FOURIER SERIES APPROXIMATION BY LINEAR OPERATORS IN L^p-NORM

Ph.D. THESIS

by

ARTI

DEPARTMENT OF MATHEMATICS
INDIAN INSTITUTE OF TECHNOLOGY ROORKEE
ROORKEE-247667 (INDIA)
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A THESIS

submitted in partial fulfilment of the requirements for the award of the degree

of

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in

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ROORKEE

CANDIDATE'S DECLARATION

I hereby certify that the work which is being presented in the thesis entitled **"FOURIER SERIES APPROXIMATION BY LINEAR OPERATORS IN L^p-NORM"** in partial fulfilment of the requirements for the award of the degree of **Doctor of Philosophy** and submitted in the Department of Mathematics, Indian Institute of Technology Roorkee, Roorkee is an authentic record of my own work carried out during a period from December, 2014 to April, 2019 under the supervision of Dr. Uaday Singh, Associate Professor, Department of Mathematics, Indian Institute of Technology Roorkee, Roorkee.

The matter presented in this thesis has not been submitted by me for the award of any other degree of this or any other institute.

(ARTI)

This is to certify that the above statement made by the candidate is correct to the best of my knowledge.

Date: April 18, 2019

(Uaday Singh)
Supervisor

Abstract

In this thesis, we study the degree of approximation of functions belonging to certain function classes through trigonometric Fourier series using summability methods. We divide the thesis into six chapters.

Chapter one is an introductory part of the thesis which deals with the upbringing of approximation theory, basic definitions and some notations which are used throughout the thesis. Literature survey and the objective of the work done is also given in this chapter.

Chapter two is about the approximation of 2π-periodic functions in the weighted Lipschitz class $W(L^p, \xi(t))$ $(p \geq 1)$ by almost summability means of their Fourier series. We also obtain a result on the approximation of conjugate functions through almost matrix means of their conjugate Fourier series, which in turn improves some of the previous results. The deviation is measured in the corresponding weighted norm. We also discuss some corollaries derived from our main results.

Chapter three deals with the approximation of functions by using Φ-method of summability of conjugate Fourier series. Here we obtained a degree of approximation of the conjugate function $\widetilde{f}$, conjugate to a 2π-periodic function f in the generalized Hölder space $H_{\alpha,p}$ $(0 < \alpha \leq 1,\ p \geq 1)$ through Borel means of the conjugate Fourier series. Our result improves some of the previous result.

In the **fourth chapter**, we obtain an estimate for the degree of approximation of functions belonging to the generalized Zygmund space Z_p^{ω} $(p \geq 1)$ through product means of Fourier series, which generalizes and improves some of the previous results. The results are obtain in terms of the moduli of continuity. We also derive some corollaries from our theorems.

In the **fifth chapter**, we obtain a quantitative estimate of Young's theorem (well known in the classical Fourier analysis) by using matrix means which generalizes the

result obtained by Mazhar and Budaiwi [76].

In the **sixth chapter**, we study the degree of approximation of 2π-periodic functions of two variables, defined on $T^2 := [-\pi, \pi] \times [-\pi, \pi]$ and belonging to certain Lipschitz classes, by means of almost Euler summability of their Fourier series. The degree of approximation so obtained depends on the modulus of continuity associated with the functions. We also derive some corollaries from our theorems for the functions of Zygmund classes.

List of Publications

Journal Papers

1. A. Rathore, U. Singh, On the degree of approximation of functions in a weighted Lipschitz class by almost matrix summability method, J. Anal. (2017), 1–13.

2. U. Singh, A. Rathore, A note on the degree of approximation of functions belonging to certain Lipschitz class by almost Riesz means, Stud. Univ. Babeş-Bolyai Math. **63**(3), (2018), 371–379.

3. A. Rathore, U. Singh, Approximation of certain bivariate functions by almost Euler means of double Fourier series, J. Inequal. Appl. **2018**, (2018), 495–504.

Communicated Papers

The following papers have been submitted for possible publication.

1. A. Rathore, U. Singh, An error estimate in the approximation through conjugate Fourier series of functions of bounded variation.

2. A. Rathore, U. Singh, Borel summability of Conjugate Fourier series in the Hölder metric.

3. A. Rathore, U. Singh, Trigonometric approximation of functions in the generalized Zygmund space.

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(Arti)

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Chapter 1

Introduction

Approximation theory is a branch of mathematical analysis which mainly concernes with the approximation of functions by simpler and more easily calculated functions (for example, polynomials or rational functions) and with estimation of an approximation error. Among all functions periodic function plays a crucial role, it has many applications in physics and engineering. Therefore, the approximation of periodic functions is an important problem.

The problem of representing a periodic function by a trigonometric series was first encountered by a French physicist and mathematician Jean-Baptiste Joseph Fourier (1768-1830) while solving the heat conduction problem. Later, different mathematicians and physicists such as Leonhard Euler, Jean le Rond d'Alembert, and Daniel Bernoulli, and others used this infinite trigonometric series of sine and cosine functions to investigate problems in vibrating strings, astronomy and some other fields of science. This infinite series of sine and cosine functions is popularly known as Fourier series or trigonometric Fourier series. Fourier series is basically a method of representing a complex periodic function by simpler functions. These simple functions are sinusoids, which are summed to produce an approximation of an original function. The theory of approximation of functions with Fourier series is generally referred as Fourier series approximation or Fourier approximation.

Fourier approximation is originated from Weierstrass approximation theorem given by Karl Weierstrass in 1885, which states that for any continuous 2π-periodic function f, there exists a sequence of trigonometric polynomials which converges uniformly to f. Intuitively, Fourier approximation is the study of error estimation of functions through their Fourier series using summability techniques. In the present thesis, we shall confine ourselves to trigonometric Fourier series only.

1.1 Fourier series

Fourier series plays an important role in designing and analyzing electrical and electronic communication systems. In an engineering system, inputs are obtained in the form of electrical signals, electromagnetic signals, magnetic waves, sound waves, vibrations, etc., and the analysis of these engineering systems are done based on Fourier coefficients.

1.1.1 Trigonometric Fourier series

Let f be a 2π-periodic Lebesgue integrable function over the interval $[-\pi, \pi]$. Then the trigonometric Fourier series of function f is defined by

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\, cos\, nx + b_n\, sin\, nx), \tag{1.1.1}$$

and the conjugate Fourier series is given by

$$\sum_{n=1}^{\infty}(b_n\, cos\, nx - a_n\, sin\, nx), \tag{1.1.2}$$

where

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, cos\, nx\, dx, \quad n = 0, 1, 2, ...,$$

and

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, sin\, nx\, dx,\ n = 1, 2, 3, ... \ .$$

are the Fourier coefficients of f.

For a Lebesgue integrable function f, the corresponding conjugate function $\widetilde{f}$

$$\widetilde{f}(x) \quad = \quad \lim_{\epsilon\to 0^+} \widetilde{f}(x;\epsilon) = \lim_{\epsilon\to 0^+} \frac{-1}{\pi}\int_{\epsilon}^{\pi} \frac{f(x+t) - f(x-t)}{2\tan(t/2)}dt, \tag{1.1.3}$$

exists almost everywhere [[159], p.51]. Let us denote by $s_n(f;x)$ and $\widetilde{s}_n(f;x)$ the n^{th} partial sums of series (1.1.1) and (1.1.2), respectively, which are given by [[159], p.49]

$$s_n(f;x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x+t)D_n(t)dt, \quad (1.1.4)$$

$$\widetilde{s}_n(f;x) = -\frac{1}{\pi}\int_{-\pi}^{\pi} f(x+t)\widetilde{D}_n(t)dt, \quad (1.1.5)$$

where $D_n(t)$ and $\widetilde{D}_n(t)$ called the Dirichlet kernel and the Dirichlet conjugate kernel, respectively, are given by

$$D_n(t) = \frac{1}{2} + \sum_{k=1}^{n}\cos(kt) = \frac{\sin(n+1/2)t}{2\sin(t/2)}, \quad (1.1.6)$$

$$\widetilde{D}_n(t) = \sum_{k=1}^{n}\sin(kt) = \frac{\cos(t/2) - \cos(n+1/2)t}{2\sin(t/2)}. \quad (1.1.7)$$

We also use the following notations throughout the thesis

$$\phi_x(t) = f(x+t) + f(x-t) - 2f(x),$$

$$\psi_x(t) = f(x+t) - f(x-t).$$

Note: It is known [151] that the conjugate Fourier series need not be a Fourier series. For example, the series

$$\sum_{n=2}^{\infty}\frac{\cos(nx)}{\log n},$$

is a Fourier series, but the associated conjugate series

$$-\sum_{n=2}^{\infty}\frac{\sin(nx)}{\log n},$$

is not a Fourier series. Hence a separate study of conjugate Fourier series (1.1.2) is justified.

1.1.2 Double Trigonometric Fourier series

Let f be a 2π-periodic function in each variable and Lebesgue integrable over the two dimensional torus $T^2 := [-\pi,\pi]\times[-\pi,\pi]$. Then the double trigonometric Fourier series of f is defined by

$$f(x,y) \sim \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\hat{f}(m,n)e^{i(mx+ny)}, \quad (1.1.8)$$

where

$$\hat{f}(m,n) = \frac{1}{(2\pi)^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f(u,v) e^{-i(mu+nv)} du dv$$

are the Fourier coefficients of the function f.

The double sequence of symmetric rectangular partial sums associated with the Fourier series of f is given by

$$s_{mn}(x,y) = \sum_{k=-m}^{m} \sum_{l=-n}^{n} \hat{f}(k,l) e^{i(kx+ly)},$$

and its integral representation is given by

$$s_{mn}(x,y) = \frac{1}{\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f(x+u, y+v) D_m(u) D_n(v) du dv, \tag{1.1.9}$$

where $D_k(t) = \frac{\sin(k+\frac{1}{2})t}{2\sin(t/2)}$ is the Dirichlet kernel.

As we know, there are functions whose Fourier series diverge almost everywhere [36] or converges slowly. In this case, one can use summability methods in order to represent the function as a trigonometric polynomial.

1.2 Summability Methods

The theory of summability is concerned with the generalization of the notion of the limit of a sequence or sum of a series which is usually affected by an auxiliary sequence of linear means of the given sequences or series. A summability method is said to be regular if it sums all convergent series to its Cauchy's sum. The most well known methods of summability are linear regular methods. Summability methods are classified into two broad categories:

1. Methods based on a sequence-to-sequence transformation, usually termed as T-method.

2. Methods based on a sequence-to-function transformation, usually termed as Φ-method.

1.2.1 T-Method

In T-method, the sequence of partial sums of an infinite series is transformed into another sequence using an infinte matrix. Such methods are usually called sequence-to-sequence transformations. Let $\sum u_n$ be an infinite series with sequence of partial sums $\{s_n\}$ and let $T \equiv (a_{n,k})$ be an infinite matrix. By the T-transform (or T-means) of sequence $\{s_n\}$, we mean the sequence

$$\tau_n = \sum_{k=0}^{\infty} a_{n,k} s_k, \quad n = 0, 1, 2, \dots .$$

Definition: The infinite series $\sum u_n$ is said to be summable to s by T-method (or T-summable to s), if

$$\lim_{n\to\infty} \tau_n = s.$$

Regularity of T-Method: The T-method is said to be regular, if

$$\lim_{n\to\infty} s_n = s \Rightarrow \lim_{n\to\infty} \tau_n = s.$$

The necessary and sufficient conditions for the matrix T to be regular [152; 135] are given as follows:

1. $\sup_n \sum_{k=0}^{\infty} |a_{n,k}| < \infty, \quad n = 0, 1, 2, \dots,$
2. $\lim_{n\to\infty} a_{n,k} = 0, \quad k = 0, 1, 2, \dots,$
3. $\lim_{n\to\infty} \sum_{k=0}^{\infty} a_{n,k} = 1.$

1.2.2 Φ-Method

In Φ-method, the sequence of partial sums of an infinite series is transformed into a function of continuous variable. Such methods are usually called sequence-to-function transformations. Let $\sum u_n$ be an infinite series with sequence of partial sums $\{s_n\}$ and let $\{\phi_n(x)\}$ be a sequence of continuous functions, by the Φ-transform (or Φ-means) of sequence $\{s_n\}$, we mean the function

$$\tau(x) = \sum_n \phi_n(x) s_n,$$

where x is a continuous parameter.

Definition: Suppose $\tau(x)$ is defined for all $x \geq 0$ and x is a continuous parameter which tends to infinity. Then, the infinite series $\sum u_n$ is said to be summable to s by Φ-method (or Φ-summable to s), if

$$\lim_{x \to \infty} \tau(x) = s.$$

Regularity of Φ-Method: The Φ-method is said to be regular, if

$$\lim_{n \to \infty} s_n = s \Rightarrow \lim_{x \to \infty} \tau(x) = s.$$

The necessary and sufficient conditions for the regularity of Φ-method [28] are as follows:

1. $\sum_n |\phi_n(x)|$ is convergent for $x \geq 0$,
2. $\sum_n |\phi_n(x)| < M$, where M is constant independent of $x > x_0$,
3. $\sum_n \phi_n(x) \to 1$ as $x \to \infty$,
4. $\phi_n(x) \to 0$ as $x \to \infty$, for every n.

1.2.3 Some Particular Cases of Summability Methods

Some of the most familiar methods of summability are those that are associated with the names of great mathematicians like Cesàro, Euler, Hausdorff, Hölder, Nörlund, Riesz, Borel, etc. We now briefly describe some important summability methods.

1. **Cesàro Method:** The Cesàro summability method is named after the famous Italian mathematician E. Cesàro [12] who, in 1890, proposed it for positive integers. Knopp [53], Hardy and Chapman [18; 19] further extended this method for $\delta > -1$.
   Let $\delta > -1$ be given. The mtrix $T \equiv (a_{n,k})$ defined by
   $$a_{n,k} = \begin{cases} A_{n-k}^{\delta-1}/A_n^{\delta}, & 0 \leq k \leq n, \\ 0, & k > n, \end{cases}$$
   where
   $$A_n^{\delta} = \frac{\Gamma(n+\delta+1)}{\Gamma(n+1)\Gamma(\delta+1)} = \sum_{k=0}^{n} A_k^{\delta-1},$$

is called Cesàro matrix of order δ and the associated matrix method is called the Cesàro method of order δ. The series $\sum u_n$ is said to be Cesàro (C,δ)-summable to s, if

$$\tau_n^\delta = \frac{1}{A_n^\delta}\sum_{k=0}^{n} A_{n-k}^{\delta-1}s_k \to s, \text{ as } n \to \infty. \tag{1.2.1}$$

This method is regular for $\delta \geq 0$.

2. **Harmonic Method:** The Harmonic summability method was introduced by M. Riesz [126] in 1924.

   The matrix $T \equiv (a_{n,k})$ defined by

$$a_{n,k} = \begin{cases} \frac{1}{(n+1-k)\log(n+1)}, & 1 \leq k \leq n; \\ 0, & k > n. \end{cases}$$

   is called Harmonic matrix and the associated matrix method is called the Harmonic method. The series $\sum u_n$ is said to be Harmonic $(H,1)$-summable to s, if

$$\tau_n = \frac{1}{\log(n+1)}\sum_{k=0}^{n}\frac{s_k}{(n+1-k)} \to s, \text{ as } n \to \infty. \tag{1.2.2}$$

   It can be verified that Harmonic method is regular. McFadden [79] proved that every series summable by Harmonic method is summable (C,δ), $\delta > 0$.

3. **Nörlund Method:** The Nörlund summability method was first introduced by G.F. Voronoi in 1902 and was rediscovered by Nörlund [113] in 1919. This method is therefore sometimes referred as the Nörlund-Voronoi method.

   Let $\{p_n\}$ be a real valued sequence with $p_0 > 0$, $P_n := \sum_{k=0}^{n} p_k$ $(n \in \mathbb{N}_0)$ and $p_{-1} = P_{-1} = 0$ be given. Then the matrix $T \equiv (a_{n,k})$ defined by

$$a_{n,k} = \begin{cases} \frac{p_{n-k}}{P_n}, & 0 \leq k \leq n; \\ 0, & k > n, \end{cases}$$

   is called Nörlund matrix and the associated matrix method is called the Nörlund method. The series $\sum u_n$ is said to be Nörlund (N,p_n)-summable to s, if

$$\tau_n = \frac{1}{P_n}\sum_{k=0}^{n} p_{n-k}s_k \to s, \text{ as } n \to \infty. \tag{1.2.3}$$

This method is regular if and only if

$$\lim_{n\to\infty}\frac{p_k}{P_n}=0\ \ \forall k,\quad \text{and}\quad \sum_{k=0}^{n}|p_k|=O(|P_n|).$$

In particular, if we take $a_{n,k}=\frac{1}{n+1}$, then the (N,p_n)-mean reduces to $(H,1)$-mean and if $a_{n,k}=\frac{\Gamma(n+\delta)}{\Gamma(n+1)\Gamma(\delta)}$, then the (N,p_n)-mean reduces to (C,δ)-mean.

4. **Riesz Method:** The Riesz summability method was introduced by Riesz [126]. Let $\{p_n\}$ be a real valued sequence with $p_0>0$, $P_n:=\sum_{k=0}^{n}p_k$ $(n\in\mathbb{N}_0)$ and $p_{-1}=P_{-1}=0$ be given. Then the matrix $T\equiv(a_{n,k})$ defined by

$$a_{n,k}=\begin{cases}\frac{p_k}{P_n}, & 0\le k\le n;\\ 0, & k>n,\end{cases}$$

is called Riesz matrix and the associated matrix method is called the Riesz method. The series $\sum u_n$ is said to be Riesz (R,p_n)-summable to s, if

$$\tau_n=\frac{1}{P_n}\sum_{k=0}^{n}p_ks_k\to s,\ \text{as}\,n\to\infty. \tag{1.2.4}$$

This method is regular if and only if

$$\lim_{n\to\infty}\frac{p_k}{P_n}=0\ \ \forall k,\quad \text{and}\quad \sum_{k=0}^{n}|p_k|=O(|P_n|).$$

5. **Euler Method:** The Euler summability method was first applied by Euler for $q=1$. Later the method was extended to arbitrary values of $q>-1$ by Knopp [53]. This method is therefore sometimes referred as the Euler-Knopp summation method.

The matrix $T\equiv(a_{n,k})$ defined by

$$a_{n,k}=\begin{cases}\binom{n}{k}\frac{q^{n-k}}{(1+q)^n}, & 0\le k\le n;\\ 0, & k>n,\end{cases}$$

is called Euler matrix and the associated matrix method is called the Euler method. The series $\sum u_n$ is said to be Euler (E,q)-summable to s, if

$$\tau_n^q=\frac{1}{(1+q)^n}\sum_{k=0}^{n}\binom{n}{k}q^{n-k}s_k\to s,\ \text{as}\,n\to\infty. \tag{1.2.5}$$

This method is regular for $q>0$.

6. **Borel Method:** The Borel summability method was introduced by Borel [11] in 1899. If we take

$$\phi_n(x) = e^{-x}\frac{x^n}{n!}, \quad 0 \leq x < \infty,$$

then the method reduces to Borel summability method. The series $\sum u_n$ is said to be Borel (B, x)-summable to s, if

$$\tau(x) = e^{-x}\sum_{n=0}^{\infty}\frac{x^n}{n!}s_n \to s, \text{ as } x \to \infty. \tag{1.2.6}$$

It can be verified that Borel method is regular. Borel methods are more powerful than the Euler methods for $q > 0$.

7. **Product of Two Summability Methods:** Let $A \equiv (a_{n,m})$ and $B \equiv (b_{n,m})$ be two summability matrices. Then the product summability method is obtained by superimposing the B-summability on A-summability to get BA-summability. The BA-means of sequence $\{s_n\}$ is defined by

$$\tau_n^{B.A} = \sum_{r=0}^{n} b_{n,r}\sum_{k=0}^{r} a_{r,k}s_k.$$

Similarly, we can define product of a ϕ-method and a T-method or product of two ϕ-methods. The product summability method is more powerful than the individual summability method.

Example Consider the infinite series $1 - 3\sum_{n=1}^{\infty}(-2)^{n-1}$.

Here $s_n = 1 - 3\sum_{k=1}^{n}(-2)^{k-1} = (-2)^n$.

The $(C, 1)$-means of the given series are

$$C_n^1 = \frac{1}{n+1}\sum_{k=0}^{n}(-2)^k = \frac{1-(-2)^{n+1}}{3(n+1)}.$$

Since $\lim_{n\to\infty} C_n^1$ does not exist, the given series is not $(C, 1)$-summable. Also $(E, 1)$-means of the given series are

$$E_n^1 = \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}(-2)^k = (-1)^n,$$

and series is not $(E, 1)$-summable. The $(C, 1)(E, 1)$-means of the given series are $(CE)_n^1 = \frac{1}{n+1}\sum_{k=0}^{n}E_k^1 = \frac{1}{n+1}\sum_{k=0}^{n}(-1)^k \to 0$ as $n \to \infty$, and the given series is $(C, 1)(E, 1)$- summable.

1.3 Basic Definitions and Notations

1. **Banach Space:** A complete normed space is called Banach space.

   Here we consider some well known examples of Banach space of functions which are used in our work.

   (i) The space of all real valued continuous functions over $T := [a, b]$ is denoted by $C(T)$ (or $C[a, b]$) and is equipped with the norm

   $$||f||_c = \sup\{|f(x)| : x \in T\}.$$

   (ii) The space of all real valued continuous functions over $T^2 := [a, b] \times [c, d]$ is denoted by $C(T^2)$ and is equipped with the norm

   $$||f||_c = \sup\{|f(x, y)| : (x, y) \in T^2\}.$$

   (iii) The space of all real valued functions integrable in the Lebesgue sense with p^{th} power [essentially bounded] over $T := [a, b]$ is denoted by $L^p(T),\ 1 \leq p < \infty[p = \infty]$ and is equipped with the norm

   $$||f||_p = \begin{cases} \left(\frac{1}{2\pi}\int_a^b |f(x)|^p\, dx\right)^{\frac{1}{p}}, & p \geq 1; \\ ess \sup\limits_{x \in T} |f(x)|, & p = \infty. \end{cases}$$

   (iv) The space of all real valued functions integrable in the Lebesgue sense with p^{th} power [essentially bounded] over $T^2 := [a, b] \times [c, d]$ is denoted by $L^p(T^2),\ 1 \leq p < \infty[p = \infty]$ and is equipped with the norm

   $$||f||_p = \begin{cases} \left(\frac{1}{2\pi}\int_c^d\int_a^b |f(x, y)|^p\, dxdy\right)^{\frac{1}{p}}, & p \geq 1; \\ ess \sup\limits_{(x,y) \in T^2} |f(x, y)|, & p = \infty. \end{cases}$$

2. **Modulus of Continuity:** For $f \in C(T)$, the functions defined by

   $$\begin{aligned} \omega_1(\delta) = \omega_1(f; \delta) &= \sup_x \sup_{|h| \leq \delta} \{|f(x + h) - f(x)|\}, \\ \omega_2(\delta) = \omega_2(f; \delta) &= \sup_x \sup_{|h| \leq \delta} \{|f(x + h) + f(x - h) - 2f(x)|\}, \end{aligned}$$

   are called the first and the second order modulus of continuity, respectively.

For $f \in C(T^2)$, the total modulus of continuity of first order is defined by

$$\omega_1(f;u,v) = \sup_{x,y} \sup_{|h|\le u, |\eta|\le v} \big\{|f(x+h,y+\eta) - f(x,y)|\big\},$$

while the two partial moduli of continuity of first order are defined by

$$\omega_{1,x}(f;u) = \omega_1(f;u,0) = \sup_{x,y} \sup_{|h|\le u} \big\{|f(x+h,y) - f(x,y)|\big\},$$
$$\omega_{1,y}(f;v) = \omega_1(f;0,v) = \sup_{x,y} \sup_{|\eta|\le v} \big\{|f(x,y+\eta) - f(x,y)|\big\}.$$

For $f \in C(T^2)$, the total modulus of continuity of second order is defined by

$$\omega_2(f;u,v) = \sup_{x,y} \sup_{|h|\le u, |\eta|\le v} \big\{|f(x+h,y+\eta) + f(x-h,y+\eta)$$
$$+ f(x+h,y-\eta) + f(x-h,y-\eta) - 4f(x,y)|\big\},$$

while the two partial moduli of continuity of second order are defined by

$$\omega_{2,x}(f;u) = \frac{1}{2}\omega_2(f;u,0) = \sup_{x,y} \sup_{|h|\le u} \{|f(x+h,y) + f(x-h,y) - 2f(x,y)|\},$$
$$\omega_{2,y}(f;v) = \frac{1}{2}\omega_2(f;0,v) = \sup_{x,y} \sup_{|\eta|\le v} \{|f(x,y+\eta) + f(x,y-\eta) - 2f(x,y)|\}.$$

3. **Integral Modulus of Continuity:** For $f \in L^p(T)$, the functions defined by

$$\omega_1^p(\delta) = \omega_1^p(f;\delta) = \sup_{|h|\le\delta} \{||f(x+h) - f(x)||_p\},$$
$$\omega_2^p(\delta) = \omega_2^p(f;\delta) = \sup_{|h|\le\delta} \{||f(x+h) + f(x-h) - 2f(x)||_p\},$$

are called the first and the second order integral modulus of continuity, respectively.

For $f \in L^p(T^2)$, the total integral modulus of continuity of first order is defined by

$$\omega_1^p(f;u,v) = \sup_{|h|\le u, |\eta|\le v} \big\{||f(x+h,y+\eta) - f(x,y)||_p\big\},$$

while the two partial integral moduli of continuity of first order are defined by

$$\omega_{1,x}^p(f;u) = \omega_1^p(f;u,0) = \sup_{|h|\le u} \big\{||f(x+h,y) - f(x,y)||_p\big\},$$
$$\omega_{1,y}^p(f;v) = \omega_1^p(f;0,v) = \sup_{|\eta|\le v} \big\{||f(x,y+\eta) - f(x,y)||_p\big\}.$$

For $f \in L^p(T^2)$, the total integral modulus of continuity of second order is defined by

$$\omega_2^p(f;u,v) = \sup_{|h|\le u, |\eta|\le v} \big\{ ||f(x+h,y+\eta)+f(x-h,y+\eta) + f(x+h,y-\eta)+f(x-h,y-\eta)-4f(x,y)||_p \big\},$$

while the two partial integral moduli of continuity of second order are defined by

$$\omega_{2,x}^p(f;u) = \frac{1}{2}\omega_2^p(f;u,0) = \sup_{|h|\le u}\{||f(x+h,y)+f(x-h,y)-2f(x,y)||_p\},$$
$$\omega_{2,y}^p(f;v) = \frac{1}{2}\omega_2^p(f;0,v) = \sup_{|\eta|\le v}\{||f(x,y+\eta)+f(x,y-\eta)-2f(x,y)||_p\}.$$

4. **Some Function Classes:**

(a) **Lipschitz Class:** First we list some function classes of the functions of one variable:

For $0<\alpha\le 1$

$Lip\alpha = \{f \in C(T) : f(x+t)-f(x) = O(t^\alpha)\}$.

For $0<\alpha\le 1$ and $p\ge 1$

$Lip(\alpha,p) = \{f \in L^p(T) : ||f(x+t)-f(x)||_p = O(t^\alpha)\}$.

For a modulus of continuity type function $\xi(t)$ and $p\ge 1$

$Lip(\xi(t),p) = \{f \in L^p(T) : ||f(x+t)-f(x)||_p = O(\xi(t))\}$.

For a modulus of continuity type function $\xi(t)$, $p\ge 1$ and $\beta\ge 0$

$W(L^p,\xi(t)) = \left\{f \in L^p(T) : ||(f(x+t)-f(x))\sin^\beta\left(\frac{x}{2}\right)||_p = O(\xi(t))\right\}$.

If $\beta=0$, then $W(L^p,\xi(t)) = Lip(\xi(t),p)$, for $\xi(t)=t^\alpha(0<\alpha\le 1)$, $Lip(\xi(t),p)=Lip(\alpha,p)$, and $Lip(\alpha,p)=Lip\alpha$ as $p\to\infty$.

Also, we have

$$Lip\alpha \subseteq Lip(\alpha,p) \text{ and } Lip(\xi(t),p)\subseteq W(L^p,\xi(t)).$$

Now we list some function classes of the functions of two variables:

For $0<\alpha\le 1$

$Lip\alpha = \{f\in C(T^2) : \omega(f;\delta) = O(\delta^\alpha)\}$.

