SOME INVESTIGATIONS IN THE AREA OF OPTIMIZATION AND IMPLICATION IN UNCERTAIN ENVIRONMENT

Ph. D. THESIS

by

VISHNU SINGH

DEPARTMENT OF MATHEMATICS
INDIAN INSTITUTE OF TECHNOLOGY ROORKEE
ROORKEE - 247 667 (INDIA)
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by

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ROORKEE

CANDIDATE'S DECLARATION

I hereby certify that the work which is being presented in the thesis entitled"**SOME INVESTIGATIONS IN THE AREA OF OPTIMIZATION AND IMPLICATION IN UNCERTAIN ENVIRONMENT**" in partial fulfillment of the requirements for the award of the Degree of Doctor of Philosophy and submitted in the Department of Mathematics of the Indian Institute of Technology Roorkee is an authentic record of my own work carried out during a period from July, 2014 to April, 2019 under the supervision of Dr. Shiv Prasad Yadav, Professor, Department of Mathematics, Indian Institute of Technology Roorkee.

The matter presented in this thesis has not been submitted by me for the award of any other degree of this or any other Institution.

Signature of the Candidate

This is to certify that the above statement made by student is correct to the best of my (our) knowledge.

Signature of Supervisor(s)

The Ph.D Viva-Voce Examination of **Mr. Vishnu Singh**, Research Scholar,has been held on

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Chairperson, SRC **Signature of External Examiner**

This is to certify that the student has made all the corrections in the thesis.

Signature of Supervisor(s) **Head of the Department**

Dated:________________

Abstract

In real-life problems such as incorporate or in industry, decision making is a continuous process. The experts and the decision-makers (DMs), usually, have to suffer with uncertainty as well as with hesitation, due to the complexity of the situations. The main reasons behind these complexities are lack of good communications with all involved persons, error in data, understanding of markets, unawareness of customers, etc.

So, the prediction of the parameters is a complex and challenging task. The classical methods encounter great difficulty in dealing with uncertainty and complexity involved in such situations. In general, the parameters of an optimization problem are considered as crisp numbers. These crisp values are determined from past occurrences which are very uncertain since the systems environment keep on changing. Therefore, some degree of uncertainty exists in such a determination. This led to the development of the fuzzy set (FS) theory by Zadeh [194]. In order to handle the insufficient information, the fuzzy approach is used to model the problem and evaluate the optimal solution. FS theory has been shown to be a useful tool to handle the situations in which the data are imprecise by attributing a degree to which a certain object belongs to a set. An FS is a generalization of an ordinary set in that it allows the degree of membership for each element to range over the unit interval [0, 1]. Thus, the membership function of an FS maps each element of the universe of discourse to its range space, which, in most cases, is assumed to be the unit interval. Using an FS approach, quantities are represented by FSs. The membership functions represent the uncertainties involved in various parameters of the problem. During the last decades, FS theory played an important role in modeling uncertain and optimization problems. Zimmermann [203] showed that the solutions of fuzzy linear programming problems (FLPPs) are always efficient. Since the FS theory came into existence, many extensions of FSs also appeared over time , e.g., L-FSs proposed by Goguen [80], interval-valued fuzzy set (IVFS) proposed by Gorzalczany [81] represents the degree of membership of an element by an interval rather than exact numerical value, intuitionistic fuzzy set (IFS) proposed by Atanassov [11] etc. One among these extensions is IFS which is playing an important role in decision making under

uncertainty and gained popularity in recent years. It helps more adequately to represent situations where DMs abstain from expressing their assessments. In this way, IFSs provide a richer tool to grasp impression and ambiguity than the conventional FSs. These characteristics of IFSs led to the extension of optimization methods in an intuitionistic fuzzy environment (IFE). An application of IFSs to optimization problems is introduced by Angelov [9]. His technique is based on maximizing the degree of membership, minimizing the degree of non-membership and the crisp model is formulated using the IF aggregation operator.

In decision making, one chooses the best alternative from the given set of feasible alternatives. There exist several processes in literature but there are mainly four stages required to choose the best alternative: (i) Evaluate the set of feasible alternatives from the given information. (ii) Determine the weight vector corresponding to alternatives or attributes which depend on DM. (iii) Aggregate alternatives by taking the weight vector given by DM. (iv) Rank the alternatives in the order of preferences and select the best one. During last decades, IFS theory played an important role in modeling uncertain and vague systems, received much attention from the researchers and meaningful results were obtained in the field of decision-making problems [138], pattern recognition [54, 143] to name a few.

There are several information measures in IFE, such as divergence measures, similarity measures, dissimilarity measures, and distance measures. They model uncertain and vague information. The inclusion between two IFSs can be measured by the concept of inclusion measure [79] and the commonality between two IFSs can be measured by the concept of similarity measure [95].

In fuzzy logic, the fuzzy implication is equally important from both the theoretical and practical points of view. From the theoretical point of view, the development of algebra is done and their properties are studied. From the practical point of view, the fuzzy implication is used to study approximate reasoning and network problems, etc. (see [19, 106]). One among the several extensions of FS is the IVFS. It has become very popular from both the theoretical and practical aspects. It has become one of the most important operators in logic [174]. The arithmetic operators in IVFS theory [55] and one can find theoretical articles concerned with different classes of interval-valued logical connectives, like, interval-valued fuzzy negations [26], interval-valued t-norms [56, 174], interval-valued fuzzy uninorms [57], interval-valued fuzzy implications [5, 28, 111]. IF t-norms and t-conorms are noted in [59]. The expression, construction, classification and several properties with applications of intuitionistic and interval-valued fuzzy implications are given in [33] and [45]. IFIs [33, 45] and IF relations [146] are studied.

The main objectives of the thesis are as follows:

(i) Modeling and analysis of optimization problems in IFE and development of algorithms for solving such problems.

(ii) Analysis of duality theory in IFE.

(iii) Analysis and development of algorithms for selecting the best alternative from the given set of feasible alternatives in decision-making problems in IFE.

(iv) Algebraic analysis of implication operators in IFE and their uses for solving distributive equations and Boolean-like laws.

The thesis is organized into eight chapters. The chapter-wise summary of the thesis is as follows:

Chapter 1 is introductory in nature. In this chapter, basic definitions of FS and IFS, various types of fuzzy and IF numbers, and their mathematical operations are introduced. A ranking function is introduced. Ranking function transforms a fuzzy or IF number into an equivalent real number. Also, basic definitions and axioms of implication operator, negation, t-norm, t-conorm in fuzzy and IF environments are introduced. It also presents a brief review of the research work done in the field of fuzzy and IF optimizations and implications.

In **Chapter 2**, the product of unrestricted LR-type IFNs is proposed. Then with the help of the proposed product, an algorithm is proposed to find the optimal solutions of unrestricted LR-type IFLPPs. A test example is given to support the proposed method and investigated the applicability of existing approaches.

In **Chapter 3**, we introduce a pair of primal-dual LPPs in IFE and prove duality results by using an aspiration level approach in which membership and non-membership functions are taken in the form of reference functions. Since the fuzzy environment and IFE cause the duality gap, we propose to investigate the impact of membership function governed by reference function on duality gap. Also, the duality gap obtained by the approach has been compared with the duality gap obtained by existing approaches.

In **Chapter 4**, the formulation of the multi-objective optimization problem (MOOP), accuracy index and value function in IFE are introduced. For resolving the mutual conflicting nature of objectives in MOOP in IFE, we introduce the membership and non-membership functions governed by reference function which do not depend on the upper and lower levels of acceptability. An efficient algorithm is developed for solving MOOP in IFE from different viewpoints, viz., optimistic, pessimistic and mixed. The optimal solution obtained by the proposed approach is compared with the solutions obtained by existing approaches.

In **Chapter 5**, the information measures are introduced in IFE to measure the uncertainty and hesitancy. We introduce and study the continuity of considered measures. Next, we prove some results that can be used to generate measures for FSs as well as for IFSs and we also prove some approaches to construct point measures from set measures in IFE. We define weight set for one and many preference orders of alternatives and investigate the properties based on these ordering. Based on the weight set, we develop the model for finding the uncertain weights corresponding to attribute. Also, we develop the model to find attribute weights in a certain environment by using attribute weights in an uncertain environment. An algorithm is developed for choosing the best alternative according to the preference orders of alternatives.

In **Chapter 6**, a new type of implication on $\mathcal{L}$, known as the residual implication, is derived from powers of continuous t-norm $\mathcal{T}$ and satisfies certain properties of residual implications by imposing some extra conditions. Moreover, some additional important properties are studied and analyzed. The solutions of Boolean-like laws in $\mathtt{I}_{\mathtt{I}\mathcal{T}}$ are obtained.

In **Chapter 7**, a new class of IFIs known as $(\mathtt{f}_{\mathtt{I}}, \omega)$-implications is introduced which is a generalized form of Yager's f-implications in IFE. Basic properties of these implications are discussed in detail. The distributive equations $\mathtt{I}_{\mathtt{I}}(\mathcal{T}(u, v), w) = \mathcal{S}(\mathtt{I}_{\mathtt{I}}(u, w), \mathtt{I}_{\mathtt{I}}(v, w))$ and $\mathtt{I}_{\mathtt{I}}(u, \mathcal{T}_1(v, w)) = \mathcal{T}_2(\mathtt{I}_{\mathtt{I}}(u, v), \mathtt{I}_{\mathtt{I}}(u, w))$ over t-representable t-norms and t-conorms generated from nilpotent and strict t-norms in IFE are discussed.

Finally, in **Chapter 8**, conclusions are drawn based on the present study and future research work is suggested in this direction.

List of Publications

1. Vishnu Singh, Shiv Prasad Yadav, Development and optimization of unrestricted LR-type intuitionistic fuzzy mathematical programming problems, *Expert Systems with Applications* 80 (2017) 147-161.

2. Vishnu Singh, Shiv Prasad Yadav, Modeling and optimization of multi-objective programming problems in intuitionistic fuzzy environment:Optimistic, pessimistic and mixed approaches, *Expert Systems with Applications* 102 (2018) 143-157.

3. Vishnu Singh, Shiv Prasad Yadav, Sujeet Kumar Singh, Duality theory in Atanassov's intuitionistic fuzzy mathematical programming problems: Optimistic, pessimistic and mixed approaches, *Annals of Operations Research* (2019) 1-40. https://doi.org/10.1007/s10479-019-03229-8.

4. Vishnu Singh, Shiv Prasad Yadav, Radko Mesiar, Information measures in Atanassov's intuitionistic fuzzy environment and their application in decision-making *(Communicated and under review).*

5. Vishnu Singh, Shiv Prasad Yadav, Radko Mesiar, Residual implications on $\mathcal{L}$ based on powers of continuous t-norm *(Communicated and under review).*

6. Vishnu Singh, Shiv Prasad Yadav, $(\mathtt{f}_{\mathtt{I}}, \omega)$-implications and distributivity of implications on $\mathcal{L}$ over t-representable t-norms: The case of strict and nilpotent t-norms *(Communicated and under review).*

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Roorkee (Vishnu Singh)

April , 2019

Dedicated

to

my parents, for their unconditional love and encouragement

&

my sisters, for always being understanding and supportive

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Symbols and Abbreviations

Symbols

$\mathbb{R}$	Set of real numbers
$\mathbb{R}^+$	Set of positive real numbers
$\forall$	for all
$\exists$	there exists
$\neg$	Negation
$\wedge$	Conjunction
$\vee$	Disjunction
$\rightarrow$	Implication
$\mathcal{I}$	Set of intuitionistic fuzzy sets
$\mathcal{I}(\mathbb{R})$	Set of intuitionistic fuzzy numbers
$\mathcal{L}$	$\{(u_1, u_2) : (u_1, u_2) \in [0,1]^2,\ u_1 + u_2 \leq 1\}$
pr_1	First projection
pr_2	Second projection
A^I_α	$\alpha-$ cut of $\tilde{A}^I$
$A^I_{(\beta)}$	$\beta-$ cut of $\tilde{A}^I$
$A^I_{(\alpha,\beta)}$	$(\alpha, \beta)-$ cut of $\tilde{A}^I$

Abbreviations

s.t.	Such that
w.r.t.	With respect to
FS	Fuzzy set
FN	Fuzzy number
TFN	Triangular fuzzy number
TrFN	Trapezoidal fuzzy number

FLPP	Fuzzy linear programming problem
MOFLPP	Multi-objective fuzzy linear programming problem
IFS	Intuitionistic fuzzy set
IFE	Intuitionistic fuzzy environment
IFN	Intuitionistic fuzzy number
TIFN	Triangular intuitionistic fuzzy number
TrIFN	Trapezoidal intuitionistic fuzzy number
IFO	Intuitionistic fuzzy optimization
IFLPP	Intuitionistic fuzzy linear programming problem
MOIFLPP	Multi-objective intuitionistic fuzzy linear programming problem
IFLPP	Intuitionistic fuzzy linear programming problem
MOIFLPP	Multi-objective intuitionistic fuzzy linear programming problem
DM	Decision maker
MCDM	Multi-criteria decision making
MADM	Multi-attribute decision making
OWA	Ordered weighted aggregation
t-norm	Triangular norm
t-conorm	Triangular conorm
R-implication	Residual implication
S-implication	Strong implication
QL-implication	Quantum logic implication
NP	Left neutrality property
OP	Ordering property
LOP	Left ordering property
ROP	Right ordering property
IP	Identity principle
EP	Exchange principle
CB	Consequent boundary
SIB	Sub-iterative Boolean law
IB	Iterative Boolean law
SBC	Strong boundary condition
LBC	Left boundary condition
RBC	Right boundary condition

General Remark: For simplicity, we have used IFE in the place of Atanassov's IFE throughout the thesis.

Chapter 1

Introduction

1.1 Fuzzy set theory

Advances in science and technology have made our modern society very complex and due to this, the decision process has become increasingly vague and hard to analyze. The human brain possesses some special characteristics that enable him to learn and reason in a vague and fuzzy environment. It has the ability to arrive at a decision based on imprecise qualitative data in contrast to formal mathematics and formed logic with imprecise and qualitative data. Modern computers also possess the capacity of thinking but can not have the human like ability. Undoubtedly, in many areas of cognition, human intelligence excels the computer intelligence of today. The developments of fuzzy concepts proposed by Zadeh [194], is a step towards the development of tools capable of handling humanistic type problems, though it may never be equal to the logic and the intelligence of men in many respects. Most of the classes of objects encountered in the real physical world are fuzzy and not sharply defined. They do not have precisely defined criteria of membership. In such classes, an object need not necessarily either belong to or not belong to a class, it may have an intermediate grade of membership. This is the concept of fuzzy set (FS) which is a class with the continuum of grades of membership. An FS is a generalization of the crisp set that has clearly defined the boundary. The FS theory is based on the idea that each element of the set can take membership value in the interval $[0, 1]$. Because of this, FS theory has a much wider scope of applicability than the crisp set theory in solving various kinds of real physical problems. FS theory is a powerful tool to deal with uncertainty. The relationships between some extensions of FS theory are noted in [60].

In this section, the definitions of FS, α-cut, fuzzy number, triangular fuzzy number, and arithmetic operations on triangular fuzzy numbers are presented.

Definition 1.1.1. *[194] Let X be a universe of discourse whose elements are denoted by x. Then an FS $\tilde{M}$ in X is defined by a set of ordered pairs*

$$\tilde{M} = \{(x, \mu_{\tilde{M}}(x)) : x \in X\},$$

where $\mu_{\tilde{M}} : X \to [0,1]$ called the membership function and $\mu_{\tilde{M}}(x)$ called the grade of membership of x being in $\tilde{M}$.

Definition 1.1.2. *[194] Let $\tilde{M}$ be a fuzzy set in X and $\alpha \in [0,1]$. Then the α-cut of the FS $\tilde{M}$ is the crisp set A_α defined by*

$$A_\alpha = \{x \in X : \mu_{\tilde{M}}(x) \geq \alpha\}.$$

Definition 1.1.3. *[204] A FS $\tilde{M} = \{(x, \mu_{\tilde{M}}(x)) : x \in \mathbb{R}\}$ is called a fuzzy number (FN) if the following conditions hold:*

(i) $\tilde{M}$ is a convex FS, i.e., $\mu_{\tilde{M}}(\lambda_1 x_1 + \lambda_2 x_2) \geq \min\{\mu_{\tilde{M}}(x_1), \mu_{\tilde{M}}(x_2)\}$, $\lambda_1 + \lambda_2 = 1$, $\lambda_1, \lambda_2 \geq 0$.

(ii) There exists only one $m \in \mathbb{R}$ such that $\mu_{\tilde{M}}(m) = 1$ (m is called the mean value of $\tilde{M}$).

(iii) $\mu_{\tilde{M}} : \mathbb{R} \to [0,1]$ is piecewise continuous function given by

$$\mu_{\tilde{M}}(x) = \begin{cases} g_1(x), & a < x < m, \\ 1, & x = m, \\ h_1(x), & m < x < b\ , \\ 0, & \textit{otherwise}. \end{cases}$$

Here m is the mean value of $\tilde{M}$; $m - a$ and $b - m$ are the left and right spreads of membership function $\mu_{\tilde{M}}$ respectively; g_1 is piecewise continuous and increasing in (a, m); h_1 is piecewise continuous and decreasing function in (m, b); (a, b) is called the support of $\tilde{M}$. The FN $\tilde{M}$ is represented by $\tilde{M} = (m; a, b)$. The set of all FNs is denoted by $\mathcal{F}(\mathbb{R})$.

Definition 1.1.4 (Arithmetic operations on FNs)**.** *[204] Let $\tilde{M}$=$(m; a, b)$, $\tilde{M}_1$=$(m_1; a_1, b_1)$ and $\tilde{M}_2$=$(m_2; a_2, b_2)$ be the FNs. Then*

(i) $\tilde{M}_1 + \tilde{M}_2$ is defined as a FN given by
$\tilde{M}_1 + \tilde{M}_2 = (m_1 + m_2; a_1 + a_2, b_1 + b_2)$;

(ii) $-\tilde{M}$ is defined as a FN given by
$-\tilde{M}$=$(-m; b, a)$;

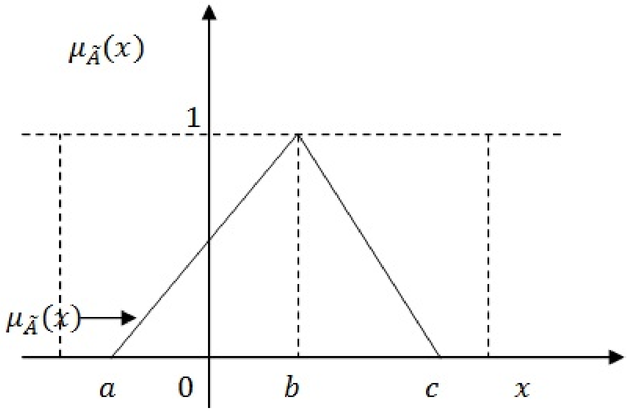

Figure 1.1: Graphical representation of a TFN.

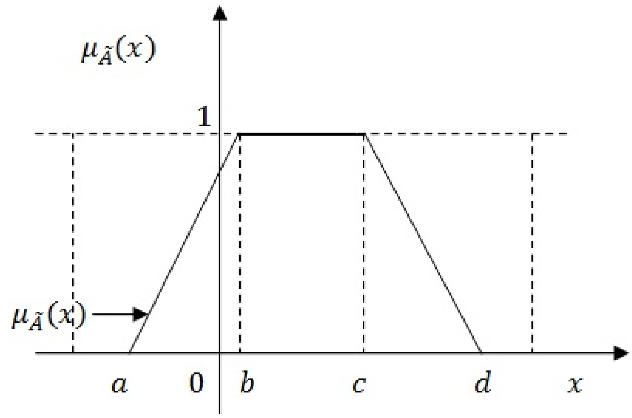

Figure 1.2: Graphical representation of a TrFN.

(iii) $\tilde{M}_1 - \tilde{M}_2$ *is defined as a FN given by*
$\tilde{M}_1 - \tilde{M}_2 = \tilde{M}_1 + (-\tilde{M}_2) = (m_1 - m_2; a_1 + b_2, b_1 + a_2)$*;*

(iv) $\tilde{M}_1 \times \tilde{M}_2$ *is defined as a FN given by*
$\tilde{M}_1 \times \tilde{M}_2 = (m; a, b)$,
where $m = m_1 m_2, a = m_1 m_2 - \min\{(m_1 - a_1)(m_2 - a_2), (m_1 - a_1)(m_2 + b_2), (m_1 + b_1)(m_2 - a_2), (m_1 + b_1)(m_2 + b_2)\}, b = \max\{(m_1 - a_1)(m_2 - a_2), (m_1 - a_1)(m_2 + b_2), (m_1 + b_1)(m_2 - a_2) - m_1 m_2$*;*

(v) $\lambda\tilde{M}$ *is defined as a FN given by*

$$\lambda\tilde{M} = \begin{cases} (\lambda m; \lambda a, \lambda b), & \lambda \geq 0, \\ (\lambda m; \lambda b, \lambda a), & \lambda < 0. \end{cases}$$

.

Definition 1.1.5. *[204] A triangular fuzzy number (TFN)* $\tilde{M}$ *is a FN with membership function* $\mu_{\tilde{M}}$ *given by*

$$\mu_{\tilde{M}}(x) = \begin{cases} \dfrac{x-a}{b-a}, & a < x \leq b, \\ \dfrac{c-x}{c-b}, & b < x \leq c, \\ 0, & \textit{otherwise}. \end{cases}$$

The TFN $\tilde{M}$ is represented by $\tilde{M} = (a, b, c)$. Its pictorial representation is given in Figure 1.1.

Definition 1.1.6 (Arithmetic operations on TFNs)**.** *[204] Let* $\tilde{M}=(a, b, c)$, $\tilde{M}_1=(a_1, b_1, c_1)$ *and* $\tilde{M}_2=(a_2, b_2, c_2)$ *be the TFNs. Then*

(i) $\tilde{M}_1 + \tilde{M}_2$ *is defined as a TFN given by*
$\tilde{M}_1 + \tilde{M}_2 = (a_1 + a_2, b_1 + b_2, c_1 + c_2)$;

(ii) $-\tilde{M}$ *is defined as a TFN given by*
$-\tilde{M}=(-c, -b, -a)$;

(iii) $\tilde{M}_1 - \tilde{M}_2$ *is defined as a TFN given by*
$\tilde{M}_1 - \tilde{M}_2 = \tilde{M}_1 + (-\tilde{M}_2) = (a_1 - c_2, b_1 - b_2, c_1 - a_2)$;

(iv) $\tilde{M}_1 \times \tilde{M}_2$ *is defined as a TFN given by*
$\tilde{M}_1 \times \tilde{M}_2 \approx (p_1, p_2, p_3)$,
where $p_1 = min\{a_1a_2, a_1c_2, c_1a_2, c_1c_2\}$, $p_2 = b_1b_2$, $p_3 = max\{a_1a_2, a_1c_2, c_1a_2, c_1c_2\}$;

(v) $\tilde{M}_1^{-1}$ *is defined as a TFN given by*
$\tilde{M}_1^{-1} = (q_1, q_2, q_3)$,
where $q_1 = min\{1/a_1, 1/c_1, 1/a_1, 1/c_1\}$, $q_2 = 1/b_1$, $q_3 = max\{1/a_1, 1/c_1, 1/a_1, 1/c_1\}$, *provided* $a_1 \neq 0$;

(vi) $\tilde{M}_1/\tilde{M}_2$ *is defined as a TFN given by*
$\tilde{M}_1/\tilde{M}_2 \approx (q_1, q_2, q_3)$,
where $q_1 = min\{a_1/a_2, a_1/c_2, c_1/a_2, c_1/c_2\}$, $q_2 = b_1/b_2$, $q_3 = max\{a_1/a_2, a_1/c_2, c_1/a_2, c_1/c_2\}$, *provided* $a_2 \neq 0.$;

(vii) $\lambda\tilde{M}$ *is defined as a TFN given by*

$$\lambda\tilde{M} = \begin{cases} (\lambda a, \lambda b, \lambda c), & \lambda \geq 0, \\ (\lambda c, \lambda b, \lambda a), & \lambda < 0. \end{cases}$$

.

Definition 1.1.7. *[204] A trapezoidal fuzzy number (TrFN)* $\tilde{M}$ *is a FS in* $\mathbb{R}$ *with membership function* $\mu_{\tilde{M}}$ *given by*

$$\mu_{\tilde{M}}(x) = \begin{cases} \dfrac{x-a}{b-a}, & a < x \leq b, \\ 1, & b < x \leq c, \\ \dfrac{d-x}{d-c}, & c < x \leq d, \\ 0, & otherwise. \end{cases}$$

The TrFN $\tilde{M}$ is denoted by $\tilde{M} = (a, b, c, d)$. Its pictorial representation is given in Figure 1.2.

Definition 1.1.8 (Arithmetic operations on TrFNs). *Let* $\tilde{M}=(a,b,c,d)$, $\tilde{M}_1=(a_1,b_1,c_1,d_1)$ *and* $\tilde{M}_2=(a_2,b_2,c_2,d_2)$ *be the TrFNs. Then*

(i) $\tilde{M}_1+\tilde{M}_2$ *is defined as a TrFN given by*
$\tilde{M}_1+\tilde{M}_2=(a_1+a_2,b_1+b_2,c_1+c_2,d_1+d_2)$;

(ii) $-\tilde{M}$ *is defined as a TrFN given by*
$-\tilde{M}=(-d,-c,-b,-a)$;

(iii) $\tilde{M}_1-\tilde{M}_2$ *is defined as a TrFN given by*
$\tilde{M}_1-\tilde{M}_2=\tilde{M}_1+(-\tilde{M}_2)=(a_1-d_2,b_1-c_2,c_1-b_2,d_1-a_2)$;

(iv) $\tilde{M}_1\times\tilde{M}_2$ *is defined as a TrFN given by*
$\tilde{M}_1\times\tilde{M}_2\approx(p_1,p_2,p_3,p_4)$,
where $p_1=min\{a_1a_2,a_1d_2,d_1a_2,d_1d_2\}$, $p_2=min\{b_1b_2,b_1c_2,c_1b_2,c_1c_2\}$,
$p_3=max\{b_1b_2,b_1c_2,c_1b_2,c_1c_2\},p_4=max\{a_1a_2,a_1d_2,d_1a_2,d_1d_2\}$;

(v) $\tilde{M}_1/\tilde{M}_2$ *is defined as a TrFN given by*
$\tilde{M}_1/\tilde{M}_2\approx(q_1,q_2,q_3,q_4)$,
where $q_1=\min\{a_1/a_2,a_1/d_2,d_1/a_2,d_1/d_2\}$, $q_2=\min\{b_1/b_2,b_1/c_2,c_1/b_2,c_1/c_2\}$,
$q_3=\max\{b_1/b_2,b_1/c_2,c_1/b_2,c_1/c_2\},q_4=\max\{a_1/a_2,a_1/d_2,d_1/a_2,d_1/d_2\}$, *provided* $a_2\neq 0$;

(vi) $\lambda\tilde{M}$ *is defined as a TrFN given by*

$$\lambda\tilde{M}=\begin{cases}(\lambda a,\lambda b,\lambda c,\lambda d), & \lambda\geq 0,\\ (\lambda d,\lambda c,\lambda b,\lambda a), & \lambda<0.\end{cases}$$

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1.2 Intuitionistic fuzzy set theory

In most of the cases of judgements, evaluation is done by human beings where certainly there are limitations of knowledge, intellectual functionaries or availability of data due to some uncontrollable factors. Naturally, every decision-maker (DM) hesitates more or less on every evaluation activity. This is the concept of intuitionistic fuzzy set (IFS) theory introduced by Atanassov [11]. It can handle both uncertainty and hesitation in parameter prediction. The major advantage of IFS over FS is that IFS separates the degree of membership and the degree of non-membership of an element in the set. IFS theory is one of the interesting generalizations of FS theory introduced by Zadeh [194]. Because of this generalization, IFS theory has a much wider scope of applicability than the usual FS theory in solving various kinds of real physical problems. In [12–14], Atanassov gave different types of operations and point operators over the IFSs. An application of IFSs in medical diagnosis is noted in [51]. Gau and Buehrer [74] gave the concept of vague sets. But, vague sets are IFSs [37]. Burillo and Bustince [32] gave a point operator, which associates a family of FSs with given IFS, and construction theorems of IFSs from one FS and from two FSs also gave results to recover the FSs used in the construction from the IFS constructed by means of different operators.

In this section, the definitions of IFS, level set, IFN, and their arithmetic operations are presented.

Definition 1.2.1. *[11] Let X be a universe of discourse. Then an IFS $\tilde{A}^I$ in X is defined by the set*

$$\tilde{A}^I = \{(x, \mu_{\tilde{A}^I}(x), \nu_{\tilde{A}^I}(x)) : x \in X\},$$

where $\mu_{\tilde{A}^I}, \nu_{\tilde{A}^I} : X \to [0,1]$ are functions such that $0 \leq \mu_{\tilde{A}^I}(x) + \nu_{\tilde{A}^I}(x) \leq 1 \; \forall \, x \in X$. The value $\mu_{\tilde{A}^I}(x)$ is called the degree of membership and $\nu_{\tilde{A}^I}(x)$ is called the degree of non-membership of $x \in X$ being in $\tilde{A}^I$. The hesitation degree of an element $x \in X$ being in $\tilde{A}^I$ is denoted by $\pi_{\tilde{A}^I}(x)$ and is defined by

$$\pi_{\tilde{A}^I}(x) = 1 - \mu_{\tilde{A}^I}(x) - \nu_{\tilde{A}^I}(x) \in [0,1] \;\; \forall \, x \in X.$$

Definition 1.2.2. *[12] Let $\tilde{A}^I_1 = \{< x, \mu_{\tilde{A}^I_1}(x), \nu_{\tilde{A}^I_1}(x) >: x \in X\}$ and $\tilde{A}^I_2 = \{< x, \mu_{\tilde{A}^I_2}(x), \nu_{\tilde{A}^I_2}(x) >: x \in X\}$ be two IFSs. Then*

(i) the standard union of $\tilde{A}^I_1$ and $\tilde{A}^I_2$ is denoted by $\tilde{A}^I_1 \cup \tilde{A}^I_2$ and is defined as an IFS given by

$$\tilde{A}^I_1 \cup \tilde{A}^I_2 = \{(x, \mu_{\tilde{A}^I_1 \cup \tilde{A}^I_2}(x), \nu_{\tilde{A}^I_1 \cup \tilde{A}^I_2}(x)) : x \in X\},$$

where $\mu_{\tilde{A}^I_1 \cup \tilde{A}^I_2}(x) = \max\{\mu_{\tilde{A}^I_1}(x), \mu_{\tilde{A}^I_2}(x)\}, \; \nu_{\tilde{A}^I_1 \cup \tilde{A}^I_2}(x) = \min\{\nu_{\tilde{A}^I_1}(x), \nu_{\tilde{A}^I_2}(x)\} \; \forall \, x \in X$;

(ii) the standard intersection of $\tilde{A}^I_1$ and $\tilde{A}^I_2$ is denoted by $\tilde{A}^I_1 \cap \tilde{A}^I_2$ and is defined as an IFS given by

$$\tilde{A}^I_1 \cap \tilde{A}^I_2 = \{(x, \mu_{\tilde{A}^I_1 \cap \tilde{A}^I_2}(x), \nu_{\tilde{A}^I_1 \cap \tilde{A}^I_2}(x)) : x \in X\},$$

where $\mu_{\tilde{A}^I_1 \cap \tilde{A}^I_2}(x) = \min\{\mu_{\tilde{A}^I_1}(x), \mu_{\tilde{A}^I_2}(x)\}, \; \nu_{\tilde{A}^I_1 \cap \tilde{A}^I_2}(x) = \max\{\nu_{\tilde{A}^I_1}(x), \nu_{\tilde{A}^I_2}(x)\} \; \forall \, x \in X$;

(iii) the standard complement of $\tilde{A}^I_1$ is denoted by $\tilde{A}'^I_1$ and is defined as an IFS given by

$$\tilde{A}'^I_1 = \{(x, \mu_{\tilde{A}'^I_1}(x), \nu_{\tilde{A}'^I_1}(x)) : x \in X\},$$

where $\mu_{\tilde{A}'^I_1}(x) = \nu_{\tilde{A}^I_1}(x), \; \nu_{\tilde{A}'^I_1}(x) = \mu_{\tilde{A}^I_1}(x) \; \forall \, x \in X$;

(iv) $\tilde{A}_1^I$ *is defined as a subset of* $\tilde{A}_2^I$*, denoted by* $\tilde{A}_1^I \subseteq \tilde{A}_2^I$*, if*
$\mu_{\tilde{A}_1^I}(x) \leq \mu_{\tilde{A}_2^I}(x)$ *and* $\nu_{\tilde{A}_1^I}(x) \geq \nu_{\tilde{A}_2^I}(x) \ \forall\, x \in X$.

Definition 1.2.3. *[127] An IFS* $\tilde{A}^I$ *in X is called a convex IFS if the following conditions hold:*

- $\mu_{\tilde{A}^I}$ *is quasi-concave over X, i.e.,*
  $\mu_{\tilde{A}^I}(\lambda_1 x_1 + \lambda_2 x_2) \geq \min(\mu_{\tilde{A}^I}(x_1), \mu_{\tilde{A}^I}(x_2)),\ \lambda_1 + \lambda_2 = 1,\ \lambda_1, \lambda_2 \geq 0,\ \forall\, x_1, x_2 \in X.$

- $\nu_{\tilde{A}^I}$ *is quasi-convex in X, i.e.,*
  $\nu_{\tilde{A}^I}(\lambda_1 x_1 + \lambda_2 x_2) \leq \max(\nu_{\tilde{A}^I}(x_1), \nu_{\tilde{A}^I}(x_2)),\ \lambda_1 + \lambda_2 = 1,\ \lambda_1, \lambda_2 \geq 0,\ \forall\, x_1, x_2 \in X.$

Definition 1.2.4. *[127] An IFS* $\tilde{A}^I = \{(x, \mu_{\tilde{A}^I}(x), \nu_{\tilde{A}^I}(x)) : x \in X\}$ *is called normal if* $\exists$ $x_1, x_2 \in X$ *such that* $\mu_{\tilde{A}^I}(x_1) = 1, \nu_{\tilde{A}^I}(x_2) = 1$.

Definition 1.2.5. *(α-cut of IFS)[127] The α-cut of an IFS* $\tilde{A}^I$ *is denoted by* A^I_α *and is defined by*

$$A^I_\alpha = \{x \in X : \mu_{\tilde{A}^I}(x) \geq \alpha\}\ \forall\, \alpha \in [0,1].$$

Remark 1.2.6. $A^I_0 = X$.

Definition 1.2.7. *(β-cut of IFS)[127] The β-cut of an IFS* $\tilde{A}^I$ *is denoted by* $A^I_{(\beta)}$ *and is defined by*

$$A^I_{(\beta)} = \{x \in X : \nu_{\tilde{A}^I}(x) \leq \beta\}\ \forall\, \beta \in [0,1].$$

Definition 1.2.8. *((α,β)-cut of IFS)[127] The (α,β)-cut of an IFS* $\tilde{A}^I$ *is denoted by* $A^I_{\alpha,\beta}$ *and is defined by*

$$A^I_{(\alpha,\beta)} = \{x \in X : \mu_{\tilde{A}^I}(x) \geq \alpha, \nu_{\tilde{A}^I}(x) \leq \beta\},\ \alpha,\beta \in [0,1];\ \alpha + \beta \leq 1.$$

Remark 1.2.9. *(i)* $A^I_{(0,1)} = X$. *(ii)* $A^I_{(\alpha,\beta)} = A^I_\alpha \cap A^I_{(\beta)}$, $\alpha + \beta \leq 1$; $\alpha, \beta \geq 0$.

Definition 1.2.10. *[66, 189] Let* $\tilde{A}^I$ *be an IFS in X. Then the score and accuracy functions of* $\tilde{A}^I$ *are denoted by* $S_{\tilde{A}^I}(x)$ *and* $A_{\tilde{A}^I}(x)$ *respectively and are defined by*

$$S_{\tilde{A}^I}(x) = \mu_{\tilde{A}^I}(x) - \nu_{\tilde{A}^I}(x),\ A_{\tilde{A}^I}(x) = \mu_{\tilde{A}^I}(x) + \nu_{\tilde{A}^I}(x)\ \ \forall\, x \in X.$$

Definition 1.2.11. *[127] An IFS* $\tilde{A}^I = \{< x, \mu_{\tilde{A}^I}(x), \nu_{\tilde{A}^I}(x) >: x \in \mathbb{R}\}$ *is called an intuitionistic fuzzy number (IFN) if the following conditions hold:*

- $\tilde{A}^I$ *is convex IFS in* $\mathbb{R}$*;*

- $\exists$ *unique* $m \in \mathbb{R}$ *such that* $\mu_{\tilde{A}^I}(m) = 1$ *(m is called the mean value of* $\tilde{A}^I$*);*

- *$\exists$ an $n \in \mathbb{R}$ such that $\nu_{\tilde{A}^I}(n) = 1$;*
- *$\mu_{\tilde{A}^I}$ and $\nu_{\tilde{A}^I}$ are piecewise continuous functions from $\mathbb{R}$ to [0, 1].*

Thus, mathematically, the membership function $\mu_{\tilde{A}^I}$ and non-membership function $\nu_{\tilde{A}^I}$ of an IFN $\tilde{A}^I$ are of the following forms:

$$\mu_{\tilde{A}^I}(x) = \begin{cases} g_1(x), & m - l < x < m, \\ 1, & x = m, \\ g_2(x), & m < x < m + r, \\ 0, & \text{otherwise}, \end{cases}$$

where g_1 and g_2 are piecewise continuous, increasing and decreasing functions in $(m-l, m)$ and $(m, m+r)$ respectively, and

$$\nu_{\tilde{A}^I}(x) = \begin{cases} h_1(x), & m - l' < x < m;\ 0 \le g_1(x) + h_1(x) \le 1, \\ 0, & x = m, \\ h_2(x), & m < x < m + r';\ 0 \le g_2(x) + h_2(x) \le 1, \\ 1, & otherwise, \end{cases}$$

where h_1 and h_2 are piecewise continuous, decreasing and increasing functions in $(m - l', m)$ and $(m, m + r')$ respectively, l is called the left spread and r is called the right spread of $\mu_{\tilde{A}^I}$, l' is called the left spread and r' is called the right spread of $\pi_{\tilde{A}^I}$. The IFN $\tilde{A}^I$ is represented by $(m; l, r; l', r')$. The graphical representation of the IFN $\tilde{A}^I$ is given in Figure 1.3.

Definition 1.2.12 (Arithmetic operations on IFNs:)**.** *[127] Let $\tilde{A}^I$=$(m; l, r; l', r')$, $\tilde{A}^I_1$=$(m_1; l_1, r_1; l'_1, r'_1)$ and $\tilde{A}^I_2$=$(m_2; l_2, r_2; l'_2, r'_2)$ be IFNs. Then*

(i) $\tilde{A}^I_1 \oplus \tilde{A}^I_2$ is defined as an IFN given by
$\tilde{A}^I_1 \oplus \tilde{A}^I_2 = (m_1 + m_2; l_1 + l_2, r_1 + r_2; l'_1 + l'_2, r'_1 + r'_2)$;

(ii) $\ominus\tilde{A}^I$ is defined as an IFN given by
$\ominus\tilde{A}_2 = (-m; r, l; r', l')$;

(iii) $\tilde{A}^I_1 \ominus \tilde{A}^I_2$ is defined as an IFN given by
$\tilde{A}^I_1 \ominus \tilde{A}^I_2 = \tilde{A}^I_1 \oplus (\ominus\tilde{A}^I_2) = (m_1 - m_2; l_1 + r_2, r_1 + l_2; l'_1 + r'_2, r'_1 + l'_2)$;

(iv) $\tilde{A}^I_1 \otimes \tilde{A}^I_2$ is defined as an IFN given by
$\tilde{A}^I_1 \otimes \tilde{A}^I_2 = (m; l, r; l', r')$,

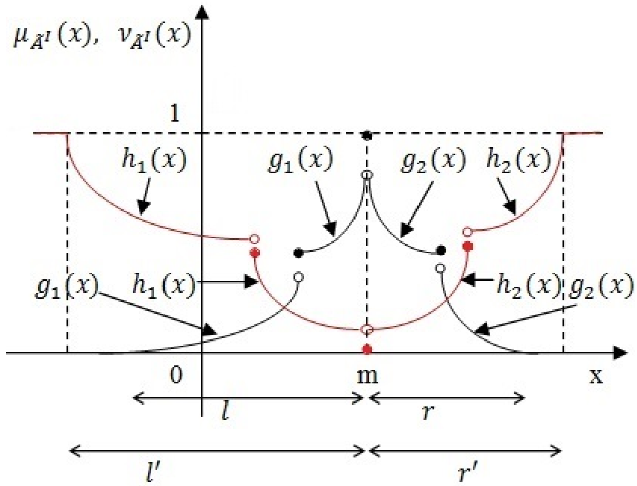

Figure 1.3: Graphical representation of an IFN.

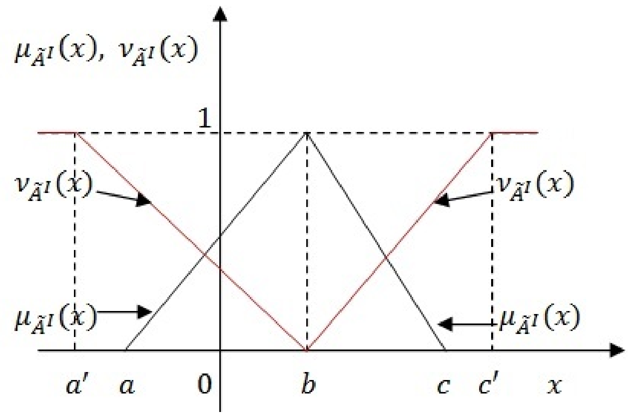

Figure 1.4: Graphical representation of a TIFN.

where $m = m_1m_2, l = m_1m_2 - \min\{(m_1-l_1)(m_2-l_2), (m_1-l_1)(m_2+r_2), (m_1+r_1)(m_2-l_2), (m_1+r_1)(m_2+r_2)\}, r = \max\{(m_1-l_1)(m_2-l_2), (m_1-l_1)(m_2+r_2), (m_1+r_1)(m_2-l_2) - m_1m_2, l' = m_1m_2 - \min\{(m_1-l_1')(m_2-l_2'), (m_1-l_1')(m_2+r_2'), (m_1+r_1')(m_2-l_2'), (m_1+r_1')(m_2+r_2')\}, r' = \max\{(m_1-l_1')(m_2-l_2'), (m_1-l_1')(m_2+r_2'), (m_1+r_1')(m_2-l_2') - m_1m_2;$

(v) $\lambda\tilde{A}^I$ *is defined as an IFN given by*

$$\lambda\tilde{A}^I = \begin{cases} (\lambda m; \lambda l, \lambda r; \lambda l', \lambda r'), & \lambda \geq 0, \\ (\lambda m; -\lambda r, -\lambda l; -\lambda r', -\lambda l'), & \lambda < 0. \end{cases}$$

.

Definition 1.2.13. *[127] An IFN $\tilde{A}^I$ denoted by $(a, b, c; a', b, c')$ is called the triangular intuitionistic fuzzy number (TIFN), if its membership function $\mu_{\tilde{A}^I}$ and non-membership function $\nu_{\tilde{A}^I}$ are given by*

$$\mu_{\tilde{A}^I}(x) = \begin{cases} \frac{x-a}{b-a}, & a < x \leq b, \\ \frac{c-x}{c-b}, & b < x \leq c, \\ 0, & otherwise, \end{cases} \quad and \quad \nu_{\tilde{A}^I}(x) = \begin{cases} \frac{b-x}{b-a'}, & a' < x \leq b, \\ \frac{x-b}{c'-b}, & b < x \leq c', \\ 1, & otherwise, \end{cases}$$

where $a' \leq a < b < c \leq c'$. b is called the mean value of $\tilde{A}^I$, $b-a$ is called the left spread and $c-b$ is called the right spread of $\mu_{\tilde{A}^I}$, $b-a'$ is called the left spread and $c'-b$ is called the right spread of $\pi_{\tilde{A}^I}$. The graphical representation of a TIFN is given in Figure 1.4.

Definition 1.2.14 (Arithmetic operations on TIFNs:). *[127] Let $\tilde{A}^I = (a, b, c; a', b, c')$, $\tilde{A}^I_1 = (a_1, b_1, c_1; a_1', b_1, c_1')$ and $\tilde{A}^I_2 = (a_2, b_2, c_2; a_2', b_2, c_2')$ be TIFNs. Then*

(i) $\tilde{A}_1^I \oplus \tilde{A}_2^I$ *is defined as a TIFN given by*
$\tilde{A}_1^I \oplus \tilde{A}_2^I = (a_1 + a_2, b_1 + b_2, c_1 + c_2; a_1' + a_2', b_1 + b_2, c_1' + c_2')$;

(ii) $\ominus\tilde{A}^I$ *is defined as a TIFN given by*
$\ominus\tilde{A}_2 = (-c_2, -b_2, -a_2; -c_2', -b_2, -a_2')$;

(iii) $\tilde{A}_1^I \ominus \tilde{A}_2^I$ *is defined as a TIFN given by*
$\tilde{A}_1^I \ominus \tilde{A}_2^I = \tilde{A}_1^I \oplus (\ominus\tilde{A}_2^I) = (a_1 - c_2, b_1 - b_2, c_1 - a_2; a_1' - c_2', b_1 - b_2, c_1' - a_2')$;

(iv) $\tilde{A}_1^I \otimes \tilde{A}_2^I$ *is defined as a TIFN given by*
$\tilde{A}_1^I \otimes \tilde{A}_2^I \approx (p_1, p_2, p_3; p_1', p_2, p_3')$, *where*
$p_1 = \min\{a_1a_2, a_1c_2, c_1a_2, c_1c_2\}$, $p_2 = b_1b_2$, $p_3 = \max\{a_1a_2, a_1c_2, c_1a_2, c_1c_2\}$, $p_1' = \min\{a_1'a_2', a_1'c_2', c_1'a_2', c_1'c_2'\}$, $p_3' = \max\{a_1'a_2', a_1'c_2', c_1'a_2', c_1'c_2'\}$;

(v) $\tilde{A}_1^I \oslash \tilde{A}_2^I$ *is defined as a TIFN given by*
$\tilde{A}_1^I \oslash \tilde{A}_2^I \approx (q_1, q_2, q_3; q_1', q_2, q_3')$, *where*
$q_1 = \min\{a_1/a_2, a_1/c_2, c_1/a_2, c_1/c_2\}$, $q_3 = \max\{a_1/a_2, a_1/c_2, c_1/a_2, c_1/c_2\}$, $q_2 = b_1/b_2$, $q_1' = \min\{a_1'/a_2', a_1'/c_2', c_1'/a_2', c_1'/c_2'\}$, $q_3' = \max\{a_1'/a_2', a_1'/c_2', c_1'/a_2', c_1'/c_2'\}$ *provided* $a_2' > 0$ *or* $c_2' < 0$;

(vi) $\lambda\tilde{A}^I$ *is defined as a TIFN given by*

$$\lambda\tilde{A}^I = \begin{cases} (\lambda a_1, \lambda b_1, \lambda c_1; \lambda a_1', \lambda b_1, \lambda c_1'), & \lambda \geq 0, \\ (\lambda c_1, \lambda b_1, \lambda a_1; \lambda c_1', \lambda b_1, \lambda a_1'), & \lambda < 0. \end{cases}$$

.

Definition 1.2.15. *[127] A trapezoidal intuitionistic fuzzy number (TrIFN)* $\tilde{A}^I$ *is an IFS in* $\mathbb{R}$, *denoted by* $\tilde{A}^I = (a, b, c, d; a', b', c', d')$, *with membership function* $\mu_{\tilde{A}^I}$ *and non-membership function* $\nu_{\tilde{A}^I}$ *given by*

$$\mu_{\tilde{A}^I}(x) = \begin{cases} \dfrac{x-a}{b-a}, & a < x \leq b, \\ 1, & b < x \leq c, \\ \dfrac{d-x}{d-c}, & c < x \leq d, \\ 0, & otherwise, \end{cases} \quad and \quad \nu_{\tilde{A}^I}(x) = \begin{cases} \dfrac{b'-x}{b'-a'}, & a' < x \leq b', \\ 0, & b' < x \leq c', \\ \dfrac{x-c'}{d'-c'}, & c' < x \leq d', \\ 1, & otherwise, \end{cases}$$

where $a' \leq a \leq b' \leq b \leq c \leq c' \leq d \leq d'$.

Particular Cases

Case 1. *If* $b' = b, c' = c$, *then* $\tilde{A}^I$ *also represents a TrIFN. It is denoted by* $\tilde{A}^I = (a, b, c, d; a', b, c, d')$.

Case 2. *If $b^{'}=b=c=c^{'}$, then $\tilde{A}^I$ represents a TIFN. It is denoted by $\tilde{A}^I=(a,b,d;a^{'},b,d^{'})$.*

Case 3. *If $a^{'}=a, b^{'}=b, c^{'}=c, d^{'}=d$, then $\tilde{A}^I$ represents TrFN. It is denoted by $\tilde{A}^I=(a,b,c,d)$.*

Case 4. *If $a^{'}=a, b^{'}=b=c^{'}=c, d^{'}=d$, then $\tilde{A}^I$ represents a TFN. It is denoted by $\tilde{A}^I=(a,b,d)$.*

Case 5. *If $a^{'}=a=b^{'}=b, c^{'}=c=d^{'}=d$, then $\tilde{A}^I$ represents the crisp interval $[b,c]$.*

Case 6. *If $a^{'}=a=b^{'}=b=c^{'}=c=d^{'}=d=m$, then $\tilde{A}^I$ represents a real number m.*

Definition 1.2.16 (Arithmetic operations on TrIFNs:). *[127] Let $\tilde{A}^I=(a,b,c,d;a^{'},b^{'},c^{'},d^{'})$, $\tilde{A}_1{}^I=(a_1,b_1,c_1,d_1;a_1^{'},b_1^{'},c_1^{'},d_1^{'})$ and $\tilde{A}_2{}^I=(a_2,b_2,c_2,d_2;a_2^{'},b_2^{'},c_2^{'},d_2^{'})$ be TrIFNs. Then*

(i) $\tilde{A}_1^I\oplus\tilde{A}_2^I$ *is defined as a TrIFN given by*
$\tilde{A}_1^I\oplus\tilde{A}_2^I=(a_1+a_2,b_1+b_2,c_1+c_2,d_1+d_2;a_1^{'}+a_2^{'},b_1^{'}+b_2^{'},c_1^{'}+c_2^{'},c_1^{'}+c_2^{'})$;

(ii) $\ominus\tilde{A}^I$ *is defined as a TrIFN given by*
$\ominus\tilde{A}_2=(-d_2,-c_2,-b_2,-a_2;-d_2^{'},-c_2^{'},-b_2^{'},-a_2^{'})$;

(iii) $\tilde{A}_1^I\ominus\tilde{A}_2^I$ *is defined as a TrIFN given by*
$\tilde{A}_1^I\ominus\tilde{A}_2^I=\tilde{A}_1^I\oplus(\ominus\tilde{A}_2^I)=(a_1-d_2,b_1-c_2,c_1-b_2,d_1-a_2;a_1^{'}-d_2^{'},b_1^{'}-c_2^{'},c_1^{'}-b_2^{'},d_1^{'}-a_2^{'})$;

(iv) $\tilde{A}_1^I\otimes\tilde{A}_2^I$ *is defined as a TrIFN given by*
$\tilde{A}_1^I\otimes\tilde{A}_2^I\approx(p_1,p_2,p_3,p_4;p_1^{'},p_2{'},p_3^{'},p_4{'})$, *where*
$p_1=\min\{a_1a_2,a_1d_2,d_1a_2,d_1d_2\}$, $p_2=\min\{b_1b_2,b_1c_2,\ c_1b_2,\ c_1c_2\}$, $p_3=\max\{b_1b_2,\ b_1c_2,\ c_1b_2,\ c_1c_2\}$, $p_4=\max\{a_1a_2,\ a_1d_2,\ d_1a_2,\ d_1d_2\}$, $p_1^{'}=\min\{a_1^{'}a_2^{'},\ a_1^{'}d_2^{'},\ d_1^{'}a_2^{'},\ d_1^{'}d_2^{'}\}$, $p_2^{'}=\min\{b_1^{'}b_2^{'},\ b_1^{'}c_2^{'},\ c_1^{'}b_2^{'},\ c_1^{'}c_2^{'}\}$, $p_3^{'}=\max\{b_1^{'}b_2^{'},\ b_1^{'}c_2^{'},\ c_1^{'}b_2^{'},\ c_1^{'}c_2^{'}\}$, $p_4^{'}=\max\{a_1^{'}a_2^{'},\ a_1^{'}d_2^{'},\ d_1^{'}a_2^{'},\ d_1^{'}d_2^{'}\}$;

(v) $\tilde{A}_1^I\oslash\tilde{A}_2^I$ *is defined as a TrIFN given by*
$\tilde{A}_1^I\oslash\tilde{A}_2^I\approx(q_1,q_2,q_3,q_4;q_1^{'},q_2{'},q_3^{'},q_4{'})$, *where*
$q_1=\min\{a_1/a_2,\ a_1/d_2,d_1/a_2,d_1/d_2\}$, $q_2=\min\{b_1/b_2,\ b_1/c_2,\ c_1/b_2,\ c_1/c_2\}$, $q_3=\max\{b_1/b_2,\ b_1/c_2,\ c_1/b_2,\ c_1/c_2\}$, $q_4=\max\{a_1/a_2,\ a_1/d_2,\ d_1/a_2,\ d_1/d_2\}$, $q_1^{'}=\min\{a_1^{'}/a_2^{'},\ a_1^{'}/d_2^{'},\ d_1^{'}/a_2^{'},\ d_1^{'}/d_2^{'}\}$, $q_2^{'}=\min\{b_1^{'}/b_2^{'},\ b_1^{'}/c_2^{'},\ c_1^{'}/b_2^{'},\ c_1^{'}/c_2^{'}\}$, $q_3^{'}=\max\{b_1^{'}/b_2^{'},\ b_1^{'}/c_2^{'},\ c_1^{'}/b_2^{'},\ c_1^{'}/c_2^{'}\}$, $q_4^{'}=\max\{a_1^{'}/a_2^{'},\ a_1^{'}/d_2^{'},\ d_1^{'}/a_2^{'},\ d_1^{'}/d_2^{'}\}$, *provided* $a_2^{'}>0$ *or* $d_2^{'}<0$;

(vi) $\lambda\tilde{A}^I$ *is defined as a TrIFN given by*

$$\lambda\tilde{A}^I=\begin{cases}(\lambda a_1,\lambda b_1,\lambda c_1,\lambda d_1;\lambda a_1^{'},\lambda b_1^{'},\lambda c_1^{'},\lambda d_1^{'}), & \lambda\geq 0,\\(\lambda d_1,\lambda c_1,\lambda b_1,\lambda a_1;\lambda d_1^{'},\lambda c_1^{'},\lambda b_1^{'},\lambda a_1^{'}), & \lambda<0.\end{cases}$$

.

