ON WELL-POSEDNESS FOR STOCHASTIC BURGERS-TYPE EQUATIONS

Ph. D. THESIS

by

VIVEK KUMAR

DEPARTMENT OF MATHEMATICS
INDIAN INSTITUTE OF TECHNOLOGY ROORKEE
ROORKEE - 247 667 (INDIA)

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INDIAN INSTITUTE OF TECHNOLOGY ROORKEE
ROORKEE

CANDIDATE'S DECLARATION

I hereby certify that the work which is being presented in the thesis entitled "**ON WELL-POSEDNESS FOR GENERALIZED STOCHASTIC BURGERS–TYPE EQUATIONS**" in partial fulfillment of the requirements for the award of the Degree of Doctor of Philosophy and submitted in the Department of mathematics of the Indian Institute of Technology Roorkee, Roorkee is an authentic record of my own work carried out during a period from July, 2014 to June, 2019 under the supervision of Dr. Anik Kumar Giri, Assistant Professor, Department of Mathematics, Indian Institute of Technology Roorkee, Roorkee.

The matter presented in this thesis has not been submitted by me for the award of any other degree of this or any other Institution.

(Vivek Kumar)

This is to certify that the above statement made by the candidate is correct to the best of my knowledge.

(Ankik Kumar Giri)
Supervisor

The Ph. D. Viva-Voce Examination of Vivek Kumar, Research Scholar, has been held on 12/06/2019.

Chairman, SRC **Signature of External Examiner**

Signature of Supervisor (s) **Head of the Department**

Date: *June 12, 2019*

Abstract

In this thesis, different classes of generalized stochastic Burgers-type equations are studied. In particular, we focus on stochastic Burgers equation and its different types of generalizations. Here, we have mainly discussed three types of generalized equations: first equation considers the polynomial types nonlinearity in place of quadratic nonlinearity, second one is equipped with the fractional differential operator or mixed fractional differential operator in place of Laplacian operator and last one mainly uses of different types of stochastic noises. The main aspects of discussion is to show the existence and uniqueness of the solution to the these equations.

The very first goal is to study of the existence of weak solutions of the one-dimensional generalized stochastic Burgers equation with polynomial nonlinearity perturbed by space time white noise with Dirichlet boundary conditions and $\alpha-$ Hölder continuous coefficient in the noise term with $\alpha \in \left[\frac{1}{2}, 1\right)$. The existence result is established by solving an equivalent martingale problem.

The second aim is to investigate the global existence and uniqueness of solutions to the one-dimensional generalized stochastic Burgers equation containing a nonlinearity of polynomial type and perturbed by cylindrical Volterra process having Dirichlet boundary conditions. In addition, we are also interested to prove that there exists an invariant measure for the same equation with the quadratic nonlinearity.

As a third task, we investigate the existence and uniqueness of solutions to the fractional Burgers-type nonlinear stochastic partial differential equation driven by cylindrical fractional Brownian motion in Hölder spaces. The existence proof relies on a finite dimensional Galerkin approximation. Moreover, the rate of convergence of the Galerkin approximation as well as fully discretization of solution are also obtained.

Finally, we address a class of stochastic nonlinear partial differential equation of Burgers-type driven by pseudo differential operator $(\Delta+\Delta_\alpha)$ where $\Delta_\alpha \equiv -(-\Delta)^{\frac{\alpha}{2}}$ with $\alpha \in (0,2)$ and which is perturbed by the fractional Brownian sheet. The existence and uniqueness of an L^p-valued (local) solution is established for the initial boundary valued problem to this equation.

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4. V. Kumar, A. K. Giri, On a mixed fractional Burgers type nonlinear equation perturbed by fractional Brownian sheet, *Submitted for publications*.

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Chapter 1

Introduction

The principle subject of this thesis is the investigation of nonlinear stochastic Burgers equation and its generalizations driven by different types of noises. In particular, we mainly focus on the existence, uniqueness and approximations to solutions of stochastic Burgers-type equations having polynomial type nonlinearity and perturbed by different types of stochastic noises. The present chapter provides a brief introduction to stochastic partial differential equations especially to Burgers equation, generalized Burgers equations followed by their stochastic counterparts i.e. stochastic Burgers equation and generalized stochastic Burgers equations. Further, we mention some existing results, methods and a brief description of new results. However, this chapter does not contain any new research work.

1.1 Overview

About three hundred years ago, Newton and Leibnitz developed the differential calculus, allowing us to model continuous time dynamical systems in mechanics, astronomy and many other areas of science. This calculus has formed the basis of the revolutionary development in science, technology, and manufacturing that the world has experienced over the last two centuries. Further, in this continuation of development of mathematical tool to model the physical phenomenon, two very popular mathematical areas, ordinary differential equations and partial differential equations have been introduced by researchers. An ordinary differential equation is basically used to model those physical phenomena which depend on any other variable for example space or time, while partial differential equation covers all those physical activities where any quantity depends on more than one variable. Partial differential equations also use to predict the nature of the system from some given initial state of the system and given external effects. Partial differential equations (PDEs) is one of the main research areas in mathematics and has applications in many areas, for example: physics, engineering, economics, and chemistry. Due to their physical significance, partial differential equations have become one of the most popular subjects among the researchers, where they study the well-posedness, assymptotic analysis, numerical approximations etc. of solutions for given PDEs, see [5, 6, 7, 11, 12, 45, 83, 89, 111, 140] and many more.

Though PDEs are used to model many physical phenomenon where the given initial data and external forces are of deterministic types. But, in many cases, initial data are found to be too sensitive that even due to a small disturbance in initial data, PDEs or ODEs refuses to model precisely the given phenomena. Similarly, if the external forces are random forces, then it is difficult to make a better realistic model in a deterministic setting. As we always try to build more realistic models, stochastic effects need to be taken into account. In the last three decades, stochastic partial differential equations (SPDEs) have been one of the most dynamic areas of stochastic processes. Stochastic effects are of central importance for mathematical models of many physical phenomena in physics, biology, economics, population dynamics, epidemiology, psychology, finance, insurance, fluid dynamics, radio astronomy, hydrology, structural mechanics, chemistry and medicine.

Let us take the following example for having a better understanding.
Suppose we have a thin wire of length l which is kept horizontally so that we can think it as an interval $[0,l]$. Consider that there is some weight which is applied to the string to vibrate it. Let $F(x,t)$ denotes the measure of weight per unit length applied towards the y-axis at place $x \in [0,l]$, at that point the position $f(t,x)$ of the wire solves the following PDE

$$\begin{cases} \frac{\partial^2 f}{\partial t^2} = k\frac{\partial^2 f}{\partial x^2} + F(t,x) & (t > 0, 0 \leq x \leq l), \\ f(0,x) = f_0(x). \end{cases} \tag{1.1}$$

The solution of (1.1) can easily be obtained by separation of variables and superposition. However, it is interesting to know that what will happen if F is a random force or disturbance? In fact, equation (1.1) does not have classical meaning when F is a random noise. So, it will be of great interest to find the solution $f(\cdot,\cdot)$ of (1.1) and then investigate the uniqueness, asymptotic behavior of solution and its dependence upon f_0, F etc. In general, SPDEs provide the solution of such type of problems.

The area of stochastic partial differential equations is very vast. Several types of mathematical studies have been done on SPDEs, for example, see [36, 51, 66, 97, 114, 117] and so on. In particular, in the present thesis, we are interested to study the following type of SPDEs

$$\frac{\partial f(t,x)}{\partial t} = \nu \frac{\partial^2 f(t,x)}{\partial x^2} + h(t,x,f(t,x)) + \frac{\partial g(t,x,f(t,x))}{\partial x} + \sigma(t,x,f(t,x))F(t,x,\omega) \tag{1.2}$$

with initial data

$$f(0,x) = f_0(x), \qquad x \in D \subset \mathbb{R}, \tag{1.3}$$

and Drichlet boundary conditions

$$f(t,x) = 0, \qquad \text{for all } \; x \in \partial D \tag{1.4}$$

where ν is the viscosity term, $F(t,x,\omega)$ is the stochastic noise, $h = h(t,x,r), g = g(t,x,r)$, and $\sigma = \sigma(t,x,r)$ are real valued functions on $\mathbb{R}^+ \times D \times \mathbb{R}$ and ∂D denotes the boundary of the bounded domain $D \subset \mathbb{R}$. The equation (1.2) covers different types of partial differential equations. For $\sigma = g = h = 0$, the equation (1.2) gives the heat equation. When $\sigma = g = 0, h \neq 0$, (1.2) becomes the reaction-diffusion equation [128]. For $g = h = 0, \sigma \neq 0$ this system gives stochastic heat equation, while for $g = 0, h \neq 0$ and $\sigma \neq 0$, (1.2) is transformed into stochastic reaction diffusion equation, see [53, 70, 77, 119, 117, 141]. In case of $\sigma = h = \nu = 0$ and $g(t,x,r) = \frac{1}{2}r^2$, (1.2) gives one of the most popular inviscid Burgers equation which is used to study shock waves and conservation laws, see [6, 11, 45, 140] and references therein. Next, for $\sigma = h = 0$ and $g(t,x,r) = -\frac{1}{2}r^2$, the equation (1.2) can be written as well known Burgers' equation which is a quasilinear partial differential equation, reads as

$$\frac{\partial f}{\partial t} = \nu \frac{\partial^2 f}{\partial x^2} - f\frac{\partial f}{\partial x} \tag{1.5}$$

with initial condition

$$f(0,x) = f_0(x), \tag{1.6}$$

where $f(t,x)$ represents the unknown velocity field which have to be decided by some given initial condition $f(0,x) = f_0(x)$. The equations (1.5)-(1.6) can be considered either on the whole real line $(x \in \mathbb{R})$ or in a bounded interval with Dirichlet or Neumann boundary conditions. There are many articles available which are dedicated to Burgers equations, see [32, 33, 73, 89, 90, 102, 103, 115, 124, 125] etc.

Burgers equation has very important role in the theory of differential equations as well as in applied mathematics. It basically shows Newton's second law and describes the relationship between the changing momentum and force on fluid elements and it is used as the simpler form of Navier-Stokes equation which represents the laminar flow as well as hydrodynamical turbulent flow. The turbulent phenomenon is well known for centuries and the first mathematical formulation is introduced in the work of Navier-Stokes. There are many research articles available for Navier-Stokes equations, theoretical as well as numerical, see [62, 63, 86, 87, 88, 116, 122] and many more. The model, shown by Navier-Stokes, is quite complicated to handle mathematically. So there was a need of some comparatively simpler mathematical model to present the phenomenon of turbulence. Thus, the established Burgers equation came into the picture. In 1948, J. M. Burgers, a Dutch researcher, presented the Burgers equation as a simple model for the dynamics of the Navier-Stokes equation in one spatial dimension [32]. This equation has been implemented in several other physical phenomenon too. Due to its wide range of applications in several fields, it has been extensively discussed by many researchers. The Burgers' equation (1.5) can be derived as a particular case of the Navier-Stokes equation

$$\frac{\partial \mathbf{f}}{\partial t} + (\mathbf{f}.\,\nabla)\mathbf{f} = -\frac{1}{\rho}\nabla P + \nu \Delta \mathbf{f} \tag{1.7}$$

in one dimension if the pressure term is omitted.

The main difficulty is that both the Navier-Stokes equation and the Burgers equation have common nonlinear term $f\dfrac{\partial f}{\partial x}$ and the diffusion term $\nu \Delta \mathbf{f}$. Though Burgers equation (1.5) has its great importance but it fails in the case of occurrence of any chaotic phenomena. To solve this problem many researchers have applied some force to the right hand side of (1.5), but this attempt was also a failure. However the situation is found to be totally different when the force is random. Several authors proposed stochastic perturbation of Burgers equation as a better model, see [35, 43, 84]. By introducing the randomness in the equations (1.6), we get the stochastic

Burgers equations. For simplicity, we consider $\nu = 1$ throughout the remaining discussions. The general stochastic Burgers' equation is a class of quasi linear stochastic PDEs and is given by (1.2) after putting $h = 0$. In particular, for $h = 0$ and $g(t,x,r) = \frac{1}{2}r^2$, the equation (1.2) is known as classic stochastic Burgers equation (SBE), [17, 51, 52] and it is given as

$$\begin{cases} \dfrac{\partial f(t,x)}{\partial t} = \frac{\partial^2 f(t,x)}{\partial x^2} + \dfrac{1}{2}\dfrac{\partial (f(t,x))^2}{\partial x} + \sigma(t,x,f(t,x))F(t,x,\omega) \\ f(0,x) = f_0(x), \qquad\qquad x \in D \subset \mathbb{R}, \end{cases} \tag{1.8}$$

with some boundary conditions. Here, $F(t,x,\omega)$ is a stochastic process (continuous or discrete) that acts as the forcing term. This turns the solution into a random field, whose properties are to be determined by the data of the problem, i.e., the random forcing term, and the initial condition. It is an important tool to model several phenomena in the area of fluid dynamics, cosmology, nonlinear acoustics, astrophysics and so on, see e.g. [51] and references therein.

Recently, several types of generalizations to the stochastic Burgers equations have been introduced. In this thesis, we mainly discussed about these types of generalizations which are mainly the stochastic counterparts of the following types of generalization of deterministic Burgers equations.

In first case, an algebraic nonlinearity has been considered in place of the quadratic nonlinearity, which is obtained by putting $F = 0, \nu = 1, h = 0$ and $g(t,x,r) = -r^p/p$, with $p \geq 2$, into (1.2) i.e.

$$\frac{\partial f}{\partial t} = \frac{\partial^2 f}{\partial x^2} - f^{p-1}\frac{\partial f}{\partial x} \tag{1.9}$$

This type of generalization has been studied in papers [8, 93, 120, 129, 138]. There are several applications of this type of equation, for instance $p = 3$, this equation has the solid nonlinear perspectives and has been utilized in many practical transport problems, for example, nonlinear waves in a medium with low-frequency pumping or absorption, transport with disturbance, wave processes in thermoelastic medium, [93, 120]. The stochastic generalization of the equation (1.9) can be written as

$$\frac{\partial f}{\partial t} = \frac{\partial^2 f}{\partial x^2} + f^{p-1}\frac{\partial f}{\partial x} + F(t,x,\omega), \tag{1.10}$$

where $p \geq 2$ is a finite integer and $F(\cdot,\cdot,\cdot)$ is the stochastic noise. This case has been studied in [68, 72, 85, 95].

Another type of generalization is by considering fractional type of differential operator or mixed fractional type of operators, in place of the Laplacian operators $\Delta = \frac{\partial^2 f}{\partial x^2}$, in particular,

$$\frac{\partial f}{\partial t} = \mathfrak{D} f + f^{p-1} \frac{\partial f}{\partial x} + F(t,x,\omega), \tag{1.11}$$

where $\mathfrak{D}$ is a fractional differential operator. Generalized Burgers equation with fractional differential operators are used in modeling of some anomalous diffusions for instance the long time behavior of the acoustic waves propagating in a gas filled cylinder and the wave propagation in viscoelastic medium, see [19, 132] and references therein.

The above two types of generalization in stochastic Burgers equation are called as generalized stochastic Burgers equation(GSBE). Further, as there is no any specific standard which control the selection of a noise term F, and the decision of a logical stochastic process truly relies upon the equation of motion in question, by taking into account of the physical significance as far as possible. In this thesis, we use different types of noises, in particular, space-time white noise [141], fractional white noise [76] and Volterra noise [47].

1.2 Existing results

There are many mathematical results available on Burgers-type equation. It is not possible to provide the details of all these results here. However, in this section we give a brief description of some existing results on Burgers equation, stochastic Burgers equation and generalized stochastic Burgers equation.

In 1915, Bateman [10] has introduced the Burgers equation first time to derive it in to some physical context. Later, Fay [58] has solved this equation in a series form. In 1948, J. M. Burgers has used the Burgers equation in mathematical modeling of turbulence phenomena and after this Burgers equation became very popular, specially in the study of fluid mechanics [32]. The name Burgers equation is given in the honor of J. M. Burgers for his contributions to make this equation popular. In 1949, Lagerstrom et al. [104] have solved the Burgers equation by linearizing it. Later, Cole [44] in 1951 and Hopf [73] in 1950, both have independently introduced a technique to transform the Burgers equation into a simple heat equation and in the honor of Cole and Hopf, this technique is known as Cole-Hopf transformation. Later, serval works on Burgers equation have been done [3, 14, 15, 22, 106, 127, 139] where they prove the existence of unique solutions, properties of solutions and its applications. To know more progress in this direction, see the latest survey paper by Bonkile et. al. [28].

Now, let us turn to the stochastic Burgers equations. In 1994, Bertini, et. al. [17] have shown the existence result for the following Cauchy problem for stochastic Burgers equation

$$\frac{\partial f(t,x)}{\partial t} = \nu \frac{\partial^2 f(t,x)}{\partial x^2} - f(t,x)\frac{\partial f(t,x)}{\partial x} + \varepsilon \eta(t,x) \tag{1.12}$$

where $(t,x) \in [0,\infty) \times [0,\infty)$, η is the white noise in the space and time and ε is the intensity of the noise. To prove existence, they have used the transformation $f(t,x) = -2\nu \partial_x \ln Z(t,x)$ as a meaningful solution to the above stochastic Burgers equation. Here $Z(t,x)$ denotes the solution of the stochastic heat equation with multiplicative half white noise. This is known as Cole-Hopf solution for the stochastic Burgers equation.

In [51], Da Prato et. al. have shown the existence and uniqueness of the global solution for stochastic Burgers' equation perturbed by the space-time white noise. In addition, the existence of the invariant measure to equation (1.12) with $\nu = \varepsilon = 1$ and η as a cylindrical white noise, is also established. In 1995, Da Prato and D. Gatarek [52] have investigated the the existence and uniqueness of the global solution to the stochastic Burgers equation perturbed by colored noise. Further, the strong Feller property and irreducibility for the corresponding transition semigroup have been shown. The above two results have been generalized by I. Gyöngy [66], in 1998, where he proved the existence, uniqueness and comparison theorems for a class of semilinear stochastic partial differential equations of type (1.2) driven by space-time white noise i.e. $F = \frac{\partial^2 W(t,x)}{\partial t \partial x}$, where W is the Brownian sheet and $\mathfrak{D} = \frac{\partial^2 f}{\partial x^2}$. However, there are many mathematical results on the existence and uniqueness of solutions to the stochastic Burgers equation with quadratic nonlinearity. But if we include the polynomial nonlinearity in the equation, then this becomes more delicate issue to discuss the existence and uniqueness of solutions. There are a very few results [72, 67, 95] which discuss the stochastic Burgers equation with polynomial nonlinearity. In [67], Gyöngy has established the existence and uniqueness of solutions for the class of quasilinear stochastic partial differential equations with polynomial nonlinearity having on the interval $[0,1]$ and driven by the white noise with respect to time only. Later, in [95], Kim has discussed about the Cauchy problem for the stochastic Burger equation with nonlinearity of polynomial type in the whole real line perturbed by the white noise with respect to time. In 2013, the existence and uniqueness of the global solution for the stochastic Burgers equation with polynomial nonlinearity driven by *Lévy process* (a stochastic process with jumps) is obtained in [72].

In these results, the coefficients of noise is mainly either constants or satisfying Lipschitz continuity. However, the case of having non-Lipschitz noise coefficient is also handled by many authors, see [27, 29, 97] and references therein. In [97], the existence of the weak solution have been shown for the classical stochastic Burgers equation (SBE) driven by space-time white noise with coefficient $\sigma = \sqrt{f(1-f)}$. Further, [29] have generalized the result of [97] in higher dimension $n > 1$ by using σ as a Hölder continuous function with exponent lies in $[1/2, 1)$ in SBE and with more general noise. For a similar type of problem, in [27], the author have proved the existence of solutions by assuming $\sigma = \sqrt{f}, \mathfrak{D} = \triangle$ and space-time white noise in SBE. These are some results available with quadratic nonlinearity having non-Lipschitz coefficients. In this thesis, Chapter 3 deals with such type of equations, where the stochastic Burgers equation is having polynomial nonlinearity and perturbed by multiplicative white noise. The coefficient of noise has been taken as a Hölder continuous function with exponent lies in $[1/2, 1)$.

In 2001, Alós et al. [2] have developed the one dimensional stochastic calculus with respect to Gaussian Volterra process, where they mainly focused on singular and regular Volterra process. The Volterra processes cover many important types of noises which may be Gaussian as well as non-Gaussian. Fractional Brownian motion, Liouville fractional Brownian motion, multifractional Brownian motion and fractional Orestein Uhlenbeck process etc, are examples of Gaussian type Volterra processes, while Rosenblat process is an example of non-Gaussian Volterra process (cf. [2, 46] and references therein). Recently, in [46], authors have studied the stochastic evolution equation driven by Volterra noise. Further, the L^p-theory of stochastic convolution integral with respect to Volterra process is developed in [47]. Motivated by the generality of Volterra process, we have studied the stochastic Burgers equations with polynomial nonlinearity and driven by the cylinderical Volterra process in Chapter 5.

The second type of generalization (1.11), where the standard Laplacian is replaced by some fractional differential operator, is also one of the interesting equations to study. In the deterministic settings, this type of generalization of Burgers equation has been well studied in many research articles, see for example [19, 101, 132]. In 2007, Breziniak and Debbi [31] have investigated the existence and uniqueness of L^2-valued solutions for the stochastic equation (1.11) having fractional Laplacian operator $\mathfrak{D} = \Delta_\alpha := (-\Delta)^{\alpha/2}$ where $\alpha \in (3/2, 2)$ and driven by multiplicative white noise F. Further, in [30], they have studied the ergodic properties of the fractional stochastic Burgers equation. The existence and uniqueness of L^2-valued local mild solutions have been shown by Truman et. al. [136] for SBE with $\mathfrak{D} = \Delta_\alpha$, $\alpha \in (0, 2]$ and having Lévy space time white noise. Later, in [82] and [144], authors have studied the multidimensional GSBE with Lévy noise under certain conditions on operator $\mathfrak{D} = (-\Delta)^{\alpha(x)/2}$, where $\alpha : \mathbb{R}^d \to (0, 2)$ is a continuous function in L^p space for $p \in (1, 2]$ and by considering at most quadratic nonlinearity. Further, the well-posedness of mild solutions to the stochastic fractional Burgers equation is studied by Zou and Wang [151]. Here, this equation contains fractional derivatives with respect to both time and space.

In recent years, the fractional Brownian sheet has become popular among the researchers studying SPDEs due to its important property of preserving long term memory and other interesting properties, see [25], [76], [78], [112], [143] and Section 2 in Chapter 6.

In 2010, the existence and uniqueness of solutions to the stochastic Burgers equation with quadratic nonlinearity driven by the fractional Brownian motion with Hurst parameter $H \in (1/4, 1)$ have been studied by Wang et. al. [142]. While, in [85], authors have considered the GSBE with third order nonlinearity perturbed by the fractional Brownian sheet with Hurst parameter $H = (H_1, H_2)$ such that $H_i \in (1/2, 1)$ for each $i = 1, 2$, where they have shown the existence and uniqueness of solutions. In addition, they have also evaluated moment estimate for the density of the solution.

Recently, in 2017, Xia et. al. [146] have studied the existence and uniqueness of the solutions to stochastic semilinear heat equations with differential operator $(\Delta + \Delta_\alpha)$ and the fractional Brownian sheet (also see [145]), where, the operator $(\Delta + \Delta_\alpha)$ represents the Lévy processes that are independent sum of diffusion processes and α-stable (rotationally symmetric) processes. This pseudo differential operator can be used in several physical problems, see [96], [121]. Chen et. al. [37, 38, 41, 40] and [39] have studied these operators in very broad way. In these articles, the main work is to find the sharp two sided estimates for the transition density of these type of processes.

The works mentioned above have motivated us to show the well-posedness of the GSBE with mixed operator $\mathfrak{D} = (\Delta + \Delta_\alpha)$, where $\alpha \in (0,2)$, and having polynomial type nonlinearity driven by the fractional Brownian sheet with Hurst parameter $H = (H_1, H_2)$ such that $H_i \in (1/2, 1)$ for each $i = 1, 2$. Such types of problems have been studied in Chapter 5.

Due to the presence of randomness and irregularity of the solution, it is very difficult to find out the analytical solution of SPDEs. Therefore, it is important to study the numerical approximation of the solution more carefully for having a good prediction of the solution to SPDEs. Several numerical method have been implmented to SPDEs: finite difference, finite element, finite volume, Galerkin approximation, particle method, Wiener chaos decomposition, splitting up method etc., see ([13, 48, 64, 65, 91, 110, 149]) and the references therein.

In [1], Alabert and Gyöngy obtained a spatial discretization to the stochastic Burgers equation in $L^2[0,1]$ for finding numerical approximations of the solution by using a finite difference scheme. Later, in [69], Hairer and Voss provided various finite difference approximations for the stochastic Burger equation and show that different finite difference formulations converge to different limiting process as the mesh size tends to zero. Further the rate of convergence obtained in [69] has been improved in [71]. Recently, Blömker and Jentzen [24] established the existence and uniqueness for more general stochastic equations with additive noise by using Galerkin approximations in finite dimension. Additionally, they have also investigated the convergence rate of Galerkin approximations to the solution. Further, Blömker et. al. [23] give the way to investigate the spectral Galerkin method for spatial discretization of stochastic Burgers equation perturbed by the coloured noise whcih is combined with the method introduced by Jentzen et. al. (2011) for time discretization. Recently, Arab and Debbi [4] have shown the existence of pathwise unique mild solutions of stochastic fractional Burgers-type equations perturbed by Wiener process in Hölder space by implementing the spectral Galerkin method. Moreover, they have calculated the rate of convergence for Galerkin spatial approximations of solutions as well as for fully discretization. These works, in [24] and [4], are the main motivation for the study taken place in chapter 6.

1.3 Contents of the thesis

The present thesis consists of seven chapters and the chapter-wise description is given below:

In the present chapter, we mainly focused on the introduction of Burgers equation along with its various generalizations and their stochastic counterparts, which are the main equations of discussion in subsequent chapters.

In Chapter 2, we collect some basic tools necessary for stochastic analysis. We mainly given the brief details of probability theory, stochastic process and Itô calculus.

In Chapter 3, weak solutions for the generalized stochastic Burgers equations having polynomial nonlinearity and a non-Lipschitz diffusion constant are constructed. This work is a generalization of the work done by Kolkovska [97] where the stochastic Burgers equation is considered with quadratic nonlinearity and stepping-stone type noises. The results are obtained by first discretizing the original problem, establishing the existence and uniqueness of weak solutions to the discretized system and then establishing the existence of weak solutions of the original problem by proving the tightness property. The content of this chapter is accepted in journal, Communications on Stochastic Analysis (COSA).

In Chapter 4, we establish the existence and uniqueness of local and global mild solution to the generalized stochastic Burgers equations perturbed by α-regular cylindrical Volterra noise. In order to get the solvability and regularity estimates for the linear system, we use the L^p-theory of stochastic convolution integral developed in [47]. We show the existence and uniqueness of a local mild solution for polynomial type of nonlinearity using contraction mapping principle, and also the global existence and uniqueness for third order nonlinear GSBE using probabilistic arguments. The biggest challenge when considering α-regular cylindrical Volterra noise is the lack of L^∞-estimate on both time and space for the stochastic convolution involving such processes. We obtained this estimate with the help of Garsia-Rodemich-Rumsey inequality. Further, we have also shown the existence of an invariant measure for quadratic nonlinear GSBE perturbed by Volterra processes of Gaussian type. In order to prove this, we adopted the method developed by G. Da Prato et al. in [51].

In Chapter 5, the existence and uniqueness of solutions to Burgers-type nonlinear stochastic partial differential equations driven by cylindrical fractional Brownian motion in Hölder space are investigated by using finite dimensional Galerkin approximation. Further, the rate of convergence of Galerkin spatial approximations and fully discretization are obtained. This work is the continuation of the work done by Z. Arab and L. Debbi [4], where they have deal the same equation but perturbed by Winner process.

In Chapter 6, we study a class of stochastic PDEs of Burgers-type involving the pseudo-differential operator $\mathfrak{D} = \Delta + \Delta_\alpha$ with $\Delta \equiv \frac{\partial^2}{\partial x^2}$ and $\Delta_\alpha \equiv -(-\Delta)^{\frac{\alpha}{2}}$ where $\alpha \in (0,2)$ and a polynomial nonlinearity. The SPDEs is driven by a fractional Brownian sheet. Here, the existence and uniqueness of L^p-valued local mild solution have been established by using a fixed point argument.

Finally, in **Chapter 7,** a few conclusions on above chapters are made. At last some open questions for the future research in the direction of nonlinear stochastic partial differential have been discussed.

Chapter 2

Preliminaries

In this chapter, we recall some existing standard results of the probability theory and stochastic analysis. These contents are mainly motivated from book by Brzezniak and Tomasz [152].

Definition 2.0.1 (Sigma-Field). Let Ω be a non-empty set and $\mathscr{F}$ be a family of subsets of Ω such that

1. the empty set $\phi \in \mathscr{F}$;
2. if for any subset $A \in \Omega$ is such that $A \in \mathscr{F}$, then the compliment of A on Ω also belongs to $\mathscr{F}$;
3. if any sequence $\{A_i\}_{i\in\mathbb{N}}$ of sets lies in $\mathscr{F}$, then $\cup_{i\in\mathbb{N}}$ also lies in $\mathscr{F}$.

Such a family of subsets $\mathscr{F}$ is known as *σ-filed* or *σ-algebra* on Ω.

Definition 2.0.2 (Events). Let $\mathscr{F}$ be a σ-field on Ω. Then, any set $A \in \mathscr{F}$ is called an event.

Definition 2.0.3 (Probability Measure). Let $\mathscr{F}$ be a σ-field of subsets of Ω. A function

$$\mathbb{P} : \mathscr{F} \to [0,1]$$

is said to be *probability measure* provided it satisfy the following three properties (Kolomgorov axioms):

1. $\mathbb{P}(A) \geq 0$ for any $A \in \mathscr{F}$;
2. $\mathbb{P}(\Omega) = 1$;

3. if $A_1, A_2, \ldots.$ are pairwise disjoint sets (i.e., $A_i \cap A_j = \phi$ for $i \neq j$) on $\mathscr{F}$, then

$$\mathbb{P}(\cup_{i\in\mathbb{N}} A_i) = \sum_{i\in\mathbb{N}} \mathbb{P}(A_i).$$

The last property is known as σ-additivity.

Definition 2.0.4 (Probability Space). The triple $(\Omega, \mathscr{F}, \mathbb{P})$ is known as a *probability space*, where Ω be any set and $\mathscr{F}$ and $\mathbb{P}$ are given by the definitions 2.0.1 and 2.0.3 respectively.

Definition 2.0.5 (Almost Surely). Let A be an event on Ω. Then, A is said to be occur almost surely (a.s.) whenever $\mathbb{P}(A) = 1$.

Properties of probability measure. The following basics properties of the probability measure can be easily verified from the definition 2.0.3,

1. for empty set $\phi, \mathbb{P}(\phi) = 0$;

2. for any $A, B \in \mathscr{F}$ with $A \subset B$, we have $\mathbb{P}(A) \leq \mathbb{P}(B)$;

3. $\mathbb{P}(A^c) = 1 - \mathbb{P}(A)$;

4. $0 \leq \mathbb{P}(A) \leq 1$;

5. Let $A_1, A_2, \ldots.$ be an sequence of events, then $\mathbb{P}(\cup_n A_n) \leq \sum_n \mathbb{P}(A_n)$;

6. $A_1 \subset A_2 \subset \cdots$ be sequence of events with $A_n \in \mathscr{F}$ then

$$\mathbb{P}(A_1 \cup A_2 \cup \cdots) = \lim_{n\to\infty} \mathbb{P}(A_n).$$

Similarly, if $A_1, A_2, \ldots.$ is an contracting sequence of events, i.e.

$$A_1 \supset A_2 \supset \cdots,$$

then

$$\mathbb{P}(A_1 \cap A_2 \cap \cdots) = \lim_{n\to\infty} \mathbb{P}(A_n).$$

Let us denote by $\mathscr{B}$ as the intersection of all σ-algebras consisting all intervals of $\mathbb{R}$.

Definition 2.0.6 (Random Variable (r.v.)). Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a given probability space and define a function $X : \Omega \to \mathbb{R}$. Then we say that the function X is random variable if it is $\mathscr{F}$ measurable i.e.

$$X^{-1}(B) := \{\omega \in \Omega : X(\omega) \in B\} \subset \mathscr{F}.$$

for each Borel set $B \subset \mathscr{B}(\mathbb{R})$.

We also express $\{X \in B\}$ in place of the event $\{\omega \in \Omega : X(\omega) \in B\}$ as a sort-hand notation.

Definition 2.0.7. Let $X : \Omega \to \mathbb{R}$, be any given r.v.. Then every such r.v. X gives rise to a probability measure

$$\mathbb{P}_X(B) = \mathbb{P}\{X \in B\}$$

on the σ-field of Borel sets $B \in \mathscr{B}(\mathbb{R})$. $\mathbb{P}_X$ is also known as the distribution of X. Further, a function $F_x : \mathbb{R} \to [0,1]$ given by

$$F_X(x) = \mathbb{P}\{X \leq x\}$$

is said to be distribution function of X.

