional Differential Equation with Deviated Argument

In this chapter the existence, uniqueness and convergence of approximate solutions of a stochastic fractional differential equation with deviated argument is studied by using analytic semigroup theory and fixed point method. Then we considered Faedo-Galerkin approximation of solution and proved some convergence results.

10.1 Introduction

The approximation of solution to a nonlinear Sobolev type evolution equation was studied by Bahuguna and Shukla [31] in a separable Hilbert space $(H, \|.\|, (.,.))$. The Faedo-Galerkin approximations of solution of a determistic problem was considered by Milleta [133]. The more general case was dealt by D. Bahuguna, S.K. Srivastava and S. Singh [32].

By far the Faedo-Galerkin approximation of solution stochastic fractional differential equation with deviated argument is neglected in literature. In an attempt to fill this gap we study the following stochastic fractional differential equation with deviated argument in a separable Hilbert space $(H, (.,.))$.

$$\begin{aligned} {}^cD_t^\beta u(t) + Au(t) &= f(t, u(t), u(h(u(t), t)))\frac{dw(t)}{dt}, \quad t \in [0, T] \\ u(0) &= u_0 \in H \end{aligned} \tag{10.1.1}$$

where $0 < \beta < 1$ and $0 < T < \infty$. ${}^cD_t^\beta$ denotes the Caputo fractional derivative of order β and $A : D(A) \subset H \to H$ is a linear operator. A and the functions f, h are defined in the hypotheses $(H1) - (H3)$.

Here we deal with two separable Hilbert spaces H and K. We assume

(H1) A is a closed, densely defined, self adjoint operator with pure point spectrum $0 \le \lambda_0 \le \lambda_1 \le \cdots \le \lambda_m \le \cdots$ with $\lambda_m \to \infty$ and $m \to \infty$ and corresponding complete orthonormal system of eigenfunctions ϕ_j such that

$$A\phi_j = \lambda_j \phi_j \ and \ < \phi_i, \phi_j >= \delta_{i,j}$$

If $(H1)$ is satisfied then $-A$ is the infinitesimal generator of an analytic semigroup $\{e^{-tA} : t \ge 0\}$ in H. We also note that $\exists$ constant C such that $\|S(t)\| \le Ce^{\omega t}$ and constants C_i's such that $\|\frac{d^i}{dt^i}S(t)\| \le C_i, \ t > 0, \ i = 1, 2$. Also $\|AS(t)\| \le Ct^{-1}$ and $\|A^\alpha S(t)\| \le C_\alpha t^{-\alpha}$.

We define the space H_α as $D(A^\alpha)$ endowed with the norm $\|.\|_\alpha$. Let $(\Omega, \mathfrak{F}, P)$ be a complete probability space endowed with complete family of right continuous increasing sub $\sigma-$ algebras $\{\mathfrak{F}_t, t \in J\}$ such that $\mathfrak{F}_t \subset \mathfrak{F}$. A $H-$ valued random variable is a $\mathcal{F}-$ measurable process.We also assume that W is a Wiener process on K with covariance operator Q. Suppose Q is symmetric, positive, linear, and bounded operator with $TrQ < \infty$. Let $K_0 = Q^{\frac{1}{2}}(K)$. The space $L_0^2 = L^2(K_0, H_\alpha)$ is a separable Hilbert space with norm $\|\psi\|_{L_0^2} = \|\psi Q^{\frac{1}{2}}\|_{L^2(K, H_\alpha)}$. Let $L^2(\Omega, \mathfrak{F}, P; H_\alpha) \equiv L^2(\Omega; H_\alpha)$ be the Banach space of all strongly measurable, square integrable, $H_\alpha-$valued random variables equipped with the norm $\|u(.)\|_{L^2}^2 = E\|u(.; w)\|_{H_\alpha}^2$. C_T^α denotes the Banach space of all continuous maps from $J = (0, T]$ into $L^2(\Omega; H_\alpha)$ which satisfy $sup_{t \in J} E\|u(t)\|_{C^\alpha}^2 < \infty$. $L_0^2(\Omega, H_\alpha) = \{f \in L^2(\Omega, H_\alpha) : f \ is \ \mathcal{F}_0 - measurable\}$ denotes an important subspace. For $0 \le \alpha < 1$ define

$$C_T^{\alpha-1} = \{u \in C_T^\alpha : \|u(t) - u(s)\|_{\alpha-1} \le L|t - s|, \forall t, s \in [0, T]\}.$$

We also assume the following hypotheses

(H2) The function $f : [O,T] \times H_\alpha \times H_{\alpha-1} \to L(K,H)$ is continuous and there exists constant a $L_f > 0$ such that
$\|f(s,u,u_1) - f(s,v,v_1)\|_Q^2 \le L_f[\|t-s\|^{\theta_1} + \|u-v\|_\alpha + \|u_1 - v_1\|_{\alpha-1}]$

(H3) The map $h : H_\alpha \times \mathcal{R}_+ \to \mathcal{R}_+$ satisfies $\|h(u,t) - h(v,s)\| \le L_h(\|u-v\|_\alpha + |t-s|^{\theta_2})$

Now let us define mild solution of (10.1.1) :

Definition 10.1.1. The mild solution of (10.1.1) is a continuous $\mathfrak{F}_t$ adapted stochastic process $u \in C_T^\alpha \cap C_T^{\alpha-1}$ which satisfies the following:

1. $u(t) \in H_\alpha$ has *Càdlàg* paths on $t \in [0,T]$.
2. $\forall t \in [0,T]$, $u(t)$ is the solution of the integral equation

$$u(t) = T_\beta(t)u_0 + \int_0^t (t-s)^{\beta-1} S_\beta(t-s) f(s, u(s), u(h(u(s),s)))dw(s), \; t \in [0,T] \tag{10.1.2}$$

where $S_\beta(t) = \int_0^\infty \zeta_\beta(\theta)S(t^\beta\theta)d\theta$; and $T_\beta(t) = q\int_0^\infty \theta\zeta_\beta(\theta)S(t^\beta\theta)d\theta$; ζ_β is a probability density function defined on $(0,\infty)$, i.e. $\zeta_\beta(\theta) \ge 0$, $\theta \in (0,\infty)$ and $\int_0^\infty \zeta_\beta(\theta)d\theta = 1$. Also $\|T_\beta(t)u\| \le C\|u\|$, $\|S_\beta(t)u\| \le \frac{\beta C}{\Gamma(1+\beta)}\|u\|$, $\|A^\alpha S_\beta(t)u\| \le \frac{\beta C_\alpha \Gamma(2-\alpha)}{\Gamma(1+\beta(1-\alpha))} t^{-\alpha\beta}\|u\|$.

Lemma 10.1.2. [147] Let $f : J \times \Omega \times \Omega \to L_0^2$ be a strongly measurable mapping with $\int_0^T E\|f(t)\|_{L_0^2}^p dt < \infty$. Then

$$E\|\int_0^t f(s)dw(s)\|^p \le l_s \int_0^t E\|f(s)\|_{L_0^2}^p ds$$

$\forall t \in [0,T]$ and $p \ge 2$ where l_s is a constant containing p and T.

l_s is incorporated into the constants in the following sections.

10.1.1 Existence and Uniqueness of Approximate Solutions

In this section we consider a sequence of approximate integrals and establish the existence and uniqueness of solution for each of the approximate integral equations. For $0 \le \alpha < 1$ and $u \in C_{T_0}^\alpha$, the hypotheses $(H2) - (H3)$, imply that $f(s,u(s),u(h(u(s),s)))$ is continuous on $[0,T_0]$. Therefore there exists a positive constant

$$N = 2L_f[T_0^{\theta_1} + 2R(1+LL_h) + LL_hT_0^{\theta_2}] + 2N_0, \quad N_0 = E\|f(0,u_0,u_0)\|^2$$

such that $\|f(s,u(s),u(h(u(s),s)))\| \le N, \quad t \in [0,T]$. Choose T_0, $0 < T_0 \le T$ such that

$$(\frac{\beta C_\alpha \Gamma(2-\alpha)}{\Gamma(1+\beta(1-\alpha))})^2 N \frac{T_0^{\beta(1-\alpha)-1}}{\beta(1-\alpha)-1} \le \frac{R}{4},$$

$$D = (\frac{\beta C_\alpha \Gamma(2-\alpha)}{\Gamma(1+\beta(1-\alpha))})^2 2L_f \frac{T_0^{\beta(1-\alpha)-1}}{2\beta(1-\alpha)-1} \le 1 \qquad (10.1.3)$$

Let

$$B_R = \{u \in \mathcal{C}_{T_0}^{\alpha} \cap \mathcal{C}_{T_0}^{\alpha-1} : u(0) = u_0, \quad \|u - u_0\|_{T_0,\alpha} \le R\}$$

It is easy to see that B_R is a closed and bounded subset of $\mathcal{C}_{T_0}^{\alpha-1}$ and complete. Let us define the operator $\mathcal{F}_n : B_R :\to B_R$ by

$$(\mathcal{F}_n u)(t) = T_\beta(t)u_0 + \int_0^t (t-s)^{\beta-1} S_\beta(t-s) f_n(s,u(s),u(h(u(s),s)))dw(s). \quad (10.1.4)$$

Theorem 10.1.3. If the hypotheses $(H1), (H2)$ *and* $(H3)$ are satisfied and $u_0 \in L_0^2(\Omega, X_\alpha)$, $0 \le \alpha < 1$, then $\exists$ a unique $u_n \in B_R$ such that $\mathcal{F}_n u_n = u_n$, $\forall\, n = 0,1,2,\cdots$, i.e., u_n satisfies the approximate integral equation

$$u_n(t) = T_\beta(t)u_0 + \int_0^t (t-s)^{\beta-1} S_\beta(t-s) f_n(s,u_n(s),u_n(h(u_n(s),s)))dw(s),$$
$$t \in [0,T] \qquad (10.1.5)$$

Proof: Step1 : We need to show that $\mathcal{F}_n u \in \mathcal{C}_{T_0}^{\alpha-1}$, $\forall u \in \mathcal{C}_{T_0}^{\alpha-1}$. It is easy to check that $\mathcal{F}_n : \mathcal{C}_T^{\alpha} \to \mathcal{C}_T^{\alpha}$. If $u \in \mathcal{C}_{T_0}^{\alpha-1}$, $0 < t_1 < t_2 < T_0$ and $0 \le \alpha < 1$ then

