EXISTENCE OF SOLUTIONS AND APPROXIMATE CONTROLLABILITY OF SOME EVOLUTION EQUATIONS

Ph. D. THESIS

by

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ROORKEE – 247 667 (INDIA)
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CANDIDATE'S DECLARATION

I hereby certify that the work which is being presented in the thesis entitled "**EXISTENCE OF SOLUTIONS AND APPROXIMATE CONTROLLABILITY OF SOME EVOLUTION EQUATIONS**" in partial fulfillment of the requirements for the award of the Degree of Doctor of Philosophy and submitted in the Department of mathematics of the Indian Institute of Technology Roorkee, Roorkee is an authentic record of my own work carried out during a period from July, 2014 to July, 2019 under the supervision of Dr. D. N. Pandey, Associate Professor, Department of Mathematics, Indian Institute of Technology Roorkee, Roorkee.

The matter presented in this thesis has not been submitted by me for the award of any other degree of this or any other Institution.

(Arshi Meraj)

This is to certify that the above statement made by the candidate is correct to the best of my knowledge.

(D. N. Pandey)
Supervisor

The Ph. D. Viva-Voce Examination of Arshi Meraj, Research Scholar, has been held on 08/07/2019.

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Signature of Supervisor (s) **Head of the Department**

Dated: ………July,2019

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Abstract

The work presented in this thesis deals with the investigation of the existence of mild solutions and approximate controllability of some fractional and integer order differential and integro-differential equations. To prove our results, we will use semigroup theory, evolution system, fixed point theorems, fractional calculus, measure of noncompactness, basic theory of functional analysis, and monotone iterative technique. The present work consists of the following eight chapters.

Chapter 1 contains a brief introduction to the problems which are discussed in the consecutive chapters and provides a motivational background to study the problems which are discussed in this thesis. Further, it contains a review of relevant literature and an outline of the thesis.

Chapter 2 contains some basic concepts of fractional calculus, functional analysis, semigroup theory and measure of noncompactness that will be required in the subsequent chapters.

Chapter 3 concerns with the study of a fractional nonlocal neutral integro-differential equation having flux type integral boundary conditions. The existence and uniqueness results are proved by using Banach and Leray-Schauder nonlinear alternative fixed point theorems.

Chapter 4 contains fractional integro-differential equations having non-instantaneous impulses. The existence result is obtained by the help of fixed point theorem and

noncompact semigroup.

Chapter 5 consists of fractional nonlocal semilinear integro-differential equations having impulsive conditions for which the impulses are not instantaneous. The approximate controllability is proved via semigroup theory, Kuratowski measure of noncmpactness and ρ-set contractive fixed point technique, without imposing the condition of Lipschitz continuity on nonlinear term as well as the condition of compactness on impulsive functions and nonlocal function.

Chapter 6 contains deformable fractional differential equations. The results of existence and approximate controllability are obtained via semigroup theory, Schauder and Banach fixed point technique.

Chapter 7 considers non-autonomous semilinear differential equations having nonlocal conditions. The existence and uniqueness are obtained via monotone iterative method with the upper and lower solutions in an ordered complete norm space, using evolution system and measure of noncompactness.

Chapter 8 extends the results of chapter 7 for non-autonomous integro-differential equations having nonlocal conditions.

The relevant references are appended at the end.

Table of Contents

Nomenclature

Notation	Description
$\mathbb{R}$	Set of Real numbers
$\mathbb{N}$	Set of Natural numbers
$\mathbb{Z}$	Set of Integers
$\mathbb{C}$	Set of Complex numbers
$B_r(x, X)$	Ball centered at x with radius r in X
$C([a,b], X)$	Set of all continuous maps from $[a,b]$ into X
$\mathcal{PC}([a,b], X)$	Set of all piecewise continuous maps from $[a,b]$ into X
$W^{m,p}([a,b], X)$	Sobolev space
$\Gamma(n)$	Euler's continuous gamma fuctions
J^q	Riemann-Liouville (in short RL) fractional integral operator of order q
${}^L\mathbf{D}^q$	Riemann-Liouville fractional differential operator of order q
${}^C\mathbf{D}^q$	Caputo fractional differential operator of order q
β	Kuratowski measure of noncompactness
χ	Hausdroff measure of noncompactness
iff	if and only if

Chapter 1

Introduction

By the evolution equations, we mean the abstract formulation of the prototype of several problems of similar kind. In an infinite dimensional space of functions, an ordinary differential equation which is generally associated with a partial differential equation modeling a physical phenomenon may be considered as an evolution equation. Laser optics, reaction diffusion equations, climate models, control theory, neural networks, heat conduction and wave propagation in materials can be modeled as abstract evolution equations in a Hilbert space or in a Banach space. In modeling a real world problem as an evolution equation, initial conditions are managed as essential conditions while the boundary conditions are treated as natural conditions because those are fused with the domain of the operator and therefore do not appear in the abstract formulation. In this way, we able to focus on certain invariant properties of a class of such problems instead of examining individual problems.

Researchers in applied mathematics and engineering have found differential equations having nonlocal conditions invaluable because of their practical applications to various physical problems such as heat conduction, nonlocal reactive transport in underground water flows in porous media, biotechnology, population dynamics, investigation of pollution process in rivers, seas etc. Particularly, introduction of nonlocal conditions into a system can improve its qualitative and quantitative characteristics. These nonlocal conditions come into picture while taking measurements at regular intervals rather than continuously over the history period. Therefore, the analysis can be presented more practically by using nonlocal conditions instead of using classical initial conditions.

A differential equation in which the highest order derivative of the unknown function appears with and without deviations is called neutral differential equation. For example,

$$z'(t) - cz'(t-\varrho) - dz(t-\varrho) = 0, \ \varrho > 0, \ c \neq 0, \tag{1.0.1}$$

is a neutral differential equation of first order. Neutral differential equations appear in problems dealing with electric networks involving lossless transmission lines.

Fractional calculus, the generalization of traditional calculus, deals with the investigation and applications of derivatives and integrals of non-integer order. In particular, in this branch of mathematics, we study the methods of solving differential equations containing fractional derivatives of the unknown function, such equations are called fractional differential equations (in short FDEs). The analysis of nonlinear oscillations of an earthquake, continuum and statistical mechanics, relaxation in fluid polymers and the modeling of visco-plasticity are some of the fields in which employment of differential equations involving fractional derivative gives a more realistic analysis of the problem considered.

Due to involvement of integral operator in the definition of fractional derivative, fractional differential operator is a non-local operator. That is, a fractional derivative at a certain point in space or time consists of information about the function at previous points in space or time, respectively. Therefore, fractional differential equations describe the hereditary and memory properties of various materials and phenomena. For example, viscoelastic materials and polymers which are related to systems with memory can be efficiently described with fractional differential equations.

Various evolutionary processes such as population dynamics, orbital transfer of satellites and sampled-data systems are characterized by the abrupt changes in their state. These abrupt changes occur for a very short interval of time and can be approximated in terms of instantaneous changes of state, i.e., impulses. Such processes can be appropriately modeled by impulsive differential equations. In past few years, the theory of impulsive differential equation have been emerged as a beneficial tool, which can precisely define mathematical model of various realistic situations, for example biological phenomena which involves optimal control models in economics, thresholds, bursting rhythm models in medicine.

Generally, the impulses start abruptly for very short duration of time that can be negligible in comparison to the overall process. But instantaneous impulses are failed to describe the certain dynamics of evolution processes in pharmacotherapy. The introduction of the drugs in the bloodstream and the resultant absorption to the body are gradual and continuous processes. To characterize these type of situations, Hernàndez and O′Regan [95] introduced new type of impulses which triggers abruptly and remains active during finite time. These type of impulsive conditions are called non-instantaneous impulsive conditions.

For most of the differential equations, it is difficult to find exact solutions in closed forms. To overcome this difficulty, many numerical and analytical techniques have been designed for example, the homotopy analysis method and the Adomian decomposition method have been applied to integrate various systems of fractional order. However, in recent years, considerable work has been done using monotone iterative technique (in short MIT), which is a productive procedure for proving existence results in a closed interval formed by the lower and upper solutions. In MIT, by choosing upper and lower solutions as two initial iterations, one may construct two monotone sequences which converge monotonically from above and below, respectively, to a solution of the problem. Ladde et al. [120] has described a comprehensive introduction to the monotone iterative techniques in their book "Monotone iterative techniques for nonlinear differential equations".

Process of influencing the behavior of a dynamical system to achieve a given goal can be described as the concept of control. Many physical systems can be controlled by manipulating their inputs based on the observation of the outputs, for example an airplane is controlled by the pilot's action based on instrument reading and visual observations. The control problem is to determine the necessary input to achieve a desired goal.

Consider the case of the vibrating system consisting of a single mass on linear spring , we obtain the differential equation of motion

$$\frac{d^2x}{dt^2} = -x$$

where x is displacement from equilibrium at time t. Can we introduce a control force u depending on x and $\frac{dx}{dt}$ so that every solution of

$$\frac{d^2x}{dt^2} = -x + u$$

returns to the rest equilibrium state $x = 0$, $\frac{dx}{dt} = 0$ after a finite duration? This is the problem of controllability.

Controllability theory originates from the famous work done by Kalman [107] which leads to very important conclusion regarding the behavior of linear and nonlinear dynamical systems. In case of infinite dimensional systems, basically there are two main concepts of controllability i.e. exact (complete) controllability and approximate controllability. Exact controllability steers the system to arbitrary desirable state while approximate controllability means that the system can be steered to arbitrary small neighborhood of desirable state. The exact controllability is stronger notion than approximate controllability, but approximate controllability is completely adequate in applications. However, in case of finite dimensional systems notions of exact and approximate controllability coincide.

1.1 Literature Survey

1.1.1 Existence of solutions

Many physical phenomena such as flow of fluid through fissured rocks [25], nonlinear oscillations of earthquake, relaxation phenomena in complex viscoelastic material [86], propagation of waves in viscoelastic media [136] can be described by fractional differential equations. Fractional derivative was first mentioned in a letter correspondence between Leibniz and L'Hospital in 1695. Later on, many famous mathematicians e.g. Euler, Laplace, Fourier, Abel, Grünwald, Riemann, Liouville, Caputo etc. provided a lot of contribution in this field. For basic theory of FDEs one may see the books [114; 140; 158; 167].

In [76], author studied existence, uniqueness and some properties of the solution of the equation

$$\frac{d^q x(t)}{dt^q} = f(t, x(t)), \quad t > 0, \tag{1.1.1}$$

where $q \in (0, 1)$. El-Sayed [77] studied existence, uniqueness, smoothness and continuation to the solution of initial value problem corresponding to (1.1.1) for $q > 0$. Delbosco [67] reduced equation (1.1.1) into an integral equation with weak singularity and applied basic techniques of nonlinear analysis to discuss the existence results. El-Sayed [78] studied the existence results for the diffusion wave equation of fractional order

$$D^q z(t) = Az(t), \quad t > 0, \quad q \in (0, 2], \tag{1.1.2}$$

where A is generator of an analytic semigroup $\mathcal{S}(t)$. Equation (1.1.2), represents the diffusion equation when $q \to 1$ and the wave equation when $q \to 2$.

In [75], El-Borai introduced mild solution in terms of probability density function with Laplace transform to Cauchy problem in a Banach space $\mathbb{X}$. El-Borai [74] studied the existence results for the FDE

$$D^q z(t) = Az(t) + F(t, B_1(t)z(t), \dots, B_m(t)z(t)) \quad t > 0,\, q \in (0, 1], \tag{1.1.3}$$

where A generates an analytic semigroup, function F satisfies uniformly Hölder continuity in t and $B_j(t), j = 1, 2, \dots m$ are densely defined closed linear operators on $\mathbb{X}$. In [153] Nyamoradi et al. investigated results for a fractional order differential inclusion via suitable fixed point theorems. In [203] and [204], Zhou obtained various results on solutions for fractional evolution equations. Later on, utilizing the concept of mild solution introduced by Zhou [203], many authors studied different type of fractional differential equations, see [1; 8; 19; 28; 40; 43; 45; 65; 71; 72; 83; 96; 135; 146; 162; 190] and the references therein.

Recently, Khalil [112] introduced a new fractional derivative named as conformable fractional derivative which seems to be a natural extension of the classical derivative. In [112], authors gave the definition of conformable fractional derivative and corresponding fractional integral also they proved the basic properties of conformable derivative, Rolle's Theorem and Mean Value Theorem for conformable derivative. For more details one may see [2; 3; 14; 15; 30; 93; 101; 154; 193; 199] and the references therein. In [205], authors defined a new derivative called as deformable derivative which is simpler than conformable derivative and ranges over wider class of functions.

The nonlocal problem was motivated by physical problems. It is used to represent mathematical models for evolution of various phenomena such as nonlocal neutral networks, nonlocal pharmacokinetics, nonlocal pollution and nonlocal combustion (see [138]). The nonlocal problem was first studied by Byszewski [37]. Deng [68] used the nonlocal conditions to describe diffusion phenomenon of an small amount of gas. Byszewski [37], Byszewski and Lakshmikantham [38] and Jackson [100] have generalized the Cauchy problem with classical initial condition to the Cauchy problem with nonlocal initial conditions. Later, many authors stududied the differential equations with nonlocal conditions, see [8; 16; 22; 28; 40; 45; 65; 70; 71; 152; 186; 187] and references therein.

Boundary value problems (in short BVPs) concerning fractional differential equations having integral boundary conditions are applicable in several fields such as thermoelasticity, chemical engineering, blood flow problem, underground water flow, cellular systems, heat transmission, plasma physics, population dynamics and so on. For more details regarding these boundary conditions, one may see the papers [6; 7; 9; 10; 35; 39; 50; 105; 106; 119].

Mouffak [142] considered neutral impulsive differential equations of first and second order to investigate the existence of solutions. He used Schaefer fixed point theorem to obtain the results. In [5], Shruti considered a class of neutral differential equations with nonlocal conditions to investigate the existence of solution. She proved the results with the help of Schauder fixed point theorem. Zhou [203] obtained various results for the existence and uniqueness associated to neutral differential equations of fractional order. Alka [41] discussed the existence of solutions for fractional neutral delay differential system having nonlocal conditions by using resolvent operator, Banach and Krasnoselskii fixed point theorems. In [202], Zhou investigated the existence, uniqueness and dependence of solution on initial value by iterative technique for linear neutral fractional differential equations with constant coefficient. Malik [137] considered non-autonomous neutral differential system of fractional order and studied the existence results via resolvent operators and Banach contraction principle. In [97] results are obtained for neutral delay differential equations via contraction mapping fixed point theorem and semigroup theory.

In the past decades, many researchers paid attention to study the differential equations with instantaneous impulses, which have been used to describe abrupt changes such as shocks, harvesting and natural disasters. Particularly, the theory of instantaneous impulsive equations have wide applications in control, mechanics, electrical engineering, biological and medical fields. In order to solve impulsive fractional differential equations there are two main approaches. The first approach (also called multiple base point approach) was introduced by Benchohra and Slimani [33] in which they considered a fractional Cauchy problem with impulsive effects, and developed the method to find the existence and uniqueness of solutions by using several fixed point theorems. The second approach (also called the single base point approach) was introduced by Fečkan et al. [81] in which they developed single base point approach to find existence results via fixed point techniques. Wang [160] considered semilinear fractional differential equation with impulsive conditions, and introduced the new concept of mild solution using semigroup, Laplace transform and probability density function. For more details, one may see the monographs [31; 122; 168], papers [42; 43; 44; 51; 63; 103; 125; 126; 170; 171; 183].

Hernández et al. [95] initially studied Cauchy problems of first order with non-instantaneous impulses. Pradeep et al. [117] considered fractional non-instantaneous impulsive system to investigate the existence results. In [116] results of [117] are extended for non-instantaneous impulsive integro-differential equations of fractional order. Muslim [147] considered second order differential equations with deviated argument having non-instantaneous impulses, and results are obtained by using strongly continuous cosine family of linear operators and Banach fixed point theorem. The recent results for evolution equations with non-instantaneous impulses can be found in [17; 18; 47; 57; 113; 117; 151; 157; 185; 191; 194] and the references therein.

Du [69] proposed a monotone iterative technique (in short MIT) for ordinary differential equations in an ordered Banach space $\mathbb{X}$. He proved the existence of minimal and maximal mild solutions lying between the lower and upper solutions. In [98] and [99], Hristova and Bainov applied MIT to functional and impulsive differential equations. In [173], Sun improved the result of Du [69] by removing the measure of noncompactness condition on function f. In [174], Sun and Zhao improved the result of [173] further and investigated the results for integro-differential equations in a weakly sequentially complete Banach space $\mathbb{X}$ to find extremal solutions. In [89], Guo and Liu used MIT to find the existence of extremal solutions of mixed type impulsive integro-differential equation. Later, Li [124] improved the result of Guo and Liu [89] by using Bellman inequality.

Chen and Li [52] used monotone iterative technique for nonlocal differential equations. Chen and Mu [56] considered impulsive integro-differential equations and obtained results via monotone iterative method. Mu [143] applied MIT for fractional differential equations. In [145], Mu and Li used MIT for impulsive differential equations of fractional order. Later on, the result has been extended for nonlocal condition by Mu [144]. Chen et al. [54] considered nonlocal impulsive semilinear system and applied perturbation method and MIT to obtain the results. In [53] a differential equation with mixed monotone nonlocal conditions of fractional order is considered to study the existence results via coupled lower and upper mild L-quasi solutions and monotone iterative technique. Kamaljeet [109] and Renu [49] used MIT for nonlocal differential equations having finite delay and for neutral differential equations having infinite delay of fractional order respectively. Chen et al. [61] used perturbation method and iterative technique for evolution equations involving non-instantaneous impulses. For more details on MIT, one may see [55; 128; 139; 177; 188; 197].

Yan [189] considered non-autonomous integro-differential equations involving nonlocal conditions to study the existence results via theory of evolution system, Banach and Schauder fixed point theorems (in short FPTs). Haloi et al. [92] studied existence, uniqueness and asymptotic stability of non-autonomous differential equations with deviated arguments via Banach fixed point theorem and theory of analytic semigroup. In [46], Alka et al. established the existence and uniqueness results for non-autonomous differential equations having iterated deviating arguments and instantaneous impulsive conditions. Borai [73] first derived the sufficient conditions for the existence of solution for a fractional non-autonomous Cauchy problem. Li [123] considered fractional non-autonomous integro-differential equations involving nonlocal condition. Chen et al. [58] generalized the results of [73] for fractional non-autonomous evolution equations with delay by using FPT with respect to ρ-set contractive operator. Nieto [32] considered second order non-autonomous evolution equations having nonlocal conditions, and established the sufficient conditions for the existence of solutions by using measure of noncompactness and Tikhonoff fixed point theorem. In [60], Chen et al. considered fractional nonlocal non-autonomous differential equations, and established the existence of mild solutions via theory of evolution families and FPT with respect to ρ-set contractive operator. In [59], authors first studied the local existence of mild solutions, then obtained a blowup alternative result for fractional non-autonomous integro-differential equation of Volterra type.

1.1.2 Controllability

The theory of controllability is introduced by Kalaman (1960) [107]. For the basic theory of controllability in finite and infinite dimensional spaces one may see [20; 26; 34; 62; 195]. Fattorini (1966) [79] studied approximate controllability of linear control system $x'(t) = Ax(t) + Bu(t)$ in infinite dimensional reflexive Banach space, by assuming that A is a linear densely defined closed operator which generates a C_0 semigroup. In [80], he discussed approximate controllability in separable Hilbert space by assuming that A is self adjoint and semibounded above operator.

Triggiani (1975) [179] considered infinite dimensional linear control system to study the approximate controllability in separable Banach space under the assumption that the operator A is bounded. He also proved that infinite dimensional linear control system can never be exactly controllable in finite time by using L_1 controls, if either the operator B is compact and the state space has a Schauder basis or range space of B is of finite dimension. In [180], he improved the results of [179] by removing the assumption that the state space has a basis. In [181], author proved that linear control system in infinite dimensional space can never achieve exact controllability in finite time by using L_1 control, if C_0 semigroup generated by A is compact.

Zhou (1983) [200] considered a class of semilinear system $x'(t) + Ax(t) = \mathcal{F}(x(t)) + Bu(t)$ in a Hilbert space, where $-A$ generates a differentiable semigroup. He assumed that the function $\mathcal{F}$ satisfies Lipschitz continuity, and obtained approximate controllability of the system by imposing an inequality condition on the range of operator B as well as approximate controllability of associated linear system. In [201], he studied approximate controllability for generalized system $x'(t) + Ax(t) = \mathcal{F}(x(t), u(t)) + Bu(t)$.

Naito (1987) [148] proved approximate controllability for semilinear system under the assumptions that $-A$ generates a C_0 semigroup, nonlinear function is uniformly bounded as well as Lipschitz continuous, and some condition on the range of operator B. In [149], he removed the condition of uniformly bounded from nonlinear function and introduced an inequality condition containing the parameters: $K > 0$ (Lipschitz constant of $\mathcal{F}$), $T > 0$ (control time), $M \geqslant 1$ (bound of C_0 semigroup), and $\|P\| \geqslant 1$ (norm of projective type operator P introduced by estimating the control efficiency of operator B) with the assumption that $\mathcal{F}(0) = 0$.

Joshi et al. (1989) [104] considered non-autonomous semilinear control system in finite dimensional space. They discussed the controllability of the system by using contraction mapping principle, monotone operator theory, and nonlinearities of both Lipschitian and non-Lipschitzian types. George (1995) [85] investigated the approximate controllability of infinite dimensional non-autonomous semilinear control system by using monotone operator theory, evolution operator, and nonlinearities of both Lipschitian and non-Lipschitzian types. Bashirov et al. (1999) [29] determined new necessary and sufficient conditions for a linear system to be approximately and exactly (complete) controllable.

Dauer et al. (2002) [64] considered semilinear system to study approximate and complete controllability. The approximate controllability is proved via compact semigroup and Schauder fixed point technique, complete controllability is attained with the help of non-compact semigroup and Banach contraction principle. Mahmudov (2003) [131] obtained the approximate controllability for semilinear deterministic and stochastic systems by using compact semigroup, properties of symmetric operators, Banach and Schauder fixed point theorems with the assumption that associated linear system is controllable. Sakthivel et al. (2010) [164] considered semilinear impulsive control system with state delay, to investigate the approximate controllability. Haloi (2017) [91] considered non-autonomous delay system having deviated arguments. He used Krasnoselskii fixed point theorem and evolution system to prove the approximate controllability.

Balachandran et al. (2009) [21] studied controllability for fractional semilinear integro-differential equations by using Banach contraction principle, semigroup theory, and theory of fractional calculus. Tai et al. (2009) [175] considered fractional semilinear neutral impulsive integro-differential system having infinite delay. They determined sufficient conditions for the controllability of the system with the help of Krasnoselskii fixed point theorem, theory of fractional calculus and semigroup. Sakthivel et al. (2011) [166] discussed approximate controllability for a class of fractional semilinear system. They obtained the results by using theory of fractional power, semigroup, and Schauder fixed point theorem. In [165] (2012), authors studied exact controllability for semilinear fractional neutral delay differential system with local and nonlocal conditions by using Banach contraction principle, semigroup theory, and approximate controllability of corresponding linear system.

Kumar et al. (2013) [118] discussed the exact controllability of fractional semilinear system. To obtain the results, they used regular integral contractor, iterative technique, and the controllability of corresponding linear system. Mahmudov et al. (2014) [132] considered fractional semilinear integro-differential system, and established sufficient conditions for the approximate controllability by the help of Schauder fixed point theorem, theory of semigroup and fractional calculus. In [23] (2014), Balasubramaniam et al. generalized the results of [132] for fractional impulsive integro-differential equations with nonlocal conditions, by assuming the compactness of impulsive and nonlocal functions in a Hilbert space. The results are proved with the help of Darbo-Sodovskii fixed point theorem, theory of semigroup and fractional calculus. Zhang et al. (2015) [198] discussed the approximate controllability of fractional impulsive integro-differential equations in a Hilbert space with the help of Krasnoselskii fixed point theorem and comapact analytic semigroup theory. In [169] (2015), authors determined sufficient conditions for the controllability of fractional semilinear non-autonomous neutral differential system. They used Krasnoselskii fixed point theorem, theory of semigroup and fractional power of operators.

In [127] (2015), Liu et al. considered fractional semilinear control system with Riemann-Liouville fractional derivative first time. They derived the expression of mild solution, and established sufficient conditions for the approximate controllability via generalized Banach contraction theorem by assuming that the semigroup is differentiable. Dong et al. (2016) [84] studied approximate controllability of semilinear impulsive fractional evolution equations via approximate technique. Kamaljeet et al. (2016) [110] considered fractional neutral semilinear integro-differential equations having nonlocal conditions and finite delay. They proved approximate controllability of the system via fractional power of operators, semigroup, and Krasnoselskii fixed point technique. Urvashi et al. (2017) [13] dealt with fractional nonlocal semilinear delay differential system. They proved exact controllability by the help of Nussbaum fixed point theorem, fractional calculus and semi group theory. Wang et al. (2017) [184] studied the approximate controllability of Sobolev-type nonlocal fractional semilinear system. For more literature on controllability, one may see [4; 27; 108; 48; 111; 102; 115; 129; 130; 133; 134; 141; 155; 159; 161; 172; 176; 182; 192] and the references therein.

1.2 Organization of Thesis

In this thesis, we divide our research work into three main parts.

In first part, we investigated the existence results to integro-differential equations of fractional order having nonlocal conditions. The main tools for this part are fixed point theorems, measure of noncompactness and semigroup theory.

In second part, we discussed the approximate controllability of differential systems with Caputo and deformable fractional derivative.

In third part, we studied the existence of extremal mild solutions to integer order non-autonomous nonlocal differential systems with the help of evolution system and monotone iterative technique.

The thesis contains following chapters.

Chapter 1 provides a brief introduction and motivational background to the problems considered in the consecutive chapters. Literature review associated to the work done in this area also has been discussed.

Chapter 2 contains some basic concepts of fractional calculus, semigroup theory, functional analysis and measure of noncompactness that are basically prerequisites for our work.

Chapter 3 concerns with the study of existence results for fractional neutral integro-differential equations having nonlocal flux type integral boundary conditions.

The contents of this chapter are published in **Malaya Journal of Matematik**.

Chapter 4 consists of fractional semilinear integro-differential equations having nonlocal and non-instantaneous impulsive conditions, and the existence result is obtained via noncompact semigroup and fixed point theorem.

The contents of this chapter are published in **Arab Journal of Mathematical Sciences (Elsevier Publications)**.

Chapter 5 contains fractional semilinear integro-differential equations having nonlocal and impulsive conditions for which the impulses are not instantaneous. The existence result and approximate controllability are obtained by using semigroup theory, Kuratowski measure of noncmpactness and ρ-set contractive fixed point theorem, without imposing the condition of Lipschitz continuity on nonlinear term as well as the condition of compactness on impulsive functions and nonlocal function.

The contents of this chapter are accepted in **Journal of Fractional Calculus and Applications**.

Chapter 6 contains deformable fractional differential equations. The existence and approximate controllability results are obtained via semigroup theory, Banach and Schauder fixed point theorems.

The contents of this chapter are accepted in **Journal of Nonlinear Evolution Equations and Applications**.

Chapter 7 consists of non-autonomous semilinear differential equations having nonlocal conditions. The existence and uniqueness results are established via monotone iterative method, using evolution system and measure of noncompactness.

The contents of this chapter are published in **Demonstratio Mathematica (DE GRUYTER)**.

Chapter 8 consists of non-autonomous integro-differential equations having nonlocal conditions. The monotone iterative method is applied to get the existence and uniqueness results.

The contents of this chapter are accepted in **Filomat**.

Chapter 2

Preliminaries

In this chapter, we provide some basic concepts of functional analysis, fractional calculus, semigroup theory and measure of noncompactness which serve as prerequisites for subsequent chapters.

2.1 Basic Concepts of Functional Analysis

Let $(\mathbb{X}, \|\cdot\|_{\mathbb{X}})$ and $(\mathbb{Y}, \|\cdot\|_{\mathbb{Y}})$ be two Banach spaces. $(\mathbb{B}(\mathbb{X},\mathbb{Y}), \|\cdot\|_{\mathbb{B}(\mathbb{X},\mathbb{Y})})$ denotes the space of all linear bounded operators from $\mathbb{X}$ to $\mathbb{Y}$ and for $\mathbb{Y} = \mathbb{X}$, we write $\mathbb{B}(\mathbb{X})$. In similar fashion, $(\mathcal{L}(\mathbb{X},\mathbb{Y}), \|\cdot\|_{\mathcal{L}(\mathbb{X},\mathbb{Y})})$ stands for the space of all linear operators from $\mathbb{X}$ to $\mathbb{Y}$. In the case $\mathbb{Y} = \mathbb{X}$, we write $\mathcal{L}(\mathbb{X})$. If A denotes the linear operator on $\mathbb{X}$, then the domain, null and range space of A are denoted by $D(A)$, $N(A)$ and $R(A)$ respectively. Let us denote $L^p(\mathcal{I},\mathbb{X})$ for Banach space of all Bochner-measurable functions $F : \mathcal{I} \to \mathbb{X}$ with the norm

$$\|F\|_{L^p(\mathcal{I},\mathbb{X})} := \begin{cases} (\int_{\mathcal{I}} \|F(s)\|_{\mathbb{X}}^p ds)^{1/p}, & 1 \le p < \infty, \\ \sup_{t\in\mathcal{I}} \|F(t)\|_{\mathbb{X}}, & p = \infty, \end{cases} \tag{2.1.1}$$

where $\mathcal{I} = (a, b)$ with $-\infty < a < b < \infty$.

The notations $C(\mathcal{I},\mathbb{X})$ and $C^m(\mathcal{I},\mathbb{X})$ stand for the space of all continuous and m-times continuously differentiable functions, respectively. Set $J = [0, T]$, $T < \infty$.

Then, $C(J,\mathbb{X})$ and $C^m(J,\mathbb{X})$ are the Banach spaces equipped with the norm

$$\|F\|_C := \sup_{t\in J}\|F(t)\|_{\mathbb{X}}, \quad \|F\|_{C^m} := \sup_{t\in J}\sum_{k=0}^{m}\|F^k(t)\|_C, \tag{2.1.2}$$

respectively.

Now, we state some important inequalities :

- **Hölder inequality**: Let $p \in [1,\infty)$ and q is such that $1/p + 1/q = 1$. Then

$$\|(FG)\|_{L^1(\mathcal{I},\mathbb{X})} \le \|F\|_{L^p(\mathcal{I},\mathbb{X})}\|G\|_{L^q(\mathcal{I},\mathbb{X})}, \tag{2.1.3}$$

  where $F \in L^p(\mathcal{I},\mathbb{X})$, $G \in L^q(\mathcal{I},\mathbb{X})$.

- **Young inequality**: Assume $F \in L^p(\mathcal{I},\mathbb{X})$, $G \in L^q(\mathcal{I},\mathbb{X})$ and $1/q + 1/p = 1/r + 1$ such that $1 \le q,p,r \le \infty$. Then

$$\|F * G\|_{L^r(\mathcal{I},\mathbb{X})} \le \|F\|_{L^p(\mathcal{I},\mathbb{X})}\|G\|_{L^q(\mathcal{I},\mathbb{X})}, \tag{2.1.4}$$

  where “ $*$ ” denotes the convolution.

- **Gronwall's inequality**: Let F and G be the non-negative continuous functions for each $t \ge t_0$ and β be a constant. Then the inequality

$$F(t) \le \beta + \int_{t_0}^{t} G(s)F(s)ds, \quad t \ge t_0, \tag{2.1.5}$$

  implies the following inequality

$$F(t) \le \beta\exp(\int_{t_0}^{t} G(s)ds), \quad t \ge t_0. \tag{2.1.6}$$

- **Generalized Gronwall's inequality** [90]: Let $F(t)$ and $G(t)$ be locally integrable nonnegative functions on $0 \leqslant t < T < \infty$ and $a \geqslant 0$, $\beta > 0$ be two constants with

$$F(t) \leqslant G(t) + a\int_0^t (t-\varrho)^{\beta-1}F(\varrho)d\varrho. \tag{2.1.7}$$

  Then

$$F(t) \leqslant G(t) + \int_0^t \left[\sum_{n=1}^{\infty}\frac{(a\Gamma(\beta))^n}{\Gamma(n\beta)}(t-\varrho)^{n\beta-1}G(\varrho)\right]d\varrho, \quad 0 \leqslant t < T. \tag{2.1.8}$$

Definition 2.1.1. *Suppose that* $I = (0,T)$*, or* $I = \mathbb{R}$*, or* $I = \mathbb{R}^+$*,* $m \in \mathbb{N}$ *and* $1 \le p < \infty$*. The Sobolev spaces* $W^{m,p}$ *is defined as*

$$\begin{aligned} W^{m,p}(I,\mathbb{X}) : &= \{F : \textit{there exists } z \in L^p(I,\mathbb{X}) : F(t) = \sum_{k=0}^{m-1} c_k \frac{t^k}{k!} \\ &+ \frac{t^{m-1}}{(m-1)!} * z(t),\ t \in I\}, \end{aligned} \tag{2.1.9}$$

where $z(t) = F^m(t)$*,* $c_k = F^k(0)$*. Also*

$$W_0^{m,p}(I,\mathbb{X}) := \{F \in W^{m,p}(I,\mathbb{X}) :\ F^k(0) = 0,\ k = 0,1,\cdots,m-1\}. \tag{2.1.10}$$

Definition 2.1.2. *The* ***Laplace transform*** *of a function* $G \in L^1(\mathbb{R}^+,\mathbb{X})$ *is defined as*

$$\widehat{G}(\lambda) := \int_0^\infty e^{-\lambda t} G(t) dt. \tag{2.1.11}$$

Definition 2.1.3. *Suppose that* $\widetilde{\mathbb{X}}$ *and* $\mathbb{X}$ *be Banach spaces. A mapping* $F : \widetilde{\mathbb{X}} \to \mathbb{X}$ *is known as* ***Lipschitz continuous*** *if there exists a constant* $L > 0$ *such that*

$$\|F(v_1) - F(v_2)\|_{\mathbb{X}} \le L\|v_1 - v_2\|_{\widetilde{\mathbb{X}}}, \quad \textit{for all } v_1,\ v_2 \in \widetilde{\mathbb{X}}. \tag{2.1.12}$$

Definition 2.1.4. *A function* $F : \widetilde{\mathbb{X}} \to \mathbb{X}$ *is said to be a* ***Hölder continuous*** *if there exist nonnegative real constants* C *and* $\theta \in (0,1]$ *such that*

$$\|F(z_1) - F(z_2)\|_{\mathbb{X}} \le C\|z_1 - z_2\|^{\theta}_{\widetilde{\mathbb{X}}} \quad \textit{for each } z_1, z_2 \in \widetilde{\mathbb{X}}. \tag{2.1.13}$$

The number θ is called the exponent of the Hölder condition. If $\theta = 1$, then the function satisfies a Lipschitz condition. If $\theta = 0$, then the function is simply bounded.

Definition 2.1.5. *A family of functions* F *defined on a set* E *in* $\mathbb{X}$ *is called* ***equicontinuous*** *on* E *if for given* $\epsilon > 0$*, we can find a* $\delta > 0$ *in a way that*

$$\|f(z_0) - f(z)\| < \epsilon \qquad \textit{whenever } \|z_0 - z\| < \delta,\ z_0,\ z \in E \ \textit{ and } f \in F. \tag{2.1.14}$$

Definition 2.1.6. *Suppose that* $F : \mathbb{X} \to \mathbb{X}$ *be a nonlinear operator. Then solution of the equation*

$$F(z) = z, \quad z \in \mathbb{X}, \tag{2.1.15}$$

is called a fixed point of operator F*.*

Definition 2.1.7. *A mapping* F *from a subset* B *of a normed space* $\mathbb{X}$ *into* $\mathbb{X}$ *is called a* ***contraction mapping*** *if we can find a number* $0 < C < 1$ *in a way that*

$$\|F(z_1) - F(z_2)\| \le C\|z_1 - z_2\|, \quad \textit{for all } z_1, z_2 \in B. \tag{2.1.16}$$

Note that contraction mappings are continuous.