Definition 2.0.8. Any r.v. $X : \Omega \to \mathbb{R}$ is integrable if

$$\int_{\Omega} |X| d\mathbb{P} < \infty.$$

Definition 2.0.9 (Expectation). If a continuous r.v. X is integrable, then we define *expectation* of X as

$$\mathbb{E}(X) = \int_{\Omega} X d\mathbb{P}$$

Definition 2.0.10 (Conditional expectation). Let X be an integrable r.v.. Then for every event $B \in \mathscr{F}$ s.t. $\mathbb{P}(B) \neq 0$, we define the *conditional expectation* of X given B as

$$\mathbb{E}(X|B) = \frac{1}{\mathbb{P}(B)} \int_B X \, d\mathbb{P}.$$

General Properties of Conditional Expectation. Conditional expectation has the following properties:

1. $\mathbb{E}(aX+b\zeta|\mathscr{G})=a\mathbb{E}(X|\mathscr{G})+b\mathbb{E}(\zeta|\mathscr{G})$ **(linearity)**;
2. $\mathbb{E}(\mathbb{E}(X|\mathscr{G}))=\mathbb{E}(X)$;
3. $\mathbb{E}(X\zeta|\mathscr{G})=X\mathbb{E}(\zeta|\mathscr{G})$ if X is $\mathscr{G}$-measurable (**taking out what is known**);
4. $\mathbb{E}(X|\mathscr{G})=\mathbb{E}(X)$ if X is independent of $\mathscr{G}$ (**an independent condition drops out**);
5. $\mathbb{E}(\mathbb{E}(X|\mathscr{G})|\mathscr{H})=\mathbb{E}(X|\mathscr{H})$ if $\mathscr{H}\subset\mathscr{G}$ **(tower property)**;
6. If $X\geq 0$, then $\mathbb{E}(X|\mathscr{G})\geq 0$ **(positivity)**.

Here a,b are arbitrary real numbers, X, ζ are integrable random variables on a probability space $(\Omega,\mathscr{F},\mathbb{P})$ and $\mathscr{G}$, $\mathscr{H}$ are σ-fields on Ω contained in $\mathscr{F}$. All equalities and inequalities in (6) hold $\mathbb{P}$-a.s.

Radon-Nikodym Theorem : Consider a probability space $(\Omega,\mathscr{F},\mathbb{P})$ and let $\mathscr{G}$ be a σ-field contained in $\mathscr{F}$. Then for every r.v. X and for each $A\in\mathscr{G}$ there exists a $\mathscr{G}$- measurable r.v. Y such that

$$\int_A X d\mathbb{P}=\int_A Y d\mathbb{P}.$$

Jensen's Inequality: Let us take a convex function $\phi:\mathbb{R}\longrightarrow\mathbb{R}$ and let X be an integrable random variable on a probability space $(\Omega,\mathscr{F},\mathbb{P})$ such that $\phi(X)$ is also integrable. Then

$$\phi\left(\mathbb{E}(X|\mathscr{G})\right)\leq\mathbb{E}\left(\phi(X)|\mathscr{G}\right)\quad a.s.,$$

for any σ-field $\mathscr{G}$ on Ω contained in $\mathscr{F}$.

Definition 2.0.11 (Filtration). Let $\mathscr{F}_1$,$\mathscr{F}_2$ ... be a sequence of σ-fields on Ω such that

$$\mathscr{F}_1\subset\mathscr{F}_2\subset\ldots\subset\mathscr{F},$$

is called a filtration. Here $\mathscr{F}_n$ consists all events A such that at time n it is possible to decide whether A has occurred or not.

Definition 2.0.12 (Martingale). A sequence X_1,X_2,... of r.v. is called a **martingale** with respect to a filtration $\mathscr{F}_1$, $\mathscr{F}_2$... if

1. X_n is integrable for each $n = 1, 2, ...$;

2. X_1,X_2,...is adapted to $\mathscr{F}_1$, $\mathscr{F}_2$...;

3. $\mathbb{E}(X_{n+1}|\mathscr{F}_n) = X_n$ a.s. for each $n = 1, 2, ...$

If in place of (3) $\mathbb{E}(X_{n+1}|\mathscr{F}_n) \leq X_n$ (respectively, $\mathbb{E}(X_{n+1}|\mathscr{F}_n) \geq X_n$) a.s. for each $n = 1, 2, ...$ holds then we say that $X_1, X_2, ...$ is a **supermartingale** (respectively **submartingale**) with respect to a filtration $\mathscr{F}_1$, $\mathscr{F}_2$....

Definition 2.0.13 (Stopping Time). A r.v. τ with values in the set $\{1, 2, ...\} \cup \{\infty\}$ is known as a **stopping time** (with respect to a filtration $\mathscr{F}_n$) if for every $n = 1, 2, ...$

$$\{\tau = n\} \in \mathscr{F}_n$$

Proposition 2.0.1 (Doob's Maximal Inequality). *Let X_n, $n \in \mathbb{N}$ be a non-negative submartingale (with respect to a filtration $\mathscr{F}_n$). Then for any $\lambda > 0$*

$$\lambda \mathbb{P}\Big(\max_{k \leq n} X_k \geq \lambda\Big) \leq \mathbb{E}\Big(X_n 1_{\{max_{k \leq n} X_k \geq \lambda\}}\Big),$$

where 1_A is the characteristics function of a set A.

Theorem 2.0.2 (Doob's maximal L^2 inequality). *Suppose that X_n,$n \in \mathbb{N}$, is a non-neative square integrable submartingale (with respect to a filtration $\mathscr{F}_n$), then*

$$\mathbb{E}\Big|\max_{k \leq n} X_k\Big|^2 \leq 4\mathbb{E}\Big|X_n\Big|^2$$

Theorem 2.0.3 (Doob's Martingale Convergence Theorem). *Suppose that X_1,X_2,...is a supermartingale (with respect to a filtration $\mathscr{F}_1$, $\mathscr{F}_2$,...) such that*

$$\sup_n \mathbb{E}(|X_n|) < \infty.$$

Then there is an integrable random variable X such that

$$\lim_{n \to \infty} X_n = X \ \ a.s.$$

Remark: *In particular, the theorem is also true for martingales as every martingale is a supermartingale. It is also true for submartingales, since X_n is a submartingale iff $-X_n$ is a supermartingale.*

Definition 2.0.14 (**Stochastic Process**). Let $T \subset \mathbb{R}$. A stochastic process is a collection of random variables $X(t)$ indexed by $t \in T$. The stochastic process $X(t)$ is said to be a discrete stochastic process if the given indexed time are discrete i.e. $T = \{1, 2, \ldots \cdots\}$. Further, if T is an interval in $\mathbb{R}$ (typically $T = \mathbb{R}^+ = [0, \infty)$), $X(t)$ is known as stochastic process in continuous time.

Definition 2.0.15 (**Sample path**). For any $\omega \in \Omega$ the map

$$T \ni t \mapsto X(t, w)$$

is known as path (or sample path) of $X(t)$.

Definition 2.0.16. A family $\mathscr{F}_t$ of σ-fields on Ω parametrized by $t \in T$, where $T \subset \mathbb{R}$, is called a filtration if

$$\mathscr{F}_s \subset \mathscr{F}_t \subset \mathscr{F}$$

for any $s, t \in T$ such that $s < t$.

Definition 2.0.17 (**Brownian Motion**). A real valued stochastic process $\{W(t), 0 \leq t < \infty\}$ is said to be Brownian motion or Wiener process if

- at $t = 0$, $W(t) = 0$ a.s;
- the paths $t \mapsto W(t)$, are almost surely continuous;
- for any $t \geq 0$, the increment in $W(t)$ is stationary and independent;
- the increment $W(t+k) - W(k)$ has the normal distribution with mean 0 and variance t for any $t > 0, k > 0$.

  The Wiener processes satisfy the following properties:

- For any $0 < s \leq t < \infty$, we get

$$\mathbb{E}\left(W(s)W(t)\right) = \min\{s, t\}$$

  and

- $$\mathbb{E}\left(|W(t) - W(s)|^2\right) = |t - s|.$$

- $W(t)$ is satisfy martingale property with respect to the filtration $\mathscr{F}_t$.

Definition 2.0.18. An n dimensional stochastic process $\mathbf{W}(\cdot) = (W_1(\cdot), \cdots, W_n(\cdot))$ is known to be an n dimensional **Wiener process** provided for each $k = 1, \ldots\ldots, n$, $W_k(t)$ are $\mathbb{R}$-valued Wiener process.

Definition 2.0.19. A stochastic process $Y(\cdot)$ is said to be follow continuous sample path $a.s.$ if for every $t, s \geq 0$

$$\mathbb{E}(|Y(t) - Y(s)|^{\rho}) \leq C|t-s|^{1+\sigma}$$

for some constants $\rho, \sigma, C \geq 0$.

Further, for each $0 < \gamma < \frac{\sigma}{\rho}$ and $T > 0$ there is a constant K which relies on constants ρ, σ, C, s.t.

$$|Y(t,\omega) - Y(s,\omega)| \leq |t-s|^{\gamma}$$

for almost every ω and $s, t \in [0, T]$.

Definition 2.0.20. We shall call $f(t), t \geq 0$ a random step process if there is a finite sequence of numbers $0 = t_0 < t_1 < \ldots < t_n$ and square integrable random variables $\eta_0, \eta_1, \ldots, \eta_{n-1}$ such that

$$f(t) = \sum_{j=0}^{n-1} \eta_j \, 1_{[t_j, t_{j+1})}, \tag{2.1}$$

where η_j is $\mathscr{F}_{t_j}$-measurable for $j = 0, 1, \ldots, n-1$. The set of random step processes will be denoted by M^2_{step} .

Definition 2.0.21. The stochastic integral of a random step process $f \in M^2_{step}$ of the form (2.1) is defined by

$$I(f) = \sum_{j=0}^{n-1} \eta_j (W(t_{j+1}) - W(t_j)).$$

Proposition 2.0.4. *For every random step process $f \in M^2_{step}$ the stochastic integral $I(f)$ is a quadratic integrable random variable, i.e. $I(f) \in L^2$, such that*

$$\mathbb{E}\Big(|I(f)|^2\Big) = \mathbb{E}\Big(\int_0^{\infty} |f(t)|^2 dt\Big).$$

Definition 2.0.22. Let us denote by M^2 the class of stochastic processes $f(t), t \geq 0$ such that

$$\mathbb{E}\Big(\int_0^{\infty} |f(t)|^2 dt\Big) < \infty$$

and there is a sequence $f_1\ , f_2\ , \ldots \in M^2_{step}$ of random step processes such that

$$\lim_{n\to\infty} \mathbb{E}\Big(\int_0^{\infty} |f(t) - f_n(t)|^2 dt\Big) = 0. \tag{2.2}$$

In this case we shall say that the sequence of random step processes $f_1\ , f_2\ , \ldots$ approximates $f \in M^2$.

Definition 2.0.23. We say $I(f) \in L^2$ the Ito stochastic integral (from 0 to ∞) of $f \in M^2$ if

$$\lim_{n\to\infty} \mathbb{E}(|I(f) - I(f_n)|^2) = 0 \tag{2.3}$$

for any sequence $f_1\ , f_2\ , \ldots \in M^2_{step}$ of random step processes that approximates $f \in M^2$, i.e . s.t. (2.2) holds . We can also express it as

$$\int_0^{\infty} f(t) dW(t)$$

in place of $I(f)$.

Definition 2.0.24. For any $T > 0$ we shall denote by M^2_T the space of all stochastic processes $f(t), t \geq 0$ such that

$$1_{[0,T)} f \in M^2$$

The Ito stochastic integral (from 0 to T) of $f \in M^2_T$ is defined by

$$I_T(f) = I(1_{[0,T)} f).$$

We shall also write

$$\int_0^{T} f(t) dW(t)$$

in place of $I_T(f)$.

Theorem 2.0.5. *The following characterizations hold for any $f, g \in M^2_t$, any $\alpha, \beta \in \mathbb{R}$, and any $0 \leq s < t$:*

1. ***linearity***

$$\int_0^t (\alpha f(r)+\beta g(r))dW(r) = \alpha \int_0^t f(r)dW(r) + \beta \int_0^t g(r)dW(r);$$

2. ***isometry***

$$\mathbb{E}\Big(\Big|\int_0^t f(r)dW(r)\Big|^2\Big) = \mathbb{E}\Big(\int_0^t |f(r)|^2 dr\Big);$$

3. ***martingale property***

$$\mathbb{E}\Big(\Big|\int_0^t f(r)dW(r)\Big|\mathscr{F}_s\Big) = \int_0^s |f(r)|dW(r);$$

Definition 2.0.25. Let $X(t)$ and $\zeta(t)$ be stochastic processes defined for $t \in T$, where $T \subset \mathbb{R}$. We call the processes are modifications(or versions) of one another provided

$$\mathbb{P}\{X(t) = \zeta(t)\} = 1 \text{ for all } t \in T.$$

Definition 2.0.26. A stochastic process $X(t)$, $t \geq 0$ is known as an Ito process provided it has almost surely continuous paths and can be expressed as

$$X(T) = X(0) + \int_0^T a(t)\ dt + \int_0^T b(t)\ dW(t) \ \text{ a.s.,} \tag{2.4}$$

where $b(t)$ is a stochastic process lying to M_T^2 for all $T > 0$ and $a(t)$ is a process measurable with respect to the filtration $\mathscr{F}_t$ s.t.

$$\int_0^T a(t)\ dt < \infty \tag{2.5}$$

for alt $T \geq 0$. The class of all adapted processes $a(t)$ satisfying (2.5) for some $T > 0$ will be represented by L_T^1.

For an Ito process $X(t)$ it is necessary to take (2.4) as

$$dX(t) = a(t)dt + b(t)dW(t) \tag{2.6}$$

and to call $dX(t)$ the stochastic differential of $X(t)$. This is popular as the Ito differential notation.

Theorem 2.0.6. *Let $F(t,x)$ be a real-valued function having continuous partial derivatives $F_t'(t,x)$, $F_x'(t,x)$ and $F_{xx}''(t,x)$ for all $t > 0$ and $x \in \mathbb{R}$. Let us also assume that the process*

$F_x'(t,W(t))$ belongs to M_T^2, for all $T > 0$. Then $F(t,W(t))$ is an Ito process such that

$$F(T,W(T)) - F(0,W(0)) = \int_0^T \Big(F_t'(t,W(t)) + \frac{1}{2}F_{xx}''(t,W(t))\Big)dt + \int_0^T F_x'(t,W(t))dW(t) \text{ a.s.} \tag{2.7}$$

In differential presentation this result can be given as

$$dF(t,W(t) = \Big(F_t'(t,W(t)) + \frac{1}{2}F_{xx}''(t,W(t))\Big)dt + F_x'(t,W(t))dW(t) \tag{2.8}$$

Definition 2.0.27. We say that an Ito process $X(t), t \geq 0$ is a solution of the given initial value problem

$$\begin{cases} dX(t) = f(X(t))dt + g(X(t))dW(t), \\ X(0) = X_0, \end{cases}$$

if $X(0)$ is an $\mathscr{F}_0$-measurable random variable, the stochastic processes $f(X(t))$ and $g(X(t))$ belong, respectively, to L_T^1 and M_T^2, and

$$X(T) = X(0) + \int_0^T f(X(t))\, dt + \int_0^T g(X(t))\, dW(t) \text{ a.s.}$$

for all $T \geq 0$.

NOTE: In this chapter only Brownian motion (Wiener process) are given in brief. However in the subsequent chapters the other type of stochastic processes (noises) are given wherever they are used.

Chapter 3

Generalized stochastic Burgers equation with non-Lipschitz diffusion coefficient

In this chapter, we study the existence of weak solutions to the one-dimensional generalized stochastic Burgers equation having polynomial type nonlinearity driven by space-time white noise with Dirichlet boundary conditions and α- Hölder continuous coefficient in noise term, where $\alpha \in [1/2, 1)$. The existence of weak solutions is shown by solving an equivalent martingale problem.

Our main contribution in the present chapter is that it generalizes the work of Kolkovska [97] by extending the quadratic nonlinearity to a class of polynomial nonlinearity and using more general diffusion coefficient σ. This work also differs from the works in [67, 95], where authors have shown the existence of mild solutions of stochastic Burgers equation with polynomial nonlinearity having white noise with respect to time and Lipschitz continuity in the diffusion coefficient σ, whereas the present work considers the α-Hölder continuity in the diffusion coefficient σ, where $\alpha \in [1/2, 1)$ along with a space-time white noise.

The structure of the chapter is the following: In the first section, we introduce the generalized stochastic Burgers equation with non-Lipschitz diffusion coefficient. In the next section, the rigorous formulation of the problem is given. In Section 3.3, the discretized form of (3.1)–(3.3) is obtained which gives a system of stochastic differential equations in finite dimension. Further, the existence of unique strong solution to this system of stochastic differential equation is established by showing the existence and pathwise uniqueness of weak solution for the system which is motivated by [60]. Next, in Section 3.4, we have shown the tightness property of the family of approximating solutions by satisfying the multi-dimensional Totoki Kolmogorov criterion. At last, in Section 3.5, the existence of weak solutions to the original problem (3.1)–(3.3) is shown by solving an equivalent martingale problem.

3.1 Introduction

In this chapter, we study the one dimensional stochastic Burgers equation with polynomial non-linearity perturbed by a space-time white noise i.e.

$$\frac{\partial f}{\partial t}(t,y) = \frac{\partial^2 f}{\partial y^2}(t,y) + \lambda \frac{\partial f^p}{\partial y}(t,y) + \sigma(f(t,y))\frac{\partial^2 W}{\partial t \partial y}(t,y) \tag{3.1}$$

with Dirichlet boundary conditions

$$f(t,0) = f(t,1) = 0, \qquad t \in [0,T], \tag{3.2}$$

and initial datum

$$f(0,y) = f_0(y), \qquad y \in (0,1), \tag{3.3}$$

where $p \geq 2$ is a fixed integer, $T > 0$, $\dfrac{\partial^2 W(t,x)}{\partial t \partial x}$ is a white noise with respect to space and time both as in [141]. Further, $\sigma = \sigma(t,x,r)$ are Borel-measurable functions on $\mathbb{R}^+ \times [0,1] \times \mathbb{R}$. If we take $\sigma(x) = \sqrt{x(1-x)}$, then the last term on right hand side of the equation (3.1) is known as the stepping stone noise and the corresponding stochastic equation is used as model continuous-time stepping stone models in population genetics, where the gene frequency in colonies is modeled by f(t,x), see [97] and references therein. In this chapter the existence of weak solutions to the equations (3.1)–(3.3) is established. The proof is mainly motivated by the technique used in Funaki [60]. With no loss of generality, we suppose that $\lambda = 1$.

3.2 Formulation of the problem

Definition 3.2.1. Let $(\Omega, \mathscr{F}, (\mathscr{F}_t), P)$ be a stochastic basis with filtration $(\mathscr{F}_t) = \{\mathscr{F}_t, t \in [0,T]\}$. Then the Brownian sheet $W(t,x) = \{W(t,x) : t \in [0,T], x \in \mathbb{R}\}$ is defined as a continuous, $(\mathscr{F}_t)$ adapted and centered Gaussian random field with covariance

$$\mathbb{E}\left(W(s,x)W(t,y)\right) = (s \wedge t)(x \wedge y)$$

in the sense of Walsh [141].

Remark 3.2.1. By the properties of W, it can be proved that white noise with respect to the filtration $(\mathscr{F}_t)$ is a martingale measure over $([0,T]\times\mathscr{B}[0,1])$, where $\mathscr{B}[0,1]$ is bounded Borel subset of $[0,1]$, see [94, 141]. Now, the equation (3.1) can be expressed in the weak sense by following [141].

Definition 3.2.2. A continuous stochastic process $\{f(t,x); t\in[0,T], x\in[0,1]\}$, which is $(\mathscr{F}_t)$-adapted, is said to be solution of equation (3.1) in a weak sense, if for every $\phi\in\mathscr{C}^2[0,1]$, such that $\phi(0)=\phi(1)=0$, and a.s. for each $t\in[0,T]$, and $x\in[0,1]$, we have

$$\int_0^1 f(t,y)\phi(y)dy = \int_0^1 f(0,y)\phi(y)dy + \int_0^t\int_0^1 f(s,y)\phi''(y)dyds \\ -\lambda\int_0^t\int_0^1 f^p(s,y)\phi'(y)dyds + \int_0^t\int_0^1 \sigma(f(s,y))\phi(y)W(ds,dx). \quad (3.4)$$

Further, we assume following conditions on σ. First condition is that the diffusion coefficient σ satisfies Hölder's continuity of order $\alpha\in[1/2,1)$ on the interval [0, 1] i.e. there exist a constant $c\geq 0$ such that

$$|\sigma(r_1)-\sigma(r_2)|\leq c|r_1-r_2|^\alpha \qquad \forall\ r_1,r_2\in[0,1], \quad (3.5)$$

and second one is

$$\sigma(0)=\sigma(1)=0. \quad (3.6)$$

Example 3.2.1. Let $\sigma:\mathbb{R}\to\mathbb{R}$ such that

$$\sigma(r) := \begin{cases} \sqrt{r(1-r)} & \text{if } 0\leq r\leq 1 \\ 0 & \text{otherwise.} \end{cases} \quad (3.7)$$

Then, σ satisfies conditions (3.5) and (3.6).

Note: Other examples of such functions can be seen in [29].

3.3 The discretization processes

Let M $\geq$ 1 be a fixed integer and define the set $\left\{\frac{i}{M}; i=0,1,\cdots,M\right\}$. On this set, consider the discretized form of (3.1)–(3.3) by applying finite difference approximation for second derivative (central difference) and first (forward difference) derivative as

$$dY(t,x_i) = \big(\Delta_M Y(t,x_i) + \nabla_M (Y^p(t,x_i))\big)dt + \sqrt{M}\sigma(Y(t,x_i))dB_i(t), \tag{3.8}$$

$$Y(t,x_0) = Y(t,x_M) = 0, \tag{3.9}$$

$$Y(0,x_i) = f_0(x_i), \tag{3.10}$$

for every $i = 1,2,\cdots,M-1$ and $t \geq 0$. Here $x_i := \left\{\frac{i}{M}\right\}$ and $\{B_i(t) : i = 1,2,\cdots M-1\}$ is the system of Brownian motions, derived from the Brownian sheet $W(t,x)$ and it is defined as

$$\mathbb{E}(B_i(t)) := 0 \quad \forall\, i = 1,2,\cdots M,$$

$$\text{and} \quad \mathbb{E}(B_i(t)B_j(s)) := \begin{cases} 0 & \text{if } i \neq j, \\ t \wedge s & \text{if } j = i, \end{cases}$$

while ∇_M and Δ_M denote the approximation of the first and second order derivative respectively with respect to the variable x in the discrete sense and defined as

$$\Delta_M Y(t,x_i) := \frac{Y(t,x_i + \frac{1}{M}) - 2Y(t,x_i) + Y(t,x_i - \frac{1}{M})}{\frac{1}{M^2}}$$

and

$$\nabla_M (Q(t,x_i)) := \frac{Q(t,x_i + \frac{1}{M}) - Q(t,x_i)}{\frac{1}{M}},$$

for all $i = 1,2,\cdots,M-1$.
By setting $Y(t,x_i \pm \frac{1}{M}) := y_{i\pm 1}(t)$, we can write (3.8) as

$$dy_i(t) = \Big(M^2[y_{i+1}(t) - 2y_i(t) + y_{i-1}(t)] + M[y_{i+1}(t)^p - y_i(t)^p]\Big)dt + \sqrt{M}\sigma(y_i(t))dB_i(t) \tag{3.11}$$

Re-writing (3.11) in more compact form as

$$dy_i(t) = \left(\sum_{j=1}^{M-1} \alpha_{ij} y_j(t) + \beta_{ij} y_j(t)^p\right) dt + \sqrt{M}\sigma(y_i(t))dB_i(t) \tag{3.12}$$

with

$$y_0(t) = y_M(t) = 0 \qquad\qquad t \in [0,T], \tag{3.13}$$

and

$$y_i(0) = f_0(i/M), \qquad\qquad 1 \le i,j \le M-1, \tag{3.14}$$

where

$$\alpha_{ij} := \begin{cases} M^2 & \text{if } j = i+1, i-1 \\ -2M^2 & \text{if } j = i \\ 0 & \text{elsewhere} \end{cases} \tag{3.15}$$

and

$$\beta_{ij} := \begin{cases} M & \text{if } j = i+1 \\ -M & \text{if } j = i \\ 0 & \text{otherwise.} \end{cases} \tag{3.16}$$

It is noticed that the drift and diffusion coefficients in (3.12) do not satisfy the Lipschitz continuity, which restrict us to apply the classical results on the existence of unique solution to equation (3.12). The following theorem contains the main result of this section.

Theorem 3.3.1. *Let $Y(0) = (y_0(0), y_1(0), \cdots, y_M(0)) \in [0,1]^{M+1}$, be some given initial random data and conditions (3.5) and (3.6) hold. Then for each $T > 0$ and any integer $M \ge 1$, the system*

$$dy_i(t) = \left(\sum_{j=1}^{M-1} \alpha_{ij} y_j(t) + \beta_{ij} y_j(t)^p \right) dt + \sqrt{M} \sigma(y_i(t)) dB_i(t) \tag{3.17}$$

$$y_0(t) = y_M(t) = 0 \qquad \forall\, t \in [0,T], \tag{3.18}$$

$$y_i(0) = y_i, \tag{3.19}$$

where $i = 1, 2, \cdots M-1$, admits a unique strong solution

$$Y(t) = (y_0(t), y_1(t), \cdots, y_M(t)) \in \mathscr{C}([0,T],[0,1]^{M+1}). \tag{3.20}$$

Proof. We consider the following modified form of stochastic differential equations (3.17)–(3.19) as

$$dy_i(t) = \left(\sum_{j=1}^{M-1} \alpha_{ij} y_j(t) + \beta_{ij} g(y_j(t)) \right) dt + \sqrt{M} K(y_i(t)) dB_i(t) \tag{3.21}$$

$$y_0(t) = y_M(t) = 0 \qquad \forall\, t \in [0,T], \tag{3.22}$$

$$y_i(0) = y_i, \tag{3.23}$$

where $i = 1, 2, \cdots, M-1$, $g : \mathbb{R} \to \mathbb{R}$ is defined as $g(x) = x^p \mathbf{1}_{\{-1 \le x \le 1\}}$ and $K : \mathbb{R} \to \mathbb{R}$ is defined as $K(x) = \sigma(x) \mathbf{1}_{\{0 \le x \le 1\}}$. Since, the coefficients of (3.21)–(3.23) are continuous and satisfies the linear growth conditions, by the [55, Theorem 3.10, Chapter 5], there exists a weak solution $Y(t)$ to (3.21)–(3.23).

Next, it is shown that for every weak solution $Y(t) = (y_0(t), y_1(t), \cdots, y_M(t))$ of (3.21)–(3.23), $y_i(t) \in [0,1]$ for every $i = 0, \cdots, M$ and $t \in [0,T]$. In order to prove this the following Lemma [61] is required.

Lemma 3.3.2. *Let Z={Z(t), t≥ 0} be a real valued semi-martingale. Suppose that there exist a function* $\rho : [0,\infty) \to [0,\infty)$ *such that* $\int_0^{\varepsilon} \frac{1}{\rho(u)} du = +\infty$ *for all* $\varepsilon \ge 0$, *and* $\int_0^t \frac{\mathbf{1}_{\{Z_s > 0\}}}{\rho(Z_s)} d\langle Z \rangle_s < \infty$ *for all* $t \ge 0$ *a.s. Then the local time of Z at zero, i.e.* $L_t^0(Z)$, *is identically zero for all t a.s..*

Let us apply the above Lemma 3.3.2 for the semi-martingale y_i and take $\rho(y_i) = y_i$, then $\int_0^{\varepsilon} \frac{1}{\rho(u)} \mathrm{du} = +\infty$ for all $\varepsilon \ge 0$, and

$$\int_0^t \frac{\mathbf{1}_{\{y_i(s) > 0\}}}{\rho(y_i(s))} d\langle y_i \rangle_s = \int_0^t \frac{\mathbf{1}_{\{y_i(s) > 0\}}}{(y_i(s))} M \sigma^2(y_i(s)) ds \quad < \infty.$$

Therefore local time $L_t^0(y_i)$ is zero. Again we use Lemma 3.3.2 and Tanaka's formula [126, Theorem 1.2 (Chapter IV)] for $(y_i(t))^- := \max\,[0, -y_i(t)]$ and summing over indices $i = 1 \cdots, M-1$, we get

$$\sum_{i=1}^{M-1}(y_i(t))^- = -\int_0^t \sum_{i=1}^{M-1}\mathbf{1}_{\{y_i(s)\le 0\}}\sum_{j=1}^{M-1}(\alpha_{ij}y_j(s)+\beta_{ij}g(y_j(s)))ds \tag{3.24}$$

$$\begin{aligned}
&= -\int_0^t \left[\sum_{i=1}^{M-1}\mathbf{1}_{\{y_i(s)\le 0\}}\sum_{j=1}^{M-1}\alpha_{ij}y_j(s)\right]ds \\
&\qquad -\int_0^t \left[\sum_{i=1}^{M-1}\mathbf{1}_{\{y_i(s)\le 0\}}\sum_{j=1}^{M-1}\beta_{ij}g(y_j(s))\right]ds \\
&\le \int_0^t \left[\sum_{i=1}^{M-1}\mathbf{1}_{\{y_i(s)\le 0\}}\sum_{j=1}^{M-1}\alpha_{ij}(y_j(s))^-\right]ds \\
&\qquad +\int_0^t \left[\sum_{i=1}^{M-1}\mathbf{1}_{\{-1\le y_i(s)\le 0\}}\sum_{j=1}^{M-1}(-\beta_{ij})(y_j(s))\right]ds \\
&\le \int_0^t \left[\sum_{i=1}^{M-1}\mathbf{1}_{\{y_i(s)\le 0\}}\sum_{j=1}^{M-1}\alpha_{ij}(y_j(s))^-\right]ds \\
&\qquad +M\int_0^t \left[\sum_{i=1}^{M-1}\mathbf{1}_{\{-1\le y_i(s)\le 0\}}\sum_{j=1}^{M-1}(y_j(s))\right]ds \\
&\le M\int_0^t \left[\sum_{i=1}^{M-1}(y_i(s))^-\right]ds.
\end{aligned} \tag{3.25}$$

Finally, Gronwall's inequality gives

$$\sum_{i=1}^{M-1}(y_i(t))^- = 0,$$

i.e. $(y_i(t))_{\{i\in\{1,2,\cdots M-1\}\}}$ is always non-negative for each $t\in[0,T]$. Again, solving the equation (3.24) for $(1-y_i(t))^-$, we can obtain $y_i(t)\le 1$ for every $t\in[0,T]$ and $1\le i\le M-1$.
Since, for $y_i(t)\in[0,1]$, the system (3.21) coincide with system (3.17), therefore, the system (3.17) has a weak solution with trajectories lies in $\mathscr{C}\left([0,T],[0,1]^{M+1}\right)$.

3.3.1 Pathwise uniqueness of the solution for the descretized equations

In order to show the pathwise uniqueness of solutions to (3.17), suppose that $Y^{(1)}=(y_0^{(1)},\cdots,y_M^{(1)})$ and $Y^{(2)}=(y_0^{(2)},\cdots,y_M^{(2)})$ are two different weak solutions of (3.17)-(3.19), with the same Brownian motion and same initial data.

Set $\qquad v_i := y_i^{(1)}-y_i^{(2)} \qquad\qquad i=1,2,\cdots,M-1$ and $t\in[0,T]$.