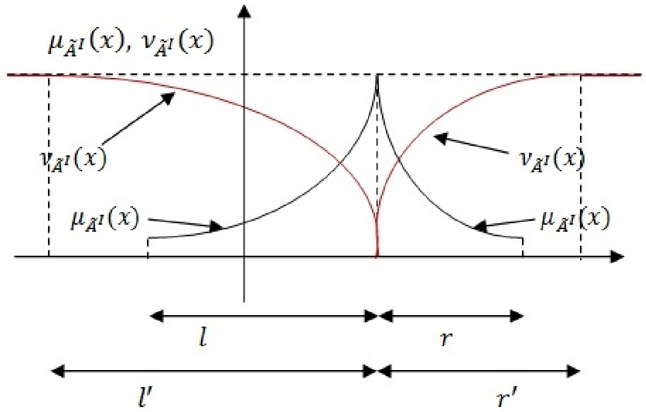

Figure 1.5: Graphical representation of an LR-Type IFN.

Definition 1.2.17. *[204] A function $f : [0,\infty) \to [0,1]$ is said to be shape function or reference function if it satisfies the following conditions:*

(i) $f(0) = 1$,

(ii) f *is continuous function on* $[0,\infty)$,

(iii) f *is decreasing on* $[0,\infty)$,

(iv) $\lim_{x\to\infty} f(x) = 0$.

Definition 1.2.18. *[83] An IFN $\tilde{A}^I$ is said to be of LR-type IFN if there exist shape functions L and R, and real constants $l > 0, r > 0, l' > 0, r' > 0$ such that its membership function $\mu_{\tilde{A}^I}$ and non-membership function $\nu_{\tilde{A}^I}$ are given by*

$$\mu_{\tilde{A}^I}(x) = \begin{cases} L\Big(\dfrac{m-x}{l}\Big), & x \le m, \\ R\Big(\dfrac{x-m}{r}\Big), & x > m, \end{cases} \quad \text{and} \quad \nu_{\tilde{A}^I}(x) = \begin{cases} 1 - L\Big(\dfrac{m-x}{l'}\Big), & x \le m, \\ 1 - R\Big(\dfrac{x-m}{r'}\Big), & x > m, \end{cases}$$

where $l \le l'$, $r \le r'$ and $0 \le \mu_{\tilde{A}^I}(x) + \nu_{\tilde{A}^I}(x) \le 1 \ \forall\, x \in \mathbb{R}$. m is called the mean value of $\tilde{A}^I$, l is called the left spread and r is called the right spread of $\mu_{\tilde{A}^I}$, l' is called the left spread and r' is called the right spread of $\pi_{\tilde{A}^I}$. The LR-type IFN is represented by $\tilde{A}^I = (m; l, r; l', r')_{LR}$. The graphical representation of an LR-type IFN is given in Figure 1.5.

Definition 1.2.19. *[83] The LR-type representation of a TIFN $\tilde{A}^I = (a, b, c; a', b, c')$ is given by $\tilde{A}^I = (b; b-a, c-b; b-a', c'-b)_{LR}$ and is defined by*

$$\mu_{\tilde{A}^I}(x) = \begin{cases} L\Big(\dfrac{b-x}{b-a}\Big), & x \le b, \\ R\Big(\dfrac{x-b}{c-b}\Big), & x > b, \end{cases} \quad \text{and} \quad \nu_{\tilde{A}^I}(x) = \begin{cases} 1 - L\Big(\dfrac{b-x}{b-a'}\Big), & x \le b, \\ 1 - R\Big(\dfrac{x-b}{c'-b}\Big), & x > b, \end{cases}$$

where $L(x) = R(x) = \max\{0, 1 - x\}$ *and* $0 \leq \mu_{\tilde{A}^I}(x) + \nu_{\tilde{A}^I}(x) \leq 1 \; \forall \, x \in \mathbb{R}$.

1.3 Fuzzy optimization

In the recent past decades, traditional optimization techniques have been successfully applied for solving a well-defined and precise structure/configuration problems. Such optimization problems are usually well-formulated when objective functions and system of constraints are precise and deterministic. Unfortunately, real-world situations are often not deterministic. There exist various types of uncertainties in social, industrial and economic systems, such as randomness of occurrence of events, imprecision and ambiguity of system data and linguistic vagueness, etc. The FS theory introduced by Zadeh [194] has been applied successfully in various fields. The use of FS theory became very rapid in the field of optimization after the pioneering work done by Bellman and Zadeh [31], defined as the FS formed by the intersection of fuzzy objective and constraint goals. According to the above definition and assuming that the constraints are "non-interactive" the logical "and" corresponds to the intersection. The "decision" in a fuzzy environment can, therefore, be viewed as the intersection of fuzzy constraints and fuzzy objective function(s). The relationship between constraints and objective functions in a fuzzy environment are therefore fully symmetric, i.e. there is no longer a difference between the former and the latter. From this point of view, Tanaka et al. [171] introduced fuzzy mathematical programming and Zimmermann [202] introduced fuzzy linear programming problem (FLPP) as conventional LP. Zimmermann [202] considered LP problems with a fuzzy goal and fuzzy constraints, used linear membership functions and the min operator as an aggregator for these functions, and assigned an equivalent LP problem to FLP. Then many authors have used FS theory in various real-life optimization problems, such as planning, scheduling, transportation, manufacturing, etc.

Since real-world problems are very complex, experts and DMs frequently do not know the values of parameters precisely. Therefore, it may be more realistic to take the knowledge of experts or DMs about the parameters as fuzzy data. Thus the multi-objective fuzzy linear programming problems (MOFLPPs) with fuzzy parameters would be viewed as more effective than the conventional one in solving real physical problems. Zimmermann [203] and Werners [184] proposed an approach for determining suitable values for the aspiration level and admissible violation of the fuzzy goal of fuzzy programming and linear programming with several objective functions. Luhandjula [122] gave compensatory operators in fuzzy linear programming with multiple objectives. Chanas [40] proposed fuzzy programming in multi-objective linear programming and it was solved by a parametric approach. Angelov [7] gave a parameterized generalization of fuzzy mathematical programming problem. Angelov [8] gave an analytical method for solving a type of fuzzy optimization problems. Stanciulescu et al. [164] gave multi-objective fuzzy linear programming problems with fuzzy decision variables. Dutta et al. [69] gave a single-period inventory model with fuzzy random variable demand. Hu et al. [94] introduced a fuzzy goal programming approach to a multi-objective optimization problem with priorities. Dutta et al. [70] gave an inventory model for single-period products with reordering opportunities under fuzzy demand. Nagar et al. [136] gave an integrated supply chain model for new products with imprecise production and supply under scenario dependent fuzzy random demand. Kumar et al. [108] gave a new method for solving fully fuzzy linear programming problems. Kaur and Kumar [104] gave Mehar's method for solving fully fuzzy linear programming problems with LR-type fuzzy parameters. Khan et al. [105] gave a simplified novel technique for solving fully fuzzy linear programming problems. Ebrahimnejad and Tavana [71] gave a novel method for solving linear programming problems with symmetric trapezoidal fuzzy numbers. Saati et al. [160] gave a fuzzy linear programming model with fuzzy parameters and decision variables. Li et al. [114] introduce dissimilarity functions and divergence measures between FSs. Ghanbari [75] introduced solutions of fuzzy LR algebraic linear systems using linear programs. Deng et al. [53] gave monotonic similarity measures between FSs and their relationship with entropy and inclusion measures. Ranjan and Singh [152] gave an aggregation approach for system efficiency evaluation of homogeneous parallel production systems. In [6, 24, 25, 84, 85, 125, 126, 150, 151, 159, 176, 185], the duality theory of fuzzy optimization problems are given in different prospectives.

1.4 Intuitionistic fuzzy optimization

Intuitionistic fuzzy optimization (IFO) is relatively a recent field for research. The deterministic linear programming models miss to accommodating any kind of imprecision or vagueness in those models. The crisp relations (inequalities or equalities) cannot always describe the constraints or objective functions completely. However, one may be able to express the constraints and objective functions in an IFS context making the model more realistic and pragmatic. Intuitionistic fuzzy linear programming (IFLP) is developed on this basis. The advantage is that this method can express the degree of acceptance and rejection of objectives and constraints. To deal with uncertainty as well as hesitation, intuitionistic fuzzy (IF) modeling seems to be more relevant which includes both uncertainty and hesitation. In IFSs, the degree of membership, the degree of non-membership and the degree of hesitancy are real values. An application of the IFS to optimization problems is given by Angelov [9]. His technique is based on maximizing the membership degree, minimizing the non-membership degree and the crisp model is formulated using the IF aggregation operator. Mahapatra and Roy [127] gave reliability evaluation using TIFNs arithmetic operations. Kumar et al. [107] gave a new approach for analyzing the fuzzy system reliability using IFN. Dubey et al. [66] introduced fuzzy linear programming (FLP) under interval uncertainty based on IFS representation. Nagoorgani and Ponnalagu [137] gave a new approach to solving the IFLP problem (IFLPP). Garg et al. [73] introduced IFO technique for solving multi-objective reliability optimization problems in interval environment. Suresh et al. [165] introduced solution technique of IFLPPs by ranking function. Li and Liu [112] gave a parameterized nonlinear programming approach to solve matrix games with payoffs of IFNs. Wan et al. [178] gave an IFLP method for logistics outsourcing provider selection. Singh and Yadav [163] introduced modeling and optimization of multi-objective non-linear programming problem in the intuitionistic fuzzy environment (IFE). Nishad and Singh [139] introduced a solution technique for multi-objective decision-making problem in IFE. Rani et al. [153] introduced Multi-objective non-linear programming problem in IFE: Optimistic and pessimistic viewpoint. Razmi et al. [156] introduced an IF goal programming approach for finding Pareto-optimal solutions to multi-objective programming problems. Zhao et al. [199] gave interactive IF methods for multilevel programming problems. Singh [162] gave IF DEA/AR model and its application to flexible manufacturing systems.

1.5 Decision-making problems in fuzzy environment

The study of classical decision theory has been approached from different perspectives, including philosophical, behavioral, biological, mathematical and computational approaches, yet a large number of challenges remain in understanding this important of higher cognition. Classic decision theory deals with

(i) a set of alternative states of nature (outcomes).

(ii) a set of alternative actions.

(iii) a relation indicating the state or outcome to be expected from each alternative action.

(iv) a utility objective function which orders the outcomes according to their desirability.

Selecting the best suitable alternative from the given set of feasible alternatives is called Multi-criteria Decision-Making (MCDM)/Multi-attribute Decision-Making (MADM). MCDM/MADM analysis has some unique characteristics such as the presence of multiple and conflicting criteria, different units of measurement among the criteria, and the presence of quite different alternatives. Incredible efforts have been spent and significant advances have been made towards the development of numerous MCDM/MADM methodologies for solving different types of problems. Classical MCDM methods along the line of multi-attribute utility theory require the determination of alternative ratings and criteria weights by eliciting the DM's judgments/preferences. Crisp values are commonly used to represent these ratings and weights, which are implicitly or explicitly aggregated by a utility function. The overall utility of an alternative represents how well the alternative satisfies the DM's objective. Alternatives with higher utilities are said to be preferred. In practical applications, alternative ratings and criteria weights cannot always be assessed precisely. Subjectiveness and vagueness are often involved which may come from various sources, including unquantifiable information, incomplete information, unobtainable information and partial ignorance. Classical MCDM methods cannot effectively handle problems with such imprecise information. Most of the decision-making problems involve uncertainty. Hence one of the most important aspects for a useful decision aid is to provide the ability to handle imprecise information such as large profit, cheap price, fast speed. For the last four decades, FS theory proposed by Zadeh [194] has been used to tackle these qualitative terms and played a significant role in decision making under uncertainty.

The application of FS theory to MCDM models provides an effective way of dealing with the subjectiveness and vagueness of the decision-making process for the general MCDM problem [145]. By using linguistic terms with fuzzy number representation, the DM can effectively express his/her subjective assessments. The DM's preference in comparing alternatives or criteria can be better modeled. Mostly fuzzy MCDM models based on classical utility theory involve two phases

- The aggregation of the fuzzy assessment with respect to all criteria for each alternative and

- The ranking of alternatives based on their aggregated overall assessments (fuzzy utilities).

Fan et al. [72] gave subsethood measure for selection of alternative in fuzzy MCDM problems. Wei et al. [183] gave a compromise weight for multi-attribute group decision making with individual preference. Park [144] introduced mathematical programming models for characterizing dominance and potential optimality when multi-attribute alternative values and weights are simultaneously incomplete. Rao et al. [154] gave preference structure on alternatives and judges in a group decision problem by a fuzzy approach. Zhang and Zhang [198] gave hybrid monotonic inclusion measure and its use in measuring similarity and distance between FSs, He et al. [93] gave T_L-transitivity of fuzzy similarity measures and Bustince et al. [39] gave grouping, overlaps, and generalized bientropic functions for fuzzy modeling of pairwise comparisons. Mesiar et al. [131] gave OWA operator for decision-making problems in fuzzy environment. It is used to order the alternatives in fuzzy decision making problems. Different types of methods for solving decision-making problems are noted in [4, 87, 88, 91].

1.6 Decision-making problems in intuitionistic fuzzy environment

Decision-making models inherently have some level of imprecision and vagueness in the estimation of model parameters. Such phenomena have been very well captured through FSs. There can be situations in which the DM is endowed with a fuzzy decision function favoring an alternative. It is worthwhile to have some measure of knowledge as to how the alternative has not been preferred. IF decision models can play a vital role in this context. Mańko [129] measured the fuzziness and the non-fuzziness of IFSs. Chen [41], and Hong and Kim [97] established the similarity measures between vague sets and between elements. Liu [115] gave new similarity measures between IFSs and between elements. Dengfeng and Chuntian [54] and Szmidt and Kacprzyk [168] gave similarity measures of IFSs and applied to pattern recognition and in supporting medical diagnostic reasoning. Also, several types of distances between IFSs and similarity measure of IFSs noted in [23, 38, 52, 78, 95, 96, 138, 143, 166, 167, 180]. It is used in decision-making problems, comparative analysis from a pattern recognition point of view and to the global comparison of images in IFE. Xu and Yager [187] introduced dynamic IF multi-attribute decision-making problems. In [90, 92, 179, 181, 186, 188, 192, 197], authors worked on solving decision-making problems in IFE with different prospectives. Montes et al. [133–135] introduced divergence measures, local divergences and entropy measures for IFSs and they applied to decision-making and pattern recognition problems. Pal et al. [142] and later Das et al. [50] gave information measures in the IF framework and their relationships. In [86, 119–121], several types of aggregation operators are given to aggregate attributes corresponding to each alternative in decision-making problems in IFE. Also, Gupta et al. [86] gave multi-attribute group decision-making based on extended TOPSIS method under interval-valued IFE.

1.7 Fuzzy implications

After Zadeh introduced the concept of FS in his pioneering work ([194], Zadeh 1965), a huge amount of work in FS theory and fuzzy logic appeared, both theoretical and applied. There are two main branches in the study of fuzzy logic, fuzzy logic in the narrow sense and fuzzy logic in the broad sense [89, 140]. Fuzzy logic in the narrow sense is a form of many-valued logic [157] constructed in the spirit of classical binary logic. It is symbolic logic concerned with syntax, semantics, axiomatization, soundness, completeness, etc. [82, 89]. Fuzzy logic in the broad sense can be seen as an extension of fuzzy logic in the narrow sense. It is a way of interpreting the natural language to model human reasoning [140].

A very important part of research in fuzzy logic (both in the narrow sense and in the broad sense) focuses on extending the classical binary logic operators negation ($\neg$), conjunction ($\wedge$), disjunction ($\vee$) and implication ($\rightarrow$) to fuzzy logic operators. The extension of implication ($\rightarrow$) to fuzzy logic is called fuzzy implication. Table 1.1 gives the truth table of the classical binary implication '$\rightarrow$'.

Table 1.1: Truth table of the classical binary implication

p	q	$p \rightarrow q$
0	0	1
0	1	1
1	0	0
1	1	1

Definition 1.7.1. *A function* $I : [0,1]^2 \rightarrow [0,1]$ *is called an fuzzy implication if for* $x, x', x'', y, y', y'' \in [0,1]$*, it satisfies the following conditions:*

- $x' < x'' \;\Rightarrow\; I(x', y) \geq I(x'', y)$*, i.e.,* $I(., y)$ *is non-increasing,*
- $y' < y'' \;\Rightarrow\; I(x, y') \leq I(x, y'')$*, i.e.,* $I(x, .)$ *is non-decreasing,*
- $I(0, 0) = 1$, $I(1, 1) = 1$, $I(1, 0) = 0$, $I(0, 1) = 1$.

Let us first have an overview of the literature on fuzzy implications in fuzzy logic in the broad sense.

The implication in classical binary logic work only on two truth values 0 and 1 while a fuzzy implication is a $[0,1]^2 \rightarrow [0,1]$ mapping. So besides the boundary condition, the first step to work on fuzzy implications is naturally to determine which fundamental requirements a fuzzy implication should fulfill. Most considerations are taken either from the point of view that a fuzzy implication is a generalization of the implication in classical binary logic [89, 140].

In the literature, there are different classes of fuzzy implications, like strong implications (S-implications), residuated implications (R-implications), quantum logic implications (QL-implications), etc. generated from the fuzzy logic operators negation, conjunction and disjunction proposed in [19, 106]. Also, Jenei [102] gave a more efficient method for defining fuzzy connectives. S-implications and QL-implications are defined respectively based on

$$p \to q = \neg p \vee q \tag{1.1}$$

$$p \to q = \neg p \vee (p \wedge q) \tag{1.2}$$

in classical binary logic, where p and q are two propositions. R-implications are defined based on the fact that the implication is residuated with and in the classical binary logic. These class of implications are widely used in different areas, like approximate reasoning, Boolean like laws, distributive equations, etc. (e.g., [19, 106]).

Besides S-, R- and QL- implications, there are many other classes of fuzzy implications which are not generated from the fuzzy logic operators negation, conjunction and disjunction, like, Yager's new class of implications J_f [21], fuzzy implications determined by aggregation operators [141], (g, min)-implications [196], fuzzy implications derived from generalized h-generators [117], residual implications derived from overlap functions [65], Fuzzy implication functions based on powers of the continuous t-norms [128, 177], etc. The Boolean-like laws over fuzzy implications, like, I(x, I(y, x)) = 1, I(x, I(y, z)) = I(I(x, y), I(x, z)) are studied in [42, 43]. Trillas and Alsina [172] studied the law $[p \wedge q \to r] = [(p \to r) \vee (q \to r)]$ in fuzzy logic. The distributivity of fuzzy implication operators over t-norm and t-conorm is studied by Balasubramaniam and Rao [22]. The concept of distributivity of fuzzy implications in [22] is extended for nilpotent or strict t-conorms by Baczyński and Jayaram [20], and for overlap and grouping functions by Qiao and Hu [148]. Jenei [103] studied continuity of left-continuous triangular norms with strong induced negations and their boundary condition. After that, the continuity of residuals of triangular norms studied by Jayaram [100].

1.8 Intuitionistic fuzzy implications

The extension of fuzzy implication to IFE when each of propositional elements lies in $\mathcal{L}$. Atanassov and Gargov [15] introduced two versions of IF propositional calculus and a version of IF predicate logic. Atanassov and Gargov [15] and later Deschrijver et al. [59] and Cornelis et al. [45] presented the definition and properties of intuitionistic fuzzy implication (IFI), IF t-norm, IF t-conorm. Also, some properties of IFIs are given and studied by Baczyński [16]. Bustince et al. [33] studied IF conditional interpretation and introduced and analyzed the properties of IFIs generated by fuzzy implications, fuzzy complications, and aggregation operator. After, Deschrijver and Kerre [62] gave triangular norms and related operators in $\mathcal{L}$. Several authors studied interval-valued implications, interval-valued aggregation operators, interval-valued t-norms, interval-valued t-conorms, interval-valued negations and different class of interval-valued implications [5, 61, 111, 118]. Van Gasse et al. [173] introduced a characterization of interval-valued residuated lattices. Bustince et al. [34] introduced the generation of interval-valued fuzzy and Atanassov's IF connectives from fuzzy connectives and from K_α operators based on laws for conjunctions and disjunctions, amplitude. Gorzalczany [81] and later Li et al. [113] gave a method of inference in approximate reasoning based on interval-valued FSs and robustness of interval-valued fuzzy inference. The distributivity of interval-valued fuzzy implications over t-representable t-norms generated from strict t-norms and nilpotent t-norms are noted in [17, 18]. Shi et al. [161] gave constructive methods for IFIs.

In this section, the definitions of IFIs, $\mathcal{R}$-implications, $(\mathcal{S},\mathcal{N})$-implications, axioms of IFIs, IF t-norms, IF t-conorms and IF negations are presented.

Definition 1.8.1 (**IFS Lattice:**, [59])**.** *Let* $\mathcal{L} = \{(u_1,u_2) : (u_1,u_2) \in [0,1]^2,\ u_1+u_2 \le 1\}$ *be an IFS and the operation* $\le_{\mathcal{L}}$ *be defined on* $\mathcal{L}$ *by*

$$(u_1,u_2) \le_{\mathcal{L}} (v_1,v_2) \Leftrightarrow u_1 \le v_1,\ u_2 \ge v_2\ \forall\, (u_1,u_2),(v_1,v_2) \in \mathcal{L}.$$

For each nonempty set $\mathcal{A} \subseteq \mathcal{L}$*, we have*

$$\sup \mathcal{A} = (\sup\{u_1 : (u_1,u_2) \in \mathcal{A}\}, \inf\{u_2 : (u_1,u_2) \in \mathcal{A}\}),$$
$$\inf \mathcal{A} = (\inf\{u_1 : (u_1,u_2) \in \mathcal{A}\}, \sup\{u_2 : (u_1,u_2) \in \mathcal{A}\}).$$

Then $(\mathcal{L},\le_{\mathcal{L}})$ *is a complete lattice [59]. Equivalently, this lattice can also be defined as an algebraic structure* $(\mathcal{L},\vee,\wedge)$*, where the join operator* $\vee$ *and the meet operator* $\wedge$ *are defined as follows:*
For $u=(u_1,u_2),\ v=(v_1,v_2) \in \mathcal{L}$*,*

$$u \vee v = (\max(u_1,v_1), \min(u_2,v_2)) \text{ and } u \wedge v = (\min(u_1,v_1), \max(u_2,v_2)).$$

Remark 1.8.2. *(i) Note that if for* $(u_1,u_2),(v_1,v_2) \in \mathcal{L},\ u_1 < v_1$ and $u_2 < v_2$*, then* u *and* v *are incomparable w.r.t.* $\le_{\mathcal{L}}$*, written as* $u\|_{\mathcal{L}}v$*.*

(ii) We denote the units $0_{\mathcal{L}} = (0,1),\ 1_{\mathcal{L}} = (1,0)$ *for the set* $\mathcal{L}$*.*

(iii) For each point $x \in X$, $\tilde{A}^I(x) = (\mu_{\tilde{A}^I}(x), \nu_{\tilde{A}^I}(x))$ is an element of $\mathcal{L}$.

Definition 1.8.3. *[109] Let $u = (u_1, u_2)$ and $v = (v_1, v_2)$ be the elements of $\mathcal{L}$ and $r > 0$ be a real number. Then*

(i) the sum of u and v is denoted by $u + v$ and is defined as an element of $\mathcal{L}$ is given by

$$u + v = (u_1 + v_1 - u_1 v_1, u_2 v_2),$$

(ii) the difference of u and v is denoted by $u - v$ and is defined as an element of $\mathcal{L}$ given by

$$u - v = \begin{cases} (\frac{u_1 - v_1}{1 - v_1}, \frac{u_2}{v_2}), & u_1 \geq v_1, u_2 \leq v_2, v_2 > 0,\ u_2(1 - v_1) \leq v_2(1 - u_1); \\ 0_{\mathcal{L}}, & otherwise, \end{cases}$$

(iii) the product of u and v is denoted by uv and is defined as an element of $\mathcal{L}$ given by

$$uv = (u_1 v_1, u_2 + v_2 - u_2 v_2),$$

(iv) the division of u by v is denoted by u/v and is defined as an element of $\mathcal{L}$ given by

$$u/v = \begin{cases} (\frac{u_1}{v_1}, \frac{u_2 - v_2}{1 - v_2}), & u_1 \leq v_1, u_2 \geq v_2, v_1 > 0,\ u_1(1 - v_2) \leq v_1(1 - u_2); \\ 0_{\mathcal{L}}, & otherwise, \end{cases}$$

(v) $ru = (1 - (1 - u_1)^r, u_2^r)$,

(vi) $u^r = (u_1^r, 1 - (1 - u_2)^r)$.

Definition 1.8.4. *[59] A function $\mathtt{I} : \mathcal{L}^2 \to \mathcal{L}$ is called an IFI if for $u, u', u'', v, v', v'' \in \mathcal{L}$, it satisfies the following conditions:*

$$u' <_{\mathcal{L}} u'' \ \Rightarrow\ \mathtt{I}_{\mathtt{I}}(u', v) \geq_{\mathcal{L}} \mathtt{I}_{\mathtt{I}}(u'', v), i.e., \mathtt{I}_{\mathtt{I}}(., v) \text{ is non-increasing} \tag{I1}$$

$$v' <_{\mathcal{L}} v'' \ \Rightarrow\ \mathtt{I}_{\mathtt{I}}(u, v') \leq_{\mathcal{L}} \mathtt{I}_{\mathtt{I}}(u, v''), i.e., \mathtt{I}_{\mathtt{I}}(u, .) \text{ is non-decreasing} \tag{I2}$$

$$\mathtt{I}_{\mathtt{I}}(0_{\mathcal{L}}, 0_{\mathcal{L}}) = 1_{\mathcal{L}}, \mathtt{I}_{\mathtt{I}}(1_{\mathcal{L}}, 1_{\mathcal{L}}) = 1_{\mathcal{L}}, \mathtt{I}_{\mathtt{I}}(1_{\mathcal{L}}, 0_{\mathcal{L}}) = 0_{\mathcal{L}}, \mathtt{I}_{\mathtt{I}}(0_{\mathcal{L}}, 1_{\mathcal{L}}) = 1_{\mathcal{L}} \tag{I3}$$

We also define the following set for further usage: $\mathcal{D} = \{(u_1, u_2) : (u_1, u_2) \in \mathcal{L},\ u_1 + u_2 = 1\}$, and the first and second projection mappings pr_1 and pr_2 on $\mathcal{L}$ defined as $pr_1(u_1, u_2) = u_1$ and $pr_2(u_1, u_2) = u_2$ for all $(u_1, u_2) \in \mathcal{L}$. Some IFIs are given in Table 1.2.

Table 1.2: List of some IFIs

Name	Formula
Zadeh 1	$\mathtt{I}_{\mathtt{I}ZD1}(u,v) =< \max(u_2,\min(u_1,v_1)),\min(u_1,v_2) >$
Zadeh 2	$\mathtt{I}_{\mathtt{I}ZD2}(u,v) =< \max(u_2,\min(u_1,v_1)),\min(u_1,\max(u_2,v_2)) >$
Gödel	$\mathtt{I}_{\mathtt{I}GD}(u,v) =< 1-(u_1-v_1).sg(u_1-v_1), v_2.sg(u_1-v_1) >$
Gaines-Rescher	$\mathtt{I}_{\mathtt{I}GR}(u,v) =< 1-sg(u_1-v_1), v_2.sg(u_1-v_1) >$
Lukasiewicz	$\mathtt{I}_{\mathtt{I}LK}(u,v) =< \min(1,u_2+v_1),\max(0,u_1+v_2-1) >$
Fodor's 1	$\mathtt{I}_{\mathtt{I}FD1}(u,v) =< \overline{sg}(u_1-v_1)+sg(u_1-v_1)\max(u_2,v_1), sg(u_1-v_1)$ $\min(u_1,v_2) >$
Reichenbach	$\mathtt{I}_{\mathtt{I}RB}(u,v) =< u_2+u_1v_1, u_1v_2 >$
Kleene-Dienes	$\mathtt{I}_{\mathtt{I}KD}(u,v) =< \max(u_2,v_1),\min(u_1,v_2) >$
Wu	$\mathtt{I}_{\mathtt{I}WU}(u,v) =< 1-(1-\min(u_2v_1.sg(u_1-v_1))),\max(u_1,v_2)sg(u_1-$ $v_1).sg(v_2-u_2) >$
Willmott	$\mathtt{I}_{\mathtt{I}WM}(u,v) =< \min(\max(u_2,v_1),\max(u_1,u_2),\max(v_1,v_2)),$ $\max(\min(u_1,v_2),\min(u_1,u_2),\min(v_1,v_2)) >$
Atanassov 1	$\mathtt{I}_{\mathtt{I}A1}(u,v) =< 1-(1-v_1).sg(u_1-v_1), v_2.sg(u_1-v_1).sg(v_2-u_2) >$
Atanassov 2	$\mathtt{I}_{\mathtt{I}A2}(u,v) =< \max(u_2,v_1), 1-\max(u_2,v_1) >$
Klir and Yuan 1	$\mathtt{I}_{\mathtt{I}KY1}(u,v) =< u_2+u_1^2v_1, u_1u_2+u_1^2v_2 >$
Klir and Yuan 2	$\mathtt{I}_{\mathtt{I}KY1}(u,v) =< v_1.\overline{sg}(1-u_1)+sg(1-u_1).(\overline{sg}(1-v_1)+u_2.sg(1-v_1)),$ $v_2.\overline{sg}(1-u_1)+u_1.sg(1-u_1).sg(1-v_1) >$
Atanassov and Kolev	$\mathtt{I}_{\mathtt{I}AK}(u,v) =< u_2+v_1-u_2v_1, u_1v_2 >$
Atanassov and Trifonov	$\mathtt{I}_{\mathtt{I}AT}(u,v) =< 1-(1-v_1).sg(u_1-v_1)-v_2.\overline{sg}(u_1-v_1).sg(v_2-u_2),$ $v_2.sg(v_2-u_2) >$

Corollary 1.8.5. *[59] IFI has the greatest and least elements $\mathtt{I}_{\mathtt{I}1_{\mathcal{L}}}$ and $\mathtt{I}_{\mathtt{I}0_{\mathcal{L}}}$ respectively given by*

$$\mathtt{I}_{\mathtt{I}1_{\mathcal{L}}}(u,v) = \begin{cases} 1_{\mathcal{L}}, & u <_{\mathcal{L}} 1_{\mathcal{L}} \text{ or } v >_{\mathcal{L}} 0_{\mathcal{L}}, \\ 0_{\mathcal{L}}, & u =_{\mathcal{L}} 1_{\mathcal{L}} \text{ and } v =_{\mathcal{L}} 0_{\mathcal{L}}, \end{cases} \quad \text{and} \quad \mathtt{I}_{\mathtt{I}0_{\mathcal{L}}}(u,v) = \begin{cases} 1_{\mathcal{L}}, & u =_{\mathcal{L}} 0_{\mathcal{L}} \text{ or } v =_{\mathcal{L}} 1_{\mathcal{L}}, \\ 0_{\mathcal{L}}, & u >_{\mathcal{L}} 0_{\mathcal{L}} \text{ and } v <_{\mathcal{L}} 1_{\mathcal{L}}, \end{cases}$$

$\forall\, u,v \in \mathcal{L}$.

Definition 1.8.6. *An IFI $\mathtt{I}_{\mathtt{I}} : \mathcal{L}^2 \to \mathcal{L}$ is said to satisfy*

the left neutrality property (NP) if $\mathtt{I}_{\mathtt{I}}(1_{\mathcal{L}}, v) = v \;\forall\, v \in \mathcal{L}$; (NP)

the ordering property (OP) if $u \leq_{\mathcal{L}} v \;\Leftrightarrow\; \mathtt{I}_{\mathtt{I}}(u,v) = 1_{\mathcal{L}} \;\forall\, u,v \in \mathcal{L}$. (OP)

the identity principle (IP) if $\mathtt{I}_{\mathtt{I}}(u,u) = 1_{\mathcal{L}} \;\forall\, u \in \mathcal{L}$. (IP)

the exchange principle (EP) if $\mathtt{I}_{\mathtt{I}}(u, \mathtt{I}_{\mathtt{I}}(v,w)) = \mathtt{I}_{\mathtt{I}}(v, \mathtt{I}_{\mathtt{I}}(u,w)) \;\forall\, u,v,w \in \mathcal{L}$. (EP)

the left ordering property (LOP) if $u \leq_{\mathcal{L}} v \;\Rightarrow\; \mathtt{I}_{\mathtt{I}}(u,v) = 1_{\mathcal{L}} \;\forall\, u,v \in \mathcal{L}$. (LOP) the right ordering property (ROP) if $\mathtt{I}_{\mathtt{I}}(u, v) = 1_{\mathcal{L}} \Rightarrow u \leq_{\mathcal{L}} v \; \forall \, u, v \in \mathcal{L}$. (ROP)

the consequent boundary (CB) if $v \leq_{\mathcal{L}} \mathtt{I}_{\mathtt{I}}(u, v) \; \forall \, u, v \in \mathcal{L}$. (CB)

the sub-iterative Boolean Law (SIB) if $\mathtt{I}_{\mathtt{I}}(u, \mathtt{I}_{\mathtt{I}}(u, v)) \geq_{\mathcal{L}} \mathtt{I}_{\mathtt{I}}(u, v) \; \forall \, u, v \in \mathcal{L}$. (SIB)

the iterative Boolean Law (IB) if $\mathtt{I}_{\mathtt{I}}(u, \mathtt{I}_{\mathtt{I}}(u, v)) = \mathtt{I}_{\mathtt{I}}(u, v) \; \forall \, u, v \in \mathcal{L}$. (IB)

the strong boundary condition (SBC) for $0_{\mathcal{L}}$, if $u \neq 0_{\mathcal{L}} \Rightarrow \mathtt{I}_{\mathtt{I}}(u, 0_{\mathcal{L}}) = 0_{\mathcal{L}} \; \forall \, u \in \mathcal{L}$. (SBC)

the left boundary condition (LBC) if $\mathtt{I}_{\mathtt{I}}(0_{\mathcal{L}}, v) = 1_{\mathcal{L}} \; \forall \, v \in \mathcal{L}$. (LBC)

the right boundary condition (RBC) if $\mathtt{I}_{\mathtt{I}}(u, 1_{\mathcal{L}}) = 1_{\mathcal{L}} \; \forall \, u \in \mathcal{L}$. (RBC)

Definition 1.8.7. *[59] A function $\mathcal{N} : \mathcal{L} \to \mathcal{L}$ is called an IF negation if*

(N1) $\mathcal{N}(0_{\mathcal{L}}) = 1_{\mathcal{L}}, \mathcal{N}(1_{\mathcal{L}}) = 0_{\mathcal{L}}$,

(N2) $\mathcal{N}$ *is non-increasing.*

Moreover, an IF negation $\mathcal{N}$ *is said to be*

(N3) strict if, in addition,

- $\mathcal{N}$ *is decreasing,*
- $\mathcal{N}$ *is continuous.*

(N4) strong if it is an involution, i.e., $\mathcal{N}(\mathcal{N}(u)) = u, \quad u \in \mathcal{L}$.

(N5) non-vanishing if $\mathcal{N}(u) = 0_{\mathcal{L}} \Leftrightarrow u = 1_{\mathcal{L}}$.

(N6) non-filling if $\mathcal{N}(u) = 1_{\mathcal{L}} \Leftrightarrow u = 0_{\mathcal{L}}$.

The standard negation is denoted by $\mathcal{N}_s$ and defined by $\mathcal{N}_s(u_1, u_2) = (u_2, u_1) \; \forall \, (u_1, u_2) \in \mathcal{L}$.

Definition 1.8.8. *[45] A function $\mathcal{T} : \mathcal{L}^2 \to \mathcal{L}$ is called a triangular norm (t-norm) if for all $u, v, w \in \mathcal{L}$, it satisfies the following conditions:*

$$\mathcal{T}(u, v) = \mathcal{T}(v, u) \tag{T1}$$

$$\mathcal{T}(u, \mathcal{T}(v, w)) = \mathcal{T}(\mathcal{T}(u, v), w) \tag{T2}$$

$$v <_{\mathcal{L}} w \Rightarrow \mathcal{T}(u, v) \leq_{\mathcal{L}} \mathcal{T}(u, w), i.e., \mathcal{T}(u, .) \text{ is non-decreasing} \tag{T3}$$

$$\mathcal{T}(u, 1_{\mathcal{L}}) = u \tag{T4}$$

Example 1.8.9. *(i) The function* $\mathcal{T}_M : \mathcal{L}^2 \to \mathcal{L}$ *given by*

$$\mathcal{T}_M(u, v) = (\min(u_1, v_1), \max(u_1, v_1)) \; \forall \, u = (u_1, u_2), v = (v_1, v_2) \in \mathcal{L}$$

is a t-norm.

(ii) The function $\mathcal{T} : \mathcal{L}^2 \to \mathcal{L}$ *given by*

$$\mathcal{T}(u, v) = (u_1 v_1, 1 - (1 - u_2)(1 - v_2)) \; \forall \, u = (u_1, u_2), v = (v_1, v_2) \in \mathcal{L}$$

is a t-norm.

(iii) The function $\mathcal{T} : \mathcal{L}^2 \to \mathcal{L}$ *given by*

$$\mathcal{T}(u, v) = (\max(0, u_1 + v_1 - 1), \min(1, u_2 + 1 - v_1, v_2 + 1 - u_1)) \; \forall \, u = (u_1, u_2), v = (v_1, v_2) \in \mathcal{L}$$

is a t-norm (see [62]).

Definition 1.8.10. *[45] A function* $\mathcal{S} : \mathcal{L}^2 \to \mathcal{L}$ *is called a triangular conorm (t-conorm) if for all* $u, v, w \in \mathcal{L}$*, it satisfies the following conditions:*

$$\mathcal{S}(u, v) = \mathcal{S}(v, u) \tag{S1}$$

$$\mathcal{S}(u, \mathcal{S}(v, w)) = \mathcal{S}(\mathcal{S}(u, v), w) \tag{S2}$$

$$v <_{\mathcal{L}} w \;\Rightarrow\; \mathcal{S}(u, v) \leq_{\mathcal{L}} \mathcal{S}(u, w), i.e., \mathcal{S}(u, .) \text{ is non-decreasing} \tag{S3}$$

$$\mathcal{S}(u, 0_{\mathcal{L}}) = u \tag{S4}$$

Example 1.8.11. *(i) The function* $\mathcal{S}_M : \mathcal{L}^2 \to \mathcal{L}$ *given by*

$$\mathcal{S}_M(u, v) = (\max(u_1, v_1), \min(u_1, v_1)) \; \forall \, u = (u_1, u_2), v = (v_1, v_2) \in \mathcal{L}$$

is a t-conorm.

(ii) The function $\mathcal{S} : \mathcal{L}^2 \to \mathcal{L}$ *given by*

$$\mathcal{S}(u, v) = (u_1 v_1, u_2 + v_2 - u_2 v_2) \; \forall \, u = (u_1, u_2), v = (v_1, v_2) \in \mathcal{L}$$

is a t-conorm.

(iii) The function $\mathcal{S} : \mathcal{L}^2 \to \mathcal{L}$ *given by*

$$\mathcal{S}(u, v) = (\min(1, u_1 + 1 - v_2, v_1 + 1 - u_2), \max(0, u_2 + v_2 - 1) \; \forall \, u = (u_1, u_2), v = (v_1, v_2) \in \mathcal{L}$$

is a t-conorm (see [62]).

Definition 1.8.12. *[45]*

(A). A t-norm $\mathcal{T}$ on $\mathcal{L}$ is called t-representable if there exists a t-norm T and a t-conorm S on [0, 1] such that

$$\mathcal{T}(u,v) = (T(u_1,v_1), S(u_2,v_2)) \text{ for } u = (u_1,u_2), v = (v_1,v_2) \in \mathcal{L}.$$

T and S are called the representants of $\mathcal{T}$. We write $\mathcal{T} = (T,S)$.

(B). A t-conorm $\mathcal{S}$ on $\mathcal{L}$ is called t-representable if there exists a t-norm T' and a t-conorm S' on [0, 1] such that

$$\mathcal{S}(u,v) = (S'(u_1,v_1), T'(u_2,v_2)) \text{ for } u = (u_1,u_2), v = (v_1,v_2) \in \mathcal{L}.$$

T' and S' are called the representants of $\mathcal{S}$. We write $\mathcal{S} = (S',T')$.

Example 1.8.13. *(i) Consider the functions $\mathcal{T} : \mathcal{L}^2 \to \mathcal{L}$ and $\mathcal{S} : \mathcal{L}^2 \to \mathcal{L}$ generated by the t-norm and t-conorm on [0, 1]. Then the mappings $\mathcal{T}$ and $\mathtt{S}$ given by*

$$\mathcal{T}(u,v) = (\min(u_1,v_1), \max(u_2,v_2)) \text{ and } \mathcal{S}(u,v) = (1-(1-u_2)(1-v_2), u_1v_1)$$

for $u = (u_1,u_2), v = (v_1,v_2) \in \mathcal{L}$ *are t-representable.*

(ii) On the other hand, t-norm $\mathcal{T}$ given in Example 1.8.9 (iii) and t-conorm $\mathcal{S}$ given in Example 1.8.11 (iii) are not representable, though both of them are continuous and Archimedean.

Definition 1.8.14. *[45] A function $\mathtt{I}_\mathtt{I} : \mathcal{L}^2 \to \mathcal{L}$ is called an $\mathcal{R}$-implication if there exists a t-norm $\mathcal{T}$ such that*

$$\mathtt{I}_\mathtt{I}(u,v) = \sup\{\gamma \in \mathcal{L} : \mathcal{T}(u,\gamma) \leq_{\mathcal{L}} v\} \; \forall \; u,v \in \mathcal{L}.$$

If $\mathtt{I}_\mathtt{I}$ is an $\mathcal{R}$-implication generated by a t-norm $\mathcal{T}$, then it is denoted by $\mathtt{I}_{\mathtt{I}\mathcal{T}}$.

Definition 1.8.15. *[45] A function $\mathtt{I}_\mathtt{I} : \mathcal{L}^2 \to \mathcal{L}$ is called an $(\mathcal{S},\mathcal{N})$-implication if there exist a t-conorm $\mathcal{S}$ and an IF negation $\mathcal{N}$ such that*

$$\mathtt{I}_\mathtt{I}(u,v) = \mathcal{S}(\mathcal{N}(u),v) \; \forall \; u,v \in \mathcal{L}.$$

If $\mathcal{N}$ is a strong IF negation, then $\mathtt{I}_\mathtt{I}$ is called a strong implication or $\mathcal{S}$-implication. Moreover, if $\mathtt{I}_\mathtt{I}$ is an $(\mathcal{S},\mathcal{N})$-implication generated by $\mathcal{S}$ and $\mathcal{N}$, then it is denoted by $\mathtt{I}_{\mathtt{I}(\mathcal{S},\mathcal{N})}$.

1.9 Organization of Thesis

This thesis is organized as follows. It consists of eight chapters. The chapter-wise summary is as follows:

The current chapter gives the literature survey related to FS and IFS theory, FS and IFS optimization problems, decision making-problems in fuzzy and IF environments, fuzzy and IFIs.

In **Chapter 2**, the product of unrestricted LR-type IFNs based on the α-cut, β-cut and (α, β)-cut is proposed. Then with the help of the proposed product, a new method is proposed to find the optimal solutions of unrestricted LR-type IFLPPs. A test example is given to support the proposed method and investigated the applicability of existing approaches.

In **Chapter 3**, we introduce a pair of primal-dual LPPs in IFE and prove duality results in IFE by using an aspiration level approach in which membership and non-membership functions are taken in the form of reference functions. Since the fuzzy environment and IFE cause the duality gap, we propose to investigate the impact of membership function governed by reference function on duality gap. This is specially meaningful for fuzzy and IF programming problems, when the primal and dual objective values may not be bounded. Finally, the duality gap obtained by the approach has been compared with the duality gap obtained by existing approaches.

Chapter 4 investigates a new approach for finding efficient solutions of the multi-objective optimization problem (MOOP) in IFE based on DM's different views, viz., optimistic, pessimistic and mixed. The point operator F_α, which transforms IFS into equivalent FS, is introduced and some desirable properties of F_α are studied. The formulation of MOOP, accuracy index and value function in IFE are introduced. For resolving the mutual conflicting nature of objectives in MOOP in IFE, we introduce the membership and non-membership functions governed by reference function which do not depend on the upper and lower levels of acceptability. Then a new method is proposed to find the efficient solutions of MOOP in IFE based on different viewpoints. Finally, a test example is given to demonstrate the practicality and effectiveness of the proposed method.

Chapter 5 considers some information measures, such as, normalized divergence measure, similarity measure, dissimilarity measure and normalized distance measure in IFE, which measure the uncertainty and hesitancy, and which can be applied to the selection of alternatives in group decision problems. We introduce and study the continuity of considered measures. Next, we prove some results that can be used to generate measures for FSs as well as for IFSs and we also prove some approaches to construct point measures from set measures in IFE. We define the weight set for one and many preference orders of alternatives. We investigate the properties and results related to the weight set. Based on the weight set, we develop the model for finding the uncertain weights corresponding to attribute. Also, we develop the model to finding positive certain weights corresponding to each attribute by using uncertain weights. Finally, an algorithm for choosing the best alternative according to the preference orders of alternatives in decision making problems is proposed and its validity is shown with the help of numerical example.

In **Chapter 6**, the powers of a t-norm $\mathcal{T}$ with identical tuple elements on $\mathcal{L}$ are introduced and their properties are studied. More specifically, a new type of implication on $\mathcal{L}$, known as the residual implication is derived from powers of continuous t-norm $\mathcal{T}$, which is denoted by $\mathtt{I}_{\mathtt{I}\mathcal{T}}$ and satisfies certain properties of residual implications by imposing some extra conditions. Moreover, some additional important properties are studied and analyzed. These altogether reveal that they do not intersect the most well-known classes of fuzzy implications. Finally, we investigate the solutions of Boolean-like laws in $\mathtt{I}_{\mathtt{I}\mathcal{T}}$.

In **Chapter 7**, a new class of IFIs known as $(\mathtt{f}_{\mathtt{I}}, \omega)$-implications is introduced which is a generalized form of Yagers f-implications in IFE. Basic properties of these implications are discussed in detail. It is shown that $(\mathtt{f}_{\mathtt{I}}, \omega)$-implications are not only the generalizations of Yagers f-implications, but also the generalizations of $\mathcal{R}$-, $(\mathcal{S}, \mathtt{N})$- and $\mathcal{QL}$-implications in IFE. The distributive equations $\mathtt{I}_{\mathtt{I}}(\mathcal{T}(u, v), w) = \mathcal{S}(\mathtt{I}_{\mathtt{I}}(u, w), \mathtt{I}_{\mathtt{I}}(v, w))$ and $\mathtt{I}_{\mathtt{I}}(u, \mathcal{T}_1(v, w)) = \mathcal{T}_2(\mathtt{I}_{\mathtt{I}}(u, v), \mathtt{I}_{\mathtt{I}}(u, w))$ over t-representable t-norms and t-conorms generated from nilpotent and strict t-norms in IF set theory are discussed.

Finally, in **Chapter 8**, conclusions are drawn based on the present study and future research work is suggested in this direction.

Chapter 2

Unrestricted LR-type intuitionistic fuzzy mathematical programming problems

In this chapter, the product of unrestricted LR-type IFNs based on the α-cut, β-cut and (α, β)-cut is proposed. Then with the help of the proposed product, a new method is proposed to find the optimal solutions of unrestricted LR-type IFLPPs. A test example is given to support the proposed method and investigated the applicability of existing approaches.

2.1 Introduction

In today's highly competitive market, the pressure on an organization is to find better ways to attain the optimal solution. In conventional optimization problems, it is assumed that the DM is sure about the precise values of data involved in the model. However, in real-world applications, all the parameters of the optimization problems may not be known precisely due to uncontrollable factors. Such type of imprecise data is well represented by a fuzzy number introduced by Zadeh [194]. Zimmermann [203] showed that the solutions of fuzzy linear programming problems (FLPPs) are always efficient. Several researchers have developed different types of methods for solving FLPPs [71], fully fuzzy linear programming problems (FFLPPs) ([105], [108]) and LR-type FFLPPs ([75], [104]).

In real life, a person may assume that an object belongs to a set to a certain degree, but it is possible that he/she is not sure about it. In other words, there may be hesitation or uncertainty about the membership degree. The main meaning is that the parameters' demand across the problem is uncertain. However, they are known to fall within a prescribed uncertainty set with some attributed degrees. In FS theory, there is no means to incorporate the hesitation in the membership degree. To incorporate the hesitation in the membership degree, IFSs proposed by Atanassov [11] can be used. It is an extension of the FS theory. Gau and Buehrer[74] introduced the concept of vague set. Bustince and Burillo [37] proved that vague sets and IFSs are the same. IFS is playing an important role in decision making under uncertainty and has gained popularity in recent years. It helps more adequately to represent situations where DMs abstain from expressing their assessments. In this way, IFSs provide a richer tool to grasp impression and ambiguity than the conventional FSs. These characteristics of IFSs led to the extension of optimization methods in IFE. An application of the IFSs to optimization problems is introduced by Angelov[9]. His technique is based on maximizing the degree of membership, minimizing the degree of non-membership and the crisp model is formulated using the IF aggregation operator. The application of IFSs in medical diagnosis is given by De et al. [51]. Mahapatra and Roy [127] introduced arithmetic operations on TIFNs and studied reliability evaluation using TIFNs. The modeling and optimization of the multi-objective non-linear programming problem in IFE are discussed in [163] with the usual approach and in [153] with different approaches such as optimistic and pessimistic approaches. Subsequently, several researchers have solved optimization problems in IFE such as matrix game with IF payoffs by using non-linear mathematical programming approach [112], IFLPP by a new method [137], IFLPPs by using the ranking function approach [165], multi-level programming problems by interactive IF methods[199], finding Pareto-optimal solutions to multi-objective programming problems by an IF goal programming approach [156].

The rest of the chapter is organized as follows: In Section 2.2, some basic definitions and arithmetic operations on LR-type IFNs are presented. In Section 2.3, the product of unrestricted LR-type IFNs is introduced. In Section 2.4, a new method is proposed to find the fuzzy optimal solution of LR-type IFLPPs. In Section 2.5, an illustrative example is given to support the proposed method and the managerial insights of this problem are discussed. In Section 2.6, the advantages of the proposed method over the existing methods are given. Concluding remarks are drawn in Section 2.7.