Definition 2.1.8. *Suppose that $\mathbb{X}$ and $\widetilde{\mathbb{X}}$ be normed linear spaces. An operator $\mathbb{T} : \widetilde{\mathbb{X}} \to \mathbb{X}$ is called* ***compact*** *if it maps every bounded subset of $\widetilde{\mathbb{X}}$ into a relatively compact subset of $\mathbb{X}$.*

Theorem 2.1.1. *(Arzela-Ascoli theorem) Let N be a compact metric space and $B \subset C(N)$. Then, B is totally bounded in $C(N)$ if and only if B equicontinuous and pointwise bounded on N.*

Theorem 2.1.2. *Let $\widetilde{\mathbb{X}}$ and $\mathbb{X}$ be two normed linear spaces. A linear operator $\mathbb{T} : \widetilde{\mathbb{X}} \to \mathbb{X}$ is compact if and only for every bounded sequence (z_n) in $\widetilde{\mathbb{X}}$ the sequence $(\mathbb{T}(z_n))$ in $\mathbb{X}$ has a convergent subsequence.*

2.2 Semigroup Theory

Definition 2.2.1. *A one parameter family $\{\mathcal{S}(t)\}_{t\geq 0}$, of bounded linear operators from Banach space $\mathbb{X}$ into $\mathbb{X}$ is called a semigroup if*

(1) $\mathcal{S}(0) = I$ *(I is the identity operator on $\mathbb{X}$).*

(2) $\mathcal{S}(t_1 + t_2) = \mathcal{S}(t_1)\mathcal{S}(t_2), \ \forall \ t_1, t_2 \geq 0.$

Definition 2.2.2. *A semigroup $\{\mathcal{S}(t)\}_{t\geq 0}$ is called strongly continuous semigroup or C_0-semigroup if*

$$\lim_{t\downarrow 0} \mathcal{S}(t)z = z, \quad \textit{for every} \ \ z \in \mathbb{X}. \tag{2.2.1}$$

Definition 2.2.3. *If $\lim_{t_1\to t_2} \| \mathcal{S}(t_1) - \mathcal{S}(t_2)\| = 0$, the semigroup $\{\mathcal{S}(t)\}_{t\geq 0}$ is called uniformly continuous semigroup.*

Definition 2.2.4. *A linear operator A on $\mathbb{X}$ defined by*

$$Az = \lim_{t\downarrow 0} \frac{\| \mathcal{S}(t)z - z\|}{t}, \ \textit{for} \ z \in D(A), \tag{2.2.2}$$

is named as infinitesimal generator of a semigroup $\{\mathcal{S}(t)\}_{t\geq 0}$ whenever the above limit exists. The domain of A is

$$D(A) = \{z \in \mathbb{X} : \lim_{t\downarrow 0} \frac{\| \mathcal{S}(t)z - z\|}{t} \ \textit{exists}\}. \tag{2.2.3}$$

Theorem 2.2.1. *If $\mathcal{S}(t)$ is a C_0-semigroup, then we can find constants $\delta \geq 0$ and $M \geq 1$ in a way that*

$$\|\mathcal{S}(t)\| \leq Me^{\delta t}, \ \ \forall \ \ t \geq 0. \tag{2.2.4}$$

Remark 2.2.2. *If $\delta = 0$, $\mathcal{S}(t)$ is called uniformly bounded, and if $M = 1$, $\mathcal{S}(t)$ is called C_0-semigroup of contractions.*

Theorem 2.2.3. *[156] Let $\mathcal{S}(t)$ be a C_0-semigroup of bounded linear operators on $\mathbb{X}$ which is generated by A. Then,*

1. *there is $a > 0$ such that $\|\mathcal{S}(t)\|$ is uniformly bounded on $[0, a]$,*
2. *for each $z \in \mathbb{X}$, $\lim_{h\to 0} \frac{1}{h}\int_t^{t+h} \mathcal{S}(s)zds = \mathcal{S}(t)z$,*
3. *for all $z \in \mathbb{X}$, $\int_0^t \mathcal{S}(s)zds \in D(A)$ and*
$$A(\int_0^t \mathcal{S}(s)zds) = \mathcal{S}(t)z - z, \tag{2.2.5}$$
4. *for $z \in D(A)$, $\mathcal{S}(t)z \in D(A)$, moreover*
$$\frac{d}{dt}\mathcal{S}(t)z = \mathcal{S}(t)Az = A\mathcal{S}(t)z, \tag{2.2.6}$$
5. *for all $z \in D(A)$,*
$$\mathcal{S}(t)z - \mathcal{S}(s)z = \int_s^t \mathcal{S}(\tau)Azd\tau = \int_s^t A\mathcal{S}(\tau)zd\tau. \tag{2.2.7}$$

Corollary 2.2.4. *[156] Suppose that A is the infinitesimal generator of a C_0-semigroup. Then, A is a linear closed operator and $D(A)$ is dense in $\mathbb{X}$.*

The resolvent set $\rho(A)$ of a linear operator A consists all $\lambda \in \mathbb{C}$ such that $(\lambda I - A)^{-1}$ exists and is a linear bounded operator on $\mathbb{X}$. $R(\lambda, A) = (\lambda I - A)^{-1}$, $\lambda \in \rho(A)$ is known as family of resolvent operators of A.

Theorem 2.2.5. *[156](**Hille-Yosida Theorem**) A linear operator A is the infinitesimal generator of an strongly continuous semigroup $\mathcal{S}(t)$ $(t \geq 0)$ satisfying $\|\mathcal{S}(t)\| \leqslant Me^{\delta t}$, iff A is densely defined in $\mathbb{X}$ and closed, $\rho(A)$ contains (δ, ∞), and for $\lambda > \delta$*
$$\|R(\lambda, A)^k\| \leq \frac{M}{(\lambda - \delta)^k}, \ \forall \ k = 1, 2, ... \tag{2.2.8}$$

Theorem 2.2.6. *[156](**Hille-Yosida Theorem for Contraction Semigroups**) Suppose that $A : D(A) \subset \mathbb{X} \to \mathbb{X}$ is a linear operator. Then, A is the infinitesimal generator of an strongly continuous semigroup of contraction iff A is densely defined in $\mathbb{X}$ and closed, $\rho(A)$ contains $\mathbb{R}^+$, and*
$$\|(\lambda I - A)^{-1}\| \leq \frac{1}{\lambda}, \text{ for all positive } \lambda. \tag{2.2.9}$$

Now, we state the following results which provide the representation of the semigroup generated by a bounded linear operator.

Theorem 2.2.7. *Suppose that A be a linear operator and densely defined in $\mathbb{X}$ which satisfies the following two conditions :*

(1) $\sum_{\mu} = \{\lambda : |arg\lambda| < \frac{\pi}{2} + \mu\} \cup \{0\} \subset \rho(A)$, *for some* $0 < \mu < \pi/2$;

(2) *there is a constant $\mathcal{L}$ in such a way that*

$$(\lambda I - A)^{-1} \leq \frac{\mathcal{L}}{|\lambda|}, \quad \lambda \in \sum_{\mu} \quad and \;\; \lambda > 0.$$

Then, A generates a C_0-semigroup $\mathcal{S}(t)$ fulfilling $\|\mathcal{S}(t)\| \leq N$ for some constant $N > 0$ and

$$\mathcal{S}(t) = \frac{1}{2\pi i} \int_{\Gamma} e^{\varrho t} (\varrho I - A)^{-1} d\varrho,$$

where Γ is a smooth curve contained in $\sum_{\mu}$ starting from $\infty e^{-i\phi}$ to $\infty e^{i\phi}$ for some $\pi/2 < \phi < \pi/2 + \mu$, the integral converges in uniform operator topology for $t > 0$.

2.3 Basic Concepts of Fractional Calculus

In literature, there are more than fifteen definitions of fractional derivative. But the most commonly used definitions are the definitions given by Caputo and Riemaan-Liouville, as these two definitions coincide with the ordinary derivative at integers.

Definition 2.3.1. *For a function $F \in L^1(\mathbb{R}^+, \mathbb{X})$, the Riemann-Liouville (in short RL) fractional integral of order $q > 0$, is defined as*

$$J^q F(t) = \frac{1}{\Gamma(q)} \int_0^t (t - \varrho)^{q-1} F(\varrho) d\varrho, \quad for \;\; t > 0. \tag{2.3.1}$$

We set $J^0 = I$ (the identity operator).

Definition 2.3.2. *For a function F, the Riemann-Liouville fractional derivative of order $q > 0$, is defined as*

$$\begin{aligned} {}^{L}\boldsymbol{D}^q F(t) &:= D^m J^{m-q} F(t) \\ &= \frac{1}{\Gamma(m-q)} \frac{d^m}{dt^m} \int_0^t (t-s)^{m-q-1} F(s) ds, \end{aligned}$$

where $t > 0$, $m = \lceil q \rceil = \min\{z \in \mathbb{Z} : z \geqslant q\}$, $D^m = \frac{d^m}{dt^m}$. We set ${}^{L}\boldsymbol{D}^0 = I$

Definition 2.3.3. *The Caputo fractional derivative of fractional order $q > 0$ of a function F is defined as*

$$\begin{aligned} {}^{C}\boldsymbol{D}^q F(t) &:= J^{m-q} D^m F(t) \\ &= \frac{1}{\Gamma(m-q)} \int_0^t (t-s)^{m-q-1} F^m(s) ds, \end{aligned}$$

for $t > 0$, $m = \lceil q \rceil$.

Comparing this definition with the Riemann-Liouville one, we see that the definition given by Caputo is more restrictive as it requires the absolute integrability of m^{th} order derivative of the function and therefore functions which are derivable in the Caputo sense are much "fewer" than those which are derivable in the Riemann-Liouville sense.

The main advantage of the Caputo derivative is that the form of initial conditions for FDEs are same as that of integer order differential equations and Caputo fractional derivative of constant function is zero, which does not hold for Riemann-Liouville definition.

2.3.1 Solutions of Caputo Fractional Differential Equations

Let us consider the homogenous problem

$$^{C}\mathbf{D}^{q}y(t) = 0, \quad t > 0. \tag{2.3.2}$$

Then solution of the above problem is

$$y(t) = d_0 + d_1 t + d_2 t^2 + \cdots + d_{n-1}t^{n-1}, \tag{2.3.3}$$

and

$$J^q \cdot {}^{C}\mathbf{D}^{q}y(t) = y(t), \tag{2.3.4}$$

where $d_i \in \mathbb{R}$, $i = 1, \cdots, n-1$ and $n = \lceil q \rceil$. For the nonhomogenous fractional differential equation

$$^{C}\mathbf{D}^{q}y(t) = F(t), \quad t \in [0, b], \ b \in \mathbb{R}^{+}. \tag{2.3.5}$$

From the equation (2.3.5), we get following integral equation

$$\begin{aligned} y(t) = {} & d_0 + d_1 t + d_2 t^2 + \cdots + d_{n-1}t^{n-1} \\ & + \frac{1}{\Gamma(q)} \int_0^t (t-\varrho)^{q-1} F(\varrho) d\varrho. \end{aligned} \tag{2.3.6}$$

Now, we consider the following fractional order problem

$$ {}^{C}\mathbf{D}^{q}y(t) = Ay(t),\ t \in (0, T], \tag{2.3.7} $$

$$ y(0) = y_0, \tag{2.3.8} $$

where $0 < q < 1$, y takes its values in $\mathbb{X}$, A is densely defined from $D(A) \subseteq \mathbb{X}$ to $\mathbb{X}$, and is a closed linear operator which generates C_0-semigroup of bounded linear operator $\mathcal{S}(t),\ t \geq 0$.

The equation (2.3.7) is equivalent to the following

$$ y(t) = y_0 + \frac{1}{\Gamma(q)} \int_0^t (t-\varrho)^{q-1} Ay(\varrho) d\varrho. \tag{2.3.9} $$

The solution to (2.3.7)-(2.3.8) is a function $y \in C([0, T], \mathbb{X})$ that satisfies the following

(i) $y(t) \in D(A)$, $t \in (0, T]$,

(ii) ${}^{C}\mathbf{D}^{q}y(t)$ exists and continuous on $(0, T]$ with $0 < q < 1$,

(iii) y satisfies the equations (2.3.7)-(2.3.8).

Applying the Laplace transformation of the equation (2.3.9) on both sides

$$ \begin{aligned} L[y(t)] &= L[y_0] + L[\frac{1}{\Gamma(q)} \int_0^t (t-\varrho)^{q-1} Ay(\varrho) d\varrho] \\ &= L[y_0] + \frac{1}{\lambda^q} AL[y(t)] \\ &= \lambda^{q-1}(\lambda^q I - A)^{-1} y_0 = \lambda^{q-1} \int_0^\infty e^{-\lambda^q \varrho} \mathcal{S}(\varrho) y_0 d\varrho. \end{aligned} \tag{2.3.10} $$

Let one-sided probability density

$$ \Phi_q(\zeta) = \frac{1}{\pi} \sum_{m=1}^{\infty} (-1)^{m-1} \zeta^{-mq-1} \frac{\Gamma(mq+1)}{n!} \sin(m\pi q),\ \zeta \in (0, \infty), \tag{2.3.11} $$

whose Laplace transform is given by

$$ \int_0^\infty e^{-\lambda\zeta} \Phi_q(\zeta) d\zeta = e^{-\lambda^q},\ \ q \in (0, 1). \tag{2.3.12} $$

Therefore, we get

$$\begin{aligned}
&\lambda^{q-1}\int_0^\infty e^{-\lambda^q s}\mathcal{S}(s)y_0 ds\\
&=\int_0^\infty q(\lambda t)^{q-1}e^{-(\lambda t)^q}\mathcal{S}(t^q)y_0 dt, \quad (\text{put } s=t^q)\\
&=\int_0^\infty \frac{-1}{\lambda}\frac{d}{dt}[e^{-(\lambda t)^q}]\mathcal{S}(t^q)y_0 dt\\
&=\int_0^\infty\int_0^\infty \zeta\Phi_q(\zeta)e^{-(\lambda t\zeta)}\mathcal{S}(t^q)y_0 d\zeta dt,\\
&=\int_0^\infty e^{-\lambda t}\Big[\int_0^\infty \Phi_q(\zeta)\mathcal{S}(t^q/\zeta^q)y_0 d\zeta\Big]dt. \qquad (2.3.13)
\end{aligned}$$

From (2.3.10) and (2.3.13), we get

$$L[y(t)]=\int_0^\infty e^{-\lambda t}\Big[\int_0^\infty \Phi_q(\zeta)\mathcal{S}(t^q/\zeta^q)y_0 d\zeta\Big]dt. \qquad (2.3.14)$$

Taking inverse Laplace transformation of above equation

$$\begin{aligned}
y(t)&=\int_0^\infty \Phi_q(\zeta)\mathcal{S}(t^q/\zeta^q)y_0 d\zeta\\
&=\int_0^\infty \Psi_q(\zeta)\mathcal{S}(t^q\zeta)y_0 d\zeta\\
&=\mathcal{S}_q(t)y_0, \qquad (2.3.15)
\end{aligned}$$

where $\Psi_q(\zeta)=\frac{1}{q}\zeta^{-1-\frac{1}{q}}\Phi_q(\zeta^{-1/q})$ with $\Psi_q(\zeta)\geq 0$, and $\int_0^\infty \Psi_q(\zeta)d\zeta=1$. Therefore, the solution of (2.3.7) is

$$y(t)=\mathcal{S}_q(t)y_0, \qquad (2.3.16)$$

where $\mathcal{S}_q(t)$ is defined by

$$\mathcal{S}_q(t)y=\int_0^\infty \Psi_q(\zeta)\mathcal{S}(t^q\zeta)y d\zeta, \ \ y\in D(A), \ t\geq 0. \qquad (2.3.17)$$

The Laplace transform of Ψ_q is given by

$$L[\Psi_q(t)]=\int_0^\infty e^{-\lambda t}\Psi_q(t)dt=F_q(\lambda)=\sum_{m=0}^\infty \frac{(-\lambda)^m}{\Gamma(qm+1)}=E_q(-\lambda), \qquad (2.3.18)$$

for $0<q<1$. Next, we consider the following inhomogeneous FDE

$$^{C}\mathbf{D}^q y(t)=Ay(t)+F(t), \quad t\in(0,T], \ \ 0\leq T<\infty, \qquad (2.3.19)$$

$$y(0)=y_0, \qquad (2.3.20)$$

where $F \in L^1([0,T],\mathbb{X})$.

Taking Laplace transformation of the equation (2.3.19) on both sides and getting

$$\begin{aligned} L[y(t)] &= \lambda^{q-1}(\lambda^q I - A)^{-1}y_0 + (\lambda^q I - A)^{-1}L[F(t)] \\ &= \lambda^{q-1}\int_0^\infty e^{-\lambda^q s}\mathcal{S}(s)y_0 ds + \int_0^\infty e^{-\lambda^q s}\mathcal{S}(s)L[F(s)]ds. \end{aligned} \tag{2.3.21}$$

Now, we estimate

$\int_0^\infty e^{-\lambda^q s}\mathcal{S}(s)L[F(s)]ds$

$$\begin{aligned} &= \int_0^\infty \int_0^\infty q t^{q-1} e^{-(\lambda t)^q}\mathcal{S}(t^q)e^{-\lambda s}F(s)dsdt \\ &= \int_0^\infty \int_0^\infty \int_0^\infty q\Phi_q(\zeta)e^{-(\lambda t\theta)}\mathcal{S}(t^q)e^{-\lambda s}t^{q-1}F(s)d\zeta dsdt \\ &= \int_0^\infty \int_0^\infty \int_0^\infty q\Phi_q(\zeta)e^{-\lambda(t+s)}\mathcal{S}(t^q/\zeta^q)\frac{t^{q-1}}{\zeta^q}F(s)d\zeta dsdt \\ &= \int_0^\infty e^{-\lambda t}\Big[q\int_0^t \int_0^\infty \Phi_q(\zeta)\mathcal{S}(\frac{(t-s)^{q-1}}{\zeta^q})F(s)\frac{(t-s)^q}{\zeta^q}d\zeta ds\Big]dt, \end{aligned}$$

Thus, we get

$$\begin{aligned} L[y(t)] = &\int_0^\infty e^{-\lambda t}\big[\int_0^\infty \Phi_q(\zeta)\mathcal{S}(t^q/\zeta^q)y_0 d\zeta\big]dt \\ &+ \int_0^\infty e^{-\lambda t}\Big[q\int_0^t \int_0^\infty \Phi_q(\zeta)\mathcal{S}(\frac{(t-s)^{q-1}}{\zeta^q})F(s)\frac{(t-s)^q}{\zeta^q}d\zeta ds\Big]dt. \end{aligned}$$

Taking inverse Laplace transformation of the above equation and obtaining

$$\begin{aligned} y(t) = &\int_0^\infty \Phi_q(\zeta)\mathcal{S}(t^q/\zeta^q)y_0 d\zeta + \\ &+ q\int_0^t \int_0^\infty \Phi_q(\zeta)\mathcal{S}(\frac{(t-s)^{q-1}}{\zeta^q})F(s)\frac{(t-s)^q}{\zeta^q}d\zeta ds \\ = &\int_0^\infty \Psi_q(\zeta)\mathcal{S}(t^q\zeta)y_0 d\zeta + q\int_0^t \int_0^\infty \zeta(t-s)^{q-1}\Psi_q(\zeta)\mathcal{S}((t-s)^{q-1}\zeta)F(s)d\zeta ds \\ = &\mathcal{P}(t)y_0 + \int_0^t (t-s)^{q-1}\mathcal{R}(t-s)F(s)ds, \end{aligned} \tag{2.3.22}$$

where, the operator $\mathcal{P}(t)$ and $\mathcal{R}(t)$ are defined by

$$\mathcal{P}(t) = \int_0^\infty \Psi_q(\zeta)\mathcal{S}(t^q\zeta)d\zeta, \quad \mathcal{R}(t) = q\int_0^\infty \zeta\Psi_q(\zeta)\mathcal{S}(t^q\zeta)d\zeta. \tag{2.3.23}$$

Remark 2.3.1. $\Psi_q(\zeta) \geqslant 0, \zeta \in (0,\infty)$, $\int_0^\infty \Psi_q(\zeta)d\zeta = 1$, $\int_0^\infty \zeta\Psi_q(\zeta)d\zeta = \frac{1}{\Gamma(1+q)}$.

Definition 2.3.4. *A function $y \in C([0,T],\mathbb{X})$ is called a solution to the problem (2.3.19)-(2.3.20) if the following integral equation*

$$y(t) = \mathcal{P}(t)y_0 + \int_0^t (t-\varrho)^{q-1}\mathcal{R}(t-\varrho)F(\varrho)d\varrho, \tag{2.3.24}$$

is verified.

Lemma 2.3.2. *[203] The operators $\{\mathcal{R}(t), t \geqslant 0\}$ and $\{\mathcal{P}(t), t \geqslant 0\}$ are linear bounded, and*

(i) $\|\mathcal{R}(t)z\| \leqslant \frac{qM}{\Gamma(1+q)}\|z\|$ *and* $\|\mathcal{P}(t)z\| \leqslant M\|z\|$, $z \in \mathbb{X}$.

(ii) *The families $\{\mathcal{R}(t) : t \geqslant 0\}$ and $\{\mathcal{P}(t) : t \geqslant 0\}$ are continuous strongly.*

(iii) *If $\mathcal{S}(t)$ is compact, then $\mathcal{R}(t)$ and $\mathcal{P}(t)$ are compact for $t > 0$.*

(iv) *If $\mathcal{S}(t)(t \geqslant 0)$ is an equicontinuous, then $\mathcal{R}(t)$ and $\mathcal{P}(t)$ are continuous for $t > 0$ by the operator norm, which means that for $0 < t' < t'' \leqslant a$*

$$\|\mathcal{R}(t'') - \mathcal{R}(t')\| \to 0 \quad \textit{and} \quad \|\mathcal{P}(t'') - \mathcal{P}(t')\| \to 0 \quad \textit{as} \quad t'' \to t'.$$

2.4 Basic Concepts of Measure of Noncompactness

Assume that $(\mathbb{X}, d)$ be a complete metric space, denote $\mathcal{B}_{\mathbb{X}}$ for the family of all bounded subsets of $\mathbb{X}$. Now, we have some notations which will be used in the subsequent chapters. If U is a subset of a metric space $(\mathbb{X}, d)$, then $\mathrm{diam}(U) = \sup\{d(y,y') : y,\ y' \in U\}$ is called the diameter of U.

Definition 2.4.1. *[178] Suppose that $\mathbb{X}$ is a complete metric space. A function $\mu : \mathcal{B}_{\mathbb{X}} \to [0,\infty)$ is said to be a* ***measure of noncompactness*** *on $\mathbb{X}$, if it satisfies the following :*

(*i*) $\mu(W) = 0$ *iff $W \in \mathcal{B}_{\mathbb{X}}$ is precompact [Regularity];*

(*ii*) $\mu(W) = \mu(\overline{W})$, *where $\overline{W}$ denotes the closure of $W \in \mathcal{B}_{\mathbb{X}}$ [Invariance under closure];*

(*iii*) $\mu(W_1 \cup W_2) = \max\{\mu(W_1), \mu(W_2)\}$, $\forall\ W_1, W_2 \in \mathcal{B}_{\mathbb{X}}$ *[Semi-additivity].*

Proposition 2.4.1. *[178] For bounded sets $W, W_1, W_2 \in \mathcal{B}_{\mathbb{X}}$, a measure of noncompactness function μ fulfills the following conditions*

(i) $\mu(W_1) \leq \mu(W_2)$, *when* $W_1 \subset W_2$, *[Monotonicity];*

(ii) $\mu(W_1 \cap W_2) \leq \min\{\mu(W_1), \mu(W_2)\}$;

(iii) $\mu(W) = 0$ *for each finite set* W, *[Non-singularity].*

Definition 2.4.2. *[178] The function $\beta : \mathcal{B}_{\mathbb{X}} \to [0, \infty)$ as following*

$$\beta(U) := \inf\{\kappa > 0 : U \subset \cup_{k=1}^{n} U_k,\ U_k \subset \mathbb{X},\ diam(U_k) < \kappa,\ k = 1, 2, \cdots, n \in \mathbb{N}\},$$

*is known as **Kuratowski measure of noncompactness**.*

Definition 2.4.3. *The **Hausdorff measure of noncompactness** $\chi : \mathcal{B}_{\mathbb{X}} \to [0, \infty)$ is defined as*

$$\chi(U) := \inf\{\kappa > 0 : U \subset \cup_{k=1}^{n} B_{r_k}(x_k),\ x_k \in \mathbb{X}, r_k < \kappa,\ k = 1, \cdots, n \in \mathbb{N}\},$$

where $B_{r_k}(x_k) = \{x \in \mathbb{X} : d(x, x_k) < r_k\}$.

Proposition 2.4.2. *Let μ denotes both Hausdorff and Kurtatowski measure of noncompactness. Then for sets $W, W_1, W_2 \in \mathcal{B}_{\mathbb{X}}$, we have*

(i) W *is relatively compact iff* $\mu(W) = 0$;

(ii) $\mu(W_1 + W_2) \leq \mu(W_1) + \mu(W_2)$ *(Algebraic semi-additivity);*

(iii) $\mu(W_1) \leq \mu(W_2)$ *when* $W_1 \subset W_2$;

(iv) $\mu(\alpha \cdot W) = |\alpha| \cdot \mu(W)$, α *is a number (Semi-homogeneity);*

(v) $\mu(W + z) = \mu(W)$ *for each* $z \in \mathbb{X}$ *(Translation invariance);*

(vi) $\mu(W) = \mu(\overline{W}) = \mu(conv(W))$, *where* $\overline{W}$ *and* $conv(W)$ *denote the closure and convex hull of* W *respectively.*

Lemma 2.4.3. *[24] Let $\mathbb{X}$ and $\mathbb{E}$ be Banach spaces and $\mathcal{F} : D(\mathcal{F}) \subset \mathbb{E} \to \mathbb{X}$ satisfies Lipschitz continuity with constant L, then $\mu(\mathcal{F}(V)) \leqslant L\mu(V)$ for any bounded subset $V \subset D(\mathcal{F})$.*

Lemma 2.4.4. *[24] For any $W \subset C([a, b], \mathbb{X})$, set $W(t) = \{w(t) : w \in W\}$. If W is equicontinuous and bounded, then $\beta(W(t))$ is continuous on $[a, b]$, moreover $\beta(W) = \sup\limits_{t \in [a,b]} \beta(W(t))$.*

Lemma 2.4.5. *[94] If $\{w_n\}_{n=1}^{\infty} \subset C([a, b], \mathbb{X})$ is a bounded sequence, then $\beta(\{w_n(t)\}_{n=1}^{\infty}) \in L^1([a, b], \mathbb{X})$ and*

$$\beta\left(\left\{\int_0^t w_n(s)ds\right\}_{n=1}^{\infty}\right) \leqslant 2\int_0^t \beta(\{w_n(s)\}_{n=1}^{\infty} ds.$$

Lemma 2.4.6. *[36] If W is bounded subset of $\mathbb{X}$, then there exists a sequence $\{w_n\}_{n=1}^{\infty} \subset W$, such that $\beta(W) \leqslant 2\beta(\{w_n\}_{n=1}^{\infty})$.*

2.5 Fixed Point Theorems

Theorem 2.5.1. ***(Banach fixed point theorem)*** *Let F be a contraction mapping from $\mathbb{X}$ into $\mathbb{X}$, where $\mathbb{X}$ is a complete metric space. Then, $\mathbb{X}$ has a unique point z such that $F(z) = z$.*

Theorem 2.5.2. ***(Leray-Schauder nonlinear alternative)*** *[87] Let B is a nonempty, convex and closed subset of a Banach space $\mathbb{X}$, and U is an open subset of B such that $0 \in U$. If $F : \overline{U} \to B$ is a compact and continuous map. Then only one of the following holds :*

(i) *F has a fixed point in $\overline{U}$,*

(ii) *there is a point $u_0 \in \partial U$ (boundary of U) for which $u_0 = \varepsilon F u_0$ holds, for some $0 < \varepsilon < 1$.*

Definition 2.5.1. *[66] Suppose that S be a nonempty subset of the Banach space $\mathbb{X}$. A continuous map $F : S \to \mathbb{X}$ is called ρ-set contractive if for some constant $\rho \in [0, 1)$, we have*

$$\beta(F(\Omega)) \leqslant \rho\beta(\Omega), \text{ for all bounded subsets } \Omega \text{ in } S.$$

Theorem 2.5.3. ***(ρ-set contraction mapping fixed point theorem)*** *[66] Let Ω be a closed bounded convex subset of a Banach space $\mathbb{X}$, and the operator $F : \Omega \to \Omega$ is ρ-set contractive, then there exists a fixed point of F in Ω.*

Theorem 2.5.4. ***(Schauder)*** *If Ω is a convex closed bounded subset of a Banach space $\mathbb{X}$, and $F : \Omega \to \Omega$ is completely continuous. Then F has a fixed point in Ω.*

For more details on fixed point theorems, we refer [12; 66; 87; 88; 150], and the references therein.

Chapter 3

Fractional Neutral Integro-Differential Equations involving Nonlocal Integral Boundary Conditions

3.1 Introduction

This chapter concerns with the following fractional integro-differential equations of neutral type having nonlocal flux type integral boundary conditions :

$$\begin{aligned} {}^{C}\mathbf{D}^{q}[x(t) - g(t, x(t))] &= f(t, x(t), \int_0^t \Upsilon(t, \varrho, x(\varrho))d\varrho),\ 0 < t < 1\,,\ 1 < q \leqslant 2, \\ x'(0) &= \alpha \int_0^{\xi} x'(\varrho)d\varrho,\ x(1) = \gamma\phi(x'(\eta)), \\ & \quad 0 \leqslant \xi, \eta \leqslant 1\,,\ \xi \neq \frac{1}{\alpha}, \end{aligned} \tag{3.1.1}$$

where f and g are real valued functions defined over $[0, 1] \times \mathbb{R} \times \mathbb{R}$ and $[0, 1] \times \mathbb{R}$ respectively, $\Upsilon : \mathbb{D} \times \mathbb{R} \to \mathbb{R}, \phi : \mathbb{R} \to \mathbb{R}$, and $\mathbb{D} = \{(\tau_1, \tau_2) : 0 \leqslant \tau_2 < \tau_1 \leqslant 1\}$. In (3.1.1), the first of nonlocal boundary condition relates the flux at the initial point 0 with the continuous distribution of flux on an interval of arbitrary length ξ, and

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the second condition expresses that the value of unknown function at final point is proportional to a nonlinear function ϕ of flux.

Motivated by the work done in [6; 9; 10; 50] for various type of integral boundary value problems, we considered the above mentioned problem (3.1.1), to investigate the existence results. The remaining part of the chapter is arranged as following: The next section contains a lemma concerning the solution of linear system associated to (3.1.1), and introduce the assumptions which are required to obtain our main results. Section 3.3 consists of the proof of the main results via Leray-Schauder nonlinear alternative and Banach contraction theorem. Finally, in section 3.4, we will present some examples to illustrate our results.

3.2 Preliminaries and Assumptions

Lemma 3.2.1. *For $\mathbb{R}$-valued function f continuous on $[0,1]$, and $g \in C^1([0,1],\mathbb{R})$, solution of the following fractional linear BVP*

$$
\begin{aligned}
{}^{C}\boldsymbol{D}^{q}[x(t)-g(t)] &= f(t), \quad t \in (0,1)\,,\, 1<q\leqslant 2 &(3.2.1)\\
x'(0) &= \alpha\int_0^{\xi} x'(\varrho)d\varrho, \quad x(1)=\gamma\phi(x'(\eta)),\ 0\leqslant \xi,\eta\leqslant 1, &(3.2.2)
\end{aligned}
$$

is given by

$$
\begin{aligned}
x(t) &= \int_0^t \frac{(t-\varrho)^{q-1}}{\Gamma(q)}f(\varrho)d\varrho + g(t) - g(1) - \int_0^1 \frac{(1-\varrho)^{q-1}}{\Gamma(q)}f(\varrho)d\varrho\\
&+\frac{t-1}{1-\alpha\xi}\left[\alpha(g(\xi)-g(0)) - g'(0) + \alpha\int_0^{\xi}\int_0^{\varrho}\frac{(\varrho-\tau)^{q-2}}{\Gamma(q-1)}f(\tau)d\tau d\varrho\right]\\
&+\gamma\phi\Bigg(\frac{\alpha}{(1-\alpha\xi)}(g(\xi)-g(0)) - \frac{1}{(1-\alpha\xi)}g'(0) + g'(\eta)\\
&+\frac{\alpha}{1-\alpha\xi}\int_0^{\xi}\int_0^{\varrho}\frac{(\varrho-\tau)^{q-2}}{\Gamma(q-1)}f(\tau)d\tau d\varrho + \int_0^{\eta}\frac{(\eta-\varrho)^{q-2}}{\Gamma(q-1)}f(\varrho)d\varrho\Bigg). \quad (3.2.3)
\end{aligned}
$$

Proof. From [158], it is known that the solution of (3.2.1) is

$$
x(t) = c_1 + c_2 t + g(t) - g(0) + \int_0^t \frac{(t-\varrho)^{q-1}}{\Gamma(q)}f(\varrho)d\varrho. \quad (3.2.4)
$$

On applying the conditions (3.2.2), we find that

$$
c_2 = \frac{1}{(1-\alpha\xi)}\left(\alpha(g(\xi)-g(0)) - g'(0) + \alpha\int_0^{\xi}\int_0^{\varrho}\frac{(\varrho-\tau)^{q-2}}{\Gamma(q-1)}f(\tau)d\tau d\varrho\right) \quad (3.2.5)
$$

and

$$c_1 = \gamma\phi\Bigg(c_2 + g'(\eta) + \int_0^\eta \frac{(\eta-\varrho)^{q-2}}{\Gamma(q-1)} f(\varrho)d\varrho\Bigg) + g(0) - g(1) - c_2 - \int_0^1 \frac{(1-\varrho)^{q-1}}{\Gamma(q)} f(\varrho)d\varrho,$$

where c_2 is given by (3.2.5). Substituting the values of c_1 and c_2 in (3.2.4), we get (3.2.3). □

Let us denote $\mathcal{C} := C([0,1],\mathbb{R})$ equipped with the norm $\|x\| = \sup_{t\in[0,1]} |x(t)|$, and $\mathcal{P} := C^1([0,1],\mathbb{R})$ equipped with norm $\|v\|_{C^1} = \sup_{t\in[0,1]}\{|v(t)|, |v'(t)|\}$.

By the help of Lemma 3.2.1, we define $F : \mathcal{P} \to \mathcal{P}$ as

$$\begin{aligned}(Fv)(t) \;=\; & \int_0^t \frac{(t-\varrho)^{q-1}}{\Gamma(q)} f(\varrho, v(\varrho), Kv(\varrho))d\varrho + g(t, v(t)) - g(1, v(1)) \\ & - \int_0^1 \frac{(1-\varrho)^{q-1}}{\Gamma(q)} f(\varrho, v(\varrho), Kv(\varrho))d\varrho \\ & + \frac{t-1}{1-\alpha\xi}\Bigg[\alpha(g(\xi, v(\xi)) - g(0, v(0))) - g'(0, v(0)) \\ & + \alpha \int_0^\xi \int_0^\varrho \frac{(\varrho-\tau)^{q-2}}{\Gamma(q-1)} f(\tau, v(\tau), Kv(\tau))d\tau d\varrho\Bigg] \\ & + \gamma\phi\Bigg(\frac{\alpha}{(1-\alpha\xi)}(g(\xi, v(\xi)) - g(0, v(0))) - \frac{1}{(1-\alpha\xi)} g'(0, v(0)) + g'(\eta, v(\eta)) \\ & + \int_0^\eta \frac{(\eta-\varrho)^{q-2}}{\Gamma(q-1)} f(\varrho, v(\varrho), Kv(\varrho))d\varrho \\ & + \frac{\alpha}{1-\alpha\xi} \int_0^\xi \int_0^\varrho \frac{(\varrho-\tau)^{q-2}}{\Gamma(q-1)} f(\tau, v(\tau), Kv(\tau))d\tau d\varrho\Bigg), \end{aligned} \tag{3.2.6}$$

where $Kv(t) := \int_0^t \Upsilon(t, \varrho, v(\varrho))d\varrho$.

Now, we state the required assumptions :

(A1) There exist continuous nondecreasing functions $\psi_1, \psi_2, \psi_3 : \mathbb{R}^+ \to \mathbb{R}^+$ and continuous positive functions p_1, p_2, p_3 defined on $[0,1]$ satisfying :

(i) $|f(\tau, y, z)| \leqslant p_1(\tau)\psi_1(\|y\|), \quad \forall\, (\tau, y, z) \in [0,1] \times \mathbb{R} \times \mathbb{R},$

(ii) $|g(\tau, z)| \leqslant p_2(\tau)\psi_2(\|z\|)$, $|g'(\tau, z)| \leqslant p_3(\tau)\psi_3(\|z\|), \quad \forall\, (\tau, z) \in [0,1] \times \mathbb{R}.$

(A2) The function ϕ is continuous, and $|\phi(v)| \leqslant |v|, \quad \forall\, v \in \mathbb{R}.$

(A3) There is a positive constant $\mathfrak{M}$ satisfying $\frac{\mathfrak{M}}{\|p\|\psi(\mathfrak{M})\Lambda_1} > 1$, where $\|p\| = \max\{\|p_i\| : i = 1, 2, 3\}$, $\Lambda_1 = \frac{1}{\Gamma(q+1)}\left[2 + |\gamma|q\eta^{q-1} + (1+|\gamma|)\frac{|\alpha|\xi^q}{|1-\alpha\xi|}\right] + 2 + |\gamma| + \frac{1+|\gamma|}{|1-\alpha\xi|} + 2(1+|\gamma|)\frac{|\alpha|}{|1-\alpha\xi|}$ and $\psi(r) = \max\{\psi_1(r), \psi_2(r), \psi_3(r)\}$.

(B1) The function f is continuous and satisfying :

$$|f(\tau, \upsilon_1, y_1) - f(\tau, \upsilon_2, y_2)| \leqslant L_1\{|\upsilon_1 - \upsilon_2| + |y_1 - y_2|\}, \quad \tau \in [0, 1], \upsilon_1, \upsilon_2, y_1, y_2 \in \mathbb{R},$$

for some positive constant L_1.

(B2) The function g is continuously differentiable and satisfying :

$$\begin{aligned} |g(\tau, \upsilon_1) - g(\tau, \upsilon_2)| \ &\leqslant \ L_2|\upsilon_1 - \upsilon_2|\,, \ |g'(\tau, \upsilon_1) - g'(\tau, \upsilon_2)| \leqslant L_3|\upsilon_1 - \upsilon_2|, \\ &\tau \in [0, 1], \upsilon_1, \upsilon_2 \in \mathbb{R}, \end{aligned}$$

for some positive constants L_2, L_3.

(B3) The function Υ is continuous and the following holds :

$$|\Upsilon(\tau, \upsilon, y_1) - \Upsilon(\tau, \upsilon, y_2)| \leqslant L_4|y_1 - y_2|, \quad \tau, \upsilon \in [0, 1], y_1, y_2 \in \mathbb{R},$$

for some positive constant L_4.

(B4) Let $\rho = \max\{\rho_1, \rho_2\} < 1$ where

$$\begin{aligned} \rho_1 \ &= \ \left(2 + (1+|\gamma|)\frac{|\alpha|\xi^q}{|1-\alpha\xi|} + |\gamma|q\eta^{q-1}\right)\frac{L_1(1+L_4)}{\Gamma(q+1)} \\ &+\left(2 + 2(1+|\gamma|)\frac{|\alpha|}{|1-\alpha\xi|}\right)L_2 + \left(|\gamma| + (1+|\gamma|)\frac{1}{|1-\alpha\xi|}\right)L_3, \end{aligned}$$

and

$$\rho_2 = \left(q + \frac{|\alpha|\xi^q}{|1-\alpha\xi|}\right)\frac{L_1(1+L_4)}{\Gamma(q+1)} + 2\frac{|\alpha|}{|1-\alpha\xi|}L_2 + \left(1 + \frac{1}{|1-\alpha\xi|}\right)L_3.$$

3.3 Existence and Uniqueness Results

The following theorem is proved via Leray-Schauder nonlinear alternative.