Then, we have

$$v_i(t) = \sum_{j=1}^{M-1} \alpha_{ij} \int_0^t v_j(s)ds + \sum_{j=1}^{M-1} \beta_{ij} \int_0^t (y_i^{(1)}(s)^p - y_i^{(2)}(s)^p)ds$$
$$+ \int_0^t \sqrt{M}\Big(\sigma(y_i^{(1)}(s)) - \sigma(y_i^{(2)}(s))\Big)dB_i(s) \qquad i = 1,2,\cdots,M-1. \quad (3.26)$$

The quadratic variation $\langle V\rangle_t$ of $v_i(t)$,

$$\langle V\rangle_t = \int_0^t \Big[\sqrt{M}\sigma(y_i^{(1)}(s)) - \sqrt{M}\sigma(y_i^{(2)}(s))\Big]^2 ds$$

satisfies

$$\int_0^t \frac{\Big[\sqrt{M}\sigma(y_i^{(1)}(s)) - \sqrt{M}\sigma(y_i^{(2)}(s))\Big]^2}{y_i^{(1)}(s) - y_i^{(2)}(s)} \mathbf{1}_{\{y_i^{(1)}(s)-y_i^{(2)}(s)>0\}} ds$$
$$= M \int_0^t \frac{\Big(\sigma(y_i^{(1)}(s)) - \sigma(y_i^{(2)}(s))\Big)^2}{y_i^{(1)}(s) - y_i^{(2)}(s)} \mathbf{1}_{\{y_i^{(1)}(s)-y_i^{(2)}(s)>0\}} ds.$$

Since $y_i^{(1)}, y_i^{(2)} \in [0,1]$, implies that $\Big(y_i^{(1)} - y_i^{(2)}\Big) \in [-1,1]$. Further, using condition (3.5) and then simplifying, we have

$$\int_0^t \frac{\Big[\sqrt{M}\sigma(y_i^{(1)}(s)) - \sqrt{M}\sigma(y_i^{(2)}(s))\Big]^2}{y_i^{(1)}(s) - y_i^{(2)}(s)} \mathbf{1}_{\{y_i^{(1)}(s)-y_i^{(2)}(s)>0\}} ds$$
$$\leq M \int_0^t \Big(y_i^{(1)}(s) - y_i^{(2)}(s)\Big)^{2\alpha-1} \mathbf{1}_{\{y_i^{(1)}(s)-y_i^{(2)}(s)>0\}} ds < 2MT < \infty. \quad (3.27)$$

Therefore, applying Lemma 3.3.2 to $v_i(t) = y_i^{(1)} - y_i^{(2)}$ with $\rho(v_i)$=v_i, we obtain that local time $L_t^0(y_i^{(1)} - y_i^{(2)}) = 0$ for all $i = 1,2,\cdots,M-1$. Using Tanaka's formula for the continuous semi-martingale $v_i(t)$, we get

$$| v_i(t) |= \sum_{j=1}^{M-1} \alpha_{ij} \int_0^t \mathrm{sgn}(v_i(s))v_i(s)ds$$
$$+ \sum_{j=1}^{M-1} \beta_{ij} \int_0^t \mathrm{sgn}(v_i(s))(y_j^{(1)}(s)^p - y_j^{(2)}(s)^p)ds$$
$$+ \sqrt{M} \int_0^t \mathrm{sgn}(v_i(s)) \Big[\sigma(y_i^{(1)}(s)) - \sigma(y_i^{(2)}(s))\Big] dB_i(s)$$

where $i = 1,2,\cdots,M-1$. Using the fact that α_{ij} and β_{ij} are bounded, summing over all $i = 1,2,\cdots,M-1$, and taking the expectation, we obtain

$$\begin{aligned}
\mathbb{E}\left(\sum_{i=1}^{M-1} \mid v_i(t) \mid\right) =& \mathbb{E}\left(\sum_{i,j=1}^{M-1} \alpha_{ij} \int_0^t v_i(s)ds\right) \\
&+ \mathbb{E}\left(\sum_{i,j=1}^{M-1} \beta_{ij} \int_0^t \operatorname{sgn}(v_i(s))(y_i^{(1)}(s)^p - y_i^{(2)}(s)^p)ds\right) \\
:=& I_1 + I_2. \qquad (3.28)
\end{aligned}$$

For I_1, we have

$$\begin{aligned}
I_1 &\leq \mathbb{E}\int_0^t \sum_{i=1}^{M-1} \mid v_i(s) \mid \left(\sum_{j=1}^{M-1} \mid \alpha_{ij} \mid\right) \\
&\leq 4M^2\,\mathbb{E}\int_0^t \sum_{i=1}^{M-1} \mid v_i(s) \mid ds. \qquad (3.29)
\end{aligned}$$

Since the values of solutions lie in interval $[0,1]$, therefore, for I_2, we estimate

$$\begin{aligned}
I_2 &\leq \mathbb{E}\int_0^t \sum_{i=1}^{M-1} \mid v_i(s) \mid \left(\sum_{j=1}^{M-1} \mid \beta_{ij} \mid\right) \\
&\leq 2M\,\mathbb{E}\int_0^t \sum_{i=1}^{M-1} \mid v_i(s) \mid ds. \qquad (3.30)
\end{aligned}$$

Inserting (3.29) and (3.30) in to (3.28) and applying Gronwall's inequality, we have

$$\mathbb{E}\left(\sum_{i=1}^{M-1} \mid v_i(t) \mid\right) = 0,$$

i.e. the weak solutions are pathwise unique. Finally, by a standard theorem of Yamada and Watanabe [148](or see [42, pages 8-9]), the existence of a unique strong solution is obtained. □

3.4 Tightness of the approximating processes

In this section, we demonstrate the tightness of the family of the strong solutions for system of stochastic differential equations (3.12)–(3.14). Let us denote the polygon approximation of $y_i(t)$ by $f_M(t,y)$ which is defined as

$$f_M(t,y) := Y\left(t, \frac{[My]+1}{M}\right)(My - [My]) + Y\left(t, \frac{[My]}{M}\right)\big([My]+1-My\big), \qquad (3.31)$$

where $t \in [0,T]$, $y \in [0,1]$ and $[y] = \frac{k}{M}$ for $\frac{i}{M} \le y < \frac{i+1}{M}$, so that we have $Y\left(t, \frac{i}{M}\right) = y_i(t) = f_M\left(t, \frac{i}{M}\right)$, for every $t \in [0,T]$ and $0 \le i \le M$.

Suppose $q_M\left(t, \frac{i}{M}, \frac{j}{M}\right)$, $t \in [0,T]$, $0 \le i, j \le M$ is the fundamental solution of the discrete heat equation such that

$$\frac{\partial}{\partial t} q_M\left(t, \frac{i}{M}, \frac{j}{M}\right) = \Delta_M q_M\left(t, \frac{i}{M}, \frac{j}{M}\right) \qquad t \ge 0, 1 \le i, j \le M-1. \tag{3.32}$$

$$q_M\left(0, \frac{i}{M}, \frac{j}{M}\right) = M\delta_{ij}, \tag{3.33}$$

with boundary conditions

$$q_M\left(t, 0, \frac{j}{M}\right) = q_M\left(t, 1, \frac{j}{M}\right) = 0 \tag{3.34}$$

for all $t \in [0,T]$, $1 \le j \le M-1$.

Then (3.12)–(3.14) can be re-written as

$$\begin{aligned} y_i(t) = &\sum_{j=1}^{M-1} \frac{1}{M} q_M\left(t, \frac{i}{M}, \frac{j}{M}\right) y_i(0) \\ &+ \int_0^t \sum_{j=1}^{M-1} \frac{1}{M} q_M\left(t-s, \frac{i}{M}, \frac{j}{M}\right) \beta_{ij}(y_i(s))^p \\ &+ \int_0^t \sum_{j=1}^{M-1} \sqrt{M} q_M\left(t-s, \frac{i}{M}, \frac{j}{M}\right) \sigma(y_i(s)) dB_i(s), \quad 1 \le i \le M-1, \end{aligned}$$

where the last integral on the right hand side represents the sum of Itô stochastic integrals. Let us define the re-scaled formulation of the Green function G_M to the heat kernel q_M in $[0,1]$ by

$$\begin{aligned} G_M\left(t, y, \frac{j}{M}\right) = &q_M\left(t, \frac{[My]+1}{M}, \frac{j}{M}\right)(My - [My]) \\ &+ q_M\left(t, \frac{[My]}{M}, \frac{j}{M}\right)([My] + 1 - My). \end{aligned} \tag{3.35}$$

Therefore the linear interpolation of the $f_M(t,y)$, for $y \in \left[\frac{i}{M}, \frac{i+1}{M}\right)$ is

$$\begin{aligned} f_M(t,y) =& \sum_{j=1}^{M-1} \frac{1}{M} G_M\left(t, \frac{i}{M}, \frac{j}{M}\right) y_i(0) \\ &+ \int_0^1 \sum_{j=1}^{M-1} \left[\frac{1}{M} q_M\left(t-s, \frac{i+1}{M}, \frac{j}{M}\right) \beta_{(i+1)j}(y_i(s))^p (My - [My])\right. \\ &\left.\frac{1}{M} q_M\left(t-s, \frac{i}{M}, \frac{j}{M}\right) \beta_{ij}(y_i(s))^p ([My]+1-My)\right] ds \\ &+ \int_0^1 \sum_{j=1}^{M-1} \sqrt{M} G_M\left(t-s, \frac{i}{M}, \frac{j}{M}\right) \sigma(y_i(s)) dB_i(s), \quad 1 \le i \le M-1, \\ :=& f_M^1(t,y) + f_M^2(t,y) + f_M^3(t,y), \end{aligned} \tag{3.36}$$

where $\{f_M^l\}$, for $l = 1,2,3$, denote the first, second and third summation on the right hand side respectively.

Proposition 3.4.1. *For every $M \ge 1$, the sequence $\{f_M(t,y) : t \in [0,T]\}$ is tight in the space $\mathscr{C}\big([0,T],A\big)$, where $A = \mathscr{C}\big([0,1],[0,1]\big)$.*

Proof. By the hypothesis (3.5), we have

$$\sigma(f_M(t,y)) \le c\min((f_M(t,y))^\alpha, (1-f_M(t,y))^\alpha), \qquad \forall\, \alpha \in [1/2,1). \tag{3.37}$$

It can easily be seen from (3.31) that $f_M \in [0,1]$ and consequently it implies through condition (3.37) that σ is also bounded by some positive constant. Further, by using the same technique as used in the proof of Lemma 2.2 and Proposition 2.1 in [60], for every $0 \le T < \infty$ and $\mu \in \mathbb{N}$, we obtain that there exists $K := K(\mu,T)$ such that

$$\mathbb{E} \mid f_M^3(t_1,x) - f_M^3(t_2,y) \mid^{2\mu} \le K\bigg(\mid t_1 - t_2 \mid^{\mu/2} + \mid x-y \mid^{\mu/2} \bigg) \tag{3.38}$$

for every $x,y \in [0,1]$ and $t_1,t_2 \in [0,T]$, and $\mu \in \mathbb{N}$ and

$$\lim_{M\to\infty} \sup_{(t,y)\in[0,T]\times[0,1]} \mid f_M^1(t,y) - f(t,y) \mid = 0. \tag{3.39}$$

Here, f represents the fundamental solution of

$$\frac{\partial f}{\partial y} = \frac{\partial^2 f}{\partial y^2}. \tag{3.40}$$

Also, $f_M^2(t,0) = f_M^2(t,1) = 0$ and

$$f_M^2\left(t,\frac{k}{M}\right) = \int_0^t \left[q_M\left(t-s,\frac{k}{M},\frac{k+1}{M}\right)y_{k+1}(s)^p - q_M\left(t-s,\frac{k}{M},\frac{k}{M}\right)y_k(s)^p\right]ds.$$

Since (3.32)–(3.34) imply that q_M is the fundamental solution of heat kernel associated to Δ_M, we have

$$\frac{\partial}{\partial t}f_M^2\left(t,\frac{k}{M}\right) = \Delta_M f_M^2\left(t,\frac{k}{M}\right) + f_M\left(t,\frac{k+1}{M}\right)^p - f_M\left(t,\frac{k}{M}\right)^p.$$

From Theorem 4.2 in [81], we obtain

$$\max_{1\le k\le M}\left|f_M^2\left(t,\frac{k}{M}\right)\right| \le e^{pt}\int_0^t \max_{1\le k\le M}\left|f_M^1\left(s,\frac{k}{M}\right) + f_M^3\left(s,\frac{k}{M}\right)\right|ds. \tag{3.41}$$

Hence, from (3.38)-(3.39), and the polygonal form of f_M^2, we conclude that for any finite $T \ge 0$ and $\mu \in \mathbb{N}$, there exists $K = K(T,\mu)$ in such a way that

$$\mathbb{E}\,|f_M^2(t_1,x) - f_M^2(t_2,y)|^{2\mu} \le K\left(|t_1-t_2|^{\mu/2} + |x-y|^{\mu/2}\right) \tag{3.42}$$

for every $t_1,t_2 \in [0,T]$, and $0 \le x,y \le 1$, and $M \in \mathbb{N}$. Substituting estimates (3.38), (3.39) and (3.42) into (3.36) and using the multidimensional Totoki-Kolmogorov criterion [134, 135] on tightness we conclude that for every $T > 0$, $f_M(t,x) \in \mathscr{C}([0,T],A)$[1] and the sequence $\{f_M(t,x), M \in \mathbb{N}\}$ is tight. □

3.5 The Weak Solution

In Section 3.4, it is shown that the sequence $f_M = \{f_M(t,y), M \ge 1\}$ is tight in $\mathscr{C}([0,T],A)$ and hence by Prokhorov's Theorem [20, page 59 (Chapter 1)], f_M is relatively compact in $\mathscr{C}([0,T],A)$, As a consequence there exist a convergent subsequence $f_M^k = \{f_M^k(t,y), M \ge 1, k \ge 1\}$ of f_M in $\mathscr{C}([0,T],A)$, which converges weakly to a stochastic process $\widetilde{f}$ in $\mathscr{C}([0,T],A)$. Applying the well known Skorohod's representation Theorem, we get another probability space $(\bar{\Omega}, \bar{\mathscr{F}}, (\bar{\mathscr{F}}_t), \bar{P})$ and a sequence of processes $\widetilde{f}_M(t,y)$ and $f(t,y)$ adapted to the filtration $\{\bar{\mathscr{F}}_t\}_{\{t\in[0,T]\}}$, in such a way that $f_M \overset{\mathrm{D}}{\equiv} \widetilde{f}_M(t,y)$, $\widetilde{f} \overset{\mathrm{D}}{\equiv} f$ and $\{\widetilde{f}_M(t,y)\}$ converges to f almost surely on compact subset of $\mathscr{C}([0,T],A)$ for any $T > 0$ as $M \to \infty$. Also f satisfies the given boundary conditions in (3.2). Next, by solving an equivalent martingale problem to (3.1)–(3.3), we show that $f(t,y)$ is the required weak solution to (3.1)–(3.3).

[1] $\mathscr{C}([0,T]\times[0,1],[0,1])$ and $\mathscr{C}([0,T],A)$ both have equal topologies (see [60, page 145]).

Proposition 3.5.1. *For every $\phi \in \mathscr{C}^2([0,1])$ such that $\phi(1) = \phi(0) = 0$, we have*

$$\mathscr{M}_\phi(t) = \int_0^1 f(t,y)\phi(y)dy - \int_0^1 f(0,y)\phi(y)dy - \int_0^t \int_0^1 f(s,y)\phi''(y)dyds + \int_0^t \int_0^1 f^p(s,y)\phi'(y)dyds \tag{3.43}$$

is a martingale with the quadratic variation

$$\langle \mathscr{M}_\phi \rangle_t = \int_0^t \int_0^1 \sigma^2(f(t,y))\phi^2(y)dyds. \tag{3.44}$$

Proof. Using the Skorohod representation theorem after multiplying both the side by $\frac{1}{M}\phi\left(\frac{k}{M}\right)$ in (3.8) and summing over all $k = 1, 2, \cdots, M-1$, we get, for fixed $M \geq 1$,

$$\begin{aligned}
\mathscr{M}_\phi^M(t) = &\sum_{k=1}^{M-1} f_M\left(t, \frac{k}{M}\right)\phi\left(\frac{k}{M}\right)\frac{1}{M} - \sum_{k=1}^{M} f_M\left(0, \frac{k}{M}\right)\phi\left(\frac{k}{M}\right)\frac{1}{M} \\
&- \int_0^t \sum_{k=1}^{M-1} \Delta_M f_M\left(t, \frac{k}{M}\right)\phi\left(\frac{k}{M}\right)\frac{1}{M} - \int_0^t \sum_{k=1}^{M-1} \nabla_M f_M^p\left(t, \frac{k}{M}\right)\phi\left(\frac{k}{M}\right)\frac{1}{M} \\
&\xrightarrow{D} \sum_{k=1}^{M-1} \widetilde{f}_M\left(t, \frac{k}{M}\right)\phi\left(\frac{k}{M}\right)\frac{1}{M} - \sum_{k=1}^{M-1} \widetilde{f}_M\left(0, \frac{k}{M}\right)\phi\left(\frac{k}{M}\right)\frac{1}{M} \\
&- \int_0^t \sum_{k=1}^{M-1} \widetilde{f}_M\left(s, \frac{k}{M}\right)\Delta_M\phi\left(\frac{k}{M}\right)\frac{1}{M} - \int_0^t \sum_{k=1}^{M-1} \widetilde{f}_M^p\left(s, \frac{k}{M}\right)\nabla_M\phi\left(\frac{k}{M}\right)\frac{1}{M} \\
= &\sum_{k=1}^{M-1} \phi\left(\frac{k}{M}\right)\frac{1}{M}\int_0^t \sqrt{M}\sigma\left(\widetilde{f}_M\left(s, \frac{k}{M}\right)\right)dB_k(s).
\end{aligned} \tag{3.45}$$

Since the right-hand side on (3.45), each integral in the summation, is an Itô integral and hence these are martingale also. Therefore, $\mathscr{M}_\phi^M(t)$ is also a martingale. Moreover, ϕ^2 is also an integrable function, therefore

$$(\mathscr{M}_\phi^M(t))^2 = \left(\sum_{k=1}^{M-1} \phi\left(\frac{k}{M}\right)\frac{1}{M}\int_0^t \sqrt{M}\sigma\left(\widetilde{f}_M\left(s, \frac{k}{M}\right)\right)dB_k(s)\right)^2$$

$$\begin{aligned}
\mathbb{E}(\mathscr{M}_\phi^M(t))^2 &= \left(\sum_{k=1}^{M-1} \phi^2\left(\frac{k}{M}\right)\frac{1}{M}\int_0^t \sigma^2\left(\widetilde{f}_M\left(s, \frac{k}{M}\right)\right)\right)ds \\
&\leq T\sum_{k=1}^{M-1} \frac{1}{M-1}\phi^2\left(\frac{k}{M}\right) \\
&< c(\phi, T),
\end{aligned} \tag{3.46}$$

where $c(\phi,t)$ is a finite constant free from M and depends only on ϕ and T. Therefore, $\mathscr{M}_\phi^M(t) \to \mathscr{M}_\phi(t)$ as $M \to \infty$, where $\mathscr{M}_\phi(t)$ is given by (3.43). Now, since the quadratic variation of $\mathscr{M}_\phi^M(t)$ is given by

$$\begin{aligned}
\langle \mathscr{M}_\phi^M(t) \rangle &= \left\langle \sum_k^{M-1} \phi\left(\frac{k}{M}\right) \frac{1}{M} \int_0^t \sqrt{M} \sigma\left(\widetilde{f}_M\left(s, \frac{k}{M}\right)\right) dB_k(s) \right\rangle_t \\
&= \int_0^t \left(\sum_k^{M-1} \frac{1}{M} \sigma^2 \left(\widetilde{f}_M\left(s, \frac{k}{M}\right)\right) \phi^2\left(\frac{k}{M}\right)\right) ds, \qquad (3.47)
\end{aligned}$$

therefore, we have

$$\lim_{M\to\infty} \langle \mathscr{M}_\phi^M(t) \rangle = \int_0^t \int_0^1 \sigma^2(f(s,y))\phi^2(y) dy ds = \langle \mathscr{M}_\phi(t) \rangle. \qquad (3.48)$$

□

Now, the main result of the present work, is as follows:

Theorem 3.5.2. *Let* $f_0 : [0,1] \to [0,1]$ *be a continuous function and* σ *satisfies the conditions (3.5)–(3.6). Then f(t,x) is a weak solution of (3.1)–(3.3).*

Proof. From Chapter 2 in Walsh [141], for the quadratic variation $\langle \mathscr{M}_\phi(t) \rangle$, we can find a martingale measure $\mathscr{M}(ds,dx)$ with quadratic variation

$$\nu(dx,dt) = \sigma(f(t,x))dtds.$$

Now, as in Kono and Siga [99], we can establish a space-time white noise $\widetilde{W}_t$ independent of $\mathscr{M}(dx,ds)$ such that

$$\begin{aligned}
W_t(\phi) = &\int_0^1 \int_0^t \frac{1}{\sigma(f(s,x))} 1_{\{f(s,x)\neq\{0,1\}\}} \phi(x) \mathscr{M}(ds,dx) \\
&+ \int_0^1 \int_0^t 1_{\{f(s,x)=\{0,1\}\}} \phi(x) \widetilde{W}(ds,dx) \qquad (3.49)
\end{aligned}$$

where W_t corresponds to the space-time white noise $W(ds,dx)$ such that

$$\mathscr{M}_t(\phi) = \int_0^1 \int_0^t \sigma(f(s,x))\phi(x) W(ds,dx) \qquad (3.50)$$

Therefore, from Proposition 3.5.1 and Definition 3.2.2, it is proved that f is the weak solution to (3.1)–(3.3).This completes the proof of Theorem 3.5.2. □

Chapter 4

On a generalized stochastic Burgers equation perturbed by Volterra noise

This chapter deals with the existence of unique local mild solution for the one-dimensional generalized stochastic Burgers equation (GSBE) containing a non-linearity of polynomial type and perturbed by α-regular cylindrical Volterra process and having Dirichlet boundary conditions. The Banach fixed point theorem is used to obtain the local solvability results. The L^∞-estimate on both time and space for the stochastic convolution involving the α-regular cylindrical Volterra process is obtained with the help of Garsia-Rodemich-Rumsey inequality. Further, the existence and uniqueness of global mild solution of GSBE up to third order nonlinearity is shown. In addition, we have also investigated the existence of the invariant measure for the same equation with quadratic nonlinearity.

Let us now provide the brief plan of this chapter: In the coming section, we introduce the generalized stochastic Burgers equation perturbed by Volterra noise. Section 4.2 provides some preliminaries, mainly about the Volterra noise, its integral representation and stochastic convolution of linear counterpart of (4.1). Brief details about the fractional Brownian motion (a most popular example of Volterra noise) and γ-radonifying operators are also mentioned in this section. Section 4.3 contains the proof of the existence and uniqueness of mild solution to the equation (4.1) subject to Volterra noise (see Theorem 4.3.9). In order to get the unique global solvability, we first consider a truncated system and establish the existence of unique global mild solution to the truncated system using contraction mapping principle (Proposition 4.3.7). In the final section, the existence of the invariant measure is shown for the equation (4.1) perturbed by Volterra noise of Gaussian type and the proof in this section is motivated by [51].

4.1 Introduction

We consider the following generalized stochastic Burgers equation (GSBE):

$$df(t,x) = \left(\frac{\partial^2 f(t,x)}{\partial x^2} + \frac{\partial g(f(t,x))}{\partial x}\right) dt + \Phi dB(t) \tag{4.1}$$

with Dirichlet boundary conditions

$$f(0,t) = f(1,t) = 0, \tag{4.2}$$

and initial datum

$$f(x,0) = f_0(x), \tag{4.3}$$

where $x \in [0,1],\ t \in [0,T]$ for any $T > 0$ and the function $g : \mathbb{R} \to \mathbb{R}$ is such that $g(r) = \frac{1}{p}r^p$ for each $p \geq 2$ is a fixed integer. Here $B(t)$ is an infinite dimensional α-regular Volterra process and Φ is a linear operator, which is defined in Section 4.2 in detail.

The Volterra processes cover many important types of noises, which may be Gaussian as well as non-Gaussian noise. Fractional Brownian motion, Liouville fractional Brownian motion, multifractional Brownian motion and fractional Orestein Uhlenbeck process etc, are examples of Gaussian type Volterra processes, while Rosenblat process is an example of non-Gaussian Volterra process (cf. [2, 46] and references therein). In 2001, Alós et al. [2] have developed the one dimensional stochastic calculus for Gaussian Volterra process, where they focused mainly on singular and regular Volterra process. Recently, in [46], authors have studied the stochastic evolution equation driven by Volterra noise. Further, the L^p-theory of stochastic convolution integral is also developed in [47]. In the following section, we will elaborate about this process in more detail.

4.2 Preliminaries

In this section, we briefly give some details about the cylindrical Volterra processes and stochastic integrals with respect to them. We refer the interested readers to the works of E. Alòs et al. [2] and P. Čoupek et al. [46, 47] for more details. In the sequel, $\mathscr{L}(H,K)$ denotes the space of all bounded linear operators from H to K, where H and K are Banach spaces.

4.2.1 Volterra Processes

Let us first give the definition and properties of Volterra processes.

Definition 4.2.1. For any $T > 0$, let $K : [0,T] \times [0,T] \to [0,\infty)$ be a measurable function. Then, K is known as α-regular Volterra kernel if it satisfies

- **Volterra**, i.e.,

  1. $K(0,0) = 0$ and $K(t,r) = 0$ *on* $\{0 \le t < r < T\}$,
  2. As $t \to r^+, K(t,r) = 0$ for all $r \ge 0$;

- and **α-regular**, i.e.,

  1. for all $r \in [0,T], K(\cdot,r) \in C^1(r,T)$, where C^1 stands for space of continuously differentiable functions,
  2. there exists an $\alpha \in (0,1/2)$ such that
  $$\left|\frac{\partial K}{\partial t}(t,r)\right| \lesssim (t-r)^{\alpha-1}\left(\frac{t}{r}\right)^{\alpha},$$

  on $\{0 < r < t \le T\}$. Here the notation $a \lesssim b$ means that there exists a constant $c > 0$ such that $a \le cb$.

Definition 4.2.2. Let $(\Omega,\mathscr{F},\mathbb{P})$ be a probability space and $\xi = (\xi(t), t \ge 0)$ be a real, centered stochastic process defined on it. We say that ξ is an α-regular Volterra process if $\xi(0) = 0$, $\mathbb{P}$-a.s., with the covariance function defined as

$$\mathbb{E}(\xi(s)\xi(t)) := R(s,t) = \int_0^{s\wedge t} K(t,r)K(s,r)dr, \;\; s,t \ge 0, \tag{4.4}$$

where K is some α-regular Volterra kernel given by Definition 4.2.1.

Remark 4.2.1. [47] An α-regular Volterra kernel satisfies the following

$$\int_0^t (K(t,r)-K(s,r))^2\,dr \lesssim (t-s)^{1+2\alpha}, \;\; 0\le s<t, \tag{4.5}$$

by using

$$(\mu\nu)^{\alpha}\int_0^{\mu\wedge\nu} r^{-2\alpha}(\mu-r)^{\alpha-1}(\nu-r)^{\alpha-1}dr = \frac{\Gamma(\alpha)\Gamma(1-2\alpha)}{\Gamma(1-\alpha)}|\mu-\nu|^{2\alpha-1}$$

which holds for $\mu,\nu\ge 0,\mu\neq\nu$. For any $T>0$, in particular, $K(t,\cdot)\in L^2([0,T])$, which validate the existence of the integral on the right of (4.4) for every $s,t\ge 0$. Further, it can be derived that ξ possesses a version with δ-Hölder continuous sample paths for each $\delta\in(0,\alpha)$, using (4.4),(4.5) and Kolomogorov continuity criterion.

4.2.2 Wiener Integral with respect to Volterra process

For $T>0$, let us introduce a linear space $\mathscr{E}$ of $\mathbb{R}$-valued deterministic step functions defined by

$$\mathscr{E} := \Bigg\{ \psi:[0,T]\to\mathbb{R}, \psi=\sum_{i=1}^{n-1}\psi_i\chi_{[t_i,t_{i+1})}(t)+\psi_n\chi_{[t_n,t_{n+1}]}(t), \\ \psi_i\in\mathbb{R}, i\in\{1,\cdots,n\}, 0=t_1<t_2<\cdots<t_{n+1}=T, n\in\mathbb{N}\Bigg\}. \tag{4.6}$$

Next, we define an operator $\mathscr{K}_T^*:\mathscr{E}\to L^2([0,T])$ by

$$(\mathscr{K}_T^*\psi)(r) := \int_r^T \psi(t)\frac{\partial K}{\partial t}(t,r)dt, \tag{4.7}$$

for every $\psi\in\mathscr{E}$. Now, we consider an α- regular Volterra process $\xi=(\xi(t),t\ge 0)$, with the kernel K and let $J_t:\mathscr{E}\to L^2(\Omega)$ be the linear operator defined as

$$\psi := \sum_{i=1}^{n-1}\psi_i\chi_{[t_i,t_{i+1})}(t)+\psi_n\chi_{[t_n,t_{n+1}]}(t) \overset{J_t}{\longmapsto} \sum_{i=1}^{n-1}\psi_i(\xi(t_{i+1})-\xi(t_i)) =: J_t(\psi).$$

Using (4.4) and (4.7), we get an Itô-type isometry for Volterra processes as

$$\|J_t(\psi)\|_{L^2(\Omega)} = \|\mathscr{K}_T^*\psi\|_{L^2([0,T])}. \tag{4.8}$$

For $u,v\in\mathscr{E}$, let us define

$$\langle u,v\rangle_{\mathscr{D}} := \langle \mathscr{K}_T^*u,\mathscr{K}_T^*v\rangle_{L^2([0,T])}. \tag{4.9}$$

Then, if $\mathscr{K}_T^*$ is one to one, the function $\langle\cdot,\cdot\rangle_{\mathscr{D}}$ induces an inner product on $\mathscr{E}$. On the other hand, if $\mathscr{K}_T^*$ is not injective, we introduce the quotient space $\widetilde{\mathscr{E}} := \mathscr{E}/\ker\mathscr{K}_T^*$ and we re-define $\mathscr{K}_T^*$ as $\mathscr{K}_T^* : \widetilde{\mathscr{E}} \to L^2([0,T])$. Next, let $\mathscr{D}$ be the completion of $\mathscr{E}$ with respect to the $\langle\cdot,\cdot\rangle_{\mathscr{D}}$. Then, this completion allows a construction of a Hilbert space $(\mathscr{D},\langle\cdot,\cdot\rangle_{\mathscr{D}})$ and also extends $\mathscr{K}_T^*$ to $\mathscr{D}$. Consequently, (4.8) implies that J_t can be extended to be an operator from $\mathscr{D}$ to $L^2(\Omega)$. The admissible space of integrands with respect to ξ is $\mathscr{D}$ and $J_t : \mathscr{D} \to L^2(\Omega)$ is the Wiener-type integral. Further, the random variable $(J_T(\psi))(\omega) = \int_0^T \psi(t)d\xi(t,\omega)$, for all $\omega \in \Omega$, denotes the stochastic integral with respect to the regular Volterra process ξ with the domain of the integrands as $\mathscr{D}$. By [46, Proposition 2.9] we have

$$\|J_t(\psi)\|^2_{L^2(\Omega)} \lesssim \int_0^T \int_0^T \psi(u)\psi(v)|u-v|^{2\alpha-1}dudv,$$

from which we get the following continuous embedding:

$$L^{\frac{2}{1+2\alpha}}(0,T) \hookrightarrow \mathscr{D}. \tag{4.10}$$

4.2.3 Examples

There are several examples of α-regular Volterra process ξ (see [46]). The most popular example is the fractional Brownian motion with Hurst parameter $H \in (1/2,1)$, which is α-regular Volterra process for $\alpha = H - 1/2$.

4.2.3.1 Fractional Brownian motion

The fractional Brownian motion was first used in 1940 by Kolmogorov [98] by another name "Wiener Helix". Later, it was Mandelbrot et al. [112] who coined first time the name fractional Brownian motion (fBm) in 1968. In this work [112], they have constructed stochastic integral with respect to fBm in terms of standard Brownian motion.

Definition 4.2.3. Let H be a constant belonging to $(0,1)$. A *fractional Brownian motion (fBm)* $(W^H(t))_{t\geq 0}$ with index H is a continuous and Gaussian process with mean zero and covariance function

$$R(t,s) = \mathbb{E}\left[W^H(t)W^H(s)\right] = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right), \tag{4.11}$$

for all $s,t \geq 0$.

From the definition it can be noticed that standard Brownian motion is a fractional Brownian motion with the index $H = \frac{1}{2}$. The index H is known as *Hurst index* which was also introduced by Mandelbrot et al. [112] after the name of the hydrologist H. E. Hurst for his pioneering works [79, 80]. From Alòs et al. [2], we have the following Wiener integral representation:

$$W^H(t) = \int_0^t K^H(t,s)dW(s), \tag{4.12}$$

where $W = \{W(t) : t \in [0,T]\}$ is a Wiener process and $K^H(t,s)$ is a kernel. If $H \in (0,1/2)$, the kernel $K^H(\cdot,\cdot)$ is given by

$$K^H(t,s) = c_H(t-s)^{H-\frac{1}{2}} + c_H\left(\frac{1}{2}-H\right)\int_s^t (u-s)^{H-\frac{3}{2}}\left(1-\left(\frac{s}{u}\right)^{\frac{1}{2}-H}\right)du, \tag{4.13}$$

where $c(H)$ is a constant given by

$$c_H = \left(\frac{2H\Gamma(\frac{3}{2}-H)}{\Gamma(H+\dfrac{1}{2})\Gamma(2-2H)}\right)^{\frac{1}{2}}. \tag{4.14}$$

If $H \in (1/2,1)$, the kernel $K^H(\cdot,\cdot)$ has a simpler expression

$$K^H(t,s) = c_H\left(H-\frac{1}{2}\right)s^{\frac{1}{2}-H}\int_s^t (u-s)^{H-\frac{3}{2}}u^{H-\frac{1}{2}}du. \tag{4.15}$$

From (4.13) and (4.15), we obtain

$$\frac{\partial K^H}{\partial t}(t,s) = c_H\left(H-\frac{1}{2}\right)(t-s)^{H-\frac{3}{2}}\left(\frac{s}{t}\right)^{\frac{1}{2}-H}. \tag{4.16}$$

Let $\mathscr{E}$ be the linear space of step functions defined on $[0,T]$ given by (4.6) and $\mathscr{H}$ be the closure of $\mathscr{E}$ with respect to scalar product

$$\langle \chi_{[0,t]}, \chi_{[0,s]}\rangle_{\mathscr{H}} = R(t,s).$$

The Wiener integral with respect to the fBm for any $\psi \in \mathscr{E}$ is defined as

$$\int_0^T \psi(s)dW^H(s) := \sum_{i=1}^{n-1} \alpha_i\left(W^H(t_{i+1}) - W^H(t_i)\right).$$

It is clear that there is an isometry between $\mathscr{E}$ and the linear space $span\{W^H(t), t \in [0,T]\} \subseteq L^2(\Omega)$, through the mapping

$$\psi = \sum_{i=1}^{n-1} \psi_i \chi_{(t_i,t_{i+1}]}(t) \overset{J_t}{\mapsto} \int_0^T \psi(s)dW^H(s).$$

This isometry is extendable to an isometry between $\mathscr{H}$ and the first Wiener chaos of the fractional Brownian motion $\overline{span}^{L^2(\Omega)}\{W^H(t), t \in [0,T]\}$ which is the closure of the $span\{W^H(t), t \in [0,T]\}$ in $L^2(\Omega)$. The image of an element $\psi \in \mathscr{H}$ by this isometry is called the *Wiener integral ψ with respect to W^H*. For simplicity, we denote the kernel K^H by K only. Let $K_\tau^* : \mathscr{E} \to L^2([0,T])$ be a linear map defined by

$$(K_\tau^* \psi)(s) := \psi(s)K(\tau,s) + \int_s^\tau (\psi(t) - \psi(s)) \frac{\partial K}{\partial t}(t,s)dt, \tag{4.17}$$

for $0 < s < \tau$ and every $H \in (0,1)$. Further, when $H > 1/2$, the operator K_τ^* has a simpler expression

$$(K_\tau^* \psi)(s) = \int_s^\tau \psi(t) \frac{\partial K}{\partial t}(t,s)dt. \tag{4.18}$$

For each $t \in [0,T]$ we can introduce K_t^* in the same fashion. Further, for $\psi \in \mathscr{E}$ and $h \in L^2([0,T])$ the following duality holds

$$\int_s^T (K_T^* \psi)(t)h(t)dt = \int_0^T \psi(t)(Kh(t))dt. \tag{4.19}$$

Identity (4.19) definitely holds when ψ belongs to $\mathscr{H}$. As a result, we dot the following connection between the Wiener integral with respect to fBm and the Itô integral with respect to the Wiener process

$$\int_0^T \psi(t)dW^H(s) = \int_0^T (K_\tau^* \psi)(s)dW(s), \tag{4.20}$$

which is true for every $\psi \in \mathscr{H}$ iff $K_\tau^*(\psi) \in L^2([0,T])$. Now, for each $s,t \in [0,T]$, one can verify the relation

$$K_T^*[\psi \chi_{[0,t]}](s) = K_T^*[\psi](s)\chi_{[0,t]}(s).$$

With the above understanding, if one defines the definite stochastic integral $\int_0^t \psi(t)dW^H(s)$, as it should be, by $\int_0^t \psi(t)\chi_{[0,t]}(t)dW^H(s)$, we obtain

$$\int_0^t \psi(t)dW^H(s) = \int_0^t (K_\tau^* \psi)(s)dW(s), \tag{4.21}$$

for any $t \in [0,T]$ and $\psi \chi_{[0,t]} \in \mathscr{H}$ iff $K_\tau^*[\psi] \in L^2([0,T])$.