For $0<\alpha,\beta\le 1$

$Lip(\alpha,\beta)=\{f\in C(T^2):\omega_{1,x}(f;u)=O(u^\alpha),\omega_{1,y}(f;v)=O(v^\beta)\}$

For $0<\alpha,\beta\leq 1$ and $p\geq 1$

$Lip(\alpha,\beta;p)=\{f\in L^p(T^2):\omega^p_{1,x}(f;u)=O(u^\alpha),\omega^p_{1,y}(f;v)=O(v^\beta)\}$.

For a positive increasing function $\psi(u,v)$ and $p>1$

$Lip(\psi(u,v);p)=\left\{f\in C(T^2):|f(x+u,y+v)-f(x,y)|\leq M\left(\frac{\psi(u,v)}{(uv)^{1/p}}\right)\right\}$.

For a positive increasing function $\psi(u,v)$ and $p>1$

$Lip(\psi(u,v))_{L^p}=\left\{f\in L^p(T^2):||f(x+u,y+v)-f(x,y)||_p\leq M\left(\frac{\psi(u,v)}{(uv)^{1/p}}\right)\right\}$.

As $p\to\infty$, $Lip(\alpha,\beta;p)=Lip(\alpha,\beta)$.

(b) **Zygmund Class:** For $0<\alpha,\ \beta\leq 2$

$Zyg(\alpha,\beta)=\{f\in C(T^2):\omega_{2,x}(f;u)=O(u^\alpha),\omega_{2,y}(f;v)=O(v^\beta)\}$.

For $0<\alpha,\ \beta\leq 2$ and $p\geq 1$

$Zyg(\alpha,\beta;p)=\{f\in L^p(T^2):\omega^p_{2,x}(f;u)=O(u^\alpha),\omega^p_{2,y}(f;v)=O(v^\beta)\}$.

As $p\to\infty$, $Zyg(\alpha,\beta;p)=Zyg(\alpha,\beta)$.

(c) **Function of Bounded Variation:** Let f be a real valued function defined on $[a,b]$. Then the total variation of f on $[a,b]$ is defined by

$$V_a^b(f)=V_f[a,b]=\sup_P\left\{\sum_{k=1}^n|f(x_i)-f(x_{i-1})|\right\},$$

where supremum is taken over all the possible partitions of $[a,b]$. If $V_a^b(f)<\infty$ then we say f is of bounded variation on $[a,b]$ and we write $f\in BV[a,b]$.

(d) **Hölder Space:** For $0<\alpha\leq 1$ and some positive constant K, the well known Hölder space H_α introduced by Prössdorf [118], is defined by

$$H_\alpha=\{f\in C(T):|f(x+t)-f(x)|=O\left(|t|^\alpha\right)\}.$$

The Hölder space H_α is a Banach space under the following norm

$$||f||_\alpha=||f(x)||_c+\sup_x\sup_{t\neq 0}\frac{|f(x+t)-f(x)|}{|t|^\alpha}.$$

For a modulus of continuity type function ω, the well known Hölder space H^ω introduced by Leindler [68], is defined by

$$H^\omega=\{f\in C(T):|f(x+t)-f(x)|=O\left(\omega(|t|)\right)\}.$$

The Hölder space H^{ω} is a Banach space under the following norm

$$||f||^{\omega} = ||f(x)||_c + \sup_x \sup_{t \neq 0} \frac{|f(x+t) - f(x)|}{\omega(|t|)}.$$

In 1996, Das et al. [21] introduced a Banach space $H_{\alpha,p}$ by generalizing H_{α}-space, defined by

$$H_{\alpha,p} = \{f \in L^p(T) : ||f(x+t) - f(x)||_p = O\left(|t|^{\alpha}\right)\}$$

with the following norm

$$||f||_{\alpha,p} = ||f(x)||_p + \sup_{t \neq 0} \frac{||f(x+t) - f(x)||_p}{|t|^{\alpha}}.$$

In 2002, Das et al. [22] introduced a Banach space H_p^{ω} by generalizing $H_{\alpha,p}$-space, defined by

$$H_p^{\omega} = \{f \in L^p(T) : ||f(x+t) - f(x)||_p = O\left(\omega(|t|)\right)\}$$

with the following norm

$$||f||_p^{\omega} = ||f(x)||_p + \sup_{t \neq 0} \frac{||f(x+t) - f(x)||_p}{\omega(|t|)}.$$

(e) **Zygmund Space:** For $0 < \alpha \leq 1$ and some positive constant K, the well known Zygmund space Z_{α}, is defined by [62; 136]

$$Z_{\alpha} = \{f \in C(T) : |f(x+t) + f(x-t) - 2f(x)| = O\left(|t|^{\alpha}\right)\}.$$

The Zygmund space Z_{α} is a Banach space under the following norm

$$||f||_{\alpha} = ||f(x)||_c + \sup_x \sup_{t \neq 0} \frac{|f(x+t) + f(x-t) - 2f(x)|}{|t|^{\alpha}}.$$

For a modulus of continuity type function ω, the well known Zygmund space Z^{ω}, is defined by

$$Z^{\omega} = \{f \in C(T) : |f(x+t) + f(x-t) - 2f(x)| = O\left(\omega(|t|)\right)\}.$$

The Zygmund space Z^{ω} is a Banach space under the following norm

$$||f||^{\omega} = ||f(x)||_c + \sup_x \sup_{t \neq 0} \frac{|f(x+t) + f(x-t) - 2f(x)|}{\omega(|t|)}.$$

Further, $Z_{\alpha,p}$-space was introduced by generalizing Z_α-space, defined by

$$Z_{\alpha,p} = \{f \in L^p(T) : ||f(x+t) + f(x-t) - 2f(x)||_p = O\left(|t|^\alpha\right)\}$$

with the following norm

$$||f||_{\alpha,p} = ||f(x)||_p + \sup_{t\neq 0} \frac{||f(x+t) + f(x-t) - 2f(x)||_p}{|t|^\alpha}.$$

Z_p^ω-space was introduced by generalizing $Z_{\alpha,p}$-space, defined by

$$Z_p^\omega = \{f \in L^p(T) : ||f(x+t) + f(x-t) - 2f(x)||_p = O\left(\omega(|t|)\right)\}$$

with the following norm

$$||f||_p^\omega = ||f(x)||_p + \sup_{t\neq 0} \frac{||f(x+t) + f(x-t) - 2f(x)||_p}{\omega(|t|)}.$$

There are many more function spaces, such as Basove spaces [3], Sobolev space [114; 156; 144], generalized Sobolev space [116] and Dunkl-Sobolev space [145] etc.

5. **Fourier Approximation:** If a function f in L^p-space is approximated by a polynomial $T_n(x)$ of degree $\leq n$ (which is either partial sums or some summability means of the Fourier series of f), then the error of approximation $E_n(f)$, in terms of n, is given by

$$E_n(f) = \min_{T_n} \| f(x) - T_n(x) \|_p .$$

The polynomial $T_n(x)$ is known as the Fourier approximant of f, and this method of approximation is called Fourier approximation. If $T_n^*(x)$ is the polynomial of best approximation, then

$$E_n(f) = \min_{T_n} \| f(x) - T_n(x) \|_p = \| f(x) - T_n^*(x) \|_p .$$

6. **Almost Convergence:** The concept of almost convergence of sequences was introduced and studied by G.G. Lorentz in 1948 [73]. A sequence $\{x_n\}$ is said to be almost convergent to a limit L, if

$$\lim_{n\to\infty} \frac{1}{n+1} \sum_{i=k}^{k+n} x_i = L, \text{ for all } k \in \mathbb{N}.$$

Móricz and Rhoades [104] extended the definition of almost convergence to double sequences of real numbers $\{x_{mn}\}$, almost converging to L; if

$$\lim_{m,n\to\infty} \frac{1}{(m+1)(n+1)} \sum_{i=k}^{k+m} \sum_{j=l}^{l+n} x_{ij} = L, \text{ for all } k, l \in \mathbb{N}.$$

1.4 Some Important Tools

To prove our results, we need the following well known results of mathematical analysis:

1. **Hölder Inequality:** For $1 \leq p \leq \infty$ and q such that $\frac{1}{p} + \frac{1}{q} = 1$, if $f \in L^p[a,b]$ and $g \in L^q[a,b]$, then $fg \in L^1[a,b]$, and

$$||fg||_1 \leq ||f||_p ||g||_q.$$

2. **Minkowski Inequality:** Let $f, g \in L^p[a,b]$, $1 \leq p \leq \infty$. Then $f+g \in L^p[a,b]$ and

$$||f+g||_p \leq ||f||_p + ||g||_p.$$

3. **Generalized Minkowski Inequality:** Let $f(u,v) \in L^p([a,b] \times [c,d])$ for $p \geq 1$. Then

$$\left\{ \int_a^b \left| \int_c^d f(u,v)dv \right|^p du \right\}^{1/p} \leq \int_c^d \left\{ \int_a^b |f(u,v)|^p du \right\}^{1/p} dv.$$

4. **Abel Lemma:** Let $\{a_n\}$ be a sequence of real numbers whose partial sums $s_n = \sum_{k=1}^n a_k$, satisfy

$$|s_n| \leq M, \quad n = 1, 2, 3, ...,$$

for some $M \in \mathbb{R}$, and let $\{b_n\}$ be a monotonic sequence of non-negative real numbers. Then
(i) if $\{b_n\}$ is a non-increasing sequence, we have

$$\sum_{k=1}^n a_k b_k \leq M b_1, \quad n = 1, 2, 3, ...,$$

(ii) if $\{b_n\}$ is a non-decreasing sequence, we have

$$\sum_{k=1}^{n} a_k b_k \leq 2M b_{n+1}, \qquad n = 1, 2, 3, \dots .$$

5. **Abel Transformation:** Let $\{a_n\}$ and $\{b_n\}$ be two sequences of real numbers. Then

$$\sum_{k=0}^{n} a_k b_k = \sum_{k=0}^{n-1} A_k \triangle b_k + b_n A_n,$$

and for $m \geq 1$

$$\sum_{k=m}^{n} a_k b_k = \sum_{k=m}^{n-1} A_k \triangle b_k + A_n b_n - A_{m-1} b_m,$$

where

$$A_k = \sum_{r=0}^{k} a_r \;\text{ and }\; \triangle b_k = b_k - b_{k+1}.$$

1.5 Literature Survey

The first significant result in the approximation theory is the Weierstrass Approximation Theorem, which is pivotal in developing the Approximation theory. The Weierstrass Approximation Theorem was introduced by Carl Weierstrass in 1885, which states that, any continuous function (continuous periodic function) defined on a closed and bounded interval can be uniformly approximated by a sequence of polynomials (trigonometric polynomials) to a desired degree of accuracy. This result has been further generalized to the L^p-space, which states that, any L^p function defined on a closed and bounded interval can be uniformly approximated by a sequence of polynomials to a desired degree of accuracy, i.e. if $f(t) \in L^p[a, b]$ $(p \geq 1)$, then for every $\epsilon > 0$ there exists a polynomial $P(t)$ such that $||f - P||_p < \epsilon$. Since then extensive research has been done in various directions concerning approximation of functions and today this is one of the richest field of Mathematics.

Lebesgue [67] obtained the result for approximation of $f \in Lip\alpha$ by the partial sums of its Fourier series and showed that $|s_n(f; x) - f(x)| = O\left(n^{-\alpha} \log n\right).$ Bernstien [5] used Cesàro summability of order 1, i.e. $(C, 1)$-means, for $f \in Lip\alpha$ and obtained $|\sigma_n(f; x) - f(x)| = O(n^{-\alpha})$, for $0 < \alpha < 1$ and $|\sigma_n(f; x) - f(x)| =$

$O(n^{-1}\log n)$, for $\alpha = 1$. Further this problem is considered by Salem and Zygmund [129] and they obtained $|\sigma_n(f;x) - f(x)| = O(n^{-\alpha})$, for monotonic $f \in Lip\alpha$. Jackson [42] determined a degree of approximation of continuous periodic function by trigonometric polynomial of order $\omega(f;n^{-1})$. Alexits [1] extended the results of Bernstien [5] to Cesáro summability of order δ, for $f \in Lip\alpha$. The Bernsein's result [5] was further improved by Alexits and Kralik [2]. The result of Jackson [42] has been studied further by Quade [119] in $Lip(\alpha,p)$-class. He studied the degree of approximation of functions by $s_n(f;x)$, $(C,1)$-means of its Fourier series and by general trigonometric polynomials. Among the other reults, Quade [119] proved the following $\|s_n(f;x) - f(x)\|_p = O(n^{-\alpha})$, for $p > 1,\ 0 < \alpha \leq 1$ and $\|s_n(f;x) - f(x)\|_p = O(n^{-\alpha}\log n)$, for $p = 1, 0 < \alpha \leq 1$. Izumi [41] studied the degree of approximation for $Lip(\alpha,p)$ functions by $s_n(f;x)$ and obtained $\|s_n(f;x) - f(x)\|_p = O\left(n^{-\alpha-1/p}\right)$, for $p > 1,\ 0 < \alpha \leq 1$ and $\alpha p > 1$.

A considerable amount of work has been done on the Euler summability of Fourier series for continuous functions and Lipschitz functions by [52; 72; 10; 39]. Sahney and Rao [128], Khan [45], Holland et.al [40] studied the approximation problem using Nörlund and Riesz means of Fourier series for $f \in Lip(\alpha,p)$ with certain conditions on p_n. Holland [38] published a survey paper in which he discussed results related to approximation by trigonometric polynomials, representation of functions by their Fourier series and representation of functions by certain summability means of their Fourier series.

The problem of apprximation of $Lip\alpha$ and $Lip(\alpha,p)$ functions by Nörlund means and general matrix means was studied further by Mohapatra and Russell [95], Chandra [14], with monotonicity condition on p_n and $a_{n,k} \leq a_{n,k+1}$ in terms of modulus of continuity. Chandra and Mohapatra [17] proved results for the absolute Nörlund summability of Fourier series. Chandra [16] obtaied a degree of approximation for $Lip(\alpha,p)$ functions by Nörlund and Riesz means with monotonic weights p_n. The result was further considered by Leinder [70] by relaxing the monotonicity assumptions on p_n. Mittal et al. [90; 91] extended the results of [16; 70] by using general matrix means. Mittal [84] proved results for F_1-effectiveness of $(C,1)T$-method. The author [84] also discussed the F_1-effectiveness of $(C,1)(N,p_n)$ and (N,p_n)-methods on which the approximation results already have been proved. Mittal and Bhardwaj [85] studied the results of [14; 16; 119] further for a linear matrix operator. Mittal and Rhoades [87] have obtained the error estimation of f through a summability matrix which does not have monotone rows. A new class $Lip(\psi(t), p)$ $(p > 1)$ of 2π-periodic functions was introduced by Khan and Ram [50], defined by, $Lip(\psi(t), p) = \{f \in C[a, b] : |f(x + t) - f(x)| \leq M \left(\psi(t)\, t^{-1/p}\right)\}$, for $0 < t < \pi$ where M is a positive number independent of x and t. They determined a degree of approximation for $f \in Lip(\psi(t), p)$ using Euler means. Nigam [110] and Nigam and Sharma [112] obtained degree of approximation of order $\left((n + 1)^{1/p}\xi(1/(n + 1))\right)$, for $f \in Lip(\xi(t), p)$ using $(E, q)(C, 1)$ and $(C, 1)(E, q)$-summability methods, respectively. Although, $\xi(t)$ does not depend on p but these results depend on p. Lal and Yadav [66] obtained a degree of approximation of $f \in W(L^p, \xi(t))$ using the $(C, 1)(E, 1)$-means. Rhoades [123; 125] extended these results to Hausdorff matrices and proved that for $f \in W(L^p, \xi(t))$, $\|H_n(f; x) - f(x)\|_p = O\left(n^{\beta+1/p}\xi(1/n)\right)$, by assuming some additional conditions on $\xi(t)$. He also derived a result for $f \in Lip\alpha$. Lal [59] initiated a study of $(C, 1)(N, p_n)$-summability of Fourier series of $f \in W(L^p, \xi(t))$ which was further improved by Singh et al. [138]. These results depend on p. Singh and Sonker [140] pointed out some remarks in the results of [59; 66; 123] and studied the problem further using Hausdorff summability means.

Various researchers studied about the approximation of conjugate functions by conjugate Fourier series using various summability methods. The problem of approximation of conjugate function by Nörlund means of conjugate Fourier series has been studied by [26; 122; 130]. Lal and Kushwaha [60], and Kęska [44] proved some useful results on $E_n(\tilde{f})$, for $f \in Lip\alpha(0 < \alpha < 1)$ using triangular matrix and Euler-Hausdorff means of conjugate Fourier series. Lal and Singh [64] and Sonker and Singh [146] proved $E_n(\tilde{f}) = O\left(n^{-\alpha+1/p}\right)$ using $(C, 1)(E, q)$-means in $Lip(\alpha, p)$-class. Lenski et. al [54; 57] also studied about approximation of conjugate functions by some linear operators.

Motivated by these results in the $Lip\alpha$, $Lip(\alpha, p)$-classes, some authors studied the problem further for $Lip(\xi(t), p)$ and $W(L^p, \xi(t))$-classes. Mittal et al. [89] approximated $\tilde{f}$, conjugate of f in $W(L^p, \xi(t))$-class by linear opertors. Rhoades [124] and

Lal and Mishra [61] proved their results for the class $Lip(\xi(t), p)$ using the Hausdorff means and Euler-Hausdorff means, respectively. The authors obtained results free from p in terms of $\xi(t)$ with a single condition on $\xi(t)$. Mishra et al. [83] used $(C,1)(N,p_n)$-means of the conjugate Fourier series for the $W(L^p, \xi(t))$-class and proved $E_n(\tilde{f}) = O\left(n^{\beta/2+p/2}\xi(1/\sqrt{n})\right)$ with additional assumptions on $\xi(t)$ and with a monotonic weight p_n. Further, Mishra et al. [80] improved this result by dropping the monotonicity on p_n which in turn generalized the results of Lal [59]. Singh and Srivastava [141] approximated the conjugate of functions belonging to $W(L^p, \xi(t))$ by Hausdorff means of conjugate Fourier series and proved $E_n(\tilde{f}) = O\left((n+1)^{\beta+1/p}\xi(1/(n+1))\right)$. The authors [141] studied the problem again to obtain the results free from p, by using the $(C,1)T$-means and proved $E_n(\tilde{f}) = O\left((n+1)^{\beta}\xi(1/(n+1))\right)$. Statistical convergence of the Fourier series of functions studied by Mursaleen et al. [4; 29; 30] using various summability means.

The idea of almost convergence for single sequence was given by Lorentz [73]. Further this concept was extended for double sequence by Moricz and Rhoades [104]. The concept of almost convergence led to the formulation of various almost summability methods. King [51] investigated the regularity conditions for the almost summability matrices. Then, Mazhar and Siddiqui [77] applied the concept of almost convergence of sequences to almost convergence of trigonometric sequences. In [107], Nanda introduced the spaces of strongly almost summable sequence spaces which happened to be complete paranormed spaces under certain conditions. Sharma and Qureshi [133] and Qureshi [120] obtained the degree of approximation of functions belonging to $Lip\alpha$-class by almost Riesz and almost Nörlund means of their Fourier series by assuming that $\{p_n\}$ is a positive non-decreasing sequence in k. Later, Qureshi [121] determined the degree of approximation of functions belonging to $Lip\alpha$-class by almost Riesz means of their conjugate Fourier series by assuming that $\{p_n\}$ is a positive non-decreasing sequence in k. Working in the same direction, Lal [58] determined the degree of approximation of functions belonging to $W(L^p, \xi(t))$-class by almost matrix summability means of their Fourier series by assuming $(a_{n,k})$ non-decreasing in k. The problem was further studied by Mittal and Mishra [86] by assuming less restrictive conditions on the matrix elements $(a_{n,k})$. Mishra [81] studied the same problem for the conjugate of $f \in W(L^p, \xi(t))$ also. However, Mishra et.

al [82] studied the same problem for $f \in Lip(\alpha, p)$ through almost Riesz means, a particular case of matrix means used by Lal [58]. Recently, Deepmala and Piscoran [24] proved a theorem on the degree of approximation for a functions belonging to $W(L^p, \xi(t))$-class using almost Riesz means of its Fourier series with non-negative, non-decreasing weights p_n.

Alexits [1] studied the approximation of functions of H_α-class by Fejér means of Fourier series in the sup norm. In 1975, Prössdorf [118] determined a degree of approximation of functions in H_α-class by Fejér means of Fourier series in Hölder metric. The result is generalized by Chandra [13], he obtained a degree of approximation using Nörlund and Riesz means of Fourier series in H_α-space. Later Mohapatra and Chandra [93] determined the degree of approximation using Euler, Borel and Taylor means of Fourier series in H_α-space. The problem was again considered by Mohapatra and Chandra [94] using the matrix means and obtained the results of Prössdorf [118] and Chandra [13] as corollaries. In 1990, Chandra [15] reconsider the problem using Borel means and obtained a better estimate than what has been obtained in [93]. Leindler [68] generalized the result of Prössdorf [118] by introducing H^ω-space. Further, the problem of degree of approximation has been studied in H^ω-space by Totik [154], [155], Mazhar and Totik [78] and Shi and Sun [134] using different methods of summability.

In 1996, $H_{\alpha,p}$-space was introduced, and several results obtained previously by Mohapatra and Chandra [94] were generalized by Das et al. [21]. In 2000, Mittal and Rhoades [88] obtained the degree of approximation of functions in H^ω-space, and several results obtained previously by Singh [137] were generalized by removing the hypothesis of monotonicity of the rows of the matrix. In 2002, Das et al. [22] studied the approximation of functions belonging to space H_p^ω through partial sums of their Fourier series. The result was further considered and various results through Cesàro, Nörlund, Riesz and generalized de la Vallée-Poussin means were obtained by Leindler [71]. Krasniqi [56] obtained a degree of approximation of function of space H_p^ω through generalized Nörlund means of their Fourier series. In 2012, a result on the degree of approximation through matrix means was obtained by Singh and Sonker [139] which is generalization of results of Leindler [71]. Further, Deger [25] obtained a result on the degree of approximation through matrix means by weakening the monotonicity conditions on $(a_{n,k})$, which is generalization of results of Leindler [71] and Singh and Sonker [139]. In 2013, Z_p^ω-space was considered, and some results on the degree of approximation of functions using matrix-Euler means were obtained by Lal and Shireen [62]. Some other forms of the generalized Zygmund classes have been investigated by Leindler [69], Móricz and Németh [100] and Móricz [99]. Recently, some theorems on the degree of approximation of functions in the Z_p^ω- space using Hausdorff means were proved by Singh et al. [136].

The classical Dirichlet-Jordan's theorem [[159], p.57] states that if f is a 2π-periodic function of bounded variation on $[-\pi, \pi]$, then $s_n(f;x) \to \frac{1}{2}\big(f(x+0) + f(x-0)\big)$ as $n \to \infty$, where $f(x+0)$ and $f(x-0)$ denote the right and left hand limit of f at x, respectively. Steèkin [147] and Natanson [108] obtained a quantitative estimate of the rate of convergence of a 2π-periodic continuous function of bounded variation by $s_n(f;x)$. A quantitative version of Dirichlet-Jordan's theorem was given by Bojanic [7]. Later, Bojanic and Mazhar [8] generalized the result of Bojanic [7] and obtained an estimate by using Nörlund-Voronoi means. Further, Bojanic and Mazhar [9] generalized the result of Bojanic [7] and obtained an estimate by using Cesáro means of order $-1 < \alpha \leq 0$. Mazhar [75] obtained an estimate of the rate of convergence of functions of bounded variation using matrix means of Fourier series, which generalized the previous results. Dubey [27] also obtained an estimate by using matrix means. The analogous criterion for convergence of the conjugate Fourier series of function of bounded variation was given by Young [[159], p.59]. Later, Mazhar and Budaiwi [76] obtained the quantitative version of Young's theorem.

G. H. Hardy laid the foundation of classical studies on the double Fourier series at the beginning of the 20^{th} century. The theory of convergence and summability of double Fourier series has been discussed by number of researchers including Young [157], Moore [96], Titchmarsh [150], Gergen [31], Sunouchi [149]. The problem of convergence of the double Fourier series of the Lebesgue integrable periodic function by Cesàro means has been studied by Marcinkiewicz [74], Gruenwald [35], Chow [20]. Sharma [131; 132] studied the convergence of double Fourier series by Harmonic means for Lebesgue periodic functions and the result is analogous to the result of Chow [20]. Herriot [37] obtained results regarding convergence of the double Fourier series by Nörlund means, which are generalization of results obtained by [74; 35]. The convergence of the double Fourier series and corresponding conjugate Fourier series of functions of bounded variation have been studied by [97; 98; 153].

Móricz and Xianlianc Shi [105] studied the rate of uniform approximation of a 2π-periodic continuous function in Lipschitz class $Lip(\alpha, \beta)$ and in the Zygmund class $Zyg(\alpha, \beta), 0 < \alpha, \beta \leq 1$, by Cesàro means of positive order of its double Fourier series. They also obtained the result for the conjugate function by using corresponding Cesàro means. Further, Móricz and Rhoades [101] obtained the rate of uniform approximation of $f(x, y)$ in the Lipα-class by the Nörlund means of its Fourier series. After that, Móricz and Rhoades [102] obtained the rate of uniform approximation of continuous function $f(x, y)$ in the Lipschitz class $Lip(\alpha, \beta)$ and in the Zygmund class $Zyg(\alpha, \beta), 0 < \alpha, \beta \leq 1$, by Nörlund means of its Fourier series. In [102], they also obtained the result for the conjugate function by using the corresponding Nörlund means. Móricz and Rhoades [103], and Mittal and Rhoades [92] generalized the results of [101], [102] and [105] for a 2π-periodic continuous function in the Lipschitz class $Lip(\alpha, \beta)$ and in the Zygmund class $Zyg(\alpha, \beta), 0 < \alpha, \beta \leq 1$, by using rectangular double matrix means of its double Fourier series. Lal ([65], [63]) obtained results for the double Fourier series using double matrix means and product matrix means. Also, Khan [47] studied the problem of approximation of functions belonging to the class $Lip(\psi(u, v); p)$ by Jackson type operator. Further, Khan and Ram [49] studied the problem of approximation functions belonging to the class $Lip(\psi(u, v); p)$ by Gauss-Weierstrass integral of the double Fourier series of $f(x, y)$. Khan et al. [48] extended the result of Khan [47] for n-dimensional Fourier series. In [55], Krasniqi obtained a result on the degree of approximation of functions of class $Lip(\psi(u, v); p)$, by Euler means of the double Fourier series of function $f(x, y)$. In fact, he generalized the result of Khan [50] for two-dimension and for n-dimensions.

In our work we shall confine our investigation to the approximation problems of periodic functions in subclasses of L^p-spaces or functions of bounded variation by means of trigonometric Fourier series using summability methods.

1.6 Objective of the Study

The objective of the present study is to fill a gap in the literature, by making some advancement in the field of Fourier approximation. The pointwise objectives are as follows:

- To study the Fourier approximation of the functions in $W(L^p, \xi(t))$-class and their conjugates using almost matrix means (Chapter 2).
- To study the Fourier approximation of the conjugates of functions belong to generalized Hölder space $H_{\alpha,p}$ using Borel exponential means (Chapter 3).
- To study the Fourier approximation of functions in generalized Zygmund space Z_p^{ω} using product means (Chapter 4).
- To study the error estimation of conjugates of functions of bounded variation, by their conjugate Fourier series using matrix means (Chapter 5).
- To introduce the function classes $Lip(\psi(u,v))_{L^p}$, $Lip(\alpha,\beta;p)$ and $Zyg(\alpha,\beta;p)$ and to determine the degree of approximation of two-variable functions belonging to them using almost Euler means (Chapter 6).

Chapter 2

Approximation of functions in a weighted Lipschitz class by almost matrix summability methods

This chapter is divided into two sections. In the first section, we discuss a problem of approximation by almost Riesz means and in the second section we generalize the problem discussed in the first section and, in addition, we also obtain a result for conjugate functions.

2.1 Definitions

Let f be a 2π-periodic function belonging to the space $L^p := L^p[0, 2\pi] (p \geq 1)$. Then the almost Riesz means of the sequence $\{s_n(f;x)\}$ are defined by

$$R_{n,m}(f;x) = \frac{1}{P_n} \sum_{k=0}^{n} p_k Q_{k,m}(f;x), \tag{2.1.1}$$

where

$$Q_{k,m}(f;x) = \frac{1}{(k+1)} \sum_{\gamma=m}^{m+k} s_\gamma(f;x). \tag{2.1.2}$$

The sequence of partial sums $\{s_n(f;x)\}$ is said to be almost Riesz summable to s, if

$$R_{n,m}(f;x) \to s \text{ as } n \to \infty,$$

uniformly with respect to m [133].