Theorem 3.3.1. *Suppose that the function f, Υ are continuous and g is continuously differentiable. Assume that the assumptions (A1)-(A3) hold. Then, there exists a solution to the boundary value problem (3.1.1).*

Proof. Observe that the operator F defined in (3.2.6) is continuous. Now, we check that the image of a bounded set in $\mathcal{P}$ under F is a bounded set in $\mathcal{P}$. Let $B_r = \{v \in \mathcal{P} : \|v\|_{C^1} \leqslant r\}$ $(r > 0)$, then for $v \in B_r$,

$$\begin{aligned}
|(Fv)(t)| \leqslant\; & \int_0^t \frac{(t-\varrho)^{q-1}}{\Gamma(q)}|f(\varrho, v(\varrho), Kv(\varrho))|d\varrho + |g(t, v(t))| + |g(1, v(1))| \\
& + \int_0^1 \frac{(1-\varrho)^{q-1}}{\Gamma(q)}|f(\varrho, v(\varrho), Kv(\varrho))|d\varrho \\
& + \frac{|t-1|}{|1-\alpha\xi|}\Big[|\alpha|(|g(\xi, v(\xi))| + |g(0, v(0))|) + |g'(0, v(0))| \\
& + |\alpha| \int_0^\xi \int_0^\varrho \frac{(\varrho-\tau)^{q-2}}{\Gamma(q-1)}|f(\tau, v(\tau), Kv(\tau))|d\tau d\varrho\Big] \\
& + |\gamma|\Big(\frac{|\alpha|}{|1-\alpha\xi|}(|g(\xi, v(\xi))| + |g(0, v(0))|) \\
& + \frac{1}{|1-\alpha\xi|}|g'(0, v(0))| + |g'(\eta, v(\eta))| \\
& + \int_0^\eta \frac{(\eta-\varrho)^{q-2}}{\Gamma(q-1)}|f(\varrho, v(\varrho), Kv(\varrho))|d\varrho \\
& + \frac{|\alpha|}{|1-\alpha\xi|} \int_0^\xi \int_0^\varrho \frac{(\varrho-\tau)^{q-2}}{\Gamma(q-1)}|f(\tau, v(\tau), Kv(\tau))|d\tau d\varrho\Big) \\
\leqslant\; & 2\|p_1\|\psi_1(r)\frac{1}{\Gamma(q+1)} + 2\|p_2\|\psi_2(r) \\
& + \frac{1}{|1-\alpha\xi|}\Big(2|\alpha|\|p_2\|\psi_2(r) + \|p_3\|\psi_3(r) + |\alpha|\|p_1\|\psi_1(r)\frac{\xi^q}{\Gamma(q+1)}\Big) \\
& + |\gamma|\Big(2\frac{|\alpha|}{|1-\alpha\xi|}\|p_2\|\psi_2(r) + \frac{1}{|1-\alpha\xi|}\|p_3\|\psi_3(r) + \|p_3\|\psi_3(r) \\
& + \|p_1\|\psi_1(r)\frac{\eta^{q-1}}{\Gamma(q)} + \|p_1\|\psi_1(r)\frac{|\alpha|\xi^q}{(|1-\alpha\xi|)\Gamma(q+1)}\Big).
\end{aligned}$$

Choosing $\psi(r) = \max\{\psi_1(r), \psi_2(r), \psi_3(r)\}$, we get

$$|(Fv)(t)| \leqslant \|p\|\psi(r)\Lambda_1. \tag{3.3.1}$$

On differentiating equation (3.2.6) with respect to t, we get

$$\begin{aligned}(Fv)'(t) &= \int_0^t \frac{(t-\varrho)^{q-2}}{\Gamma(q-1)} f(\varrho, v(\varrho), Kv(\varrho))d\varrho \\ &+g'(t,v(t)) + \frac{1}{1-\alpha\xi}\Big[\alpha(g(\xi,v(\xi)) - g(0,v(0))) - g'(0,v(0)) \\ &+\alpha \int_0^\xi \int_0^\varrho \frac{(\varrho-\tau)^{q-2}}{\Gamma(q-1)} f(\tau, v(\tau), Kv(\tau))d\tau d\varrho\Big], \qquad (3.3.2)\end{aligned}$$

by (3.3.2), for each $v \in B_r$ we have

$$\begin{aligned}|(Fv)'(t)| &\leqslant \|p_1\|\psi_1(r)\frac{1}{\Gamma(q)} + \|p_3\|\psi_3(r) + \frac{1}{|1-\alpha\xi|}\bigg(2|\alpha|\|p_2\|\psi_2(r) + \|p_3\|\psi_3(r) \\ &+|\alpha|\|p_1\|\psi_1(r)\frac{\xi^q}{\Gamma(q+1)}\bigg) \\ &\leqslant \|p\|\psi(r)\bigg(1 + \frac{1}{\Gamma(q)} + \frac{1+2|\alpha|}{|1-\alpha\xi|} + \frac{|\alpha|\xi^q}{(|1-\alpha\xi|)\Gamma(q+1)}\bigg),\end{aligned}$$

by denoting $\Lambda_2 = \bigg(1 + \frac{1}{\Gamma(q)} + \frac{1+2|\alpha|}{|1-\alpha\xi|} + \frac{|\alpha|\xi^q}{(|1-\alpha\xi|)\Gamma(q+1)}\bigg)$, we have

$$|(Fv)'(t)| \leqslant \|p\|\psi(r)\Lambda_2, \qquad (3.3.3)$$

observe that $\Lambda_2 \leqslant \Lambda_1$. Thus by (3.3.1) and (3.3.3), we have

$$\|Fv\|_{C^1} \leqslant \|p\|\psi(r)\Lambda_1. \qquad (3.3.4)$$

Next, we check that bounded sets are taken into equicontinuous sets of $\mathcal{P}$ by F. For

$t_1 < t_2$ in $[0,1]$ and $v \in B_r$

$$\begin{aligned}
|(Fv)(t_2)-(Fv)(t_1)| \leqslant& \left|\int_0^{t_2}\frac{(t_2-\varrho)^{q-1}}{\Gamma(q)}f(\varrho,v(\varrho),Kv(\varrho))d\varrho\right.\\
&\left.-\int_0^{t_1}\frac{(t_1-\varrho)^{q-1}}{\Gamma(q)}f(\varrho,v(\varrho),Kv(\varrho))d\varrho\right|\\
&+|g(t_2,v(t_2))-g(t_1,v(t_1))|\\
&+\frac{|t_2-t_1|}{|1-\alpha\xi|}\Big|\alpha(g(\xi,v(\xi))-g(0,v(0)))-g'(0,v(0))\\
&+\alpha\int_0^{\xi}\int_0^{\varrho}\frac{(\varrho-\tau)^{q-2}}{\Gamma(q-1)}f(\tau,v(\tau),Kv(\tau))d\tau d\varrho\Big|\\
\leqslant& \frac{1}{\Gamma(q)}\int_0^{t_1}[(t_2-\varrho)^{q-1}-(t_1-\varrho)^{q-1}]|f(\varrho,v(\varrho),Kv(\varrho))|d\varrho\\
&+\frac{1}{\Gamma(q)}\int_{t_1}^{t_2}(t_2-\varrho)^{q-1}|f(\varrho,v(\varrho),Kv(\varrho))|d\varrho\\
&+|g(t_2,v(t_2))-g(t_1,v(t_1))|\\
&+\frac{|t_2-t_1|}{|1-\alpha\xi|}\Big|\alpha(g(\xi,v(\xi))-g(0,v(0)))-g'(0,v(0))\\
&+\alpha\int_0^{\xi}\int_0^{\varrho}\frac{(\varrho-\tau)^{q-2}}{\Gamma(q-1)}f(\tau,v(\tau),Kv(\tau))d\tau d\varrho\Big|. \qquad (3.3.5)
\end{aligned}$$

Therefore, $|(Fv)(t_2)-(Fv)(t_1)| \to 0$ independently of $v \in B_r$ as $t_2 \to t_1$, similarly for the derivative term

$$\begin{aligned}
|(Fv)'(t_2)-(Fv)'(t_1)| \leqslant& \left|\int_0^{t_2}\frac{(t_2-\varrho)^{q-2}}{\Gamma(q-1)}f(\varrho,v(\varrho),Kv(\varrho))d\varrho\right.\\
&\left.-\int_0^{t_1}\frac{(t_1-\varrho)^{q-2}}{\Gamma(q-1)}f(\varrho,v(\varrho),Kv(\varrho))d\varrho\right|\\
&+|g'(t_2,v(t_2))-g'(t_1,v(t_1))|\\
\leqslant& \frac{1}{\Gamma(q-1)}\int_0^{t_1}[(t_2-\varrho)^{q-2}-(t_1-\varrho)^{q-2}]\\
&|f(\varrho,v(\varrho),Kv(\varrho))|d\varrho\\
&+\frac{1}{\Gamma(q-1)}\int_{t_1}^{t_2}(t_2-\varrho)^{q-2}|f(\varrho,v(\varrho),Kv(\varrho))|d\varrho\\
&+|g'(t_2,v(t_2))-g'(t_1,v(t_1))|. \qquad (3.3.6)
\end{aligned}$$

So, $|(Fv)'(t_2)-(Fv)'(t_1)| \to 0$ independently of $v \in B_r$ as $t_2 \to t_1$, therefore by (3.3.5), (3.3.6) and Arzela-Ascoli's theorem it follows that F is completely continuous.

Now, let $v = \lambda Fv$ where $\lambda \in (0,1)$, then $\|v\|_{C^1} < \|Fv\|_{C^1}$. Using (3.3.4), we have $\|v\|_{C_1} \leqslant \|p\|\psi(\|v\|_{C_1})\Lambda_1$. Consequently, we have $\frac{\|v\|_{C_1}}{\|p\|\psi(\|v\|_{C_1})\Lambda_1} \leqslant 1$. By assumption

$(A3)$, we have positive constant $\mathfrak{M}$ with $\|v\|_{C_1} \neq \mathfrak{M}$. Suppose that $S = \{v \in \mathcal{P} : \|v\|_{C^1} < \mathfrak{M}\}$. Observe that $F : \overline{S} \to \mathcal{P}$ is continuous and compact. Moreover, there is no $v \in \partial S$, satisfying $v = \lambda F v$ for some $\lambda \in (0, 1)$. Thus, Leray-Schauder nonlinear alternative theorem implies that F has a fixed point $v \in \overline{S}$, hence a solution to the problem (3.1.1). □

Theorem 3.3.2. *Assume that the hypotheses $(A2)$, $(B1)$-$(B4)$ hold. Then, the system (3.1.1) has a unique solution.*

Proof. For $t \in [0, 1]$ and $v, w \in \mathcal{P}$, we estimate

$$
\begin{aligned}
|(Fv)(t) - (Fw)(t)| \leqslant\ & \int_0^t \frac{(t-\varrho)^{q-1}}{\Gamma(q)} |f(\varrho, v(\varrho), Kv(\varrho)) - f(\varrho, w(\varrho), Kw(\varrho))| d\varrho \\
& + |g(t, v(t)) - g(t, w(t))| + |g(1, v(1)) - g(1, w(1))| \\
& + \int_0^1 \frac{(1-\varrho)^{q-1}}{\Gamma(q)} |f(\varrho, v(\varrho), Kv(\varrho)) - f(\varrho, w(\varrho), Kw(\varrho))| d\varrho \\
& + \frac{1}{|1-\alpha\xi|} \Bigg[|\alpha| (|g(\xi, v(\xi)) - g(\xi, w(\xi))| \\
& + |\alpha| (|g(0, v(0)) - g(0, w(0))|) + |g'(0, v(0)) - g'(0, w(0))| \\
& + |\alpha| \int_0^\xi \int_0^\varrho \frac{(\varrho-\tau)^{q-2}}{\Gamma(q-1)} |f(\tau, v(\tau), Kv(\tau)) \\
& - f(\tau, w(\tau), Kw(\tau))| d\tau d\varrho \Bigg] \\
& + |\gamma| \Bigg(\frac{|\alpha|}{|1-\alpha\xi|} (|g(\xi, v(\xi)) - g(\xi, w(\xi))| \\
& + |g(0, v(0)) - g(0, w(0))|) + \frac{1}{|1-\alpha\xi|} |g'(0, v(0)) \\
& - g'(0, w(0))| + |g'(\eta, v(\eta)) - g'(\eta, w(\eta))| \\
& + \int_0^\eta \frac{(\eta-\varrho)^{q-2}}{\Gamma(q-1)} |f(\varrho, v(\varrho), Kv(\varrho)) \\
& - f(\varrho, w(\varrho), Kw(\varrho))| d\varrho \\
& + \frac{|\alpha|}{|1-\alpha\xi|} \int_0^\xi \int_0^\varrho \frac{(\varrho-\tau)^{q-2}}{\Gamma(q-1)} |f(\tau, v(\tau), Kv(\tau)) \\
& - f(\tau, w(\tau), Kw(\tau))| d\tau d\varrho \Bigg)
\end{aligned}
$$

$$
\begin{aligned}
&\leqslant \Bigg[\frac{2}{\Gamma(q+1)}L_1(1+L_4)+2L_2 \\
&\quad +\frac{1}{|1-\alpha\xi|}\left(2|\alpha|L_2+L_3+|\alpha|L_1(1+L_4)\frac{\xi^q}{\Gamma(q+1)}\right) \\
&\quad +|\gamma|\Bigg(2\frac{|\alpha|}{|1-\alpha\xi|}L_2+\frac{1}{|1-\alpha\xi|}L_3+L_3 \\
&\quad +L_1(1+L_4)\frac{\eta^{q-1}}{\Gamma(q}+L_1(1+L_4)\frac{|\alpha|\xi^q}{(|1-\alpha\xi|)\Gamma(q+1)}\Bigg) \\
&\quad \Bigg]\|v-w\| \\
&\leqslant \rho_1\|v-w\|. \qquad (3.3.7)
\end{aligned}
$$

Similarly, for the derivative term we have

$$
\begin{aligned}
|(Fv)'(t)-(Fw)'(t)| &\leqslant \int_0^t \frac{(t-\varrho)^{q-2}}{\Gamma(q-1)}|f(\varrho,v(\varrho),Kv(\varrho))-f(\varrho,w(\varrho),Kw(\varrho))|d\varrho \\
&\quad +|g'(t,v(t))-g'(t,w(t))|+\frac{1}{|1-\alpha\xi|}\Bigg[|\alpha||g(\xi,v(\xi)) \\
&\quad -g(\xi,w(\xi))|+|\alpha||g(0,v(0))-g(0,w(0))|+|g'(0,v(0)) \\
&\quad -g'(0,w(0))|+|\alpha|\int_0^\xi\int_0^\varrho \frac{(\varrho-\tau)^{q-2}}{\Gamma(q-1)}|f(\tau,v(\tau),Kv(\tau)) \\
&\quad -f(\tau,w(\tau),Kw(\tau))|d\tau d\varrho\Bigg] \\
&\leqslant \Bigg[\frac{1}{\Gamma(q)}L_1(1+L_4)+L_3 \\
&\quad +\frac{1}{|1-\alpha\xi|}\left(2|\alpha|L_2+L_3+|\alpha|L_1(1+L_4)\frac{\xi^q}{\Gamma(q+1)}\right) \\
&\quad \Bigg]\|v-w\| \\
&\leqslant \rho_2\|v-w\|. \qquad (3.3.8)
\end{aligned}
$$

By (3.3.7) and (3.3.8), we estimate

$$
\|Fv-Fw\|_{C^1}<\rho\|v-w\|_{C^1}, \quad v,w\in\mathcal{P},
$$

since $\rho<1$ by assumption $(B4)$, consequently F is a contraction. By Banach contraction fixed point theorem, we conclude that there exists a unique solution to the problem (3.1.1). □

3.4 Examples

Example(1): Consider the following fractional BVP

$$\begin{cases} {}^{C}\mathbf{D}^{\frac{3}{2}}[x(t) - \frac{e^{-t}}{35(1+11e^{t})}x(t)] = \frac{1}{(t+4)^2}|x(t)| + \frac{1}{16}\int_0^t \frac{e^{-\varrho}}{9}\frac{|x(t)|}{1+|x(t)|}d\varrho, \quad t \in (0,1); \\ x'(0) = \frac{1}{2}\int_0^{\frac{1}{3}} x'(\varrho)d\varrho, \ x(1) = \frac{1}{3}\phi(x'(\frac{3}{4})). \end{cases} \tag{3.4.1}$$

Here, $q = \frac{3}{2}$, $\alpha = \frac{1}{2}$, $\xi = \frac{1}{3}$, $\gamma = \frac{1}{3}$, $\eta = \frac{3}{4}$ and

$$\phi(v) = \begin{cases} \sqrt{|v|}, & |v| \geqslant 1; \\ v^2, & |v| < 1. \end{cases} \tag{3.4.2}$$

With the given values, we find that $\Lambda_1 \approx 9.45$. Clearly,

$$f(t, x, Kx) = \frac{1}{(t+4)^2}|x(t)| + \frac{1}{16}Kx(t),$$

where $Kx(t) = \int_0^t \frac{e^{-s}}{9}\frac{|x(t)|}{1+|x(t)|}ds$ and $g(t, x) = \frac{e^{-t}}{35(1+11e^{t})}x(t)$.

$$\begin{aligned} |f(t, x, Kx)| &\leqslant \frac{1}{(t+4)^2}|x(t)| + \frac{1}{16}\int_0^t \frac{1}{9}e^{-s}ds \\ &\leqslant \frac{1}{16}(\|x\|_{C^1} + \frac{1}{9}), \\ |g(t, x)| &\leqslant \frac{e^{-t}}{35(1+11e^{t})}\|x\|_{C^1}, \\ g'(t, x) &= \frac{1}{35}\left[\frac{e^{-t}(x' - x) - 22x + 11x'}{(1+11e^{t})^2}\right], \end{aligned}$$

therefore, we have

$$|g'(t, x)| \leqslant \frac{1}{(1+11e^{t})^2}\|x\|_{C^1}.$$

Hence, $p_1(t) = \frac{1}{16}$, $p_2(t) = \frac{e^{-t}}{35(1+11e^{t})}$, $p_3(t) = \frac{1}{(1+11e^{t})^2}$, $\psi_1(r) = r + \frac{1}{9}$, $\psi_2(r) = r$, $\psi_3(r) = r$, and $\|p\| = \frac{1}{16}$, $\psi(r) = r + \frac{1}{9}$. Now using the condition in $(A3)$ that is $\frac{\mathfrak{M}}{\|p\|\psi(\mathfrak{M})\Lambda_1} > 1$, we find that $\mathfrak{M} > 0.1603$. Hence, Theorem 3.3.1 tells that the BVP (3.4.1) has a solution.

Example(2): Consider the following fractional BVP

$$\begin{cases} {}^{C}\mathbf{D}^{\frac{3}{2}}[x(t) - \frac{1}{9}e^{-t}x(t)] = \frac{1}{(t+6)^2}\frac{|x(t)|}{1+|x(t)|} + \frac{1}{36}\int_0^t e^{\frac{-1}{5}x(s)}ds, \quad t \in (0,1); \\ x'(0) = \frac{1}{2}\int_0^{\frac{1}{3}} x'(s)ds, \ x(1) = \frac{1}{3}\phi(x'(\frac{3}{4})). \end{cases} \tag{3.4.3}$$

Here, $q = \frac{3}{2}$, $\alpha = \frac{1}{2}$, $\xi = \frac{1}{3}$, $\gamma = \frac{1}{3}$, $\eta = \frac{3}{4}$ and ϕ is given as (3.4.2). Clearly, $f(t,x,Kx) = \frac{1}{(t+6)^2}\frac{|x(t)|}{1+|x(t)|} + \frac{1}{36}Kx(t)$, where $Kx(t) = \int_0^t e^{\frac{-1}{5}x(s)}ds$ and $g(t,x) = \frac{1}{9}e^{-t}x(t)$.

$$\begin{aligned} |\Upsilon(t,s,v(s)) - \Upsilon(t,s,w(s))| &= |e^{\frac{-1}{5}v(s)} - e^{\frac{-1}{5}w(s)}| \\ &\leqslant \frac{1}{5}\|v-w\|_{C^1}, \end{aligned}$$

$$\begin{aligned} |f(t,v,Kv) - f(t,w,Kw)| &\leqslant \frac{1}{(t+6)^2}\frac{\|v-w\|}{(1+\|v\|)(1+\|w\|)} + \frac{1}{36}\|Kv-Kw\| \\ &\leqslant \frac{1}{36}\Big[\|v-w\|_{C^1} + \|Kv-Kw\|_{C^1}\Big], \\ |g(t,v) - g(t,w)| &\leqslant \frac{1}{9}\|v-w\|_{C^1}, \\ g'(t,v) &= \frac{1}{9}e^{-t}(v'-v), \\ |g'(t,v) - g'(t,w)| &\leqslant \frac{1}{9}\Big[\|v'-w'\| + \|v-w\|\Big] \\ &\leqslant \frac{2}{9}\|v-w\|_{C^1}. \end{aligned}$$

Therefore $L_1 = \frac{1}{36}$, $L_2 = \frac{1}{9}$, $L_3 = \frac{2}{9}$, $L_4 = \frac{1}{5}$, and we get $\rho_1 = 0.893 < 1$ and $\rho_2 = 0.662 < 1$. Hence, by Theorem 3.3.2, we have a unique solution to the BVP (3.4.3).

Chapter 4

Fractional Non-Instantaneous Impulsive Integro-Differential Equations with Nonlocal Conditions

4.1 Introduction

In [95], Hernández et al. introduced first order non-instantaneous impulsive differential equations and the concepts of solutions for it. They established the existence of solutions by the help of compact semigroup and fixed point theorems. Further, Kumar et al. [117] generalized the problem of [95] for fractional order non-instantaneous impulsive evolution equations. Chen et al. [57] established the existence of mild solutions for first order non-instantaneous impulsive differential equations by using noncompact semigroup, Kuratowski measure of noncompactness and ρ-set contraction mapping fixed point theorem.

Therefore inspired by the above works, we considered the following abstract integro-differential systems of fractional order with non-instantaneous impulses and

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nonlocal conditions in a Banach space $\mathbb{X}$:

$$\begin{aligned} {}^{C}\mathbf{D}^{q}x(t) + Ax(t) &= f\bigg(t, x(t), \int_0^t \mathfrak{K}(t,s)x(s)ds\bigg), \quad t \in \cup_{k=0}^{m}(s_k, t_{k+1}], \\ x(t) &= \gamma_k(t, x(t)), \quad t \in \cup_{k=1}^{m}(t_k, s_k], \\ x(0) + g(x) &= x_0, \end{aligned} \tag{4.1.1}$$

where $0 < q < 1$, A is closed linear operator defined on $D(A) \subset \mathbb{X}$, $-A$ generates an equicontinuous and uniformly bounded C_0 semigroup $\mathcal{S}(t)(t \geqslant 0)$ on $\mathbb{X}$, $J = [0, a]$, $0 < t_1 < t_2 < \ldots < t_m < t_{m+1} := a$, $s_0 := 0$ and $s_k \in (t_k, t_{k+1})$ for each $k = 1, 2, \ldots, m$, f and g are $\mathbb{X}$-valued functions defined over $J \times \mathbb{X} \times \mathbb{X}$ and $\mathcal{PC}(J, \mathbb{X})$ respectively, $\gamma_k : (t_k, s_k] \times \mathbb{X} \to \mathbb{X}$ are non-instantaneous impulsive functions for all $k = 1, 2, \ldots, m$, $\mathfrak{K}$ belongs to $C(\mathbb{D}, \mathbb{R}^+)$ with $\mathbb{D} := \{(\tau, \upsilon) : 0 \leqslant \upsilon < \tau \leqslant a\}$ and $x_0 \in \mathbb{X}$.

We conclude this section by summarizing the contents of this chapter. Section 4.2 consists of some basic notations, definitions, preliminary lemmas and assumptions. Section 4.3 contains the proof of main result without assuming that semigroup is noncompact. Section 4.4 consists of an example.

4.2 Preliminaries and Assumptions

Suppose that $\mathbb{X}$ be a Banach space and $\mathcal{PC}(J, \mathbb{X}) = \{v : J \to \mathbb{X} : v$ is continuous at $t \neq t_k$, $v(t_{k-}) = v(t_k)$ and $v(t_{k+})$ exists for all $k = 1, 2, \ldots .m\}$, which is a Banach space, we use θ to denote the zero function in $\mathcal{PC}(J, \mathbb{X})$. For each finite constant $r > 0$, let $B_r = \{v \in \mathcal{PC}(J, \mathbb{X}) : \|v\| \leqslant r\}$. Denote $\mathfrak{F}v(t) := \int_0^t \mathfrak{K}(t, \varrho)v(\varrho)d\varrho$, and let $G^* = \sup\limits_{t \in J} \displaystyle\int_0^t \mathfrak{K}(t, \varrho)d\varrho < \infty$. Let $M = \sup_{t \in J} \|\mathcal{S}(t)\|_{\mathbb{B}(\mathbb{X})}$, note that $M \geqslant 1$. A C_0-semigroup $\mathcal{S}(t)(t \geqslant 0)$ is called equicontinuous if for every $t > 0$, the operator $\mathcal{S}(t)$ is continuous by the operator norm.

Definition 4.2.1. *([117]) A function $v \in \mathcal{PC}(J, \mathbb{X})$ is known as a mild solution to* *the system* (4.1.1) *if* $v(0) = x_0 - g(v)$, $v(t) = \gamma_k(t, v(t))$ *for all* $t \in \cup_{k=1}^{m}(t_k, s_k]$, *and*

$$v(t) = \begin{cases} \mathcal{P}(t)(x_0 - g(v)) + \int_0^t (t-\varrho)^{q-1}\mathcal{R}(t-\varrho) f(\varrho, v(\varrho), \mathfrak{F}v(\varrho))d\varrho, & t \in (0, t_1]; \\ \mathcal{P}(t-s_k)\gamma_k(s_k, v(s_k)) + \int_{s_k}^t (t-\varrho)^{q-1}\mathcal{R}(t-\varrho) f(\varrho, v(\varrho), \mathfrak{F}v(\varrho))d\varrho, \\ \quad t \in (s_k, t_{k+1}], \; k = 1, 2, \ldots, m. \end{cases}$$

Let us introduce the assumptions which are needed to prove our main result :

(H1) The function $f(t, \cdot, \cdot)$ is continuous on $\mathbb{X} \times \mathbb{X}$ for each $t \in J$, and $f(\cdot, z, v)$ is Lebesgue measurable on J for each $(z, v) \in \mathbb{X} \times \mathbb{X}$.

(H2) For some nondecreasing continuous function $\psi : [0, \infty) \to (0, \infty)$, and function $\phi \in L^{\frac{1}{q_1}}(J, \mathbb{R}^+)$ where $q_1 \in (0, q)$, the following holds

$$\|f(t, z, y)\| \leqslant \phi(t)\psi(\|z\|), \quad \forall\, z\,,\, y \in \mathbb{X}\,;\, t \in J.$$

(H3) g is continuous, and

$$\|g(z) - g(v)\| \leqslant \alpha^*\|z - v\|, \quad \forall\, z\,,\, v \in \mathcal{PC}(J, \mathbb{X}),$$

for some constant $\alpha^* > 0$.

(H4) $\gamma_k : [t_k, s_k] \times \mathbb{X} \to \mathbb{X}$ are continuous and there exist constants $K_{\gamma_k} > 0$, $k = 1, 2, \ldots, m$ such that

$$\|\gamma_k(t, x) - \gamma_k(t, y)\| \leqslant K_{\gamma_k}\|x - y\|, \quad \forall\, x\,,\, y \in \mathbb{X}\,;\, t \in [t_k, s_k].$$

(H5) For any countable sets D_1, $D_2 \subset \mathbb{X}$, assume that

$$\beta(f(t, D_1, D_2)) \leqslant L_k\beta(D_1) + N_k\beta(D_2), \quad \forall\, t \in (s_k, t_{k+1}],\, k = 0, 1, 2, \ldots, m,$$

for some positive constants L_k and N_k, $k = 0, 1, 2, \ldots, m$.

Let us denote :

$$K = \max_{k=1,2,\ldots,m} K_{\gamma_k}, \quad K^* = \max\{K, \alpha^*\},$$
$$L = \max_{k=0,1,2,\ldots,m} (L_k + N_k G^*)(t_{k+1} - s_k)^q. \tag{4.2.1}$$

4.3 Main Result

Theorem 4.3.1. *If the semigroup $\mathcal{S}(t)(t \geqslant 0)$ is equicontinuous, the functions $g(\theta)$ and $\gamma_k(\cdot, \theta)$ are bounded for $k = 1, 2, \ldots, m$, and the assumptions $(H1)$-$(H5)$ are satisfied. Then the system (4.1.1) has a PC- mild solution, provided that*

$$\max\{\Lambda_1, \Lambda_2\} < 1, \tag{4.3.1}$$

where $\Lambda_1 = M(\alpha^* + K)$ *and* $\Lambda_2 = M(K^* + \frac{4L}{\Gamma(q+1)})$.

Proof. Let us define an operator F on $\mathcal{PC}(J, \mathbb{X})$ as

$$(Fx)(t) = (F_1x)(t) + (F_2x)(t), \tag{4.3.2}$$

where

$$(F_1x)(t) = \begin{cases} \mathcal{P}(t)(x_0 - g(x)), & t \in [0, t_1]; \\ \gamma_k(t, x(t)), & t \in (t_k, s_k],\ k = 1, 2, \ldots, m, \\ \mathcal{P}(t - s_k)\gamma_k(s_k, x(s_k)), & t \in (s_k, t_{k+1}],\ k = 1, 2, \ldots, m. \end{cases} \tag{4.3.3}$$

$$(F_2x)(t) = \begin{cases} \int_{s_k}^t (t-\varrho)^{q-1}\mathcal{R}(t-\varrho) f(\varrho, x(\varrho), \mathfrak{F}x(\varrho))d\varrho, & \\ \qquad t \in (s_k, t_{k+1}],\ k = 0, 1, 2, \ldots, m, & \\ 0, & \text{otherwise.} \end{cases} \tag{4.3.4}$$

It is easy to see that F is well defined. From the Definition 4.2.1, one can easily see that the PC-mild solution of the system (4.1.1) is a fixed point F.

Let $x \in B_R$ for some $R > 0$, $q_2 = \frac{q-1}{1-q_1} \in (-1, 0)$ and $M_1 = \psi(R)\|\phi\|_{L^{\frac{1}{q_1}}(J,\mathbb{R}^+)}$, by using Hölder inequality and $(H2)$, we obtain

$$\begin{aligned} \int_0^t \|(t-\varrho)^{q-1} f(\varrho, x(\varrho), \mathfrak{F}x(\varrho))\| d\varrho &\leqslant \left(\int_0^t (t-\varrho)^{q_2} d\varrho\right)^{1-q_1} \psi(R)\|\phi\|_{L^{\frac{1}{q_1}}(J,\mathbb{R}^+)} \\ &\leqslant \frac{M_1}{(1+q_2)^{1-q_1}} a^{(1+q_2)(1-q_1)}. \end{aligned} \tag{4.3.5}$$

For our convenience, we divide the proof into following steps :

Step I: We prove that $F(B_R) \subset B_R$, for some $R > 0$.
If this is not true, then for each $r > 0$, there will exist $x_r \in B_r$ and $t_r \in J$ such that $\|(Fx_r(t_r))\| > r$. If $t_r \in [0, t_1]$, then by (4.3.2), (4.3.5), and $(H3)$ we have

$$\begin{aligned} \|(Fx_r)(t_r)\| &\leqslant M(\|x_0\| + \alpha^*\|x_r - \theta\| + \|g(\theta)\|) + \frac{MM_1}{\Gamma(q)(1+q_2)^{1-q_1}} a^{(1+q_2)(1-q_1)} \\ &\leqslant M(\alpha^* r + \|x_0\| + \|g(\theta)\|) + \frac{MM_1}{\Gamma(q)(1+q_2)^{1-q_1}} a^{(1+q_2)(1-q_1)}. \end{aligned} \tag{4.3.6}$$

If $t_r \in (t_k, s_k],\ k = 1, 2, \ldots, m$, then by (4.3.3) and $(H4)$, we obtain

$$\begin{aligned} \|(Fx_r)(t_r)\| &= \|\gamma_k(t_r, x_r(t_r))\| \\ &\leqslant K_{\gamma_k}\|x_r(t_r)\| + \|\gamma_k(t_r, \theta)\| \\ &\leqslant K_{\gamma_k} r + \alpha, \end{aligned} \tag{4.3.7}$$

where $\alpha = \max_{k=1,2,\ldots,m}\{\sup_{t\in J}\|\gamma_k(t,\theta)\|\}$. If $t_r \in (s_k, t_{k+1}],\ k = 1, 2, \ldots, m$, then by (4.3.2), (4.3.5), and $(H4)$ we have

$$\begin{aligned} \|(Fx_r)(t_r)\| &\leqslant M(K_{\gamma_k} r + \alpha) + M\int_{s_k}^{t_r}(t_r - s)^{q-1}\|f(s, x_r(s), \mathfrak{F}x_r(s))\|ds \\ &\leqslant M(K_{\gamma_k} r + \alpha) + \frac{MM_1}{\Gamma(q)(1+q_2)^{1-q_1}}a^{(1+q_2)(1-q_1)} \end{aligned} \tag{4.3.8}$$

Combining (4.3.6), (4.3.7) and (4.3.8) with the fact $r < \|(Fx_r)(t_r)\|$, we obtain

$$r < \|(Fx_r)(t_r)\| \leqslant M(\alpha^* r + \|x_0\| + \|g(\theta)\|) + M(Kr + \alpha) + \frac{MM_1}{\Gamma(q)(1+q_2)^{1-q_1}}a^{(1+q_2)(1-q_1)}. \tag{4.3.9}$$

Dividing by r to the both sides of (4.3.9) and taking limit $r \to \infty$, we get

$$1 \leqslant M(\alpha^* + K), \tag{4.3.10}$$

which gives contradiction to (4.3.1).

Step II: We claim that $F_1 : B_R \to B_R$ is Lipschitz continuous.
For $t \in [0, t_1]$ and $v, z \in B_R$, using (4.3.3) and $(H3)$ we have

$$\|(F_1v)(t) - (F_1z)(t)\| \leqslant M\|g(v) - g(z)\| \leqslant M\alpha^*\|v - z\|. \tag{4.3.11}$$

If $t \in (t_k, s_k],\ k = 1, 2, \ldots, m$ and $v, z \in B_R$, by (4.3.3) and the assumption $(H4)$, we obtain

$$\|(F_1v)(t) - (F_1z)(t)\| \leqslant K_{\gamma_k}\|v(t) - z(t)\| \leqslant MK\|v - z\|. \tag{4.3.12}$$

For $t \in (s_k, t_{k+1}],\ k = 1, 2, \ldots, m$ and $v, z \in B_R$, using $(H4)$, we get

$$\begin{aligned} \|(F_1v)(t) - (F_1z)(t)\| &\leqslant M\|\gamma_k(s_k, v(s_k)) - \gamma_k(s_k, z(s_k))\| \\ &\leqslant MK\|v - z\|. \end{aligned} \tag{4.3.13}$$

From (4.3.11), (4.3.12) and (4.3.13), we obtain

$$\|F_1v - F_1z\| \leqslant MK^*\|v - z\|. \tag{4.3.14}$$

Step III: In this step, we prove that F_2 is continuous on B_R.
Let $\{v_n\}$ be a convergent sequence in B_R with $\lim_{n\to\infty} v_n = v$. Since, f is continuous with respect to second and third variables, for each $\varrho \in J$, we have

$$\lim_{n\to\infty} f(\varrho, v_n(\varrho), \mathfrak{F}v_n(\varrho)) = f(\varrho, v(\varrho), \mathfrak{F}v(\varrho)). \tag{4.3.15}$$

So, we can conclude that

$$\sup_{\varrho\in J}\|f(\varrho,v_n(\varrho),\mathfrak{F}v_n(\varrho))-f(\varrho,v(\varrho),\mathfrak{F}v(\varrho))\|\to 0 \quad \text{as} \quad n\to\infty. \tag{4.3.16}$$

For $\varrho\in[s_k,t]$ and $t\in(s_k,t_{k+1}],\ k=0,1,2,\ldots,m$, by (4.3.15) and (4.3.16), we get

$$\begin{aligned}\|(F_2v_n)(t)-(F_2v)(t)\| &\leqslant \frac{M}{\Gamma(q)}\int_{s_k}^{t}(t-\varrho)^{q-1}\|f(\varrho,v_n(\varrho),\mathfrak{F}v_n(\varrho))\\ &\quad -f(\varrho,v(\varrho),\mathfrak{F}v(\varrho))\|d\varrho\\ &\leqslant \frac{Ma^q}{\Gamma(q+1)}\sup_{\varrho\in J}\|f(\varrho,v_n(\varrho),\mathfrak{F}v_n(\varrho))-f(\varrho,v(\varrho),\mathfrak{F}v(\varrho))\|\\ &\to 0 \quad \text{as} \quad n\to\infty. \end{aligned}\tag{4.3.17}$$

Hence,

$$\|F_2v_n-F_2v\|\to 0 \quad \text{as} \quad n\to\infty, \tag{4.3.18}$$

which means that F_2 is continuous on B_R.

