DIRECT AND INVERSE PROBLEMS FOR CERTAIN SUMSETS IN ADDITIVE NUMBER THEORY

Ph.D. THESIS

by

JAGANNATH BHANJA

DEPARTMENT OF MATHEMATICS
INDIAN INSTITUTE OF TECHNOLOGY ROORKEE
ROORKEE – 247 667 (INDIA)
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ROORKEE

CANDIDATE'S DECLARATION

I hereby certify that the work which is being presented in the thesis entitled "**DIRECT AND INVERSE PROBLEMS FOR CERTAIN SUMSETS IN ADDITIVE NUMBER THEORY**" in partial fulfilment of the requirements for the award of the Degree of Doctor of Philosophy and submitted in the Department of Mathematics of the Indian Institute of Technology Roorkee, Roorkee is an authentic record of my own work carried out during a period from July, 2014 to April, 2019 under the supervision of Dr. Ram Krishna Pandey, Assistant Professor, Department of Mathematics, Indian Institute of Technology Roorkee, Roorkee.

The matter presented in this thesis has not been submitted by me for the award of any other degree of this or any other Institution.

(JAGANNATH BHANJA)

This is to certify that the above statement made by the candidate is correct to the best of my knowledge.

(Ram Krishna Pandey)
Supervisor

Date:

Dedicated

to

My Parents

Abstract

The present thesis deals with the study of direct and inverse problems for certain sumsets in additive number theory. Let A and B be two nonempty finite sets of integers. Let h and r be two positive integers. The first sumset considered is the sumset of the form $A + r \cdot B$, called the sum of dilates of the sets A and B. The second sumset considered is the h-fold generalized sumset $h^{(\gamma)}A$ with $\gamma \geq 1$ an integer, which is a generalization of the h-fold sumset hA and the h-fold restricted sumset $h^{\wedge}A$. The third sumset considered is the h-fold signed sumset $h_{\pm}A$. The fourth sumset considered is the h-fold restricted signed sumset $h^{\wedge}_{\pm}A$. The last sumset considered are the subset and subsequence sums, where the subset sums are actually the unions of restricted sumsets and the subsequence sums are the unions of generalized sumsets.

The sumset $A + r \cdot B := \{a + rb : a \in A, b \in B\}$ is called the *sum of dilates* of A and B. For $r = 1$, the sum of dilates $A + r \cdot B$ coincides with the *Minkowski sumset* $A + B := \{a + b : a \in A, b \in B\}$. The direct problem for the sum of dilates $A + r \cdot B$ is to find the minimum number of elements in $A + r \cdot B$ in terms of number of elements in the sets A and B. The inverse problem for $A + r \cdot B$ is to find the structure of the finite sets A and B for which $|A + r \cdot B|$ is minimal. In this thesis, we solve both direct and inverse problems for $A + r \cdot B$.

Let $A = \{a_0, a_1, \ldots, a_{k-1}\}$ be a nonempty finite set of integers. The *h-fold sumset* hA is the set of all sums of h elements of A, and the *h-fold restricted sumset* $h^{\wedge}A$ is the set of all sums of h distinct elements of A. More precisely,

$$hA := \left\{ \sum_{i=0}^{k-1} \lambda_i a_i : \lambda_i \in \mathbb{N} \text{ for } i = 0, 1, \ldots, k-1 \text{ and } \sum_{i=0}^{k-1} \lambda_i = h \right\},$$

and

$$h^{\wedge}A := \left\{ \sum_{i=0}^{k-1} \lambda_i a_i : \lambda_i \in \{0, 1\} \text{ for } i = 0, 1, \ldots, k-1 \text{ and } \sum_{i=0}^{k-1} \lambda_i = h \right\},$$

where $\mathbb{N}$ denotes the set of nonnegative integers, and $1 \leq h \leq k$ in case of $h^{\wedge}A$.

We define the *h-fold signed sumset* of A, denoted by $h_{\pm}A$, by

$$h_{\pm}A := \left\{ \sum_{i=0}^{k-1} \lambda_i a_i : \lambda_i \in \mathbb{Z} \text{ for } i = 0, 1, \ldots, k-1 \text{ and } \sum_{i=0}^{k-1} |\lambda_i| = h \right\}.$$

We also define the *h-fold restricted signed sumset* of A, denoted by $h^{\wedge}_{\pm}A$, by

$$h^{\wedge}_{\pm}A := \left\{ \sum_{i=0}^{k-1} \lambda_i a_i : \lambda_i \in \{-1, 0, 1\} \text{ for } i = 0, 1, \ldots, k-1 \text{ and } \sum_{i=0}^{k-1} |\lambda_i| = h \right\},$$

where $1 \leq h \leq k$.

The direct problem for the sumset $h_{\pm}A$ (similarly for $h^{\wedge}_{\pm}A$) is to find the minimum number of elements in $h_{\pm}A$ (respectively, $h^{\wedge}_{\pm}A$) in terms of number of elements in A. The inverse problem for $h_{\pm}A$ (similarly for $h^{\wedge}_{\pm}A$) is to determine the structure of the finite set A for which $|h_{\pm}A|$ (respectively, $|h^{\wedge}_{\pm}A|$) is minimal. In this thesis, we study the direct and inverse problems for both the sumsets $h_{\pm}A$ and $h^{\wedge}_{\pm}A$.

In the next part of the thesis, we consider the following generalized sumset. As the name suggests, this sumset generalizes both regular sumset hA and restricted sumset $h^{\wedge}A$. For a nonempty finite set A of k integers, and for positive integers h, γ with $1 \leq \gamma \leq h \leq k\gamma$, the *h-fold generalized sumset* $h^{(\gamma)}A$ is defined by

$$h^{(\gamma)}A := \left\{ \sum_{i=0}^{k-1} \lambda_i a_i : \lambda_i \in \{0, 1, \ldots, \gamma\} \text{ for } i = 0, 1, \ldots, k-1 \text{ and } \sum_{i=0}^{k-1} \lambda_i = h \right\}.$$

Clearly, the h-fold sumset hA and the h-fold restricted sumset $h^{\wedge}A$ are particular cases of the h-fold generalized sumset $h^{(\gamma)}A$ for $\gamma = h$ and $\gamma = 1$, respectively.

Let $A = \{0, 1, \ldots, k-2, k-1+b\}$, where b is a nonnegative integer. We investigate the behaviour of $|h^{(\gamma)}A|$ with respect to b, by finding the exact cardinality of $h^{(\gamma)}A$.

Let A be a nonempty finite set of k integers. Given a subset B of A, the sum of all elements of B is called the *subset sum* of B. Let $S(A)$ be the set of all subset sums of A. The *subsequence sum* of a given sequence $\mathscr{A}$ of integers is defined in a similar way.

We consider the following subset and subsequence sums with some restriction on the number of elements of the set A (or sequence $\mathscr{A}$). For a nonnegative integer α ($\leq k$), we define $S_{\alpha}(A)$ to be the set of subset sums of all subsets of A that are of the size at least α. More precisely,

$$S_{\alpha}(A) := \left\{ \sum_{b \in B} b : B \subset A,\ |B| \geq \alpha \right\}.$$

Similarly, for a nonempty sequence $\mathscr{A} = (\underbrace{a_0, \ldots, a_0}_{r \text{ copies}}, \underbrace{a_1, \ldots, a_1}_{r \text{ copies}}, \ldots, \underbrace{a_{k-1}, \ldots, a_{k-1}}_{r \text{ copies}})$ of k distinct integers each repeating exactly r (≥ 1) times, and for a nonnegative integer α ($\leq rk$), we define $S_\alpha(r, \mathscr{A})$ to be the set of subsequence sums of all subsequences of $\mathscr{A}$ that are of the size at least α. More precisely,

$$S_\alpha(r, \mathscr{A}) := \left\{ \sum_{b \in \mathscr{B}} b : \mathscr{B} \text{ is a subsequence of } \mathscr{A} \text{ with } |\mathscr{B}| \geq \alpha \right\},$$

where $|\mathscr{B}|$ is the number of terms in the subsequence $\mathscr{B}$.

We find the minimum cardinality of the set of subset sums $S_\alpha(A)$ and the set of subsequence sums $S_\alpha(r, \mathscr{A})$. We also find the structure of the finite set A (or sequence $\mathscr{A}$) of integers for which $|S_\alpha(A)|$ (or $|S_\alpha(r, \mathscr{A})|$) is minimal.

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Chapter 1

Introduction

1.1 Basic definitions

Additive number theory is primarily the study of sumsets of subsets of additive abelian groups. An additive abelian group is a commutative group under addition. Let G be an additive abelian group. Let $h \geq 2$ be an integer. Let $A_1, A_2, \ldots, A_h$ be nonempty subsets of G. The *Minkowski sumset* or the *regular sumset* or simply, the *sumset* $A_1 + A_2 + \cdots + A_h$ is defined by $A_1 + A_2 + \cdots + A_h := \{a_1 + a_2 + \cdots + a_h : a_i \in A_i \text{ for } i = 1, 2, \ldots, h\}$. Similarly, the *restricted sumset* $A_1 \hat{+} A_2 \hat{+} \cdots \hat{+} A_h$ is defined by $A_1 \hat{+} A_2 \hat{+} \cdots \hat{+} A_h := \{a_1 + a_2 + \cdots + a_h : a_i \in A_i \text{ for } i = 1, 2, \ldots, h, \text{ and } a_i \neq a_j \text{ for } i \neq j\}$. If $A_i = A$ for $i = 1, 2, \ldots, h$, then the sumset $A_1 + A_2 + \cdots + A_h$ is denoted by hA and the restricted sumset $A_1 \hat{+} A_2 \hat{+} \cdots \hat{+} A_h$ is denoted by $h^{\wedge}A$. Thus, the *h-fold sumset* hA is the set of all sums of h elements of A, and the *h-fold restricted sumset* $h^{\wedge}A$ is the set of all sums of h distinct elements of A.

Let $A = \{a_0, a_1, \ldots, a_{k-1}\}$ be a nonempty subset of G. Let h and γ be positive integers such that $1 \leq \gamma \leq h \leq k\gamma$. Observe that, in the sumset hA an element of the set A appearing in a h-fold sum may be repeated at most h times, while in the sumset $h^{\wedge}A$ an element of the set A may repeat at most once. Consider the following sumset, that generalizes both regular sumset and restricted sumset. The *h-fold generalized sumset*, denoted by $h^{(\gamma)}A$ is defined by

$$h^{(\gamma)}A := \left\{ \sum_{i=0}^{k-1} \lambda_i a_i : 0 \leq \lambda_i \leq \gamma \text{ for } i = 0, 1, \ldots, k-1 \text{ and } \sum_{i=0}^{k-1} \lambda_i = h \right\}.$$

So, the h-fold generalized sumset $h^{(\gamma)}A$ is the set of all sums of h elements of A, where each element appearing in a h-fold sum may be repeated at most γ times. Therefore, hA and $h^{\wedge}A$ are particular cases of $h^{(\gamma)}A$ for $\gamma = h$ and $\gamma = 1$, respectively.

Let $h \geq 2$, and let $A_1, A_2, \ldots, A_h$ be nonempty subsets of G. Let $\alpha_1, \alpha_2, \ldots, \alpha_h$ be positive integers. The sumset $\alpha_1 \cdot A_1 + \alpha_2 \cdot A_2 + \cdots + \alpha_h \cdot A_h := \{\alpha_1 a_1 + \alpha_2 a_2 + \cdots + \alpha_h a_h : a_i \in A_i \text{ for } i = 1, 2, \ldots, h\}$ is called the *sum of dilates* of the sets $A_1, A_2, \ldots, A_h$. Clearly, if $\alpha_i = 1$ for $i = 1, 2, \ldots, h$, then the sum of dilates $\alpha_1 \cdot A_1 + \alpha_2 \cdot A_2 + \cdots + \alpha_h \cdot A_h$ coincides with the Minkowski sumset $A_1 + A_2 + \cdots + A_h$.

There are some other special type of sumsets, for example, subset sums and subsequence sums. These sumsets are defined as follows. Let A be a nonempty finite subset of G. Given a subset B of A, the sum of all elements of B is called the *subset sum* of B. The set of all subset sums of A is defined by

$$S(A) := \left\{ \sum_{b \in B} b : B \subset A \right\},$$

where $s(\emptyset) = 0$.

The subsequence sum of a given sequence of elements of G is defined in a similar way. Let $\mathscr{A} = (\underbrace{a_0, \ldots, a_0}_{r_0 \text{ copies}}, \underbrace{a_1, \ldots, a_1}_{r_1 \text{ copies}}, \ldots, \underbrace{a_{k-1}, \ldots, a_{k-1}}_{r_{k-1} \text{ copies}})$ be a nonempty sequence in G with k distinct elements, where $r_i \geq 1$ for $i = 0, 1, \ldots, k-1$. Given a subsequence $\mathscr{B}$ of $\mathscr{A}$, the sum of all terms of $\mathscr{B}$ is called the *subsequence sum* of $\mathscr{B}$. The set of all subsequence sums of $\mathscr{A}$ is defined by

$$S(\bar{r}, \mathscr{A}) := \left\{ \sum_{b \in \mathscr{B}} b : \mathscr{B} \text{ is a subsequence of } \mathscr{A} \right\},$$

where $\bar{r} = (r_0, r_1, \ldots, r_{k-1})$.

Two main problems associated with these sumsets are the direct and inverse problems. A *direct problem* is a problem where we have the information about the set(s) or sequence(s) and we try to describe the sumset. An example of a direct theorem is *Lagrange's four-square theorem*, which states that "every nonnegative integer can be written as the sum of four squares". Thus, if A is the set of all nonnegative squares, then the sumset $4A$ is the set of all nonnegative integers. An *inverse problem* is a problem where we have the information about the sumset, from which we try to determine the structure and the properties of the underlying set(s) or sequence(s). An example of an inverse theorem is "if A is a nonempty finite set of k integers such that the 2-fold sumset $2A$ contains exactly $2k-1$ integers, then A is an arithmetic progression".

1.2 Motivation and objectives

Let $r \geq 1$ be an integer. Let A and B be nonempty finite sets of integers. The sumset $A + r \cdot B := \{a + rb : a \in A, b \in B\}$ is called the *sum of dilates* of A and B. If $r = 1$, then the sum of dilates $A + r \cdot B$ coincides with the Minkowski sumset $A + B$. For $B = A$ we have $A + r \cdot A \subset A + rA = (r+1)A$. Moreover, if A is an arithmetic progression with $|A| \geq r$, then $A + r \cdot A = (r+1)A$. Further, inverse results for sums of dilates in integers have a natural connection with inverse results in noncommutative groups such as the *Baumslag–Solitar* group $BS(1,n) := \langle a, b : ab = ba^n \rangle$ (see [39, 40]). So, it is a natural problem to find the minimum cardinality of the sum of dilates $A + r \cdot B$ in terms of cardinalities of A and B. It is also equally important to describe the sets A and B for which the minimum cardinality of $A + r \cdot B$ is achieved. In fact, several results about the minimum cardinality of the sum of dilates and its inverse that if the minimum cardinality is achieved, then the characterization of individual sets have been obtained by now. For example, for $r = 1, 2, 3$ the direct and inverse problems for $A + r \cdot A$ are completely settled (see [27, 39, 82]). We study similar direct and inverse problems for the sum of dilates $A + r \cdot B$ in Chapter 2.

Let $h \geq 1$, and let $A = \{a_0, a_1, \ldots, a_{k-1}\}$ be a finite subset of an additive abelian group G. The h-fold sumset hA and the h-fold restricted sumset $h^{\wedge}A$ respectively, are

$$hA := \left\{ \sum_{i=0}^{k-1} \lambda_i a_i : \lambda_i \in \mathbb{N} \text{ for } i = 0, 1, \ldots, k-1 \text{ and } \sum_{i=0}^{k-1} \lambda_i = h \right\},$$

and

$$h^{\wedge}A := \left\{ \sum_{i=0}^{k-1} \lambda_i a_i : \lambda_i \in \{0,1\} \text{ for } i = 0, 1, \ldots, k-1 \text{ and } \sum_{i=0}^{k-1} \lambda_i = h \right\},$$

where $\mathbb{N} = \{0, 1, 2, \ldots\}$, and $1 \leq h \leq k$ in case of $h^{\wedge}A$.

Observe that, in the sumsets hA and $h^{\wedge}A$ the variables λ_i only assume nonnegative integer values. On the other hand, we define the following h-fold sumset in which λ_i may take negative integer values. Define the *h-fold signed sumset* of A, denoted by $h_{\pm}A$, by

$$h_{\pm}A := \left\{ \sum_{i=0}^{k-1} \lambda_i a_i : \lambda_i \in \mathbb{Z} \text{ for } i = 0, 1, \ldots, k-1 \text{ and } \sum_{i=0}^{k-1} |\lambda_i| = h \right\}.$$

Clearly, $hA \cup h(-A) \subset h_{\pm}A \subset h(A \cup -A)$. Thus, if A is a symmetric set, i.e., for all $a \in A$, $-a \in A$, then $h_{\pm}A = hA$. Further, the signed sumset have a connection with some other problems in number theory, such as the "independence number" of a subset A of a group G (see [11]), and the "diameter" of G with respect to the subset A (see [63, 64]). So, it is natural to find the minimum cardinality of $h_{\pm}A$ in terms of cardinality of A, and to classify the underlying sets for which the minimum cardinality of $h_{\pm}A$ is achieved. Recently, Bajnok and Matzke [9, 10] have studied the direct and inverse problems for $h_{\pm}A$ in some finite abelian groups. Inspired by Bajnok and Matzke's results, we study both direct and inverse problems for $h_{\pm}A$ in the group of integers in Chapter 3.

Motivated by this signed sumset $h_{\pm}A$, we further define the *h-fold restricted signed sumset* of A, denoted by $h^{\wedge}_{\pm}A$, by

$$h^{\wedge}_{\pm}A := \left\{ \sum_{i=0}^{k-1} \lambda_i a_i : \lambda_i \in \{-1,0,1\} \text{ for } i = 0,1,\ldots,k-1 \text{ and } \sum_{i=0}^{k-1} |\lambda_i| = h \right\},$$

where $1 \leq h \leq k$. Similar to the signed sumset $h_{\pm}A$, we study the direct and inverse problems for $h^{\wedge}_{\pm}A$ in the group of integers in Chapter 4.

Let A be a nonempty finite set of k integers. Let h, γ be positive integers such that $1 \leq \gamma \leq h \leq k\gamma$. Let $m = \lfloor h/\gamma \rfloor$. Finding the exact cardinality of h-fold sumsets hA, $h^{\wedge}A$ and $h^{(\gamma)}A$ of a given set A is a difficult problem. But, it may be comparatively easy to find the exact cardinality of these sumsets in some special cases. For example, if $A = \{0,1,\ldots,k-1\}$, then $hA = \{0,1,\ldots,h(k-1)\}$, $h^{\wedge}A = \{\frac{h(h-1)}{2}, \frac{h(h-1)}{2}+1,\ldots,hk-\frac{h(h+1)}{2}\}$, and $h^{(\gamma)}A = \{\frac{m\gamma(m-1)}{2} + (h-m\gamma)m, \frac{m\gamma(m-1)}{2}+(h-m\gamma)m+1,\ldots,mk\gamma-\frac{m\gamma(m+1)}{2}+(h-m\gamma)(k-m-1)\}$. Hence, $|hA| = hk-h+1$, $|h^{\wedge}A| = hk-h^2+1$, and $|h^{(\gamma)}A| = m\gamma(k-m)+(h-m\gamma)(k-2m-1)+1$. In 1996, Nathanson [81] proved that, "if $A = \{0,1,\ldots,k-2,k-1+b\}$, where b is a nonnegative integer, then $|hA|$ is a strictly increasing piecewise-linear function of b for $0 \leq b \leq (h-1)(k-2)$ and that $|hA|$ is constant for $b \geq (h-1)(k-2)$". It is natural to derive a Nathanson's type theorems for the restricted sumset $h^{\wedge}A$ and the generalized sumset $h^{(\gamma)}A$ too. We study these problems in Chapter 5.

The subset and subsequence sums are fundamental in additive number theory, in particular, in the study of the zero-sum constants, such as *Noether number*, *Davenport constant* and some variations of these constants [1, 2, 3, 4, 5, 6, 13, 14, 16, 17, 18, 28, 36, 41, 42, 43, 55, 83, 93]. It is necessary to bound the subset and subsequence sums in order to ensure the existence of a nontrivial zero-subset sum or nontrivial zero-subsequence sum of a set (or sequence). In these problems, apart from the regular subset and subsequence sums, the subset and subsequence sums with some restriction on the number of elements have been appeared several times (see [20, 44, 45, 46, 49, 50, 54]). Very recently, Balandraud [12] have studied similar subset sums and obtained the minimum cardinality of these subset sums in finite fields. This motivates us to study the following subset sums, as that considered by Balandraud, in the group of integers.

Let A be a nonempty set of k integers. For a nonnegative integer α ($\leq k$), let $S_\alpha(A)$ be the set of subset sums of all subsets of A that are of the size at least α. More precisely,

$$S_\alpha(A) := \left\{ \sum_{b \in B} b : B \subset A,\ |B| \geq \alpha \right\},$$

Clearly, $S_\alpha(A) \subset S(A)$. Moreover, $S_\alpha(A)$ may be considered as a generalization of the usual subset sums $S(A)$, as for $\alpha = 0$, we have $S_\alpha(A) = S(A)$. Therefore, it makes sense to study both direct and inverse problems for $S_\alpha(A)$.

Since a finite set is a particular case of a finite sequence, the subset sum problems may be viewed as a particular case of the subsequence sum problems. This motivates us to study the analogues subsequence sums of the subset sums $S_\alpha(A)$. We study these subset and subsequence sum problems in Chapter 6.

1.3 Notation

Throughout the thesis, we follow the following notation. Let $\mathbb{N} = \{0, 1, 2, \ldots\}$. For a finite set S, let $|S|$ be the number of elements in S. Let $\bar{S}$ denote the complement of S in G. For an integer c, let $c \cdot S = \{cs : s \in S\}$, $c + S = \{c + s : s \in S\}$ and $c - S = \{c - s : s \in S\}$. We say that the set S is symmetric, if for all $s \in S$, $-s \in S$. For a real number x, we use the standard notation $\lfloor x \rfloor$ for the greatest integer less than or equal to x, and $\lceil x \rceil$ for the smallest integer greater than or equal to x. We also agree with the convention that $\binom{a}{b} = 0$, if a and b are two nonnegative integers with $a < b$. For any two integers a, b ($b \geq a$), we let the interval of integers $[a, b] = (a, a+1, \ldots, b)$. For a nonempty set $A = \{a_0, a_1, \ldots, a_{k-1}\}$ of integers with $a_0 < a_1 < \cdots < a_{k-1}$, we let $d(A) := \gcd(a_1 - a_0, a_2 - a_0, \ldots, a_{k-1} - a_0)$, $\ell(A) := \max(A) - \min(A)$, the length of A, and $h_A := \ell(A) + 1 - |A|$ the number of holes in A.

Let $\mathscr{A} = (\underbrace{a_0, \ldots, a_0}_{r_0 \text{ copies}}, \underbrace{a_1, \ldots, a_1}_{r_1 \text{ copies}}, \ldots, \underbrace{a_{k-1}, \ldots, a_{k-1}}_{r_{k-1} \text{ copies}})$ be a nonempty sequence in G with k distinct elements, where $r_i \geq 1$ for $i = 0, 1, \ldots, k-1$. We denote this sequence alternatively by

$\mathscr{A} = (a_0, a_1, \ldots, a_{k-1})_{\bar{r}}$, where $\bar{r} = (r_0, r_1, \ldots, r_{k-1})$ be the ordered k-tuple. For a positive integer c and for a sequence $\mathscr{A} = (a_0, a_1, \ldots, a_{k-1})_{\bar{r}}$, we let

$$c \cdot \mathscr{A} = (ca_0, ca_1, \ldots, ca_{k-1})_{\bar{r}}.$$

For integers a, b $(b \geq a)$, we let the sequence interval $[a,b]_{\bar{r}}$ to be the sequence $(a, a+1, \ldots, b)_{\bar{r}}$. Some other specific notation are introduced at appropriate places.

1.4 Some preliminary results

The direct and inverse problems for the sumsets (mentioned in section 1.1 and 1.2) are extensively studied in the past. The first result in this direction dates back to 1813, and is due to Cauchy [24]. But, the result of Cauchy was not familiar to the mathematical community, until Davenport [29] rediscovered Cauchy's result in 1935 (see also [30]). Then, it started getting attention of several mathematicians, who are mainly responsible for developing the subject called *Additive number theory*, includes Chowla [25], Kneser [65], Vosper [105], Erdös [37], Heilbronn [37], Mann [73, 74], Freiman [38], Kemperman [62], Szemerédy [98, 99], Dias da Silva [32], Hamidoune [32], Lev [67, 71], Alon [7, 8], Nathanson [81], Ruzsa [7, 8], and many others. Most of these classical results may be found in the text books of Nathanson [81] and Mann [74]. Freiman's monograph [38] provides a structural approach to these problems in additive number theory. Tao and Vu's [100] book *Additive Combinatorics* provides analytical aspects of these problems. While these books are mainly responsible to popularize the subject of additive number theory, there are other books of Geroldinger and Halter-Koch [47], Geroldinger and Ruzsa [48], Grynkiewicz [51], Halberstam and Roth [52], which are also equally informative for this subject. In the last few decades, there have been remarkable progress in the subject of additive number theory. Now, it is one of the active area of research. Bellow, we discuss some direct and inverse results in additive number theory related to our work in the present thesis.

1.4.1 Direct and inverse results for Minkowski sumset

The direct problem for the h-fold sumset hA is to find the minimum number of elements in hA in terms of number of elements in A. The inverse problem for hA is to find the structure of the finite set A for which $|hA|$ is minimal. The direct and inverse problems for the sumset

$A_1 + A_2 + \cdots + A_h$ are defined in a similar way. The following direct theorem is for the sumset hA, when A is a finite set of integers.

Theorem 1.4.1. *[81, Theorem 1.3] Let $h \geq 1$, and let A be a nonempty finite set of integers. Then*

$$|hA| \geq h|A| - h + 1.$$

This theorem is a particular case of the following direct theorem for the sumset $A_1 + A_2 + \cdots + A_h$.

Theorem 1.4.2. *[81, Theorem 1.4] Let $h \geq 1$, and let $A_1, A_2, \ldots, A_h$ be nonempty finite sets of integers. Then*

$$|A_1 + A_2 + \cdots + A_h| \geq |A_1| + |A_2| + \cdots + |A_h| - h + 1.$$

The next theorem is an inverse theorem for hA in the group of integers.

Theorem 1.4.3. *[81, Theorem 1.6] Let $h \geq 2$. Let A be a nonempty finite set of integers such that*

$$|hA| = h|A| - h + 1.$$

Then A is an arithmetic progression.

This theorem is a particular case of the following inverse theorem for the sumset $A_1 + A_2 + \cdots + A_h$.

Theorem 1.4.4. *[81, Theorem 1.5] Let $h \geq 2$. Let $A_1, A_2, \ldots, A_h$ be nonempty finite sets of integers such that*

$$|A_1 + A_2 + \cdots + A_h| = |A_1| + |A_2| + \cdots + |A_h| - h + 1.$$

Then $A_1, A_2, \ldots, A_h$ are arithmetic progressions with the same common difference.

The following theorem due to Cauchy [24] in 1813, is believed to be one of the oldest and classical theorem in additive number theory, which finds the minimum cardinality of the sumset $A + B$, where A and B are nonempty subsets of residue classes modulo a prime. In 1935, Davenport [29] rediscovered Cauchy's result, and later, in 1947, Davenport acknowledged Cauchy's work (see [30]). This result is now known as the *Cauchy-Davenport theorem*.

Theorem 1.4.5 (Cauchy-Davenport [24, 29])**.** *Let q be a prime number, and let A, B be two nonempty subsets of the group $\mathbb{Z}/q\mathbb{Z}$. Then*

$$|A+B| \geq \min\{q, |A|+|B|-1\}.$$

The following theorem is the h-fold generalization of this theorem.

Theorem 1.4.6. *[81, Theorem 2.3] Let $h \geq 2$, and q be a prime number. Let A be a nonempty subset of the group $\mathbb{Z}/q\mathbb{Z}$. Then*

$$|hA| \geq \min\{q, h|A|-h+1\}.$$

Immediately, after Davenport, Chowla [25] extended the Cauchy-Davenport theorem to the group $\mathbb{Z}/m\mathbb{Z}$, where m may be a composite integer. Several generalizations of this famous Cauchy-Davenport theorem are available now. Some of them are due to Pillai [86], Shatrowsky [94], Kneser [65], Pollard [90, 91], Mann [73], Hamidoune [53], Devos et al. [31], Karoly [60], and Wheeler [107].

The following theorem due to Vosper [105] is the inverse theorem of the Cauchy-Davenport theorem.

Theorem 1.4.7 (Vosper [105])**.** *Let q be a prime number, and let A and B be nonempty subsets of $\mathbb{Z}/q\mathbb{Z}$ with $A+B \neq \mathbb{Z}/q\mathbb{Z}$. Then*

$$|A+B| = \min\{q, |A|+|B|-1\}$$

if and only if at least one of the following three conditions holds:

(i) $\min(|A|, |B|) = 1$,

(ii) $|A+B| = q-1$ *and* $B = \overline{c-A}$*, where* $\{c\} = (\mathbb{Z}/q\mathbb{Z}) \setminus (A+B)$*,*

(iii) A and B are arithmetic progressions with the same common difference.

Some generalizations of Vosper's theorem are due to Kemperman [62], Brailovsky and Freiman [21], Karoly [60], and Hamidoune [55, 56].

Several partial results about the minimum cardinality of the sumsets and its inverse that if the minimum cardinality is achieved, then the characterization of individual sets have been obtained by now. Eliahou, Kervaire and Plagne (see [34, 35, 87, 88, 89]) finally settled the direct problem by obtaining the minimum cardinality of h-fold sumset hA in abelian groups. The theorem of Eliahou, Kervaire and Plagne is given bellow.

Theorem 1.4.8 (Eliahou, Kervaire and Plagne [35]). *Let G be an abelian group of order n, and let A be a nonempty subset of G. Then*

$$|hA| \geq \min\{(h\lceil |A|/d\rceil - h + 1)\cdot d :\ d \in D(n)\},$$

where $D(n)$ is the set of positive divisors of n.

The problem of finding the structure of sets A and B such that $|A+B| \leq f(|A|,|B|)$, where $f(|A|,|B|)$ is a small diversion from the usual lower bound, is called an *extended inverse problem* for the sumset $A+B$. The following theorem is an example of an extended inverse theorem, popularly known as the *Freiman's* $3k-4$ *theorem* [38, 81].

Theorem 1.4.9 (Freiman [38]). *Let A be a finite set of k (≥ 3) integers. If $|2A| = 2k-1+b \leq 3k-4$, then A is a subset of an arithmetic progression of length $k+b \leq 2k-3$.*

The following theorem is a combined result of Lev and Smeliansky [71], and Stanchescu [95], which generalizes the Freiman's $3k-4$ theorem to the sumset $A+B$.

Theorem 1.4.10 (Lev, Smeliansky and Stanchescu [71, 95]). *Let A and B be finite subsets of $\mathbb{N}$ such that $0 \in A\cap B$. Define*

$$\delta_{A,B} = \begin{cases} 1, & \text{if } \ell(A) = \ell(B); \\ 0, & \text{if } \ell(A) \neq \ell(B). \end{cases}$$

Then the following statements hold:

(i) If $\ell(A) = \max(\ell(A),\ell(B)) \geq |A|+|B|-1-\delta_{A,B}$ and $d(A) = 1$, then

$$|A+B| \geq |A|+2|B|-2-\delta_{A,B}.$$

(ii) If $\max(\ell(A),\ell(B)) \leq |A|+|B|-2-\delta_{A,B}$, then

$$|A+B| \geq (|A|+|B|-1)+\max(h_A,h_B) = \max(\ell(A)+|B|,\ell(B)+|A|).$$

1.4.2 Direct and inverse results for restricted sumset

The direct problem for the h-fold restricted sumset $h^{\wedge}A$ is to find the minimum number of elements in $h^{\wedge}A$ in terms of number of elements in A. The inverse problem for $h^{\wedge}A$ is to find the structure of the finite set A for which $|h^{\wedge}A|$ is minimal. The following theorem due to Nathanson [80] is a direct theorem for $h^{\wedge}A$ in the group of integers.