Remark 4.2.2. The fBm is a Gaussian type α-regular Volterra process. Lioville fractional Brownian motion, multifractional Brownian motion, fractional Orestein-Uhenbeck processes are other examples of α-regular Volterra process which are Gaussian processes as well. On the other hand, the Rosenblatt process is an example of non Gaussian Volterra process (cf. [2, 46, 47], etc).

4.2.4 Cylindrical Volterra processes

Generally, by following the standard approach, in [50], for the space time white noise, we can define cylindrical Volterra processes. In order to do this, let us take U as a real separable Hilbert space.

Definition 4.2.4. [47] Let $\{e_n\}$ be an orthonormal basis of U and $\{\xi_n\}$ be mutually uncorrelated (not necessarily independent) one dimensional α-regular Volterra processes with kernel K. An α-regular cylindrical Volterra process $B = (B(t), t \in [0,T])$ is defined by the sum

$$B(t) := B_x(t) = \sum_{n\in\mathbb{N}} e_n(x)\xi_n(t), \tag{4.22}$$

for $x \in U$.

Now, we provide Khinchin-Kahane inequality [57], which is used in later sections.

Lemma 4.2.1. ***Khinchin-Kahane inequality:*** *For any $p > 1$ and any random function S, we define*

$$\|S\|_p = (\mathbb{E}|S|^p)^{\frac{1}{p}}, \qquad p > 1.$$

Then the inequality,

$$\|S\|_p \leq C(p,q)\|S\|_q, \qquad 1 < q < p, \tag{4.23}$$

holds for the sum

$$S = \sum_{n=1}^{\infty} a_n \eta_n,$$

where $\{\eta_n\}$ is sequence of Rademacher random variables (i.e. η_n is an independent random variable which follows the Bernoulli distribution and takes values ± 1 with probability $\frac{1}{2}$ each).

4.2.5 γ-Radonifying operators

In this subsection, we recall the definition of the γ-Radonifying operators and an example of the coefficient of α-regular cylindrical Volterra process appearing in (4.1). Let E be a separable Banach space and $\{\gamma_n\}_{n\geq 1}$ be an independent standard Gaussian sequence on some probability space $(\widetilde{\Omega}, \widetilde{\mathscr{F}}, \widetilde{\mathbb{P}})$.

Definition 4.2.5. [118] A linear operator $R : U \to E$ is said to be *γ-radonifying operator* if for some orthonormal basis $\{h_n\}_{n\geq 1}$ of U, the Gaussian sum $\sum_{n\geq 1} \gamma_n R h_n$ converges in $L^2(\widetilde{\Omega}, E)$. Denote $\gamma(U,E)$, the space of all γ-radonifying operators from U to E. The space $\gamma(U,E)$ forms a Banach space with respect to the norm

$$\|R\|^2_{\gamma(U,E)} := \mathbb{E}\left\| \sum_{n\geq 1} \gamma_n R h_n \right\|^2_E. \tag{4.24}$$

Let us now give an example of the γ-radonifying operator considered in this paper.

Example 4.2.2. Let A be the self-adjoint and unbounded operator on $L^2([0,1])$ defined as

$$Au := \frac{\partial^2 u}{\partial x^2}$$

with domain $D(A) = \{u \in H^2([0,1]) : u(0) = u(1) = 0\}$. The eigenvalues and the corresponding eigenvectors of A are given by

$$\widetilde{\lambda}_n = -n^2\pi^2, \;\; n = 1,2\ldots, \;\; \text{and} \;\; e_n(x) = \sqrt{\frac{2}{\pi}} \sin\sqrt{\lambda_n} x, \;\; n = 1,2\ldots,$$

where $\lambda_n = -\widetilde{\lambda}_n$. Now, let us take $\Phi = A^{-\nu}$, for some $\nu > 1/4$. Then we show that Φ is a γ-radonifying operator from $U = L^2([0,1])$ into $E = L^p(0,1)$. Indeed, if $(\gamma_n)_{n\geq 1}$ is an independent sequence of Gaussian random variables, then by applying Hölder's inequality, stochastic

Fubini's theorem and then Kahane-Khinchin inequality (4.23), we have

$$\begin{aligned}
\|\Phi\|^2_{\gamma(U,E)} = \|A^{-\nu}\|^2_{\gamma(U,E)} &= \sup_{m\geq 1} \mathbb{E}\left\|\sum_{n=1}^{m} \gamma_n A^{-\nu} e_n\right\|^2_{L^p} \\
&= \sup_{m\geq 1} \mathbb{E}\left(\int_0^1 \left|\sum_{n=1}^{m} \gamma_n A^{-\nu} e_n(x)\right|^p dx\right)^{2/p} \\
&\leq \sup_{m\geq 1} \left(\mathbb{E}\left(\int_0^1 \left|\sum_{n=1}^{m} \gamma_n A^{-\nu} e_n(x)\right|^p dx\right)\right)^{2/p} \\
&= \sup_{m\geq 1} \left(\int_0^1 \mathbb{E}\left|\sum_{n=1}^{m} \gamma_n A^{-\nu} e_n(x)\right|^p dx\right)^{2/p} \\
&\leq C \sup_{m\geq 1} \left(\int_0^1 \left(\mathbb{E}\left|\sum_{n=1}^{m} \gamma_n A^{-\nu} e_n(x)\right|^2\right)^{p/2} dx\right)^{2/p} \\
&= C \sup_{m\geq 1} \left(\int_0^1 \left(\sum_{n=1}^{m} \mathbb{E}[(\gamma_n)^2](\lambda_n^{-\nu} e_n(x))^2\right)^{p/2} dx\right)^{2/p} \\
&\leq C \sup_{m\geq 1} \left(\int_0^1 \left(\sum_{n=1}^{m} (\pi^2 n^2)^{-2\nu}\right)^{p/2} dx\right)^{2/p} \\
&\leq C \sum_{n=1}^{\infty} (\pi n)^{-4\nu},
\end{aligned}$$

which converges only if $\nu > 1/4$.

4.2.6 Stochastic integration with respect to Volterra processes

In this subsection, we define the stochastic integration with respect to Volterra processes in L^p-spaces, following the methods developed in [47]. For that, we first define the stochastic integral $I_T(G)$ with respect to the cylindrical Volterra process B which is in Definition 4.2.4.

Definition 4.2.6. Let $T > 0$ and (D, μ) be a seperable σ-finite measurable space. We define an operator $Q \in \mathscr{L}(U, L^p(D; \mathscr{D}))$ as elementary operator if for every $u \in U, t \in [0, T]$, and μ-almost every $x \in D$ the equation

$$[Qu](x)(t) = \sum_{l=1}^{n} h_l(t) \langle u, e_l \rangle_U f_l(x) \tag{4.25}$$

holds, where $n \in \mathbb{N}$, $\{e_l\}$ is an orthonormal basis of U, $\{h_l\} \in C^1(0, T)$(a space of continuously differentiable functions), and $\{f_l\} \in L^p(D)$.

The integral of Q with respect to the Volterra process is defined by a linear operator $I_T(Q)$ as

$$I_T(Q) := \sum_{l=1}^{n} \left(\int_0^T h_l(t) d\xi_l(t) \right) f_l, \tag{4.26}$$

By the following Proposition in [47, Proposition 3.1], it can be proved that the natural space of integrands is the space $\gamma(U, L^p(D;\mathscr{D}))$.

Proposition 4.2.3. *Let $1 \leq p < \infty$ and $G : U \to L^p(D;\mathscr{D})$ be a bounded linear operator. Then, G is stochastically integrable if and only if $G \in \gamma(U, L^p(D;\mathscr{D}))$, and in this case for all $1 \leq q < \infty$*

$$\|I_T(G)\|_{L^q(\Omega;L^p)} \eqsim \|G\|_{\gamma(U,L^p(D,\mathscr{D}))} \tag{4.27}$$

holds, where the symbol $a \eqsim b$ means that there exists positive constants k_1, k_2 such that $k_1 b \leq a \leq k_2 b$.

Remark 4.2.3. [47] The stochastic integral of $Q \in \gamma(U, L^p(D;\mathscr{D}))$ with respect to cylindrical Volterra process B can be defined as

$$\int_0^T Q(t) dB(t) := I_T(Q) = \sum_n \int_0^T Q(t) e_n d\xi_n(t).$$

Corollary 4.2.4. *Let $\frac{2}{1+2\alpha} \leq p < \infty$. The space $L^{\frac{2}{1+2\alpha}}([0,T];\gamma(U,L^p))$ is continuously embedded in $\gamma(U, L^p(D;\mathscr{D}))$ and*

$$\|I_T(G)\|_{L^q(\Omega;L^p)} \eqsim \|G\|_{L^{\frac{2}{1+2\alpha}}([0,T];\gamma(U,L^p(D;\mathscr{D})))}$$

holds for every $1 \leq q < \infty$ and $G \in L^{\frac{2}{1+2\alpha}}([0,T];\gamma(U,L^p(D;\mathscr{D})))$.

4.3 Existence and Uniqueness

Let A represents the unbounded self adjoint operator on $L^2([0,1])$ by

$$Af = \frac{\partial^2}{\partial x^2} f(t,x), \tag{4.28}$$

and domain of A is given as

$$D(A) := \left\{ f \in H^2([0,1]) : f(0) = f(1) = 0 \right\}. \tag{4.29}$$

Let $\{e^{tA}\}_{t\geq 0}$ denotes the semigroup generated by A in $L^2([0,1])$. Then, we can extend e^{tA} to an analytic, strongly continuous semigroup in $L^p([0,1])$ for any $p\geq 1$, which we still denote by $\{e^{tA}\}_{t\geq 0}$, (see [108, 123]). Let $\{e_k\}$ be a complete orthonormal system on $L^2([0,1])$ which diagonalize A and $\{\widetilde{\lambda}_k\}$ be the corresponding eigenvalues. Then, we know that

$$\lambda_k = -\widetilde{\lambda}_k = \pi^2 k^2,\ k=1,2,\ldots. \tag{4.30}$$

and

$$e_k(x) = \sqrt{\frac{2}{\pi}}\sin\sqrt{\lambda_k}x,\ k=1,2,\ldots. \tag{4.31}$$

Now equation (4.1)-(4.3) can be re-written in the form of abstract stochastic differential equation

$$\begin{cases} df(t,x) = \left(Af(t,x) + \dfrac{\partial g(f(t,x))}{\partial x}\right)dt + \Phi dB_x(t), \\ f(0,x) = f_0(x), \end{cases} \tag{4.32}$$

where Φ is a γ-radonifying operator and Φ has eigenvalues μ_k corresponding to eigenvectors $\{e_k\}$, such that

$$\Phi e_k = \mu_k e_k, \quad \text{for all } k\in\mathbb{N},\ \ \mu_k > 0, \tag{4.33}$$

where $\{e_k\}$ is a complete orthonormal system on $L^2([0,1])$ given by (4.31).

Let $(\Omega,\mathscr{F},\mathbb{P})$ be a given complete probability space equipped with an increasing family of sub-sigma fields $\{\mathscr{F}_t\}_{0\leq t\leq T}$ of $\mathscr{F}$ satisfying usual conditions. Now, we formulate the mild solution to equations (4.1)-(4.3) by means of the following definitions.

Definition 4.3.1. An $L^p([0,1])$-valued and $\mathscr{F}_t$-adapted stochastic process $f:[0,\infty)\times\mathbb{R}\times\Omega\to\mathbb{R}$ is a mild solution to (4.1)-(4.3) if for any $T>0$, $f(t,x):=f(t,x,\cdot)$ satisfies the following integral equation

$$f(t,x) = e^{tA}f_0(x) + \int_0^t e^{(t-s)A}\frac{\partial g(f(s,x))}{\partial x}ds + \int_0^t e^{(t-s)A}\Phi dB_x(s) \tag{4.34}$$

$\mathbb{P}$-a.s., for each $t\in[0,T]$.

Definition 4.3.2. We define an $\mathscr{F}_t$-adapted stochastic process f as a local mild solution to (4.1)-(4.3) if there exists an $\mathscr{F}_t$-adapted stopping time $\tau\in[0,T]$ such that $\{f(t,x)\}_{t\leq\tau}$ is a mild solution of (4.1)-(4.3) in the sense of Definition 4.3.1.

In order to establish the existence and uniqueness of mild solution to the equation (4.32), following assumptions are essentially required.

Assumption 1. For any $2 \leq p < \infty$,

(A1) $\Phi \in \gamma\big(L^2([0,1]), L^p([0,1])\big)$ for every $t \geq 0$,

(A2) $f_0 \in L^\rho(\Omega, L^p([0,1]))$ for some $\rho \geq p$.

Before turning to establish the solvability results of (4.1)-(4.3), we consider the linear stochastic differential equation corresponding to (4.1)-(4.3) as

$$\begin{cases} df(t,x) = Af(t,x)dt + \Phi dB_x(t), \\ f(0,x) = f_0(x), \end{cases} \tag{4.35}$$

Remembering Definition 4.3.1, the mild solution to (4.35), if it exists, can be written as

$$f(t,x) = e^{tA} f_0(x) + \int_0^t e^{(t-s)A} \Phi dB_x(s). \tag{4.36}$$

The following proposition gives the existence of a mild solution to (4.35) in the form of (4.36).

Proposition 4.3.1. *Let (A1),(A2) holds and $t \in [0, T_0]$ where $T_0 > 0$. Then, the $\mathscr{F}_t$-adapted solution $f(t,x)$, given by (4.36), is well defined in $L^p([0,1])$ and mean square right continuous.*

For the proof we refer to [47, Proposition 4.1].

Proposition 4.3.2. *Let us define*

$$Z(t,x) := \int_0^t e^{(t-s)A} \Phi dB_x(s),$$

then

$$\mathbb{E}\left[\|Z(t)\|_{L^p}^2\right] < +\infty, \tag{4.37}$$

for all $t \in [0,T]$.

Proof. By following the proof [47, Proposition 4.1], we have

$$\mathbb{E}\left[\|Z(t)\|_{L^p}^2\right] \lesssim \left(\int_0^t \|e^{(t-s)A}\Phi\|_{\gamma(L^2,L^p)}^{\frac{2}{1+2\alpha}} ds\right)^{1+2\alpha} = \left(\int_0^t \|\Phi\|_{\gamma(L^2,L^p)}^{\frac{2}{1+2\alpha}} ds\right)^{1+2\alpha} < +\infty,$$

which completes the proof. ∎

Remark 4.3.1. In the case of fractional Brownian motion with $H > 1/2$ the Proposition 4.3.1 holds if $\frac{2}{1+2\alpha} \leq p < \infty$, i.e., if $1 \leq pH < \infty$.

In the following lemma, we discuss about the regularity of the process $Z(t)$. In order to get that, let us first set

$$Z(t) := Z(t,x),\ t \geq 0 \text{ and } x \in [0,1],$$

where $Z(\cdot,\cdot)$ has the following representation

$$Z(t,x) = \sum_{k=1}^{\infty} \left(\int_0^t e^{(t-s)A} \Phi d\xi_k(s) \right) e_k(x), \tag{4.38}$$

for $t \geq 0$ and $x \in [0,1]$.

Lemma 4.3.3. *Let (4.31), (4.30) and (4.33) hold and for any* $\delta \in (0,\beta)$,

$$\sum_{k=1}^{\infty} \frac{\mu_k^2}{\lambda_k^{1+2\alpha-2\delta}} < +\infty, \tag{4.39}$$

for some $\beta > 0$. *Then there exist positive constants* C_1 *and* C_2 *such that*

$$\mathbb{E}\,|Z(t,x) - Z(t,y)|^2 \leq C_1 |x-y|^{2\delta} \tag{4.40}$$

and

$$\mathbb{E}\,|Z(t,x) - Z(s,x)|^2 \leq C_2 (|t-s|^{2\delta} \vee |t-s|^{(1+2\alpha)\delta}), \tag{4.41}$$

for every $t,s \geq 0$ *and* $x,y \in [0,1]$.

Proof. Let the inequality (4.39) holds. We prove the following steps.

Step 1. Let us first establish that the series (4.38) converges in $L^2(\Omega,\mathscr{F},\mathbb{P})$. Indeed, for $(t,x) \in [0,T] \times [0,1]$, we have

$$\begin{aligned}\mathbb{E}|Z(t,x)|^2 &= \mathbb{E}\left| \sum_{k=1}^{\infty} \left(\int_0^t e^{(t-s)A} \Phi d\xi_k(s) \right) e_k(x) \right|^2 \\ &= \mathbb{E}\left| \sum_{k=1}^{\infty} \mu_k e_k(x) \left(\int_0^t e^{-(t-s)\lambda_k} d\xi_k(s) \right) \right|^2,\end{aligned}$$

using (4.33). Now, using the fact that sequence $\{\xi_k\}$ consists of mutually uncorrelated α-regular Volterra process and considering the symmetry in (4.8) and the embedding (4.10), we have

$$\begin{aligned}\mathbb{E}|Z(t,x)|^2 &= \sum_{k=1}^{\infty} \mu_k^2 |e_k(x)|^2 \mathbb{E}\left|\int_0^t e^{-(t-s)\lambda_k} d\xi_k(s)\right|^2 \\ &= \sum_{k=1}^{\infty} \mu_k^2 |e_k(x)|^2 \|e^{-\lambda_k(t-\cdot)}\|_{\mathscr{D}}^2 \le C \sum_{k=1}^{\infty} \mu_k^2 |e_k(x)|^2 \|e^{-\lambda_k(t-\cdot)}\|^2_{L^{\frac{2}{1+2\alpha}}}.\end{aligned}$$

Since, $|e_k(x)|^2 \le \frac{2}{\pi}$, and

$$\|e^{-\lambda_k(t-\cdot)}\|^2_{L^{\frac{2}{1+2\alpha}}} = \left(\int_0^t e^{-\frac{2\lambda_k}{1+2\alpha}(t-s)} ds\right)^{1+2\alpha} = \left(\frac{1-e^{-\frac{2\lambda_k}{1+2\alpha}(t)}}{\frac{2\lambda_k}{1+2\alpha}}\right)^{1+2\alpha} \le \left(\frac{1+2\alpha}{2\lambda_k}\right)^{1+2\alpha},$$

for $t \in [0,T]$, we have

$$\mathbb{E}|Z(t,x)|^2 \le C \sum_{k=1}^{\infty} \mu_k^2 \left(\frac{1+2\alpha}{2\lambda_k}\right)^{1+2\alpha} \le C \sum_{k=1}^{\infty} \frac{\mu_k^2}{\lambda_k^{1+2\alpha}} < +\infty, \tag{4.42}$$

using (4.39). Using the Kahane-Khinchin inequality [57] (see Lemma 4.2.1, also see [47, Proposition 2.1]), we have

$$\mathbb{E}|Z(t,x)|^{2q} \le C(q)\left(\mathbb{E}|Z(t,x)|^2\right)^q,$$

for any $q \in [0,\infty)$. By the virtue of (4.42), we get

$$\sup_{(t,x)\in[0,T]\times[0,1]} \mathbb{E}|Z(t,x)|^{2q} < +\infty. \tag{4.43}$$

Step 2. Next, from (4.31), we have

$$|e_k(x) - e_k(y)| \le C(\sqrt{\lambda_k})|x-y|,$$

for any $k \in \mathbb{N}$ and $x, y \in [0,1]$. From the above inequality we have

$$|e_k(x) - e_k(y)| \le C2^{1-\delta} \lambda_k^{\frac{\delta}{2}} |x-y|^{\delta}, \tag{4.44}$$

for all $\delta \in [0,1]$. Therefore, we have

$$\begin{aligned}\mathbb{E}|Z(t,x) - Z(t,y)|^2 &= \sum_{k=1}^{\infty} \mu_k^2 |e_k(x) - e_k(y)|^2 \mathbb{E}\left|\int_0^t e^{-(t-s)\lambda_k} d\xi_k(s)\right|^2 \\ &\le C \sum_{k=1}^{\infty} \frac{\mu_k^2}{\lambda_k^{1+2\alpha}} \left(1 - e^{-\frac{2\lambda_k}{1+2\alpha}(t)}\right)^{1+2\alpha} |e_k(x) - e_k(y)|^2 \\ &\le C \sum_{k=1}^{\infty} \frac{\mu_k^2}{\lambda_k^{1+2\alpha-\delta}} |x-y|^{2\delta} \\ &\le C|x-y|^{2\delta}, \end{aligned} \tag{4.45}$$

using (4.39). This gives the estimate (4.40).

Step 3. Now, fix any $T > 0$, and let $0 < s < t < T$. Then, we have

$$\begin{aligned}
\mathbb{E}|Z(t,x) - Z(s,x)|^2 &= \sum_{k=1}^{\infty} \mu_k^2 |e_k(x)|^2 \mathbb{E} \left| \left(\int_0^t e^{-(t-r)\lambda_k} - \int_0^s e^{-(s-r)\lambda_k} \right) d\xi_k(r) \right|^2 \\
&\le 2 \sum_{k=1}^{\infty} \mu_k^2 |e_k(x)|^2 \mathbb{E} \left| \int_s^t e^{-(t-r)\lambda_k} d\xi_k(r) \right|^2 \\
&\quad + 2 \sum_{k=1}^{\infty} \mu_k^2 |e_k(x)|^2 \mathbb{E} \left| \int_0^s (e^{-(t-r)\lambda_k} - e^{-(s-r)\lambda_k}) d\xi_k(r) \right|^2 \\
&=: 2(S_1 + S_2), \qquad (4.46)
\end{aligned}$$

where S_1 and S_2 are the final two terms appearing in the right hand side of (4.46). Note that for any $\delta \in [0,1]$ and all $x \ge 0, y \ge 0$, we have

$$|e^{-x} - e^{-y}| \le |x-y|^{\delta}.$$

Let us estimate S_1 as

$$\begin{aligned}
S_1 &= \sum_{k=1}^{\infty} \mu_k^2 \|e^{-(t-\cdot)\lambda_k}\|_{\mathscr{D}}^2 \le C \sum_{k=1}^{\infty} \mu_k^2 \|e^{-(t-\cdot)\lambda_k}\|^2_{L^{\frac{1}{1+2\alpha}}([s,t])} \\
&\le C \sum_{k=1}^{\infty} \frac{\mu_k^2}{\lambda_k^{1+2\alpha}} \left(1 - e^{\frac{2\lambda_k}{1+2\alpha}(t-s)} \right)^{1+2\alpha} \\
&\le C \sum_{k=1}^{\infty} \frac{\mu_k^2}{\lambda_k^{1+2\alpha}} \left(\lambda_k^{(1+2\alpha)\delta} |t-s|^{(1+2\alpha)\delta} \right) \\
&= C \sum_{k=1}^{\infty} \frac{\mu_k^2}{\lambda_k^{(1+2\alpha)-(1+2\alpha)\delta}} |t-s|^{(1+2\alpha)\delta} \\
&\le C |t-s|^{(1+2\alpha)\delta}, \qquad (4.47)
\end{aligned}$$

since by (4.39)

$$\sum_{k=1}^{\infty} \frac{\mu_k^2}{\lambda_k^{1+2\alpha-(1+2\alpha)\delta}} < +\infty.$$

In order to estimate S_2, we first note that

$$\begin{aligned}\left\|e^{-\lambda_k(t-\cdot)}-e^{-\lambda_k(s-\cdot)}\right\|^2_{L^{\frac{2}{1+2\alpha}}([0,s])} &= \left(\int_0^s \left|e^{-\lambda_k(t-r)}-e^{-\lambda_k(s-r)}\right|^{\frac{2}{1+2\alpha}} dr\right)^{1+2\alpha}\\ &= \left|e^{-\lambda_k(t-s)}-1\right|^2 \left(\int_0^s e^{-\frac{2\lambda_k}{1+2\alpha}(s-r)}dr\right)^{1+2\alpha}\\ &\le \left({\lambda_k}^{2\delta}|t-s|^{2\delta}\right)\left(\frac{1-e^{-\frac{2\lambda_k}{1+2\alpha}(s)}}{\frac{2\lambda_k}{1+2\alpha}}\right)^{1+2\alpha}\\ &\le C\frac{1}{\lambda_k^{1+2\alpha-2\delta}}|t-s|^{2\delta}.\end{aligned}$$

Thus, S_2 is evaluated as

$$\begin{aligned}S_2 = \sum_{k=1}^{\infty}\mu_k^2\left\|e^{-\lambda_k(t-\cdot)}-e^{-\lambda_k(t-\cdot)}\right\|^2_{\mathscr{D}} &\le C\sum_{k=1}^{\infty}\mu_k^2\left\|e^{-\lambda_k(t-\cdot)}-e^{-\lambda_k(t-\cdot)}\right\|^2_{L^{\frac{2}{1+2\alpha}}}\\ \le C\sum_{k=1}^{\infty}\frac{\mu_k^2}{\lambda_k^{1+2\alpha-2\delta}}|t-s|^{2\delta} \le C|t-s|^{2\delta}, \end{aligned}\tag{4.48}$$

using (4.39). Substituting (4.47) and (4.48) into (4.46), we infer that

$$\mathbb{E}|Z(t,x)-Z(s,x)|^2 \le C_1\{|t-s|^{2\delta}\vee|t-s|^{(1+2\alpha)\delta}\},\tag{4.49}$$

for some constant $C>1$ and which completes the proof of (4.41). □

The following inequality, known as Garsia-Rodemich-Rumsey inequality [75], is used to find the regularity of $Z(\cdot,\cdot)$.

Theorem 4.3.4. *Garsia-Rodemich-Rumsey Inequality Let φ and ψ be continuous, strictly increasing functions on $[0,\infty)$ such that*

$$\varphi(0)=\psi(0)=0 \ \textit{and} \ \lim_{t\to\infty}\psi(t)=\infty.\tag{4.50}$$

For any given $T>0$ and $\phi\in C([0,T],\mathbb{R})$, if there is a constant M such that

$$\int_0^T\int_0^T \psi\left(\frac{\phi(t)-\phi(s)}{\varphi(|t-s|)}\right)dsdt \le M,$$

then for all $s,t\in[0,T]$, we have

$$|\phi(t)-\phi(s)| \le C\int_0^{|t-s|}\psi^{-1}\left(\frac{4M}{u^2}\right)\phi(du).$$

Corollary 4.3.5. *Let $X := \{X_t, 0 \leq t \leq T\}$ be a stochastic process satisfying that all the moments of the X exists and there are constants $\eta > 0$, and $p_0 > 1$ such that*

$$\mathbb{E}\left(|X_t - X_s|^p\right) \leq C_p |t-s|^{\eta p}, \text{ for all } 0 \leq s,t \leq T \text{ and for all } p \geq p_0.$$

Then for any $\beta \in (0,\eta)$, the process X has a version which is Hölder continuous with exponent η.

With the help of above Corollary 4.3.5, we state the following Lemma which gives the bounds on the L^∞-norm with respect to space and time both.

Lemma 4.3.6. *For each $q \in (1,\infty)$, we have*

$$\mathbb{E}\left[\|Z\|^{2q}_{L^\infty([0,T]\times[0,1])}\right] < +\infty \tag{4.51}$$

Proof. We have

$$\mathbb{E}|Z(t,x) - Z(s,y)|^{2q} \leq \quad C\left(\mathbb{E}|Z(t,x) - Z(t,y)|^{2q} + \mathbb{E}|Z(t,x) - Z(s,x)|^{2q}\right). \tag{4.52}$$

From (4.45) and (4.49) and using Kahane-Khinchin inequality (see Lemma 4.2.1, also see [47, Proposition 2.1]), we have

$$\mathbb{E}|Z(t,x) - Z(t,y)|^{2q} \leq C(\mathbb{E}|Z(t,x) - Z(t,y)|^2)^q \leq C\left(|x-y|^2\right)^{q\delta},$$

for all $x,y \in [0,1]$ and

$$\mathbb{E}|Z(t,x) - Z(s,x)|^{2q} \leq C(\mathbb{E}|Z(t,x) - Z(s,x)|^2)^q \leq C\left(|t-s|^2 \vee |t-s|^{1+2\alpha}\right)^{q\delta},$$

for all $0 < s \leq t$. Also, the inequality (4.43), shows that all the moments of $Z(\cdot,\cdot)$ exists. Therefore, by Corollary 4.3.5, the process $Z(\cdot,\cdot)$ has a version which is Hölder continuous with exponent δ. Consequently, Lemma 4.3.6 holds for any $(t,x) \in [0,T] \times [0,1]$. □

Remark 4.3.2. For $\Phi = A^{-\upsilon}$, with $\upsilon > 1/4$, the condition (4.39) gives

$$\sum_{k=1}^{\infty} \frac{(-\pi^2k^2)^{-2\alpha}}{(-\pi^2k^2)^{1+2\alpha-2\delta}} \leq C\sum_{k=1}^{\infty} \frac{1}{k^{2(1+2\alpha+2\upsilon-2\delta)}} = C\sum_{k=1}^{\infty} \frac{1}{k^{1+(1+4\alpha+4\upsilon-4\delta)}},$$

which converges if $(1+4\alpha+4\upsilon-4\delta) > 0$ i.e. $\delta < \frac{1}{4} + \alpha + \upsilon$. Therefore, in this case we get $\beta = \frac{1}{4} + \alpha + \upsilon$, such that $\delta \in (0,\beta)$.

Now we establish the existence and uniqueness of the local mild solution to equations (4.1)-(4.3).

Let the continuous function $\Theta_n : [0,\infty) \to [0,1]$ be defined as

$$\Theta_n(x) = \begin{cases} 1, & x \le n, \\ n+1-x, & n < x \le n+1, \\ 0, & x > n+1, \end{cases} \tag{4.53}$$

where n is a natural number. With the help of Θ_n, for each fixed n, we introduce the following truncated stochastic integral equation:

$$f_n(t,x) = e^{tA} f_0(x) + \int_0^t e^{(t-s)A} \Theta_n\left(\|f_n(s,\cdot)\|_{L^p}\right) \frac{\partial g(f_n(s,x))}{\partial x} ds + \int_0^t e^{(t-s)A} \Phi dB_x(s). \tag{4.54}$$

Proposition 4.3.7. *Let **(A1)**, **(A2)** and Lemma 4.3.6 hold. Let us consider the function $g(r) = \frac{1}{p} r^p$ in equation (4.54). Then for any fixed $n \in \mathbb{N}$ and for each $2 \le p < \infty$, equation (4.54) has a unique L^p-valued and $\mathscr{F}_t$-adapted continuous solution such that*

$$\sup_{t\in[0,T]} \mathbb{E}\left[\|f_n(t)\|_{L^p}^{\rho}\right] < +\infty, \tag{4.55}$$

for each $\rho \ge p$ and $T > 0$.

Proof. Let us construct a space $\mathscr{B}_{p,\rho}$ consisting of all the $L^p([0,1])$-valued and $\mathscr{F}_t$-adapted stochastic processes with the norm defined by

$$\|f\|_{p,\rho} := \sup_{t\in[0,T]} \left(\mathbb{E}\|f(t)\|_{L^p}^{\rho}\right)^{\frac{1}{\rho}}. \tag{4.56}$$

Clearly, $(\mathscr{B}_{p,\rho}, \|\cdot\|_{p,\rho})$ is a Banach space. Next, for each fixed $n \in \mathbb{N}$, let us define an operator $\mathscr{T}$ as

$$(\mathscr{T} f_n)(t,x) := e^{tA} f_0(x) + \frac{1}{p} \int_0^t e^{(t-s)A} \Theta_n\left(\|f_n(s)\|_{L^p}\right) \frac{\partial (f_n)^p(s,x)}{\partial x} ds + \int_0^t e^{(t-s)A} \Phi dB_x(s). \tag{4.57}$$

We show the existence of a unique local mild solution to (4.54) by showing that $\mathscr{T}$ is a contraction on the Banach space $(\mathscr{B}_{p,\rho}, \|\cdot\|_{p,\rho})$.