$$\begin{aligned}
&E\|\mathcal{F}_n u(t_2) - \mathcal{F}_n u(t_1)\|_{\alpha-1}^2 \\
&\le 3E\|[T_\beta(t_2) - T_\beta(t_1)]u_0\|_{\alpha-1}^2 \\
&\quad + 3E\|\int_{t_1}^{t_2} (t_2-s)^{\beta-1} A^{\alpha-1} S_\beta(t_2-s) f_n(s,u(s),u(h(u(s),s)))dw(s)\|_Q^2 \\
&\quad + 3E\|\int_0^{t_1} A[(t_2-s)^{\beta-1} S_\beta(t_2-s) - (t_1-s)^{\beta-1} S_\beta(t_1-s)] \\
&\quad A^{\alpha-2} \times f_n(s,u(s),u(h(u(s),s)))dw(s)\|_Q \\
&\le 3E\|[T_\beta(t_2) - T_\beta(t_1)]u_0\|_{\alpha-1}^2 + 3\frac{\beta^2 C_\alpha^2 \Gamma^2(2-\alpha)}{\Gamma^2(1+\beta(1-\alpha))} \int_{t_1}^{t_2} \|(t_2-s)^{2\beta(1-\alpha)-2}\| \\
&\quad \times \|A^{-1}\|^2 E\|f_n(s,u(s),u(h(u(s),s)))\|^2 ds \\
&\quad + 3\int_0^{t_1} \|A[(t_2-s)^{\beta-1} S_\beta(t_2-s) - (t_1-s)^{\beta-1} S_\beta(t_1-s)] \\
&\quad \times \|A^{\alpha-2}\|^2 E\|f_n(s,u(s),u(h(u(s),s)))\|^2 ds \qquad (10.1.6)
\end{aligned}$$

$\forall u \in H$, we can write

$$[S(t_2^\beta\theta) - S(t_1^\beta\theta)]u = \int_{t_1}^{t_2} \frac{d}{dt} S(t^\beta\theta)udt = \int_{t_1}^{t_2} \theta\beta t^{\beta-1} AS(t^\beta\theta)dt.$$

The first term of (10.1.6) can be estimated as follows

$$\begin{aligned}\|[T_\beta(t_2) - T_\beta(t_1)]u_0\|_{\alpha-1}^2 &\leq (\int_0^\infty \zeta_\beta(\theta)\|S(t_2^\beta\theta) - S(t_1^\beta\theta)\|\|A^{\alpha-1}u_0\|d\theta)^2 \\ &\leq (\int_0^\infty \zeta_\beta(\theta)[\int_{t_1}^{t_2} \|\frac{d}{dt}S(t^\beta\theta)\|dt]\|u_0\|_\alpha d\theta)^2 \\ &\leq C_1^2\|u_0\|_{\alpha-1}^2(t_2 - t_1)^2 \qquad (10.1.7)\end{aligned}$$

For the second term of (10.1.6) we get the following estimate

$$\begin{aligned}&\int_{t_1}^{t_2} (t_2 - s)^{2\beta(1-\alpha)-2} E\|f_n(s, u(s), u(h(u(s), s)))\|^2 ds \\ &\leq \frac{N(t_2 - t_1)^{2\beta(1-\alpha)-1}}{2\beta(1-\alpha) - 1} \qquad (10.1.8)\end{aligned}$$

For the third term we will use the following estimate

$$\begin{aligned}&\int_0^{t_1} \|A[(t_2 - s)^{\beta-1}S_\beta(t_2 - s) - (t_1 - s)^{\beta-1}S_\beta(t_1 - s)]\|^2 \\ &\quad \times \|A^{\alpha-2}\|^2 E\|f_n(s, u(s), u(h(u(s), s)))\|^2 ds \\ &\leq \int_0^{t_1} (\int_0^\infty \zeta_\beta(\theta)\|[\frac{d}{dt}S((t-s)^\beta\theta)|_{t=t_2} - \frac{d}{dt}S((t-s)^\beta\theta)|_{t=t_1}]\|d\theta)^2 \\ &\quad \times E\|f(s, u(s), u(h(u(s), s)))\|^2 ds \\ &\leq \int_0^{t_1} (\int_0^\infty \zeta_\beta(\theta)[\int_{t_1}^{t_2} \|A^{\alpha-2}\frac{d^2}{dt^2}S((t-s)^\beta\theta)\|dt]d\theta)^2 N ds \\ &\leq C_2^2\|A^{\alpha-2}\|^2(t_2 - t_1)^2 NT_0 \qquad (10.1.9)\end{aligned}$$

Hence from inequalities (10.1.7)-(10.1.9) we see that the map $\mathcal{F}_n : \mathcal{C}_{T_0}^{\alpha-1} \to \mathcal{C}_{T_0}^{\alpha-1}$ is well-defined. Now we prove that $\mathcal{F}_n : B_R \to B_R$. So for $t \in [0, T_0]$ and $u \in B_R$.

$$\begin{aligned}
&E\|(\mathcal{F}_n u)(t) - u_0\|_\alpha^2 \\
&\leq 2E\|(T_\beta(t) - I)u_0\|_\alpha^2 \\
&\quad + 2E\|\int_0^t (t-s)^{\beta-1} S_\beta(t-s) f(s, u(s), u(h(u(s), s)))dw(s)\|_Q^2 \\
&\leq 2E\|(T_\beta(t) - I)u_0\|_\alpha^2 + 2(\frac{\beta C_\alpha \Gamma(2-\alpha)}{\Gamma(1+\beta(1-\alpha))})^2 \int_0^t \|(t_2 - s)^{2\beta(1-\alpha)-2}\|^2 \\
&\quad \times E\|f_n(s, u(s), u(h(u(s), s)))\|^2 ds \\
&\leq \frac{R}{2} + 2(\frac{\beta C_\alpha \Gamma(2-\alpha)}{\Gamma(1+\beta(1-\alpha))})^2 N \frac{T_0^{\beta(1-\alpha)-1}}{\beta(1-\alpha)-1} \leq \frac{R}{2} + \frac{R}{2} = R
\end{aligned}$$