2.2 Basic definitions and arithmetic operations

Definition 2.2.1. *Let* $\tilde{A}^I = \{(x, \mu_{\tilde{A}^I}(x), \nu_{\tilde{A}^I}(x)) : x \in \mathbb{R}\}$ *be an IFN. Then the operators* $C_\mu : \mathcal{I}(\mathbb{R}) \to \mathbb{R}^+$ *and* $C_\nu : \mathcal{I}(\mathbb{R}) \to \mathbb{R}^+$ *of an* $\tilde{A}^I$ *are defined by*

$$C_\mu(\tilde{A}^I) = \int_{m-l}^{m} \mu_{\tilde{A}^I}(x)dx + \frac{1}{2}\int_{m-l}^{m+r} \mu_{\tilde{A}^I}(x)dx \tag{2.1}$$

$$C_\nu(\tilde{A}^I) = \int_{m-l'}^{m} \nu_{\tilde{A}^I}(x)dx + \frac{1}{2}\int_{m-l'}^{m+r'} \nu_{\tilde{A}^I}(x)dx \tag{2.2}$$

Lemma 2.2.2. *Let* $A^I_\alpha = [A^I_{L\alpha}, A^I_{R\alpha}]$, $\alpha \in (0,1]$, *and* $A^I_\beta = [A^I_{L\beta}, A^I_{R\beta}]$, $\beta \in [0,1)$, *be the* α*-cut and* β*-cut respectively of an IFN* $\tilde{A}^I = \{(x, \mu_{\tilde{A}^I}(x), \nu_{\tilde{A}^I}(x)) : x \in \mathbb{R}\}$. *Then*

$$C_\mu(\tilde{A}^I) = \int_0^1 A^I_{L\alpha}d\alpha + \frac{1}{2}\int_0^1 (A^I_{R\alpha} - A^I_{L\alpha})d\alpha \tag{2.3}$$

$$C_\nu(\tilde{A}^I) = \int_0^1 A^I_{L\beta}d\beta + \frac{1}{2}\int_0^1 (A^I_{R\beta} - A^I_{L\beta})d\beta \tag{2.4}$$

Proof. There are several approaches to prove (2.3) and (2.4). The simplest way is assessing the area of the curves $A^I_{L\alpha}$, $A^I_{L\beta}$ and between the curves $A^I_{L\alpha}$ and $A^I_{R\alpha}$; $A^I_{L\beta}$ and $A^I_{R\beta}$.

Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be piecewise continuous functions on a segment $[a, b]$ such that $f(x) \geq g(x)\ \forall\, x \in [a, b]$. Then the area of the curve $y = f(x)$ and the area between curves $y = f(x)$ and $y = g(x)$ (according to classical results of math analysis, e.g., [19]) are

$$S(f) = \int_a^b f(x)dx \ \text{ and } \ S(f-g) = \int_a^b (f(x) - g(x))dx \tag{2.5}$$

respectively.

Determination of the area under the curve $\mu_{\tilde{A}^I}(x)$, $x \in (m-l, m)$, and half the area under the curve $\mu_{\tilde{A}^I}(x)$, $x \in (m-l, m+r)$, according to (2.1), includes all the possible curves for values $\mu_{\tilde{A}^I}(m-l)$, $\mu_{\tilde{A}^I}(m+r)$, e.g., if $\mu_{\tilde{A}^I}(m+r) = \alpha_2 > 0$. For the latest case,

$$A^I_{R\alpha} = m + r, \quad \alpha \in [0, \alpha_2]$$

Let function $\alpha = \mu_{\tilde{A}^I}(x)$ has a discontinuous in the point x_0. According to the property of $\mu_{\tilde{A}^I}(x)$, $\mu_{\tilde{A}^I}(x)$ is a piecewise upper continuous function, therefore

$$\alpha_0 = \mu_{\tilde{A}^I}(x_0) = \max\left(\alpha_1 = \lim_{x \to x_0^-} \mu_{\tilde{A}^I}(x), \alpha_2 = \lim_{x \to x_0^+} \mu_{\tilde{A}^I}(x)\right).$$

Assume $\alpha_0 = \alpha_1$, then (as $\mu_{\tilde{A}^I}(x)$ is upper-continuous) $A^I_{R\alpha} = x_0$ for $\alpha \in (\alpha_2, \alpha_1]$. Thus, as in segment in [0, 1], $A^I_{L\alpha}$ is a non-decreasing piecewise lower-continuous function, and $A^I_{R\alpha}$ is a non-increasing piecewise upper-continuous function. The sum of the areas, i.e., the area of the curve $x = A^I_{L\alpha}$ and the half of area between the curves $x = A^I_{R\alpha}$ and $x = A^I_{L\alpha}$ are assessed based on the approach (2.5) is the required result (2.3).

Similarly, we can prove the result (2.4). □

Definition 2.2.3. *Let $\tilde{A}^I \in \mathcal{I}(\mathbb{R})$. Then the score and accuracy indices of $\tilde{A}^I$ are denoted by $I_S(\tilde{A}^I)$ and $I_A(\tilde{A}^I)$ respectively and are defined by*

$$I_S(\tilde{A}^I) = |C_\mu(\tilde{A}^I) - C_\nu(\tilde{A}^I)|, \;\; I_A(\tilde{A}^I) = C_\mu(\tilde{A}^I) + C_\nu(\tilde{A}^I).$$

Definition 2.2.4. *An LR-type IFN $\tilde{A}^I = (m; l, r; l', r')_{LR}$ is called LR-type unrestricted IFN if m is any real number.*

Definition 2.2.5. *An LR-type IFN $\tilde{A}^I = (m; l, r; l', r')_{LR}$ is called non-positive if $m + r' \leq 0$ and non-negative if $m - l' \geq 0$.*

Definition 2.2.6. *An LR-type IFN $\tilde{A}^I = (m; l, r; l', r')_{LR}$ is called negative if $m + r' < 0$ and positive if $m - l' > 0$.*

Theorem 2.2.7. *Let $\tilde{A}^I = (a, b, c; a', b, c')$ be a TIFN. Then its (i) $\alpha-$cut A^I_α, (ii) $\beta-$cut $A^I_{(\beta)}$ are given by*

$$A^I_\alpha = [a + (b-a)\alpha, c - (c-b)\alpha] \tag{2.6}$$

$$A^I_{(\beta)} = [b - (b-a')\beta, b + (c'-b)\beta] \tag{2.7}$$

for $\alpha \in (0, 1], \beta \in [0, 1)$.

Proof. (i). For $\alpha \in (0, 1]$,

$$\begin{aligned} \mu_{\tilde{A}^I}(x) \geq \alpha \;\; &\Rightarrow \;\frac{x-a}{b-a} \geq \alpha, \frac{c-x}{c-b} \geq \alpha \\ &\Rightarrow \; x \geq a + (b-a)\alpha, \; x \leq c - (c-b)\alpha \\ &\Rightarrow \; a + (b-a)\alpha \leq x \leq c - (c-b)\alpha \end{aligned}$$

Therefore,

$$A^I_\alpha = [a + (b-a)\alpha, c - (c-b)\alpha].$$

(ii). Now, for $\beta \in [0, 1)$,

$$\begin{aligned} \nu_{\tilde{A}^I}(x) \leq \beta \;\; &\Rightarrow \;\frac{b-x}{b-a'} \leq \beta, \frac{x-b}{c'-b} \leq \beta \\ &\Rightarrow \; x \geq b - (b-a')\beta, \; x \leq b + (c'-b)\beta \\ &\Rightarrow \; b - (b-a')\beta \leq x \leq b + (c'-b)\beta \end{aligned}$$

Therefore,

$$A^I_{(\beta)} = [b - (b-a')\beta, b + (c'-b)\beta].$$

$\square$

Theorem 2.2.8. *Let $\tilde{A}^I = (a, b, c; a', b, c')$ be a TIFN. Then its $(\alpha, \beta)-$cut $A^I_{(\alpha,\beta)}$ is given by*

$$A^I_{(\alpha,\beta)} = [a + (b - a)\alpha, c - (c - a)\alpha] \cap [b - (b - a')\beta, b + (c' - b)\beta] \tag{2.8}$$

for $\alpha, \beta \in (0, 1),\ \alpha + \beta \leq 1$.

Proof. For $\alpha \in (0, 1]$, the α-cut of $\tilde{A}^I$ is given by (see (2.6))

$$A^I_\alpha = [a + (b - a)\alpha, c - (c - b)\alpha].$$

For $\beta \in [0, 1), \alpha + \beta \leq 1$, the β-cut of $\tilde{A}^I$ is given by (see (2.7))

$$A^I_{(\beta)} = [b - (b - a')\beta, b + (c' - b)\beta].$$

Therefore, by Definition 1.2.8,

$$A^I_{(\alpha,\beta)} = A^I_\alpha \cap A^I_{(\beta)} = [a + (b - a)\alpha, c - (c - a)\alpha] \cap [b - (b - a')\beta, b + (c' - b)\beta]$$

for $\alpha, \beta \in (0, 1),\ \alpha + \beta \leq 1$. □

Theorem 2.2.9. *Let $\tilde{A}^I = (m; l, r; l', r')_{LR}$ be an LR-type IFN. Then its (i) $\alpha-$cut A^I_α, $\beta-$cut $A^I_{(\beta)}$ are given by*

$$A^I_\alpha = [m - lL^{-1}(\alpha), m + rR^{-1}(\alpha)], \tag{2.9}$$

$$A^I_{(\beta)} = [m - l'L^{-1}(1 - \beta), m + r'R^{-1}(1 - \beta)] \tag{2.10}$$

for $\alpha \in (0, 1], \beta \in [0, 1)$.

Proof. (i) For $\alpha \in (0, 1]$,

$$\begin{aligned}
\mu_{\tilde{A}^I}(x) \geq \alpha &\Rightarrow L\Big(\frac{m - x}{l}\Big) \geq \alpha, R\Big(\frac{x - m}{r}\Big) \geq \alpha \\
&\Rightarrow \frac{m - x}{l} \leq L^{-1}(\alpha), \frac{x - m}{r} \leq R^{-1}(\alpha) \quad (\because \text{ L and R are decreasing functions}) \\
&\Rightarrow x \geq m - lL^{-1}(\alpha),\ x \leq m + rR^{-1}(\alpha) \\
&\Rightarrow m - lL^{-1}(\alpha) \leq x \leq m + rR^{-1}(\alpha)
\end{aligned}$$

Therefore,

$$A^I_\alpha = [m - lL^{-1}(\alpha), m + rR^{-1}(\alpha)]. \tag{2.11}$$

(ii) Now, for $\beta \in [0, 1)$ and $\alpha + \beta \leq 1$,

$$\begin{aligned}
\nu_{\tilde{A}^I}(x) \leq \beta &\Rightarrow 1 - L\Big(\frac{m - x}{l'}\Big) \leq \beta, 1 - R\Big(\frac{x - m}{r'}\Big) \leq \beta \\
&\Rightarrow x \geq m - l'L^{-1}(1 - \beta),\ x \leq m + rR^{-1}(1 - \beta) \\
&\Rightarrow m - l'L^{-1}(1 - \beta) \leq x \leq m + rR^{-1}(1 - \beta)
\end{aligned}$$

Therefore,

$$A^I_{(\beta)} = [m - l'L^{-1}(1-\beta), m + r'R^{-1}(1-\beta)]. \tag{2.12}$$

□

Theorem 2.2.10. *Let* $\tilde{A}^I = (m; l, r; l', r')_{LR}$ *be an LR-type IFN. Then its* $(\alpha, \beta)-$*cut* $A^I_{(\alpha,\beta)}$ *is given by*

$$A^I_{(\alpha,\beta)} = [m - lL^{-1}(\alpha), m + rR^{-1}(\alpha)] \cap [m - l'L^{-1}(1-\beta), m + r'R^{-1}(1-\beta)] \tag{2.13}$$

for $\alpha, \beta \in (0, 1),\ \alpha + \beta \leq 1$.

Proof. For $\alpha \in (0, 1]$, the α-cut of $\tilde{A}^I$ is given by (see (2.9))

$$A^I_{\alpha} = [m - lL^{-1}(\alpha), m + rR^{-1}(\alpha)]$$

For $\beta \in [0, 1), \alpha + \beta \leq 1$, the β-cut of $\tilde{A}^I$ is given by (see (2.10))

$$A^I_{(\beta)} = [m - l'L^{-1}(1-\beta), m + r'R^{-1}(1-\beta)].$$

Therefore, by Definition 1.2.8,

$$A^I_{(\alpha,\beta)} = A^I_{\alpha} \cap A^I_{(\beta)} = [m - lL^{-1}(\alpha), m + rR^{-1}(\alpha)] \cap [m - l'L^{-1}(1-\beta), m + r'R^{-1}(1-\beta)]$$

for $\alpha, \beta \in (0, 1),\ \alpha + \beta \leq 1$. □

Remark 2.2.11. *Let* $\tilde{A}^I$=$(m; l, r; l', r')_{LR}$ *be an LR-type IFN. Then, based on Definition 2.2.3, the score and accuracy indices of* $\tilde{A}^I$ *are denoted by* $I_S(\tilde{A}^I)$ *and* $I_A(\tilde{A}^I)$*, respectively and are given by*

$$\begin{aligned} I_S(\tilde{A}^I) =& \Big| \frac{1}{2}\int_0^1 \{(m - lL^{-1}(\alpha)) + (m + rR^{-1}(\alpha))\} d\alpha \\ & - \frac{1}{2}\int_0^1 \{(m - l'L^{-1}(1-\beta)) + (m + r'R^{-1}(1-\beta))\} d\beta \Big| \end{aligned}$$

and

$$\begin{aligned} I_A(\tilde{A}^I) =& \frac{1}{2}\int_0^1 \{(m - lL^{-1}(\alpha)) + (m + rR^{-1}(\alpha))\} d\alpha \\ & + \frac{1}{2}\int_0^1 \{(m - l'L^{-1}(1-\beta)) + (m + r'R^{-1}(1-\beta))\} d\beta. \end{aligned}$$

Theorem 2.2.12. *(Score index of LR-type TIFN) Let* $\tilde{A}^I$=$(b; b-a, c-b; b-a', c'-b)_{LR}$ *be an LR-type TIFN. Then*

$$I_S(\tilde{A}^I) = \Big| \frac{(a+c) - (a'+c')}{4} \Big|.$$

Proof. By Remark 2.2.11,

$$I_S(\tilde{A}^I) = \Big|\frac{1}{2}\int_0^1 \{(b-(b-a)L^{-1}(\alpha)) + (b+(c-b)R^{-1}(\alpha))\}d\alpha$$
$$-\frac{1}{2}\int_0^1 \{(b-(b-a')L^{-1}(1-\beta)) + (b+(c'-b)R^{-1}(1-\beta))\}d\beta\Big| \tag{2.14}$$

By Definition 1.2.19 for a TIFN,

$$L(x) = R(x) = \max\{0, 1-x\}\ \forall\ x \geq 0.$$

Therefore,

$$L(\alpha) = 1-\alpha \qquad (\because 0 < \alpha \leq 1)$$
$$\Rightarrow L^{-1}(\alpha) = 1-\alpha \tag{2.15}$$

Similarly,

$$L^{-1}(1-\beta) = \beta,\ R^{-1}(\alpha) = 1-\alpha,\ R^{-1}(1-\beta) = \beta \tag{2.16}$$

Using (2.15) and (2.16) in (2.14), we have

$$I_S(\tilde{A}^I) = \Big|\frac{1}{2}\int_0^1 \{(b-(b-a)(1-\alpha)) + (b+(c-b)(1-\alpha))\}d\alpha$$
$$-\frac{1}{2}\int_0^1 \{(b-(b-a')\beta) + (b+(c'-b)\beta)\}d\beta\Big|$$
$$= \Big|\frac{(a+c)-(a'+c')}{4}\Big|.$$

□

Theorem 2.2.13. *(Accuracy index of LR-type TIFN) Let* $\tilde{A}^I$=$(b; b-a, c-b; b-a', c'-b)_{LR}$ *be an LR-type TIFN. Then*

$$I_A(\tilde{A}^I) = \frac{(a+c+4b+a'+c')}{4}.$$

Proof. By Remark 2.2.11,

$$I_A(\tilde{A}^I) = \frac{1}{2}\int_0^1 \{(b-(b-a)L^{-1}(\alpha)) + (b+(c-b)R^{-1}(\alpha))\}d\alpha$$
$$+\frac{1}{2}\int_0^1 \{(b-(b-a')L^{-1}(1-\beta)) + (b+(c'-b)R^{-1}(1-\beta))\}d\beta. \tag{2.17}$$

Using (2.15) and (2.16) in (2.17), we have

$$\begin{aligned} I_A(\tilde{A}^I) &= \frac{1}{2}\int_0^1 \{(b-(b-a)(1-\alpha)) + (b+(c-b)(1-\alpha))\}d\alpha \\ &+ \frac{1}{2}\int_0^1 \{(b-(b-a')\beta) + (b+(c'-b)\beta)\}d\beta \\ &= \frac{(a+c+4b+a'+c')}{4}. \end{aligned}$$

□

Definition 2.2.14. *Let* $\tilde{A}_1^I=(m_1;l_1,r_1;l_1',r_1')_{LR}$ *and* $\tilde{A}_2^I=(m_2;l_2,r_2;l_2',r_2')_{LR}$ *be two LR-type IFNs.*

(A) *Then*

(i) $\tilde{A}_1^I$ *is defined as less than* $\tilde{A}_2^I$, *written as* $\tilde{A}_1^I \prec \tilde{A}_2^I$, *if* $I_S(\tilde{A}_1^I) < I_S(\tilde{A}_2^I)$,

(ii) $\tilde{A}_1^I$ *is defined as greater than* $\tilde{A}_2^I$, *written as* $\tilde{A}_1^I \succ \tilde{A}_2^I$, *if* $I_S(\tilde{A}_1^I) > I_S(\tilde{A}_2^I)$.

(B) *Let* $I_S(\tilde{A}_1^I) = I_S(\tilde{A}_2^I)$. *Then*

(i) $\tilde{A}_1^I$ *is defined as less than* $\tilde{A}_2^I$, *written as* $\tilde{A}_1^I \prec \tilde{A}_2^I$, *if* $I_A(\tilde{A}_1^I) < I_A(\tilde{A}_2^I)$,

(ii) $\tilde{A}_1^I$ *is defined as greater than* $\tilde{A}_2^I$, *written as* $\tilde{A}_1^I \succ \tilde{A}_2^I$, *if* $I_A(\tilde{A}_1^I) > I_A(\tilde{A}_2^I)$,

(iii) $\tilde{A}_1^I$ *is defined as equal to* $\tilde{A}_2^I$, *written as* $\tilde{A}_1^I \approx \tilde{A}_2^I$, *if* $I_A(\tilde{A}_1^I) = I_A(\tilde{A}_2^I)$.

2.2.1 Arithmetic Operations on LR-type IFNs

In this subsection, the arithmetic operations on LR-type IFNs are presented.

Proposition 2.2.15. *Let* $\tilde{A}_1^I=(m_1;l_1,r_1;l_1',r_1')_{LR}$ *and* $\tilde{A}_2^I=(m_2;l_2,r_2;l_2',r_2')_{LR}$ *be two LR-type IFNs. Then*

$$\tilde{A}_1^I \oplus \tilde{A}_2^I = (m_1+m_2;l_1+l_2,r_1+r_2;l_1'+l_2',r_1'+r_2')_{LR},$$

where $0 < l_1+l_2 \le l_1'+l_2'$ *and* $0 < r_1+r_2 \le r_1'+r_2'$.

Proof. The α and β-cuts of $\tilde{A}_1^I$ and $\tilde{A}_2^I$ are given by

$$A_{1\alpha}^I = [m_1 - l_1L^{-1}(\alpha), m_1 + r_1R^{-1}(\alpha)],\ A_{2\alpha}^I = [m_2 - l_2L^{-1}(\alpha), m_2 + r_2R^{-1}(\alpha)] \tag{2.18}$$

$$A_{1(\beta)}^I = [m_1 - l_1'L^{-1}(1-\beta), m_1 + r_1'R^{-1}(1-\beta)],\ A_{2(\beta)}^I = [m_2 - l_2'L^{-1}(1-\beta), m_2 + r_2'R^{-1}(1-\beta)] \tag{2.19}$$

respectively.

From (2.18), we get

$$(\tilde{A}_1^I \oplus \tilde{A}_2^I)_\alpha = A_{1\alpha}^I + A_{2\alpha}^I = [m_1 - l_1 L^{-1}(\alpha), m_1 + r_1 R^{-1}(\alpha)] + [m_2 - l_2 L^{-1}(\alpha), m_2 + r_2 R^{-1}(\alpha)]$$
$$= [m_1 + m_2 - (l_1 + l_2) L^{-1}(\alpha), m_1 + m_2 + (r_1 + r_2) R^{-1}(\alpha)] \quad (2.20)$$

Since L and R are decreasing functions on $[0, \infty)$ with $L(0) = R(0) = 1$, $\exists$ a unique $\alpha_0 \in (0, 1]$ such that $L^{-1}(\alpha_0) = R^{-1}(\alpha_0) = 1$. Hence

$$(\tilde{A}_1^I \oplus \tilde{A}_2^I)_{\alpha_0} = [m_1 + m_2 - (l_1 + l_2), m_1 + m_2 + r_1 + r_2] \quad (2.21)$$

(2.21) gives left and right spreads of membership function of $\tilde{A}_1^I \oplus \tilde{A}_2^I$ are $l_1 + l_2$ and $r_1 + r_2$ respectively.

Putting $\alpha = 1$ in (2.20), we obtain the modal point of $\tilde{A}_1^I \oplus \tilde{A}_2^I$ given by

$$(\tilde{A}_1^I \oplus \tilde{A}_2^I)_1 = [m_1 + m_2, m_1 + m_2] = m_1 + m_2 \quad (2.22)$$

From (2.19), we get

$$(\tilde{A}_1^I \oplus \tilde{A}_2^I)_{(\beta)} = A_{1(\beta)}^I + A_{2(\beta)}^I = [m_1 - l_1' L^{-1}(1-\beta), m_1 + r_1' R^{-1}(1-\beta)]$$
$$+ [m_2 - l_2' L^{-1}(1-\beta), m_2 + r_2' R^{-1}(1-\beta)]$$
$$= [m_1 + m_2 - (l_1' + l_2') L^{-1}(1-\beta), m_1 + m_2 + (r_1' + r_2') R^{-1}(1-\beta)].$$

Since L and R are decreasing functions on $[0, \infty)$ with $L(0) = R(0) = 1$, $\exists$ a unique $\beta_0 \in [0, 1)$ such that $L^{-1}(1-\beta_0) = R^{-1}(1-\beta_0) = 1$. Hence

$$(\tilde{A}_1^I \oplus \tilde{A}_2^I)_{(\beta_0)} = [m_1 + m_2 - l_1' - l_2', m_1 + m_2 + r_1' + r_2'] \quad (2.23)$$

(2.23) gives left and right spreads of non-membership function of $\tilde{A}_1^I \oplus \tilde{A}_2^I$ are $l_1' + l_2'$ and $r_1' + r_2'$ respectively.

Since $\tilde{A}_1^I$ and $\tilde{A}_2^I$ are an LR-type IFNs,

$$0 < l_1 \leq l_1', 0 \leq l_2 \leq l_2', 0 < r_1 \leq r_1', 0 < r_2 \leq r_2'.$$

Thus, $0 < l_1 + l_2 \leq l_1' + l_2'$ and $0 < r_1 + r_2 \leq r_1' + r_2'$.

From (2.21), (2.22) and (2.23), we have

$$\tilde{A}_1^I \oplus \tilde{A}_2^I = (m_1 + m_2; l_1 + l_2, r_1 + r_2; l_1' + l_2', r_1' + r_2')_{LR},$$

where $0 < l_1 + l_2 \leq l_1' + l_2'$ and $0 < r_1 + r_2 \leq r_1' + r_2'$. $\square$

Proposition 2.2.16. *Let* $\tilde{A}_1^I$=$(m_1; l_1, r_1; l_1', r_1')_{LR}$ *and* $\tilde{A}_2^I$=$(m_2; l_2, r_2; l_2', r_2')_{LR}$ *be two LR-type IFNs. Then*

$$\tilde{A}_1^I \ominus \tilde{A}_2^I = (m_1 - m_2; l_1 + r_2, l_2 + r_1; l_1' + r_2', l_2' + r_1')_{LR},$$

where $0 < l_1 + r_2 \leq l_1' + r_2'$ *and* $0 < l_2 + r_1 \leq l_2' + r_1'$.

Proof. The α and β-cuts of $\tilde{A}_1^I$ and $\tilde{A}_2^I$ are given by

$$A_{1\alpha}^I = [m_1 - l_1L^{-1}(\alpha), m_1 + r_1R^{-1}(\alpha)],\ A_{2\alpha}^I = [m_2 - l_2L^{-1}(\alpha), m_2 + r_2R^{-1}(\alpha)] \tag{2.24}$$

$$A_{1(\beta)}^I = [m_1 - l_1'L^{-1}(1-\beta), m_1 + r_1'R^{-1}(1-\beta)],\ A_{2(\beta)}^I = [m_2 - l_2'L^{-1}(1-\beta), m_2 + r_2'R^{-1}(1-\beta)] \tag{2.25}$$

respectively.
From (2.24), we get

$$\begin{aligned}(\tilde{A}_1^I \ominus \tilde{A}_2^I)_\alpha = A_{1\alpha}^I - A_{2\alpha}^I &= [m_1 - l_1L^{-1}(\alpha), m_1 + r_1R^{-1}(\alpha)] - [m_2 - l_2L^{-1}(\alpha), m_2 + r_2R^{-1}(\alpha)] \\ &= [m_1 - l_1L^{-1}(\alpha) - m_2 - r_2R^{-1}(\alpha), m_1 + r_1R^{-1}(\alpha) - m_2 + l_2L^{-1}(\alpha)]\end{aligned} \tag{2.26}$$

Since L and R are decreasing functions on $[0, \infty)$ with $L(0) = R(0) = 1$, $\exists$ a unique $\alpha_0 \in (0, 1]$ such that $L^{-1}(\alpha_0) = R^{-1}(\alpha_0) = 1$. Hence

$$(\tilde{A}_1^I \ominus \tilde{A}_2^I)_{\alpha_0} = [m_1 - m_2 - l_1 - r_2, m_1 - m_2 + l_2 + r_1] \tag{2.27}$$

(2.27) gives left and right spreads of membership function of $\tilde{A}_1^I \ominus \tilde{A}_2^I$ are $l_1 + r_2$ and $l_2 + r_1$ respectively.

Putting $\alpha = 1$ in (2.26), we obtain the modal point of $\tilde{A}_1^I \ominus \tilde{A}_2^I$ given by

$$(\tilde{A}_1^I \ominus \tilde{A}_2^I)_1 = [m_1 - m_2, m_1 - m_2] = m_1 - m_2 \tag{2.28}$$

From (2.25), we get

$$\begin{aligned}(\tilde{A}_1^I \ominus \tilde{A}_2^I)_{(\beta)} = A_{1(\beta)}^I - A_{2(\beta)}^I =& [m_1 - l_1'L^{-1}(1-\beta), m_1 + r_1'R^{-1}(1-\beta)] \\ &- [m_2 - l_2'L^{-1}(1-\beta), m_2 + r_2'R^{-1}(1-\beta)] \\ =& [m_1 - l_1'L^{-1}(1-\beta) - m_2 - r_2'R^{-1}(1-\beta), \\ & m_1 + r_1'R^{-1}(1-\beta) - m_2 + l_2'L^{-1}(1-\beta)]\end{aligned}$$

Since L and R are decreasing functions on $[0, \infty)$ with $L(0) = R(0) = 1$, $\exists$ a unique $\beta_0 \in [0, 1)$ such that $L^{-1}(1-\beta_0) = R^{-1}(1-\beta_0) = 1$. Hence

$$(\tilde{A}_1^I \ominus \tilde{A}_2^I)_{(\beta_0)} = [m_1 - m_2 - l_1' - r_2', m_1 - m_2 + l_2' + r_1'] \tag{2.29}$$

(2.29) gives left and right spreads of non-membership function of $\tilde{A}_1^I \ominus \tilde{A}_2^I$ are $l'_1 + r'_2$ and $l'_2 + r'_1$ respectively.

Since $\tilde{A}_1^I$ and $\tilde{A}_2^I$ are an LR-type IFNs,

$$0 < l_1 \leq l'_1, 0 < l_2 \leq l'_2, 0 < r_1 \leq r'_1 \text{ and } 0 < r_2 \leq r'_2.$$

Thus, $0 < l_1 + r_2 \leq l'_1 + r'_2$ and $0 < l_2 + r_1 \leq l'_2 + r'_1$.
From (2.27), (2.28) and (2.29), we have

$$\tilde{A}_1^I \ominus \tilde{A}_2^I = (m_1 - m_2; l_1 + r_2, l_2 + r_1; l'_1 + r'_2, l'_2 + r'_1)_{LR},$$

where $0 < l_1 + r_2 \leq l'_1 + r'_2$ and $0 < l_2 + r_1 \leq l'_2 + r'_1$. □

Proposition 2.2.17. *Let* $\tilde{A}_1^I=(m_1; l_1, r_1; l'_1, r'_1)_{LR}$ *and* $\tilde{A}_2^I=(m_2; l_2, r_2; l'_2, r'_2)_{LR}$ *be two non-negative LR-type IFNs. Then*

$$\tilde{A}_1^I \odot \tilde{A}_2^I = (m_1 m_2; m_1 l_2 + m_2 l_1 - l_1 l_2, m_1 r_2 + m_2 r_1 + r_1 r_2; m_1 l'_2 + m_2 l'_1 - l'_1 l'_2, m_1 r'_2 + m_2 r'_1 + r'_1 r'_2)_{LR},$$

where $0 < m_1 l_2 + m_2 l_1 - l_1 l_2 \leq m_1 l'_2 + m_2 l'_1 - l'_1 l'_2$ *and* $0 < m_1 r_2 + m_2 r_1 + r_1 r_2 \leq m_1 r'_2 + m_2 r'_1 + r'_1 r'_2$.

Proof. The α and β-cuts of $\tilde{A}_1^I$ and $\tilde{A}_2^I$ are given by

$$A^I_{1\alpha} = [m_1 - l_1 L^{-1}(\alpha), m_1 + r_1 R^{-1}(\alpha)],\ A^I_{2\alpha} = [m_2 - l_2 L^{-1}(\alpha), m_2 + r_2 R^{-1}(\alpha)] \quad (2.30)$$

$$A^I_{1(\beta)} = [m_1 - l'_1 L^{-1}(1-\beta), m_1 + r'_1 R^{-1}(1-\beta)],\ A^I_{2(\beta)} = [m_2 - l'_2 L^{-1}(1-\beta), m_2 + r'_2 R^{-1}(1-\beta)] \quad (2.31)$$

respectively.
Since $\tilde{A}_1^I$ and $\tilde{A}_2^I$ are non-negative,

$$m_1 - l'_1 \geq 0 \text{ and } m_2 - l'_2 \geq 0.$$

Thus, $m_1 - l'_1 L^{-1}(1-\beta) \geq 0$ and $m_2 - l'_2 L^{-1}(1-\beta) \geq 0\ \forall\, \beta \in [0,1)$.
From (2.30), we get

$$\begin{aligned}(\tilde{A}_1^I \odot \tilde{A}_2^I)_\alpha = A^I_{1\alpha} A^I_{2\alpha} &= [m_1 - l_1 L^{-1}(\alpha), m_1 + r_1 R^{-1}(\alpha)][m_2 - l_2 L^{-1}(\alpha), m_2 + r_2 R^{-1}(\alpha)] \\ &= [(m_1 - l_1 L^{-1}(\alpha))(m_2 - l_2 L^{-1}(\alpha)), (m_1 + r_1 R^{-1}(\alpha))(m_2 + r_2 R^{-1}(\alpha))]\end{aligned} \quad (2.32)$$

Since L and R are decreasing functions on $[0, \infty)$ with $L(0) = R(0) = 1$, $\exists$ a unique $\alpha_0 \in (0,1]$ such that $L^{-1}(\alpha_0) = R^{-1}(\alpha_0) = 1$. Hence

$$\begin{aligned}(\tilde{A}_1^I \odot \tilde{A}_2^I)_{\alpha_0} &= [(m_1 - l_1)(m_2 - l_2), (m_1 + r_1)(m_2 + r_2)] \\ &= [m_1 m_2 - m_2 l_1 - m_1 l_2 + l_1 l_2, m_1 m_2 + m_2 r_1 + m_1 r_2 + r_1 r_2]\end{aligned} \quad (2.33)$$

(2.33) gives left and right spreads of membership function of $\tilde{A}_1^I \odot \tilde{A}_2^I$ are $m_2l_1 + m_1l_2 - l_1l_2$ and $m_2r_1 + m_1r_2 + r_1r_2$ respectively.

Putting $\alpha = 1$ in (2.32), we obtain the modal point of $\tilde{A}_1^I \odot \tilde{A}_2^I$ given by

$$(\tilde{A}_1^I \odot \tilde{A}_2^I)_1 = [m_1m_2, m_1m_2] = m_1m_2 \tag{2.34}$$

From (2.31), we get

$$\begin{aligned}(\tilde{A}_1^I \odot \tilde{A}_2^I)_{(\beta)} = A^I_{1(\beta)}A^I_{2(\beta)} =&[m_1 - l_1'L^{-1}(1-\beta), m_1 + r_1'R^{-1}(1-\beta)]\\ &+[m_2 - l_2'L^{-1}(1-\beta), m_2 + r_2'R^{-1}(1-\beta)]\\ &=[(m_1 - l_1'L^{-1}(1-\beta))(m_2 - l_2'L^{-1}(1-\beta)),\\ &\quad(m_1 + r_1'R^{-1}(1-\beta))(m_2 + r_2'R^{-1}(1-\beta))].\end{aligned}$$

Since L and R are decreasing functions on $[0, \infty)$ with $L(0) = R(0) = 1$, $\exists$ a unique $\beta_0 \in [0, 1)$ such that $L^{-1}(1-\beta_0) = R^{-1}(1-\beta_0) = 1$. Hence

$$\begin{aligned}(\tilde{A}_1^I \odot \tilde{A}_2^I)_{(\beta_0)} &= [(m_1 - l_1')(m_2 - l_2'), (m_1 + r_1')(m_2 + r_2')]\\ &= [m_1m_2 - m_1l_2' - m_2l_1' + l_1'l_2', m_1m_2 + m_1r_2' + m_2r_1' + r_1'r_2'] \qquad (2.35)\end{aligned}$$

(2.35) gives left and right spreads of non-membership function of $\tilde{A}_1^I \odot \tilde{A}_2^I$ are $m_1l_2' + m_2l_1' - l_1'l_2'$ and $m_1r_2' + m_2r_1' + r_1'r_2'$ respectively.

Since $\tilde{A}_1^I$ and $\tilde{A}_2^I$ are non-negative LR-type IFNs,

$$\begin{aligned}&m_1 - l_1' \geq 0, m_2 - l_2' \geq 0, 0 < l_1 \leq l_1', 0 < l_2 \leq l_2', 0 < r_1 \leq r_1', 0 < r_2 \leq r_2', m_1 - l_1'\\ &\leq m_1 - l_1, m_2 - l_2' \leq m_2 - l_2, m_1 + r_1 \leq m_1 + r_1', m_2 + r_2 \leq m_2 + r_2'\\ \Rightarrow\ &(m_1 - l_1')(m_2 - l_2') \leq (m_1 - l_1)(m_2 - l_2),\ (m_1 + r_1)(m_2 + r_2) \leq (m_1 + r_1')(m_2 + r_2')\\ \Rightarrow\ &m_1m_2 - m_2l_1' - m_1l_2' + l_1'l_2' \leq m_1m_2 - m_2l_1 - m_1l_2 + l_1l_2,\ m_1m_2 + m_2r_1 + m_1r_2\\ &+ r_1r_2 \leq m_1m_2 + m_2r_1' + m_1r_2' + r_1'r_2'\\ \Rightarrow\ &-m_2l_1' - m_1l_2' + l_1'l_2' \leq -m_2l_1 - m_1l_2 + l_1l_2,\\ &m_2r_1 + m_1r_2 + r_1r_2 \leq m_2r_1' + m_1r_2' + r_1'r_2'\\ \Rightarrow\ &m_2l_1' + m_1l_2' - l_1'l_2' \geq m_2l_1 + m_1l_2 - l_1l_2,\ m_2r_1 + m_1r_2 + r_1r_2 \leq m_2r_1' + m_1r_2' + r_1'r_2'\end{aligned}$$

Clearly, $(m_2l_1 + m_1l_2 - l_1l_2) = l_1(m_2 - l_2) + m_1l_2 > 0$, $(m_2r_1 + m_1r_2 + r_1r_2) > 0$ $(\because \tilde{A}_1^I$ and $\tilde{A}_2^I$ are non-negative)

From (2.33), (2.34) and (2.35), we have

$$\tilde{A}_1^I \odot \tilde{A}_2^I = (m_1m_2; m_1l_2 + m_2l_1 - l_1l_2, m_1r_2 + m_2r_1 + r_1r_2; m_1l_2' + m_2l_1' - l_1'l_2', m_1r_2' + m_2r_1' + r_1'r_2')_{LR},$$

where $0 < m_1l_2 + m_2l_1 - l_1l_2 \leq m_1l'_2 + m_2l'_1 - l'_1l'_2$ and $0 < m_1r_2 + m_2r_1 + r_1r_2 \leq m_1r'_2 + m_2r'_1 + r'_1r'_2$. □

Proposition 2.2.18. *Let* $\tilde{A}^I_1 = (m_1; l_1, r_1; l'_1, r'_1)_{LR}$ *be non-positive and* $\tilde{A}^I_2 = (m_2; l_2, r_2; l'_2, r'_2)_{LR}$ *be non-negative LR-type IFNs. Then*

$$\tilde{A}^I_1 \odot \tilde{A}^I_2 = (m_1m_2; m_2l_1 - m_1l_1 + r_2r_2, m_2r_1 - m_1l_2 - r_1l_2; m_2l'_1 - m_1l'_1 + r'_2r'_2, m_2r'_1 - m_1l'_2 - r'_1l'_2)_{LR},$$

where $0 < m_2l_1 - m_1l_1 + r_2r_2 \leq m_2l'_1 - m_1l'_1 + r'_2r'_2$ *and* $0 < m_2r_1 - m_1l_2 - r_1l_2 \leq m_2r'_1 - m_1l'_2 - r'_1l'_2$.

Proof. The α and β-cuts of $\tilde{A}^I_1$ and $\tilde{A}^I_2$ are given by

$$A^I_{1\alpha} = [m_1 - l_1L^{-1}(\alpha), m_1 + r_1R^{-1}(\alpha)],\ A^I_{2\alpha} = [m_2 - l_2L^{-1}(\alpha), m_2 + r_2R^{-1}(\alpha)] \tag{2.36}$$

$$A^I_{1(\beta)} = [m_1 - l'_1L^{-1}(1-\beta), m_1 + r'_1R^{-1}(1-\beta)],\ A^I_{2(\beta)} = [m_2 - l'_2L^{-1}(1-\beta), m_2 + r'_2R^{-1}(1-\beta)] \tag{2.37}$$

respectively.

Since $\tilde{A}^I_1$ is non-positive and $\tilde{A}^I_2$ is non-negative,

$$m_1 + r'_1 \leq 0 \text{ and } m_2 - l'_2 \geq 0.$$

Thus, $m_1 + r'_1R^{-1}(1-\beta) \leq 0$ and $m_2 - l'_2L^{-1}(1-\beta) \geq 0\ \forall\ \beta \in [0, 1)$.

From (2.36), we get

$$\begin{aligned}(\tilde{A}^I_1 \odot \tilde{A}^I_2)_\alpha = A^I_{1\alpha}A^I_{2\alpha} &= [m_1 - l_1L^{-1}(\alpha), m_1 + r_1R^{-1}(\alpha)][m_2 - l_2L^{-1}(\alpha), m_2 + r_2R^{-1}(\alpha)] \\ &= [(m_1 - l_1L^{-1}(\alpha))(m_2 + r_2R^{-1}(\alpha)), \\ &\quad (m_1 + r_1R^{-1}(\alpha))(m_2 - l_2L^{-1}(\alpha))]\end{aligned} \tag{2.38}$$

Since L and R are decreasing functions on $[0, \infty)$ with $L(0) = R(0) = 1$, $\exists$ a unique $\alpha_0 \in (0, 1]$ such that $L^{-1}(\alpha_0) = R^{-1}(\alpha_0) = 1$. Hence

$$\begin{aligned}(\tilde{A}^I_1 \odot \tilde{A}^I_2)_{\alpha_0} &= [(m_1 - l_1)(m_2 + r_2), (m_1 + r_1)(m_2 - l_2)] \\ &= [m_1m_2 - m_2l_1 + m_1r_2 - l_1r_2, m_1m_2 + m_2r_1 - m_1l_2 - r_1l_2]\end{aligned} \tag{2.39}$$

(2.39) gives left and right spreads of membership function of $\tilde{A}^I_1 \odot \tilde{A}^I_2$ are $m_2l_1 - m_1r_2 + l_1r_2$ and $m_2r_1 - m_1l_2 - r_1l_2$ respectively.

Putting $\alpha = 1$ in (2.29), we obtain the modal point of $\tilde{A}^I_1 \odot \tilde{A}^I_2$ given by

$$(\tilde{A}^I_1 \odot \tilde{A}^I_2)_1 = [m_1m_2, m_1m_2] = m_1m_2 \tag{2.40}$$

From (2.37), we get

$$
\begin{aligned}
(\tilde{A}_1^I \odot \tilde{A}_2^I)_{(\beta)} = A^I_{1(\beta)}A^I_{2(\beta)} =&[m_1 - l_1' L^{-1}(1-\beta), m_1 + r_1' R^{-1}(1-\beta)]\\
&+[m_2 - l_2' L^{-1}(1-\beta), m_2 + r_2' R^{-1}(1-\beta)]\\
=&[(m_1 - l_1' L^{-1}(1-\beta))(m_2 + r_2' R^{-1}(1-\beta)),\\
&(m_1 + r_1' R^{-1}(1-\beta))(m_2 - l_2' L^{-1}(1-\beta))]
\end{aligned}
$$

Since L and R are decreasing functions on $[0,\infty)$ with $L(0) = R(0) = 1$, $\exists$ a unique $\beta_0 \in [0,1)$ such that $L^{-1}(1-\beta_0) = R^{-1}(1-\beta_0) = 1$. Hence

$$
\begin{aligned}
(\tilde{A}_1^I \odot \tilde{A}_2^I)_{(\beta_0)} &= [(m_1 - l_1')(m_2 + r_2'), (m_1 + r_1')(m_2 - l_2')]\\
&= [m_1m_2 - m_2l_1' + m_1r_2' - l_1'r_2', m_1m_2 + m_2r_1' - m_1l_2' - r_1'l_2'] \qquad (2.41)
\end{aligned}
$$

(2.41) gives left and right spreads of non-membership function of $\tilde{A}_1^I \odot \tilde{A}_2^I$ are $m_2l_1' - m_1r_2' + l_1'r_2'$ and $m_2r_1' - m_1l_2' - r_1'l_2'$ respectively. Since $\tilde{A}_1^I$ is non-positive and $\tilde{A}_2^I$ is non-negative LR-type IFNs,

$$
\begin{aligned}
& m_1 + r_1' \le 0, m_2 - l_2' \ge 0, 0 < l_1 \le l_1', 0 < l_2 \le l_2', 0 < r_1 \le r_1', 0 < r_2 \le r_2', m_1 - l_1' \le m_1\\
& - l_1 < 0, 0 \le m_2 - l_2' \le m_2 - l_2, m_1 + r_1 \le m_1 + r_1' \le 0, 0 \le m_2 + r_2 \le m_2 + r_2'.\\
\Rightarrow & -(m_1 - l_1')(m_2 + r_2') \ge -(m_1 - l_1)(m_2 + r_2) \ge 0,\\
& -(m_1 + r_1)(m_2 - l_2) \ge -(m_1 + r_1')(m_2 - l_2') \ge 0\\
\Rightarrow & -m_1m_2 + m_2l_1' - m_1l_1' + r_2'r_2' \ge -m_1m_2 + m_2l_1 - m_1l_1 + r_2r_2 \ge 0,\\
& -m_1m_2 - m_2r_1 + m_1l_2 + r_1l_2 \ge -m_1m_2 - m_2r_1' + m_1l_2' + r_1'l_2'\\
\Rightarrow & m_2l_1' - m_1l_1' + r_2'r_2' \ge m_2l_1 - m_1l_1 + r_2r_2 \ge 0,\\
& m_2r_1 - m_1l_2 - r_1l_2 \le m_2r_1' - m_1l_2' - r_1'l_2'
\end{aligned}
$$

Clearly, $m_2r_1 - m_1l_2 - r_1l_2 = r_1(m_2 - l_2) - m_1l_2 > 0 \quad (\because r_1 > 0, l_2 > 0, m_2 - l_2 \ge 0$, and $m_1 \le 0$)

From (2.39), (2.40) and (2.41), we have

$$\tilde{A}_1^I \odot \tilde{A}_2^I = (m_1m_2; m_2l_1 - m_1l_1 + r_2r_2, m_2r_1 - m_1l_2 - r_1l_2; m_2l_1' - m_1l_1' + r_2'r_2', m_2r_1' - m_1l_2' - r_1'l_2')_{LR},$$

where $0 < m_2l_1 - m_1l_1 + r_2r_2 \le m_2l_1' - m_1l_1' + r_2'r_2'$ and $0 < m_2r_1 - m_1l_2 - r_1l_2 \le m_2r_1' - m_1l_2' - r_1'l_2'$. □

Proposition 2.2.19. *Let $\tilde{A}_1^I$=$(m_1; l_1, r_1; l_1', r_1')_{LR}$ be non-negative and $\tilde{A}_2^I$=$(m_2; l_2, r_2; l_2', r_2')_{LR}$ be non-positive LR-type IFNs. Then*

$$\tilde{A}_1^I \odot \tilde{A}_2^I \approx (m_1m_2; m_1l_2 - m_2r_1 + r_1l_2, m_1r_2 - m_2l_1 - l_1r_2; m_1l_2' - m_2r_1' + r_1'l_2', m_1r_2' - m_2l_1' - l_1'r_2')_{LR},$$

where $0 < m_1l_2 - m_2r_1 + r_1l_2 \leq m_1l'_2 - m_2r'_1 + r'_1l'_2$ *and* $0 < m_1r_2 - m_2l_1 - l_1r_2 \leq m_1r'_2 - m_2l'_1 - l'_1r'_2$.

Proof. The same as Proposition 2.2.18. □

Proposition 2.2.20. *Let* $\tilde{A}^I_1 = (m_1; l_1, r_1; l'_1, r'_1)_{LR}$ *and* $\tilde{A}^I_2 = (m_2; l_2, r_2; l'_2, r'_2)_{LR}$ *be both non-positive LR-type IFNs. Then*

$$\tilde{A}^I_1 \odot \tilde{A}^I_2 \approx (m_1m_2; -m_1r_2 - m_2r_1 - r_1r_2, -m_1l_2 - m_2l_1 + l_1l_2; -m_1r'_2 - m_2r'_1 - r'_1r'_2, -m_1l'_2 - m_2l'_1 + l'_1l'_2)_{LR},$$

where $0 < -m_1r_2 - m_2r_1 - r_1r_2 \leq -m_1r'_2 - m_2r'_1 - r'_1r'_2$ *and* $0 < -m_1l_2 - m_2l_1 + l_1l_2 \leq -m_1l'_2 - m_2l'_1 + l'_1l'_2$.

Proof. The same as Propositions 2.2.18 and 2.2.15. □

Proposition 2.2.21. *Let* $\tilde{A}^I = (m; l, r; l', r')_{LR}$ *be non-negative LR-type IFN and* λ *be any real number. Then*

$$\lambda\tilde{A}^I = \begin{cases} (\lambda m; \lambda l, \lambda r; \lambda l', \lambda r')_{LR}, & \lambda \geq 0, \\ (\lambda m; -\lambda r, -\lambda l; -\lambda r', -\lambda l')_{LR}, & \lambda < 0. \end{cases}$$

Proof. The same as Propositions 2.2.15 and 2.2.17. □

2.3 Proposed product for unrestricted LR-type IFNs

In this section, product for unrestricted LR-type IFNs is proposed.

Theorem 2.3.1. *Let* $\tilde{A}^I_1 = (m_1; l_1, r_1; l'_1, r'_1)_{LR}$ *be LR-type IFN, where* $m_1 - l'_1 < 0$, $m_1 - l_1 \geq 0$ *and* $\tilde{A}^I_2 = (m_2; l_2, r_2; l'_2, r'_2)_{LR}$ *be another LR-type IFN, where* $m_2 - l'_2$, $m_2 - l_2$, m_2, $m_2 + r_2$, $m_2 + r'_2$ *are real numbers. Then* $\tilde{A}^I_1 \odot \tilde{A}^I_2 \approx (m; l, r; l', r')_{LR}$, *where* $m = m_1m_2$, $l = m_1m_2 - \min\{m_1m_2 - m_1l_2 - m_2l_1 + l_1l_2, m_1m_2 - m_1l_2 + m_2r_1 - l_2r_1\}$, $r = \max\{m_1m_2 + m_1r_2 + m_2r_1 + r_1r_2, m_1m_2 + m_1r_2 - m_2l_1 - l_1r_2\} - m_1m_2$, $l' = m_1m_2 - \min\{m_1m_2 - m_2l'_1 + m_1r'_2 - l'_1r'_2, m_1m_2 + m_2r'_1 - m_1l'_2 - l'_2r'_1\}$, $r' = \max\{m_1m_2 - m_2l'_1 - m_1l'_2 + l'_1l'_2, m_1m_2 + m_2r'_1 + m_1r'_2 + r'_1r'_2\} - m_1m_2$, *and* $0 < l \leq l', 0 < r \leq r'$.

Proof. Let $\tilde{A}^I_1 = (m_1; l_1, r_1; l'_1, r'_1)_{LR}$ and $\tilde{A}^I_2 = (m_2; l_2, r_2; l'_2, r'_2)_{LR}$ be two LR-type IFNs in which $m_1 - l'_1 < 0$, $m_1 - l_1 \geq 0$, and $m_2 - l'_2, m_2 - l_2, m_2, m_2 + r_2, m_2 + r'_2$ are real numbers. Then the α and β-cuts of $\tilde{A}^I_1$ and $\tilde{A}^I_2$ are given by

$$A^I_{1\alpha} = [m_1 - l_1L^{-1}(\alpha), m_1 + r_1R^{-1}(\alpha)],\ A^I_{2\alpha} = [m_2 - l_2L^{-1}(\alpha), m_2 + r_2R^{-1}(\alpha)] \tag{2.42}$$

$$A^I_{1(\beta)} = [m_1 - l'_1L^{-1}(1-\beta), m_1 + r'_1R^{-1}(1-\beta)],\ A^I_{2(\beta)} = [m_2 - l'_2L^{-1}(1-\beta), m_2 + r'_2R^{-1}(1-\beta)] \tag{2.43}$$

respectively.
Since $m_1 - l_1' < 0$ and $m_1 - l_1 \geq 0$, $m_1 - l_1 L^{-1}(\alpha) \geq 0$ for $\alpha \in (0,1]$, $m_1 - l_1' L^{-1}(1-\beta) \leq 0$ for $\beta \leq (1 - L(\frac{m_1}{l_1'}))$, $m_1 - l_1' L^{-1}(1-\beta) \geq 0$ for $\beta \geq (1 - L(\frac{m_1}{l_1'}))$.
To find the product of $\tilde{A}_1^I$ and $\tilde{A}_2^I$, we need to consider the following six cases:
Case 1. If $m_2 - l_2' \geq 0$, then $m_2 - l_2 L^{-1}(\alpha) \geq 0$ and $m_2 - l_2' L^{-1}(1-\beta) \geq 0$ $\forall \alpha \in (0,1], \beta \in [0,1)$.

(a) If $m_1 - l_1' L^{-1}(1-\beta) \leq 0$ and $m_2 - l_2' L^{-1}(1-\beta) \geq 0$, then

$$\begin{aligned}(\tilde{A}_1^I \odot \tilde{A}_2^I)_\alpha = A_{1\alpha}^I A_{2\alpha}^I =& [m_1 - l_1 L^{-1}(\alpha), m_1 + r_1 R^{-1}(\alpha)][m_2 - l_2 L^{-1}(\alpha), m_2 + r_2 R^{-1}(\alpha)] \\ =& [(m_1 - l_1 L^{-1}(\alpha))(m_2 - l_2 L^{-1}(\alpha)), \\ & (m_1 + r_1 R^{-1}(\alpha))(m_2 + r_2 R^{-1}(\alpha))] \end{aligned} \quad (2.44)$$

Since L and R are decreasing functions on $[0,\infty)$ with $L(0) = R(0) = 1$, $\exists$ a unique $\alpha_0 \in (0,1]$ such that $L^{-1}(\alpha_0) = R^{-1}(\alpha_0) = 1$. Hence

$$\begin{aligned}(\tilde{A}_1^I \odot \tilde{A}_2^I)_{\alpha_0} &= [(m_1 - l_1)(m_2 - l_2), (m_1 + r_1)(m_2 + r_2)] \\ &= [m_1 m_2 - m_2 l_1 - m_1 l_2 + l_1 l_2, m_1 m_2 + m_2 r_1 + m_1 r_2 + r_1 r_2] \end{aligned} \quad (2.45)$$

(2.45) gives left and right spreads of membership function of $\tilde{A}_1^I \odot \tilde{A}_2^I$ are $m_2 l_1 + m_1 l_2 - l_1 l_2$ and $m_2 r_1 + m_1 r_2 + r_1 r_2$ respectively.