Step IV: Now, we show $\{F_2v : v\in B_R\}$ is equicontinuous. For any $v\in B_R$ and $s_k\leqslant t<\tau\leqslant t_{k+1}$ for $k=0,1,2,\ldots,m$

$$\begin{aligned}\|(F_2v)(\tau)-(F_2v)(t)\| &= \|\int_{s_k}^{\tau}(\tau-\varrho)^{q-1}\mathcal{R}(\tau-\varrho)f(\varrho,v(\varrho),\mathfrak{F}v(\varrho))d\varrho\\ &\quad -\int_{s_k}^{t}(t-\varrho)^{q-1}\mathcal{R}(t-\varrho)f(\varrho,v(\varrho),\mathfrak{F}v(\varrho))d\varrho\|\\ &\leqslant \|\int_{t}^{\tau}(\tau-\varrho)^{q-1}\mathcal{R}(\tau-\varrho)f(\varrho,v(\varrho),\mathfrak{F}v(\varrho))d\varrho\|\\ &\quad +\|\int_{s_k}^{t}[(\tau-\varrho)^{q-1}-(t-\varrho)^{q-1}]\mathcal{R}(\tau-\varrho)\\ &\quad f(\varrho,v(\varrho),\mathfrak{F}v(\varrho))d\varrho\|\\ &\quad +\|\int_{s_k}^{t}(t-\varrho)^{q-1}[\mathcal{R}(\tau-\varrho)-\mathcal{R}(t-\varrho)]\\ &\quad f(\varrho,v(\varrho),\mathfrak{F}v(\varrho))d\varrho\|\\ &= I_1+I_2+I_3,\end{aligned}$$

where,

$$\begin{aligned}I_1 &= \|\int_{t}^{\tau}(\tau-\varrho)^{q-1}\mathcal{R}(\tau-\varrho)f(\varrho,v(\varrho),\mathfrak{F}v(\varrho))d\varrho\|,\\ I_2 &= \|\int_{s_k}^{t}[(\tau-\varrho)^{q-1}-(t-\varrho)^{q-1}]\mathcal{R}(\tau-\varrho)f(\varrho,v(\varrho),\mathfrak{F}v(\varrho))d\varrho\|,\\ I_3 &= \|\int_{s_k}^{t}(t-\varrho)^{q-1}[\mathcal{R}(\tau-\varrho)-\mathcal{R}(t-\varrho)]f(\varrho,v(\varrho),\mathfrak{F}v(\varrho))d\varrho\|.\end{aligned}$$

Now, we only need to check that I_1, I_2 and I_3 tend to 0 independently of $v \in B_R$ when $\tau \to t$. By (4.3.5), we have

$$I_1 \leqslant \frac{M_1 M}{\Gamma(q)(1+q_2)^{1-q_1}}(\tau - t)^{(1+q_2)(1-q_1)} \to 0 \quad \text{as} \quad \tau \to t.$$

For I_2, by $(H2)$, Lemma 2.3.2, Hölder inequality, and [196], we get that

$$\begin{aligned}
I_2 &\leqslant \frac{M}{\Gamma(q)}\bigg(\int_{s_k}^{t}[(\tau-\varrho)^{q-1}-(t-\varrho)^{q-1}]^{\frac{1}{1-q_1}}d\varrho\bigg)^{1-q_1}\psi(R)\|\phi\|_{L^{\frac{1}{q_1}}(J,\mathbb{R})}\\
&\leqslant \frac{M_1 M}{\Gamma(q)}\bigg(\int_{s_k}^{t}[(t-\varrho)^{q_2}-(\tau-\varrho)^{q_2}]d\varrho\bigg)^{1-q_1}\\
&\leqslant \frac{M_1 M}{\Gamma(q)(1+q_2)^{1-q_1}}[(t-s_k)^{1+q_2}+(\tau-s_k)^{1+q_2}-(\tau-t)^{1+q_2}]^{1-q_1}\\
&\leqslant \frac{M_1 M}{\Gamma(q)(1+q_2)^{1-q_1}}(\tau-t)^{(1+q_2)(1-q_1)} \to 0 \quad \text{as} \quad \tau \to t.
\end{aligned}$$

For $t = s_k$, $I_3 = 0$. For $t > s_k$ and $\epsilon > 0$, by $(H2)$, equicontinuity of $\mathcal{S}(t)$, and Lemma 2.3.2, we estimate

$$\begin{aligned}
I_3 &\leqslant \|\int_{s_k}^{t-\epsilon}(t-\varrho)^{q-1}[\mathcal{R}(\tau-\varrho)-\mathcal{R}(t-\varrho)]f(\varrho,v(\varrho),\mathfrak{F}v(\varrho))d\varrho\|\\
&\quad+\|\int_{t-\epsilon}^{t}(t-\varrho)^{q-1}[\mathcal{R}(\tau-\varrho)-\mathcal{R}(t-\varrho)]f(\varrho,v(\varrho),\mathfrak{F}v(\varrho))d\varrho\|\\
&\leqslant \int_{s_k}^{t-\epsilon}\|(t-\varrho)^{q-1}f(\varrho,v(\varrho),\mathfrak{F}v(\varrho))\|d\varrho \sup_{\varrho\in[s_k,t-\epsilon]}\|\mathcal{R}(\tau-\varrho)-\mathcal{R}(t-\varrho)\|\\
&\quad+\frac{2M}{\Gamma(q)}\int_{t-\epsilon}^{t}\|(t-\varrho)^{q-1}f(\varrho,v(\varrho),\mathfrak{F}v(\varrho))\|d\varrho\\
&\leqslant \frac{M_1}{(1+q_2)^{1-q_1}}((t-s_k)^{1+q_2}-\epsilon^{1+q_2})^{1-q_1}\sup_{\varrho\in[s_k,t-\epsilon]}\|\mathcal{R}(\tau-\varrho)-\mathcal{R}(t-\varrho)\|\\
&\quad+\frac{2M_1 M}{\Gamma(q)(1+q_2)^{1-q_1}}\epsilon^{(1+q_2)(1-q_1)} \to 0 \quad \text{as} \quad \tau \to t.
\end{aligned}$$

As a result, $\|(F_2v)(\tau)-(F_2v)(t)\| \to 0$ independently of $v \in B_R$ as $\tau \to t$, hence $\{F_2(B_R)\}$ is equicontinuous.

Step V: We show that $F : B_R \to B_R$ is a ρ-set contractive map.
For any bounded set $D \subset B_R$, by Lemma 2.4.6, we have a countable set $D_0 = \{v_n\} \subset D$ such that

$$\beta(F_2(D)) \leqslant 2\beta(F_2(D_0)). \tag{4.3.19}$$

Since $F_2(D_0) \subset F_2(B_R)$ is bounded and equicontinuous, by Lemma 2.4.4, we get

$$\beta(F_2(D_0)) = \max_{t\in[s_k,t_{k+1}],k=0,1,2,\dots,m}\beta(F_2(D_0)(t)). \tag{4.3.20}$$

For every $t \in [s_k, t_{k+1}]$, $k = 0, 1, 2, \ldots, m$, by the assumption $(H5)$, Lemma 2.4.5, and (4.2.1), we get

$$\begin{aligned}
\beta(F_2(D_0)(t)) &= \beta\Big(\Big\{\int_{s_k}^{t}(t-\varrho)^{q-1}\mathcal{R}(t-\varrho)f(\varrho, v_n(\varrho), \mathfrak{F}v_n(\varrho))d\varrho\Big\}\Big) \\
&\leqslant \frac{2M}{\Gamma(q)}\int_{s_k}^{t}(t-\varrho)^{q-1}\beta(\{f(\varrho, v_n(\varrho), \mathfrak{F}v_n(\varrho))\})d\varrho \\
&\leqslant \frac{2M}{\Gamma(q)}\int_{s_k}^{t}(t-\varrho)^{q-1}[L_k\beta(D_0(\varrho)) + N_k\beta(\mathfrak{F}D_0(\varrho))]d\varrho. \quad (4.3.21)
\end{aligned}$$

Meanwhile, we have

$$\beta(\mathfrak{F}D_0(\varrho)) \leqslant \beta(\mathfrak{F}D_0) \leqslant \|\mathfrak{F}\|\beta(D_0) \leqslant G^*\beta(D_0) \leqslant G^*\beta(D). \quad (4.3.22)$$

Therefore,

$$\begin{aligned}
\beta(F_2(D_0)(t)) &\leqslant \frac{2M}{\Gamma(q+1)}(L_k + N_kG^*)(t_{k+1} - s_k)^q\beta(D) \\
&\leqslant \frac{2ML}{\Gamma(q+1)}\beta(D). \quad (4.3.23)
\end{aligned}$$

From (4.3.19) and (4.3.23), we obtain

$$\beta(F_2(D)) \leqslant \frac{4ML}{\Gamma(q+1)}\beta(D). \quad (4.3.24)$$

From Lemma 2.4.3 and (4.3.14), for any bounded set $D \subset B_R$,

$$\beta(F_1(D)) \leqslant MK^*\beta(D). \quad (4.3.25)$$

Therefore, by (4.3.24) and (4.3.25), we obtain

$$\begin{aligned}
\beta(F(D)) &\leqslant \beta(F_1(D)) + \beta(F_2(D)) \\
&\leqslant M\Big(K^* + \frac{4L}{\Gamma(q+1)}\Big)\beta(D) = \Lambda_2\beta(D). \quad (4.3.26)
\end{aligned}$$

Now combining (4.3.26) with (4.3.1) and Definition 2.5.1, we get that $F : B_R \to B_R$ is a ρ-set-contractive map with $\rho = \Lambda_2$. Hence, Theorem 2.5.3 implies that F has a fixed point in B_R, hence a PC-mild solution to (4.1.1). □

4.4 Example

Consider a partial differential system with non-instantaneous impulses and nonlocal conditions of fractional order as :

$$\begin{cases} {}^{C}\mathbf{D}^{\frac{1}{2}}v(t,y)+\frac{\partial^2}{\partial y^2}v(t,y)=\frac{1}{25}\frac{e^{-t}}{1+e^t}v(t,y)+\int_0^t\frac{1}{50}e^{-s}v(s,y)ds, \\ \qquad y\in(0,1),t\in(0,\frac{1}{3}]\cup(\frac{2}{3},1]; \\ v(t,0)=v(t,1)=0, \quad t\in[0,1]; \\ v(t,y)=\frac{e^{-(t-\frac{1}{3})}}{4}\frac{|v(t,y)|}{1+|v(t,y|}, \quad y\in(0,1),t\in(\frac{1}{3},\frac{2}{3}]; \\ v(0,y)+\displaystyle\sum_{i=1}^{2}\frac{1}{3^i}v(\frac{1}{i},y)=x_0(y), \quad y\in[0,1]. \end{cases} \tag{4.4.1}$$

Let $\mathbb{X}=L^2[0,1]$, $x_0(y)\in\mathbb{X}$ and $Av=v''$ with $D(A)=\{v\in\mathbb{X}:v,v'$ are absolutely continuous and $v''\in\mathbb{X},v(0)=v(1)=0\}$. By [156], $-A$ is generates an equicontinuous C_0-semigroup $\mathcal{S}(t)(t\geqslant 0)$ on $\mathbb{X}$, with $\|\mathcal{S}(t)\|\leqslant 1$, for any $t\geqslant 0$. Let $a=t_2=1$, $t_0=s_0=0$, $t_1=\frac{1}{3}$, $s_1=\frac{2}{3}$. By putting

$$\begin{aligned} v(t) &= v(t,\cdot), \\ f(t,v(t),\mathfrak{F}v(t)) &= \frac{1}{25}\frac{e^{-t}}{1+e^t}v(t,\cdot)+\int_0^t\frac{1}{50}e^{-s}v(\varrho,\cdot)d\varrho, \\ \mathfrak{F}v(t) &= \int_0^t\frac{1}{50}e^{-\varrho}v(\varrho,\cdot)d\varrho, \\ \gamma_1(t,v(t)) &= \frac{e^{-(t-\frac{1}{3})}}{4}\frac{|v(t,\cdot)|}{1+|v(t,\cdot)|}, \\ g(v) &= \sum_{i=1}^{2}\frac{1}{3^i}v(\frac{1}{i},\cdot), \end{aligned}$$

the parabolic partial differential equation (4.4.1) can be transformed into the abstract form (4.1.1) for $m=1$. Observe that the assumptions $(H1)$-$(H5)$ and condition (4.3.1) hold with

$$q=\frac{1}{2},\ M=1,\ \phi(t)=\frac{1}{25}\frac{e^{-t}}{1+e^t}+\frac{1}{50},\ \psi(r)=r,$$
$$\alpha^*=\frac{4}{9},\ K=K_{\gamma_1}=\frac{1}{4},\ L=0.02,\ \Lambda_1=0.69<1,\ \Lambda_2=0.53<1.$$

Therefore, Theorem 4.3.1 is applicable, so the system (4.4.1) has a PC-mild solution.

Chapter 5

Approximate Controllability of Nonlocal Non-Instantaneous Impulsive Integro-Differential Equations of Fractional Order

5.1 Introduction

Controllability is one of the most important issue in mathematical control theory and engineering. The problem of controllability for various kinds of differential, integro-differential equations and impulsive differential equations are studied. In case of controllability, the literature on abstract impulsive differential equations consists of problems involving instantaneous impulses. In [23], Balasubramaniam studied the approximate controllability of impulsive integro-differential system of fractional order having nonlocal conditions, by assuming the compactness of impulsive and nonlocal functions in a Hilbert space. Zhang [198] considered fractional impulsive integro-differential equations in a Hilbert space to study the approximate controllability. Dong et al. [84] also considered fractional impulsive evolution equations

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having nonlocal conditions, and studied the approximate controllability via approximate technique.

This chapter deals with the approximate controllability of a certain class of abstract fractional evolution equations of the form :

$$\begin{aligned} {}^{C}\mathbf{D}^{q}x(t) &= Ax(t)+f(t,x(t),\mathcal{H}x(t))+Bu(t), \quad t\in\cup_{k=0}^{m}(s_k,t_{k+1}], \\ x(t) &= \gamma_k(t,x(t)), \quad t\in\cup_{k=1}^{m}(t_k,s_k], \\ x(0)+g(x) &= x_0, \end{aligned} \tag{5.1.1}$$

where $0<q<1$, $J=[0,b]$, the variable x assumes values in a reflexive separable Banach space $\mathbb{X}$, A is linear closed operator defined on $D(A)\subset\mathbb{X}$, and generates a C_0 semigroup $\mathcal{S}(t)(t\geqslant 0)$ on $\mathbb{X}$, $u\in L^2(J,U)$ is control function, B is $\mathbb{X}$-valued bounded linear operator defined on U, which is a Banach space. $0<t_1<t_2<\ldots<t_m<t_{m+1}:=b$, $s_0:=0$, $s_k\in(t_k,t_{k+1})$; $k=1,2,\ldots,m$, f and g are given $\mathbb{X}$-valued functions defined on $J\times\mathbb{X}\times\mathbb{X}$ and $\mathcal{PC}(J,\mathbb{X})$ respectively, $\gamma_k:(t_k,s_k]\times\mathbb{X}\to\mathbb{X}$ are non-instantaneous impulsive functions for all $k=1,2,\ldots,m$, $\mathcal{H}x(t):=\int_0^t h(t,\varrho,x(\varrho))d\varrho$, $h:\mathbb{D}\times\mathbb{X}\to\mathbb{X}$ is continuous function where $\mathbb{D}:=\{(\tau,\upsilon):0\leqslant\upsilon<\tau\leqslant b\}$ and $x_0\in\mathbb{X}$.

To the best of our knowledge, there is no work yet reported on the approximate controllability of fractional non-instantaneous impulsive integro-differential equations. Therefore inspired by this fact, we consider the system (5.1.1) to investigate the approximate controllability via Kuratowski measure of noncompactness and ρ-set contraction mapping fixed point theorem without assuming the compactness condition on impulsive and nonlocal functions.

The remaining part of the chapter is arranged as following: Section 5.2 consists of basic definitions, notations and theorems. Section 5.3 concerns with the main results. Finally, section 5.4 deals with an example to illustrate our results.

5.2 Preliminaries and Assumptions

Let $\mathbb{X}$ be a separable reflexive Banach space, and $\mathcal{PC}(J,\mathbb{X}) = \{v : J \to \mathbb{X} : v \text{ is continuous at } t \neq t_k,\ v(t_{k-}) = v(t_k) \text{ and } v(t_{k+}) \text{ exists for all } k = 1,2,\ldots.m\}$, which is a complete normed space with supremum norm. Let $B_r = \{v \in \mathcal{PC}(J,\mathbb{X}) : \|v\| \leqslant r\}\,(r > \theta)$, we use θ to denote the zero function in $\mathcal{PC}(J,\mathbb{X})$. Let $M = \sup_{t\in J}\|\mathcal{S}(t)\|_{\mathbb{B}(\mathbb{X})}$, note that $M \geqslant 1$.

Definition 5.2.1. *([117]) A function $v \in \mathcal{PC}(J,\mathbb{X})$ is known as a mild solution of the problem (5.1.1) if for any $u \in L^2(J,U)$, v satisfies $v(0) = x_0 - g(v)$, $v(t) = \gamma_k(t, v(t))$ for all $t \in \cup_{k=1}^{m}(t_k, s_k]$, and*

$$v(t) = \begin{cases} \mathcal{P}(t)(x_0 - g(v)) + \int_0^t (t-\varrho)^{q-1}\mathcal{R}(t-\varrho)[f(\varrho, v(\varrho), \mathcal{H}v(\varrho)) \\ \qquad +Bu(\varrho)]d\varrho,\ t \in (0, t_1]; \\ \mathcal{P}(t-s_k)\gamma_k(s_k, v(s_k)) + \int_{s_k}^t (t-\varrho)^{q-1}\mathcal{R}(t-\varrho)[f(\varrho, v(\varrho), \mathcal{H}v(\varrho)) \\ \qquad +Bu(\varrho)]d\varrho,\ t \in \cup_{k=1}^{m}(s_k, t_{k+1}]. \end{cases} \tag{5.2.1}$$

Let $x^b(x_0, u)$ be the value of state at time b for the initial value x_0 and control u. Denote $\mathbf{R}(b, x_0) = \{x^b(x_0, u) : u \in L^2(J,U)\}$, which is reachable set for (5.1.1) at time b.

Definition 5.2.2. *([132]) If $\overline{\mathbf{R}(b, x_0)} = \mathbb{X}$, the system (5.1.1) is called approximately controllable on J.*

Now, consider the linear control system of fractional order

$$\begin{aligned} {}^C D^q x(t) &= Ax(t) + Bu(t), \quad t \in J, \\ x(0) &= x_0. \end{aligned} \tag{5.2.2}$$

The controllability and resolvent operators associated with linear system (5.2.2) are mentioned below :

$$\Gamma_0^b = \int_0^b (b-\varrho)^{q-1}\mathcal{R}(b-\varrho)BB^*\mathcal{R}^*(b-\varrho)d\varrho, \tag{5.2.3}$$

$$R(\lambda, \Gamma_0^b) = (\lambda I + \Gamma_0^b)^{-1},\ \lambda > 0, \tag{5.2.4}$$

respectively, where $(\cdot)^*$ denotes the adjoint operator of $(\cdot)$. Observe that the operator Γ_0^b is linear and bounded. Now, we state the following hypothesis :

(H0) $\lambda R(\lambda, \Gamma_0^b) \to 0$ as $\lambda \to 0^+$ in the strong operator topology.

Theorem 5.2.1. *([131]) Let $\mathbb{Z}$ is a reflexive separable complete norm space and $\mathbb{Z}^*$ denotes dual space of $\mathbb{Z}$, $\Gamma : \mathbb{Z}^* \to \mathbb{Z}$ is a symmetric map. Then the statements given below are equivalent :*

(i) *Γ is positive map, which means $\langle v^*, \Gamma v^* \rangle > 0$ for all nonzero $v^* \in \mathbb{Z}^*$.*

(ii) *$\lambda(\lambda I + \Gamma \mathfrak{J})^{-1}(v)$ converges to zero strongly as $\lambda \to 0^+$, $\forall v \in \mathbb{Z}$, here $\mathfrak{J}$ is the duality map from $\mathbb{Z} \to \mathbb{Z}^*$.*

Lemma 5.2.2. *([132]) The linear control system (5.2.2) is approximately controllable on J iff (H0) satisfies.*

Proof. The system (5.2.2) is approximately controllable on J iff $\langle v, \Gamma_0^b v \rangle > 0$, for each nonzero $v \in X$ (see Theorem 4.1.7 of [62]), so the lemma is straightforward consequence of Theorem 5.2.1. □

Remark 5.2.3. *Notice that the system (5.2.2) is approximately controllable on J iff $\langle v, \Gamma_0^b v \rangle = \int_0^b (b - \varrho)^{q-1} \| B^* \mathcal{R}^* (b - \varrho) v \|^2 d\varrho > 0$, for all nonzero $v \in \mathbb{X}$, this is further equivalent to $B^* \mathcal{R}^* (b - \varrho) v = 0$, $0 \leqslant \varrho < b \Longrightarrow v = 0$.*

Now, let us state the basic assumptions which are useful to prove our main results:

(H1) $\mathcal{S}(t)(t > 0)$ is a compact semigroup.

(H2) The function $f(t, \cdot, \cdot)$ is continuous on $\mathbb{X} \times \mathbb{X}$, for every fixed $t \in J$, and $f(\cdot, v, z)$ is Lebesgue measurable on J, for all $(v, z) \in \mathbb{X} \times \mathbb{X}$.

(H3) There exist a nondecreasing continuous function ψ from $\mathbb{R}^+$ to $\mathbb{R}^+$, a constant $0 < q_1 < q$, and a function $\phi \in L^{\frac{1}{q_1}}(J, \mathbb{R}^+)$ satisfying

$$\| f(t, v, z) \| \leqslant \phi(t) \psi(\| v \|), \quad \forall \, v \, , \, z \in \mathbb{X} \, ; \, t \in J.$$

(H4) The function g is continuous and

$$\| g(v) - g(z) \| \leqslant \alpha \| v - z \|, \quad \forall \, v \, , \, z \in \mathcal{PC}(J, \mathbb{X}),$$

for some constant $\alpha > 0$.

(H5) γ_k are continuous and

$$\|\gamma_k(t,v)-\gamma_k(t,z)\| \leqslant K_{\gamma_k}\|v-z\|, \quad \forall\, v\,,\, z\in\mathbb{X}\,;\, t\in[t_k,s_k],$$

for some constants $K_{\gamma_k}>0\,,\ k=1,2,\ldots,m.$

For our convenience, we use the following notations :

$$\begin{aligned}
K &= \max_{k=1,2,\ldots,m} K_{\gamma_k}, \quad K_1=\max\{K,\alpha\}, \quad M_B:=\|B\|,\\
\overline{M} &= \frac{b^q M^2 (M_B)^2}{q\lambda(\Gamma(q))^2}, \quad q_2=\frac{q-1}{1-q_1}\in(-1,0), \quad M_1=\psi(R)\|\phi\|_{L^{\frac{1}{q_1}}(J,\mathbb{R}^+)},\\
M_b &= \frac{MM_1}{\Gamma(q)(1+q_2)^{1-q_1}} b^{(1+q_2)(1-q_1)}. \qquad (5.2.5)
\end{aligned}$$

For any $v\in\mathcal{PC}(J,\mathbb{X})$, we choose the control function for the nonlinear system (5.1.1) as given below :

$$u(t)=u_\lambda(t,v)=B^*\mathcal{R}^*(b-t)R(\lambda,\Gamma_0^b)p(v), \qquad (5.2.6)$$

where

$$p(v)=\begin{cases} x^b-\mathcal{P}(b)(x_0-g(v))-\int_0^b(b-\varrho)^{q-1}\mathcal{R}(b-\varrho)f(\varrho,v(\varrho),\mathcal{H}v(\varrho))d\varrho, \ t\in(0,t_1],\\ x^b-\mathcal{P}(b-s_k)\gamma_k(s_k,v(s_k))-\int_{s_k}^b(b-\varrho)^{q-1}\mathcal{R}(b-\varrho)f(\varrho,v(\varrho),\mathcal{H}v(\varrho))d\varrho,\\ \qquad t\in\cup_{k=1}^m(s_k,t_{k+1}].\end{cases}$$

By using the control function (5.2.6), for any $\lambda>0$ define the operator F_λ on $\mathcal{PC}(J,\mathbb{X})$ as following :

$$(F_\lambda v)(t)=(\Phi_\lambda v)(t)+(\widetilde{\Phi_\lambda} v)(t), \qquad (5.2.7)$$

where

$$(\Phi_\lambda v)(t)=\begin{cases}\mathcal{P}(t)(x_0-g(v)), & t\in[0,t_1],\\ \gamma_k(t,v(t)), & t\in\cup_{k=1}^m(t_k,s_k],\\ \mathcal{P}(t-s_k)\gamma_k(s_k,v(s_k)), & t\in\cup_{k=1}^m(s_k,t_{k+1}],\end{cases} \qquad (5.2.8)$$

$$(\widetilde{\Phi_\lambda} v)(t)=\begin{cases}\int_{s_k}^t(t-\varrho)^{q-1}\mathcal{R}(t-\varrho)[f(\varrho,v(\varrho),\mathcal{H}v(\varrho))+Bu_\lambda(\varrho,v)]d\varrho,\\ \qquad t\in\cup_{k=0}^m(s_k,t_{k+1}],\\ 0, \qquad \text{otherwise}.\end{cases} \qquad (5.2.9)$$

5.3 Main Results

5.3.1 Existence of mild solutions

Theorem 5.3.1. *Let the hypotheses* $(H1)$*-*$(H5)$ *are satisfied, the functions* $g(\theta)$ *and* $\gamma_k(\cdot,\theta)$ *are bounded for* $k = 1, 2, \ldots, m$. *Then the system* (5.1.1) *has a* PC*- mild solution, provided that*

$$\rho := MK_1 < 1. \tag{5.3.1}$$

Proof. First let us observe that, for $v \in B_R$ $(R > 0)$ with the help of $(H3)$ and Hölder inequality, we have

$$\begin{aligned}\int_0^t \|(t-\varrho)^{q-1} f(\varrho, v(\varrho), \mathcal{H}v(\varrho))\| d\varrho &\leqslant \left(\int_0^t (t-\varrho)^{q_2} d\varrho\right)^{1-q_1} \psi(R)\|\phi\|_{L^{\frac{1}{q_1}}(J,\mathbb{R}^+)} \\ &\leqslant \frac{M_1}{(1+q_2)^{1-q_1}} b^{(1+q_2)(1-q_1)}. \end{aligned} \tag{5.3.2}$$

We divide the proof into following steps :

Step 1: For any $\lambda > 0$, we want to show that there exists a constant $R = R(\lambda) > 0$, satisfying $F_\lambda(B_R) \subset B_R$. Let $v \in B_r$ for any positive constant r. If $t \in [0, t_1]$, then by using (5.2.6) and (5.3.2), we have

$$\begin{aligned} u_\lambda(t,v) &= B^*\mathcal{R}^*(b-t)R(\lambda,\Gamma_0^b)\Big[x^b - \mathcal{P}(b)(x_0 - g(v)) \\ &\quad - \int_0^b (b-\varrho)^{q-1}\mathcal{R}(b-\varrho) f(\varrho, v(\varrho), \mathcal{H}v(\varrho)) d\varrho\Big] \\ \|u_\lambda(t,v)\| &\leqslant \frac{MM_B}{\lambda\Gamma(q)}\Big[\|x^b\| + M(\|x_0\| + \alpha\|v-\theta\| + \|g(\theta)\|) + M_b\Big] \\ &\leqslant \frac{MM_B}{\lambda\Gamma(q)}\Big[\|x^b\| + M(\alpha r + \|x_0\| + \|g(\theta)\|) + M_b\Big], \end{aligned} \tag{5.3.3}$$

and from (5.2.7), (5.3.3), we obtain

$$\begin{aligned} (F_\lambda v)(t) &= \mathcal{P}(t)(x_0 - g(v)) + \int_0^t (t-\varrho)^{q-1}\mathcal{R}(t-\varrho) f(\varrho, v(\varrho), \mathcal{H}v(\varrho)) d\varrho \\ &\quad + \int_0^t (t-\varrho)^{q-1}\mathcal{R}(t-\varrho) B u_\lambda(\varrho, v) d\varrho \\ \|(F_\lambda v)(t)\| &\leqslant M(\alpha r + \|x_0\| + \|g(\theta)\|) + M_b + \int_0^t (t-\varrho)^{q-1}\|\mathcal{R}(t-\varrho)\| \\ &\quad \|Bu_\lambda(\varrho, v)\| d\varrho \\ &\leqslant M(\alpha r + \|x_0\| + \|g(\theta)\|) + M_b \\ &\quad + \frac{b^q M^2 (M_B)^2}{q\lambda(\Gamma(q))^2}\Big[\|x^b\| + M(\alpha r + \|x_0\| + \|g(\theta)\|) + M_b\Big]. \end{aligned} \tag{5.3.4}$$

For $t \in (t_k, s_k]$; $k = 1, 2, \ldots, m$, then by (5.2.7) and $(H5)$, we estimate

$$\begin{aligned} \|(F_\lambda v)(t)\| &= \|\gamma_k(t, v(t))\| \\ &\leqslant K_{\gamma_k}\|v(t)\| + \|\gamma_k(t, \theta)\| \\ &\leqslant Kr + \alpha^* \leqslant M(Kr + \alpha^*), \end{aligned} \tag{5.3.5}$$

where $\alpha^* = \max_{k=1,2,\ldots,m}\{\sup_{t\in J}\|\gamma_k(t,\theta)\|\}$. If $t \in (s_k, t_{k+1}]$; $k = 1, 2, \ldots, m$ then (5.2.6), (5.2.7) and (5.3.2) yield the following estimations

$$\|u_\lambda(t, v)\| \leqslant \frac{MM_B}{\lambda\Gamma(q)}\Big[\|x^b\| + M(Kr + \alpha^*) + M_b\Big], \tag{5.3.6}$$

$$\begin{aligned} \|(F_\lambda v)(t)\| &\leqslant M(Kr + \alpha^*) + M_b + \frac{b^q M^2 (M_B)^2}{q\lambda(\Gamma(q))^2} \\ &\quad \Big[\|x^b\| + M(Kr + \alpha^*) + M_b\Big]. \end{aligned} \tag{5.3.7}$$

Combining (5.3.4), (5.3.5) and (5.3.7), we obtain

$$\begin{aligned} \|(F_\lambda v)(t)\| &\leqslant M_b + M(Kr + \alpha^*) + M(\alpha r + \|x_0\| + \|g(\theta)\|) + \overline{M}M(Kr + \alpha^*) \\ &\quad + \overline{M}\Big[\|x^b\| + M(\alpha r + \|x_0\| + \|g(\theta)\|) + M_b\Big]. \end{aligned} \tag{5.3.8}$$

Then, we get that for large enough $R > 0$, $F_\lambda(B_R) \subset B_R$ holds.

Step 2: We show that $\Phi_\lambda : B_R \to B_R$ is Lipschitz continuous. Let $v, z \in B_R$, for $t \in [0, t_1]$, using (5.2.8) and $(H4)$

$$\|(\Phi_\lambda v)(t) - (\Phi_\lambda z)(t)\| \leqslant M\|g(v) - g(z)\| \leqslant M\alpha\|v - z\|, \tag{5.3.9}$$

for $t \in (t_k, s_k]$, $k = 1, 2, \ldots, m$, by (5.2.8) and the assumption $(H5)$

$$\|(\Phi_\lambda v)(t) - (\Phi_\lambda z)(t)\| \leqslant K_{\gamma_k}\|v(t) - z(t)\| \leqslant MK\|v - z\|, \tag{5.3.10}$$

for $t \in (s_k, t_{k+1}]$, $k = 1, 2, \ldots, m$, using $(H5)$, we have

$$\begin{aligned} \|(\Phi_\lambda v)(t) - (\Phi_\lambda z)(t)\| &\leqslant M\|\gamma_k(s_k, v(s_k)) - \gamma_k(s_k, z(s_k))\| \\ &\leqslant MK\|v - z\|. \end{aligned} \tag{5.3.11}$$

From (5.3.9), (5.3.10) and (5.3.11), we obtain

$$\|\Phi_\lambda v - \Phi_\lambda z\| \leqslant MK_1\|v - z\|. \tag{5.3.12}$$

Step 3: Let $\{v_n\}$ be a sequence in B_R such that $\lim_{n\to\infty} v_n = v$ in B_R. Since, f is continuous with respect to second and third variables, for each $\varrho \in J$, we have

$$\lim_{n\to\infty} f(\varrho, v_n(\varrho), \mathcal{H}v_n(\varrho)) = f(\varrho, v(\varrho), \mathcal{H}v(\varrho)). \tag{5.3.13}$$

So, we can conclude that

$$\sup_{\varrho\in J}\|f(\varrho,v_n(\varrho),\mathcal{H}v_n(\varrho))-f(\varrho,v(\varrho),\mathcal{H}v(\varrho))\|\to 0 \quad \text{as} \quad n\to\infty. \tag{5.3.14}$$

For $t\in(s_k,t_{k+1}]$, $(H5)$ and (5.3.14) yield the following

$$\begin{aligned}\|p(v_n)-p(v)\| &\leqslant M\|\gamma_k(s_k,v_n(s_k))-\gamma_k(s_k,v(s_k))\| \\ &\quad +\frac{M}{\Gamma(q)}\int_{s_k}^{b}(b-\varrho)^{q-1}\|f(\varrho,v_n(\varrho),\mathcal{H}v_n(\varrho))-f(\varrho,v(\varrho),\mathcal{H}v(\varrho))\|d\varrho \\ &\leqslant M\|\gamma_k(s_k,v_n(s_k))-\gamma_k(s_k,v(s_k))\| \\ &\quad +\frac{Mb^q}{\Gamma(q+1)}\sup_{\varrho\in J}\|f(\varrho,v_n(\varrho),\mathcal{H}v_n(\varrho))-f(\varrho,v(\varrho),\mathcal{H}v(\varrho))\| \\ &\to 0 \quad \text{as} \quad n\to\infty. \end{aligned}\tag{5.3.15}$$

Therefore, (5.2.6) and (5.3.15) imply that

$$\|u_\lambda(\varrho,v_n)-u_\lambda(\varrho,v)\|\to 0 \quad \text{as} \quad n\to\infty, \tag{5.3.16}$$

also (5.2.9), (5.3.14) and (5.3.16) yield

$$\begin{aligned}\|(\widetilde{\Phi_\lambda}v_n)(t)-(\widetilde{\Phi_\lambda}v)(t)\| &\leqslant \frac{M}{\Gamma(q)}\int_{s_k}^{t}(t-\varrho)^{q-1}\|f(\varrho,v_n(\varrho),\mathcal{H}v_n(\varrho)) \\ &\quad -f(\varrho,v(\varrho),\mathcal{H}v(\varrho))\|d\varrho \\ &\quad +\frac{M}{\Gamma(q)}\int_{s_k}^{t}(t-\varrho)^{q-1}\|B\|\|u_\lambda(\varrho,v_n)-u_\lambda(\varrho,v)\|d\varrho \\ &\leqslant \frac{Mb^q}{\Gamma(q+1)}\sup_{\varrho\in J}\|f(\varrho,v_n(\varrho),\mathcal{H}v_n(\varrho))-f(\varrho,v(\varrho),\mathcal{H}v(\varrho))\| \\ &\quad +\frac{b^qMM_B}{\Gamma(q+1)}\sup_{\varrho\in J}\|u_\lambda(\varrho,v_n)-u_\lambda(\varrho,v)\| \\ &\to 0 \quad \text{as} \quad n\to\infty, \end{aligned}\tag{5.3.17}$$

which means that $\widetilde{\Phi_\lambda}$ is continuous in B_R.