Now we show that $\mathcal{F}_n$ is a contraction map by using (10.1.3) in last but one inequality. $\forall u, v \in B_R$

$$\begin{aligned}
E\|(\mathcal{F}_n u)(t) - (\mathcal{F}_n v)(t)\|_\alpha^2 &= E\|\int_0^t (t-s)^{\beta-1} A^\alpha S_\beta(t-s) \\
&\quad \times [f(s, u(s), u(h(u(s), s))) - f(s, v(s), v(h(v(s), s)))dw(s)]\|_Q^2 \\
&\leq (\frac{\beta C_\alpha \Gamma(2-\alpha)}{\Gamma(1+\beta(1-\alpha))})^2 \int_0^t (t_2 - s)^{2\beta(1-\alpha)-2} \\
&\quad \times E\|f(s, u(s), u(h(u(s), s))) - f(s, v(s), v(h(v(s), s)))\|^2 ds \\
&\leq (\frac{\beta C_\alpha \Gamma(2-\alpha)}{\Gamma(1+\beta(1-\alpha))})^2 2L_f(1 + 2LLh)\|u - v\|_\alpha^2 \frac{T^2\beta(1-\alpha)-1}{2\beta(1-\alpha)-1} \\
&\leq \|u - v\|_\alpha^2.
\end{aligned}$$

This implies that there exists a unique fixed point u_n of $\mathcal{F}_n$ which is the unique approximate mild solution of (10.1.1)

Lemma 10.1.4. Let $(H1) - (H3)$ hold. If $u_0 \in L_0^2(\Omega, D(A^\alpha))$, $\forall 0 < \alpha < \eta < 1$, then $u_n(t) \in D(A^\gamma)$ for all $t \in [0, T_0]$ with $0 < \gamma < \eta < 1$. Also if $u_0 \in D(A)$, then $u_n(t) \in D(A^\gamma)$ $\forall t \in [0, T_0]$, where $0 < \gamma < \eta < 1$.

Proof:By Theorem (10.1.3) we get the existence of a unique $u_n \in B_R$, satisfying (10.1.5). Theorem 2.6.13 of [146] implies for $t > 0$, $0 \leq \gamma < 1$, $S(t) : H \to D(A^\gamma)$ and for $0 \leq \gamma < \eta < 1$, $D(A^\eta) \subset D(A^\gamma)$. It is easy to see that Holder continuity of u_n can be proved using the similar arguments from (10.1.6)-(10.1.9). Also from

Theorem 1.2.4 in [146], we have $S(t)u \in D(A)$ if $u \in D(A)$. The result follows from these facts and that $D(A) \subset D(A^\gamma)$ for $0 \leq \gamma < 1$.

Lemma 10.1.5. Let $(H1)-(H3)$ hold and $u_0 \in L_0^2(\Omega, X_\alpha)$. Then for any $t_0 \in (0, T_0]$ $\exists$ a constant U_{t_0}, independent of n such that $E\|u_n(t)\|_\gamma^2 \leq U_{t_0}$ $\forall t \in [t_0, T_0]$, $n = 1, 2, \cdots$. Also if $u_0 \in L_0^2(\Omega, D(A))$ then $\exists$ constant U_0 independent of n such that $E\|u_n(t)\|_\gamma^2 \leq U_0$ $\forall t \in [t_0, T_0]$, $n = 1, 2, \cdots$, $\forall\, 0 < \gamma \leq 1$.

Proof: Let $u_0 \in L_0^2(\Omega, H_\alpha)$. Applying A^γ on both sides of (10.1.4)

$$\begin{aligned} & E\|u_n(t)\|_\gamma^2 \\ & \leq 2E\|T_\beta(t)u_0\|_\gamma^2 + 2\|\int_0^t (t-s)^{\beta-1} S_\beta(t-s) f_n(s, u(s), u(h(u(s), s)))dw(s)\|_Q^2 \\ & \leq 2C_\gamma^2 t_0^{-2\gamma\beta}\|u_0\|^2 + (\frac{\beta C_\gamma \Gamma(2-\gamma)}{\Gamma(1+\beta(1-\gamma))})^2 \frac{N(T_0)^{2\beta(1-\gamma)-1}}{2\beta(1-\gamma)-1} = U_{t_0}. \end{aligned}$$

Also if $u_0 \in L_0^2(\Omega, D(A))$, then we have that $u_0 \in L_0^2(\Omega, D(A^\gamma))$ for $0 \leq \gamma < 1$. Hence,

$$\begin{aligned} & E\|u_n(t)\|_\gamma^2 \\ & \leq 2E\|T_\beta(t)u_0\|_\gamma^2 + 2\|\int_0^t (t-s)^{\beta-1} S_\beta(t-s) f_n(s, u(s), u(h(u(s), s)))dw(s)\|_Q^2 \\ & \leq 2C'^2\|u_0\|^2 + (\frac{\beta C_\gamma \Gamma(2-\gamma)}{\Gamma(1+\beta(1-\gamma))})^2 \frac{N(T_0)^{2\beta(1-\gamma)-1}}{2\beta(1-\gamma)-1} = U_0. \end{aligned}$$

Hence proved.

10.1.2 Convergence of Solutions

In this section the convergence of the solution $u_n \in H_\alpha$ of the approximate integral equation (10.1.5) to a unique solution u of (10.1.2), is discussed.