From (2.43), we get

$$\begin{aligned}(\tilde{A}_1^I \odot \tilde{A}_2^I)_{(\beta)} = A_{1(\beta)}^I A_{2(\beta)}^I =& [m_1 - l_1' L^{-1}(1-\beta), m_1 + r_1' R^{-1}(1-\beta)] \\ & [m_2 - l_2' L^{-1}(1-\beta), m_2 + r_2' R^{-1}(1-\beta)] \\ =& [(m_1 - l_1' L^{-1}(1-\beta))(m_2 + r_2' R^{-1}(1-\beta)), \\ & (m_1 + r_1' R^{-1}(1-\beta))(m_2 + r_2' R^{-1}(1-\beta))] \end{aligned} \quad (2.46)$$

Since L and R are decreasing functions on $[0,\infty)$ with $L(0) = R(0) = 1$, $\exists$ a unique $\beta_0 \in [0,1)$ such that $L^{-1}(1-\beta_0) = R^{-1}(1-\beta_0) = 1$. Hence

$$\begin{aligned}(\tilde{A}_1^I \odot \tilde{A}_2^I)_{(\beta_0)} &= [(m_1 - l_1')(m_2 + r_2'), (m_1 + r_1')(m_2 + r_2')] \\ &= [m_1 m_2 - m_2 l_1' + m_1 r_2' - l_1' r_2', m_1 m_2 + m_2 r_1' + m_1 r_2' + r_1' r_2'] \end{aligned} \quad (2.47)$$

(2.47) gives left and right spreads of non-membership function of $\tilde{A}_1^I \odot \tilde{A}_2^I$ are $m_2 l_1' - m_1 r_2' + l_1' r_2'$ and $m_2 r_1' + m_1 r_2' + r_1' r_2'$ respectively.

(b) If $m_1 - l_1' L^{-1}(1-\beta) \geq 0$ and $m_2 - l_2' L^{-1}(1-\beta) \geq 0$, then

$$\begin{aligned}(\tilde{A}_1^I \odot \tilde{A}_2^I)_\alpha = A_{1\alpha}^I A_{2\alpha}^I =&[m_1 - l_1 L^{-1}(\alpha), m_1 + r_1 R^{-1}(\alpha)][m_2 - l_2 L^{-1}(\alpha), m_2 + r_2 R^{-1}(\alpha)]\\ =&[(m_1 - l_1 L^{-1}(\alpha))(m_2 - l_2 L^{-1}(\alpha)),\\ &(m_1 + r_1 R^{-1}(\alpha))(m_2 + r_2 R^{-1}(\alpha))]\end{aligned} \tag{2.48}$$

Putting $\alpha = 1$ in (2.48), we obtain the modal point of $\tilde{A}_1^I \odot \tilde{A}_2^I$ given by

$$(\tilde{A}_1^I \odot \tilde{A}_2^I)_1 = [m_1 m_2, m_1 m_2] = m_1 m_2 \tag{2.49}$$

Since $m_1 - l_1' \leq 0$, $m_1 - l_1 \geq 0$, $m_2 - l_2' \geq 0$, $0 < l_1 \leq l_1'$, $0 < l_2 \leq l_2'$, $0 < r_1 \leq r_1'$, $0 < r_2 \leq r_2'$,
$(m_1 - l_1') \leq (m_1 - l_1)$, $(m_2 - l_2') \leq (m_2 - l_2)$, $(m_1 + r_1) \leq (m_1 + r_1')$ and $(m_2 + r_2) \leq (m_2 + r_2')$
$\Rightarrow (m_1 - l_1')(m_2 + r_2') \leq (m_1 - l_1')(m_2 - l_2') \leq (m_1 - l_1)(m_2 - l_2)$ and $(m_1 + r_1)(m_2 + r_2) \leq (m_1 + r_1')(m_2 + r_2')$ $\quad (\because m_2 - l_2' \leq m_2 + r_2', m_1 - l_1' \leq 0\ )$
$\Rightarrow m_1 m_2 - m_2 l_1' + m_1 r_2' - l_1' r_2' \leq m_1 m_2 - m_2 l_1 - m_1 l_2 + l_1 l_2$ and $m_1 m_2 + m_2 r_1 + m_1 r_2 + r_1 r_2 \leq m_1 m_2 + m_2 r_1' + m_1 r_2' + r_1' r_2'$
$\Rightarrow m_1 m_2 - (m_1 m_2 - m_2 l_1' + m_1 r_2' - l_1' r_2') \geq m_1 m_2 - (m_1 m_2 - m_2 l_1 - m_1 l_2 + l_1 l_2)$ and $(m_1 m_2 + m_2 r_1 + m_1 r_2 + r_1 r_2) - m_1 m_2 \leq (m_1 m_2 + m_2 r_1' + m_1 r_2' + r_1' r_2') - m_1 m_2$
Clearly, $m_1 m_2 - (m_1 m_2 - m_2 l_1 - m_1 l_2 + l_1 l_2) = l_2(m_1 - l_1) + m_2 l_1 > 0$, $(m_1 m_2 + m_2 r_1 + m_1 r_2 + r_1 r_2) - m_1 m_2 > 0$.
From (2.45), (2.47) and (2.49) of Case 1, we get

$$\begin{aligned}\tilde{A}_1^I \odot \tilde{A}_2^I \approx (&m_1 m_2; m_1 m_2 - (m_1 m_2 - m_2 l_1 - m_1 l_2 + l_1 l_2), (m_1 m_2 + m_2 r_1 + m_1 r_2 + r_1 r_2) - m_1 m_2;\\ &m_1 m_2 - (m_1 m_2 - m_2 l_1' + m_1 r_2' - l_1' r_2'), (m_1 m_2 + m_2 r_1' + m_1 r_2' + r_1' r_2') - m_1 m_2)_{LR},\end{aligned}$$

where $0 < m_1 m_2 - (m_1 m_2 - m_2 l_1 - m_1 l_2 + l_1 l_2) \leq m_1 m_2 - (m_1 m_2 - m_2 l_1' + m_1 r_2' - l_1' r_2')$ and $0 < (m_1 m_2 + m_2 r_1 + m_1 r_2 + r_1 r_2) - m_1 m_2 \leq (m_1 m_2 + m_2 r_1' + m_1 r_2' + r_1' r_2') - m_1 m_2$.

Case 2. If $m_2 - l_2' < 0$ and $m_2 - l_2 \geq 0$, then $m_2 - l_2' L^{-1}(1-\beta) \geq 0$ for $\beta \geq (1 - L(\frac{m_2}{l_2'}))$ and $m_2 - l_2' L^{-1}(1-\beta) \leq 0$ for $\beta \leq (1 - L(\frac{m_2}{l_2'}))$.

(a) If $m_1 - l_1' L^{-1}(1-\beta) \leq 0$ and $m_2 - l_2' L^{-1}(1-\beta) \leq 0$, then

$$\begin{aligned}(\tilde{A}_1^I \odot \tilde{A}_2^I)_\alpha = A_{1\alpha}^I A_{2\alpha}^I =&[m_1 - l_1 L^{-1}(\alpha), m_1 + r_1 R^{-1}(\alpha)][m_2 - l_2 L^{-1}(\alpha), m_2 + r_2 R^{-1}(\alpha)]\\ =&[(m_1 - l_1 L^{-1}(\alpha))(m_2 - l_2 L^{-1}(\alpha)),\\ &(m_1 + r_1 R^{-1}(\alpha))(m_2 + r_2 R^{-1}(\alpha))]\end{aligned} \tag{2.50}$$

Since L and R are decreasing functions on $[0,\infty)$ with $L(0) = R(0) = 1$, $\exists$ a unique $\alpha_0 \in (0,1]$ such that $L^{-1}(\alpha_0) = R^{-1}(\alpha_0) = 1$. Hence

$$\begin{aligned}(\tilde{A}_1^I \odot \tilde{A}_2^I)_{\alpha_0} &= [(m_1 - l_1)(m_2 - l_2), (m_1 + r_1)(m_2 + r_2)] \\ &= [m_1m_2 - m_2l_1 - m_1l_2 + l_1l_2, m_1m_2 + m_2r_1 + m_1r_2 + r_1r_2] \qquad (2.51)\end{aligned}$$

(2.51) gives left and right spreads of membership function of $\tilde{A}_1^I \odot \tilde{A}_2^I$ are $m_2l_1 + m_1l_2 - l_1l_2$ and $m_2r_1 + m_1r_2 + r_1r_2$ respectively.

From (2.43), we get

$$\begin{aligned}(\tilde{A}_1^I \odot \tilde{A}_2^I)_{(\beta)} = A^I_{1(\beta)}A^I_{2(\beta)} =&[m_1 - l_1'L^{-1}(1-\beta), m_1 + r_1'R^{-1}(1-\beta)] \\ &[m_2 - l_2'L^{-1}(1-\beta), m_2 + r_2'R^{-1}(1-\beta)] \\ =&[\min\{(m_1 - l_1'L^{-1}(1-\beta))(m_2 + r_2'R^{-1}(1-\beta)), \\ &(m_1 + r_1'R^{-1}(1-\beta))(m_2 - l_2'L^{-1}(1-\beta))\}, \\ &\max\{(m_1 - l_1'L^{-1}(1-\beta))(m_2 - l_2'L^{-1}(1-\beta)), \\ &(m_1 + r_1'R^{-1}(1-\beta))(m_2 + r_2'R^{-1}(1-\beta))\}] \qquad (2.52)\end{aligned}$$

Since L and R are decreasing functions on $[0,\infty)$ with $L(0) = R(0) = 1$, $\exists$ a unique $\beta_0 \in [0,1)$ such that $L^{-1}(1-\beta_0) = R^{-1}(1-\beta_0) = 1$. Hence

$$\begin{aligned}(\tilde{A}_1^I \odot \tilde{A}_2^I)_{(\beta_0)} =&[\min\{(m_1 - l_1')(m_2 + r_2'), (m_1 + r_1')(m_2 - l_2')\}, \\ &\max\{(m_1 - l_1')(m_2 - l_2'), (m_1 + r_1')(m_2 + r_2')\}] \\ &[\min\{m_1m_2 - m_2l_1' + m_1r_2' - l_1'r_2', \\ &m_1m_2 + m_2r_1' - m_1l_2' - r_1'l_2'\}, \max\{m_1m_2 - m_2l_1' - m_1l_2' \\ &+ l_1'l_2', m_1m_2 + m_2r_1' + m_1r_2' + r_1'r_2'\}] \qquad (2.53)\end{aligned}$$

(2.53) gives left and right spreads of non-membership function of $\tilde{A}_1^I \odot \tilde{A}_2^I$ are $m_1m_2 - \min\{m_1m_2 - m_2l_1' + m_1r_2' - l_1'r_2', m_1m_2 + m_2r_1' - m_1l_2' - r_1'l_2'\}$ and $\max\{m_1m_2 - m_2l_1' - m_1l_2' + l_1'l_2', m_1m_2 + m_2r_1' + m_1r_2' + r_1'r_2'\} - m_1m_2$ respectively.

(b) If $m_1 - l_1'L^{-1}(1-\beta) \geq 0$ and $m_2 - l_2'L^{-1}(1-\beta) \geq 0$, then

$$\begin{aligned}(\tilde{A}_1^I \odot \tilde{A}_2^I)_{\alpha} = A^I_{1\alpha}A^I_{2\alpha} =&[m_1 - l_1L^{-1}(\alpha), m_1 + r_1R^{-1}(\alpha)][m_2 - l_2L^{-1}(\alpha), m_2 + r_2R^{-1}(\alpha)] \\ =& [(m_1 - l_1L^{-1}(\alpha))(m_2 - l_2L^{-1}(\alpha)), \\ &(m_1 + r_1R^{-1}(\alpha))(m_2 + r_2R^{-1}(\alpha))] \qquad (2.54)\end{aligned}$$

Putting $\alpha = 1$ in (2.54), we obtain the modal point of $\tilde{A}_1^I \odot \tilde{A}_2^I$ given by

$$(\tilde{A}_1^I \odot \tilde{A}_2^I)_1 = [m_1m_2, m_1m_2] = m_1m_2 \tag{2.55}$$

Since $m_1 - l_1' < 0$, $m_1 - l_1 \geq 0$, $m_2 - l_2' < 0$, $m_2 - l_2' \geq 0$, $0 < l_1 \leq l_1'$, $0 < l_2 \leq l_2'$, $0 < r_1 \leq r_1'$ and $0 < r_2 \leq r_2'$,

$$(m_1 - l_1') \leq (m_1 - l_1), (m_2 - l_2') \leq (m_2 - l_2), (m_1 + r_1) \leq (m_1 + r_1'), (m_2 + r_2) \leq (m_2 + r_2').$$

$$\begin{aligned}
&\Rightarrow -(m_1 - l_1')(m_2 + r_2') \geq -(m_1 - l_1)(m_2 - l_2),\ (m_1 - l_1)(m_2 - l_2) \geq (m_1 - l_1)(m_2 - l_2') \\
&\quad \geq (m_1 + r_1')(m_2 - l_2') \qquad\qquad (\because m_2 - l_2 \leq m_2 + r_2, m_2 - l_2' \leq 0, m_1 - l_1 \geq 0) \\
&\Rightarrow m_1m_2 - m_2l_1' + m_1r_2' - l_1'r_2' \leq m_1m_2 - m_2l_1 - m_1l_2 + l_1l_2,\ m_1m_2 - m_2l_1 - \\
&\quad m_1l_2 + l_1l_2 \geq m_1m_2 + m_2r_1' - m_1l_2' - r_1'l_2' \\
&\Rightarrow m_1m_2 - (m_1m_2 - m_2l_1 - m_1l_2 + l_1l_2) \leq m_1m_2 - \min\{m_1m_2 - m_2l_1' + m_1r_2' - l_1'r_2', \\
&\quad m_1m_2 + m_2r_1' - m_1l_2' - r_1'l_2'\} \qquad (2.56)
\end{aligned}$$

Now, if $(m_1 + r_1')(m_2 + r_2') \leq (m_1 - l_1')(m_2 - l_2')$, then

$$(m_1 + r_1)(m_2 + r_2) \leq (m_1 + r_1')(m_2 + r_2') \leq (m_1 - l_1')(m_2 - l_2') \tag{2.57}$$

If $(m_1 + r_1')(m_2 + r_2') \geq (m_1 - l_1')(m_2 - l_2')$, then

$$(m_1 + r_1)(m_2 + r_2) \leq (m_1 + r_1')(m_2 + r_2') \tag{2.58}$$

From (2.57) and (2.58), we get
$(m_1m_2 + m_2r_1 + m_1r_2 + r_1r_2) - m_1m_2 \leq \max\{m_1m_2 - m_2l_1' - m_1l_2' + l_1'l_2', m_1m_2 + m_2r_1' + m_1r_2' + r_1'r_2'\} - m_1m_2$.
Clearly, $m_1m_2 - (m_1m_2 - m_2l_1 - m_1l_2 + l_1l_2) = l_2(m_1 - l_1) + m_2l_1 > 0$, $(m_1m_2 + m_2r_1 + m_1r_2 + r_1r_2) - m_1m_2 > 0$.
From (2.51), (2.53) and (2.55) of Case 2, we get

$$\begin{aligned}
\tilde{A}_1^I \odot \tilde{A}_2^I \approx &(m_1m_2; m_1m_2 - (m_1m_2 - m_2l_1 - m_1l_2 + l_1l_2), (m_1m_2 + m_2r_1 + m_1r_2 + r_1r_2) - \\
&m_1m_2; m_1m_2 - \min\{m_1m_2 - m_2l_1' + m_1r_2' - l_1'r_2', m_1m_2 + m_2r_1' - m_1l_2' - r_1'l_2'\}, \\
&\max\{m_1m_2 - m_2l_1' - m_1l_2' + l_1'l_2', m_1m_2 + m_2r_1' + m_1r_2' + r_1'r_2'\} - m_1m_2)_{LR},
\end{aligned}$$

where $0 < m_1m_2 - (m_1m_2 - m_2l_1 - m_1l_2 + l_1l_2) \leq m_1m_2\text{-}\min\{m_1m_2\text{-}m_2l_1'\text{+}m_1r_2'\text{-}l_1'r_2'$, $m_1m_2\text{+}m_2r_1' - m_1l_2' - r_1'l_2'\}$ and $0 < (m_1m_2 + m_2r_1 + m_1r_2 + r_1r_2) - m_1m_2 \leq \max\{m_1m_2\text{-}$ $m_2l_1'\text{-}m_1l_2'\text{+}l_1'l_2'$, $m_1m_2 + m_2r_1' + m_1r_2' + r_1'r_2'\} - m_1m_2$.

Case 3. If $m_2 - l_2 < 0$ and $m_2 \geq 0$, then $m_2 - l_2 L^{-1}(\alpha) \geq 0$ for $\alpha \leq L(\frac{m_2}{l_2})$ and $m_2 - l_2 L^{-1}(\alpha) \leq 0$ for $\alpha \geq L(\frac{m_2}{l_2})$.

(a) If $m_1 - l_1 L^{-1}(\alpha) \geq 0$, $m_2 - l_2 L^{-1}(\alpha) \leq 0$, $m_1 - l_1' L^{-1}(1-\beta) \leq 0$ and $m_2 - l_2' L^{-1}(1-\beta) \leq 0$, then

$$\begin{aligned}(\tilde{A}_1^I \odot \tilde{A}_2^I)_\alpha = A_{1\alpha}^I A_{2\alpha}^I =& [m_1 - l_1 L^{-1}(\alpha), m_1 + r_1 R^{-1}(\alpha)][m_2 - l_2 L^{-1}(\alpha), m_2 + r_2 R^{-1}(\alpha)]\\ =& [(m_1 + r_1 R^{-1}(\alpha))(m_2 - l_2 L^{-1}(\alpha)),\\ & (m_1 + r_1 R^{-1}(\alpha))(m_2 + r_2 R^{-1}(\alpha))] \qquad (2.59)\end{aligned}$$

Since L and R are decreasing functions on $[0, \infty)$ with $L(0) = R(0) = 1$, $\exists$ a unique $\alpha_0 \in (0, 1]$ such that $L^{-1}(\alpha_0) = R^{-1}(\alpha_0) = 1$. Hence

$$\begin{aligned}(\tilde{A}_1^I \odot \tilde{A}_2^I)_{\alpha_0} &= [(m_1 + r_1)(m_2 - l_2), (m_1 + r_1)(m_2 + r_2)]\\ &= [m_1 m_2 + m_2 r_1 - m_1 l_2 - r_1 l_2, m_1 m_2 + m_2 r_1 + m_1 r_2 + r_1 r_2] \qquad (2.60)\end{aligned}$$

(2.60) gives left and right spreads of membership function of $\tilde{A}_1^I \odot \tilde{A}_2^I$ are $-m_2 r_1 + m_1 l_2 + r_1 l_2$ and $m_2 r_1 + m_1 r_2 + r_1 r_2$ respectively.

From (2.43), we get

$$\begin{aligned}(\tilde{A}_1^I \odot \tilde{A}_2^I)_{(\beta)} = A_{1(\beta)}^I A_{2(\beta)}^I =& [m_1 - l_1' L^{-1}(1-\beta), m_1 + r_1' R^{-1}(1-\beta)]\\ & [m_2 - l_2' L^{-1}(1-\beta), m_2 + r_2' R^{-1}(1-\beta)]\\ =& [\min\{(m_1 - l_1' L^{-1}(1-\beta))(m_2 + r_2' R^{-1}(1-\beta)),\\ & (m_1 + r_1' R^{-1}(1-\beta))(m_2 - l_2' L^{-1}(1-\beta))\},\\ & \max\{(m_1 - l_1' L^{-1}(1-\beta))(m_2 - l_2' L^{-1}(1-\beta)),\\ & (m_1 + r_1' R^{-1}(1-\beta))(m_2 + r_2' R^{-1}(1-\beta))\}] \qquad (2.61)\end{aligned}$$

Since L and R are decreasing functions on $[0, \infty)$ with $L(0) = R(0) = 1$, $\exists$ a unique $\beta_0 \in [0, 1)$ such that $L^{-1}(1-\beta_0) = R^{-1}(1-\beta_0) = 1$. Hence

$$\begin{aligned}(\tilde{A}_1^I \odot \tilde{A}_2^I)_{(\beta_0)} =& [\min\{(m_1 - l_1')(m_2 + r_2'), (m_1 + r_1')(m_2 - l_2')\},\\ & \max\{(m_1 - l_1')(m_2 - l_2'), (m_1 + r_1')(m_2 + r_2')\}]\\ =& [\min\{(m_1 m_2 - m_2 l_1' + m_1 r_2' - l_1' r_2', m_1 m_2 + m_2 r_1' - m_1 l_2' - r_1' l_2'\},\\ & \max\{m_1 m_2 - m_2 l_1' - m_1 l_2' + l_1' l_2', m_1 m_2 + m_2 r_1' + m_1 r_2' + r_1' r_2'\}] \qquad (2.62)\end{aligned}$$

(2.62) gives left and right spreads of non-membership function of $\tilde{A}_1^I \odot \tilde{A}_2^I$ are $m_1 m_2$-$\min\{(m_1 m_2$-$m_2 l_1'$+$m_1 r_2'$-$l_1' r_2'$, $m_1 m_2$+$m_2 r_1'$-$m_1 l_2'$-$r_1' l_2'\}$ and $\max\{m_1 m_2$-$m_2 l_1'$-$m_1 l_2'$+$l_1' l_2'$, $m_1 m_2$ +$m_2 r_1'$+$m_1 r_2'$+$r_1' r_2'\}$-$m_1 m_2$ respectively.

(b) If $m_1 - l_1 L^{-1}(\alpha) \geq 0$, $m_2 - l_2 L^{-1}(\alpha) \geq 0$, $m_1 - l'_1 L^{-1}(1-\beta) \geq 0$ and $m_2 - l'_2 L^{-1}(1-\beta) \geq 0$, then

$$\begin{aligned}(\tilde{A}^I_1 \odot \tilde{A}^I_2)_\alpha = A^I_{1\alpha} A^I_{2\alpha} =& [m_1 - l_1 L^{-1}(\alpha), m_1 + r_1 R^{-1}(\alpha)][m_2 - l_2 L^{-1}(\alpha), m_2 + r_2 R^{-1}(\alpha)] \\ =& [(m_1 - l_1 L^{-1}(\alpha))(m_2 - l_2 L^{-1}(\alpha)), \\ & (m_1 + r_1 R^{-1}(\alpha))(m_2 + r_2 R^{-1}(\alpha))] \end{aligned} \tag{2.63}$$

Putting $\alpha = 1$ in (2.63), we obtain the modal point of $\tilde{A}^I_1 \odot \tilde{A}^I_2$ given by

$$(\tilde{A}^I_1 \odot \tilde{A}^I_2)_1 = [m_1 m_2, m_1 m_2] = m_1 m_2 \tag{2.64}$$

Since $m_1 - l'_1 < 0$, $m_1 - l_1 \geq 0$, $m_2 - l_2 < 0$, $m_2 \geq 0$, $0 < l_1 \leq l'_1$, $0 < l_2 \leq l'_2$, $0 < r_1 \leq r'_1$, $0 < r_2 \leq r'_2$,

$(m_1 - l'_1) \leq (m_1 - l_1)$, $(m_2 - l'_2) \leq (m_2 - l_2)$, $(m_1 + r_1) \leq (m_1 + r'_1)$ and $(m_2 + r_2) \leq (m_2 + r'_2)$.

$\Rightarrow -(m_1 - l'_1)(m_2 + r'_2) \geq -(m_1 - l_1)(m_2 + r'_2) \geq -(m_1 + r_1)(m_2 + r'_2) \geq -(m_1 + r_1)(m_2 - l_2), -(m_2 - l'_2)(m_1 + r'_1) \geq -(m_1 + r_1)(m_2 - l_2) \quad (\because m_2 - l_2 \leq 0, m_1 - l_1 \geq 0)$

$\Rightarrow m_1 m_2 - m_2 l'_1 + m_1 r'_2 - l'_1 r'_2 \leq m_1 m_2 + m_2 r_1 - m_1 l_2 - r_1 l_2$,

$m_1 m_2 + m_2 r_1 - m_1 l_2 - r_1 l_2 \geq m_1 m_2 + m_2 r'_1 - m_1 l'_2 - r'_1 l'_2$

$$\Rightarrow m_1 m_2 - (m_1 m_2 + m_2 r_1 - m_1 l_2 - r_1 l_2) \leq m_1 m_2 - \min\{m_1 m_2 - m_2 l'_1 + m_1 r'_2 - l'_1 r'_2, m_1 m_2 + m_2 r'_1 - m_1 l'_2 - r'_1 l'_2\} \tag{2.65}$$

Now, if $(m_1 + r'_1)(m_2 + r'_2) \leq (m_1 - l'_1)(m_2 - l'_2)$, then

$$(m_1 + r_1)(m_2 + r_2) \leq (m_1 + r'_1)(m_2 + r'_2) \leq (m_1 - l'_1)(m_2 - l'_2) \tag{2.66}$$

If $(m_1 + r'_1)(m_2 + r'_2) \geq (m_1 - l'_1)(m_2 - l'_2)$, then

$$(m_1 + r_1)(m_2 + r_2) \leq (m_1 + r'_1)(m_2 + r'_2) \tag{2.67}$$

From (2.66) and (2.67), we get

$(m_1 m_2 + m_2 r_1 + m_1 r_2 + r_1 r_2) - m_1 m_2 \leq \max\{m_1 m_2 - m_2 l'_1 - m_1 l'_2 + l'_1 l'_2, m_1 m_2 + m_2 r'_1 + m_1 r'_2 + r'_1 r'_2\} - m_1 m_2$.

Clearly, $m_1 m_2 - (m_1 m_2 + m_2 r_1 - m_1 l_2 - r_1 l_2) = r_1(l_2 - m_2) + m_1 l_2 > 0$, $(m_1 m_2 + m_2 r_1 + m_1 r_2 + r_1 r_2) - m_1 m_2 > 0$.

From (2.60), (2.62) and (2.64) of Case 3, we get

$$\begin{aligned}\tilde{A}^I_1 \odot \tilde{A}^I_2 \approx& (m_1 m_2; m_1 m_2 - (m_1 m_2 + m_2 r_1 - m_1 l_2 - r_1 l_2), (m_1 m_2 + m_2 r_1 + m_1 r_2 + r_1 r_2) \\ & - m_1 m_2; m_1 m_2 - \min\{m_1 m_2 - m_2 l'_1 + m_1 r'_2 - l'_1 r'_2, m_1 m_2 + m_2 r'_1 - m_1 l'_2 \\ & - r'_1 l'_2\}, \max\{m_1 m_2 - m_2 l'_1 - m_1 l'_2 + l'_1 l'_2, m_1 m_2 + m_2 r'_1 + m_1 r'_2 + r'_1 r'_2 - m_1 m_2\} \\ & - m_1 m_2)_{LR},\end{aligned}$$

where $0 < m_1m_2 - (m_1m_2 + m_2r_1 - m_1l_2 - r_1l_2) \leq m_1m_2 - \min\{m_1m_2 - m_2l'_1 + m_1r'_2 - l'_1r'_2, m_1m_2 + m_2r'_1 - m_1l'_2 - r'_1l'_2\}$ and $0 < (m_1m_2 + m_2r_1 + m_1r_2 + r_1r_2) - m_1m_2 \leq \max\{m_1m_2 - m_2l'_1 - m_1l'_2 + l'_1l'_2, m_1m_2 + m_2r'_1 + m_1r'_2 + r'_1r'_2 - m_1m_2\} - m_1m_2$.

Case 4. If $m_2 < 0$ and $m_2 + r_2 \geq 0$, then $m_2 - l_2L^{-1}(\alpha) \leq 0$ for every $\alpha \in (0,1]$, $m_2 + r_2R^{-1}(\alpha) \geq 0$ for $\alpha \leq R(\frac{m_2}{-r_2})$, $m_2 + r_2R^{-1}(\alpha) \leq 0$ for $\alpha \geq R(\frac{m_2}{-r_2})$, $m_2 - l'_2L^{-1}(1-\beta) \leq 0$ for every $\beta \in [0,1)$, $m_2 + r'_2R^{-1}(1-\beta) \leq 0$ for $\beta \leq (1 - R(\frac{m_2}{-r'_2}))$ and $m_2 + r'_2R^{-1}(1-\beta) \geq 0$ for $\beta \geq (1 - R(\frac{m_2}{-r'_2}))$.

(a) If $m_2 + r_2R^{-1}(\alpha) \geq 0$, $m_2 + r'_2R^{-1}(1-\beta) \geq 0$ and $m_1 - l'_1L^{-1}(1-\beta) \leq 0$, then

$$\begin{aligned}(\tilde{A}_1^I \odot \tilde{A}_2^I)_\alpha = A^I_{1\alpha}A^I_{2\alpha} &= [m_1 - l_1L^{-1}(\alpha), m_1 + r_1R^{-1}(\alpha)][m_2 - l_2L^{-1}(\alpha), m_2 + r_2R^{-1}(\alpha)] \\ &= [(m_1 + r_1R^{-1}(\alpha))(m_2 - l_2L^{-1}(\alpha)), (m_1 + r_1R^{-1}(\alpha))(m_2 + r_2R^{-1}(\alpha))]\end{aligned} \tag{2.68}$$

Since L and R are decreasing functions on $[0,\infty)$ with $L(0) = R(0) = 1$, $\exists$ a unique $\alpha_0 \in (0,1]$ such that $L^{-1}(\alpha_0) = R^{-1}(\alpha_0) = 1$. Hence

$$\begin{aligned}(\tilde{A}_1^I \odot \tilde{A}_2^I)_{\alpha_0} &= [(m_1 + r_1)(m_2 - l_2), (m_1 + r_1)(m_2 + r_2)] \\ &= [m_1m_2 + m_2r_1 - m_1l_2 - r_1l_2, m_1m_2 + m_2r_1 + m_1r_2 + r_1r_2]\end{aligned} \tag{2.69}$$

(2.69) gives left and right spreads of membership function of $\tilde{A}_1^I \odot \tilde{A}_2^I$ are $-m_2r_1 + m_1l_2 + r_1l_2$ and $m_2r_1 + m_1r_2 + r_1r_2$ respectively.

From (2.43), we get

$$\begin{aligned}(\tilde{A}_1^I \odot \tilde{A}_2^I)_{(\beta)} = A^I_{1(\beta)}A^I_{2(\beta)} =& [m_1 - l'_1L^{-1}(1-\beta), m_1 + r'_1R^{-1}(1-\beta)] \\ & [m_2 - l'_2L^{-1}(1-\beta), m_2 + r'_2R^{-1}(1-\beta)] \\ =& [\min\{(m_1 - l'_1L^{-1}(1-\beta))(m_2 + r'_2R^{-1}(1-\beta)), \\ & (m_1 + r'_1R^{-1}(1-\beta))(m_2 - l'_2L^{-1}(1-\beta))\}, \\ & \max\{(m_1 - l'_1L^{-1}(1-\beta))(m_2 - l'_2L^{-1}(1-\beta)), \\ & (m_1 + r'_1R^{-1}(1-\beta))(m_2 + r'_2R^{-1}(1-\beta))\}]\end{aligned} \tag{2.70}$$

Since L and R are decreasing functions on $[0,\infty)$ with $L(0) = R(0) = 1$, $\exists$ a unique

$\beta_0 \in [0,1)$ such that $L^{-1}(1-\beta_0) = R^{-1}(1-\beta_0) = 1$. Hence

$$\begin{aligned}(\tilde{A}_1^I \odot \tilde{A}_2^I)_{(\beta_0)} =&[\min\{(m_1-l_1')(m_2+r_2'),(m_1+r_1')(m_2-l_2')\},\\ &\max\{(m_1-l_1')(m_2-l_2'),(m_1+r_1')(m_2+r_2')\}]\\ =&[\min\{(m_1m_2-m_2l_1'+m_1r_2'-l_1'r_2',\\ &m_1m_2+m_2r_1'-m_1l_2'-r_1'l_2'\},\\ &\max\{m_1m_2-m_2l_1'-m_1l_2'+l_1'l_2',\\ &m_1m_2+m_2r_1'+m_1r_2'+r_1'r_2'\}]\end{aligned} \tag{2.71}$$

(2.71) gives left and right spreads of non-membership function of $\tilde{A}_1^I \odot \tilde{A}_2^I$ are m_1m_2-$\min\{(m_1m_2$-m_2l_1'+m_1r_2'-$l_1'r_2'$, m_1m_2+m_2r_1'-m_1l_2'-$r_1'l_2'\}$ and $\max\{m_1m_2$-m_2l_1'-m_1l_2'+$l_1'l_2'$, m_1m_2 +m_2r_1'+m_1r_2'+$r_1'r_2'\}$-m_1m_2 respectively.

(b) If $m_2 + r_2R^{-1}(\alpha) \le 0$, $m_2 + r_2'R^{-1}(1-\beta) \le 0$ and $m_1 - l_1'L^{-1}(1-\beta) \ge 0$, then

$$\begin{aligned}(\tilde{A}_1^I \odot \tilde{A}_2^I)_\alpha &= A_{1\alpha}^I A_{2\alpha}^I = [m_1-l_1L^{-1}(\alpha), m_1+r_1R^{-1}(\alpha)][m_2-l_2L^{-1}(\alpha), m_2+r_2R^{-1}(\alpha)]\\ &= [(m_1+r_1R^{-1}(\alpha))(m_2-l_2L^{-1}(\alpha)),(m_1-l_1L^{-1}(\alpha))(m_2+r_2R^{-1}(\alpha))]\end{aligned} \tag{2.72}$$

Putting $\alpha = 1$ in (2.72), we obtain the modal point of $\tilde{A}_1^I \odot \tilde{A}_2^I$ given by

$$(\tilde{A}_1^I \odot \tilde{A}_2^I)_1 = [m_1m_2, m_1m_2] = m_1m_2 \tag{2.73}$$

Since $m_1 - l_1' < 0$, $m_1 - l_1 \ge 0$, $m_2 < 0$, $m_2 + r_2 \ge 0$, $0 < l_1 \le l_1'$, $0 < l_2 \le l_2'$, $0 < r_1 \le r_1'$, $0 < r_2 \le r_2'$,
$(m_1-l_1') \le (m_1-l_1)$, $(m_2-l_2') \le (m_2-l_2)$, $(m_1+r_1) \le (m_1+r_1')$, and $(m_2+r_2) \le (m_2+r_2')$.

$$\begin{aligned}&\Rightarrow -(m_1-l_1')(m_2+r_2') \ge -(m_1-l_1)(m_2+r_2') \ge -(m_1+r_1)(m_2+r_2') \ge -(m_1+r_1)(m_2\\ &\quad -l_2), -(m_2-l_2')(m_1+r_1') \ge -(m_1+r_1)(m_2-l_2) \quad (\because m_2 \le 0, m_1-l_1 \ge 0)\\ &\Rightarrow m_1m_2-m_2l_1'+m_1r_2'-l_1'r_2' \le m_1m_2+m_2r_1-m_1l_2-r_1l_2,\\ &\quad m_1m_2+m_2r_1-m_1l_2-r_1l_2 \ge m_1m_2+m_2r_1'-m_1l_2'-r_1'l_2'\\ &\Rightarrow m_1m_2-(m_1m_2+m_2r_1-m_1l_2-r_1l_2) \le m_1m_2-\min\{m_1m_2-m_2l_1'+m_1r_2'-l_1'r_2',\\ &\quad m_1m_2+m_2r_1'-m_1l_2'-r_1'l_2'\}\end{aligned} \tag{2.74}$$

Now, if $(m_1+r_1')(m_2+r_2') \le (m_1-l_1')(m_2-l_2')$, then

$$(m_1+r_1)(m_2+r_2) \le (m_1+r_1')(m_2+r_2') \le (m_1-l_1')(m_2-l_2') \tag{2.75}$$

If $(m_1+r_1')(m_2+r_2') \geq (m_1-l_1')(m_2-l_2')$, then

$$(m_1+r_1)(m_2+r_2) \leq (m_1+r_1')(m_2+r_2') \tag{2.76}$$

From (2.75) and (2.76), we get
$(m_1m_2+m_2r_1+m_1r_2+r_1r_2)-m_1m_2 \leq \max\{m_1m_2-m_2l_1'-m_1l_2'+l_1'l_2', m_1m_2+m_2r_1'+m_1r_2'+r_1'r_2'\}-m_1m_2$.
Clearly, $m_1m_2-(m_1m_2+m_2r_1-m_1l_2-r_1l_2)=r_1(l_2-m_2)+m_1l_2>0$, $(m_1m_2+m_2r_1+m_1r_2+r_1r_2)-m_1m_2>0$.
From (2.69), (2.71) and (2.73) of Case 4, we get

$$\begin{aligned}\tilde{A}_1^I \odot \tilde{A}_2^I \approx &(m_1m_2; m_1m_2-(m_1m_2+m_2r_1-m_1l_2-r_1l_2), (m_1m_2+m_2r_1+m_1r_2+r_1r_2)-\\ &m_1m_2; m_1m_2-\min\{m_1m_2-m_2l_1'+m_1r_2'-l_1'r_2', m_1m_2+m_2r_1'-m_1l_2'-r_1'l_2'\},\\ &\max\{m_1m_2-m_2l_1'-m_1l_2'+l_1'l_2', m_1m_2+m_2r_1'+m_1r_2'+r_1'r_2'\}-m_1m_2)_{LR},\end{aligned}$$

where $0<m_1m_2-(m_1m_2+m_2r_1-m_1l_2-r_1l_2) \leq m_1m_2-\min\{m_1m_2-m_2l_1'+m_1r_2'-l_1'r_2', m_1m_2+m_2r_1'-m_1l_2'-r_1'l_2'\}$ and $0<(m_1m_2+m_2r_1+m_1r_2+r_1r_2)-m_1m_2 \leq \max\{m_1m_2-m_2l_1'-m_1l_2'+l_1'l_2', m_1m_2+m_2r_1'+m_1r_2'+r_1'r_2'\}-m_1m_2$.

Case 5. If $m_2+r_2<0$ and $m_2+r_2' \geq 0$, then $m_2+r_2R^{-1}(\alpha) \leq 0$ for every $\alpha \in (0,1]$, $m_2+r_2'R^{-1}(1-\beta) \leq 0$ for $\beta \leq (1-R(\frac{m_2}{-r_2}))$ and $m_2+r_2'R^{-1}(1-\beta) \geq 0$ for $\beta \geq (1-R(\frac{m_2}{-r_2}))$.

(a) If $m_2+r_2R^{-1}(\alpha) \leq 0$, $m_2+r_2'R^{-1}(1-\beta) \geq 0$ and $m_1-l_1'L^{-1}(1-\beta) \leq 0$, then

$$\begin{aligned}(\tilde{A}_1^I \odot \tilde{A}_2^I)_\alpha &= A_{1\alpha}^I A_{2\alpha}^I = [m_1-l_1L^{-1}(\alpha), m_1+r_1R^{-1}(\alpha)][m_2-l_2L^{-1}(\alpha), m_2+r_2R^{-1}(\alpha)]\\ &= [(m_1+r_1R^{-1}(\alpha))(m_2-l_2L^{-1}(\alpha)), (m_1-l_1L^{-1}(\alpha))(m_2+r_2R^{-1}(\alpha))] \quad (2.77)\end{aligned}$$

Since L and R are decreasing functions on $[0,\infty)$ with $L(0)=R(0)=1$, $\exists$ a unique $\alpha_0 \in (0,1]$ such that $L^{-1}(\alpha_0)=R^{-1}(\alpha_0)=1$. Hence

$$\begin{aligned}(\tilde{A}_1^I \odot \tilde{A}_2^I)_{\alpha_0} &= [(m_1+r_1)(m_2-l_2), (m_1-l_1)(m_2+r_2)]\\ &= [m_1m_2+m_2r_1-m_1l_2-r_1l_2, m_1m_2-m_2l_1+m_1r_2-l_1r_2] \quad (2.78)\end{aligned}$$

(2.78) gives left and right spreads of membership function of $\tilde{A}_1^I \odot \tilde{A}_2^I$ are $-m_2r_1+m_1l_2+r_1l_2$ and $-m_2l_1+m_1r_2-l_1r_2$ respectively.

From (2.43), we get

$$(\tilde{A}_1^I \odot \tilde{A}_2^I)_{(\beta)} = A_{1(\beta)}^I A_{2(\beta)}^I = [m_1 - l_1' L^{-1}(1-\beta), m_1 + r_1' R^{-1}(1-\beta)]$$
$$[m_2 - l_2' L^{-1}(1-\beta), m_2 + r_2' R^{-1}(1-\beta)]$$
$$= [\min\{(m_1 - l_1' L^{-1}(1-\beta))(m_2 + r_2' R^{-1}(1-\beta)),$$
$$(m_1 + r_1' R^{-1}(1-\beta))(m_2 - l_2' L^{-1}(1-\beta))\},$$
$$\max\{(m_1 - l_1' L^{-1}(1-\beta))(m_2 - l_2' L^{-1}(1-\beta)),$$
$$(m_1 + r_1' R^{-1}(1-\beta))(m_2 + r_2' R^{-1}(1-\beta))\}] \quad (2.79)$$

Since L and R are decreasing functions on $[0,\infty)$ with $L(0) = R(0) = 1$, $\exists$ a unique $\beta_0 \in [0,1)$ such that $L^{-1}(1-\beta_0) = R^{-1}(1-\beta_0) = 1$. Hence

$$(\tilde{A}_1^I \odot \tilde{A}_2^I)_{(\beta_0)} = [\min\{(m_1 - l_1')(m_2 + r_2'), (m_1 + r_1')(m_2 - l_2')\},$$
$$\max\{(m_1 - l_1')(m_2 - l_2'), (m_1 + r_1')(m_2 + r_2')\}]$$
$$= [\min\{(m_1 m_2 - m_2 l_1' + m_1 r_2' - l_1' r_2',$$
$$m_1 m_2 + m_2 r_1' - m_1 l_2' - r_1' l_2'\},$$
$$\max\{m_1 m_2 - m_2 l_1' - m_1 l_2' + l_1' l_2',$$
$$m_1 m_2 + m_2 r_1' + m_1 r_2' + r_1' r_2'\}] \quad (2.80)$$

(2.80) gives left and right spreads of non-membership function of $\tilde{A}_1^I \odot \tilde{A}_2^I$ are m_1m_2-$\min\{(m_1m_2$-m_2l_1'+m_1r_2'-$l_1'r_2'$, m_1m_2+m_2r_1'-m_1l_2'-$r_1'l_2'\}$ and $\max\{m_1m_2$-m_2l_1'-m_1l_2'+$l_1'l_2'$, m_1m_2 +m_2r_1'+m_1r_2'+$r_1'r_2'\}$-m_1m_2 respectively.

(b) If $m_2 + r_2 R^{-1}(\alpha) \le 0$, $m_2 + r_2' R^{-1}(1-\beta) \le 0$ and $m_1 - l_1' L^{-1}(1-\beta) \ge 0$, then

$$A_{1\alpha}^I A_{2\alpha}^I = A_{1\alpha}^I A_{2\alpha}^I = [m_1 - l_1 L^{-1}(\alpha), m_1 + r_1 R^{-1}(\alpha)][m_2 - l_2 L^{-1}(\alpha), m_2 + r_2 R^{-1}(\alpha)]$$
$$= [(m_1 + r_1 R^{-1}(\alpha))(m_2 - l_2 L^{-1}(\alpha)), (m_1 - l_1 L^{-1}(\alpha))(m_2 + r_2 R^{-1}(\alpha))] \quad (2.81)$$

Putting $\alpha = 1$ in (2.77), we obtain the modal point of $\tilde{A}_1^I \odot \tilde{A}_2^I$ given by

$$(\tilde{A}_1^I \odot \tilde{A}_2^I)_1 = [m_1 m_2, m_1 m_2] = m_1 m_2 \quad (2.82)$$

Since $m_1 - l_1' < 0$, $m_1 - l_1 \ge 0$, $m_2 + r_2 < 0$, $m_2 + r_2' \ge 0$, $0 < l_1 \le l_1'$, $0 < l_2 \le l_2'$, $0 < r_1 \le r_1'$, $0 < r_2 \le r_2'$, $(m_1 - l_1') \le (m_1 - l_1)$, $(m_2 - l_2') \le (m_2 - l_2)$, $(m_1 + r_1) \le (m_1 + r_1')$, and $(m_2 + r_2) \leq (m_2 + r_2')$.

$$\Rightarrow -(m_1 - l_1')(m_2 + r_2') \geq -(m_1 - l_1)(m_2 + r_2') \geq -(m_1 + r_1)(m_2 + r_2') \geq -(m_1 + r_1)(m_2 - l_2), -(m_2 - l_2')(m_1 + r_1') \geq -(m_1 + r_1)(m_2 - l_2) \quad (\because m_2 + r_2 \leq 0, m_1 - l_1 \geq 0)$$

$$\Rightarrow m_1m_2 - m_2l_1' + m_1r_2' - l_1'r_2' \leq m_1m_2 + m_2r_1 - m_1l_2 - r_1l_2,$$

$$m_1m_2 + m_2r_1 - m_1l_2 - r_1l_2 \geq m_1m_2 + m_2r_1' - m_1l_2' - r_1'l_2'$$

$$\Rightarrow m_1m_2 - (m_1m_2 + m_2r_1 - m_1l_2 - r_1l_2) \leq m_1m_2 - \min\{m_1m_2 - m_2l_1' + m_1r_2' - l_1'r_2', m_1m_2 + m_2r_1' - m_1l_2' - r_1'l_2'\} \tag{2.83}$$

Now, if $(m_1 + r_1')(m_2 + r_2') \leq (m_1 - l_1')(m_2 - l_2')$, then

$$(m_1 - l_1)(m_2 + r_2) \leq (m_1 - l_1')(m_2 + r_2) \leq (m_1 - l_1')(m_2 - l_2') \quad (\because m_2 + r_2 \leq 0, m_1 - l_1 \geq 0) \tag{2.84}$$

If $(m_1 + r_1')(m_2 + r_2') \geq (m_1 - l_1')(m_2 - l_2')$, then

$$(m_1 - l_1)(m_2 + r_2) \leq (m_1 - l_1')(m_2 + r_2) \leq (m_1 - l_1')(m_2 - l_2') \leq (m_1 + r_1')(m_2 + r_2') \tag{2.85}$$

From (2.84) and (2.85), we get
$(m_1m_2 - m_2l_1 + m_1r_2 - l_1r_2) - m_1m_2 \leq \max\{m_1m_2 - m_2l_1' - m_1l_2' + l_1'l_2', m_1m_2 + m_2r_1' + m_1r_2' + r_1'r_2'\} - m_1m_2$.
Clearly, $m_1m_2 - (m_1m_2 + m_2r_1 - m_1l_2 - r_1l_2) = r_1(l_2 - m_2) + m_1l_2 > 0$, $(m_1m_2 - m_2l_1 + m_1r_2 - l_1r_2) - m_1m_2 = -l_1(m_2 + r_2) + m_1r_2 > 0$.
From (2.78), (2.80) and (2.82) of Case 5, we get

$$\begin{aligned}\tilde{A}_1^I \odot \tilde{A}_2^I \approx &(m_1m_2; m_1m_2 - (m_1m_2 + m_2r_1 - m_1l_2 - r_1l_2), (m_1m_2 - m_2l_1 + m_1r_2 - l_1r_2) - \\ &m_1m_2; m_1m_2 - \min\{m_1m_2 - m_2l_1' + m_1r_2' - l_1'r_2', m_1m_2 + m_2r_1' - m_1l_2' - r_1'l_2'\}, \\ &\max\{m_1m_2 - m_2l_1' - m_1l_2' + l_1'l_2', m_1m_2 + m_2r_1' + m_1r_2' + r_1'r_2'\} - m_1m_2)_{LR},\end{aligned}$$

where $0 < m_1m_2 - (m_1m_2 + m_2r_1 - m_1l_2 - r_1l_2) \leq m_1m_2 - \min\{m_1m_2 - m_2l_1' + m_1r_2' - l_1'r_2', m_1m_2 + m_2r_1' - m_1l_2' - r_1'l_2'\}$ and $0 < (m_1m_2 - m_2l_1 + m_1r_2 - l_1r_2) - m_1m_2 \leq \max\{m_1m_2 - m_2l_1' - m_1l_2' + l_1'l_2', m_1m_2 + m_2r_1' + m_1r_2' + r_1'r_2'\} - m_1m_2$.

Case 6. If $m_2 + r_2' < 0$, then $m_2 + r_2'R^{-1}(1-\beta) \leq 0$ for every $\beta \in [0, 1)$.

(a) If $m_1 - l_1'L^{-1}(1-\beta) \leq 0$ and $m_2 + r_2'R^{-1}(1-\beta) \leq 0$, then

$$\begin{aligned}(\tilde{A}_1^I \odot \tilde{A}_2^I)_\alpha &= A_{1\alpha}^I A_{2\alpha}^I = [m_1 - l_1L^{-1}(\alpha), m_1 + r_1R^{-1}(\alpha)][m_2 - l_2L^{-1}(\alpha), m_2 + r_2R^{-1}(\alpha)] \\ &= [(m_1 + r_1R^{-1}(\alpha))(m_2 - l_2L^{-1}(\alpha)), (m_1 - l_1L^{-1}(\alpha))(m_2 + r_2R^{-1}(\alpha))]\end{aligned} \tag{2.86}$$

Since L and R are decreasing functions on $[0,\infty)$ with $L(0) = R(0) = 1$, $\exists$ a unique $\alpha_0 \in (0,1]$ such that $L^{-1}(\alpha_0) = R^{-1}(\alpha_0) = 1$. Hence

$$\begin{aligned}(\tilde{A}_1^I \odot \tilde{A}_2^I)_{\alpha_0} &= [(m_1+r_1)(m_2-l_2), (m_1-l_1)(m_2+r_2)] \\ &= [m_1m_2 + m_2r_1 - m_1l_2 - r_1l_2, m_1m_2 - m_2l_1 + m_1r_2 - l_1r_2] \end{aligned} \tag{2.87}$$

(2.87) gives left and right spreads of membership function of $\tilde{A}_1^I \odot \tilde{A}_2^I$ are $-m_2r_1+m_1l_2+r_1l_2$ and $-m_2l_1+m_1r_2-l_1r_2$ respectively. From (2.43), we get

$$\begin{aligned}(\tilde{A}_1^I \odot \tilde{A}_2^I)_{(\beta)} = A^I_{1(\beta)}A^I_{2(\beta)} =& [m_1 - l_1'L^{-1}(1-\beta), m_1 + r_1'R^{-1}(1-\beta)] \\ & [m_2 - l_2'L^{-1}(1-\beta), m_2 + r_2'R^{-1}(1-\beta)] \\ =& [(m_1 + r_1'R^{-1}(1-\beta))(m_2 - l_2'L^{-1}(1-\beta)), \\ & (m_1 - l_1'L^{-1}(1-\beta))(m_2 - l_2'L^{-1}(1-\beta))] \end{aligned} \tag{2.88}$$

Since L and R are decreasing functions on $[0,\infty)$ with $L(0) = R(0) = 1$, $\exists$ a unique $\beta_0 \in [0,1)$ such that $L^{-1}(1-\beta_0) = R^{-1}(1-\beta_0) = 1$. Hence

$$\begin{aligned}(\tilde{A}_1^I \odot \tilde{A}_2^I)_{(\beta_0)} &= [(m_1+r_1')(m_2-l_2'), (m_1-l_1')(m_2-l_2')] \\ &= [m_1m_2 + m_2r_1' - m_1l_2' - r_1'l_2', m_1m_2 - m_2l_1' - m_1l_2' + l_1'l_2'] \end{aligned} \tag{2.89}$$

(2.89) gives left and right spreads of non-membership function of $\tilde{A}_1^I \odot \tilde{A}_2^I$ are $-m_2r_1' + m_1l_2' + r_1'l_2'$ and $-m_2l_1' - m_1l_2' + l_1'l_2'$ respectively.