Theorem 1.4.11. *[80, Theorem 1] Let A be a nonempty finite set of integers, and let* $1 \leq h \leq |A|$. *Then*

$$|h^{\wedge}A| \geq h|A| - h^2 + 1.$$

This lower bound is best possible.

The following theorem due to Nathanson [80] is an inverse theorem for $h^{\wedge}A$ in the group of integers.

Theorem 1.4.12. *[80, Theorem 2] Let A be a finite set of integers with* $|A| \geq 5$. *Let* $2 \leq h \leq |A| - 2$. *If*

$$|h^{\wedge}A| = h|A| - h^2 + 1,$$

then A is an arithmetic progression.

Here, we note that not all extremal sets, i.e., the sets where the minimum cardinality of $h^{\wedge}A$ is achieved, are arithmetic progressions.

The following theorem due to Dias da Silva and Hamidoune [32] is a direct theorem for the restricted sumset $h^{\wedge}A$ in the group $\mathbb{Z}/q\mathbb{Z}$, where q is a prime number.

Theorem 1.4.13 (Dias da Silva and Hamidoune [32])**.** *Let A be a nonempty subset of the group* $\mathbb{Z}/q\mathbb{Z}$*, and let* $1 \leq h \leq |A|$. *Then*

$$|h^{\wedge}A| \geq \min\{q, h|A| - h^2 + 1\}.$$

This theorem is popular as the *Erdös-Heilbronn Conjecture*, as it was first conjectured by Erdös and Heilbronn [37] in 1964. Three decades later, in 1994, Dias da Silva and Hamidoune [32] first proved this conjecture in its general form using some ideas from the exterior algebra and representation theory. A year later, it was re-proved by Alon, Nathanson and Ruzsa [7, 8] using the polynomial method. In the last few years, a large number of articles have been published concerning possible extensions and generalizations of the Erdös-Heilbronn conjecture. Some of them are due to Hou and Sun [57], Liu and Sun [72], Pan and Sun [84, 85], Sun and Zhao [96], Lev [68, 69, 70], Karolyi [59], Balister and Wheeler [15].

Till date, several attempts are made towards the inverse theorem of the Erdös-Heilbronn conjecture. Some of them are due to Karolyi [61], Vu and Wood [106], and Bilu, Lev and Ruzsa [19].

1.4.3 Direct and inverse results for generalized sumset

The direct problem for the h-fold generalized sumset $h^{(\gamma)}A$ is to find the minimum number of elements in $h^{(\gamma)}A$ in terms of number of elements in A. The inverse problem for $h^{(\gamma)}A$ is to find the structure of the finite set A for which $|h^{(\gamma)}A|$ is minimal. The following two theorems are direct and inverse theorems for $h^{(\gamma)}A$, respectively, and are due to Mistri and Pandey [75].

Theorem 1.4.14. *[75, Theorem 2.1] Let A be a finite set of k integers. Let γ and h be positive integers such that $1 \leq \gamma \leq h \leq k\gamma$. Set $m = \lfloor h/\gamma \rfloor$. Then*

$$|h^{(\gamma)}A| \geq m\gamma(k-m) + (h-m\gamma)(k-2m-1) + 1.$$

Theorem 1.4.15. *[75, Theorem 3.1, Theorem 3.2] Let $k \geq 3$. Let γ and $h \geq 2$ be integers such that $1 \leq \gamma \leq h \leq k\gamma - 2$ and $(k,h,\gamma) \neq (4,2,1)$. Set $m = \lfloor h/\gamma \rfloor$. If A is a finite set of k integers such that*

$$|h^{(\gamma)}A| = m\gamma(k-m) + (h-m\gamma)(k-2m-1) + 1,$$

then A is an arithmetic progression.

Clearly, Theorem 1.4.1 and Theorem 1.4.11 are particular cases of Theorem 1.4.14, and Theorem 1.4.3 and Theorem 1.4.12 are particular cases of Theorem 1.4.15. Furthermore, similar to the restricted sumset $h^{\wedge}A$, in genaralized sumset $h^{(\gamma)}A$ also, not all extremal sets are arithmetic progressions.

The following theorem due to Monopoli [79], is a direct theorem for $h^{(\gamma)}A$ in the group $\mathbb{Z}/q\mathbb{Z}$, where q is a prime number.

Theorem 1.4.16. *[79, Theorem 1.3] Let A be a nonempty subset of the group $\mathbb{Z}/q\mathbb{Z}$ with $|A| = k$. Let γ and h be positive integers such that $1 \leq \gamma \leq h \leq k\gamma$. Set $m = \lfloor h/\gamma \rfloor$. Then*

$$|h^{(\gamma)}A| \geq \min\{q, m\gamma(k-m) + (h-m\gamma)(k-2m-1)+1\}.$$

1.4.4 Direct and inverse results for sum of dilates

The direct problem for the sum of dilates $A + r \cdot B$ is to find the minimum number of elements in $A + r \cdot B$ in terms of number of elements in A and B. The inverse problem for $A + r \cdot B$ is to find the structure of the finite sets A and B for which $|A + r \cdot B|$ is minimal. Till date, several direct and inverse results for the sum of dilates $A + r \cdot B$ are known, here we mention few of them.

For $r = 1$, the sum of dilates $A + r \cdot B$ coincides with the sumset $A + B$. So, the direct and inverse theorems for the sum of dilates $A + 1 \cdot B$ are the direct and inverse theorems for the sumset $A + B$. For $r = 2$ and $B = A$, Nathanson [82], Cilleruelo et al. [27], and Freiman et al. [39] completely solved the direct problem by showing $|A + 2 \cdot A| \geq 3|A| - 2$. Further, Nathanson [82] settled the inverse problem for $r = 2$. Their results can be summarized by the following theorem.

Theorem 1.4.17. *[27, Theorem 1.1] For any nonempty finite set A, we have $|A+2\cdot A| \geq 3|A|-2$. Furthermore, if $|A+2\cdot A| = 3|A|-2$, then A is an arithmetic progression or a singleton.*

The sharp lower bound and the description of the extremal set(s) for the case $r = 3$ have been settled by Cilleruelo et al. [27]. In particular, they proved the following theorem.

Theorem 1.4.18. *[27, Theorem 1.2] For any nonempty finite set A, we have $|A+3\cdot A| \geq 4|A|-4$. Furthermore, if the equality holds, then $A = \{0,1,3\}$ or $A = \{0,1,4\}$ or $A = 3\cdot\{0,1,\ldots,n\} + \{0,1\}$ or A is an affine transformation of any of these sets.*

The following theorem due to Du et al. [33], solves the direct problem for the sum of dilates $A + 4 \cdot A$.

Theorem 1.4.19. *[33, Theorem 3] For any finite set A of integers with $|A| \geq 5$, we have*

$$|A+4\cdot A| \geq 5|A|-6.$$

For $r \geq 3$, Nathanson [82] obtained the uniform lower bound, "$|A + r \cdot A| \geq \left\lfloor \frac{7}{2}|A| - \frac{5}{2} \right\rfloor$". Later, Freiman et al. [39] extended Nathanson's result by proving the following theorem.

Theorem 1.4.20. *[39, Theorem 5] Let A be a nonempty finite set of integers, and let $r \geq 3$. Then*

$$|A + r \cdot A| \geq 4|A| - 4.$$

Several other direct and inverse results for $A + r \cdot A$ are available for large sets and fixed r. For example, Cilleruelo et al. [26] proved the following theorem.

Theorem 1.4.21. *[26, Corollary 1.3] Let r be a prime number. Let A be a finite set of integers with $|A| \geq 3(r-1)^2(r-1)!$. Then*

$$|A + r \cdot A| \geq (r+1)|A| - \left\lceil \frac{r(r+2)}{4} \right\rceil.$$

Moreover, up to affine transformations, equality holds only if $A = r \cdot \{0, 1, \ldots, n\} + \{0, 1, \ldots, (r-1)/2\}$, for some n.

Later, Du et al. [33], generalized this result to product of primes and to powers of a fixed prime.

1.4.5 Direct and inverse results for subset and subsequence sums

The direct problem for the subset sums $S(A)$ is to find the minimum number of elements in $S(A)$ in terms of number of elements in A. The inverse problem for $S(A)$ is to determine the structure of the finite set A for which $|S(A)|$ is minimal. For a nonempty finite sequence $\mathscr{A}$ of integers, the direct and inverse problems for the subsequence sums $S(\bar{r}, \mathscr{A})$ are defined in a similar way. The following theorem due to Nathanson [80] is a direct theorem for $S(A)$ in the group of integers.

Theorem 1.4.22. *[80, Theorem 3] If A is a nonempty finite set of positive integers, then*

$$|S(A)| \geq \binom{|A|+1}{2} + 1.$$

If A is a nonempty finite set of nonnegative integers with $0 \in A$, then

$$|S(A)| \geq \binom{|A|}{2} + 1.$$

These lower bounds are best possible.

The following inverse theorem of Nathanson [80] classifies all possible sets where the lower bound for $S(A)$ is exact.

Theorem 1.4.23. *[80, Theorem 5] Let A be a finite set of positive integers. If* $|A| \geq 2$ *and* $|S(A)| = \binom{|A|+1}{2} + 1$*, then*

$$A = d \cdot [1, |A|],$$

for some positive integer d.

Let A be a finite set of nonnegative integers with $0 \in A$*. If* $|A| \geq 3$ *and* $|S(A)| = \binom{|A|}{2} + 1$*, then*

$$A = d \cdot [0, |A| - 1],$$

for some positive integer d.

The aforementioned direct and inverse theorems for the subset sums $S(A)$ have been generalized to the following direct and inverse theorems for the subsequence sums $S(\bar{r}, \mathscr{A})$ by Mistri and Pandey [77].

Theorem 1.4.24. *[77, Theorem 3.1] Let* $k \geq 2$*. Let* $\mathscr{A} = (a_1, a_2, \ldots, a_k)_{\bar{r}}$ *be a nonempty sequence of integers with* $a_1 < a_2 < \cdots < a_k$ *and* $\bar{r} = (r_1, r_2, \ldots, r_k)$*, where* $r_i \geq 1$ *for* $i = 1, 2, \ldots, k$*. If* $a_1 > 0$*, then*

$$|S(\bar{r}, \mathscr{A})| \geq \sum_{i=1}^{k} i r_i + 1.$$

If $a_1 = 0$*, then*

$$|S(\bar{r}, \mathscr{A})| \geq \sum_{i=1}^{k} (i-1) r_i + 1.$$

These lower bounds are best possible.

Theorem 1.4.25. *[77, Theorem 3.2] Let* $k \geq 5$*, and let* $\bar{r} = (r_1, r_2, \ldots, r_k)$*, where* $r_i \geq 1$ *for* $i = 1, 2, \ldots, k$*. If* $\mathscr{A} = (a_1, a_2, \ldots, a_k)_{\bar{r}}$ *is a nonempty sequence of integers with* $0 < a_1 < a_2 < \cdots < a_k$*, and* $|S(\bar{r}, \mathscr{A})| = \sum_{i=1}^{k} i r_i + 1$*, then*

$$\mathscr{A} = a_1 \cdot [1, k]_{\bar{r}}.$$

If $\mathscr{A} = (a_1, a_2, \ldots, a_k)_{\bar{r}}$ *is a nonempty sequence of integers with* $0 = a_1 < a_2 < \cdots < a_k$*, and* $|S(\bar{r}, \mathscr{A})| = \sum_{i=1}^{k} (i-1) r_i + 1$*, then*

$$\mathscr{A} = a_2 \cdot [0, k-1]_{\bar{r}}.$$

Clearly, Theorem 1.4.22 and Theorem 1.4.23 are particular cases of Theorem 1.4.24 and Theorem 1.4.25, respectively, for $\bar{r} = (1,1,\dots,1)$.

Very recently, Jiang and Li [58] have settled the direct and inverse problems for $S(\bar{r},\mathscr{A})$ in the remaining case, i.e., when the sequence $\mathscr{A}$ contains positive integers, negative integers and/or zero. The following two theorems due to Jiang and Li [58], are direct theorems for the subsequence sums $S(\bar{r},\mathscr{A})$.

Theorem 1.4.26. *[58, Theorem 3.1] Let k, l be two positive integers and* $\bar{r} = (r_{-l},\dots,r_{-1},r_1,\dots,r_k)$, *with* $r_i \geq 1$ *for* $i = -l,\dots,-1,1,\dots,k$. *Let* $\mathscr{A} = (a_{-l},\dots,a_{-1},a_1,\dots,a_k)_{\bar{r}}$ *be a nonempty sequence of integers with* $a_{-l} < \cdots < a_{-1} < 0 < a_1 < \cdots < a_k$. *Then*

$$|S(\bar{r},\mathscr{A})| \geq \sum_{i=1}^{k} ir_i + \sum_{i=1}^{l} ir_{-i} + 1.$$

This lower bound is best possible.

Theorem 1.4.27. *[58, Corollary 3.2] Let k, l be two positive integers and* $\bar{r} = (r_{-l},r_{-l+1},\dots,r_k)$, *with* $r_i \geq 1$ *for* $i = -l,-l+1,\dots,k$. *Let* $\mathscr{A} = (a_{-l},a_{-l+1},\dots,a_k)_{\bar{r}}$ *be a nonempty sequence of integers with* $a_{-l} < a_{-l+1} < \cdots < a_{-1} < a_0 = 0 < a_1 < \cdots < a_k$. *Then*

$$|S(\bar{r},\mathscr{A})| \geq \sum_{i=1}^{k} ir_i + \sum_{i=1}^{l} ir_{-i} + 1.$$

This lower bound is best possible.

The following two theorems due to Jiang and Li [58], are inverse theorems for the subsequence sums $S(\bar{r},\mathscr{A})$.

Theorem 1.4.28. *[58, Theorem 3.3] Let k, l be two positive integers with* $k \geq 2$ *or* $l \geq 2$, *and* $\bar{r} = (r_{-l},\dots,r_{-1},r_1,\dots,r_k)$, *with* $r_i \geq 1$ *for* $i = -l,\dots,-1,1,\dots,k$. *Let* $\mathscr{A} = (a_{-l},\dots,a_{-1},a_1,\dots,a_k)_{\bar{r}}$ *be a nonempty sequence of integers with* $a_{-l} < \cdots < a_{-1} < 0 < a_1 < \cdots < a_k$. *If* $|S(\bar{r},\mathscr{A})| = \sum_{i=1}^{k} ir_i + \sum_{i=1}^{l} ir_{-i} + 1$, *then*

$$\mathscr{A} = a_1 \cdot \left([-l,-1]_{\bar{\mathbf{r}''}} \cup [1,k]_{\bar{\mathbf{r}'}}\right),$$

where $\bar{\mathbf{r}'} = (r_1,\dots,r_k)$ *and* $\bar{\mathbf{r}''} = (r_{-l},\dots,r_{-1})$.

Theorem 1.4.29. *[58, Corollary 3.5] Let k, l be two positive integers and* $\bar{r} = (r_{-l},r_{-l+1},\dots,r_k)$, *with* $r_i \geq 1$ *for* $i = -l,-l+1,\dots,k$. *Let* $\mathscr{A} = (a_{-l},a_{-l+1},\dots,a_k)_{\bar{r}}$ *be a nonempty sequence of* *integers with* $a_{-l} < a_{-l+1} < \cdots < a_{-1} < a_0 = 0 < a_1 < \cdots < a_k$. *If* $|S(\bar{r},\mathscr{A})| = \sum_{i=1}^{k} ir_i + \sum_{i=1}^{l} ir_{-i} + 1$, *then*

$$\mathscr{A} = a_1 \cdot [-l,k]_{\bar{r}}.$$

1.5 Results and the overview of the contents

In Chapter 2, we solve some direct and inverse problems for the sum of dilates $A + r \cdot B$, where A, B are finite sets of integers and r is a positive integer. In particular, we present a new proof of the direct theorem, Theorem 1.4.20 due to Freiman et al. [39], for the sum of dilates $A + r \cdot A$. Our method of proof of this theorem is elementary and self-contained.

We also generalize the following extended inverse theorem of Freiman et al. [39] to the sum of dilates $A + 2 \cdot B$ of two sets A and B.

Theorem 1.5.1. *[39, Theorem 4] Let A be a finite set of integers with* $|A| \geq 3$. *If* $|A + 2 \cdot A| < 4|A| - 4$, *then A is a subset of an arithmetic progression of length* $|A + 2 \cdot A| - 2|A| + 2 \leq 2|A| - 3$.

In Chapter 3, we solve both direct and inverse problems for the signed sumset $h_{\pm}A$ in the group of integers. One of the theorem we prove is the following direct theorem, when A contains only positive integers.

Theorem 1.5.2. *Let* $h \geq 1$. *Let A be a nonempty finite set of positive integers. If* $h \leq 2$, *then*

$$|h_{\pm}A| \geq 2(h|A| - h + 1).$$

If $h \geq 3$, *then*

$$|h_{\pm}A| \geq 2h|A| - h + 1.$$

These lower bounds are best possible.

We also prove the following inverse theorem for $h_{\pm}A$, when A contains only positive integers.

Theorem 1.5.3. *Let* $h \geq 2$. *Let A be a finite set of positive integers with* $|A| \geq 3$. *If* $|h_{\pm}A| = 2(h|A| - h + 1)$, *then* $h = 2$ *and*

$$A = d \cdot \{1, 3, \ldots, 2|A| - 1\},$$

for some positive integer d.

If $h \geq 3$ and $|h_{\pm}A| = 2h|A| - h + 1$, then

$$A = d \cdot \{1, 3, \ldots, 2|A| - 1\},$$

for some positive integer d.

Similar direct and inverse theorems for $h_{\pm}A$ have been proved, when the set A contains (i) nonnegative integers with $0 \in A$ (ii) arbitrary integers.

In Chapter 4, we study both direct and inverse problems for the restricted signed sumset $h^{\wedge}_{\pm}A$ in the group of integers in two separate cases, such as the set A contains (i) only positive integers, (ii) nonnegative integers with $0 \in A$. For $h = 1, 2$ and $|A|$, we prove the direct and inverse theorems for $h^{\wedge}_{\pm}A$. For $3 \leq h \leq |A| - 1$, we conjecture similar direct and inverse results for $h^{\wedge}_{\pm}A$. We also verify our conjectures for the case $h = 3$.

In Chapter 5, we find the exact cardinality of the generalized sumset $h^{(\gamma)}A$ for $A = \{0, 1, \ldots, k-2, k-1+b\} = [0, k-2] \cup \{k-1+b\}$, where b is a nonnegative integer. We prove that "$|h^{(\gamma)}A|$ is a strictly increasing linear function of b for $0 \leq b \leq N_1$ and is a strictly increasing, piecewise-linear function of b for $N_1 \leq b \leq N_2$ and that $|h^{(\gamma)}A|$ is constant for $b \geq N_2$, for some positive integers N_1 and N_2". Our result is analogues to the Nathanson's result for the regular h-fold sumset hA (see [81]), which says that "$|hA|$ is a strictly increasing piecewise-linear function of b for $0 \leq b \leq (h-1)(k-2)$ and that $|hA|$ is constant for $b \geq (h-1)(k-2)$". Further, as a corollary of our result, we obtain a similar result for the restricted sumset $h^{\wedge}A$, which says "$|h^{\wedge}A|$ is a strictly increasing linear function of b for $0 \leq b \leq N$ and that $|h^{\wedge}A|$ is constant for $b \geq N$, for some positive integer N".

In Chapter 6, we consider the subset sums $S_{\alpha}(A)$ of a finite set A of k integers and nonnegative integer $\alpha \leq k$. We find the minimum cardinality of $S_{\alpha}(A)$ in terms of number of elements in A and α. We also find the structure of the finite set A of integers for which $|S_{\alpha}(A)|$ is minimal. Further, we generalize the subset sums $S_{\alpha}(A)$ by defining the following subsequence sums. For a nonempty sequence $\mathscr{A} = (a_0, a_1, \ldots, a_{k-1})_r$ of k distinct integers each repeating exactly r times, and for a nonnegative integer α $(\leq rk)$, we define

$$S_{\alpha}(r, \mathscr{A}) := \{s(\mathscr{B}) : \mathscr{B} \text{ is a subsequence of } \mathscr{A} \text{ with } |\mathscr{B}| \geq \alpha\},$$

where $|\mathscr{B}|$ is the number of terms in the subsequence $\mathscr{B}$.

Similar to the subset sums $S_\alpha(A)$, we find the minimum cardinality of the subsequence sums $S_\alpha(r,\mathscr{A})$ also. We also find the structure of the finite sequence $\mathscr{A}$ of integers for which $|S_\alpha(r,\mathscr{A})|$ is minimal.

We conclude the thesis in Chapter 7, with a summary of results proved in this thesis. We also propose some unsolved problems for future work.

Chapter 2

Direct and inverse problems for sum of dilates

In this chapter, we study both direct and inverse problems for the sum of dilates $A + r \cdot B$, where A, B are finite sets of integers and r is a positive integer. In particular, when $B = A$, we prove a direct theorem which gives a uniform lower bound for the sum of dilates $A + r \cdot A$, for all $r \geq 3$. We also prove a Freiman's $3k - 4$ type theorem for the sum of dilates $A + 2 \cdot B$.

2.1 Introduction

Let A and B be nonempty finite sets of integers. Let $r \geq 1$ be an integer. The sumset $A + r \cdot B := \{a + rb : a \in A, b \in B\}$ is called the *sum of dilates* of A and B. For $r = 1$, the sum of dilates $A + r \cdot B$ coincides with the Minkowski sumset $A + B$. The direct problem for $A + r \cdot B$ is to find the minimum number of elements in $A + r \cdot B$ in terms of number of elements in A and B. The inverse problem for $A + r \cdot B$ is to find the structure of the finite sets A and B for which $|A + r \cdot B|$ is minimal. Several direct and inverse results for the sum of dilates $A + r \cdot B$ have been obtained by now, few of them are mentioned in Chapter 1. In particular, Freiman et al. [39] proved the following direct theorem in 2014.

Theorem 2.1.1. *[39, Theorem 5] Let A be a nonempty finite set of integers, and let $r \geq 3$. Then*

$$|A + r \cdot A| \geq 4|A| - 4.$$

In this chapter, we present a new, self contained, elementary proof of this theorem.

An extended inverse problem for the sum of dilates $A+r\cdot B$ is the problem of finding the structure of the finite sets A and B such that $|A+r\cdot B|\leq f(r,|A|,|B|)$, where $f(r,|A|,|B|)$ is a small diversion from the usual lower bound. For example, when $r=1$, we have the celebrated Freiman's $3k-4$ theorem, i.e., Theorem 1.4.9. In the same essence, Freiman et al. [39] obtained the following extended inverse theorem for the sum of dilates $A+2\cdot A$.

Theorem 2.1.2. *[39, Theorem 4] Let A be a finite set of integers with $|A|\geq 3$. If $|A+2\cdot A|<4|A|-4$, then A is a subset of an arithmetic progression of length $|A+2\cdot A|-2|A|+2\leq 2|A|-3$.*

In this chapter, we also prove the following theorem, which generalizes Theorem 2.1.2 to the sum of dilates $A+2\cdot B$ of two sets A and B.

Theorem 2.1.3. *Let A and B be two nonempty finite sets of integers with $|A|\geq 3$ such that*

(i) $d(A)=d(B)=1$,

(ii) $\ell(A)\leq \ell(B)$, *and*

(iii) $h_A\leq h_B$.

If

$$|A+2\cdot B|=|A|+2(|B|-1)+h<2(|A|+|B|-2), \tag{2.1}$$

then both A and B are subsets of arithmetic progressions of length $|B|+h=|A+2\cdot B|-|A|-|B|+2\leq |A|+|B|-3$.

Further, at the end of this chapter, we present some examples which show that the conditions (ii) and (iii) of Theorem 2.1.3 are sufficient but not necessary.

2.2 Direct problem

Proof of Theorem 2.1.1. Let $A=\{a_0,a_1,\ldots,a_{k-1}\}$, where $a_0<a_1<\cdots<a_{k-1}$. Then

$$\begin{aligned} a_0+ra_0<a_1+ra_0<a_0+ra_1<a_1+ra_1<a_2+ra_1<a_1+ra_2<\cdots< \\ a_{k-2}+ra_{k-1}<a_{k-1}+ra_{k-1}. \end{aligned} \tag{2.2}$$

For each $0\leq i\leq k-2$, there are three distinct integers in the above list, satisfying $a_i+ra_i<a_{i+1}+ra_i<a_i+ra_{i+1}$, with one extra integer $a_{k-1}+ra_{k-1}$. So, we have $|A+r\cdot A|\geq 3|A|-2$.

It remains to exhibit $|A|-2$ more integers of $A+r\cdot A$. Consider the following string of six consecutive integers of (2.2), for each $i=1,2,\ldots,k-2$,

$$a_{i-1}+ra_{i-1}<a_i+ra_{i-1}<a_{i-1}+ra_i<a_i+ra_i<a_{i+1}+ra_i<a_i+ra_{i+1}. \tag{2.3}$$

We claim that for each string of type (2.3) there exists an extra element of $A+r\cdot A$ which is not in the list (2.3) but in between $a_{i-1}+ra_{i-1}$ and a_i+ra_{i+1}. Once the claim is established, we have the theorem. To prove the claim, we first check

Subclaim: for every sequence of type (2.3), either

$$a_i+ra_{i-1}<a_{i+1}+ra_{i-1}<a_{i-1}+ra_i,$$

or

$$a_{i+1}+ra_i<a_{i-1}+ra_{i+1}<a_i+ra_{i+1}.$$

Since for each $1\leq i\leq k-2$, $a_i+ra_{i-1}<a_{i+1}+ra_{i-1}$ and $a_{i-1}+ra_{i+1}<a_i+ra_{i+1}$, so we only need to show that either $a_{i+1}+ra_{i-1}<a_{i-1}+ra_i$ or $a_{i+1}+ra_i<a_{i-1}+ra_{i+1}$.

If $a_{i+1}-a_i<(r-1)(a_i-a_{i-1})$, then clearly $a_{i+1}+ra_{i-1}<a_{i-1}+ra_i$. If not, then $a_{i+1}-a_i\geq(r-1)(a_i-a_{i-1})$, which implies

$$(r-1)(a_{i+1}-a_i)\geq(r-1)^2(a_i-a_{i-1})>a_i-a_{i-1}.$$

This is equivalent to

$$a_{i-1}+ra_{i+1}>a_{i+1}+ra_i.$$

Hence, the subclaim is proved.

Next, we show that for any two consecutive strings of six integers of the form (2.3) one can always find two distinct elements of $A+r\cdot A$, that have not been previously included in (2.2). This proves our main claim. Consider two consecutive strings of six integers

$$a_{i-1}+ra_{i-1}<a_i+ra_{i-1}<a_{i-1}+ra_i<a_i+ra_i<a_{i+1}+ra_i<a_i+ra_{i+1}, \tag{2.4}$$

and

$$a_i+ra_i<a_{i+1}+ra_i<a_i+ra_{i+1}<a_{i+1}+ra_{i+1}<a_{i+2}+ra_{i+1}<a_{i+1}+ra_{i+2}. \tag{2.5}$$

By the subclaim, there exist x_1,x_2 in $A+r\cdot A$ such that $a_{i-1}+ra_{i-1}<x_1<a_i+ra_{i+1}$ and $a_i+ra_i<x_2<a_{i+1}+ra_{i+2}$, with x_1,x_2 not in the lists (2.4) or (2.5). We show that either

$x_1 \neq x_2$ or there exists an $x_3 \neq x_1 (= x_2)$ such that $x_3 \in A + r \cdot A$ and it lies between $a_{i-1} + ra_{i-1}$ and $a_{i+1} + ra_{i+2}$.

Any one of the following four cases may arise.

Case 1: If $a_{i+1} - a_i < (r-1)(a_i - a_{i-1})$ and $a_{i+2} - a_{i+1} < (r-1)(a_{i+1} - a_i)$, then we get two new distinct elements $x_1 = a_{i+1} + ra_{i-1}$ and $x_2 = a_{i+2} + ra_i$ of $A + r \cdot A$, not previously included in (2.4) or (2.5).

Case 2: If $a_{i+1} - a_i < (r-1)(a_i - a_{i-1})$ and $a_{i+2} - a_{i+1} \geq (r-1)(a_{i+1} - a_i)$, then we get two elements $x_1 = a_{i+1} + ra_{i-1}$ and $x_2 = a_i + ra_{i+2}$ of $A + r \cdot A$, not previously included in (2.4) or (2.5). Clearly, $x_1 \neq x_2$, because

$$a_{i+1} + ra_{i-1} < a_{i+1} + ra_i < a_{i+2} + ra_i < a_i + ra_{i+2}.$$

Case 3: If $a_{i+1} - a_i \geq (r-1)(a_i - a_{i-1})$ and $a_{i+2} - a_{i+1} \geq (r-1)(a_{i+1} - a_i)$, then also we get two new distinct elements $x_1 = a_{i-1} + ra_{i+1}$ and $x_2 = a_i + ra_{i+2}$ of $A + r \cdot A$, not previously included in (2.4) or (2.5).

Case 4: Let $a_{i+1} - a_i \geq (r-1)(a_i - a_{i-1})$ and $a_{i+2} - a_{i+1} < (r-1)(a_{i+1} - a_i)$. Then we have integers $a_{i-1} + ra_{i+1}$, $a_{i+2} + ra_i$ such that $a_{i+1} + ra_i < a_{i-1} + ra_{i+1} < a_i + ra_{i+1}$ and $a_{i+1} + ra_i < a_{i+2} + ra_i < a_i + ra_{i+1}$.

If $a_{i-1} + ra_{i+1} \neq a_{i+2} + ra_i$, then we get two distinct extra elements $x_1 = a_{i-1} + ra_{i+1}$ and $x_2 = a_{i+2} + ra_i$, not previously included in (2.4) or (2.5).

If not, then

$$x_1 = a_{i-1} + ra_{i+1} = a_{i+2} + ra_i = x_2. \tag{2.6}$$

In this case, we show that there exists a new integer $x_3 = a_i + ra_{i+2}$ in the list (2.5), which is different from $x_1 = x_2$. Clearly, $x_3 > x_2 = x_1$. So we only need to show that $a_i + ra_{i+2} > a_{i+2} + ra_{i+1}$, or

$$(r-1)a_{i+2} - ra_{i+1} + a_i > 0.$$

Using (2.6) together with the inequality $a_{i+1} - a_i \geq (r-1)(a_i - a_{i-1})$, we get

$$\begin{aligned}(r-1)a_{i+2} - ra_{i+1} + a_i &= (r-1)(ra_{i+1} - ra_i + a_{i-1}) - ra_{i+1} + a_i \\ &= (r^2 - 2r)a_{i+1} - (r^2 - r - 1)a_i + (r-1)a_{i-1} \\ &\geq (r^2 - 2r)a_{i+1} - (r^2 - r - 1)a_i - a_{i+1} + ra_i \\ &= (r^2 - 2r - 1)(a_{i+1} - a_i).\end{aligned}$$

Since $r \geq 3$ and $a_{i+1} - a_i > 0$, we get $(r^2 - 2r - 1)(a_{i+1} - a_i) > 0$.

Thus, in each case, we get two distinct elements of $A + r \cdot A$, which are not in (2.4) or (2.5). Hence, we get $|A| - 2$ extra distinct elements of $A + r \cdot A$, which are not in (2.2). Hence, $|A + r \cdot A| \geq 4|A| - 4$. This completes the proof of the theorem. □

2.3 Inverse problem

We start with the following simple corollary of [27, Lemma 2].