Let $T \equiv (a_{n,k})$ be an infinite regular triangular matrix. We assume that $\{a_{n,k}\}$ be a non-negative and monotonic sequence in k, and $A_{n,\tau} = \sum_{k=\tau}^{n} a_{n,k}$, with $A_{n,0} = 1$. The almost T-means of the sequence $s_n(f;x)$ and $\widetilde{s}_n(f;x)$ are defined by

$$\tau_{n,m}(f;x) = \sum_{k=0}^{n} a_{n,k} S_{k,m}(f;x), \tag{2.1.3}$$

and

$$\widetilde{\tau}_{n,m}(f;x) = \sum_{k=0}^{n} a_{n,k} \widetilde{S}_{k,m}(f;x), \tag{2.1.4}$$

where

$$S_{k,m}(f;x) = \frac{1}{(k+1)} \sum_{\nu=m}^{m+k} s_\nu(f;x), \quad \text{and} \quad \widetilde{S}_{k,m}(f;x) = \frac{1}{(k+1)} \sum_{\nu=m}^{m+k} \widetilde{s}_\nu(f;x).$$

The sequences $\{s_n(f;x)\}$ and $\{\widetilde{s}_n(f;x)\}$ are said to be almost T-summable, if

$$\tau_{n,m}(f;x) \to s \text{ as } n \to \infty, \quad \widetilde{\tau}_{n,m}(f;x) \to s' \text{ as } n \to \infty,$$

uniformly with respect to m [133], wher s and s' are finite numbers.

For a modulus of continuity type function $\xi(t)$, $p \geq 1$ and $\beta \geq 0$, a function $f \in W(L^p, \xi(t))$, if

$$\left\| (f(x+t) - f(x))\, sin^\beta \left(\frac{x}{2}\right) \right\|_p = O\left(\xi(t)\right), \quad t > 0. \tag{2.1.5}$$

Khan [46] was the first to use the weight function of the form $\sin^{\beta p}(x/2)$.

2.2 Approximation of functions by almost Riesz summability methods

Recently, Deepmala and Piscoran [24] obtained a result on the degree of approximation of functions belnging to $W(L^p, \xi(t))(p \geq 1)$-class using almost Riesz means of its Fourier series with non-negative, non-decreasing weights p_n. They proved the following theorem:

Theorem 2.2.1. *[24] Assume f is a 2π-periodic function and integrable in the sense of Lebesgue over $[0, 2\pi]$. Then the degree of approximation of $f \in W(L^p, \xi(t))$ $(p \geq$*

1)*-class with* $0 \le \beta \le 1-1/p$ *by an almost Riesz means of its Fourier series is given by*

$$||R_{n,m}(f;x) - f(x)||_p = O\left(P_n^{\beta+1/p}\xi(P_n^{-1})\right), \quad \forall n > 0, \tag{2.2.1}$$

provided that the positive increasing function $\xi(t)$ *has the following properties:*

$$\xi(t)/t \textit{ is non-increasing in } t, \tag{2.2.2}$$

$$\left(\int_0^{\pi/P_n} \left(\frac{|\phi_x(t)|}{\xi(t)}\right)^p sin^{\beta p}(t/2)dt\right)^{1/p} = O(1), \tag{2.2.3}$$

and

$$\left(\int_{\pi/P_n}^{\pi} \left(\frac{t^{-\delta}|\phi_x(t)|}{\xi(t)}\right)^p dt\right)^{1/p} = O(P_n^{\delta}), \tag{2.2.4}$$

where δ *is an arbitrary number such that* $(\beta-\delta)q-1 > 0,\ p^{-1}+q^{-1} = 1,\ 1 \le p \le \infty$, $\phi_x(t) = f(x+t)-2f(x)+f(x-t)$, *and conditions* (2.2.3) *and* (2.2.4) *hold uniformly in* x.

Remark 2.2.1. We note that in the statement of Theorem 2.2.1, the authors have taken $p \ge 1$, but in the proof they have used the Hölder's inequality for $p > 1$. Therefore, the proof is not valid for $p = 1$.

Remark 2.2.2. For $p = \infty$, conditions (2.2.3) and (2.2.4) will not hold in the present form.

Remark 2.2.3. Using the remarks of Zhang [[158], p.1140], we note that the assumptions $0 \le \beta \le 1 - 1/p$ with $1/p + 1/q = 1$ and $(\beta - \delta)q - 1 > 0$ of Theorem 2.2.1 imply that $\delta < 0$. In this case, from condition (2.2.4), Theorem 2.2.1 is true for the function f which is a constant almost everywhere and thus the result is trivial.

2.2.1 Reformulation of the Problem and Main Result

Motivated by the above remarks, we reconsider the problem of Theorem 2.2.1 and note that the authors defined the function class $W(L^p, \xi(t))(p \ge 1)$ with the weight function $sin^{\beta p}(x/2)$ whereas the deviation $||\tau_{n,m}(f;x) - f(x)||_p$ is measured in the ordinary L_p-norm. Actually, the function class $W(L^p, \xi(t))$ defined in (2.1.5) is a subclass of the weighted $L^p[0, 2\pi]$-space with the weight function $sin^{\beta p}(x/2)$. So it is pertinent to measure the deviation in the weighted norm defined by

$$||f||_{p,\beta} = \left(\frac{1}{2\pi} \int_0^{2\pi} |f(x)|^p sin^{\beta p}(x/2) dx \right)^{1/p}, \ p \geq 1. \tag{2.2.5}$$

We reformulate the problem of Theorem 2.2.1 for the almost Riesz means and measure the deviation in the weighted norm defined in (2.2.5). More precisely, we prove:

Theorem 2.2.2. *Let f be a 2π-periodic function in $W(L^p, \xi(t))(p \geq 1)$-class and let $\{p_n\}$ be a non-negative, monotonic sequence such that*

$$(n+1)\max\{p_0, p_n\} = O(P_n). \tag{2.2.6}$$

Then the degree of approximation of f by almost Riesz means of its Fourier series is given by

$$||R_{n,m}(f;x) - f(x)||_{p,\beta} = O\left(\xi\left(\frac{\pi}{n+1} \right) + (n+1)^{-\sigma} \right), \tag{2.2.7}$$

where

$$t^{-\sigma}\xi(t) \text{ is non-decreasing for some } 0 < \sigma < 1. \tag{2.2.8}$$

Note that the conditions (2.2.3) and (2.2.4) of Theorem 2.2.1 have been relaxed in Theorem 2.2.2. Also, we prove the theorem for both non-decreasing and non-increasing sequence $\{p_n\}$ with condition (2.2.6).

2.2.2 Lemma

We require the following lemma for the proof of our main result.

Lemma 2.2.3. *Let $L_{n,m}(t) = \frac{1}{2\pi} \sum_{k=0}^{n} \frac{p_k}{(k+1)P_n} \frac{\sin((k+2m+1)\frac{t}{2}) \sin((k+1)\frac{t}{2})}{\sin^2(\frac{t}{2})}$. Then*

$$(i) \ L_{n,m}(t) = O(n+1), \quad \text{for } 0 < t \leq \frac{\pi}{(n+1)}.$$

$$(ii) \ L_{n,m}(t) = O\left(\frac{1}{(n+1)t^2} \right), \quad \text{for } \frac{\pi}{(n+1)} < t \leq \pi.$$

Proof.(i) For $0 < t \leq \frac{\pi}{n+1}$, using $\sin(t/2) \geq t/\pi$ and $\sin nt \leq n \sin t$, we have

$$\begin{aligned}
|L_{n,m}(t)| &= \left| \frac{1}{2\pi} \sum_{k=0}^{n} \frac{p_k}{(k+1)P_n} \frac{\sin((k+2m+1)\frac{t}{2}) \sin((k+1)\frac{t}{2})}{\sin^2(\frac{t}{2})} \right| \\
&\leq \frac{1}{2\pi P_n} \sum_{k=0}^{n} \frac{p_k}{(k+1)} \frac{(k+1)(k+2m+1)\sin^2(\frac{t}{2})}{\sin^2(\frac{t}{2})} \\
&= \frac{1}{2\pi P_n} \sum_{k=0}^{n} p_k (k+2m+1) \\
&= O(n+1).
\end{aligned}$$

(ii) For $\pi/(n+1) < t \leq \pi$, using $\sin(t/2) \geq t/\pi$ and $\sin nt \leq n \sin t$, we have

$$\begin{aligned}
|L_{n,m}(t)| &= \left| \frac{1}{2\pi} \sum_{k=0}^{n} \frac{p_k}{(k+1)P_n} \frac{\sin((k+2m+1)\frac{t}{2}) \sin((k+1)\frac{t}{2})}{\sin^2(\frac{t}{2})} \right| \\
&\leq \frac{1}{2\pi P_n} \left| \sum_{k=0}^{n} \frac{p_k}{(k+1)} \frac{(k+1)\sin(\frac{t}{2})\sin((k+2m+1)\frac{t}{2})}{\sin(\frac{t}{2})} \frac{\pi}{t} \right| \\
&= \frac{1}{2\pi P_n t} \left| \sum_{k=0}^{n} p_k \sin\left((k+2m+1)\frac{t}{2}\right) \right|.
\end{aligned}$$

Then, using condition (2.2.6), monotonicity of $\{p_n\}$ and Abel's lemma, we have

$$\left|L_{n,m}(t)\right| = O\left(\frac{1}{(n+1)t^2}\right). \qquad \left(\text{Since } \left| \sum_{k=0}^{n} \sin((k+2m+1)t/2) \right| = O(1/t)\right)$$

2.2.3 Proof of Theorem 2.2.2

Using the integral representation of $Q_{k,m}(f;x)$ and definition of $R_{n,m}(f;x)$ given in (2.1.1), we have

$$\begin{aligned}
R_{n,m}(f;x) - f(x) &= \frac{1}{P_n} \sum_{k=0}^{n} p_k \{Q_{k,m}(f;x) - f(x)\} \\
&= \frac{1}{2\pi P_n} \int_0^{\pi} \phi_x(t) \sum_{k=0}^{n} \frac{p_k}{(k+1)} \frac{\cos mt - \cos(k+m+1)t}{2\sin^2(\frac{t}{2})} dt \\
&= \frac{1}{2\pi P_n} \int_0^{\pi} \phi_x(t) \sum_{k=0}^{n} \frac{p_k}{(k+1)} \frac{\sin((k+2m+1)t/2)\sin((k+1)t/2)}{\sin^2(\frac{t}{2})} dt \\
&= \int_0^{\pi} \phi_x(t) L_{n,m}(t) dt,
\end{aligned}$$

which on applying the generalized Minkowski inequality gives

$$||R_{n,m}(f;x)-f(x)||_{p,\beta} = \left\{\frac{1}{2\pi}\int_0^{2\pi}\left|\int_0^{\pi}\phi_x(t)L_{n,m}(t)dt\right|^p sin^{\beta p}(x/2)dx\right\}^{1/p}$$
$$\leq \int_0^{\pi}\left(\frac{1}{2\pi}\int_0^{2\pi}|\phi_x(t)|^p sin^{\beta p}(x/2)dx\right)^{1/p}|L_{n,m}(t)|dt.$$

Since $\phi_x(t) \in W(L^p, \xi(t))$ due to $f \in W(L^p, \xi(t))$ [[142], p.3, Note 1], we have

$$||R_{n,m}(f;x)-f(x)||_{p,\beta} = \int_0^{\pi} O(\xi(t))|L_{n,m}(t)|dt$$
$$= O(1)\Bigg[\int_0^{\pi/(n+1)}\xi(t)|L_{n,m}(t)|dt + \int_{\pi/(n+1)}^{\pi}\xi(t)|L_{n,m}(t)|dt\Bigg]$$
$$= I_1 + I_2 \text{ say.} \tag{2.2.9}$$

Using Lemma 2.2.3 part(i), increasing nature of $\xi(t)$ and the mean value theorem for integrals, we have

$$I_1 = O(1)\int_0^{\pi/(n+1)}\xi(t)|L_{n,m}(t)|dt = O(1)\int_0^{\pi/(n+1)}(n+1)\xi(t)dt$$
$$= O\bigg(\xi\Big(\frac{\pi}{n+1}\Big)\bigg). \tag{2.2.10}$$

Using Lemma 2.2.3 part(ii), condition (2.2.8) and the mean value theorem for integrals, we have

$$I_2 = O(1)\int_{\pi/(n+1)}^{\pi}\xi(t)|L_{n,m}(t)|dt$$
$$= O(1)\frac{1}{(n+1)}\int_{\pi/(n+1)}^{\pi}\frac{t^{\sigma}}{t^2}\frac{\xi(t)}{t^{\sigma}}dt = O(1)\frac{\xi(\pi)\pi^{-\sigma}}{(n+1)}\left(\frac{\pi}{n+1}\right)^{\sigma-1}$$
$$= O\big((n+1)^{-\sigma}\big). \tag{2.2.11}$$

Collecting (2.2.9) – (2.2.11), we have

$$||R_{n,m}(f;x)-f(x)||_{p,\beta} = O\bigg(\xi\left(\frac{\pi}{n+1}\right) + (n+1)^{-\sigma}\bigg).$$

Which completes the proof of Theorem 2.2.2.

2.3 Approximation of functions and their conjugates by almost matrix summability methods

Lal [58] determined the degree of approximation of functions in $W(L^p, \xi(t))$-class by almost matrix summability means of its Fourier series by assuming $(a_{n,k})$ non-decreasing in k. He obtained the following result:

Theorem 2.3.1. *[58] Let* $T \equiv (a_{n,k})$ *be an infinite regular triangular matrix such that* $\{a_{n,k}\}$ *is non-negative and non-decreasing sequence in* k, $A_{n,\tau} = \sum_{k=0}^{\tau} a_{n,n-k}$, *and* $A_{n,n} = 1 \; \forall \; n$. *If* $f(x)$ *is a* 2π*-periodic function belonging to the class* $W(L^p, \xi(t))(p \geq 1)$ *then its degree of approximation by* $\tau_{n,m}(f;x) = \sum_{k=0}^{n} \frac{a_{n,n-k}}{n-k+1} \sum_{\nu=m}^{n-k+m} s_\nu$, *i.e., almost matrix means of its Fourier series is given by*

$$||\tau_{n,m}(f;x) - f(x)||_p = O\big(\xi(1/n)n^{\beta+\frac{1}{p}}\big),$$

provided $\xi(t)$ *satisfies the following conditions*

$$\left\{\int_0^{1/n} \left(\frac{t|\phi_x(t)|}{\xi(t)}\right)^p \sin^{\beta p} t dt\right\}^{\frac{1}{p}} = O\left(\frac{1}{n}\right), \tag{2.3.1}$$

$$\left\{\int_{1/n}^{\pi} \left(\frac{t^{-\delta}|\phi_x(t)|}{\xi(t)}\right)^p dt\right\}^{\frac{1}{p}} = O(n^\delta), \tag{2.3.2}$$

where δ *is an arbitrary number such that* $q(1-\delta) - 1 > 0$, *conditions* (2.3.1) *and* (2.3.2) *hold uniformly in* x, *and* $p^{-1} + q^{-1} = 1$.

For more details about conditions (2.3.1) and (2.3.2), one can see [45]. The above problem was further studied by Mittal and Mishra [86] by dropping the monotonicity condition on the matrix elements $(a_{n,k})$. They obtained the following result:

Theorem 2.3.2. *[86] Let* $T \equiv (a_{n,k})$ *be an infinite regular triangular matrix such that* $\{a_{n,k}\}$ *is non-negative,* $A_{n,k} = \sum_{r=k}^{n} a_{n,r}$, *and* $A_{n,0} = 1 \; \forall \; n$. *If* $f(x)$ *is a* 2π*-periodic function belonging to weighted* $W(L^p, \xi(t))(p \geq 1)$*-class, then its degree of approximation by* $\tau_{n,m}(f;x)$, *i.e., almost matrix means of its Fourier series is given by*

$$||\tau_{n,m}(f;x) - f(x)||_p = O\big(\xi(1/(n+1))(n+1)^{\beta+\frac{1}{p}}\big),$$

provided $\xi(t)$ *satisfies the following conditions*

$$\left\{\int_0^{\pi/(n+1)} \left(\frac{t|\phi_x(t)|}{\xi(t)}\right)^p \sin^{\beta p} t dt\right\}^{\frac{1}{p}} = O\left(\frac{1}{n+1}\right), \tag{2.3.3}$$

$$\left\{\int_{\pi/(n+1)}^{\pi} \left(\frac{t^{-\delta}|\phi_x(t)|}{\xi(t)}\right)^p dt\right\}^{\frac{1}{p}} = O((n+1)^\delta), \tag{2.3.4}$$

uniformly in x, *and* $p^{-1} + q^{-1} = 1$, *where* δ *is an arbitrary number such that* $q(1-\delta) - 1 > 0$, *and* $\xi(t)/t$ *is decreasing function of* t, *holds.*

Mishra et. al [80] and Mishra [81] also studied the same problem for conjugate of $f \in W(L^p, \xi(t))$. However, Mishra et. al [80] studied the problem for $f \in Lip(\alpha, p)$, a subclass of $W(L^p, \xi(t))$ through almost Riesz means, a particular case of matrix means used by Lal [58].

Remark 2.3.1. We note that in the statement of Theorem 2.3.1 and Theorem 2.3.2, the authors have taken $p \geq 1$, but in the proofs they used the Hölder's inequality for $p > 1$. Therefore, the proofs do not work for $p = 1$.

Remark 2.3.2. While proving the theorems, the authors [58; 86] obtained the integral of the form

$$\begin{aligned} I_1 &= O(\xi(1/n))\left\{ \int_0^{1/n} \left(\frac{dt}{t^{(1+\beta)q}} \right) \right\}^{\frac{1}{q}} \\ &= O(\xi(1/n) n^{\beta+1-\frac{1}{q}}) = O(\xi(1/n) n^{\beta+\frac{1}{p}}), \end{aligned}$$

the above calculation is correct only if $\beta < -\frac{1}{p}$, which implies $\beta < 0$. This contradicts the definition of weighted Lipschitz class $W(L^p, \xi(t))$. In fact, for $\beta \geq 0$ the above integral is not convergent.

Further, in the proof of Theorem 2.3.1, the author [[58], p.73] obtained the integral of the form

$$\begin{aligned} I_2 &= \left\{ \int_{1/n}^{\pi} \left| \frac{t^{-\delta} \sin^\beta t \phi_x(t)}{\xi(t)} \right|^p dt \right\}^{\frac{1}{p}} \left\{ \int_{1/n}^{\pi} \left(\frac{\xi(t)}{t^{-\delta}} \frac{M_{n,m}(t)}{\sin^\beta t} \right)^q dt \right\}^{\frac{1}{q}} \\ &= \left\{ \int_{1/n}^{\pi} \left| \frac{t^{-\delta} \phi_x(t)}{\xi(t)} \right|^p dt \right\}^{\frac{1}{p}} \left\{ \int_{1/n}^{\pi} \left(\frac{\xi(t)}{t^{\beta+1-\delta}} A_{n,\tau} \right)^q dt \right\}^{\frac{1}{q}}. \end{aligned}$$

The author then make the substitution $y = 1/t$ to obtain

$$\begin{aligned} I_2 &= O(n^\delta)\left\{ \int_{1/\pi}^{n} \left(\frac{\xi(1/y)}{y^{\delta-\beta-1}} A_{n,y} \right)^q \frac{dy}{y^2} \right\}^{\frac{1}{q}} \\ &= O(n^\delta \xi(1/n) A_{n,\tau})\left\{ \int_{1/\pi}^{n} \frac{dy}{y^{q(\delta-\beta-1)+2}} \right\}^{\frac{1}{q}} \\ &= O(\xi(1/n) n^{\beta+\frac{1}{p}}). \end{aligned}$$

In the above calculation at the second step, the author has used the inequality $\pi|\sin t| \geq 2|t|$ for $(1/n) < t < \pi$. But this ineqaulity holds only for $|t| \leq \pi/2$.

Then, the author applied mean value theorem for integrals by assuming $\xi(1/y)$ as a non-decreasing function. But, $\xi(1/y)$ is non-increasing function as $\xi(y)$ is increasing function. Therefore, the evaluation of the integral I_2 appears to be incorrect.

However, the author [86] used $(\delta-\beta)q+1<0$ instead of $q(1-\delta)-1>0$, in the evaluation of I_2.

2.3.1 Reformulation of the Problems and Main Results

Motivated by the above remarks, we reconsider the problems and obtain the deviation $||\tau_{n,m}(f;x)-f(x)||$ in the weighted norm. We also relaxed the conditions on $\xi(t)$ used in the papers [58] and [86]. More precisely, we prove the following theorems:

Theorem 2.3.3. *Let f be a 2π-periodic function in $W(L^p,\xi(t))(p\geq 1)$-class and let $\{a_{n,k}\}$ be a non-negative, monotonic sequence in k such that*

$$(n+1)\max\{a_{n,0},a_{n,n}\}=O(1). \tag{2.3.5}$$

Then the degree of approximation of f by almost matrix means of its Fourier series is given by

$$||\tau_{n,m}(f;x)-f(x)||_{p,\beta}=O\big(\xi(\pi/(n+1))+(n+1)^{-\sigma}\big), \tag{2.3.6}$$

where

$$t^{-\sigma}\xi(t) \text{ is non-decreasing for some } 0<\sigma<1. \tag{2.3.7}$$

Theorem 2.3.4. *Let f be a 2π-periodic function in $W(L^p,\xi(t))(p\geq 1)$-class and let $\{a_{n,k}\}$ be a non-negative, monotonic sequence in k with (2.3.5). Then the degree of approximation of $\widetilde{f}$, conjugate of f, by almost matrix means of its conjugate Fourier series is given by*

$$||\widetilde{\tau}_{n,m}(f;x)-\widetilde{f}(x)||_{p,\beta}=O\big(\xi(\pi/(n+1))+(n+1)^{-\sigma}\big),$$

where $\xi(t)$ satisfies the condition (2.3.7).

2.3.2 Lemmas

We require the following lemmas for the proof of our main results.

Lemma 2.3.5. *Let $M_{n,m}(t)=\frac{1}{2\pi}\sum_{k=0}^{n}a_{n,k}\frac{\sin((k+2m+1)t/2)\sin((k+1)t/2)}{(k+1)\sin^2(t/2)}$. Then*

(i) $M_{n,m}(t)=O(n+1)$, *for* $0<t\leq(\pi/(n+1))$.

(ii) $M_{n,m}(t)=O(1/(n+1)t^2)$, *for* $(\pi/(n+1))<t\leq\pi$.

Proof.(i) Using $\sin(t/2) \geq (t/\pi)$ and $\sin nt \leq n \sin t$, for $0 < t \leq (\pi/(n+1))$. We have

$$\begin{aligned}
|M_{n,m}(t)| &= \left| \frac{1}{2\pi} \sum_{k=0}^{n} \frac{a_{n,k}}{(k+1)} \frac{\sin((k+2m+1)t/2)\sin((k+1)t/2)}{\sin^2(t/2)} \right| \\
&\leq \frac{1}{2\pi} \sum_{k=0}^{n} \frac{a_{n,k}}{(k+1)} \frac{(k+1)(k+2m+1)\sin^2(t/2)}{\sin^2(t/2)} \\
&= \frac{1}{2\pi} \sum_{k=0}^{n} a_{n,k}(k+2m+1) \\
&= O(n+1).
\end{aligned}$$

(ii) Using $\sin(t/2) \geq (t/\pi)$ and $\sin nt \leq n \sin t$, for $(\pi/(n+1)) < t \leq \pi$. We have

$$\begin{aligned}
|M_{n,m}(t)| &= \left| \frac{1}{2\pi} \sum_{k=0}^{n} \frac{a_{n,k}}{(k+1)} \frac{\sin((k+2m+1)t/2)\sin((k+1)t/2)}{\sin^2(t/2)} \right| \\
&\leq \frac{1}{2\pi} \left| \sum_{k=0}^{n} \frac{a_{n,k}}{(k+1)} \frac{(k+1)\sin(t/2)\sin((k+2m+1)t/2)}{\sin(t/2)} \frac{\pi}{t} \right| \\
&= \frac{1}{2t} \left| \sum_{k=0}^{n} a_{n,k} \sin((k+2m+1)t/2) \right|.
\end{aligned}$$

Now, using condition (2.3.5), monotonicity of $(a_{n,k})$ and Abel's lemma, we have

$$M_{n,m}(t) = O(1/(n+1)t^2). \qquad \left(\text{Since} \left| \sum_{k=0}^{n} \sin((k+2m+1)t/2) \right| = O(1/t) \right)$$

Lemma 2.3.6. *Let* $\widetilde{M}_{n,m}(t) = \frac{1}{2\pi} \sum_{k=0}^{n} a_{n,k} \frac{\cos((k+2m+1)t/2)\sin((k+1)t/2)}{(k+1)\sin^2(t/2)}$. *Then*

(*i*) $\widetilde{M}_{n,m}(t) = O(1/t)$, *for* $0 < t \leq (\pi/(n+1))$.

(*ii*) $\widetilde{M}_{n,m}(t) = O(1/(n+1)t^2)$, *for* $(\pi/(n+1)) < t \leq \pi$.

Proof.(i) Using $\sin(t/2) \geq (t/\pi)$ and $\cos nt \leq 1$, for $0 < t \leq (\pi/(n+1))$.

We have

$$\left|\widetilde{M}_{n,m}(t)\right| = \frac{1}{2\pi}\left|\sum_{k=0}^{n}\frac{a_{n,k}}{(k+1)}\frac{\cos((k+2m+1)t/2)\sin((k+1)t/2)}{\sin^2(t/2)}\right|$$
$$\leq \frac{1}{2\pi}\sum_{k=0}^{n}\frac{a_{n,k}}{(k+1)}\frac{(k+1)\sin(t/2)}{\sin^2(t/2)}$$
$$= O(1/t)\sum_{k=0}^{n}a_{n,k}$$
$$= O(1/t).$$

(ii) Using $\sin(t/2) \geq (t/\pi)$ and $\cos nt \leq 1$, for $(\pi/(n+1)) < t \leq \pi$.