(b) If $m_1 - l_1'L^{-1}(1-\beta) \geq 0$ and $m_2 + r_2'R^{-1}(1-\beta) \leq 0$, then

$$\begin{aligned}(\tilde{A}_1^I \odot \tilde{A}_2^I)_{\alpha} = A^I_{1\alpha}A^I_{2\alpha} =& [m_1 - l_1L^{-1}(\alpha), m_1 + r_1R^{-1}(\alpha)] \\ & [m_2 - l_2L^{-1}(\alpha), m_2 + r_2R^{-1}(\alpha)] \\ =& [(m_1 + r_1R^{-1}(\alpha))(m_2 - l_2L^{-1}(\alpha)), \\ & (m_1 - l_1L^{-1}(\alpha))(m_2 + r_2R^{-1}(\alpha))] \end{aligned} \tag{2.90}$$

Putting $\alpha = 1$ in (2.90), we obtain the modal point of $\tilde{A}_1^I \odot \tilde{A}_2^I$ given by

$$(\tilde{A}_1^I \odot \tilde{A}_2^I)_1 = [m_1m_2, m_1m_2] = m_1m_2 \tag{2.91}$$

Since $m_1 - l_1' \leq 0$, $m_1 - l_1 \geq 0$, $m_2 + r_2' < 0$, $0 < l_2 \leq l_2'$, $0 < r_1 \leq r_1'$, $0 < r_2 \leq r_2'$,

$m_1 - l'_1 \le m_1 - l_1$, $m_2 - l'_2 \le m_2 - l_2$, $m_1 + r_1 \le m_1 + r'_1$, and $m_2 + r_2 \le m_2 + r'_2$.

$$\Rightarrow \; -(m_1 + r'_1)(m_2 - l'_2) \ge -(m_1 + r_1)(m_2 - l_2), (m_1 - l_1)(m_2 + r_2) \le (m_1 - l'_1)(m_2 - l'_2)$$
$$(\because m_2 + r'_2 < 0, m_1 - l'_1 < 0, m_1 - l_1 \ge 0)$$
$$\Rightarrow m_1 m_2 + m_2 r'_1 - m_1 l'_2 - r'_1 l'_2 \le m_1 m_2 + m_2 r_1 - m_1 l_2 - r_1 l_2,$$
$$m_1 m_2 - m_2 l_1 + m_1 r_2 - l_1 r_2 \le m_1 m_2 - m_2 l'_1 - m_1 l'_2 + l'_1 l'_2$$
$$\Rightarrow m_1 m_2 - (m_1 m_2 + m_2 r'_1 - m_1 l'_2 - r'_1 l'_2) \ge m_1 m_2 - (m_1 m_2 + m_2 r_1 - m_1 l_2 - r_1 l_2),$$
$$(m_1 m_2 - m_2 l_1 + m_1 r_2 - l_1 r_2) - m_1 m_2 \le (m_1 m_2 - m_2 l'_1 - m_1 l'_2 + l'_1 l'_2) - m_1 m_2$$

Clearly, $m_1 m_2 - (m_1 m_2 + m_2 r_1 - m_1 l_2 - r_1 l_2) = r_1(l_2 - m_2) + m_1 l_2 > 0$, $(m_1 m_2 - m_2 l_1 + m_1 r_2 - l_1 r_2) - m_1 m_2 = -l_1(m_2 + r_2) + m_1 r_2 > 0$.

From (2.87), (2.89) and (2.91) of Case 6, we get

$$\begin{aligned}\tilde{A}^I_1 \odot \tilde{A}^I_2 \approx &(m_1 m_2; m_1 m_2 - (m_1 m_2 - m_1 l_2 + m_2 r_1 - r_1 l_2), (m_1 m_2 + m_1 r_2 - m_2 l_1 - l_1 r_2) \\ &- m_1 m_2; m_1 m_2 - (m_1 m_2 - m_1 l'_2 + m_2 r'_1 - r'_1 l'_2), (m_1 m_2 - m_2 l'_1 - m_1 l'_2 + l'_1 l'_2) \\ &- m_1 m_2)_{LR},\end{aligned}$$

where $0 < m_1 m_2 - (m_1 m_2 - m_1 l_2 + m_2 r_1 - r_1 l_2) \le m_1 m_2 - (m_1 m_2 - m_1 l'_2 + m_2 r'_1 - r'_1 l'_2)$ and $0 < (m_1 m_2 + m_1 r_2 - m_2 l_1 - l_1 r_2) - m_1 m_2 \le (m_1 m_2 - m_2 l'_1 - m_1 l'_2 + l'_1 l'_2) - m_1 m_2$.

From all six cases, we have
$\tilde{A}^I_1 \odot \tilde{A}^I_2 \approx (m; l, r; l', r')_{LR}$, where $m = m_1 m_2$, $l = m_1 m_2 - \min\{m_1 m_2 - m_1 l_2 - m_2 l_1 + l_1 l_2, m_1 m_2 - m_1 l_2 + m_2 r_1 - l_2 r_1\}$, $r = \max\{m_1 m_2 + m_1 r_2 + m_2 r_1 + r_1 r_2, m_1 m_2 + m_1 r_2 - m_2 l_1 - l_1 r_2\} - m_1 m_2$, $l' = m_1 m_2 - \min\{m_1 m_2 - m_2 l'_1 + m_1 r'_2 - l'_1 r'_2, m_1 m_2 + m_2 r'_1 - m_1 l'_2 - l'_2 r'_1\}$, $r' = \max\{m_1 m_2 - m_2 l'_1 - m_1 l'_2 + l'_1 l'_2, m_1 m_2 + m_2 r'_1 + m_1 r'_2 + r'_1 r'_2\} - m_1 m_2$, and $0 < l \le l', 0 < r \le r'$. □

Theorem 2.3.2. *Let $\tilde{A}^I_1$=$(m_1; l_1, r_1; l'_1, r'_1)_{LR}$ be LR-type IFN, where $m_1 - l_1 < 0$, $m_1 \ge 0$ and $\tilde{A}^I_2$=$(m_2; l_2, r_2; l'_2, r'_2)_{LR}$ be another LR-type IFN, where $m_2 - l'_2$, $m_2 - l_2$, m_2, $m_2 + r_2$, $m_2 + r'_2$ are real numbers. Then $\tilde{A}^I_1 \odot \tilde{A}^I_2 \approx (m; l, r; l', r')_{LR}$, where $m = m_1 m_2$, $l = m_1 m_2 - \min\{m_1 m_2 - m_2 l_1 + m_1 r_2 - l_1 r_2, m_1 m_2 + m_2 r_1 - m_1 l_2 - l_2 r_1\}$, $r = \max\{m_1 m_2 - m_2 l_1 - m_1 l_2 + l_1 l_2, m_1 m_2 + m_2 r_1 + m_1 r_2 + r_1 r_2\} - m_1 m_2$, $l' = m_1 m_2 - \min\{m_1 m_2 - m_2 l'_1 + m_1 r'_2 - l'_1 r'_2, m_1 m_2 + m_2 r'_1 - m_1 l'_2 - l'_2 r'_1\}$, $r' = \max\{m_1 m_2 - m_2 l'_1 - m_1 l'_2 + l'_1 l'_2, m_1 m_2 + m_2 r'_1 + m_1 r'_2 + r'_1 r'_2\} - m_1 m_2$, and $0 < l \le l', 0 < r \le r'$.*

Proof. The same as Theorem 2.3.1. □

Theorem 2.3.3. *Let $\tilde{A}^I_1$=$(m_1; l_1, r_1; l'_1, r'_1)_{LR}$ be LR-type IFN, where $m_1 < 0$, $m_1 + r_1 \ge 0$ and $\tilde{A}^I_2$=$(m_2; l_2, r_2; l'_2, r'_2)_{LR}$ be another LR-type IFN, where $m_2 - l'_2$, $m_2 - l_2$, m_2, $m_2 + r_2$, $m_2 + r'_2$*

are real numbers. Then $\tilde{A}_1^I \odot \tilde{A}_2^I \approx (m; l, r; l', r')_{LR}$*, where* $m = m_1m_2$, $l = m_1m_2 - \min\{m_1m_2 - m_2l_1 + m_1r_2 - l_1r_2, m_1m_2 + m_2r_1 - m_1l_2 - l_2r_1\}$, $r = \max\{m_1m_2 - m_2l_1 - m_1l_2 + l_1l_2, m_1m_2 + m_2r_1 + m_1r_2 + r_1r_2\} - m_1m_2$, $l' = m_1m_2 - \min\{m_1m_2 - m_2l_1' + m_1r_2' - l_1'r_2', m_1m_2 + m_2r_1' - m_1l_2' - l_2'r_1'\}$, $r' = \max\{m_1m_2 - m_2l_1' - m_1l_2' + l_1'l_2', m_1m_2 + m_2r_1' + m_1r_2' + r_1'r_2'\} - m_1m_2$*, and* $0 < l \leq l', 0 < r \leq r'$.

Proof. The same as Theorem 2.3.1. □

Theorem 2.3.4. *Let* $\tilde{A}_1^I = (m_1; l_1, r_1; l_1', r_1')_{LR}$ *be LR-type IFN, where* $m_1 + r_1 < 0$, $m_1 + r_1' \geq 0$ *and* $\tilde{A}_2^I = (m_2; l_2, r_2; l_2', r_2')_{LR}$ *be another LR-type IFN, where* $m_2 - l_2'$, $m_2 - l_2$, m_2, $m_2 + r_2$, $m_2 + r_2'$ *are real numbers. Then* $\tilde{A}_1^I \odot \tilde{A}_2^I \approx (m; l, r; l', r')_{LR}$*, where* $m = m_1m_2$, $l = m_1m_2 - \min\{m_1m_2 - m_2l_1 + m_1r_2 - l_1r_2, m_1m_2 + m_1r_2 + m_2r_1 + r_1r_2\}$, $r = \max\{m_1m_2 + m_2r_1 - m_1l_2 - l_2r_1, m_1m_2 - m_1l_2 - m_2l_1 + l_1l_2\} - m_1m_2$, $l' = m_1m_2 - \min\{m_1m_2 - m_2l_1' + m_1r_2' - l_1'r_2', m_1m_2 + m_2r_1' - m_1l_2' - l_2'r_1'\}$, $r' = \max\{m_1m_2 - m_2l_1' - m_1l_2' + l_1'l_2', m_1m_2 + m_2r_1' + m_1r_2' + r_1'r_2'\} - m_1m_2$*, and* $0 < l \leq l', 0 < r \leq r'$.

Proof. The same as Theorem 2.3.1. □

Theorem 2.3.5. *Let* $\tilde{A}_1^I = (m_1; l_1, r_1; l_1', r_1')_{LR}$ *be LR-type IFN, where* $m_1 + r_1' < 0$ *and* $\tilde{A}_2^I = (m_2; l_2, r_2; l_2', r_2')_{LR}$ *be another LR-type IFN, where* $m_2 - l_2'$, $m_2 - l_2$, m_2, $m_2 + r_2$, $m_2 + r_2'$ *are real numbers. Then* $\tilde{A}_1^I \odot \tilde{A}_2^I \approx (m; l, r; l', r')_{LR}$*, where* $m = m_1m_2$, $l = m_1m_2 - \min\{m_1m_2 - m_2l_1 + m_1r_2 - l_1r_2, m_1m_2 + m_1r_2 + m_2r_1 + r_1r_2\}$, $r = \max\{m_1m_2 + m_2r_1 - m_1l_2 - l_2r_1, m_1m_2 - m_1l_2 - m_2l_1 + l_1l_2\} - m_1m_2$, $l' = m_1m_2 - \min\{m_1m_2 - m_2l_1' + m_1r_2' - l_1'r_2', m_1m_2 + m_1r_2' + m_2r_1' + r_1'r_2'\}$, $r' = \max\{m_1m_2 + m_2r_1' - m_1l_2' - l_2'r_1', m_1m_2 - m_2l_1' - m_1l_2' + l_1'l_2'\} - m_1m_2$*, and* $0 < l \leq l', 0 < r \leq r'$.

Proof. The same as Theorem 2.3.1. □

Theorem 2.3.6. *Let* $\tilde{A}_1^I = (m_1; l_1, r_1; l_1', r_1')_{LR}$ *be LR-type IFN, where* $m_1 - l_1' \geq 0$ *and* $\tilde{A}_2^I = (m_2; l_2, r_2; l_2', r_2')_{LR}$ *be another LR-type IFN, where* $m_2 - l_2'$, $m_2 - l_2$, m_2, $m_2 + r_2$, $m_2 + r_2'$ *are real numbers. Then* $\tilde{A}_1^I \odot \tilde{A}_2^I \approx (m; l, r; l', r')_{LR}$*, where* $m = m_1m_2$, $l = m_1m_2 - \min\{m_1m_2 - m_1l_2 - m_2l_1 + l_1l_2, m_1m_2 - m_1l_2 + m_2r_1 - l_2r_1\}$, $r = \max\{m_1m_2 + m_1r_2 + m_2r_1 + r_1r_2, m_1m_2 + m_1r_2 - m_2l_1 - l_1r_2\} - m_1m_2$, $l' = m_1m_2 - \min\{m_1m_2 - m_1l_2' - m_2l_1' + l_1'l_2', m_1m_2 - m_1l_2' + m_2r_1' - l_2'r_1'\}$, $r' = \max\{m_1m_2 + m_1r_2' + m_2r_1' + r_1'r_2', m_1m_2 + m_1r_2' - m_2l_1' - l_1'r_2'\} - m_1m_2$*, and* $0 < l \leq l', 0 < r \leq r'$.

Proof. The same as Theorem 2.3.1. □

Example 2.3.7. *Let* $\tilde{A}_1^I = (2; 4, 5; 5, 6)_{LR}$ *and* $\tilde{A}_2^I = (5; 4, 2; 7, 3)_{LR}$*. Then to find the product of* $\tilde{A}_1^I$ *and* $\tilde{A}_2^I$.

The product of $\tilde{A}_1^I$ *and* $\tilde{A}_2^I$, *i.e.,* $\tilde{A}_1^I \odot \tilde{A}_2^I$ *is*

$$\tilde{A}_1^I \odot \tilde{A}_2^I = (10; 24, 39; 34, 54)_{LR}$$

Example 2.3.8. *Let* $\tilde{A}_1^I = (-2; 1, 2; 1, 3)_{LR}$ *and* $\tilde{A}_2^I = (5; 3, 2; 6, 3)_{LR}$. *Then to find the product of* $\tilde{A}_1^I$ *and* $\tilde{A}_2^I$.

The product of $\tilde{A}_1^I$ *and* $\tilde{A}_2^I$, *i.e.,* $\tilde{A}_1^I \odot \tilde{A}_2^I$ *is*

$$\tilde{A}_1^I \odot \tilde{A}_2^I = (-10; 11, 10; 14, 18)_{LR}$$

Thus, it is clear that the proposed product is better to find the product of $\tilde{A}_1^I$ and $\tilde{A}_2^I$, when the signs of $\tilde{A}_1^I$ and $\tilde{A}_2^I$ are known or not known.

2.4 Proposed Method for Fully Intuitionistic Fuzzy Programming Problem (FIFPP)

In this section, a new method is proposed to find the optimal solution of FIFPP with LR-type intuitionistic fuzzy parameters.

The problem is defined by

$$\begin{aligned}
\min/\max \quad & \tilde{z}^I = \sum_{j=1}^{n} \tilde{c}_j^I \odot \tilde{x}_j^I, \\
\text{subject to} \quad & \sum_{j=1}^{n} \tilde{a}_{ij}^I \odot \tilde{x}_j^I \prec, \approx, \succ \tilde{b}_i^I, i = 1, 2, 3, \ldots, m, \\
& \tilde{x}_j^I \text{ is LR-type unrestricted IFN}, \; j = 1, 2, \ldots, n.
\end{aligned}$$

The steps of the proposed method are as follows:

Step 1. Let $\tilde{a}_{ij}^I = (a_{ij}; \gamma_{ij}, \delta_{ij}; \gamma'_{ij}, \delta'_{ij})_{LR}$, $\tilde{x}_j^I = (x_j; \rho_j, \sigma_j; \rho'_j, \sigma'_j)_{LR}$, $\tilde{b}_i^I = (b_i; \eta_i, \vartheta_i; \eta'_i, \vartheta'_i)_{LR}$ and $\tilde{c}_j^I = (c_j; \kappa_j, \lambda_j; \kappa'_j, \lambda'_j)_{LR}$.

Then FIFPP is reduced to

$$\begin{aligned}
\min/\max \quad & \tilde{z}^I = \sum_{j=1}^{n} ((c_j; \kappa_j, \lambda_j; \kappa'_j, \lambda'_j)_{LR} \odot (x_j; \rho_j, \sigma_j; \rho'_j, \sigma'_j)_{LR}), \\
\text{subject to} \quad & \sum_{j=1}^{n} ((a_{ij}; \gamma_{ij}, \delta_{ij}; \gamma'_{ij}, \delta'_{ij})_{LR} \odot (x_j; \rho_j, \sigma_j; \rho'_j, \sigma'_j)_{LR}) \prec, \approx, \succ (b_i; \eta_i, \vartheta_i; \eta'_i, \vartheta'_i)_{LR}, \\
& i = 1, 2, \ldots, m, \\
& (x_j; \rho_j, \sigma_j; \rho'_j, \sigma'_j)_{LR} \text{ is LR-type unrestricted IFN}, \; j = 1, 2, \ldots, n.
\end{aligned}$$

Step 2. Let $(c_j; \kappa_j, \lambda_j; \kappa'_j, \lambda'_j)_{LR} \odot (x_j; \rho_j, \sigma_j; \rho'_j, \sigma'_j)_{LR} = (s_j; \tau_j, \omega_j; \tau'_j, \omega'_j)_{LR}$ and $(a_{ij};\ \gamma_{ij},\ \delta_{ij};\ \gamma'_{ij},\ \delta'_{ij}\ )_{LR} \odot (x_j;\ \rho_j,\ \sigma_j;\ \rho'_j, \sigma'_j)_{LR} = (m_{ij}; l_{ij}, r_{ij}; l'_{ij}, r'_{ij})_{LR}$.

Then LR-type FIFPP in Step 1 is transformed to the following problem:

$$\begin{aligned}
\text{min/max}\quad & \tilde{z}^I = \sum_{j=1}^{n} (s_j; \tau_j, \omega_j; \tau'_j, \omega'_j)_{LR},\\
\text{subject to}\quad & \sum_{j=1}^{n} (m_{ij}; l_{ij}, r_{ij}; l'_{ij}, r'_{ij})_{LR} \prec, \approx, \succ (b_i; \eta_i, \vartheta_i; \eta'_i, \vartheta'_i)_{LR}, i = 1, 2, \ldots, m,\\
& (s_j; \tau_j, \omega_j; \tau'_j, \omega'_j)_{LR} \text{ and } (m_{ij}; l_{ij}, r_{ij}; l'_{ij}, r'_{ij})_{LR} \text{ are LR-type unrestricted IFNs},\\
& i = 1, 2, \ldots, m,\ j = 1, 2, \ldots, n.
\end{aligned}$$

Simplifying the above problem, we have

$$\begin{aligned}
\text{min/max}\quad & \tilde{z}^I = (\sum_{j=1}^{n} s_j; \sum_{j=1}^{n} \tau_j, \sum_{j=1}^{n} \omega_j; \sum_{j=1}^{n} \tau'_j, \sum_{j=1}^{n} \omega'_j)_{LR},\\
\text{subject to}\quad & (\sum_{j=1}^{n} m_{ij}; \sum_{j=1}^{n} l_{ij}, \sum_{j=1}^{n} r_{ij}; \sum_{j=1}^{n} l'_{ij}, \sum_{j=1}^{n} r'_{ij})_{LR} \prec, \approx, \succ (b_i; \eta_i, \vartheta_i; \eta'_i, \vartheta'_i)_{LR},\\
& i = 1, 2, \ldots, m,\\
& (\sum_{j=1}^{n} s_j; \sum_{j=1}^{n} \tau_j, \sum_{j=1}^{n} \omega_j; \sum_{j=1}^{n} \tau'_j, \sum_{j=1}^{n} \omega'_j)_{LR} \text{ and } (\sum_{j=1}^{n} m_{ij}; \sum_{j=1}^{n} l_{ij}, \sum_{j=1}^{n} r_{ij};\\
& \sum_{j=1}^{n} l'_{ij}, \sum_{j=1}^{n} r'_{ij})_{LR} \text{ are LR-type unrestricted IFNs},\ i = 1, 2, \ldots, m.
\end{aligned}$$

Step 3. Applying score and accuracy indices and solve the following programming problems

$$\begin{aligned}
\text{min/max}\quad & w_1(I_S(\sum_{j=1}^{n} s_j; \sum_{j=1}^{n} \tau_j, \sum_{j=1}^{n} \omega_j; \sum_{j=1}^{n} \tau'_j, \sum_{j=1}^{n} \omega'_j)_{LR}+\\
& w_2(I_A(\sum_{j=1}^{n} s_j; \sum_{j=1}^{n} \tau_j, \sum_{j=1}^{n} \omega_j; \sum_{j=1}^{n} \tau'_j, \sum_{j=1}^{n} \omega'_j)_{LR})\\
\text{subject to}\quad & I_A(\sum_{j=1}^{n} m_{ij}; \sum_{j=1}^{n} l_{ij}, \sum_{j=1}^{n} r_{ij}; \sum_{j=1}^{n} l'_{ij}, \sum_{j=1}^{n} r'_{ij})_{LR} <, =, > I_A(b_i; \eta_i, \vartheta_i; \eta'_i, \vartheta'_i)_{LR},\\
& i = 1, 2, \ldots, m,\\
& w_1 + w_2 = 1, w_1, w_2 \geq 0,\\
& \sum_{j=1}^{n} \tau'_j \geq \sum_{j=1}^{n} \tau_j, \sum_{j=1}^{n} \omega'_j \geq \sum_{j=1}^{n} \omega_j, \sum_{j=1}^{n} \tau_j > 0, \sum_{j=1}^{n} \omega_j > 0,
\end{aligned}$$

$$\sum_{j=1}^{n} l'_{ij} \geq \sum_{j=1}^{n} l_{ij}, \sum_{j=1}^{n} r'_{ij} \geq \sum_{j=1}^{n} r_{ij}, \sum_{j=1}^{n} l_{ij} \geq 0, \sum_{j=1}^{n} r_{ij} \geq 0,$$
$$\sum_{j=1}^{n} s_j \text{ and } \sum_{j=1}^{n} m_{ij}\ (i = 1, 2, \ldots, m) \text{ are real numbers.}$$

Step 4. Solve the non-linear programming problem obtained in Step 3, to find the optimal values $s_j^*, \tau_j^*, \omega_j^*, \tau_j^{'*}, \omega_j^{'*}$ and put their values in Step 2, i.e., in $(c_j; \kappa_j, \lambda_j; \kappa'_j, \lambda'_j)_{LR} \odot (x_j; \rho_j, \sigma_j; \rho'_j, \sigma'_j)_{LR} = (s_j; \tau_j, \omega_j; \tau'_j, \omega'_j)_{LR}$ to find the optimal value of $\tilde{x}_j^I = (x_j; \rho_j, \sigma_j; \rho'_j, \sigma'_j)_{LR}$. Let it be $\tilde{x}_j^{*I} = (x_j^*; \rho_j^*, \sigma_j^*; \rho_j^{'*}, \sigma_j^{'*})_{LR}$.

Step 5. Find the fuzzy optimal value of LR-type FIFPP by putting the optimal value $\tilde{x}_j^{*I}$ in $\tilde{z}^I = \sum_{j=1}^{n} \tilde{c}_j^I \odot \tilde{x}_j^I$.

2.5 Numerical Example

Let us consider the following problem: A manufacturing company plans to produce a product for next three consecutive months: January, February and March. The demands of the product for January, February and March are around 520, 700 and 600 units respectively. However, the demands fluctuate according to situations of the market. The company has 10 employees. But to meet the fluctuating demands, company needs to hire or fire temporary employees. The extra cost of hiring or firing is around Rs. 200 which also fluctuates according to availability of workers at the market. A permanent worker can produce around 12 units per month and a temporary worker, having lack of work efficiencies, only can produce around 10 units per month. It is obvious that these production units depend on various uncontrollable factors such as efficiency of machines, availability of resources, and efficiency of workers etc. The company can produce more than needed in any month and carry the surplus over to the next month at a holding cost of around Rs. 50 per unit per month. The general manager (GM) wants to develop an optimal hiring/firing policy to minimize the total cost.

Mathematical Model: It is clear from the problem that the parameters involve uncertainty as well as hesitation both due to the fluctuating nature of the market. So, it is more reliable to denote the parameters with general IFNs (LR-type IFNs). The GM estimates the parameters as given below by the past experiences and discussion with other fellow members.
The demand parameters:

$$\tilde{520}^I = (520; 2, 3; 3, 5)_{LR},\ \tilde{700}^I = (700; 3, 4; 4, 5)_{LR} \text{ and } \tilde{600}^I = (600; 35, 23; 68, 55)_{LR}.$$

The cost parameters:

$$\tilde{50}^I = (50; 4, 5; 5, 6)_{LR} \text{ and } \tilde{200}^I = (200; 5, 6; 6, 7)_{LR}$$

The production parameters:

$$\tilde{12}^I = (12; 1, 2; 2, 3)_{LR} \text{ and } \tilde{10}^I = (10; 1, 1; 2, 2)_{LR}.$$

Now,

Demand for January = $\tilde{520}^I \ominus 10(\tilde{12}^I) \approx (520; 2, 3; 3, 5)_{LR} \ominus 10(12; 1, 2; 2, 3)_{LR} \approx$ (400; 22, 13; 33, 25)$_{LR}$ units

Demand for February = $\tilde{700}^I \ominus 10(\tilde{12}^I) \approx (700; 3, 4; 4, 5)_{LR} \ominus 10(12; 1, 2; 2, 3)_{LR} \approx$ (580; 23, 14; 34, 25)$_{LR}$ units

Demand for March = $\tilde{600}^I \ominus 10(\tilde{12}^I) \approx (600; 35, 23; 68, 55)_{LR} \ominus 10(12; 1, 2; 2, 3)_{LR} \approx$ (480; 55, 33; 98, 75)$_{LR}$ units.

For i= 1, 2, 3,

$\tilde{x}_i^I = (x_i; \alpha_i, \beta_i; \alpha_i', \beta_i')_{LR}$ = Number of temporary workers at the start of Month 'i' after any hiring or firing

$\tilde{s}_i^I = (s_i; \rho_i, \sigma_i; \rho_i', \sigma_i')_{LR}$ = Number of temporary workers hired or fired at the start of Month 'i'

$\tilde{y}_i^I = (y_i; \gamma_i, \delta_i; \gamma_i', \delta_i')_{LR}$ = Units of ending inventory for Month 'i'.

If temporary workers are hired, then $\tilde{s}_i^I$ is nonnegative and if fired, then $\tilde{s}_i^I$ is non-positive. Thus $\tilde{s}_i^I$ is unrestricted LR-type IFN. Clearly, $\tilde{x}_i^I$, and $\tilde{y}_i^I$ are nonnegative. The objective of the problem is to minimize the sum of total cost of hiring or firing and the total cost of holding inventory from one month to the next.

Inventory holding cost = $\tilde{50}^I \odot (\tilde{y}_1^I \oplus \tilde{y}_2^I)$ ($\because \tilde{y}_3^I$ in the optimal solution)

Cost of hiring or firing = $\tilde{200}^I \odot (\tilde{s}_1^I \oplus \tilde{s}_2^I \oplus \tilde{s}_3^I)$

Thus the inventory constraints are

$$10\tilde{x}_1^I \approx (400; 22, 13; 33, 25)_{LR} \oplus \tilde{y}_1^I,$$
$$\tilde{y}_1^I \oplus 10\tilde{x}_2^I \approx (580; 23, 14; 34, 25)_{LR} \oplus \tilde{y}_2^I,$$
$$\tilde{y}_2^I \oplus 10\tilde{x}_3^I \approx (480; 55, 33; 98, 75)_{LR},$$

where $\tilde{x}_1^I$, $\tilde{x}_2^I$, $\tilde{x}_3^I$, $\tilde{y}_1^I$, and $\tilde{y}_2^I$ are nonnegative LR-type IFNs.

Now, other constraints depend on the model deal with inventory and hiring or firing are as

$$\tilde{x}_1^I \approx \tilde{s}_1^I,$$
$$\tilde{x}_2^I \approx \tilde{x}_1^I \oplus \tilde{s}_2^I,$$
$$\tilde{x}_3^I \approx \tilde{x}_2^I \oplus \tilde{s}_3^I,$$

where $\tilde{x}_1^I$, $\tilde{x}_2^I$ and $\tilde{x}_3^I$ are nonnegative LR-type IFNs and $\tilde{s}_1^I$, $\tilde{s}_2^I$ and $\tilde{s}_3^I$ are unrestricted LR-type IFNs.

The complete model is

$$\min\ \tilde{z}^I = \tilde{50}^I \odot (\tilde{y}_1^I \oplus \tilde{y}_2^I) \oplus \tilde{200}^I \odot (\tilde{s}_1^I \oplus \tilde{s}_2^I \oplus \tilde{s}_3^I),$$

subject to

$$10\tilde{x}_1^I \approx (400; 22, 13; 33, 25)_{LR} \oplus \tilde{y}_1^I,$$
$$\tilde{y}_1^I \oplus 10\tilde{x}_2^I \approx (580; 23, 14; 34, 25)_{LR} \oplus \tilde{y}_2^I,$$
$$\tilde{y}_2^I \oplus 10\tilde{x}_3^I \approx (480; 55, 33; 98, 75)_{LR},$$
$$\tilde{x}_1^I \approx \tilde{s}_1^I,$$
$$\tilde{x}_2^I \approx \tilde{x}_1^I \oplus \tilde{s}_2^I,$$
$$\tilde{x}_3^I \approx \tilde{x}_2^I \oplus \tilde{s}_3^I,$$

where $\tilde{x}_1^I$, $\tilde{x}_2^I$, $\tilde{x}_3^I$, $\tilde{y}_1^I$, and $\tilde{y}_2^I$ are nonnegative LR-type IFNs and $\tilde{s}_1^I$, $\tilde{s}_2^I$ and $\tilde{s}_3^I$ are unrestricted LR-type IFNs.

Applying the proposed product, we get

$\min\ \tilde{z}^I = (50y_1+50y_2; 46\gamma_1+46\gamma_2+4y_1+4y_2, 45\delta_1+45\delta_2+5y_1+5y_2; 45\gamma_1'+45\gamma_2'+5y_1+5y_2, 44\delta_1'+44\delta_2'+6y_1+6y_2)_{LR} \oplus (200s_1+200s_2+200s_3; 200s_1+200s_2+200s_3-\min\{195s_1+195s_2+195s_3-195\rho_1-195\rho_2-195\rho_3, 206s_1+206s_2+206s_3-206\rho_1-206\rho_2-206\rho_3\}, \max\{206s_1+206s_2+206s_3+206\sigma_1+206\sigma_2+206\sigma_3, 195s_1+195s_2+195s_3+195\sigma_1+195\sigma_2+195\sigma_3\}-200s_1-200s_2-200s_3; 200s_1+200s_2+200s_3-\min\{194s_1+194s_2+194s_3-194\rho_1'-194\rho_2'-194\rho_3', 207s_1+207s_2+207s_3-207\rho_1'-207\rho_2'-207\rho_3'\}, \max\{207s_1+207s_2+207s_3+207\sigma_1'+207\sigma_2'+207\sigma_3', 194s_1+194s_2+194s_3+194\sigma_1'+194\sigma_2'+194\sigma_3'\}-200s_1-200s_2-200s_3)_{LR}$,

subject to

$(10x_1; 10\alpha_1, 10\beta_1; 10\alpha_1', 10\beta_1')_{LR} \approx (400+y_1; 22+\gamma_1, 13+\delta_1; 33+\gamma_1', 25+\delta_1')_{LR}$,

$(y_1+10x_2; \gamma_1+10\alpha_2, \delta_1+10\beta_2; \gamma_1'+10\alpha_2', \delta_1'+10\beta_2')_{LR} \approx (580+y_2; 23+\gamma_2, 14+\delta_2; 34+\gamma_2', 25+\delta_2')_{LR}$,

$(y_2+10x_3; \gamma_2+10\alpha_3, \delta_2+10\beta_3; \gamma_2'+10\alpha_3', \delta_2'+10\beta_3')_{LR} \approx (480; 55, 33; 98, 75)_{LR}$,

$(x_1; \alpha_1, \beta_1; \alpha_1', \beta_1')_{LR} \approx (s_1; \rho_1, \sigma_1; \rho_1', \sigma_1')_{LR}$,

$(x_2; \alpha_2, \beta_2; \alpha_2', \beta_2')_{LR} \approx (x_1+s_2; \alpha_1+\rho_2, \beta_1+\sigma_2; \alpha_1'+\rho_2', \beta_1'+\sigma_2')_{LR}$,

$(x_3; \alpha_3, \beta_3; \alpha_3', \beta_3')_{LR} \approx (x_2+s_3; \alpha_2+\rho_3, \beta_2+\sigma_3; \alpha_2'+\rho_3', \beta_2'+\sigma_3')_{LR}$,

where $(x_1; \alpha_1, \beta_1; \alpha_1', \beta_1')_{LR}$, $(x_2; \alpha_2, \beta_2; \alpha_2', \beta_2')_{LR}$, $(x_3; \alpha_3, \beta_3; \alpha_3', \beta_3')_{LR}$, $(y_1; \gamma_1, \delta_1; \gamma_1', \delta_1')_{LR}$, and $(y_2; \gamma_2, \delta_2; \gamma_2', \delta_2')_{LR}$ are nonnegative LR-type IFNs and $(s_1; \rho_1, \sigma_1; \rho_1', \sigma_1')_{LR}$, $(s_2; \rho_2, \sigma_2; \rho_2', \sigma_2')_{LR}$, and $(s_3; \rho_3, \sigma_3; \rho_3', \sigma_3')_{LR}$ are unrestricted LR-type IFNs.

Using $\min(a,b)=\frac{1}{2}(a+b)-\frac{1}{2}|(a-b)|$, $\max(a,b)=\frac{1}{2}(a+b)+\frac{1}{2}|(a-b)|$, we get

$\min\ \tilde{z}^I = (50y_1+50y_2+200s_1+200s_2+200s_3; 46\gamma_1+46\gamma_2+4y_1+4y_2-\frac{1}{2}(s_1+s_2+s_3)+\frac{401}{2}(\rho_1+\rho_2+\rho_3)+\frac{11}{2}|s_1+s_2+s_3-\rho_1-\rho_2-\rho_3|, 45\delta_1+45\delta_2+5y_1+5y_2+\frac{1}{2}(s_1+s_2+s_3)+\frac{401}{2}(\sigma_1+\sigma_2+\sigma_3)+\frac{11}{2}|s_1+$

$s_2+s_3+\sigma_1+\sigma_2+\sigma_3|; 45\gamma_1'+45\gamma_2'+5y_1+5y_2-\frac{1}{2}(s_1+s_2+s_3)+\frac{401}{2}(\rho_1'+\rho_2'+\rho_3')+\frac{13}{2}|s_1+s_2+s_3-\rho_1'-\rho_2'-\rho_3'|, 44\delta_1'+44\delta_2'+6y_1+6y_2+\frac{1}{2}(s_1+s_2+s_3)+\frac{401}{2}(\sigma_1'+\sigma_2'+\sigma_3')+\frac{13}{2}|s_1+s_2+s_3+\sigma_1'+\sigma_2'+\sigma_3'|)_{LR}$,
subject to
$(10x_1; 10\alpha_1, 10\beta_1; 10\alpha_1', 10\beta_1')_{LR} \approx (400 + y_1; 22 + \gamma_1, 13 + \delta_1; 33 + \gamma_1', 25 + \delta_1')_{LR}$,
$(y_1+10x_2; \gamma_1+10\alpha_2, \delta_1+10\beta_2; \gamma_1'+10\alpha_2', \delta_1'+10\beta_2')_{LR} \approx (580+y_2; 23+\gamma_2, 14+\delta_2; 34+\gamma_2', 25+\delta_2')_{LR}$,
$(y_2 + 10x_3; \gamma_2 + 10\alpha_3, \delta_2 + 10\beta_3; \gamma_2' + 10\alpha_3', \delta_2' + 10\beta_3')_{LR} \approx (480; 55, 33; 98, 75)_{LR}$,
$(x_1; \alpha_1, \beta_1; \alpha_1', \beta_1')_{LR} \approx (s_1; \rho_1, \sigma_1; \rho_1', \sigma_1')_{LR}$,
$(x_2; \alpha_2, \beta_2; \alpha_2', \beta_2')_{LR} \approx (x_1 + s_2; \alpha_1 + \rho_2, \beta_1 + \sigma_2; \alpha_1' + \rho_2', \beta_1' + \sigma_2')_{LR}$,
$(x_3; \alpha_3, \beta_3; \alpha_3', \beta_3')_{LR} \approx (x_2 + s_3; \alpha_2 + \rho_3, \beta_2 + \sigma_3; \alpha_2' + \rho_3', \beta_2' + \sigma_3')_{LR}$,
where $(x_1; \alpha_1, \beta_1; \alpha_1', \beta_1')_{LR}$, $(x_2; \alpha_2, \beta_2; \alpha_2', \beta_2')_{LR}$, $(x_3; \alpha_3, \beta_3; \alpha_3', \beta_3')_{LR}$, $(y_1; \gamma_1, \delta_1; \gamma_1', \delta_1')_{LR}$, and $(y_2; \gamma_2, \delta_2; \gamma_2', \delta_2')_{LR}$ are nonnegative LR-type IFNs and $(s_1; \rho_1, \sigma_1; \rho_1', \sigma_1')_{LR}$, $(s_2; \rho_2, \sigma_2; \rho_2', \sigma_2')_{LR}$ and $(s_3; \rho_3, \sigma_3; \rho_3', \sigma_3')_{LR}$ are unrestricted LR-type IFNs.
Taking $L(x) = R(x) = \max\{0, (1 - x)\}$, and applying proposed method, we get

$$\tilde{x}_1^I = (39.75; 5.04 \times 10^{-7}, 5.0 \times 10^{-7}; 6.05 \times 10^{-7}, 6.10 \times 10^{-7})_{LR},$$
$$\tilde{x}_2^I = (57.79; 0.21, 0.22; 0.29, 0.30)_{LR},$$
$$\tilde{x}_3^I = (47.41; 0.15, 0.32; 0.20, 0.43)_{LR},$$
$$\tilde{y}_1^I = (0; 0, 0.25; 4.64, 1.09)_{LR},$$
$$\tilde{y}_2^I = (0.16; 0, 0.45; 4.24, 0.63)_{LR},$$
$$\tilde{s}_1^I = (41.20; 0, 2.30; 16.78, 2.88)_{LR},$$
$$\tilde{s}_2^I = (19.42; 0, 2.44; 16.22, 2.78)_{LR},$$
$$\tilde{s}_3^I = (-8.89; 0, 2.64; 16.85, 2.65)_{LR}.$$

The optimal solution is

$$\tilde{z}^I = (10354.25; 259.41, 1841.10; 10210.68, 2158.05)_{LR}.$$

The solution calls for hiring around 40 temporary employees in January, again hiring around 18 temporary employees in February and firing around 11 temporary employees in March. No further hiring or firing is recommended until the end of March.

2.5.1 Managerial insights

By the discussions of the proposed models and results obtained, we find that the proposed models allow uses of different types of LR functions. It is useful from a managerial point of view to understand how the optimal solutions will be affected by changes in LR functions.

Table 2.1: Advantage of proposed method

Existing models	Proposed model
1. The existing methods ([9], [189]) can be used only for solving IFLPPs in which all the decision variables are represented by non-negative crisp parameters.	**1.** The proposed method can also be used to solve IFLPPs in which all the decision variables are represented by non-negative crisp parameters.
2. The existing methods ([137], [165]) can be used for solving FIFLPPs in which all the decision variables are represented by non-negative IFNs.	**2.** The proposed method can also be used to solve FIFLPPs in which all the decision variables are represented by non-negative IFNs.
3. The existing models ([9], [10], [137], [165], [189]) do not define the product of unrestricted LR-type IFNs.	**3.** The proposed model has introduced the definition of the product of unrestricted LR-type IFNs.
4. The existing methods ([9], [137], [165], [189]) can not solve FIFLPPs in which some or all the decision variables are represented by unrestricted LR-type IFNs.	**4.** The proposed method can be used to solve FIFLPPs in which some or all the decision variables are represented by unrestricted LR-type IFNs.

As clear from the discussion of Numerical Ex., the manager may see what he/she should do for better running of the manufacturing company, i.e., when and how many employees should be hired/fired.

2.6 Advantages of the proposed method over existing methods

The advantages of the proposed method over existing methods ([9], [10], [137], [165], [189]) for solving FIFLPPs are summarized in Table 2.1.

2.7 Concluding remarks

In this chapter, firstly we defined the product of unrestricted LR-type IFNs with the help of α-cut, β-cut, (α, β)- cut. We also defined the score and accuracy indices of LR-type IFNs and derived some results based on these indices. Subsequently, a new method is proposed for solving unrestricted LR-type FIFPPs. It can be observed that all the FIFLPPs which can be solved by the existing methods [9, 137, 165, 189] can also be solved by the proposed method. However, there exist several FIFLPPs which can not be solved by the existing methods but can be solved by the proposed method. Hence, the proposed method is better than the existing methods [9, 137, 165, 189] for solving FIFLPPs.

Chapter 3

Duality theory in intuitionistic fuzzy mathematical programming problems: Optimistic, pessimistic and mixed approaches

In this chapter, we introduce a pair of primal-dual LPPs in IFE and prove duality results by using an aspiration level approach in which membership and non-membership functions are taken in the form of reference functions. Since fuzzy and IF environments cause duality gap, we propose to investigate the impact of membership function governed by reference function on the duality gap. This is specially meaningful for fuzzy and IF programming problems, when the primal and dual objective values may not be bounded. Finally, the duality gap obtained by the approach is compared with the duality gap obtained by existing approaches.

3.1 Introduction

In decision-making problems, crisp programming problem (CPP) plays an important role for solving real-world problems. In CPP, if the number of constraints is greater than the number of unknowns, then such problems can not be easily solved. In this situation, the duality theory is useful. Duality means an optimization problem can be observed from either of two viewpoints, the primal or dual problem. Generally, the value of objective function at a feasible solution of primal problem (min) is an upper bound to the set of values of objective function at the feasible solutions of dual problem (max), i.e., the objective function value at a feasible solution of primal problem and the objective function value at a feasible solution of dual problem may or may not be equal. Their difference is called the duality gap. Duality gap is zero for convex CPP; otherwise, it may or may not be zero. Input data used in CPP have complete certainty. However, real-world situations are characterized by imprecision (fuzziness) rather than exactness. FS theory developed by Zadeh [194] explains uncertainty. Bellman and Zadeh [31] gave an idea to use FS theory in decision-making problems. A number of researchers worked on modeling and applications of FS theory. Zimmermann [204] introduced the applications of FS theory to real-life problems. Luhandjula and Rangoaga [123] gave a method for solving fuzzy multi-objective programming problem.

Primal and dual objective values of the fuzzy optimization problems may be unbounded. Duality theory developed by Rodder and Zimmermann [159] is useful for solving such type of problems using aspiration level approach. Several researchers worked on solving fuzzy programming problems using fuzzy duality theory [24, 176, 185]. Ramik [150, 151] introduced some new concepts and results, possibility and necessary relations of duality in fuzzy linear programming. Mahdavi-Amiri and Nasseri [125, 126] gave duality theory for solving fuzzy LPPs using ranking and dual simplex method. The solutions of fuzzy primal-dual programming problems and duality gap affected by taking different types of membership functions such as linear or non-linear membership functions are studied. The most likely non-linear membership function is exponential membership function because it has flexibility in changing shape parameters. Gupta and Mehlawat [85] studied Bector-Chandra [24] type duality in a fuzzy programming problem with exponential membership function. Gupta and Danger [84] applied duality results of Gupta and Mehlawat [85] model to fuzzy quadratic programming problem. Alidaee and Wang[6] introduced the zero duality gap in the surrogate constraint optimization problem.

FS theory deals with uncertainties by assigning a degree of association, called the membership degree or degree of belongingness, to an element x in the universal set X being $\tilde{A}$. The degree by which an element x is not in $\tilde{A}$, is described as the non-membership degree or degree of non-belongingness. If the sum of the degree of membership and degree of non-membership at each point is 1, then the system contains only uncertainty which is explained by FS theory. If the sum lies between 0 and 1 at each point, then system contains uncertainty as well as hesitation. The degree of hesitation is given by 1- (degree of membership + degree of non-membership). For example, when several scientists evaluate the strength of soil, some scientists are claiming the soil strength as 'good', some scientists are claiming the soil strength as 'bad', and some scientists fail to claim the soil strength as 'good' or 'bad'. The claim in favor of 'good' describes the membership degree and the claim in favor of 'bad' describes the non-membership degree, while the claim neither in favor of 'good' nor 'bad' describes the hesitation degree. Thus, the FS theory explains the uncertainty but no hesitation. A new type of FS theory known as IFS theory developed by Atanassov [11] explains uncertainty as well as hesitation by assigning membership and nonmembership functions. Atanassov [12, 13] studied operations and properties of IFSs. Angelov [9] applied IFS theory to optimization problems. Several researchers applied of IFS theory to matrix game with IF goals and IF payoffs [1, 3]. The fuzzy linear programming under interval uncertainty based on IFS is given in [66]. Aggarwal and Khan [2] studied Angelov's model in IFE for solving IF programming problems.

3.1.1 Motivation

Let $\mathbb{R}^n$ be the set of n tuples of real number (n dimensional Euclidean space). Consider the following crisp primal (CP1) dual (CD1) pair of LPPs given by

$$(CP1) \qquad \max\,(c^T x) \text{ subject to } Ax \leq b, \;\; x \geq 0,$$
$$(CD1) \qquad \min(b^T y) \text{ subject to } A^T y \geq c, \;\; y \geq 0$$

where $c, x \in \mathbb{R}^n$, $b, y \in \mathbb{R}^m$ and $A = (a_{ij})_{m\times n}$, and A^T denotes the transpose of A.
Bector and Chandra[24] studied (CP1) and (CD1) in the fuzzy environment by taking linear membership functions and proved to establish the duality relationship between them. After that Gupta and Mehlawat [85] also studied (CP1) and (CD1) in the fuzzy environment by taking exponential membership functions and calculated duality results using an aspiration level approach and investigated the duality gap. Dubey et al. [66] used the pessimistic, optimistic and mixed approaches for solving fuzzy linear programming under interval uncertainty based on IFS representation and gave the significance of these approaches for solving IF LPPs. Aggarwal et al.[1] proposed intuitionistic fuzzy primal (IFP) and intuitionistic fuzzy dual (IFD) problems given by

(IFP) Find $x \in \mathbb{R}^n$ such that,

$$c^T x(IF) \succeq z_0,$$
$$Ax(IF) \preceq b,$$
$$x \geq 0,$$

(IFD) Find $y \in \mathbb{R}^m$ such that,

$$b^T y(IF) \preceq w_0,$$
$$A^T y(IF) \succeq c,$$
$$y \geq 0,$$

where " $\preceq$" and " $\succeq$" represent the less than or equal to and greater than or equal to in the sense of IFE respectively; z_0 and w_0 represent the aspiration levels for the IFP and IFD objective functions respectively. They solved IF LPPs by taking the concept of primal (IFP) and dual (IFD) problems.

The above facts have motivated us to take up the study of a pair of primal-dual LPPs in IFE with different approaches, like pessimistic, optimistic and mixed.

The rest of the chapter is organized as follows: In Section 3.2, some basic concepts related to IFSs, and meaning of fuzzy and IF inequality are presented. Duality in IF linear programming under the pessimistic approach is introduced in Section 3.3. In Section 3.4, duality in IF linear programming under the optimistic approach is introduced. Duality in IF linear programming under the mixed approach is introduced in Section 3.5. In Section 3.6, two numerical examples are given to verify the duality results. Concluding remarks are discussed in Section 3.7.

3.2 Basic concept

In this section, some basic concepts are presented which are helpful in understanding this chapter.

3.2.1 Fuzzy inequality $(x \succeq a)$

The inequality $x \succeq a$, read as "x is greater than or equal to a in the fuzzy sense" [66]. The logical meaning of $x \succeq a$ in terms of membership function is that if $x \geq a$ then the inequality is always satisfied, i.e., the degree of membership is 1, if $x \leq a-p$, where $p > 0$ is the maximum tolerance (decided by the decision-maker), then $\succeq$ is never satisfied, i.e., the degree of membership is 0, and if $x \in (a-p, a)$ then the inequality is governed by a piecewise continuously increasing membership function. Thus, mathematically, the fuzzy inequality $x \succeq a$ is defined as follows:

$$\mu(x) = \begin{cases} 1, & x \geq a, \\ f(x), & a-p < x < a, \\ 0, & x \leq a-p, \end{cases}$$

where f is the piecewise continuous and increasing function in (a-p, a). The inequality should be read in term of tolerance as "x is greater than or equal to a in fuzzy sense with tolerance p". The graphical representation of $x \succeq a$ is depicted in Figure 3.1.

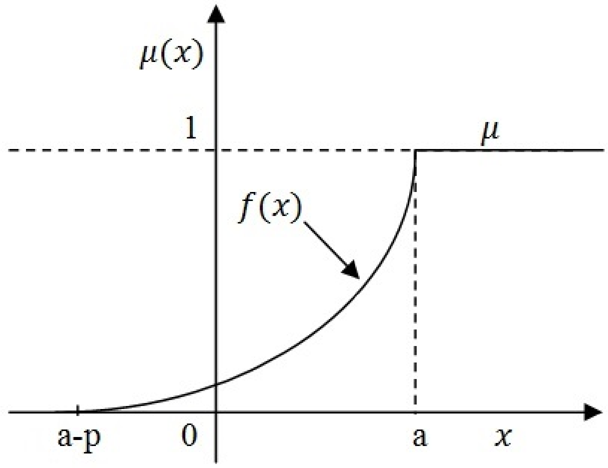

Figure 3.1: Graphical representation of $x \succeq a$ in fuzzy approach.

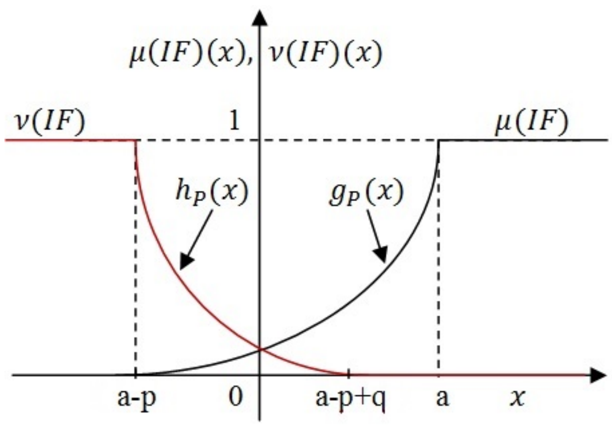

Figure 3.2: Graphical representation of $x(IF) \succeq a$ in pessimistic approach.

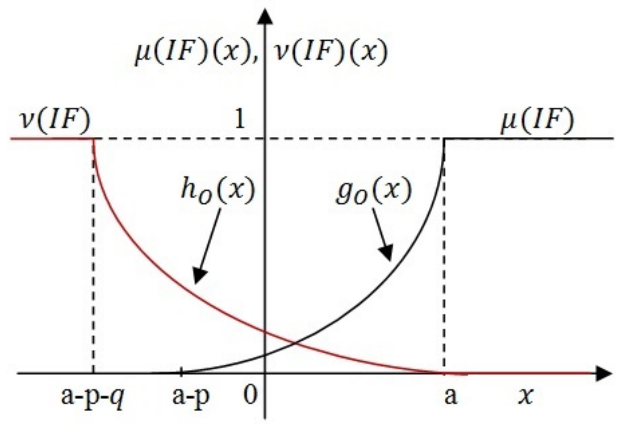

Figure 3.3: Graphical representation of $x(IF) \succeq a$ in optimistic approach.

3.2.2 IF inequality $(x \succeq a)$

It is denoted as $x(IF) \succeq a$ in IF sense [66]. Though there is no general idea to describe the meaning of $x(IF) \succeq a$, but three approaches are most likely to explain the meaning of $x(IF) \succeq a$. These are named as the pessimistic approach, optimistic approach and mixed approach.

3.2.2.1 The pessimistic approach

In this approach, the decision maker (DM) is possibly ready for extra acceptance, i.e., if the degree of rejection of x is zero, the DM is not ready to accept fully. If there exist tolerances p, q, $0 < q < p$ (depend on DM), its membership function $\mu(IF)$ and non-membership function

$\nu(IF)$ are defined as follows:

$$\mu(IF)(x) = \begin{cases} 1, & x \geq a, \\ g_P(x), & a-p < x \leq a, \\ 0, & x < a-p. \end{cases} \quad \nu(IF)(x) = \begin{cases} 1, & x \leq a-p, \\ h_P(x), & a-p < x \leq a-p+q; \\ & 0 \leq g_P(x) + h_P(x) \leq 1, \\ 0, & x > a-p+q, \end{cases}$$

where g_P and h_P are piecewise continuous, increasing and decreasing in (a-p, a) and (a-p, a-p+q) respectively. It is noted, in the interval (a-p+q, a), the degree of rejection is zero but the degree of acceptance is not one. Figure 3.2 shows graphical representation of $x(IF) \succeq a$ in pessimistic approach.