Corollary 2.3.1. *Let A be a finite set of integers with $|A| \geq 3$ and $d(A) = 1$. Then for any nonempty finite set B of integers, we have*

$$|A + 2 \cdot B| \geq |A| + 2(|B| - 1).$$

Proof. Since the translation of A does not change the cardinality of $A + 2 \cdot B$, without loss of generality we may assume that $A \subset \mathbb{N}$ and $0 \in A$. Let $\hat{A}$ denote the natural projection of A onto $\mathbb{Z}/2\mathbb{Z}$. Since $0 \in A$ and $d(A) = 1$, we have $|\hat{A}| = 2$. Hence, by [27, Lemma 2],

$$|A + 2 \cdot B| \geq |A| + 2(|B| - 1).$$

□

Proof of Theorem 2.1.3. Let $A = \{a_0, a_1, \ldots, a_{k-1}\}$ and $B = \{b_0, b_1, \ldots, b_{l-1}\}$, where $a_0 < a_1 < \cdots < a_{k-1}$ and $b_0 < b_1 < \cdots < b_{l-1}$. Since translating A or B does not change the cardinality of $A + 2 \cdot B$, without loss of generality we may assume that $a_0 = b_0 = 0$. Hence $\ell(A) = a_{k-1}$ and $\ell(B) = b_{l-1}$. Now, write

$$A = A_0 \cup A_1,$$

where $A_0 \subset 2\mathbb{Z}$ and $A_1 \subset 2\mathbb{Z}+1$. Since $0 \in A_0 \subset A$ and $d(A) = 1$, it follows that both A_0 and A_1 are nonempty. Let $|A_0| = m$ and $|A_1| = n$. So $m,n \geq 1$ and $k = m+n$. Let

$$A_0 = \{0 = 2x_0 < 2x_1 < \cdots < 2x_{m-1}\},$$
$$A_0^* = \frac{1}{2} \cdot A_0 = \{0 = x_0 < x_1 < \cdots < x_{m-1}\},$$
$$A_1 = \{2y_0 + 1 < 2y_1 + 1 < \cdots < 2y_{n-1} + 1\},$$
$$A_1^* = \frac{1}{2} \cdot (A_1 - 1) - y_0 = \{0 < y_1 - y_0 < y_2 - y_0 < \cdots < y_{n-1} - y_0\}.$$

Then

$$\ell(A_0^*) = x_{m-1} < a_{k-1} = \ell(A) \text{ and } \ell(A_1^*) = y_{n-1} - y_0 < a_{k-1} = \ell(A).$$

So,

$$\begin{aligned} |A + 2 \cdot B| &= |(A_0 \cup A_1) + 2 \cdot B| \\ &= |A_0 + 2 \cdot B| + |A_1 + 2 \cdot B| \\ &= |2 \cdot A_0^* + 2 \cdot B| + |2 \cdot (A_1^* + y_0) + 1 + 2 \cdot B| \\ &= |A_0^* + B| + |A_1^* + B|. \end{aligned} \tag{2.7}$$

We will use two inequalities, stated as Claim 1 and Claim 2 below.

Claim 1.

$$\ell(B) \leq l + \max(m,n) - 2 \leq k + l - 3. \tag{2.8}$$

Since $\ell(B) \geq \ell(A) > \ell(A_0^*)$ and $\ell(B) \geq \ell(A) > \ell(A_1^*)$, we have $\delta_{B,A_0^*} = \delta_{B,A_1^*} = 0$.

Suppose first that $m \leq n$. If the claim is false, then

$$\ell(B) \geq l + n - 1 = |B| + |A_1^*| - 1 \geq l + m - 1 = |B| + |A_0^*| - 1.$$

Hence from Theorem 1.4.10(i), we have

$$|B + A_0^*| \geq |B| + 2(|A_0^*| - 1) = l + 2m - 2 \text{ and } |B + A_1^*| \geq |B| + 2(|A_1^*| - 1) = l + 2n - 2. \tag{2.9}$$

Using (2.7) and (2.9), we get, $|A + 2 \cdot B| \geq 2(|A| + |B| - 2)$, which contradicts our hypothesis (2.1).

Similarly, if $n \leq m$ and $\ell(B) \geq l + m - 1 \geq l + n - 1$, then $d(A) = 1$ and Theorem 1.4.10(i) again imply the inequalities (2.9), which together with (2.7) yields $|A + 2 \cdot B| \geq (|A| + |B| - 2)$, a contradiction.

Hence, $\ell(B) \leq l + \max(m,n) - 2$. Since $k = m+n$ and $m,n \geq 1$, it follows that $\max(m,n) \leq k-1$ and hence $\ell(B) \leq l + \max(m,n) - 2 \leq k + l - 3$.

Claim 2.

$$|A + 2\cdot B| \geq |A| + 2(|B| - 1) + h_B.$$

For the proof of Claim 2, we consider two cases.

Case 1: Suppose $m \leq n$. By (2.8), we have $\ell(B) \leq l + n - 2$. So, it follows from Theorem 1.4.10(ii) that

$$|B + A_1^*| \geq (|B| + |A_1^*| - 1) + h_B.$$

Therefore,

$$\begin{aligned} |A + 2\cdot B| &= |A_0^* + B| + |A_1^* + B| \\ &\geq (|B| + |A_0^*| - 1) + (|B| + |A_1^*| - 1) + h_B \\ &= (m + l - 1) + (n + l - 1) + h_B \\ &= |A| + 2(|B| - 1) + h_B. \end{aligned}$$

Case 2: Suppose that $n < m$. By (2.8), we have $\ell(B) \leq l + m - 2$. It follows from Theorem 1.4.10(ii) that

$$|B + A_0^*| \geq (m + l - 1) + h_B = (|B| + |A_0^*| - 1) + h_B.$$

Therefore,

$$\begin{aligned} |A + 2\cdot B| &= |A_0^* + B| + |A_1^* + B| \\ &\geq (|B| + |A_0^*| - 1) + h_B + (|B| + |A_1^*| - 1) \\ &= (m + l - 1) + h_B + (n + l - 1) \\ &= |A| + 2(|B| - 1) + h_B. \end{aligned}$$

In both the cases, we see that

$$0 \leq h_B \leq |A + 2\cdot B| - (|A| + 2|B| - 2) = h \leq k - 3 = |A| - 3.$$

Thus, B is a subset of the arithmetic progression $\{0, 1, \ldots, b_{l-1}\}$ of size $b_{l-1} + 1 = l + h_B \leq l + h \leq k + l - 3$.

Similarly, since $\ell(A) \leq \ell(B)$, the set A is a subset of an arithmetic progression of size $k + l - 3$. This completes the proof of the theorem. □

2.4 Remarks

The following remarks show that the conditions (ii) and (iii) of Theorem 2.1.3 are sufficient but not necessary.

Remark 2.4.1. The condition $\ell(A) \leq \ell(B)$ is not necessary, as can be seen by the following example.

Example 2.4.1. Let $k \geq 3$ be a fixed integer. Let $A = [0, 2k-1] \cup \{2k+1\}$ and $B = [0, 2k-2] \cup \{2k\}$. Clearly, $d(A) = d(B) = 1$, $h_A = 1 = h_B$ and $\ell(A) = 2k+1 > 2k = \ell(B)$. Since $4k \leq 6k-4$ for all $k \geq 2$, we have

$$\begin{aligned} A + 2 \cdot B &= ([0, 2k-1] \cup \{2k+1\}) + (\{0, 2, 4, \ldots, 4k-4\} \cup \{4k\}) \\ &= [0, 6k-5] \cup \{2k+1, 2k+3, \ldots, 6k-3\} \cup [4k, 6k-1] \cup \{6k+1\} \\ &= [0, 6k-1] \cup \{6k+1\}. \end{aligned}$$

Thus, $|A+2\cdot B| = 6k+1 < 8k-2 = 2(|A|+|B|-2)$ and the sets A and B are subsets of arithmetic progressions of length at most $2k+2 = |A+2\cdot B| - |A| - |B| + 2 \leq 4k-2 = |A|+|B|-3$.

Remark 2.4.2. The condition $h_A \leq h_B$ is not necessary, as can be seen by the following example.

Example 2.4.2. Let $k \geq 6$ be an integer. Let $A = [0, k-2] \cup \{k+1\}$ and $B = [0, k-2] \cup \{k, k+1\}$. Clearly in this case, $d(A) = d(B) = 1$, $\ell(A) = k+1 = \ell(B)$ and $h_A = 2 > 1 = h_B$. Since $2k \leq 3k-5$ for all $k \geq 5$, we have

$$\begin{aligned} A + 2 \cdot B &= ([0, k-2] \cup \{k+1\}) + (\{0, 2, 4, \ldots, 2k-4\} \cup \{2k, 2k+2\}) \\ &= [0, 3k-6] \cup \{k+1, k+3, \ldots, 3k-3\} \cup [2k, 3k-2] \cup [2k+2, 3k] \\ &\quad \cup \{3k+1, 3k+3\} \\ &= [0, 3k] \cup \{3k+1, 3k+3\}. \end{aligned}$$

Thus, $|A+2\cdot B| = 3k+3 < 4k-2 = 2(|A|+|B|-2)$ and the sets A and B are subsets of arithmetic progressions of length at most $k+4 = |A+2\cdot B| - |A| - |B| + 2 \leq 2k-2 = |A|+|B|-3$.

Chapter 3

Direct and inverse problems for signed sumset $h_{\pm}A$

In this chapter, we study the h-fold signed sumset $h_{\pm}A$. We solve both direct and inverse problems for the sumset $h_{\pm}A$ by considering three different cases, namely (i) A contains only positive integers, (ii) A contains nonnegative integers with $0 \in A$, and (iii) A contains arbitrary integers.

3.1 Introduction

Let G be an additive abelian group, and let $A = \{a_0, a_1, \ldots, a_{k-1}\}$ be a nonempty subset of G. Let $h \geq 1$ be an integer. Recall that, the h-fold sumset hA is defined by

$$hA := \left\{ \sum_{i=0}^{k-1} \lambda_i a_i : \lambda_i \in \mathbb{N} \text{ for } i = 0, 1, \ldots, k-1 \text{ and } \sum_{i=0}^{k-1} \lambda_i = h \right\}.$$

Define the *h-fold signed sumset* of A, denoted by $h_{\pm}A$, by

$$h_{\pm}A := \left\{ \sum_{i=0}^{k-1} \lambda_i a_i : \lambda_i \in \mathbb{Z} \text{ for } i = 0, 1, \ldots, k-1 \text{ and } \sum_{i=0}^{k-1} |\lambda_i| = h \right\}.$$

Clearly,

$$hA \cup h(-A) \subset h_{\pm}A \subset h\left(A \cup (-A)\right),$$

and for any integer α,

$$h_{\pm}(\alpha \cdot A) = \alpha \cdot (h_{\pm}A).$$

The signed sumset $h_{\pm}A$ was first appeared in the work of Bajnok and Ruzsa [11] in the context of the "independence number" of a subset A of G and in the work of Klopsch and Lev [63, 64] in the context of the "diameter" of G with respect to the subset A. The first systematic and point centric study appeared in the work of Bajnok and Matzke [9] in which they studied the minimum cardinality of h-fold signed sumset $h_{\pm}A$ of subsets of a finite abelian group. In particular, they proved that the minimum cardinality of $h_{\pm}A$ is the same as the minimum cardinality of hA, when A is a subset of a finite cyclic group. One year later, they [10] obtained results regarding cases where the minimum cardinality of $h_{\pm}A$ coincide with the minimum cardinality of hA, when A is a subset of an elementary abelian group.

The direct problem for $h_{\pm}A$ is to find the minimum number of elements in $h_{\pm}A$ in terms of number of elements in A. The inverse problem for $h_{\pm}A$ is to find the structure of the finite set A for which $|h_{\pm}A|$ is minimal.

The direct and inverse theorems for hA are well established in the group of integers, which are mentioned in Chapter 1. In this chapter, we solve similar direct and inverse problems for $h_{\pm}A$ in the group of integers. This study is done by considering three different cases, viz.; (i) A contains only positive integers, (ii) A contains nonnegative integers with $0 \in A$, and (iii) A contains arbitrary integers, in the sections 3.2, 3.3 and 3.4, respectively.

3.2 A contains only positive integers

Theorem 3.2.1. *Let $h \geq 1$, and let A be a set of k positive integers. We have*

$$|h_{\pm}A| \geq 2(hk - h + 1).$$

This lower bound is best possible for $h \leq 2$.

Proof. Let $A = \{a_0, a_1, \ldots, a_{k-1}\}$, where $0 < a_0 < a_1 < \cdots < a_{k-1}$. The sumset $h_{\pm}A$ contains at least the following $2(hk-h+1)$ integers.

$$\begin{aligned} ha_0 &< (h-1)a_0+a_1 < (h-2)a_0+2a_1 < \cdots < a_0+(h-1)a_1 < ha_1 \\ &< (h-1)a_1+a_2 < (h-2)a_1+2a_2 < \cdots < a_1+(h-1)a_2 < ha_2 \\ &\;\vdots \\ &< (h-1)a_{k-2}+a_{k-1} < (h-2)a_{k-2}+2a_{k-1} < \cdots < a_{k-2}+(h-1)a_{k-1} \\ &< ha_{k-1} \end{aligned} \tag{3.1}$$

and

$$\begin{aligned} -ha_{k-1} &< -(h-1)a_{k-1}-a_{k-2} < \cdots < -a_{k-1}-(h-1)a_{k-2} < -ha_{k-2} \\ &< -(h-1)a_{k-2}-a_{k-3} < \cdots < -a_{k-2}-(h-1)a_{k-3} < -ha_{k-3} \\ &\;\vdots \\ &< -(h-1)a_1-a_0 < \cdots < -a_1-(h-1)a_0 < -ha_0. \end{aligned} \tag{3.2}$$

Hence,

$$|h_{\pm}A| \geq 2(hk-h+1).$$

Next, we show that this lower bound is best possible. If $h=1$, then $|1_{\pm}A| = 2k$. Hence the lower bound is tight for every finite set A. Next, let $h=2$ and $A = \{1,3,5,\ldots,2k-1\}$. Then

$$2_{\pm}A = \{-(4k-2),\ldots,-4,-2,2,4,\ldots,(4k-2)\}.$$

Hence, $|2\underline{+}A| = 4k-2$. This completes the proof of the theorem. □

Theorem 3.2.2. *Let $h \geq 2$, and let A be a set of k positive integers. If $|h_{\pm}A| = 2(hk-h+1)$, then $h=2$ and $A = d \cdot \{1,3,\ldots,2k-1\}$, for some positive integer d.*

Proof. Let $A = \{a_0,a_1,\ldots,a_{k-1}\}$, where $0 < a_0 < a_1 < \cdots < a_{k-1}$. Since $|h_{\pm}A| = 2(hk-h+1)$, it follows from Theorem 3.2.1, that the sumset $h_{\pm}A$ consists precisely the integers listed in (3.1) and (3.2). For each $i = 1,2,\ldots,k-2$, we have

$$a_{i-1}+(h-1)a_i < ha_i < (h-1)a_i+a_{i+1}.$$

Also,

$$a_{i-1}+(h-1)a_i < a_{i-1}+(h-2)a_i+a_{i+1} < (h-1)a_i+a_{i+1}.$$

Thus,

$$ha_i = a_{i-1} + (h-2)a_i + a_{i+1}.$$

This is equivalent to

$$a_{i+1} - a_i = a_i - a_{i-1}.$$

Therefore, the set A is an arithmetic progression, i.e., $a_i - a_{i-1} = d$, for some $d > 0$ and for all $1 \leq i \leq k-1$.

Again,

$$-ha_1 < -(h-1)a_1 - a_0 < -(h-1)a_1 + a_0 < -(h-2)a_1 + 2a_0 < \cdots < \\ -a_1 + (h-1)a_0 < ha_0. \tag{3.3}$$

Thus, from (3.1), (3.2) and (3.3), it follows that, for $i = 1, 2, \ldots, h-1$,

$$-(h-i)a_1 + ia_0 = -(h-i-1)a_1 - (i+1)a_0.$$

So, the common difference $d = a_1 - a_0 = 2ia_0$, for $i = 1, 2, \ldots, h-1$. This is possible, only if $h = 2$. Hence

$$A = d \cdot \{1, 3, \ldots, 2k-1\}.$$

This completes the proof of the theorem. □

Theorem 3.2.3. *Let $h \geq 3$, and let A be a set of k (≥ 3) positive integers. Then*

$$|h_{\pm}A| \geq 2hk - h + 1. \tag{3.4}$$

This lower bound is best possible.

The above theorem does not hold for $k = 2$, as it can be seen by taking $A = \{1, 2\}$, $h = 3$; $A = \{1, 3\}$, $h = 4$; and $A = \{2, 3\}$, $h = 5$.

Further, if $A = \{a_0, a_1\}$, where $0 < a_0 < a_1$ and $h < \frac{a_0 + a_1}{2a_0}$, we observe in the following remark that $|h_{\pm}A| = 4h$.

Remark 3.2.1. Let $A = \{a_0, a_1\}$, where $0 < a_0 < a_1$. Let $3 \leq h < \frac{a_0 + a_1}{2a_0}$. Then, every summand in $h_{\pm}A$ is either of the form $(h-i)a_0 + ia_1$, or $(h-i)a_0 - ia_1$, or $-(h-i)a_0 + ia_1$, or $-(h-i)a_0 - ia_1$, where $0 \leq i \leq h$. Hence, the maximum possibility of integers in $h_{\pm}A$ is $4h$, i.e.,

$$|h_{\pm}A| \leq 4h. \tag{3.5}$$

On the other hand, as $h < \frac{a_0+a_1}{2a_0}$, i.e., $0 < (2h-1)a_0 < a_1$, we have

$$ha_0 < -(h-1)a_0 + a_1 < (h-1)a_0 + a_1 < -(h-2)a_0 + 2a_1 < (h-2)a_0 + 2a_1 < \cdots < -a_0 + (h-1)a_1 < a_0 + (h-1)a_1 < ha_1.$$

Since each of the above $2h$ signed h-fold summand is positive and in $h_{\pm}A$, their negatives are also in $h_{\pm}A$. Hence, $|h_{\pm}A| \geq 4h$. This together with (3.5) give $|h_{\pm}A| = 4h$.

Proof of Theorem 3.2.3. Let $A = \{a_0, a_1, \ldots, a_{k-1}\}$, where $0 < a_0 < a_1 < \cdots < a_{k-1}$. From Theorem 3.2.1, it follows that the sumset $h_{\pm}A$ contains at least $2(hk-h+1)$ integers listed in (3.1) and (3.2). So, it remains to show at least $(h-1)$ extra integers in $h_{\pm}A$ different from the integers in (3.1) and (3.2). To show this, we consider three cases depending on $a_2 - a_1 < a_1 - a_0$, $a_2 - a_1 > a_1 - a_0$, and $a_2 - a_1 = a_1 - a_0$. Except in a subcase of the last case, namely, $a_2 - a_1 = a_1 - a_0 = 2a_0$, which will lead to present the example for the best possible bound, we show much more extra summands than $h-1$ in $h_{\pm}A$.

Case 1. ($a_2 - a_1 < a_1 - a_0$, i.e., $a_2 < 2a_1 - a_0$). Consider the following sequence of integers, which is taken from (3.1).

$$(h-1)a_0 + a_1 < (h-2)a_0 + 2a_1 < (h-3)a_0 + 3a_1 < \cdots < a_0 + (h-1)a_1 < ha_1 \tag{3.6}$$

We shall insert an extra signed h-fold summand between each pair of successive integers of (3.6) as follows:

$$(h-1)a_0 + a_1 < (h-1)a_0 + a_2 < (h-2)a_0 + 2a_1 < (h-2)a_0 + a_1 + a_2 < (h-3)a_0 + 3a_1 < (h-3)a_0 + 2a_1 + a_2 < (h-4)a_0 + 4a_1 < \cdots < 2a_0 + (h-3)a_1 + a_2 < a_0 + (h-1)a_1 < a_0 + (h-2)a_1 + a_2 < ha_1.$$

Thus, we get $h-1$ extra positive integers of $h_{\pm}A$. Similarly, taking the negatives of these $h-1$ summands, we get another set of $h-1$ integers of $h_{\pm}A$. Hence, we get a total of at least $2(h-1)$ extra integers of $h_{\pm}A$, not already listed in (3.1) and (3.2).

Case 2. ($a_2 - a_1 > a_1 - a_0$, i.e., $2a_1 < a_2 + a_0$). Similar to the Case 1, we have

$$ha_1 < (h-2)a_1 + a_2 + a_0 < (h-1)a_1 + a_2 < (h-3)a_1 + 2a_2 + a_0 < (h-2)a_1 + 2a_2 < (h-4)a_1 + 3a_2 + a_0 < (h-3)a_1 + 3a_2 < \cdots < (h-1)a_2 + a_0 < a_1 + (h-1)a_2.$$

So, we get $h-1$ extra summands in $h_{\pm}A$ between ha_1 and $a_1+(h-1)a_2$. Hence, taking negatives of these $h-1$ positive summands, we get a total of at least $2(h-1)$ extra integers of $h_{\pm}A$.

Case 3. ($a_2-a_1=a_1-a_0$, i.e., a_0, a_1, a_2 are in arithmetic progression). Let $a_1=a_0+d$, $a_2=a_0+2d$, for some positive integer d.

Subcase 1. ($d>2a_0$). Consider the following integers of (3.1)

$$ha_0<(h-1)a_0+a_1<(h-2)a_0+2a_1<\cdots<a_0+(h-1)a_1<ha_1<(h-1)a_1+a_2$$
$$<(h-2)a_1+2a_2<\cdots<a_1+(h-1)a_2<ha_2.$$

Rewrite the list as

$$ha_0<ha_0+d<ha_0+2d<\cdots<ha_0+(h-1)d<ha_0+hd<ha_0+(h+1)d$$
$$<ha_0+(h+2)d<\cdots<ha_0+(2h-1)d<ha_0+2hd.$$

For each $i=0,1,\ldots,h-2$, we insert an extra summand between ha_0+2id and $ha_0+(2i+1)d$. We have

$$ha_0+2id<(h-2)a_0+(2i+1)d=(h-2-i)a_0-a_1+(i+1)a_2<ha_0+(2i+1)d.$$

Each of these $h-1$ extra signed h-fold summand $(h-2-i)a_0-a_1+(i+1)a_2$, is positive. So, we get $h-1$ extra positive integers of $h_{\pm}A$. The negatives of these $h-1$ integers are also signed h-fold summands, hence are in the set $h_{\pm}A$ and different from the summands in (3.2). Hence, we get at least $2(h-1)$ extra integers of $h_{\pm}A$, which are not listed in (3.1) and (3.2).

Subcase 2. ($d<2a_0$). We use induction argument on h to write $\lfloor \frac{h}{2} \rfloor$ extra positive integers of $h_{\pm}A$.

If $h=3$, then

$$a_0<a_2-a_1+a_0=a_0+d<3a_0.$$

If $h=4$, then

$$2a_0<a_2-a_1+2a_0=2a_0+d<4a_0,$$

and

$$0<-a_1+3a_0=2a_0-d<2a_0.$$

If $h=5$, then

$$3a_0 < a_2 - a_1 + 3a_0 = 3a_0 + d < 5a_0,$$

and

$$a_0 < -a_1 + 4a_0 = 3a_0 - d < 3a_0.$$

If $h=6$, then

$$4a_0 < a_2 - a_1 + 4a_0 = 4a_0 + d < 6a_0,$$

$$2a_0 < -a_1 + 5a_0 = 4a_0 - d < 4a_0,$$

and

$$0 < 2a_2 - 3a_1 + a_0 = d < 2a_0.$$

In all the above cases we get exactly $\lfloor \frac{h}{2} \rfloor$ number of extra positive signed h-fold summands, which are not included in (3.1) and (3.2). Now, let $h \geq 7$ and assume that the result is true for $h-1$. If $h = 4k+1$ or $h = 4k+3$ for some $k \geq 1$, then $\lfloor \frac{h}{2} \rfloor = \lfloor \frac{h-1}{2} \rfloor = \frac{h-1}{2}$. By the induction hypothesis, $\lfloor \frac{h-1}{2} \rfloor$ extra positive integers as signed $(h-1)$-fold summands may be obtained in $(h-1)_{\pm}A$. Adding a single copy of a_0 to all these $(h-1)$-fold summands, we can obtain $\lfloor \frac{h-1}{2} \rfloor (= \lfloor \frac{h}{2} \rfloor)$ extra positive signed h-fold summands. This completes the induction in this case.

Now, let $h = 4k$, $k \geq 1$. Then $\lfloor \frac{h-1}{2} \rfloor$ extra positive integers may be obtained from the $\lfloor \frac{h-1}{2} \rfloor$ extra positive summands of $(h-1)$-fold signed sumset of A by just adding a_0 to it and one more summand is given by $0 < (k-1)a_2 - (2k-1)a_1 + (k+2)a_0 = 2a_0 - d < 2a_0$. Hence, we get $\lfloor \frac{h}{2} \rfloor$ extra positive integers.

Similarly, if $h = 4k+2$, $k \geq 1$, then $\lfloor \frac{h-1}{2} \rfloor$ extra positive integers may be obtained from the $\lfloor \frac{h-1}{2} \rfloor$ extra positive summands of $(h-1)$-fold signed sumset of A by just adding a_0 to it and one more summand is given by $0 < (k+1)a_2 - (2k+1)a_1 + ka_0 = d < 2a_0$.

Since, the negatives of these $\lfloor \frac{h}{2} \rfloor$ integers are also in the set $h_{\pm}A$. Hence, we get a total of at least $2\lfloor \frac{h}{2} \rfloor$ extra integers in $h_{\pm}A$.

Further, in both the above subcases 1 and 2, we get even more $2\lfloor \frac{h}{3} \rfloor$ integers. Let m be the largest integer such that $3m \leq h$, i.e., $m = \lfloor \frac{h}{3} \rfloor$ or $h = 3m + \varepsilon$, $\varepsilon \in \{0,1,2\}$. Then,

$$(h-3)a_0 + 2a_1 - a_2 = (h-2)a_0,$$

$$(h-6)a_0+4a_1-2a_2=(h-4)a_0,$$

$$(h-9)a_0+6a_1-3a_2=(h-6)a_0,$$

$$\vdots$$

$$\varepsilon a_0+2ma_1-ma_2=(m+\varepsilon)a_0.$$

So, there are $m=\lfloor\frac{h}{3}\rfloor$ further extra positive signed h-fold summands which are multiples of a_0, between 0 and ha_0. Thus, including negatives of these integers we get, $2m=2\lfloor\frac{h}{3}\rfloor$ even more extra integers in both the subcases $d>2a_0$ and $d<2a_0$. Hence, in both the subcases 1 and 2, we get a total of at least $2(\lfloor\frac{h}{2}\rfloor+\lfloor\frac{h}{3}\rfloor)$ extra signed h-fold summands neither included in (3.1) nor in (3.2).

Subcase 3. ($d=2a_0$). In this case we show that $-(h-2)a_0,-(h-4)a_0,-(h-6)a_0,\ldots,(h-6)a_0,(h-4)a_0,(h-2)a_0$ are signed h-fold summands, which are neither included in (3.1) nor in (3.2). Clearly, their number is $h-1$.

If $h=3$, then $2a_1-a_2=a_0$, and $a_2-2a_1=-a_0$. So, we get $(h-1)=2$ distinct integers which are previously not included.

Now, let $h\geq 4$. Rewrite the summands of (3.1), which are between ha_0 and ha_1 as follows:

$$(h-1)a_0+a_1<(h-2)a_0+2a_1<(h-3)a_0+3a_1<\cdots<a_0+(h-1)a_1. \qquad (3.7)$$

Adding $-(a_1+a_2)$ to the first three successive integers $(h-1)a_0+a_1$, $(h-2)a_0+2a_1$, $(h-3)a_0+3a_1$ of (3.7), we get

$$(h-1)a_0+a_1-(a_1+a_2)=(h-1)a_0-a_2=(h-6)a_0,$$

$$(h-2)a_0+2a_1-(a_1+a_2)=(h-2)a_0+a_1-a_2=(h-4)a_0,$$

and

$$(h-3)a_0+3a_1-(a_1+a_2)=(h-3)a_0+2a_1-a_2=(h-2)a_0.$$

Now leave the first term of (3.7) and add $-2(a_1+a_2)$ to the next three successive integers $(h-2)a_0+2a_1$, $(h-3)a_0+3a_1$, $(h-4)a_0+4a_1$ of (3.7), we get

$$(h-2)a_0+2a_1-2(a_1+a_2)=(h-2)a_0-2a_2=(h-12)a_0,$$

$$(h-3)a_0+3a_1-2(a_1+a_2)=(h-3)a_0+a_1-2a_2=(h-10)a_0,$$

and

$$(h-4)a_0+4a_1-2(a_1+a_2)=(h-4)a_0+2a_1-2a_2=(h-8)a_0.$$

We continue this process up to the last triplet $3a_0+(h-3)a_1$, $2a_0 + (h-2)a_1$, $a_0+(h-1)a_1$ of (3.7) by adding $-(h-3)(a_1+a_2)$, to get

$$3a_0+(h-3)a_1-(h-3)(a_1+a_2)=3a_0-(h-3)a_2=-(5h-18)a_0,$$

$$2a_0+(h-2)a_1-(h-3)(a_1+a_2)=2a_0+a_1-(h-3)a_2=-(5h-20)a_0,$$

and

$$a_0+(h-1)a_1-(h-3)(a_1+a_2)=a_0+2a_1-(h-3)a_2=-(5h-22)a_0.$$

The above process covers all the $h-1$ integers $-(h-2)a_0, -(h-4)a_0, -(h-6)a_0, \ldots, (h-6)a_0, (h-4)a_0, (h-2)a_0$ as signed h-fold summands with some other possible negative integers which are already counted in (3.2). One may stop this process till one gets $-(h-2)a_0$. Thus, we get exactly $h-1$ extra integers of $h_{\pm}A$, not already included in (3.1) and (3.2).

Thus, in all the above cases 1, 2 and 3, we get at least $h-1$ extra integers of $h_{\pm}A$, which are not included in (3.1) and (3.2). Hence,

$$|h_{\pm}A| \geq 2hk-h+1.$$

Next, we show that the lower bound in (3.4) is best possible. Let $A=\{1,3,5,\ldots,(2k-1)\}$ for some integer $k \geq 1$. If h is even, then

$$h_{\pm}A \subset \{-h(2k-1),\ldots,-4,-2,0,2,4,\ldots,h(2k-1)\}.$$

If h is odd, then

$$h_{\pm}A \subset \{-h(2k-1),\ldots,-5,-3,-1,1,3,5,\ldots,h(2k-1)\}.$$

In both these cases, $|h_{\pm}A| \leq 2hk-h+1$. Hence, together with (3.4), we get, $|h_{\pm}A| = 2hk-h+1$. This completes the proof of the theorem. □

Theorem 3.2.4. *Let $h \geq 3$, and let A be a set of k (≥ 3) positive integers. If $|h_{\pm}A| = 2hk-h+1$, then $A = d\cdot\{1,3,\ldots,2k-1\}$, for some positive integer d.*

Proof. Let $A = \{a_0, a_1, \ldots, a_{k-1}\}$, where $0 < a_0 < a_1 < \cdots < a_{k-1}$. Since $|h_{\pm}A| = 2hk - h + 1$, from the proof of Theorem 3.2.3 it follows that $a_2 - a_1 = a_1 - a_0 = d = 2a_0$. Again, by the similar argument used in Theorem 3.2.2, we get, for each $i = 1, 2, \ldots, k-2$

$$a_{i-1} + (h-1)a_i < ha_i < (h-1)a_i + a_{i+1},$$

and

$$a_{i-1} + (h-1)a_i < a_{i-1} + (h-2)a_i + a_{i+1} < (h-1)a_i + a_{i+1}.$$

Thus,

$$ha_i = a_{i-1} + (h-2)a_i + a_{i+1}.$$

This is equivalent to

$$a_{i+1} - a_i = a_i - a_{i-1}.$$

Therefore, the set A is an arithmetic progression, and hence

$$A = d \cdot \{1, 3, \ldots, 2k-1\}.$$

This completes the proof of the theorem. □

3.3 A contains nonnegative integers with $0 \in A$

Theorem 3.3.1. *Let $h \geq 1$, and let A be a set of k nonnegative integers with $0 \in A$. Then*

$$|h_{\pm}A| \geq 2hk - 2h + 1. \tag{3.8}$$

This lower bound is best possible.