We have

$$\left|\widetilde{M}_{n,m}(t)\right| = \frac{1}{2\pi}\left|\sum_{k=0}^{n}\frac{a_{n,k}}{(k+1)}\frac{\cos((k+2m+1)t/2)\sin((k+1)t/2)}{\sin^2(t/2)}\right|$$
$$\leq \frac{1}{2\pi}\left|\sum_{k=0}^{n}\frac{a_{n,k}}{(k+1)}\frac{(k+1)\sin(t/2)\cos((k+2m+1)t/2)}{\sin(t/2)}\frac{\pi}{t}\right|$$
$$= \frac{1}{2t}\left|\sum_{k=0}^{n}a_{n,k}\cos((k+2m+1)t/2)\right|.$$

Now, using condition (2.3.5), monotonicity of $(a_{n,k})$ and Abel's lemma, we have

$$\widetilde{M}_{n,m}(t) = O(1/(n+1)t^2). \qquad \left(\text{Since } \left|\sum_{k=0}^{n}\cos((k+2m+1)t/2)\right| = O(1/t)\right)$$

2.3.3 Proof of Main Results

Proof of Theorem 2.3.3. Using the integral representation of $S_{k,m}(f;x)$ and definition of $\tau_{n,m}(f;x)$ given in (2.1.3), we have

$$\tau_{n,m}(f;x) - f(x) = \sum_{k=0}^{n}a_{n,k}\{S_{k,m}(f;x) - f(x)\}$$
$$= \frac{1}{2\pi}\int_0^{\pi}\phi_x(t)\sum_{k=0}^{n}\frac{a_{n,k}}{(k+1)}\frac{[\cos(mt) - \cos((k+m+1)t)]}{2\sin^2(t/2)}dt$$
$$= \frac{1}{2\pi}\int_0^{\pi}\phi_x(t)\sum_{k=0}^{n}\frac{a_{n,k}}{(k+1)}\frac{\sin((k+2m+1)t/2)\sin((k+1)t/2)}{\sin^2(t/2)}dt$$
$$= \int_0^{\pi}\phi_x(t)M_{n,m}(t)dt,$$

which on applying the generalized Minkowski inequality gives

$$\begin{aligned}||\tau_{n,m}(f;x)-f(x)||_{p,\beta} &= \left\{\frac{1}{2\pi}\int_0^{2\pi}\left|\int_0^{\pi}\phi_x(t)M_{n,m}(t)dt\right|^p sin^{\beta p}(x/2)dx\right\}^{1/p}\\ &\leq \int_0^{\pi}\left(\frac{1}{2\pi}\int_0^{2\pi}|\phi_x(t)|^p\sin^{\beta p}(x/2)dx\right)^{1/p}|M_{n,m}(t)|dt.\end{aligned}$$

Since $\phi_x(t)\in W(L^p,\xi(t))$ due to $f\in W(L^p,\xi(t))$ [[142], p.3, Note 1], we have

$$\begin{aligned}||\tau_{n,m}(f;x)-f(x)||_{p,\beta} &= \int_0^{\pi}O(\xi(t))|M_{n,m}(t)|dt\\ &= O(1)\left[\int_0^{\pi/(n+1)}+\int_{\pi/(n+1)}^{\pi}\right]\xi(t)|M_{n,m}(t)|dt\\ &= I_1+I_2 \text{ say.}\end{aligned}\tag{2.3.8}$$

Using Lemma 2.3.5 part(i), increasing nature of $\xi(t)$ and the mean value theorem for integrals, we have

$$\begin{aligned}I_1 &= O(1)\left[\int_0^{\pi/(n+1)}\xi(t)|M_{n,m}(t)|dt\right]=O(n+1)\left[\int_0^{\pi/(n+1)}\xi(t)dt\right]\\ &= O(\xi(\pi/(n+1))).\end{aligned}\tag{2.3.9}$$

Using Lemma 2.3.5 part(ii), condition (2.3.7) and the mean value theorem for integrals, we have

$$\begin{aligned}I_2 &= O(1)\left[\int_{\pi/(n+1)}^{\pi}\xi(t)|M_{n,m}(t)|dt\right]=O\left(\frac{1}{n+1}\right)\left[\int_{\pi/(n+1)}^{\pi}\frac{\xi(t)}{t^2}\right]\\ &= O\left(\frac{1}{n+1}\right)\left[\int_{\pi/(n+1)}^{\pi}\frac{t^{\sigma}}{t^2}\frac{\xi(t)}{t^{\sigma}}dt\right]=O\left(\frac{1}{n+1}\right)\left[\xi(\pi)\pi^{-\sigma}\left(\frac{\pi}{n+1}\right)^{\sigma-1}\right]\\ &= O((n+1)^{-\sigma}).\end{aligned}\tag{2.3.10}$$

Collecting (2.3.8)-(2.3.10), we have

$$||\tau_{n,m}(f;x)-f(x)||_{p,\beta}=O\big(\xi(\pi/(n+1))+(n+1)^{-\sigma}\big).$$

Which completes the proof of Theorem 2.3.3.

Proof of Theorem 2.3.4. Using the integral representation of $\widetilde{S}_{k,m}(f;x)$ and the definition of $\widetilde{\tau}_{n,m}(f;x)$ given in (2.1.4), we have

$$\begin{aligned}\widetilde{\tau}_{n,m}(f;x)-\widetilde{f}(x) &= \sum_{k=0}^{n} a_{n,k}\{\widetilde{S}_{k,m}(f;x)-\widetilde{f}(x)\}\\ &= \frac{1}{2\pi}\int_0^{\pi}\psi_x(t)\sum_{k=0}^{n}\frac{a_{n,k}}{(k+1)}\frac{[\sin((m+k+1)t)-\sin(mt)]}{2\sin^2(t/2)}dt\\ &= \frac{1}{2\pi}\int_0^{\pi}\psi_x(t)\sum_{k=0}^{n}\frac{a_{n,k}}{(k+1)}\frac{\cos((k+2m+1)t/2)\sin((k+1)t/2)}{\sin^2(t/2)}dt\\ &= \int_0^{\pi}\psi_x(t)\widetilde{M}_{n,m}(t)dt,\end{aligned}$$

which on applying the generalized Minkowski inequality gives

$$\begin{aligned}||\widetilde{\tau}_{n,m}(f;x)-\widetilde{f}(x)||_{p,\beta} &= \left\{\frac{1}{2\pi}\int_0^{2\pi}\left|\int_0^{\pi}\psi_x(t)\widetilde{M}_{n,m}(t)dt\right|^p sin^{\beta p}(x/2)dx\right\}^{1/p}\\ &\leq \int_0^{\pi}\left(\frac{1}{2\pi}\int_0^{2\pi}|\psi_x(t)|^p\sin^{\beta p}(x/2)dx\right)^{1/p}|\widetilde{M}_{n,m}(t)|dt.\end{aligned}$$

Since $\psi_x(t)\in W(L^p,\xi(t))$ due to $f\in W(L^p,\xi(t))$ [[142], p.3, Note 1], we have

$$\begin{aligned}&= \int_0^{\pi}O(\xi(t))|\widetilde{M}_{n,m}(t)|dt\\ &= O(1)\left[\int_0^{\pi/(n+1)}+\int_{\pi/(n+1)}^{\pi}\right]\xi(t)|\widetilde{M}_{n,m}(t)|dt\\ &= I_1+I_2 \ \ \text{say.}\end{aligned}\tag{2.3.11}$$

Using Lemma 2.3.6 part(i), condition (2.3.7) and the mean value theorem for integrals, we have

$$\begin{aligned}I_1 &= O(1)\left[\int_0^{\pi/(n+1)}\xi(t)|\widetilde{M}_{n,m}(t)|dt\right] = O(1)\left[\int_0^{\pi/(n+1)}\frac{\xi(t)}{t}dt\right]\\ &= O(1)\left[\int_0^{\pi/(n+1)}\frac{t^{\sigma}\xi(t)}{t^{1+\sigma}}dt\right]\\ &= O((n+1)^{\sigma}\xi(\pi/(n+1)))\left[\int_0^{\pi/(n+1)}t^{\sigma-1}dt\right]\\ &= O(\xi(\pi/(n+1))).\end{aligned}\tag{2.3.12}$$

Using Lemma 2.3.6 part(ii), condition (2.3.7) and the mean value theorems for integrals, we have

$$
\begin{aligned}
I_2 &= O(1)\left[\int_{\pi/(n+1)}^{\pi} \xi(t)|\widetilde{M}_{n,m}(t)|dt\right] = O\left(\frac{1}{n+1}\right)\left[\int_{\pi/(n+1)}^{\pi} \frac{\xi(t)}{t^2}\right] \\
&= O\left(\frac{1}{n+1}\right)\left[\int_{\pi/(n+1)}^{\pi} \frac{t^{\sigma}\xi(t)}{t^{2+\sigma}}dt\right] = O\left(\frac{1}{n+1}\right)\left[\xi(\pi)\pi^{-\sigma}\left(\frac{\pi}{n+1}\right)^{\sigma-1}\right] \\
&= O((n+1)^{-\sigma}). \qquad (2.3.13)
\end{aligned}
$$

Collecting (2.3.11)-(2.3.13), we have

$$||\widetilde{\tau}_{n,m}(f;x) - \widetilde{f}(x)||_{p,\beta} = O\big(\xi(\pi/(n+1)) + (n+1)^{-\sigma}\big).$$

Which completes the proof of Theorem 2.3.4.

2.3.4 Corollaries

We can derive the following corollaries from Theorem 2.3.3.

Corollary 2.3.1. The weighted class $W(L^p, \xi(t))$ becomes $Lip(\xi(t), p)$, for $\beta = 0$. Thus for $f \in Lip(\xi(t), p)$,

$$||\tau_{n,m}(f;x) - f(x)||_p = O\left(\xi\left(\frac{\pi}{n+1}\right) + (n+1)^{-\sigma}\right),$$

where $t^{-\sigma}\xi(t)$ is non-decreasing for some $0 < \sigma < 1$.

Corollary 2.3.2. The weighted class $W(L^p, \xi(t))$ becomes $Lip(\alpha, p)$, for $\beta = 0$ and $\xi(t) = t^{\alpha}$. In this case, the function $t^{-\sigma}\xi(t) = t^{\alpha-\sigma}$ is increasing for $0 < \sigma < \alpha \leq 1$. Thus for $f \in Lip(\alpha, p)$,

$$||\tau_{n,m}(f;x) - f(x)||_p = O\big((n+1)^{-\sigma}\big).$$

However, we can obtain the degree of approximation of a function $f \in Lip(\alpha, p)$ independently as under:
Putting $\xi(t) = t^{\alpha}$ in (2.3.8), we have

$$I_1 = O(1)\left[\int_0^{\pi/(n+1)} (n+1)t^{\alpha}dt\right] = O((n+1)^{-\alpha}),\ 0 < \alpha \leq 1, \qquad (2.3.14)$$

and

$$I_2 = O(1/(n+1))\left[\int_{\pi/(n+1)}^{\pi} \frac{t^{\alpha}}{t^2}dt\right] = \begin{cases} O((n+1)^{-\alpha}), & 0 < \alpha < 1; \\ O\big(\frac{\log(n+1)}{n+1}\big), & \alpha = 1. \end{cases} \qquad (2.3.15)$$

Combining (2.3.14) and (2.3.15), we have

$$||\tau_{n,m}(f;x) - f(x)||_p = \begin{cases} O((n+1)^{-\alpha}), & 0 < \alpha < 1; \\ O\big(\frac{\log(n+1)}{n+1}\big), & \alpha = 1. \end{cases}$$

Corollary 2.3.3. The weighted class $W(L^p, \xi(t))$ becomes $Lip\alpha$, for $\beta = 0$, $\xi(t) = t^\alpha$, and $p \to \infty$. Since

$$\lim_{p\to\infty} ||\tau_{n,m}(f;x) - f(x)||_p = ||\tau_{n,m}(f;x) - f(x)||_\infty ,$$

using Corollary 2.3.2, we have

$$||\tau_{n,m}(f;x) - f(x)||_\infty = O\big((n+1)^{-\sigma}\big), \quad 0 < \sigma < \alpha \leq 1.$$

Independently, we can obtain

$$||\tau_{n,m}(f;x) - f(x)||_\infty = \begin{cases} O((n+1)^{-\alpha}), & 0 < \alpha < 1; \\ O\big(\frac{\log(n+1)}{n+1}\big), & \alpha = 1. \end{cases}$$

Similar corollaries can be derived from Theorem 2.2.2 and Theorem 2.3.4, also.

Chapter 3

Borel summability of conjugate Fourier series in the Hölder metric

3.1 Definitions

Let f be a 2π-periodic function and Lebesgue integrable over $[0, 2\pi]$. Then the Borel exponential means of $\{\widetilde{s}_n(f;x)\}$ are defined by [[36], p.182]

$$\widetilde{B}_r(f;x) = e^{-r}\sum_{n=0}^{\infty}\frac{r^n}{n!}\widetilde{s}_n(f;x) \qquad (r > 0), \tag{3.1.1}$$

and the sequence $\{\widetilde{s}_n(f;x)\}$ is said to be Borel summable to s if

$$\widetilde{B}_r(f;x) = e^{-r}\sum_{n=0}^{\infty}\frac{r^n}{n!}\widetilde{s}_n(f;x) \to s \quad \text{as} \;\; r \to \infty.$$

In 1996, Das et al. [21] introduced a Banach space $H_{\alpha,p}$, by generalizing H_α-space, defined by

$$H_{\alpha,p} = \{f \in L^p[0, 2\pi] : ||f(x+t) - f(x)||_p \leq K|t|^\alpha\}$$

with the following norm

$$||f||_{\alpha,p} = ||f(x)||_p + \sup_{t\neq 0}\frac{||f(x+t) - f(x)||_p}{|t|^\alpha}. \tag{3.1.2}$$

The metric induced by the above norm is called generalized Hölder metric.

Note: Let $0 \leq \beta < \alpha \leq 1$ and $p \geq 1$. Then

$$||f||_{\beta,p} \leq (2\pi)^{\alpha-\beta}||f||_{\alpha,p}$$

which implies

$$H_{\alpha,p} \subseteq H_{\beta,p} \subseteq L^p, \quad p \geq 1.$$

3.2 Some Known Results

Borel summability of Fourier series has been studied by Stone [148], Sinvhal [143], Sahney and Kathal [127; 43], Mohapatra and Chandra [93; 15], Das et al. [23], among others. Recently, Nigam and Hadish [111] studied a problem of approximation of conjugate functions through product summability means. Here, we recall one of the result of Nigam and Hadish [111]

Theorem 3.2.1. *Let $f \in H_{\alpha,p}$. Then for $0 \leq \beta < \alpha \leq 1$, we have*

$$||\widetilde{T}_n(f;x) - \widetilde{f}(x)||_{\beta,p} = \begin{cases} O\left(\log(e(n+1))(n+1)^{\beta-\alpha}\right), & \text{if } 0 \leq \beta < \alpha < 1; \\ O\left(\frac{\log(e(n+1))\log(\pi(n+1))}{(n+1)}\right), & \text{if } \beta = 0, \alpha = 1, \end{cases}$$

where $\widetilde{T}_n(f;x)$ are the $T(C,1)$-means of the sequence $\{\widetilde{s}_n(f;x)\}$.

To the best of our knowledge, the degree of approximation of conjugate function by Borel means in Hölder metric has not been investigated so far. This motivated us to work in this direction. Our result improves Theorem 3.2.1.

3.3 Main Result

We prove the following theorem:

Theorem 3.3.1. *Let $f \in H_{\alpha,p}$. Then for $0 \leq \beta < \alpha \leq 1$, we have*

$$||\widetilde{B}_r(f;x) - \widetilde{f}(x)||_{\beta,p} = O\left(r^{-(\alpha-\beta)\sigma}\right) + O\left(r^{-(1-2\sigma)\delta}\right). \tag{3.3.1}$$

3.4 Lemmas

We require the following lemmas to prove our main result:

Lemma 3.4.1. *Let $f \in H_{\alpha,p}$ and $0 \leq \beta < \alpha \leq 1$. Then for $0 < t \leq \pi$ and $p \geq 1$*

$$(i) \qquad ||\psi_x(t)||_p = O(t^\alpha).$$

$$(ii) \qquad ||\psi_{x+y}(t) - \psi_x(t)||_p = O(1) \begin{cases} t^\alpha \\ |y|^\alpha \end{cases} .$$

$$(iii) \qquad ||\psi_{x+y}(t) - \psi_x(t)||_p = O\big(|y|^\beta t^{\alpha-\beta}\big).$$

Proof. (i) We omit this proof as it is an immediate consequence of definition of $H_{\alpha,p}$.

(ii) We have

$$\psi_{x+y}(t) - \psi_x(t) = \frac{1}{2}\{f(x+y+t) - f(x+y-t)\} - \frac{1}{2}\{f(x+t) - f(x-t)\},$$

which on applying the Minkowski inequality gives

$$\begin{aligned} ||\psi_{x+y}(t) - \psi_x(t)||_p &\leq \frac{1}{2}||f(x+y+t) - f(x+t)||_p + \frac{1}{2}||f(x+y-t) - f(x-t)||_p \\ &= O(t^\alpha). \end{aligned} \tag{3.4.1}$$

Now, writing

$$\psi_{x+y}(t) - \psi_x(t) = \frac{1}{2}\{f(x+y+t) - f(x+t)\} - \frac{1}{2}\{f(x+y-t) - f(x-t)\},$$

and proceeding as above, we get

$$||\psi_{x+y}(t) - \psi_x(t)||_p = O(|y|^\alpha). \tag{3.4.2}$$

(iii) From (3.4.1), we have

$$||\psi_{x+y}(t) - \psi_x(t)||_p = O(t^\alpha) = O\big(t^\beta t^{\alpha-\beta}\big) = O\big(|y|^\beta t^{\alpha-\beta}\big), \qquad \text{for } t \leq |y|. \tag{3.4.3}$$

If $t \geq |y|$, then

$$|y|^{\alpha-\beta} \leq t^{\alpha-\beta} \quad (since \alpha > \beta).$$

Which implies

$$\begin{aligned} ||\psi_{x+y}(t) - \psi_x(t)||_p &= O(|y|^\alpha) \quad \text{(using (3.4.2))} \\ &= O\big(|y|^\beta |y|^{\alpha-\beta}\big) \\ &= O\big(|y|^\beta t^{\alpha-\beta}\big). \end{aligned} \tag{3.4.4}$$

□

Lemma 3.4.2. *Let $0 < t \leq \pi$. Then for $c = 2/\pi^2$*

$$e^{-r(1-\cos t)} = O\big(e^{-rct^2}\big). \tag{3.4.5}$$

For the proof one can see [[23], Lemma 3].

3.5 Proof of Theorem 3.3.1

By using (1.1.3) and (1.1.5), we can write

$$\widetilde{s}_n(f;x) - \widetilde{f}(x) = \frac{1}{\pi}\int_0^{\pi} \psi_x(t)\frac{\cos(n+1/2)t}{\sin(t/2)}dt. \tag{3.5.1}$$

We write

$$\widetilde{\tau}_r(x) = \widetilde{B}_r(f;x) - \widetilde{f}(x). \tag{3.5.2}$$

therefore, from (3.1.1) and (3.5.1), we have

$$\begin{aligned}
\widetilde{\tau}_r(x) &= e^{-r}\sum_{n=0}^{\infty}\frac{r^n}{n!}\widetilde{s}_n(f;x) - \widetilde{f}(x) \\
&= e^{-r}\sum_{n=0}^{\infty}\frac{r^n}{n!}\big\{\widetilde{s}_n(f;x) - \widetilde{f}(x)\big\} \\
&= e^{-r}\sum_{n=0}^{\infty}\frac{r^n}{n!}\left\{\frac{1}{\pi}\int_0^{\pi}\psi_x(t)\frac{\cos(n+1/2)t}{\sin(t/2)}dt\right\} \\
&= \frac{1}{\pi}\int_0^{\pi}\frac{\psi_x(t)}{\sin(t/2)}\left\{e^{-r}\sum_{n=0}^{\infty}\frac{r^n}{n!}\cos(n+1/2)t\right\}dt. \qquad (3.5.3)
\end{aligned}$$

Let

$$M_n(t) \;:=\; e^{-r}\sum_{n=0}^{\infty}\frac{r^n}{n!}\cos(n+1/2)t. \tag{3.5.4}$$

Then, we have the following expression for $M_n(t)$

$$
\begin{aligned}
M_n(t) &= Re\left\{e^{-r}\sum_{n=0}^{\infty}\frac{r^n}{n!}e^{i(n+1/2)t}\right\} \\
&= e^{-r}Re\left\{e^{it/2}\sum_{n=0}^{\infty}\frac{r^n}{n!}e^{int}\right\} \\
&= e^{-r}Re\left\{e^{it/2}\sum_{n=0}^{\infty}\frac{(re^{it})^n}{n!}\right\} \\
&= e^{-r}Re\left\{e^{it/2}e^{re^{it}}\right\} \\
&= e^{-r}e^{r\cos t}\cos\big(t/2+r\sin t\big) \\
&= e^{-r(1-\cos t)}\cos\big(t/2+r\sin t\big). \qquad (3.5.5)
\end{aligned}
$$

Using (3.5.5) in (3.5.3), and then on applying the generalized Minkowski inequality, we get

$$
\begin{aligned}
\|\widetilde{\tau}_r(x)\|_p &\le \frac{1}{\pi}\int_0^{\pi}\frac{\|\psi_x(t)\|_p}{\sin(t/2)}e^{-r(1-\cos t)}\big|\cos\big(t/2+r\sin t\big)\big|dt \\
&= \frac{1}{\pi}\left\{\int_0^{\frac{1}{r^\sigma}}+\int_{\frac{1}{r^\sigma}}^{\pi}\right\}\frac{\|\psi_x(t)\|_p}{\sin(t/2)}e^{-r(1-\cos t)}\big|\cos\big(t/2+r\sin t\big)\big|dt \\
&= I_1+I_2 \quad \text{say.} \qquad (3.5.6)
\end{aligned}
$$

Using Lemma 3.4.1 part (i), $|\cos t|\le 1$ and $\sin(t/2)\ge t/\pi$, we have

$$
\begin{aligned}
I_1 &= \frac{1}{\pi}\int_0^{\frac{1}{r^\sigma}}\frac{\|\psi_x(t)\|_p}{\sin(t/2)}e^{-r(1-\cos t)}\big|\cos\big(t/2+r\sin t\big)\big|dt \\
&= O(1)\int_0^{\frac{1}{r^\sigma}}\frac{t^\alpha}{t}e^{-r(1-\cos t)}dt \\
&= O(1)\int_0^{\frac{1}{r^\sigma}}t^{\alpha-1}dt \\
&= O\left(r^{-\alpha\sigma}\right). \qquad (3.5.7)
\end{aligned}
$$

Using Lemma 3.4.1 part (i), Lemma 3.4.2, $|\cos t|\le 1$ and $\sin(t/2)\ge t/\pi$, we have

$$
\begin{aligned}
I_2 &= \frac{1}{\pi}\int_{\frac{1}{r^\sigma}}^{\pi}\frac{\|\psi_x(t)\|_p}{\sin(t/2)}e^{-r(1-\cos t)}\big|\cos\big(t/2+r\sin t\big)\big|dt \\
&= O(1)\int_{\frac{1}{r^\sigma}}^{\pi}\frac{t^\alpha}{t}e^{-rct^2}dt \\
&= O\left(e^{\frac{-rc}{r^{2\sigma}}}\right)\int_{\frac{1}{r^\sigma}}^{\pi}t^{\alpha-1}dt \\
&= O\left(r^{-(1-2\sigma)\delta}\right), \qquad \delta>0,\ \text{however large.} \qquad (3.5.8)
\end{aligned}
$$

Collecting (3.5.6)-(3.5.8), we get

$$||\widetilde{\tau}_r(x)||_p \;=\; O\left(r^{-\alpha\sigma}\right) + O\left(r^{-(1-2\sigma)\delta}\right). \tag{3.5.9}$$

Now, we consider

$$\widetilde{\tau}_r(x) - \widetilde{\tau}_r(y) \;=\; \frac{1}{2\pi}\int_0^{\pi} \frac{\psi_{x+y}(t) - \psi_x(t)}{\sin(t/2)} e^{-r(1-\cos t)} \cos\big(t/2 + r\sin t\big)dt.$$

Applying generalized Minkowski inequality, we obtain

$$\begin{aligned}
||\widetilde{\tau}_r(x) - \widetilde{\tau}_r(y)||_p &\le \frac{1}{2\pi}\int_0^{\pi} \frac{||\psi_{x+y}(t) - \psi_x(t)||_p}{\sin(t/2)} e^{-r(1-\cos t)} \big|\cos\big(t/2 + r\sin t\big)\big|dt \\
&= \frac{1}{\pi}\left\{\int_0^{\frac{1}{r^\sigma}} + \int_{\frac{1}{r^\sigma}}^{\pi}\right\} \frac{||\psi_{x+y}(t) - \psi_x(t)||_p}{\sin(t/2)} e^{-r(1-\cos t)} \big|\cos\big(t/2 + r\sin t\big)\big|dt \\
&= J_1 + J_2. \qquad (3.5.10)
\end{aligned}$$

Using Lemma 3.4.1 part (iii), $|\cos t| \le 1$ and $\sin(t/2) \ge t/\pi$, we have

$$\begin{aligned}
J_1 &= \frac{1}{\pi}\int_0^{\frac{1}{r^\sigma}} \frac{||\psi_{x+y}(t) - \psi_x(t)||_p}{\sin(t/2)} e^{-r(1-\cos t)} \big|\cos\big(t/2 + r\sin t\big)\big|dt \\
&\le O(1)\int_0^{\frac{1}{r^\sigma}} \frac{|y|^\beta t^{\alpha-\beta}}{t} e^{-r(1-\cos t)}dt \\
&= O\big(|y|^\beta\big)\int_0^{\frac{1}{r^\sigma}} t^{\alpha-\beta-1}dt \\
&= O\left(|y|^\beta r^{-(\alpha-\beta)\sigma}\right). \qquad (3.5.11)
\end{aligned}$$

Using Lemma 3.4.1 part (iii), Lemma 3.4.2, $|\cos t| \le 1$ and $\sin(t/2) \ge t/\pi$, we have

$$\begin{aligned}
J_2 &= \frac{1}{\pi}\int_{\frac{1}{r^\sigma}}^{\pi} \frac{||\psi_{x+y}(t) - \psi_x(t)||_p}{\sin(t/2)} e^{-r(1-\cos t)} \big|\cos\big(t/2 + r\sin t\big)\big|dt \\
&= O(1)\int_{\frac{1}{r^\sigma}}^{\pi} \frac{|y|^\beta t^{\alpha-\beta}}{t} e^{-rct^2}dt \\
&= O\left(|y|^\beta e^{-cr^{1-2\sigma}}\right)\int_{\frac{1}{r^\sigma}}^{\pi} t^{\alpha-\beta-1}dt \\
&= O\left(|y|^\beta r^{-(1-2\sigma)\delta}\right), \qquad \delta > 0, \text{ however large.} \qquad (3.5.12)
\end{aligned}$$

Collecting (3.5.10)-(3.5.12), we get

$$||\widetilde{\tau}_r(x) - \widetilde{\tau}_r(y)||_p \;=\; O\left(|y|^\beta r^{-(\alpha-\beta)\sigma}\right) + O\left(|y|^\beta r^{-(1-2\sigma)\delta}\right). \tag{3.5.13}$$

Using the definition of the generalized Hölder norm (3.1.2), (3.5.9), and (3.5.13), we obtain our desired estimate

$$\begin{aligned} ||\widetilde{\tau}_r(x)||_{\beta,p} &= ||\widetilde{\tau}_r(x)||_p + \sup_{y\neq 0} \frac{||\widetilde{\tau}_r(x) - \widetilde{\tau}_r(y)||_p}{|y|^\beta} \\ &= O\left(r^{-\alpha\sigma}\right) + O\left(r^{-(1-2\sigma)\delta}\right) + O\left(r^{-(\alpha-\beta)\sigma}\right) + O\left(r^{-(1-2\sigma)\delta}\right) \\ &= O\left(r^{-(\alpha-\beta)\sigma}\right) + O\left(r^{-(1-2\sigma)\delta}\right). \end{aligned} \tag{3.5.14}$$

This completes the proof.

3.6 Corollary

The space $H_{\alpha,p}$ becomes H_α, as $p \to \infty$. Thus, we have the following corollary:

Corollary 3.6.1. Let $f \in H_\alpha$. Then for $0 \leq \beta < \alpha \leq 1$, we have

$$||\widetilde{B}_r(f;x) - \widetilde{f}(x)||_\beta = O\left(r^{-(\alpha-\beta)\sigma}\right) + O\left(r^{-(1-2\sigma)\delta}\right).$$

Chapter 4

Trigonometric approximation of functions in the generalized Zygmund space

4.1 Definitions

Let f be a 2π-periodic Lebesgue integrable function. Also let $T \equiv (a_{n,k})$ be an infinite regular triangular matrix with real entries. The sequence-to-sequence transformation

$$t_n(f;x) = \sum_{k=0}^{n} a_{n,k} s_k(f;x) \tag{4.1.1}$$

defines the T-means of the sequence $s_n(f;x)$. The Fourier series 1.1.1 is said to be T-summable to s, if $t_n(f;x) \to s$ as $n \to \infty$.
If we take

$$a_{n,k} = \begin{cases} \frac{1}{n+1}, & 0 \le k \le n; \\ 0, & k > n, \end{cases}$$

then the T-means reduces to $(C,1)$-means (Cesàro means of first order)

$$\sigma_n(f;x) = \frac{1}{n+1} \sum_{k=0}^{n} s_k(f;x). \tag{4.1.2}$$

The product of T-means with $(C,1)$-means defines $T(C,1)$-means

$$\tau_n(f;x) = \sum_{k=0}^{n} a_{n,k}\sigma_k(f;x). \tag{4.1.3}$$

A non-negative sequence $\mathbf{a} \equiv \{a_{n,k}\}$ is called almost monotonically decreasing (or almost monotonically increasing) with respect to k, if there exists a constant K, depending on the sequence $\mathbf{a}$ only such that $a_{n,p} \leq K a_{n,m}$, for all $p \geq m$ (or $a_{n,p} \leq K a_{n,m}$, for all $p \leq m$) and we write $\mathbf{a}$ ∈AMDS (or $\mathbf{a}$ ∈AMIS). We also use the notation $a_n << b_n$ if there exist a positive constant K such that $a_n \leq K b_n$, $\forall n$. Let us recall the definitions of moduli of continuity:

For $f \in C[0,2\pi]$, the first and second order modulus of continuity, respectively, are defined by

$$\begin{aligned} \omega^1(\delta;f) &= \sup_x \sup_{0<|h|\leq\delta} |f(x+h)-f(x)|, \\ \omega^2(\delta;f) &= \sup_x \sup_{0<|h|\leq\delta} |f(x+h)+f(x-h)-2f(x)|. \end{aligned}$$

For $f \in L^p[0,2\pi] (p \geq 1)$, the first and second order integral modulus of continuity, respectively, are defined by

$$\begin{aligned} \omega_p^1(\delta;f) &= \sup_{0<|h|\leq\delta} ||f(x+h)-f(x)||_p, \\ \omega_p^2(\delta;f) &= \sup_{0<|h|\leq\delta} ||f(x+h)+f(x-h)-2f(x)||_p. \end{aligned}$$

It is clear that $\omega^1(\delta;f)$, $\omega^2(\delta;f)$, $\omega_p^1(\delta;f)$ and $\omega_p^2(\delta;f)$ are non-decreasing functions in δ and that

$$\omega^2(\delta;f) \leq 2\omega^1(\delta;f), \qquad \omega_p^2(\delta;f) \leq 2\omega_p^1(\delta;f). \tag{4.1.4}$$

Any real valued function $\omega(\delta)$ defined on the interval $[0,2\pi]$ is said to be modulus of continuity type function if it possesses the following properties:

$\omega(\delta)$ is non-decreasing, continuous, $\omega(0) = 0$, and

$$\omega(\delta_1+\delta_2) \leq \omega(\delta_1)+\omega(\delta_2).$$

The following function spaces are well known in the literature.