3.2.2.2 The optimistic approach

In this approach, the DM takes a flexible way about rejection. Specially, if the degree of acceptance of x is zero, the DM do not reject fully. Therefore, mathematically, for tolerances $p, q > 0$, the membership function $\mu(IF)$ and non-membership function $\nu(IF)$ are defined as follows:

$$\mu(IF)(x) = \begin{cases} 1, & x \geq a, \\ g_O(x), & a-p < x \leq a, \\ 0, & x < a-p. \end{cases} \quad \nu(IF)(x) = \begin{cases} 1, & x \leq a-p-q, \\ h_O(x), & a-p-q < x \leq a; \\ & 0 \leq g_o(x) + h_o(x) \leq 1, \\ 0, & x > a, \end{cases}$$

where g_O and h_O are piecewise continuous, increasing and decreasing in (a-p, a) and (a-p-q, a) respectively. It is noted that, in the interval (a-p-q, a-p), the degree of acceptance is zero but the degree of rejection is not one. The optimistic approach representation of $x(IF) \succeq a$ is shown in Figure 3.3.

3.2.2.3 The mixed approach

In this approach, the DM is not flexible to reject and is not capable for extra acceptance. Thus, mathematically, for tolerances $p, q, r > 0$, $r > q$ and $r < p + q$, the membership function $\mu(IF)$

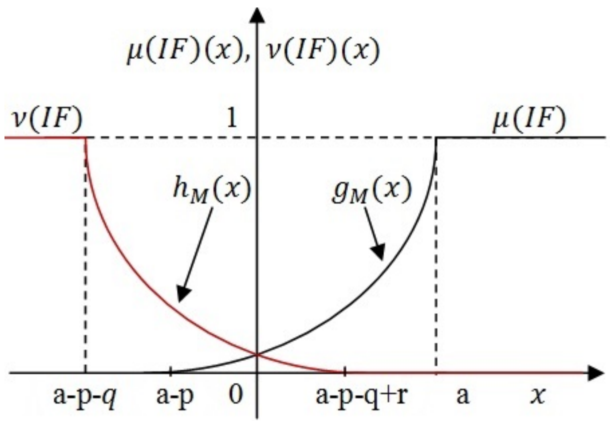

Figure 3.4: Graphical representation of $x \succeq a$ in mixed approach.

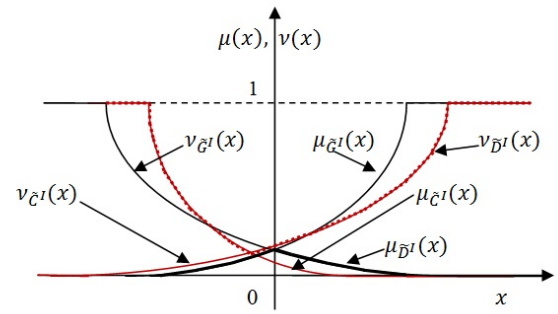

Figure 3.5: Graphical representation of decision making in IFE.

and non-membership function $\nu(IF)$ are defined as follows:

$$\mu(IF)(x) = \begin{cases} 1, & x \geq a, \\ g_M(x), & a-p < x < a, \\ 0, & x \leq a-p. \end{cases} \quad \nu(IF)(x) = \begin{cases} 1, & x \leq a-p-q, \\ h_M(x), & a-p-q < x \leq a-p-q+r; \\ & 0 \leq g_m(x)+h_m(x) \leq 1, \\ 0, & x > a-p-q+r, \end{cases}$$

where g_M and h_M are piecewise continuous, increasing and decreasing in (a-p, a) and (a-p-q, a-p-q+r) respectively. The graphical representation of $x(IF) \succeq a$ in mixed approach is shown in Figure 3.4.

3.2.3 Decision making in IFE

Let X be a universe of discourse. Let $\tilde{G}^I$=$\{(x, \mu_{\tilde{G}^I}(x), \nu_{\tilde{G}^I}(x)) : x \in X\}$ and $\tilde{C}^I$=$\{(x, \mu_{\tilde{C}^I}(x), \nu_{\tilde{C}^I}(x)) : x \in X\}$ be the goal and constraint respectively in IFE. Then the decision $\tilde{D}^I = \tilde{G}^I \cap \tilde{C}^I$ is an IFS defined as $\tilde{D}^I = \{(x, \mu_{\tilde{D}^I}(x), \nu_{\tilde{D}^I}(x)) : x \in X\}$, where $\mu_{\tilde{D}^I}(x) = \min\{\mu_{\tilde{G}^I}(x), \mu_{\tilde{C}^I}(x)\}$, and $\nu_{\tilde{D}^I}(x) = \max\{\nu_{\tilde{G}^I}(x), \nu_{\tilde{C}^I}(x)\}$. The relation between $\tilde{G}^I, \tilde{C}^I$ and $\tilde{D}^I$ is depicted in Figure 3.5. More generally, suppose that we have 'm' goals $\tilde{G}^I_i$=$\{(x, \mu_{\tilde{G}^I_i}(x), \nu_{\tilde{G}^I_i}(x)) : x \in X\}$, $i = 1, 2, \ldots, m$, and 'n' constraints $\tilde{C}^I_j$=$\{(x, \mu_{\tilde{C}^I_j}(x), \nu_{\tilde{C}^I_j}(x)) : x \in X\}$, $j = 1, 2, \ldots, n$. Then the decision $\tilde{D}^I = (\tilde{G}^I_1 \cap \tilde{G}^I_2 \cap \ldots \cap \tilde{G}^I_m) \cap (\tilde{C}^I_1 \cap \tilde{C}^I_2 \cap \ldots \cap \tilde{C}^I_n)$ is an IFS defined by

$$\tilde{D}^I = \{(x, \mu_{\tilde{D}^I}(x), \nu_{\tilde{D}^I}(x)) : x \in X\},$$

where $\mu_{\tilde{D}^I}(x) = \min_{i,j}\{\mu_{\tilde{G}^I_i}(x), \mu_{\tilde{C}^I_j}(x)\}$ and $\nu_{\tilde{D}^I}(x) = \max_{i,j}\{\nu_{\tilde{G}^I_i}(x), \nu_{\tilde{C}^I_j}(x)\}$ (see [31]).

Definition 3.2.1. *Let* $S_{\tilde{D}^I}(x) = \mu_{\tilde{D}^I}(x) - \nu_{\tilde{D}^I}(x),\ x \in X$ *be the score function of the IFS* $\tilde{D}^I$. *An* $\check{x} \in X$ *is said to be an optimal decision in IFE if* $S_{\tilde{A}^I}(\check{x}) \geq S_{\tilde{A}^I}(x)\ \forall\ x \in X$, *i.e.,* $S_{\tilde{A}^I}(\check{x}) = \max_{x \in X} S_{\tilde{A}^I}(x)$.

Let α and β denote the minimum degree of acceptance and the maximum degree of rejection respectively. Then the IF decision problem is transformed into the following crisp optimization problem [9]:

$$\begin{aligned} \max \quad & (\alpha - \beta), \\ \text{subject to} \quad & \mu_{\tilde{G}_i^I}(x) \geq \alpha,\ \ \nu_{\tilde{G}_i^I}(x) \leq \beta,\ i = 1, 2, \ldots, m, \\ & \mu_{\tilde{C}_j^I}(x) \geq \alpha,\ \ \nu_{\tilde{C}_j^I}(x) \leq \beta,\ j = 1, 2, \ldots, n, \\ & \alpha \geq \beta \geq 0, \alpha + \beta \leq 1,\ x \in \mathbb{R}^n. \end{aligned}$$

3.3 Duality in IF programming under pessimistic approach

Let p_0 and q_0 be the tolerances corresponding to membership function and non-membership function of the objective function respectively, where $0 < q_0 < p_0$. Let p_i and q_i be the tolerances corresponding to membership function and non-membership function of the ith constraint respectively, where $0 < q_i < p_i$ for $i = 1, 2, \ldots, m$.

We take the following form of the membership function as well as non-membership function governed by reference functions for the objective function and all the constraints of IFP problem:

$$\mu_P(c^T x) = \begin{cases} 1, & c^T x \geq z_0, \\ L_0\left(\frac{z_0 - c^T x}{p_0}\right), & z_0 - p_0 < \\ & c^T x < z_0, \\ 0, & otherwise, \end{cases} \qquad \nu_P(c^T x) = \begin{cases} 1, & c^T x \leq z_0 - p_0, \\ R_0\left(\frac{c^T x - (z_0 - p_0)}{q_0}\right), & z_0 - p_0 < c^T x \\ & < z_0 - p_0 + q_0, \\ 0, & otherwise, \end{cases}$$

$$\mu_P(A_i x) = \begin{cases} 1, & A_i x \leq b_i, \\ R_i\left(\frac{A_i x - b_i}{p_i}\right), & b_i < A_i x \\ & < b_i + p_i, \\ 0, & otherwise, \end{cases} \qquad \nu_P(A_i x) = \begin{cases} 1, & A_i x \geq b_i + p_i, \\ L_i\left(\frac{b_i + p_i - A_i x}{q_i}\right), & b_i + p_i - q_i < \\ & A_i x < b_i + p_i, \\ 0, & otherwise, \end{cases}$$

where L_0 and R_0 are reference functions corresponding to the primal objective function such that $L_0\left(\frac{z_0 - c^T x}{p_0}\right) +$

$R_0\left(\frac{c^Tx-(z_0-p_0)}{q_0}\right) \leq 1$ for $z_0 - p_0 < c^Tx < z_0 - p_0 + q_0$; L_i and R_i are reference functions corresponding to the ith primal constraint such that $R_i\left(\frac{A_ix-b_i}{p_i}\right)+L_i\left(\frac{b_i+p_i-A_ix}{q_i}\right) \leq 1$ for $b_i+p_i-q_i < A_ix < b_i+p_i$, $i = 0, 1, 2, \ldots, m$; and A_i is the ith row of A, $i = 1, 2, \ldots, m$.

Let s_0 and t_0 be the tolerances corresponding to the membership function and non-membership function of the objective function respectively, where $0 < t_0 < s_0$. Let s_j and t_j be the tolerances corresponding to the membership function and non-membership function of the jth dual constraint respectively, where $0 < t_j < s_j$ for $j = 1, 2, \ldots, n$.

We take the following form of the membership function as well as non-membership function governed by reference functions for the objective function and all the constraints of IFD problem:

$$\mu_P(b^Ty) = \begin{cases} 1, & b^Ty \leq w_0, \\ R_0\left(\frac{b^Ty-w_0}{s_0}\right), & w_0 < b^Ty \\ & < w_0+s_0, \\ 0, & otherwise, \end{cases} \qquad \nu_P(b^Ty) = \begin{cases} 1, & b^Ty \geq w_0+s_0, \\ L_0\left(\frac{w_0+s_0-b^Ty}{t_0}\right), & w_0+s_0-t_0 < \\ & b^Ty < w_0+s_0, \\ 0, & otherwise, \end{cases}$$

$$\mu_P(A_j^Ty) = \begin{cases} 1, & A_j^Ty \geq c_j, \\ L_j\left(\dfrac{c_j - A_j^Ty}{s_j}\right), & c_j - s_j < \\ & A_j^Ty < c_j, \\ 0, & otherwise, \end{cases} \qquad \nu_P(A_j^Ty) = \begin{cases} 1, & A_j^Ty \leq c_j - s_j, \\ R_j\left(\frac{A_j^Ty-(c_j-s_j)}{t_j}\right), & c_j - s_j < A_j^Ty \\ & < c_j - s_j + t_j, \\ 0, & otherwise, \end{cases}$$

where L_0 and R_0 are reference functions corresponding to the dual objective function such that $R_0\left(\frac{b^Ty-w_0}{s_0}\right)+L_0\left(\frac{w_0+s_0-b^Ty}{t_0}\right) \leq 1$ for $w_0 + s_0 - t_0 < b^Ty < w_0 + s_0$; L_j and R_j are reference functions corresponding to the jth dual constraint such that $L_j\left(\dfrac{c_j - A_j^Ty}{s_j}\right)+R_j\left(\frac{A_j^Ty-(c_j-s_j)}{t_j}\right) \leq 1$ for $c_j-s_j < A_j^Ty < c_j-s_j+t_j$, $j = 0, 1, 2, \ldots, n$; and A_j is the jth column of A, $j = 1, 2, \ldots, n$.

Let α_1, β_1 be the minimum degree of acceptance and maximum degree of rejection of IFP problem. Angelov [9] transformed the IFP problem into the equivalent crisp primal problem

(CP2) as given below:

$$
\begin{aligned}
&\text{(CP2)}\ \ \max\ (\alpha_1-\beta_1)\\
&\text{subject to}\quad L_0\Big(\frac{z_0-c^Tx}{p_0}\Big)\ge\alpha_1,\\
&\qquad R_i\Big(\frac{A_ix-b_i}{p_i}\Big)\ge\alpha_1,\ \ i=1,2,\dots,m,\\
&\qquad R_0\Big(\frac{c^Tx-(z_0-p_0)}{q_0}\Big)\le\beta_1,\\
&\qquad L_i\Big(\frac{b_i+p_i-A_ix}{q_i}\Big)\le\beta_1,\ \ i=1,2,\dots,m,\\
&\qquad \alpha_1\ge\beta_1\ge0,\alpha_1+\beta_1\le1,\ x\in\mathbb{R}^n.
\end{aligned}
$$

Simplifying the above problem, we get

$$
\begin{aligned}
&\text{(CP3)}\ \ \max\ (\alpha_1-\beta_1)\\
&\text{subject to}\ \ z_0-c^Tx-p_0L_0^{-1}(\alpha_1)\le0, && (3.1)\\
&\qquad A_ix-b_i-p_iR_i^{-1}(\alpha_1)\le0,\quad i=1,2,\dots,m, && (3.2)\\
&\qquad c^Tx-z_0+p_0-q_0R_0^{-1}(\beta_1)\ge0, && (3.3)\\
&\qquad b_i+p_i-A_ix-q_iL_i^{-1}(\beta_1)\ge0,\quad i=1,2,\dots,m, && (3.4)\\
&\qquad \alpha_1\ge\beta_1\ge0,\alpha_1+\beta_1\le1,x\ge0.
\end{aligned}
$$

Let α_2 and β_2 be the minimum degree of acceptance and maximum degree of rejection of IFD problem. The equivalent crisp dual problem (CD2) of the IFD problem as given below:

$$
\begin{aligned}
&\text{(CD2)}\ \ \max\ (\alpha_2-\beta_2)\\
&\text{subject to}\quad R_0\Big(\frac{b^Ty-w_0}{s_0}\Big)\ge\alpha_2,\\
&\qquad L_j\Big(\frac{c_j-A_j^Ty}{s_j}\Big)\ge\alpha_2,\ \ j=1,2,\dots,n),\\
&\qquad L_0\Big(\frac{w_0+s_0-b^Ty}{t_0}\Big)\le\beta_2,\\
&\qquad R_j\Big(\frac{A_j^Ty-(c_j-s_j)}{t_j}\Big)\le\beta_2,\ \ j=1,2,\dots,n,\\
&\qquad \alpha_2\ge\beta_2\ge0,\alpha_2+\beta_2\le1,\ y\in\mathbb{R}^m
\end{aligned}
$$

Simplifying the above problem, we get

$$\text{(CD3)} \;\; \max\ (\alpha_2 - \beta_2)$$

$$\text{subject to } \; b^T y - w_0 - s_0 R_0^{-1}(\alpha_2) \le 0, \tag{3.5}$$

$$c_j - A_j^T y - s_j L_j^{-1}(\alpha_2) \le 0, \quad j = 1, 2, \ldots, n, \tag{3.6}$$

$$w_0 + s_0 - b^T y - t_0 L_0^{-1}(\beta_2) \ge 0, \tag{3.7}$$

$$A_j^T y - (c_j - s_j) - t_j R_j^{-1}(\beta_2) \ge 0, \quad j = 1, 2, \ldots, n, \tag{3.8}$$

$$\alpha_2 \ge \beta_2 \ge 0, \alpha_2 + \beta_2 \le 1, y \ge 0.$$

The membership and non-membership functions, governed by reference functions, are flexible because of the reference functions, which depend on DM.

Theorem 3.3.1. *(Modified weak duality) Let* (x, α_1, β_1) *and* (y, α_2, β_2) *be feasible solutions of (CP3) and (CD3) respectively. Then*

(i) $\displaystyle\sum_{i=1}^{m} R_i^{-1}(\alpha_1) p_i y_i + \sum_{j=1}^{n} L_j^{-1}(\alpha_2) s_j x_j \ge c^T x - b^T y,$

(ii) $\displaystyle\sum_{i=1}^{m} L_i^{-1}(\beta_1) q_i y_i + \sum_{j=1}^{n} R_j^{-1}(\beta_2) t_j x_j \le (b + p)^T y - (c - s)^T x.$

Proof. From (3.2), we have

$$\begin{aligned} A_i x - b_i - p_i R_i^{-1}(\alpha_1) \le 0 \;\; &\Rightarrow \;\; p_i R_i^{-1}(\alpha_1) \ge A_i x - b_i \\ &\Rightarrow \;\; \sum_{i=1}^{m} R_i^{-1}(\alpha_1) p_i y_i \ge \sum_{i=1}^{m} (A_i x - b_i) y_i \\ &\Rightarrow \;\; \sum_{i=1}^{m} R_i^{-1}(\alpha_1) p_i y_i \ge (x^T A^T y - b^T y) \end{aligned} \tag{3.9}$$

From (3.6), we have

$$\begin{aligned} c_j - A_j^T y - s_j L_j^{-1}(\alpha_2) \le 0 \;\; &\Rightarrow \;\; s_j L_j^{-1}(\alpha_2) \ge c_j - A_j^T y \\ &\Rightarrow \;\; \sum_{j=1}^{n} L_j^{-1}(\alpha_2) s_j x_j \ge \sum_{j=1}^{n} (c_j - A_j^T y) x_j \\ &\Rightarrow \;\; \sum_{j=1}^{n} L_j^{-1}(\alpha_2) s_j x_j \ge (c^T x - y^T A x) \end{aligned} \tag{3.10}$$

Adding (3.9) and (3.10), we have

$$\sum_{i=1}^{m} R_i^{-1}(\alpha_1) p_i y_i + \sum_{j=1}^{n} L_j^{-1}(\alpha_2) s_j x_j \ge (c^T x - b^T y) \qquad (\because y^T A x = (y^T A x)^T = x^T A^T y) \tag{3.11}$$

which is the condition (i).

From (3.4), we get

$$\sum_{i=1}^{m} L_i^{-1}(\beta_1)q_i y_i \leq (b+p)^T y - x^T A^T y \tag{3.12}$$

From (3.8), we get

$$\sum_{j=1}^{n} R_j^{-1}(\beta_2)t_j x_j \leq y^T A x - (c-s)^T x \tag{3.13}$$

Adding (3.12) and (3.13), we have

$$\sum_{i=1}^{m} L_i^{-1}(\beta_1)q_i y_i + \sum_{j=1}^{n} R_j^{-1}(\beta_2)t_j x_j \leq (b+p)^T y - (c-s)^T x \tag{3.14}$$

which is the condition (ii). □

Remark 3.3.2. *Adding (3.1) and (3.5), we get*

$$L_0^{-1}(\alpha_1)p_0 + R_0^{-1}(\alpha_2)s_0 \geq b^T y - c^T x + z_0 - w_0 \tag{3.15}$$

The (3.15) relates the relative difference of aspiration level z_0 of $c^T x$ and w_0 of $b^T y$ corresponding to membership function in terms of their tolerance levels p_0 and s_0 respectively.
Adding (3.3) and (3.7), we get

$$R_0^{-1}(\beta_1)q_0 + L_0^{-1}(\beta_2)t_0 \leq c^T x - b^T y + w_0 - z_0 + p_0 + s_0 \tag{3.16}$$

The (3.16) relates the relative difference of aspiration level z_0 of $c^T x$ and w_0 of $b^T y$ corresponding to non-membership function in terms of their tolerance levels p_0, s_0, q_0 and t_0 respectively.

Remark 3.3.3. *Putting $\alpha_1 = 1, \beta_1 = 0, \alpha_2 = 1, \beta_2 = 0$ $(\because 0 \leq \beta_1 \leq \alpha_1, \alpha_1 + \beta_1 \leq 1, 0 \leq \beta_2 \leq \alpha_2, \alpha_2 + \beta_2 \leq 1)$, we get*

$$c^T x \leq b^T y \; \forall \; x \geq 0, y \geq 0 \tag{3.17}$$

$$q^T y + t^T x \leq -c^T x + s^T x + b^T y + p^T y \; \forall \; x \geq 0, y \geq 0 \tag{3.18}$$

$$z_0 - c^T x \leq w_0 - b^T y \; \forall \; x \geq 0, y \geq 0 \tag{3.19}$$

$$q_0 + t_0 \leq c^T x - b^T y - z_0 + w_0 + p_0 + s_0 \; \forall \; x \geq 0, y \geq 0 \tag{3.20}$$

(3.17) represents the weak duality in CPPs. If $0 < \alpha_1, \alpha_2, \beta_1, \beta_2 < 1$, then the Theorem 3.3.1 shows the weak duality in IFE.

Theorem 3.3.4. *Let $(\breve{x}, \breve{\alpha}_1, \breve{\beta}_1)$ and $(\breve{y}, \breve{\alpha}_2, \breve{\beta}_2)$ be feasible solutions of (CP3) and (CD3) respectively such that*

(i) $\sum_{i=1}^{m} R_i^{-1}(\breve{\alpha}_1) p_i \breve{y}_i + \sum_{j=1}^{n} L_j^{-1}(\breve{\alpha}_2) s_j \breve{x}_j = c^T \breve{x} - b^T \breve{y},$

(ii) $\sum_{i=1}^{m} L_i^{-1}(\breve{\beta}_1) q_i \breve{y}_i + \sum_{j=1}^{n} R_j^{-1}(\breve{\beta}_2) t_j \breve{x}_j = (b+p)^T \breve{y} - (c-s)^T \breve{x},$

(iii) $L_0^{-1}(\breve{\alpha}_1) p_0 + R_0^{-1}(\breve{\alpha}_2) s_0 = b^T \breve{y} - c^T \breve{x} + z_0 - w_0,$

(iv) $R_0^{-1}(\breve{\beta}_1) q_0 + L_0^{-1}(\breve{\beta}_2) t_0 = c^T \breve{x} - b^T \breve{y} + w_0 - z_0 + p_0 + s_0,$

(v) *the aspiration levels z_0 and w_0 satisfy $z_0 - w_0 \leq 0$,*

(vi) $q^T \breve{y} + t^T \breve{x} + q_0 + t_0 \leq s^T \breve{x} + p^T \breve{y} - z_0 + w_0 + p_0 + q_0.$

Then $(\breve{x}, \breve{\alpha}_1, \breve{\beta}_1)$ and $(\breve{y}, \breve{\alpha}_2, \breve{\beta}_2)$ are the optimal solutions of (CP3) and (CD3) respectively.

Proof. The proof of this theorem consists of two parts:
1^{st} **part:** Let (x, α_1, β_1) and (y, α_2, β_2) be any feasible solutions of (CP3) and (CD3) respectively. Then by Theorem 3.3.1, we have

$$\sum_{i=1}^{m} R_i^{-1}(\alpha_1) p_i y_i + \sum_{j=1}^{n} L_j^{-1}(\alpha_2) s_j x_j - (c^T x - b^T y) \geq 0 \tag{3.21}$$

From hypothesis (i), we have

$$\sum_{i=1}^{m} R_i^{-1}(\breve{\alpha}_1) p_i \breve{y}_i + \sum_{j=1}^{n} L_j^{-1}(\breve{\alpha}_2) s_j \breve{x}_j - (c^T \breve{x} - b^T \breve{y}) = 0 \tag{3.22}$$

(3.21) and (3.22) imply that for any feasible solutions (x, α_1, β_1) and (y, α_2, β_2) of (CP3) and (CD3) respectively, we get

$$\sum_{i=1}^{m} R_i^{-1}(\breve{\alpha}_1) p_i \breve{y}_i + \sum_{j=1}^{n} L_j^{-1}(\breve{\alpha}_2) s_j \breve{x}_j - (c^T \breve{x} - b^T \breve{y}) \leq \sum_{i=1}^{m} R_i^{-1}(\alpha_1) p_i y_i + \sum_{j=1}^{n} L_j^{-1}(\alpha_2) s_j x_j - (c^T x - b^T y).$$

$\Rightarrow$ $(\breve{x}, \breve{\alpha}_1, \breve{y}, \breve{\alpha}_2)$ is the optimal solution of crisp primal problem $(CP4)$ given below, whose maximum objective value is zero.

$$\text{(CP4)} \qquad \max \Big[-\sum_{i=1}^{m} R_i^{-1}(\alpha_1) p_i y_i - \sum_{j=1}^{n} L_j^{-1}(\alpha_2) s_j x_j + (c^T x - b^T y) \Big],$$

$$\begin{aligned} \text{subject to } & p_0 L_0^{-1}(\alpha_1) \geq z_0 - c^T x), \\ & s_0 R_0^{-1}(\alpha_2) \geq b^T y - w_0), \\ & p_i R_i^{-1}(\alpha_1) \geq A_i x - b_i, \quad i = 1, 2, \ldots, m, \\ & s_j L_j^{-1}(\alpha_2) \geq c_j - A_j^T y, \quad j = 1, 2, \ldots, n, \\ & \alpha_1 \leq 1, \alpha_2 \leq 1, x \geq 0, y \geq 0, \alpha_1 \geq 0, \alpha_2 \geq 0. \end{aligned}$$

From hypothesis (i), we have

$$\sum_{i=1}^{m} R_i^{-1}(\check{\alpha}_1) p_i \check{y}_i + \sum_{j=1}^{n} L_j^{-1}(\check{\alpha}_2) s_j \check{x}_j - (c^T \check{x} - b^T \check{y}) = 0 \tag{3.23}$$

From hypothesis (iii), we have

$$L_0^{-1}(\check{\alpha}_1) p_0 + R_0^{-1}(\check{\alpha}_2) s_0 = b^T \check{y} - c^T \check{x} + z_0 - w_0 \tag{3.24}$$

Adding (3.23) and (3.24), we get

$$L_0^{-1}(\check{\alpha}_1) p_0 + \sum_{i=1}^{m} R_i^{-1}(\check{\alpha}_1) p_i \check{y}_i + R_0^{-1}(\check{\alpha}_2) s_0 + \sum_{j=1}^{n} L_j^{-1}(\check{\alpha}_2) s_j \check{x}_j + (w_0 - z_0) = 0 \tag{3.25}$$

Each term in (3.25) is non-negative, we get

$$\sum_{i=1}^{m} R_i^{-1}(\check{\alpha}_1) p_i \check{y}_i = 0, \sum_{j=1}^{n} L_j^{-1}(\check{\alpha}_2) s_j \check{x}_j = 0, L_0^{-1}(\check{\alpha}_1) p_0 = 0, R_0^{-1}(\check{\alpha}_2) s_0 = 0.$$

Since $L_0^{-1}(\alpha_1) p_0 \geq 0, R_0^{-1}(\alpha_2) s_0 \geq 0$,

$$L_0^{-1}(\alpha_1) p_0 \geq L_0^{-1}(\check{\alpha}_1) p_0, R_0^{-1}(\alpha_2) s_0 \geq R_0^{-1}(\check{\alpha}_2) s_0 \tag{3.26}$$

Since L_0, R_0 are the reference functions,

$$\alpha_1 \leq \check{\alpha}_1, \alpha_2 \leq \check{\alpha}_2 \tag{3.27}$$

2^{nd} **part:** Let (x, α_1, β_1) and (y, α_2, β_2) be (CP1)-feasible and (CD1)-feasible respectively. Then by Theorem 3.3.1, we have

$$\sum_{i=1}^{m} L_i^{-1}(\beta_1) q_i y_i + \sum_{j=1}^{n} R_j^{-1}(\beta_2) t_j x_j - ((b + p)^T y - (c - s)^T x) \leq 0 \tag{3.28}$$

From hypothesis (ii), we have

$$\sum_{i=1}^{m} L_i^{-1}(\check{\beta}_1)q_i\check{y}_i + \sum_{j=1}^{n} R_j^{-1}(\check{\beta}_2)t_j\check{x}_j - ((b+p)^T\check{y} - (c-s)^T\check{x}) = 0 \tag{3.29}$$

(3.28) and (3.29) imply that for any feasible solutions (x, α_1, β_1) and (y, α_2, β_2) of (CP3) and (CD3) respectively, we get

$$\sum_{i=1}^{m} L_i^{-1}(\beta_1)q_iy_i + \sum_{j=1}^{n} R_j^{-1}(\beta_2)t_jx_j - ((b+p)^Ty - (c-s)^Tx) \leq \sum_{i=1}^{m} L_i^{-1}(\check{\beta}_1)q_i\check{y}_i + \sum_{j=1}^{n} R_j^{-1}(\check{\beta}_2)t_j\check{x}_j - ((b+p)^T\check{y} - (c-s)^T\check{x}) \tag{3.30}$$

$\Rightarrow$ $(\check{x}, \check{\beta}_1, \check{y}, \check{\beta}_2)$ is the optimal of the crisp dual problem $(CD4)$ given below, whose maximum objective value is zero.

$$\text{(CD4)} \qquad \max\Big[\sum_{i=1}^{m} L_i^{-1}(\beta_1)q_iy_i + \sum_{j=1}^{n} R_j^{-1}(\beta_2)t_jx_j - ((b+p)^Ty - (c-s)^Tx)\Big],$$

$$\begin{aligned}
\text{subject to } \; & q_0R_0^{-1}(\beta_1) \leq c^Tx - z_0 + p_0, \\
& t_0L_0^{-1}(\beta_2) \leq w_0 + s_0 - b^Ty, \\
& q_iL_i^{-1}(\beta_1) \leq b_i + p_i - A_ix, \;\; i = 1, 2, \ldots, m, \\
& t_jR_j^{-1}(\beta_2) \leq A_j^Ty - (c_j - s_j), \;\; j = 1, 2, \ldots, n, \\
& \beta_1 \leq 1, \beta_2 \leq 1, x \geq 0, y \geq 0, \beta_1 \geq 0, \beta_2 \geq 0.
\end{aligned}$$

From hypothesis (ii), we have

$$\sum_{i=1}^{m} L_i^{-1}(\check{\beta}_1)q_i\check{y}_i + \sum_{j=1}^{n} R_j^{-1}(\check{\beta}_2)t_j\check{x}_j - ((b+p)^T\check{y} - (c-s)^T\check{x}) = 0 \tag{3.31}$$

From hypothesis (iv), we have

$$R_0^{-1}(\check{\beta}_1)q_0 + L_0^{-1}(\check{\beta}_2)t_0 - (c^T\check{x} - b^T\check{y} + w_0 - z_0 + p_0 + s_0) = 0 \tag{3.32}$$

Adding (3.31) and (3.32), we get

$$\begin{aligned}
& R_0^{-1}(\check{\beta}_1)q_0 + \sum_{i=1}^{m} L_i^{-1}(\check{\beta}_1)q_i\check{y}_i + L_0^{-1}(\check{\beta}_2)t_0 + \sum_{j=1}^{n} R_j^{-1}(\check{\beta}_2)t_j\check{x}_j \\
& - p^T\check{y} - s^T\check{x} + (z_0 - w_0) - (p_0 + s_0) = 0.
\end{aligned}$$

$$\begin{aligned}
\Rightarrow \;\; & (1 - R_0^{-1}(\check{\beta}_1))q_0 + \sum_{i=1}^{m}(1 - L_i^{-1}(\check{\beta}_1))q_i\check{y}_i + (1 - L_0^{-1}(\check{\beta}_2))t_0 + \sum_{j=1}^{n}(1 - R_j^{-1}(\check{\beta}_2))t_j\check{x}_j \\
& - q^T\check{y} - t^T\check{x} - q_0 - t_0 + p^T\check{y} + s^T\check{x} - (z_0 - w_0) + (p_0 + s_0) = 0
\end{aligned} \tag{3.33}$$

Each term in (3.33) is non-negative, we get

$$(1 - R_0^{-1}(\check{\beta}_1))q_0 = 0, \qquad \sum_{i=1}^{m}(1 - L_i^{-1}(\check{\beta}_1))q_i\check{y}_i = 0,$$
$$(1 - L_0^{-1}(\check{\beta}_2))t_0 = 0, \qquad \sum_{j=1}^{n}(1 - R_j^{-1}(\check{\beta}_2))t_j\check{x}_j = 0.$$

Since $(1 - R_0^{-1}(\beta_1))q_0 \geq 0,\ (1 - L_0^{-1}(\beta_2))t_0 \geq 0,$

$$(1 - R_0^{-1}(\beta_1))q_0 \geq (1 - R_0^{-1}(\check{\beta}_1))q_0, \quad (1 - L_0^{-1}(\beta_2))t_0 \geq (1 - L_0^{-1}(\check{\beta}_2))t_0 \tag{3.34}$$

Since L_0, R_0 are the reference functions,

$$\beta_1 \geq \check{\beta}_1, \beta_2 \geq \check{\beta}_2 \tag{3.35}$$

From (3.27) and (3.35), we have

$$\alpha_1 - \alpha_2 \leq \check{\alpha}_1 - \check{\alpha}_2 \quad \text{and} \quad \beta_1 - \beta_2 \leq \check{\beta}_1 - \check{\beta}_2.$$

This proves the result. □

3.4 Duality in IF programming under optimistic approach

Let $p_0 > 0$ and $q_0 > 0$ be the tolerances corresponding to the membership and non-membership functions of the primal objective function respectively. Let $p_i > 0$ and $q_i > 0$ be the tolerances corresponding to the membership and non-membership functions of the ith primal constraint respectively for $i = 1, 2, \ldots, m$.

We take the following forms of the membership function as well as non-membership function governed by reference functions for the objective function and all the constraints of the IFP problem:

$$\mu_O(c^Tx) = \begin{cases} 1, & c^Tx \geq z_0, \\ L_0\Big(\dfrac{z_0 - c^Tx}{p_0}\Big), & z_0 - p_0 < c^Tx < z_0, \\ 0, & otherwise, \end{cases} \qquad \nu_O(c^Tx) = \begin{cases} 1, & c^Tx \leq z_0 - p_0, \\ R_0\Big(\dfrac{c^Tx + p_0 + q_0 - z_0}{p_0 + q_0}\Big), & z_0 - p_0 - q_0 < c^Tx < z_0, \\ 0, & otherwise, \end{cases}$$

$$\mu_O(A_ix) = \begin{cases} 1, & A_ix \le b_i, \\ R_i\Big(\dfrac{A_ix - b_i}{p_i}\Big), & b_i < A_ix \\ & < b_i + p_i, \\ 0, & otherwise, \end{cases} \qquad \nu_O(A_ix) = \begin{cases} 1, & A_ix \ge b_i + \\ & p_i + q_i, \\ L_i\Big(\dfrac{b_i + p_i + q_i - A_ix}{p_i + q_i}\Big), & b_i < A_ix < \\ & b_i + p_i + q_i, \\ 0, & otherwise, \end{cases}$$

where L_0 and R_0 are reference functions corresponding to objective function such that $L_0\Big(\dfrac{z_0 - c^Tx}{p_0}\Big)$ $+R_0\Big(\dfrac{c^Tx + p_0 + q_0 - z_0}{p_0 + q_0}\Big) \le 1$ for $z_0 - p_0 - q_0 < c^Tx < z_0$; L_i and R_i are reference functions corresponding to the ith primal constraint such that $R_i\Big(\dfrac{A_ix - b_i}{p_i}\Big) + L_i\Big(\dfrac{b_i + p_i + q_i - A_ix}{p_i + q_i}\Big) \le 1$ for $b_i < A_ix < b_i + p_i + q_i$, $i = 0, 1, 2, \ldots, m$; where A_i is the ith row of A, $i = 1, 2, \ldots, m$.

Let $s_0 > 0$ and $t_0 > 0$ be the tolerances corresponding to the membership and non-membership functions of the objective function respectively. Let $s_j > 0$ and $t_j > 0$ be the tolerances corresponding to the membership and non-membership functions of the jth dual constraint respectively for $j = 1, 2, \ldots, n$.

We take the following form of the membership function as well as non-membership function governed by reference functions for the objective function and all the constraints of the IFD problem:

$$\mu_O(b^Ty) = \begin{cases} 1, & b^Ty \le w_0, \\ R_0\Big(\dfrac{b^Ty - w_0}{s_0}\Big), & w_0 < b^Ty \\ & < w_0 + s_0, \\ 0, & otherwise, \end{cases} \qquad \nu_O(b^Ty) = \begin{cases} 1, & b^Ty \ge w_0 + \\ & s_0 + t_0, \\ L_0\Big(\dfrac{w_0 + s_0 + t_0 - b^Ty}{s_0 + t_0}\Big), & w_0 < b^Ty < \\ & w_0 + s_0 + t_0, \\ 0, & otherwise, \end{cases}$$

$$\mu_O(A_j^Ty) = \begin{cases} 1, & A_j^Ty \ge c_j, \\ L_j\Big(\dfrac{c_j - A_j^Ty}{s_j}\Big), & c_j - s_j < \\ & A_j^Ty < c_j, \\ 0, & otherwise, \end{cases} \qquad \nu_O(A_j^Ty) = \begin{cases} 1, & A_j^Ty \le c_j \\ & -s_j - t_j, \\ R_j\Big(\dfrac{A_j^Ty + s_j + t_j - c_j}{s_j + t_j}\Big), & c_j - s_j - \\ & t_j < A_j^Ty \\ & < c_j, \\ 0, & otherwise, \end{cases}$$

where L_0 and R_0 are reference functions corresponding to objective function such that $R_0\Big(\dfrac{b^Ty - w_0}{s_0}\Big)$ $+L_0\Big(\dfrac{w_0 + s_0 + t_0 - b^Ty}{s_0 + t_0}\Big) \leq 1$ for $w_0 < b^Ty < w_0 + s_0 + t_0$; L_j and R_j are reference functions corresponding to constraints such that $L_j\Big(\dfrac{c_j - A_j^Ty}{s_j}\Big)+R_j\Big(\dfrac{A_j^Ty + s_j + t_j - c_j}{s_j + t_j}\Big) \leq 1$ for $c_j - s_j - t_j < A_j^Ty < c_j$, $j = 0, 1, 2, \ldots, n$; and A_j is the jth row of A, $j = 1, 2, \ldots, n$.

Let α_1, β_1 be the minimum degree of acceptance and maximum degree of rejection of the IFP problem. Angelov [9] transformed the IFP problem into the equivalent crisp primal problem (CP5) as given below:

$$
\begin{aligned}
&\text{(CP5)} \quad \max\ (\alpha_1 - \beta_1) \\
&\text{subject to} \quad L_0\Big(\frac{z_0 - c^Tx}{p_0}\Big) \geq \alpha_1, \\
&\qquad R_i\Big(\frac{A_ix - b_i}{p_i}\Big) \geq \alpha_1, \;\; i = 1, 2, \ldots, m, \\
&\qquad R_0\Big(\frac{c^Tx + p_0 + q_0 - z_0}{p_0 + q_0}\Big) \leq \beta_1, \\
&\qquad L_i\Big(\frac{b_i + p_i + q_i - A_ix}{p_i + q_i}\Big) \leq \beta_1, \;\; i = 1, 2, \ldots, m, \\
&\qquad \alpha_1 \geq \beta_1 \geq 0, \alpha_1 + \beta_1 \leq 1, \;\; x \in \mathbb{R}^n.
\end{aligned}
$$

Simplifying the above problem, we get

$$\text{(CP6)} \quad \max\ (\alpha_1 - \beta_1)$$

$$\text{subject to } z_0 - c^Tx - p_0L_0^{-1}(\alpha_1) \leq 0, \tag{3.36}$$

$$A_ix - b_i - p_iR_i^{-1}(\alpha_1) \leq 0, \quad i = 1, 2, \ldots, m, \tag{3.37}$$

$$c^Tx - z_0 + p_0 + q_0 - (p_0 + q_0)R_0^{-1}(\beta_1) \geq 0, \tag{3.38}$$

$$b_i + p_i + q_i - A_ix - (p_i + q_i)L_i^{-1}(\beta_1) \geq 0, \quad i = 1, 2, \ldots, m, \tag{3.39}$$

$$\alpha_1 \geq \beta_1 \geq 0, \alpha_1 + \beta_1 \leq 1, x \geq 0.$$

Let α_2 and β_2 be the minimum degree of acceptance and maximum degree of rejection of the

IFD problem. The equivalent crisp dual problem (CD5) of the IFD problem as given below:

$$
\begin{aligned}
\text{(CD5)} \quad & \max\ (\alpha_2 - \beta_2) \\
\text{subject to} \quad & R_0\Big(\frac{b^T y - w_0}{s_0}\Big) \geq \alpha_2, \\
& L_j\Big(\frac{c_j - A_j^T y}{s_j}\Big) \geq \alpha_2, \;\; j = 1, 2, \ldots, n, \\
& L_0\Big(\frac{w_0 + s_0 + t_0 - b^T y}{s_0 + t_0}\Big) \leq \beta_2, \\
& R_j\Big(\frac{A_j^T y + s_j + t_j - c_j}{s_j + t_j}\Big) \leq \beta_2, \;\; j = 1, 2, \ldots, n, \\
& \alpha_2 \geq \beta_2 \geq 0, \alpha_2 + \beta_2 \leq 1, \; y \in \mathbb{R}^m
\end{aligned}
$$

Simplifying the above problem, we get

$$
\begin{aligned}
\text{(CD6)} \quad & \max\ (\alpha_2 - \beta_2) \\
\text{subject to} \; & b^T y - w_0 - s_0 R_0^{-1}(\alpha_2) \leq 0, && (3.40) \\
& c_j - A_j^T y - s_j L_j^{-1}(\alpha_2) \leq 0, \quad j = 1, 2, \ldots, n && (3.41) \\
& w_0 + s_0 + t_0 - b^T y - (s_0 + t_0) L_0^{-1}(\beta_2) \geq 0, && (3.42) \\
& A_j^T y - (c_j - s_j - t_j) - (s_j + t_j) R_j^{-1}(\beta_2) \geq 0, \quad j = 1, 2, \ldots, n, && (3.43) \\
& \alpha_2 \geq \beta_2 \geq 0, \alpha_2 + \beta_2 \leq 1, y \geq 0.
\end{aligned}
$$

The membership and non-membership functions, governed by reference functions, are flexible because of the reference functions, which depend on DM.

Theorem 3.4.1. *(Modified weak duality) Let (x, α_1, β_1) and (y, α_2, β_2) be feasible solutions of (CP6) and (CD6) respectively. Then*

(i) $\sum_{i=1}^{m} R_i^{-1}(\alpha_1) p_i y_i + \sum_{j=1}^{n} L_j^{-1}(\alpha_2) s_j x_j \geq c^T x - b^T y,$

(ii) $\sum_{i=1}^{m} L_i^{-1}(\beta_1)(p_i + q_i) y_i + \sum_{j=1}^{n} R_j^{-1}(\beta_2)(t_j + s_j) x_j \leq (b + p + q)^T y - (c - s - t)^T x.$

Proof. The same as Theorem 3.3.1. □

Theorem 3.4.2. *Let $(\check{x}, \check{\alpha}_1, \check{\beta}_1)$ and $(\check{y}, \check{\alpha}_2, \check{\beta}_2)$ be feasible solutions of (CP6) and (CD6) respectively such that*

(i) $\sum_{i=1}^{m} R_i^{-1}(\check{\alpha}_1) p_i \check{y}_i + \sum_{j=1}^{n} L_j^{-1}(\check{\alpha}_2) s_j \check{x}_j = c^T \check{x} - b^T \check{y},$

(ii) $\sum_{i=1}^{m} L_i^{-1}(\check{\beta}_1)(p_i+q_i)\check{y}_i + \sum_{j=1}^{n} R_j^{-1}(\check{\beta}_2)(t_j+s_j)\check{x}_j = (b+p+q)^T\check{y} - (c-s-t)^T\check{x},$

(iii) $L_0^{-1}(\check{\alpha}_1)p_0 + R_0^{-1}(\check{\alpha}_2)s_0 = b^T\check{y} - c^T\check{x} + z_0 - w_0,$

(iv) $R_0^{-1}(\check{\beta}_1)(p_0+q_0) + L_0^{-1}(\check{\beta}_2)(t_0+s_0) = c^T\check{x} - b^T\check{y} + w_0 - z_0 + p_0 + s_0 + q_0 + t_0,$

(v) *the aspiration levels* z_0 *and* w_0 *satisfy* $z_0 - w_0 \leq 0,$

(vi) $q^T\check{y} + t^T\check{x} + q_0 + t_0 \leq s^T\check{x} - z_0 + w_0 + p^T\check{y} + p_0 + q_0.$

Then $(\check{x}, \check{\alpha}_1, \check{\beta}_1)$ *and* $(\check{y}, \check{\alpha}_2, \check{\beta}_2)$ *are the optimal solutions of (CP6) and (CD6) respectively.*

Proof. The same as Theorem 3.3.4. □

3.5 Duality in IF programming under mixed approach

Let p_0, q_0 and r_0 be the tolerances corresponding to the membership and non-membership functions of the primal objective function respectively, where $0 < q_0 < r_0$ and $r_0 < p_0 + q_0$. Let p_i, q_i and r_i be the tolerances corresponding to the membership function and non-membership functions of the ith primal constraint respectively, where $0 < q_i < r_i$ and $r_i < p_i + q_i$ for $i = 1, 2, \ldots, m$.

We take the following forms of the membership function as well as non-membership function governed by reference functions for the objective function and all the constraints of the IFP problem:

$$\mu_M(c^Tx) = \begin{cases} 1, & c^Tx \geq z_0, \\ L_0\Big(\dfrac{z_0 - c^Tx}{p_0}\Big), & z_0 - p_0 < c^Tx < z_0, \\ 0, & otherwise, \end{cases} \qquad \nu_M(c^Tx) = \begin{cases} 1, & c^Tx \leq z_0 - p_0 - q_0, \\ R_0\Big(\dfrac{c^Tx + p_0 + q_0 - z_0}{r_0}\Big), & z_0 - p_0 - q_0 < c^Tx < z_0 - p_0 - q_0 + r_0, \\ 0, & otherwise, \end{cases}$$

$$\mu_M(A_ix) = \begin{cases} 1, & A_ix \le b_i, \\ R_i\Big(\dfrac{A_ix - b_i}{p_i}\Big)\Big), & b_i < A_ix \\ & < b_i + p_i, \\ 0, & otherwise, \end{cases} \qquad \nu_M(A_ix) = \begin{cases} 1, & A_ix \ge b_i+ \\ & p_i + q_i, \\ L_i\Big(\dfrac{b_i + p_i + q_i - A_ix}{r_i}\Big), & b_i + p_i + q_i \\ & -r_i < A_ix < \\ & b_i + p_i + q_i, \\ 0, & otherwise, \end{cases}$$

where L_0 and R_0 are reference functions corresponding to the objective function such that $L_0\Big(\dfrac{z_0 - c^Tx}{p_0}\Big) + R_0\Big(\dfrac{c^Tx + p_0 + q_0 - z_0}{r_0}\Big) \le 1$ for $z_0 - p_0 - q_0 < c^Tx < z_0 - p_0 - q_0 + r_0$; L_j and R_j are reference functions corresponding to constraints such that $R_i\Big(\dfrac{A_ix - b_i}{p_i}\Big)\Big) + L_i\Big(\dfrac{b_i + p_i + q_i - A_ix}{r_i}\Big)$ ≤ 1 for $b_i + p_i + q_i - r_i < A_ix < b_i + p_i + q_i$, $i = 0, 1, 2, \ldots, m$; and A_i is the ith row of A, $i = 1, 2, \ldots, m$.

Let s_0, t_0 and u_0 be the tolerances corresponding to the membership and non-membership functions of the dual objective function respectively, where $0 < t_0 < u_0$ and $u_0 < s_0 + t_0$. Let s_j, t_j and u_j be the tolerances corresponding to the membership and non-membership functions of the jth dual constraint respectively, where $0 < t_j < u_j$ and $u_j < s_j + t_j$ for $j = 1, 2, \ldots, n$.

We take the following forms of the membership function as well as non-membership functions governed by reference functions for the objective function and all the constraints of the IFD problem:

$$\mu_M(b^Ty) = \begin{cases} 1, & b^Ty \le w_0, \\ R_0\Big(\dfrac{b^Ty - w_0}{s_0}\Big), & w_0 < b^Ty \\ & < w_0 + s_0, \\ 0, & otherwise, \end{cases} \qquad \nu_M(b^Ty) = \begin{cases} 1, & b^Ty \ge w_0+ \\ & s_0 + t_0, \\ L_0\Big(\dfrac{w_0 + s_0 + t_0 - b^Ty}{u_0}\Big), & w_0 + s_0 + t_0 \\ & -u_0 < b^Ty < \\ & w_0 + s_0 + t_0, \\ 0, & otherwise, \end{cases}$$

$$\mu_M(A_j^Ty) = \begin{cases} 1, & A_j^Ty \geq c_j, \\ L_j\Big(\dfrac{c_j - A_j^Ty}{s_j}\Big), & c_j - s_j < A_j^Ty < c_j, \\ 0, & otherwise, \end{cases} \qquad \nu_M(A_j^Ty) = \begin{cases} 1, & A_j^Ty \leq c_j - s_j - t_j, \\ R_j\Big(\dfrac{A_j^Ty + s_j + t_j - c_j}{u_j}\Big), & c_j - s_j - t_j < A_j^Ty < c_j - s_j - t_j + u_j, \\ 0, & otherwise, \end{cases}$$

where L_0 and R_0 are reference functions corresponding to the objective function such that $R_0\Big(\dfrac{b^Ty - w_0}{s_0}\Big) + L_0\Big(\dfrac{w_0 + s_0 + t_0 - b^Ty}{u_0}\Big) \leq 1$ for $w_0+s_0+t_0-u_0b^Ty < w_0+s_0+t_0$; L_j and R_j are reference functions corresponding to constraints such that $L_j\Big(\dfrac{c_j - A_j^Ty}{s_j}\Big) + R_j\Big(\dfrac{A_j^Ty + s_j + t_j - c_j}{u_j}\Big) \leq 1$ for $c_j - s_j - t_j < A_j^Ty < c_j - s_j - t_j + u_j$, $j = 0, 1, 2, \ldots, n$; and A_j is the jth row of A, $j = 1, 2, \ldots, n$.

Let α_1, β_1 be the minimum degree of acceptance and maximum degree of rejection of the IFP problem. Angelov [9] transformed the IFP problem into the equivalent crisp primal problem (CP7) as given below:

$$\begin{aligned} \text{(CP7)} \quad & \max\ (\alpha_1 - \beta_1) \\ \text{subject to} \quad & L_0\Big(\frac{z_0 - c^Tx}{p_0}\Big) \geq \alpha_1, \\ & R_i\Big(\frac{A_ix - b_i}{p_i}\Big) \geq \alpha_1, \ \ i = 1, 2, \ldots, m, \\ & R_0\Big(\frac{c^Tx + p_0 + q_0 - z_0}{r_0}\Big) \leq \beta_1, \\ & L_i\Big(\frac{b_i + p_i + q_i - A_ix}{r_i}\Big) \leq \beta_1, \ \ i = 1, 2, \ldots, m, \\ & \alpha_1 \geq \beta_1 \geq 0, \alpha_1 + \beta_1 \leq 1, \ x \in \mathbb{R}^n. \end{aligned}$$

Simplifying the above problem, we get

$$\text{(CP8)} \quad \max\ (\alpha_1 - \beta_1)$$

$$\text{subject to} \quad z_0 - c^Tx - p_0L_0^{-1}(\alpha_1) \leq 0, \tag{3.44}$$

$$A_ix - b_i - p_iR_i^{-1}(\alpha_1) \leq 0, \quad i = 1, 2, \ldots, m, \tag{3.45}$$

$$c^Tx - z_0 + p_0 + q_0 - r_0R_0^{-1}(\beta_1) \geq 0, \tag{3.46}$$

$$b_i + p_i + q_i - A_ix - r_iL_i^{-1}(\beta_1) \geq 0, \quad i = 1, 2, \ldots, m, \tag{3.47}$$

$$\alpha_1 \geq \beta_1 \geq 0, \alpha_1 + \beta_1 \leq 1, x \geq 0.$$

Let α_2 and β_2 be the minimum degree of acceptance and maximum degree of rejection of the IFD problem. The equivalent crisp dual problem (CD7) of the IFD problem as given below:

$$\begin{aligned}
\text{(CD7)} \quad & \max\ (\alpha_2 - \beta_2) \\
\text{subject to} \quad & R_0\Big(\frac{b^T y - w_0}{s_0}\Big) \geq \alpha_2, \\
& L_j\Big(\frac{c_j - A_j^T y}{s_j}\Big) \geq \alpha_2, \ \ j = 1, 2, \ldots, n, \\
& L_0\Big(\frac{w_0 + s_0 + t_0 - b^T y}{u_0}\Big) \leq \beta_2, \\
& R_j\Big(\frac{A_j^T y + s_j + t_j - c_j}{u_j}\Big) \leq \beta_2, \ \ j = 1, 2, \ldots, n, \\
& \alpha_2 \geq \beta_2 \geq 0, \alpha_2 + \beta_2 \leq 1, \ y \in \mathbb{R}^m
\end{aligned}$$

Simplifying the above problem, we get

$$\text{(CD8)} \quad \max\ (\alpha_2 - \beta_2)$$

$$\text{subject to } b^T y - w_0 - s_0 R_0^{-1}(\alpha_2) \leq 0, \tag{3.48}$$

$$c_j - A_j^T y - s_j L_j^{-1}(\alpha_2) \leq 0, \quad j = 1, 2, \ldots, n, \tag{3.49}$$

$$w_0 + s_0 + t_0 - b^T y - u_0 L_0^{-1}(\beta_2) \geq 0, \tag{3.50}$$

$$A_j^T y - (c_j - s_j - t_j) - u_j R_j^{-1}(\beta_2) \geq 0, \quad j = 1, 2, \ldots, n, \tag{3.51}$$

$$\alpha_2 \geq \beta_2 \geq 0, \alpha_2 + \beta_2 \leq 1, y \geq 0.$$

The membership and non-membership functions, governed by reference functions, are flexible because of the reference functions, which depend on DM.