Theorem 10.1.6. Let the hypotheses $(H1)-(H3)$ hold and if $u_0 \in L_0^2(\Omega, H_\alpha)$ then $\forall t_0 \in (0, T]$,

$$\lim_{m\to\infty} \sup_{\{n \geq M, t_0 \leq t \leq T_0\}} \|u_n(t) - u_m(t)\|_\alpha = 0.$$

Proof: Let $0 < \alpha < \gamma < \eta$. For $t_0 \in (0, T_0]$

$$\begin{aligned} &E\|f_n(t, u_n(t), u_n(h(u_n(t), t))) - f_m(t, u_m(t), u_m(h(u_m(t), t)))\|^2 \\ &\quad \leq 2E\|f_n(t, u_n(t), u_n(h(u_n(t), t))) - f_n(t, u_m(t), u_m(h(u_m(t), t)))\|^2 \\ &\quad \leq 2E\|f_n(t, u_m(t), u_m(h(u_m(t), t))) - f_m(t, u_m(t), u_m(h(u_m(t), t)))\|^2 \\ &\quad \leq 2(2L_f(1 + 2LL_h)[E\|u_n - u_m\|_\alpha^2 + E\|(P^n - P^m)u_m(t)\|_\alpha^2]) \qquad (10.1.10) \end{aligned}$$

Now,

$$E\|(P^n - P^m)u_m(t)\|^2 \leq E\|A^{\alpha-\gamma}(P^n - P^m)A^\gamma u_m(t)\|^2 \leq \frac{1}{\lambda_m^{2(\gamma-\alpha)}} E\|A^\gamma u_m(t)\|^2$$

Then we have

$$\begin{aligned} &E\|f_n(t, u_n(t), u_n(h(u_n(t), t))) - f_m(t, u_m(t), u_m(h(u_m(t), t)))\|^2 \\ &\quad \leq 2(2L_f(1 + 2LL_h)[E\|u_n - u_m\|_\alpha^2 + \frac{1}{\lambda_m^{2(\gamma-\alpha)}} E\|A^\gamma u_m(t)\|^2]) \end{aligned}$$

For $0 < t_0' < t_0$

$$\begin{aligned} E\|u_n(t) - u_m(t)\|_\alpha^2 \leq 2(&\int_0^{t_0'} + \int_{t_0'}^t)\|(t-s)^{\beta-1}A^\alpha S_\beta(t-s)\|^2 \\ &\times E\|f_n(t, u_n(t), u_n(h(u_n(t), t))) - f_m(t, u_m(t), u_m(h(u_m(t), t)))\|^2 ds \end{aligned} \qquad (10.1.11)$$

The estimate of first integral of the above inequality is

$$\begin{aligned} &E\|u_n(t) - u_m(t)\|_\alpha^2 \\ &\quad \leq \int_0^{t_0'} \|(t-s)^{\beta-1}A^\alpha S_\beta(t-s)\|^2 \\ &\qquad \times E\|f_n(t, u_n(t), u_n(h(u_n(t), t))) - f_m(t, u_m(t), u_m(h(u_m(t), t)))\|^2 ds \\ &\quad \leq (\frac{\beta C_\gamma \Gamma(2-\gamma)}{\Gamma(1+\beta(1-\gamma))})^2 \frac{2N(t_0 - \delta_1 t_0')^{2\beta(1-\gamma)-2}}{2\beta(1-\gamma)-1} t_0', \quad 0 < \delta < 1 \qquad (10.1.12) \end{aligned}$$

The estimate of second integral is

$$E\|u_n(t)-u_m(t)\|_\alpha^2 \leq \int_{t_0'}^t \|(t-s)^{\beta-1}A^\alpha S_\beta(t-s)\|^2$$
$$\times E\|f_n(t,u_n(t),u_n(h(u_n(t),t))) - f_m(t,u_m(t),u_m(h(u_m(t),t)))\|^2 ds$$
$$\leq (\frac{\beta C_\gamma \Gamma(2-\gamma)}{\Gamma(1+\beta(1-\gamma))})^2 \int_{t_0'}^t (t-s)^{2\beta(\alpha-1)-2}$$
$$\times 4L_f(1+2LL_h)[E\|u_n-u_m\|_\alpha^2 + \frac{E\|A^\gamma u_m(s)\|^2}{\lambda^2(\gamma-\alpha)}]ds$$
$$\leq 4L_f(1+2LL_h)(\frac{\beta C_\gamma \Gamma(2-\gamma)}{\Gamma(1+\beta(1-\gamma))})^2 [\int_{t_0'}^t (t-s)^{2\beta(\alpha-1)-2}$$
$$\times E\|u_n-u_m\|_\alpha^2 ds + \frac{U_{t_0}}{\lambda_m^{2(\gamma-\alpha)}} \frac{T_0^{2\beta(1-\alpha)-1}}{2\beta(1-\alpha)-1}] \tag{10.1.13}$$

Substituting inequalities (10.1.12),(10.1.13) in (10.1.11) we get

$$E\|u_n(t)-u_m(t)\|_\alpha^2$$
$$\leq (\frac{\beta C_\gamma \Gamma(2-\gamma)}{\Gamma(1+\beta(1-\gamma))})^2 \frac{4N(t_0-\delta_1 t_0')^{2\beta(1-\gamma)-2}}{2\beta(1-\gamma)-1} t_0'$$
$$+ 8L_f(1+2LL_h)(\frac{\beta C_\gamma \Gamma(2-\gamma)}{\Gamma(1+\beta(1-\gamma))})^2 [\int_{t_0'}^t (t-s)^{2\beta(\alpha-1)-2}$$
$$\times E\|u_n-u_m\|_\alpha^2 ds + \frac{U_{t_0}}{\lambda_m^{2(\gamma-\alpha)}} \frac{T_0^{2\beta(1-\alpha)-1}}{2\beta(1-\alpha)-1}]$$

By using Gronwall's inequality, there exists a constant D such that

$$E\|u_n(t)-u_m(t)\|_\alpha^2 \leq [(\frac{\beta C_\gamma \Gamma(2-\gamma)}{\Gamma(1+\beta(1-\gamma))})^2 \frac{4N(t_0-\delta_1 t_0')^{2\beta(1-\gamma)-2}}{2\beta(1-\gamma)-1} t_0'$$
$$+ 8L_f(1+2LL_h)(\frac{\beta C_\gamma \Gamma(2-\gamma)}{\Gamma(1+\beta(1-\gamma))})^2 \frac{U_{t_0}}{\lambda_m^{2(\gamma-\alpha)}} \frac{T_0^{2\beta(1-\alpha)-1}}{2\beta(1-\alpha)-1}] \times D$$