Das et al. [9] defined the following space

$$H_p^\omega = \left\{ f \in L^p[0,2\pi], p \geq 1 : \omega_p^1(\delta;f) = O\left(\omega(\delta)\right) \right\}$$

with the norm

$$||f||_p^\omega = ||f||_p + \sup_{t\neq 0} \frac{||f(x+t) - f(x)||_p}{\omega(|t|)} \tag{4.1.5}$$

where ω is a modulus of continuity type function.
If we take $\omega(t) = t^\alpha$, $0 < \alpha \leq 1$, then H_p^ω reduces to $H_{\alpha,p}$-space (with the norm $||.||_p^\omega$ replaced by $||.||_{\alpha,p}$) which is introduced earlier by Das et al. [10].
The space H_p^ω was further generalized and the new space was defined as [62; 136]:

$$Z_p^\omega = \left\{ f \in L^p[0, 2\pi], p \geq 1 : \omega_p^2(\delta; f) = O\left(\omega(\delta)\right) \right\}$$

with the norm

$$||f||_p^\omega = ||f(x)||_p + \sup_{t\neq 0} \frac{||f(x+t) + f(x-t) - 2f(x)||_p}{\omega(|t|)}, \tag{4.1.6}$$

where ω is a modulus of continuity type function.
If we take $\omega(t) = t^\alpha$, $0 < \alpha \leq 2$, then Z_p^ω reduces to $Z_{\alpha,p}$ space (with the norm $||.||_p^\omega$ replaced by $||.||_{\alpha,p}$) [62].
The completeness of H_p^ω and Z_p^ω can be discussed by considering the completeness of L^p $(p \geq 1)$, and hence the spaces H_p^ω and Z_p^ω are Banach spaces under their respective norms. The space H_p^ω is called the generalized Hölder space and the corresponding norm is called the generalized Hölder norm. The space Z_p^ω is called the generalized Zygmund space and the corresponding norm is called the generalized Zygmund norm.

Remark 4.1.1. Given ω and ν, if $\omega(t)/\nu(t)$ is nondecreasing, then

$$H_p^\omega \subseteq H_p^\nu \subseteq L^p, \quad p \geq 1,$$
$$Z_p^\omega \subseteq Z_p^\nu \subseteq L^p, \quad p \geq 1.$$

Remark 4.1.2. Given α and β,

$$H_{\alpha,p} \subseteq H_{\beta,p} \subseteq L^p, \quad p \geq 1, 0 < \beta < \alpha \leq 1,$$
$$Z_{\alpha,p} \subseteq Z_{\beta,p} \subseteq L^p, \quad p \geq 1, 0 < \beta < \alpha \leq 2.$$

Remark 4.1.3. For α and ω, from (4.1.4), it is clear that

$$H_{\alpha,p} \subseteq Z_{\alpha,p}, \quad p \geq 1$$
$$H_p^\omega \subseteq Z_p^\omega, \quad p \geq 1.$$

4.2 Some Known Results

In 2002, the degree of approximation of functions in the H_p^ω-space through partial sums of the Fourier series was studied by Das et al. [22]. The problem was further studied through Cesàro, Nörlund, Riesz and generalized de la Vallée-Poussin means by Leindler [71]. The results on the degree of approximation through matrix means was obtained by Singh and Sonker [139] and Deger [25] which are generalization of the results of Leindler [71].

In 2013, some results on the degree of approximation of functions in Z_p^ω-space using Matrix-Euler means were obtained by Lal and Shireen [62]. Recently, some theorems on the degree of approximation of functions in the Z_p^ω-space using Hausdorff matrix means were proved by Singh et al. [136].

Remark 4.2.1. We observe that the several authors including Leindler [71], Singh and Sonker [139], Lal and Shireen [62] and Singh et. al [136] determined estimates for degree of approximation of the following order

$$||T_n(f;x) - f(x)||_p^\nu = O\left(\frac{\omega(1/n)}{\nu(1/n)}\log n\right); \quad f \in Z_p^\omega \text{ or } f \in H_p^\omega,$$

where $T_n(f;x)$ are some summability means of the sequence $\{s_n(f;x)\}$.
Here we consider the problem of degree of approximation of functions in the generalized Zygmund class Z_p^ω by $T(C,1)$-means and obtain an estimate, which generalize and improve some of the previous results.

4.3 Main Result

We prove the following theorem:

Theorem 4.3.1. *Let ω and ν be moduli of continuity such that $\omega(t)/\nu(t)$ is non-decreasing. Moreover, let the function*

$$\gamma(t) := t^{-\epsilon}\frac{\omega(t)}{\nu(t)} \tag{4.3.1}$$

be non-increasing for some $0 < \epsilon < 1$. Let $T \equiv (a_{n,k})$ be lower triangular regular matrix with non-negative entries. If one of the following condition is satisfied:

(i) $\{a_{n,k}\}$ *is AMIS,*

(ii) $\{a_{n,k}\}$ *is AMDS and* $(n+1)a_{n,0} << 1$*,*

then for $f \in Z_p^{\omega}$ $(p \geq 1)$

$$||\tau_n(f;x) - f(x)||_p^{\nu} \quad << \quad \frac{\omega(1/n)}{\nu(1/n)}, \qquad \forall n \geq 1, \tag{4.3.2}$$

where $\tau_n(f;x)$ *denote the* $T(C,1)$*-means of the sequence* $\{s_k(f;x)\}$*.*

4.4 Lemmas

We need the following lemmas for the proof of our theorem:

Lemma 4.4.1. *[[62], p.8] Let* $f \in Z_p^{\omega}$*,* ω *and* ν *be defined as in Theorem 4.3.1. Then for* $0 < t \leq \pi$ *and* $p \geq 1$

$$(i)||\phi_x(t)||_p = O(\omega(t)).$$

$$(ii)||\phi_{x+y}(t) + \phi_{x-y}(t) - 2\phi_x(t)||_p = O(1)\begin{cases} (\omega(t)), \\ (\omega(|y|)). \end{cases}$$

$$(iii)||\phi_{x+y}(t) + \phi_{x-y}(t) - 2\phi_x(t)||_p = O(1)\nu(|y|)\frac{\omega(t)}{\nu(t)},$$

$$\text{where} \qquad \phi_x(t) = f(x+t) + f(x-t) - 2f(x).$$

Lemma 4.4.2. *If the conditions of Theorem 4.3.1 are satisfied with some* $0 < \epsilon < 1$*. Then, for* $f \in Z_p^{\omega}$ $(p \geq 1)$*, we have*

$$||\sigma_n(f;x) - f(x)||_p^{\nu} \quad << \quad \frac{\omega(1/n)}{\nu(1/n)}, \qquad \forall n \geq 1. \tag{4.4.1}$$

Proof. The $(C,1)$ transform of the sequence $\{s_n(f;x)\}$, denoted by $\sigma_n(f;x)$, is given by

$$\sigma_n(f;x) = \frac{1}{n+1}\sum_{k=0}^{n} s_k(f;x).$$

Using (4.1.2), we can write

$$\begin{aligned} l_n(x) &:= \sigma_n(f;x) - f(x) \\ &= \frac{2}{\pi}\int_0^{\pi} \phi_x(t)\left(\frac{1}{n+1}\sum_{k=0}^{n} D_k(t)\right) dt \\ &= \frac{2}{\pi}\int_0^{\pi} \phi_x(t)\frac{2}{n+1}\left(\frac{\sin(n+1)t/2}{2\sin t/2}\right)^2 dt \\ &= \frac{2}{\pi}\int_0^{\pi} \phi_x(t)K_n(t)dt \\ &= \frac{2}{\pi}\int_0^{1/n} \phi_x(t)K_n(t)dt + \frac{2}{\pi}\int_{1/n}^{\pi} \phi_x(t)K_n(t)dt. \end{aligned}$$

Using $\sin nt \leq n\sin t$, $\sin t \leq 1$ and $\sin(t/2) < t/\pi$, we obtain the following estimates

$$|K_n(t)| \leq n+1 \text{ and } |K_n(t)| \leq \frac{\pi^2}{2(n+1)t^2}. \tag{4.4.2}$$

Using the generalized Minkowski inequality, Lemma 4.4.1 and (4.4.2), we obtain

$$\begin{aligned} ||l_n(x)||_p &\leq \frac{2}{\pi}\int_0^{1/n} ||\phi_x(t)||_p|K_n(t)|dt + \frac{2}{\pi}\int_{1/n}^{\pi} ||\phi_x(t)||_p|K_n(t)|dt \\ &<< \frac{2}{\pi}\int_0^{1/n} \omega(t)(n+1)dt + \frac{2}{\pi}\int_{1/n}^{\pi} \omega(t)\frac{\pi^2}{2(n+1)t^2}dt \\ &<< \frac{2(n+1)}{\pi}\int_0^{1/n} \omega(t)dt + \frac{\pi}{n+1}\int_{1/n}^{\pi} \omega(t)\frac{1}{t^2}dt. \end{aligned}$$

Using non-decreasing nature of $\omega(t)$, $\nu(t)$ and non-increasing nature of $\gamma(t)$, we have

$$\begin{aligned} ||l_n(x)||_p &<< \frac{2(n+1)}{\pi}\omega(1/n)\int_0^{1/n} dt + \frac{\pi\nu(\pi)}{n+1}\int_{1/n}^{\pi} \frac{\omega(t)}{t^{\epsilon}\nu(t)}\frac{1}{t^{2-\epsilon}}dt \\ &<< \omega(1/n) + \frac{\pi}{n+1}\int_{1/n}^{\pi} \gamma(t)\frac{1}{t^{2-\epsilon}}dt \\ &<< \omega(1/n) + \frac{\gamma(1/n)}{n+1}n^{1-\epsilon} \\ &<< \omega(1/n) + \frac{\omega(1/n)}{\nu(1/n)} \\ &<< \frac{\omega(1/n)}{\nu(1/n)}. \end{aligned} \tag{4.4.3}$$

Now, we consider

$$l_n(x+y) + l_n(x-y) - 2l_n(x) = \frac{2}{\pi}\int_0^{\pi} \{\phi_{x+y}(t) + \phi_{x-y}(t) - 2\phi_x(t)\}K_n(t)dt.$$

Using the generalized Minkowski inequality, Lemma 4.4.1, we obtain

$$\begin{aligned}
&||l_n(x+y)+l_n(x-y)-2l_n(x)||_p \\
&\qquad \le \frac{2}{\pi}\int_0^\pi ||\phi_{x+y}(t)+\phi_{x-y}(t)-2\phi_x(t)||_p|K_n(t)|dt \\
&\qquad << \frac{2}{\pi}\int_0^\pi \frac{\omega(t)}{\nu(t)}\nu(|y|)|K_n(t)|dt \\
&\qquad << \nu(|y|)\left\{\int_0^{1/n}+\int_{1/n}^\pi\right\}\frac{\omega(t)}{\nu(t)}|K_n(t)|dt.
\end{aligned}$$

Using (4.4.2), non-decreasing nature of $\frac{\omega(t)}{\nu(t)}$ and non-increasing nature of $\gamma(t)$, we get

$$\begin{aligned}
&||l_n(x+y)+l_n(x-y)-2l_n(x)||_p \\
&\qquad << \nu(|y|)\Big\{\int_0^{1/n}\frac{\omega(t)}{\nu(t)}(n+1)dt+\int_{1/n}^\pi \frac{\omega(t)}{\nu(t)}\frac{\pi^2}{2(n+1)t^2}dt\Big\} \\
&\qquad << \nu(|y|)\Big\{(n+1)\frac{\omega(1/n)}{\nu(1/n)}\int_0^{1/n}dt+\frac{1}{n+1}\int_{1/n}^\pi \gamma(t)\frac{1}{t^{2-\epsilon}}dt\Big\} \\
&\qquad << \nu(|y|)\Big\{\frac{\omega(1/n)}{\nu(1/n)}+\frac{\gamma(1/n)}{n+1}n^{1-\epsilon}\Big\} \\
&\qquad << \nu(|y|)\frac{\omega(1/n)}{\nu(1/n)}. \qquad (4.4.4)
\end{aligned}$$

Using the definition of Zygmund norm given in (4.1.6), we have

$$||l_n(x)||_p^\nu = ||l_n(x)||_p+\sup_{y\neq 0}\frac{||l_n(x+y)+l_n(x-y)-2l_n(x)||_p}{\nu(|y|)}.$$

Using (4.4.3) and (4.4.4), we obtain our required result

$$||l_n(x)||_p^\nu = ||\sigma_n(f;x)-f(x)||_p^\nu \quad << \quad \frac{\omega(1/n)}{\nu(1/n)}, \qquad \forall n\ge 1.$$

If we take $\omega(t)=t^\alpha$ and $\nu(t)=t^\beta$ then the class Z_p^ω reduces to $Z_{\alpha,p}$. In this case $\frac{\omega(t)}{\nu(t)}=t^{\alpha-\beta}$ is non-decreasing function for $0<\beta<\alpha\le 2$. Thus, Lemma 4.4.2 reduces to the following corollary

Corollary 4.4.1. If $f\in Z_{\alpha,p}$ $(p\ge 1)$, then

$$||\sigma_n(f;x)-f(x)||_{\beta,p} \quad << \quad \begin{cases} n^{\beta-\alpha}, & 0<\alpha-\beta<1; \\ n^{-1}, & 1<\alpha-\beta<2; \\ \frac{\log n}{n}, & \alpha-\beta=1. \end{cases}$$

Here we recall the results of Leindler [71] and Quade [119]:

Theorem 4.3.1.[[71], p.56] If the conditions of Theorem 4.3.1 are satisfied with some $0 < \epsilon < 1$. Then, for $f \in H_p^\omega$ $(p \geq 1)$, we have

$$||\sigma_n(f;x) - f(x)||_p^\nu \quad << \quad \frac{\omega(1/n)}{\nu(1/n)} \log n, \qquad \forall n \geq 2.$$

Remark 4.4.1. We note that the Lemma 4.4.2 improves and generalizes Theorem 4.3.1.

Theorem 4.3.2.[[119], p.541] If $f \in Lip(\alpha, p)$ $0 < \alpha \leq 1$, then

$$(i) \text{ if } p > 1\text{or if } p = 1, \alpha < 1, ||\sigma_n(f;x) - f(x)||_p = O(n^{-\alpha}).$$

$$(ii) \text{ if } p = \alpha = 1, ||\sigma_n(f;x) - f(x)||_p = O\left(\frac{\log n}{n}\right).$$

Remark 4.4.2. We note that the Corollary 4.4.1 partially includes Theorem 4.3.2.

4.5 Proof of Theorem 4.3.1.

Since the function given in (4.3.1) is non-increasing, the sequence

$$\gamma_n := \gamma(1/n) = n^\epsilon \frac{\omega(1/n)}{\nu(1/n)}$$

is non-decreasing, which implies that

$$\frac{\omega(1/n)}{\nu(1/n)} >> n^{-\epsilon} >> \frac{1}{n}, \qquad 0 < \epsilon \leq 1. \tag{4.5.1}$$

It is given that, the function $\frac{\omega(t)}{\nu(t)}$ is non-decreasing, the sequence $\frac{\omega(1/n)}{\nu(1/n)}$ is non-increasing. Then, for $r := [n/2]$, we have

$$\frac{\omega(1/r)}{\nu(1/r)} << \frac{\omega(1/n)}{\nu(1/n)}. \tag{4.5.2}$$

Now, we shall discuss the proof.

Proof of part (i). We have

$$\begin{aligned} \tau_n(f;x) - f(x) &= \sum_{k=0}^{n} a_{n,k}\sigma_k(f;x) - f(x) \\ &= \sum_{k=0}^{n} a_{n,k}\left(\sigma_k(f;x) - f(x)\right) = \sum_{k=0}^{n} a_{n,k} l_k(x). \end{aligned}$$

Using $\{a_{n,k}\} \in$ AMIS, non-decreasing nature of $\frac{\omega(t)}{\nu(t)}$ and non-increasing nature of $\gamma(t)$, we have

$$\begin{aligned}
||\tau_n(f;x) - f(x)||_p^{\nu} &\leq \sum_{k=0}^{n} a_{n,k}||l_k(x)||_p^{\nu} \\
&= \left(\sum_{k=0}^{r} + \sum_{k=r+1}^{n}\right) a_{n,k}||l_k(x)||_p^{\nu} \\
&<< a_{n,r}\left(1 + \sum_{k=0}^{r} \frac{\omega(1/k)}{\nu(1/k)}\right) + \sum_{k=r+1}^{n} a_{n,k}\frac{\omega(1/k)}{\nu(1/k)} \quad \text{(by (4.4.1))} \\
&<< a_{n,r}\left(1 + r^{\epsilon}\frac{\omega(1/r)}{\nu(1/r)}\sum_{k=0}^{r} k^{-\epsilon}\right) + \frac{\omega(1/r)}{\nu(1/r)}\sum_{k=r+1}^{n} a_{n,k} \\
&<< a_{n,r}\left(1 + r^{\epsilon}\frac{\omega(1/n)}{\nu(1/n)}r^{1-\epsilon}\right) + \frac{\omega(1/n)}{\nu(1/n)}\sum_{k=0}^{n} a_{n,k} \quad \text{(by (4.5.2))}.
\end{aligned}$$

Since $\{a_{n,k}\} \in$ AMIS implies that $1 = \sum_{k=0}^{n} a_{n,k} > \sum_{k=r}^{n} a_{n,k} >> (n-r+1)a_{n,r} >> na_{n,r}$, we have

$$\begin{aligned}
||\tau_n(f;x) - f(x)||_p^{\nu} &<< \frac{1}{n}\left\{1 + \frac{\omega(1/n)}{\nu(1/n)}n\right\} + \frac{\omega(1/n)}{\nu(1/n)} \\
&<< \frac{\omega(1/n)}{\nu(1/n)}, \quad \forall n \geq 1 \quad \text{(by (4.5.1))}.
\end{aligned}$$

Proof of part (ii). Using $\{a_{n,k}\} \in$ AMDS, non-decreasing nature of $\frac{\omega(t)}{\nu(t)}$ and non-increasing nature of $\gamma(t)$, we have

$$\begin{aligned}
||\tau_n(f;x) - f(x)||_p^{\nu} &\leq \sum_{k=0}^{n} a_{n,k}||l_n(x)||_p^{\nu} \\
&= \left(\sum_{k=0}^{r} + \sum_{k=r+1}^{n}\right) a_{n,k}||l_n(x)||_p^{\nu} \\
&<< a_{n,0}\left\{1 + \sum_{k=0}^{r} \frac{\omega(1/k)}{\nu(1/k)}\right\} + \sum_{k=r+1}^{n} a_{n,k}\frac{\omega(1/k)}{\nu(1/k)} \quad \text{(by (4.4.1))} \\
&<< a_{n,0}\left\{1 + r^{\epsilon}\frac{\omega(1/r)}{\nu(1/r)}\sum_{k=0}^{r} k^{-\epsilon}\right\} + \frac{\omega(1/r)}{\nu(1/r)}\sum_{k=r+1}^{n} a_{n,k} \\
&<< a_{n,0}\left\{1 + \frac{\omega(1/n)}{\nu(1/n)}r\right\} + \frac{\omega(1/n)}{\nu(1/n)} \quad \text{(by (4.5.2))}.
\end{aligned}$$

Since $(n+1)a_{n,0} << 1$, we have

$$\begin{aligned} ||\tau_n(f;x) - f(x)||_p^\nu &<< \frac{1}{n}\left\{1 + \frac{\omega(1/n)}{\nu(1/n)}n\right\} + \frac{\omega(1/n)}{\nu(1/n)} \\ &<< \frac{\omega(1/n)}{\nu(1/n)}, \quad \forall n \geq 1 \quad \text{(by (4.5.1))}. \end{aligned}$$

This completes the proof.

Chapter 5

An error estimate in the approximation through conjugate Fourier series of functions of bounded variation

5.1 Some Known Results

The well known Dirichlet-Jordan theorem [[159], p.57] states that:

Theorem 5.1.1. *If $f \in BV[-\pi, \pi]$, then*

$$\lim_{n\to\infty} \left(s_n(f;x) - \frac{1}{2}\big(f(x+0) + f(x-0)\big) \right) = 0. \tag{5.1.1}$$

In particular, $s_n(f;x)$ converges to $f(x)$ at every point of continuity of f.

A quantitative version of Dirichlet-Jordan's theorem was given by Bojanic [7], Bojanic and Mazhar [8], Bojanic and Mazhar [9] and Mazhar [75] by using different summability means of the Fourier series.
The analogous criterion for convergence of the conjugate Fourier series of function of bounded variation was given by Young [[159], p.59]. He proved

Theorem 5.1.2. *If $f \in BV[-\pi,\pi]$, then a necessary and sufficient condition for the convergence of the conjugate Fourier series at x is the existence of the integral*

$$\widetilde{f}(x) = \lim_{\epsilon\to 0^+} \widetilde{f}(x;\epsilon) = \lim_{\epsilon\to 0^+} \left\{ \frac{-2}{\pi} \int_\epsilon^\pi \frac{\psi_x(t)}{2\tan(t/2)} dt \right\},$$

which represents then the conjugate Fourier series of f.

Later, Mazhar and Budaiwi [76] obtained the quantitative version of Young's theorem, by proving

$$\left| \widetilde{s}_n(f;x) - \widetilde{f}\left(x;\frac{\pi}{n}\right) \right| \le \frac{9}{n} \sum_{k=1}^{n} V_0^{\pi/k}(\psi_x), \tag{5.1.2}$$

where $V_a^b(f) = V_f[a,b] :=$ the total variation of f on $[a,b]$, $-\pi \le a < b \le \pi$.

5.2 Main Result

The aim of this paper is to generalize the result of Mazhar and Budaiwi [76] by determining an estimate of the rate of convergence using matrix means, i.e., $\widetilde{T}_n(f;x) = \sum_{k=0}^n a_{n,k}\widetilde{s}_k(f;x)$. Here we assume that $\{a_{n,k}\}$ is a non-negative and monotonic sequence in k, and $A_{n,\tau} = \sum_{k=\tau}^n a_{n,k}$ with $A_{n,0} = 1$. We also assume that $\{a_{n,n}\}$ is a non-increasing sequence in n.

Theorem 5.2.1. *Let $f \in BV[-\pi,\pi]$. Then the error estimate in the approximation of $\widetilde{f}$, conjugate of f, by the matrix means of its conjugate Fourier series is given by*
(i) if $\{a_{n,k}\}$ is a non-increasing sequence in k, then

$$\begin{aligned}\left|\widetilde{T}_n(f;x) - \widetilde{f}(x;\delta)\right| &\le 2a_{n,n} \sum_{k=1}^{n} V_0^{\pi a_{k,k}}(\psi_x) \\ &+ a_{n,0} \sum_{k=0}^{n-1} V_0^{\pi a_{k,k}}(\psi_x) \left(\frac{1}{a_{k+1,k+1}} - \frac{1}{a_{k,k}} \right),\end{aligned}$$

(ii) if $\{a_{n,k}\}$ is a non-decreasing sequence in k, then

$$\begin{aligned}\left|\widetilde{T}_n(f;x) - \widetilde{f}(x;\delta)\right| &\le 2a_{n,n} \sum_{k=1}^{n} V_0^{\pi a_{k,k}}(\psi_x) \\ &+ 2a_{n,n} \sum_{k=0}^{n-1} V_0^{\pi a_{k,k}}(\psi_x) \left(\frac{1}{a_{k+1,k+1}} - \frac{1}{a_{k,k}} \right),\end{aligned}$$

where $\delta = \pi a_{n,n} \to 0$ as $n \to \infty$.

5.3 Lemma

We shall require the following lemma for the proof of our theorem.