Proof. Let $A = \{a_0, a_1, \ldots, a_{k-1}\}$, where $0 = a_0 < a_1 < \cdots < a_{k-1}$. From (3.1) and (3.2), it is clear that $h_{\pm}A$ contains at least $hk - h$ positive integers $(h-i)a_j + ia_{j+1}$ and $hk - h$ negative integers $-(h-i)a_j - ia_{j+1}$, for $0 \leq i \leq h$, $1 \leq j \leq k-2$, and one extra integer zero. Thus,

$$|h_{\pm}A| \geq 2hk - 2h + 1.$$

Next, we show that the lower bound in (3.8) is best possible. Let $A = \{0, 1, 2, \ldots, k-1\} = [0, k-1]$. The smallest integer of $h_{\pm}A$ is $-h(k-1)$ and the largest element of $h_{\pm}A$ is $h(k-1)$. Therefore,

$$h_{\pm}A \subset [-h(k-1), h(k-1)].$$

So

$$|h_{\pm}A| \leq 2h(k-1)+1.$$

This inequality together with (3.8), implies

$$|h_{\pm}A| = 2hk-2h+1.$$

This completes the proof of the theorem. □

Theorem 3.3.2. *Let $h \geq 2$, and let A be a set of k nonnegative integers with $0 \in A$. Then $|h_{\pm}A| = 2hk-2h+1$ if and only if $A = d \cdot [0, k-1]$, for some positive integer d.*

Proof. Let $A = \{a_0, a_1, \ldots, a_{k-1}\}$, where $0 = a_0 < a_1 < \cdots < a_{k-1}$. Since $|h_{\pm}A| = 2hk-2h+1$, it follows from Theorem 3.3.1 that the sumset $h_{\pm}A$ consists precisely the integers listed in equation (3.1) and (3.2). By the similar argument as used in Theorem 3.2.2 and Theorem 3.2.4, we obtain that the set A is an arithmetic progression. Hence $A = d \cdot [0, k-1]$. □

3.4 A contains arbitrary integers

Theorem 3.4.1. *Let $h \geq 1$, and let A be a set of k integers. Then*

$$|h_{\pm}A| \geq hk-h+1. \tag{3.9}$$

This lower bound is best possible.

Proof. The lower bound is trivial and it follows from (3.1). To see that the lower bound is optimal, consider the interval of integers $A = \left[-\lfloor \frac{k}{2} \rfloor, \lfloor \frac{k}{2} \rfloor\right]$, where $k \geq 3$ is an odd integer. Then,

$$h_{\pm}A \subset \left[-h\left\lfloor \frac{k}{2} \right\rfloor, h\left\lfloor \frac{k}{2} \right\rfloor\right].$$

Thus,

$$|h_{\pm}A| \leq 2h\left\lfloor \frac{k}{2} \right\rfloor + 1 = (k-1)h+1 = hk-h+1.$$

This inequality together with (3.9) gives $|h_{\pm}A| = hk-h+1$. This completes the proof of the theorem. □

Theorem 3.4.2. *Let $h \geq 2$, and let A be a set of k (≥ 2) integers such that $|h_{\pm}A| = hk-h+1$. Then A is a symmetric set, which is also an arithmetic progression.*

Proof. Let $A = \{a_0, a_1, \ldots, a_{k-1}\}$, where $a_0 < a_1 < \cdots < a_{k-1}$. Let $|h_{\pm}A| = hk - h + 1$. Since $hA \subset h_{\pm}A$, Theorem 1.4.1 implies that $hA = h_{\pm}A$. Thus, by Theorem 1.4.3 the set A is in arithmetic progression. Again, since $|h_{\pm}A| = hk - h + 1$, the sumset $h_{\pm}A$ contains precisely the $(hk - h + 1)$ integers listed in (3.1). It also contains the $(hk - h + 1)$ integers listed in (3.2). Thus, for all $i = 0, 1, \ldots, k-1$, we have

$$ha_i = -ha_{k-1-i}.$$

This is equivalent to $a_i = -a_{k-1-i}$, for all $i = 0, 1, \ldots, k-1$. This completes the proof of the theorem. $\square$

Chapter 4

Direct and inverse problems for restricted signed sumset $h^{\wedge}_{\pm}A$

In Chapter 3, we studied both direct and inverse problems for the signed sumset $h_{\pm}A$ in the group of integers. In this chapter, we consider the signed sumset $h_{\pm}A$ by restricting the weights of a h-fold sum to at most the absolute value 1. We study both direct and inverse problems for this sumset in the group of integers.

4.1 Introduction

Let G be an additive abelian group, and let $A = \{a_0, a_1, \ldots, a_{k-1}\}$ be a nonempty subset of G. Let $h \geq 1$. Recall that, the h-fold sumset hA, the h-fold restricted sumset $h^{\wedge}A$, and the h-fold signed sumset $h_{\pm}A$ of the set A are defined by

$$hA := \left\{ \sum_{i=0}^{k-1} \lambda_i a_i : \lambda_i \in \mathbb{N} \text{ for } i = 0, 1, \ldots, k-1 \text{ and } \sum_{i=0}^{k-1} \lambda_i = h \right\},$$

$$h^{\wedge}A := \left\{ \sum_{i=0}^{k-1} \lambda_i a_i : \lambda_i \in \{0, 1\} \text{ for } i = 0, 1, \ldots, k-1 \text{ and } \sum_{i=0}^{k-1} \lambda_i = h \right\},$$

and

$$h_{\pm}A := \left\{ \sum_{i=0}^{k-1} \lambda_i a_i : \lambda_i \in \mathbb{Z} \text{ for } i = 0, 1, \ldots, k-1 \text{ and } \sum_{i=0}^{k-1} |\lambda_i| = h \right\},$$

where $1 \leq h \leq k$ in case of $h^{\wedge}A$.

We define the *h-fold restricted signed sumset* of A (for $1 \le h \le k$), denoted by $h^{\wedge}_{\pm}A$, by

$$h^{\wedge}_{\pm}A := \left\{ \sum_{i=0}^{k-1} \lambda_i a_i : \lambda_i \in \{-1,0,1\} \text{ for } i = 0,1,\ldots,k-1 \text{ and } \sum_{i=0}^{k-1} |\lambda_i| = h \right\}.$$

Clearly,

$$h^{\wedge}A \cup h^{\wedge}(-A) \subset h^{\wedge}_{\pm}A.$$

Also, for an integer α, we have

$$h^{\wedge}_{\pm}(\alpha \cdot A) = \alpha \cdot (h^{\wedge}_{\pm}A).$$

The direct problem for $h^{\wedge}_{\pm}A$ is to find the minimum number of elements in $h^{\wedge}_{\pm}A$ in terms of number of elements in A. The inverse problem for $h^{\wedge}_{\pm}A$ is to find the structure of the finite set A for which $|h^{\wedge}_{\pm}A|$ is minimal.

The direct and inverse theorems for hA and $h^{\wedge}A$ are well settled in the group of integers, which are mentioned in Chapter 1. We settled the direct and inverse theorems for $h_{\pm}A$ in the group of integers in Chapter 3. In this chapter, we prove similar direct and inverse theorems for $h^{\wedge}_{\pm}A$ in the group of integers.

4.2 *A* contains only positive integers

Theorem 4.2.1. *Let A be a set of k positive integers, and let $1 \le h \le k$. Then*

$$|h^{\wedge}_{\pm}A| \ge 2(hk - h^2) + \binom{h+1}{2} + 1. \tag{4.1}$$

This lower bound is best possible for $h = 1$, 2 and k.

Proof. Let $A = \{a_0, a_1, \ldots, a_{k-1}\}$, where $0 < a_0 < a_1 < \cdots < a_{k-1}$. For $i = 0,1,\ldots,k-h-1$ and $j = 0,1,\ldots,h$, let

$$s_{i,j} := \sum_{\substack{l=0 \\ l \ne h-j}}^{h} a_{i+l}. \tag{4.2}$$

Let

$$s_{k-h,0} := \sum_{l=0}^{h-1} a_{k-h+l}. \tag{4.3}$$

Each $s_{i,j}$ is a sum of h distinct elements of A, and hence it is in $h^{\wedge}_{\pm}A$. Moreover, for $i = 0,1,\ldots,k-h-1$ and $j = 0,1,\ldots,h-1$, we have

$$s_{i,j} < s_{i,j+1} \quad \text{and} \quad s_{i,h} = s_{i+1,0}.$$

Thus, we get at least $hk-h^2+1$ positive integers in $h^{\wedge}_{\pm}A$. Since $h^{\wedge}_{\pm}A$ is symmetric, the inverses of these $hk-h^2+1$ integers are also in $h^{\wedge}_{\pm}A$ with $-s_{0,0} < s_{0,0}$. So, we get $2(hk-h^2+1)$ integers in $h^{\wedge}_{\pm}A$.

For $i = 0,1,\dots,h-1$ and $j = 0,1,\dots,h-i-1$, define the sequence of integers

$$t_{i,j} := \sum_{\substack{l=0 \\ l\neq j}}^{h-i-1} (-a_l) + a_j + \sum_{m=1}^{i} a_{h-m}. \tag{4.4}$$

Clearly, each $t_{i,j} \in h^{\wedge}_{\pm}A$. Moreover, for $j = 0,1,\dots,h-i-2$, we have

$$t_{i,j} < t_{i,j+1},$$

and for $i = 0,1,\dots,h-2$, we have

$$t_{i,h-i-1} < t_{i+1,0}.$$

Also,

$$-s_{0,0} < t_{0,0} \quad \text{and} \quad t_{h-1,0} = s_{0,0}.$$

Therefore, we get $\binom{h+1}{2}-1$ more integers in $h^{\wedge}_{\pm}A$ which are listed in (4.4). Further, these elements are different from the elements in (4.2) and (4.3). Hence, we get

$$|h^{\wedge}_{\pm}A| \geq 2(hk-h^2) + \binom{h+1}{2} + 1.$$

Next, we show that the lower bound in (4.1) is best possible for $h = 1$, 2 and k.

Let $h = 1$. Then for any finite set A of k positive integers $|1^{\wedge}_{\pm}A| = 2k$ and $2(hk-h^2) + \binom{h+1}{2} + 1 = 2k$.

Now, let $h = 2$ and $A = \{1,3,5,\dots,2k-1\}$. Then

$$2^{\wedge}_{\pm}A = \{-(4k-4), -(4k-6), \dots, -2, 2, 4, \dots, 4k-4\},$$

and hence $|2^{\wedge}_{\pm}A| = 4k-4 = 2(hk-h^2) + \binom{h+1}{2} + 1$.

Finally, let $h = k$ and $A = [1,k]$. It is easy to see that $k^{\wedge}_{\pm}A$ contains either odd integers or even integers. Since $k^{\wedge}_{\pm}A \subset \left[-\binom{k+1}{2}, \binom{k+1}{2}\right]$, we get

$$|k^{\wedge}_{\pm}A| \leq \binom{k+1}{2} + 1.$$

This together with (4.1) give $|k^{\wedge}_{\pm}A| = \binom{k+1}{2} + 1 = 2(hk-h^2) + \binom{h+1}{2} + 1$. This completes the proof of the theorem. $\square$

The next two theorems give the inverse results for the cases $h = 2$ and $h = k$, respectively. For $h = 1$, any set with k elements is extremal, i.e., where the lower bound is achieved.

Theorem 4.2.2. *Let A be a set of k (≥ 2) positive integers such that $|2^{\wedge}_{\pm}A| = 4k-4$. Then $A = \{a_0, a_1\}$ with $a_0 < a_1$, if $k = 2$, and $A = d \cdot \{1, 3, \ldots, 2k-1\}$, for some positive integer d, if $k \geq 3$.*

Proof. Let $A = \{a_0, a_1, \ldots, a_{k-1}\}$, where $0 < a_0 < a_1 < \cdots < a_{k-1}$. Let

$$|2^{\wedge}_{\pm}A| = 4k-4.$$

First, let $k = 2$. Then

$$2^{\wedge}_{\pm}A = \{a_0 + a_1, a_0 - a_1, -a_0 + a_1, -a_0 - a_1\},$$

where

$$-a_0 - a_1 < a_0 - a_1 < -a_0 + a_1 < a_0 + a_1.$$

Thus, for every set A of two positive integers $|2^{\wedge}_{\pm}A| = 4 = 4k-4$.

Next, let $k = 3$. Then

$$2^{\wedge}_{\pm}A = \{a_0 + a_1, a_0 - a_1, -a_0 + a_1, -a_0 - a_1, a_0 + a_2, a_0 - a_2, -a_0 + a_2, -a_0 - a_2, a_1 + a_2,$$
$$a_1 - a_2, -a_1 + a_2, -a_1 - a_2\},$$

where

$$-a_1 - a_2 < -a_0 - a_2 < -a_0 - a_1 < a_0 - a_1 < -a_0 + a_1 < a_0 + a_1 < a_0 + a_2 < a_1 + a_2. \quad (4.5)$$

If $|2^{\wedge}_{\pm}A| = 4k-4 = 8$, then $2^{\wedge}_{\pm}A$ contains precisely the integers listed in (4.5). Since

$$-a_0 - a_2 < a_0 - a_2 < a_0 - a_1,$$

we get $a_0 - a_2 = -a_0 - a_1$, i.e., $a_2 - a_1 = 2a_0$.

Similarly, as

$$a_0 - a_1 < a_2 - a_1 < a_2 - a_0 = a_0 + a_1,$$

we get $a_2 - a_1 = a_1 - a_0$. Hence, $A = a_0 \cdot \{1, 3, 5\}$.

Now, let $k = 4$. Then

$$\begin{aligned} 2^{\wedge}_{\pm}A = \{&a_0+a_1, a_0-a_1, -a_0+a_1, -a_0-a_1, a_0+a_2, a_0-a_2, -a_0+a_2, -a_0-a_2, a_0+a_3,\\ &a_0-a_3, -a_0+a_3, -a_0-a_3, a_1+a_2, a_1-a_2, -a_1+a_2, -a_1-a_2, a_1+a_3, a_1-a_3,\\ &-a_1+a_3, -a_1-a_3, a_2+a_3, a_2-a_3, -a_2+a_3, -a_2-a_3\}, \end{aligned}$$

where

$$\begin{aligned} &-a_2-a_3 < -a_1-a_3 < -a_1-a_2 < -a_0-a_2 < -a_0-a_1 < a_0-a_1 < -a_0+a_1\\ &< a_0+a_1 < a_0+a_2 < a_1+a_2 < a_1+a_3 < a_2+a_3. \end{aligned} \tag{4.6}$$

If $|2^{\wedge}_{\pm}A| = 4k-4 = 12$, then $2^{\wedge}_{\pm}A$ contains precisely the integers listed in (4.6). Since

$$a_0+a_2 < a_0+a_3 < a_1+a_3,$$

it follows from (4.6) that $a_0+a_3 = a_1+a_2$, which is equivalent to $a_3-a_2 = a_1-a_0$.

Similarly, since

$$-a_0+a_1 < -a_0+a_2 < a_0+a_2,$$

we have $-a_0+a_2 = a_0+a_1$, or $a_2-a_1 = 2a_0$.

We also have

$$-a_1-a_2 = -a_0-a_3 < a_0-a_3 < a_0-a_2 = -a_0-a_1.$$

Therefore, $a_0-a_3 = -a_0-a_2$, or $a_3-a_2 = 2a_0$. Hence, $A = a_0 \cdot \{1,3,5,7\}$ is the extremal set for all $a_0 > 0$.

Finally, let $k \geq 5$, and $|2^{\wedge}_{\pm}A| = 4k-4$. From Theorem 4.2.1 it follows that the sumset $h^{\wedge}_{\pm}A$ contains precisely the integers listed in (4.2), (4.3) and (4.4), for $h = 2$. Since $2^{\wedge}A \subset [a_0+a_1, a_{k-2}+a_{k-1}]$ and there are exactly $2k-3$ integers in (4.2) and (4.3) between a_0+a_1 and $a_{k-2}+a_{k-1}$, Theorem 1.4.12 implies that the set A is in arithmetic progression. That is, the common difference $d = a_1-a_0 = a_2-a_1 = \cdots = a_{k-1}-a_{k-2}$.

Again, since

$$-a_0-a_2 < -a_0-a_1 < a_0-a_1,$$

and

$$-a_0-a_2 < a_0-a_2 < a_0-a_1,$$

we get $a_2 - a_1 = 2a_0$. Hence $A = a_0 \cdot \{1, 3, \ldots, 2k-1\}$. This completes the proof of the theorem. $\square$

Theorem 4.2.3. *Let A be a set of k (≥ 3) positive integers such that*

$$|k^{\wedge}_{\pm}A| = \binom{k+1}{2} + 1.$$

Then $A = \{a_0, a_1, a_0 + a_1\}$ with $a_0 < a_1$, if $k = 3$, and $A = d \cdot [1, k]$ for some positive integer d, if $k \geq 4$.

Proof. First, let $k = 3$ and $A = \{a_0, a_1, a_2\}$, where $0 < a_0 < a_1 < a_2$. Then

$$\begin{aligned} 3^{\wedge}_{\pm}A = \{&a_0 + a_1 + a_2, a_0 + a_1 - a_2, a_0 - a_1 + a_2, a_0 - a_1 - a_2, -a_0 + a_1 + a_2, -a_0 + a_1 - a_2, \\ &-a_0 - a_1 + a_2, -a_0 - a_1 - a_2\}, \end{aligned}$$

where, we have

$$\begin{aligned} -a_0 - a_1 - a_2 &< a_0 - a_1 - a_2 < -a_0 + a_1 - a_2 < -a_0 - a_1 + a_2 < a_0 - a_1 + a_2 \\ &< -a_0 + a_1 + a_2 < a_0 + a_1 + a_2. \end{aligned} \tag{4.7}$$

If $|3^{\wedge}_{\pm}A| = \binom{4}{2} + 1 = 7$, then $3^{\wedge}_{\pm}A$ contains precisely the seven integers in (4.7). Since

$$-a_0 + a_1 - a_2 < a_0 + a_1 - a_2 < a_0 - a_1 + a_2,$$

we have $a_0 + a_1 - a_2 = -a_0 - a_1 + a_2$, i.e., $a_2 - a_1 = a_0$. Hence, $A = \{a_0, a_1, a_0 + a_1\}$ is an extremal set.

Next, let $k = 4$ and $A = \{a_0, a_1, a_2, a_3\}$, where $0 < a_0 < a_1 < a_2 < a_3$. Let $|4^{\wedge}_{\pm}A| = \binom{5}{2} + 1 = 11$. Then $4^{\wedge}_{\pm}A$ contains precisely the following sequence of integers written in an increasing order.

$$\begin{aligned} -a_0 - a_1 - a_2 - a_3 &< a_0 - a_1 - a_2 - a_3 < -a_0 + a_1 - a_2 - a_3 < -a_0 - a_1 + a_2 - a_3 \\ &< -a_0 - a_1 - a_2 + a_3 < a_0 - a_1 - a_2 + a_3 < -a_0 + a_1 - a_2 + a_3 < -a_0 - a_1 + a_2 + a_3 \\ &< a_0 - a_1 + a_2 + a_3 < -a_0 + a_1 + a_2 + a_3 < a_0 + a_1 + a_2 + a_3. \end{aligned} \tag{4.8}$$

Since the sumset $4^{\wedge}_{\pm}A$ is symmetric, from (4.8) it follows that

$$-a_0 - a_1 + a_2 - a_3 = -(-a_0 - a_1 + a_2 + a_3),$$

$$-a_0 - a_1 - a_2 + a_3 = -(-a_0 + a_1 - a_2 + a_3),$$

and

$$a_0 - a_1 - a_2 + a_3 = 0.$$

These above three equations give $a_3 - a_2 = a_2 - a_1 = a_1 - a_0 = a_0$. Hence, $A = a_0 \cdot \{1,2,3,4\}$.

Finally, let $k \geq 5$ and $A = \{a_0, a_1, \ldots, a_{k-1}\}$, where $0 < a_0 < a_1 < \cdots < a_{k-1}$. Let

$$|k^{\wedge}_{\pm}A| = \binom{k+1}{2} + 1.$$

Then, $k^{\wedge}_{\pm}A$ contains precisely the integers listed in (4.4) (for $h = k$), with one more integer $-a_0 - a_1 - \cdots - a_{k-1}$. For $j = 1,2,\ldots,k-1$, set

$$u_j = a_0 + \sum_{\substack{l=1 \\ l \neq j}}^{k-1} (-a_l) + a_j. \tag{4.9}$$

Clearly,

$$t_{0,1} < u_1 < u_2 < \cdots < u_{k-2} < u_{k-1} = t_{1,0}.$$

So, there are exactly $k-2$ distinct integers in (4.9) between $t_{0,1}$ and $t_{1,0}$. Therefore, (4.4) and (4.9) implies that, for $j = 1,2,\ldots,k-2$,

$$t_{0,j+1} = u_j.$$

This is equivalent to $a_{j+1} - a_j = a_0$, for $j = 1,2,\ldots,k-2$. That is

$$a_{k-1} - a_{k-2} = \cdots = a_3 - a_2 = a_2 - a_1 = a_0.$$

Again, since $k^{\wedge}_{\pm}A$ is symmetric, we have $-t_{0,0} = t_{k-3,0}$, i.e.,

$$-(-a_0 - a_1 - a_2 - a_3 + a_4 - \cdots - a_{k-1}) = a_0 - a_1 - a_2 + a_3 + a_4 + \cdots + a_{k-1}.$$

In other words,

$$a_4 = a_1 + a_2.$$

Since $a_3 - a_2 = a_0$, we get $a_4 - a_3 = a_1 - a_0$. Hence, $A = a_0 \cdot [1,k]$. This completes the proof of the theorem. □

For $h \geq 3$, we believe that the sumset $h^{\wedge}_{\pm}A$ contains at least $2hk - h^2 + 1$ integers. So we conjecture that

Conjecture 4.2.4. *Let A be a set of k (≥ 4) positive integers, and let $3 \leq h \leq k-1$. Then*

$$|h^{\wedge}_{\pm}A| \geq 2hk - h^2 + 1.$$

This lower bound is best possible.

The following example confirms the conjecture in a special case. Also in Theorem 4.2.5, we prove the conjecture for $h = 3$. Moreover, we also give the inverse result in this case.

Example 1 (Super increasing sequence). Let $A = \{a_0, a_1, \ldots, a_{k-1}\}$, where $k \geq 6$, $a_0 > 0$, and $a_i > \sum_{j=0}^{i-1} a_j$, for $i = 1, 2, \ldots, k-1$.

Let $h \geq 5$. Clearly, the sumset $h^{\wedge}_{\pm}A$ contains at least $2(hk - h^2) + \binom{h+1}{2} + 1$ integers, which are listed in (4.2), (4.3) and (4.4).

For $j = 1, 2, \ldots, h-2$, consider the integers $-2a_0 + s_{0,j}$. Clearly

$$-2a_0 + s_{0,j} = -a_0 + \sum_{\substack{l=1 \\ l \neq h-j}}^{h} a_l \in h^{\wedge}_{\pm}A,$$

and

$$s_{0,j-1} < -2a_0 + s_{0,j} < s_{0,j}.$$

So, we get $h-2$ extra positive integers $h^{\wedge}_{\pm}A$, which are not present in (4.2), (4.3) and (4.4).

Since

$$-s_{0,j} < -(-2a_0 + s_{0,j}) < -s_{0,j-1},$$

we get $h-2$ further extra integers in $h^{\wedge}_{\pm}A$.

For $j = 2, 3, \ldots, h-3$, consider the integers

$$t_{0,h-j-1} < -t_{j,h-j-2} < -t_{j,h-j-3} < \cdots < -t_{j,0} < -t_{j-1,h-j} < t_{0,h-j}. \tag{4.10}$$

Then, for each $j = 2, 3, \ldots, h-3$, we get $h-j$ extra integers. Therefore, we get $3 + 4 + \cdots + (h-2) = \binom{h}{2} - h - 2$ more integers in $h^{\wedge}_{\pm}A$ which are listed in (4.10) and never counted before. We also get one more integer, i.e., $-t_{h-3,2}$ such that $t_{0,1} < -t_{h-3,2} < t_{0,2}$. So, we get $2(h-2) + \binom{h}{2} - h - 2 + 1 = \binom{h}{2} + (h-5)$ extra integers. Hence, by and large, we have

$$|h^{\wedge}_{\pm}A| \geq 2hk - h^2 + h - 4 \geq 2hk - h^2 + 1.$$

Theorem 4.2.5. *Let A be a set of k* (≥ 4) *positive integers. Then*

$$|3^{\wedge}_{\pm}A| \geq 6k-8. \tag{4.11}$$

Moreover, if $|3^{\wedge}_{\pm}A| = 6k-8$*, then* $A = d\cdot\{1,3,5,\ldots,2k-1\}$ *for some positive integer d.*

Proof. Let $A = \{a_0, a_1, \ldots, a_{k-1}\}$, where $0 < a_0 < a_1 < \cdots < a_{k-1}$. From Theorem 4.2.1, we have $|3^{\wedge}_{\pm}A| \geq 6k-11$.

Next, we show that there exist at least three extra integers in $3^{\wedge}_{\pm}A$ which are not counted in Theorem 4.2.1. Consider the following thirteen integers of $3^{\wedge}_{\pm}A$:

$$\begin{aligned} &-a_1-a_2-a_3 < -a_0-a_2-a_3 < -a_0-a_1-a_3 < -a_0-a_1-a_2 < a_0-a_1-a_2 \\ &\quad < -a_0+a_1-a_2 < -a_0-a_1+a_2 < a_0-a_1+a_2 < -a_0+a_1+a_2 < a_0+a_1+a_2 \\ &\quad < a_0+a_1+a_3 < a_0+a_2+a_3 < a_1+a_2+a_3. \end{aligned} \tag{4.12}$$

We exhibit at least three extra integers between $-a_1-a_2-a_3$ and $a_1+a_2+a_3$ in all possible cases.

Case 1: Let $a_3-a_2 < a_3-a_1 < 2a_0$. Then, we get at least two extra positive integers $-a_0+a_1+a_3$ and $-a_0+a_2+a_3$ which are not present in (4.12) such that

$$-a_0+a_1+a_2 < -a_0+a_1+a_3 < -a_0+a_2+a_3 < a_0+a_1+a_2.$$

Case 2: Let $a_3-a_2 < 2a_0 < a_3-a_1$. Then, we get at least two extra positive integers $-a_0-a_1+a_3$ and $-a_0+a_1+a_3$ which are not present in (4.12) such that

$$\begin{aligned} &-a_0-a_1+a_2 < -a_0-a_1+a_3 < a_0-a_1+a_2 < -a_0+a_1+a_2 < -a_0+a_1+a_3 \\ &\quad < a_0+a_1+a_2. \end{aligned}$$

Case 3: Let $a_3-a_1 > a_3-a_2 > 2a_0$. Then, we get an extra positive integer $-a_0+a_1+a_3$ such that

$$a_0+a_1+a_2 < -a_0+a_1+a_3 < a_0+a_1+a_3.$$

To exhibit one further extra positive integer consider the following subcases

Subcase (i) $(a_2-a_1 < 2a_0)$. We get one more extra positive integer $-a_0+a_2+a_3$ such that

$$a_0+a_1+a_2 < -a_0+a_1+a_3 < -a_0+a_2+a_3 < a_0+a_1+a_3.$$

Subcase (ii) $(a_2 - a_1 > 2a_0)$. We get one more extra positive integer $-a_0 + a_2 + a_3$ such that

$$a_0 + a_1 + a_2 < -a_0 + a_1 + a_3 < a_0 + a_1 + a_3 < -a_0 + a_2 + a_3 < a_0 + a_2 + a_3.$$

Subcase (iii) $(a_2 - a_1 = 2a_0)$. In this subcase, we get two positive integers $a_0 - a_1 + a_3$ and $a_0 - a_2 + a_3$ such that

$$a_0 - a_1 + a_2 = 3a_0 < a_0 - a_2 + a_3 < a_0 - a_1 + a_3 < -a_0 + a_1 + a_3 < a_0 + a_1 + a_3.$$

But, we already have

$$a_0 - a_1 + a_2 < -a_0 + a_1 + a_2 < a_0 + a_1 + a_2 < -a_0 + a_1 + a_3 < a_0 + a_1 + a_3.$$

Thus, except in the cases $a_0 - a_2 + a_3 = -a_0 + a_1 + a_2$ and $a_0 - a_1 + a_3 = a_0 + a_1 + a_2$, we get at least one extra positive integer and hence we are done.

So, let

$$a_0 - a_2 + a_3 = -a_0 + a_1 + a_2,$$

and

$$a_0 - a_1 + a_3 = a_0 + a_1 + a_2.$$

These two equations imply

$$2(a_2 - a_0) = a_3 - a_1 = a_1 + a_2.$$

Consider the integer $-a_0 - a_2 + a_3$. We have

$$-a_0 - a_1 + a_2 = a_0 < -a_0 - a_2 + a_3 < -a_0 - a_1 + a_3 = -a_0 + a_1 + a_2.$$

If $-a_0 - a_2 + a_3 \neq a_0 - a_1 + a_2$, then we are done, as we get one extra positive integer. Otherwise, let

$$-a_0 - a_2 + a_3 = a_0 - a_1 + a_2,$$

or

$$a_3 - a_2 = 2a_0 - a_1 + a_2 = 4a_0.$$

Therefore, we have

$$a_3 - a_1 = a_3 - a_2 + a_2 - a_1 = 6a_0,$$

and

$$a_2 - a_0 = \frac{1}{2}(a_3 - a_1) = 3a_0.$$

Solving these equations we get $a_1 = 2a_0$, $a_2 = 4a_0$ and $a_3 = 8a_0$. Thus, we get one extra positive integer $-a_1 + a_2 + a_3$ such that

$$-a_0 + a_1 + a_3 = 9a_0 < 10a_0 = -a_1 + a_2 + a_3 < 11a_0 = a_0 + a_1 + a_3.$$

Hence, we get at least two extra positive integers in every case.

Case 4: Let $a_3 - a_2 < a_3 - a_1 = 2a_0$. Then we get at least two extra positive integers $-a_0 - a_1 + a_3$ and $-a_0 + a_1 + a_3$ which are not present in (4.12) such that

$$\begin{aligned} &-a_0 - a_1 + a_2 < -a_0 - a_1 + a_3 = a_0 < a_0 - a_1 + a_2 < -a_0 + a_1 + a_2 < -a_0 + a_1 + a_3 \\ &\quad < a_0 + a_1 + a_2. \end{aligned}$$

Case 5: Let $a_3 - a_1 > a_3 - a_2 = 2a_0$. We consider the following three subcases:

Subcase (i) Let $a_2 - a_1 < 2a_0$. Then, we get at least two extra positive integers $-a_0 - a_2 + a_3$ and $-a_0 + a_2 + a_3$ such that

$$\begin{aligned} &-a_0 - a_1 + a_2 < a_0 = -a_0 - a_2 + a_3 < a_0 - a_1 + a_2 < -a_0 + a_1 + a_2 < a_0 + a_1 + a_2 \\ &\quad < -a_0 + a_2 + a_3 < a_0 + a_1 + a_3. \end{aligned}$$

Subcase (ii) Let $a_2 - a_1 > 2a_0$. Then, we get two extra positive integers $-a_0 - a_2 + a_3$ and $-a_0 + a_2 + a_3$ such that

$$\begin{aligned} &a_0 + a_1 - a_2 < -a_0 < a_0 = -a_0 - a_2 + a_3 < -a_0 - a_1 + a_2 < a_0 - a_1 + a_2 < -a_0 + a_1 + a_2 \\ &\quad < a_0 + a_1 + a_2 < a_0 + a_1 + a_3 < -a_0 + a_2 + a_3 < a_0 + a_2 + a_3. \end{aligned}$$

Subcase (iii) Let $a_2 - a_1 = 2a_0$. We get an extra positive integer $a_1 - a_2 + a_3$ such that

$$a_0 - a_1 + a_2 = 3a_0 < 2a_0 + a_1 = a_1 - a_2 + a_3 < a_0 + 2a_1 = -a_0 + a_1 + a_2.$$

If $a_1 - a_0 > 2a_0$, then we get one more extra positive integer $a_0 - a_1 + a_3$ such that

$$a_0 - a_1 + a_2 < a_0 - a_1 + a_3 < -2a_0 + a_3 = a_1 - a_2 + a_3 < -a_0 + a_1 + a_2.$$

If $a_1 - a_0 < 2a_0$, then we get one more extra positive integer $-a_1 + a_2 + a_3$ such that

$$a_0 - a_1 + a_2 < a_1 - a_2 + a_3 < -a_0 + a_1 + a_2 < a_0 + a_1 + a_2 < -a_1 + a_2 + a_3 < a_0 + a_1 + a_3.$$

Let $a_1 - a_0 = 2a_0$. Then, the integer $-a_0 - a_1 + a_2 = a_0$ is positive. So, the inverse of this integer gives one more extra integer with

$$-a_0 + a_1 - a_2 < a_0 + a_1 - a_2 < -a_0 - a_1 + a_2 < a_0 - a_1 + a_2.$$

From the above discussion, we conclude that except in the case $a_1 - a_0 = a_2 - a_1 = a_3 - a_2 = 2a_0$, we get at least two extra positive integers in $3^{\wedge}_{\pm}A$, which are not present in (4.12). Since, the inverses of these integers are negative, we get two more extra integers. So, total we get at least four extra integers in $3^{\wedge}_{\pm}A$, which are not included in (4.12). In case $a_1 - a_0 = a_2 - a_1 = a_3 - a_2 = 2a_0$, we get at least three extra integers. Therefore, in each case we get at least three extra integers in $3^{\wedge}_{\pm}A$, which are not present in (4.12). Hence, $|3^{\wedge}_{\pm}A| \geq 6k - 8$. This establishes (4.11).