Step 4: We claim that $\widetilde{\Phi_\lambda}:B_R\to B_R$ is compact, this result will be proved by using Arzela-Ascoli theorem. For this we need to prove :
(i): For any $t\in J$, the set $\{(\widetilde{\Phi_\lambda}v)(t):v\in B_R\}$ is relatively compact in $\mathbb{X}$. For $t\notin(s_k,t_{k+1}]$, $k=0,1,2,\dots,m$, obviously the set $\{(\widetilde{\Phi_\lambda}v)(t):v\in B_R\}=\{0\}$ which is compact in $\mathbb{X}$. Let $t\in(s_k,t_{k+1}]$, $k=0,1,2,\dots,m$ be fixed. For any $\varepsilon\in(s_k,t)$

and $\delta > 0$, we define an operator $\widetilde{\Phi_\lambda}^{\varepsilon,\delta}$ on B_R as following :

$$\begin{aligned}
(\widetilde{\Phi_\lambda}^{\varepsilon,\delta} v)(t) &= q\int_{s_k}^{t-\varepsilon}\int_{\delta}^{\infty}\varpi(t-\varrho)^{q-1}\Psi_q(\varpi)\mathcal{S}((t-\varrho)^q\varpi)[f(\varrho,v(\varrho),\mathcal{H}v(\varrho))\\
&\quad +Bu_\lambda(\varrho,v)]d\varpi d\varrho\\
&= \mathcal{S}(\varepsilon^q\delta)q\int_{s_k}^{t-\varepsilon}\int_{\delta}^{\infty}\varpi(t-\varrho)^{q-1}\Psi_q(\varpi)\mathcal{S}((t-\varrho)^q\varpi-\varepsilon^q\delta)\\
&\quad [f(\varrho,v(\varrho),\mathcal{H}v(\varrho))+Bu_\lambda(\varrho,v)]d\varpi d\varrho\\
&:= \mathcal{S}(\varepsilon^q\delta)y(t,\varepsilon).
\end{aligned}$$

Since $y(t,\varepsilon)$ is bounded on B_R and $\mathcal{S}(\varepsilon^q\delta)(\varepsilon^q\delta > 0)$ is compact on $\mathbb{X}$, we conclude that the set $\{(\widetilde{\Phi_\lambda}^{\varepsilon,\delta} v)(t) : v \in B_r\}$ is relatively compact in $\mathbb{X}$. Now, observe that

$$\begin{aligned}
\|(\widetilde{\Phi_\lambda} v)(t)-(\widetilde{\Phi_\lambda}^{\varepsilon,\delta} v)(t)\| &= q\Bigg\|\int_{s_k}^{t}\int_{0}^{\delta}\varpi(t-\varrho)^{q-1}\Psi_q(\varpi)\mathcal{S}((t-\varrho)^q\varpi)\\
&\quad [f(\varrho,v(\varrho),\mathcal{H}v(\varrho))+Bu_\lambda(\varrho,v)]d\varpi d\varrho\\
&\quad +\int_{s_k}^{t}\int_{\delta}^{\infty}\varpi(t-\varrho)^{q-1}\Psi_q(\varpi)\mathcal{S}((t-\varrho)^q\varpi)\\
&\quad [f(\varrho,v(\varrho),\mathcal{H}v(\varrho))+Bu_\lambda(\varrho,v)]d\varpi d\varrho\\
&\quad -\int_{s_k}^{t-\varepsilon}\int_{\delta}^{\infty}\varpi(t-\varrho)^{q-1}\Psi_q(\varpi)\mathcal{S}((t-\varrho)^q\varpi)\\
&\quad [f(\varrho,v(\varrho),\mathcal{H}v(\varrho))+Bu_\lambda(\varrho,v)]d\varpi d\varrho\Bigg\|\\
&= q\Bigg\|\int_{s_k}^{t}\int_{0}^{\delta}\varpi(t-\varrho)^{q-1}\Psi_q(\varpi)\mathcal{S}((t-\varrho)^q\varpi)\\
&\quad [f(\varrho,v(\varrho),\mathcal{H}v(\varrho))+Bu_\lambda(\varrho,v)]d\varpi d\varrho\\
&\quad +\int_{t-\epsilon}^{t}\int_{\delta}^{\infty}\varpi(t-\varrho)^{q-1}\Psi_q(\varpi)\mathcal{S}((t-\varrho)^q\varpi)\\
&\quad [f(\varrho,v(\varrho),\mathcal{H}v(\varrho))+Bu_\lambda(\varrho,v)]d\varpi d\varrho\Bigg\|\\
&\leqslant q(I_1+I_2), \qquad (5.3.18)
\end{aligned}$$

where

$$\begin{aligned}
I_1 &= \|\int_{s_k}^{t}\int_{0}^{\delta}\varpi(t-\varrho)^{q-1}\Psi_q(\varpi)\mathcal{S}((t-\varrho)^q\varpi)[f(\varrho,v(\varrho),\mathcal{H}v(\varrho))\\
&\quad +Bu_\lambda(\varrho,v)]d\varpi d\varrho\|,\\
I_2 &= \|\int_{t-\epsilon}^{t}\int_{\delta}^{\infty}\varpi(t-\varrho)^{q-1}\Psi_q(\varpi)\mathcal{S}((t-\varrho)^q\varpi)[f(\varrho,v(\varrho),\mathcal{H}v(\varrho))\\
&\quad +Bu_\lambda(\varrho,v)]d\varpi d\varrho\|.
\end{aligned}$$

Now, by (5.3.2) and (5.3.6) we have

$$\begin{aligned} I_1 &\leqslant M\bigg(\int_0^\delta \varpi\Psi_q(\varpi)d\varpi\bigg)\bigg[\int_{s_k}^t (t-\varrho)^{q-1}\|f(\varrho,v(\varrho),\mathcal{H}v(\varrho))\|d\varrho + M_B\|u_\lambda\|\frac{b^q}{q}\bigg] \\ &\leqslant M\bigg(\int_0^\delta \varpi\Psi_q(\varpi)d\varpi\bigg)\bigg[\frac{M_1}{(1+q_2)^{1-q_1}}b^{(1+q_2)(1-q_1)} \\ &\quad +\frac{M(M_B)^2}{\lambda\Gamma(q)}[\|x^b\|+M(Kr+\alpha^*)+M_b]\frac{b^q}{q}\bigg]. \end{aligned} \tag{5.3.19}$$

Similarly using Remark 2.3.1, we can obtain

$$\begin{aligned} I_2 &\leqslant M\bigg(\int_\delta^\infty \varpi\Psi_q(\varpi)d\varpi\bigg)\bigg[\frac{M_1}{(1+q_2)^{1-q_1}}\epsilon^{(1+q_2)(1-q_1)} + \frac{M(M_B)^2}{\lambda\Gamma(q)} \\ &\quad [\|x^b\|+M(Kr+\alpha^*)+M_b]\frac{\epsilon^q}{q}\bigg] \\ &\leqslant \frac{M}{\Gamma(q+1)}\bigg[\frac{M_1}{(1+q_2)^{1-q_1}}\epsilon^{(1+q_2)(1-q_1)} + \frac{M(M_B)^2}{\lambda\Gamma(q)} \\ &\quad [\|x^b\|+M(Kr+\alpha^*)+M_b]\frac{\epsilon^q}{q}\bigg]. \end{aligned} \tag{5.3.20}$$

Therefore by (5.3.18), (5.3.19), and (5.3.20) we conclude that

$$\|(\widetilde{\Phi_\lambda}v)(t) - (\widetilde{\Phi_\lambda}^{\varepsilon,\delta}v)(t)\| \to 0 \quad as \quad \varepsilon\to 0, \delta\to 0.$$

This implies that the set $\{(\widetilde{\Phi_\lambda}v)(t) : v \in B_R\}$ *is relatively compact in* $\mathbb{X}$ *for* $t \in (s_k, t_{k+1}]$, $k = 0, 1, 2, \ldots, m$.
(ii): The family of functions $\{\widetilde{\Phi_\lambda}v : v \in B_R\}$ *is equicontinuous. For any* $v \in B_R$

and $s_k \leqslant t' < t'' \leqslant t_{k+1}$ *for* $k = 0, 1, 2, \ldots, m,$ *we have*

$$\begin{aligned}
\|(\widetilde{\Phi_\lambda} v)(t'') - (\widetilde{\Phi_\lambda} v)(t')\| &\leqslant \left\| \int_{t'}^{t''} (t''-\varrho)^{q-1}\mathcal{R}(t''-\varrho) f(\varrho, v(\varrho), \mathcal{H}v(\varrho))d\varrho \right\| \\
&+ \left\| \int_{t'}^{t''} (t''-\varrho)^{q-1}\mathcal{R}(t''-\varrho) Bu_\lambda(\varrho, v) d\varrho \right\| \\
&+ \left\| \int_{s_k}^{t'} [(t''-\varrho)^{q-1} - (t'-\varrho)^{q-1}]\mathcal{R}(t''-\varrho) \right. \\
&\left. f(\varrho, v(\varrho), \mathcal{H}v(\varrho))d\varrho \right\| \\
&+ \left\| \int_{s_k}^{t'} [(t''-\varrho)^{q-1} - (t'-\varrho)^{q-1}]\mathcal{R}(t''-\varrho) \right. \\
&\left. Bu_\lambda(\varrho, v) d\varrho \right\| \\
&+ \left\| \int_{s_k}^{t'} (t'-\varrho)^{q-1}[\mathcal{R}(t''-\varrho) - \mathcal{R}(t'-\varrho)] \right. \\
&\left. f(\varrho, v(\varrho), \mathcal{H}v(\varrho))d\varrho \right\| \\
&+ \left\| \int_{s_k}^{t'} (t'-\varrho)^{q-1}[\mathcal{R}(t''-\varrho) - \mathcal{R}(t'-\varrho)] \right. \\
&\left. Bu_\lambda(\varrho, v) d\varrho \right\| \\
&= J_1 + J_2 + J_3 + J_4 + J_5 + J_6,
\end{aligned}$$

Now, we only need to check that J_1, J_2, J_3, J_4, J_5 *and* J_6 *tends to* 0 *independently of* $v \in B_R$ *when* $t'' \to t'$*. By* (5.3.2)*, we have*

$$\begin{aligned}
J_1 &\leqslant \frac{M_1 M}{\Gamma(q)(1+q_2)^{1-q_1}}(t''-t')^{(1+q_2)(1-q_1)} \to 0 \quad as \quad t'' \to t', \\
J_2 &\leqslant \frac{M M_B}{\Gamma(q+1)}(t''-t')^q \|u_\lambda\| \to 0 \quad as \quad t'' \to t'.
\end{aligned}$$

By (H3), Lemma 2.3.2, and Hölder inequality, we get that

$$
\begin{aligned}
J_3 &\leqslant \frac{M}{\Gamma(q)}\left(\int_{s_k}^{t'}[(t''-\varrho)^{q-1}-(t'-\varrho)^{q-1}]^{\frac{1}{1-q_1}}d\varrho\right)^{1-q_1}\psi(R)\|\phi\|_{L^{\frac{1}{q_1}}(J,\mathbb{R})}\\
&\leqslant \frac{M_1M}{\Gamma(q)}\left(\int_{s_k}^{t'}[(t'-\varrho)^{q_2}-(t''-\varrho)^{q_2}]d\varrho\right)^{1-q_1}\\
&\leqslant \frac{M_1M}{\Gamma(q)(1+q_2)^{1-q_1}}[(t'-s_k)^{1+q_2}+(t''-s_k)^{1+q_2}-(t''-t')^{1+q_2}]^{1-q_1}\\
&\leqslant \frac{M_1M}{\Gamma(q)(1+q_2)^{1-q_1}}(t''-t')^{(1+q_2)(1-q_1)}\to 0 \quad as \quad t''\to t',
\end{aligned}
$$

and

$$
J_4 \leqslant \frac{MM_B}{\Gamma(q+1)}\left[(t''-s_k)^q-(t'-s_k)^q-(t''-t')^q\right]\|u_\lambda\|\to 0 \quad as \quad t''\to t'.
$$

For $t'=s_k$, it is easy to see that $J_5=0$. For $t'>s_k$ and $\epsilon>0$ (small enough), by Lemma 2.3.2 and (H3), we obtain

$$
\begin{aligned}
J_5 &\leqslant \|\int_{s_k}^{t'-\epsilon}(t'-\varrho)^{q-1}[\mathcal{R}(t''-\varrho)-\mathcal{R}(t'-\varrho)]f(\varrho,v(\varrho),\mathcal{H}v(\varrho))d\varrho\|\\
&\quad+\|\int_{t'-\epsilon}^{t'}(t'-\varrho)^{q-1}[\mathcal{R}(t''-\varrho)-\mathcal{R}(t'-\varrho)]f(\varrho,v(\varrho),\mathcal{H}v(\varrho))d\varrho\|\\
&\leqslant \int_{s_k}^{t'-\epsilon}\|(t'-\varrho)^{q-1}f(\varrho,v(\varrho),\mathcal{H}v(\varrho))\|d\varrho\\
&\quad\sup_{\varrho\in[s_k,t'-\epsilon]}\|\mathcal{R}(t''-\varrho)-\mathcal{R}(t'-\varrho)\|\\
&\quad+\frac{2M}{\Gamma(q)}\int_{t'-\epsilon}^{t'}\|(t'-\varrho)^{q-1}f(\varrho,v(\varrho),\mathcal{H}v(\varrho))\|d\varrho\\
&\leqslant \frac{M_1}{(1+q_2)^{1-q_1}}((t'-s_k)^{1+q_2}-\epsilon^{1+q_2})^{1-q_1}\\
&\quad\sup_{\varrho\in[s_k,t'-\epsilon]}\|\mathcal{R}(t''-\varrho)-\mathcal{R}(t'-\varrho)\|\\
&\quad+\frac{2M_1M}{\Gamma(q)(1+q_2)^{1-q_1}}\epsilon^{(1+q_2)(1-q_1)}\to 0 \quad as \quad t''\to t',\ \epsilon\to 0,
\end{aligned}
$$

similarly

$$
\begin{aligned}
J_6 &\leqslant \frac{M_B}{q}[(t'-s_k)^q-\epsilon^q]\|u_\lambda\|\sup_{\varrho\in[s_k,t'-\epsilon]}\|\mathcal{R}(t''-\varrho)-\mathcal{R}(t'-\varrho)\|\\
&\quad+\frac{2MM_B}{\Gamma(q+1)}\epsilon^q\|u_\lambda\|\to 0 \quad as \quad t''\to t',\ \epsilon\to 0.
\end{aligned}
$$

So, $\|(\widetilde{\Phi_\lambda}v)(t'')-(\widetilde{\Phi_\lambda}v)(t')\|\to 0$ independently of $v\in B_R$ as $t''\to t'$, hence $\{\widetilde{\Phi_\lambda}(B_R)\}$ is equicontinuous. So, $\widetilde{\Phi_\lambda}$ is compact on B_R by Arzela-Ascoli theorem.

Step 5: *We show that F_λ is ρ-set contractive map. For any bounded set $D \subset B_R$, by Lemma 2.4.6, we have a countable set $D_0 = \{v_n\} \subset D$ such that*

$$\beta(\widetilde{\Phi_\lambda}(D)) \leqslant 2\beta(\widetilde{\Phi_\lambda}(D_0)).$$

Since $\widetilde{\Phi_\lambda}(D_0) \subset \widetilde{\Phi_\lambda}(B_R)$ is bounded and equicontinuous, by Lemma 2.4.4 we obtain

$$\beta(\widetilde{\Phi_\lambda}(D_0)) = \max_{t\in[s_k,t_{k+1}],k=0,1,2,\ldots,m} \beta(\widetilde{\Phi_\lambda}(D_0)(t)).$$

By Proposition 2.4.2 (i) and Step 4 (i), we have $\beta(\widetilde{\Phi_\lambda}(D_0)(t)) = 0$ for all $t \in J$, therefore $\beta(\widetilde{\Phi_\lambda}(D)) = 0$. From (5.3.12) and Lemma 2.4.3, we know that for any bounded set $D \subset B_R$

$$\beta(\Phi_\lambda(D)) \leqslant MK_1\beta(D).$$

Thus, by Proposition 2.4.2 (ii)

$$\begin{aligned} \beta(F_\lambda(D)) &\leqslant \beta(\Phi_\lambda(D)) + \beta(\widetilde{\Phi_\lambda}(D)) \\ &\leqslant MK_1\beta(D) = \rho\beta(D). \end{aligned} \tag{5.3.21}$$

Now combining (5.3.21) with (5.3.1) and Definition 2.5.1, we conclude that $F_\lambda : B_R \to B_R$ is a ρ-set-contractive map with $\rho = MK_1$. Hence from Theorem 2.5.3, it follows that there exists a fixed point of F_λ in B_R, hence a PC-mild solution of (5.1.1). □

5.3.2 Approximate controllability

Theorem 5.3.2. *Assume that the assumptions $(H0)$-$(H5)$ are satisfied. Moreover, assume that the functions f, g, γ_k $(k = 1, 2, \ldots, m)$ are uniformly bounded by positive constants L_1, L_2 and $N_k(k = 1, 2, \ldots, m)$. Then, the system (5.1.1) is approximately controllable on J.*

Proof. Let x_λ be a mild solution of the problem (5.1.1) under the control

$$u_\lambda(t, x_\lambda) = B^*\mathcal{R}^*(b - t)R(\lambda, \Gamma_0^b)p(x_\lambda),$$

and satisfies the equality

$$x_\lambda(b) = x^b - \lambda R(\lambda, \Gamma_0^b)p(x_\lambda), \tag{5.3.22}$$

where

$$p(x_\lambda) = \begin{cases} x^b - \mathcal{P}(b)(x_0 - g(x_\lambda)) - \int_0^b (b-\varrho)^{q-1}\mathcal{R}(b-\varrho)f(\varrho, x_\lambda(\varrho), \mathcal{H}x_\lambda(\varrho))d\varrho, \\ \qquad t \in (0, t_1], \\ x^b - \mathcal{P}(b - s_k)\gamma_k(s_k, x_\lambda(s_k)) - \int_{s_k}^b (b-\varrho)^{q-1}\mathcal{R}(b-\varrho) \\ f(\varrho, x_\lambda(\varrho), \mathcal{H}x_\lambda(\varrho))d\varrho, \qquad t \in \cup_{k=1}^m (s_k, t_{k+1}]. \end{cases}$$

Since, $\mathcal{P}(t)(t>0)$ is compact, and g is uniformly bounded, we see that there exists a subsequence of $\{\mathcal{P}(b)g(x_\lambda) : \lambda > 0\}$, still denoted by it, converges to some $x_g \in \mathbb{X}$ as $\lambda \to 0$. Similarly there exists a subsequence of $\{\mathcal{P}(b-s_k)\gamma_k(s_k, x_\lambda(s_k)) : \lambda > 0\}$, still denoted by it, converges to some $x_{\gamma_k} \in \mathbb{X}$ as $\lambda \to 0$. Since f is uniformly bounded, we have

$$\int_0^b \|f(\varrho, x_\lambda(\varrho), \mathcal{H}x_\lambda(\varrho))\|^2 d\varrho \leqslant L_1^2 b.$$

Hence the sequence $f(\cdot, x_\lambda(\cdot), \mathcal{H}x_\lambda(\cdot))$ is bounded in $L^2(J,\mathbb{X})$. Then there exists a subsequence of $\{f(\cdot, x_\lambda(\cdot), \mathcal{H}x_\lambda(\cdot)) : \lambda > 0\}$, still denoted by it, converges weakly to some $f(\cdot) \in L^2(J,\mathbb{X})$. Define

$$\omega = \begin{cases} x^b - \mathcal{P}(b)(x_0) + x_g - \int_0^b (b-\varrho)^{q-1}\mathcal{R}(b-\varrho)f(\varrho)d\varrho, & t \in (0,t_1]; \\ x^b - x_{\gamma_k} - \int_{s_k}^b (b-\varrho)^{q-1}\mathcal{R}(b-\varrho)f(\varrho)d\varrho, & t \in \cup_{k=1}^m (s_k, t_{k+1}]. \end{cases}$$

It follows that for $t \in (0,t_1]$ and $t \in (s_k, t_{k+1}]$, $k = 1,2,\ldots,m$,

$$\|p(x_\lambda) - \omega\| \to 0 \quad \text{as} \quad \lambda \to 0^+, \tag{5.3.23}$$

because of compactness of the operator (see [166])

$$l(\cdot) \to \int_0^{\cdot} (\cdot - s)^{q-1}\mathcal{R}(\cdot - \varrho)l(\varrho)d\varrho : L^2(J,\mathbb{X}) \to C(J,\mathbb{X}).$$

Then, from (5.3.22), (5.3.23), and $(H0)$, we obtain

$$\begin{aligned} \|x_\lambda(b) - x^b\| &\leqslant \|\lambda R(\lambda, \Gamma_0^b)p(x_\lambda)\| \\ &\leqslant \|\lambda R(\lambda, \Gamma_0^b)\omega\| + \|\lambda R(\lambda, \Gamma_0^b)\|\|p(x_\lambda) - \omega\| \\ &\leqslant \|\lambda R(\lambda, \Gamma_0^b)\omega\| + \|p(x_\lambda) - \omega\| \to 0 \text{ as } \lambda \to 0^+. \end{aligned}$$

It concludes that the system (5.1.1) is approximately controllable on J. □

5.4 Example

Consider a control system as following :

$$\begin{cases} {}^C\mathbf{D}^{\frac{1}{2}} v(t,y) = \frac{\partial^2}{\partial y^2} v(t,y) + u(t,y) + \frac{1}{25}\frac{e^{-t}}{1+e^t}\frac{|v(t,y)|}{1+|v(t,y)|} + \int_0^t \frac{1}{50} e^{-\varrho}\frac{|v(\varrho,y)|}{1+|v(\varrho,y)|}d\varrho, \\ \qquad y \in (0,1), t \in (0,\frac{1}{3}] \cup (\frac{2}{3}, 1], \\ v(t,0) = v(t,1) = 0, \quad t \in [0,1], \\ v(t,y) = \frac{e^{-(t-\frac{1}{3})}}{4}\frac{|v(t,y)|}{1+|v(t,y)|}, \quad y \in (0,1), t \in (\frac{1}{3}, \frac{2}{3}], \\ v(0,y) + \sum_{i=1}^{2} \frac{1}{3^i}\frac{v(\frac{1}{i}, y)}{1+v(\frac{1}{i}, y)} = x_0(y), \quad y \in [0,1], \end{cases} \tag{5.4.1}$$

where $\mathbb{X} = U = L^2[0,1]$, $J = [0,1]$, $x_0(y) \in \mathbb{X}$. Define $Av = v''$ with

$$D(A) = \{v \in \mathbb{X} : v, v' \text{ are absolutely continuous and } v'' \in \mathbb{X}, v(0) = v(1) = 0\}.$$

Then

$$Av = \sum_{n=1}^{\infty} -n^2 < v, \psi_n > \psi_n,\ v \in D(A), \tag{5.4.2}$$

where $\psi_n(y) = \sqrt{\frac{2}{\pi}}\sin(ny)$, $0 \leqslant y \leqslant 1$, $n = 1, 2, \ldots$, and A is generator of a compact semigroup $\mathcal{S}(t)(t > 0)$ on $\mathbb{X}$, and

$$\mathcal{S}(t)v = \sum_{n=1}^{\infty} e^{-n^2 t} < v, \psi_n > \psi_n,\ v \in \mathbb{X}, \tag{5.4.3}$$

with $\|\mathcal{S}(t)\| \leqslant 1$, for any $t \geqslant 0$. Let $b = t_2 = 1, t_0 = s_0 = 0, t_1 = \frac{1}{3}, s_1 = \frac{2}{3}$. Put $v(t) = v(t,\cdot)$, means $v(t)(y) = v(t,y)$, $t, y \in [0,1]$. Let $u(t) = u(t,\cdot)$ is continuous, and $Bu(t) = u(t,\cdot)$. Further

$$\begin{aligned} f(t, v(t), \mathcal{H}v(t)) &= \frac{1}{25}\frac{e^{-t}}{1+e^t}\frac{|v(t,\cdot)|}{1+|v(t,\cdot)|} + \int_0^t \frac{1}{50}e^{-\varrho}\frac{|v(\varrho,\cdot)|}{1+|v(\varrho,\cdot)|}d\varrho, \\ \mathcal{H}v(t) &= \int_0^t \frac{1}{50}e^{-\varrho}\frac{|v(\varrho,\cdot)|}{1+|v(\varrho,\cdot)|}d\varrho, \\ \gamma_1(t, v(t)) &= \frac{e^{-(t-\frac{1}{3})}}{4}\frac{|v(t,\cdot)|}{1+|v(t,\cdot)|}, \\ g(v) &= \sum_{i=1}^{2}\frac{1}{3^i}\frac{v(\frac{1}{i},\cdot)}{1+v(\frac{1}{i},\cdot)}. \end{aligned}$$

Then the system (5.4.1) can be rewritten into the abstract form of (5.1.1) for $m = 1$. Easily we can verify that the assumptions $(H1)$-$(H5)$ as well as condition (5.3.1) hold with

$$q = \frac{1}{2},\ M = 1,\ \phi(t) = \frac{1}{25}\frac{e^{-t}}{1+e^t} + \frac{1}{50},\ \psi(r) = r,$$
$$\alpha = \frac{4}{9},\ K_{\gamma_1} = \frac{1}{4},\ \rho = \frac{4}{9} < 1.$$

Also f, g and γ_1 are uniformly bounded with $L_1 = \frac{3}{50}$, $L_2 = \frac{4}{9}$, $N_1 = \frac{1}{4}$ respectively. Moreover, $(H0)$ also holds based on the argument in [132]. Thus by Theorem 5.3.2, the system (5.4.1) is approximately controllable.

Chapter 6

Fractional Evolution Equations with Deformable Fractional Derivative

6.1 Introduction

There are several generalization of the notions of derivative to fractional derivative so far, however only few of them become popular namely Riemann-Liouville, Caputo, Caputo-Fabrizio , Hilfer (Riemann-Liouville generalized), and Hadamard fractional derivatives. These definitions of fractional derivatives are based on integral form. Khalil [112], first introduced limit based definition of fractional derivative, and named it conformable fractional derivative, which looks a generalization of classical derivative and satisfies the classical properties that is linearity, product rule, quotient rule, Rolle's theorem, and mean value theorem. However this definition lacks to include zero and negative numbers. In [205], another limit based definition of fractional derivative is introduced by the authors and named as deformable fractional derivative, which is simpler than conformable fractional derivative and ranges

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over a wide class of functions, and also satisfies the classical properties.

Motivated by the fact that there is no work yet available on deformable fractional differential equations, in this chapter, first we discuss the existence and uniqueness of mild solution for the following fractional system

$$\begin{aligned} D^{\alpha}x(t) &= Ax(t) + f(t, x(t)), \quad t \in J, \\ x(0) &= x_0. \end{aligned} \tag{6.1.1}$$

In recent years, controllability of various fractional differential equations, fractional integro-differential equations, fractional impulsive evolution equations has been studied where the fractional derivative is taken as either Caputo, Riemann-Liouville or Hilfer sense. However the controllability of fractional differential equations with local fractional derivative is still untreated topic. Motivated by this fact, next we establish the sufficient conditions for the approximate controllability of the following abstract deformable control system

$$\begin{aligned} D^{\alpha}x(t) &= Ax(t) + f(t, x(t)) + Bu(t), \quad t \in J, \\ x(0) &= x_0, \end{aligned} \tag{6.1.2}$$

where D^{α} denotes the deformable fractional derivative with order $\alpha \in (0, 1)$ which is introduced by Zulfeqarr et. al. [205]. The operator A defined from $D(A) \subset \mathbb{X}$ to $\mathbb{X}$ generates a C_0 semigroup $\mathcal{S}(t)(t \geqslant 0)$ on a Banach space $\mathbb{X}$, $x_0 \in \mathbb{X}$, $J = [0, b]$, $b > 0$ is a constant, $u \in L^2(J, U)$, $B : U \to \mathbb{X}$ is a bounded linear operator, U is a Hilbert space, and $f : J \times \mathbb{X} \to \mathbb{X}$ be a given function satisfying certain assumptions.

The remaining part of the chapter is organized as following: Section 6.2 contains some basic notations, definitions and theorems. Section 6.3 is further subdivided into two subsections. In first part, we will obtain the expression for mild solutions for the system (6.1.1) and discuss the sufficient conditions for the existence and uniqueness of mild solution. In second part, we will prove that the system (6.1.2) is approximately controllable on J. Finally, in section 6.4, we will present some examples to illustrate our results.

6.2 Preliminaries

Now, we recall definition of deformable fractional derivative, its fractional integral and some theorems, which are needed to prove our results. For more details regarding deformable fractional derivative, we refer [205].

Definition 6.2.1. *The deformable fractional derivative of order α for a function $f:(a,b)\to\mathbb{R}$ is given as*

$$D^{\alpha}f(t)=\lim_{\epsilon\to 0}\frac{(1+\epsilon\gamma)f(t+\epsilon\alpha)-f(t)}{\epsilon},$$

where $\alpha+\gamma=1$, $0\leqslant\alpha\leqslant 1$.

Remark 6.2.1. *If $\alpha=0$, $D^{0}f(t)=f(t)$, and if $\alpha=1$, $Df(t)=f'(t)$.*

Definition 6.2.2. *The α-fractional integral of a continuous function f on $[a,b]$ is defined as*

$$I_{a}^{\alpha}f(t)=\frac{1}{\alpha}e^{\frac{-\gamma}{\alpha}t}\int_{a}^{t}e^{\frac{\gamma}{\alpha}s}f(s)ds,\quad where\ \alpha+\gamma=1,\ \alpha\in(0,1].$$

Theorem 6.2.2. *A differentiable function f at a point $t\in(a,b)$ is always α-differentiable at that point for any α. Moreover,*

$$D^{\alpha}f(t)=\gamma f(t)+\alpha Df(t),$$

where $Df(t)=\frac{d}{dt}f(t)$.

Theorem 6.2.3. *Let a function f is defined on (a,b), then f is α-differentiable for any $\alpha\in(0,1]$ if and only if f is differentiable*

Theorem 6.2.4. *The operators D^{α} and I^{α} posses the following properties*

(i) $D^{\alpha}(af+bg)=aD^{\alpha}f+bD^{\alpha}g$. *(Linearity)*

(ii) $D^{\alpha_1}.D^{\alpha_2}=D^{\alpha_2}.D^{\alpha_1}$. *(Commutativity)*

(iii) $D^{\alpha}k=\gamma k$, *where k is a constant.*

(iv) $D^{\alpha}(f.g)=(D^{\alpha}f).g+\alpha f.Dg$.

(v) $I_{a}^{\alpha}(bf+cg)=bI_{a}^{\alpha}f+cI_{a}^{\alpha}g$. *(Linearity)*

(vi) $I_{a}^{\alpha_1}I_{a}^{\alpha_2}=I_{a}^{\alpha_2}I_{a}^{\alpha_1}$. *(Commutativity)*

Theorem 6.2.5. *(Inverse Property) If f is a function defined on $[a, b]$ and is continuous also, then $I_a^\alpha f$ is α-differentiable in (a, b). In fact, we have*

$$D^\alpha(I_a^\alpha f(t)) = f(t).$$

Conversely, suppose g is a continuous anti-α-derivative of f over (a, b), that is $g = D^\alpha f$, then

$$I_a^\alpha(D^\alpha f(t)) = I_a^\alpha(g(t)) = f(t) - e^{\frac{\gamma}{\alpha}(a-t)}f(a).$$

6.3 Main Results

6.3.1 Existence and uniqueness

Let $C(J, \mathbb{X})$ be the Banach space of all continuous maps from J to $\mathbb{X}$ with supremum norm. Denote $M = \sup_{t \in J} \|\mathcal{S}(t)\|_{\mathbb{B}(\mathbb{X})}$, note that $M \geqslant 1$.

Lemma 6.3.1. *Let A generates a C_0-semigroup $\mathcal{S}(t)(t \geqslant 0)$ on $\mathbb{X}$, then*

$$D^\alpha \mathcal{S}(t)x = (\gamma I + \alpha A)\mathcal{S}(t)x, \;\; x \in D(A).$$

Proof. By [156], we know that $\mathcal{S}(t)x \in D(A)$ and $\frac{d}{dt}\mathcal{S}(t)x = A\mathcal{S}(t)x = \mathcal{S}(t)Ax$, for $x \in D(A)$. Now using Theorem 6.2.2, we get

$$\begin{aligned} D^\alpha \mathcal{S}(t)x &= \gamma \mathcal{S}(t)x + \alpha \frac{d}{dt}\mathcal{S}(t)x \\ &= \gamma \mathcal{S}(t)x + \alpha A\mathcal{S}(t)x \\ &= (\gamma I + \alpha A)\mathcal{S}(t)x. \end{aligned}$$

□

Consider the following linear deformable fractional Cauchy problem

$$\begin{aligned} D^\alpha x(t) &= Ax(t), \quad t \in J, \\ x(0) &= x_0. \end{aligned} \tag{6.3.1}$$

Definition 6.3.1. *A function v is called solution to the problem (6.3.1), if the following hold*

(i) $v \in C(J, \mathbb{X})$, and $v(t) \in D(A)$ for all $t \in J$,

(ii) $D^\alpha v$ exists and continuous on J,

(iii) v satisfies (6.3.1).

Theorem 6.3.2. *Let A be infinitesimal generator of a C_0-semigroup $\mathcal{S}(t)(t \geqslant 0)$. If $x_0 \in D(A)$, then $e^{\frac{-\gamma}{\alpha}t}\mathcal{S}(\frac{t}{\alpha})x_0$ is a solution to the problem (6.3.1).*

Proof. Let $x(t) = e^{\frac{-\gamma}{\alpha}t}\mathcal{S}(\frac{t}{\alpha})x_0$. Since $x_0 \in D(A)$, $x(t)$ is differentiable. Now using Lemma 6.3.1, Theorem 6.2.2, and Theorem 6.2.4 we get

$$\begin{aligned} D^\alpha x(t) &= \left[(D^\alpha e^{\frac{-\gamma}{\alpha}t})\mathcal{S}(\frac{t}{\alpha}) + \alpha e^{\frac{-\gamma}{\alpha}t}\frac{d}{dt}\mathcal{S}(\frac{t}{\alpha})\right]x_0 \\ &= \left[(\gamma e^{\frac{-\gamma}{\alpha}t} + \alpha(\frac{-\gamma}{\alpha})e^{\frac{-\gamma}{\alpha}t})\mathcal{S}(\frac{t}{\alpha}) + e^{\frac{-\gamma}{\alpha}t}A\mathcal{S}(\frac{t}{\alpha})\right]x_0 \\ &= Ae^{\frac{-\gamma}{\alpha}t}\mathcal{S}(\frac{t}{\alpha})x_0 \\ &= Ax(t). \end{aligned}$$

□

Now, we consider the inhomogeneous initial value problem

$$\begin{aligned} D^\alpha x(t) &= Ax(t) + f(t), \quad t \in J, \\ x(0) &= x_0, \end{aligned} \tag{6.3.2}$$

where A generates a C_0-semigroup, $x_0 \in \mathbb{X}$, and $f : J \to \mathbb{X}$ be a suitable function.

Theorem 6.3.3. *Let x is a solution to the problem (6.3.2) and $f \in L^1(J, \mathbb{X})$, then x satisfies*

$$x(t) = e^{\frac{-\gamma}{\alpha}t}\mathcal{S}(\frac{t}{\alpha})x_0 + \frac{1}{\alpha}e^{\frac{-\gamma}{\alpha}t}\int_0^t e^{\frac{\gamma}{\alpha}\varrho}\mathcal{S}(\frac{t-\varrho}{\alpha})f(\varrho)d\varrho.$$

Proof. Since x be a solution of the problem (6.3.2), so we have

$$D^\alpha x(t) = Ax(t) + f(t). \tag{6.3.3}$$

Also by Theorem 6.2.2, we know

$$D^\alpha x(t) = \gamma x(t) + \alpha x'(t). \tag{6.3.4}$$

From (6.3.3) and (6.3.4) easily we can conclude that

$$x'(t) = \frac{1}{\alpha}[(A - \gamma I)x(t) + f(t)]. \tag{6.3.5}$$

Let $h(\varrho) = \mathcal{S}(\frac{t-\varrho}{\alpha})x(\varrho)$, $0 \leqslant \varrho \leqslant t$, since $x(\varrho) \in D(A)$ therefore h is differentiable and hence α-differentiable by Theorem 6.2.2. Now using Theorem 6.2.4, Lemma

6.3.1, and (6.3.5) we get

$$\begin{aligned}
D^\alpha h(\varrho) &= \Big(D^\alpha \mathcal{S}(\frac{t-\varrho}{\alpha})\Big)x(\varrho) + \alpha \mathcal{S}(\frac{t-\varrho}{\alpha})x'(\varrho) \\
&= \Big[\gamma \mathcal{S}(\frac{t-\varrho}{\alpha}) + \alpha \frac{d}{d\varrho}\mathcal{S}(\frac{t-\varrho}{\alpha})\Big]x(\varrho) + \mathcal{S}(\frac{t-\varrho}{\alpha})[(A-\gamma I)x(\varrho) + f(\varrho)] \\
&= \Big[\gamma \mathcal{S}(\frac{t-\varrho}{\alpha}) - A\mathcal{S}(\frac{t-\varrho}{\alpha})\Big]x(\varrho) + A\mathcal{S}(\frac{t-\varrho}{\alpha})x(\varrho) - \gamma \mathcal{S}(\frac{t-\varrho}{\alpha})x(\varrho) \\
&\quad + \mathcal{S}(\frac{t-\varrho}{\alpha})f(\varrho) \\
&= \mathcal{S}(\frac{t-\varrho}{\alpha})f(\varrho). \qquad (6.3.6)
\end{aligned}$$

By Theorem 6.2.5, we obtain

$$\begin{aligned}
I^\alpha(D^\alpha h(t)) &= h(t) - e^{\frac{-\gamma}{\alpha}t}h(0) \\
&= x(t) - e^{\frac{-\gamma}{\alpha}t}\mathcal{S}(\frac{t}{\alpha})x_0. \qquad (6.3.7)
\end{aligned}$$

Since $f \in L^1(J, \mathbb{X})$, so $\mathcal{S}(\frac{t-\varrho}{\alpha})f(\varrho)$ is integrable. Integrating (6.3.6)

$$I^\alpha(D^\alpha h(t)) = \frac{1}{\alpha}e^{\frac{-\gamma}{\alpha}t}\int_0^t e^{\frac{\gamma}{\alpha}\varrho}\mathcal{S}(\frac{t-\varrho}{\alpha})f(\varrho)d\varrho. \qquad (6.3.8)$$

From (6.3.7) and (6.3.8), we obtain

$$x(t) = e^{\frac{-\gamma}{\alpha}t}\mathcal{S}(\frac{t}{\alpha})x_0 + \frac{1}{\alpha}e^{\frac{-\gamma}{\alpha}t}\int_0^t e^{\frac{\gamma}{\alpha}\varrho}\mathcal{S}(\frac{t-\varrho}{\alpha})f(\varrho)d\varrho.$$

□

Theorem 6.3.4. *Let A generates a C_0-semigroup $\mathcal{S}(t)$ and $f \in C(J, \mathbb{X})$. If $f(\varrho) \in D(A)$ for $0 < \varrho < t$ and $Af(\varrho) \in L^1(J, \mathbb{X})$, then for every $x_0 \in D(A)$ the function $x : J \to \mathbb{X}$ defined by*

$$x(t) = e^{\frac{-\gamma}{\alpha}t}\mathcal{S}(\frac{t}{\alpha})x_0 + \frac{1}{\alpha}e^{\frac{-\gamma}{\alpha}t}\int_0^t e^{\frac{\gamma}{\alpha}\varrho}\mathcal{S}(\frac{t-\varrho}{\alpha})f(\varrho)d\varrho, \qquad (6.3.9)$$

is a solution to the problem (6.3.2).

Proof. Let $u(t) = e^{\frac{-\gamma}{\alpha}t}\mathcal{S}(\frac{t}{\alpha})x_0$ and $v(t) = \frac{1}{\alpha}e^{\frac{-\gamma}{\alpha}t}\int_0^t e^{\frac{\gamma}{\alpha}\varrho}\mathcal{S}(\frac{t-\varrho}{\alpha})f(\varrho)d\varrho$, therefore $x(t)$ given in (6.3.9) is rewritten as $x(t) = u(t) + v(t)$. Since $x_0 \in D(A)$, $u(t)$ is differentiable and by Theorem 6.3.2, we know $D^\alpha u(t) = Au(t)$. From the assumptions it is easy to conclude that $v(t)$ is differentiable, and

$$\begin{aligned}
v'(t) &= \frac{1}{\alpha}(\frac{-\gamma}{\alpha})e^{\frac{-\gamma}{\alpha}t}\int_0^t e^{\frac{\gamma}{\alpha}\varrho}\mathcal{S}(\frac{t-\varrho}{\alpha})f(\varrho)d\varrho + \frac{1}{\alpha}e^{\frac{-\gamma}{\alpha}t}\frac{d}{dt}\int_0^t e^{\frac{\gamma}{\alpha}\varrho}\mathcal{S}(\frac{t-\varrho}{\alpha})f(\varrho)d\varrho \\
&= \frac{-\gamma}{\alpha^2}e^{\frac{-\gamma}{\alpha}t}\int_0^t e^{\frac{\gamma}{\alpha}\varrho}\mathcal{S}(\frac{t-\varrho}{\alpha})f(\varrho)d\varrho + \frac{1}{\alpha}e^{\frac{-\gamma}{\alpha}t}\Big[\int_0^t e^{\frac{\gamma}{\alpha}\varrho}\frac{\partial}{\partial t}\mathcal{S}(\frac{t-\varrho}{\alpha})f(\varrho)d\varrho \\
&\quad + e^{\frac{\gamma}{\alpha}t}f(t)\Big] \\
&= \frac{-\gamma}{\alpha^2}e^{\frac{-\gamma}{\alpha}t}\int_0^t e^{\frac{\gamma}{\alpha}\varrho}\mathcal{S}(\frac{t-\varrho}{\alpha})f(\varrho)d\varrho + \frac{1}{\alpha}e^{\frac{-\gamma}{\alpha}t}\Big[\int_0^t e^{\frac{\gamma}{\alpha}\varrho}\frac{1}{\alpha}A\mathcal{S}(\frac{t-\varrho}{\alpha})f(\varrho)d\varrho \\
&\quad + e^{\frac{\gamma}{\alpha}t}f(t)\Big] \\
&= \frac{-\gamma}{\alpha^2}e^{\frac{-\gamma}{\alpha}t}\int_0^t e^{\frac{\gamma}{\alpha}\varrho}\mathcal{S}(\frac{t-\varrho}{\alpha})f(\varrho)d\varrho + \frac{1}{\alpha^2}Ae^{\frac{-\gamma}{\alpha}t}\int_0^t e^{\frac{\gamma}{\alpha}\varrho}\mathcal{S}(\frac{t-\varrho}{\alpha})f(\varrho)d\varrho + \frac{1}{\alpha}f(t) \\
&= \frac{1}{\alpha}\Big(-\gamma v(t) + Av(t) + f(t)\Big). \qquad (6.3.10)
\end{aligned}$$

Now, by Theorem 6.2.2 and (6.3.10) we get

$$\begin{aligned}
D^\alpha v(t) &= \gamma v(t) + \alpha v'(t) \\
&= \gamma v(t) + \Big(-\gamma v(t) + Av(t) + f(t)\Big) \\
&= Av(t) + f(t). \qquad (6.3.11)
\end{aligned}$$

So,

$$\begin{aligned}
D^\alpha x(t) &= D^\alpha u(t) + D^\alpha v(t) \\
&= Au(t) + Av(t) + f(t) \\
&= Ax(t) + f(t), \qquad (6.3.12)
\end{aligned}$$

also, $x(0) = u(0) + v(0) = x_0$. Thus $x(t)$ given by (6.3.9) is a solution to inhomogeneous problem (6.3.2). □

Now, we will study the semilinear initial value problem (6.1.1).