Theorem 3.5.1. *(Modified weak duality) Let (x, α_1, β_1) and (y, α_2, β_2) be feasible solutions of (CP8) and (CD8) respectively. Then,*

(i) $\displaystyle\sum_{i=1}^{m} R_i^{-1}(\alpha_1) p_i y_i + \sum_{j=1}^{n} L_j^{-1}(\alpha_2) s_j x_j \geq c^T x - b^T y,$

(ii) $\displaystyle\sum_{i=1}^{m} L_i^{-1}(\beta_1) r_i y_i + \sum_{j=1}^{n} R_j^{-1}(\beta_2) u_j x_j \leq (b + p + q)^T y - (c - s - t)^T x.$

Proof. The same as Theorem 3.3.1. □

Theorem 3.5.2. *Let $(\check{x}, \check{\alpha}_1, \check{\beta}_1)$ and $(\check{y}, \check{\alpha}_2, \check{\beta}_2)$ be feasible solutions of (CP8) and (CD8) respectively such that* *(i)* $\sum_{i=1}^{m} R_i^{-1}(\check{\alpha}_1)p_i\check{y}_i + \sum_{j=1}^{n} L_j^{-1}(\check{\alpha}_2)s_j\check{x}_j = c^T\check{x} - b^T\check{y},$

(ii) $\sum_{i=1}^{m} L_i^{-1}(\check{\beta}_1)r_i\check{y}_i + \sum_{j=1}^{n} R_j^{-1}(\check{\beta}_2)u_j\check{x}_j = (b+p+q)^T\check{y} - (c-s-t)^T\check{x},$

(iii) $L_0^{-1}(\check{\alpha}_1)p_0 + R_0^{-1}(\check{\alpha}_2)s_0 = b^T\check{y} - c^T\check{x} + z_0 - w_0,$

(iv) $R_0^{-1}(\check{\beta}_1)r_0 + L_0^{-1}(\check{\beta}_2)u_0 = c^T\check{x} - b^T\check{y} + w_0 - z_0 + p_0 + s_0 + q_0 + t_0,$

(v) the aspiration levels z_0 *and* w_0 *satisfy* $z_0 - w_0 \leq 0,$

(vi) $q^T\check{y} + t^T\check{x} + q_0 + t_0 \leq s^T\check{x} - z_0 + w_0 + p^T\check{y} + p_0 + q_0.$

Then $(\check{x}, \check{\alpha}_1, \check{\beta}_1)$ *and* $(\check{y}, \check{\alpha}_2, \check{\beta}_2)$ *are the optimal solutions of (CP8) and (CD8) respectively.*

Proof. The same as Theorem 3.3.4. □

3.6 Numerical example

Example 3.6.1. *Consider the following primal-dual pair of LPPs ([24, 85]) as given below:*

$$\text{(CP)} \quad \max\ 2x \quad \textit{subject to} \quad x \leq 1,\ x \geq 0,$$

$$\text{(CD)} \quad \textit{Min } y \quad \textit{subject to} \quad y \geq 2,\ y \geq 0.$$

Optimistic point of view: Let us apply optimistic approach. Now taking $z_0 = 1$, $L_0(x) = \frac{\exp(-\rho_0 x)-\exp(-\rho_0)}{1-\exp(-\rho_0)}$, $R_0(x) = \frac{\exp(-\sigma_0 x)-\exp(-\sigma_0)}{1-\exp(-\sigma_0)}$, $L_i(x) = \frac{\exp(-\sigma_i x)-\exp(-\sigma_i)}{1-\exp(-\sigma_i)}$, $R_i(x) = \frac{\exp(-\rho_i x)-\exp(-\rho_i)}{1-\exp(-\rho_i)}$, where ρ_i, σ_i, $0 < \rho_i, \sigma_i < \infty$, $i = 0, 1, 2, \ldots, m$, are the shape parameters that measure the degree of vagueness, for (CP6), we get the following problem (CP1):

$$\begin{aligned}
\text{(CP1)}\ \ & \max\ (\alpha_1 - \beta_1) \\
\text{subject to}\ \ & p_0 \ln[\alpha_1(1 - e^{-\rho_0}) + e^{-\rho_0}] \leq \rho_0(2x - 1), \\
& p_1 \ln[\alpha_1(1 - e^{-\rho_1}) + e^{-\rho_1}] \leq \rho_1(1 - x), \\
& q_0 \ln[\beta_1(1 - e^{-\sigma_0}) + e^{-\sigma_0}] + \sigma_0(2x - 1 + p_0 + q_0) \geq 0, \\
& q_1 \ln[\beta_1(1 - e^{-\sigma_1}) + e^{-\sigma_1}] + \sigma_1(1 - x + p_1 + q_1) \geq 0, \\
& \alpha_1 \geq \beta_1 \geq 0, \alpha_1 + \beta_1 \leq 1, x \geq 0.
\end{aligned}$$

Using Mathematica 9.0, we solve (CP1) by taking $p_0 = 3$, $p_1 = 5$, $q_0 = 2, q_1 = 4$. The optimal solution of (CP1) is obtained as $\check{x} = 0.99526$, $\check{\alpha}_1 = 1$, $\check{\beta}_1 = 0$ with $\max(\alpha_1 - \beta_1)$=$(\check{\alpha}_1 - \check{\beta}_1) = 1$

for $\rho_0 = 1$, $\rho_1 = 2$, $\sigma_0 = 1$, $\sigma_1 = 2$. Therefore, the optimal solution of (CP) is $\breve{x} = 0.99526$ with $\max 2x = 2\breve{x} = 1.990520$.

Now again taking $w_0 = 1$, $L_0(x) = \frac{\exp(-\psi_0 x)-\exp(-\psi_0)}{1-\exp(-\psi_0)}$, $R_0(x) = \frac{\exp(-\phi_0 x)-\exp(-\phi_0)}{1-\exp(-\phi_0)}$, $L_j(x) = \frac{\exp(-\phi_j x)-\exp(-\phi_j)}{1-\exp(-\phi_j)}$, $R_j(x) = \frac{\exp(-\psi_j x)-\exp(-\psi_j)}{1-\exp(-\psi_j)}$, where ϕ_j, ψ_j, $0 < \phi_j, \psi_j < \infty$, $j = 0, 1, 2, \ldots, n$, are the shape parameters that measure the degree of vagueness, for (CD6), we get the following problem (CD1):

$$
\begin{aligned}
\text{(DP1)}\ \ & \max\ (\alpha_2 - \beta_2) \\
\text{subject to}\ \ & s_0 \ln[\alpha_2(1-e^{-\phi_0}) + e^{-\phi_0}] \leq \phi_0(1-y), \\
& s_1 \ln[\alpha_2(1-e^{-\phi_1}) + e^{-\phi_1}] \leq \phi_1(y-2), \\
& (s_0+t_0) \ln[\beta_2(1-e^{-\psi_0}) + e^{-\psi_0}] + \psi_0(1+y+s_0+t_0) \geq 0, \\
& (s_1+t_1) \ln[\beta_2(1-e^{-\psi_1}) + e^{-\psi_1}] + \psi_1(y-2+s_1+t_1) \geq 0, \\
& \alpha_2 \geq \beta_2 \geq 0, \alpha_2 + \beta_2 \leq 1, y \geq 0.
\end{aligned}
$$

Using Mathematica 9.0, we solve (DP1) for taking $s_0 = 3, s_1 = 5$, $t_0 = 2, t_1 = 4$. The optimal solution of (DP1) is obtained as $\breve{y} = 1.45984$, $\breve{\alpha_2} = 0.775288$, $\breve{\beta_2} = 0.019986$ with $\max(\alpha_2 - \beta_2) = (\breve{\alpha_2} - \breve{\beta_2}) = 0.755302$ by taking $\phi_0 = 1$, $\phi_1 = 2$, $\psi_0 = 1$, $\psi_1 = 2$. Therefore, the optimal solution of (DP) is $\breve{y} = 1.45984$ with $\min y = \breve{y} = 1.45984$. For these optimal solutions, both

Table 3.1: Optimistic solutions result

Primal problem								Dual problem								Duality
ρ_0	ρ_1	σ_0	σ_1	α_1	β_1	x	$c^T x$	ϕ_0	ϕ_1	ψ_0	ψ_1	α_2	β_2	w	$b^T w$	gap
1	2	1	2	1	0	0.99526	1.990520	1	2	1	2	0.775288	0.019986	1.45984	1.45984	0.00687108
2	2	2	2	1	0	0.996817	1.993630	2	2	2	2	0.744179	0.023321	1.375	1.3375	0.004381
3	2	3	2	1	0	0.995358	1.990720	3	2	3	2	0.720569	0.025994	1.308590	1.308590	0.006068
3	5	3	5	1	0	0.990885	1.981770	3	5	3	5	0.595151	0.002244	1.485630	1.485630	0.013539
4	5	4	5	1	0	0.989429	1.978860	4	5	4	5	0.559571	0.002555	1.424700	1.424700	0.015054
5	5	5	5	1	0	0.982416	1.964830	5	5	5	5	0.527522	0.007620	1.379690	1.379690	0.037268
6	6	6	6	1	0	0.980898	1.961800	6	6	6	6	0.47105	0.001284	1.375000	1.375000	0.026262
6	8	6	8	1	0	0.994417	1.988830	6	8	6	8	0.410353	0.000228	1.443590	1.443590	0.008063
7	8	7	8	1	0	0.989763	1.979530	7	8	7	8	0.386719	0.000233	1.406550	1.406550	0.014398
8	5	8	5	1	0	0.976224	1.952450	8	5	8	5	0.480635	0.003367	1.274610	1.274610	0.030306

the hypotheses of Theorem 3.3.1; Inequalities (3.15) and (3.16) are satisfied. The solutions of the fuzzy primal-dual problems with linear membership function, noted in [24] are as follows: Fuzzy primal optimal solution is x = 0.5 and optimal value of objective function is 1, and fuzzy dual optimal solution is y = 1.25 and optimal value of objective function is 1.25. The solutions of the fuzzy primal-dual problems with exponential membership function, noted in [85] are as follows: Fuzzy primal optimal solution is x = 0.9856906 and optimal value of objective function is 1.971381; and fuzzy dual optimal solution is y = 1.194403 and optimal value of objective function is 1.194403. The duality gaps in [24] and [85] are 0.625 and 0.017088 respectively.

Using the proposed approach with exponential membership function in IFE, the duality gap is 0.006871 (the difference between the right hand side and left hand side of the condition (i) of Theorem 3.3.1). Small duality gap implies that the primal and dual objective values in fuzzy and IFE are closer to each other for a given tolerance. The solutions of primal and dual problems in IFE with different values of shape parameters ρ_0, ρ_1, σ_0, σ_1, ϕ_0, ϕ_1, ψ_0, ψ_1 are given in Table 3.1.

Pessimistic point of view: Let us apply pessimistic approach. Now taking $z_0 = 1$, $L_0(x) = \frac{\exp(-\rho_0 x)-\exp(-\rho_0)}{1-\exp(-\rho_0)}$, $R_0(x) = \frac{\exp(-\sigma_0 x)-\exp(-\sigma_0)}{1-\exp(-\sigma_0)}$, $L_i(x) = \frac{\exp(-\sigma_i x)-\exp(-\sigma_i)}{1-\exp(-\sigma_i)}$, $R_i(x) = \frac{\exp(-\rho_i x)-\exp(-\rho_i)}{1-\exp(-\rho_i)}$, where ρ_i, σ_i, $0 < \rho_i, \sigma_i < \infty$, $i = 0, 1, 2, \ldots, m$, are the shape parameters that measure the degree of vagueness, for (CP3), we get the following problem (CP1):

$$
\begin{aligned}
\text{(CP1)}\ \ &\max\ (\alpha_1 - \beta_1)\\
\text{subject to}\ \ & p_0 \ln[\alpha_1(1-e^{-\rho_0}) + e^{-\rho_0}] \le \rho_0(2x-1),\\
& p_1 \ln[\alpha_1(1-e^{-\rho_1}) + e^{-\rho_1}] \le \rho_1(1-x),\\
& q_0 \ln[\beta_1(1-e^{-\sigma_0}) + e^{-\sigma_0}] + \sigma_0(2x-1+p_0) \ge 0,\\
& q_1 \ln[\beta_1(1-e^{-\sigma_1}) + e^{-\sigma_1}] + (1+p_1-x) \ge 0,\\
& \alpha_1 \ge \beta_1 \ge 0, \alpha_1 + \beta_1 \le 1, x \ge 0.
\end{aligned}
$$

Now again taking $w_0 = 1$, $L_0(x) = \frac{\exp(-\psi_0 x)-\exp(-\psi_0)}{1-\exp(-\psi_0)}$, $R_0(x) = \frac{\exp(-\phi_0 x)-\exp(-\phi_0)}{1-\exp(-\phi_0)}$, $L_j(x) = \frac{\exp(-\phi_j x)-\exp(-\phi_j)}{1-\exp(-\phi_j)}$, $R_j(x) = \frac{\exp(-\psi_j x)-\exp(-\psi_j)}{1-\exp(-\psi_j)}$, where ϕ_j, ψ_j, $0 < \phi_j, \psi_j < \infty$, $j = 0, 1, 2, \ldots, n$, are the shape parameters that measure the degree of vagueness, for (CD3), we get the following problem (CD1):

$$
\begin{aligned}
\text{(DP1)}\ \ &\max\ (\alpha_2 - \beta_2)\\
\text{subject to}\ \ & s_0 \ln[\alpha_2(1-e^{-\phi_0}) + e^{-\phi_0}] \le \phi_0(1-y),\\
& s_1 \ln[\alpha_2(1-e^{-\phi_1}) + e^{-\phi_1}] \le \phi_1(y-2),\\
& (s_0+t_0) \ln[\beta_2(1-e^{-\psi_0}) + e^{-\psi_0}] + \psi_0(1+s_0+y) \ge 0,\\
& (s_1+t_1) \ln[\beta_2(1-e^{-\psi_1}) + e^{-\psi_1}] + \psi_1(y-2+s_1) \ge 0,\\
& \alpha_2 \ge \beta_2 \ge 0, \alpha_2 + \beta_2 \le 1, y \ge 0.
\end{aligned}
$$

Using Mathematica 9.0, we solve (CP1) and (DP1) for $p_0 = 3, p_1 = 5$, $q_0 = 2, q_1 = 4$, $s_0 = 3, s_1 = 5$, $t_0 = 2, t_1 = 4$. The optimal solutions of (CP1) and (DP1) problems with different values of shape parameters ρ_0, ρ_1, σ_0, σ_1, ϕ_0, ϕ_1, ψ_0, ψ_1 are given in Table 3.2.

Mixed point of view: Let us apply mixed approach. Now taking $z_0 = 1$, $L_0(x) = \frac{\exp(-\rho_0 x)-\exp(-\rho_0)}{1-\exp(-\rho_0)}$, $R_0(x) = \frac{\exp(-\sigma_0 x)-\exp(-\sigma_0)}{1-\exp(-\sigma_0)}$, $L_i(x) = \frac{\exp(-\sigma_i x)-\exp(-\sigma_i)}{1-\exp(-\sigma_i)}$, $R_i(x) = \frac{\exp(-\rho_i x)-\exp(-\rho_i)}{1-\exp(-\rho_i)}$,

Table 3.2: Pessimistic solutions result

Primal problem								Dual problem								Duality
ρ_0	ρ_1	σ_0	σ_1	α_1	β_1	x	$c^T x$	ϕ_0	ϕ_1	ψ_0	ψ_1	α_2	β_2	w	$b^T w$	gap
1	2	1	2	1	0	0.989852	1.979700	1	2	1	2	0.775271	0	1.459800	1.459800	0.014775
2	2	2	2	1	0	0.989852	1.979700	2	2	2	2	0.744201	0	1.375020	1.375020	0.013918
3	2	3	2	1	0	0.989852	1.979700	3	2	3	2	0.720560	0	1.308560	1.308560	0.013251
3	5	3	5	1	0	0.970791	1.941580	3	5	3	5	0.595151	0	1.485630	1.485630	0.043393
4	5	4	5	1	0	0.960719	1.921440	4	5	4	5	0.559571	0	1.424700	1.424700	0.055957
5	5	5	5	1	0	0.951880	1.903760	5	5	5	5	0.532109	0	1.375000	1.375000	0.066165
6	6	6	6	1	0	0.953798	1.907620	6	6	6	6	0.471055	0	1.375000	1.375000	0.063525
6	8	6	8	1	0	0.99389	1.998780	6	8	6	8	0.410353	0	1.443590	1.443590	0.008798
7	8	7	8	1	0	0.981114	1.962230	7	8	7	8	0.386719	0	1.406550	1.406550	0.026565
8	5	8	5	1	0	0.941133	1.882270	8	5	8	5	0.480635	0	1.274610	1.274610	0.075031

where ρ_i, σ_i, $0 < \rho_i, \sigma_i < \infty$, $i = 0, 1, 2, \ldots, m$, are the shape parameters that measure the degree of vagueness, for (CP8), we get the following problem (CP1):

$$\begin{aligned}
\text{(CP1)}\ \ &\max\ (\alpha_1 - \beta_1)\\
\text{subject to}\ \ &p_0 \ln[\alpha_1(1 - e^{-\rho_0}) + e^{-\rho_0}] \leq \rho_0(2x - 1),\\
&p_1 \ln[\alpha_1(1 - e^{-\rho_1}) + e^{-\rho_1}] \leq \rho_1(1 - x),\\
&r_0 \ln[\beta_1(1 - e^{-\sigma_0}) + e^{-\sigma_0}] + \sigma_0(2x - 1 + p_0 + q_0) \geq 0,\\
&r_1 \ln[\beta_1(1 - e^{-\sigma_1}) + e^{-\sigma_1}] + \sigma_1(1 - x + p_1 + q_1) \geq 0,\\
&\alpha_1 \geq \beta_1 \geq 0, \alpha_1 + \beta_1 \leq 1, x \geq 0.
\end{aligned}$$

Now again taking $w_0 = 1$, $L_0(x) = \frac{\exp(-\psi_0 x) - \exp(-\psi_0)}{1 - \exp(-\psi_0)}$, $R_0(x) = \frac{\exp(-\phi_0 x) - \exp(-\phi_0)}{1 - \exp(-\phi_0)}$, $L_j(x) = \frac{\exp(-\phi_j x) - \exp(-\phi_j)}{1 - \exp(-\phi_j)}$, $R_j(x) = \frac{\exp(-\psi_j x) - \exp(-\psi_j)}{1 - \exp(-\psi_j)}$, where ϕ_j, ψ_j, $0 < \phi_j, \psi_j < \infty$, $j = 0, 1, 2, \ldots, n$, are the shape parameters that measure the degree of vagueness, for (CD8), we get the following problem (CD1):

$$\begin{aligned}
\text{(DP1)}\ \ &\max\ (\alpha_2 - \beta_2)\\
\text{subject to}\ \ &s_0 \ln[\alpha_2(1 - e^{-\phi_0}) + e^{-\phi_0}] \leq \phi_0(1 - y),\\
&s_1 \ln[\alpha_2(1 - e^{-\phi_1}) + e^{-\phi_1}] \leq \phi_1(y - 2),\\
&u_0 \ln[\beta_2(1 - e^{-\psi_0}) + e^{-\psi_0}] + \psi_0(1 + y + s_0 + t_0) \geq 0,\\
&u_1 \ln[\beta_2(1 - e^{-\psi_1}) + e^{-\psi_1}] + \psi_1(y - 2 + s_1 + t_1) \geq 0,\\
&\alpha_2 \geq \beta_2 \geq 0, \alpha_2 + \beta_2 \leq 1, y \geq 0.
\end{aligned}$$

Using Mathematica 9.0, we solve (CP1) and (DP1) for $p_0 = 3, p_1 = 5$, $q_0 = 2, q_1 = 4, r_0 = 4, r_1 = 8$, $s_0 = 3, s_1 = 5$, $t_0 = 2, t_1 = 4, u_0 = 4, u_1 = 8$. The optimal solutions of (CP1) and (DP1) problems with different values of shape parameters ρ_0, ρ_1, σ_0, σ_1, ϕ_0, ϕ_1, ψ_0, ψ_1 are given in Table 3.3.

Table 3.3: Mixed solutions result

Primal problem								Dual problem								Duality
ρ_0	ρ_1	σ_0	σ_1	α_1	β_1	x	c^Tx	ϕ_0	ϕ_1	ψ_0	ψ_1	α_2	β_2	w	b^Tw	gap
1	2	1	2	1	0	0.989852	1.979700	1	2	1	2	0.775271	0	1.459800	1.459800	0.014775
2	2	2	2	1	0	0.989852	1.979700	2	2	2	2	0.744201	0	1.375020	1.375020	0.013918
3	2	3	2	1	0	0.989852	1.979700	3	2	3	2	0.720560	0	1.308560	1.308560	0.013251
3	5	3	5	1	0	0.959388	1.918780	3	5	3	5	0.595151	0	1.485630	1.485630	0.060328
4	5	4	5	1	0	0.961500	1.923000	4	5	4	5	0.559571	0	1.424700	1.424700	0.054847
5	5	5	5	1	0	0.960324	1.920650	5	5	5	5	0.532115	0	1.375000	1.375000	0.054542
6	6	6	6	1	0	0.945316	1.890630	6	6	6	6	0.471055	0	1.375000	1.375000	0.045173
6	8	6	8	1	0	0.993890	1.998780	6	8	6	8	0.410353	0	1.443590	1.443590	0.008798
7	8	7	8	1	0	0.981114	1.962230	7	8	7	8	0.386719	0	1.406550	1.406550	0.026565
8	5	8	5	1	0	0.927700	1.855400	8	5	8	5	0.480635	0	1.274610	1.274610	0.092157

Example 3.6.2. *Consider the following primal-dual pair of LPPs ([24, 85]) as given below:*

$$\textit{(CP)} \quad \max\ 3x_1 + 4x_2 \quad \textit{subject to} \quad 4x_1 + 2x_2 \leq 8,\ 3x_1 + 5x_2 \geq 18,\ x_1,\ x_2 \geq 0,$$

$$\textit{(CD)} \quad \textit{Min}\ 8y_1 + 18y_2 \quad \textit{subject to} \quad 4y_1 + 3y_2 \geq 3,\ 2y_1 + 5y_2 \geq 4,\ y_1, y_2 \geq 0.$$

Optimistic point of view: Let us apply optimistic approach. Now taking $z_0 = 10$, $L_0(x) = \frac{\exp(-\rho_0 x) - \exp(-\rho_0)}{1-\exp(-\rho_0)}$, $R_0(x) = \frac{\exp(-\sigma_0 x) - \exp(-\sigma_0)}{1-\exp(-\sigma_0)}$, $L_i(x) = \frac{\exp(-\sigma_i x) - \exp(-\sigma_i)}{1-\exp(-\sigma_i)}$, $R_i(x) = \frac{\exp(-\rho_i x) - \exp(-\rho_i)}{1-\exp(-\rho_i)}$, where ρ_i, σ_i, $0 < \rho_i, \sigma_i < \infty$, $i = 0, 1, 2, \ldots, m$, are the shape parameters that measure the degree of vagueness, for (CP6), we get the following problem (CP1):

$$\begin{aligned}
\text{(CP1)}\ \ &\max\ (\alpha_1 - \beta_1)\\
\text{subject to}\ \ & p_0 \ln[\alpha_1(1 - e^{-\rho_0}) + e^{-\rho_0}] \leq \rho_0(3x_1 + 4x_2 - 10),\\
& p_1 \ln[\alpha_1(1 - e^{-\rho_1}) + e^{-\rho_1}] \leq \rho_1(8 - (4x_1 + 2x_2)),\\
& p_2 \ln[\alpha_1(1 - e^{-\rho_2}) + e^{-\rho_2}] \leq \rho_2(18 - (3x_1 + 5x_2)),\\
& (p_0 + q_0) \ln[\beta_1(1 - e^{-\sigma_0}) + e^{-\sigma_0}] + \sigma_0(3x_1 + 4x_2 - 10 + p_0 + q_0) \geq 0,\\
& (p_1 + q_1) \ln[\beta_1(1 - e^{-\sigma_1}) + e^{-\sigma_1}] + (8 + p_1 + q_1 - (4x_1 + 2x_2)) \geq 0,\\
& (p_2 + q_2) \ln[\beta_1(1 - e^{-\sigma_2}) + e^{-\sigma_2}] + (18 + p_2 + q_2 - (3x_1 + 5x_2)) \geq 0,\\
& \alpha_1 \geq \beta_1 \geq 0, \alpha_1 + \beta_1 \leq 1, x_1, x_2 \geq 0.
\end{aligned}$$

Using Mathematica 9.0, we solve (CP1) by taking $p_0 = 4, p_1 = 5, p_2 = 6$, $q_0 = 2, q_1 = 3, q_2 = 4$. The optimal solution of (CP1) is obtained as $\breve{x} = 0.99526$, $\breve{\alpha_1} = 1$, $\breve{\beta_1} = 0$ with $\max(\alpha_1 - \beta_1) = (\breve{\alpha_1} - \breve{\beta_1}) = 1$ for $\rho_0 = 1$, $\rho_1 = 2$, $\sigma_0 = 1$, $\sigma_1 = 2$. Therefore, the optimal solution of (CP) is $\breve{x} = 0.99526$ with $\max 2x = 2\breve{x} = 1.990520$.

Now again taking $w_0 = 15$, $L_0(x) = \frac{\exp(-\psi_0 x) - \exp(-\psi_0)}{1-\exp(-\psi_0)}$, $R_0(x) = \frac{\exp(-\phi_0 x) - \exp(-\phi_0)}{1-\exp(-\phi_0)}$, $L_j(x) = \frac{\exp(-\phi_j x) - \exp(-\phi_j)}{1-\exp(-\phi_j)}$, $R_j(x) = \frac{\exp(-\psi_j x) - \exp(-\psi_j)}{1-\exp(-\psi_j)}$, where ϕ_j, ψ_j, $0 < \phi_j, \psi_j < \infty$, $j = 0, 1, 2, \ldots, n$, are the shape parameters that measure the degree of vagueness, for (CD6), we get the following problem (CD1):

$$\begin{aligned}
\text{(DP1)}\ \ \max\ & (\alpha_2 - \beta_2) \\
\text{subject to}\ \ & s_0 \ln[\alpha_2(1-e^{-\phi_0}) + e^{-\phi_0}] \leq \phi_0(15 - (8y_1 + 18y_2)), \\
& s_1 \ln[\alpha_2(1-e^{-\phi_1}) + e^{-\phi_1}] \leq \phi_1(4y_1 + 3y_2 - 3), \\
& s_2 \ln[\alpha_2(1-e^{-\phi_2}) + e^{-\phi_2}] \leq \phi_2(2y_1 + 5y_2 - 4), \\
& (s_0 + t_0) \ln[\beta_2(1-e^{-\psi_0}) + e^{-\psi_0}] + \psi_0(15 + s_0 + t_0 - (8y_1 + 18y_2)) \geq 0, \\
& (s_1 + t_1) \ln[\beta_2(1-e^{-\psi_1}) + e^{-\psi_1}] + \psi_1(4y_1 + 3y_2 - 3 + s_1 + t_1) \geq 0, \\
& (s_2 + t_2) \ln[\beta_2(1-e^{-\psi_2}) + e^{-\psi_2}] + \psi_2(2y_1 + 5y_2 - 4 + s_2 + t_2) \geq 0, \\
& \alpha_2 \geq \beta_2 \geq 0, \alpha_2 + \beta_2 \leq 1, y_1, y_2 \geq 0.
\end{aligned}$$

Using Mathematica 9.0, we solve (DP1) for taking $s_0 = 3, s_1 = 4, s_2 = 5,\ t_0 = 2, t_1 = 3, t_2 = 4$. The optimal solution of (DP1) is obtained as $\check{y} = 1.45984$, $\check{\alpha_2} = 0.775288$, $\check{\beta_2} = 0.019986$ with $\max(\alpha_2 - \beta_2)$=$(\check{\alpha_2} - \check{\beta_2}) = 0.755302$ by taking $\phi_0 = 1$, $\phi_1 = 2$, $\psi_0 = 1$, $\psi_1 = 2$.

Table 3.4: Optimistic solutions result

Primal problem											Dual problem											Duality
ρ_0	ρ_1	ρ_2	σ_0	σ_1	σ_2	α_1	β_1	x_1	x_2	$c^T x$	ϕ_0	ϕ_1	ϕ_2	ψ_0	ψ_1	ψ_2	α_2	β_2	y_1	y_2	$b^T w$	gap
1	1	2	1	1	2	1	0	0.270886	3.435380	14.554200	1	1	2	1	1	2	1	0	0.722114	0.511742	14.988300	0.434100
1	2	2	1	2	2	1	0	0.265412	3.439120	14.552700	1	2	2	1	2	2	1	0	0.239925	0.723052	14.934300	0.381600
2	3	2	2	3	2	1	0	0.267347	3.437700	14.552800	2	3	2	2	3	2	1	0	0.353971	0.666587	14.830300	0.277500
2	3	5	2	3	5	1	0	0.273479	3.433530	14.554600	2	3	5	2	3	5	1	0	0.396253	0.650036	14.870700	0.316100
3	4	5	3	4	5	1	0	0.271385	3.434840	14.553500	3	4	5	3	4	5	1	0	0.414076	0.643125	14.888900	0.335400
3	5	5	3	5	5	1	0	0.271042	3.435100	14.553500	3	5	5	3	5	5	1	0	0.391158	0.655541	14.929000	0.375500
4	6	6	4	6	6	1	0	0.272029	3.434510	14.554100	4	6	6	4	6	6	1	0	0.347476	0.672630	14.887100	0.333000
4	6	7	4	6	7	1	0	0.266263	3.438830	14.554100	4	6	7	4	6	7	1	0	0.337171	0.671615	14.786400	0.232300
5	8	7	5	8	7	1	0	0.273619	3.433270	14.553900	5	8	7	5	8	7	1	0	0.439962	0.630069	14.860900	0.307000
5	8	5	5	8	5	1	0	0.274344	3.432700	14.553800	5	8	5	5	8	5	1	0	0.379998	0.657275	14.870900	0.317100

The solutions of primal and dual problems in IFE with different values of shape parameters ρ_0, ρ_1, ρ_2, σ_0, σ_1, σ_2, ϕ_0, ϕ_1, ϕ_2, ψ_0, ψ_1, ψ_2 are given in Table 3.4.

Pessimistic point of view: Let us apply pessimistic approach. Now taking $z_0 = 10$, $L_0(x) = \frac{\exp(-\rho_0 x) - \exp(-\rho_0)}{1 - \exp(-\rho_0)}$, $R_0(x) = \frac{\exp(-\sigma_0 x) - \exp(-\sigma_0)}{1 - \exp(-\sigma_0)}$, $L_i(x) = \frac{\exp(-\sigma_i x) - \exp(-\sigma_i)}{1 - \exp(-\sigma_i)}$, $R_i(x) = \frac{\exp(-\rho_i x) - \exp(-\rho_i)}{1 - \exp(-\rho_i)}$, where $\rho_i, \sigma_i, 0 < \rho_i, \sigma_i < \infty$, $i = 0, 1, 2, \ldots, m$, are the shape parameters that measure the degree of vagueness, for (CP3), we get the following problem (CP1):

$$\begin{aligned}
\text{(CP1)}\ \ &\max\ (\alpha_1-\beta_1)\\
\text{subject to}\ \ & p_0\ln[\alpha_1(1-e^{-\rho_0})+e^{-\rho_0}]\le \rho_0(3x_1+4x_2-10),\\
& p_1\ln[\alpha_1(1-e^{-\rho_1})+e^{-\rho_1}]\le \rho_1(8-(4x_1+2x_2)),\\
& p_2\ln[\alpha_1(1-e^{-\rho_2})+e^{-\rho_2}]\le \rho_2(18-(3x_1+5x_2)),\\
& q_0\ln[\beta_1(1-e^{-\sigma_0})+e^{-\sigma_0}]+\sigma_0(3x_1+4x_2-10+p_0)\ge 0,\\
& q_1\ln[\beta_1(1-e^{-\sigma_1})+e^{-\sigma_1}]+(8+p_1-(4x_1+2x_2))\ge 0,\\
& q_2\ln[\beta_1(1-e^{-\sigma_2})+e^{-\sigma_2}]+(18+p_2-(3x_1+5x_2))\ge 0,\\
& \alpha_1\ge\beta_1\ge 0,\alpha_1+\beta_1\le 1,x_1,x_2\ge 0.
\end{aligned}$$

Now again taking $w_0 = 15$, $L_0(x) = \frac{\exp(-\psi_0 x)-\exp(-\psi_0)}{1-\exp(-\psi_0)}$, $R_0(x) = \frac{\exp(-\phi_0 x)-\exp(-\phi_0)}{1-\exp(-\phi_0)}$, $L_j(x) = \frac{\exp(-\phi_j x)-\exp(-\phi_j)}{1-\exp(-\phi_j)}$, $R_j(x) = \frac{\exp(-\psi_j x)-\exp(-\psi_j)}{1-\exp(-\psi_j)}$, where ϕ_j, ψ_j, $0 < \phi_j, \psi_j < \infty$, $j = 0, 1, 2, \ldots, n$, are the shape parameters that measure the degree of vagueness, for (CD3), we get the following problem (CD1):

$$\begin{aligned}
\text{(DP1)}\ \ &\max\ (\alpha_2-\beta_2)\\
\text{subject to}\ \ & s_0\ln[\alpha_2(1-e^{-\phi_0})+e^{-\phi_0}]\le \phi_0(15-(8y_1+18y_2)),\\
& s_1\ln[\alpha_2(1-e^{-\phi_1})+e^{-\phi_1}]\le \phi_1(4y_1+3y_2-3),\\
& s_2\ln[\alpha_2(1-e^{-\phi_2})+e^{-\phi_2}]\le \phi_2(2y_1+5y_2-4),\\
& (s_0+t_0)\ln[\beta_2(1-e^{-\psi_0})+e^{-\psi_0}]+\psi_0(15+s_0-(8y_1+18y_2))\ge 0,\\
& (s_1+t_1)\ln[\beta_2(1-e^{-\psi_1})+e^{-\psi_1}]+\psi_1(4y_1+3y_2-3+s_1)\ge 0,\\
& (s_2+t_2)\ln[\beta_2(1-e^{-\psi_2})+e^{-\psi_2}]+\psi_2(2y_1+5y_2-4+s_2)\ge 0,\\
& \alpha_2\ge\beta_2\ge 0,\alpha_2+\beta_2\le 1,y_1,y_2\ge 0.
\end{aligned}$$

Using Mathematica 9.0, we solve (CP1) and (DP1) for $p_0 = 4, p_1 = 5, p_2 = 6$, $q_0 = 2, q_1 = 3, q_2 = 4$, $s_0 = 3, s_1 = 4, s_2 = 5$, $t_0 = 2, t_1 = 3, t_2 = 4$. The optimal solutions of (CP1) and (DP1) problems with different values of shape parameters ρ_0, ρ_1, ρ_2, σ_0, σ_1, σ_2, ϕ_0, ϕ_1, ϕ_2, ψ_0, ψ_1, ψ_2 are given in Table 3.5.

Mixed point of view: Let us apply mixed approach. Now taking $z_0 = 10$, $L_0(x) = \frac{\exp(-\rho_0 x)-\exp(-\rho_0)}{1-\exp(-\rho_0)}$, $R_0(x) = \frac{\exp(-\sigma_0 x)-\exp(-\sigma_0)}{1-\exp(-\sigma_0)}$, $L_i(x) = \frac{\exp(-\sigma_i x)-\exp(-\sigma_i)}{1-\exp(-\sigma_i)}$, $R_i(x) = \frac{\exp(-\rho_i x)-\exp(-\rho_i)}{1-\exp(-\rho_i)}$, where ρ_i, σ_i, $0 < \rho_i, \sigma_i < \infty$, $i = 0, 1, 2, \ldots, m$, are the shape parameters that measure the degree

Table 3.5: Pessimistic solutions result

Primal problem											Dual problem											Duality
ρ_0	ρ_1	ρ_2	σ_0	σ_1	σ_2	α_1	β_1	x_1	x_2	c^Tx	ϕ_0	ϕ_1	ϕ_2	ψ_0	ψ_1	ψ_2	α_2	β_2	y_1	y_2	b^Tw	gap
1	1	2	1	1	2	1	0	0.235549	3.452670	14.517300	1	1	2	1	1	2	1	0	0.196655	0.745495	14.992200	0.474900
1	2	2	1	2	2	1	0	0.235549	3.452670	14.517300	1	2	2	1	2	2	1	0	0.378862	0.656541	14.848600	0.331300
2	3	2	2	3	2	1	0	0.252991	3.236590	13.705300	2	3	2	2	3	2	1	0	0.236272	0.727201	14.979800	1.274500
2	3	5	2	3	5	1	0	0.102331	3.375990	13.811000	2	3	5	2	3	5	1	0	0.300433	0.694823	14.910300	1.099300
3	4	5	3	4	5	1	0	0.135635	3.412880	14.058400	3	4	5	3	4	5	1	0	0.511328	0.603710	14.957400	0.899000
3	5	5	3	5	5	1	0	0.135635	3.412880	14.058400	3	5	5	3	5	5	1	0	0.489998	0.609919	14.898500	0.840100
4	6	6	4	6	6	1	0	0.032308	3.378290	13.610100	4	6	6	4	6	6	1	0	0.460125	0.621723	14.872000	1.261900
4	6	7	4	6	7	1	0	0.000475	3.403730	13.616300	4	6	7	4	6	7	1	0	0.413595	0.639218	14.814700	1.198400
5	8	7	5	8	7	1	0	0.025300	3.583920	14.411600	5	8	7	5	8	7	1	0	0.457740	0.627858	14.963400	0.551800
5	8	5	5	8	5	1	0	0.037432	3.573770	14.407400	5	8	5	5	8	5	1	0	0.644420	0.543996	14.947300	0.539900

of vagueness, for (CP8), we get the following problem (CP1):

$$
\begin{aligned}
\text{(CP1)}\ \ & \max\ (\alpha_1-\beta_1)\\
\text{subject to}\ \ & p_0\ln[\alpha_1(1-e^{-\rho_0})+e^{-\rho_0}]\le\rho_0(3x_1+4x_2-10),\\
& p_1\ln[\alpha_1(1-e^{-\rho_1})+e^{-\rho_1}]\le\rho_1(8-(4x_1+2x_2)),\\
& p_2\ln[\alpha_1(1-e^{-\rho_2})+e^{-\rho_2}]\le\rho_2(18-(3x_1+5x_2)),\\
& r_0\ln[\beta_1(1-e^{-\sigma_0})+e^{-\sigma_0}]+\sigma_0(3x_1+4x_2-10+p_0+q_0)\ge 0,\\
& r_1\ln[\beta_1(1-e^{-\sigma_1})+e^{-\sigma_1}]+(8+p_1+q_1-(4x_1+2x_2))\ge 0,\\
& r_2\ln[\beta_1(1-e^{-\sigma_2})+e^{-\sigma_2}]+(18+p_2+q_2-(3x_1+5x_2))\ge 0,\\
& \alpha_1\ge\beta_1\ge 0,\alpha_1+\beta_1\le 1,x_1,x_2\ge 0.
\end{aligned}
$$

Now again taking $w_0=15$, $L_0(x)=\frac{\exp(-\psi_0 x)-\exp(-\psi_0)}{1-\exp(-\psi_0)}$, $R_0(x)=\frac{\exp(-\phi_0 x)-\exp(-\phi_0)}{1-\exp(-\phi_0)}$, $L_j(x)=\frac{\exp(-\phi_j x)-\exp(-\phi_j)}{1-\exp(-\phi_j)}$, $R_j(x)=\frac{\exp(-\psi_j x)-\exp(-\psi_j)}{1-\exp(-\psi_j)}$, where ϕ_j,ψ_j, $0<\phi_j,\psi_j<\infty$, $j=0,1,2,\ldots,n$, are the shape parameters that measure the degree of vagueness, for (CD8), we get the following problem (CD1):

$$
\begin{aligned}
\text{(DP1)}\ \ & \max\ (\alpha_2-\beta_2)\\
\text{subject to}\ \ & s_0\ln[\alpha_2(1-e^{-\phi_0})+e^{-\phi_0}]\le\phi_0(15-(8y_1+18y_2)),\\
& s_1\ln[\alpha_2(1-e^{-\phi_1})+e^{-\phi_1}]\le\phi_1(4y_1+3y_2-3),\\
& s_2\ln[\alpha_2(1-e^{-\phi_2})+e^{-\phi_2}]\le\phi_2(2y_1+5y_2-4),\\
& u_0\ln[\beta_2(1-e^{-\psi_0})+e^{-\psi_0}]+\psi_0(15+s_0+t_0-(8y_1+18y_2))\ge 0,\\
& u_1\ln[\beta_2(1-e^{-\psi_1})+e^{-\psi_1}]+\psi_1(4y_1+3y_2-3+s_1+t_1)\ge 0,\\
& u_2\ln[\beta_2(1-e^{-\psi_2})+e^{-\psi_2}]+\psi_2(2y_1+5y_2-4+s_2+t_2)\ge 0,\\
& \alpha_2\ge\beta_2\ge 0,\alpha_2+\beta_2\le 1,y_1,y_2\ge 0.
\end{aligned}
$$

Using Mathematica 9.0, we solve (CP1) and (DP1) for $p_0=4,p_1=5,p_2=6,\ q_0=2,q_1=3,q_2=4,r_0=4,r_1=6,r_2=8,\ s_0=3,s_1=4,s_2=5,\ t_0=2,t_1=3,t_2=4,u_0=4,u_1=6,u_2=8$. The optimal solutions of (CP1) and (DP1) problems with different values of shape parameters ρ_0, ρ_1, ρ_2, σ_0, σ_1, σ_2, ϕ_0, ϕ_1, ϕ_2, ψ_0, ψ_1, ψ_2 are given in Table 3.6. The above examples (Examples 3.6.1

Table 3.6: Mixed solutions result

Primal problem											Dual problem											Duality
ρ_0	ρ_1	ρ_2	σ_0	σ_1	σ_2	α_1	β_1	x_1	x_2	$c^T x$	ϕ_0	ϕ_1	ϕ_2	ψ_0	ψ_1	ψ_2	α_2	β_2	y_1	y_2	$b^T w$	gap
1	1	2	1	1	2	1	0	0.235549	3.452670	14.517300	1	1	2	1	1	2	1	0	0.196655	0.745495	14.992200	0.474900
1	2	2	1	2	2	1	0	0.235549	3.452670	14.517300	1	2	2	1	2	2	1	0	0.378862	0.656541	14.848600	0.331300
2	3	2	2	3	2	1	0	0.252991	3.236590	13.705300	2	3	2	2	3	2	1	0	0.236272	0.727201	14.979800	1.274500
2	3	5	2	3	5	1	0	0.102331	3.375990	13.811000	2	3	5	2	3	5	1	0	0.451977	0.625542	14.875600	1.064600
3	4	5	3	4	5	1	0	0.135635	3.412880	14.058400	3	4	5	3	4	5	1	0	0.462231	0.622055	14.894800	0.836400
3	5	5	3	5	5	1	0	0.135635	3.412880	14.058400	3	5	5	3	5	5	1	0	0.579639	0.571855	14.930500	0.872100
4	6	6	4	6	6	1	0	0.032308	3.378290	13.610100	4	6	6	4	6	6	1	0	0.260274	0.713838	14.931300	1.321200
4	6	7	4	6	7	1	0	0.000475	3.403730	13.616300	4	6	7	4	6	7	1	0	0.419299	0.637305	14.825900	1.209600
5	8	7	5	8	7	1	0	0.025300	3.583920	14.411600	5	8	7	5	8	7	1	0	0.450945	0.626384	14.882500	0.470900
5	8	5	5	8	5	1	0	0.037432	3.573770	14.407400	5	8	5	5	8	5	1	0	0.421774	0.639529	14.885700	0.478300

and 3.6.2) are solved with different approaches like optimistic, pessimistic and mixed approaches in IFE. The results of Examples 1 and 2 are depicted in Tables 3.1-3.6 in different approaches. Based on the solutions with optimistic, pessimistic and mixed approaches in Examples 1 and 2, it is clear that optimistic solutions are good compared to pessimistic and mixed solutions due to the duality gaps. The duality gaps in optimistic solutions are very less as compared to pessimistic and mixed solutions. Also, the duality gaps in optimistic approaches for Examples 3.6.1 and 3.6.2 in IFE are very small compared to the methods given in [24, 85] in a fuzzy environment.

3.7 Concluding remarks

In this chapter, we extended the primal-dual theories discussed in [24, 85] in IFE by taking membership and non-membership functions governed by reference functions in different approaches, viz., pessimistic, optimistic and mixed. This extended theory carries a significant contribution to scholarly research because it breaks new ground to allow researchers to further seek and obtain the importance of uncertainty as well as hesitation in general LPP. We have also compared duality gaps by illustrative numerical examples with different approaches and existing approaches (see Tables 3.1-3.6). From Tables 3.1-3.6, we observe that, for the given numerical Examples 3.6.1 and 3.6.2, the solutions are better in case of an optimistic approach in terms of the duality gap.

Chapter 4

Multi-objective programming problems in intuitionistic fuzzy environment: Optimistic, pessimistic and mixed approaches

This chapter investigates a new approach for finding efficient solutions of the multi-objective optimization problem (MOOP) in IFE based on DM's different views, viz., optimistic, pessimistic and mixed. The point operator F_α, which transforms IFS into equivalent FS, is introduced and some desirable properties of F_α are studied. After that, the formulation of MOOP, accuracy index and value function in IFE are introduced. For resolving the mutual conflicting nature of objectives in MOOP in IFE, we introduce the membership and non-membership functions governed by reference function which do not depend on the upper and lower levels of acceptability. Then a new method is proposed to find the efficient solutions of MOOP in IFE based on different viewpoints. Finally, a test example is given to demonstrate the practicality and effectiveness of the proposed method.

4.1 Introduction

MOOPs are concerned with mathematical optimization problems involving more than one objective function to be optimized simultaneously. MOOPs have been successfully applied to different fields such as science, engineering, economics, and logistics where optimal decisions need to be taken in the presence of conflicting objectives. But most of the decisions such as engineering or management decisions are generally made through available data and information that are mostly vague, imprecise and uncertain in nature. Also, the subjective characteristics of the alternatives are generally uncertain and need to be evaluated by the DM who is under time pressure and has insufficient knowledge and judgments. The nature of this kind of vagueness and uncertainty is fuzzy rather than random, especially when subjective assessments are involved in the decision-making process. To cope up with such situation, FS theory proposed by Zadeh [194] has been a powerful tool for handling the uncertainties and vagueness of the data by assigning a degree, called the membership degree. During the last decades, FS theory played an important role in modeling uncertain and optimization problems. Zimmermann [203] gave a method for solving fuzzy programming with several objective functions.

Several approaches have been proposed in the literature for solving fuzzy linear programming with multiple objectives, such as compensatory operators technique [122], defuzzification approach [164], a parametric approach [40], goal programming approach [94] to name a few. There are some real-life situations where DM has hesitation in deciding membership grade. In FS theory, there is no means to incorporate this hesitation. To incorporate the hesitation in the membership degree, Atanassov [11] proposed IFS. IFS is an extension of FS [194] and is characterized by a membership degree, a non-membership degree, and a hesitancy degree. Gau and Buehrer [74] introduced the concept of vague set. But Bustince and Burillo [37] proved that vague sets are IFSs. In IFSs, the degree of membership, the degree of non-membership and the degree of hesitancy are real values. An application of the IFS to optimization problems is given by Angelov [9]. His technique is based on maximizing the membership degree, minimizing the non-membership degree and the crisp model is formulated using the IF aggregation operator [31]. For solving optimization problems in IF, ranking function plays the key role, which transforms IFN to crisp number. In [137, 165], the authors gave a ranking function for IFN and solved IFLPPs by using the ranking function. Nishad and Singh [139] solved a real-life MOLPP in IFE. The IF optimization technique for solving multi-objective reliability optimization problems in interval environment is given in [73]. In [66], Dubey et al. proposed optimistic, pessimistic and mixed approaches to solving the IFLPP. Singh and Yadav [163] developed the modeling and optimization of the MONLPP in IFE. Based on the model in [163], Rani et al. [153] solved the model with different approaches such as optimistic and pessimistic. The development of interactive IF methods for solving multilevel programming problems noted in [199]. An IF goal programming approach for finding Pareto-optimal solutions to multi-objective programming problems is given in [156].

The rest of the chapter is organized as follows. Section 4.2 introduces the basic concept of accuracy index and useful theorems relevant to the proposed work. In Section 4.3, the mathematical formulation of MOOP in IFE and its properties are presented. The limitations of the existing methods for finding an optimal solution of the MOOP in IFE are pointed out in Section 4.4. In Section 4.5, a new method for finding efficient solutions of the MOOP in IFE with different viewpoints, viz., optimistic, pessimistic and mixed is proposed. In Section 4.6, an illustrative example is given to demonstrate the practicality and effectiveness of the proposed method. The comparative study of the proposed method with existing methods is given in Section 4.7. In Section 4.8, the advantages of the proposed method with existing methods are given. Finally, we conclude this paper in Section 4.9.