Moreover, if $|3^{\wedge}_{\pm}A| = 6k - 8$, then $a_1 - a_0 = a_2 - a_1 = a_3 - a_2 = 2a_0$.

Now, let $|3^{\wedge}_{\pm}A| = 6k - 8$. If $k = 4$, then we are done, as $A = \{a_0, 3a_0, 5a_0, 7a_0\} = a_0 \cdot \{1, 3, 5, 7\}$.

Let $k \geq 5$, and let $A' = A \setminus \{a_0\}$. Therefore, A' is a nonempty set of $k - 1$ positive integers with $3^{\wedge}A' \subset [a_1 + a_2 + a_3, a_{k-3} + a_{k-2} + a_{k-1}]$. Since $|3^{\wedge}_{\pm}A| = 6k - 8$, from the above proof it follows that $|3^{\wedge}A'| = 3k - 11$. Thus, Theorem 1.4.12 implies that the set A' is in arithmetic progression, i.e.,

$$a_{k-1} - a_{k-2} = \cdots = a_2 - a_1 = d.$$

Hence

$$A = a_0 \cdot \{1, 3, 5, \ldots, 2k - 1\}.$$

This completes the proof of the theorem. □

Now, we conjecture the inverse result as follows:

Conjecture 4.2.6. *Let A be a set of k (≥ 4) positive integers, and let $3 \leq h \leq k - 1$. If $|h^{\wedge}_{\pm}A| = 2hk - h^2 + 1$, then $A = d \cdot \{1, 3, \ldots, 2k - 1\}$, for some positive integer d.*

Theorem 4.2.5 confirms Conjecture 4.2.6 for $h = 3$.

4.3 A contains nonnegative integers with $0 \in A$

Theorem 4.3.1. *Let A be a set of k nonnegative integers with $0 \in A$, and let $1 \leq h \leq k$. Then*

$$|h^{\wedge}_{\pm}A| \geq 2(hk - h^2) + \binom{h}{2} + 1. \tag{4.13}$$

This lower bound is best possible for $h = 1$, 2 and k.

Proof. Let $A = \{a_0, a_1, \ldots, a_{k-1}\}$, where $0 = a_0 < a_1 < \cdots < a_{k-1}$. From (4.2) and (4.3), it follows that $h^{\wedge}_{\pm}A$ contains at least $hk - h^2 + 1$ positive integers and hence including their inverses, $h^{\wedge}_{\pm}A$ contains at least $2(hk - h^2 + 1)$ integers.

Again, since $a_0 = 0$, from (4.4) it follows that, for $i = 0, 1, \ldots, h-2$, we have $t_{i,h-i-1} = t_{i+1,0}$, $-s_{0,0} = t_{0,0}$ and $t_{h-1,0} = s_{0,0}$. Thus, we get $\binom{h}{2} - 1$ extra integers in $h^{\wedge}_{\pm}A$ from the list (4.4). Hence

$$|h^{\wedge}_{\pm}A| \geq 2(hk - h^2) + \binom{h}{2} + 1.$$

Next, we show that the lower bound in (4.13) is best possible for $h = 1$, 2 and k.

If $h = 1$, then for any finite set A of k nonnegative integers with $0 \in A$, we have $|1^{\wedge}_{\pm}A| = 2k - 1$ and $2(hk - h^2) + \binom{h}{2} + 1 = 2k - 1$.

Now, let $h = 2$ and $A = [0, k-1]$. Then

$$2^{\wedge}_{\pm}A = [-(2k-3), (2k-3)] \setminus \{0\}.$$

So, $|2^{\wedge}_{\pm}A| = 4k - 6 = 2(hk - h^2) + \binom{h}{2} + 1$.

Finally, let $h = k$ and $A = [0, k-1]$. Then, it is easy to see that $k^{\wedge}_{\pm}A$ contains either odd integers or even integers. Since $k^{\wedge}_{\pm}A \subset \left[-\binom{k}{2}, \binom{k}{2}\right]$, we get

$$|k^{\wedge}_{\pm}A| \leq \binom{k}{2} + 1.$$

This together with (4.13) give $|k^{\wedge}_{\pm}A| = \binom{k}{2} + 1 = 2(hk - h^2) + \binom{h}{2} + 1$. This completes the proof of the theorem. □

We now give inverse results for $h = 2$ and $h = k$ in theorems 4.3.2 and 4.3.3, respectively.

Theorem 4.3.2. *Let A be a set of k (≥ 2) nonnegative integers with $0 \in A$ and $|2^{\wedge}_{\pm}A| = 4k - 6$. Then $A = \{0, a\}$ with $a > 0$, if $k = 2$, and $A = d \cdot [0, k-1]$ for some positive integer d, if $k \geq 3$.*

Proof. Let $A = \{0, a_1, a_2, \ldots, a_{k-1}\}$, where $0 < a_1 < a_2 < \cdots < a_{k-1}$. Let

$$|2^{\wedge}_{\pm}A| = 4k-6.$$

First, let $k = 2$. Then $2^{\wedge}_{\pm}A = \{a_1, -a_1\}$. So, $|2^{\wedge}_{\pm}A| = 2 = \binom{2}{2} + 1$. Thus, every set A of two integers with $0 \in A$ is an extremal set.

Next, let $k = 3$. Then

$$2^{\wedge}_{\pm}A = \{a_1, -a_1, a_2, -a_2, a_1+a_2, a_1-a_2, -a_1+a_2, -a_1-a_2\},$$

where

$$-a_1 - a_2 < -a_2 < -a_1 < a_1 < a_2 < a_1 + a_2. \tag{4.14}$$

If $|2^{\wedge}_{\pm}A| = 6 = 4k-6$, then $2^{\wedge}_{\pm}A$ contains precisely the integers listed in (4.14). Since

$$-a_2 < a_1 - a_2 < a_1,$$

from (4.14) it follows that $a_1 - a_2 = -a_1$, i.e., $a_2 - a_1 = a_1$. Hence, $A = \{0, a_1, 2a_1\}$.

Now, let $k = 4$. Then

$$2^{\wedge}_{\pm}A = \{a_1, -a_1, a_2, -a_2, a_3, -a_3, a_1+a_2, a_1-a_2, -a_1+a_2, -a_1-a_2, a_1+a_3, a_1-a_3,$$
$$-a_1+a_3, -a_1-a_3, a_2+a_3, a_2-a_3, -a_2+a_3, -a_2-a_3\},$$

where

$$-a_2-a_3 < -a_1-a_3 < -a_1-a_2 < -a_2 < -a_1 < a_1 < a_2 < a_1+a_2 < a_1+a_3 < a_2+a_3. \tag{4.15}$$

If $|2^{\wedge}_{\pm}A| = 10 = 4k-6$, then $2^{\wedge}_{\pm}A$ contains precisely the integers listed in (4.15). Since

$$a_2 < a_3 < a_1 + a_3,$$

from (4.15) it follows that $a_3 = a_1 + a_2$, or $a_3 - a_2 = a_1$.

Similarly,

$$-a_2 < a_1 - a_2 < a_1$$

imply $a_1 - a_2 = -a_1$, or $a_2 - a_1 = a_1$. Hence, $A = \{0, a_1, 2a_1, 3a_1\}$.

Finally, let $k \geq 5$, and $|2^{\wedge}_{\pm}A| = 4k-6$. From Theorem 1.4.11 we know that $|2^{\wedge}A| \geq 2k-3$, and since $2^{\wedge}A \cap (-2^{\wedge}A) = \emptyset$, we get $|2^{\wedge}A| = 2k-3$. Therefore, by Theorem 1.4.12, the set A is an arithmetic progression with the common difference $a_{k-1} - a_{k-2} = \cdots = a_1 - a_0 = a_1$. Hence, $A = a_1 \cdot [0, k-1]$. This completes the proof of the theorem. □

Theorem 4.3.3. *Let A be a set of k (≥ 3) nonnegative integers with $0 \in A$ and $|k^{\wedge}_{\pm}A| = \binom{k}{2} + 1$. Then*

$$A = \begin{cases} \{0, a_1, a_2\},\ 0 < a_1 < a_2,\ \ \textit{if } k = 3; \\ \{0, a_1, a_2, a_1 + a_2\},\ 0 < a_1 < a_2,\ \ \textit{if } k = 4; \\ d \cdot [0, k-1],\ \textit{for some positive integer } d,\ \ \textit{if } k \geq 5. \end{cases}$$

Proof. Let $A = \{0, a_1, a_2, \ldots, a_{k-1}\}$, where $0 < a_1 < a_2 < \cdots < a_{k-1}$. Let

$$|k^{\wedge}_{\pm}A| = \binom{k}{2} + 1.$$

First, let $k = 3$. Then

$$3^{\wedge}_{\pm}A = \{a_1 + a_2, a_1 - a_2, -a_1 + a_2, -a_1 - a_2\},$$

where

$$-a_1 - a_2 < a_1 - a_2 < -a_1 + a_2 < a_1 + a_2.$$

So, $|3^{\wedge}_{\pm}A| = 4 = \binom{3}{2} + 1$. Thus, A is an extremal set.

Next, let $k = 4$. Then

$$\begin{aligned} 4^{\wedge}_{\pm}A = \{&a_1 + a_2 + a_3, a_1 + a_2 - a_3, a_1 - a_2 + a_3, a_1 - a_2 - a_3, -a_1 + a_2 + a_3, -a_1 + a_2 - a_3, \\ &- a_1 - a_2 + a_3, -a_1 - a_2 - a_3\}, \end{aligned}$$

where

$$\begin{aligned} &- a_1 - a_2 - a_3 < a_1 - a_2 - a_3 < -a_1 + a_2 - a_3 < -a_1 - a_2 + a_3 < a_1 - a_2 + a_3 \\ &\quad < -a_1 + a_2 + a_3 < a_1 + a_2 + a_3. \end{aligned} \tag{4.16}$$

If $|4^{\wedge}_{\pm}A| = \binom{4}{2} + 1 = 7$, then $4^{\wedge}_{\pm}A$ contains precisely the above seven integers in (4.16). Since

$$-a_1 + a_2 - a_3 < a_1 + a_2 - a_3 < a_1 - a_2 + a_3,$$

we have $a_1 + a_2 - a_3 = -a_1 - a_2 + a_3$, i.e., $a_3 - a_2 = a_1$. Hence, $A = \{0, a_1, a_2, a_1 + a_2\}$ is an extremal set.

Finally, let $k \geq 5$, and $|k^{\wedge}_{\pm}A| = \binom{k}{2} + 1$. Let $A' = A \setminus \{0\}$. So, A' is a nonempty set of $k-1$ positive integers with $k^{\wedge}_{\pm}A = (k-1)^{\wedge}_{\pm}A'$. Since $|(k-1)^{\wedge}_{\pm}A'| = |k^{\wedge}_{\pm}A| = \binom{k}{2} + 1$, Theorem 4.2.3

implies that the set A' is in arithmetic progression with the common difference a_1, the smallest element in A'. Hence $A = a_1 \cdot \{0,1,2,\ldots,k-1\} = a_1 \cdot [0,k-1]$. This completes the proof of the theorem. □

For $h \geq 3$, we believe that the sumset $h^{\wedge}_{\pm}A$ contains at least $2hk - h(h+1) + 1$ integers. So, we conjecture that

Conjecture 4.3.4. *Let A be a set of k (≥ 5) nonnegative integers with $0 \in A$, and let $3 \leq h \leq k-1$. Then*

$$|h^{\wedge}_{\pm}A| \geq 2hk - h(h+1) + 1.$$

This lower bound is best possible.

We confirm Conjecture 4.3.4 for $h = 3$. Moreover, we also give the inverse result in this case.

Theorem 4.3.5. *Let A be a set of k (≥ 5) nonnegative integers with $0 \in A$. Then*

$$|3^{\wedge}_{\pm}A| \geq 6k - 11. \tag{4.17}$$

Moreover, if $|3^{\wedge}_{\pm}A| = 6k - 11$, then $A = d \cdot [0,k-1]$.

Proof. Let $A = \{0, a_1, a_2, \ldots, a_{k-1}\}$, where $0 < a_1 < a_2 < \cdots < a_{k-1}$. From Theorem 4.3.1, it follows that $|3^{\wedge}_{\pm}A| \geq 6k - 14$.

Next, we show that there exists at least three extra integers in $3^{\wedge}_{\pm}A$ which are not counted in Theorem 4.3.1. Consider the following twelve integers of $3^{\wedge}_{\pm}A$:

$$\begin{aligned} & -a_1 - a_2 - a_4 < -a_1 - a_2 - a_3 < -a_2 - a_3 < -a_1 - a_3 < -a_1 - a_2 < a_1 - a_2 < -a_1 + a_2 \\ & < a_1 + a_2 < a_1 + a_3 < a_2 + a_3 < a_1 + a_2 + a_3 < a_1 + a_2 + a_4. \end{aligned} \tag{4.18}$$

We exhibit at least three extra integers in between $-a_1 - a_2 - a_4$ and $a_1 + a_2 + a_4$ in all possible cases.

Case 1: Let $a_3 - a_2 < a_1$. Then, we have

$$a_1 - a_2 < -a_2 + a_3 < -a_1 + a_3 < a_1 + a_2,$$

and

$$a_1 - a_2 < -a_1 + a_2 < -a_1 + a_3.$$

If $-a_2+a_3 \neq -a_1+a_2$, then we get two extra positive integers $-a_2+a_3$ and $-a_1+a_3$. So, let $-a_2+a_3 = -a_1+a_2$. If $a_3-a_1 < a_1$, then we get two extra positive integers $-a_1+a_3$ and $-a_1+a_2+a_3$ such that

$$-a_1+a_2 < -a_1+a_3 < -a_1+a_2+a_3 < a_1+a_2.$$

If $a_3-a_1 > a_1$, then we get two extra positive integers $-a_1+a_3$ and $-a_1+a_2+a_3$ such that

$$-a_1+a_2 < -a_1+a_3 < a_1+a_2 < -a_1+a_2+a_3 < a_1+a_3.$$

If $a_3-a_1 = a_1$, then also we get two extra positive integers $-a_1+a_3$ and $a_1-a_2+a_3$ such that

$$-a_1+a_2 < -a_1+a_3 < a_1-a_2+a_3 < a_1+a_2.$$

Case 2: Let $a_3-a_2 = a_1$. Then, by similar arguments to Case 1, unless $-a_2+a_3 = -a_1+a_2$, we get two extra positive integers $-a_2+a_3$ and $-a_1+a_3$.

Let $-a_2+a_3 = -a_1+a_2$. Then we get an extra positive integer $-a_1+a_3$ such that

$$-a_1+a_2 < -a_1+a_3 < a_1+a_2.$$

Again, we get one more extra integer $-a_1-a_2+a_3 = 0$ such that

$$a_1-a_2 < -a_1-a_2+a_3 < -a_1+a_2.$$

Case 3: Let $a_3-a_2 > a_1$. So, $a_3-a_1 > a_1$.

Subcase (i). Let $-a_1+a_3 < a_1+a_2$. Unless $-a_2+a_3 = -a_1+a_2$, we get two extra positive integers $-a_2+a_3$ and $-a_1+a_3$ which are not included in (4.18).

Let $-a_2+a_3 = -a_1+a_2$. Then also we get two extra positive integers $-a_1+a_3$ and $-a_1+a_2+a_3$ such that

$$-a_1+a_2 < -a_1+a_3 < a_1+a_2 < a_1+a_3 < -a_1+a_2+a_3 < a_2+a_3.$$

Subcase (ii). Let $-a_1+a_3 > a_1+a_2$. Then, we get an extra positive integer $-a_1+a_3$ such that

$$a_1+a_2 < -a_1+a_3 < a_1+a_3.$$

If $-a_2+a_3 \neq -a_1+a_2$ and $-a_2+a_3 \neq a_1+a_2$, then we are done as we get one more extra positive integer $-a_2+a_3$.

If $-a_2+a_3=-a_1+a_2$, then we get an extra positive integer $-a_1-a_2+a_3$ such that

$$a_1-a_2<-a_1-a_2+a_3<-a_1+a_2.$$

If $-a_2+a_3=a_1+a_2$, then also we are done as we get an extra positive integer $-a_1-a_2+a_3$ such that

$$-a_1+a_2<-a_1-a_2+a_3<a_1+a_2.$$

Subcase (iii). Let $-a_1+a_3=a_1+a_2$. If $-a_2+a_3<-a_1+a_2$, then we get two extra positive integers $-a_2+a_3$ and $-a_1-a_2+a_3$ such that

$$a_1-a_2<-a_1-a_2+a_3<-a_2+a_3<-a_1+a_2.$$

If $-a_2+a_3=-a_1+a_2$, then $a_2=3a_1$ and $a_3=5a_1$. We get two extra positive integers $-a_1-a_2+a_3$ and $a_1-a_2+a_3$ such that

$$a_1-a_2<-a_1-a_2+a_3<-a_1+a_2<a_1-a_2+a_3<a_1+a_2.$$

Now, let $-a_2+a_3>-a_1+a_2$. Then we get an extra positive integer $-a_2+a_3$ such that

$$-a_1+a_2<-a_2+a_3<-a_1+a_3=a_1+a_2.$$

If $a_2-a_1\neq a_1$, then $-a_1+a_2+a_3\neq a_1+a_3$. So, we get one more extra positive integer $-a_1+a_2+a_3$ such that

$$a_1+a_2=-a_1+a_3<-a_1+a_2+a_3<a_2+a_3.$$

Let $a_2-a_1=a_1$. So, $a_2=2a_1$ and $a_3=4a_1$. If $a_4-a_3>a_1$, then we get an extra positive integer a_2+a_4 such that

$$a_1+a_2+a_3<a_2+a_4<a_1+a_2+a_4.$$

If $a_4-a_3<a_1$, then we get an extra positive integer a_2+a_4 such that

$$a_2+a_3<a_2+a_4<a_1+a_2+a_3.$$

If $a_4-a_3=a_1$, then also we get an extra positive integer $a_1-a_2+a_4$ such that

$$a_1+a_2<a_1-a_2+a_4<a_1+a_3.$$

Thus, in Case 1 and Case 3, we get at least two extra positive integers. As the inverses of these extra integers are also in $3_{\pm}^{\wedge}A$, so we get four extra integers in these two cases, which are not present in (4.18). In Case 2, we get at least three extra integers. Therefore, in each case we get at least three extra integers in $3_{\pm}^{\wedge}A$ which are not present in (4.18). Hence

$$|3_{\pm}^{\wedge}A| \geq 6k - 11.$$

This establishes (4.17).

Now, let $|3_{\pm}^{\wedge}A| = 6k - 11$. From the above discussion it is clear that we are in Case 2 with $a_3 - a_2 = a_2 - a_1 = a_1$.

Let $A' = A \setminus \{0\}$. Then, A' is a nonempty set of $k-1$ positive integers with $3^{\wedge}A' \subset [a_1 + a_2 + a_3, a_{k-3} + a_{k-2} + a_{k-1}]$. Since $|3_{\pm}^{\wedge}A| = 6k - 11$, it follows from the above discussion that $|3^{\wedge}A'| = 3k - 11$. Thus, Theorem 1.4.12 implies that the set A' is in arithmetic progression, i.e.,

$$a_{k-1} - a_{k-2} = \cdots = a_2 - a_1 = d.$$

Hence, $A = a_1 \cdot \{0, 1, 2, \ldots, k-1\}$. This completes the proof of the theorem. □

We observe in the following theorem that the minimum requirement of five elements in the set A in Theorem 4.3.5 is the best possible.

Theorem 4.3.6. *Let A be a set of four nonnegative integers with* $0 \in A$. *Then*

$$|3_{\pm}^{\wedge}A| \geq 12. \tag{4.19}$$

Moreover, if $|3_{\pm}^{\wedge}A| = 12$*, then* $A = d \cdot \{0, 1, 2, 4\}$.

Proof. Let $A = \{0, a_1, a_2, a_3\}$, where $0 < a_1 < a_2 < a_3$. From Theorem 4.3.1, it follows that $3_{\pm}^{\wedge}A$ contains at least the following ten integers.

$$\begin{aligned} &-a_1 - a_2 - a_3 < -a_2 - a_3 < -a_1 - a_3 < -a_1 - a_2 < a_1 - a_2 < -a_1 + a_2 < a_1 + a_2 \\ &\quad < a_1 + a_3 < a_2 + a_3 < a_1 + a_2 + a_3. \end{aligned} \tag{4.20}$$

Again, from the proof of Theorem 4.3.5, it follows that the sumset $3_{\pm}^{\wedge}A$ contains at least three extra integers, except when $a_2 = 2a_1$, $a_3 = 4a_1$. In the case $a_2 = 2a_1$, $a_3 = 4a_1$, we get two extra integers. Therefore, we always get two extra integers in $3_{\pm}^{\wedge}A$ which are not present in (4.20). Hence $|3_{\pm}^{\wedge}A| \geq 12$. This establishes (4.19). Moreover, if $|3_{\pm}^{\wedge}A| = 12$, then we have $a_2 = 2a_1$ and $a_3 = 4a_1$. Hence $A = a_1 \cdot \{0, 1, 2, 4\}$. This completes the proof of the theorem. □

We finally conjecture the inverse problem as follows:

Conjecture 4.3.7. *Let A be a set of k (≥ 5) nonnegative integers with $0 \in A$, and let $3 \leq h \leq k-1$. If $|h^{\wedge}_{\pm}A| = 2hk - h(h+1) + 1$, then $A = d \cdot [0, k-1]$ for some positive integer d.*

Theorem 4.3.5 confirms Conjecture 4.3.7 for $h = 3$.

Chapter 5

Counting the number of elements in $h^{(\gamma)}A$: A special case

Let $k \geq 2$. Let h and γ be positive integers such that $1 \leq \gamma \leq h \leq \gamma k$. Set $m = \lfloor h/\gamma \rfloor$. Let $A = [0, k-1]$ be an interval. Then the h-fold generalized sumset $h^{(\gamma)}A = [\frac{m\gamma(m-1)}{2} + (h - m\gamma)m, m\gamma\left(k - \frac{m+1}{2}\right) + (h - m\gamma)(k - m - 1)]$. Hence, $|h^{(\gamma)}A| = m\gamma(k-m) + (h - m\gamma)(k - 2m - 1) + 1$. In this chapter, we find $|h^{(\gamma)}A|$ for $A = \{0, 1, \ldots, k-2, k-1+b\} = [0, k-2] \cup \{k-1+b\}$, which is an almost interval, where b is a nonnegative integer.

5.1 Introduction

Let A be a nonempty finite set of integers. Let h and γ be positive integers such that $\gamma \leq h$. Recall that, the h-fold generalized sumset $h^{(\gamma)}A$ is the set of all sums of h elements of A, where each element appearing in a sum may be repeated at most γ times. So, the regular sumset hA and the restricted sumset $h^{\wedge}A$ are special cases of the generalized sumset $h^{(\gamma)}A$, for $\gamma = h$ and $\gamma = 1$, respectively.

Let $A = [0, k-1]$. Then $hA = [0, h(k-1)]$, and hence $|hA| = hk - h + 1$. Now, let $A = \{0, 1, \ldots, k-2, k-1+b\} = [0, k-2] \cup \{k-1+b\}$, where b is a nonnegative integer. Nathanson [81], proved that "$|hA|$ is a strictly increasing piecewise-linear function of b for $0 \leq b \leq (h-1)(k-2)$ and that $|hA|$ is constant for $b \geq (h-1)(k-2)$". The following theorem gives the precise statement of Nathanson's result.

Theorem 5.1.1. *[81, Theorem 1.11] Let $h \geq 2$ and $k \geq 3$. For $b \geq 0$, let $A = [0, k-2] \cup \{k-1+b\}$ and $b = q(k-2) + r$, where $q \geq 0$ and $0 \leq r \leq k-3$.*
If $b \leq (h-1)(k-2)$, then

$$|hA| = hk - (h-1) + \frac{q(2h-q-1)(k-2)}{2} + (h-q-1)r.$$

If $b \geq (h-1)(k-2)$, then

$$|hA| = hk - (h-1) + \frac{h(h-1)(k-2)}{2}.$$

In this chapter, we prove an analogue of Nathanson's theorem to the general h-fold sumset $h^{(\gamma)}A$. Furthermore, as a particular case, we obtain a similar result for the restricted sumset $h^{\wedge}A$.

Let h and γ be integers such that $1 \leq \gamma \leq h \leq \gamma k$. Let m be a positive integer such that $h = m\gamma + \varepsilon$, where $0 \leq \varepsilon < \gamma$. Let $h \geq 2$ and $k \geq m+2$. For $b \geq 0$, let $A = [0, k-2] \cup \{k-1+b\}$. For $l = 0, 1, \ldots, \varepsilon$, the smallest element of $h^{(\gamma)}A$ including l copies of $\{k-1+b\}$ is

$$\begin{aligned} \gamma \cdot 0 + \gamma \cdot 1 + \cdots + \gamma \cdot (m-1) + (\varepsilon - l) \cdot m + l \cdot (k-1+b) \\ = \frac{m\gamma(m-1)}{2} + \varepsilon m + l(k-1+b-m), \end{aligned}$$

and the largest element of $h^{(\gamma)}A$ including l copies of $\{k-1+b\}$ is

$$\begin{aligned} \gamma \cdot (k-2) + \gamma \cdot (k-3) + \cdots + \gamma \cdot (k-m-1) + (\varepsilon - l) \cdot (k-m-2) + l \cdot (k-1+b) \\ = m\gamma \left(k - 2 - \frac{(m-1)}{2} \right) + \varepsilon(k-m-2) + l(b+m+1). \end{aligned}$$

For $l' = \varepsilon+1, \varepsilon+2, \ldots, \gamma$, the smallest element of $h^{(\gamma)}A$ including l' copies of $\{k-1+b\}$ is

$$\begin{aligned} \gamma \cdot 0 + \gamma \cdot 1 + \cdots + \gamma \cdot (m-2) + (\gamma + \varepsilon - l') \cdot (m-1) + l' \cdot (k-1+b) \\ = \frac{m\gamma(m-1)}{2} + \varepsilon(m-1) + l'(k-m+b), \end{aligned}$$

and the largest element of $h^{(\gamma)}A$ including l' copies of $\{k-1+b\}$ is

$$\begin{aligned} \gamma \cdot (k-2) + \gamma \cdot (k-3) + \cdots + \gamma \cdot (k-m) + (\gamma + \varepsilon - l') \cdot (k-m-1) + l' \cdot (k-1+b) \\ = m\gamma \left(k - 2 - \frac{(m-1)}{2} \right) + \varepsilon(k-m-1) + l'(b+m). \end{aligned}$$

Thus,

$$h^{(\gamma)}A = \bigcup_{l=0}^{\varepsilon} I_l \bigcup_{l'=\varepsilon+1}^{\gamma} J_{l'},$$

where

$$I_l = [\frac{m\gamma(m-1)}{2} + \varepsilon m + l(k-1+b-m), m\gamma(k-2-\frac{(m-1)}{2}) + \varepsilon(k-m-2) + l(b+m+1)],$$

and

$$J_{l'} = [\frac{m\gamma(m-1)}{2} + \varepsilon(m-1) + l'(k+b-m), m\gamma(k-2-\frac{(m-1)}{2}) + \varepsilon(k-m-1) + l'(b+m)].$$

We have,

$$| I_l | = m\gamma(k-m-1) + 1 + (\varepsilon - l)(k-2m-2),$$

and

$$| J_{l'} | = m\gamma(k-m-1) + 1 + (\varepsilon - l')(k-2m).$$

Since $k \geq m+2 \geq 3$, the intervals I_l and $J_{l'}$ are shifting towards right in the sense that the sequence of the left end points and the sequence of right end points are increasing as l and l' increase, respectively. Also the left end point of $J_{\varepsilon+1}$ is greater than the left end point of I_ε and the right end point of $J_{\varepsilon+1}$ is greater than the right end point of I_ε.

For $l = 1, \ldots, \varepsilon$, the set $I_{l-1} \cup I_l$ is the interval,

$$[\frac{m\gamma(m-1)}{2} + \varepsilon m + (l-1)(k-1+b-m), m\gamma(k-2-\frac{(m-1)}{2}) + \varepsilon(k-m-2) + l(b+m+1)]$$

if and only if

$$\begin{aligned} b &\leq m\gamma(k-m-1) - m + (\varepsilon - l)(k-2m-2) \\ &= (k\gamma - m\gamma - 1)(m-1) + \gamma(k-2m) - 1 + (\varepsilon - l)(k-2m-2). \end{aligned}$$

For $l' = \varepsilon+2, \ldots, \gamma$, the set $J_{l'-1} \cup J_{l'}$ is the interval,

$$[\frac{m\gamma(m-1)}{2} + \varepsilon(m-1) + (l'-1)(k+b-m), m\gamma(k-2-\frac{(m-1)}{2}) + \varepsilon(k-m-1) + l'(b+m)]$$

if and only if

$$\begin{aligned} b &\leq m\gamma(k-m-1) - m + 1 - (l' - \varepsilon)(k-2m) \\ &= (k\gamma - m\gamma - 1)(m-1) + (\gamma + \varepsilon - l')(k-2m). \end{aligned}$$

Also, $I_\varepsilon \cup J_{\varepsilon+1}$ is the interval

$$[\frac{m\gamma(m-1)}{2} + \varepsilon(k-1+b), m\gamma(k-2-\frac{(m-1)}{2}) + \varepsilon(k-1+b) + b + m]$$

if and only if

$$b \leq (m\gamma - 1)(k - m - 1) = (k\gamma - m\gamma - 1)(m - 1) + (\gamma - 1)(k - 2m).$$

We find the cardinality of $h^{(\gamma)}A$ according as $k \geq 2m+2$, $m+2 \leq k \leq 2m$, and $k = 2m+1$ in Sections 5.2, 5.3 and 5.4, respectively. In section 5.5, as concluding remarks, we conclude that $|h^{\wedge}A|$ is a strictly increasing linear function of b for $0 \leq b \leq (h-1)(k-h-1)$ and $|h^{\wedge}A|$ is constant for $b \geq (h-1)(k-h-1)$. Further, we also conclude that all the results are also valid for almost arithmetic progressions.