Step 1. First, we show that the operator $\mathscr{T}$ defined by (4.57) is well defined, i.e., $\mathscr{T}$ maps $(\mathscr{B}_{p,\rho}, \|\cdot\|_{p,\rho})$ into itself. Taking $L^p([0,1])$ norm on both sides of (4.57), we have

$$\|(\mathscr{T} f_n)(t)\|_{L^p} \leq \|e^{tA} f_0\|_{L^p} + \left\| \frac{1}{p} \int_0^t e^{(t-s)A} \Theta_n \left(\|f_n(s)\|_{L^p}\right) \frac{\partial f_n^p(s,x)}{\partial x} ds \right\|_{L^p} + \left\| \int_0^t e^{(t-s)A} \Phi dB_x(s) \right\|_{L^p}$$
$$=: I_1 + I_2 + I_3, \tag{4.58}$$

where I_1, I_2 and I_3 are given by

$$I_1 := \|e^{tA} f_0\|_{L^p},$$

$$I_2 := \left\| \frac{1}{p} \int_0^t e^{(t-s)A} \Theta_n \left(\|f_n(s)\|_{L^p}\right) \frac{\partial f_n^p}{\partial x} ds \right\|_{L^p},$$

and

$$I_3 := \left\| \int_0^t e^{(t-s)A} \Phi dB_x(s) \right\|_{L^p}.$$

Using the contraction property of semigroup $\{e^{tA}\}_{t\geq 0}$, we obtain

$$I_1 = \|e^{tA} f_0\|_{L^p} \leq \|f_0\|_{L^p},$$

and by **(A2)**, we have

$$\mathbb{E}[I_1^\rho] < +\infty. \tag{4.59}$$

Now, I_2 can be estimated as

$$I_2 \leq \frac{1}{p} \int_0^t \left\| e^{(t-s)A} \Theta_n \left(\|f_n(s)\|_{L^p}\right) \frac{\partial f_n^p(s)}{\partial x} \right\|_{L^p} ds. \tag{4.60}$$

In order to estimate the term inside the integral in the inequality (5.56), the following Sobolev embedding is needed

$$\|f\|_{L^{q_1}} \leq \|f\|_{W^{k,q_2}}, \text{ whenever } k < \frac{1}{q_2}, \tag{4.61}$$

where $\frac{1}{q_1} = \frac{1}{q_2} - k$ [56, Chapter 5]. We also need a smoothing property of the Laplacian, i.e., for any $r_1 \leq r_2$ in $\mathbb{R}$, and $\theta \geq 1$, e^{tA} maps $W^{r_1,\theta}([0,1])$ into $W^{r_2,\theta}([0,1])$, for all $t > 0$. Moreover the following estimate holds

$$\|e^{tA} u\|_{W^{r_2,\theta}([0,1])} \leq C_1 (t^{\frac{r_1 - r_2}{2}} + 1) \|u\|_{W^{r_1,\theta}([0,1])}, \tag{4.62}$$

for all $u \in W^{r_1,\theta}([0,1])$, where $C_1 = C_1(r_1,r_2,\theta)$ is a positive constant (see in [128, 51]). Applying (4.61) with $q_1 = p, q_2 = 1, k = \frac{p-1}{p}$ and then using the smoothing property (4.62) with $r_1 = -1, r_2 = \frac{p-1}{p}, \theta = 1$, we evaluate

$$\begin{aligned}\left\|e^{(t-s)A}\Theta_n\left(\|f_n(t)\|_{L^p}\right)\frac{\partial f_n^p(t)}{\partial x}\right\|_{L^p} &\leq C\left\|e^{(t-s)A}\Theta_n\left(\|f_n(t)\|_{L^p}\right)\frac{\partial f_n^p(t)}{\partial x}\right\|_{W^{\frac{p-1}{p},1}} \\ &\leq C(t-s)^{\frac{1-2p}{2p}}\left\|\Theta_n\left(\|f_n(t)\|_{L^p}\right)\frac{\partial f_n^p(t)}{\partial x}\right\|_{W^{-1,1}} \\ &= C(t-s)^{\frac{1-2p}{2p}}\Theta_n\left(\|f_n(t)\|_{L^p}\right)\|f_n^p(t)\|_{L^1} \\ &= C(t-s)^{\frac{1-2p}{2p}}\Theta_n\left(\|f_n(t)\|_{L^p}\right)\|f_n(t)\|_{L^p}^p \\ &\leq C(n+1)^p(t-s)^{\frac{1-2p}{2p}}, \end{aligned} \tag{4.63}$$

where in the last step we have used the definition of Θ_n. Substituting the estimate (4.63) in (5.56), we obtain

$$I_2 \leq C(n+1)^p\int_0^t (t-s)^{\frac{1-2p}{2p}}ds \leq C(n+1)^p T^{\frac{1}{2p}},$$

and hence we have

$$\mathbb{E}[I_2^\rho] < +\infty. \tag{4.64}$$

Further, implementing the Corollary 4.2.4 and then **(A1)**, we have

$$\begin{aligned}\mathbb{E}(I_3)^\rho =& \mathbb{E}\left\|\int_0^t e^{(t-s)A}\Phi dB_x(s)\right\|_{L^p}^\rho \leq \left(\int_0^t \left\|e^{(t-s)A}\Phi\right\|_{\gamma(L^2,L^p)}^{\frac{2}{1+2\alpha}} ds\right)^{1+2\alpha} \\ &\leq t^{1+2\alpha}\|\Phi\|_{\gamma(L^2,L^p)}^2 < +\infty. \end{aligned} \tag{4.65}$$

Using estimates (4.59), (4.64) and (4.65) in the inequality (4.58), we get

$$\mathbb{E}\left[\|(\mathscr{T}f_n)(t)\|_{L^p}^\rho\right] < +\infty. \tag{4.66}$$

Step 2. Next, we show that the map $\mathscr{T}$ is a contraction on $\mathscr{B}_{p,\rho}$. Then Banach fixed point theorem gives the unique solution to equation (4.54). Let f_n and h_n be two different members of $\mathscr{B}_{p,\rho}$ with same initial data. Then, we have

$$\begin{aligned}&\|(\mathscr{T}f_n)(t) - (\mathscr{T}h_n)(t)\|_{L^p} \\ &\quad = C\int_0^t \left\|e^{(t-s)A}\left(\Theta_n\left(\|f_n(s)\|_{L^p}\right)\frac{\partial f_n^p(s)}{\partial x} - \Theta_n\left(\|h_n(s)\|_{L^p}\right)\frac{\partial h_n^p(s)}{\partial x}\right)\right\|_{L^p} ds \\ &\quad \leq C\int_0^t (t-s)^{\frac{1-2p}{2p}}\left\|\Theta_n\left(\|f_n(s)\|_{L^p}\right)f_n^p(s) - \Theta_n\left(\|h_n(s)\|_{L^p}\right)h_n^p(s)\right\|_{L^1} ds, \end{aligned} \tag{4.67}$$

by following the same argument as in the proof of (4.63). Now, we consider

$$
\begin{aligned}
&\|\Theta_n(\|f_n(t)\|_{L^p}) f_n^p(t) - \Theta_n(\|h_n(t)\|_{L^p}) h_n^p(t)\|_{L^1} \\
&\leq \|\Theta_n(\|f_n(t)\|_{L^p})(f_n^p(t) - h_n^p(t))\|_{L^1} + (\Theta_n(\|f_n(t)\|_{L^p}) - \Theta_n(\|h_n(t)\|_{L^p}))\|h_n^p(t)\|_{L^1} \\
&=: \Psi_1 + \Psi_2, \qquad (4.68)
\end{aligned}
$$

where Ψ_1 and Ψ_2 are given by

$$
\begin{aligned}
\Psi_1 &:= \|\Theta_n(\|f_n(t)\|_{L^p})(f_n^p(t) - h_n^p(t))\|_{L^1} \\
\Psi_2 &:= (\Theta_n(\|f_n(t)\|_{L^p}) - \Theta_n(\|h_n(t)\|_{L^p}))\|h_n^p(t)\|_{L^1}.
\end{aligned}
$$

Without loss of generality, let us take

$$
\|f_n(t)\|_{L^p} \geq \|h_n(t)\|_{L^p}. \qquad (4.69)
$$

Now, for I_1, we have

$$
\Psi_1 = \|\Theta_n(\|f_n(t)\|_{L^p})(f_n^p(t) - h_n^p(t))\|_{L^1} = \Theta_n(\|f_n(t)\|_{L^p})\|f_n^p(t) - h_n^p(t)\|_{L^1}, \qquad (4.70)
$$

where

$$
\|(f_n(t))^p - (h_n(t))^p)\|_{L^1} = \left\|(f_n(t) - h_n(t))\sum_{j=0}^{p-1}(f_n(t))^{p-j-1}(h_n(t))^j\right\|_{L^1}.
$$

Using Hölder inequality and (4.69), we obtain

$$
\begin{aligned}
\|(f_n(t))^p - (h_n(t))^p)\|_{L^1} &\leq \|(f_n(t) - h_n(t))\|_{L^p}\sum_{j=0}^{p-1}\left\|f_n^{p-j-1}(t)h_n^j(t)\right\|_{L^{\frac{p}{p-1}}} \\
&\leq \|(f_n(t) - h_n(t))\|_{L^p}\sum_{j=0}^{p-1}\left\|f_n^{p-j-1}(t)h_n^j(t)\right\|_{L^{\frac{p}{p-1}}} \\
&\leq \|(f_n(t) - h_n(t))\|_{L^p}\sum_{j=0}^{p-1}\left\|f_n^{p-j-1}(t)\right\|_{L^{\frac{p}{p-j-1}}}\left\|h_n^j(t)\right\|_{L^{\frac{p}{j}}} \\
&= \|(f_n(t) - h_n(t))\|_{L^p}\sum_{j=0}^{p-1}\|f_n(t)\|_{L^p}^{p-j-1}\|h_n(t)\|_{L^p}^j \\
&\leq \|(f_n(t) - h_n(t))\|_{L^p}\sum_{j=0}^{p-1}\|f_n(t)\|_{L^p}^{p-1}. \qquad (4.71)
\end{aligned}
$$

Using the estimate (4.71) in (4.70), and then applying definition of Θ_n, we get

$$
\Psi_1 \leq p(n+1)^{p-1}\|(f_n(t) - h_n(t))\|_{L^p}. \qquad (4.72)
$$

Now, we simplify Ψ_2 as

$$\Psi_2 = \left(\Theta_n\left(\|f_n(t)\|_{L^p}\right) - \Theta_n\left(\|h_n(t)\|_{L^p}\right)\right)\|h_n(t)\|_{L^1}^p \leq 0, \tag{4.73}$$

using (4.69). The following six cases are considered to obtain (4.73):

Case 1: $\|f_n(t)\|_{L^p} < n$ and $\|h_n(t)\|_{L^p} < n$,

Case 2: $n < \|f_n(t)\|_{L^p} < n+1$ and $\|h_n(t)\|_{L^p} < n$,

Case 3: $n < \|f_n(t)\|_{L^p} < n+1$ and $n < \|h_n(t)\|_{L^p} < n+1$,

Case 4: $\|f_n(t)\|_{L^p} > n+1$ and $\|h_n(t)\|_{L^p} < n$,

Case 5: $\|f_n(t)\|_{L^p} > n+1$ and $n < \|h_n(t)\|_{L^p} < n+1$ and

Case 6: $\|f_n(t)\|_{L^p} > n+1$ and $\|h_n(t)\|_{L^p} > n+1$.

Therefore, combining (4.72), (4.73) and (4.68), we obtain

$$\begin{aligned}
\|\Theta_n\left(\|f_n(t)\|_{L^p}\right) f_n^p(t) - \Theta_n\left(\|h_n(t)\|_{L^p}\right) h_n^p(t)\|_{L^1} &\leq (p(n+1)^{p-1})(\|f_n(t) - h_n(t)\|_{L^p}) \\
&= \widetilde{C}_n \|f_n(t) - h_n(t)\|_{L^p},
\end{aligned} \tag{4.74}$$

where $\widetilde{C}_n = p(n+1)^{p-1}$. Inserting the estimate (4.74) into (4.67), we get

$$\|(\mathscr{T} f_n)(t) - (\mathscr{T} h_n)(t)\|_{L^p} \leq C\widetilde{C} \int_0^t (t-s)^{\frac{1-2p}{2p}} \|f_n(s) - h_n(s)\|_{L^p} ds. \tag{4.75}$$

Next, using Hölder's inequality and definition of the norm $\|\cdot\|_{p,\rho}$, we evaluate

$$\begin{aligned}
&\mathbb{E}\|(\mathscr{T} f_n)(t) - (\mathscr{T} h_n)(t)\|_{L^p}^\rho \\
&\leq C\widetilde{C}_n \mathbb{E}\left(\int_0^t (t-s)^{\frac{1-2p}{2p}} \|f_n(s) - h_n(s)\|_{L^p} ds\right)^\rho \\
&\leq C\widetilde{C}_n \mathbb{E}\left[\left(\int_0^t (t-s)^{\frac{1-2p}{2p}} ds\right)^{\frac{\rho-1}{\rho}} \left(\int_0^t (t-s)^{\frac{1-2p}{2p}} \|f_n(s) - h_n(s)\|_{L^p}^\rho ds\right)^{\frac{1}{\rho}}\right]^\rho \\
&\leq C\widetilde{C}_n t^{\frac{\rho-1}{2p}} \int_0^t (t-s)^{\frac{1-2p}{2p}} \mathbb{E}\left[\|f_n(s) - h_n(s)\|_{L^p}^\rho\right] ds \\
&\leq C\widetilde{C}_n t^{\frac{\rho}{2p}} \|f_n - h_n\|_{p,\rho},
\end{aligned}$$

holds for every $t \in [0,T]$. Therefore, we have

$$\|(\mathscr{T} f_n) - (\mathscr{T} g_n)\|_{p,\rho} \leq C\widetilde{C}_n T^{\frac{\rho}{2p}} \|f_n - h_n\|_{p,\rho},$$

i.e. the operator $\mathscr{T}$ is a contraction on the Banach space $\mathscr{B}_{p,\rho}$ if $T < (C\widetilde{C}_n)^{-\frac{\rho}{2p}}$. Hence by Banach fixed point theorem, there exists the unique solution satisfying the equation (4.54) in the interval $[0, (C\widetilde{C}_n)^{-\frac{\rho}{2p}})$. Thus, there exists a unique local mild solution to the equation (4.54).

Step 3. Let us now show that the local mild solution to the equation (4.54) is global. Let us take $\Upsilon = C\widetilde{C}_n$. Observe that the constants appearing in Υ do not depends on the initial data. Therefore, replacing the given initial data, i.e., $f(0,\cdot) = f_0(\cdot)$ at $t = 0$, by next initial condition $f(t_1,\cdot)$ at time $t = t_1$ where $t_1 = (C\widetilde{C}_n)^{-\frac{\rho}{2p}}$, we get a local mild solution of equation (4.54) in the interval $[t_1,t_2)$ for some $t_2 > t_1$, following the same way as in Steps 1 and 2. Proceeding with the same strategy, one can construct a unique mild solution to the equation (4.54) in the entire interval $[0,T]$, for arbitrary $T > 0$, and hence the proof is complete. □

The local mild solution to the system (4.1)-(4.3) is constructed by using the unique mild solution of the truncated equation (4.54) obtained in the previous Proposition.

Proposition 4.3.8. *Let $2 \leq p < \infty$ and the initial function $f_0 : [0,1] \times \Omega \to \mathbb{R}$ be $L^p([0,1])$-valued and $\mathscr{F}_0$- measurable. Let* ***(A1)*** *and* ***(A2)*** *hold, and Lemma 4.3.6 be satisfied. Then there exists a unique local mild solution $f(t) := f(t,\cdot,\cdot)$ to the system (4.1)-(4.3) such that*

$$\sup_{t\in[0,T]} \mathbb{E}\left[\|f(t\wedge\tau)\|_{L^p}^{\rho}\right] < +\infty, \tag{4.76}$$

holds for each $\tau \geq p$ and $T > 0$, where ρ is an $\mathscr{F}_t$-adapted stopping time.

Proof. For each $n \in \mathbb{N}$, let us define a sequence of $\mathscr{F}_t$-adapted sequence of stopping times

$$\tau_n(\omega) := \inf\{t \in [0,T]; \|f_n(t,\omega)\|_{L^p} dx \geq n\},$$

for each $\omega \in \Omega$, where f_n is the unique mild solution to the equation (4.54). Clearly, for each $m \geq n$, using the fact that f_n is the unique mild solution to the equation (4.54), we infer that

$$f_m(t,x) = f_n(t,x),\ \mathbb{P}\text{-a.s., for all } (t,x) \in [0, T\wedge\tau_m\wedge\tau_n) \times [0,1]. \tag{4.77}$$

But from the definition of τ_n, it is also immediate that $\tau_n \leq \tau_m$, $\mathbb{P}$-a.s. for each $m \geq n$. Since $T > 0$ is arbitrary, from (4.77), we also have

$$f_m(t,x) = f_n(t,x),\ \mathbb{P}\text{-a.s., for all } (t,x) \in [0, \tau_n) \times [0,1]. \tag{4.78}$$

Since $\tau_n \leq \tau_m$, for $n \leq m$, using Zorn's lemma, we obtain a maximal element $\tau = \lim_{n\to\infty} \tau_n$, $\mathbb{P}$-a.s. Let us now define

$$f(t,\cdot) := \lim_{n\to\infty} f_n(t,\cdot)$$

for all $t \in [0,\tau)$. Then the Proposition 4.3.7 assures the existence of a unique local solution f to equations (4.1)-(4.3) in the interval $[0,\tau)$ in the sense of Definition 6.3.2. □

The local solution obtained above will be the global solution, if $\tau = \lim_{n\to\infty} \tau_n = \infty$, $\mathbb{P}$-a.s. This requirement is fulfilled by the following theorem. However, in Theorem 4.3.9, we restrict ourselves to third order nonlinearity only.

Theorem 4.3.9. *Let (A1) and (A2) hold true and* $g(r) = \frac{r^a}{a}$, *where* $a = 2,3$. *Then for any* $p \geq a$ *and* $T > 0$, *there exists a unique* $\mathscr{F}_t$*-adapted mild solution* $f(t,x) \in L^\infty([0,T], L^p[0,1])$, $\mathbb{P}$*-a.s., for every* $(t,x) \in [0,T] \times [0,1]$ *to the system (4.1)-(4.3) with* $\mathbb{P}$*-a.s., continuous trajectories.*

Proof. Let $p \geq a$ and n be any arbitrary natural number. Then, following similar arguments in [85, Proposition 3.2], we have

$$\sup_{n\in\mathbb{N}} \mathbb{E}\left[\sup_{t\in[0,T]} \|f_n(t,\cdot)\|_{L^p}^{\rho}\right] < +\infty.$$

Therefore, for any $T > 0$, using Markov's inequality, we have

$$\begin{aligned}\mathbb{P}\Big\{\omega \in \Omega : \tau_n \leq T\Big\} &= \mathbb{P}\left\{\omega \in \Omega : \sup_{t\in[0,T]} \|f_n(t,\cdot)\|_{L^p} \geq n\right\} = \mathbb{P}\left\{\omega \in \Omega : \sup_{t\in[0,T]} \|f_n(t,\cdot)\|_{L^p}^{\rho} \geq n^{\rho}\right\} \\ &\leq \mathbb{E}\left(\sup_{t\in[0,T]} \|f_n(t,\cdot)\|_{L^p}^{\rho}\right) n^{-\rho}\end{aligned}$$

Passing the limit $n \to \infty$ implies that $\mathbb{P}\{\omega \in \Omega : \tau_n \leq T\} \to 0$. Consequently, $\tau = \infty$, $\mathbb{P}$-a.s., and the proof of Theorem 4.3.9 is completed. □

4.4 Invariant Measure

In the above section, we have shown the existence and uniqueness of the global mild solution to the equation (4.1) having third order nonlinearity. Due to some technical difficulties, here we restrict ourselves to the second order nonlinearity only while showing the existence of an invariant measure. That is, the existence of an invariant measure is shown for the equation:

$$df(t,x) = \left(\frac{\partial^2 f(t,x)}{\partial x^2} + \frac{1}{2} \frac{\partial f^2}{\partial x}(t,x) \right) dt + \Phi dB_x(t). \tag{4.79}$$

In (4.79), we take $B_x(t)$ as the α-regular cylindrical Volterra process of Gaussian type process. In order to prove the existence of an invariant measure of the equation (4.1), we extend the noise $B_x(t)$ on the right hand side (4.1) to whole real line $\mathbb{R}$ by taking another α-cylindrical Volterra process $V_x(t), t \geq 0$, such that

$$B_x(t) = V_x(-t) \;\text{ for all } t \leq 0, \text{ and } x \in [0,1]. \tag{4.80}$$

Here the process $V_x(t)$ is assumed to be independent of $B_x(t)$. Now, we introduce a process $\{f_\lambda\}_{\lambda \geq 0}$ as

$$\begin{cases} df_\lambda = \left(Af_\lambda + \frac{1}{2}\frac{\partial}{\partial x} f_\lambda^2 \right) dt + \Phi dB_x(t), \\ f_\lambda(-\lambda) = 0. \end{cases} \tag{4.81}$$

Thanks to Theorem 4.3.9, which assures the existence of a unique mild solution to the system (4.81). To show the existence of an invariant measure, it is sufficient to prove that the family of laws $\{\mathscr{L}(f_\lambda(0))\}_{\lambda \geq 0}$ is tight. In order to achieve this goal, it is sufficient to prove that law of the solution is bounded in probability in $W^{\sigma,2}([0,1])$ for some $\sigma > 0$. Further, using the Sobolev embedding of $W^{\sigma,2}([0,1])$ into $L^2([0,1])$ for all $\sigma > 0$, we get our desired result. For any $\beta > 0$, let us introduce the processes Z_A^β and g_λ as

$$\begin{cases} dZ_A^\beta(t) = (A-\beta) Z_A^\beta(t) dt + \Phi dB_x(t), \\ Z_A^\beta(0) = z_0, \end{cases} \tag{4.82}$$

where

$$z_0 = \int_{-\infty}^{0} e^{-s(A-\beta)} \Phi dB_x(s),$$

and

$$\begin{cases} \frac{dg_\lambda(t)}{dt} = \left(Ag_\lambda + \frac{1}{2}\frac{\partial}{\partial x}(g_\lambda + Z_A^\beta)^2 \right) + \beta Z_A^\beta(t), \\ g_\lambda(-\lambda) = -Z_A^\beta(-\lambda). \end{cases} \tag{4.83}$$

The solution Z_A^β to (4.82) can be written as a modified convolution

$$Z_A^\beta(t) = \int_{-\infty}^{t} e^{(t-s)(A-\beta)}\Phi dB_x(s). \tag{4.84}$$

On the other hand, if we suppose $g_\lambda(t)$ is the difference of the two processes f_λ and Z_A^β, i.e.,

$$g_\lambda(t) = f_\lambda(t) - Z_A^\beta(t), \ \ t \geq -\lambda,$$

then, it can be noticed that f_λ is the solution of (4.81) if g_λ is the solution of (4.83). Remember that the Volterra noise $B_x(t)$ in (4.79) is of Gaussian type and for this type of noise, we have following results:

1. $Z_A^\beta(t)$ is bounded in probability in $W^{\frac{1}{4},2}([0,1])$,

2. $Z_A^\beta(t)$ is a stationary process,

3. $Z_A^\beta(t)$ is ergodic process.

These results can be verified in similar way as in [49, 50, 51].

Lemma 4.4.1. *Let the assumptions **(A1)** and **(A2)** hold. For any $\varepsilon > 0$ and $\sigma > 0$, there exist $\beta > 0$ depending on ε and σ such that*

$$\mathbb{E}\left(|(-A)^\sigma Z_A^\beta(t)|^2_{L^2([0,1])}\right) < \varepsilon, \tag{4.85}$$

provided

$$\sum_{k=1}^{\infty} \frac{\mu_k^2(\pi k)^{4\sigma}}{(\lambda_k+\beta)^{1+2\alpha}} < +\infty. \tag{4.86}$$

Proof. Using the Definition 4.2.4, we know that

$$B_x(t) = \sum_{k=1}^{\infty} \xi_k(t)e_k(x),$$

and hence we have

$$Z_A^\beta(t) = \sum_{k=1}^{\infty}\left(\int_{-\infty}^{t} e^{(t-s)(A-\beta)}\Phi d\xi_k(s)\right)e_k(x).$$

Since ξ_k are uncorrelated, we find, by using (4.10),

$$
\begin{aligned}
&\mathbb{E}\left(|(-A)^{\sigma}Z_A^{\beta}(t)|_{L^2}^2\right)\\
&=\mathbb{E}\left(\left|(-A)^{\sigma}\sum_{k=1}^{\infty}\left(\int_{-\infty}^{t}e^{(t-s)(A-\beta)}\Phi d\xi_k(s)\right)e_k(\cdot)\right|_{L^2}^2\right)\\
&=\mathbb{E}\left(\left|\sum_{k=1}^{\infty}(-A)^{\sigma}\mu_k e_k(\cdot)\left(\int_{-\infty}^{t}e^{-(t-s)(\lambda_k+\beta)}d\xi_k(s)\right)\right|_{L^2}^2\right)\\
&=\sum_{k=1}^{\infty}\mathbb{E}\left(\left|\mu_k(-A)^{\sigma}\left(\int_{-\infty}^{t}e^{-(t-s)(\lambda_k+\beta)}d\xi_k(s)\right)\right|^2\left(\int_0^1|e_k(x)|^2dx\right)\right)\\
&\le C\sum_{k=1}^{\infty}\mu_k^2(\pi k)^{4\sigma}\mathbb{E}\left|\int_{-\infty}^{t}e^{-(t-s)(\lambda_k+\beta)}d\xi_k(s)\right|^2\\
&=\sum_{k=1}^{\infty}\mu_k^2(\pi k)^{4\sigma}\left\|e^{-(t-\cdot)(\lambda_k+\beta)}\right\|_{\mathscr{D}(-\infty,t)}^2\\
&\le\sum_{k=1}^{\infty}\mu_k^2(\pi k)^{4\sigma}\left\|e^{-(t-\cdot)(\lambda_k+\beta)}\right\|_{L^{\frac{2}{1+2\alpha}}(-\infty,t)}^2
\end{aligned}
\tag{4.87}
$$

Now, it can be easily seen that

$$
\left\|e^{-(t-\cdot)(\lambda_k+\beta)}\right\|_{L^{\frac{2}{1+2\alpha}}(-\infty,t)}^2=\left(\int_{-\infty}^{t}|e^{-(t-s)(\lambda_k+\beta)}|^{\frac{2}{1+2\alpha}}ds\right)^{1+2\alpha}\le C\frac{1}{(\lambda_k+\beta)^{1+2\alpha}}. \tag{4.88}
$$

Substituting (4.88) into (4.87), we get

$$
\mathbb{E}\left(|(-A)^{\sigma}Z_A^{\beta}(t)|_{L^2}^2\right)\quad\le C\textstyle\sum_{k=1}^{\infty}\frac{\mu_k^2(\pi k)^{4\sigma}}{(\lambda_k+\beta)^{1+2\alpha}}\le\varepsilon, \tag{4.89}
$$

provided

$$
\sum_{k=1}^{\infty}\frac{\mu_k^2(\pi k)^{4\sigma}}{(\lambda_k+\beta)^{1+2\alpha}}<+\infty,\ \text{ for some }\ \sigma>0. \tag{4.90}
$$

□

Remark 4.4.1. For $\Phi=A^{-\nu}$, for $\nu>\frac{1}{4}$, we know that $\mu_k=\pi^{-\nu}k^{-\nu}$, and

$$
\sum_{k=1}^{\infty}\frac{\mu_k^2(\pi k)^{4\sigma}}{(\lambda_k+\beta)^{1+2\alpha}}=\sum_{k=1}^{\infty}\frac{(\pi^{-\nu}k^{-\nu})^2(\pi k)^{4\sigma}}{(\pi k)^{2+4\alpha}}\le C\sum_{k=1}^{\infty}\frac{1}{\lambda_k^{2+4\alpha+2\nu-4\sigma}}<+\infty,
$$

if $\sigma<\frac{1}{4}+\frac{2\alpha+\nu}{2}$.

Using Lemma 4.4.1, the following theorem, which gives the existence of an invariant measure for the equation (4.79), can be proved in a similar way as in the proof of [51, Theorem 4.1].

Theorem 4.4.2. *Let the assumptions* ***(A1)*** *and* ***(A2)*** *hold and the operator* Φ *is chosen in such way that, for any* $\sigma \in [0, 1/4)$*, condition (4.85) holds. Then there exists an invariant measure for the equation (4.79).*

Chapter 5

On a fractional stochastic Burgers type nonlinear equation perturbed by a cylindrical fractional Brownian motion in Hölder space

In this Chapter, we study the existence and uniqueness of the fractional stochastic Burgers-type equation driven by cylindrical fractional Brownian motion in Hölder space. This work is the continuation of [4], where the authors have proved the same type of problem with Wiener process. Therefore, many parts of this chapter is mainly motivated by [4]. Moreover, it is shown that the same rate of convergence for Galerkin approximations can be obtained by considering the cylindrical fractional Brownian motion as in the case of Wiener process [4].

This Chapter is organized in the following manner: In coming section, we first introduce the problem and then recall a theorem from [24, Theorem 3.1] which is essentially required for proving the main result. Section 2 consists of all required definitions and we also recall here some estimates from [4]. In Section 3, the formulation and main results on solutions to the problem are given. Section 4 deals with all estimates on stochastic terms. At the end, in Section 5, we provide the proof of theorems given in Section 3.

5.1 Introduction

In this chapter, we deal with the following class of nonlinear fractional stochastic Burgers-type equations

$$\begin{cases} df(t) = [-A^{\frac{\alpha}{2}} f(t) + G(f(t))]dt + dB^h(t), & t \in [0,T] \\ f(0) = f_0, \end{cases} \tag{5.1}$$

where $T > 0$, $A = -\Delta$ is the Laplacian with Dirichlet boundary conditions. The operator G is defined as $G(g(t,x)) = \partial_x g(t,x),\ \ x \in (0,1)$ and the $B^h(t)$ is the cylindrical fractional Brownian motion for all $t \in [0,T]$. Fractional Burgers equations are used in several model related to anomalous diffusions, for examples, diffusion in complex, propagation of acoustic waves in gas-filled tube [101, 132] and references therein. Recently, a similar type of problem (equation (5.1)) with Wiener process has discussed by Z. Arab and L. Debbi [4], where they have generalized the work of Blömker and Jentzen [24] by considering more general nonlinearity and the fractional Laplacian in place of standard Laplacian. The works done in [24] and [4] have motivated us to study the fractional Burgers-type equations driven by cylindrical fractional Brownian motions. Now, we recall an important result, which is the base of our work.

Theorem 5.1.1. *[24, Theorem 3.1] Let $T \in (0,\infty)$ and $(\Omega,\mathscr{F},P)$ be some given probability space. Consider U,V be as two $\mathbb{R}$-Banach spaces and let $P_N : V \to V$ be a sequence of linear bounded operators.*

Let $\mathscr{L}(U,V)$ represents the space of bounded linear operators on U and V. Consider that the following assumptions are true:

- ***Assumption** 1. Let $\eta \in [0,1)$ and $\gamma \in (0,\infty)$ are two real valued constants and let $S : (0,T] \to \mathscr{L}(U,V)$ be a strongly continuous map satisfying*

$$\sup_{t\in(0,T]} \left(t^{\eta} \left|S(t)\right|_{\mathscr{L}(U,V)} \right) < \infty, \tag{5.2}$$

$$\sup_{N\in\mathbb{N}} \sup_{t\in(0,T]} \left(N^{\gamma} t^{\eta} \left|S(t) - P_N S(t)\right|_{\mathscr{L}(U,V)} \right) < \infty. \tag{5.3}$$

- ***Assumption** 2. Let $G : V \to U$ be a mapping which satisfies*

$$\sup_{|f|_V,|g|_V \le r, f\neq g} \frac{|G(f) - G(g)|_U}{|f - g|_V} < \infty \tag{5.4}$$

  for every $r \in (0,\infty)$.

- ***Assumption*** 3. *Let* $O : [0,T] \times \Omega \to V$ *be a stochastic process with continuous simple paths and*

$$\sup_{N\in\mathbb{N}} \sup_{t\in(0,T]} (N^{\gamma}|(1-P_N)O_t(\omega)|_V) < \infty, \text{ for every } \omega, \tag{5.5}$$

where $\gamma \in (0,\infty)$ *is given in Assumption* 1.

- ***Assumption*** 4. *Let* $X^N : [0,T] \times \Omega \to V, \ \ N \in \mathbb{N}$ *be a sequence of stochastic processes with continuous sample paths and with*

$$\sup_{N\in\mathbb{N}} \sup_{t\in(0,T]} \left(\left|X_t^N(\omega)\right|_V\right) < \infty, \tag{5.6}$$

$$X_t^N(\omega) = \int_0^t P_N S(t-s) G(X_s^N(\omega))ds + P_N(O_t(\omega)), \tag{5.7}$$

for every $\omega \in \Omega, \ \ t \in [0,T]$ *and every* $N \in \mathbb{N}$.

Then there exists a unique stochastic process $X : [0,T] \times \Omega \to V$, *with continuous sample paths such that*

$$X_t(\omega) = \int_0^t S(t-s)G(X_s(\omega))ds + O_t(\omega), \tag{5.8}$$

for every $\omega \in \Omega, \ \ t \in [0,T]$. *Moreover, there exists an* $\mathscr{F}/\mathscr{B}([0,\infty))$*-measurable mapping* $C : \Omega \to [0,\infty)$, *such that*

$$\sup_{t\in(0,T]} \left|X_t(\omega) - X_t^N(\omega)\right|_V \le C(\omega)N^{-\gamma}, \tag{5.9}$$

holds for every $N \in \mathbb{N}$ *and every* $\omega \in \Omega$, *where* γ *is given in Assumption* 1.