Definition 6.3.2. *A continuous solution of the following integral equation*

$$x(t) = e^{\frac{-\gamma}{\alpha}t}\mathcal{S}(\frac{t}{\alpha})x_0 + \frac{1}{\alpha}e^{\frac{-\gamma}{\alpha}t}\int_0^t e^{\frac{\gamma}{\alpha}\varrho}\mathcal{S}(\frac{t-\varrho}{\alpha})f(\varrho, x(\varrho))d\varrho, \qquad (6.3.13)$$

is named as mild solution to the problem (6.1.1).

Now, we state the basic assumptions required to prove the main result :

(HA) A generates a C_0-semigroup of bounded linear operators $\mathcal{S}(t)(t \geqslant 0)$ on $\mathbb{X}$.

(Hf) The function f from $J \times \mathbb{X}$ to $\mathbb{X}$ is continuous and satisfies

$$\|f(t,v) - f(t,z)\| \leqslant L\|v - z\|, \quad \forall\, v\,,\, z \in \mathbb{X},$$

for some positive constant L.

Theorem 6.3.5. *Let the hypotheses (HA) and (Hf) are satisfied, then for every $x_0 \in \mathbb{X}$, the system (6.1.1) has a unique mild solution, provided that $\frac{ML}{\gamma} < 1$.*

Proof. Let us define a map $F : C(J,\mathbb{X}) \to C(J,\mathbb{X})$ as

$$(Fx)(t) = e^{\frac{-\gamma}{\alpha}t}\mathcal{S}(\frac{t}{\alpha})x_0 + \frac{1}{\alpha}e^{\frac{-\gamma}{\alpha}t}\int_0^t e^{\frac{\gamma}{\alpha}\varrho}\mathcal{S}(\frac{t-\varrho}{\alpha})f(\varrho, x(\varrho))d\varrho, \quad t \in J.$$

Let z, $v \in C(J,\mathbb{X})$, it follows readily by the definition of F

$$\begin{aligned}
\|(Fz)(t) - (Fv)(t)\| &\leqslant \frac{1}{\alpha}e^{\frac{-\gamma}{\alpha}t}\int_0^t e^{\frac{\gamma}{\alpha}\varrho}\|\mathcal{S}(\frac{t-\varrho}{\alpha})\|\|f(\varrho, z(\varrho)) - f(\varrho, v(\varrho))\|d\varrho \\
&\leqslant \frac{ML}{\alpha}e^{\frac{-\gamma}{\alpha}t}\bigg(\int_0^t e^{\frac{\gamma}{\alpha}\varrho}d\varrho\bigg)\|z - v\| \\
&= \frac{ML}{\gamma}e^{\frac{-\gamma}{\alpha}t}\bigg[e^{\frac{\gamma}{\alpha}t} - 1\bigg]\|z - v\| \\
&= \frac{ML}{\gamma}\bigg[1 - e^{\frac{-\gamma}{\alpha}t}\bigg]\|z - v\| \\
&\leqslant \frac{ML}{\gamma}\|z - v\|.
\end{aligned}$$

Hence, F is contraction map. Banach contraction fixed point theorem implies that F has a unique fixed point $x \in C(J,\mathbb{X})$, hence a unique mild solution to the system (6.1.1).

□

6.3.2 Approximate controllability

Throughout this subsection, we consider $\mathbb{X}$ is a Hilbert space.

Definition 6.3.3. *A function $v \in C(J,\mathbb{X})$ is known as a mild solution of (6.1.2), if for any $u \in L^2(J,\mathbb{X})$, the following holds*

$$v(t) = e^{\frac{-\gamma}{\alpha}t}\mathcal{S}(\frac{t}{\alpha})x_0 + \frac{1}{\alpha}e^{\frac{-\gamma}{\alpha}t}\int_0^t e^{\frac{\gamma}{\alpha}\varrho}\mathcal{S}(\frac{t-\varrho}{\alpha})[f(\varrho, v(\varrho)) + Bu(\varrho)]d\varrho, \quad t \in J.$$

Let $x^b(x_0, u)$ be the state value of (6.1.2) at time b with respect to the initial value x_0 and control u. Let $\mathbf{R}(b, x_0) = \{x^b(x_0, u) : u \in L^2(J, U)\}$, which is named as reachable set to the system (6.1.2) at time b.

Definition 6.3.4. *([132]) If $\overline{\mathbf{R}(b, x_0)} = \mathbb{X}$, the system (6.1.2) is called approximately controllable on J.*

Now, Consider the linear control system corresponding to (6.1.2)

$$\begin{aligned} D^\alpha x(t) &= Ax(t) + Bu(t), \quad t \in J, \\ x(0) &= x_0. \end{aligned} \tag{6.3.14}$$

Definition 6.3.5. **(a)** *A controllability map for the system (6.3.14) on J is a linear bounded map $\mathfrak{B}^b : L^2(J, U) \to \mathbb{X}$, defined as*

$$\mathfrak{B}^b u := \frac{1}{\alpha} e^{\frac{-\gamma}{\alpha} b} \int_0^b e^{\frac{\gamma}{\alpha}\varrho} \mathcal{S}(\frac{b-\varrho}{\alpha}) Bu(\varrho) d\varrho. \tag{6.3.15}$$

(b) *The system (6.3.14) is called approximately controllable on J, if the range space of $\mathfrak{B}^b$ is dense in $\mathbb{X}$ i.e.*

$$\overline{ran \mathfrak{B}^b} = \mathbb{X}.$$

(c) *The controllability gramian of (6.3.14) on J is defined by*

$$\Gamma_0^b := \mathfrak{B}^b (\mathfrak{B}^b)^*. \tag{6.3.16}$$

Lemma 6.3.6. *The controllability map and controllability gramian satisfy the following :*

(a) $(\mathfrak{B}^b)^* x(\varrho) = B^* \mathcal{S}^*(\frac{b-\varrho}{\alpha}) x$, *for* $\varrho \in J$, $x \in \mathbb{X}$.

(b) $\Gamma_0^b \in \mathbb{B}(\mathbb{X})$, *is symmetric, and has the representation*

$$\Gamma_0^b = \frac{1}{\alpha} e^{\frac{-\gamma}{\alpha} b} \int_0^b e^{\frac{\gamma}{\alpha}\varrho} \mathcal{S}(\frac{b-\varrho}{\alpha}) BB^* \mathcal{S}^*(\frac{b-\varrho}{\alpha}) d\varrho, \tag{6.3.17}$$

and $\Gamma_0^b \geqslant 0$, where $(\cdot)^*$ denotes the adjoint operator of $(\cdot)$.

Proof. **(a)** : The way of proof is based on [62] (Lemma 4.1.4, page 144). For $x \in \mathbb{X}$ and $u \in L^2(J, U)$

$$\begin{aligned} \langle u, (\mathfrak{B}^b)^* x \rangle &= \langle \mathfrak{B}^b u, x \rangle \\ &= \langle \frac{1}{\alpha} e^{\frac{-\gamma}{\alpha} b} \int_0^b e^{\frac{\gamma}{\alpha}\varrho} \mathcal{S}(\frac{b-\varrho}{\alpha}) Bu(\varrho) d\varrho, x \rangle \\ &= \frac{1}{\alpha} e^{\frac{-\gamma}{\alpha} b} \int_0^b \langle e^{\frac{\gamma}{\alpha}\varrho} \mathcal{S}(\frac{b-\varrho}{\alpha}) Bu(\varrho), x \rangle d\varrho \\ &= \frac{1}{\alpha} e^{\frac{-\gamma}{\alpha} b} \int_0^b e^{\frac{\gamma}{\alpha}\varrho} \langle u(\varrho), B^* \mathcal{S}^*(\frac{b-\varrho}{\alpha}) x \rangle d\varrho, \end{aligned}$$

and this proves (**a**).

(**b**) : From (6.3.16), it is easy to see Γ_0^b is symmetric, and $\Gamma_0^b \geqslant 0$. Equation (6.3.17) follows easily by (6.3.15), (6.3.16) and (**a**). □

Theorem 6.3.7. *([62]) The system (6.3.14) is approximately controllable on J iff any one of the following hold :*

(i) *Γ_0^b is positive operator, that means $\langle \Gamma_0^b x, x\rangle > 0$, for all $0 \neq x \in \mathbb{X}$.*

(ii) *$ker(\mathfrak{B}^b)^* = \{0\}$.*

(iii) *$B^*\mathcal{S}^*(\frac{b-t}{\alpha})x = 0$ on $J \Longrightarrow x = 0$.*

Theorem 6.3.8. *([132]) Let $\mathbb{Z}$ be a reflexive separable complete norm space and $\mathbb{Z}^*$ denotes the dual space of $\mathbb{Z}$, $\Gamma : \mathbb{Z}^* \to \mathbb{Z}$ is a symmetric map, then the statements given below are equivalent :*

(i) *$\Gamma : \mathbb{Z}^* \to \mathbb{Z}$ is positive.*

(ii) *$\lambda(\lambda I + \Gamma\mathfrak{J})^{-1}(v)$ converges to zero strongly as $\lambda \to 0^+$, $\forall v \in \mathbb{Z}$. Here $\mathfrak{J}$ is the duality map from $\mathbb{Z} \to \mathbb{Z}^*$.*

Lemma 6.3.9. *The linear control system (6.3.14) is approximately controllable on J iff $\lambda R(\lambda, \Gamma_0^b) \to 0$ as $\lambda \to 0^+$ in strong operator topology, where $R(\lambda, \Gamma_0^b) = (\lambda I + \Gamma_0^b)^{-1}$.*

Proof. The proof is straightforward from Theorem 6.3.7 and Theorem 6.3.8. □

To investigate the approximate controllability of (6.1.2), we impose the following assumptions :

(H1) A generates a C_0-semigroup $\mathcal{S}(t)(t \geqslant 0)$ on $\mathbb{X}$, and $\mathcal{S}(t)(t > 0)$ is compact.

(H2) The function f is continuous on $\mathbb{X}$ for each $t \in J$, and is Lebesgue measurable on J for all $x \in \mathbb{X}$.

(H3) There exists a function $\phi \in L^{\frac{1}{\alpha_1}}(J, \mathbb{R}^+)$ where $\alpha_1 \in (0, \alpha)$ such that

$$\|f(t,z)\| \leqslant \phi(t), \quad \forall\, z \in \mathbb{X}\,,\ t \in J.$$

(H4) The linear control system (6.3.14) is approximately controllable on J.

We will use the following notations :

$$\begin{aligned} B_r &= \{v \in C(J,\mathbb{X}) : \|v\| \leqslant r\}, \text{ for each finite constant } r > 0, \\ M_B &= \|B\|, \;\; M_\phi = \|\phi\|_{L^{\frac{1}{\alpha_1}}(J,\mathbb{R}^+)}, \;\; q = \frac{1}{1-\alpha_1}, \;\; N = \frac{1}{\alpha}\left(\frac{\alpha}{q\gamma}\right)^{\frac{1}{q}} M_\phi. \end{aligned}$$

For an arbitrary function $v \in C(J,\mathbb{X})$, we consider the control function for the nonlinear system (6.1.2) as :

$$u(t) = u_\lambda(t,v) = B^*\mathcal{S}^*(\frac{b-t}{\alpha})R(\lambda,\Gamma_0^b)p(v), \tag{6.3.18}$$

where

$$p(v) = x^b - e^{\frac{-\gamma}{\alpha}b}\mathcal{S}(\frac{b}{\alpha})x_0 - \frac{1}{\alpha}e^{\frac{-\gamma}{\alpha}b}\int_0^b e^{\frac{\gamma}{\alpha}\varrho}\mathcal{S}(\frac{b-\varrho}{\alpha})f(\varrho,v(\varrho))d\varrho. \tag{6.3.19}$$

For any $\lambda > 0$, define $F_\lambda : C(J,\mathbb{X}) \to C(J,\mathbb{X})$ as following :

$$(F_\lambda v)(t) = e^{\frac{-\gamma}{\alpha}t}\mathcal{S}(\frac{t}{\alpha})x_0 + \frac{1}{\alpha}e^{\frac{-\gamma}{\alpha}t}\int_0^t e^{\frac{\gamma}{\alpha}\varrho}\mathcal{S}(\frac{t-\varrho}{\alpha})[f(\varrho,v(\varrho)) + Bu_\lambda(\varrho,v)]d\varrho. \tag{6.3.20}$$

Lemma 6.3.10. *If the assumptions $(H1)$-$(H3)$ hold, then for any $t \in J$ we have*

(i) $\frac{1}{\alpha}e^{\frac{-\gamma}{\alpha}t}\int_0^t e^{\frac{\gamma}{\alpha}\varrho}\|\mathcal{S}(\frac{t-\varrho}{\alpha})f(\varrho,v(\varrho))\|d\varrho \leqslant MN.$

(ii) $\|u_\lambda(t,v)\| \leqslant \frac{MM_B}{\lambda}\Big[\|x^b\| + M(\|x_0\| + N)\Big].$

Proof. (i): By $(H3)$ and Hölder inequality, we get

$$\begin{aligned} \frac{1}{\alpha}e^{\frac{-\gamma}{\alpha}t}\int_0^t e^{\frac{\gamma}{\alpha}\varrho}\|\mathcal{S}(\frac{t-\varrho}{\alpha})f(\varrho,v(\varrho))\|d\varrho &\leqslant \frac{M}{\alpha}e^{\frac{-\gamma}{\alpha}t}\int_0^t e^{\frac{\gamma}{\alpha}\varrho}\phi(\varrho)d\varrho \\ &\leqslant \frac{M}{\alpha}e^{\frac{-\gamma}{\alpha}t}\left(\int_0^t e^{\frac{q\gamma\varrho}{\alpha}}d\varrho\right)^{\frac{1}{q}}\|\phi\|_{L^{\frac{1}{\alpha_1}}(J,\mathbb{R}^+)} \\ &\leqslant \frac{1}{\alpha}\left(\frac{\alpha}{q\gamma}\right)^{\frac{1}{q}}e^{\frac{-\gamma}{\alpha}t}([e^{\frac{q\gamma t}{\alpha}} - 1])^{\frac{1}{q}}MM_\phi \\ &\leqslant \frac{1}{\alpha}\left(\frac{\alpha}{q\gamma}\right)^{\frac{1}{q}}e^{\frac{-\gamma}{\alpha}t}e^{\frac{\gamma}{\alpha}t}MM_\phi \\ &\leqslant \frac{1}{\alpha}\left(\frac{\alpha}{q\gamma}\right)^{\frac{1}{q}}MM_\phi = MN. \end{aligned}$$

(ii): Using (6.3.18), (6.3.19), and (i) we obtain

$$\begin{aligned}
\|u_\lambda(t,v)\| &\leqslant \|B^*\mathcal{S}^*(\frac{b-t}{\alpha})R(\lambda,\Gamma_0^b)p(v)\| \\
&\leqslant \frac{MM_B}{\lambda}\|p(v)\| \\
&\leqslant \frac{MM_B}{\lambda}\Bigg[\|x^b\| + M\|x_0\| + \frac{1}{\alpha}e^{\frac{-\gamma}{\alpha}b}\int_0^b e^{\frac{\gamma}{\alpha}\varrho}\|\mathcal{S}(\frac{b-\varrho}{\alpha})f(\varrho,v(\varrho))\|d\varrho\Bigg] \\
&\leqslant \frac{MM_B}{\lambda}\Bigg[\|x^b\| + M(\|x_0\|+N)\Bigg].
\end{aligned}$$

□

Theorem 6.3.11. *If the hypotheses* $(H1)$-$(H3)$ *hold, then the fractional semilinear control system* (6.1.2) *has a mild solution.*

Proof. For convenience, we divide the proof into several steps :

Step I: For given $\lambda > 0$, we want to show that there exists a constant $R = R(\lambda) > 0$, satisfying $F_\lambda(B_R) \subset B_R$. Let $v \in B_r$, for any positive constant r. If $t \in J$, then by using Lemma 6.3.10, we have

$$\begin{aligned}
\|(F_\lambda v)(t)\| &\leqslant e^{\frac{-\gamma}{\alpha}t}\|\mathcal{S}(\frac{t}{\alpha})x_0\| + \frac{1}{\alpha}e^{\frac{-\gamma}{\alpha}t}\int_0^t e^{\frac{\gamma}{\alpha}\varrho}\|\mathcal{S}(\frac{t-\varrho}{\alpha})[f(\varrho,v(\varrho)) + Bu_\lambda(\varrho,v)]\|d\varrho \\
&\leqslant M\|x_0\| + \frac{1}{\alpha}e^{\frac{-\gamma}{\alpha}t}\int_0^t e^{\frac{\gamma}{\alpha}\varrho}\|\mathcal{S}(\frac{t-\varrho}{\alpha})f(\varrho,v(\varrho))\|d\varrho \\
&\quad + \frac{1}{\alpha}e^{\frac{-\gamma}{\alpha}t}\int_0^t e^{\frac{\gamma}{\alpha}\varrho}\|\mathcal{S}(\frac{t-\varrho}{\alpha})Bu_\lambda(\varrho,v)\|d\varrho \\
&\leqslant M\|x_0\| + MN + \frac{1}{\gamma}MM_Be^{\frac{-\gamma}{\alpha}t}[e^{\frac{\gamma}{\alpha}t}-1]\|u_\lambda\| \\
&\leqslant M(\|x_0\|+N) + \frac{M^2(M_B)^2}{\lambda\gamma}\Bigg[\|x^b\| + M(\|x_0\|+N)\Bigg].
\end{aligned}$$

This implies that for large enough $R > 0$, $F_\lambda(B_R) \subset B_R$ holds.

Step II: For any $t \in J$ the set $\{(F_\lambda v)(t) : v \in B_R\}$ is relatively compact in $\mathbb{X}$. If $t = 0$, clearly $\{(F_\lambda v)(0) : v \in B_R\} = \{x_0\}$, which is compact in $\mathbb{X}$. Let $0 < t \leqslant b$ and $0 < \varepsilon < t$. For $v \in B_R$, define

$$\begin{aligned}
(F_\lambda^\varepsilon v)(t) &= e^{\frac{-\gamma}{\alpha}t}\mathcal{S}(\frac{t}{\alpha})x_0 + \frac{1}{\alpha}e^{\frac{-\gamma}{\alpha}t}\int_0^{t-\varepsilon} e^{\frac{\gamma}{\alpha}\varrho}\mathcal{S}(\frac{t-\varrho}{\alpha})[f(\varrho,v(\varrho)) + Bu_\lambda(\varrho,v)]d\varrho \\
&= e^{\frac{-\gamma}{\alpha}t}\mathcal{S}(\frac{t}{\alpha})x_0 + \frac{1}{\alpha}e^{\frac{-\gamma}{\alpha}t}\mathcal{S}(\frac{\varepsilon}{\alpha})\int_0^{t-\varepsilon} e^{\frac{\gamma}{\alpha}\varrho}\mathcal{S}(\frac{t-\varrho-\varepsilon}{\alpha})[f(\varrho,v(\varrho)) \\
&\quad + Bu_\lambda(\varrho,v)]d\varrho \\
&= e^{\frac{-\gamma}{\alpha}t}\mathcal{S}(\frac{t}{\alpha})x_0 + \frac{1}{\alpha}e^{\frac{-\gamma}{\alpha}t}\mathcal{S}(\frac{\varepsilon}{\alpha})y(t,\varepsilon),
\end{aligned}$$

since $y(t,\varepsilon)$ is bounded on B_R and $\mathcal{S}(t)$ is compact for $(t>0)$, we obtain that the set $\{(F_\lambda^\varepsilon v)(t): v\in B_R\}$ is relatively compact in $\mathbb{X}$. Also, observe that

$$\begin{aligned}\|(F_\lambda v)(t)-(F_\lambda^\varepsilon v)(t)\| &= \|\frac{1}{\alpha}e^{\frac{-\gamma}{\alpha}t}\int_{t-\varepsilon}^{t} e^{\frac{\gamma}{\alpha}\varrho}\mathcal{S}(\frac{t-\varrho}{\alpha})[f(\varrho,v(\varrho))+Bu_\lambda(\varrho,v)]d\varrho\| \\ &\leqslant I_1+I_2, \qquad (6.3.21)\end{aligned}$$

where

$$\begin{aligned} I_1 &= \|\frac{1}{\alpha}e^{\frac{-\gamma}{\alpha}t}\int_{t-\varepsilon}^{t} e^{\frac{\gamma}{\alpha}\varrho}\mathcal{S}(\frac{t-\varrho}{\alpha})f(\varrho,v(\varrho))d\varrho\|,\\ I_2 &= \|\frac{1}{\alpha}e^{\frac{-\gamma}{\alpha}t}\int_{t-\varepsilon}^{t} e^{\frac{\gamma}{\alpha}\varrho}\mathcal{S}(\frac{t-\varrho}{\alpha})Bu_\lambda(\varrho,v)d\varrho\|.\end{aligned}$$

Now, proceeding in the same way as Lemma 6.3.10 yields

$$\begin{aligned} I_1 &\leqslant MN[e^{\frac{\gamma q}{\alpha}t}-e^{\frac{\gamma q}{\alpha}(t-\varepsilon)}]^{\frac{1}{q}}, \qquad (6.3.22)\\ I_2 &\leqslant \frac{M^2(M_B)^2}{\lambda\gamma}\left[\|x^b\|+M(\|x_0\|+N)\right][e^{\frac{\gamma}{\alpha}t}-e^{\frac{\gamma}{\alpha}(t-\varepsilon)}]. \qquad (6.3.23)\end{aligned}$$

Therefore, by (6.3.21), (6.3.22) and (6.3.23) we conclude that

$$\|(F_\lambda v)(t)-(F_\lambda^\varepsilon v)(t)\|\to 0 \quad \text{as} \quad \varepsilon\to 0.$$

This implies that the set the set $\{(F_\lambda v)(t):\ v\in B_R\},\ \ t\in(0,b]$ is relatively compact in $\mathbb{X}$.

Step III: The family of functions $\{F_\lambda v: v\in B_R\}$ is equicontinuous on J. For any $0\leqslant t_1<t_2\leqslant b$, and $v\in B_R$

$$\begin{aligned}\|(F_\lambda v)(t_2)-(F_\lambda v)(t_1)\| &\leqslant \left\|e^{\frac{-\gamma}{\alpha}t_1}[\mathcal{S}(\frac{t_2}{\alpha})-\mathcal{S}(\frac{t_1}{\alpha})]x_0\right\| \\ &\quad+\left\|\frac{1}{\alpha}e^{\frac{-\gamma}{\alpha}t_2}\int_{t_1}^{t_2} e^{\frac{\gamma}{\alpha}\varrho}\mathcal{S}(\frac{t_2-\varrho}{\alpha})f(\varrho,v(\varrho))d\varrho\right\| \\ &\quad+\left\|\frac{1}{\alpha}e^{\frac{-\gamma}{\alpha}t_2}\int_{t_1}^{t_2} e^{\frac{\gamma}{\alpha}\varrho}\mathcal{S}(\frac{t_2-\varrho}{\alpha})Bu_\lambda(\varrho,v)d\varrho\right\| \\ &\quad+\left\|\frac{1}{\alpha}e^{\frac{-\gamma}{\alpha}t_1}\int_{0}^{t_1} e^{\frac{\gamma}{\alpha}\varrho}[\mathcal{S}(\frac{t_2-\varrho}{\alpha})-\mathcal{S}(\frac{t_1-\varrho}{\alpha})]f(\varrho,v(\varrho))d\varrho\right\| \\ &\quad+\left\|\frac{1}{\alpha}e^{\frac{-\gamma}{\alpha}t_1}\int_{0}^{t_1} e^{\frac{\gamma}{\alpha}\varrho}[\mathcal{S}(\frac{t_2-\varrho}{\alpha})-\mathcal{S}(\frac{t_1-\varrho}{\alpha})]Bu_\lambda(\varrho,v)d\varrho\right\| \\ &= J_1+J_2+J_3+J_4+J_5.\end{aligned}$$

Now, using Lemma 6.3.10 we get

$$\begin{aligned} J_1 &\leqslant \left\| \mathcal{S}(\frac{t_2}{\alpha}) - \mathcal{S}(\frac{t_1}{\alpha}) \right\| \|x_0\|, \\ J_2 &\leqslant \frac{MM_\phi}{\alpha} \left(\frac{\alpha}{\gamma q} \right)^{\frac{1}{q}} [e^{\frac{\gamma q}{\alpha} t_2} - e^{\frac{\gamma q}{\alpha} t_1}]^{\frac{1}{q}}, \\ J_3 &\leqslant \frac{M^2 (M_B)^2}{\lambda \gamma} \left[\|x^b\| + M(\|x_0\| + N) \right] [e^{\frac{\gamma}{\alpha} t_2} - e^{\frac{\gamma}{\alpha} t_1}]. \end{aligned}$$

For $t_1 = 0$, it is easy to see that $J_4 = 0$. For $t_1 > 0$ and $\epsilon > 0$ (small enough), we obtain

$$\begin{aligned} J_4 &\leqslant \left\| \frac{1}{\alpha} e^{\frac{-\gamma}{\alpha} t_1} \int_0^{t_1 - \epsilon} e^{\frac{\gamma}{\alpha}\varrho} [\mathcal{S}(\frac{t_2 - \varrho}{\alpha}) - \mathcal{S}(\frac{t_1 - \varrho}{\alpha})] f(\varrho, v(\varrho)) d\varrho \right\| \\ &\quad + \left\| \frac{1}{\alpha} e^{\frac{-\gamma}{\alpha} t_1} \int_{t_1 - \epsilon}^{t_1} e^{\frac{\gamma}{\alpha}\varrho} [\mathcal{S}(\frac{t_2 - \varrho}{\alpha}) - \mathcal{S}(\frac{t_1 - \varrho}{\alpha})] f(\varrho, v(\varrho)) d\varrho \right\| \\ &\leqslant \frac{1}{\alpha} e^{\frac{-\gamma}{\alpha} t_1} \int_0^{t_1 - \epsilon} e^{\frac{\gamma}{\alpha}\varrho} \phi(\varrho) d\varrho \sup_{\varrho \in [0, t_1 - \epsilon]} \left\| \mathcal{S}(\frac{t_2 - \varrho}{\alpha}) - \mathcal{S}(\frac{t_1 - \varrho}{\alpha}) \right\| \\ &\quad + \frac{2M}{\alpha} e^{\frac{-\gamma}{\alpha} t_1} \int_{t_1 - \epsilon}^{t_1} e^{\frac{\gamma}{\alpha}\varrho} \phi(\varrho) d\varrho \\ &\leqslant N e^{\frac{-\gamma}{\alpha}\epsilon} \sup_{\varrho \in [0, t_1 - \epsilon]} \left\| \mathcal{S}(\frac{t_2 - \varrho}{\alpha}) - \mathcal{S}(\frac{t_1 - \varrho}{\alpha}) \right\| + 2MN [e^{\frac{\gamma q}{\alpha} t_1} - e^{\frac{\gamma q}{\alpha}(t_1 - \epsilon)}]^{\frac{1}{q}}, \end{aligned}$$

similarly

$$\begin{aligned} J_5 &\leqslant \frac{M_B}{\gamma} [e^{\frac{-\gamma}{\alpha}\epsilon} - e^{\frac{-\gamma}{\alpha} t_1}] \|u_\lambda\| \sup_{\varrho \in [0, t_1 - \epsilon]} \left\| \mathcal{S}(\frac{t_2 - \varrho}{\alpha}) - \mathcal{S}(\frac{t_1 - \varrho}{\alpha}) \right\| \\ &\quad + \frac{2MM_B}{\gamma} [e^{\frac{\gamma}{\alpha} t_1} - e^{\frac{\gamma}{\alpha}(t_1 - \epsilon)}] \|u_\lambda\|. \end{aligned}$$

Using $(H1)$, it is clear that $J_i \to 0\,(i = 1, 2, 3, 4, 5)$ as $t_2 \to t_1$, $\epsilon \to 0$. As a result $\|(F_\lambda v)(t_2) - (F_\lambda v)(t_1)\| \to 0$ independently of $v \in B_R$ as $t_2 \to t_1$, which means that $\{F_\lambda(B_R)\}$ is equicontinuous.

Thus, combining $StepII$ and $StepIII$, we conclude F_λ is compact on B_R by Arzela-Ascoli theorem.

Step IV: F_λ is continuous in B_R. Let $\{v_n\}$ be a sequence which converges to v in B_R. Since, f is continuous in second variable, for each $\varrho \in J$, we have

$$\lim_{n \to \infty} f(\varrho, v_n(\varrho)) = f(\varrho, v(\varrho)). \tag{6.3.24}$$

So, we can conclude that

$$\sup_{\varrho \in J} \|f(\varrho, v_n(\varrho)) - f(\varrho, v(\varrho))\| \to 0 \quad \text{as} \quad n \to \infty. \tag{6.3.25}$$

From (6.3.19), and (6.3.25) we obtain that

$$\begin{aligned}
\|p(v_n)-p(v)\| &= \left\| \frac{1}{\alpha} e^{\frac{-\gamma}{\alpha}b} \int_0^b e^{\frac{\gamma}{\alpha}\varrho} \mathcal{S}(\frac{b-\varrho}{\alpha})[f(\varrho,v_n(\varrho))-f(\varrho,v(\varrho))]d\varrho \right\| \\
&\leqslant \frac{M}{\alpha} e^{\frac{-\gamma}{\alpha}b} \left(\int_0^b e^{\frac{\gamma}{\alpha}\varrho} d\varrho \right) \sup_{\varrho\in J} \|f(\varrho,v_n(\varrho))-f(\varrho,v(\varrho))\| \\
&\leqslant \frac{M}{\gamma}[1-e^{\frac{-\gamma}{\alpha}b}] \sup_{\varrho\in J} \|f(\varrho,v_n(\varrho))-f(\varrho,v(\varrho))\| \\
&\leqslant \frac{M}{\gamma} \sup_{\varrho\in J} \|f(\varrho,v_n(\varrho))-f(\varrho,v(\varrho))\| \\
&\to 0 \quad \text{as} \quad n\to\infty, \qquad (6.3.26)
\end{aligned}$$

therefore, (6.3.18) and (6.3.26) imply that

$$\|u_\lambda(\varrho,v_n)-u_\lambda(\varrho,v)\| \to 0 \quad \text{as} \quad n\to\infty, \qquad (6.3.27)$$

and (6.3.25), (6.3.27) yield

$$\begin{aligned}
\|(F_\lambda v_n)(t)-(F_\lambda v)(t)\| &\leqslant \frac{M}{\alpha} e^{\frac{-\gamma}{\alpha}t} \int_0^t e^{\frac{\gamma}{\alpha}\varrho} \|f(\varrho,v_n(\varrho))-f(\varrho,v(\varrho))\| d\varrho \\
&\quad + \frac{MM_B}{\alpha} e^{\frac{-\gamma}{\alpha}t} \int_0^t e^{\frac{\gamma}{\alpha}\varrho} \|u_\lambda(\varrho,v_n(\varrho))-u_\lambda(\varrho,v(\varrho))\| d\varrho \\
&\leqslant \frac{M}{\gamma} \sup_{\varrho\in J} \|f(\varrho,v_n(\varrho))-f(\varrho,v(\varrho))\| \\
&\quad + \frac{MM_B}{\gamma} \sup_{\varrho\in J} \|u_\lambda(\varrho,v_n(\varrho))-u_\lambda(\varrho,v(\varrho))\| \\
&\to 0 \quad \text{as} \quad n\to\infty, \qquad (6.3.28)
\end{aligned}$$

which means that F_λ is continuous in B_R.

Hence, by Schauder fixed point theorem there exists a fixed point of F_λ, thus a mild solution to (6.1.2). □

Theorem 6.3.12. *Suppose that the hypotheses* $(H1)$-$(H4)$ *hold. Moreover, assume that f is uniformly bounded by a positive constant K. Then, the fractional semilinear system* (6.1.2) *is approximately controllable on J.*

Proof. Let x_λ is a mild solution to the problem (6.1.2) with the control

$$u_\lambda(t,x_\lambda) = B^*\mathcal{S}^*(\frac{b-t}{\alpha})R(\lambda,\Gamma_0^b)p(x_\lambda),$$

where

$$p(x_\lambda) = x^b - e^{\frac{-\gamma}{\alpha}b}\mathcal{S}(\frac{b}{\alpha})x_0 - \frac{1}{\alpha}e^{\frac{-\gamma}{\alpha}b}\int_0^b e^{\frac{\gamma}{\alpha}\varrho}\mathcal{S}(\frac{b-\varrho}{\alpha})f(\varrho,x_\lambda(\varrho))d\varrho,$$

and satisfies

$$\begin{aligned} x_\lambda(b) &= e^{\frac{-\gamma}{\alpha}b}\mathcal{S}(\frac{b}{\alpha})x_0+\frac{1}{\alpha}e^{\frac{-\gamma}{\alpha}b}\int_0^b e^{\frac{\gamma}{\alpha}\varrho}\mathcal{S}(\frac{b-\varrho}{\alpha})[f(\varrho,x_\lambda(\varrho))+Bu_\lambda(\varrho,x_\lambda)]d\varrho \\ &= x^b-p(x_\lambda)+\left(\frac{1}{\alpha}e^{\frac{-\gamma}{\alpha}b}\int_0^b e^{\frac{\gamma}{\alpha}\varrho}\mathcal{S}(\frac{b-\varrho}{\alpha})BB^*\mathcal{S}^*(\frac{b-\varrho}{\alpha})d\varrho\right)R(\lambda,\Gamma_0^b)p(x_\lambda) \\ &= x^b-p(x_\lambda)+\Gamma_0^bR(\lambda,\Gamma_0^b)p(x_\lambda) \\ &= x^b-\lambda R(\lambda,\Gamma_0^b)p(x_\lambda). \end{aligned} \tag{6.3.29}$$

Since, f is uniformly bounded, we have

$$\int_0^b\|f(\varrho,x_\lambda(\varrho))\|^2d\varrho\leqslant K^2b.$$

Hence, the sequence $f(\cdot,x_\lambda(\cdot))$ is bounded in $L^2(J,\mathbb{X})$. So, we have a subsequence of $\{f(\cdot,x_\lambda(\cdot)) : \lambda>0\}$, still denoted by itself, converges weakly to some $f(\cdot)\in L^2(J,\mathbb{X})$. Define

$$\omega=x^b-e^{\frac{-\gamma}{\alpha}b}\mathcal{S}(\frac{b}{\alpha})x_0-\frac{1}{\alpha}e^{\frac{-\gamma}{\alpha}b}\int_0^b e^{\frac{\gamma}{\alpha}\varrho}\mathcal{S}(\frac{b-\varrho}{\alpha})f(\varrho)d\varrho.$$

It follows that

$$\begin{aligned} \|p(x_\lambda)-\omega\| &= \left\|\frac{1}{\alpha}e^{\frac{-\gamma}{\alpha}b}\int_0^b e^{\frac{\gamma}{\alpha}\varrho}\mathcal{S}(\frac{b-\varrho}{\alpha})[f(\varrho,x_\lambda(\varrho))-f(\varrho)]d\varrho\right\| \\ &\leqslant \sup_{t\in J}\left\|\frac{1}{\alpha}e^{\frac{-\gamma}{\alpha}t}\int_0^t e^{\frac{\gamma}{\alpha}\varrho}\mathcal{S}(\frac{t-\varrho}{\alpha})[f(\varrho,x_\lambda(\varrho))-f(\varrho)]d\varrho\right\|. \end{aligned}$$

By the proof of Theorem 6.3.11, it is easy to observe that the operator

$$l(\cdot)\to\frac{1}{\alpha}e^{\frac{-\gamma}{\alpha}(\cdot)}\int_0^{\cdot} e^{\frac{\gamma}{\alpha}\varrho}\mathcal{S}(\frac{\cdot-\varrho}{\alpha})l(\varrho)d\varrho : L^2(J,\mathbb{X})\to C(J,\mathbb{X}),$$

is compact, consequently

$$\|p(x_\lambda)-\omega\|\to 0 \quad \text{as} \quad \lambda\to 0^+. \tag{6.3.30}$$

Then, from (6.3.29), (6.3.30), and Lemma 6.3.9, we obtain

$$\begin{aligned} \|x_\lambda(b)-x^b\| &\leqslant \|\lambda R(\lambda,\Gamma_0^b)p(x_\lambda)\| \\ &\leqslant \|\lambda R(\lambda,\Gamma_0^b)\omega\|+\|\lambda R(\lambda,\Gamma_0^b)\|\|p(x_\lambda)-\omega\| \\ &\leqslant \|\lambda R(\lambda,\Gamma_0^b)\omega\|+\|p(x_\lambda)-\omega\|\to 0 \text{ as } \lambda\to 0^+. \end{aligned}$$

It proves the approximate controllability of (6.1.2). □

6.4 Examples

Example(1): Consider the deformable fractional partial differential equation as given below

$$\begin{cases} D^{\frac{1}{2}}x(t,v) = \frac{\partial^2}{\partial v^2}x(t,v) + \frac{1}{4}\sin t\frac{|x(t,v)|}{1+|x(t,v)|}, & v \in (0,1), 0 < t < 1; \\ x(t,1) = x(t,0) = 0, & 0 \leqslant t \leqslant 1; \\ x(0,v) = x_0(v), & v \in [0,1], \end{cases} \tag{6.4.1}$$

where $\mathbb{X} = L^2[0,1]$, $x_0(v) \in \mathbb{X}$. Define $Aw = w''$ with

$$D(A) = \{w \in \mathbb{X} : w, w' \text{ are absolutely continuous and } w'' \in \mathbb{X}, w(0) = w(1) = 0\}.$$

Then

$$Aw = \sum_{n=1}^{\infty} -n^2 < w, e_n > e_n, \;\; w \in D(A), \tag{6.4.2}$$

where $e_n(v) = \sqrt{\frac{2}{\pi}}\sin(nv)$, $0 \leqslant v \leqslant 1$, $n = 1, 2, \ldots$. It is well known that A generates a C_0-semigroup $\mathcal{S}(t)(t \geqslant 0)$, on $\mathbb{X}$ and is expressed as

$$\mathcal{S}(t)w = \sum_{n=1}^{\infty} e^{-n^2 t} < w, e_n > e_n, \;\; w \in \mathbb{X}, \tag{6.4.3}$$

with $\|\mathcal{S}(t)\| \leqslant 1$, for any $t \geqslant 0$. Put $x(t) = x(t,\cdot)$, that means $x(t)(v) = x(t,v)$, $0 \leqslant t, v \leqslant 1$. Further

$$f(t, x(t)) = \frac{1}{4}\sin t\frac{|x(t,\cdot)|}{1 + |x(t,\cdot)|},$$

then, the system (6.4.1) can be rewritten into the abstract form of (6.1.1).

$$\begin{aligned} \|f(t,x_1) - f(t,x_2)\| &\leqslant \frac{1}{4}\frac{\|x_1 - x_2\|}{(1 + \|x_1\|)(1 + \|x_2\|)} \\ &\leqslant \frac{1}{4}\|x_1 - x_2\|. \end{aligned}$$

Therefore $L = \frac{1}{4}$, also we have $\alpha = \gamma = \frac{1}{2}$ and $M = 1$. So $\frac{ML}{\gamma} = \frac{1}{2} < 1$. Hence the required assumptions for Theorem 6.3.5 are fulfilled, and we have a unique mild solution for the system (6.4.1).