Let $m \to \infty$. Taking supremum over $[t_0, T_0]$ we get the following inequality.

$$E\|u_n(t)-u_m(t)\|_\alpha^2 \leq [(\frac{\beta C_\gamma \Gamma(2-\gamma)}{\Gamma(1+\beta(1-\gamma))})^2 \frac{4N(t_0-\delta_1 t_0')^{2\beta(1-\gamma)-2}}{2\beta(1-\gamma)-1} t_0'] \times D$$

Since t_0' is arbitrary, the right hand side can be made infinitesimally small by choosing t_0' sufficiently small. Thus the lemma is proved.

Corollary 10.1.7. If $u_0 \in D(A)$, then $\lim_{m\to\infty} \sup_{\{n\geq m,\ 0\leq t\leq T_0\}} E\|u_n(t)-u_m(t)\|_\alpha^2 = 0$

Proof: By using Lemma (10.1.4) and Lemma (10.1.5) we can take $t_0 = 0$ in the proof of Theorem (10.1.6) and hence the corollary follows.

Theorem 10.1.8. Let us assume that $(H1) - (H3)$ are satisfied and suppose $u_0 \in L_0^2(\Omega, X_\alpha)$. Then for $t \in [0, T_0]$, $\exists$ a unique function $u_n \in B_R$ where
$u_n(t) = T_\beta u_0 + \int_0^t (t-s)^{\beta-1} S_\beta(t-s) f_n(s, u_n(s), u_n(h_n(u_n(s), s)))dw(s)$,
and $u(t) \in B_R$, where
$u(t) = T_\beta u_0 + \int_0^t (t-s)^{\beta-1} S_\beta(t-s) f(s, u(s), u(h(u(s), s)))dw(s), t \in [0, T_0]$, such that $u_n \to u$ as $n \to \infty$ in B_R and u satisfies (10.1.2) on $[0, T_0]$.

Proof: By using above Corollary, Theorem 10.1.3 and Theorem 10.1.6 it is to see that $\exists$ $u(t) \in B_R$ such that
$\lim_{n\to\infty} E\|u_n(t) - u(t)\|_\alpha^2 = 0$ on $[0, T_0]$. Now

$$\begin{aligned} E\|u_n(t) - T_\beta u_0 + \int_{t_0}^t (t-s)^{\beta-1} S_\beta(t-s) f_n(s, u_n(s), u_n(h_n(u_n(s), s)))dw(s)\|^2 \\ \leq E\| \int_0^{t_0} (t-s)^{\beta-1} S_\beta(t-s) f_n(s, u_n(s), u_n(h_n(u_n(s), s)))dw(s)\|^2 \\ \leq (\frac{\beta C}{\Gamma(1+\beta)})^2 N \frac{T_0^{2\beta-2}}{2\beta-2} t_0 \end{aligned} \quad (10.1.14)$$

Let $n \to \infty$ then
$E\|u_n(t) - T_\beta u_0 + \int_{t_0}^t (t-s)^{\beta-1} S_\beta(t-s) f_n(s, u_n(s), u_n(h_n(u_n(s), s)))dw(s)\|^2$
$\leq (\frac{\beta C}{\Gamma(1+\beta)})^2 N \frac{T_0^{2\beta-2}}{2\beta-2} t_0$ and since t_0 is arbitrary we conclude $u(t)$ satisfies (10.1.2). Uniqueness follows easily from Theorem 10.1.3, 10.1.6 and Gronwall's inequality.

10.1.3 Faedo-Galerkin Approximations

For any $0 \leq T_0 \leq T$, there exists a unique $u \in C_{T_0}^\alpha$ satisfying the integral equation $u(t) = T_\beta u_0 + \int_0^t (t-s)^{\beta-1} S_\beta(t-s) f(s, u(s), u(h(u(s), s)))dw(s)$, $t \in [0, T_0]$. This follows previous section Also, $\exists$ a unique solution $u_n \in C_{T_0}^\alpha$ of the approximate integral equation
$u_n(t) = T_\beta u_0 + \int_0^t (t-s)^{\beta-1} S_\beta(t-s) f_n(s, u_n(s), u_n(h(u_n(s), s)))dw(s)$, $t \in [0, T_0]$.
Faedo-Galerkin approximation $\bar{u}_n = P^n u_n$ is given by
$P^n u_n(t) = \bar{u}_n(t) = T_\beta(t) P^n u_0$
$+ \int_0^t (t-s)^{\beta-1} S_\beta(t-s) P^n f(s, u_n(s), u_n(h(u_n(s), s)))dw(s), t \in [0, T_0]$. If the solution $u(t)$ to (10.1.2) exists on $[0, T_0]$ then it can be expressed as

$u(t) = \sum_{i=0}^{\infty} \alpha_i(t)\phi_i$, where $\alpha_i(t) = (u(t), \phi_i)$ for $i = 0, 1, 2, 3, \cdots$ and

$\bar{u}_n(t) = \sum_{i=0}^{n} \alpha_i^n(t)\phi_i$, where $\alpha_i^n(t) = (\bar{u}_n(t), \phi_i)$ for $i = 0, 1, 2, 3, \cdots$.

As a consequence of Theorem 10.1.3 and Theorem 10.1.6, we have the following result.