Lemma 5.3.1. *If* $\beta_n(u) = \int_u^\pi \sum_{k=0}^n a_{n,k} \frac{\cos((k+1/2)t)}{2\sin(t/2)} dt$, *then*

$$|\beta_n(u)| \leq \begin{cases} a_{n,0}\frac{\pi^2}{2}\left(\frac{1}{u} - \frac{1}{\pi}\right), & \text{if } \{a_{n,k}\} \text{ is non-increasing in } k; \\ a_{n,n}\pi^2\left(\frac{1}{u} - \frac{1}{\pi}\right), & \text{if } \{a_{n,k}\} \text{ is non-decreasing in } k. \end{cases}$$

Proof. Using $\sin(t/2) \geq t/\pi$, condition of monotonicity of $\{a_{n,k}\}$ and Abel's Lemma, we have

$$\begin{aligned} \left|\sum_{k=0}^n a_{n,k} \frac{\cos\big((k+1/2)t\big)}{2\sin(t/2)}\right| &= \frac{1}{2\sin(t/2)}\left|\sum_{k=0}^n a_{n,k}\cos\big((k+1/2)t\big)\right| \\ &\leq \frac{\pi}{2t}\left|\sum_{k=0}^n a_{n,k}\cos\big((k+1/2)t\big)\right| \\ &\leq \begin{cases} \frac{\pi^2}{2t^2}a_{n,0}, & \text{if } a_{n,k} \text{ is non-increasing in } k; \\ \frac{\pi^2}{t^2}a_{n,n}, & \text{if } a_{n,k} \text{ is non-decreasing in } k. \end{cases} \end{aligned}$$

If $\{a_{n,k}\}$ is a non-increasing sequence, then we have

$$\begin{aligned} |\beta_n(u)| &\leq \int_u^\pi \left|\sum_{k=0}^n a_{n,k}\frac{\cos\big((k+1/2)t\big)}{2\sin(t/2)}\right| dt \\ &\leq \int_u^\pi a_{n,0}\frac{\pi^2}{2t^2}dt \\ &= a_{n,0}\frac{\pi^2}{2}\left(\frac{1}{u} - \frac{1}{\pi}\right). \end{aligned}$$

If $\{a_{n,k}\}$ is a non-decreasing sequence, then we have

$$\begin{aligned} |\beta_n(u)| &\leq \int_u^\pi \left|\sum_{k=0}^n a_{n,k}\frac{\cos\big((k+1/2)t\big)}{2\sin(t/2)}\right| dt \\ &\leq \int_u^\pi a_{n,n}\frac{\pi^2}{t^2}dt \\ &= a_{n,n}\pi^2\left(\frac{1}{u} - \frac{1}{\pi}\right). \end{aligned}$$

□

5.4 Proof of Theorem 5.2.1.

Using the integral representation of $\widetilde{s}_n(f;x)$ and the definition of $\widetilde{f}$, we have

$$\begin{aligned}
\widetilde{T}_n(f;x) - \widetilde{f}(x;\delta) &= -\frac{2}{\pi}\int_0^{\pi} \psi_x(t)\Big(\sum_{k=0}^{n} a_{n,k}\widetilde{D}_k(t)\Big)dt + \frac{2}{\pi}\int_{\delta}^{\pi} \frac{\psi_x(t)}{2\tan(t/2)}dt \\
&= -\frac{2}{\pi}\int_0^{\delta} \psi_x(t)\Big(\sum_{k=0}^{n} a_{n,k}\widetilde{D}_k(t)\Big)dt \\
&+ \frac{2}{\pi}\int_{\delta}^{\pi} \psi_x(t)\sum_{k=0}^{n} a_{n,k}\frac{\cos\big((k+1/2)t\big)}{2\sin(t/2)}dt \\
&= I_1 + I_2, \quad \text{say.} \qquad (5.4.1)
\end{aligned}$$

Using $|\widetilde{D}_k(t)| \le k$ and the hypothesis that the row sum $A_{n,0}$ is 1, we have

$$\Big|\sum_{k=0}^{n} a_{n,k}\widetilde{D}_k(t)\Big| \le \sum_{k=0}^{n} a_{n,k}k \le n.$$

Using the above estimate, we have

$$\begin{aligned}
|I_1| &= \Big| -\frac{2}{\pi}\int_0^{\delta} \psi_x(t)\Big(\sum_{k=0}^{n} a_{n,k}\widetilde{D}_k(t)\Big)dt\Big| \\
&\le \frac{2}{\pi}\int_0^{\delta} |\psi_x(t)|\Big|\sum_{k=0}^{n} a_{n,k}\widetilde{D}_k(t)\Big|dt \\
&\le \frac{2}{\pi}\int_0^{\delta} n|\psi_x(t) - \psi_x(0)|dt \\
&\le \frac{2}{\pi}n\int_0^{\delta} V_0^t(\psi_x)dt \\
&\le \frac{2}{\pi}n\delta V_0^{\delta}(\psi_x). \qquad (5.4.2)
\end{aligned}$$

Now consider the second integral, using integration by parts, we get

$$\begin{aligned}
I_2 &= \frac{2}{\pi}\int_{\delta}^{\pi} \psi_x(t)\sum_{k=0}^{n} a_{n,k}\frac{\cos\big((k+1/2)t\big)}{2\sin(t/2)}dt \\
&= \frac{2}{\pi}[\psi_x(t)\beta_n(t)]_{\delta}^{\pi} - \frac{2}{\pi}\int_{\delta}^{\pi} \beta_n(t)\frac{d}{dt}\{\psi_x(t)\} \\
&= -\frac{2}{\pi}\psi_x(\delta)\beta_n(\delta) - \frac{2}{\pi}\int_{\delta}^{\pi} \beta_n(t)d\psi_x(t).
\end{aligned}$$

If $\{a_{n,k}\}$ is non-increasing sequence, then using Lemma 5.3.1

$$
\begin{aligned}
|I_2| &\leq \frac{2}{\pi}|\psi_x(\delta)||\beta_n(\delta)| + \frac{2}{\pi}\int_\delta^\pi |\beta_n(t)||d\psi_x(t)| \\
&\leq a_{n,0}\pi\left(\frac{1}{\delta} - \frac{1}{\pi}\right)|\psi_x(\delta) - \psi_x(0)| + \pi a_{n,0}\int_\delta^\pi \left(\frac{1}{t} - \frac{1}{\pi}\right)|dV_0^t(\psi_x)| \\
&\leq a_{n,0}\frac{\pi}{\delta}V_0^\delta(\psi_x) - a_{n,0}V_0^\delta(\psi_x) - a_{n,0}V_0^\pi(\psi_x) + a_{n,0}V_0^\delta(\psi_x) \\
&+ \pi a_{n,0}\int_\delta^\pi \frac{1}{t}|dV_0^t(\psi_x)| \\
&= a_{n,0}\frac{\pi}{\delta}V_0^\delta(\psi_x) - a_{n,0}V_0^\pi(\psi_x) + a_{n,0}V_0^\pi(\psi_x) - a_{n,0}\frac{\pi}{\delta}V_0^\delta(\psi_x) \\
&+ \pi a_{n,0}\int_\delta^\pi \frac{1}{t^2}V_0^t(\psi_x)dt \\
&= \pi a_{n,0}\int_\delta^\pi \frac{1}{t^2}V_0^t(\psi_x)dt. \qquad (5.4.3)
\end{aligned}
$$

Collecting (5.4.1)-(5.4.3), we get

$$
\left|\widetilde{T}_n(f;x) - \widetilde{f}(x;\delta)\right| \leq \frac{2}{\pi}n\delta V_0^\delta(\psi_x) + \pi a_{n,0}\int_\delta^\pi \frac{1}{t^2}V_0^t(\psi_x)dt.
$$

Substituting $t = \frac{\pi}{s}$ in the above equation, we get

$$
\left|\widetilde{T}_n(f;x) - \widetilde{f}(x;\delta)\right| \leq \frac{2}{\pi}n\delta V_0^\delta(\psi_x) + a_{n,0}\int_1^{\pi/\delta} V_0^{\pi/s}(\psi_x)ds.
$$

Choosing $\delta = \pi\lambda_{n,n}$, we have

$$
\begin{aligned}
\left|\widetilde{T}_n(f;x) - \widetilde{f}(x;\delta)\right| &\leq 2na_{n,n}V_0^{\pi a_{n,n}}(\psi_x) + a_{n,0}\int_1^{1/a_{n,n}} V_0^{\pi/s}(\psi_x)ds \\
&= 2na_{n,n}V_0^{\pi a_{n,n}}(\psi_x) + a_{n,0}\sum_{k=0}^{n-1}\int_{1/a_{k,k}}^{1/a_{k+1,k+1}} V_0^{\pi/s}(\psi_x)ds.
\end{aligned}
$$

Using the hypothesis that $\{a_{n,n}\}$ is non-increasing and $V_0^{\pi/s}(\psi_x)$ is a non-increasing function of s, we have

$$
\begin{aligned}
\left|\widetilde{T}_n(f;x) - \widetilde{f}(x;\delta)\right| &\leq 2na_{n,n}V_0^{\pi a_{n,n}}(\psi_x) + a_{n,0}\sum_{k=0}^{n-1}V_0^{\pi a_{k,k}}(\psi_x)\int_{1/a_{k,k}}^{1/a_{k+1,k+1}} ds \\
&= 2na_{n,n}V_0^{\pi a_{n,n}}(\psi_x) + a_{n,0}\sum_{k=0}^{n-1}V_0^{\pi a_{k,k}}(\psi_x)\left(\frac{1}{\lambda_{k+1,k+1}} - \frac{1}{a_{k,k}}\right) \\
&\leq 2a_{n,n}\sum_{k=1}^{n}V_0^{\pi a_{k,k}}(\psi_x) + a_{n,0}\sum_{k=0}^{n-1}V_0^{\pi a_{k,k}}(\psi_x)\left(\frac{1}{a_{k+1,k+1}} - \frac{1}{a_{k,k}}\right).
\end{aligned}
$$

This completes the proof of part (i).

If $\{a_{n,k}\}$ is non-decreasing sequence, then we proceed in the similar manner and get

$$\begin{aligned}\left|\widetilde{T}_n(f;x)-\widetilde{f}(x;\delta)\right| &\leq 2na_{n,n}V_0^{\pi a_{n,n}}(\psi_x)+2a_{n,n}\sum_{k=0}^{n-1}V_0^{\pi a_{k,k}}(\psi_x)\left(\frac{1}{a_{k+1,k+1}}-\frac{1}{a_{k,k}}\right)\\ &\leq 2a_{n,n}\sum_{k=1}^{n}V_0^{\pi a_{k,k}}(\psi_x)+2a_{n,n}\sum_{k=0}^{n-1}V_0^{\pi a_{k,k}}(\psi_x)\left(\frac{1}{a_{k+1,k+1}}-\frac{1}{a_{k,k}}\right).\end{aligned}$$

This completes the proof of part (ii).

Further, if f is continuous function, then using $V_0^{\eta}(\psi_x)\leq V_{x-\eta}^{x+\eta}(f)\leq 2\omega_{V(f)}(\eta)$, our result reduces to:

(i) if $\{a_{n,k}\}$ is a non-increasing sequence in k, then

$$\begin{aligned}\left|\widetilde{T}_n(f;x)-\widetilde{f}(x;\delta)\right| &\leq 4a_{n,n}\sum_{k=1}^{n}\omega_{V(f)}(\pi a_{k,k})\\ &+ 2a_{n,0}\sum_{k=0}^{n-1}\omega_{V(f)}(\pi a_{k,k})\left(\frac{1}{a_{k+1,k+1}}-\frac{1}{a_{k,k}}\right).\end{aligned}$$

(ii) if $\{a_{n,k}\}$ is a non-decreasing sequence in k, then

$$\begin{aligned}\left|\widetilde{T}_n(f;x)-\widetilde{f}(x;\delta)\right| &\leq 4a_{n,n}\sum_{k=1}^{n}\omega_{V(f)}(\pi a_{k,k})\\ &+ 4a_{n,n}\sum_{k=0}^{n-1}\omega_{V(f)}(\pi a_{k,k})\left(\frac{1}{a_{k+1,k+1}}-\frac{1}{a_{k,k}}\right).\end{aligned}$$

5.5 Corollaries

Let $\widetilde{N}_n(f;x)$ denotes the n^{th} term of the Nörlund means of $\{\widetilde{s}_n(f;x)\}$. Then $\widetilde{N}_n(f;x)=\sum_{k=0}^{n}\frac{p_{n-k}}{P_k}\widetilde{s}_k(f;x)$. Here we assume $\{p_n\}$ to be a non-negative and monotonic sequence in n. The diagonal entries $a_{n,n}=\frac{p_0}{P_n}$, which is non-increasing sequence, therefore the hypothesis that $a_{n,n}$ is a non-increasing sequence in n, is satisfied automatically.

Then the Theorem 5.2.1 reduces to the following corollary:

Corollary 5.5.1. Let $f\in BV[-\pi,\pi]$. Then

(i) if $\{p_n\}$ is a non-decreasing sequence, then

$$\left|\widetilde{N}_n(f;x)-\widetilde{f}(x;\delta)\right| \leq \frac{2}{P_n}\sum_{k=1}^{n}V_0^{\pi/P_k}(\psi_x)+\frac{p_n}{P_n}\sum_{k=0}^{n-1}p_{k+1}V_0^{\pi/P_k}(\psi_x).$$

(ii) if $\{p_n\}$ is a non-increasing sequence, then

$$\left|\widetilde{N}_n(f;x) - \widetilde{f}(x;\delta)\right| \;\leq\; \frac{2}{P_n}\sum_{k=1}^{n} V_0^{\pi/P_k}(\psi_x) + \frac{2}{P_n}\sum_{k=0}^{n-1} p_{k+1} V_0^{\pi/P_k}(\psi_x).$$

This is equivalent to the result for Nörlund-Voronoi means of Fourier series [8]. For continuous function of bounded variation, we have $V_0^{\eta}(\psi_x) \leq V_{x-\eta}^{x+\eta}(f) \leq 2\omega_{V(f)}(\eta)$. Thus above corollary becomes

Corollary 5.5.2. Let $f \in BV[-\pi,\pi]$ and be a continuous function. Then
(i) if $\{p_n\}$ is a non-decreasing sequence, then

$$\left|\widetilde{N}_n(f;x) - \widetilde{f}(x;\delta)\right| \;\leq\; \frac{4}{P_n}\sum_{k=1}^{n} \omega_{V(f)}(\pi/P_k) + \frac{2p_n}{P_n}\sum_{k=0}^{n-1} p_{k+1}\omega_{V(f)}(\pi/P_k).$$

(ii) if $\{p_n\}$ is a non-increasing sequence, then

$$\left|\widetilde{N}_n(f;x) - \widetilde{f}(x;\delta)\right| \;\leq\; \frac{4}{P_n}\sum_{k=1}^{n} \omega_{V(f)}(\pi/P_k) + \frac{4}{P_n}\sum_{k=0}^{n-1} p_{k+1}\omega_{V(f)}(\pi/P_k).$$

Similar corollaries can be obtained for Riesz means of the conjugate Fourier series.

Chapter 6

Approximation of certain bivariate functions by almost Euler means of double Fourier series

6.1 Definitions

Let $f(x, y)$ be a 2π-periodic function in each variable and Lebesgue integrable over the two dimensional torus $T^2 := [-\pi, \pi] \times [-\pi, \pi]$. Then the Euler means $E_{mn}(x, y)$ of the sequence $\{s_{kl}(x, y)\}$ are defined by

$$E_{mn}(x,y) = \frac{1}{(1+q_1)^m(1+q_2)^n} \sum_{k=0}^{m} \sum_{l=0}^{n} \binom{m}{k}\binom{n}{l} {q_1}^{m-k} {q_2}^{n-l} s_{kl}(x,y), \; q_1, q_2 > 0, \tag{6.1.1}$$

and the almost Euler means of the sequence $\{s_{kl}(x, y)\}$ are defined by

$$\tau_{mn}^{rs}(x,y) = \frac{1}{(1+q_1)^m(1+q_2)^n} \sum_{k=0}^{m} \sum_{l=0}^{n} \binom{m}{k}\binom{n}{l} {q_1}^{m-k} {q_2}^{n-l} S_{kl}^{rs}(x,y), \tag{6.1.2}$$

where

$$S_{kl}^{rs}(x,y) = \frac{1}{(k+1)(l+1)} \sum_{\gamma=r}^{r+k} \sum_{\mu=s}^{s+l} s_{\gamma\mu}(x,y).$$

Here, we generalize the definitions of $Lip(\alpha, \beta)$ and $Zyg(\alpha, \beta)$ given in [92] and [105], respectively, by introducing a new Lipschitz class $Lip(\alpha, \beta; p)$ and a Zygmund class

$Zyg(\alpha, \beta; p)$. The Lipschitz class $Lip(\alpha, \beta; p)$ $(p \geq 1)$, for $0 < \alpha, \beta \leq 1$ is defined as

$$Lip(\alpha, \beta; p) = \{f \in L^p(T^2) \mid \omega^p_{1,x}(f; u) = O(u^\alpha), \omega^p_{1,y}(f; v) = O(v^\beta)\}. \tag{6.1.3}$$

We also use the notion of the integral modulus of smoothness. For $0 < \alpha,\ \beta \leq 2$, the Zygmund class $Zyg(\alpha, \beta; p)$ $(p \geq 1)$ is defined as

$$Zyg(\alpha, \beta; p) = \{f \in L^p(T^2) \mid \omega^p_{2,x}(f; u) = O(u^\alpha), \omega^p_{2,y}(f; v) = O(v^\beta)\}. \tag{6.1.4}$$

We know that $\omega^p_2(f; u, v)$, $\omega^p_{2,x}(f; u)$ and $\omega^p_{2,y}(f; v)$ are non-decreasing functions in u and v and, that

$$\begin{aligned} 2\max\{\omega^p_{2,x}(f; u), \omega^p_{2,y}(f; v)\} \leq \omega^p_2(f; u, v) \leq 2\{\omega^p_{2,x}(f; u) + \omega^p_{2,y}(f; v)\}, \\ \text{and } \omega^p_{2,x}(f; u) \leq 2\omega^p_{1,x}(f; u),\ \omega^p_{2,y}(f; v) \leq 2\omega^p_{1,y}(f; v). \end{aligned} \tag{6.1.5}$$

From (6.1.5) it is clear that $Lip(\alpha, \beta; p) \subseteq Zyg(\alpha, \beta; p)$, for $0 < \alpha, \beta \leq 1$, and similar to one-dimensional case, $Lip(\alpha, \beta; p) = Zyg(\alpha, \beta; p)$, for $0 < \alpha, \beta < 1$, but $Lip(\alpha, \beta; p) \neq Zyg(\alpha, \beta; p)$, for $\max(\alpha, \beta) = 1$ [[159], p.44].

Let $\omega(\delta)$ be a nondecreasing function of $\delta \geq 0$. Then $\omega(\delta)$ is of the first kind [92] if

$$\int_\delta^\pi \frac{\omega(u)}{u^2} du = O\left\{\frac{\omega(\delta)}{\delta}\right\}, \qquad 0 < \delta \leq \pi, \tag{6.1.6}$$

and $\omega(\delta)$ is of the second kind [92] if

$$\int_\delta^\pi \frac{\omega(u)}{u^2} du = O\left\{\frac{\omega(\delta)}{\delta} \log \frac{\pi}{\delta}\right\}, \qquad 0 < \delta \leq \pi. \tag{6.1.7}$$

Here we also generalize the definition of $Lip(\psi(u, v); p)$ $(p > 1)$-class [47; 48; 49] by introducing a new Lipschitz class $Lip(\psi(u, v))_{L^p}$ $(p > 1)$ defined as

$$||f(x + u, y + v) - f(x, y)||_p \leq M\left(\frac{\psi(u, v)}{(uv)^{1/p}}\right). \tag{6.1.8}$$

Throughout the chapter we shall use the following notations

$$\phi_{x,y}(u,v) = \{f(x+u,y+v) + f(x-u,y+v) + f(x+u,y-v) + f(x-u,y-v) - 4f(x,y)\},$$

$$S_k^r(u) = \frac{\sin((k+1)\frac{u}{2})\sin((k+2r+1)\frac{u}{2})}{\sin^2(u/2)} = \sum_{\gamma=r}^{r+k} D_\gamma(u), \tag{6.1.9}$$

$$S_l^s(v) = \frac{\sin((l+1)\frac{v}{2})\sin((l+2s+1)\frac{v}{2})}{\sin^2(v/2)} = \sum_{\mu=s}^{s+l} D_\mu(v), \tag{6.1.10}$$

$$R_m^r(u) = \sum_{k=0}^{m} \binom{m}{k} \frac{q_1^{m-k}}{(k+1)} S_k^r(u), \quad q_1 > 0, \tag{6.1.11}$$

$$R_n^s(v) = \sum_{l=0}^{n} \binom{n}{l} \frac{q_2^{n-l}}{(l+1)} S_l^s(v), \quad q_2 > 0. \tag{6.1.12}$$

Note 1. Note that $\phi_{x,y}(u,v)$ satisfies the following inequalities:

$$|\phi_{x,y}(u,v)| \leq 2\big(\omega_{2,x}(f;u) + \omega_{2,y}(f;v)\big), \tag{6.1.13}$$

$$\text{and } ||\phi_{x,y}(u,v)||_p \leq 2\big(\omega_{2,x}^p(f;u) + \omega_{2,y}^p(f;v)\big). \tag{6.1.14}$$

Clearly

$$\begin{aligned}
||\phi_{x,y}(u,v)||_p &= ||f(x+u,y+v) + f(x-u,y+v) + f(x+u,y-v) \\
&\quad + f(x-u,y-v) - 4f(x,y)||_p \\
&\leq ||f(x+u,y+v) + f(x-u,y+v) - 2f(x,y+v)||_p \\
&\quad + ||f(x+u,y-v) + f(x-u,y-v) - 2f(x,y-v)||_p \\
&\quad + 2||f(x,y+v) + f(x,y-v) - 2f(x,y)||_p \\
&\leq \sup_{|h|\leq u} \{||f(x+h,y+v) + f(x-h,y+v) - 2f(x,y+v)||_p\} \\
&\quad + \sup_{|h|\leq u} \{||f(x+h,y-v) + f(x-h,y-v) - 2f(x,y-v)||_p\} \\
&\quad + 2\sup_{|k|\leq v} \{||f(x,y+k) + f(x,y-k) - 2f(x,y)||_p\} \\
&\leq 2\big(\omega_{2,x}^p(f;u) + \omega_{2,y}^p(f;v)\big).
\end{aligned}$$

6.2 Some Known Results

Móricz and Xianlianc Shi [105], Móricz and Rhoades [101], Móricz and Rhoades [102] and Mittal and Rhoades [92] studied the rate of uniform approximation of a 2π-periodic continuous function $f(x,y)$ in Lipschitz class $Lip(\alpha,\beta)$ and in Zygmund class $Zyg(\alpha,\beta), 0<\alpha,\beta\leq 1$, by considering different summability means. Also, the problem of approximation of a 2π-periodic function $f(x,y)$ in $Lip(\psi(u,v);p)$ $(p>1)$-class has been studied by Krasniqi [55] and Khan et. al [48] by using Euler means and Jackson type operator, respectively.

6.3 Main Results

In this chapter, we study the problem in more generalized function classes defined in Section 6.1 and determine the degree of approximation by almost Euler means of the double Fourier series. We prove the following theorems:

Theorem 6.3.1. *Let $f(x,y)$ be a 2π-periodic function in each variable and belongs to $L^p(T^2)$ $(1\leq p<\infty)$. Then the degree of approximation of $f(x,y)$ by almost Euler means of its double Fourier series is given by:*
(i) If both $\omega^p_{2,x}$ and $\omega^p_{2,y}$ are of the first kind, then

$$||\tau^{rs}_{mn}(x,y)-f(x,y)||_p = O\left(\omega^p_{2,x}\left(f;\frac{1}{m+1}\right)+\omega^p_{2,y}\left(f;\frac{1}{n+1}\right)\right).$$

(ii) If $\omega^p_{2,x}$ is of the first kind and $\omega^p_{2,y}$ is of the second kind, then

$$||\tau^{rs}_{mn}(x,y)-f(x,y)||_p = O\left(\omega^p_{2,x}\left(f;\frac{1}{m+1}\right)+\log(\pi(n+1))\omega^p_{2,y}\left(f;\frac{1}{n+1}\right)\right).$$

(iii) If $\omega^p_{2,x}$ is of the second kind and $\omega^p_{2,y}$ is of the first kind, then

$$||\tau^{rs}_{mn}(x,y)-f(x,y)||_p = O\left(\log(\pi(m+1))\omega^p_{2,x}\left(f;\frac{1}{m+1}\right)+\omega^p_{2,y}\left(f;\frac{1}{n+1}\right)\right).$$

(iv) If both $\omega^p_{2,x}$ and $\omega^p_{2,y}$ are of the second kind, then

$$\begin{aligned}||\tau^{rs}_{mn}(x,y)-f(x,y)||_p =& O\left(\log(\pi(m+1))\omega^p_{2,x}\left(f;\frac{1}{m+1}\right)+\log(\pi(n+1))\right.\\ &\left.\omega^p_{2,y}\left(f;\frac{1}{n+1}\right)\right).\end{aligned}$$

For $p = \infty$, the partial integral moduli of smoothness $\omega^p_{2,x}$ and $\omega^p_{2,y}$ reduce to the moduli of smoothness $\omega_{2,x}$ and $\omega_{2,y}$, respectively. Thus, for $p = \infty$, we have the following theorem:

Theorem 6.3.2. *Let $f(x, y)$ be a 2π-periodic function in each variable and belongs to $L^\infty(T^2)$. Then the degree of approximation of $f(x, y)$ by almost Euler means of its double Fourier series is given by:*
(i) If both $\omega_{2,x}$ and $\omega_{2,y}$ are of the first kind, then

$$||\tau^{rs}_{mn}(x, y) - f(x, y)||_\infty = O\left(\omega_{2,x}\left(f; \frac{1}{m+1}\right) + \omega_{2,y}\left(f; \frac{1}{n+1}\right)\right).$$

(ii) If $\omega_{2,x}$ is of the first kind and $\omega_{2,y}$ is of the second kind, then

$$||\tau^{rs}_{mn}(x, y) - f(x, y)||_\infty = O\left(\omega_{2,x}\left(f; \frac{1}{m+1}\right) + \log(\pi(n+1))\omega_{2,y}\left(f; \frac{1}{n+1}\right)\right).$$

(iii) If $\omega_{2,x}$ is of the second kind and $\omega_{2,y}$ is of the first kind, then

$$||\tau^{rs}_{mn}(x, y) - f(x, y)||_\infty = O\left(\log(\pi(m+1))\omega_{2,x}\left(f; \frac{1}{m+1}\right) + \omega_{2,y}\left(f; \frac{1}{n+1}\right)\right).$$

(iv) If both $\omega_{2,x}$ and $\omega_{2,y}$ are of the second kind, then

$$\begin{aligned}||\tau^{rs}_{mn}(x, y) - f(x, y)||_\infty =& O\left(\log(\pi(m+1))\omega_{2,x}\left(f; \frac{1}{m+1}\right) + \log(\pi(n+1))\right.\\ &\left.\omega_{2,y}\left(f; \frac{1}{n+1}\right)\right).\end{aligned}$$

Theorem 6.3.3. *Let $f(x, y)$ be a 2π-periodic function in each variable and belongs to the class $Lip(\psi(u, v))_{L^p}$ $(p > 1)$. If the positive increasing function $\psi(u, v)$ satisfies the condition*

$$(uv)^{-\sigma}\psi(u, v) \text{ is non-decreasing for some } 1/p < \sigma < 1, \tag{6.3.1}$$

then the degree of approximation of $f(x, y)$ by almost Euler means of its double Fourier series is given by

$$\begin{aligned}||\tau^{rs}_{mn}(x, y) - f(x, y)||_p = O\Bigg(&\big((m+1)(n+1)\big)^{1/p}\Bigg(\psi\left(\frac{1}{m+1}, \frac{1}{n+1}\right) + (n+1)^{-\sigma}\\ &\psi\left(\frac{1}{m+1}, \pi\right) + (m+1)^{-\sigma}\psi\left(\pi, \frac{1}{n+1}\right) + \big((m+1)(n+1)\big)^{-\sigma}\Bigg)\Bigg).\end{aligned}$$

For $p=\infty$, the class $Lip(\psi(u,v))_{L^p}$ reduces to the class $Lip(\psi(u,v))_{L^\infty}$, defined as

$$|f(x+u,y+v)-f(x,y)| \leq M\psi(u,v).$$

Thus, for $p=\infty$ we have the following theorem:

Theorem 6.3.4. *Let $f(x,y)$ be a 2π-periodic function in each variable and belongs to the class $Lip(\psi(u,v))_{L^\infty}$. If the positive increasing function $\psi(u,v)$ satisfies the condition*

$$(uv)^{-\sigma}\psi(u,v) \text{ is non-decreasing for some } 0<\sigma<1, \tag{6.3.2}$$

then the degree of approximation of $f(x,y)$ by almost Euler means of its double Fourier series is given by

$$||\tau_{mn}^{rs}(x,y)-f(x,y)||_p = O\Bigg(\psi\left(\frac{1}{m+1},\frac{1}{n+1}\right)+(n+1)^{-\sigma}\psi\left(\frac{1}{m+1},\pi\right)$$
$$+(m+1)^{-\sigma}\psi\left(\pi,\frac{1}{n+1}\right)+\big((m+1)(n+1)\big)^{-\sigma}\Bigg).$$

6.4 Lemmas

The following lemmas are required to prove the main results:

Lemma 6.4.1. *Let $R_m^r(u)$ and $R_n^s(v)$ be given by* (6.1.11) *and* (6.1.12), *respectively. Then*

$$(i) R_m^r(u) = O\big((1+q_1)^m(m+1)\big), \quad \text{for } 0<u\leq\frac{1}{m+1}.$$
$$(ii) R_n^s(v) = O\big((1+q_2)^n(n+1)\big), \quad \text{for } 0<v\leq\frac{1}{n+1}.$$

Proof: (i) For $0<u\leq\frac{1}{m+1}$, using $\sin(u/2)\geq u/\pi$ and $\sin mu\leq m\sin u$, we

have

$$
\begin{aligned}
|R_m^r(u)| &= \left| \sum_{k=0}^{m} \binom{m}{k} \frac{q_1^{m-k}}{(k+1)} S_k^r(u) \right| \\
&= \left| \sum_{k=0}^{m} \binom{m}{k} \frac{q_1^{m-k}}{(k+1)} \frac{\sin((k+1)\frac{u}{2}) \sin((k+2r+1)\frac{u}{2})}{\sin^2(u/2)} \right| \\
&\leq \sum_{k=0}^{m} \binom{m}{k} \frac{q_1^{m-k}}{(k+1)} \frac{(k+1)(k+2r+1) \sin(\frac{u}{2}) \sin(\frac{u}{2})}{\sin^2(u/2)} \\
&= \sum_{k=0}^{m} \binom{m}{k} q_1^{m-k}(k+2r+1) \\
&= (1+q_1)^m(m+2r+1) \\
&= O\big((1+q_1)^m(m+1)\big). \qquad (6.4.1)
\end{aligned}
$$

(ii) It can be proved similar to part (i).

Lemma 6.4.2. *Let $R_m^r(u)$ and $R_n^s(v)$ be given by* (6.1.11) *and* (6.1.12), *respectively. Then*

$$
(i) R_m^r(u) = O\left(\frac{(1+q_1)^m}{(m+1)u^2} \right), \quad \textit{for } \frac{1}{m+1} < u \leq \pi.
$$

$$
(ii) R_n^s(v) = O\left(\frac{(1+q_2)^n}{(n+1)v^2} \right), \quad \textit{for } \frac{1}{n+1} < v \leq \pi.
$$

Proof: (i) For $\frac{1}{m+1} < u \leq \pi$, using $\sin(u/2) \geq u/\pi$ and $\sin u \leq 1$, we have

$$
\begin{aligned}
|R_m^r(u)| &= \left| \sum_{k=0}^{m} \binom{m}{k} \frac{q_1^{m-k}}{(k+1)} S_k^r(u) \right| \\
&= \left| \sum_{k=0}^{m} \binom{m}{k} \frac{q_1^{m-k}}{(k+1)} \frac{\sin((k+1)\frac{u}{2}) \sin((k+2r+1)\frac{u}{2})}{\sin^2(u/2)} \right| \\
&\leq \sum_{k=0}^{m} \binom{m}{k} \frac{q_1^{m-k}}{(k+1)} \frac{\pi^2}{u^2} \\
&= \frac{\pi^2}{(m+1)u^2} \sum_{k=0}^{m} \binom{m+1}{k+1} q_1^{m-k} \\
&= \frac{\pi^2}{(m+1)u^2} \left((1+q_1)^{m+1} - q_1^{m+1}\right) \\
&= O\left(\frac{(1+q_1)^m}{(m+1)u^2} \right). \qquad (6.4.2)
\end{aligned}
$$

(ii) It can be proved similar to part (i).