5.2 The Case $k \geq 2(m+1)$

Theorem 5.2.1. *Let $k \geq 2(m+1)$ and $M = m\gamma(k-m) + \varepsilon(k-2m-1) + 1$. If $0 \leq b \leq (k\gamma - m\gamma - 1)(m-1) + \varepsilon(k-2m)$, then $|h^{(\gamma)}A| = M + b\gamma$.*
If $(k\gamma - m\gamma - 1)(m-1) + \varepsilon(k-2m) < b \leq (k\gamma - m\gamma - 1)(m-1) + (\gamma - 1)(k-2m)$, let $b = (k\gamma - m\gamma - 1)(m-1) + \varepsilon(k-2m) + q(k-2m) + s$, where $q \geq 0$ and $0 \leq s < (k-2m)$, then

$$|h^{(\gamma)}A| = M + \gamma\{(k\gamma - m\gamma - 1)(m-1) + \varepsilon(k-2m)\} + \frac{q(2\gamma - q - 1)(k-2m)}{2} + (\gamma - q - 1)s.$$

If $(k\gamma - m\gamma - 1)(m-1) + (\gamma - 1)(k-2m) < b \leq (k\gamma - m\gamma - 1)(m-1) + \gamma(k-2m) - 1$, then

$$\begin{aligned}|h^{(\gamma)}A| &= M + (\gamma - \varepsilon)\{(k\gamma - m\gamma - 1)(m-1) + (\gamma - 1)(k-2m)\} \\ &\quad - \frac{(\gamma - \varepsilon)(\gamma - \varepsilon - 1)(k-2m)}{2} + \varepsilon b.\end{aligned}$$

If $k > 2m+2$ and $(k\gamma - m\gamma - 1)(m-1) + \gamma(k-2m) - 1 < b \leq (k\gamma - m\gamma - 1)(m-1) + \gamma(k-2m) - 1 + (\varepsilon - 1)(k-2m-2)$, let $b = (k\gamma - m\gamma - 1)(m-1) + \gamma(k-2m) - 1 + q(k-2m-2) + s$, where $q \geq 0$ and $0 \leq s < (k-2m-2)$, then

$$\begin{aligned}|h^{(\gamma)}A| &= M + \gamma\{(k\gamma - m\gamma - 1)(m-1) + \gamma(k-2m) - 1\} - (\gamma - \varepsilon)(k-2m-1) \\ &\quad - \frac{(\gamma - \varepsilon)(\gamma - \varepsilon - 1)(k-2m)}{2} + \frac{q(2h - 2m\gamma - q - 1)(k-2m-2)}{2} + (\varepsilon - q - 1)s.\end{aligned}$$

If $b > (k\gamma - m\gamma - 1)(m-1) + \gamma(k-2m) - 1 + (\varepsilon - 1)(k-2m-2)$, then

$$\begin{aligned}|h^{(\gamma)}A| &= M + \gamma\{(k\gamma - m\gamma - 1)(m-1) + \gamma(k-2m) - 1\} - (\gamma - \varepsilon)(k-2m-1) \\ &\quad - \frac{(\gamma - \varepsilon)(\gamma - \varepsilon - 1)(k-2m)}{2} + \frac{\varepsilon(\varepsilon - 1)(k-2m-2)}{2}.\end{aligned}$$

Proof. For $k \geq 2(m+1)$, we have the following

$$\begin{aligned}
0 &\leq (k\gamma - m\gamma - 1)(m-1) + \varepsilon(k-2m) \\
&< (k\gamma - m\gamma - 1)(m-1) + (\varepsilon+1)(k-2m) \\
&\vdots \\
&< (k\gamma - m\gamma - 1)(m-1) + (\gamma-2)(k-2m) \\
&< (k\gamma - m\gamma - 1)(m-1) + (\gamma-1)(k-2m) \\
&< (k\gamma - m\gamma - 1)(m-1) + \gamma(k-2m) - 1 \\
&\leq (k\gamma - m\gamma - 1)(m-1) + \gamma(k-2m) - 1 + (k-2m-2) \\
&\vdots \\
&\leq (k\gamma - m\gamma - 1)(m-1) + \gamma(k-2m) - 1 + (\varepsilon-1)(k-2m-2).
\end{aligned}$$

Case 1. Let $0 \leq b \leq (k\gamma - m\gamma - 1)(m-1) + \varepsilon(k-2m)$. Clearly, $I_0 \cup \cdots \cup I_\varepsilon \cup J_{\varepsilon+1} \cup \cdots \cup J_\gamma$ is the interval $[m\gamma(m-1)/2 + \varepsilon m, m\gamma(k-2-(m-1)/2) + \varepsilon(k-m-1) + \gamma(b+m)]$. Therefore,

$$|h^{(\gamma)}A| = M + b\gamma.$$

Case 2. If $(k\gamma - m\gamma - 1)(m-1) + \varepsilon(k-2m) < b \leq (k\gamma - m\gamma - 1)(m-1) + (\gamma-1)(k-2m)$, then there exists a unique $t \in [\varepsilon-1, \gamma-1]$ such that $(k\gamma - m\gamma - 1)(m-1) + (\gamma+\varepsilon-t-1)(k-2m) < b \leq (k\gamma - m\gamma - 1)(m-1) + (\gamma+\varepsilon-t)(k-2m)$. So $I_0 \cup I_1 \cup \cdots \cup I_\varepsilon \cup J_{\varepsilon+1} \cup \cdots \cup J_t$ is the interval $J = [m\gamma(m-1)/2 + \varepsilon m, m\gamma(k-2-(m-1)/2) + \varepsilon(k-m-1) + t(b+m)]$ and the intervals $J, J_{t+1}, \ldots, J_\gamma$ are pairwise disjoint. Hence,

$$\begin{aligned}
|h^{(\gamma)}A| &= |J| + \sum_{l'=t+1}^{\gamma} |J_{l'}| \\
&= M + tb + (\gamma-t)\{(k\gamma - m\gamma - 1)(m-1) + \varepsilon(k-2m)\} + \frac{(\gamma-t)(\gamma-t-1)(k-2m)}{2}.
\end{aligned}$$

Let $b = (k\gamma - m\gamma - 1)(m-1) + \varepsilon(k-2m) + q(k-2m) + s$, where $q \geq 0$ and $0 \leq s < (k-2m)$. If $s = 0$, then $t = \gamma - q$. Hence,

$$|h^{(\gamma)}A| = M + \gamma\{(k\gamma - m\gamma - 1)(m-1) + \varepsilon(k-2m)\} + \frac{q(2\gamma-q-1)(k-2m)}{2}.$$

If $0 < s < (k-2m)$, then $q = \gamma - t - 1$ or $t = \gamma - q - 1$. Hence,

$$|h^{(\gamma)}A| = M + \gamma\{(k\gamma - m\gamma - 1)(m-1) + \varepsilon(k-2m)\} + \frac{q(2\gamma-q-1)(k-2m)}{2} + (\gamma-q-1)s.$$

Case 3. Let $(k\gamma-m\gamma-1)(m-1)+(\gamma-1)(k-2m)<b\leq(k\gamma-m\gamma-1)(m-1)+\gamma(k-2m)-1$. Clearly, $I_0\cup I_1\cup\cdots\cup I_\varepsilon$ is the interval $J=[m\gamma(m-1)/2+\varepsilon m, m\gamma(k-2-(m-1)/2)+\varepsilon(k-m-2)+\varepsilon(b+m+1)]$. Moreover, the intervals $J,J_{\varepsilon+1},\dots,J_\gamma$ are pairwise disjoint. Hence,

$$\begin{aligned}|h^{(\gamma)}A| &= |J|+\sum_{l'=\varepsilon+1}^{\gamma}|J_{l'}|\\ &= M+(\gamma-\varepsilon)\{(k\gamma-m\gamma-1)(m-1)+(\gamma-1)(k-2m)\}\\ &\quad -\frac{(\gamma-\varepsilon)(\gamma-\varepsilon-1)(k-2m)}{2}+\varepsilon b.\end{aligned}$$

Case 4. If $k=2m+2$, then we arrive in Case 3. So, let $k>2(m+1)$ and $(k\gamma-m\gamma-1)(m-1)+\gamma(k-2m)-1<b\leq(k\gamma-m\gamma-1)(m-1)+\gamma(k-2m)-1+(\varepsilon-1)(k-2m-2)$. There exists a unique $t\in[1,\varepsilon-1]$ such that $(k\gamma-m\gamma-1)(m-1)+\gamma(k-2m)-1+(\varepsilon-t-1)(k-2m-2)<b\leq(k\gamma-m\gamma-1)(m-1)+\gamma(k-2m)-1+(\varepsilon-t)(k-2m-2)$. So $I_0\cup I_1\cup\cdots\cup I_t$ is the interval $J=[m\gamma(m-1)/2+\varepsilon m, m\gamma(k-2-(m-1)/2)+\varepsilon(k-m-2)+t(b+m+1)]$. Clearly, the intervals $J,I_{t+1},\dots,I_\varepsilon,J_{\varepsilon+1},\dots,J_\gamma$ are pairwise disjoint. Hence,

$$\begin{aligned}|h^{(\gamma)}A| &= |J|+\sum_{l=t+1}^{\varepsilon}|I_l|+\sum_{l'=\varepsilon+1}^{\gamma}|J_{l'}|\\ &= M+(\gamma-t)\{m\gamma(k-m-1)-m\}+tb-\frac{(\gamma-\varepsilon)(\gamma-\varepsilon-1)(k-2m)}{2}\\ &\quad -(\gamma-\varepsilon)(k-2m-1)+\frac{(\varepsilon-t)(\varepsilon-t-1)(k-2m-2)}{2}.\end{aligned}$$

Let $b=(k\gamma-m\gamma-1)(m-1)+\gamma(k-2m)-1+q(k-2m-2)+s$, where $q\geq 0$ and $0\leq s<(k-2m-2)$. If $s=0$, then $q=\varepsilon-t$. Hence,

$$\begin{aligned}|h^{(\gamma)}A| &= M+\gamma\{(k\gamma-m\gamma-1)(m-1)+\gamma(k-2m)-1\}-(\gamma-\varepsilon)(k-2m-1)\\ &\quad -\frac{(\gamma-\varepsilon)(\gamma-\varepsilon-1)(k-2m)}{2}+\frac{q(2h-2m\gamma-q-1)(k-2m-2)}{2}.\end{aligned}$$

If $0<s<(k-2m-2)$, then $q=\varepsilon-t-1$. Hence,

$$\begin{aligned}|h^{(\gamma)}A| &= M+\gamma\{(k\gamma-m\gamma-1)(m-1)+\gamma(k-2m)-1\}-(\gamma-\varepsilon)(k-2m-1)\\ &\quad -\frac{(\gamma-\varepsilon)(\gamma-\varepsilon-1)(k-2m)}{2}+\frac{q(2h-2m\gamma-q-1)(k-2m-2)}{2}+(\varepsilon-q-1)s.\end{aligned}$$

Case 5. Let $b>(k\gamma-m\gamma-1)(m-1)+\gamma(k-2m)-1+(\varepsilon-1)(k-2m-2)$. Clearly, $I_0,I_1,\dots,I_\varepsilon,J_{\varepsilon+1},\dots,J_\gamma$

are pairwise disjoint intervals. Hence,

$$\begin{aligned}|h^{(\gamma)}A| &= \sum_{l=0}^{\varepsilon} |I_l| + \sum_{l'=\varepsilon+1}^{\gamma} |J_{l'}| \\ &= M+\gamma\{(k\gamma-m\gamma-1)(m-1)+\gamma(k-2m)-1\}-(\gamma-\varepsilon)(k-2m-1) \\ &\quad -\frac{(\gamma-\varepsilon)(\gamma-\varepsilon-1)(k-2m)}{2}+\frac{\varepsilon(\varepsilon-1)(k-2m-2)}{2}.\end{aligned}$$

This completes the proof of the theorem. □

5.3 The Case $m+2 \le k \le 2m$

Theorem 5.3.1. *Let $m+2 \le k \le 2m$. If $0 \le b \le (k\gamma-m\gamma-1)(m-1)+\gamma(k-2m)-1+(\varepsilon-1)(k-2m-2)$, then $|h^{(\gamma)}A| = M+b\gamma$.*

If $(k\gamma-m\gamma-1)(m-1)+\gamma(k-2m)-1+(\varepsilon-1)(k-2m-2) < b \le (k\gamma-m\gamma-1)(m-1)+\gamma(k-2m)-1$, let $b = (k\gamma-m\gamma-1)(m-1)+\gamma(k-2m)-1+(\varepsilon-1)(k-2m-2)+q(2m+2-k)+s$, where $q \ge 0$ and $0 \le s < (2m+2-k)$, then

$$\begin{aligned}|h^{(\gamma)}A| &= M+\gamma\{(k\gamma-m\gamma-1)(m-1)+\gamma(k-2m)-1+(\varepsilon-1)(k-2m-2)\} \\ &\quad -\frac{q(2\gamma-q-1)(k-2m-2)}{2}+(\gamma-q-1)s.\end{aligned}$$

If $(k\gamma-m\gamma-1)(m-1)+\gamma(k-2m)-1 < b \le (k\gamma-m\gamma-1)(m-1)+(\gamma-1)(k-2m)$, then

$$\begin{aligned}|h^{(\gamma)}A| &= M+\varepsilon\{(k\gamma-m\gamma-1)(m-1)+\gamma(k-2m)-1+(\varepsilon-1)(k-2m-2)\} \\ &\quad -\frac{\varepsilon(\varepsilon-1)(k-2m-2)}{2}+(\gamma-\varepsilon)b.\end{aligned}$$

If $k<2m$ and $(k\gamma-m\gamma-1)(m-1)+(\gamma-1)(k-2m) < b \le (k\gamma-m\gamma-1)(m-1)+\varepsilon(k-2m)$, let $b = (k\gamma-m\gamma-1)(m-1)+(\gamma-1)(k-2m)+q(2m-k)+s$, where $q \ge 0$ and $0 \le s < (2m-k)$, then

$$\begin{aligned}|h^{(\gamma)}A| &= M+\gamma\{(k\gamma-m\gamma-1)(m-1)+(\gamma-1)(k-2m)\}+\frac{\varepsilon(\varepsilon-1)(k-2m-2)}{2} \\ &\quad +\varepsilon(k-2m-1)-\frac{q(2\gamma-2\varepsilon-q-1)(k-2m)}{2}+(\gamma-\varepsilon-q-1)s.\end{aligned}$$

If $b > (k\gamma-m\gamma-1)(m-1)+\varepsilon(k-2m)$, then

$$\begin{aligned}|h^{(\gamma)}A| &= M+\gamma\{(k\gamma-m\gamma-1)(m-1)+(\gamma-1)(k-2m)\}-\frac{(\gamma-\varepsilon)(\gamma-\varepsilon-1)(k-2m)}{2} \\ &\quad +\varepsilon(k-2m-1)+\frac{\varepsilon(\varepsilon-1)(k-2m-2)}{2}.\end{aligned}$$

Proof. For $k \leq 2m$, we have

$$\begin{aligned}
0 &\leq (k\gamma - m\gamma - 1)(m-1) + \gamma(k-2m) - 1 + (\varepsilon - 1)(k-2m-2) \\
&< (k\gamma - m\gamma - 1)(m-1) + \gamma(k-2m) - 1 + (\varepsilon - 2)(k-2m-2) \\
&\vdots \\
&< (k\gamma - m\gamma - 1)(m-1) + \gamma(k-2m) - 1 \\
&< (k\gamma - m\gamma - 1)(m-1) + (\gamma - 1)(k-2m) \\
&\leq (k\gamma - m\gamma - 1)(m-1) + (\gamma - 2)(k-2m) \\
&\vdots \\
&\leq (k\gamma - m\gamma - 1)(m-1) + \varepsilon(k-2m).
\end{aligned}$$

Case 1. Let $0 \leq b \leq (k\gamma - m\gamma - 1)(m-1) + \gamma(k-2m) - 1 + (\varepsilon - 1)(k-2m-2)$. Clearly, $I_0 \cup \cdots \cup I_\varepsilon \cup J_{\varepsilon+1} \cup \cdots \cup J_\gamma$ is the interval $[m\gamma(m-1)/2 + \varepsilon m, m\gamma(k-2-(m-1)/2) + \varepsilon(k-m-1) + \gamma(b+m)]$. Hence,

$$|h^{(\gamma)}A| = M + b\gamma.$$

Case 2. Let $(k\gamma - m\gamma - 1)(m-1) + \gamma(k-2m) - 1 + (\varepsilon - 1)(k-2m-2) < b \leq (k\gamma - m\gamma - 1)(m-1) + \gamma(k-2m) - 1$. There exists a unique $t \in [1, \varepsilon - 1]$ such that $(k\gamma - m\gamma - 1)(m-1) + \gamma(k-2m) - 1 + (\varepsilon - t)(k-2m-2) < b \leq (k\gamma - m\gamma - 1)(m-1) + \gamma(k-2m) - 1 + (\varepsilon - t - 1)(k-2m-2)$. So $I_0, I_1, \ldots, I_{t-1}, J$ are pairwise disjoint intervals, where $J = I_t \cup \cdots \cup I_\varepsilon \cup J_{\varepsilon+1} \cup \cdots \cup J_\gamma = [m\gamma(m-1)/2 + \varepsilon m + t(k-1+b-m), m\gamma(k-2-(m-1)/2) + \varepsilon(k-m-1) + \gamma(b+m)]$. Hence,

$$\begin{aligned}
|h^{(\gamma)}A| &= |J| + \sum_{l=0}^{t-1} |I_l| \\
&= M + (\gamma - t)b + t\{(k\gamma - m\gamma - 1)(m-1) + \gamma(k-2m) - 1 + (\varepsilon - 1)(k-2m-2)\} \\
&\quad - \frac{t(t-1)(k-2m-2)}{2}.
\end{aligned}$$

Let $b = (k\gamma - m\gamma - 1)(m-1) + \gamma(k-2m) - 1 + (\varepsilon - 1)(k-2m-2) + q(2m+2-k) + s$, where $q \geq 0$ and $0 \leq s < (2m+2-k)$. If $s = 0$, then $t = q$. Hence,

$$\begin{aligned}
|h^{(\gamma)}A| &= M + \gamma\{(k\gamma - m\gamma - 1)(m-1) + \gamma(k-2m) - 1 + (\varepsilon - 1)(k-2m-2)\} \\
&\quad - \frac{q(2\gamma - q - 1)(k-2m-2)}{2}.
\end{aligned}$$

If $0 < s < (2m+2-k)$, then $t = q+1$. Hence,

$$|h^{(\gamma)}A| = M + \gamma\{(k\gamma - m\gamma - 1)(m-1) + \gamma(k-2m) - 1 + (\varepsilon - 1)(k-2m-2)\} - \frac{q(2\gamma - q - 1)(k-2m-2)}{2} + (\gamma - q - 1)s.$$

Case 3. Let $(k\gamma - m\gamma - 1)(m-1) + \gamma(k-2m) - 1 < b \leq (k\gamma - m\gamma - 1)(m-1) + (\gamma - 1)(k-2m)$. Clearly, $I_0, I_1, \ldots, I_{\varepsilon-1}, J$ are pairwise disjoint intervals, where $J = I_\varepsilon \cup J_{\varepsilon+1} \cup \cdots \cup J_\gamma = [m\gamma(m-1)/2 + \varepsilon(k-1+b), m\gamma(k-2-(m-1)/2) + \varepsilon(k-m-1) + \gamma(b+m)]$. Thus,

$$\begin{aligned} |h^{(\gamma)}A| &= |J| + \sum_{l=0}^{\varepsilon-1} |I_l| \\ &= M + \varepsilon\{(k\gamma - m\gamma - 1)(m-1) + \gamma(k-2m) - 1 + (\varepsilon - 1)(k-2m-2)\} \\ &\quad - \frac{\varepsilon(\varepsilon-1)(k-2m-2)}{2} + (\gamma - \varepsilon)b. \end{aligned}$$

Case 4. If $k = 2m$, then we are in Case 3. Therefore, we let $k < 2m$ and $(k\gamma - m\gamma - 1)(m-1) + (\gamma - 1)(k-2m) < b \leq (k\gamma - m\gamma - 1)(m-1) + \varepsilon(k-2m)$. There exists a unique $t \in [\varepsilon+1, \gamma-1]$ such that $(k\gamma - m\gamma - 1)(m-1) + (\gamma + \varepsilon - t)(k-2m) < b \leq (k\gamma - m\gamma - 1)(m-1) + (\gamma + \varepsilon - t - 1)(k-2m)$. So $I_0, I_1, \ldots, I_\varepsilon, J_{\varepsilon+1}, \ldots, J_{t-1}, J$ are pairwise disjoint intervals, where $J = J_t \cup \cdots \cup J_\gamma = [m\gamma(m-1)/2 + \varepsilon(m-1) + t(k+b-m), m\gamma(k-2-(m-1)/2) + \varepsilon(k-m-1) + \gamma(b+m)]$. Therefore,

$$\begin{aligned} |h^{(\gamma)}A| &= |J| + \sum_{l=0}^{\varepsilon} |I_l| + \sum_{l'=\varepsilon+1}^{t-1} |J_{l'}| \\ &= M + (\gamma - t)b + t\{(k\gamma - m\gamma - 1)(m-1) + (\gamma - 1)(k-2m)\} + \frac{\varepsilon(\varepsilon-1)(k-2m-2)}{2} \\ &\quad + \varepsilon(k-2m-1) - \frac{(t-\varepsilon)(t-\varepsilon-1)(k-2m)}{2}. \end{aligned}$$

Let $b = (k\gamma - m\gamma - 1)(m-1) + (\gamma - 1)(k-2m) + q(2m-k) + s$, where $q \geq 0$ and $0 \leq s < (2m-k)$. If $s = 0$, then $t - \varepsilon = q$. Hence,

$$|h^{(\gamma)}A| = M + \gamma\{(k\gamma - m\gamma - 1)(m-1) + (\gamma - 1)(k-2m)\} + \frac{\varepsilon(\varepsilon-1)(k-2m-2)}{2} + \varepsilon(k-2m-1) - \frac{q(2\gamma - 2\varepsilon - q - 1)(k-2m)}{2}.$$

If $0 < s < (2m-k)$, then $t - \varepsilon - 1 = q$. Hence,

$$|h^{(\gamma)}A| = M + \gamma\{(k\gamma - m\gamma - 1)(m-1) + (\gamma - 1)(k-2m)\} + \frac{\varepsilon(\varepsilon-1)(k-2m-2)}{2} + \varepsilon(k-2m-1) - \frac{q(2\gamma - 2\varepsilon - q - 1)(k-2m)}{2} + (\gamma - \varepsilon - q - 1)s.$$

Case 5. Let $b > (k\gamma - m\gamma - 1)(m-1) + \varepsilon(k-2m)$. Clearly, $I_0, I_1, \ldots, I_\varepsilon, J_{\varepsilon+1}, \ldots, J_\gamma$ are pairwise disjoint intervals. Hence,

$$|h^{(\gamma)}A| = M + \gamma\{(k\gamma - m\gamma - 1)(m-1) + (\gamma-1)(k-2m)\} - \frac{(\gamma-\varepsilon)(\gamma-\varepsilon-1)(k-2m)}{2} + \varepsilon(k-2m-1) + \frac{\varepsilon(\varepsilon-1)(k-2m-2)}{2}.$$

This completes the proof of the theorem. □

5.4 The Case $k = 2m+1$

Theorem 5.4.1. *Let $k = 2m+1$. If $0 \le \varepsilon \le \gamma/2$ with $0 \le b \le (m\gamma+\gamma-1)(m-1)+\varepsilon$, then $|h^{(\gamma)}A| = M + b\gamma$.*

If $\varepsilon < \gamma/2$ with $(m\gamma+\gamma-1)(m-1)+\varepsilon < b \le (m\gamma+\gamma-1)(m-1)+\gamma-\varepsilon$, then

$$|h^{(\gamma)}A| = M + \gamma\{(m\gamma+\gamma-1)(m-1)+\varepsilon\} + \frac{q(2\gamma-q-1)}{2},$$

where $b = (m\gamma+\gamma-1)(m-1)+\varepsilon+q$, $0 < q \le \gamma - 2\varepsilon$.

If $(m\gamma+\gamma-1)(m-1)+\gamma-\varepsilon < b \le (m\gamma+\gamma-1)(m-1)+\gamma-2$, let $b = (m\gamma+\gamma-1)(m-1)+\gamma-\varepsilon+q$, where $0 \le q \le \varepsilon - 2$, then

$$|h^{(\gamma)}A| = M + \gamma\{(m\gamma+\gamma-1)(m-1)+\gamma-1-(\varepsilon-1)\} + \frac{q(2\gamma-q-1)}{2} - \frac{(\gamma-2\varepsilon+q)(\gamma-2\varepsilon+q+1)}{2}.$$

If $\varepsilon > \gamma/2$ with $0 \le b \le (m\gamma+\gamma-1)(m-1)+\gamma-\varepsilon$, then $|h^{(\gamma)}A| = M + b\gamma$.

If $(m\gamma+\gamma-1)(m-1)+\gamma-\varepsilon < b \le (m\gamma+\gamma-1)(m-1)+\varepsilon$, then

$$|h^{(\gamma)}A| = M + \gamma\{(m\gamma+\gamma-1)(m-1)+\gamma-\varepsilon\} + \frac{q(2\gamma-q-1)}{2},$$

where $b = (m\gamma+\gamma-1)(m-1)+\gamma-\varepsilon+q$, $0 < q \le 2\varepsilon-\gamma+1$.

If $(m\gamma+\gamma-1)(m-1)+\varepsilon < b \le (m\gamma+\gamma-1)(m-1)+\gamma-2$, let $b = (m\gamma+\gamma-1)(m-1)+\varepsilon+q$, where $0 \le q \le \gamma-\varepsilon-2$, then

$$|h^{(\gamma)}A| = M + \gamma\{(m\gamma+\gamma-1)(m-1)+\varepsilon\} + \frac{q(2\gamma-q-1)}{2} - \frac{(\gamma-2\varepsilon-q)(\gamma-2\varepsilon-q-1)}{2}.$$

If $b \ge (m\gamma+r-1)(\gamma-1)+\gamma-1$, then

$$|h^{(\gamma)}A| = M + \gamma\{(m\gamma+\gamma-1)(m-1)+\gamma-1\} - \frac{\varepsilon(\varepsilon-1)}{2} - \frac{(\gamma-\varepsilon)(\gamma-\varepsilon-1)}{2}.$$

Proof. If $b=0$, then $h^{(\gamma)}A=[0,h(k-1)]$. Hence, $|h^{(\gamma)}A|=M$. Now let $b>0$. For $k=2m+1$, we have the following two sets of integers with their corresponding orders:

$$\begin{aligned} 0 &< (m\gamma+\gamma-1)(m-1)+\gamma-\varepsilon \\ &< (m\gamma+\gamma-1)(m-1)+\gamma-\varepsilon+1 \\ &\vdots \\ &< (m\gamma+\gamma-1)(m-1)+\gamma-1, \end{aligned}$$

and

$$\begin{aligned} 0 &< (m\gamma+\gamma-1)(m-1)+\varepsilon \\ &< (m\gamma+\gamma-1)(m-1)+\varepsilon+1 \\ &\vdots \\ &< (m\gamma+\gamma-1)(m-1)+\gamma-2. \end{aligned}$$

We have $(m\gamma+\gamma-1)(m-1)+\varepsilon \le (m\gamma+\gamma-1)(m-1)+\gamma-\varepsilon$ if and only if $0\le\varepsilon\le\gamma/2$. Assume that $0\le\varepsilon\le\gamma/2$.

Case 1. Let $0<b\le(m\gamma+\gamma-1)(m-1)+\varepsilon$. Clearly, $I_0\cup\cdots\cup I_\varepsilon\cup J_{\varepsilon+1}\cup\cdots\cup J_\gamma$ is the interval $[m\gamma(m-1)/2+\varepsilon m, m\gamma(2m-1-(m-1)/2)+\varepsilon m+\gamma(b+m)]$. Hence,

$$|h^{(\gamma)}A|=M+b\gamma.$$

Case 2. Since $\varepsilon=\gamma/2$ is covered in Case 1, let $\varepsilon<\gamma/2$ and $(m\gamma+\gamma-1)(m-1)+\varepsilon<b\le(m\gamma+\gamma-1)(m-1)+\gamma-\varepsilon$. There exists a unique $t\in[2\varepsilon,\gamma-1]$ such that $(m\gamma+\gamma-1)(m-1)+(\gamma+\varepsilon-t-1)<b\le(m\gamma+\gamma-1)(m-1)+(\gamma+\varepsilon-t)$. So, $I_0\cup I_1\cup\cdots\cup I_\varepsilon\cup J_{\varepsilon+1}\cup\cdots\cup J_t$ is the interval $J=[m\gamma(m-1)/2+\varepsilon m, m\gamma(2m-1-(m-1)/2)+\varepsilon m+t(b+m)]$. Moreover, the intervals $J,J_{t+1},\ldots,J_\gamma$ are pairwise disjoint. Therefore,

$$\begin{aligned} |h^{(\gamma)}A| &= |J|+\sum_{l'=t+1}^{\gamma}|J_{l'}| \\ &= M+tb+(\gamma-t)\{(m\gamma+\gamma-1)(m-1)+\varepsilon\}+\frac{(\gamma-t)(\gamma-t-1)}{2}. \end{aligned}$$

Let $b=(m\gamma+\gamma-1)(m-1)+\varepsilon+q$, where $0<q\le\gamma-2\varepsilon$. So $\gamma-t=q$. Hence,

$$|h^{(\gamma)}A|=M+\gamma\{(m\gamma+\gamma-1)(m-1)+\varepsilon\}+\frac{q(2\gamma-q-1)}{2}.$$

Case 3. Let $(m\gamma+\gamma-1)(m-1)+\gamma-\varepsilon < b \leq (m\gamma+\gamma-1)(m-1)+\gamma-2$. There exists a unique $t\in[1,\varepsilon-2]$ such that $(m\gamma+\gamma-1)(m-1)+\gamma-\varepsilon-1+t < b \leq (m\gamma+\gamma-1)(m-1)+\gamma-\varepsilon+t$. So, $I_t\cup\cdots\cup I_\varepsilon\cup J_{\varepsilon+1}\cup\cdots\cup J_{2\varepsilon-t}$ is the interval $J=[m\gamma(m-1)/2+\varepsilon m+t(b+m), m\gamma(2m-1-(m-1)/2)+\varepsilon m+(2\varepsilon-t)(b+m)]$. Moreover the intervals $I_0, I_1,\ldots,I_{t-1},J,J_{2\varepsilon-1},\ldots,J_\gamma$ are pairwise disjoint. Hence,

$$\begin{aligned}|h^{(\gamma)}A| &= |J|+\sum_{l=0}^{t-1}|I_l|+\sum_{l'=2\varepsilon-t+1}^{r}|J_{l'}|\\ &= m^2\gamma+1-t(b+m-m^2\gamma-1)-\frac{t(2\varepsilon-t+1)}{2}+\gamma(m^2r+1)\\ &\quad+(2\varepsilon-t)(b+m-m^2\gamma-1)-\frac{(\gamma-t+1)(\gamma-2\varepsilon+t)}{2}.\end{aligned}$$

Let $b=(m\gamma+\gamma-1)(m-1)+\gamma-\varepsilon+q$, where $0\leq q\leq \varepsilon-2$. So $q=t$. Hence,

$$|h^{(\gamma)}A| = M+\gamma\{(m\gamma+\gamma-1)(m-1)+\gamma-\varepsilon\}+\frac{q(2\gamma-q-1)}{2}-\frac{(\gamma-2\varepsilon+q)(\gamma-2\varepsilon+q+1)}{2}.$$

Now assume that $\varepsilon>\gamma/2$.