Theorem 10.1.9. Let us assume that $(H1) - (H3)$ are satisfied and suppose $u_0 \in L_0^2(\Omega, X_\alpha)$. Then for $t \in [0, T_0]$, $\exists$ a unique function $u_n \in B_R$ where
$u_n(t) = T_\beta P^n u_0 + \int_0^t (t-s)^{\beta-1} S_\beta(t-s) P^n f_n(s, u_n(s), u_n(h(u_n(s), s))) dw(s)$,
and $u(t) \in B_R$, where
$u(t) = T_\beta u_0 + \int_0^t (t-s)^{\beta-1} S_\beta(t-s) f(s, u(s), u(h(u(s), s))) dw(s), t \in [0, T_0]$, such that $u_n \to u$ as $n \to \infty$ in B_R and u satisfies (10.1.2) on $[0, T_0]$.

Now the convergence of $\alpha_i^n(t) \to \alpha_i(t)$ is shown. It is easily seen that
$A^\alpha[u(t) - \bar{u}_n(t)] = A^\alpha\left[\sum_{i=0}^{n}\{\alpha_i(t) - \alpha_i^n(t)\}\phi_i\right] + A^\alpha \sum_{i=n+1}^{\infty} \alpha_i(t)\phi_i$
$= \sum_{i=0}^{n} \lambda_i^\alpha\{\alpha_i(t) - \alpha_i^n(t)\}\phi_i + \sum_{i=n+1}^{\infty} \lambda_i^\alpha \alpha_i(t)\phi_i$. Thus we have
$E\|A^\alpha[u(t) - \bar{u}_n(t)\|^2 \geq \sum_{i=0}^{n} \lambda_i^{2\alpha} E|\alpha_i(t) - \alpha_i^n(t)|^2$.

Theorem 10.1.10. Let us assume $(H1) - (H3)$ hold.
(i) If $u_0 \in L_0^2(\Omega, X_\alpha)$ then $\lim_{n\to\infty} \sup_{t\in[t_0,T_0]} \left[\sum_{i=0}^{n} \lambda_i(t)^{2\alpha} E\|\alpha_i(t) - \alpha_i^n(t)\|^2\right] = 0$

(ii) If $u_0 \in L_0^2(\Omega, D(A))$ then $\lim_{n\to\infty} \sup_{t\in[0,T_0]} \left[\sum_{i=0}^{n} \lambda_i(t)^{2\alpha} E\|\alpha_i(t) - \alpha_i^n(t)\|^2\right] = 0$

The theorem 10.1.10 follows from the facts mentioned above the theorem.

Corollary 10.1.11. Let us assume $(H1) - (H3)$ hold.
(i) If $u_0 \in L_0^2(\Omega, X_\alpha)$ then $\lim_{n\to\infty} \sup_{t\in[t_0,T_0], n\geq m} E\|A^\alpha[\bar{u}_n(t) - \bar{u}_m(t)]\|^2 = 0$
(ii) If $u_0 \in L_0^2(\Omega, D(A))$ then $\lim_{n\to\infty} \sup_{t\in[0,T_0], n\geq m} E\|A^\alpha[\bar{u}_n(t) - \bar{u}_m(t)]\|^2 = 0$

Proof:

$$\begin{aligned}
E\|A^\alpha[\bar{u}_n(t) - \bar{u}_m(t)]\|^2 &= E\|P^n u_n(t) - P^m u_m(t)\|_\alpha^2 \\
&\leq 2E\|P^n[u_n(t) - u_m(t)]\|_\alpha^2 + 2E\|(P^n - P^m)y_m(t)\|_\alpha^2 \\
&\leq 2E\|[u_n(t) - u_m(t)]\|_\alpha^2 + 2\frac{1}{\lambda_m^{\gamma-\alpha}} E\|A^\gamma u_m(t)\|^2
\end{aligned}$$

Then the result (i) follows from theorem 10.1.6 and result (ii) follows from corollary 10.1.7.

10.1.4 Example

Consider the following stochastic fractional differential equation with deviating argument. Suppose for $t \geq 0,\ x \in (0,1), 0 < \beta \leq 1$

$$^{c}D^{\beta}v_t(t,x) = v_{xx}(t,x) + F(t, v(t,x), v(h(t, v(t,x))))\frac{dw(t)}{dt},$$
$$v(t,x) = v_0,\ t = 0,\ x \in (0,1) \quad and \quad v(t,0) = v(t,1) = 0,\ t \geq 0 \quad (10.1.15)$$

Let F is an appropriate Holder continuous function satisfying $(H2)$ in $L_0^2(K,(0,1))$. w is a standard $L^2(0,1)$ valued Weiner process.

Let us define $A = -\frac{d^2}{dx^2}$, $f := F$, $v(t,x) = u(t)$ and assume $\alpha = 1/2$. Let $D(A) = H_0^1(0,1) \cap H^2(0,1)$, $D(A^{1/2}) = H_0^1(0,1)$, i.e. the Banach space endowed with the norm

$$\|x\|_{1/2} := \|A^{1/2}x\|,\ x \in D(A^{1/2}).$$

We denote this space by $X_{1/2}$.