6.5 Proof of Main Results

Proof of Theorem 2.1: Using the integral representation of $s_{kl}(x,y)$ given in (1.1.9), we have

$$s_{kl}(x,y) - f(x,y) = \frac{1}{4\pi^2}\int_0^\pi\int_0^\pi \phi_{x,y}(u,v)D_k(u)D_l(v)dudv.$$

Therefore,

$$\begin{aligned} S_{kl}^{rs}(x,y) - f(x,y) &= \frac{1}{(k+1)(l+1)}\sum_{\gamma=r}^{r+k}\sum_{\mu=s}^{s+l}\big(s_{\gamma\mu}(x,y) - f(x,y)\big) \\ &= \frac{1}{(k+1)(l+1)}\sum_{\gamma=r}^{r+k}\sum_{\mu=s}^{s+l}\int_0^\pi\int_0^\pi \frac{\phi_{x,y}(u,v)}{4\pi^2}D_\gamma(u)D_\mu(v)dudv \\ &= \int_0^\pi\int_0^\pi \frac{\phi_{x,y}(u,v)}{4\pi^2(k+1)(l+1)}\left(\sum_{\gamma=r}^{r+k}D_\gamma(u)\right)\left(\sum_{\mu=s}^{s+l}D_\mu(v)\right)dudv. \end{aligned}$$

Now

$$\begin{aligned} &\tau_{mn}^{rs}(x,y) - f(x,y) \\ &= \left\{\frac{1}{(1+q_1)^m(1+q_2)^n}\sum_{k=0}^{m}\sum_{l=0}^{n}\binom{m}{k}\binom{n}{l}{q_1}^{m-k}{q_2}^{n-l}S_{kl}^{rs}(x,y)\right\} - f(x,y) \\ &= \frac{1}{4\pi^2}\int_0^\pi\int_0^\pi \frac{\phi_{x,y}(u,v)}{(1+q_1)^m(1+q_2)^n}\left(\sum_{k=0}^{m}\frac{\binom{m}{k}q_1^{m-k}S_k^r(u)}{(k+1)}\right)\left(\sum_{l=0}^{n}\frac{\binom{n}{l}q_2^{n-l}S_l^s(v)}{(l+1)}\right)dudv \\ &= \frac{1}{4\pi^2}\int_0^\pi\int_0^\pi \phi_{x,y}(u,v)\frac{R_m^r(u)}{(1+q_1)^m}\frac{R_n^s(v)}{(1+q_2)^n}dudv, \end{aligned}$$

which on applying the generalized Minkowski inequality, gives

$$
\begin{aligned}
&||\tau_{mn}^{rs}(x,y)-f(x,y)||_p \\
&=\left|\left|\frac{1}{4\pi^2}\int_0^{\pi}\int_0^{\pi}\phi_{x,y}(u,v)\frac{R_m^r(u)}{(1+q_1)^m}\frac{R_n^s(v)}{(1+q_2)^n}dudv\right|\right|_p \\
&=\left\{\frac{1}{2\pi}\int_0^{2\pi}\int_0^{2\pi}\left|\frac{1}{4\pi^2}\int_0^{\pi}\int_0^{\pi}\phi_{x,y}(u,v)\frac{R_m^r(u)}{(1+q_1)^m}\frac{R_n^s(v)}{(1+q_2)^n}dudv\right|^p dxdy\right\}^{1/p} \\
&\le \frac{1}{4\pi^2}\int_0^{\pi}\int_0^{\pi}||\phi_{x,y}(u,v)||_p\frac{|R_m^r(u)|}{(1+q_1)^m}\frac{|R_n^s(v)|}{(1+q_2)^n}dudv \\
&=\frac{1}{\pi^2}\left\{\int_0^{\frac{1}{(m+1)}}\int_0^{\frac{1}{(n+1)}}+\int_{\frac{1}{(m+1)}}^{\pi}\int_0^{\frac{1}{(n+1)}}+\int_0^{\frac{1}{(m+1)}}\int_{\frac{1}{(n+1)}}^{\pi}+\int_{\frac{1}{(m+1)}}^{\pi}\int_{\frac{1}{(n+1)}}^{\pi}\right\} \\
&\quad ||\phi_{x,y}(u,v)||_p\frac{|R_m^r(u)|}{(1+q_1)^m}\frac{|R_n^s(v)|}{(1+q_2)^n}dudv \\
&=\frac{1}{4\pi^2}\{I_1+I_2+I_3+I_4\},\quad \text{say.}
\end{aligned}
\tag{6.5.1}
$$

Proof of part (i)**:** Using Lemma 6.4.1 and (6.1.14), we have

$$
\begin{aligned}
I_1 &= \int_0^{\frac{1}{(m+1)}}\int_0^{\frac{1}{(n+1)}}||\phi_{x,y}(u,v)||_p\frac{|R_m^r(u)|}{(1+q_1)^m}\frac{|R_n^s(v)|}{(1+q_2)^n}dudv \\
&= O\big((m+1)(n+1)\big)\int_0^{\frac{1}{(m+1)}}\int_0^{\frac{1}{(n+1)}}\big(\omega_{2,x}^p(f;u)+\omega_{2,y}^p(f;v)\big)dudv \\
&= O\left(\omega_{2,x}^p\left(f;\frac{1}{m+1}\right)+\omega_{2,y}^p\left(f;\frac{1}{n+1}\right)\right).
\end{aligned}
\tag{6.5.2}
$$

Using Lemma 6.4.1, Lemma 6.4.2, (6.1.6) and (6.1.14), we have

$$
\begin{aligned}
I_2 &= \int_{\frac{1}{(m+1)}}^{\pi}\int_0^{\frac{1}{(n+1)}}||\phi_{x,y}(u,v)||_p\frac{|R_m^r(u)|}{(1+q_1)^m}\frac{|R_n^s(v)|}{(1+q_2)^n}dudv \\
&= O\left(\frac{n+1}{m+1}\right)\int_{\frac{1}{(m+1)}}^{\pi}\int_0^{\frac{1}{(n+1)}}\big(\omega_{2,x}^p(f;u)+\omega_{2,y}^p(f;v)\big)\frac{dudv}{u^2} \\
&= O\left(\frac{n+1}{m+1}\right)\int_{\frac{1}{(m+1)}}^{\pi}\left(\frac{1}{n+1}\right)\left(\omega_{2,x}^p(f;u)+\omega_{2,y}^p\left(f;\frac{1}{n+1}\right)\right)\frac{du}{u^2} \\
&= O\left(\frac{1}{m+1}\right)\left\{\int_{\frac{1}{(m+1)}}^{\pi}\frac{\omega_{2,x}^p(f;u)}{u^2}du+\omega_{2,y}^p\left(f;\frac{1}{n+1}\right)\int_{\frac{1}{(m+1)}}^{\pi}\frac{du}{u^2}\right\} \\
&= O\left(\omega_{2,x}^p\left(f;\frac{1}{m+1}\right)+\omega_{2,y}^p\left(f;\frac{1}{n+1}\right)\right).
\end{aligned}
\tag{6.5.3}
$$

Similarly, we have

$$
I_3 = O\left(\omega_{2,x}^p\left(f;\frac{1}{m+1}\right)+\omega_{2,y}^p\left(f;\frac{1}{n+1}\right)\right). \tag{6.5.4}
$$

Using Lemma 6.4.2, (6.1.6) and (6.1.14), we have

$$
\begin{aligned}
I_4 &= \int_{\frac{1}{(m+1)}}^{\pi}\int_{\frac{1}{(n+1)}}^{\pi} ||\phi_{x,y}(u,v)||_p \frac{|R_m^r(u)|}{(1+q_1)^m}\frac{|R_n^s(v)|}{(1+q_2)^n} dudv \\
&= O\biggl(\frac{1}{(m+1)(n+1)}\biggr)\int_{\frac{1}{(m+1)}}^{\pi}\int_{\frac{1}{(n+1)}}^{\pi} \bigl(\omega_{2,x}^p(f;u)+\omega_{2,y}^p(f;v)\bigr)\frac{dudv}{u^2v^2} \\
&= O\biggl(\frac{1}{(m+1)(n+1)}\biggr)\int_{\frac{1}{(m+1)}}^{\pi} (n+1)\biggl(\omega_{2,x}^p(f;u)+\omega_{2,y}^p\biggl(f;\frac{1}{n+1}\biggr)\biggr)\frac{du}{u^2} \\
&= O\biggl(\frac{1}{m+1}\biggr)\biggl\{\int_{\frac{1}{(m+1)}}^{\pi}\frac{\omega_{2,x}^p(f;u)}{u^2}du + \omega_{2,y}^p\biggl(f;\frac{1}{n+1}\biggr)\int_{\frac{1}{(m+1)}}^{\pi}\frac{du}{u^2}\biggr\} \\
&= O\left(\omega_{2,x}^p\left(f;\frac{1}{m+1}\right)+\omega_{2,y}^p\left(f;\frac{1}{n+1}\right)\right). \qquad (6.5.5)
\end{aligned}
$$

Collecting (6.5.1) – (6.5.5), we have

$$
||\tau_{mn}^{rs}(x,y)-f(x,y)||_p = O\left(\omega_{2,x}^p\left(f;\frac{1}{m+1}\right)+\omega_{2,y}^p\left(f;\frac{1}{n+1}\right)\right),
$$

which proves part (i).

Proof of part (ii)**:** Using (6.5.2) and (6.5.3), we have

$$
I_1 = O\left(\omega_{2,x}^p\left(f;\frac{1}{m+1}\right)+\omega_{2,y}^p\left(f;\frac{1}{n+1}\right)\right), \qquad (6.5.6)
$$

$$
I_2 = O\left(\omega_{2,x}^p\left(f;\frac{1}{m+1}\right)+\omega_{2,y}^p\left(f;\frac{1}{n+1}\right)\right). \qquad (6.5.7)
$$

Using Lemma 6.4.1, Lemma 6.4.2, (6.1.7) and (6.1.14), we have

$$
\begin{aligned}
I_3 &= \int_0^{\frac{1}{(m+1)}}\int_{\frac{1}{(n+1)}}^{\pi} ||\phi_{x,y}(u,v)||_p \frac{|R_m^r(u)|}{(1+q_1)^m}\frac{|R_n^s(v)|}{(1+q_2)^n} dudv \\
&= O\biggl(\frac{m+1}{n+1}\biggr)\int_0^{\frac{1}{(m+1)}}\int_{\frac{1}{(n+1)}}^{\pi} \bigl(\omega_{2,x}^p(f;u)+\omega_{2,y}^p(f;v)\bigr)\frac{dudv}{v^2} \\
&= O\biggl(\frac{m+1}{n+1}\biggr)\int_0^{\frac{1}{(m+1)}} (n+1)\biggl[\omega_{2,x}^p(f;u)+\log(\pi(n+1))\omega_{2,y}^p\biggl(f;\frac{1}{n+1}\biggr)\biggr]du \\
&= O(m+1)\biggl\{\int_0^{\frac{1}{(m+1)}}\omega_{2,x}^p(f;u)du + \log(\pi(n+1))\omega_{2,y}^p\biggl(f;\frac{1}{n+1}\biggr)\int_0^{\frac{1}{(m+1)}} du\biggr\} \\
&= O\left(\omega_{2,x}^p\left(f;\frac{1}{m+1}\right)+\log(\pi(n+1))\omega_{2,y}^p\left(f;\frac{1}{n+1}\right)\right). \qquad (6.5.8)
\end{aligned}
$$

Using Lemma 6.4.2, (6.1.6), (6.1.7) and (6.1.14), we have

$$\begin{aligned}
I_4 &= \int_{\frac{1}{(m+1)}}^{\pi} \int_{\frac{1}{(n+1)}}^{\pi} ||\phi_{x,y}(u,v)||_p \frac{|R_m^r(u)|}{(1+q_1)^m} \frac{|R_n^s(v)|}{(1+q_2)^n} dudv \\
&= O\left(\frac{1}{(m+1)(n+1)}\right) \int_{\frac{1}{(m+1)}}^{\pi} \int_{\frac{1}{(n+1)}}^{\pi} \left(\omega_{2,x}^p(f;u) + \omega_{2,y}^p(f;v)\right) \frac{dudv}{u^2v^2} \\
&= O\left(\frac{1}{(m+1)(n+1)}\right) \int_{\frac{1}{(m+1)}}^{\pi} (n+1)\left[\omega_{2,x}^p(f;u) + \log(\pi(n+1))\omega_{2,y}^p\left(f;\frac{1}{n+1}\right)\right] \frac{du}{u^2} \\
&= O\left(\frac{1}{m+1}\right)\left\{\int_{\frac{1}{(m+1)}}^{\pi} \frac{\omega_{2,x}^p(f;u)}{u^2} du + \log(\pi(n+1))\omega_{2,y}^p\left(f;\frac{1}{n+1}\right) \int_{\frac{1}{(m+1)}}^{\pi} \frac{du}{u^2}\right\} \\
&= O\left(\omega_{2,x}^p\left(f;\frac{1}{m+1}\right) + \log(\pi(n+1))\omega_{2,y}^p\left(f;\frac{1}{n+1}\right)\right). \qquad (6.5.9)
\end{aligned}$$

Collecting (6.5.1), (6.5.6)-(6.5.9), we have

$$||\tau_{mn}^{rs}(x,y) - f(x,y)||_p = O\left(\omega_{2,x}^p\left(f;\frac{1}{m+1}\right) + \log(\pi(n+1))\omega_{2,y}^p\left(f;\frac{1}{n+1}\right)\right),$$

which proves part (ii).

In the similar manner, we can prove part (iii) and part (iv).

Proof of Theorem 2.2: We have

$$\begin{aligned}
&|\tau_{mn}^{rs}(x,y) - f(x,y)| \\
&= \left|\frac{1}{4\pi^2}\int_0^{\pi}\int_0^{\pi} \phi_{x,y}(u,v)\frac{R_m^r(u)}{(1+q_1)^m}\frac{R_n^s(v)}{(1+q_2)^n}dudv\right| \\
&\leq \frac{1}{4\pi^2}\int_0^{\pi}\int_0^{\pi} |\phi_{x,y}(u,v)|\frac{|R_m^r(u)|}{(1+q_1)^m}\frac{|R_n^s(v)|}{(1+q_2)^n}dudv \\
&= \frac{1}{4\pi^2}\left[\int_0^{\frac{1}{(m+1)}}\int_0^{\frac{1}{(n+1)}} + \int_{\frac{1}{(m+1)}}^{\pi}\int_0^{\frac{1}{(n+1)}} + \int_0^{\frac{1}{(m+1)}}\int_{\frac{1}{(n+1)}}^{\pi}\right. \\
&\qquad \left. + \int_{\frac{1}{(m+1)}}^{\pi}\int_{\frac{1}{(n+1)}}^{\pi}\right]|\phi_{x,y}(u,v)|\frac{|R_m^r(u)|}{(1+q_1)^m}\frac{|R_n^s(v)|}{(1+q_2)^n}dudv \\
&= \frac{1}{4\pi^2}\{I_1 + I_2 + I_3 + I_4\}, \quad \text{say.}
\end{aligned}$$

Using (6.1.13) and following the proof of Theorem 2.1 with supremum norm, we will get the required result.

Proof of Theorem 2.3: Following the proof of Theorem 2.1, using the generalized

Minkowski inequality and the fact that $\phi_{x,y}(u,v) \in Lip(\psi(u,v))_{L^p}(p > 1)$, we have

$$
\begin{aligned}
&||\tau_{mn}^{rs}(x,y) - f(x,y)||_p \\
&= \left|\left| \frac{1}{4\pi^2}\int_0^\pi \int_0^\pi \phi_{x,y}(u,v)\frac{R_m^r(u)}{(1+q_1)^m}\frac{R_n^s(v)}{(1+q_2)^n}dudv \right|\right|_p \\
&\leq \frac{1}{4\pi^2}\int_0^\pi \int_0^\pi ||\phi_{x,y}(u,v)||_p \frac{|R_m^r(u)|}{(1+q_1)^m}\frac{|R_n^s(v)|}{(1+q_2)^n}dudv \\
&= \frac{1}{4\pi^2}\left[\int_0^{\frac{1}{(m+1)}}\int_0^{\frac{1}{(n+1)}} + \int_{\frac{1}{(m+1)}}^\pi \int_0^{\frac{1}{(n+1)}} + \int_0^{\frac{1}{(m+1)}}\int_{\frac{1}{(n+1)}}^\pi + \int_{\frac{1}{(m+1)}}^\pi\int_{\frac{1}{(n+1)}}^\pi\right] \qquad (6.5.10) \\
&\quad M\frac{\psi(u,v)}{(uv)^{1/p}}\frac{|R_m^r(u)|}{(1+q_1)^m}\frac{|R_n^s(v)|}{(1+q_2)^n}dudv \\
&= \frac{1}{4\pi^2}\{I_1 + I_2 + I_3 + I_4\}, \quad \text{say.} \qquad (6.5.11)
\end{aligned}
$$

Using Lemma 6.4.1, we have

$$
\begin{aligned}
I_1 &\leq \int_0^{1/(m+1)}\int_0^{1/(n+1)} M\frac{\psi(u,v)}{(uv)^{1/p}}\frac{|R_m^r(u)|}{(1+q_1)^m}\frac{|R_n^s(v)|}{(1+q_2)^n}dudv \\
&= O\big((m+1)(n+1)\big)\psi\left(\frac{1}{m+1},\frac{1}{n+1}\right)\int_0^{1/(m+1)}\int_0^{1/(n+1)}(uv)^{-1/p}dudv \\
&= O\left(\psi\left(\frac{1}{m+1},\frac{1}{n+1}\right)\big((m+1)(n+1)\big)^{1/p}\right). \qquad (6.5.12)
\end{aligned}
$$

Using Lemma 6.4.1 and Lemma 6.4.2, we have

$$
\begin{aligned}
I_2 &\leq \int_0^{1/(m+1)}\int_{1/(n+1)}^\pi M\frac{\psi(u,v)}{(uv)^{1/p}}\frac{|R_m^r(u)|}{(1+q_1)^m}\frac{|R_n^s(v)|}{(1+q_2)^n}dudv \\
&= \int_0^{1/(m+1)}\int_{1/(n+1)}^\pi M\frac{(uv)^{-\sigma}\psi(u,v)}{(uv)^{1/p-\sigma}}\frac{|R_m^r(u)|}{(1+q_1)^m}\frac{|R_n^s(v)|}{(1+q_2)^n}dudv \\
&= O\left(\frac{m+1}{n+1}\right)\left(\frac{\psi(\frac{1}{m+1},\pi)}{(\frac{1}{m+1}.\pi)^\sigma}\right)\int_0^{1/(m+1)}\int_{1/(n+1)}^\pi \frac{(uv)^\sigma}{v^2(uv)^{1/p}}dudv \\
&= O\left(\psi\left(\frac{1}{m+1},\pi\right)(m+1)^{1/p}(n+1)^{1/p-\sigma}\right). \qquad (6.5.13)
\end{aligned}
$$

Similarly, we have

$$
I_3 = O\left(\psi\left(\pi,\frac{1}{n+1}\right)(m+1)^{1/p-\sigma}(n+1)^{1/p}\right). \qquad (6.5.14)
$$

Using Lemma 6.4.1 and Lemma 6.4.2, we have

$$
\begin{aligned}
I_4 &\leq \int_{1/(m+1)}^{\pi}\int_{1/(n+1)}^{\pi} M\frac{\psi(u,v)}{(uv)^{1/p}}\frac{|R_m^r(u)|}{(1+q_1)^m}\frac{|R_n^s(v)|}{(1+q_2)^n}dudv \\
&= \int_{1/(m+1)}^{\pi}\int_{1/(n+1)}^{\pi} M\frac{(uv)^{-\sigma}\psi(u,v)}{(uv)^{1/p-\sigma}}\frac{|R_m^r(u)|}{(1+q_1)^m}\frac{|R_n^s(v)|}{(1+q_2)^n}dudv \\
&= O\bigg(\frac{1}{(m+1)(n+1)}\bigg)\bigg(\frac{\psi(\pi,\pi)}{(\pi)^{2\sigma}}\bigg)\int_{1/(m+1)}^{\pi}\int_{1/(n+1)}^{\pi}\frac{(uv)^{\sigma}}{(uv)^{2+1/p}}dudv \\
&= O\big((m+1)(n+1)\big)^{1/p-\sigma}. \qquad (6.5.15)
\end{aligned}
$$

Collecting (6.5.11)-(6.5.15), we have

$$
\begin{aligned}
||\tau_{mn}^{rs}(x,y)-f(x,y)||_p = O\Bigg(&\big((m+1)(n+1)\big)^{1/p}\Bigg(\psi\bigg(\frac{1}{m+1},\frac{1}{n+1}\bigg)+(n+1)^{-\sigma} \\
&\psi\bigg(\frac{1}{m+1},\pi\bigg)+(m+1)^{-\sigma}\psi\bigg(\pi,\frac{1}{n+1}\bigg)+\big((m+1)(n+1)\big)^{-\sigma}\Bigg)\Bigg).
\end{aligned}
$$

Proof of Theorem 2.4: We have

$$
\begin{aligned}
&|\tau_{mn}^{rs}(x,y)-f(x,y)| \\
&= \left|\frac{1}{4\pi^2}\int_0^{\pi}\int_0^{\pi}\phi_{x,y}(u,v)\frac{R_m^r(u)}{(1+q_1)^m}\frac{R_n^s(v)}{(1+q_2)^n}dudv\right| \\
&\leq \frac{1}{4\pi^2}\int_0^{\pi}\int_0^{\pi}|\phi_{x,y}(u,v)|\frac{|R_m^r(u)|}{(1+q_1)^m}\frac{|R_n^s(v)|}{(1+q_2)^n}dudv
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{4\pi^2}\Bigg[\int_0^{\frac{1}{(m+1)}}\int_0^{\frac{1}{(n+1)}}+\int_{\frac{1}{(m+1)}}^{\pi}\int_0^{\frac{1}{(n+1)}}+\int_0^{\frac{1}{(m+1)}}\int_{\frac{1}{(n+1)}}^{\pi}+\int_{\frac{1}{(m+1)}}^{\pi}\int_{\frac{1}{(n+1)}}^{\pi}\Bigg] \\
&\quad M\psi(u,v)\frac{|R_m^r(u)|}{(1+q_1)^m}\frac{|R_n^s(v)|}{(1+q_2)^n}dudv \\
&= \frac{1}{4\pi^2}\{I_1+I_2+I_3+I_4\}, \quad \text{say.} \qquad (6.5.16)
\end{aligned}
$$

Now we follow the proof of Theorem 2.3 with supremum norm to get the result.

6.6 Corollaries

If $f \in Zyg(\alpha,\beta;p)$, then

$$\omega_{2,x}^p(f;u)=O(u^{\alpha}) \text{ and } \omega_{2,y}^p(f;v)=O(v^{\beta}).$$

For $0 < \alpha, \beta < 1$,

$$\int_{\delta}^{\pi} \frac{u^{\alpha}}{u^2} du = O\big(\delta^{\alpha-1}\big) \text{ and } \int_{\delta}^{\pi} \frac{v^{\beta}}{v^2} dv = O\big(\delta^{\beta-1}\big),$$

which implies that u^{α} and v^{β} are of the first kind.
For $\alpha = \beta = 1$,

$$\int_{\delta}^{\pi} \frac{u^{\alpha}}{u^2} du = O\big(\log \frac{\pi}{\delta}\big) \text{ and } \int_{\delta}^{\pi} \frac{v^{\beta}}{v^2} dv = O\big(\log \frac{\pi}{\delta}\big),$$

which implies that u^{α} and v^{β} are of the second kind.
Thus, Theorem 2.1 reduces to the following corollary:

Corollary 6.6.1. If $f \in Zyg(\alpha, \beta; p)$, then

$$||\tau_{mn}^{rs}(x,y) - f(x,y)||_p = \begin{cases} O\big((m+1)^{-\alpha} + (n+1)^{-\beta}\big), & 0 < \alpha, \beta < 1; \\ O\big((m+1)^{-\alpha} + \frac{\log(n+1)}{n+1}\big), & 0 < \alpha < 1, \beta = 1; \\ O\big(\frac{\log(m+1)}{m+1} + (n+1)^{-\beta}\big), & \alpha = 1, 0 < \beta < 1; \\ O\big(\frac{\log(m+1)}{m+1} + \frac{\log(n+1)}{n+1}\big), & \alpha = 1, \beta = 1. \end{cases}$$

For $p = \infty$, the Zygmund class $Zyg(\alpha, \beta; p)$ reduces to $Zyg(\alpha, \beta)$. In this case, from Theorem 2.2, we have the following corollary:

Corollary 6.6.2. If $f \in Zyg(\alpha, \beta)$, then

$$||\tau_{mn}^{rs}(x,y) - f(x,y)||_{\infty} = \begin{cases} O\big((m+1)^{-\alpha} + (n+1)^{-\beta}\big), & 0 < \alpha, \beta < 1; \\ O\big((m+1)^{-\alpha} + \frac{\log(n+1)}{n+1}\big), & 0 < \alpha < 1, \beta = 1; \\ O\big(\frac{\log(m+1)}{m+1} + (n+1)^{-\beta}\big), & \alpha = 1, 0 < \beta < 1; \\ O\big(\frac{\log(m+1)}{m+1} + \frac{\log(n+1)}{n+1}\big), & \alpha = 1, \beta = 1. \end{cases}$$

Conclusions and Future Scope

In the present thesis, we aimed to determine the degree of approximation of functions belonging to Lipschitz classes: $Lip\alpha$, $Lip(\alpha, p)$, $Lip(\xi(t), p)$ and $W(L^p, \xi(t))$, $p \geq 1$, and their conjugates using almost triangular matrix means. Functions belonging to Zygmund space and Hölder space and their conjugates are also considered to obtain degree of approximation by product means and Borel means of theie trigonometric Fourier series and their conjugate Fourier series, respectively. We also obtained the error estimates for conjugates of functions of bounded variation using triangular matrix means. Double Fourier series is also used to estimate the degree of approximation of f belonging to $Lip(\alpha, \beta; p)$, $Zyg(\alpha, \beta; p)$ and $Lip(\psi(u, v))_{L^p}$ using almost Euler means. Some corollaries of the results are also discussed to justify that our results extend and improve some of the earlier results, and contribute significantly to the literature.

During this study we observed that this work can be extended in several directions. Some of the possible options for future work are listed below:

- We can study the statistical convergence of Fourier series of functions belonging to different function classes using summability methods.
- The work of this thesis can be extended to the other Fourier series such as Walsh [6; 32] and Bessel [[109], pp.775,812] Fourier series.
- We can extend our study to approximation of double conjugate Fourier series by using summability methods. We can also extend our work to the double Walsh-Fourier series [33; 34; 106].
- Our work can be extended to obtain the results on approximation properties of non-periodic functions by Fourier transform [117; 115].

Bibliography

[1] Alexits, G.: 1961, *Convergence problems of orthogonal series*, Translated from the German by I. Földer. International Series of Monographs in Pure and Applied Mathematics, Vol. 20, Pergamon Press, New York-Oxford-Paris.

[2] Alexits, G. and Králik, D.: 1964, Über den Annäherungsgrad der Approximation im starken Sinne von stetigen Funktionen, *Magyar Tud. Akad. Mat. Kutató Int. Közl.* **8**, 317–327 (1964).

[3] Almira, J. M. and Luther, U.: 2004, Generalized approximation spaces and applications, *Math. Nachr.* **263/264**, 3–35.

[4] Alotaibi, A. and Mursaleen, M.: 2012, *A*-statistical summability of Fourier series and Walsh-Fourier series, *Appl. Math. Inf. Sci.* **6**(3), 535–538.

[5] Bernstein, S. N. and Surl: 1912, Ordre de la meilleure approximation des fonctions par des polynomes de degre donne, *Mem. Cl. Acad. Roy. Belg.* **4**, 11–104.