Case 4. Let $0<b\leq(m\gamma+\gamma-1)(m-1)+\gamma-\varepsilon$. Clearly, $I_0\cup\cdots\cup I_\varepsilon\cup J_{\varepsilon+1}\cup\cdots\cup J_\gamma$ is the interval $[m\gamma(m-1)/2+\varepsilon m, m\gamma(2m-1-(m-1)/2)+\varepsilon m+\gamma(b+m)]$. Therefore,

$$|h^{(\gamma)}A| = M+b\gamma.$$

Case 5. Let $(m\gamma+\gamma-1)(m-1)+\gamma-\varepsilon < b \leq (m\gamma+\gamma-1)(m-1)+\varepsilon$. There exists a unique $t\in[1,2\varepsilon-\gamma]$ such that $(m\gamma+\gamma-1)(m-1)+\gamma-\varepsilon+t-1 < b\leq (m\gamma+\gamma-1)(m-1)+\gamma-\varepsilon+t$. So, $I_t\cup I_{t+1}\cup\cdots\cup I_\varepsilon\cup J_{\varepsilon+1}\cup\cdots\cup J_\gamma$ is the interval $J=[m\gamma(m-1)/2+\varepsilon m+t(b+m), m\gamma(2m-1-(m-1)/2)+\varepsilon m+\gamma(b+m)]$. Moreover, the intervals $I_0,I_1,\ldots,I_{t-1},J$ are pairwise disjoint. Hence,

$$\begin{aligned}|h^{(\gamma)}A| &= |J|+\sum_{l=0}^{t-1}|I_l|\\ &= M+b\gamma-t(b+m-m^2\gamma-1)-\frac{t(2\varepsilon-t+1)}{2}.\end{aligned}$$

Let $b=(m\gamma+\gamma-1)(m-1)+\gamma-\varepsilon+q$, where $0<q\leq 2\varepsilon-\gamma+1$. So $t=q$. Therefore,

$$|h^{(\gamma)}A| = M+\gamma\{(m\gamma+\gamma-1)(m-1)+\gamma-\varepsilon\}+\frac{q(2\gamma-q-1)}{2}.$$

Case 6. Let $(m\gamma+\gamma-1)(m-1)+\varepsilon < b\leq (m\gamma+\gamma-1)(m-1)+\gamma-2$. There exists a unique $t\in[\varepsilon+2,\gamma-1]$ such that $(m\gamma+\gamma-1)(m-1)+(\gamma+\varepsilon-t-1) < b \leq (m\gamma+\gamma-1)(m-1)+$

$(\gamma+\varepsilon-t)$. So $I_{2\varepsilon-t}\cup\cdots\cup I_{\varepsilon}\cup J_{\varepsilon+1}\cup\cdots\cup J_t$ is the interval $J=[m\gamma(m-1)/2+\varepsilon m+(2\varepsilon-t)(b+m), m\gamma(2m-1-(m-1)/2)+\varepsilon m+t(b+m)]$. Moreover the intervals $I_0, I_1, \ldots, I_{2\varepsilon-t-1}, J, J_{t+1}, \ldots, J_\gamma$ are pairwise disjoint. Hence,

$$\begin{aligned}
|h^{(\gamma)}A| &= |J| + \sum_{l=0}^{2\varepsilon-t-1} |I_l| + \sum_{l'=t+1}^{\gamma} |J_{l'}| \\
&= m^2\gamma+1-(2\varepsilon-t)(b+m-m^2\gamma-1) - \frac{(2\varepsilon-t)(t+1)}{2} + \gamma(m^2\gamma+1) \\
&\quad + t(b+m-m^2\gamma-1) - \frac{(\gamma-t)(\gamma-2\varepsilon+t+1)}{2}.
\end{aligned}$$

Let $b=(m\gamma+\gamma-1)(m-1)+\varepsilon+q$, where $0\le q\le \gamma-\varepsilon-2$. So $q=\gamma-t$. Hence,

$$|h^{(\gamma)}A| = M+\gamma\{(m\gamma+\gamma-1)(m-1)+\varepsilon\} + \frac{q(2\gamma-q-1)}{2} - \frac{(\gamma-2\varepsilon-q)(\gamma-2\varepsilon-q-1)}{2}.$$

Case 7. Let $b=(m\gamma+\gamma-1)(m-1)+\gamma-1$. So $I_0, I_1, \ldots, I_{\varepsilon-2}, J, J_{\varepsilon+2}, \ldots, J_\gamma$ are pairwise disjoint intervals, where $J=I_{\varepsilon-1}\cup I_\varepsilon\cup J_{\varepsilon+1}=[m\gamma(m-1)/2+\varepsilon m+(\varepsilon-1)(b+m), m\gamma(2m-1-(m-1)/2)+\varepsilon m+(\varepsilon+1)(b+m)]$. Therefore,

$$\begin{aligned}
|h^{(\gamma)}A| &= |J| + \sum_{l=0}^{\varepsilon-2} |I_l| + \sum_{l'=\varepsilon+2}^{\gamma} |J_{l'}| \\
&= M+\gamma\{(m\gamma+\gamma-1)(m-1)+\gamma-1\} - \frac{\varepsilon(\varepsilon-1)}{2} - \frac{(\gamma-\varepsilon)(\gamma-\varepsilon-1)}{2}.
\end{aligned}$$

Case 8. Let $b>(m\gamma+\gamma-1)(m-1)+\gamma-1$. Clearly, $I_0, I_1, \ldots, I_\varepsilon, J_{\varepsilon+1}, \cdots, J_\gamma$ are pairwise disjoint intervals. Hence,

$$\begin{aligned}
|h^{(\gamma)}A| &= \sum_{l=0}^{\varepsilon} |I_l| + \sum_{l'=\varepsilon+1}^{\gamma} |J_{l'}| \\
&= M+\gamma\{(m\gamma+\gamma-1)(m-1)+\gamma-1\} - \frac{\varepsilon(\varepsilon-1)}{2} - \frac{(\gamma-\varepsilon)(\gamma-\varepsilon-1)}{2}.
\end{aligned}$$

This completes the proof of the theorem. □

5.5 Remarks

1. If $\varepsilon=0$ and $\gamma=h$, we get Theorem 5.1.1 of Nathanson.

2. If $\gamma=1$, we get the following;

Let $k \geq 3$ and $2 \leq h \leq k$. For $b \geq 0$, let $A = [0, k-2] \cup \{k-1+b\}$. If $h \leq k-1$ and $b \leq (h-1)(k-h-1)$, then

$$|h^{\wedge}A| = hk - h^2 + 1 + b.$$

If $b > (h-1)(k-h-1)$, then

$$|h^{\wedge}A| = hk - h^2 + 1 + (h-1)(k-h-1).$$

If $h = k$, then $|h^{\wedge}A| = 1$.

Thus, we observe that $|h^{\wedge}A|$ is a strictly increasing linear function of b for $0 \leq b \leq (h-1)(k-h-1)$ and $|h^{\wedge}A|$ is constant for $b \geq (h-1)(k-h-1)$.

3. Let $A = \{a, a+d, a+2d, \ldots, a+(k-2)d, a+(k-1+b)d\}$ is almost an arithmetic progression, where b is a nonnegative integer. Then $A = a + d \cdot ([0, k-2] \cup \{k-1+b\})$. Set $A' = [0, k-2] \cup \{k-1+b\}$. Then $h^{(\gamma)}A = \{ha\} + d \cdot h^{(\gamma)}A'$. Thus, $|h^{(\gamma)}A| = |h^{(\gamma)}A'|$. Hence, our result also holds for the set A.

4. For $\varepsilon = 0$, i.e., $h = m\gamma$ one requires to consider only two cases; $k \geq 2m$ and $k < 2m$. In both the cases $|h^{(\gamma)}A|$ is a strictly increasing linear function of b for $0 \leq b \leq N_1$ and is a strictly increasing piecewise-linear function of b for $N_1 \leq b \leq N_2$ and $|h^{(\gamma)}A|$ is constant for $b \geq N_2$; $N_1 = (k\gamma - m\gamma - 1)(m-1)$, $N_2 = (k\gamma - m\gamma - 1)(m-1) + (\gamma - 1)(k-2m)$, if $k \geq 2m$ and $N_1 = (k\gamma - m\gamma - 1)(m-1) + (\gamma - 1)(k-2m)$, $N_2 = (k\gamma - m\gamma - 1)(m-1)$, if $k < 2m$.

Chapter 6

Direct and inverse problems for certain subset and subsequence sums

In this chapter, we consider certain subset and subsequence sums in the group of integers. We solve the direct and inverse problems for the subset sums. We also solve the direct and inverse problems for the subsequence sums, when all the distinct integers of the sequence have the same multiplicity. Moreover, as corollaries of direct and inverse results for these subset and subsequence sums we obtain already established direct and inverse results for the regular subset and subsequence sums.

6.1 Introduction

Let A be a nonempty finite set of integers. Given a subset B of A, the sum of all elements of B is called the *subset sum* of B. Let $S(A)$ be the set of all subset sums of A, i.e.,

$$S(A) := \left\{ \sum_{b \in B} b : B \subset A \right\},$$

where $s(\emptyset) = 0$.

The subsequence sum of a given sequence of integers is defined in a similar way. Let $\mathscr{A} = (a_0, a_1, \ldots, a_{k-1})_{\bar{r}}$ be a nonempty sequence of k distinct integers with repetition $\bar{r} = (r_0, r_1, \ldots, r_{k-1})$. Given a subsequence $\mathscr{B}$ of $\mathscr{A}$, the sum of all terms of $\mathscr{B}$ is called the *subsequence sum* of $\mathscr{B}$.

Let $S(\bar{r},\mathscr{A})$ be the set of all subsequence sums of $\mathscr{A}$, i.e.,

$$S(\bar{r},\mathscr{A}) := \left\{ \sum_{b\in\mathscr{B}} b : \mathscr{B} \text{ is a subsequence of } \mathscr{A} \right\}.$$

The direct problem for $S(A)$ is to find the minimum number of elements in $S(A)$ in terms of number of elements in A. The inverse problem for $S(A)$ is to find the structure of the finite set A for which $|S(A)|$ is minimal. The direct and inverse problems for the subsequence sums $S(\bar{r},\mathscr{A})$ of the sequence $\mathscr{A} = (a_0, a_1, \ldots, a_{k-1})_{\bar{r}}$ are defined in a similar way.

The direct and inverse theorems for the subset and subsequence sums are well established in the group of integers (see [58, 76, 77, 78, 80]). In a recent study, Balandraud [12] finds the minimum cardinality of certain subset sums with some restriction on the number of elements, in finite fields. We study the same subset sums in the group of integers, as that considered by Balandraud. We also study the analogues subsequence sums.

Definition 6.1.1 (Balandraud [12])**.** Let A be a nonempty finite set of k integers. Let $\alpha \in [0,k]$ be an integer. We define $S_\alpha(A)$ to be the set of subset sums of all subsets of A that are of the size at least α, and $S^\alpha(A)$ to be the set of subset sums of all subsets of A that are of the size at most $k-\alpha$. More precisely,

$$S_\alpha(A) := \left\{ \sum_{b\in B} b : B \subset A,\ |B| \geq \alpha \right\},$$

and

$$S^\alpha(A) := \left\{ \sum_{b\in B} b : B \subset A,\ |B| \leq k-\alpha \right\}.$$

It is easy to see that these subset sums have the following properties:

- If $\alpha = 0$, then $S_0(A) = S^0(A) = S(A)$.

- For every $\alpha \in [0,k]$, we have the symmetric relation

$$S_\alpha(A) = \sum_{a\in A} a - S^\alpha(A).$$

  Thus, $|S_\alpha(A)| = |S^\alpha(A)|$.

- If $\alpha \leq \alpha'$, then $S_{\alpha'}(A) \subset S_\alpha(A)$ and $S^{\alpha'}(A) \subset S^\alpha(A)$.

Definition 6.1.2. Let $\mathscr{A} = (a_0, a_1, \ldots, a_{k-1})_{\bar{r}}$ be a nonempty sequence of k distinct integers with repetition $\bar{r} = (r_0, r_1, \ldots, r_{k-1})$. Let $\alpha \in \left[0, \sum_{i=0}^{k-1} r_i\right]$ be an integer. We define $S_\alpha(\bar{r}, \mathscr{A})$ to be the set of subsequence sums of all subsequences of $\mathscr{A}$ that are of the size at least α, and $S^\alpha(\bar{r}, \mathscr{A})$ to be the set of subsequence sums of all subsequences of $\mathscr{A}$ that are of the size at most $\sum_{i=0}^{k-1} r_i - \alpha$. More precisely,

$$S_\alpha(\bar{r}, \mathscr{A}) := \left\{ \sum_{b \in \mathscr{B}} b : \mathscr{B} \text{ is a subsequence of } \mathscr{A} \text{ with } |\mathscr{B}| \geq \alpha \right\},$$

and

$$S^\alpha(\bar{r}, \mathscr{A}) := \left\{ \sum_{b \in \mathscr{B}} b : \mathscr{B} \text{ is a subsequence of } \mathscr{A} \text{ with } |\mathscr{B}| \leq \sum_{i=0}^{k-1} r_i - \alpha \right\}.$$

These subsequence sums also satisfy similar properties as that satisfied by the aforementioned subset sums.

- If $\alpha = 0$, then $S_0(\bar{r}, \mathscr{A}) = S^0(\bar{r}, \mathscr{A}) = S(\bar{r}, \mathscr{A})$.
- For every $\alpha \in \left[0, \sum_{i=0}^{k-1} r_i\right]$, we have
$$S_\alpha(\bar{r}, \mathscr{A}) = \sum_{a \in \mathscr{A}} a - S^\alpha(\bar{r}, \mathscr{A}).$$
Thus, $|S_\alpha(\bar{r}, \mathscr{A})| = |S^\alpha(\bar{r}, \mathscr{A})|$.
- If $\alpha \leq \alpha'$, then $S_{\alpha'}(\bar{r}, \mathscr{A}) \subset S_\alpha(\bar{r}, \mathscr{A})$ and $S^{\alpha'}(\bar{r}, \mathscr{A}) \subset S^\alpha(\bar{r}, \mathscr{A})$.

If $r_i = r$ for $i = 0, 1, \ldots, k-1$, then we use the notation $S_\alpha(r, \mathscr{A})$ for $S_\alpha(\bar{r}, \mathscr{A})$ and $S^\alpha(r, \mathscr{A})$ for $S^\alpha(\bar{r}, \mathscr{A})$.

The direct problem for $S_\alpha(A)$ is to find the minimum number of elements in $S_\alpha(A)$ in terms of number of elements in A and α. The inverse problem for $S_\alpha(A)$ is to find the structure of the finite set A for which $|S_\alpha(A)|$ is minimal. Similarly, the direct problem for $S_\alpha(\bar{r}, \mathscr{A})$ is to find the minimum number of elements in $S_\alpha(\bar{r}, \mathscr{A})$ in terms of number of distinct elements in $\mathscr{A}$ and α. The inverse problem for $S_\alpha(\bar{r}, \mathscr{A})$ is to find the structure of the finite sequence $\mathscr{A}$ for which $|S_\alpha(\bar{r}, \mathscr{A})|$ is minimal.

In this chapter, we solve both direct and inverse problems for the set of subset sums $S_\alpha(A)$ in Section 6.2. We also solve both direct and inverse problems for the set of subsequence sums $S_\alpha(r, \mathscr{A})$ in Section 6.3. Furthermore, as particular cases of our results we obtain the direct and inverse results of Nathanson [80] on regular subset sums, and the direct and inverse results of Mistri and Pandey [77] on regular subsequence sums.

6.2 Subset sum

Theorem 6.2.1. *Let $k \geq 1$ and $\alpha \in [0,k]$. Let A be a set of k positive integers. Then*

$$|S_\alpha(A)| \geq \binom{k+1}{2} - \binom{\alpha+1}{2} + 1. \tag{6.1}$$

This lower bound is best possible.

Proof. First, let $k = 1$. Then $A = \{a\}$ for some integer $a > 0$. Clearly, $S_0(A) = \{0, a\}$ and $S_1(A) = \{a\}$. Thus, (6.1) holds for $k = 1$. So, we may assume that $k \geq 2$.

Let $A = \{a_0, a_1, \ldots, a_{k-1}\}$, where $0 < a_0 < a_1 < \cdots < a_{k-1}$. If $\alpha = k$, then $S_\alpha(A) = \{a_0 + a_1 + \cdots + a_{k-1}\}$, and hence $|S_\alpha(A)| = 1$. This satisfies (6.1).

So, let $0 \leq \alpha \leq k-1$. For $h = 1, 2, \ldots, k-\alpha$, define

$$B_h := \{a_i + a_{k-\alpha-h+1} + a_{k-\alpha-h+2} + \cdots + a_{k-\alpha-1} : \ i = 0, 1, \ldots, k-\alpha-h\}. \tag{6.2}$$

Each element of B_h is a sum of at most $k-\alpha$ distinct elements of A. Therefore, $B_h \subset S^\alpha(A)$. Moreover, for $h = 1, 2, \ldots, k-\alpha-1$, we have

$$\begin{aligned} \max(B_h) &= a_{k-\alpha-h} + a_{k-\alpha-h+1} + \cdots + a_{k-\alpha-1} \\ &< a_0 + a_{k-\alpha-h} + a_{k-\alpha-h+1} + \cdots + a_{k-\alpha-1} = \min(B_{h+1}). \end{aligned}$$

Therefore, the sets $B_1, B_2, \ldots, B_{k-\alpha}$ are pairwise disjoint.

For $i = 0, 1, \ldots, \alpha-1$ and $j = 0, 1, \ldots, k-\alpha$, set

$$s_{i,j} := \sum_{\substack{l=0 \\ l \neq k-\alpha-j}}^{k-\alpha} a_{i+l}, \tag{6.3}$$

and

$$s_{\alpha,0} := \sum_{l=0}^{k-\alpha-1} a_{\alpha+l}. \tag{6.4}$$

Each of these sums in (6.3) and (6.4) is a sum of $k-\alpha$ distinct elements of A, and hence $s_{i,j} \in S^\alpha(A)$. Moreover, for $i = 0, 1, \ldots, \alpha-1$ and $j = 0, 1, \ldots, k-\alpha-1$, we have

$$s_{i,j} < s_{i,j+1},$$

and

$$s_{i,k-\alpha} = s_{i+1,0}.$$

Therefore,

$$s_{i,0} < s_{i,1} < \cdots < s_{i,k-\alpha-1} < s_{i,k-\alpha} = s_{i+1,0}.$$

Thus, the total number of integers mentioned in (6.3) and (6.4) is $\alpha(k-\alpha)+1$.

Since, $\max(B_{k-\alpha}) = s_{0,0}$, and $0 \in S^{\alpha}(A)$, we have

$$\begin{aligned}
|S_{\alpha}(A)| &= |S^{\alpha}(A)| \\
&\geq |\bigcup_{h=1}^{k-\alpha} B_h| + \alpha(k-\alpha)+1 \\
&= \sum_{h=1}^{k-\alpha} |B_h| + \alpha(k-\alpha)+1 \\
&= \sum_{h=1}^{k-\alpha} (k-\alpha-h+1) + \alpha(k-\alpha)+1 \\
&= \binom{k+1}{2} - \binom{\alpha+1}{2} + 1.
\end{aligned}$$

Next, we show that the lower bound in (6.1) is best possible.

Let $k \geq 2$ and $A = [1,k]$. Then

$$\begin{aligned}
S^{\alpha}(A) &\subset [0, k+(k-1)+\cdots+(\alpha+1)] \\
&= \left[0, \binom{k+1}{2} - \binom{\alpha+1}{2}\right].
\end{aligned}$$

Therefore,

$$|S^{\alpha}(A)| \leq \binom{k+1}{2} - \binom{\alpha+1}{2} + 1.$$

This together with (6.1) gives

$$|S_{\alpha}(A)| = |S^{\alpha}(A)| = \binom{k+1}{2} - \binom{\alpha+1}{2} + 1.$$

This completes the proof of theorem. □

Corollary 6.2.2. *Let $k \geq 2$ and $\alpha \in [0,k]$. Let A be a set of k nonnegative integers with $0 \in A$. Then*

$$|S_{\alpha}(A)| \geq \binom{k}{2} - \binom{\alpha}{2} + 1. \tag{6.5}$$

This lower bound is best possible.

Proof. Let $A = \{a_0, a_1, \dots, a_{k-1}\}$, where $0 = a_0 < a_1 < \cdots < a_{k-1}$. Let $A' = A \setminus \{0\}$. So, A' is a nonempty set of $k-1$ positive integers. It is easy to see that if $\alpha = 0$, then

$$S^0(A) = S(A) = S(A') = S^0(A'),$$

and if $\alpha \geq 1$, then

$$S^\alpha(A) = S^{\alpha-1}(A').$$

Hence, by Theorem 6.2.1, we have

$$|S_0(A)| = |S^0(A)| = |S^0(A')| \geq \binom{k}{2} + 1,$$

and for $\alpha \geq 1$, we have

$$\begin{aligned} |S_\alpha(A)| &= |S^\alpha(A)| \\ &= |S^{\alpha-1}(A')| \\ &\geq \binom{k}{2} - \binom{\alpha}{2} + 1. \end{aligned}$$

Next, we show that the lower bound in (6.5) is best possible.

Let $k \geq 3$, and $A = [0, k-1]$. Then

$$\begin{aligned} S^\alpha(A) &\subset [0, (k-1) + (k-2) + \cdots + \alpha] \\ &= \left[0, \binom{k}{2} - \binom{\alpha}{2}\right]. \end{aligned}$$

Therefore,

$$|S^\alpha(A)| \leq \binom{k}{2} - \binom{\alpha}{2} + 1.$$

This together with (6.5) gives

$$|S_\alpha(A)| = |S^\alpha(A)| = \binom{k}{2} - \binom{\alpha}{2} + 1.$$

This completes the proof of the corollary. □

As a consequence of Theorem 6.2.1 and Corollary 6.2.2, for $\alpha = 0$, we obtain Theorem 1.4.22.

Theorem 6.2.3. *Let $k \geq 4$ and $0 \leq \alpha \leq k-2$. Let A be a set of k positive integers such that*

$$|S_\alpha(A)| = \binom{k+1}{2} - \binom{\alpha+1}{2} + 1.$$

Then $A = d \cdot [1, k]$ for some positive integer d.

Proof. Let $A = \{a_0, a_1, \ldots, a_{k-1}\}$, where $0 < a_0 < a_1 < \cdots < a_{k-1}$. Let

$$|S^{\alpha}(A)| = |S_{\alpha}(A)| = \binom{k+1}{2} - \binom{\alpha+1}{2} + 1.$$

Then, Theorem 6.2.1 implies that $S^{\alpha}(A)$ contains precisely the integers listed in (6.2), (6.3) and (6.4), with one more integer, 0. For $h = 1, 2, \ldots, k-\alpha-1$, we have

$$\begin{aligned}
&a_{k-\alpha-h-1} + a_{k-\alpha-h+1} + a_{k-\alpha-h+2} + \cdots + a_{k-\alpha-1} \\
&\quad < a_{k-\alpha-h} + a_{k-\alpha-h+1} + a_{k-\alpha-h+2} + \cdots + a_{k-\alpha-1} = \max(B_h) \\
&\quad < a_0 + a_{k-\alpha-h} + a_{k-\alpha-h+1} + a_{k-\alpha-h+2} + \cdots + a_{k-\alpha-1} = \min(B_{h+1}),
\end{aligned}$$

and

$$\begin{aligned}
&a_{k-\alpha-h-1} + a_{k-\alpha-h+1} + a_{k-\alpha-h+2} + \cdots + a_{k-\alpha-1} \\
&\quad < a_0 + a_{k-\alpha-h-1} + a_{k-\alpha-h+1} + a_{k-\alpha-h+2} + \cdots + a_{k-\alpha-1} \\
&\quad < a_0 + a_{k-\alpha-h} + a_{k-\alpha-h+1} + a_{k-\alpha-h+2} + \cdots + a_{k-\alpha-1} = \min(B_{h+1}).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\max(B_h) &= a_{k-\alpha-h} + a_{k-\alpha-h+1} + a_{k-\alpha-h+2} + \cdots + a_{k-\alpha-1} \\
&= a_0 + a_{k-\alpha-h-1} + a_{k-\alpha-h+1} + a_{k-\alpha-h+2} + \cdots + a_{k-\alpha-1}.
\end{aligned}$$

That is

$$a_{k-\alpha-h} - a_{k-\alpha-h-1} = a_0$$

for $h = 1, 2, \ldots, k-\alpha-1$. Thus

$$a_{k-\alpha-1} - a_{k-\alpha-2} = a_{k-\alpha-2} - a_{k-\alpha-3} = \cdots = a_1 - a_0 = a_0. \tag{6.6}$$

If $\alpha = 0$, then we are done. So, we may assume that $\alpha \geq 1$. Then

$$\begin{aligned}
\max(B_{h-1}) &= a_1 + a_2 + a_3 + \cdots + a_{k-\alpha-1} \\
&< a_0 + a_1 + a_2 + \cdots + a_{k-\alpha-1} = \min(B_h) = \max(B_h) = s_{0,0} \\
&< a_0 + a_1 + a_2 + \cdots + a_{k-\alpha-2} + a_{k-\alpha} = s_{0,1},
\end{aligned}$$

and

$$\begin{aligned}\max(B_{h-1}) &= a_1 + a_2 + a_3 + \cdots + a_{k-\alpha-1} \\ &< a_1 + a_2 + a_3 + \cdots + a_{k-\alpha-2} + a_{k-\alpha} \\ &< a_0 + a_1 + a_2 + \cdots + a_{k-\alpha-2} + a_{k-\alpha} = s_{0,1}.\end{aligned}$$

Therefore,

$$\begin{aligned}\max(B_h) &= a_0 + a_1 + a_2 + \cdots + a_{k-\alpha-1} \\ &= a_1 + a_2 + a_3 + \cdots + a_{k-\alpha-2} + a_{k-\alpha}.\end{aligned}$$

That is

$$a_{k-\alpha} - a_{k-\alpha-1} = a_0. \tag{6.7}$$

Again, if $\alpha = 1$, then we are done, as the result follows from (6.6) and (6.7). So, let $\alpha \geq 2$. Then, for $i = 1, 2, \ldots, \alpha - 1$, we have

$$\begin{aligned}s_{i-1,k-\alpha-1} &= a_{i-1} + a_{i+1} + a_{i+2} + \cdots + a_{i+k-\alpha-1} \\ &< a_i + a_{i+1} + a_{i+2} + \cdots + a_{i+k-\alpha-1} = s_{i-1,k-\alpha} = s_{i,0} \\ &< a_i + a_{i+1} + a_{i+2} + \cdots + a_{i+k-\alpha-2} + a_{i+k-\alpha} = s_{i,1},\end{aligned}$$

and

$$\begin{aligned}s_{i-1,k-\alpha-1} &= a_{i-1} + a_{i+1} + a_{i+2} + \cdots + a_{i+k-\alpha-1} \\ &< a_{i-1} + a_{i+1} + a_{i+2} + \cdots + a_{i+k-\alpha-2} + a_{i+k-\alpha} \\ &< a_i + a_{i+1} + a_{i+2} + \cdots + a_{i+k-\alpha-2} + a_{i+k-\alpha} = s_{i,1}.\end{aligned}$$

Therefore

$$\begin{aligned}s_{i,0} &= a_i + a_{i+1} + a_{i+2} + \cdots + a_{i+k-\alpha-1} \\ &= a_{i-1} + a_{i+1} + a_{i+2} + \cdots + a_{i+k-\alpha-2} + a_{i+k-\alpha}.\end{aligned}$$

That is

$$a_{i+k-\alpha} - a_{i+k-\alpha-1} = a_i - a_{i-1}. \tag{6.8}$$

Since, $\alpha \leq k-2$, we get $i + k - \alpha \geq i + 2$. Hence, from (6.6), (6.7) and (6.8) it follows that

$$a_{k-1} - a_{k-2} = \cdots = a_1 - a_0 = a_0.$$

This completes the proof of the theorem. □

Corollary 6.2.4. *Let $k \geq 5$ and $0 \leq \alpha \leq k-2$. Let A be a set of k nonnegative integers with $0 \in A$ and*

$$|S_\alpha(A)| = \binom{k}{2} - \binom{\alpha}{2} + 1.$$

Then $A = d \cdot [0, k-1]$ for some positive integer d.

Proof. Let $A = \{a_0, a_1, \ldots, a_{k-1}\}$, where $0 = a_0 < a_1 < \cdots < a_{k-1}$. Let

$$|S^\alpha(A)| = |S_\alpha(A)| = \binom{k}{2} - \binom{\alpha}{2} + 1.$$

Let $A' = A \setminus \{0\}$. So, A' is a nonempty set of $k-1$ positive integers. First, let $\alpha = 0$. Since $S^0(A) = S(A) = S(A') = S^0(A')$, we have

$$|S^0(A')| = |S^0(A)| = \binom{k}{2} + 1.$$

Therefore, by Theorem 6.2.3, the set A' is an arithmetic progression with the common difference a_1, the smallest integer in A'. Hence, A is an arithmetic progression with the common difference a_1, i.e., $A = a_1 \cdot [0, k-1]$.

Now, let $\alpha \geq 1$. Since $S^\alpha(A) = S^{\alpha-1}(A')$, we have

$$|S^{\alpha-1}(A')| = |S^\alpha(A)| = \binom{k}{2} - \binom{\alpha}{2} + 1.$$

Therefore, by Theorem 6.2.3, the set A' is an arithmetic progression with the common difference a_1. Hence, A is an arithmetic progression with the common difference a_1, i.e., $A = a_1 \cdot [0, k-1]$. This completes the proof of the corollary. □

As a consequence of Theorem 6.2.3 and Corollary 6.2.4, for $\alpha = 0$, we obtain Theorem 1.4.23.

6.3 Subsequence sum

Let $\mathscr{A} = (a_0, a_1, \ldots, a_{k-1})_r$ be a nonempty sequence of k distinct nonnegative integers each repeating exactly r times. Let $0 \leq \alpha \leq rk$ be an integer. If $\alpha = rk$, then $S_\alpha(r, \mathscr{A}) = \{ra_0 + ra_1 + \cdots + ra_{k-1}\}$, and hence $|S_\alpha(r, \mathscr{A})| = 1$. So, we assume that $0 \leq \alpha \leq rk - 1$.

Theorem 6.3.1. *Let $k \geq 2$, $r \geq 1$ and $0 \leq \alpha < rk$. Let $m \in [1,k]$ be an integer such that $(m-1)r \leq \alpha < mr$. If $\mathscr{A}$ is a nonempty sequence of k distinct positive integers each repeating exactly r times, then*

$$|S_\alpha(r,\mathscr{A})| \geq r\left[\binom{k+1}{2} - \binom{m+1}{2}\right] + m(mr-\alpha) + 1. \tag{6.9}$$

This lower bound is best possible.

Proof. Let $\mathscr{A} = (a_0, a_1, \ldots, a_{k-1})_r$, where $0 < a_0 < a_1 < \cdots < a_{k-1}$, $k \geq 2$ and $r \geq 1$. First, let $m = k$. Since $(k-1)r \leq \alpha < rk$, we have $0 < rk - \alpha \leq r$. Define

$$A_0 := \{ia_0 : \ i = 1, 2, \ldots, rk-\alpha\}, \tag{6.10}$$

and for $j = 1, 2, \ldots, k-1$, set

$$A_j := \{(rk-\alpha-i)a_{j-1} + ia_j : \ i = 1, 2, \ldots, rk-\alpha\}. \tag{6.11}$$

Clearly, for $j = 0, 1, \ldots, k-1$, the sets A_j are subsets of $S^\alpha(r,\mathscr{A})$, with $\max(A_j) < \min(A_{j+1})$ for $j = 0, 1, 2, \ldots, k-2$. Therefore, the sets $A_0, A_1, \ldots, A_{k-1}$ are pairwise disjoint. Since $0 \in S^\alpha(r,\mathscr{A})$, but $0 \notin A_j$ for $j = 0, 1, 2, \ldots, k-1$, we have

$$\begin{aligned}
|S_\alpha(r,\mathscr{A})| &= |S^\alpha(r,\mathscr{A})| \\
&\geq \Big|\bigcup_{j=0}^{k-1} A_j\Big| + 1 \\
&= \sum_{j=0}^{k-1} |A_j| + 1 \\
&= k(rk-\alpha) + 1 \\
&= r\left[\binom{k+1}{2} - \binom{m+1}{2}\right] + m(mr-\alpha) + 1.
\end{aligned}$$

Thus, (6.9) is true for $m = k$.