Also denote $C_t^{1/2} = C(t,0; D(A^{1/2}))$ endowed with sup norm

$$\|x\|_{t,1/2} := \sup_{0 \leq s \leq t} \|x(s)\|_{1/2},\ x \in C_t^{1/2}.$$

When $v \in D(A),\ \lambda \in \mathbb{R}$ with $Av = -v'' = \lambda v$ we have $< Av, v >=< \lambda v, v >$, i.e.

$$< -v'', v >= \|v'\|_{L^2}^2 = \lambda \|v\|_{L^2}^2.$$

Therefore the solution v of $Av = \lambda v$ is of the form

$$v(x) = C\cos(\sqrt{\lambda}x) + D\sin(\sqrt{\lambda}x)$$

From the conditions $v(0) = v(1) = 0$ imply that $C = 0$ and $\lambda = \lambda_n = n^2\pi^2, n \in \mathbb{N}$. So, for each n the solution is

$$v_n(x) = D\sin(\sqrt{\lambda_n}x).$$

Also note that $< v_n, v_m >= 0$ for $n \neq m$ and $< v_n, v_n >= 1$. Therefore $D = \sqrt{2}$. For $v \in D(A),\ \exists$ a sequence of real numbers $\{a_n\}$ such that

$$v(x) = \sum_{n \in \mathbf{N}} a_n v_n(x),\ \sum_{n \in \mathbf{N}} (a_n)^2 < \infty,\ \sum_{n \in \mathbf{N}} (\lambda)^2 (a_n)^2.$$

So, $A^{1/2}v(x) = \sum_{n\in\mathbf{N}} \sqrt{\lambda_n} a_n v_n(x)$, with $v \in D(A^{1/2})$.
$X_{-1/2} = H^1(0,1)$ is a Sobolev space of negative index with equivalent norm $\|.\|_{-1/2} = \sum_{n=1}^{\infty} \| < ., v_n > \|^2$. Then (10.1.15) can be reformulated into (10.1.1). Now from theorems (10.1.3),(10.1.6) we can similarly prove the existence, uniqueness and approximation of the mild solution of (10.1.15).

10.2 Conclusion

The existence, uniqueness and convergence of approximate solutions of a stochastic fractional differential equation with deviated argument is established. Then Faedo-Galerkin approximation of solution is considered and some convergence results are proved.

List of Publications

Journals

1. Sanjukta Das, D. N. Pandey, and N. Sukavanam, Exact Controllability of an Impulsive Semilinear System with Deviated Argument in a Banach Space, Journal of Difference Equations, Hindawi, volume 2014 (2014), Article ID 461086, 6 pages, http://dx.doi.org/10.1155/2014/461086.

2. Sanjukta Das, D. N. Pandey and N. Sukavanam, Approximate Controllability of a Functional Differential Equation with Deviated Argument, Nonlinear Dynamics and Systems Theory, Infor Math, volume 14, no. 3, (2014), 265-277.

3. D. N. Pandey, Sanjukta Das, N. Sukavanam, Existence of Solution for a Second-Order Neutral Differential Equation with State Dependent Delay and Non-instantaneous Impulses, International Journal of Nonlinear Science, World Scientific, volume 18, no. 2, (2014), 145-155.

4. Sanjukta Das, D. N. Pandey and N. Sukavanam, Existence of Solution of Impulsive Second-Order Neutral Integro-Differential Equation with State Delay, in revision in Journal of Integral Equations and Applications, Rocky Mountain Mathematics Consortium, http://rmmc.asu.edu/.

5. Sanjukta Das, D. N. Pandey and N. Sukavanam, Approximate Controllability of a Second Order Neutral Differential Equation with State Dependent Delay, Differential Equations and Dynamical Systems, Springer, DOI 10.1007/s12591-014-0218-6, (2014).

6. Sanjukta Das, D. N. Pandey and N. Sukavanam, Existence of Solution and Approximate Controllability for Neutral Differential Equation with State Dependent Delay, International Journal of Partial Differential Equations, Hindawi, volume 2014 (2014), Article ID 787092, 12 pages, http://dx.doi.org/10.1155/2014/787092.

7. Sanjukta Das, D. N. Pandey, and N. Sukavanam, Approximate Controllability of a Fractional Neutral System with Deviated Argument in Banach Space, Differential Equations and Dynamical Systems, Springer, DOI: 10.1007/s12591-015-0237-y, (2015).

8. Sanjukta Das, D. N. Pandey, and N. Sukavanam, Approximate controllability of an impulsive neutral fractional stochastic differential equation with deviated argument and infinite delay, NONLINEAR STUDIES - www.nonlinearstudies.com. volume 22, no. 1, (2015), 1-16, CSP - Cambridge, UK; - Florida, USA.

9. Sanjukta Das, D. N. Pandey, and N. Sukavanam, Approximations of Solutions of a Fractional Stochastic Differential Equations with Deviated Argument, accepted for publication in Journal of Fractional Calculus and Applications in (2015).

10. Sanjukta Das, D. N. Pandey, and N. Sukavanam, Approximations of Solutions to Neutral Retarded Integro-differential Equations, accepted to appear in Journal of Nonlinear Evolution Equations in (2015).

11. Sanjukta Das, D. N. Pandey, and N. Sukavanam, Approximate Controllability of an Impulsive Stochastic Delay Differential Equations, to appear in Journal of Advanced Research in Dynamical and Control Systems, (JARDCS), jardcs-Oct-21-2014-cf8eb309 in (2015).

Conferences

1. Sanjukta Das, D. N. Pandey, N. Sukavanam Existence results for a partial neutral differential equation with deviated argument in a Banach Space in International Conference on Recent Trends in Algebra and Analysis with Applications(ICRTAA-2014) held at Department of Mathematics, Aligarh Muslim University, Aligarh, India during February 12-14, 2014.

2. Sanjukta Das, D. N. Pandey, and N. Sukavanam, Approximations of Solutions of a Stochastic Differential Equations, accepted for publication in Springer Proceedings of International Conference on Recent Trends in Mathematical Analysis and Applications, (ICRTMAA) held at IIT Roorkee on 21-23 December 2014.

3. Sanjukta Das and D. N. Pandey, Approximate Controllability of an Impulsive Stochastic Differential Equation with Deviating Argument, SIAM Dynamical System-15, Utah, USA, May 17-21, 2015.

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