In [4], authors have proved the Theorem 5.1.1 for their problem by satisfying all the Assumptions $1-4$. In order to prove Theorem 5.1.1 for problem (5.1), Assumption 1 and Assumption 2 will hold in the same manner as in [4]. Therefore, to prove Theorem 5.1.1, we need to only show the validation of Assumptions 3 and 4 for equation (5.1).

5.2 Preliminaries

5.2.1 Functions spaces and important inequalities

Let X,Y be two Banach spaces, we denote the notation $|\cdot|_X$ to express the norm on X and $\mathscr{L}(X,Y)$ denotes the space of linear bounded operators defined on X and Y having norm $|\cdot$

$|_{\mathscr{L}(X,Y)}$. The other functions spaces, we have mainly used in this chapter, are following.

Space of continuous functions: We define the space of continuous functions as

$$C := \{\phi : \phi \text{ is a bounded and continuous function s.t. } |\phi|_C := \sup_{x\in\mathbb{R}} |\phi(x)| < \infty\}. \tag{5.10}$$

Hölder Spaces: For $\delta \in (0,1)$, we define

$$C^\delta := \left\{ \phi \in C, \text{ such that } |f|_{C^\delta} := |\phi|_C + \sup_{x,y\in\mathbb{R}, x\neq y} \frac{|\phi(x)-\phi(y)|}{|x-y|^\delta} < \infty \right\}. \tag{5.11}$$

Sobolev Spaces: For $1 \leq p < \infty$ and $m \in \mathbb{N}$,

$$W_p^m := \left\{ \phi \in L^p, \text{ such that } |\phi|^p_{W_p^m} := \sum_{k=0}^m |D^k\phi|^p_{L^p} \right\} < \infty, \tag{5.12}$$

where $D^k\phi$ represents the derivative of ϕ of order k in the distributional sense.

Fractional Sobolev Spaces: Let $1 \leq p < \infty$, and $s > 0$ be a non-integer. Then, we define

$$W_s^p := \left\{ \phi \in W_p^{[s]}, \text{ such that } |f|^p_{W_p^{[s]}} + \sum_{k=0}^{[s]} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{|D^k\phi(x) - D^k\phi(y)|^p}{|x-y|^{1+\{s\}p}} dxdy < \infty \right\}, \tag{5.13}$$

where $[s]$ is integer part of s and $\{s\}$ is fractional part of s.

Note: For the case $p = 2$, the space W_s^2 is a Hilbert space and we denote it by H_2^s.

Lemma 5.2.1. *For every $\gamma > 0$ there exists $C_\gamma > 0$, such that $x^\gamma e^{-x} \leq C_\gamma$.*

5.2.2 Some properties on the operator A

Let us denote A_q as a part of the operator A on Lebesgue space $L^q(0,1)$, $1 < q < \infty$, but for Hilbert space $L^2(0,1)$, we will denote A_q by A. Let $(e^{-A_qt})_{t\geq 0}$ is a semigroup generated by A_q. Then A_q, is densely defined, has a bounded inverse and further the semigroup $(e^{-A_qt})_{t\geq 0}$ is analytic on $L^q(0,1)$ for any $q \geq 2$, see [133]. We define the fractional power of $A_q^\beta, \beta \in \mathbb{R}$ as the following:

Definition 5.2.1. The fractional power A_q^β, of A_q, is defined as the inverse of

$$A_q^{-\beta} := \frac{1}{\Gamma(\beta)} \int_0^\infty s^{\beta-1} e^{-sA_q} ds. \tag{5.14}$$

Here the integral, on the right hand side of (5.14), is Dunford integral and it converges in the uniform topology.

Recall that $A : D(A) \to L^2(0,1)$ is an isomorphism, and further A^{-1}, the inverse of A is self adjoint, where $D(A)$ denotes the domain of A [133, page 283 and 303]. Further, the compactness of $D(A)$ on $L^2(0,1)$ implies that A^{-1} is compact on $L^2(0,1)$. Therefore, there exists an orthonormal basis $(e_j)_{j\in\mathbb{N}} \subset D(A)$ corresponding to eigenvalues $(\lambda_j)_{j\in\mathbb{N}}$ of A, which also consists the eigenfunctions of A^{-1}. Now, from [59], we have following results:

Lemma 5.2.2. *Let the operator A is positive, self adjoint and dense in $L^2(0,1)$. Applying the spectral decomposition, we introduce positive and negative fractional powers $A^{\beta/2}, \beta \in \mathbb{R}$. In particular, if the spectrum of A is reduced to the discrete one, we get a classic expression for $(A^{\beta/2}, D(A^{\beta/2}))$. Moreover, let $\beta \geq 0$, then*

$$D(A^{\beta/2}) = \{\phi \in L^2(0,1) : |\phi|_{D(A^{\beta/2})} := \sum_{j\in\mathbb{N}} \lambda_j^{\beta} \langle \phi, e_j \rangle^2 < \infty\},$$

$$A^{\beta/2}\phi = \sum_{j\in\mathbb{N}} \lambda_j^{\beta/2} \langle \phi, e_j \rangle e_j, \quad \forall\, \phi \in D(A^{\beta/2}).$$

The operator $A_q^{\alpha/2}$ is an infinitesimal generator and it generate an analytic semigroup $(e^{-A_q^{\alpha/2}t})_{t\geq 0}$ on $L^q(0,1)$. Further, for any $\beta \geq 0$, we have

$$\left| A_q^{\beta/2} e^{-A_q^{\alpha/2}t} \right|_{\mathscr{L}(L^q)} \leq ct^{-\frac{\beta}{\alpha}}$$

and

$$(e^{-A^{\alpha/2}t}\phi)(x) := \sum_{k=1}^{\infty} e^{\lambda_k^{\alpha/2}t} \langle \phi, e_k \rangle e_k(x), \;\; \forall \phi \in L^2(0,1).$$

5.2.3 Definition and assumptions on the operator G

Let us define the spaces $V := C^{\delta}(0,1)$, for some $\delta \in (0,1)$ and $U := H_2^{-\alpha/2}(0,1)$. Now, let us assume that there exists an operator $G : V \to U$ such that it satisfies the Assumption 2 of Theorem 5.1.1. In this work, we consider the following example of such an operator G :

$$G(f)(x) = \frac{dg(f(x))}{dx}, \;\; \text{where} \;\; g : H_2^{1-\alpha/2}(0,1) \to C^{\delta}(0,1). \tag{5.15}$$

In this chapter our g in (5.15) is defined as

$$g(x) := a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n, \tag{5.16}$$

where $a_0, a_1, \cdots, a_n \neq 0 \in \mathbb{R}, n \in \mathbb{N} \cup \{0\}$. Here for $g(x) = a_2 x^2$, with $a_n = 0$ for all $n \in \mathbb{N} \cup \{0\} - \{2\}$, the operator G attains the quadratic nonlinearity and then equation (5.1) gives the fractional stochastic Burgers equation. Further, for $g(x) = a_p x^p$, with $a_n = 0$ for all $n \in \mathbb{N} \cup \{0\} - \{p\}$, $p \geq 2$, the equation (5.1) becomes generalized stochastic Burgers equations which has been studied in previous chapters.

Lemma 5.2.3. *[4, Lemma 2.10] Let $1 < \alpha \leq 2, \delta \in (1 - \frac{\alpha}{2}, 1)$ and G is defined as in (5.15). Then the mapping*

$$G : C^\delta(0,1) \to H_2^{-\alpha/2}(0,1)$$

with

$$f \mapsto G(f) = \frac{dg(f(x))}{dx},$$

is well defined. Further, let there exists positive constant K_R for all $R > 0$, such that for any $\phi, \psi \in C^\delta(0,1)$, with $|\phi|_{C^\delta(0,1)}, |\psi|_{C^\delta(0,1)} < R$, we have following inequality

$$|G(\phi) - G(\psi)|_{H_2^{-\alpha/2}(0,1)} \leq K_R |\phi - \psi|_{C^\delta(0,1)}. \tag{5.17}$$

In the above lemma, putting $\psi = 0$, we get

$$|G(\phi)|_{H_2^{-\alpha/2}(0,1)} \leq K_R (1 + |\phi|_{C^\delta(0,1)}). \tag{5.18}$$

for any $R > 0$ and ϕ with $|\phi|_{C^\delta(0,1)} < R$.
Now, let $V := C^\delta(0,1)$, and $U := H_2^{-\alpha/2}(0,1)$, G is given by (5.15) for any polynomial function g. Then, due to Lemma 5.2.3, we have following results:

Corollary 5.2.4. *Let $1 < \alpha \leq 2$, and $\delta \in (1 - \frac{\alpha}{2}, 1)$. Then by Lemma 5.2.3, Assumption 2 of Theorem 5.1.1 is satisfied for any such α and δ.*

Corollary 5.2.5. *Let $\alpha \in \left(\frac{3}{2}, 2\right], \delta \in \left(1 - \frac{\alpha}{2}, \frac{\alpha - 1}{2}\right)$. Then by [4, Lemma 2.4] and Lemma 5.2.3, Assumption 1 and Assumption 2 are simultaneously satisfied.*

5.2.4 Definition of stochastic noise

Let $\left(\Omega, \mathscr{F}, P, \mathscr{F}_{t \in [0,T]}\right)$ be a stochastic basis. Let h be Hurst index lies in $(0,1)$. A *fractional Brownian motion (fBm)* $(W^h(t))_{t \geq 0}$ with index h is a continuous and centered Gaussian process with covariance function

$$R(t,s) = \mathbb{E}\left[W^h(t)W^h(s)\right] = \frac{1}{2}\left(t^{2h} + s^{2h} - |t-s|^{2h}\right), \tag{5.19}$$

for every $s,t \geq 0$.

From the definition, it can be noticed that the standard Brownian motion is a fractional Brownian motion with the Hurst index $h = \frac{1}{2}$. More details can be found in Chapter 4. Now, following the standard approach in [50], we define the cylindrical fractional Brownian motion $B^h(t)$ as the sum of an infinite series:

$$B^h(t) := \sum_{k=1}^{\infty} W_k^h(t)e_k, \tag{5.20}$$

P-a.s., where $\left(W_k^h\right)$ are real independent fBms and $\left(e_k(\cdot) = \sqrt{2}\sin k\pi(\cdot)\right)$ is an orthonormal basis in the space $L^2(0,1)$. Again following the [50], we introduce the following Ornstein-Uhlenbeck stochastic process

$$Z(t) := \int_0^t e^{-A^{\frac{\alpha}{2}}(t-s)} B^h(ds). \tag{5.21}$$

Then by following [142, 147], it can be shown that $Z(t)$ is well defined for $\alpha > 1$. Thanks to [113], we have the following embedding results for the fractional Brownian motion W^h for any $h \in (\frac{1}{2}, 1)$.

Theorem 5.2.6. *Let W^h be the fractional Brownian motion with $h > 1/2$, then for every $t \in [0.T]$ and for every $r > 0$ there exists a constant $C(h,r)$ such that*

$$\mathbb{E}\left(\int_0^t f(s)dW^h(s)\right)^r \leq C(h,r)\|f\|^r_{L^{\frac{1}{h}}(0,t)}. \tag{5.22}$$

5.2.5 Galerkin Approximations

Definition 5.2.2. Let us fix $N \geq 1$. Then denote, by P_N, the Galerkin projection on the finite dimensional space H_N generated by the first N eigenvectors $(e_k)_{k=1}^N$ i.e. for $\delta > 0$, $\phi \in C^\delta(0,1) \subset L^2(0,1)$ and for all $x \in [0,1]$,

$$P_N\phi(x) = \sum_{k=1}^{N} \langle \phi, e_k \rangle e_k(x). \tag{5.23}$$

Now, we state the following result based on the above definition which is taken from [4].

Lemma 5.2.7. *[4, Lemma 2.14]*

- *P_N and semigroup $e^{-tA^{\alpha/2}}$ commute to each other.*
- *Let $\delta \in [0,1)$ and $\eta > \delta + \frac{1}{2}$, then there exists $C_{\delta,\eta} > 0$ such that*

$$\|P_N\|_{\mathscr{L}(H_2^\eta, C^\delta)} \leq C_{\delta,\eta}.$$

- *Let $\beta \leq \gamma \in \mathbb{R}$, then there exists $C_{\gamma,\beta} > 0$, such that*

$$\|1 - P_N\|_{\mathscr{L}(H_2^\gamma, H_2^\beta)} \leq C_{\gamma,\beta} N^{-(\gamma-\beta)}.$$

- *Let $\alpha \in (1,2]$ and $\delta \in [0, \frac{\alpha-1}{2})$, then there exists $C_{\alpha,\delta} > 0$, s.t.*

$$N^{\left(\frac{\alpha-1}{2}-\delta\right)} \|(1-P_N)e^{-A^{\alpha/2}t}\|_{\mathscr{L}(H_2^{\alpha/2}, C^\delta)} \leq C_{\alpha,\delta}.$$

- *Let $\beta \in \mathbb{R}$, then there exists $C_\beta > 0$ such that*

$$\|P_N\|_{\mathscr{L}(H_2^\beta)} \leq C_\beta. \tag{5.24}$$

Applying Definition 5.2.2, we discretize the equation (5.1) in the following form

$$\begin{cases} df_N(t) = [-A^{\alpha/2} f_N(t) + P_N G(f_N(t))]dt + dB_N^h(t), \quad t \in [0,T], \\ f(0) = P_N f_0, \end{cases} \tag{5.25}$$

where

$$B_N^h(t) = \sum_{k=1}^{N} W_k^h(t) e_k. \tag{5.26}$$

Fully Discretization: Here we discretize the equation (5.1) with respect to space and time. For this, let us fix $M \geq 1$ and discretizing the equation (5.25) into the uniform step subdivision on time interval $[0,T]$, with step size $\triangle t = \frac{T}{M}$ and then we define $t_m = m \triangle t$, for $m = 1, 2, \ldots, M$. By this construction, we get the sequence of random variables $\left(f_{N,M}^m\right)_{m=0}^{M}$, which satisfies the following equation

$$\begin{cases} f_{N,M}^0 = P_N f_0, \\ f_{N,M}^{m+1} := e^{-A^{\alpha/2} T/M} \left(f_{N,M}^m + \frac{T}{M} (P_N F)(f_{N,M}^m) \right) \\ \qquad\qquad + P_N \left(Z((m+1)\frac{T}{M}) - e^{-A^{\alpha/2} T/M} Z((m)\frac{T}{M}) \right). \end{cases} \tag{5.27}$$

5.3 Definitions of Solutions and Main Results

Definition 5.3.1. Let E be UMD Banach space of type-2 and H be a Hilbert space. Let us assume that the initial data f_0 such that $f_0 : \Omega \to E$ be $\mathscr{F}_0$ measurable. Then a process $\{f(t) : t \in [0,T]\}$, which is $\mathscr{F}_t$ adapted and E-valued, is said to be a mild solution to the equation (5.1) if

1. for all $t \in [0,T], s \mapsto e^{-(t-s)A^{\frac{\alpha}{2}}} G(f(s))$ is in $L^0(\Omega, L^1([0,T];E))$ (a complete metric space),

2. for all $t \in [0,T], s \mapsto e^{-(t-s)A^{\frac{\alpha}{2}}} I$ is H-strongly measurable and adapted and in γ-Radonifying space; $R(H,E)$,

3. for all $t \in [0,T]$, almost surely

$$f(t) = e^{-A^{\alpha/2}t} f_0 + \int_0^t e^{-A^{\alpha/2}(t-s)} G(f(s))ds + \int_0^t e^{-A^{\alpha/2}(t-s)} dB^h(s). \tag{5.28}$$

Definition 5.3.2. Equation (5.1) is said to have pathwise unique solution if for any two solutions $\{f^1(t) : t \in [0,T]\}$ and $\{f^2(t) : t \in [0,T]\}$ to equation (5.1) having same initial data f_0, we have

$$P(f^1(t) = f^2(t) \text{ for all } t \in [0,T]) = 1. \tag{5.29}$$

Hypothesis $\mathscr{H}$: For $\delta \in [0,1)$ and $\beta > \delta + 1/2$, we have $f_0 : \Omega \to H_2^\beta(0,1)$ is a $\mathscr{F}_0$ measurable random variable.

Remark: In [4], four main results [4, Theorem 3.3- Theorem 3.6] are shown. We also have a target to prove the same types of results for our problem. The result [4, Theorem 3.3] also holds for f_N given by equation (5.25) with some Hölder continuity less than $1/2$. Therefore, we need to mainly prove here the [4, Theorem 3.4- Theorem 3.6] for the equation (5.1). Let us first give the result for existence of solution to (5.25).

Theorem 5.3.1. *Let $T > 0$ and f_0 satisfies the Hypothesis $\mathscr{H}$. Then*

1. *for $\alpha \in \left(\frac{3}{2}, 2\right), \delta \in \left(1 - \frac{\alpha}{2}, \frac{\alpha-2}{2}\right)$ equation (5.25) has a unique L^2-mild solution $f_N := (f_N(t), t \in [0,T])$ satisfies $\sup_N \sup_{t\in[0,T]} |f_N(t)|_{L^2} < \infty$.*

2. *for $\frac{7}{4} < \alpha < 2$ and $\delta \in \left(1 - \frac{\alpha}{2}, \frac{\alpha-2}{2}\right)$, then f_N is Hölder continuous of index $\left(\left(\frac{\alpha-1-2\delta}{2\alpha}\right) \vee \left(h\left(\frac{2\alpha h-1-2\delta}{\alpha}\right)\right)\right)-$.*

Now, let $T>0$ be finite, $\alpha \in (\frac{7}{4},2), \delta \in (1-\frac{\alpha}{2}, \frac{2\alpha-3}{2})$ and f_0 satisfy the Hypothesis $\mathscr{H}$. Then, the main results of this chapter are stated as follows:

Theorem 5.3.2. *The equation (5.1) with initial condition f_0, has a unique mild solution $f: [0,T]\times\Omega \to C^{\delta}(0,1)$. Further it is almost surely satisfy Hölder continuity of order $\frac{1}{2}-$.*

Theorem 5.3.3. *There exists a $\mathscr{F}/\mathscr{B}([0,\infty))$-measurable mapping $C:\Omega\to\mathbb{R}$ such that*

$$\sup_{t\in[0,T]} |f(t)-f_N(t)|_{C^{\delta}} \le C(\omega) N^{-\left(\frac{\alpha-1}{2}-\delta\right)-} \quad a.s., \tag{5.30}$$

where f is the solution to equation (5.1) with initial data f_0 and f_N is the solution of Galerkin approximations obtained in equation (5.25).

Theorem 5.3.4. *There exists a $\mathscr{F}/\mathscr{B}([0,\infty))$-measurable mapping $C:\Omega\to\mathbb{R}$ such that*

$$\sup_{t_m\in[0,T]} |f(t_m)-f^m_{N,M}|_{C^{\delta}} \le C(\omega)\left((\triangle t)^{\frac{\alpha-1-2\delta}{2\alpha}} + N^{-\left(\frac{\alpha-1}{2}-\delta\right)-}\right) \quad a.s., \tag{5.31}$$

for every $N,M\ge 1$.

5.4 Some Estimates on Stochastic Terms

Let us define the Galerkin projection of Orenstein-Uhlenbeck process $Z(t)$ by using equation (5.21) and equation (5.26) as

$$Z_N(t) := P_N Z(t) = \int_0^t e^{-A^{\alpha/2}(t-s)} B_N^h(ds) = \sum_{k=1}^{N} \int_0^t e^{-A^{\alpha/2}(t-s)} dW_N^h(s) e_k \tag{5.32}$$

Now, we extend the result [4, Lemma 4.1] for $Z_N(t)$ defined in (5.32) by stating the following Lemma

Lemma 5.4.1. *Let $\alpha\in(1,2], 0<\beta<\frac{\alpha-1}{2}, q\ge 2$ and let $p_0 > \frac{2\alpha}{2\alpha h-1-2\beta}$ be fixed. Then for $p\ge 1$ there exists a constant $C_{\alpha,\beta,q,p,h}>0$ such that*

$$\sup_{N\in\mathbb{N}} \mathbb{E}\left[|Z_N|^p_{C_tH^{\beta}_q} + N^{p\left(\frac{2\alpha h-1}{2}-\left(\beta+\frac{\alpha}{p_0}\right)\right)-} |(1-P_N)Z|^p_{C_tH^{\beta}_q} + |Z|^p_{C_tH^{\beta}_q}\right] < C_{\alpha,\beta,q,p,h}. \tag{5.33}$$

Proof. In order to prove Lemma 5.4.1, it is sufficient to show the middle term on the left hand side of the inequality (5.33) is bounded, as other terms can be handled analogously. This can be shown in following two steps. In first step, we show that for $\alpha \in (1,2], 0<\beta<\frac{\alpha-1}{2}, q\geq 2$ and let $p_0 > \frac{2\alpha}{2\alpha h-1-2\beta}$ there exist $C_{\alpha,\beta,q,p,h}>0$ and $\xi_p \in (0, 2\alpha\gamma h-1)$ such that

$$\sup_{N\in\mathbb{N}} \mathbb{E}[N^{\xi_p p}|(1-P_N)Z|^p_{C_t H^\beta_q}] < C_{\alpha,\beta,q,p,h}, \tag{5.34}$$

and then in second step it is proved that the estimate (5.34) holds for every $p\geq 1$.

Step 1. By Definitions 5.21 and 5.23, we have

$$(1-P_N)Z(t) = \int_0^t e^{A^{\alpha/2}(t-s)}(1-P_N)B^h(ds). \tag{5.35}$$

Using the factorization method [50, page 129-130], we represent $(1-P_N)Z$ as

$$(1-P_N)Z(t) = \int_0^t (t-s)^{\nu-1} e^{A^{\alpha/2}(t-s)} Y^N(s)ds, \tag{5.36}$$

$$Y^N(t) = \int_0^t (s-\sigma)^{-\nu} e^{A^{\alpha/2}(s-\sigma)} B^h(d\sigma), \tag{5.37}$$

where $\nu\in(0,1)$. Thanks to [4, Lemma 4.1], following the proof of this lemma, for any $\nu\in(0,1)$ such that $\frac{1}{p}+\frac{\beta}{\alpha}<\nu<1$, we get

$$\mathbb{E}|(1-P_N)Z(t)|^p_{C_t H^q_\beta} \leq C T^{(1+(\nu-1-\beta/\alpha)\frac{p}{p-1})(p-1)} \mathbb{E}\int_0^T |Y^N(s)|^p_{L^q} ds. \tag{5.38}$$

Now, using Theorem 5.2.6, the estimate $|e_k|_{L^q}\leq 1$ and the fact that $\{W^h_k\}$ are independent and centered Gaussian variables, we have

$$\begin{aligned}
\mathbb{E}\int_0^T |Y^N(s)|^p_{L^q} =& \mathbb{E}\int_0^T \left|\int_0^t (s-\sigma)^{-\nu} e^{A^{\alpha/2}(s-\sigma)} B^h(d\sigma)\right|^p_{L^q} ds \\
&= \mathbb{E}\int_0^T \Big|\int_0^s (s-\sigma)^{-\nu}\left(\sum_{k=N+1}^\infty e^{-\lambda_k^{\alpha/2}(s-\sigma)} dW^h(\sigma)\right) e_k(x)\Big|^p_{L^q} ds \\
&\leq \mathbb{E}\int_0^T \left(\int_0^s \sum_{k=N+1}^\infty (s-\sigma)^{-\nu} e^{-\lambda_k^{\alpha/2}(s-\sigma)} dW^h(\sigma)\right)^p ds \\
&\leq C(h)\int_0^T \left(\sum_{k=N+1}^\infty \left|(s-\cdot)^{-\nu} e^{-\lambda_k^{\alpha/2}(s-\cdot)}\right|^2_{L^{1/h}(0,s)}\right)^{p/2}.
\end{aligned} \tag{5.39}$$

Next, by using the fact that for every $\gamma>0$, there exist $C_\gamma>0$ such that $x^\gamma e^{-x}\leq C_\gamma$, we estimate

$$\begin{aligned}
\left|(s-\cdot)^{-\nu} e^{-\lambda_k^{\alpha/2}(s-\cdot)}\right|^2_{L^{1/h}(0,s)} &= \left(\int_0^s (s-\sigma)^{-\nu/h} e_k^{\frac{-\lambda^{\alpha/2}(s-\sigma)}{h}} d\sigma\right)^{2h} \\
&\leq C(h)\left(\int_0^s (s-\sigma)^{-\nu/h}\lambda_k^{-\gamma\alpha/2}(s-\sigma)^{-\gamma} d\sigma\right)^{2h} \\
&= C(h)\lambda_k^{-\gamma\alpha h}\left(\int_0^s (s-\sigma)^{-\nu/h-\gamma} d\sigma\right)^{2h}.
\end{aligned} \tag{5.40}$$

Therefore, by substituting (5.40) into (5.39), for $\frac{1}{2\alpha h} < \gamma < (1-\frac{\nu}{h})$, we have

$$\begin{aligned}
\mathbb{E}\int_0^T |Y^N(s)|_{L^q}^p \le C(h)\int_0^T \left(\sum_{k=N+1}^{\infty} \lambda_k^{-\gamma\alpha h}\left(\int_0^s (s-\sigma)^{-\nu/h-\gamma} d\sigma\right)^{2h}\right)^{p/2} ds \\
= C(h)\int_0^T \left(\sum_{k=N+1}^{\infty} \lambda_k^{-\gamma\alpha h} s^{(1-\nu/h-\gamma)2h}\right)^{p/2} ds \\
= C(h,\alpha,\gamma)\left(\int_0^T s^{(1-\nu/h-\gamma)pH} ds\right)\left(\sum_{k=N+1}^{\infty} k^{-2\gamma\alpha h}\right)^{p/2}. \qquad (5.41)
\end{aligned}$$

Now, let $\xi \in (0, 2\alpha\gamma h - 1)$. Then we can write

$$\sum_{k=N+1}^{\infty} k^{-2\gamma\alpha h} \le N^{-\xi} \sum_{k=N+1}^{\infty} k^{-2\gamma\alpha h+\xi}. \qquad (5.42)$$

Using this inequality in (5.41), we obtain

$$\begin{aligned}
\mathbb{E}\int_0^T |Y^N(s)|_{L^q}^p \le \quad C(h,\alpha,\gamma,\beta) T^{(1-\nu/h-\gamma)pH+1} N^{-\xi}\left(\sum_{k=N+1}^{\infty} k^{-2\gamma\alpha h+\xi}\right)^{p/2} \\
\le C(h,\alpha,\gamma,\beta,T) N^{-\xi p/2}. \qquad (5.43)
\end{aligned}$$

Now combining the estimates (5.38) and (5.43), and assuming that $\frac{1}{p}+\frac{\beta}{\alpha} < \nu < h - \frac{1}{2\alpha}$, we have

$$\mathbb{E}|(1-P_N)Z(t)|_{C_t H_\beta^q}^p \le C(h,\alpha,\gamma,\beta,T) N^{-\xi p/2}. \qquad (5.44)$$

Step 2: Now following the proof of [4, Lemma 4.1], the estimate (5.34) is true for any $p \ge 1$. The other terms in the estimate (5.33), can be estimated in a similar manner. This completes the proof of the Lemma 5.4.1. □

Remark 5.4.1. The result obtained in Lemma 5.4.1 gives the same result [4, Lemma 4.1] by putting $h = \frac{1}{2}$ in (5.33) i.e. the result in the case of Wiener process.

Corollary 5.4.2. *Let $\alpha \in (1,2], 0 < \delta < \frac{\alpha-1}{2}$ and $p_0 > \frac{2\alpha}{2\alpha h - 1 - 2\beta}$. Then for $p \ge 1$, there exists $C_{\alpha,\delta,p} > 0$, such that*

$$\sup_{N\in\mathbb{N}} \mathbb{E}\left[|Z_N|_{C_t C^\delta}^p + N^{p\left(\frac{2\alpha h-1}{2}-\left(\delta+\frac{\alpha+1}{p_0}\right)\right)-}|(1-P_N)Z|_{C_t C^\delta}^p + |Z|_{C_t C^\delta}^q\right] < C_{\alpha,\delta,h,p}. \qquad (5.45)$$

Proof. let $\delta < \frac{\alpha-1}{2}, p_0 > \frac{2\alpha}{2\alpha h-1-2\beta}$ and $q = p$. Then the estimate (5.33) is also true for $\beta = \delta + \frac{1}{p_0}$. Putting these values in (5.33), and using the embedding $h_{p_0}^{\delta+\frac{1}{p_0}} \hookrightarrow C^\delta$, we obtain the estimate (5.45). □

Corollary 5.4.3. *Let $\alpha \in (1,2]$ and $0 < \delta < \frac{\alpha-1}{2}$. Then there exists a finite positive random variable $C_{\alpha,\delta,h}$ such that*

$$\sup_{N\in\mathbb{N}}[N^{\left(\frac{2\alpha h-1}{2}-\delta\right)-}|(1-P_N)Z|_{C_tC^\delta}] < C_{\alpha,\delta,h}(\omega) \quad a.s.. \tag{5.46}$$

Proof. Fractional Brownian motions are Gaussian processes and for Gaussian processes we have following result.

Let $\tau > 0, (C_p)_{p\geq 1} \subset [0,\infty)$ and let $(W_n)_{n\in\mathbb{N}}$ be a sequence of random variables such that

$$(\mathbb{E}|Z_n|^p)^{\frac{1}{p}} \leq C_p n^{-\tau}, \tag{5.47}$$

for every $p \geq 1$ and all $n \in \mathbb{N}$. Then

$$P\left(\sup_{n\in\mathbb{N}}(n^{\tau-\varepsilon}|Z_n|) < \infty\right) = 1 \tag{5.48}$$

for every $\varepsilon \in (0,\tau)$.

Using result (5.47) and Lemma (5.4.1), we obtain (5.46). □

Lemma 5.4.4. *Let $\alpha \in (1,2]$ and $0 < \delta < \frac{\alpha-1}{2}$. Then there exists a finite positive random variable $C_{\alpha,\delta,h}$, such that*

$$\sup_{N\in\mathbb{N}}\sup_{t\in[0,T]}|Z_N(t,\omega)|_{C^\delta} < C_{\alpha,\delta,h}(\omega). \tag{5.49}$$

Proof. Using the fact that for any $\delta \in (0,1]$, there exists a constant $l_\delta > 0$ (independent of k) such that $|e_k|_{C^\delta} \leq l_\delta k^\delta$, we calculate

$$\begin{aligned}
|Z_N(t,\omega)|_{C^\delta} &= \left|\sum_{k=1}^{N}\left(\int_0^t e^{(t-s)}\lambda_k^{\alpha/2}dW^h(s)\right)(\omega)e_k(x)\right|_{C^\delta} \\
&\leq l_\delta \sum_{k=1}^{N}\left(\int_0^t (k\pi)^\delta e^{(t-s)}(k\pi)^\alpha dW^h(s)\right)(\omega) \\
&= C_{\alpha,\delta,h}(t,\omega,h),
\end{aligned} \tag{5.50}$$

where $C_{\alpha,\delta}(t,\omega,h)$ is defined as

$$C_{\alpha,\delta,h}(t,\omega) := \sum_{k=1}^{N}\left(\int_0^t (k\pi)^\delta e^{(t-s)}(k\pi)^\alpha dW^h(s)\right)(\omega).$$

Here the process $C_{\alpha,\delta,h}(t,\omega)$ is well defined as there exists $\gamma > 0$ in Lemma 5.2.1 which satisfies $\frac{1}{\alpha}\left(1+\frac{\delta}{h}\right) < \gamma < 1$ such that $\sum_{k=1}^{N}\left(\int_0^t (k\pi)^{\delta/h} e^{(t-s)(k\pi)^{\alpha}/h} ds\right)^{2h} < \infty$. Moreover, $C_{\alpha,\delta,h}(\cdot,\omega)$ has continuous trajectories on [0,T]. Therefore, we get the random variable:

$$C_{\alpha,\delta,h}(\omega) := \sup_{t\in[0,T]} C_{\alpha,\delta}(t,\omega) \tag{5.51}$$

which is positive and finite and we have,

$$\sup_{t\in[0,T]} |Z_N(t,\omega)|_{C^{\delta}} \le C_{\alpha,\delta,h}(\omega). \tag{5.52}$$

□

Lemma 5.4.5. *Let $W^h : [0,T]\times\Omega \to \mathbb{R}$ be a fractional Brownian motion with $h \ge 1/2$. Then for every $\tau \in [0,1], \lambda \in (0,\infty)$ and $t_1, t_2 \in [0,T]$, we have*

$$\mathbb{E}\left(\int_0^{t_2} e^{-\lambda(t_2-s)} dW^h(s) - \int_0^{t_1} e^{-\lambda(t_1-s)} dW^h(s)\right) \\ \le C(h,t)\lambda^{-2(h-\tau)}\{|t_2-t_1|^{2\tau} \vee |t_2-t_1|^{2h\tau}\}. \tag{5.53}$$

Proof. Without loss of generality, let $0 \le t_1 \le t_2$. Then, we have

$$\begin{aligned} &\mathbb{E}\left(\int_0^{t_2} e^{-\lambda(t_2-s)} dW^h(s) - \int_0^{t_1} e^{-\lambda(t_1-s)} dW^h(s)\right)^2 \\ &= \mathbb{E}\left(\int_0^{t_1}\left(e^{-\lambda(t_2-s)} - e^{-\lambda(t_1-s)}\right) dW^h(s) + \int_{t_1}^{t_2} e^{-\lambda(t_2-s)} dW^h(s)\right)^2 \\ &\le \mathbb{E}\left(\int_0^{t_1}\left(e^{-\lambda(t_2-s)} - e^{-\lambda(t_1-s)}\right) dW^h(s)\right)^2 + \mathbb{E}\left(\int_{t_1}^{t_2} e^{-\lambda(t_2-s)} dW^h(s)\right)^2 \\ &=: I_1 + I_2, \end{aligned} \tag{5.54}$$

where the second last inequality in (5.54) is obtained from the fact that fractional Brownian motions are independent centered Gaussian processes. Now, by using Theorem 5.2.6, I_1 is estimated as

$$\begin{aligned} I_1 &= \mathbb{E}\left(\int_0^{t_1}\left(e^{-\lambda(t_2-s)} - e^{-\lambda(t_1-s)}\right) dW^h(s)\right)^2 \\ &\le C(h)\left(\int_0^{t_1}\left(e^{-\lambda(t_2-s)} - e^{-\lambda(t_1-s)}\right)^{\frac{1}{h}} dW^h(s)\right)^{2h} \\ &= C(h)\left|e^{-\lambda(t_2-t_1)} - 1\right|^2 \left(\int_0^{t_1} e^{-\frac{\lambda}{h}(t_1-s)} ds\right)^{2h} \\ &\le C(h)\lambda^{2\tau}|t_2-t_1|^{2\tau} \times \lambda^{-2h}\left(1 - e^{-\frac{\lambda}{h}(t_1)}\right)^{2h} \\ &\le C(h,t)\lambda^{-2(h-\tau)}|t_2-t_1|^{2\tau}. \end{aligned} \tag{5.55}$$

Similarly, we can estimate I_2 as

$$I_2 = \mathbb{E}\left(\int_{t_1}^{t_2} e^{-\lambda(t_2-s)} dW^h(s)\right)^2 \leq C(h)\left(\int_{t_1}^{t_2} e^{-\frac{\lambda}{h}(t_2-s)} ds\right)^{2h}$$
$$\leq C(h)\lambda^{-2h}\left(1-e^{-\frac{\lambda}{h}(t_2-t_1)}\right)^{2h}$$
$$\leq C(h)\lambda^{-2h(1-\tau)}|t_2-t_1|^{2h\tau}. \tag{5.56}$$

□

Inserting estimates (5.55) and (5.56) into (5.54), we get

$$\mathbb{E}\left(\int_0^{t_2} e^{-\lambda(t_2-s)} dW^h(s) - \int_0^{t_1} e^{-\lambda(t_1-s)} dW^h(s)\right)^2$$
$$\leq C(h)\lambda^{-2(h-\tau)}\{|t_2-t_1|^{2\tau} \vee |t_2-t_1|^{2h\tau}\}. \tag{5.57}$$

Lemma 5.4.6. *Let $\alpha \in (1,2]$, $0 \leq \delta < \frac{\alpha-1}{2}$ and fix $N \in \mathbb{N}$. Then the stochastic process $Z_N(t) : [0,T] \times \Omega \to C^\delta(0,1)$ has Hölder continuous sample path of degree $\frac{1}{2}-$.*

Proof. This lemma can be proved in a similar by following Lemma 5.4.5 and then using the same techniques as in [4, Lemma 4.5]. □

Corollary 5.4.7. *Let $\alpha \in (1,2]$ and $0 \leq \delta < \frac{\alpha-1}{2}$. The Ornstein-Uhlenbeck stochastic process Z has a continuous sample version, we still denote by $Z : [0,T] \times \Omega \to C^\delta(0,1)$ with Hölder continuous sample paths of degree $\left(h\left(\frac{2\alpha h-1-2\delta}{\alpha}\right)\right)-$.*

5.5 Proof of Theorems:

Proof of Theorem 5.3.1: Let us first write the equation (5.25) into following form

$$f_N(t) = e^{-tA^{\alpha/2}} P_N f_0 + \int_0^t e^{-(t-s)A^{\alpha/2}} P_N F(f_N(s)) ds + Z_N(t) \quad t \in [0,T], \tag{5.58}$$

then the existence of f_N can be established in a similar way to [4, Theorem 3.3].