Example(2): Let a control system governed by a deformable fractional partial differential equation as

$$\begin{cases} D^{\frac{1}{2}}x(t,z) = \frac{\partial^2}{\partial z^2}x(t,z) + u(t,z) + \frac{1}{8}\frac{e^{-t}}{1+e^t}\frac{|x(t,z)|}{1+|x(t,z)|}, & z \in (0,1), t \in (0,b]; \\ x(t,0) = x(t,1) = 0, & t \in [0,b]; \\ x(0,z) = x_0(z), & z \in [0,1], \end{cases} \tag{6.4.4}$$

where $\mathbb{X} = U = L^2[0,1]$, $x_0(z) \in \mathbb{X}$, $J = [0,b]$. Define $Aw = w''$ with

$$D(A) = \{w \in \mathbb{X} : w, w' \text{ are absolutely continuous and } w'' \in \mathbb{X}, w(0) = w(1) = 0\}.$$

Then, A generates a compact semigroup $\mathcal{S}(t)(t > 0)$ given by expression (6.4.3), clearly assumption $(H1)$ holds. Let $B : U \to \mathbb{X}$ is defined as $Bu(t) = u(t,\cdot)$, which is linear bounded operator. It is easy to observe that the assumptions $(H2)$ and $(H3)$ hold with $\phi(t) = \frac{e^{-t}}{1+e^t}$ and $K = \frac{1}{8}$.

By Theorem 6.3.7, the linear system corresponding to (6.4.4) is approximately controllable on J iff

$$B^*\mathcal{S}^*(\frac{b-t}{\alpha})x = 0, \ t \in J \Longrightarrow x = 0. \tag{6.4.5}$$

Using (6.4.3), observe that

$$B^*\mathcal{S}^*(\frac{b-t}{\alpha})x = \sum_{n=1}^{\infty} e^{-n^2(\frac{b-t}{\alpha})} < x, e_n > e_n, \ x \in \mathbb{X}, \ t \in J.$$

Therefore the condition (6.4.5) hold, and hence the assumption $(H4)$. Thus by Theorem 6.3.12, we have approximate controllability of the system (6.4.4) on J.

Chapter 7

Monotone Iterative Technique for Non-Autonomous Nonlocal Differential Equations

7.1 Introduction

Monotone iterative technique is an effective method to find the existence and uniqueness of mild solutions. Using this method, we get monotone sequences of approximate solutions that converge to maximal and minimal mild solutions. Du [121], first used this technique to find extremal mild solutions for a differential equation. Recently, in [49; 52; 56; 109] MIT has been used to established the existence and uniqueness results for various differential systems. The technique has been used for autonomous systems till now. Motivated by this fact, in this chapter, we will study monotone iterative method to find the existence and uniqueness of extremal mild solutions for the following non-autonomous system in an ordered Banach space $\mathbb{X}$:

$$x'(t) + \mathbb{A}(t)x(t) = \mathcal{F}(t, x(t)), \quad 0 < t \leqslant b, \tag{7.1.1}$$

$$x(0) = \sum_{i=1}^{k} \xi_i x(t_i) + x_0, \tag{7.1.2}$$

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where $\mathbb{A}(t) : D(\mathbb{A}(t)) \subset \mathbb{X} \to \mathbb{X}$ is a linear operator, $J = [0, b]$, f is a given function from $J \times \mathbb{X}$ to $\mathbb{X}$ satisfying certain assumptions, $t_i \in (0, b)$, for $i < j$, $t_i < t_j$, $0 \neq \xi_i \in \mathbb{R}$, $i = 1, 2, \ldots, k \in \mathbb{N}$, $x_0 \in \mathbb{X}$.

We organize the chapter as following: in section 7.2, we will recall some basic definitions, notations and theorems. In section 7.3, we will show the existence of extremal mild solutions for the system (7.1.1)-(7.1.2), and also we will show the uniqueness of extremal mild solutions with the help of evolution system and Kuratowski measure of noncompactness. Section 7.4 contains an example to illustrate our results.

7.2 Preliminaries

Let $(\mathbb{X}, \|\cdot\|, \leqslant)$ be a partially ordered complete norm space, $\mathcal{P} = \{x \in \mathbb{X} | \quad x \geqslant 0\}$ (0 is the zero of $\mathbb{X}$) is a positive cone of $\mathbb{X}$. The cone $\mathcal{P}$ is called normal if there is a real number $\mathcal{N} > 0$ such that $0 \leqslant x_1 \leqslant x_2 \Rightarrow \|x_1\| \leqslant \mathcal{N}\|x_2\|$, for all $x_1, x_2 \in \mathbb{X}$, the smallest value of such $\mathcal{N}$ is called normal constant. For $x_1, x_2 \in C(J, \mathbb{X})$, $x_1 \leqslant x_2 \Leftrightarrow x_1(t) \leqslant x_2(t)$, $t \in J$, we denote the intervals $[\nu, \omega] = \{x \in C(J, \mathbb{X}) : \nu \leqslant x \leqslant \omega\}$ in $C(J, \mathbb{X})$, and $[\nu(t), \omega(t)] = \{x \in \mathbb{X} : \nu(t) \leqslant x \leqslant \omega(t)\}(t \in J)$ in $\mathbb{X}$. Let us denote $C^1(J, \mathbb{X}) = \{x \in C(J, \mathbb{X}) : x' \text{ exists on } J, x' \in C(J, \mathbb{X}), \ x(t) \in D(A) \ (t \geqslant 0)\}$. For more details related to ordered spaces and cone one may see [11; 66; 163].

First, we recall the definition and some basic properties of evolution system. For more details, we refer [82] and [156].

Definition 7.2.1. *[156] Let $\mathbb{X}$ be a Banach space. A two parameter family of bounded linear operators $\mathbb{U}(t, s), 0 \leqslant s \leqslant t \leqslant b$ on $\mathbb{X}$ is called an evolution system, if the following conditions are satisfied :*

1. *$\mathbb{U}(s, s) = I$, where I is the identity operator.*

2. *$\mathbb{U}(\tau, \rho)\mathbb{U}(\rho, s) = \mathbb{U}(\tau, s)$ for $0 \leqslant s \leqslant \rho \leqslant \tau \leqslant b$.*

3. *$(t, s) \to \mathbb{U}(t, s)$ is strongly continuous for $0 \leqslant s \leqslant t \leqslant b$.*

For the family of linear operators $\{\mathbb{A}(t) : t \in J\}$ on $\mathbb{X}$, we impose the following assumptions :

(A1) The domain of $\mathbb{A}(t)$ is dense and independent of t, and $\mathbb{A}(t)$ is a closed operator.

(A2) The resolvent of $\mathbb{A}(t)$ exists for $Re(\vartheta) \leqslant 0$, $t \in J$, and $\| R(\vartheta, \mathbb{A}(t))\| \leqslant \frac{\varsigma}{|\vartheta|+1}$, for some constant $\varsigma > 0$.

(A3) There are positive constants K, and $\rho \in (0, 1]$ such that

$$\|[\mathbb{A}(\tau_1) - \mathbb{A}(\tau_2)]\mathbb{A}^{-1}(\tau_3)\| \leqslant K|\tau_1 - \tau_2|^{\rho}, \text{ for any } \tau_1, \tau_2, \tau_3 \in J.$$

The above assumptions imply that, $-\mathbb{A}(t)$ generates a unique evolution system $\mathbb{U}(t, s)(0 \leqslant s \leqslant t \leqslant b)$ of linear bounded operators on $\mathbb{X}$, which satisfies the following properties (see [82; 156]) :

(i) $\mathbb{U}(t_1, t_2) \in \mathbb{B}(\mathbb{X})$ the space of bounded linear operators on $\mathbb{X}$, and is continuous strongly for $0 \leqslant t_2 \leqslant t_1 \leqslant b$.

(ii) $\mathbb{U}(t_1, t_2)x \in D(\mathbb{A})$, $x \in \mathbb{X}$, $0 \leqslant t_2 \leqslant t_1 \leqslant b$.

(iii) $\mathbb{U}(t_1, t_2)U(t_2, t_3) = \mathbb{U}(t_1, t_3)$, $0 \leqslant t_3 \leqslant t_2 \leqslant t_1 \leqslant b$.

(iv) $\mathbb{U}(\tau, \tau) = I$ the identity operator on $\mathbb{X}$, for each $\tau \in J$.

(v) There is a positive constant $\mathcal{M}$ such that $\|\mathbb{U}(t_1, t_2)\| \leqslant \mathcal{M}$, $0 \leqslant t_2 \leqslant t_1 \leqslant b$.

(vi) For each fixed s, $\{\mathbb{U}(\tau, s), s \leqslant \tau\}$ is continuous in τ in uniform operator topology.

(vii) The derivative $\frac{\partial \mathbb{U}(\tau,s)}{\partial \tau}$ exists in strong operator topology for $0 \leqslant s < \tau \leqslant b$, and strongly continuous in τ, where $0 \leqslant s < \tau \leqslant b$.

(viii) $\frac{\partial \mathbb{U}(\tau,s)}{\partial \tau} + \mathbb{A}(\tau)\mathbb{U}(\tau, s) = 0$, $0 \leqslant s < \tau \leqslant b$.

Lemma 7.2.1. *([82]: Lemma 14.1) Let assumptions (A1)-(A3) hold. If* $0 \leqslant \gamma \leqslant 1$, $0 \leqslant \varpi \leqslant \delta < 1 + \alpha$, $0 < \delta - \gamma \leqslant 1$, *then for any* $0 \leqslant \tau < t < t + \triangle t \leqslant t_0$, $0 \leqslant \zeta \leqslant t_0$,

$$\|\mathbb{A}^{\gamma}(\zeta)[\mathbb{U}(t+\triangle t, \tau) - \mathbb{U}(t,\tau)]\mathbb{A}^{-\varpi}(\tau)\| \leqslant C(\varpi,\gamma,\delta)(\triangle t)^{\delta-\gamma}|t-\tau|^{\varpi-\delta}.$$

Definition 7.2.2. *An evolution system* $\mathbb{U}(t,s)$ *is called positive if* $\mathbb{U}(t,s)y \geqslant 0$, *for* $y \in \mathcal{P}$ *and* $0 \leqslant s \leqslant t \leqslant b$.

Definition 7.2.3. $\omega_0 \in C^1(J, \mathbb{X})$ *is called lower solution for the system* (7.1.1)-(7.1.2), *if*

$$\begin{aligned} \omega_0'(t) + \mathbb{A}(t)\omega_0(t) &\leqslant \mathcal{F}(t, \omega_0(t)), \quad 0 < t \leqslant b, \\ \omega_0(0) &\leqslant \sum_{i=1}^{k} \xi_i \omega_0(t_i) + x_0. \end{aligned} \tag{7.2.1}$$

If the inequalities of (7.2.1) *are opposite, solution is known as upper solution.*

Theorem 7.2.2. *([82; 156]) Suppose that the assumptions (A1)-(A3) hold and f satisfies uniform Hölder continuity on J with exponent* $\gamma \in (0, 1]$, *then the unique solution of the following linear Cauchy problem*

$$\begin{aligned} x'(t) + \mathbb{A}(t)x(t) &= \mathcal{F}(t), \quad 0 < t \leqslant b, \\ x(0) &= x_0 \in \mathbb{X}, \end{aligned} \tag{7.2.2}$$

is

$$x(t) = \mathbb{U}(t,0)x_0 + \int_0^t \mathbb{U}(t,\varrho)\mathcal{F}(\varrho)d\varrho. \tag{7.2.3}$$

7.3 Main Results

Let us introduce our basic assumptions :

(H0) The evolution system $\mathbb{U}(t,s)$ is positive.

(H1) $\xi_i > 0$, $i = 1, 2, \ldots, k$ and $\mathcal{M}\sum_{i=1}^{k} \xi_i < 1$.

Assumption $(H1)$ implies that $\|\sum_{i=1}^{k} \xi_i \mathbb{U}(t_i, 0)\| \leqslant \mathcal{M}\sum_{i=1}^{k} \xi_i < 1$, by operator spectrum theorem, we know that

$$\mathcal{B} := \left(I - \sum_{i=1}^{k} \xi_i \mathbb{U}(t_i, 0)\right)^{-1} \tag{7.3.1}$$

exists, bounded and can be expressed as

$$\mathcal{B}=\sum_{j=0}^{\infty}\left(\sum_{i=1}^{k}\xi_i\mathbb{U}(t_i,0)\right)^j. \tag{7.3.2}$$

Since $\mathbb{U}(t,s)$ is positive, so is $\mathcal{B}$ and

$$\|\mathcal{B}\|\leqslant\sum_{j=0}^{\infty}\|\sum_{i=1}^{k}\xi_i\mathbb{U}(t_i,0)\|^j\leqslant\sum_{j=0}^{\infty}\left(\mathcal{M}\sum_{i=1}^{k}\xi_i\right)^j=\frac{1}{1-\mathcal{M}\sum_{i=1}^{k}\xi_i}. \tag{7.3.3}$$

Definition 7.3.1. *A function $x\in C(J,\mathbb{X})$ is called mild solution of the system (7.1.1)-(7.1.2), if x satisfies the following integral equation :*

$$\begin{aligned}x(t) &= \mathbb{U}(t,0)\mathcal{B}x_0+\sum_{i=1}^{k}\xi_i\mathbb{U}(t,0)\mathcal{B}\int_0^{t_i}\mathbb{U}(t_i,\varrho)\mathcal{F}(\varrho,x(\varrho))d\varrho\\ &\quad+\int_0^t\mathbb{U}(t,\varrho)\mathcal{F}(\varrho,x(\varrho))d\varrho. \end{aligned}\tag{7.3.4}$$

Lemma 7.3.1. *If the assumptions $(A1)$-$(A3)$ and $(H1)$ hold and $\mathcal{F}\in C(J\times\mathbb{X},\mathbb{X})$, then a mild solution of (7.1.1)-(7.1.2) is given as*

$$\begin{aligned}x(t) &= \mathbb{U}(t,0)\mathcal{B}x_0+\sum_{i=1}^{k}\xi_i\mathbb{U}(t,0)\mathcal{B}\int_0^{t_i}\mathbb{U}(t_i,\varrho)\mathcal{F}(\varrho,x(\varrho))d\varrho\\ &\quad+\int_0^t\mathbb{U}(t,\varrho)\mathcal{F}(\varrho,x(\varrho))d\varrho. \end{aligned}\tag{7.3.5}$$

Proof. By Theorem 7.2.2, the mild solution of problem (7.1.1)-(7.1.2) can be expressed as

$$x(t) = \mathbb{U}(t,0)x(0)+\int_0^t\mathbb{U}(t,\varrho)\mathcal{F}(\varrho,x(\varrho))d\varrho. \tag{7.3.6}$$

So,

$$x(t_i) = \mathbb{U}(t_i,0)x(0)+\int_0^{t_i}\mathbb{U}(t_i,\varrho)\mathcal{F}(\varrho,x(\varrho))d\varrho,\quad i=1,2,3,\ldots,k. \tag{7.3.7}$$

By using (7.3.7) in (7.1.2), we get

$$x(0) = \sum_{i=1}^{k}\xi_i[\mathbb{U}(t_i,0)x(0)+\int_0^{t_i}\mathbb{U}(t_i,\varrho)\mathcal{F}(\varrho,x(\varrho))]d\varrho+x_0.$$

Now, assumption $(H1)$ and (7.3.1) imply that

$$x(0) = \mathcal{B}x_0+\sum_{i=1}^{k}\xi_i\mathcal{B}\int_0^{t_i}\mathbb{U}(t_i,\varrho)\mathcal{F}(\varrho,x(\varrho))d\varrho. \tag{7.3.8}$$

Hence (7.3.6) and (7.3.8) imply that (7.3.5) is satisfied. □

Let us define $\mathcal{Q} : C(J, \mathbb{X}) \to C(J, \mathbb{X})$ as :

$$\begin{aligned} \mathcal{Q}x(t) &= \mathbb{U}(t,0)\mathcal{B}x_0 + \sum_{i=1}^{k} \xi_i \mathbb{U}(t,0)\mathcal{B} \int_0^{t_i} \mathbb{U}(t_i, \varrho)\mathcal{F}(\varrho, x(\varrho))d\varrho \\ &\quad + \int_0^t \mathbb{U}(t,\varrho)\mathcal{F}(\varrho, x(\varrho))d\varrho. \end{aligned} \tag{7.3.9}$$

To prove that the system (7.1.1)-(7.1.2) has a mild solution, we need to show the operator $\mathcal{Q}$ has a fixed point.

Theorem 7.3.2. *Suppose $\mathbb{X}$ is a partially ordered complete norm space with normal positive cone $\mathcal{P}$, the assumptions $(H0), (H1)$ and $(A1)$-$(A3)$ hold, $\mathcal{F}$ is continuous from $J \times \mathbb{X} \to \mathbb{X}$, $x_0 \in \mathbb{X}$, and ω_0, $\nu_0 \in C^1(J, \mathbb{X})$ are lower and upper solutions respectively for (7.1.1)-(7.1.2). Moreover, assume the following :*

(H2) *For $t \in J$, and $y_1, y_2 \in \mathbb{X}$ with $\omega_0(t) \leqslant y_1 \leqslant y_2 \leqslant \nu_0(t)$;*

$$\mathcal{F}(t, y_1) \leqslant \mathcal{F}(t, y_2).$$

(H3) *There is a constant $\mathcal{L} > 0$ such that for $t \in J$, and monotone sequence $\{x_n\}$ in $[\omega_0(t), \nu_0(t)]$;*

$$\beta(\{\mathcal{F}(t, x_n)\}) \leqslant \mathcal{L}\beta(\{x_n\}).$$

(H4) *Suppose $\omega_n = \mathcal{Q}\omega_{n-1}$ and $\nu_n = \mathcal{Q}\nu_{n-1}$, $n \in \mathbb{N}$ with $\{\omega_n(0)\}$ and $\{\nu_n(0)\}$ are convergent.*

Then, the system (7.1.1)-(7.1.2) has extremal mild solutions in the interval $[\omega_0, \nu_0]$, provided that $K := 2\mathcal{M}\mathcal{L}b < 1$.

Proof. Let us denote $D = [\omega_0, \nu_0]$. For any $x \in D$, $(H2)$ implies

$$\mathcal{F}(t, \omega_0(t)) \leqslant \mathcal{F}(t, x(t)) \leqslant \mathcal{F}(t, \nu_0(t)).$$

Therefore, from the normality of $\mathcal{P}$ we get a constant $c > 0$, such that

$$\|\mathcal{F}(t, x(t))\| \leqslant c, \ \ x \in D. \tag{7.3.10}$$

First, we will prove that the map $\mathcal{Q} : D \to C(J, \mathbb{X})$ is continuous. Let $\{x_n\}$ be a sequence in D such that $x_n \to x \in D$. Since $\mathcal{F}$ is continuous, so $\mathcal{F}(\varrho, x_n(\varrho)) \to \mathcal{F}(\varrho, x(\varrho))$ for $\varrho \in J$, and from (7.3.10) we get that $\|\mathcal{F}(\varrho, x_n(\varrho)) - \mathcal{F}(\varrho, x(\varrho))\| \leqslant 2c$.

So, by Lebesgue dominated convergence theorem, we estimate

$$\begin{aligned}
\|\mathcal{Q}x_n(t)-\mathcal{Q}x(t)\| &\leqslant \sum_{i=1}^{k}\xi_i\|\mathbb{U}(t,0)\|\|\mathcal{B}\|\int_0^{t_i}\|\mathbb{U}(t_i,\varrho)\|\|\mathcal{F}(\varrho,x_n(\varrho))-\mathcal{F}(\varrho,x(\varrho))\|d\varrho \\
&\quad +\int_0^{t}\|\mathbb{U}(t,\varrho)\|\|\mathcal{F}(\varrho,x_n(\varrho))-\mathcal{F}(\varrho,x(\varrho))\|d\varrho. \\
&\leqslant \mathcal{M}^2\|\mathcal{B}\|\sum_{i=1}^{k}\xi_i\int_0^{t_i}\|\mathcal{F}(\varrho,x_n(\varrho))-\mathcal{F}(\varrho,x(\varrho))\|d\varrho \\
&\quad +\mathcal{M}\int_0^{t}\|\mathcal{F}(\varrho,x_n(\varrho))-\mathcal{F}(\varrho,x(\varrho))\|d\varrho \\
&\to 0 \quad \text{as} \quad n\to\infty.
\end{aligned}$$

Thus $\mathcal{Q}$ is continuous map on D.

Next, we will prove $\mathcal{Q}: D\to D$ is monotone increasing. Let $x_1,x_2\in D,\ x_1\leqslant x_2$. Since $\mathbb{U}(t,s)$ is positive evolution system and the hypothesis $(H2)$ holds, it is easy to see $\mathcal{Q}x_1\leqslant\mathcal{Q}x_2$. Suppose $\omega_0'(\eta)+\mathbb{A}(\eta)\omega_0(\eta)=h(\eta)$, Definition 7.2.3, implies $h(\eta)\leqslant\mathcal{F}(\eta,\omega_0(\eta))$, for $\eta\in J$. From Lemma 7.3.1 and the positivity of evolution system, we have

$$\begin{aligned}
\omega_0(\eta) &= \mathbb{U}(\eta,0)\mathcal{B}x_0+\sum_{i=1}^{k}\xi_i\mathbb{U}(\eta,0)\mathcal{B}\int_0^{t_i}\mathbb{U}(t_i,\varrho)h(\varrho)d\varrho \\
&\quad +\int_0^{\eta}\mathbb{U}(\eta,\varrho)h(\varrho)d\varrho \\
&\leqslant \mathbb{U}(\eta,0)\mathcal{B}x_0+\sum_{i=1}^{k}\xi_i\mathbb{U}(\eta,0)\mathcal{B}\int_0^{t_i}\mathbb{U}(t_i,\varrho)\mathcal{F}(\varrho,\omega_0(\varrho))d\varrho \\
&\quad +\int_0^{\eta}\mathbb{U}(\eta,\varrho)\mathcal{F}(\varrho,\omega_0(\varrho))d\varrho \\
&= \mathcal{Q}\omega_0(\eta).
\end{aligned}$$

Hence, $\omega_0\leqslant\mathcal{Q}\omega_0$. In the same way, we get $\mathcal{Q}\nu_0\leqslant\nu_0$. For $u\in D$, we have $\omega_0\leqslant\mathcal{Q}\omega_0\leqslant\mathcal{Q}u\leqslant\mathcal{Q}\nu_0\leqslant\nu_0$. Therefore, $\mathcal{Q}: D\to D$ is monotone increasing.

Now we will show $\mathcal{Q}(D)$ is equicontinuous on J. For $x\in D$ and $\eta_1,\eta_2\in J$ with

$\eta_1 < \eta_2$ we have

$$\begin{aligned}
\|\mathcal{Q}x(\eta_2) - \mathcal{Q}x(\eta_1)\| &\leqslant \|\mathbb{U}(\eta_2,0) - \mathbb{U}(\eta_1,0)\|\|\mathcal{B}\|\|x_0\| \\
&\quad + \|\mathcal{B}\|\Bigg(\sum_{i=1}^{k}\xi_i\int_0^{t_i}\|\mathbb{U}(t_i,\varrho)\|\|\mathcal{F}(\varrho,x(\varrho))\|d\varrho\Bigg) \\
&\quad \|\mathbb{U}(\eta_2,0) - \mathbb{U}(\eta_1,0)\| + \int_0^{\eta_1}\|\mathbb{U}(\eta_2,\varrho) - \mathbb{U}(\eta_1,\varrho)\| \\
&\quad \|\mathcal{F}(\varrho,x(\varrho))\|d\varrho + \int_{\eta_1}^{\eta_2}\|\mathbb{U}(\eta_2,\varrho)\|\|\mathcal{F}(\varrho,x(\varrho))\|d\varrho \\
&\leqslant \|\mathbb{U}(\eta_2,0) - \mathbb{U}(\eta_1,0)\|\|\mathcal{B}\|\|x_0\| \\
&\quad + \mathcal{M}\|\mathcal{B}\|bc\Bigg(\sum_{i=1}^{k}\xi_i\Bigg)\|\mathbb{U}(\eta_2,0) - \mathbb{U}(\eta_1,0)\| \\
&\quad + \int_0^{\eta_1}\|\mathbb{U}(\eta_2,\varrho) - \mathbb{U}(\eta_1,\varrho)\|\|\mathcal{F}(\varrho,x(\varrho))\|d\varrho + \mathcal{M}c(\eta_2-\eta_1). \\
&\leqslant I_1 + I_2 + I_3 + I_4.
\end{aligned}$$

For $\eta_1 = 0$, it is easy to see that $I_3 = 0$. For $\eta_1 > 0$ and $\epsilon > 0$ small enough, we obtain

$$\begin{aligned}
I_3 &\leqslant \int_0^{\eta_1-\epsilon}\|\mathbb{U}(\eta_2,\varrho) - \mathbb{U}(\eta_1,\varrho)\|\|\mathcal{F}(\varrho,x(\varrho))\|d\varrho \\
&\quad + \int_{\eta_1-\epsilon}^{\eta_1}\|\mathbb{U}(\eta_2,\varrho) - \mathbb{U}(\eta_1,\varrho)\|\|\mathcal{F}(\varrho,x(\varrho))\|d\varrho \\
&\leqslant c(\eta_1-\epsilon)\sup_{\varrho\in[0,\eta_1-\epsilon]}\|\mathbb{U}(\eta_2,\varrho) - \mathbb{U}(\eta_1,\varrho)\| + 2\mathcal{M}c\epsilon. \\
&\rightarrow 0 \quad \text{as} \quad \eta_2 \rightarrow \eta_1,\ \epsilon \rightarrow 0,
\end{aligned}$$

using the continuity of $\{U(\eta,\varrho) : \eta > \varrho\}$ in uniform operator topology. By using Lemma 7.2.1, it is easy to see that $I_1 \rightarrow 0$, $I_2 \rightarrow 0$ as $\eta_2 \rightarrow \eta_1$. Also it is clear from the expression of I_4 that $I_4 \rightarrow 0$ as $\eta_2 \rightarrow \eta_1$. As a result $\|Qx(\eta_2) - Qx(\eta_1)\| \rightarrow 0$ as $\eta_2 \rightarrow \eta_1$, independently of $x \in D$. Hence $Q(D)$ is equicontinuous on J.

Now we define the sequences

$$\omega_n = \mathcal{Q}\omega_{n-1} \quad \text{and} \quad \nu_n = \mathcal{Q}\nu_{n-1}, \quad n \in \mathbb{N}, \tag{7.3.11}$$

monotonicity of $\mathcal{Q}$ implies

$$\omega_0 \leqslant \omega_1 \leqslant \cdots \omega_n \leqslant \cdots \leqslant \nu_n \leqslant \cdots \leqslant \nu_1 \leqslant \nu_0. \tag{7.3.12}$$

Let $S = \{\omega_n\}$ and $S_0 = \{\omega_{n-1}\}$. Then $S_0 = S \cup \{\omega_0\}$ and $\beta(S_0(t)) = \beta(S(t))$, $t \in J$. Since the sequence $\{\omega_n(0)\}$ is convergent, so $\beta(\{\omega_n(0)\}) = 0$. From Lemma 2.4.5,

$(H3)$, (7.3.9) and (7.3.11), we get

$$\begin{aligned}
\beta(S(t)) &= \beta(\mathcal{Q}(S_0(t))) \\
&= \beta\Bigg(\Bigg\{\mathbb{U}(t,0)\mathcal{B}x_0 + \sum_{i=1}^{k}\xi_i\mathbb{U}(t,0)\mathcal{B}\int_0^{t_i}\mathbb{U}(t_i,\varrho)\mathcal{F}(\varrho,\omega_{n-1}(\varrho))d\varrho \\
&\quad + \int_0^t \mathbb{U}(t,\varrho)\mathcal{F}(\varrho,\omega_{n-1}(\varrho))d\varrho\Bigg\}\Bigg) \\
&\leqslant \mathcal{M}\beta(\{\omega_n(0)\}) + 2\mathcal{M}\int_0^t \beta(\mathcal{F}(\varrho,\omega_{n-1}(\varrho)))d\varrho \\
&\leqslant 2\mathcal{M}\mathcal{L}\int_0^t \beta(\{\omega_{n-1}(\varrho)\})d\varrho \\
&\leqslant 2\mathcal{M}\mathcal{L}\int_0^t \beta(S_0(\varrho))d\varrho = 2\mathcal{M}\mathcal{L}\int_0^t \beta(S(\varrho))d\varrho. \qquad (7.3.13)
\end{aligned}$$

Since $\{\mathcal{Q}\omega_{n-1}\}$ i.e. $\{\omega_n\}$ is equicontinuous, by Lemma 2.4.4 and (7.3.13), we obtain

$$\begin{aligned}
\beta(S) &= \sup_{t\in J}\beta(S(t)) \\
&\leqslant 2\mathcal{M}\mathcal{L}b\sup_{t\in J}\beta(S(t)) = 2\mathcal{M}\mathcal{L}b\beta(S) = K\beta(S).
\end{aligned}$$

Since $K < 1$, therefore $\beta(S) = 0$. Hence the set S is relatively compact in D, so there is a convergent subsequence of $\{\omega_n\}$ in D. From (7.3.12), it is easy to see that $\{\omega_n\}$ itself is a convergent sequence. Let $\omega_n \to \omega^*$ as $n \to \infty$. By (7.3.9) and (7.3.11)

$$\begin{aligned}
\omega_n(t) &= \mathcal{Q}\omega_{n-1}(t) \\
&= \mathbb{U}(t,0)\mathcal{B}x_0 + \sum_{i=1}^{k}\xi_i\mathbb{U}(t,0)\mathcal{B}\int_0^{t_i}\mathbb{U}(t_i,\varrho)\mathcal{F}(\varrho,\omega_{n-1}(\varrho))d\varrho \\
&\quad + \int_0^t \mathbb{U}(t,\varrho)\mathcal{F}(\varrho,\omega_{n-1}(\varrho))d\varrho.
\end{aligned}$$

Let $n \to \infty$ and using Lebesgue dominated convergence theorem, we get $\omega^* = \mathcal{Q}\omega^*$ and $\omega^* \in C(J,\mathbb{X})$. Hence ω^* is a mild solution for (7.1.1)-(7.1.2). In the same way there is $\nu^* \in C(J,\mathbb{X})$ with $\nu_n \to \nu^*$ as $n \to \infty$, and $\nu^* = \mathcal{Q}\nu^*$. If $x \in D$ and $x = \mathcal{Q}x$, then $\omega_1 = \mathcal{Q}\omega_0 \leqslant \mathcal{Q}x = x \leqslant \mathcal{Q}\nu_0 = \nu_1$. From the process of induction $\omega_n \leqslant x \leqslant \nu_n$, and $\omega_0 \leqslant \omega^* \leqslant x \leqslant \nu^* \leqslant \nu_0$ as $n \to \infty$. That means ω^* is the minimal and ν^* is the maximal mild solution for (7.1.1)-(7.1.2) in $[\omega_0, \nu_0]$. □

Corollary 7.3.3. *Suppose $\mathbb{X}$ is a partially ordered and weakly sequentially complete Banach space with normal positive cone $\mathcal{P}$, the assumptions $(H0), (H1), (H2)$ and $(A1)$-$(A3)$ hold, $\mathcal{F}$ is continuous from $J \times \mathbb{X} \to \mathbb{X}$, $x_0 \in \mathbb{X}$, and ω_0, $\nu_0 \in C^1(J,\mathbb{X})$ are lower and upper solutions respectively for (7.1.1)-(7.1.2), then the system (7.1.1)-(7.1.2) has extremal mild solutions in the interval $[\omega_0, \nu_0]$.*

Proof. Since the assumptions $(H1),(H2)$ hold, by the proof of Theorem 7.3.2 we have the sequences $\{\omega_n(t)\}$ and $\{\nu_n(t)\}$ defined by (7.3.11) satisfying (7.3.12). So, $\{\omega_n(t)\}$ and $\{\nu_n(t)\}$ are monotone and bounded, hence are precompact on $\mathbb{X}$ for $\mathbb{X}$ is weakly sequentially complete Banach space. Therefore, the sequences are convergent uniformly. Let $\omega^*(t) = \lim_{n\to\infty} \omega_n(t)$ and $\nu^*(t) = \lim_{n\to\infty} \nu_n(t)$, for $t \in J$. By Lebesgue dominated convergence theorem, we get $\omega^* = \mathcal{Q}\omega^*$ and $\nu^* = \mathcal{Q}\nu^*$ with $\omega^*, \nu^* \in C(J,\mathbb{X})$. Hence ω^* and ν^* are a mild solutions for (7.1.1)-(7.1.2). If $x \in D$ and $x = \mathcal{Q}x$, then $\omega_1 = \mathcal{Q}\omega_0 \leqslant \mathcal{Q}x = x \leqslant \mathcal{Q}\nu_0 = \nu_1$. From the process of induction $\omega_n \leqslant x \leqslant \nu_n$, and $\omega_0 \leqslant \omega^* \leqslant x \leqslant \nu^* \leqslant \nu_0$ as $n \to \infty$. That means ω^* is the minimal and ν^* is the maximal mild solution for (7.1.1)-(7.1.2). □

Theorem 7.3.4. *Suppose $\mathbb{X}$ is a partially ordered complete norm space, with normal positive cone $\mathcal{P}$ and normal constant $\mathcal{N}$, the assumptions $(H0),(H1),(H2),(H4)$ and $(A1)$-$(A3)$ hold, $\mathcal{F} \in C(J\times\mathbb{X},\mathbb{X})$, $x_0 \in \mathbb{X}$, and ω_0, $\nu_0 \in C^1(J,\mathbb{X})$ are lower and upper solutions respectively for (7.1.1)-(7.1.2). Moreover, assume the following :*

(H5) *There is a constant $\mathcal{L}_1 > 0$ such that, for $t \in J$, and $z_1, z_2 \in \mathbb{X}$ with $\omega_0(t) \leqslant z_1 \leqslant z_2 \leqslant \nu_0(t)$,*

$$\mathcal{F}(t,z_2) - \mathcal{F}(t,z_1) \leqslant \mathcal{L}_1(z_2 - z_1).$$

Then, the system (7.1.1)-(7.1.2) has a unique mild solution in $[\omega_0,\nu_0]$, provided that

$$K_1 := \frac{2 - \mathcal{M}\sum_{i=1}^{k}\xi_i}{1 - \mathcal{M}\sum_{i=1}^{k}\xi_i}\mathcal{N}\mathcal{L}_1\mathcal{M}b < 1.$$

Proof. Let $\{x_n\} \subset \mathbb{X}$ be an increasing monotone sequence, and $n, m \in \mathbb{N}$ with $n > m$. $(H2)$ and $(H5)$ imply

$$0 \leqslant \mathcal{F}(t,x_n) - \mathcal{F}(t,x_m) \leqslant \mathcal{L}_1(x_n - x_m).$$

From the normality of $\mathcal{P}$

$$\|\mathcal{F}(t,x_n) - \mathcal{F}(t,x_m)\| \leqslant \mathcal{N}\mathcal{L}_1\|(x_n - x_m)\|. \tag{7.3.14}$$

So by Lemma 2.4.3, we get

$$\beta(\{\mathcal{F}(t,x_n)\}) \leqslant \mathcal{N}\mathcal{L}_1\beta(\{x_n\}).$$

Hence the assumption $(H3)$ holds, and Theorem 7.3.2 is applicable. Therefore, (7.1.1)-(7.1.2) has minimal mild solution ω^* and maximal mild solutions ν^* in $[\omega_0,\nu_0]$.