[6] Blahota, I. and Nagy, K.: 2018, Approximation by Θ-means of Walsh-Fourier series, *Anal. Math.* **44**(1), 57–71.

[7] Bojanić, R.: 1979, An estimate of the rate of convergence for Fourier series of functions of bounded variation, *Publ. Inst. Math. (Beograd)* **26**(40), 57–60.

[8] Bojanić, R. and Mazhar, S. M.: 1980, An estimate of the rate of convergence of the Nörlund-Voronoi means of the Fourier series of functions of bounded variation, *Approximation theory, III (Proc. Conf., Univ. Texas, Austin, Tex., 1980)*, Academic Press, New York-London, pp. 243–248.

[9] Bojanić, R. and Mazhar, S. M.: 1988, An estimate of the rate of convergence of Cesáro means of Fourier series of functions of bounded variation, *Mathematical Analysis and its Applications*, Elsevier, pp. 17–22.

[10] Bollinger, R. C.: 1971, On Euler summability, *Math. Student* **39**, 308–310.

[11] Borel, E.: 1899, Mémoire sur les séries divergentes, *Ann. Scient. Éc. Norm. Sup.* **16**(3), 9–131.

[12] Cesáro, E.: 1890, Sur la multiplication des series, *Bull. Sci. Math.* **14**, 114–120.

[13] Chandra, P.: 1982, On the generalised Fejér means in the metric of Hölder space, *Math. Nachr.* **109**(1), 39–45.

[14] Chandra, P.: 1988, On the degree of approximation of a class of functions by means of Fourier series, *Acta Math. Hungar.* **52**(3-4), 199–205.

[15] Chandra, P.: 1990, Degree of approximation of functions in the Hölder metric by Borel's means, *J. Math. Anal. Appl.* **149**(1), 236–248.

[16] Chandra, P.: 2002, Trigonometric approximation of functions in L_p-norm, *J. Math. Anal. Appl.* **275**(1), 13–26.

[17] Chandra, P. and Mohapatra, R. N.: 1979, Absolute Nörlund summability factors of Fourier series, *Tamkang J. Math.* **10**(2), 245–252.

[18] Chapman, S.: 1910, On non-integrable orders of summability of series and integrals, *Proc. Lond. Math. Soc.* **2**, 369–409.

[19] Chapman, S. and Hardy, G. H.: 1911, A general view of the theory at summable series, *Q. J. Math.* **42**, 181–215.

[20] Chow, Y. S.: 1954, On the Cesàro summability of double Fourier series, *Tohoku Math. J.* **5**, 277–283.

[21] Das, G., Ghosh, T. and Ray, B. K.: 1996, Degree of approximation of functions by their Fourier series in the generalized Hölder metric, *Proc. Indian Acad. Sci.* **106**(2), 139–153.

[22] Das, G., Nath, A. and Ray, B. K.: 2002, An estimate of the rate of convergence of Fourier series in the generalized Hölder metric, *Anal. Appl. (Ujjain, 1999)*, Narosa, New Delhi, pp. 43–60.

[23] Das, G., Ojha, A. K. and Ray, B. K.: 1998, Degree of approximation of functions associated with Hardy-Littlewood series in the Hölder metric by Borel means, *J. Math. Anal. Appl.* **219**(2), 279–293.

[24] Deepmala and Piscoran, L. I.: 2016, Approximation of signals (functions) belonging to certain Lipschitz classes by almost Riesz means of its Fourier series, *J. Inequal. Appl.* **2016**, 1–10. Paper No. 163.

[25] Değer, U.: 2016, A note on degree of approximation by matrix means in generalized Hölder metric, *Ukraïn. Mat. Zh.* **68**(4), 485–494.

[26] Dikshit, H. P.: 1962, The Nörlund summability of the conjugate series of a Fourier series, *Rend. Circ. Mat. Palermo (2)* **11**, 217–224.

[27] Dubey, S. N.: 2011, On the latest progeny of the rate of convergence of Fourier series of functions of bounded variation, *Int. J. Appl. Math.* **3**(1), 47–56.

[28] Dutta, H. and Rhoades, B. E. (eds): 2016, *Current Topics in Summability Theory and Applications*, Springer Singapore.

[29] Edely, O. H. H. and Mursaleen, M.: 2009, On statistical A-summability, *Math. Comput. Modelling* **49**(3-4), 672–680.

[30] Edely, O. H. H., Mursaleen, M. and Khan, A.: 2013, Approximation for periodic functions via weighted statistical convergence, *Appl. Math. Comput.* **219**(15), 8231–8236.

[31] Gergen, J. J.: 1933, Convergence criteria for double Fourier series, *Trans. Amer. Math. Soc.* **35**(1), 29–63.

[32] Goginava, U.: 2002, On the approximation properties of Cesàro means of negative order of Walsh-Fourier series, *J. Approx. Theory* **115**(1), 9–20.

[33] Goginava, U.: 2003, On Cesàro means of double trigonometric Fourier series, *Math. Notes* **74**(3-4), 477–482.

[34] Goginava, U.: 2004, Approximation properties of (C, α) means of double Walsh-Fourier series, *Anal. Theory Appl.* **20**(1), 77–98.

[35] Grünwald, G.: 1939, Zur Summabilitätstheorie der Fourierschen Doppelreihe, *Proc. Cambridge Philos. Soc.* **35**, 343–350.

[36] Hardy, G. H.: 1949, *Divergent Series*, Oxford, at the Clarendon Press.

[37] Herriot, J. G.: 1942, Nörlund summability of double Fourier series, *Trans. Amer. Math. Soc.* **52**, 72–94.

[38] Holland, A. S. B.: 1981, A survey of degree of approximation of continuous functions, *SIAM Rev.* **23**(3), 344–379.

[39] Holland, A. S. B. and Sahney, B. N.: 1976, On degree of approximation by Euler $(E,\, q)$-means, *Studia Sci. Math. Hungar.* **11**(3-4), 431–435.

[40] Holland, A. S. B., Sahney, B. N. and Tzimbalario, J.: 1976, On degree of approximation of a class of functions by means of Fourier series, *Acta Sci. Math. (Szeged)* **38**(1-2), 69–72.

[41] Izumi, S.: 1950, Notes on Fourier analysis. XXI. On the degree of approximation of the partial sums of a Fourier series, *J. London Math. Soc.* **25**, 240–242.

[42] Jackson, D.: 1994, *The theory of approximation*, Vol. 11 of *American Mathematical Society Colloquium Publications*, American Mathematical Society, Providence, RI. Reprint of the 1930 original.

[43] Kathal, P. D. and Sahney, B. N.: 1967, Borel summability of Fourier series, *Boll. Un. Mat. Ital. (3)* **22**, 179–182.

[44] Kęska, S.: 2016, The degree of approximation by Hausdorff means of a conjugate Fourier series, *Ann. Univ. Mariae Curie-Skłodowska Sect. A* **70**(2), 63–82.

[45] Khan, H. H.: 1973/74, On the degree of approximation to a function belonging to weighted $(L^p,\, \psi(t))$ class, *Aligarh Bull. Math.* **3/4**, 83–88.

[46] Khan, H. H.: 1982, A note on a theorem of Izumi, *Comm. Fac. Sci. Univ. Ankara Sér.* A_1 *Math.* **31**(14), 123–127.

[47] Khan, H. H.: 1994, On the degree of approximation by Jackson type operators (IC–94/344), *IAEA* **26**(4), 1–4.

[48] Khan, H. H., Mishra, V. N. and Khan, I. A.: 2013, An extension of the degree of approximation by Jackson type operators, *IJSER* **4**(9), 997–1000.

[49] Khan, H. H. and Ram, G.: 2000, On the degree of approximation by Gauss Weierstrass integrals, *Int. J. Math. Math. Sci.* **23**(9), 645–649.

[50] Khan, H. H. and Ram, G.: 2003, On the degree of approximation, *Facta Univ. Ser. Math. Inform.* (18), 47–57.

[51] King, J. P.: 1966, Almost summable sequences, *Proc. Amer. Math. Soc.* **17**, 1219–1225.

[52] Knopp, K.: 1922, Über das Eulersche Summierungsverfahren, *Math. Z.* **15**(1), 226–253.

[53] Knopp, K.: 1955, Nörlund-vergahren für functonen, *Math. Z.* **63**, 39–52.

[54] Kranz, R., Ł enski, W. and Szal, B.: 2012, On the degrees of approximation of functions belonging to $L^p(\tilde{\omega})_\beta$ class by matrix means of conjugate Fourier series, *Math. Inequal. Appl.* **15**(3), 717–732.

[55] Krasniqi, X. Z.: 2009, On the degree of approximation of functions from Lipschitz classes, *Appl. Math. Sci.* **3**(6), 277–286.

[56] Krasniqi, X. Z.: 2011, On the degree of approximation by Fourier series of functions from the Banach space $H_p^{(\omega)}$, $p \geq 1$, in generalized Hölder metric, *Int. Math. Forum* **6**(13), 613–625.

[57] Ł enski, W. o. and Szal, B.: 2014, Approximation of conjugate functions by general linear operators of their Fourier series at the Lebesgue points, *Demonstr. Math.* **47**(4), 878–892.

[58] Lal, S.: 2004, On the approximation of function belonging to weighted $(L^p, \xi(t))$ class by almost matrix summability method of its Fourier series, *Tamkang J. Math.* **35**(1), 67–76.

[59] Lal, S.: 2009, Approximation of functions belonging to the generalized Lipschitz class by $C^1 \cdot N_p$ summability method of Fourier series, *Appl. Math. Comput.* **209**(2), 346–350.

[60] Lal, S. and Kushwaha, J. K.: 2009, Approximation of conjugate of functions belonging to the generalized Lipschitz class by lower triangular matrix means, *Int. J. Math. Anal. (Ruse)* **3**(21), 1031–1041.

[61] Lal, S. and Mishra, A.: 2013, Euler-Hausdorff matrix summability operator and trigonometric approximation of the conjugate of a function belonging to the generalized Lipschitz class, *J. Inequal. Appl.* **2013**(1), 1–14.

[62] Lal, S. and Shireen: 2013, Best approximation of functions of generalized Zygmund class by matrix-Euler summability means of Fourier series, *Bull. Math. Anal. Appl.* **5**(4), 1–13.

[63] Lal, S. and Singh, H. P.: 2009, Double matrix summability of double Fourier series, *Int. J. Math. Anal. (Ruse)* **3**(34), 1669–1681.

[64] Lal, S. and Singh, P. N.: 2002, Degree of approximation of conjugate of $Lip(\alpha, p)$ function by $(C, 1)(E, 1)$ means of conjugate series of a Fourier series, *Tamkang J. Math.* **33**(3), 269–274.

[65] Lal, S. and Tripathi, V. N.: 2003, On the study of double Fourier series by double matrix summability methods, *Tamkang J. Math.* **34**(1), 1–16.

[66] Lal, S. and Yadav, K. N. S.: 2001, On degree of approximation of function belonging to the Lipschitz class by $(C, 1)(E, 1)$ means of its Fourier series, *Bull. Calcutta Math. Soc.* **93**(3), 191–196.

[67] Lebesgue, H.: 1910, Sur la représentation trigonométrique approchée des fonctions satisfaisant à une condition de Lipschitz, *Bull. Soc. Math. France* **38**, 184–210.

[68] Leindler, L.: 1979, Generalizations of Prössdorf's theorems, *Studia Sci. Math. Hungar.* **14**(4), 431–439.

[69] Leindler, L.: 1981, Strong approximation and generalized Zygmund class, *Acta Sci. Math. (Szeged)* **43**(3-4), 301–309.

[70] Leindler, L.: 2005, Trigonometric approximation in L_p-norm, *J. Math. Anal. Appl.* **302**(1), 129–136.

[71] Leindler, L.: 2009, A relaxed estimate of the degree of approximation by Fourier series in generalized Hölder metric, *Anal. Math.* **35**(1), 51–60.

[72] Lorch, L.: 1952, The Lebesgue constants for $(E, 1)$ summation of Fourier series, *Duke Math. J.* **19**, 45–50.

[73] Lorentz, G. G.: 1948, A contribution to the theory of divergent sequences, *Acta Math.* **80**, 167–190.

[74] Marcinkiewicz, J.: 1939, Sur une méthode remarquable de sommation des séries doubles de Fourier, *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* **8**(2), 149–160.

[75] Mazhar, S. M.: 1982, An estimate of the rate of convergence of the triangular matrix means of the Fourier series of functions of bounded variation, *Collect. Math.* **33**(2), 187–193.

[76] Mazhar, S. M. and Al-Budaiwi, A.: 1987, An estimate of the rate of convergence of the conjugate Fourier series of functions of bounded variation, *Acta Math. Hungar.* **49**(3-4), 377–380.

[77] Mazhar, S. M. and Siddiqi, A. H.: 1969, On the almost summability of a trigonometric sequence, *Acta Math. Acad. Sci. Hungar.* **20**, 21–24.

[78] Mazhar, S. M. and Totik, V.: 1990, Approximation of continuous functions by T-means of Fourier series, *J. Approx. Theory* **60**(2), 174–182.

[79] McFadden, L.: 1942, Absolute Nörlund summability, *Duke Math. J.* **9**, 168–207.

[80] Mishra, L. N., Mishra, V. N., Khatri, K. and Deepmala: 2014, On the trigonometric approximation of signals belonging to generalized weighted Lipschitz $W(L^r, \xi(t))(r \geq 1)$-class by matrix $(C^1 \cdot N_p)$ operator of conjugate series of its Fourier series, *Appl. Math. Comput.* **237**, 252–263.

[81] Mishra, V. N.: 2009, On the degree of approximation of signals(functions) belnging to weighted $W(L^p, \xi(t)), (p \geq 1)$-class by almost matrix summability method of its conjugate Fourier series, *Int. J. Appl. Math. Mech.* **5**(7), 16–27.

[82] Mishra, V. N., Khan, H. H., Khan, I. A. and Mishra, L. N.: 2014, On the degree of approximation of signals $Lip(\alpha, r), (r \geq 1)$ class by almost Riesz means of its Fourier series, *J. Class. Anal.* **4**(1), 79–87.

[83] Mishra, V. N., Sonavane, V. and Mishra, L. N.: 2013, L_r-approximation of signals (functions) belonging to weighted $W(L_r, \xi(t))$-class by $C^1 \cdot N_p$ summability method of conjugate series of its Fourier series, *J. Inequal. Appl.* pp. 2013:440, 15.

[84] Mittal, M. L.: 1998, A sufficient condition for (F_1)-effectiveness of the C^1T-method, *J. Math. Anal. Appl.* **220**(2), 434–450.

[85] Mittal, M. L. and Bhardwaj, N.: 1998, Approximation of functions by linear matrix operators, *Approximation theory IX, Vol. I. (Nashville, TN, 1998)*, Innov. Appl. Math., Vanderbilt Univ. Press, Nashville, TN, pp. 263–270.

[86] Mittal, M. L. and Mishra, V. N.: 2008, Approximation of Signals (functions) belonging to the weighted-class by almost matrix summability method of its Fourier series, *International J. of Math. Sci. & Engg. Appls. (IJMSEA)* **2**(4), 285–294.

[87] Mittal, M. L. and Rhoades, B. E.: 1999, On the degree of approximation of continuous functions by using linear operators on their Fourier series, *Int. J. Math. Game Theory Algebra* **9**(4), 259–267.

[88] Mittal, M. L. and Rhoades, B. E.: 2000, Degree of approximation to functions in a normed space, *J. Comput. Anal. Appl.* **2**(1), 1–10.

[89] Mittal, M. L., Rhoades, B. E. and Mishra, V. N.: 2006, Approximation of signals (functions) belonging to the weighted $W(L_p, \xi(t))$-class by linear operators, *Int. J. Math. Math. Sci.* **2006**(18). Art. ID 53538.

[90] Mittal, M. L., Rhoades, B. E., Mishra, V. N. and Singh, U.: 2007, Using infinite matrices to approximate functions of class $Lip(\alpha, p)$ using trigonometric polynomials, *J. Math. Anal. Appl.* **326**(1), 667–676.

[91] Mittal, M. L., Rhoades, B. E., Sonker, S. and Singh, U.: 2011, Approximation of signals of class $Lip(\alpha, p)$ by linear operators, *Appl. Math. Comput.* **217**(9), 4483–4489.

[92] Mittal, N. L. and Rhoades, B. E.: 1999, Approximation by matrix means of double Fourier series to continuous functions in two variables, *Rad. Mat.* **9**(1), 77–99.

[93] Mohapatra, R. N. and Chandra, P.: 1982, Hölder continuous functions and their Euler, Borel and Taylor means, *Math. Chronicle* **11**(1-2), 81–96.

[94] Mohapatra, R. N. and Chandra, P.: 1983, Degree of approximation of functions in the Hölder metric, *Acta Math. Hungar.* **41**(1-2), 67–76.

[95] Mohapatra, R. N. and Russell, D. C.: 1983, Some direct and inverse theorems in approximation of functions, *J. Austral. Math. Soc. Ser. A* **34**(2), 143–154.

[96] Moore, E. H.: 1896, Concerning transcendentally transcendental functions, *Math. Ann.* **48**(1-2), 49–74.

[97] Móricz, F.: 1992, A quantitative version of the Dirichlet-Jordan test for double Fourier series, *J. Approx. Theory* **71**(3), 344–358.

[98] Móricz, F.: 1995, A quantitative version of the Young test for the convergence of conjugate series, *J. Approx. Theory* **81**(2), 207–216.

[99] Móricz, F.: 2010, Enlarged Lipschitz and Zygmund classes of functions and Fourier transforms, *East J. Approx.* **16**(3), 259–271.

[100] Móricz, F. and Németh, J.: 2007, Generalized Zygmund classes of functions and strong approximation by Fourier series, *Acta Sci. Math. (Szeged)* **73**(3-4), 637–647.

[101] Móricz, F. and Rhoades, B. E.: 1987, Approximation by Nörlund means of double Fourier series for Lipschitz functions, *J. Approx. Theory* **50**(4), 341–358.

[102] Móricz, F. and Rhoades, B. E.: 1987, Approximation by Nörlund means of double Fourier series to continuous functions in two variables, *Constr. Approx.* **3**(3), 281–296.

[103] Móricz, F. and Rhoades, B. E.: 1987, Approximation to continuous functions by triangular summability methods, *Analysis* **7**(2), 113–127.

[104] Móricz, F. and Rhodes, B. E.: 1998, Almost convergence of double sequences and strong regularity of summability matrices, *Math. Proc. Camb. Phil. Soc.* **104**, 283–294.

[105] Móricz, F. and Shi, X. L.: 1987, Approximation to continuous functions by Cesàro means of double Fourier series and conjugate series, *J. Approx. Theory* **49**(4), 346–377.

[106] Nagy, K.: 2012, Approximation by Nörlund means of double Walsh-Fourier series for Lipschitz functions, *Math. Inequal. Appl.* **15**(2), 301–322.

[107] Nanda, S.: 1976, Some sequence spaces and almost convergence, *J. Austral. Math. Soc. Ser. A* **22**(4), 446–455.

[108] Natanson, G. I.: 1972, The Fourier series of continuous functions of bounded variation, *Vestnik Leningrad. Univ.* (7), 154–155.

[109] Neil, P. V. O.: 2003, *Advanced Engineering Mathematics*, Thomson Books/Cole. USA.

[110] Nigam, H. K.: 2010, On degree of approximation of a function belonging to $Lip(\xi(t), r)$ class by $(E, q)(C, 1)$ product means of Fourier series, *Commun. Appl. Anal.* **14**(4), 607–614.

[111] Nigam, H. K. and Hadish, M.: 2018, Best approximation of functions in generalized Hölder class, *J. Inequal. Appl.* pp. Paper No. 276, 15.

[112] Nigam, H. K. and Sharma, K.: 2011, Degree of approximation of a class of function by $(C, 1)(E, q)$ means of Fourier series, *IAENG Int. J. Appl. Math.* **41**(2), 128–132.

[113] Nörlund, N. E.: 1919, Sur une application des functions permutables, *Lunds Univ. Arssk.* **16**, 1–10.

[114] Pathak, R. S. and Prasad, A.: 2006, A generalized pseudo-differential operator on Gel'fand-Shilov space and Sobolev space, *Indian J. Pure Appl. Math.* **37**(4), 223–235.

[115] Pathak, R. S. and Upadhyay, S. K.: 1994, W^p-spaces and Fourier transform, *Proc. Amer. Math. Soc.* **121**(3), 733–738.

[116] Prasad, A. and Kumar, P.: 2016, The continuous fractional wavelet transform on a generalized Sobolev space, *Int. J. Wavelets Multiresolut. Inf. Process.* **14**(6). Art. ID 1650046.

[117] Prasad, A., Manna, S., Mahato, A. and Singh, V. K.: 2014, The generalized continuous wavelet transform associated with the fractional Fourier transform, *J. Comput. Appl. Math.* **259**, 660–671.

[118] Prössdorf, S.: 1975, Zur Konvergenz der Fourierreihen hölderstetiger Funktionen, *Math. Nachr.* **69**, 7–14.

[119] Quade, E. S.: 1937, Trigonometric approximation in the mean, *Duke Math. J.* **3**(3), 529–543.

[120] Qureshi, K.: 1981, On the degree of approximation of a periodic function f by almost Nörlund means, *Tamkang J. Math.* **12**(1), 35–38.

[121] Qureshi, K.: 1982, On the degree of approximation of a periodic function f by almost Riesz means of its conjugate series, *Indian J. Pure Appl. Math.* **13**(10), 1136–1139.

[122] Rajagopal, C. T.: 1963, On the Nörlund summability of Fourier series, *Proc. Cambridge Philos. Soc.* **59**, 47–53.

[123] Rhoades, B. E.: 2001, On the degree of approximation of functions belonging to the weighted $(L^p, \xi(t))$ class by Hausdorff means, *Tamkang J. Math.* **32**(4), 305–314.

[124] Rhoades, B. E.: 2014, The degree of approximation of functions, and their conjugates, belonging to several general Lipschitz classes by Hausdorff matrix means of the Fourier series and conjugate series of a Fourier series, *Tamkang J. Math.* **45**(4), 389–395.

[125] Rhoades, B. E., Ozkoklu, K. and Albayrak, I.: 2011, On the degree of approximation of functions belonging to a Lipschitz class by Hausdorff means of its Fourier series, *Appl. Math. Comput.* **217**(16), 6868–6871.

[126] Riesz, M.: 1924, Sur L'Equivalence de Certaines Methodes de Sommation, *Proc. London Math. Soc. (2)* **22**, 412–419.

[127] Sahney, B. N.: 1961, A note on the Borel summability of Fourier series, *Boll. Un. Mat. Ital. (3)* **16**, 44–47.

[128] Sahney, B. N. and Venugopal Rao, V.: 1972, Error bounds in the approximation of functions, *Bull. Austral. Math. Soc.* **6**, 11–18.

[129] Salem, R. and Zygmund, A.: 1946, The approximation by partial sums of Fourier series, *Trans. Amer. Math. Soc.* **59**, 14–22.

[130] Saxena, A.: 1964, On Nörlund summability of the conjugate series of a Fourier series, *Rend. Circ. Mat. Palermo (2)* **13**, 222–228.

[131] Sharma, P. L.: 1958, On the harmonic summability of double Fourier series, *Proc. Amer. Math. Soc.* **9**, 979–986.

[132] Sharma, P. L.: 1961, On the harmonic summability of double Fourier series, *Ann. Mat. Pura Appl. (4)* **56**, 159–175.

[133] Sharma, P. L. and Qureshi, K.: 1980, On the degree of approximation of a periodic function f by almost Riesz means, *Ranchi Univ. Math. J.* **11**, 29–33.

[134] Shi, X. L. and Sun, Q. Y.: 1994, Approximation of periodic continuous functions by logarithmic means of Fourier series, *Acta Math. Hungar.* **63**(2), 103–112.

[135] Silverman, L. L.: 1913, *On the definition of the sum of a divergent series*, Vol. 1, University of Missouri.

[136] Singh, M. V., Mittal, M. L. and Rhoades, B. E.: 2017, Approximation of functions in the generalized Zygmund class using Hausdorff means, *J. Inequal. Appl.* **2017**(1), 1–11.

[137] Singh, T.: 1992, Degree of approximation to functions in a normed space, *Publ. Math. Debrecen* **40**(3-4), 261–271.

[138] Singh, U., Mittal, M. L. and Sonker, S.: 2012, Trigonometric approximation of signals (functions) belonging to $W(L^r, \xi(t))$ class by matrix $(C^1 \cdot N_p)$ operator, *Int. J. Math. Math. Sci.* **2012**. Art. ID 964101.

[139] Singh, U. and Sonker, S.: 2012, Degree of approximation of function $f \in H_p^{(w)}$ class in generalized Hölder metric by matrix means, *Mathematical modelling and scientific computation*, Vol. 283 of *Commun. Comput. Inf. Sci.*, Springer, Heidelberg, pp. 1–10.

[140] Singh, U. and Sonker, S.: 2013, Trigonometric approximation of signals (functions) belonging to weighted $(L^p, \xi(t))$-class by Hausdorff means, *J. Appl. Funct. Anal.* **8**(1), 37–44.

[141] Singh, U. and Srivastava, S. K.: 2014, Approximation of conjugate of functions belonging to weighted Lipschitz class $W(L^p, \xi(t))$ by Hausdorff means of conjugate Fourier series, *J. Comput. Appl. Math.* **259**(part B), 633–640.

[142] Singh, U. and Srivastava, S. K.: 2014, Trigonometric approximation of functions belonging to certain Lipschitz classes by $C^1 \cdot T$ operator, *Asian-Eur. J. Math.* **7**(4). Art. ID 1450064.

[143] Sinvhal, S. D.: 1955, Borel summability of the conjugate series of a derived Fourier series, *Duke Math. J.* **22**, 445–450.

[144] Sivaramakrishnan, C., Sukumar, D. and Dogga, V. N.: 2019, On the images of Sobolev spaces under the Schrödinger semigroup, *Adv. Pure Appl. Math.* **10**(1), 65–79.

[145] Sivaramakrishnan, C., Sukumar, D. and Naidu, D. V.: 2019, On the images of Dunkl–Sobolev spaces under the Schrödinger semigroup associated to Dunkl operators, *J. Pseudo-Differ. Oper. Appl.* **10**(1), 93–120.

[146] Sonker, S. and Singh, U.: 2012, Degree of approximation of the conjugate of signals (functions) belonging to $Lip(\alpha, r)$-class by $(C, 1)(E, q)$ means of conjugate trigonometric Fourier series, *J. Inequal. Appl.* **2012**(1), 1–8.

[147] Steékin, S. B.: 1952, On the approximation of continuous functions by Fourier sums, *Uspehi Mat. Nauk.* **7**, 139–141.

[148] Stone, M. H.: 1926, The Borel Summability of Fourier Series, *Amer. J. Math.* **48**(2), 101–112.

[149] Sunouchi, G.: 1952, Convergence criteria for Fourier series, *Tohoku Math. J. (2)* **4**(2), 187–193.

[150] Titchmarsh, E. C.: 1924, The double Fourier series of a discontinuous function, *Proc. Royal Soc. Lond.* **106**(737), 299–314.

[151] Titchmarsh, E. C.: 1939, *The theory of functions*, second edn, Oxford University Press, Oxford.

[152] Toeplitz, O.: 1911, Ãœber allgemeine lineare mittelbildungen., *Prace Matematyczno-Fizyczne* **22**(1), 113–119.

[153] Topolewska, M.: 1995, On the degree of convergence of Borel and Euler means for double Fourier series of functions of bounded variation in Hardy sense, *Math. Bohem.* **120**(1), 1–12.

[154] Totik, V.: 1980, On the strong approximation of Fourier series, *Acta Math. Acad. Sci. Hungar.* **35**(1-2), 151–172.

[155] Totik, V.: 1981, Strong approximation by the de la Vallée-Poussin and Abel means of Fourier series, *J. Indian Math. Soc. (N.S.)* **45**(1-4), 85–108.

[156] Upadhyay, S. K.: 2010, Pseudo-differential operators on $W^{\Omega}(C^n)$-space, *J. Int. Acad. Phys. Sci.* **14**(1), 53–60.

[157] Young, W. H.: 1913, On Multiple Fourier Series, *Proc. London Math. Soc. (2)* **11**, 133–184.

[158] Zhang, R. J.: 2014, On the trigonometric approximation of the generalized weighted Lipschitz class, *Appl. Math. Comput.* **247**, 1139–1140.

[159] Zygmund, A.: 2002, *Trigonometric series*, third edn, Cambridge University Press, Cambridge.