Proof of Theorem 5.3.2 and Theorem 5.3.3: In order to prove these theorems, we have to verify Assumptions 1-4 in Theorem 5.1.1. Now, thanks to [4], following the proof of [4, Theorem 3.4 and Theorem 3.5], we are left to verify the Assumption 3 of Theorem 5.1.1 and then Assumption 4 will be followed in the same way as in [4, Theorem 3.3]. By Corollary 5.4.7, $Z(t)$

is continuous. The process $e^{-A^{\alpha/2t}}f_0 : \Omega \to C^\delta(0,1)$ is continuous and consequently the process $O(t) := e^{-A^{\alpha/2t}}f_0 + Z(t) : \Omega \to C^\delta(0,1)$ is also continuous. Further due to Hypothesis ($\mathscr{H}$) and the fact that for $\alpha < 2+\delta, H_2^\beta(0,1) \hookrightarrow H_{\delta+\frac{1}{2}(0,1)} \hookrightarrow H_2^{\alpha/2}(0,1)$, we have

$$|(1-P_N)e^{-A^{\alpha/2}t}f_0(\omega)|_{C^\delta} \leq C_{\alpha,\delta}(\omega)N^{-(\frac{\alpha-1}{2}-\delta)+}|f_0(\omega)|_{H_2^\beta}. \tag{5.59}$$

P-a.s.. Now, by Corollary 5.4.3 and (5.59), we show that the process $O(t)$ in Theorem 5.1.1, satisfies

$$\sup_{N\in\mathbb{N}}\sup_{t\in[0,T]} N^{-(\frac{\alpha-1}{2}-\delta)-}|(1-P_N)O(t,\omega)|_{C^\delta} < \infty. \tag{5.60}$$

And thus it is proved that the Assumption 3 is also valid for problem (5.1).

Proof of the Theorem 5.3.4: This Theorem can be easily proved by following the proof of Theorem 3.6 in [4], and the proof of Theorem 5.3.3.

Remark 5.5.1. The rate of convergence obtained here is same as in [4]. However, this can be improved by considering more regularity in the initial data f_0. This work is in progress.

Chapter 6

On a mixed fractional Burgers type nonlinear equation perturbed by fractional Brownian sheet

In this paper, we discuss a class of stochastic nonlinear partial differential equation of Burgers-type driven by pseudo differential operator $(\Delta+\Delta_\alpha)$ for $\alpha \in (0,2)$ and the fractional Brownian sheet. The existence and uniqueness of an L^p-valued (local) solution is established for the initial boundary valued problem to the equation.

This chapter is formulated according to the following plan: In the next section, we describe the mixed fractional Burgers-type nonlinear equation perturbed by the fractional Brownian sheet which we study in this chapter. Section 5.2 deals with some preliminaries consisting of basic properties to the fractional Brownian sheet and pseudo differential operators. In Section 5.3, a valid formulation of the solution to the equation (6.1) is given. Later, the existence and uniqueness of the local solution is shown in the sense of definition (6.15).

6.1 Introduction

Our aim is to study the following class of stochastic partial differential equation of Burgers type with polynomial nonlinearity perturbed by mixed fractional Laplacian operators and fractional

Brownian sheet

$$\begin{cases} \frac{\partial f}{\partial t}(t,x) = & \mathfrak{D} f(t,x) + \frac{\partial q}{\partial x}(t,x,f(t,x)) + g(t,x,f(t,x)) + \frac{\partial^2 W^H}{\partial t \partial x}(t,x), \\ f(0,x) = & f_0(x) \qquad x \in D, \\ f(t,x) = & 0, \qquad \text{for every } x \in \partial D \text{ and } t > 0, \end{cases} \tag{6.1}$$

where $(t,x) \in ([0,\infty),D)$, $D \subset \mathbb{R}$ is a bounded domain with boundary ∂D, $\mathfrak{D} = \Delta + \Delta_\alpha$ with $\Delta \equiv \frac{\partial^2}{\partial x^2}$ and $\Delta_\alpha \equiv -(-\Delta)^{\frac{\alpha}{2}}$ where $\alpha \in (0,2)$. Functions $q,g : \mathbb{R}_+ \times D \times \mathbb{R} \to \mathbb{R}$, are non-linear measurable, and W^H is the two parameter fractional Brownian sheet with Hurst index $H = (H_1,H_2)$ where $H_i \in (1/2,1)$ for each $i = 1,2$.

In recent years, the fractional Brownian sheet became popular among the researchers studying the stochastic partial differential equations, due to its important property of preserving long term memory and other interesting properties, see [25, 76, 78, 112, 143] and Section 2 in the present work. On the other hand, the operator $(\Delta + \Delta_\alpha)$ represents the Lévy processes that are independent sum of diffusion processes and α-stable (rotationally symmetric) processes. This pseudo differential operator can be used in several physical problems [96, 121].

In this paper, we particularly study an important generalization of the SBE by considering fractional Brownian sheet with Hurst index $H = (H_1,H_2)$ where $H_i \in (1/2,1)$ for each $i = 1,2$, and the pseudo differential operator $\mathfrak{D} = \Delta + \Delta_\alpha$ where $\alpha \in (0,2)$ together with polynomial type nonlinearity in q as given in assumptions **(H3)** and **(H5)**. Here the main novelty is to establish the existence and uniqueness of $L^p(D)$-valued local solution to (6.1) for $p \in [2,\infty)$. For this purpose, a fixed point argument is adapted which is motivated from [137] and [144].

Note: In this chapter, we use $C_j > 0, j = 1 \cdots, 6$, to indicate constants whose values are fixed throughout the paper and C is used to denote a positive constant which may depend on p, α, H_1, H_2 and its exact value is unimportant which may change from one appearance to another.

6.2 Preliminaries

6.2.1 Fractional Brownian sheet

Let us give the following definition of the two parameter fractional Brownian sheet which is followed by ([76, 113, 112, 131], and [143]).

Definition 6.2.1. Let $(\Omega, \mathscr{F}, \mathbb{P}, \{\mathscr{F}_t\}_{t\geq 0})$ be a given probability space. For $T > 0$, an one dimensional double-parameter $\mathscr{F}_t$-adapted fractional Brownian sheet $W^H = \{W^H(t,x), (t,x) \in [0,T] \times D\}$ with indices $H = (H_1, H_2)$, where $H_i \in (0,1)$ for $i \in \{1,2\}$, is a centered Gaussian random field defined on $(\Omega, \mathscr{F}, \mathbb{P})$ with $W^H(0,0) = 0$ and covariance

$$\mathbf{E}[W^H(t,x)W^H(s,y)] = R_{H_1}(s,t)R_{H_2}(x,y),$$

for all $s,t > 0$ and $x,y \in D$, where

$$R_{H_i}(a,b) = \frac{1}{2}(a^{2H_i} + b^{2H_i} - |a-b|^{2H_i}) \quad i = 1,2; \text{ and } a,b \in \mathbb{R}.$$

Let $\psi(t,s,x,y) = 4H_1H_2(2H_1-1)(2H_2-1)|t-s|^{2H_1-2}|x-y|^{2H_2-2}$ for any $s,t \in [0,T]$ and $x,y \in D$. Let us introduce the following function space

$$\mathbb{L}^2_\psi = \Big\{ f : [0,T] \times D \to \mathbb{R}; \|f\|^2_\psi := \int_{[0,t]^2} \int_{D^2} \psi(u,v,x,y) \\ f(u,x)f(v,y)dydxdvdu < \infty \Big\}. \tag{6.2}$$

It can be easily shown that $\mathbb{L}^2_\psi$ is a Hilbert space and for each $f \in \mathbb{L}^2_\psi$, there exists a sequence $\{f_n\}$, defined as

$$f_n = \sum_{k=1}^{k_n} a_{n,k} \mathbf{1}_{(x_{n,k},y_{n,k}] \times (s_{n,k},t_{n,k}]} \quad n = 1,2,\cdots,$$

where $a_{n,k} \in \mathbb{R}, (s_{n,k},t_{n,k}) \in [0,T] \times [0,T], (x_{n,k},y_{n,k}) \in D \times D$ and $1 < k < k_n$, such that as $n \to \infty$, we have $f_n \to f \in \mathbb{L}^2_\psi$.

Let $\mathscr{E}$ denotes the set of the all simple functions f_n on $[0,T] \times D$. Then $\mathscr{E}$ is linear subspace and dense in $\mathbb{L}^2_\psi$, see [76]. Also, for any $f_n \in \mathscr{E}$, define

$$\int_0^t \int_{\mathbb{D}} f_n(x,s) W^H(dx,ds) = \sum_k^{k_n} a_{k,n} W^H_{(s_{n,k},t_{n,k}] \times (x_{n,k},y_{n,k}]},$$

where $W^H(dx,ds)$ is equivalent to $\dot{W}^H(x,s)dxds$. Since $\mathscr{E}$ is dense in $\mathbb{L}^2_\psi$, for each $f \in \mathbb{L}^2_\psi$ there exists a sequence of simple functions $\{f_n\}$, such that

$$\int_0^t \int_{\mathbb{D}} f(x,s) W^H(dx,ds) = \lim_{n\to\infty} \int_0^t \int_{\mathbb{D}} f_n(x,s) W^H(dx,ds).$$

More detail can be found in [143]. Now, based on the above information, we have the following proposition.

Proposition 6.2.1. *For $f, g \in \mathbb{L}^2_\psi$ we have*

$$\mathbb{E}\int_0^t \int_D f(s,x)W^H(dx,ds) = 0,$$

and

$$\mathbb{E}\int_0^t \int_D f(s,x)W^H(dx,ds)\int_0^t \int_D g(s,x)W^H(dx,ds)$$
$$= \int_{[0,t]^2}\int_{D^2} \psi(u,v,x,y)f(u,x)g(v,y)dydxdvdu.$$

Next, we have the following embedding result followed by [113, Theorem 1.1] and [107, Lemma 3.2].

Proposition 6.2.2. *For $H > \frac{1}{2}$, we have following the embedding property*

$$L^{\frac{1}{H}}([0,T]\times D) \subset \mathbb{L}^2_\psi.$$

6.2.2 Pseudo differential operator

In equation (6.1), the operator $\Delta + \Delta_\alpha$ is a pseudo differential operator which generates a Lévy process $X \in \mathbb{R}^d$, as a sum of a diffusion process and an independent symmetric α- stable process where $\alpha \in (0,2)$. Here by a symmetric α- stable process, we mean a process $Y^\alpha = \{Y^\alpha_t, t \geq 0\}$ which is Lévy

$$\mathbb{E}\left[e^{i\xi(Y^\alpha_t - Y^\alpha_0)}\right] = e^{-t(|\xi|)^\alpha},$$

for every $\xi \in \mathbb{R}$.

Recall that if $\mathscr{L}$ denotes a class of second order elliptic operators on $\mathbb{R}^d$, then there is a diffusion process **B** (Brownian motion) in $\mathbb{R}^d$, associated with $\mathscr{L}$ in such a way that $\mathscr{L}$ is an infinitesimal generator of **B** and vice-versa. We say the transition density G of **B** as the fundamental solution of the the equation

$$\frac{\partial f}{\partial t} = \mathscr{L} f.$$

We also know G as Green function or heat kernel. The diffusion process **B** is the continuous Markov process and the differential operator $\mathscr{L}$ can be treated as an infinitesimal generator of this continuous Markov process. But, in case of discontinuous Markov processes, the infinitesimal generator need not to be a differential operator. For example, a rotationally symmetric α

-stable process with $\alpha \in (0,2)$ is a discontinuous Markov process with infinitesimal generator as fractional Laplacian $\Delta_\alpha = -(-\Delta)^{\alpha/2}$, which is no longer a differential operator. Indeed, it is a non-local (pseudo-differential) operator, defined by

$$\Delta_\alpha h = lim_{\varepsilon \to 0} \int_{\mathscr{D}} (h(y) - h(z)) \frac{\mathscr{A}(d,\alpha)}{|y-z|^{d+\alpha}},$$

where $\mathscr{D} := \{y \in \mathbb{R}^d : |y-z| > \varepsilon\}$ and $\mathscr{A}(d,\alpha) := \alpha z^{\alpha-1}\pi^{-d/2}\Gamma(\frac{d+\alpha}{2})/\Gamma(1-\frac{\alpha}{2})$, where Γ is a standard Gamma function.

Let Y^α be a α -stable process with $\alpha \in (0,2)$ with infinitesimal generator Δ_α and $\mathbf{B}$ is the Brownian motion independent of Y^α and with infinitesimal generator Δ.
Further, we consider a Lévy process $\{Y_t, t \geq 0\}$ as an independent sum of Y^α and $\mathbf{B}$, i.e.

$$Y_t = \mathbf{B} + Y^\alpha.$$

Then the infinitesimal generator of Y is $\Delta + \Delta_\alpha$ and

$$\mathbb{E}[e^{i\eta(Y_t - Y_0)}] = e^{\{-t(|\eta|^2 + |\eta|^\alpha)\}},$$

for every $\eta \in \mathbb{R}$. Now, let K_α be the fundamental solution of the equation

$$\frac{\partial f}{\partial t} = (\Delta + \Delta_\alpha) f. \tag{6.3}$$

Then, by following Chen et. al. [40], (see also [39, 130]), there exist constants $C_i \geq 1, \; i = 1,2,$ such that, for all $(t,s,x,y) \in (0,\infty) \times (0,\infty) \times \mathbb{R}^d \times \mathbb{R}^d, d \geq 1,$

$$\begin{aligned} C_1^{-1} &\left((t-s)^{\frac{-d}{2}} exp\left(-\frac{C_2|x-y|^2}{(t-s)} \right) + (t-s)^{\frac{-d}{2}} \wedge \frac{t-s}{|x-y|^{d+\alpha}} \right) \\ &\leq K_\alpha(t-s,x,y) \leq C_1 \left((t-s)^{\frac{-d}{2}} exp\left(-\frac{|x-y|^2}{C_2(t-s)} \right) + (t-s)^{\frac{-d}{2}} \wedge \frac{t-s}{|x-y|^{d+\alpha}} \right), \end{aligned} \tag{6.4}$$

where for $(\rho_1, \rho_2) \in \mathbb{R}^d \times \mathbb{R}^d, \rho_1 \wedge \rho_2 := \min\{\rho_1, \rho_2\}$.

Let us give some notation of functions which will be used further in this paper. For $T > 0, x, y \in \mathbb{R}^d,$ where $d \geq 1$ and $0 < \beta = C_2^{-1}$,

$$K^d(t,x,y) := t^{-d/2} exp\left(-\frac{\beta|x-y|^2}{t} \right), \tag{6.5}$$

$$Q^d(t,x,y) := K^d(t,x,y) + t^{-d/2} \wedge \frac{t}{|x-y|^{d+\alpha}}. \tag{6.6}$$

The following lemma is due to [39]. It gives an estimate on Q^d.

Lemma 6.2.3. *Let $Q^d(t,w), d \geq 1$ satisfies*

$$Q^d(t,w) = t^{-d/2} exp\left(-\frac{\beta|w|^2}{t}\right) + t^{-d/2} \wedge \frac{t}{|w|^{d+\alpha}}, \tag{6.7}$$

for all $t > 0$ and $w \in \mathbb{R}$. Then there exists a constant $C_3 = C_3(\alpha, T, \beta)$ such that

$$Q^d(t,w) \overset{C_3}{\asymp} K^d(t,w) + \frac{t}{|w|^{d+\alpha}} \mathbf{1}_{\{|w|^2 \geq t\}}, \tag{6.8}$$

where for any A, B, the notation $A \overset{c}{\asymp} B$ means that there exist a constant $c > 0$ such that $c^{-1}B \leq A \leq cB$ and

$$K^d(t,w) := t^{-d/2} exp\left(-\frac{\beta|w|^2}{t}\right).$$

Further from [39, Theorem 2.2] and Lemma 6.2.3, we have the following important theorem.

Theorem 6.2.4. *For any $T > 0$, there exists a positive constant $C_4 = C_4(d, \alpha, T)$, such that for all $0 < t \leq T$ and $w \in \mathbb{R}^d$,*

$$2\pi C_1^{-1} Q^{d+2}(t,w)|w| \leq |\triangledown_w K_\alpha(t,w)| \leq C_4 Q^{d+1}(t,w). \tag{6.9}$$

It should be explicitly mention that, throughout in this paper, we deal with one dimensional case i.e. $d = 1$ and on a bounded domain $D \subset \mathbb{R}$.

Let $t > s > 0$ and $f : [0,\infty) \times \mathbb{D} \to \mathbb{R}$ be given. We introduce the operators $(\mathfrak{I}_{y^i})$, for $i = 0, 1$, as

$$(\mathfrak{I}_{y^0} f)(t,x)) := \int_0^t \int_D K_\alpha(t-s,x,y) f(s,y) ds dy, \tag{6.10}$$

and

$$(\mathfrak{I}_{y^1} f)(t,x)) := \int_0^t \int_D \frac{\partial K_\alpha}{\partial y}(t-s,x,y) f(s,y) dy ds. \tag{6.11}$$

Lemma 6.2.5. *Let for $t > s > 0$, $(x,y) \in D \times D$, and K_α is given by the inequality (6.4). Then, for each $1 < p < \infty$ and $\alpha \in (0,2)$, there exist positive constants $C, C_5 = C_5(C_1, C_2)$ and $C_6 = C_6(C_1, C_2, C_4)$ such that*

$$\|(\mathfrak{I}_{y^0} f)(t,\cdot))\|_{L^p} \leq CC_5 \int_0^t ((t-s)^{\frac{1}{2p}-\frac{1}{2}} \vee (t-s)^{\frac{1}{2}}) \|f(s,\cdot)\|_{L^1} ds, \tag{6.12}$$

and

$$\|(\mathfrak{I}_{y^1} f)(t,\cdot))\|_{L^p} \leq CC_6 \int_0^t ((t-s)^{\frac{1}{2p}-1} \vee (t-s)^{\frac{-\alpha}{2}}) \|f(s,\cdot)\|_{L^1} ds, \tag{6.13}$$

where $a \vee b = \max(a,b)$.

Proof. Let us define the following domain as

$$D_x := \{z \in D : |x-z| < (t-s)^{\frac{3}{2(1+\alpha)}}\},$$

such that

$$D = D_x \cup \overline{D_x},$$

where $\overline{D_x}$ is the compliment of the D_x. Further, define

$$K_{1\alpha} := K_1 + (t-s)^{-\frac{1}{2}}$$
$$K_{2\alpha} := K_1 + \frac{t-s}{|x-y|^{1+\alpha}},$$

for all $t > s > 0$ and $(x,y) \in D \times D$. Here K_1 is given by (6.5) for $d = 1$ and $\beta = {C_2}^{-1}$.

Therefore, the upper bound of K_α in (6.4) can be written as

$$K_\alpha \leq C_1(K_{1\alpha} + K_{2\alpha}). \tag{6.14}$$

By inserting Minkowoski inequality (6.42), in (6.10), we have

$$\begin{aligned}
\|(\mathfrak{J}_{y^0} f)(t,\cdot)\|_{L^p} &\leq \int_0^t \left\{ \int_D \left| \int_D K_\alpha(t-s,x,y) f(s,y) dy \right|^p dx \right\}^{\frac{1}{p}} ds \\
&\leq C_1 \int_0^t \left\{ \int_D \left| \int_D (K_{1\alpha} + K_{2\alpha}) f(s,y) dy \right|^p dx \right\}^{\frac{1}{p}} ds \\
&\leq C_1 \Bigg(\int_0^t \left\{ \int_D \left| \int_{D_x} (K_1 + (t-s)^{-\frac{1}{2}}) f(s,y) dy \right|^p dx \right\}^{\frac{1}{p}} ds \\
&+ \int_0^t \left\{ \int_D \left| \int_{\overline{D_x}} (K_1 + \frac{t-s}{|x-y|^{1+\alpha}}) f(s,y) dy \right|^p dx \right\}^{\frac{1}{p}} ds \Bigg) \\
&= C_1 \Bigg(2 \int_0^t \left\{ \int_D \left| \int_D (t-s)^{-\frac{1}{2}} exp\left(-\frac{C_2|x-y|^2}{(t-s)} \right) f(s,y) dy \right|^p dx \right\}^{\frac{1}{p}} ds \\
&\quad + \int_0^t \left\{ \int_D \left| \int_{D_x} ((t-s)^{-\frac{1}{2}}) f(s,y) dy \right|^p dx \right\}^{\frac{1}{p}} ds \\
&\quad + \int_0^t \left\{ \int_D \left| \int_{\overline{D_x}} \left(\frac{t-s}{|x-y|^{1+\alpha}} \right) f(s,y) dy \right|^p dx \right\}^{\frac{1}{p}} ds \Bigg).
\end{aligned}$$

Using Young's inequality for convolution with $r = p$ and $q = 1$ (see (6.43)), and applying the fact that

$$\frac{1}{|x-y|^{1+\alpha}} \leq \frac{1}{(t-s)^{\frac{3}{2}}},$$

for all $t > s > 0$ on $\overline{D_x}$, we have

$$\begin{aligned}\|(\mathfrak{I}_{y^0} f)(t,\cdot)\|_{L^p} &\le CC_5\Bigg(\int_0^t (t-s)^{\frac{1}{2p}-\frac{1}{2}}\|f(s,\cdot)\|_{L^1}ds + \int_0^t (t-s)^{-\frac{1}{2}}|f(s,\cdot)|_{L^1}ds \\ &\qquad + \int_0^t (t-s)^{-\frac{1}{2}}\|f(s,\cdot)\|_{L^1}ds\Bigg) \\ &\le CC_5\int_0^t ((t-s)^{\frac{1}{2p}-\frac{1}{2}} + 2(t-s)^{-\frac{1}{2}})\|f(s,\cdot)\|_{L^1}ds \\ &\le CC_5\int_0^t ((t-s)^{\frac{1}{2p}-\frac{1}{2}} \vee (t-s)^{-\frac{1}{2}})\|f(s,\cdot)\|_{L^1}ds.\end{aligned}$$

Thus we obtained (6.12). Next, in order to prove (6.13), let us apply Minkowoski inequality (6.42) to have

$$\begin{aligned}\|(\mathfrak{I}_{y^1} f)(t,\cdot)\|_{L^p} &\le \int_0^t \left\{\int_D \left|\int_D \frac{\partial K_\alpha}{\partial y}(t-s,x,y) f(s,y)dy\right|^p dx\right\}^{\frac{1}{p}} ds \\ &\le \int_0^t \left\{\int_D \left|\int_D \left|\frac{\partial K_\alpha}{\partial y}(t-s,x,y)\right| |f(s,y)|dy\right|^p dx\right\}^{\frac{1}{p}} ds.\end{aligned}$$

Applying Lemma 6.2.3 and Theorem 6.2.4 for $d = 1$, we have

$$\begin{aligned}\|(\mathfrak{I}_{y^1} f)(t,\cdot)\|_{L^p} &\le C_4\int_0^t \left\{\int_D \left(\int_D \left|(t-s)^{-1} exp\left(-\frac{|x-y|^2}{C_2|t-s|}\right)\right| |f(s,y)|\,dy\right)^p dx\right\}^{\frac{1}{p}} ds \\ &\quad + \int_0^t \left\{\int_D \left(\int_{\{|x-y|^2 \ge (t-s)\}} \frac{(t-s)}{|x-y|^{2+\alpha}} |f(s,y)|\,dy\right)^p dx\right\}^{\frac{1}{p}} ds \\ &\le C_4\int_0^t \left\{\int_D \left(\int_D \left|(t-s)^{-1} exp\left(-\frac{|x-y|^2}{C_2|t-s|}\right)\right| |f(s,y)|\,dy\right)^p dx\right\}^{\frac{1}{p}} ds \\ &\quad + \int_0^t \left\{\int_D \left(\int_{\{|x-y|^2 \ge (t-s)\}} (t-s)^{-\alpha/2} |f(s,y)|\,dy\right)^p dx\right\}^{\frac{1}{p}} ds,\end{aligned}$$

where in the last integral on the right hand side, we have used the fact that $|x-y|^2 \ge (t-s)$, which implies that $\frac{(t-s)}{|x-y|^{2+\alpha}} \le (t-s)^{-\alpha/2}$. Thus, by using Young's inequality, we obtain

$$\|(\mathfrak{I}_{y^1} f)(t,\cdot)\|_{L^p} \le CC_6\int_0^t ((t-s)^{\frac{1}{2p}-1} \vee (t-s)^{-\alpha/2})\|f(s,\cdot)\|_{L^1}ds,$$

which is (6.13). $\square$

6.3 Existence and Uniqueness

For a arbitrary fixed $T > 0$, let us introduce $\mathfrak{B}_{T,p} := \mathfrak{B}_{T,p}(D)$, consisting the collection of all $\mathscr{F}_t$-adapted functions $f : [0,\infty) \times D \times \Omega \to L^p(D)$ such that

$$\|f\|_{T,p}^p := \sup_{t\in[0,T]} \mathbb{E}\left[\|f(t,\cdot,\cdot)\|_{L^p}^p\right] < \infty.$$

It can easily be shown that $\mathfrak{B}_{T,p}$ is a Banach space equipped with norm $\|\cdot\|_{T,p}$. Next, following assumptions are required to establish the existence and uniqueness of $L^p(D)$-valued local solution to (6.1).

Assumption 2. **(H1):** The functions $q, g : [0,\infty) \times D \times \mathbb{R} \to \mathbb{R}$ are measurable functions,

(H2) : the function $g : [0,\infty) \times D \times \mathbb{R} \to \mathbb{R}$ satisfies the following growth condition,

$$|g(t,x,r)| \le C(1+|r|),$$

(H3): the function $q(t,x,r)$ satisfies the following type of nonlinearity condition

$$|q(t,x,r)| \le C\sum_{n=2}^{p} (a_{p-n}(x))^{(p-n)}|r|^n,$$

where p is finite and for every $n = 2,\cdots,p, a_n \in L^p(D)$,

(H4): for any real valued functions $r_1 \in \mathbb{R}$ and $r_2 \in \mathbb{R}$, we have

$$|g(t,x,r_1) - g(t,x,r_2)| \le C|r_1 - r_2|,$$

and

(H5)

$$|q(t,x,r_1) - q(t,x,r_2)| \le C\sum_{n=2}^{p} |(a_{p-n}(x))|^{(p-n)}|r_1^n - r_2^n|,$$

where $C > 0$ is a constant.

Now, we formulate the mild solution (in the sense of Walsh [141]) by means of the following definitions.

Definition 6.3.1. An $L^p(D)$-valued and $\mathscr{F}_t$-adapted stochastic process $f : [0,\infty) \times D \times \Omega \to \mathbb{R}$ is a solution to (6.1) if for any $T > 0$, $f(t,x) := f(t,x,\cdot)$ satisfies the following integral equation

$$\begin{aligned} f(t,x) = \int_D K_\alpha(t,x,y)f_0(y)dy + \int_0^t\int_D K_\alpha(t-s,x,y)g(t,x,f(s,y))dyds \\ - \int_0^t\int_D \frac{\partial K_\alpha}{\partial y}(t-s,x,y)q(t,x,f(s,y))dyds \\ + \int_0^t\int_D K_\alpha(t-s,x,y)W^H(dy,ds) \quad a.s., \end{aligned} \tag{6.15}$$

for each $t \in [0,T]$, and where K_α is the fundamental solution of the equation (6.3) defined in Subsection 2.2.

Definition 6.3.2. We define an $\mathscr{F}_t$-adapted stochastic process f as a local solution of (6.1) if there exists an $\mathscr{F}_t$-adapted stopping time τ such that $\{f(t,x)\}_{t\leq\tau}$ is a solution of (6.1).

Let us state the following Lemma which will play an important role in showing our main results.