By (7.3.9), $(H5)$, and the positivity of the operator $\mathbb{U}(t,s)$, we get

$$\begin{aligned} 0 &\leqslant \nu^*(t)-\omega^*(t)=\mathcal{Q}\nu^*(t)-\mathcal{Q}\omega^*(t) \\ &= \sum_{i=1}^{k}\xi_i\mathbb{U}(t,0)\mathcal{B}\int_0^{t_i}\mathbb{U}(t_i,\varrho)[\mathcal{F}(\varrho,\nu^*(\varrho))-\mathcal{F}(\varrho,\omega^*(\varrho))]d\varrho \\ &\quad +\int_0^t\mathbb{U}(t,\varrho)[\mathcal{F}(\varrho,\nu^*(\varrho))-\mathcal{F}(\varrho,\omega^*(\varrho))]d\varrho \\ &\leqslant \mathcal{L}_1\Bigg[\sum_{i=1}^{k}\xi_i\mathbb{U}(t,0)\mathcal{B}\int_0^{t_i}\mathbb{U}(t_i,\varrho)(\nu^*(\varrho)-\omega^*(\varrho))d\varrho \\ &\quad +\int_0^t\mathbb{U}(t,\varrho)(\nu^*(\varrho)-\omega^*(\varrho))d\varrho\Bigg]. \end{aligned}$$

Now using $(H1)$, (7.3.3), and the normality of positive cone, we get

$$\begin{aligned} \|\nu^*-\omega^*\| &\leqslant \mathcal{N}\mathcal{L}_1\Bigg[\Bigg(\sum_{i=1}^{k}\xi_i\Bigg)\mathcal{M}^2\|\mathcal{B}\|\int_0^b\|\nu^*(\varrho)-\omega^*(\varrho)\|d\varrho \\ &\quad +\mathcal{M}\int_0^b\|\nu^*(\varrho)-\omega^*(\varrho)\|d\varrho\Bigg] \\ &\leqslant \mathcal{N}\mathcal{L}_1\mathcal{M}b(\|\mathcal{B}\|+1)\|\nu^*-\omega^*\| \\ &\leqslant \frac{2-\mathcal{M}\sum_{i=1}^{k}\xi_i}{1-\mathcal{M}\sum_{i=1}^{k}\xi_i}\mathcal{N}\mathcal{L}_1\mathcal{M}b\|\nu^*-\omega^*\| = K_1\|\nu^*-\omega^*\|. \end{aligned}$$

Since $K_1<1$, so $\|\nu^*-\omega^*\|=0$, i.e. $\nu^*(t)=\omega^*(t)$ for all $t\in J$. Hence $\nu^*=\omega^*$ is the unique mild solution for (7.1.1)-(7.1.2) in $[\omega_0,\nu_0]$. □

7.4 Example

Now we consider an example to show how our abstract results can be applied to a concrete problem. Consider the following partial differential equation :

$$\begin{cases} x'(t,z)+a(t,z)\frac{\partial^2}{\partial z^2}x(t,z)=\frac{1}{4}t^2x(t,z), & z\in[0,\pi],\quad t\in J=[0,b]; \\ x(t,0)=0,\ x(t,\pi)=0 & t\in J; \\ x(0,z)=\sum_{i=1}^{k}\xi_i x(t_i,z)+x_0(z), & z\in[0,\pi], \end{cases} \tag{7.4.1}$$

where $\mathbb{X}=L^2([0,b]\times[0,\pi],\mathbb{R})$, $x_0(z)\in\mathbb{X}$, $a(t,z)$ is continuous function and satisfies uniformly Hölder continuity in t, and $0<t_1<t_2<\ldots<t_i<b$, $\xi_i\neq 0$ $(i=$

$1, 2, \ldots k)$ are real numbers. Define

$$\mathbb{A}(t)x(t,z) = a(t,z)\frac{\partial^2}{\partial z^2}x(t,z), \tag{7.4.2}$$

with domain

$$\mathcal{D}(\mathbb{A}) = \{w \in \mathbb{X} : w, \frac{\partial w}{\partial z} \text{ are absolutely continuous}, \frac{\partial^2 w}{\partial z^2} \in \mathbb{X}, w(0) = w(\pi) = 0\}.$$

Then, $-\mathbb{A}(t)$ generates a positive evolution system of bounded linear operators $\mathbb{U}(t,s)$ on $\mathbb{X}$ with $\|\mathbb{U}(t,s)\| \leqslant \mathcal{M}$ and satisfies the conditions $(A1)$-$(A3)$ (see [156]). Put $x(t)(z) = x(t,z)$, $t \in [0,b], z \in [0,\pi]$ and $f(t,x(t))(z) = \frac{1}{4}t^2 x(t,z)$, then the system (7.4.1) can be rewritten into the abstract form of (7.1.1)-(7.1.2). It is easy to check that assumption $(H2)$ holds. Now, assume the following :

1. $x_0(z) \geqslant 0$ for $z \in [0,\pi]$, and there exists a function $v(t,z) \geqslant 0$ such that

$$\begin{aligned} v'(t,z) + \mathbb{A}(t)v(t,z) &\geqslant \frac{1}{4}t^2 v(t,z), \quad t \in J,\ z \in [0,\pi], \\ v(t,0) &= v(t,\pi) = 0, \quad t \in J, \\ v(0,z) &\geqslant \sum_{i=1}^{k} \xi_i v(t_i,z) + x_0(z), \quad z \in [0,\pi]. \end{aligned}$$

2. $\xi_i > 0$, $i = 1, 2, \ldots, k$ and $\mathcal{M}\sum_{i=1}^{k} \xi_i < 1$.

From the above assumptions it is clear that $(H1)$ holds, and $\omega_0 = 0$ and $\nu_0 = v(t,z)$ are the lower and upper solutions for the system (7.4.1). So, by Corollary 7.3.3, we conclude that the minimal and maximal mild solutions for (7.4.1) exist between the lower solution 0 and upper solution v.

Chapter 8

Monotone Iterative Technique for Non-Autonomous Nonlocal Integro-Differential Equations

8.1 Introduction

In this chapter, we extend the result of chapter 7 for the following non-autonomous integro-differential system involving nonlocal condition in an ordered Banach space $\mathbb{X}$:

$$\begin{aligned} x'(t) + \mathbb{A}(t)x(t) &= \mathcal{F}\Big(t, x(t), \int_0^t k(t,s)x(s)ds\Big), \quad t \in (0,b], \\ x(0) &= x_0 + \mathcal{G}(x), \end{aligned} \tag{8.1.1}$$

where $\mathbb{A}(t)$ is an $\mathbb{X}$-valued linear operator defined on $D(\mathbb{A}(t)) \subset \mathbb{X}$, $x_0 \in \mathbb{X}$, $\mathcal{F}$ is $\mathbb{X}$-valued function defined over $J \times \mathbb{X} \times \mathbb{X}$, $\mathcal{G}$ is $\mathbb{X}$-valued function defined over $C(J, \mathbb{X})$ with $J = [0, b]$, and k is continuous function from $\mathbb{D}$ to $\mathbb{R}^+$ where $\mathbb{D} := \{(\tau, s) : 0 \leqslant s \leqslant \tau \leqslant b\}$.

We organize the chapter as following. In section 8.2, we will recall some basic theory. In section 8.3, we will establish the existence of extremal mild solutions for

The contents of this chapter are accepted in **Filomat** as Arshi Meraj, Dwijendra Narain Pandey: "Existence and uniqueness of extremal mild solutions for non-autonomous nonlocal integro-differential equations via monotone iterative technique".

the system (8.1.1), and also we will show the uniqueness of extremal mild solutions with the help of evolution system and Kuratowski measure of noncompactness by MIT. In last section, we will discuss an example to illustrate our results.

8.2 Preliminaries

Let $(\mathbb{X}, \|\cdot\|, \leqslant)$ is a partially ordered complete norm space, $\mathcal{P} = \{x \in \mathbb{X} : x \geqslant 0\}$ (0 is the zero element of $\mathbb{X}$) is a positive cone of $\mathbb{X}$. The cone $\mathcal{P}$ is known as normal if there is a real number $\mathcal{N} > 0$ such that $0 \leqslant x_1 \leqslant x_2 \Rightarrow \|x_1\| \leqslant \mathcal{N}\|x_2\|$, for all $x_1, x_2 \in \mathbb{X}$, the smallest value of such $\mathcal{N}$ is called normal constant. For $x_1, x_2 \in C(J, \mathbb{X})$, $x_1 \leqslant x_2 \Leftrightarrow x_1(t) \leqslant x_2(t)$, $\forall\, t \in J$. For ν, $\omega \in C(J, \mathbb{X})$ with $\nu \leqslant \omega$, we will use the notation $[\nu, \omega] := \{x \in C(J, \mathbb{X}) : \nu \leqslant x \leqslant \omega\}$ for an interval in $C(J, \mathbb{X})$, and $[\nu(t), \omega(t)] := \{x \in \mathbb{X} : \nu(t) \leqslant x \leqslant \omega(t)\}(t \in J)$ for an interval in $\mathbb{X}$. Let us denote $C^1(J, \mathbb{X}) = \{x \in C(J, \mathbb{X}) : x' \text{ exists on } J\ , x' \in C(J, \mathbb{X}),\ x(t) \in D(\mathbb{A})\ (t \geqslant 0)\}$. For our convenience, we denote $\mathcal{K}x(t) := \int_0^t k(t, s)x(s)ds$, and $K^* := \sup\limits_{(t,s)\in\mathbb{D}} k(t, s)$.

First, we recall the definition and some basic properties of evolution system. For more details, we refer [82] and [156].

Definition 8.2.1. *([156]) Let $\mathbb{X}$ be a Banach space. A two parameter family of bounded linear operators $\mathbb{U}(t_1, t_2), 0 \leqslant t_2 \leqslant t_1 \leqslant b$ on X is known as evolution system, if :*

1. *$\mathbb{U}(s, s) = I$, where I is the identity operator.*
2. *$\mathbb{U}(t_1, t_2)\mathbb{U}(t_2, t_3) = \mathbb{U}(t_1, t_3)$ for $0 \leqslant t_3 \leqslant t_2 \leqslant t_1 \leqslant b$.*
3. *$(t_1, t_2) \to \mathbb{U}(t_1, t_2)$ is strongly continuous for $0 \leqslant t_2 \leqslant t_1 \leqslant b$.*

For the family of linear operators $\{\mathbb{A}(t) : t \in J\}$ on $\mathbb{X}$, we impose the restrictions given below :

(A1) The operator $\mathbb{A}(t)$ is closed, the domain of $\mathbb{A}(t)$ is independent of t, and dense in $\mathbb{X}$.

(A2) The resolvent of $\mathbb{A}(t)$ exists for $Re(\vartheta) \leqslant 0$, $t \in J$, and $\|R(\vartheta, \mathbb{A}(t))\| \leqslant \frac{\varsigma}{|\vartheta|+1}$, for some positive constant ς.

(A3) There exist positive constants K, and $\rho \in (0,1]$ such that $\|[\mathbb{A}(\tau_1)-\mathbb{A}(\tau_2)]\mathbb{A}^{-1}(\tau_3)\| \leqslant K|\tau_1-\tau_2|^{\rho}$ for any $\tau_1,\tau_2,\tau_3 \in J$.

Theorem 8.2.1. *([156]) Suppose that the assumptions (A1)-(A3) hold, then* $-\mathbb{A}(t)$ *generates a unique evolution system* $\{\mathbb{U}(t_1,t_2) : 0 \leqslant t_2 \leqslant t_1 \leqslant b\}$, *which satisfies the following properties :*

(i) There exists a positive constant $\mathcal{M}$ *such that* $\|\mathbb{U}(t_1,t_2)\| \leqslant \mathcal{M}$, $0 \leqslant t_2 \leqslant t_1 \leqslant b$.

(ii) For $0 \leqslant t_2 < t_1 \leqslant b$, *the derivative* $\frac{\partial \mathbb{U}(t_1,t_2)}{\partial t_1}$ *exists in strong operator topology, is strongly continuous, and belongs to* $\mathbb{B}(\mathbb{X})$ *(set of all bounded linear operators on* $\mathbb{X}$*). Moreover,*

$$\frac{\partial \mathbb{U}(t_1,t_2)}{\partial t_1} + \mathbb{A}(t_1)\mathbb{U}(t_1,t_2) = 0,\ 0 \leqslant t_2 < t_1 \leqslant b.$$

Proposition 8.2.2. *([189]) The family of operators* $\{\mathbb{U}(t_1,t_2),\ t_2 < t_1\}$ *is continuous in* t_1 *uniformly for* t_2 *with respect to operator norm.*

Theorem 8.2.3. *([156]) Suppose that the assumptions (A1)-(A3) hold and* $\mathcal{F}$ *satisfies uniform Hölder continuity on* J *with exponent* $\alpha \in (0,1]$, *then the unique solution of the following linear Cauchy problem*

$$\begin{aligned} x'(t) + \mathbb{A}(t)x(t) &= \mathcal{F}(t), \quad 0 < t \leqslant b, \\ x(0) &= x_0 \in \mathbb{X}, \end{aligned} \tag{8.2.1}$$

is given as

$$x(t) = \mathbb{U}(t,0)x_0 + \int_0^t \mathbb{U}(t,\eta)\mathcal{F}(\eta)d\eta. \tag{8.2.2}$$

Definition 8.2.2. *A mild solution of (8.1.1) is a function* $x \in C(J,\mathbb{X})$ *satisfying the following integral equation*

$$x(\varrho) = \mathbb{U}(\varrho,0)(x_0+\mathcal{G}(x)) + \int_0^{\varrho} \mathbb{U}(\varrho,\eta)\mathcal{F}(\eta,x(\eta),\mathcal{K}x(\eta))d\eta,\ \varrho \in J.$$

Definition 8.2.3. *An evolution system* $\mathbb{U}(t,s)$ *is called positive if* $\mathbb{U}(t,s)y \geqslant 0$, *for all* $y \in \mathcal{P}$ *and* $0 \leqslant s \leqslant t \leqslant b$.

Definition 8.2.4. $\omega_0 \in C^1(J,\mathbb{X})$ *is called lower solution for the system (8.1.1), if*

$$\begin{aligned} \omega_0'(t) + \mathbb{A}(t)\omega_0(t) &\leqslant \mathcal{F}\Big(t,\omega_0(t),\int_0^t k(t,s)\omega_0(s)ds\Big), \quad t \in (0,b], \\ \omega_0(0) &\leqslant x_0 + \mathcal{G}(\omega_0). \end{aligned} \tag{8.2.3}$$

If the inequalities of (8.2.3) are opposite, solution is known as upper solution.

8.3 Main Results

First, we will show the existence of extremal mild solutions for (8.1.1), then the uniqueness will be discussed.

Let us define $\mathcal{Q} : C(J, \mathbb{X}) \to C(J, \mathbb{X})$ in the following way :

$$\mathcal{Q}x(\varrho) = \mathbb{U}(\varrho, 0)(x_0 + \mathcal{G}(x)) + \int_0^{\varrho} \mathbb{U}(\varrho, \eta)\mathcal{F}(\eta, x(\eta), \mathcal{K}x(\eta))d\eta. \quad (8.3.1)$$

To prove that the system (8.1.1) has a mild solution, we need to show the operator $\mathcal{Q}$ has a fixed point.

Theorem 8.3.1. *Suppose $\mathbb{X}$ is a partially ordered complete norm space with normal positive cone $\mathcal{P}$, the assumptions (A1)-(A3) hold, the evolution system $\mathbb{U}(t, s)$ is positive, $\mathcal{F}$ is continuous from $J \times \mathbb{X} \times \mathbb{X} \to \mathbb{X}$, $x_0 \in \mathbb{X}$, and ω_0, $\nu_0 \in C^1(J, \mathbb{X})$ with $\omega_0 \leqslant \nu_0$ are lower and upper solutions respectively for (8.1.1). Moreover, assume the following :*

(H1) *For $t \in J$, we have*

$$\mathcal{F}(t, y_1, x_1) \leqslant \mathcal{F}(t, y_2, x_2),$$

where $y_1, y_2 \in \mathbb{X}$ with $\omega_0(t) \leqslant y_1 \leqslant y_2 \leqslant \nu_0(t)$, and $\mathcal{K}\omega_0(t) \leqslant x_1 \leqslant x_2 \leqslant \mathcal{K}\nu_0(t)$.

(H2) *There exists a constant $\mathcal{L} > 0$ such that for all $t \in J$,*

$$\beta(\{\mathcal{F}(t, y_n, x_n)\}) \leqslant \mathcal{L}(\beta(\{y_n\}) + \beta(\{x_n\})),$$

where $\{y_n\} \subset [\omega_0(t), \nu_0(t)]$ and $\{x_n\} \subset [\mathcal{K}\omega_0(t), \mathcal{K}\nu_0(t)]$ are monotone increasing or decreasing sequences.

(H3) *$\mathcal{G} : C(J, \mathbb{X}) \to \mathbb{X}$ is a continuous increasing compact function.*

Then, the system (8.1.1) has extremal mild solutions in the interval $[\omega_0, \nu_0]$, provided that

$$\Lambda_1 := 2\mathcal{M}\mathcal{L}b(1 + bK^*) < 1.$$

Proof. Let us denote $I = [\omega_0, \nu_0]$. For any $x \in I$, $(H1)$ implies

$$\mathcal{F}(\varrho, \omega_0(\varrho), \mathcal{K}\omega_0(\varrho)) \leqslant \mathcal{F}(\varrho, x(\varrho), \mathcal{K}x(\varrho)) \leqslant \mathcal{F}(\varrho, \nu_0(\varrho), \mathcal{K}\nu_0(\varrho)).$$

Therefore, from the normality of $\mathcal{P}$ we get a constant $c > 0$, such that

$$\|\mathcal{F}(\varrho, x(\varrho), \mathcal{K}x(\varrho))\| \leqslant c, \ \ x \in I. \quad (8.3.2)$$

First, we will prove that the map $\mathcal{Q}: I \to C(J, \mathbb{X})$ is continuous. Let $\{x_n\}$ be a sequence in I such that $x_n \to x \in I$. Since $\mathcal{G}$, $\mathcal{F}$ are continuous, so $\mathcal{G}x_n \to \mathcal{G}x$, and $\mathcal{F}(\varrho, x_n(\varrho), \mathcal{K}x_n(\varrho)) \to \mathcal{F}(\varrho, x(\varrho), \mathcal{K}x(\varrho))$ for $\varrho \in J$, and from (8.3.2) we get that $\|\mathcal{F}(\varrho, x_n(\varrho), \mathcal{K}x_n(\varrho)) - \mathcal{F}(\varrho, x(\varrho), \mathcal{K}x(\varrho))\| \leqslant 2c$. So, by Lebesgue dominated convergence theorem, we estimate

$$\begin{aligned} \|\mathcal{Q}x_n(t) - \mathcal{Q}x(t)\| &\leqslant \mathcal{M}\|\mathcal{G}x_n - \mathcal{G}x\| \\ &\quad + \mathcal{M}\int_0^t \|\mathcal{F}(\varrho, x_n(\varrho), \mathcal{K}x_n(\varrho)) - \mathcal{F}(\varrho, x(\varrho), \mathcal{K}x(\varrho))\| d\varrho \\ &\to 0 \quad \text{as} \quad n \to \infty. \end{aligned}$$

Thus $\mathcal{Q}$ is continuous map on I.

Next, we will prove $\mathcal{Q}: I \to I$ is monotone increasing. Let $x_1, x_2 \in I$, $x_1 \leqslant x_2$. Using the positivity of $\mathbb{U}(t,s)$, the hypotheses $(H1)$ and $(H3)$, it is easy to see that $\mathcal{Q}x_1 \leqslant \mathcal{Q}x_2$. Suppose $\omega_0'(\eta) + \mathbb{A}(\eta)\omega_0(\eta) = h(\eta)$, Definition 8.2.4 implies $h(\eta) \leqslant \mathcal{F}(\eta, \omega_0(\eta), \mathcal{K}\omega_0(\eta))$ for $\eta \in J$, and $\omega_0(0) \leqslant x_0 + \mathcal{G}(\omega_0)$. Therefore, for any $t \in J$, Theorem 8.2.3 yields

$$\begin{aligned} \omega_0(t) &= \mathbb{U}(t,0)\omega_0(0) + \int_0^t \mathbb{U}(t,\eta)h(\eta)d\eta \\ &\leqslant \mathbb{U}(t,0)(x_0 + \mathcal{G}(\omega_0)) + \int_0^t \mathbb{U}(t,\eta)\mathcal{F}(\eta, \omega_0(\eta), \mathcal{K}\omega_0(\eta))d\eta \\ &= \mathcal{Q}\omega_0(t). \end{aligned}$$

Hence, $\omega_0 \leqslant \mathcal{Q}\omega_0$. In the same way, we get $\mathcal{Q}\nu_0 \leqslant \nu_0$. Let $u \in I$, so we have $\omega_0 \leqslant \mathcal{Q}\omega_0 \leqslant \mathcal{Q}u \leqslant \mathcal{Q}\nu_0 \leqslant \nu_0$, that means $\mathcal{Q}u \in I$. Therefore, $\mathcal{Q}: I \to I$ is monotone increasing.

Now, we will show $\mathcal{Q}(I)$ is equicontinuous on J. For $x \in I$ and $\eta_1, \eta_2 \in J$ with $\eta_1 < \eta_2$, we have

$$\begin{aligned} \|\mathcal{Q}x(\eta_2) - \mathcal{Q}x(\eta_1)\| &\leqslant \|\mathbb{U}(\eta_2,0) - \mathbb{U}(\eta_1,0)\|\|x_0 + \mathcal{G}x\| \\ &\quad + \int_0^{\eta_1} \|\mathbb{U}(\eta_2,\varrho) - \mathbb{U}(\eta_1,\varrho)\|\|\mathcal{F}(\varrho, x(\varrho), \mathcal{K}x(\varrho))\| d\varrho \\ &\quad + \int_{\eta_1}^{\eta_2} \|\mathbb{U}(\eta_2,\varrho)\|\|\mathcal{F}(\varrho, x(\varrho), \mathcal{K}x(\varrho))\| d\varrho \\ &\leqslant I_1 + I_2 + I_3. \end{aligned}$$

For $\eta_1 = 0$, it is easy to see that $I_2 = 0$. For $\eta_1 > 0$ and $\epsilon > 0$ small enough, we obtain

$$\begin{aligned} I_2 &\leqslant \int_0^{\eta_1 - \epsilon} \|\mathbb{U}(\eta_2,\varrho) - \mathbb{U}(\eta_1,\varrho)\|\|\mathcal{F}(\varrho, x(\varrho), \mathcal{K}x(\varrho))\| d\varrho \\ &\quad + \int_{\eta_1 - \epsilon}^{\eta_1} \|\mathbb{U}(\eta_2,\varrho) - \mathbb{U}(\eta_1,\varrho)\|\|\mathcal{F}(\varrho, x(\varrho), \mathcal{K}x(\varrho))\| d\varrho \\ &\leqslant c(\eta_1 - \epsilon) \sup_{\varrho \in [0, \eta_1 - \epsilon]} \|\mathbb{U}(\eta_2,\varrho) - \mathbb{U}(\eta_1,\varrho)\| + 2\mathcal{M}c\epsilon. \\ &\to 0 \quad \text{as} \quad \eta_2 \to \eta_1,\ \epsilon \to 0, \end{aligned}$$

by using the continuity of $\{\mathbb{U}(\eta,\varrho):\varrho<\eta\}$ in η in uniform operator topology. Also It is clear from the expression of I_1, I_3 that $I_1 \to 0$, $I_3 \to 0$ as $\eta_2 \to \eta_1$. As a result $\|Qx(\eta_2)-Qx(\eta_1)\| \to 0$ as $\eta_2 \to \eta_1$, independently of $x \in I$. Hence $Q(I)$ is equicontinuous on J.

Now we define the sequences

$$\omega_n = \mathcal{Q}\omega_{n-1} \quad \text{and} \quad \nu_n = \mathcal{Q}\nu_{n-1}, \quad n \in \mathbb{N}, \tag{8.3.3}$$

monotonicity of $\mathcal{Q}$ implies

$$\omega_0 \leqslant \omega_1 \leqslant \cdots \omega_n \leqslant \cdots \leqslant \nu_n \leqslant \cdots \leqslant \nu_1 \leqslant \nu_0. \tag{8.3.4}$$

Let $S=\{\omega_n\}$ and $S_0=\{\omega_{n-1}\}$. Then $S_0 = S\cup\{\omega_0\}$ and $\beta(S_0(t))=\beta(S(t))$, $t\in J$. Observe that $\beta(\mathbb{U}(t,0)(x_0)) = 0 = \beta(\mathbb{U}(t,0)\mathcal{G}(\omega_{n-1}))$ for $\{x_0\}$ is compact set, $\mathcal{G}$ is compact map and $\mathbb{U}(t,0)$ is bounded. Also, with the help of Lemma 2.4.4 and Lemma 2.4.5, we observe that

$$\begin{aligned}
\beta\Big(\{\mathcal{K}\omega_{n-1}(\eta)\}\Big) &= \beta\Big(\int_0^\eta k(\eta,s)\omega_{n-1}(s)ds\Big)\\
&\leqslant K^*\beta\Big(\int_0^\eta \omega_{n-1}(s)ds\Big)\\
&\leqslant 2K^*\int_0^\eta \beta(\omega_{n-1}(s))ds\\
&\leqslant 2K^*\eta \sup_{s\in[0,\eta]} \beta(S_0(s)).
\end{aligned}$$

Now, from Lemma 2.4.5, $(H2)$, $(H3)$, (8.3.1) and (8.3.3), we get

$$\begin{aligned}
\beta(S(t)) &= \beta(\mathcal{Q}(S_0(t)))\\
&= \beta\Big(\mathbb{U}(t,0)(x_0+\mathcal{G}(\omega_{n-1})) + \int_0^t \mathbb{U}(t,\eta)\mathcal{F}(\eta,\omega_{n-1}(\eta),\mathcal{K}\omega_{n-1}(\eta))d\eta\Big)\\
&\leqslant \beta(\mathbb{U}(t,0)x_0) + \beta(\mathbb{U}(t,0)\mathcal{G}(\omega_{n-1})) + 2\mathcal{M}\int_0^t \beta\Big(\mathcal{F}(\eta,\omega_{n-1}(\eta),\mathcal{K}\omega_{n-1}(\eta))d\eta\Big)\\
&\leqslant 2\mathcal{M}\mathcal{L}\int_0^t \Big[\beta(\{\omega_{n-1}(\eta)\}) + \beta(\{\mathcal{K}\omega_{n-1}(\eta)\})\Big]d\eta\\
&\leqslant 2\mathcal{M}\mathcal{L}b(1+bK^*)\sup_{t\in J}\beta(S(t)).
\end{aligned} \tag{8.3.5}$$

Since $\{\mathcal{Q}\omega_{n-1}\}$ i.e. $\{\omega_n\}$ is equicontinuous, by Lemma 2.4.4 and (8.3.5), we obtain

$$\begin{aligned}
\beta(S) &= \sup_{t\in J}\beta(S(t))\\
&\leqslant 2\mathcal{M}\mathcal{L}b(1+bK^*)\sup_{t\in J}\beta(S(t)) = 2\mathcal{M}\mathcal{L}b(1+bK^*)\beta(S) = \Lambda_1\beta(S).
\end{aligned}$$

Since $\Lambda_1 < 1$, therefore $\beta(S) = 0$. Hence the set S is relatively compact in I, so there exists a convergent subsequence of $\{\omega_n\}$ in I. From (8.3.4), it is easy to see that $\{\omega_n\}$ itself is a convergent sequence, let $\omega_n \to \omega^*$ as $n \to \infty$. By (8.3.1) and (8.3.3)

$$\begin{aligned} \omega_n(t) &= \mathcal{Q}\omega_{n-1}(t) \\ &= \mathbb{U}(t,0)(x_0 + \mathcal{G}(\omega_{n-1})) \\ &\quad + \int_0^t \mathbb{U}(t,\eta)\mathcal{F}(\eta, \omega_{n-1}(\eta), \mathcal{K}\omega_{n-1}(\eta))d\eta. \end{aligned} \tag{8.3.6}$$

In (8.3.6), let $n \to \infty$ and use Lebesgue dominated convergence theorem, we get

$$\omega^*(t) = \mathbb{U}(t,0)(x_0 + \mathcal{G}(\omega^*)) + \int_0^t \mathbb{U}(t,\eta)\mathcal{F}(\eta, \omega^*(\eta), \mathcal{K}\omega^*(\eta))d\eta.$$

So, $\omega^* = \mathcal{Q}\omega^*$ and $\omega^* \in C(J, \mathbb{X})$. Hence ω^* is a mild solution for (8.1.1). In the same way there exists $\nu^* \in C(J, \mathbb{X})$ with $\nu_n \to \nu^*$ as $n \to \infty$, and $\nu^* = \mathcal{Q}\nu^*$. Now we show ω^*, ν^* are extremal mild solutions. Let $x \in I$ and $x = \mathcal{Q}x$, then $\omega_1 = \mathcal{Q}\omega_0 \leqslant \mathcal{Q}x = x \leqslant \mathcal{Q}\nu_0 = \nu_1$. From the process of induction $\omega_n \leqslant x \leqslant \nu_n$, and $\omega_0 \leqslant \omega^* \leqslant x \leqslant \nu^* \leqslant \nu_0$ as $n \to \infty$. That means ω^* is the minimal and ν^* is the maximal mild solution for (8.1.1) in $[\omega_0, \nu_0]$. □

Theorem 8.3.2. *Suppose $\mathbb{X}$ is a partially ordered complete norm space, with normal positive cone $\mathcal{P}$ and normal constant $\mathcal{N}$, the assumptions $(H1), (H3), (A1)$-$(A3)$ hold, the evolution system $\mathbb{U}(t,s)(0 \leqslant s \leqslant t \leqslant b)$ is positive, $\mathcal{F}$ is continuous from $J \times \mathbb{X} \times \mathbb{X}$ to $\mathbb{X}$, $x_0 \in \mathbb{X}$, and ω_0, $\nu_0 \in C^1(J, \mathbb{X})$ with $\omega_0 \leqslant \nu_0$ are lower and upper solutions respectively for (8.1.1). Moreover, assume the following :*

(H4) *There is a constant $\mathcal{L}_1 > 0$ such that for $t \in J$*

$$\mathcal{F}(t, y_2, x_2) - \mathcal{F}(t, y_1, x_1) \leqslant \mathcal{L}_1[(y_2 - y_1) + (x_2 - x_1)],$$

where $y_1, y_2 \in \mathbb{X}$ with $\omega_0(t) \leqslant y_1 \leqslant y_2 \leqslant \nu_0(t)$, and $\mathcal{K}\omega_0(t) \leqslant x_1 \leqslant x_2 \leqslant \mathcal{K}\nu_0(t)$.

(H5) *There exists a constant $\mathcal{L}_2 > 0$ such that*

$$\mathcal{G}(y) - \mathcal{G}(x) \leqslant \mathcal{L}_2(y - x), \text{ for } x, y \in I \text{ with } x \leqslant y.$$

Then, the system (8.1.1) has a unique mild solution in $[\omega_0, \nu_0]$, provided that

$$\Lambda_2 := \mathcal{N}\mathcal{M}\Big[\mathcal{L}_2 + \mathcal{L}_1 b(1 + bK^*)\Big] < 1.$$

Proof. Let $\{y_n\} \subset [\omega_0(t), \nu_0(t)]$ and $\{x_n\} \subset [\mathcal{K}\omega_0(t), \mathcal{K}\nu_0(t)]$ be increasing monotone sequences. For $t \in J$ and $n, m \in \mathbb{N}$ with $n > m$, the assumptions $(H1)$ and $(H4)$ imply

$$0 \leqslant \mathcal{F}(t, y_n, x_n) - \mathcal{F}(t, y_m, x_m) \leqslant \mathcal{L}_1[(y_n - y_m) + (x_n - x_m)].$$

Since the positive cone is normal, therefore

$$\|\mathcal{F}(t,y_n,x_n)-\mathcal{F}(t,y_m,x_m)\| \leqslant \mathcal{N}\mathcal{L}_1\|(y_n-y_m)+(x_n-x_m)\|. \tag{8.3.7}$$

So by Lemma 2.4.3, we get

$$\mu(\{\mathcal{F}(t,y_n,x_n)\}) \leqslant \mathcal{N}\mathcal{L}_1(\mu(\{y_n\})+\mu(\{x_n\})).$$

Hence the assumption $(H2)$ holds, and Theorem 8.3.1 is applicable. Therefore (8.1.1) has minimal mild solution ω^* and maximal mild solutions ν^* in $[\omega_0,\nu_0]$. From (8.3.1), $(H4)$, $(H5)$, and the positivity of the operator $\mathbb{U}(t,s)$, we get

$$\begin{aligned}
0 &\leqslant \nu^*(t)-\omega^*(t)=\mathcal{Q}\nu^*(t)-\mathcal{Q}\omega^*(t)\\
&= \mathbb{U}(t,0)(\mathcal{G}(\nu^*)-\mathcal{G}(\omega^*))+\int_0^t \mathbb{U}(t,\eta)[\mathcal{F}(\eta,\nu^*(\eta),\mathcal{K}\nu^*(\eta))\\
&\quad -\mathcal{F}(\eta,\omega^*(\eta),\mathcal{K}\omega^*(\eta))]d\eta\\
&\leqslant \mathcal{L}_2\mathbb{U}(t,0)(\nu^*-\omega^*)+\mathcal{L}_1\int_0^t \mathbb{U}(t,\eta)\Big[(\nu^*(\eta)-\omega^*(\eta))\\
&\quad +(\mathcal{K}\nu^*(\eta)-\mathcal{K}\omega^*(\eta))\Big]d\eta.
\end{aligned}$$

Since the positive cone is normal, therefore

$$\begin{aligned}
\|\nu^*-\omega^*\| &\leqslant \mathcal{N}\Big[\mathcal{L}_2\mathcal{M}\|\nu^*-\omega^*\|+\mathcal{M}\mathcal{L}_1 b\Big(\|\nu^*-\omega^*\|+\|\mathcal{K}\nu^*-\mathcal{K}\omega^*\|\Big)\Big]\\
&\leqslant \mathcal{N}\mathcal{M}\Big[\mathcal{L}_2+\mathcal{L}_1 b(1+bK^*)\Big]\|\nu^*-\omega^*\| = \Lambda_2\|\nu^*-\omega^*\|.
\end{aligned}$$

Since $\Lambda_2<1$, so $\|\nu^*-\omega^*\|=0$, i.e. $\nu^*(t)=\omega^*(t),\ \ \forall\, t\in J$. Thus $\nu^*=\omega^*$ is the unique mild solution for (8.1.1) in $[\omega_0,\nu_0]$. □

8.4 Example

Now we consider an example to show how our abstract results can be applied to a concrete problem. Consider the following partial differential equation :

$$\begin{cases}
x'(t,z)+a(t,z)\frac{\partial^2}{\partial z^2}x(t,z)=\frac{1}{25}\frac{e^{-t}}{1+e^t}x(t,z)+\int_0^t \frac{1}{50}e^{-s}x(s,z)ds,\\
\qquad z\in[0,\pi], \quad t\in J=[0,b],\\
x(t,0)=0,\ x(t,\pi)=0, \qquad t\in J,\\
x(0,z)=\frac{e^{x(t,z)}}{1+e^{x(t,z)}}+x_0(z), \quad z\in[0,\pi],
\end{cases} \tag{8.4.1}$$

where $\mathbb{X} = L^2([0,b]\times[0,\pi],\mathbb{R})$, $x_0(z)\in\mathbb{X}$, $a(t,z)$ is continuous function and satisfies uniform Hölder continuity in t. Define

$$\mathbb{A}(t)x(t,z) = a(t,z)\frac{\partial^2}{\partial z^2}x(t,z), \tag{8.4.2}$$

with domain

$$\mathcal{D}(\mathbb{A}) = \{w\in\mathbb{X} : w,\ \frac{\partial w}{\partial z} \text{ are absolutely continuous, } \frac{\partial^2 w}{\partial z^2}\in\mathbb{X},\ w(0)=w(\pi)=0\}.$$

Then, $-\mathbb{A}(t)$ generates a positive evolution system of bounded linear operators $\mathbb{U}(t,s)$ on $\mathbb{X}$ and satisfies the conditions $(A1)$-$(A3)$ (see [156]). Put

$$\begin{aligned} x(t)(z) &= x(t,z),\ t\in[0,b],\ z\in[0,\pi],\\ \mathcal{F}(t,x(t),\mathcal{K}x(t))(z) &= \frac{1}{25}\frac{e^{-t}}{1+e^t}x(t,z) + \int_0^t \frac{1}{50}e^{-s}x(s,z)ds,\\ (\mathcal{K}x(t))(z) &= \int_0^t \frac{1}{50}e^{-s}x(s,z)ds,\\ (\mathcal{G}x(t))(z) &= \frac{e^{x(t,z)}}{1+e^{x(t,z)}}. \end{aligned} \tag{8.4.3}$$

Then the system (8.4.1) can be rewritten into the abstract form of (8.1.1). Now, assume that $x_0(z)\geqslant 0$ for $z\in[0,\pi]$, and there exists a function $v(t,z)\geqslant 0$ such that

$$\begin{aligned} v'(t,z)+\mathbb{A}(t)v(t,z) &\geqslant \mathcal{F}\Big(t,v(t,z),\mathcal{K}v(t,z)\Big), \quad t\in J,\ z\in[0,\pi],\\ v(t,0) &= v(t,\pi)=0, \quad t\in J,\\ v(0,z) &\geqslant \mathcal{G}(v(z))+x_0(z), \quad z\in[0,\pi]. \end{aligned}$$

From the above assumptions, we have $\omega_0 = 0$ and $\nu_0 = v(t,z)$ are the lower and upper solutions for the system (8.4.1). By (8.4.3), it is easy to verify that the assumptions $(H1)$ and $(H3)$ hold. Suppose $\{x_n\}\subset[\omega_0(t),\nu_0(t)]$ be a monotone increasing sequence. For $n\leqslant m$

$$\begin{aligned} \|\mathcal{F}(t,x_m,\mathcal{K}x_m)-\mathcal{F}(t,x_n,\mathcal{K}x_n)\| &\leqslant \frac{1}{25}\Big(\|x_m-x_n\|+\|\mathcal{K}x_m-\mathcal{K}x_n\|\Big), \text{ hence}\\ \beta\Big(\mathcal{F}(t,x_n,\mathcal{K}x_n)\Big) &\leqslant \frac{1}{25}\Big(\beta(\{x_n\})+\beta(\{\mathcal{K}x_n\})\Big). \end{aligned}$$

Therefore, assumption $(H2)$ is satisfied. So, by Theorem 8.3.1, we conclude that the minimal and maximal mild solutions for (8.4.1) exist between the lower solution 0 and upper solution v.

List of Publications

List of papers published/accepted in International Refereed Journals

1. A. Meraj, D.N. Pandey, "Existence result for neutral fractional integrodifferential equations with nonlocal integral boundary conditions", *Malaya Journal of Matematik*, **6** (1) (2018), 21-27.

2. A. Meraj, D.N. Pandey, "Existence of mild solutions for fractional non instantaneous impulsive integro-differential equations with nonlocal conditions", *Arab Journal of Mathematical Sciences*, DOI: 10.1016/j.ajmsc.2018.11.002.

3. A. Meraj, D.N. Pandey, "Approximate controllability of fractional integro-differential evolution equations with nonlocal and non-instantaneous impulsive conditions", *Journal of Fractional Calculus and Applications*, **10** (2) (2019), 3-17.

4. A. Meraj, D.N. Pandey, "Existence and uniqueness of mild solution and approximate controllability of fractional evolution equations with deformable fractional derivative", *Journal of Nonlinear Evolution Equations and Applications* (Accepted).

5. A. Meraj, D.N. Pandey, "Monotone iterative technique for non-autonomous semilinear differential equations with nonlocal condition", *Demonstratio Mathematica*, **52** (1) (2019), 29-39.

6. A. Meraj, D.N. Pandey, "Existence and uniqueness of extremal mild solutions

for non-autonomous nonlocal integro-differential equations via monotone iterative technique”, *Filomat* (Accepted).

7. A. Meraj, D.N. Pandey, “Approximate controllability of non-autonomous Sobolev type integro-differential equations having nonlocal and non-instantaneous impulsive conditions”, *Indian Journal of Pure and Applied Mathematics* (Accepted).

List of papers communicated in International Refereed Journals

1. A. Meraj, D.N. Pandey, “Monotone iterative technique for non-autonomous semilinear differential equations with non-instantaneous impulses”, communicated. (*Springer Book Chapter* through the ICAME’18)

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