Now, let $1 \leq m \leq k-1$. For $j = 1, 2, \ldots, mr-\alpha$, define

$$B_j := \{a_i + (j-1)a_{k-m} : \ i = 0, 1, \ldots, k-m\}, \tag{6.12}$$

and for $l = 1, 2, \ldots, k-m$ and $j = 1, 2, \ldots, r$, define also

$$B_{lr+j} := \left\{a_i + (j-1)a_{k-m-l} + \sum_{t=m+1}^{m+l-1} ra_{k-t} + (mr-\alpha)a_{k-m} : \ i = 0, 1, \ldots, k-m-l\right\}. \tag{6.13}$$

Next, for $l = 1, 2, \dots, k-m$, define

$$C_l := \left\{ \sum_{\substack{j=0 \\ j \neq k-m-l \\ j \neq k-m-l+1}}^{k-m} ra_j + (r-i)a_{k-m-l} + (mr-\alpha+i)a_{k-m-l+1} \,:\, i = 0, 1, \dots, r-mr+\alpha \right\}. \tag{6.14}$$

Clearly, the sets defined in (6.12), (6.13) and (6.14) are subsets of $S^{\alpha}(r, \mathscr{A})$, with

$$\max(B_j) < \min(B_{j+1}) \;\text{ for } j = 1, 2, \dots, mr-\alpha-1,$$

$$\max(B_{mr-\alpha}) < \min(B_{r+1}),$$

$$\max(B_j) < \min(B_{j+1}) \;\text{ for } j = r+1, r+2, \dots, (k-m+1)r-1,$$

$$\max(B_{(k-m+1)r}) = \min(C_1),$$

and

$$\max(C_l) = \min(C_{l+1}) \;\text{ for } l = 1, 2, \dots, k-m-1.$$

If $m = 1$, i.e., $0 \leq \alpha < r$, then the largest integer of the set $S^{\alpha}(r, \mathscr{A})$ is $(r-\alpha)a_0 + ra_1 + ra_2 + \cdots + ra_{k-1} = \max(C_{k-m})$. Observe that $0 \in S^{\alpha}(r, \mathscr{A})$, and $0 < a_0 = \min(B_1)$. Thus, by (6.12), (6.13) and (6.14) we have

$$\begin{aligned}
|S_{\alpha}(r, \mathscr{A})| &= |S^{\alpha}(r, \mathscr{A})| \\
&\geq \sum_{j=1}^{mr-\alpha}(k-m+1) + \sum_{l=1}^{k-m}\sum_{j=1}^{r}(k-m-l+1) + \sum_{l=1}^{k-m}(r-mr+\alpha) + 1 \\
&= (mr-\alpha)(k-m+1) + \frac{r(k-m+1)(k-m)}{2} + (r-mr+\alpha)(k-m) + 1 \\
&= (r-\alpha)k + \frac{rk(k-1)}{2} + \alpha(k-1) + 1 \\
&= \frac{rk(k+1)}{2} - \alpha + 1 \\
&= r\left[\binom{k+1}{2} - \binom{m+1}{2}\right] + m(mr-\alpha) + 1.
\end{aligned}$$

So, (6.9) is true for $m = 1$.

Now, let $2 \leq m \leq k-1$. For $x = 0, 1, \dots, m-2$ and $l = 0, 1, \dots, k-m-1$, define

$$D_l^x := \left\{ (mr-\alpha)a_x + \sum_{\substack{j=x+1 \\ j \neq k-m+x-l \\ j \neq k-m+x-l+1}}^{k-m+x+1} ra_j + (r-i)a_{k-m+x-l} + ia_{k-m+x-l+1} \,:\, i = 1, 2, \dots, r \right\}, \tag{6.15}$$

and

$$D_{k-m}^{x} := \left\{ (mr-\alpha-i)a_x + ia_{x+1} + \sum_{j=x+2}^{k-m+x+1} ra_j : \ i = 1,2,\ldots,mr-\alpha \right\}. \tag{6.16}$$

The sets defined in (6.15) and (6.16) are subsets of $S^{\alpha}(r,\mathscr{A})$, with

$$\max(C_{k-m}) < \min(D_0^0),$$

$$\max(D_l^x) < \min(D_{l+1}^x) \ \text{ for } l = 0,1,\ldots,k-m-1,$$

and

$$\max(D_{k-m}^x) < \min(D_0^{x+1}) \ \text{ for } x = 0,1,\ldots,m-3.$$

Hence, by (6.12), (6.13), (6.14), (6.15) and (6.16) we have

$$\begin{aligned}
|S_{\alpha}(r,\mathscr{A})| &= |S^{\alpha}(r,\mathscr{A})| \\
&\geq \sum_{j=1}^{mr-\alpha}(k-m+1) + \sum_{l=1}^{k-m}\sum_{j=1}^{r}(k-m-l+1) + \sum_{l=1}^{k-m}(r-mr+\alpha) \\
&\quad + \sum_{x=0}^{m-2}\sum_{l=0}^{k-m-1} r + \sum_{x=0}^{m-2}(mr-\alpha) + 1 \\
&= (mr-\alpha)(k-m+1) + \frac{r(k-m+1)(k-m)}{2} + (r-mr+\alpha)(k-m) \\
&\quad + (m-1)(k-m)r + (m-1)(mr-\alpha) + 1 \\
&= r\left[\binom{k+1}{2} - \binom{m+1}{2}\right] + m(mr-\alpha) + 1.
\end{aligned}$$

Hence, (6.9) holds for all $1 \leq m \leq k$.

Next, we show that the lower bound in (6.9) is best possible.

Let $k \geq 2$ and $\mathscr{A} = [1,k]_r$. Then

$$S^{\alpha}(r,\mathscr{A}) \subset [0, r(k+(k-1)+\cdots+(m+1)) + (mr-\alpha)m].$$

Therefore

$$|S^{\alpha}(r,\mathscr{A})| \leq r\left[\binom{k+1}{2} - \binom{m+1}{2}\right] + m(mr-\alpha) + 1.$$

This together with (6.9) gives

$$|S_{\alpha}(r,\mathscr{A})| = |S^{\alpha}(r,\mathscr{A})| = r\left[\binom{k+1}{2} - \binom{m+1}{2}\right] + m(mr-\alpha) + 1.$$

This completes the proof of the theorem. □

Corollary 6.3.2. *Let $k \geq 3$, $r \geq 1$ and $0 \leq \alpha < rk$. Let $m \in [1,k]$ be an integer such that $(m-1)r \leq \alpha < mr$. If $\mathscr{A}$ is a nonempty sequence of k distinct nonnegative integers each repeating exactly r times and $0 \in \mathscr{A}$, then*

$$|S_\alpha(r,\mathscr{A})| \geq r\left[\binom{k}{2} - \binom{m}{2}\right] + (m-1)(mr-\alpha) + 1. \tag{6.17}$$

This lower bound is best possible.

Proof. Let $\mathscr{A} = (a_0, a_1, \ldots, a_{k-1})_r$, where $0 = a_0 < a_1 < \cdots < a_{k-1}$ and $r \geq 1$. Let $\mathscr{A}' = \mathscr{A} \setminus \{0\}$. So, $\mathscr{A}'$ is a nonempty sequence of $k-1$ distinct positive integers each repeating exactly r times.

First, let $m = 1$, i.e., $0 \leq \alpha < r$. Then

$$S^\alpha(r,\mathscr{A}) = S^0(r,\mathscr{A}'). \tag{6.18}$$

Hence, by Theorem 6.3.1 we have

$$\begin{aligned} |S_\alpha(r,\mathscr{A})| &= |S^\alpha(r,\mathscr{A})| \\ &= |S^0(r,\mathscr{A}')| \\ &\geq r\binom{k}{2} + 1. \end{aligned}$$

Now, let $m \geq 2$, i.e., $r \leq \alpha < rk$. Clearly, $(m-1)r \leq \alpha < mr$ implies that $(m-2)r \leq \alpha - r < (m-1)r$. Thus,

$$S^\alpha(r,\mathscr{A}) = S^{\alpha-r}(r,\mathscr{A}'). \tag{6.19}$$

Hence, by Theorem 6.3.1 we have

$$\begin{aligned} |S_\alpha(r,\mathscr{A})| &= |S^\alpha(r,\mathscr{A})| \\ &= |S^{\alpha-r}(r,\mathscr{A}')| \\ &\geq r\left[\binom{k}{2} - \binom{m}{2}\right] + (m-1)[(m-1)r - (\alpha - r)] + 1 \\ &= r\left[\binom{k}{2} - \binom{m}{2}\right] + (m-1)(mr-\alpha) + 1. \end{aligned}$$

Next, we show that the lower bound in (6.17) is best possible.

Let $k \geq 3$, and $\mathscr{A} = [0, k-1]_r$. Then

$$S^\alpha(r,\mathscr{A}) \subset [0, r((k-1) + (k-2) + \cdots + m) + (mr-\alpha)(m-1)].$$

Therefore
$$|S^{\alpha}(r,\mathscr{A})| \le r\left[\binom{k}{2}-\binom{m}{2}\right]+(m-1)(mr-\alpha)+1.$$
This together with (6.17) gives
$$|S_{\alpha}(r,\mathscr{A})| = |S^{\alpha}(r,\mathscr{A})| = r\left[\binom{k}{2}-\binom{m}{2}\right]+(m-1)(mr-\alpha)+1.$$
This completes the proof of the corollary. □

As a consequence of Theorem 6.3.1 and Corollary 6.3.2, for $\alpha = 0$, we obtain the following corollary, which is a particular case of Theorem 1.4.24.

Corollary 6.3.3. *[77, Theorem 2.1] Let $k \ge 3$ and $r \ge 1$. Let $\mathscr{A}$ be a nonempty sequence of k distinct positive integers each repeating exactly r times. Then*
$$|S(r,\mathscr{A})| \ge r\binom{k+1}{2}+1. \tag{6.20}$$

Let $\mathscr{A}$ be a nonempty sequence of k distinct nonnegative integers each repeating exactly r times and $0 \in \mathscr{A}$. Then
$$|S(r,\mathscr{A})| \ge r\binom{k}{2}+1. \tag{6.21}$$
The lower bounds in (6.20) and (6.21) are best possible.

Theorem 6.3.4. *Let $k \ge 4$, $r \ge 1$ and $0 \le \alpha \le rk-2$. Let $1 \le m \le k$ be an integer such that $(m-1)r \le \alpha < mr$. If $\mathscr{A}$ is a nonempty sequence of k distinct positive integers each repeating exactly r times such that*
$$|S_{\alpha}(r,\mathscr{A})| = r\left[\binom{k+1}{2}-\binom{m+1}{2}\right]+m(mr-\alpha)+1,$$
then $\mathscr{A} = d\cdot[1,k]_r$ for some positive integer d.

Proof. Let $\mathscr{A} = (a_0,a_1,\ldots,a_{k-1})_r$, where $0 < a_0 < a_1 < \cdots < a_{k-1}$ and $r \ge 1$. Let
$$|S^{\alpha}(r,\mathscr{A})| = |S_{\alpha}(r,\mathscr{A})| = r\left[\binom{k+1}{2}-\binom{m+1}{2}\right]+m(mr-\alpha)+1. \tag{6.22}$$
First, let $m = k$, i.e., $(k-1)r \le \alpha \le rk-2$. Then, equation (6.22) and Theorem 6.3.1 implies that $S^{\alpha}(r,\mathscr{A})$ contains precisely the integers listed in (6.10) and (6.11) with one more integer, 0. We have
$$(rk-\alpha-1)a_0 < (rk-\alpha)a_0 = \max(A_0) < (rk-\alpha-1)a_0+a_1 = \min(A_1),$$

and

$$(rk-\alpha-1)a_0 < (rk-\alpha-2)a_0+a_1 < (rk-\alpha-1)a_0+a_1 = \min(A_1).$$

Thus,

$$(rk-\alpha-2)a_0+a_1 = (rk-\alpha)a_0.$$

That is

$$a_1 - a_0 = a_0. \tag{6.23}$$

Again, for $j = 1,2,\ldots,k-2$, we have

$$a_{j-1}+(rk-\alpha-1)a_j < (rk-\alpha)a_j = \max(A_j) < (rk-\alpha-1)a_j+a_{j+1} = \min(A_{j+1}),$$

and

$$a_{j-1}+(rk-\alpha-1)a_j < a_{j-1}+(rk-\alpha-2)a_j+a_{j+1} < (rk-\alpha-1)a_j+a_{j+1} = \min(A_{j+1}).$$

Therefore,

$$a_{j-1}+(rk-\alpha-2)a_j+a_{j+1} = (rk-\alpha)a_j \text{ for } j = 1,2,\ldots,k-2.$$

That is

$$a_{j+1}-a_j = a_j - a_{j-1} \text{ for } j = 1,2,\ldots,k-2.$$

In other words

$$a_{k-1}-a_{k-2} = \cdots = a_1 - a_0. \tag{6.24}$$

Hence, from (6.23) and (6.24) it follows that $\mathscr{A} = a_0 \cdot [1,k]_r$.

Now, let $1 \le m \le k-1$, i.e., $0 \le \alpha < r(k-1) \le rk-2$. Equation (6.22) and Theorem 6.3.1 implies that $S^{\alpha}(r,\mathscr{A})$ contains precisely the integers listed in (6.12), (6.13), (6.14), (6.15) and (6.16) with one more integer, 0. We have

$$\begin{aligned} a_{k-m-1}+(mr-\alpha-1)a_{k-m} &< (mr-\alpha)a_{k-m} = \max(B_{mr-\alpha}) \\ &< a_0+(mr-\alpha)a_{k-m} = \min(B_{r+1}), \end{aligned}$$

and

$$\begin{aligned} a_{k-m-1}+(mr-\alpha-1)a_{k-m} &< a_0+a_{k-m-1}+(mr-\alpha-1)a_{k-m} \\ &< a_0+(mr-\alpha)a_{k-m} = \min(B_{r+1}). \end{aligned}$$

Therefore,

$$a_0 + a_{k-m-1} + (mr - \alpha - 1)a_{k-m} = (mr - \alpha)a_{k-m}.$$

That is

$$a_{k-m} - a_{k-m-1} = a_0. \tag{6.25}$$

Similarly, for $m \leq k-2$ and $l = 1, 2, \ldots, k-m-1$, we have

$$\begin{aligned}
&a_{k-m-l-1} + (r-1)a_{k-m-l} + r(a_{k-m-l+1} + \cdots + a_{k-m-1}) + (mr-\alpha)a_{k-m} \\
&< r(a_{k-m-l} + a_{k-m-l+1} + \cdots + a_{k-m-1}) + (mr-\alpha)a_{k-m} = \max(B_{(l+1)r}) \\
&< a_0 + r(a_{k-m-l} + a_{k-m-l+1} + \cdots + a_{k-m-1}) + (mr-\alpha)a_{k-m} = \min(B_{(l+1)r+1}),
\end{aligned}$$

and

$$\begin{aligned}
&a_{k-m-l-1} + (r-1)a_{k-m-l} + r(a_{k-m-l+1} + \cdots + a_{k-m-1}) + (mr-\alpha)a_{k-m} \\
&< a_0 + a_{k-m-l-1} + (r-1)a_{k-m-l} + r(a_{k-m-l+1} + \cdots + a_{k-m-1}) + (mr-\alpha)a_{k-m} \\
&< a_0 + r(a_{k-m-l} + a_{k-m-l+1} + \cdots + a_{k-m-1}) + (mr-\alpha)a_{k-m} = \min(B_{(l+1)r+1}).
\end{aligned}$$

Therefore,

$$\begin{aligned}
&a_0 + a_{k-m-l-1} + (r-1)a_{k-m-l} + r(a_{k-m-l+1} + \cdots + a_{k-m-1}) + (mr-\alpha)a_{k-m} \\
&\qquad = r(a_{k-m-l} + a_{k-m-l+1} + \cdots + a_{k-m-1}) + (mr-\alpha)a_{k-m}.
\end{aligned}$$

That is

$$a_{k-m-l} - a_{k-m-l-1} = a_0 \text{ for } l = 1, 2, \ldots, k-m-1.$$

In other words

$$a_{k-m-1} - a_{k-m-2} = a_{k-m-2} - a_{k-m-3} = \cdots = a_1 - a_0 = a_0. \tag{6.26}$$

If $m = 1$, then (6.25) and (6.26) imply that $\mathscr{A} = a_0 \cdot [1, k]$. Hence, we are done. So, we may assume that $2 \leq m \leq k-1$. For $x = 0, 1, \ldots, m-2$, we have

$$\begin{aligned}
&(mr-\alpha)a_x + r(a_{x+1} + a_{x+2} + \cdots + a_{k-m+x-1}) + a_{k-m+x} + (r-1)a_{k-m+x+1} \\
&< (mr-\alpha)a_x + r(a_{x+1} + a_{x+2} + \cdots + a_{k-m+x-1}) + ra_{k-m+x+1} = \max(D_0^x) \\
&< (mr-\alpha)a_x + r(a_{x+1} + a_{x+2} + \cdots + a_{k-m+x-2}) + (r-1)a_{k-m+x-1} + a_{k-m+x} \\
&\quad + ra_{k-m+x+1} \\
&= \min(D_1^x),
\end{aligned}$$

and

$$
\begin{aligned}
&(mr-\alpha)a_x + r(a_{x+1}+a_{x+2}+\cdots+a_{k-m+x-1}) + a_{k-m+x} + (r-1)a_{k-m+x+1} \\
&< (mr-\alpha)a_x + r(a_{x+1}+a_{x+2}+\cdots+a_{k-m+x-2}) + (r-1)a_{k-m+x-1} + 2a_{k-m+x} \\
&\quad + (r-1)a_{k-m+x+1} \\
&< (mr-\alpha)a_x + r(a_{x+1}+a_{x+2}+\cdots+a_{k-m+x-2}) + (r-1)a_{k-m+x-1} + a_{k-m+x} \\
&\quad + ra_{k-m+x+1} \\
&= \min(D_1^x).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&(mr-\alpha)a_x + r(a_{x+1}+a_{x+2}+\cdots+a_{k-m+x-2}) + (r-1)a_{k-m+x-1} + 2a_{k-m+x} \\
&\quad + (r-1)a_{k-m+x+1} \\
&= (mr-\alpha)a_x + r(a_{x+1}+a_{x+2}+\cdots+a_{k-m+x-1}) + ra_{k-m+x+1}.
\end{aligned}
$$

That is

$$a_{k-m+x+1} - a_{k-m+x} = a_{k-m+x} - a_{k-m+x-1} \quad \text{for } x = 0,1,\ldots,m-2.$$

In other words

$$a_{k-1} - a_{k-2} = \cdots = a_{k-m+1} - a_{k-m} = a_{k-m} - a_{k-m-1}. \tag{6.27}$$

Hence, for $m = k-1$ the result follows from (6.25) and (6.27), and for $2 \leq m \leq k-2$ the result follows from (6.25), (6.26) and (6.27). This completes the proof of the theorem. □

Corollary 6.3.5. *Let $k \geq 5$, $r \geq 1$ and $0 \leq \alpha \leq rk-2$. Let $1 \leq m \leq k$ be an integer such that $(m-1)r \leq \alpha < mr$. If $\mathscr{A}$ is a nonempty sequence of k distinct nonnegative integers each repeating exactly r times and $0 \in \mathscr{A}$, such that*

$$|S_\alpha(r,\mathscr{A})| = r\left[\binom{k}{2} - \binom{m}{2}\right] + (m-1)(mr-\alpha) + 1,$$

then $\mathscr{A} = d \cdot [0,k-1]_r$ for some positive integer d.

Proof. Let $\mathscr{A} = (a_0,a_1,\ldots,a_{k-1})_r$, where $0 = a_0 < a_1 < \cdots < a_{k-1}$ and $r \geq 1$. Let

$$|S^\alpha(r,\mathscr{A})| = |S_\alpha(r,\mathscr{A})| = r\left[\binom{k}{2} - \binom{m}{2}\right] + (m-1)(mr-\alpha) + 1.$$

Let $\mathscr{A}' = \mathscr{A} \setminus \{0\}$. So, $\mathscr{A}'$ is a nonempty sequence of $k-1$ distinct positive integers each repeating exactly r times. First, let $m=1$, i.e., $0 \le \alpha < r$. Let

$$|S^{\alpha}(r,\mathscr{A})| = r\binom{k}{2}+1.$$

By (6.18), we have $S^{\alpha}(r,\mathscr{A}) = S(r,\mathscr{A}')$. Therefore,

$$|S(r,\mathscr{A}')| = r\binom{k}{2}+1.$$

Hence, by Theorem 6.3.4 (for $\alpha = 0$), the set $\mathscr{A}'$ is an arithmetic progression with the common difference a_1, the smallest integer in $\mathscr{A}$. Hence, $\mathscr{A}$ is an arithmetic progression with the common difference a_1, i.e., $\mathscr{A} = a_1 \cdot [0,k-1]_r$.

Now, let $2 \le m \le k$, i.e., $r \le \alpha < rk$. Thus, by (6.19) we have $S^{\alpha}(r,\mathscr{A}) = S^{\alpha-r}(r,\mathscr{A}')$. Therefore,

$$|S^{\alpha}(r,\mathscr{A})| = r\left[\binom{k}{2}-\binom{m}{2}\right]+(m-1)(mr-\alpha)+1$$

implies that

$$\begin{aligned}|S^{\alpha-r}(r,\mathscr{A}')| &= r\left[\binom{k}{2}-\binom{m}{2}\right]+(m-1)(mr-\alpha)+1\\ &= r\left[\binom{k}{2}-\binom{m}{2}\right]+(m-1)[(m-1)r-(\alpha-r)]+1.\end{aligned}$$

Hence, by Theorem 6.3.4 (for $\alpha - r$), the set $\mathscr{A}'$ is an arithmetic progression with the common difference a_1, the smallest integer in $\mathscr{A}$. Hence, $\mathscr{A}$ is an arithmetic progression with the common difference a_1, i.e., $\mathscr{A} = a_1 \cdot [0,k-1]_r$. This completes the proof of the corollary. □

As a consequence of Theorem 6.3.4 and Corollary 6.3.5, for $\alpha = 0$, we obtain the following corollary, which is a particular case of Theorem 1.4.25.

Corollary 6.3.6. *[77, Theorem 2.3] Let $k \ge 5$ and $r \ge 1$. If $\mathscr{A}$ is a nonempty sequence of k distinct positive integers each repeating exactly r times such that*

$$|S(r,\mathscr{A})| = r\binom{k+1}{2}+1,$$

then $\mathscr{A} = d \cdot [1,k]_r$ for some positive integer d.

If $\mathscr{A}$ is a nonempty sequence of k distinct nonnegative integers each repeating exactly r times and $0 \in \mathscr{A}$ such that

$$|S(r,\mathscr{A})| = r\binom{k}{2} + 1,$$

then $\mathscr{A} = d \cdot [0, k-1]_r$ for some positive integer d.

Chapter 7

Conclusions and future scope

7.1 Conclusions

The work done in this thesis are mainly concerned about new results on direct and inverse problems for certain sumsets in the group of integers. The results of the present thesis can be summarized by the following notes.

The following conclusions can be drawn from Chapter 2:

- The sum of dilates $A + r \cdot A$ contains at least $4k - 4$ distinct integers, for all $r \geq 3$ and for every finite set A of k integers.
- The Freiman's $3k - 4$ type theorem for $A + 2 \cdot A$ can be extended to $A + 2 \cdot B$, under some conditions on the sets A and B.
- The conditions under which the Freiman's $3k - 4$ type theorem for $A + 2 \cdot B$ holds are necessary but not sufficient.

The following conclusions can be drawn from Chapter 3:

- If A is a set of k positive integers, then the h-fold signed sumset $h_{\pm}A$ contains at least $2(hk - h + 1)$ distinct integers. Moreover, if $h \geq 2$ and this lower bound is exact, then $h = 2$ and $A = d \cdot \{1, 3, \ldots, 2k - 1\}$ for some positive integer d.
- For $h \geq 3$, this bound can be improved to $2hk - h + 1$. Moreover, if $|h_{\pm}A| = 2hk - h + 1$, then $A = d \cdot \{1, 3, \ldots, 2k - 1\}$ for some positive integer d.

- Similar direct and inverse results also holds for $h_{\pm}A$, when A contains (i) nonnegative integers with $0 \in A$, and (ii) arbitrary integers.

The following conclusions can be drawn from Chapter 4:

- If A is a set of k positive integers, then the h-fold restricted signed sumset $h^{\wedge}_{\pm}A$ contains at least $2(hk-h^2)+\binom{h+1}{2}+1$ distinct integers. This bound is optimal for $h=1$, 2 and k.

- If $k \geq 4$ and $|2^{\wedge}_{\pm}A| = 4k-4$, then $A = d \cdot \{1,3,\ldots,2k-1\}$ for some positive integer d.

- If $k \geq 4$ and $|k^{\wedge}_{\pm}A| = \binom{k+1}{2}+1$, then $A = d \cdot [1,k]$ for some positive integer d.

- If A is a set of k (≥ 5) positive integers, then the lower bound $6k-11$ for the sumset $3^{\wedge}_{\pm}A$ can be improved to $6k-8$. Moreover, if $|3^{\wedge}_{\pm}A| = 6k-8$, then $A = d \cdot \{1,3,\ldots,2k-1\}$ for some positive integer d.

- Similar direct and inverse results also holds for $h^{\wedge}_{\pm}A$, when A contains nonnegative integers with $0 \in A$.

The following conclusions can be drawn from Chapter 5:

- If $A = \{0,1,\ldots,k-2,k-1+b\} = [0,k-2] \cup \{k-1+b\}$, where b is a nonnegative integer, then $|h^{(\gamma)}A|$ is a strictly increasing linear function of b for $0 \leq b \leq N_1$ and is a strictly increasing, piecewise-linear function of b for $N_1 \leq b \leq N_2$ and that $|h^{(\gamma)}A|$ is constant for $b \geq N_2$, for some positive integers N_1 and N_2.

- A similar result also holds for the restricted sumset $h^{\wedge}A$, which states that $|h^{\wedge}A|$ is a strictly increasing linear function of b for $0 \leq b \leq N$ and that $|h^{\wedge}A|$ is constant for $b \geq N$, for some positive integer N.

The following conclusions can be drawn from Chapter 6:

- If A is a set of k positive integers and $\alpha \in [0,k]$, then the subset sums $S_{\alpha}(A)$ contains at least $\binom{k+1}{2}-\binom{\alpha+1}{2}+1$ distinct integers. Moreover, if this lower bound is exact with $k \geq 4$ and $0 \leq \alpha \leq k-2$, then $A = d \cdot [1,k]$ for some positive integer d.

- If A is a set of k nonnegative integers with $0 \in A$ and $\alpha \in [0,k]$, then $S_{\alpha}(A)$ contains at least $\binom{k}{2}-\binom{\alpha}{2}+1$ distinct integers. Moreover, if this lower bound is exact with $k \geq 5$ and $0 \leq \alpha \leq k-2$, then $A = d \cdot [0,k-1]$ for some positive integer d.

- Similar direct and inverse results also holds for the subsequence sums $S_{\alpha}(r,\mathscr{A})$ in both the cases, when the sequence $\mathscr{A}$ contains (i) only positive integers, and (ii) nonnegative integers with $0 \in \mathscr{A}$.

7.2 Future plan

This thesis is focused on the direct and inverse problems for certain sumsets in the group of integers. There are several unsolved problems related to the sumsets considered in this thesis. For example, the sumsets considered in Chapter 3 and 4 are certainly new and not much known about these sumsets. The same can be said for the subset and subsequence sums considered in Chapter 6. It would be interesting to extend the study of these sumsets in to finite abelian groups. Bellow, we discuss some unsolved problems which we shall try to solve in the near future.

7.2.1 Some unsolved problems from the thesis

As mentioned in the Conclusion section, in Chapter 4, we settled the direct and inverse theorems for the h-fold restricted signed sumset $h^{\wedge}_{\pm}A$ in the group of integers in the cases $h = 1,2$ and k. In all other cases, i.e., for $3 \leq h \leq k-1$, we conjectured the following direct and inverse results.

Conjecture 7.2.1. *Let $k \geq 5$ and $3 \leq h \leq k-1$. If A is a set of k positive integers, then*

$$|h^{\wedge}_{\pm}A| \geq 2hk - h^2 + 1.$$

If A is a set of k nonnegative integers with $0 \in A$, then

$$|h^{\wedge}_{\pm}A| \geq 2hk - h(h+1) + 1.$$

These lower bounds are best possible.

Conjecture 7.2.2. *Let $k \geq 5$ and $3 \leq h \leq k-1$. If A is a set of k positive integers such that $|h^{\wedge}_{\pm}A| = 2hk - h^2 + 1$, then*

$$A = d \cdot \{1, 3, \ldots, 2k-1\},$$

for some positive integer d.

If A is a set of k nonnegative integers with $0 \in A$ and $|h^{\wedge}_{\pm}A| = 2hk - h(h+1) + 1$, then

$$A = d \cdot [0, k-1],$$

for some positive integer d.

We also verified the conjectures 7.2.1 and 7.2.2, for the case $h = 3$ by solving the direct and inverse problems for $3^{\wedge}_{\pm}A$, that are Theorem 4.2.5 and Theorem 4.3.5, respectively. We observed that, the technique used for the case $h = 3$ is also applicable for $h \geq 4$, but the computational complexity increases with the increasing value of h. So, it would be interesting to furnish a new technique which will establish our conjectures in all the cases.

7.2.2 Generalized signed sumset

Let G be an additive abelian group. Let $A = \{a_0, a_1, \ldots, a_{k-1}\}$ be a nonempty subset of G. Let $h \geq 1$. Recall that, the h-fold signed sumset $h_{\pm}A$ and the h-fold restricted signed sumset $h^{\wedge}_{\pm}A$ are defined by

$$h_{\pm}A = \left\{ \sum_{i=0}^{k-1} \lambda_i a_i : \lambda_i \in \mathbb{Z} \text{ for } i = 0, 1, \ldots, k-1 \text{ and } \sum_{i=0}^{k-1} |\lambda_i| = h \right\},$$

and

$$h^{\wedge}_{\pm}A = \left\{ \sum_{i=0}^{k-1} \lambda_i a_i : \lambda_i \in \{-1, 0, 1\} \text{ for } i = 0, 1, \ldots, k-1 \text{ and } \sum_{i=0}^{k-1} |\lambda_i| = h \right\},$$

respectively.

Observe that, in the sumset $h_{\pm}A$ the variables λ_i can assume any integer value between $-h$ and h, while in the sumset $h^{\wedge}_{\pm}A$ the variables λ_i can assume the integer values -1, 0 and 1. One can generalize these two sumsets by defining a h-fold sumset, where in a h-fold sum the variables λ_i can assume any integer value between $-\gamma$ and γ, where $1 \leq \gamma \leq h$. For integers h, γ with $1 \leq \gamma \leq h \leq k\gamma$, define the *h-fold generalized signed sumset* of A, denoted by $h^{(\gamma)}_{\pm}A$, by

$$h^{(\gamma)}_{\pm}A := \left\{ \sum_{i=0}^{k-1} \lambda_i a_i : \lambda_i \in [-\gamma, \gamma] \text{ for } i = 0, 1, \ldots, k-1 \text{ and } \sum_{i=0}^{k-1} |\lambda_i| = h \right\}.$$

So, the signed sumset $h_{\pm}A$ and the restricted signed sumset $h^{\wedge}_{\pm}A$ are particular cases of the generalized signed sumset $h^{(\gamma)}_{\pm}A$, for $\gamma = h$ and $\gamma = 1$, respectively. Therefore, the generalized signed sumset provides a unified theory for the signed sumset and restricted signed sumset. It would be interesting to study both direct and inverse problems for this generalized signed sumset similar to those in Chapter 3 and 4.

7.2.3 Other combinatorial problems

I would like to study an interesting diophantine problem due to Frobenius. The *Frobenius Problem* is to determine the largest positive integer that is not representable as a nonnegative integer combination of a given set of positive integers that are coprime. More formally, given a finite set $A = \{a_1, \dots, a_k\}$ of positive integers with $\gcd(a_1, \dots, a_k) = 1$, let $\Gamma(A) := \{a_1x_1 + \cdots + a_kx_k : x_i \geq 0\}$. It is well known that $\Gamma^c(A) := \mathbb{N} \setminus \Gamma(A)$ is finite. The Frobenius number of A is defined by $g(A) := \max \Gamma^c(A)$. It is also equally important to determine the number $n(A) := |\Gamma^c(A)|$. Although, it was Sylvester [97] who first showed that $g(a_1, a_2) = (a_1 - 1)(a_2 - 1) - 1$ and $n(a_1, a_2) = \frac{1}{2}(a_1 - 1)(a_2 - 1)$, it was Frobenius who was mainly responsible to give a recognition and it is after him that the problem is also named. Determining the exact value of $g(A)$ and $n(A)$ is a difficult problem in general; there is no general formula for $|A| > 2$. There are only a few cases other than when $|A| = 2$ where $g(A)$ or $n(A)$ have been determined (see [22, 92, 101, 102, 103, 104]).

I would also like to study the diophantine equations arising from some well-known sequences, such as Fibonacci sequence, Lucas sequence, and Pell sequence etc. One can see for example, the Fibonacci numbers that are representable in the form $x^l \pm x^m \pm 1$ considered in [66]. See also [23] for some other diophantine equations arising from Pell's and Pell-Lucas sequences.

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