Lemma 6.3.1. *Let $a_n \in L^p(D)$. Setting*

$$(F(Z))(t,x) := \sum_{n=2}^{p} a_{p-n}^{p-n} Z^n, \tag{6.16}$$

for a given function $Z : [0,\infty) \times D \to \mathbb{R}$. Then, we have

$$\|F(Z)(t,\cdot)\|_{L^1} \leq \sum_{n=2}^{p} \|a_{p-n}\|_{L^p}^{p-n} \|Z\|_{L^p}^{n}. \tag{6.17}$$

Further, let $Z_1, Z_2 : [0,\infty) \times D \to \mathbb{R}$. Then, we have

$$\|(F(Z_1) - F(Z_2))(t,\cdot)\|_{L^1} \leq C \sum_{n=2}^{p} \|a_{p-n}\|_{L^p}^{p-n} \left(\|Z_1\|_{L^p}^{n-1} + \|Z_2\|_{L^p}^{n-1}\right) \|Z_1 - Z_2\|_{L^p}. \tag{6.18}$$

Proof. Using Hölder's inequality, we have

$$\begin{aligned}
\|F(Z)(t,\cdot)\|_{L_1} &\leq \sum_{n=2}^{p} \|a_{p-n}^{p-n}\|_{L^{\frac{p}{p-n}}} \|Z^n\|_{L^{\frac{p}{n}}} \\
&= \sum_{n=2}^{p} \|a_{p-n}\|_{L^p}^{p-n} \|Z\|^n{}_{L^p},
\end{aligned}$$

which is (6.17). Next, we have

$$|(F(Z_1) - F(Z_2))(t,\cdot)| = \left| \sum_{n=2}^{p} a_{p-n}^{p-n} (Z_1^n - Z_2^n) \right|.$$

Taking L^1 norm on the both side and then applying Hölder's inequality, we get

$$\begin{aligned}
\|(F(Z_1) - F(Z_2))(t,\cdot)\|_{L^1} &\leq \sum_{n=2}^{p} \|a_{p-n}^{p-n}\|_{L^{\frac{p}{p-n}}} \|Z_1^n - Z_2^n\|_{L^{\frac{p}{n}}} \\
&= \sum_{n=2}^{p} \|a_{p-n}\|_{L^p}^{p-n} \|Z_1^n - Z_2^n\|_{L^{\frac{p}{n}}}.
\end{aligned} \tag{6.19}$$

Now

$$
\begin{aligned}
\|Z_1^n - Z_2^n\|_{L^{\frac{p}{n}}} &= \left\| (Z_1 - Z_2) \sum_{m=0}^{n-1} Z_1^{n-m-1} Z_2^m \right\|_{L^{\frac{p}{n}}} \\
&\le \|Z_1 - Z_2\|_{L^p} \sum_{m=0}^{n-1} \left\| Z_1^{n-m-1} Z_2^m \right\|_{L^{\frac{p}{n-1}}} \\
&= \|Z_1 - Z_2\|_{L^p} \left(\|Z_1\|_{L^p}^{n-1} + \sum_{m=1}^{n-2} \left\| Z_1^{n-m-1} \right\|_{L^{\frac{p}{n-m-1}}} \|Z_2^m\|_{L^{\frac{p}{m}}} + \|Z_2\|_{L^p}^{n-1} \right). \quad (6.20)
\end{aligned}
$$

Recalling the generalized Young's inequality, i.e for non-negative u and v, one has

$$
uv \le r^{\frac{1}{r}} \left(\frac{u^p}{p} + \frac{v^q}{q} \right)^{\frac{1}{r}}, \tag{6.21}
$$

where $1 \le p, q < \infty$ such that $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$.
Using (6.21) in the summation on the right hand side to inequality (6.20) with $r = \frac{1}{n-1}, p = \frac{1}{n-m-1}$ and $q = \frac{1}{m}$, we get

$$
\begin{aligned}
\|Z_1^n - Z_2^n\|_{L^{\frac{p}{n}}} \le \|Z_1 - Z_2\|_{L^p} \Bigg(\|Z_1\|_{L^p}^{n-1} + \sum_{m=1}^{n-2} (n-1)^{\frac{1}{n-1}} \big\{ (n-m-1) \|Z_1\|_{L^p} \\
+ m \|Z_2\|_{L^p} \big\}^{n-1} + \|Z_2\|_{L^p}^{n-1} \Bigg) \\
\le C \|Z_1 - Z_2\|_{L^p} \left(\|Z_1\|_{L^P}^{n-1} + \|Z_2\|_{L^P}^{n-1} \right). \quad (6.22)
\end{aligned}
$$

Using the estimate (6.22) in to (6.19), we finally obtain (6.18). □

Now, in order to formulate the main result of this paper, we introduce a mapping Λ_N from $L^p(D)$ to $L^p(D)$ such that for each fixed $N \in \mathbb{N}$, we have

$$
\Lambda_N(f) := \begin{cases} f, & \|f\|_{L^p} \le N, \\ \frac{Nf}{\|f\|_{L^p}}, & \|f\|_{L^p} \ge N. \end{cases} \tag{6.23}
$$

Then, for any fixed $N \in \mathbb{N}$, we have

$$
\|\Lambda_N(f)\|_{L^p} \le N, \tag{6.24}
$$

and there exists a constant $L > 0$ (see Lemma 6.4.1) such that

$$
\|\Lambda_N(f_1) - \Lambda_N(f_2)\|_{L^p} \le L \|f_1 - f_2\|_{L^p}, \tag{6.25}
$$

for every $f_1, f_2 \in L^p(D)$.

Theorem 6.3.2. *Let $2 \leq p < \infty$ and the initial function $f_0 : D \times \Omega \to \mathbb{R}$ is $L^p(\Omega, L^p(D))$-valued and $\mathscr{F}_0$- measurable. Also, if **(H1)** $-$ **(H5)** hold, then there exists an unique local solution $f(t) := f(t,\cdot,\cdot)$ to (6.1), which satisfies*

$$\sup_{t\in[0,T]} \mathbb{E}\left[\|f(t\wedge\tau)\|_{L^p}^p\right] < \infty, \tag{6.26}$$

where τ is a $\mathscr{F}_t$-adapted stopping time.

Proof. Let us give the truncated stochastic integral equation associated with Λ_N for any fixed $N \in \mathbb{N}$,

$$\begin{aligned} f(t,x) = \int_D K_\alpha(t,x,y) f_0(y) dy + \int_0^t \int_D K_\alpha(t-s,x,y) g(t,x,\Lambda_N f(s,y)) dy ds \\ - \int_0^t \int_D \frac{\partial K_\alpha}{\partial y}(t-s,x,y) q(t,x,\Lambda_N f(s,y)) dy ds \\ + \int_0^t \int_D K_\alpha(t-s,x,y) W^H(dy,ds) \quad a.s.. \end{aligned} \tag{6.27}$$

Next, we define the mapping $\mathscr{T}$ on the Banach space $\mathfrak{B}_{T,p}$ as

$$(\mathscr{T} f)(t,x) := \int_D K_\alpha(t,x,y) f_0(y) dy + (\mathscr{T}_1 f)(t,x) - (\mathscr{T}_2 f)(t,x) + (\mathscr{T}_3 f)(t,x) \tag{6.28}$$

where

$$(\mathscr{T}_1 f)(t,x) = \int_0^t \int_D K_\alpha(t-s,x,y) g(t,x,\Lambda_N f(s,y)) dy ds,$$

$$(\mathscr{T}_2 f)(t,x) = \int_0^t \int_D \frac{\partial K_\alpha}{\partial y}(t-s,x,y) q(t,x,\Lambda_N f(s,y)) dy ds,$$

$$(\mathscr{T}_3 f)(t,x) = \int_0^t \int_D K_\alpha(t-s,x,y) W^H(dy,ds).$$

The proof of the Theorem 6.3.2 relies on the fixed point argument i.e. it is required to show that $\mathscr{T}$ is a contraction mapping on the Banach space $\mathfrak{B}_{T,p}$. First it is shown that $\mathscr{T}$ maps $\mathfrak{B}_{T,p}$ into itself i.e. $\|(\mathscr{T} f)(t,x)\|_{T,p} < \infty$. From Lemma 6.2.5, **(H2)** and (6.24), we estimate

$$\begin{aligned} \|(\mathscr{T}_1 f)(t)\|_{L^p} &\leq CC_5 \int_0^t ((t-s)^{\frac{1}{2p}-\frac{1}{2}} \vee (t-s)^{-\frac{1}{2}}) \|g(s,y,\Lambda_n f(s,y))\|_{L^1} ds \\ &\leq CC_5 \int_0^t ((t-s)^{\frac{1}{2p}-\frac{1}{2}} \vee (t-s)^{-\frac{1}{2}}) (1 + \|\Lambda_n f(s,y))\|_{L^p}) ds \\ &\leq CC_5 (t^{\frac{1}{2p}+\frac{1}{2}} \vee (t)^{\frac{1}{2}})(1+N) \\ &< \infty, \end{aligned} \tag{6.29}$$

for each $t \in [0,T]$. Next, using **(H3)**, Lemma 6.2.5 and then by applying (6.17), we have

$$\begin{aligned}
\|(\mathscr{T}_2 f)(t)\|_{L^p} &\leq CC_6 \int_0^t ((t-s)^{\frac{1}{2p}-1} \vee (t-s)^{-\frac{\alpha}{2}}) \|q(s,y,\Lambda_n f(s,y))\|_{L^1} ds \\
&\leq CC_6 \int_0^t ((t-s)^{\frac{1}{2p}-1} \vee (t-s)^{-\frac{\alpha}{2}}) \sum_{n=2}^{p} \|(a_{p-n}(\cdot))^{(p-n)} |\Lambda_n f(s,\cdot)|^n\|_{L^1} \\
&\leq CC_6 \int_0^t ((t-s)^{\frac{1}{2p}-1} \vee (t-s)^{-\frac{\alpha}{2}}) \sum_{n=2}^{p} \|(a_{p-n}(\cdot))\|_{L^p}^{(p-n)} \|\Lambda_n f(s,\cdot)\|_{L^p}^n \\
&\leq CC_6 \int_0^t ((t-s)^{\frac{1}{2p}-1} \vee (t-s)^{-\frac{\alpha}{2}}) \sum_{n=2}^{p} \|(a_{p-n}(\cdot))\|_{L^p}^{(p-n)} N^n \\
&\leq CC_6 (t^{\frac{1}{2p}} \vee (t-s)^{1-\frac{\alpha}{2}}) \sum_{n=0}^{p} \|(a_{p-n}(\cdot))\|_{L^p}^{(p-n)} N^n \\
&< \infty. \qquad (6.30)
\end{aligned}$$

Next, for $(\mathscr{T}_3 f)$, using isometry property given in Proposition 6.2.1 and then applying Hölder's inequality and Hardy-Littlewood inequality (see [131, Theorem 1(page 119)]) repeatedly, we have

$$\begin{aligned}
\mathbb{E}\|(\mathscr{T}_3 f)(t)\|_{L^p}^p &= \mathbb{E} \int_D \left| \int_0^t \int_D K_\alpha(t-s,x,y) W^H(dy,ds) \right|^p dx \\
&\leq C \int_D \bigg| \int_0^t \int_0^t ds_1 ds_2 \int_{D^2} K_\alpha(t-s_1,x,y_1) \\
&\qquad \psi(s_1,s_2,y_1 y_2) K_\alpha(t-s_2,x,y_2) dy_1 dy_2 \bigg|^{\frac{p}{2}} dx \\
&\leq C \bigg(\int_D \bigg(\int_0^t \int_0^t |s_1-s_2|^{2H_1-2} ds_1 ds_2 \int_{D^2} |y_1-y_2|^{2H_2-2} \\
&\qquad K_\alpha(t-s_1,x,y_1) K_\alpha(t-s_2,x,y_2) dy_1 dy_2 \bigg)^{\frac{p}{2}} dx \bigg) \\
&\leq C \bigg(\int_D \bigg(\int_0^t \int_0^t |s_1-s_2|^{2H_1-2} \|K_\alpha(t-s_1,x,\cdot)\|_{L^{\frac{1}{H_2}}} \\
&\qquad \|K_\alpha(t-s_2,x,\cdot)\|_{L^{\frac{1}{H_2}}} ds_1 ds_2 \bigg)^{\frac{p}{2}} dx \bigg) \\
&\leq C \bigg(\int_D \bigg(\int_0^T \Big(\|K_\alpha(t-s,x,\cdot)\|_{L^{\frac{1}{H_2}}} \Big)^{\frac{1}{H_1}} ds \bigg)^{pH_1} dx \bigg). \qquad (6.31)
\end{aligned}$$

Now, for any $t > s > 0$ and $(x,y) \in D \times D$, we have

$$\begin{aligned}
\|K_\alpha(t-s,x,\cdot)\|_{L^{\frac{1}{H_2}}} &= \left(\int_D |K_\alpha(t-s,x,y)|^{\frac{1}{H_2}} dy\right)^{H_2} \\
&\le C_1\left(\int_D \left((t-s)^{-\frac{1}{2}} exp\left(-\frac{|x-y|^2}{C_2(t-s)}\right) + (t-s)^{-\frac{1}{2}} \wedge \frac{(t-s)}{|x-y|^{1+\alpha}}\right)^{\frac{1}{H_2}} dy\right)^{H_2} \\
&\le CC_1\left(\int_D \left((t-s)^{-\frac{1}{2}} exp\left(-\frac{|x-y|^2}{C_2(t-s)}\right)\right)^{\frac{1}{H_2}} dy\right. \\
&\qquad\left. + \int_D \left((t-s)^{-\frac{1}{2}} \wedge \frac{(t-s)}{|x-y|^{1+\alpha}} dy\right)^{\frac{1}{H_2}}\right)^{H_2} \\
&\le CC_1\left((t-s)^{\frac{H_2-1}{2}} + (t-s)^{-\frac{1}{2}+\frac{3H_2}{2(1+\alpha)}}\right). \qquad (6.32)
\end{aligned}$$

Substituting (6.32) into (6.31) and then simplifying, we obtain

$$\mathbb{E}\|(\mathscr{T}_3 f)(t)\|_{L^p}^p < \infty. \qquad (6.33)$$

By using the estimates (6.29), (6.30), (6.33) and considering that $f_0 \in L^p(\Omega, L^p(D))$, we have

$$\|(\mathscr{T} f)(t)\|_{T,p} < \infty. \qquad (6.34)$$

Thus the mapping $\mathscr{T}$ maps $\mathfrak{B}_{T,p}$ into itself. Next, we prove that $\mathscr{T}$ is a contraction. Then the Banach fixed point theorem implies the existence and uniqueness of solution to equation (6.27) for each fixed N. Let us consider an arbitrary fixed real number $\lambda > 0$. For any $L^p(D)$ valued and $\mathscr{F}_t$-adapted stochastic process $f : [0,T] \times D \times \Omega \to \mathbb{R}$ with initial condition f_0, we define

$$\|f\|_{p,\lambda}^p := \sup_{t\in[0,T]} e^{-\lambda pt} \mathbb{E}\left[\|f(t,\cdot,\cdot)\|_{L^p}^p\right].$$

Clearly, $\|.\|_{p,\lambda}$ defines a norm. Let $\mathfrak{B}_{p,\lambda}$ be the space of all $\mathscr{F}_t$-adapted stochastic processes $f : [0,T] \times D \times \Omega \to \mathbb{R}$ such that

$$\|f\|_{p,\lambda}^p < \infty.$$

Then $\left(\mathfrak{B}_{p,\lambda}, \|\cdot\|_{p,\lambda}\right)$ is a Banach space and the new norm $\|\cdot\|_{p,\lambda}$ is equivalent to the previous norm $\|\cdot\|_{T,p}$ for each fixed $\lambda > 0$. Now, let $f_i \in \mathfrak{B}_{p,\lambda}$, for $i = 1,2$. Then (6.34) implies that $(\mathscr{T}_i f) \in \mathfrak{B}_{p,\lambda}$ for each $i = 1,2$. Next, using (6.28), we have

$$\|(\mathscr{T} f_1)(t) - (\mathscr{T} f_2)(t)\|_{L^p} = \left\| \sum_{j=1}^{2} (\mathscr{T}_j f_1)(t,\cdot) - (\mathscr{T}_j f_2)(t,\cdot)\right\|_{L^p}, \qquad (6.35)$$

where

$$\|(\mathscr{T}_1 f_1)(t)-(\mathscr{T}_1 f_2)(t)\|_{L^p} = \Big\| \int_0^t \int_D K_\alpha(t-s,\cdot,y)(g(t,\cdot,\Lambda_N f_1(s,y)) - g(t,\cdot,\Lambda_N f_2(s,y)))dyds \Big\|_{L^p}, \quad (6.36)$$

and

$$\|(\mathscr{T}_2 f_1)(t)-(\mathscr{T}_2 f_2)(t)\|_{L^p} = \Big\| \int_0^t \int_D \frac{\partial K_\alpha}{\partial y}(t-s,\cdot,y)(q(t,\cdot,\Lambda_N f_1(s,y)) - q(t,\cdot,\Lambda_N f_2(s,y)))dyds \Big\|_{L^p}. \quad (6.37)$$

We first solve (6.37) and following the same way, (6.36) can be solved. Now, using **(H-5)** in (6.37), we obtain

$$\|(\mathscr{T}_2 f_1)(t)-(\mathscr{T}_2 f_2)(t)\|_{L^p} \leq C\Big\| \int_0^t \int_D \frac{\partial K_\alpha}{\partial y}(t-s,\cdot,y) \sum_{n=2}^{p} \big|a_{p-n}(\cdot)\big|^{(p-n)} |(\Lambda_n f_1(s,\cdot))^n - (\Lambda_n f_2(s,\cdot))^n| \Big\|_{L^p} ds.$$

Using Lemma 6.2.5 and then by Lemma 6.18, we get

$$\begin{aligned}\|(\mathscr{T}_2 f_1)(t)-(\mathscr{T}_2 f_2)(t)\|_{L^p} &\leq CC_6 \int_0^t ((t-s)^{\frac{1}{2p}-1} \vee (t-s)^{-\frac{\alpha}{2}}) \sum_{n=2}^{p} \|(a_{p-n}(\cdot))\|_{L^p}^{p-n} \left(\|\Lambda_n f_1(s,\cdot)\|_{L^p}^{n-1} + \|\Lambda_n f_2(s,.)\|_{L^p}^{n-1}\right) \\ &\quad \times \|\Lambda_n f_1(s,\cdot) - \Lambda_n f_2(s,\cdot)\|_{L^p} ds,\end{aligned}$$

inserting the estimate (6.24) and (6.25) into above inequality, we have

$$\begin{aligned}\|(\mathscr{T}_2 f_1)(t)-(\mathscr{T}_2 f_2)(t)\|_{L^p} &\leq CC_6 L \int_0^t ((t-s)^{\frac{1}{2p}-1} \vee (t-s)^{-\frac{\alpha}{2}}) \sum_{n=2}^{p} \|(a_{p-n}(\cdot))\|_{L^p}^{p-n} \left(2N^{n-1}\right) \|f_1(s,\cdot) - f_2(s,\cdot)\|_{L^p} ds \\ &\leq CC_6 L \int_0^t ((t-s)^{\frac{1}{2p}-1} \vee (t-s)^{-\frac{\alpha}{2}}) \|f_1(s,\cdot) - f_2(s,\cdot)\|_{L^p} ds.\end{aligned}$$

Here, in last integral we have used the fact that $N \in \mathbb{N}$ is fixed and $a_n \in L^p(D)$ for each $n = 2, \cdots, p$. Now, taking p^{th} power on the both sides, and then applying Hölder's inequality, we evaluate

$$\begin{aligned}\|(\mathscr{T}_2 f_1)(t)-(\mathscr{T}_2 f_2)(t)\|_{L^p}^p &\leq C(C_6 L)^p \left(\int_0^t ((t-s)^{\frac{1}{2p}-1} \vee (t-s)^{-\frac{\alpha}{2}}) \|f_1(s,\cdot) - f_2(s,\cdot)\|_{L^p} ds \right)^p \\ &\leq C(C_6 L)^p \int_0^t ((t-s)^{\frac{1}{2p}-1} \vee (t-s)^{-\frac{\alpha}{2}}) \|f_1(s,\cdot) - f_2(s,\cdot)\|_{L^p}^p ds.\end{aligned}$$

Finally, the definition of the norm $\|\cdot\|_{p,\lambda}$, we have

$$\begin{aligned}
&\|(\mathscr{T}_2 f_1)(t)-(\mathscr{T}_2 f_2)(t)\|_{p,\lambda}^p \\
&\leq C(C_6L)^p\int_0^t e^{-\lambda p(t-s)}((t-s)^{\frac{1}{2p}-1}\vee(t-s)^{-\frac{\alpha}{2}})e^{-\lambda ps}\mathbb{E}[\|f_1(s,\cdot)-f_2(s,\cdot)\|_{L^p}^p]ds \\
&\leq C(C_6L)^p\|f_1-f_2\|_{p,\lambda}^p\int_0^t e^{-\lambda ps}(s^{\frac{1}{2p}-1}\vee s^{-\frac{\alpha}{2}})ds. \qquad (6.38)
\end{aligned}$$

Following the similar calculation, we obtain

$$\|(\mathscr{T}_1 f_1)(t)-(\mathscr{T}_1 f_2)(t)\|_{p,\lambda}^p \leq C(C_5L)^p\|f_1-f_2\|_{p,\lambda}^p\int_0^t e^{-\lambda ps}(s^{\frac{1}{2p}-\frac{1}{2}}\vee s^{-\frac{\alpha}{2}})ds. \qquad (6.39)$$

Substitution the estimates (6.38) and (6.39) in equation (6.35), we get

$$\begin{aligned}
\|(\mathscr{T} f_1)(t)-(\mathscr{T} f_2)(t)\|_{p,\lambda}^p &\leq C(C_5L)^p\|f_1-f_2\|_{p,\lambda}^p\int_0^t e^{-\lambda ps}(s^{\frac{1}{2p}-\frac{1}{2}}\vee s^{-\frac{\alpha}{2}})ds \\
&\quad + C(C_6L)^p\|f_1-f_2\|_{p,\lambda}^p\int_0^t e^{-\lambda ps}(s^{\frac{1}{2p}-1}\vee s^{-\frac{\alpha}{2}})ds \\
&= C(L)^p\left(\int_0^t e^{-\lambda ps}\left((C_5)^p(s^{\frac{1}{2p}-\frac{1}{2}}\vee s^{-\frac{\alpha}{2}})+(C_6)^p(s^{\frac{1}{2p}-1}\vee s^{-\frac{\alpha}{2}})\right)ds\right)\|f_1-f_2\|_{p,\lambda}^p.
\end{aligned}$$

Since $\lambda>0$ is an arbitrary constant, for some suitable $\lambda>0$, there exists a $0<\eta<1$ such that

$$C(L)^p\left(\int_0^t e^{-\lambda ps}\left((C_5)^p(s^{\frac{1}{2p}-\frac{1}{2}}\vee s^{-\frac{\alpha}{2}})+(C_6)^p(s^{\frac{1}{2p}-1}\vee s^{-\frac{\alpha}{2}})\right)ds\right)<\eta<1.$$

Therefore, we have

$$\|(\mathscr{T} f_1)(t)-(\mathscr{T} f_2)(t)\|_{p,\lambda}^p<\eta\|f_1-f_2\|_{p,\lambda}^p, \qquad (6.40)$$

where $\eta<1$. This shows that the mapping $\mathscr{T}:\mathfrak{B}_{p,\lambda}\to\mathfrak{B}_{p,\lambda}$ is a contraction. Consequently, by Banach fixed point theorem, there exist a unique fixed point for $\mathscr{T}$ in $\mathfrak{B}_{p,\lambda}$ and this fixed point is indeed the solution to the equation (6.27).

Further, it is shown that the solution for equation (6.27), obtained above gives the unique local solution to equation (6.1) by following the similar technique established in [137]. Let f_N be an unique solution to the equation (6.27) for each $N\in\mathbb{N}$. Now, we define the $\mathscr{F}_t$-adapted stochastic stopping time

$$\tau_N := \inf\left\{t\in[0,T]:\int_D|f_N(t,x,\omega)|^p dx\geq N^p\right\}, \qquad for\ every\ \omega\in\Omega.$$

It is observed that the sequence $\{\tau_N\}_{N\in\mathbb{N}} \subset [0,T]$ is increasing $\mathbb{P}$- a.s. and therefore $\lim_{N\to\infty}\tau_N$ exists. Let us denote this by

$$\tau := \lim_{N\to\infty} \tau_N(\omega) \qquad \forall \omega \in \Omega.$$

Now, the contraction property of $\mathscr{T}$ implies that

$$f_M(t,x,\cdot) = f_N(t,x,\cdot) \qquad \forall (t,x) \in [0,\tau_N) \times D,$$

holds $\mathbb{P}$-a.s., for each $M \geq N$. Therefore, for any $N \in \mathbb{N}$ we define

$$f(t,x,\omega) := f_N(t,x,\omega) \qquad \forall (t,x,\omega) \in [0,\tau_N) \times D \times \Omega.$$

Thus, by passing the limit $N \to \infty$, we obtained the stochastic process

$$\{f(t,x,\omega) : (t,x,\omega) \in [0,\tau) \times D \times \Omega\},$$

as a local solution of (6.1) in the sense of the Definition 6.3.2 of the local solution.
At last, the uniqueness of local solutions to (6.1) is investigated. Let us assume that f_1 and f_2 are any two local solutions to (6.1). Then for each $N \in \mathbb{N}$, f_1 and f_2 satisfy the equation (6.27). But, we have already proved that (6.27) has a unique solution, we get

$$f_1(t,x,\omega) = f_2(t,x,\omega) \qquad\qquad (t,x,\omega) \in [0,\tau_N) \times D \times \Omega.$$

Finally, the proof can be concluded by passing the limit $N \to \infty$. □

Theorem 6.3.2 mainly gives the unique local solution to equation (6.1) due to the presence of polynomial type nonlinearity in q and the mixed fractional operators. But, if we take $q = 0$ in **(H3)** and **(H5)**, we get stochastic heat equation driven by mixed fractional operators $(\Delta + \Delta_\alpha)$ and fractional Brownian sheet. Indeed for this, we have the following result

Corollary 6.3.3. *Let (H1)-(H5) holds with $q(t,x,r) = 0$ for every $(t,x) \in [0,\infty) \times D$ and $r \in \mathbb{R}$. Then there exists an unique global solution to the equation (6.1) satisfying*

$$\sup_{t\in[0,T]} \mathbb{E}\|f(t,\cdot,\cdot)\|_{L^p}^p < \infty.$$

Proof. This can be shown similar to Theorem 6.3.2. □

Remark 6.3.1. In this article, we have proved our result in one dimension because for dimension $d > 1$ estimate (6.13) does not hold due to the condition (6.2.3). Therefore, it restricts us up to one dimension. However, in the case of stochastic heat equation i.e. for $q = 0$, we can extend the result of Corollary 6.3.3 up to two dimensions.

6.4 Appendix

Lemma 6.4.1. *The map $\Lambda_N : L^p \to L^p$, defined in (6.23) is globally Lipschitz.*

Proof. The mapping Λ_N is given as

$$\Lambda_N(\xi) := \begin{cases} \xi, & \|\xi\|_{L^p} \leq N, \\ \frac{N\xi}{\|\xi\|_{L^p}}, & \|\xi\|_{L^p} > N. \end{cases} \tag{6.41}$$

Let $\xi_1, \xi_2 \in L^p(D)$. Then we have the following cases:

1. $\|\xi_1\|_{L^p} < N$ and $\|\xi_2\|_{L^p} < N$. By (6.41), we have

$$\|\Lambda_N(f_1) - \Lambda_N(f_2)\|_{L^p} < \|\xi_1 - \xi_2\|_{L^p}.$$

2. $\|\xi_1\|_{L^p} < N$ and $\|\xi_2\|_{L^p} > N$. By (6.41), we have

$$\begin{aligned}
\|\Lambda_N(\xi_1) - \Lambda_N(\xi_2)\|_{L^p} &= \left\| \xi_1 - \frac{N\xi_2}{\|\xi_2\|_{L^p}} \right\|_{L^p} \\
&= \frac{1}{\|\xi_2\|_{L^p}} \|\xi_1 \|\xi_2\|_{L^p} - N\xi_2\|_{L^p} \\
&= \frac{1}{\|\xi_2\|_{L^p}} \|\xi_1 \|\xi_2\|_{L^p} - N\xi_1 + N\xi_1 - N\xi_2\|_{L^p} \\
&\leq \frac{1}{\|\xi_2\|_{L^p}} \{\|\xi_1(\|\xi_2\|_{L^p} - N)\|_{L^p} + N\|\xi_1 - \xi_2\|_{L^p}\} \\
&= \frac{1}{\|\xi_2\|_{L^p}} \{\|\xi_1\|_{L^p}(\|\xi_2\|_{L^p} - N) + N\|\xi_1 - \xi_2\|_{L^p}\} \\
&\leq \frac{1}{\|\xi_2\|_{L^p}} \{\|\xi_1\|_{L^p}(\|\xi_2\|_{L^p} - \|\xi_1\|_{L^p}) + N\|\xi_1 - \xi_2\|_{L^p}\} \\
&\leq \frac{1}{\|\xi_2\|_{L^p}} \{\|\xi_1\|_{L^p}(\|\xi_2 - \xi_1\|_{L^p}) + N\|\xi_1 - \xi_2\|_{L^p}\} \\
&\leq \frac{N + \|\xi_1\|_{L^p}}{\|\xi_2\|_{L^p}} \|\xi_1 - \xi_2\|_{L^p} \\
&\leq 2\|\xi_1 - \xi_2\|_{L^p}.
\end{aligned}$$

3. Let $\|\xi_1\|_{L^p} > N$ and $\|\xi_2\|_{L^p} < N$. This case can be solved as similar as done in above case 2.

4. Let $\|\xi_1\|_{L^p} > N$ and $\|\xi_2\|_{L^p} > N$. Then by (6.41), we have

$$\begin{aligned}
\|\Lambda_N(\xi_1) - \Lambda_N(\xi_2)\|_{L^p} &= \left\| \frac{N\xi_1}{\|\xi_1\|_{L^p}} - \frac{N\xi_2}{\|\xi_2\|_{L^p}} \right\| \\
&= \frac{N}{\|xi_1\|_{L^p}\|\xi_2\|_{L^p}} \big\| \xi_1\|\xi_2\|_{L^p} - \xi_2\|\xi_1\|_{L^p} \big\|_{L^p} \\
&= \frac{N}{\|\xi_1\|_{L^p}\|\xi_2\|_{L^p}} \big\| \xi_1\|\xi_2\|_{L^p} - \xi_2\|\xi_2\|_{L^p} + \xi_2\|\xi_2\|_{L^p} - \xi_2\|\xi_1\|_{L^p} \big\|_{L^p} \\
&= \frac{N}{\|\xi_1\|_{L^p}\|\xi_2\|_{L^p}} \big\| \|\xi_2\|_{L^p}(\xi_1 - \xi_2) + \xi_2(\|\xi_2\|_{L^p} - \|\xi_1\|_{L^p}) \big\|_{L^p} \\
&\leq \frac{N}{\|\xi_1\|_{L^p}\|\xi_2\|_{L^p}} \big(\|\xi_2\|_{L^p}\|\xi_1 - \xi_2\|_{L^p} + \|\xi_2\|_{L^p}(\|\xi_2\|_{L^p} - \|\xi_1\|_{L^p}) \big) \\
&\leq \frac{N}{\|\xi_1\|_{L^p}\|\xi_2\|_{L^p}} \big(\|\xi_2\|_{L^p}\|\xi_1 - \xi_2\|_{L^p} + \|\xi_2\|_{L^p}(\|\xi_2 - \xi_1\|_{L^p}) \big) \\
&= \frac{2N\|\xi_2\|_{L^p}}{\|\xi_1\|_{L^p}\|\xi_2\|_{L^p}} \|\xi_1 - \xi_2\|_{L^p} \\
&= \frac{2N}{\|\xi_1\|_{L^p}} \|\xi_1 - \xi_2\|_{L^p} \\
&\leq 2\|\xi_1 - \xi_2\|_{L^p}.
\end{aligned}$$

□

At the end, the following inequalities are given.

6.4.1 Minkowoski's Inequality in integral form.

[105, p. 47] Let $1 \leq p < \infty$ and $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is Lebesgue measurable. Then

$$\left(\int_{\mathbb{R}} \left| \int_{\mathbb{R}} f(x,y) dx \right|^p dy \right)^{\frac{1}{p}} \leq \int_{\mathbb{R}} \left(\int_{\mathbb{R}} |f(x,y)|^p dy \right)^{\frac{1}{p}} dx. \tag{6.42}$$

6.4.2 Young's Inequality for Convolution

[105, p. 99] Let $p, q, r \in \mathbb{R}$ such that $1 \leq p, q, r < \infty$ and

$$1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{q}.$$

Let $f \in L^p(\mathbb{R})$ and $g \in L^q(\mathbb{R})$. Then the convolution $f * g \in L^r(\mathbb{R})$ and we have the following inequality

$$\|f * g\|_{L^r} \leq \|f\|_{L^p}\|g\|_{L^q}. \tag{6.43}$$

Chapter 7

Conclusions and Future Scope

The present thesis is devoted to study the existence and uniqueness of the solution to four different generalized stochastic Burgers-type equations.

First of all, the existence of weak solutions was shown for the one-dimensional generalized stochastic Burgers equation having polynomial nonlinearity perturbed by space-time white noise with Dirichlet boundary conditions and non-Lipschitz coefficient in the noise term. The existence result was investigated by solving an equivalent martingale problem.

Next, the existence and uniqueness of local and global mild solution is shown to the one-dimensional generalized stochastic Burgers equation (GSBE) containing a non-linearity of polynomial type and perturbed by α-regular cylindrical Volterra noise having Dirichlet boundary conditions. In order to get the solvability and regularity estimates for the linear system, the L^p-theory of stochastic convolution integral, developed in [47], was used. It is shown that there exists a unique local mild solution for polynomial type of nonlinearity using contraction mapping principle, and also the global existence and uniqueness for third order nonlinear GSBE using probabilistic arguments. The biggest challenge, when considering α-regular cylindrical Volterra noise, was the absence of L^∞-estimate on both time and space for the stochastic convolution involving such processes. We obtained this estimate with the help of Garsia-Rodemich-Rumsey inequality. Further, we have also assured the existence of an invariant measure for quadratic nonlinear GSBE perturbed by Volterra processes of Gaussian type. In order to prove this, we adopted the method developed by G. Da Prato et al. in [51].

Further, the existence and uniqueness of solutions to the fractional Burgers-type nonlinear stochastic partial differential equation driven by cylindrical fractional Brownian motion in Hölder spaces were shown by using a finite dimensional Galerkin approximation. Moreover, the rate of convergence of the Galerkin approximation as well as fully discretization of solution were also achieved.

Finally, we considered a class of stochastic nonlinear partial differential equation of Burgers -type driven by pseudo differential operator $(\Delta+\Delta_\alpha)$ for $\alpha \in (0,2)$, and perturbed by the fractional Brownian sheet. The existence and uniqueness of an L^p-valued (local) solution were established for the initial boundary value problem to this equation.

At the end, we propose the following open problems which we found while working for this thesis.

- In Chapter 3, we have shown only the existence of weak solutions, while the uniqueness of this problem is still an open problem. We have tried to work out for uniqueness but due to some technical difficulties, we could not prove this. A similar type of problem has been considered in [27], where the authors claims that the uniqueness is still an open problem due to some technical difficulties.

- In Chapter 6, we have shown only the local existence and uniqueness of the problem. Therefore, the existence and uniqueness of global solution to the given equation is still an open problem.

- In the Chapter 6, we have proved our result for one dimensional problem because in case of multi-dimension, the estimate (6.13) does not hold due to the condition (6.2.3) which restricted us up to one-dimensional problem only. Therefore, a future work would be to solve the problem (6.1) in Chapter 6 in multi-dimension.

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