4.2 Accuracy index

Let $\tilde{A}^I$ be an IFN with (α,β)-cut $A^I_{(\alpha,\beta)} = [A^I_{L\alpha}, A^I_{R\alpha}] \cap [A^I_{L(\beta)}, A^I_{R(\beta)}]$. Then the accuracy index of $\tilde{A}^I$ is denoted by $I_A(\tilde{A}^I)$ and is given by

$$I_A(\tilde{A}^I) = \frac{\int_0^1 (A^I_{L\alpha} + A^I_{R\alpha})d\alpha + \int_0^1 (A^I_{L(\beta)} + A^I_{R(\beta)})d\beta}{2}.$$

Theorem 4.2.1. *The accuracy index I_A of an IFN is a linear function.*

Proof. Let $\tilde{A}^I_1$ and $\tilde{A}^I_2$ be two IFNs with $(\alpha,\beta)-$cuts $A^I_{1(\alpha,\beta)} = [A^I_{1L\alpha}, A^I_{1R\alpha}] \cap [A^I_{1L(\beta)}, A^I_{1R(\beta)}]$ and $A^I_{2(\alpha,\beta)} = [A^I_{2L\alpha}, A^I_{2R\alpha}] \cap [A^I_{2L(\beta)}, A^I_{2R(\beta)}]$ respectively. We have the following four cases:
Case 1. $\lambda_1, \lambda_2 \geq 0$. $\lambda_1\tilde{A}^I_1 \oplus \lambda_2\tilde{A}^I_2$ is an IFN with $(\alpha,\beta)-$cut $[\lambda_1 A^I_{1L\alpha} + \lambda_2 A^I_{2L\alpha}, \lambda_1 A^I_{1R\alpha} + \lambda_2 A^I_{2R\alpha}] \cap [\lambda_1 A^I_{1L(\beta)} + \lambda_2 A^I_{2L(\beta)}, \lambda_1 A^I_{1R(\beta)} + \lambda_2 A^I_{2R(\beta)}]$. Therefore,

$$\begin{aligned}
I_A(\lambda_1\tilde{A}^I_1 \oplus \lambda_2\tilde{A}^I_2) =& \frac{\int_0^1 (\lambda_1 A^I_{1L\alpha} + \lambda_2 A^I_{2L\alpha} + \lambda_1 A^I_{1R\alpha} + \lambda_2 A^I_{2R\alpha})d\alpha + \int_0^1 (\lambda_1 A^I_{1L(\beta)} + \lambda_2 A^I_{2L(\beta)} + \lambda_1 A^I_{1R(\beta)} + \lambda_2 A^I_{2R(\beta)})d\beta}{2}\\
=&\lambda_1 \frac{\int_0^1 (A^I_{1L\alpha} + A^I_{1R\alpha})d\alpha + \int_0^1 (A^I_{1L(\beta)} + A^I_{1R(\beta)})d\beta}{2} + \lambda_2 \frac{\int_0^1 (A^I_{2L\alpha} + A^I_{2R\alpha})d\alpha + \int_0^1 (A^I_{2L(\beta)} + A^I_{2R(\beta)})d\beta}{2}\\
=&\lambda_1 I_A(\tilde{A}^I_1) + \lambda_2 I_A(\tilde{A}^I_2).
\end{aligned}$$

Case 2. $\lambda_1 \leq 0, \lambda_2 \geq 0$. $\lambda_1\tilde{A}^I_1 \oplus \lambda_2\tilde{A}^I_2$ is an IFN with $(\alpha,\beta)-$cut is $[\lambda_1 A^I_{1R\alpha} + \lambda_2 A^I_{2L\alpha}, \lambda_1 A^I_{1L\alpha} + \lambda_2 A^I_{2R\alpha}] \cap [\lambda_1 A^I_{1R(\beta)} + \lambda_2 A^I_{2L(\beta)}, \lambda_1 A^I_{1L(\beta)} + \lambda_2 A^I_{2R(\beta)}]$. Therefore,

$$\begin{aligned}
I_A(\lambda_1\tilde{A}^I_1 \oplus \lambda_2\tilde{A}^I_2) =& \frac{\int_0^1 (\lambda_1 A^I_{1L\alpha} + \lambda_2 A^I_{2L\alpha} + \lambda_1 A^I_{1R\alpha} + \lambda_2 A^I_{2R\alpha})d\alpha + \int_0^1 (\lambda_1 A^I_{1L(\beta)} + \lambda_2 A^I_{2L(\beta)} + \lambda_1 A^I_{1R(\beta)} + \lambda_2 A^I_{2R(\beta)})d\beta}{2}\\
=&\lambda_1 \frac{\int_0^1 (A^I_{1L\alpha} + A^I_{1R\alpha})d\alpha + \int_0^1 (A^I_{1L(\beta)} + A^I_{1R(\beta)})d\beta}{2} + \lambda_2 \frac{\int_0^1 (A^I_{2L\alpha} + A^I_{2R\alpha})d\alpha + \int_0^1 (A^I_{2L(\beta)} + A^I_{2R(\beta)})d\beta}{2}\\
=&\lambda_1 I_A(\tilde{A}^I_1) + \lambda_2 I_A(\tilde{A}^I_2).
\end{aligned}$$

Case 3. $\lambda_1 \geq 0, \lambda_2 \leq 0$. $\lambda_1 \tilde{A}_1^I \oplus \lambda_2 \tilde{A}_2^I$ is an IFN with $(\alpha,\beta)-$cut is $[\lambda_1 A_{1L\alpha}^I + \lambda_2 A_{2R\alpha}^I, \lambda_1 A_{1R\alpha}^I + \lambda_2 A_{2L\alpha}^I] \cap [\lambda_1 A_{1L(\beta)}^I + \lambda_2 A_{2R(\beta)}^I, \lambda_1 A_{1R(\beta)}^I + \lambda_2 A_{2L(\beta)}^I]$. Therefore,

$$\begin{aligned} I_A(\lambda_1 \tilde{A}_1^I \oplus \lambda_2 \tilde{A}_2^I) =& \frac{\int_0^1 (\lambda_1 A_{1L\alpha}^I + \lambda_2 A_{2L\alpha}^I + \lambda_1 A_{1R\alpha}^I + \lambda_2 A_{2R\alpha}^I) d\alpha + \int_0^1 (\lambda_1 A_{1L(\beta)}^I + \lambda_2 A_{2L(\beta)}^I + \lambda_1 A_{1R(\beta)}^I + \lambda_2 A_{2R(\beta)}^I) d\beta}{2} \\ =& \lambda_1 \frac{\int_0^1 (A_{1L\alpha}^I + A_{1R\alpha}^I) d\alpha + \int_0^1 (A_{1L(\beta)}^I + A_{1R(\beta)}^I) d\beta}{2} + \lambda_2 \frac{\int_0^1 (A_{2L\alpha}^I + A_{2R\alpha}^I) d\alpha + \int_0^1 (A_{2L(\beta)}^I + A_{2R(\beta)}^I) d\beta}{2} \\ =& \lambda_1 I_A(\tilde{A}_1^I) + \lambda_2 I_A(\tilde{A}_2^I). \end{aligned}$$

Case 4. $\lambda_1 \leq 0, \lambda_2 \leq 0$. $\lambda_1 \tilde{A}_1^I \oplus \lambda_2 \tilde{A}_2^I$ is an IFN with $(\alpha,\beta)-$cut is $[\lambda_1 A_{1R\alpha}^I + \lambda_2 A_{2R\alpha}^I, \lambda_1 A_{1L\alpha}^I + \lambda_2 A_{2L\alpha}^I] \cap [\lambda_1 A_{1R(\beta)}^I + \lambda_2 A_{2R(\beta)}^I, \lambda_1 A_{1L(\beta)}^I + \lambda_2 A_{2L(\beta)}^I]$. Therefore,

$$\begin{aligned} A(\lambda_1 \tilde{A}_1^I \oplus \lambda_2 \tilde{A}_2^I) =& \frac{\int_0^1 (\lambda_1 A_{1L\alpha}^I + \lambda_2 A_{2L\alpha}^I + \lambda_1 A_{1R\alpha}^I + \lambda_2 A_{2R\alpha}^I) d\alpha + \int_0^1 (\lambda_1 A_{1L(\beta)}^I + \lambda_2 A_{2L(\beta)}^I + \lambda_1 A_{1R(\beta)}^I + \lambda_2 A_{2R(\beta)}^I) d\beta}{2} \\ =& \lambda_1 \frac{\int_0^1 (A_{1L\alpha}^I + A_{1R\alpha}^I) d\alpha + \int_0^1 (A_{1L(\beta)}^I + A_{1R(\beta)}^I) d\beta}{2} + \lambda_2 \frac{\int_0^1 (A_{2L\alpha}^I + A_{2R\alpha}^I) d\alpha + \int_0^1 (A_{2L(\beta)}^I + A_{2R(\beta)}^I) d\beta}{2} \\ =& \lambda_1 I_A(\tilde{A}_1^I) + \lambda_2 I_A(\tilde{A}_2^I). \end{aligned}$$

From Cases 1, 2, 3 and 4, we conclude that $\forall \lambda_1, \lambda_2 \in \mathbb{R}$,

$$I_A(\lambda_1 \tilde{A}_1^I \oplus \lambda_2 \tilde{A}_2^I) = \lambda_1 I_A(\tilde{A}_1^I) + \lambda_2 I_A(\tilde{A}_2^I)$$

Therefore, A is a linear function. □

Definition 4.2.2. *Let $\tilde{A}_1^I$ and $\tilde{A}_2^I$ be two IFNs. Then*

1. *$\tilde{A}_1^I$ is defined as less than $\tilde{A}_2^I$ and is written as $\tilde{A}_1^I \prec \tilde{A}_2^I$ if $I_A(\tilde{A}_1^I) < I_A(\tilde{A}_2^I)$,*

2. *$\tilde{A}_1^I$ is defined as greater than $\tilde{A}_2^I$ and is written as $\tilde{A}_1^I \succ \tilde{A}_2^I$ if $I_A(\tilde{A}_1^I) > I_A(\tilde{A}_2^I)$,*

3. *$\tilde{A}_1^I$ is defined as equal to $\tilde{A}_2^I$ and is written as $\tilde{A}_1^I \approx \tilde{A}_2^I$ if $I_A(\tilde{A}_1^I) = I_A(\tilde{A}_2^I)$.*

Definition 4.2.3. *A square IF matrix $\tilde{H}^I = (\tilde{a}_{ij}^I)_{n\times n}$ is called symmetric if*

$$I_A(\tilde{a}_{ij}^I) = I_A(\tilde{a}_{ji}^I) \ \ \forall\, i,j = 1,2,\ldots,n.$$

Geometrical interpretation of IFS

In Figure 4.1, an expert is represented by a point P having coordinates $(\mu_{\tilde{A}^I}(x), \nu_{\tilde{A}^I}(x), \pi_{\tilde{A}^I}(x))$ and experts A and B having co-ordinates (1, 0, 0) and (0, 1, 0) represent full acceptance and full rejection of an idea respectively. The experts placed on the line segment AB decide their points of view; their hesitation degrees are equal to zero on the line segment AB, so each expert is convinced to the extent $\mu_{\tilde{A}^I}(x)$ against $\nu_{\tilde{A}^I}(x)$, and $\mu_{\tilde{A}^I}(x) + \nu_{\tilde{A}^I}(x) = 1$; the line segment AB represents a FS. Expert C having co-ordinates (0, 0, 1) is absolutely hesitant, i.e., undecided - he/she is the most open to the influence of the arguments presented. Triangular region ABO (Figure 4.2) is an orthogonal projection of the tetrahedron OABC (Figure 4.1). An element of an IFS has coordinates $(\mu_{\tilde{A}^I}(x), \nu_{\tilde{A}^I}(x), \pi_{\tilde{A}^I}(x))$, therefore the most natural representation of an IFS is a tetrahedron bounded by $\mu_{\tilde{A}^I}(x) = 0$, $\nu_{\tilde{A}^I}(x) = 0$, $\pi_{\tilde{A}^I}(x) = 0$ and $\mu_{\tilde{A}^I}(x)+\nu_{\tilde{A}^I}(x)+\pi_{\tilde{A}^I}(x) = 1\ \forall\ x \in X$. Hence, the tetrahedron OABC (Figure 4.1) represents an IFS.

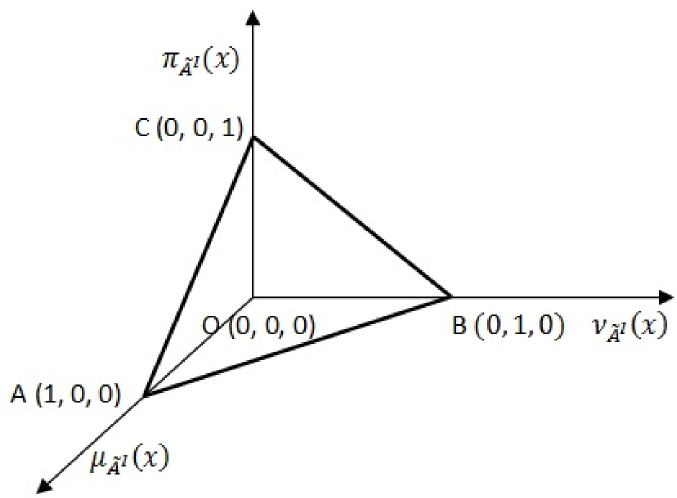

Figure 4.1: A three-dimension representation of an IFS $\tilde{A}^I$.

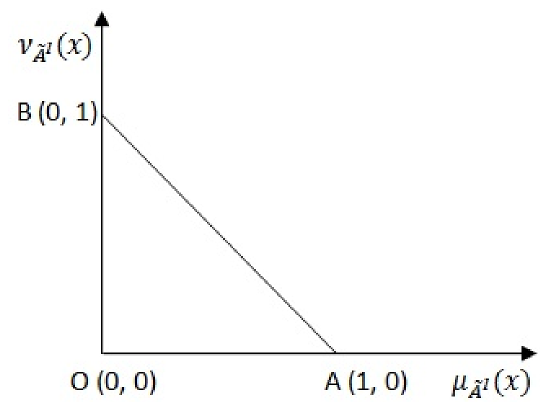

Figure 4.2: An orthogonal projection of an IFS $\tilde{A}^I$.

Distance between two IFSs:

Definition 4.2.4. *(1). ([167]) Let* $\tilde{A}_1^I = \{(x_j, \mu_{\tilde{A}_1^I}(x_j), \nu_{\tilde{A}_1^I}(x_j)) : x_j \in X\}$ *and* $\tilde{A}_2^I = \{(x_j, \mu_{\tilde{A}_2^I}(x_j), \nu_{\tilde{A}_2^I}(x_j)) : x_j \in X\}$ *be two IFSs in* $X = \{x_1, \ldots, x_j, \ldots, x_n\}$*. Then the distance between* $\tilde{A}_1^I$ *and* $\tilde{A}_2^I$ *is denoted by* $d(\tilde{A}_1^I, \tilde{A}_2^I)$ *and is defined by*

$$d(\tilde{A}_1^I, \tilde{A}_2^I) = \frac{1}{2n}\sum_{j=1}^{n}(|\mu_{\tilde{A}_1^I}(x_j) - \mu_{\tilde{A}_2^I}(x_j)| + |\nu_{\tilde{A}_1^I}(x_j) - \nu_{\tilde{A}_2^I}(x_j)| + |\pi_{\tilde{A}_1^I}(x_j) - \pi_{\tilde{A}_2^I}(x_j)|) \quad (4.1)$$

(2). ([168]) A similarity measure of IFSs $\tilde{A}_1^I$ *and* $\tilde{A}_2^I$ *is denoted by* $\vartheta_1(\tilde{A}_1^I, \tilde{A}_2^I)$ *and is defined by*

$$\vartheta_1(\tilde{A}_1^I, \tilde{A}_2^I) = \frac{d(\tilde{A}_1^I, \tilde{A}_2^I)}{d(\tilde{A}_1^I, \tilde{A}_2'^I)} \quad (4.2)$$

The above formula considers not only the distance between two IFSs but also reflects the fact that the compared IFSs are more similar or more dissimilar. Xu and Yager [188] improved ([167], [168]) results and developed the following similarity measure:

$$\vartheta_2(\tilde{A}_1^I, \tilde{A}_2^I) = 1 - \frac{d(\tilde{A}_1^I, \tilde{A}_2^I)}{d(\tilde{A}_1^I, \tilde{A}_2^I) + d(\tilde{A}_1^I, \tilde{A}_2'^I)} = \frac{d(\tilde{A}_1^I, \tilde{A}_2'^I)}{d(\tilde{A}_1^I, \tilde{A}_2^I) + d(\tilde{A}_1^I, \tilde{A}_2'^I)} \quad (4.3)$$

Atanassov [12, 14] established a way of transforming an IFS into an ordinary FS by introducing an operator D_α with parameter α, where $\alpha \in [0,1]$. This operator is the Atanassov's point operator and is defined as follows:

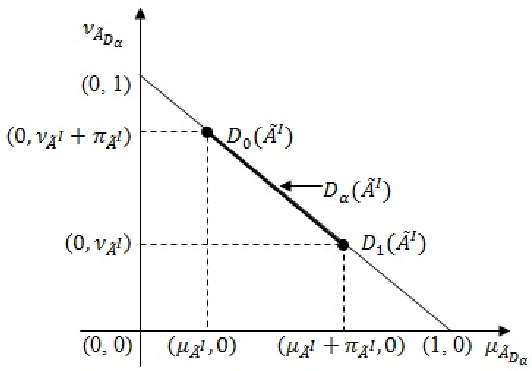

Figure 4.3: A geometrical interpretation of $D_\alpha(\tilde{A}^I)$ and $\mu_{D_\alpha}(x) + \nu_{D_\alpha}(x) = 1,\ x \in X$.

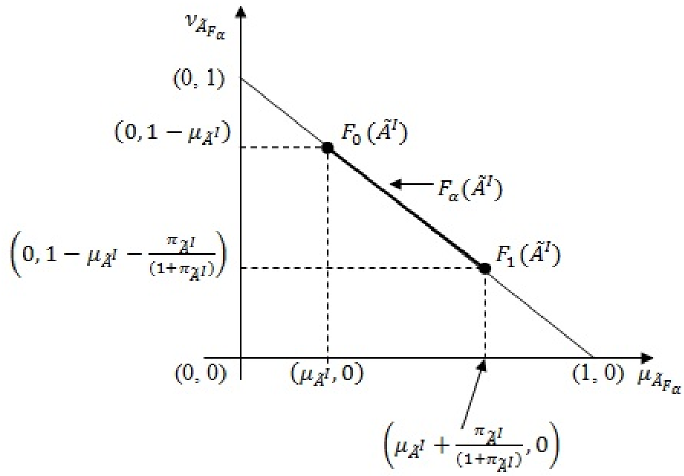

Figure 4.4: A geometrical interpretation of $F_\alpha(\tilde{A}^I)$ and $\mu_{F_\alpha}(x) + \nu_{F_\alpha}(x) = 1,\ x \in X$.

Definition 4.2.5. *[12, 14] Let $\mathcal{I}$ and $\mathcal{F}$ be the sets of IFSs and FSs on X. Then for each $x \in X$ and parameter $\alpha \in [0,1]$, $D_\alpha : \mathcal{I} \to \mathcal{F}$ is defined by*

$$D_\alpha(\tilde{A}^I) = \tilde{A}_{D_\alpha} \in \mathcal{F}\ \forall\ \tilde{A}^I \in \mathcal{I},$$

where $\tilde{A}_{D_\alpha} \in \mathcal{F}$ is defined by the membership function $\mu_{\tilde{A}_{D_\alpha}}$ given by

$$\mu_{\tilde{A}_{D_\alpha}}(x) = \mu_{\tilde{A}^I}(x) + \alpha\pi_{\tilde{A}^I}(x)\ \ \forall\ x \in X.$$

Thus

$$D_\alpha(\tilde{A}^I) = \{(x, \mu_{\tilde{A}^I}(x) + \alpha\pi_{\tilde{A}^I}(x)) : x \in X\},\ \alpha \in [0,1].$$

The geometrical interpretation of the FS $D_\alpha(\tilde{A}^I)$, $\alpha \in [0,1]$ is shown in Figure 4.3. This figure shows that every IFS $\tilde{A}^I$ is mapped onto the diagonal of unit triangular disc through the Atanassov's point operator D_α, $\alpha \in [0,1]$.

In addition, this study introduces a new point operator F_α to transform an IFS into FS with parameter α, where $\alpha \in [0,1]$ and is defined as follows:

Definition 4.2.6. *Let $\mathcal{I}$ and $\mathcal{F}$ be the sets of IFSs and FSs on X. Then for each $x \in X$ and parameter $\alpha \in [0,1]$, $F_\alpha : \mathcal{I} \to \mathcal{F}$ is defined by*

$$F_\alpha(\tilde{A}^I) = \tilde{A}_{F_\alpha} \in \mathcal{F}\ \forall\ \tilde{A}^I \in \mathcal{I},$$

where $\tilde{A}_{F_\alpha} \in \mathcal{F}$ *is defined by the membership function* $\mu_{\tilde{A}_{F_\alpha}}$ *given by*

$$\begin{aligned}
\mu_{\tilde{A}_{F_\alpha}}(x) &= \mu_{\tilde{A}^I}(x) + \frac{\alpha(1 - d((\mu_{\tilde{A}^I}(x), \nu_{\tilde{A}^I}(x), \pi_{\tilde{A}^I}(x)), (0,0,1)))}{d((\mu_{\tilde{A}^I}(x), \nu_{\tilde{A}^I}(x), \pi_{\tilde{A}^I}(x)), (1,0,0)) + d((\mu_{\tilde{A}^I}(x), \nu_{\tilde{A}^I}(x), \pi_{\tilde{A}^I}(x)), (0,1,0))} \\
&= \mu_{\tilde{A}^I}(x) + \frac{\alpha(1 - \frac{1}{2}(|\mu_{\tilde{A}^I}(x) - 0| + |\nu_{\tilde{A}^I}(x) - 0| + |\pi_{\tilde{A}^I}(x) - 1|))}{\frac{1}{2}(|\mu_{\tilde{A}^I}(x) - 1| + |\nu_{\tilde{A}^I}(x) - 0| + |\pi_{\tilde{A}^I}(x) - 0|) + \frac{1}{2}(|\mu_{\tilde{A}^I}(x) - 0| + |\nu_{\tilde{A}^I}(x) - 1| + |\pi_{\tilde{A}^I}(x) - 0|)} \\
&= \mu_{\tilde{A}^I}(x) + \left(\frac{\alpha \pi_{\tilde{A}^I}(x)}{1 + \pi_{\tilde{A}^I}(x)}\right) \ \forall\, x \in X.
\end{aligned}$$

Thus

$$\tilde{A}_{F_\alpha} = F_\alpha(\tilde{A}^I) = \left\{\left(x, \mu_{\tilde{A}^I}(x) + \left(\frac{\alpha \pi_{\tilde{A}^I}(x)}{1 + \pi_{\tilde{A}^I}(x)}\right)\right) : x \in X\right\},\ \alpha \in [0,1].$$

Figure 4.4 presents a convenient geometrical interpretation of the FS $F_\alpha(\tilde{A}^I)$. This figure shows that every IFS $\tilde{A}^I$ is mapped onto the diagonal of unit triangular disc through the point operator F_α, $\alpha \in [0,1]$. This operator is called **optimistic point operator** because of the non-decreasing degree of membership, and non-increasing degrees of non-membership and hesitation simultaneously. The family of all FSs associated with $\tilde{A}^I$ by the operator F_α will be denoted by $\{F_\alpha(\tilde{A}^I)\}_{\alpha \in [0,1]}$. Now we shall prove $\{F_\alpha(\tilde{A}^I)\}_{\alpha \in [0,1]}$ is a totally ordered family of FSs.

Lemma 4.2.7. *If* $\alpha_1 \leq \alpha_2$ *with* $\alpha_1, \alpha_2 \in [0,1]$*, then* $F_{\alpha_1}(\tilde{A}^I) \subseteq F_{\alpha_2}(\tilde{A}^I)$*.*

Proof. Since $\mu_{\tilde{A}^I}(x), \nu_{\tilde{A}^I}(x), \pi_{\tilde{A}^I}(x) \in [0,1]$ for all $x \in X$ and $\alpha_1 \leq \alpha_2$ with $\alpha_1, \alpha_2 \in [0,1]$, by Definition 4.2.6, we have

$$\begin{aligned}
& \mu_{\tilde{A}^I}(x) + \left(\frac{\alpha_1 \pi_{\tilde{A}^I}(x)}{1 + \pi_{\tilde{A}^I}(x)}\right) \leq \mu_{\tilde{A}^I}(x) + \left(\frac{\alpha_2 \pi_{\tilde{A}^I}(x)}{1 + \pi_{\tilde{A}^I}(x)}\right) \\
\Rightarrow\ & \mu_{F_{\alpha_1}(\tilde{A}^I)}(x) \leq \mu_{F_{\alpha_2}(\tilde{A}^I)}(x)\ \forall\, x \in X \\
\Rightarrow\ & F_{\alpha_1}(\tilde{A}^I) \subseteq F_{\alpha_2}(\tilde{A}^I).
\end{aligned}$$

□

Lemma 4.2.7 implies that the membership function of the FS $F_\alpha(\tilde{A}^I)$ increases as the α-value increases.

4.3 Mathematical formulation of MOOP in IFE

A conventional crisp MOOP is given by

$$\begin{aligned}
\max \quad & \phi_k(x; c_k),\ k = 1, 2, \ldots, K_1, \\
\min \quad & \phi_k(x; c_k),\ k = K_1 + 1, K_1 + 2, \ldots, K, \\
\text{subject to} \quad & \psi_l(x; a_l) \leq b_l,\ l = 1, 2, \ldots, L_1, \\
& \psi_l(x; a_l) \geq b_l,\ l = L_1 + 1, L_1 + 2, \ldots, L_2, \\
& \psi_l(x; a_l) = b_l,\ l = L_2 + 1, L_2 + 1, \ldots, L, \\
& x \geq 0,
\end{aligned} \tag{4.4}$$

where $\phi_k(.; c_k), k = 1, 2, \ldots, K$ are real valued linear or quadratic convex functions, i.e., $\phi_k(x; c_k) = \sum_{i=1}^{n} c_k^i x_i = c_k^T x$ or $\sum_{j=1}^{n} \sum_{i=1}^{n} c_k^{ij} x_i x_j = x^T H x,\ c_k = [c_k^1, c_k^2, \ldots, c_k^n]^T$ and $H = (c_k^{ij})_{n \times n}$ is real valued symmetric positive semi-definite matrix; $\psi_l(.; a_l), k = 1, 2, \ldots, L$ are real valued linear functions, i.e., $\phi_k(x; a_l) = \sum_{i=1}^{n} a_l^i x_i = a_l^T x$ and x is n-tuple decision vector $x = [x_1, x_2, ..., x_n]^T$.

Definition 4.3.1. *Let Ω be the set of all feasible solutions of Problem (4.4). Then $x^* \in \Omega$ is said to be the efficient solution if there is no $x \in \Omega$ s.t. $\phi_k(x; c_k) \geq \phi_k(x^*; c_k), k = 1, 2, \ldots, K_1$ and $\phi_k(x; c_k) > \phi_k(x^*; c_k)$ for at least one $k \in \{1, 2, \ldots, K_1\}$; and $\phi_k(x; c_k) \leq \phi_k(x^*; c_k), k = K_1 + 1, K_1 + 2, \ldots, K$ and $\phi_k(x; c_k) < \phi_k(x^*; c_k)$ for at least one $k \in \{K_1 + 1, K_1 + 2, \ldots, K\}$.*

Definition 4.3.2. *An $\bar{x} \in \Omega$ is said to dominate $x \in \Omega$ if $\phi_k(\bar{x}; c_k) \geq \phi_k(x; c_k), k = 1, 2, \ldots, K_1$ and $\phi_k(\bar{x}; c_k) > \phi_k(x; c_k)$ for at least one $k \in \{1, 2, \ldots, K_1\}$; and $\phi_k(\bar{x}; c_k) \leq \phi_k(x; c_k), k = K_1 + 1, K_1 + 2, \ldots, K$ and $\phi_k(x; c_k) < \phi_k(x^*; c_k)$ for at least one $k \in \{K_1 + 1, K_1 + 2, \ldots, K\}$.*

In a crisp MOOP, the objective functions as well as constraints data are certain. But in most of the real world practical problems such as transportation, production, planning etc. the objective functions as well as constraints data or information are mostly vague/imprecise and hesitant in nature. For dealing with such a problem, we use IFS concept. An intuitionistic fuzzy MOOP (IFMOOP) is given below:

$$\begin{aligned}
\max \quad & \tilde{\phi}_k^I(x; \tilde{c}_k^I),\ k = 1, 2, \ldots, K_1, \\
\min \quad & \tilde{\phi}_k^I(x; \tilde{c}_k^I),\ k = K_1 + 1, K_1 + 2, \ldots, K, \\
\text{subject to} \quad & \tilde{\psi}_l^I(x; \tilde{a}_l^I) \preceq \tilde{b}_l^I,\ l = 1, 2, \ldots, L_1, \\
& \tilde{\psi}_l^I(x; \tilde{a}_l^I) \succeq \tilde{b}_l^I,\ l = L_1 + 1, L_1 + 2, \ldots, L_2, \\
& \tilde{\psi}_l^I(x; \tilde{a}_l^I) \approx \tilde{b}_l^I,\ l = L_2 + 1, L_2 + 2, \ldots, L, \\
& x \geq 0,
\end{aligned} \tag{4.5}$$

where $\tilde{\phi}_k^I(.;\tilde{c}_k^I), k=1,2,\ldots,K$ are IF functions and of the form $\tilde{\phi}_k^I(x;\tilde{c}_k^I)=(\tilde{c}_k^I)^T x$, or $x^T\tilde{H}^I x$, $c_k=[(\tilde{c}_k^I)^1,(\tilde{c}_k^I)^2,\ldots,(\tilde{c}_k^I)^n]^T$ and $\tilde{H}^I=((\tilde{c}_k^I)^{ij})_{n\times n}$ is symmetric positive semi-definite matrix in IFE (see Definition 4.2.3); $\tilde{\psi}_l^I(.;\tilde{a}_k^I), k=1,2,\ldots,L$ are IF functions and of the form $\tilde{\psi}_l^I(x;\tilde{a}_k^I)=(\tilde{a}_l^I)^T x$ and x is n-tuple decision vector $x=[x_1,x_2,...,x_n]^T$.
Using the accuracy index I_A, Problem (4.5) is transformed to the following crisp MOOP:

$$\begin{aligned}
\max \quad & \phi_k(x;I_A(\tilde{c}_k^I)),\ k=1,2,\ldots,K_1,\\
\min \quad & \phi_k(x;I_A(\tilde{c}_k^I)),\ k=K_1+1,K_1+2,\ldots,K,\\
\text{subject to} \quad & \psi_l(x;I_A(\tilde{a}_l^I))\le A(\tilde{b}_l^I),\ l=1,2,\ldots,L_1,\\
& \psi_l(x;I_A(\tilde{a}_l^I))\ge A(\tilde{b}_l^I),\ l=L_1+1,L_1+2,\ldots,L_2,\\
& \psi_l(x;I_A(\tilde{a}_l^I))= A(\tilde{b}_l^I),\ l=L_2+1,L_2+2,\ldots,L,\\
& x\ge 0.
\end{aligned} \tag{4.6}$$

Let $c_k'=I_A(\tilde{c}_k^I), k=1,2,\ldots,K;\ a_l'=I_A(\tilde{a}_l^I), l=1,2,\ldots,L$ and $b_l'=I_A(\tilde{b}_l^I), l=1,2,\ldots,L$. Then Problem (4.6) becomes

$$\begin{aligned}
\max \quad & \phi_k(x;c_k'),\ k=1,2,\ldots,K_1,\\
\min \quad & \phi_k(x;c_k'),\ k=K_1+1,K_1+2,\ldots,K,\\
\text{subject to} \quad & \psi_l(x;a_l')\le b_l',\ l=1,2,\ldots,L_1,\\
& \psi_l(x;a_l')\ge b_l',\ l=L_1+1,L_1+2,\ldots,L_2,\\
& \psi_l(x;a_l')= b_l',\ l=L_2+1,L_2+2,\ldots,L,\\
& x\ge 0.
\end{aligned} \tag{4.7}$$

Theorem 4.3.3. *An efficient solution $x^*=[x_1^*,x_2^*,\ldots,x_n^*]^T$ of Problem (4.7) is also an efficient solution of Problem (4.5).*

Proof. Since $x^*=[x_1^*,x_2^*,...,x_n^*]^T$ is an efficient solution of Problem (4.7), $x^*=[x_1^*,x_2^*,\ldots,x_n^*]^T$ is also its feasible solution, i.e.,

$$\begin{aligned}
& \psi_l(x^*;a_l')\le b_l',\ l=1,2,\ldots,L_1,\\
& \psi_l(x^*;a_l')\ge b_l',\ l=L_1+1,L_1+2,\ldots,L_2,\\
& \psi_l(x^*;a_l')= b_l',\ l=L_2+1,L_2+2,\ldots,L,\\
& x\ge 0,
\end{aligned} \tag{4.8}$$

i.e.,

$$\begin{aligned}
&\tilde{\psi}_l^I(x^*; I_A(\tilde{a}_l^I)) \leq A(\tilde{b}_l^I),\ l = 1, 2, \ldots, L_1,\\
&\tilde{\psi}_l^I(x^*; I_A(\tilde{a}_l^I)) \geq A(\tilde{b}_l^I),\ l = L_1+1, L_1+2, \ldots, L_2,\\
&\tilde{\psi}_l^I(x^*; I_A(\tilde{a}_l^I)) = A(\tilde{b}_l^I),\ l = L_2+1, L_2+2, \ldots, L,\\
&x^* \geq 0,
\end{aligned} \tag{4.9}$$

Since I_A is linear,

$$\begin{aligned}
&\tilde{\psi}_l^I(x^*; \tilde{a}_l^I) \preceq \tilde{b}_l^I,\ l = 1, 2, \ldots, L_1,\\
&\tilde{\psi}_l^I(x^*; \tilde{a}_l^I) \succeq \tilde{b}_l^I,\ l = L_1+1, L_1+2, \ldots, L_2,\\
&\tilde{\psi}_l^I(x^*; \tilde{a}_l^I) \approx \tilde{b}_l^I,\ l = L_2+1, L_2+2, \ldots, L,\\
&x^* \geq 0,
\end{aligned} \tag{4.10}$$

which implies that x^* is a feasible solution of Problem (4.5).

Next, since x^* is an efficient for Problem (4.7), $\nexists$ any $x = [x_1, x_2, \ldots, x_n]^T$ s.t. $\phi_k(x; c_k') \geq \phi_k(x^*; c_k'), k = 1, 2, \ldots, K_1$ and $\phi_k(x; c_k') > \phi_k(x^*; c_k')$ for at least one $k \in \{1, 2, \ldots, K_1\}$; and $\phi_k(x; c_k') \leq \phi_k(x^*; c_k'), k = K_1+1, K_1+2, \ldots, K$ and $\phi_k(x; c_k') < \phi_k(x^*; c_k')$ for at least one $k \in \{K_1+1, K_1+2, \ldots, K\}$. We have no x s.t. $\tilde{\phi}_k^I(x; I_A(\tilde{c}_k^I)) \geq \tilde{\phi}_k^I(x^*; I_A(\tilde{c}_k^I)), k = 1, 2, \ldots, K_1$ and $\tilde{\phi}_k^I(x; I_A(\tilde{c}_k^I)) > \tilde{\phi}_k^I(x^*; I_A(\tilde{c}_k^I))$ for at least one $k \in \{1, 2, \ldots, K_1\}$; and $\tilde{\phi}_k^I(x; I_A(\tilde{c}_k^I)) \leq \tilde{\phi}_k^I(x^*; I_A(\tilde{c}_k^I)), k = K_1+1, K_1+2, \ldots, K$ and $\tilde{\phi}_k^I(x; I_A(\tilde{c}_k^I)) < \tilde{\phi}_k^I(x^*; I_A(\tilde{c}_k^I))$ for at least one $k \in \{K_1+1, K_1+2, \ldots, K\}$. Since I_A is linear, we have no x s.t. $\tilde{\phi}_k^I(x; \tilde{c}_k^I) \geq \tilde{\phi}_k^I(x^*; \tilde{c}_k^I), k = 1, 2, \ldots, K_1$ and $\tilde{\phi}_k^I(x; \tilde{c}_k^I) > \tilde{\phi}_k^I(x^*; \tilde{c}_k^I)$ for at least one $k \in \{1, 2, \ldots, K_1\}$; and $\tilde{\phi}_k^I(x; I_A(c_k)) \leq \tilde{\phi}_k^I(x^*; I_A(c_k)), k = K_1+1, K_1+2, \ldots, K$ and $\tilde{\phi}_k^I(x; \tilde{c}_k^I) < \tilde{\phi}_k^I(x^*; \tilde{c}_k^I)$ for at least one $k \in \{K_1+1, K_1+2, \ldots, K\}$.

Hence x^* is an efficient solution of Problem (4.5). □

4.4 Existing models for finding an optimal solution of the MOOP in IFE

Several papers in the literature have concentrated on developing programming models for determining an IF efficient compromise solution to the MOOPs.

As the most pioneering attempt, Angelov [8] proposed a model to achieve this purpose. The model proposed by Angelov [8] is defined as follows:

$$
\begin{aligned}
\max \quad & (\lambda - \lambda'), \\
\text{subject to} \quad & \mu_{U_k}(\phi_k(x; c'_k)) \geq \lambda, \ k = 1, 2, \dots, K_1, \\
& \nu_{U_k}(\phi_k(x; c'_k)) \leq \lambda', \ k = 1, 2, \dots, K_1, \\
& \mu_{L_k}(\phi_k(x; c'_k)) \geq \lambda, \ k = K_1 + 1, K_1 + 2, \dots, K, \\
& \nu_{L_k}(\phi_k(x; c'_k)) \leq \lambda', \ k = K_1 + 1, K_1 + 2, \dots, K, \\
& \psi_l(x; a'_l) \leq b'_l, \ l = 1, 2, \dots, L_1, \\
& \psi_l(x; a'_l) \geq b'_l, \ l = L_1 + 1, L_1 + 2, \dots, L_2, \\
& \psi_l(x; a'_l) = b'_l, \ l = L_2 + 1, L_2 + 2, \dots, L, \\
& 0 \leq \lambda' \leq \lambda, \ \lambda + \lambda' \leq 1, \\
& x \geq 0.
\end{aligned}
\tag{4.11}
$$

The model presented in (4.11) is based on a straightforward extension of the model given by Bellman and Zadeh [31] in fuzzy environment. This model associates a value function

$$f_{\tilde{D}^I}(x) = \mu_{\tilde{D}^I}(x) - \nu_{\tilde{D}^I}(x) \tag{4.12}$$

with each decision $\tilde{D}^I = \{< x, \mu_{\tilde{D}^I}(x), \nu_{\tilde{D}^I}(x) >: x \in X\}$. Then it obtains an IF efficient solution $x^* \in X$ in a way such that $f_{\tilde{D}^I}(x^*) = \max_{x \in X}\{f_{\tilde{D}^I}(x)\}$. Yager [189] pointed out the drawback of the value function $f_{\tilde{D}^I}(x)$ used in model (4.11). Consider two alternatives x and y, where $\mu_{\tilde{D}^I}(x) = 0.48, \nu_{\tilde{D}^I}(x) = 0.50, \mu_{\tilde{D}^I}(y) = 0.1$ and $\nu_{\tilde{D}^I}(y) = 0$. Using the value function $f_{\tilde{D}^I}(x)$, we get $f_{\tilde{D}^I}(x) = -0.02$ and $f_{\tilde{D}^I}(y) = 0.1$. So the alternative y would be selected over the alternative x. In this case, the membership degree of the alternative y is 0.1 and the hesitation degree about this alternative is 0.9; while the membership degree of the alternative x is 0.48 and Yager [189] provides an alternative approach that overcomes such a difficulty. The value function proposed by Yager [189] is defined as follows:

$$g_{\tilde{D}^I}(x) = \mu_{\tilde{D}^I}(x) + \alpha \pi_{\tilde{D}^I}(x) \tag{4.13}$$

where $\alpha \in [0, 1]$ is the fuzzification parameter. A large value for the fuzzification parameter α results in resolving hesitancy in favor of membership degree. On the contrary, a small value for the fuzzification parameter α results in resolving hesitancy in favor of non-membership degree. When $\alpha = 1/2$, the same degree of emphasis is placed on the membership and non-membership degrees.

Motivated by the approach proposed by Yager [189] and Dubey et al. [66] defined another programming model for determining an IF efficient compromise solution to MOOPs. The model proposed by Dubey et al. [66] and Rani et al. [153] is given as follows:

$$\begin{aligned}
\max \quad & \lambda, \\
\text{subject to} \quad & (g_\omega)_{U_{k1}}(\phi_k(x;c_k')) + M\delta_{U_k} \geq \lambda,\ k = 1,2,\ldots,K_1, \\
& (g_\omega)_{U_{k2}}(\phi_k(x;c_k')) + M(1-\delta_{U_k}) \geq \lambda,\ k = 1,2,\ldots,K_1, \\
& (g_\omega)_{L_{k1}}(\phi_k(x;c_k')) + M\delta_{L_k} \geq \lambda,\ k = K_1+1, K_1+2,\ldots,K, \\
& (g_\omega)_{L_{k2}}(\phi_k(x;c_k')) + M(1-\delta_{L_k}) \geq \lambda,\ k = K_1+1, K_1+2,\ldots,K, \\
& \psi_l(x;a_l') \leq b_l',\ l = 1,2,\ldots,L_1, \\
& \psi_l(x;a_l') \geq b_l',\ l = L_1+1, L_1+2,\ldots,L_2, \\
& \psi_l(x;a_l') = b_l',\ l = L_2+1, L_2+2,\ldots,L, \\
& 0 \leq \lambda \leq 1,\ 0 \leq \delta_{L_k} \leq \delta_{U_k} \leq 1, \\
& x \geq 0,
\end{aligned} \tag{4.14}$$

where

$$(g_\omega)_{U_k}(\phi_k(x;c_k')) = \mu_{U_k}(\phi_k(x;c_k')) + \alpha\pi_{U_k}(\phi_k(x;c_k')),\ k = 1,2,\ldots,K_1 \tag{4.15}$$

$$(g_\omega)_{L_k}(\phi_k(x;c_k')) = \mu_{L_k}(\phi_k(x;c_k')) + \alpha\pi_{L_k}(\phi_k(x;c_k')),\ k = K_1+1, K_1+2,\ldots,K \tag{4.16}$$

$0 \leq \alpha \leq 1$, M is a large positive value, $(g_\omega)_{U_{k1}}(\phi_k(x;c_k'))$ and $(g_\omega)_{U_{k2}}(\phi_k(x;c_k'))$ denote the segments of the membership function $(g_\omega)_{U_k}(\phi_k(x;c_k'))$) in descending order of the related sub-domains, $(g_\omega)_{L_{k1}}(\phi_k(x;c_k'))$ and $(g_\omega)_{L_{k2}}(\phi_k(x;c_k'))$ denote the segments of the membership function $(g_\omega)_{L_k}(\phi_k(x;c_k'))$) in ascending order of the related sub-domains.

This model associates the value function given (4.13). Then it obtains an IF efficient solution $x^* \in X$ in a way such that $g_{\tilde{D}^I}(x^*) = \max_{x\in X}\{g_{\tilde{D}^I}(x)\}$. We find the drawback of the value function $g_{\tilde{D}^I}(x)$ for determining an IF efficient compromise solution to MOOPs as explained below. Consider two alternatives x and y, where $\mu_{\tilde{D}^I}(x) = 0.4, \nu_{\tilde{D}^I}(x) = 0.3, \mu_{\tilde{D}^I}(y) = 0.6,\ \nu_{\tilde{D}^I}(y) = 0.35$ and $\alpha = 1$. Using the value function $g_{\tilde{D}^I}(x)$, we get $g_{\tilde{D}^I}(x) = 0.7$ and $g_{\tilde{D}^I}(y) = 0.65$. So the alternative x would be selected over the alternative y. In this case, the membership degree of alternative x is 0.4 and the hesitation degree about this alternative is 0.3; while the membership degree of alternative y is 0.6 and we propose an alternative approach that overcomes such a difficulty. The value function proposed is defined as follows

$$h_{\tilde{D}^I}(x) = \mu_{\tilde{D}^I}(x) + \left(\frac{\alpha\pi_{\tilde{D}^I}(x)}{(1+\pi_{\tilde{D}^I}(x))}\right) \tag{4.17}$$

where $\alpha \in [0,1]$ is a fuzzification parameter. When the membership function and parameter α increase while the nonmembership function decreases, then $h_{\tilde{D}^I}(x)$ attains maximum value and in that situation results in resolving hesitancy in favor of membership degree.

Lemma 4.4.1. *If the parameter $\alpha \in (0,1]$ increases, then the value function h also increases.*

Proof. The same as Lemma 4.2.7. □

Lemma 4.4.2. *The value function defined in (4.17) always lies in [0, 1] for every parameter $\alpha \in (0,1]$.*

Proof. Let $\tilde{D}^I = \{< x, \mu_{\tilde{D}^I}(x), \nu_{\tilde{D}^I}(x) >: x \in X\}$ be IFS. Then

$$0 \leq \mu_{\tilde{D}^I}(x), \nu_{\tilde{D}^I}(x) \leq 1 \text{ and } 0 \leq \mu_{\tilde{D}^I}(x) + \nu_{\tilde{D}^I}(x) \leq 1 \;\forall\, x \in X.$$

For every $\alpha \in (0,1]$ and $x \in X$, we have

$$0 \leq \alpha\pi_{\tilde{D}^I}(x) \leq (1 - \mu_{\tilde{D}^I}(x)) \tag{4.18}$$

$$0 \leq \mu_{\tilde{D}^I}(x)\pi_{\tilde{D}^I}(x) \leq \pi_{\tilde{D}^I}(x) \tag{4.19}$$

From (4.18) and (4.19), we have

$$\begin{aligned}
&0 \leq \mu_{\tilde{D}^I}(x)\pi_{\tilde{D}^I}(x) + \alpha\pi_{\tilde{D}^I}(x) \leq \pi_{\tilde{D}^I}(x) + (1 - \mu_{\tilde{D}^I}(x)) \\
\Rightarrow\; &0 \leq \mu_{\tilde{D}^I}(x)(1 + \pi_{\tilde{D}^I}(x)) + \alpha\pi_{\tilde{D}^I}(x) \leq (1 + \pi_{\tilde{D}^I}(x)) \\
\Rightarrow\; &0 \leq \mu_{\tilde{D}^I}(x) + \frac{\alpha\pi_{\tilde{D}^I}(x)}{(1 + \pi_{\tilde{D}^I}(x))} \leq 1 \\
\Rightarrow\; &0 \leq h_{\tilde{D}^I}(x) \leq 1,\; \forall\, x \in X. \qquad (4.20)
\end{aligned}$$

(4.20) shows that the value function defined in (4.17) for any IFS and any arbitrary $\alpha \in [0,1]$ lies in [0, 1]. □

Theorem 4.4.3. *The IFS which has the larger membership degree and the smaller non-membership degree should be given priority on the basis of value function h.*

Proof. Let $\tilde{A}_1^I = \{< x, \mu_{\tilde{A}_1^I}(x), \nu_{\tilde{A}_1^I}(x) >: x \in X\}$ and $\tilde{A}_2^I = \{< x, \mu_{\tilde{A}_2^I}(x), \nu_{\tilde{A}_2^I}(x) >: x \in X\}$ be two IFSs s.t. $\mu_{\tilde{A}_1^I}(x) > \mu_{\tilde{A}_2^I}(x)$ and $\nu_{\tilde{A}_1^I}(x) < \nu_{\tilde{A}_2^I}(x)$ for each $x \in X$. Then by (4.7), we have

$$h_{\tilde{A}_1^I}(x) = \mu_{\tilde{A}_1^I}(x) + \frac{\alpha\pi_{\tilde{A}_1^I}(x)(x)}{(1 + \pi_{\tilde{A}_1^I}(x))} \text{ and } h_{\tilde{A}_2^I}(x) = \mu_{\tilde{A}_2^I}(x) + \frac{\alpha\pi_{\tilde{A}_2^I}(x)}{(1 + \pi_{\tilde{A}_2^I}(x))} \;\forall\, \alpha \in (0,1]$$

Let $\mu_{\tilde{A}_1^I}(x) - \mu_{\tilde{A}_2^I}(x) = \Delta_1(x)$ and $\nu_{\tilde{A}_2^I}(x) - \nu_{\tilde{A}_1^I}(x) = \Delta_2(x)$. Then we have

$$\begin{aligned}
h_{\tilde{A}_1^I}(x) = \mu_{\tilde{A}_1^I}(x) + \frac{\alpha(1 - \mu_{\tilde{A}_1^I}(x) - \nu_{\tilde{A}_1^I}(x))}{(2 - \mu_{\tilde{A}_1^I}(x) - \nu_{\tilde{A}_1^I}(x))} &> \mu_{\tilde{A}_1^I}(x) + \frac{\alpha(1 - \mu_{\tilde{A}_1^I}(x) - \nu_{\tilde{A}_1^I}(x))}{(2 - (\mu_{\tilde{A}_1^I}(x) - \Delta_1(x)) - \nu_{\tilde{A}_1^I}(x))} \\
&= \mu_{\tilde{A}_1^I}(x) + \frac{\alpha(1 - \mu_{\tilde{A}_1^I}(x) - \nu_{\tilde{A}_1^I}(x))}{(2 - \mu_{\tilde{A}_2^I}(x) - \nu_{\tilde{A}_1^I}(x))} \qquad (4.21)
\end{aligned}$$

$$h_{\tilde{A}_2^I}(x) = \mu_{\tilde{A}_2^I}(x) + \frac{\alpha(1-\mu_{\tilde{A}_2^I}(x)-\nu_{\tilde{A}_2^I}(x))}{(2-\mu_{\tilde{A}_2^I}(x)-\nu_{\tilde{A}_2^I}(x))} \;=\; \mu_{\tilde{A}_2^I}(x)+\alpha-\frac{\alpha}{(2-\mu_{\tilde{A}_2^I}(x)-\nu_{\tilde{A}_2^I}(x))}$$
$$< \mu_{\tilde{A}_2^I}(x)+\alpha-\frac{\alpha}{(2-\mu_{\tilde{A}_2^I}(x)-(\nu_{\tilde{A}_2^I}(x)-\Delta_2(x)))} \;=\; \mu_{\tilde{A}_2^I}(x)+\alpha-\frac{\alpha}{(2-\mu_{\tilde{A}_2^I}(x)-\nu_{\tilde{A}_1^I}(x))}$$
$$= \mu_{\tilde{A}_2^I}(x)+\frac{\alpha(1-\mu_{\tilde{A}_2^I}(x)-\nu_{\tilde{A}_1^I}(x))}{(2-\mu_{\tilde{A}_2^I}(x)-\nu_{\tilde{A}_1^I}(x))} < \; \mu_{\tilde{A}_1^I}(x)+\frac{\alpha(1-\mu_{\tilde{A}_1^I}(x)-\nu_{\tilde{A}_1^I}(x))}{(2-\mu_{\tilde{A}_2^I}(x)-\nu_{\tilde{A}_1^I}(x))} \tag{4.22}$$

From (4.18) and (4.22), we obtain

$$h_{\tilde{A}_1^I}(x) \;>\; \mu_{\tilde{A}_1^I}(x)+\frac{\alpha(1-\mu_{\tilde{A}_1^I}(x)-\nu_{\tilde{A}_1^I}(x))}{(2-\mu_{\tilde{A}_2^I}(x)-\nu_{\tilde{A}_1^I}(x))} \;>\; h_{\tilde{A}_2^I}(x) \qquad (\because\; \mu_{\tilde{A}_1^I}(x)>\mu_{\tilde{A}_2^I}(x),\; \nu_{\tilde{A}_1^I}(x)<\nu_{\tilde{A}_2^I}(x))$$
$$\Rightarrow\; h_{\tilde{A}_1^I}(x) \;>\; h_{\tilde{A}_2^I}(x)$$ □

4.5 The proposed solution approach

In the literature, several authors solved IF MOOPs by using linear [139], S-shaped [66] and parabolic ([163], [153] membership functions and utilized in a decision making process. These membership functions are commonly used because of their simplicity. It is defined by fixing two points, the upper and lower levels of acceptability of the decision variables. If general IFS theory is considered, then such types of assumption is not justified always. Thus the justification in the assumption is desirable according to fuzziness of the data. If the IFS theory is used to model real decision making process and an assertion is made that the resulting models are the real models, then some kinds of empirical justification for this assumption is necessary. From this point of view, we have considered LR-type IFN for different approaches viz. optimistic, pessimistic and mixed.

Let L_k be the aspiration level of achievement and U_k be the acceptable level of achievement for the kth objective function ϕ_k, $k = 1, 2, 3, \ldots, K$.

In order to find an efficient solution of an MOOP, different approaches such as optimistic, pessimistic and mixed corresponding to objective functions have been taken. For each approach and each objective, a membership and a non-membership functions have been constructed and are described briefly as follows:

(i) Max objectives, (ii) Min objectives.

Case 1: Max objectives Firstly, consider K_1 objectives $\phi_k(.; c'_k)$; $k = 1, \ldots, K_1$ to be maximized. The degree of DM's satisfaction increases as each objective value approaches its respective upper bound and he/she is fully satisfied if all the objectives assume their upper bounds. But it is quite common that, practically, attaining of exact values of these upper bounds is uncertain. Let $q_k > 0$ be the respective tolerance for $\phi_k(.; c'_k)$. For this, based on the decision and judgement of the DM, the degrees of attainability and non-attainability of the upper bound U_k, have been interpreted in three different ways- the optimistic, pessimistic and mixed approaches.

The optimistic approach:

In this approach, the DM takes a flexible way about rejection. Specially, if the degree of acceptance of x is zero, the DM do not reject fully. Then the membership function μ_{U_k} and the non-membership function ν_{U_k} of the fuzzy goal assigned to the kth objective function $\phi_k(.;c'_k)$ in optimistic approach for maximization problem are of the following forms (see Figure 4.5)

$$\mu_{U_k}(\phi_k(x;c'_k)) = \begin{cases} 0, & \phi_k(x;c'_k) < l_k, \\ L\left(\frac{U_k-\phi_k(x;c'_k)}{\alpha_k}\right), & l_k \leq \phi_k(x;c'_k) \leq U_k, \\ 1, & \phi_k(x;c'_k) \geq U_k, \end{cases} \tag{4.23A}$$

and

$$\nu_{U_k}(\phi_k(x;c'_k)) = \begin{cases} 1, & \phi_k(x;c'_k) \leq L_k - q_k, \\ R\left(\frac{\phi_k(x;c'_k)-(L_k-q_k)}{\beta_k}\right), & L_k - q_k \leq \phi_k(x;c'_k) \leq r_k, \\ 0, & \phi_k(x;c'_k) > r_k, \end{cases} \tag{4.23B}$$

where L and R are the reference functions. The values of l_k and r_k are the real numbers such that $L_k \leq l_k < r_k \leq U_k$. We assume that $\mu_{U_k}(l_k) = \delta_k$ and $\nu_{U_k}(r_k) = \eta_k$ (see Figure 7), where δ_k and η_k are the real numbers such that $0 \leq \delta_k < 1,\ 0 \leq \eta_k < 1$ and $0 \leq \delta_k + \eta_k < 1$. Therefore, α_k and β_k must fulfil the following relations:

$$\alpha_k = \frac{U_k - l_k}{L^{-1}(\delta_k)} \quad \text{and} \quad \beta_k = \frac{r_k - (L_k - q_k)}{R^{-1}(\eta_k)} \tag{4.23C}$$

where L^{-1} and R^{-1} are the inverse functions of L and R, respectively, in the proper interval. The possible shapes of μ_{U_k} and ν_{U_k} are shown in Figure 4.5. From it, it is observed that there is an interval $(L_k - q_k, l_k)$ in which the membership degree of achieving the goal is zero but the non-membership degree is not one.

The pessimistic approach:

In this approach, the DM is possibly ready for extra acceptance, i.e., if the degree of rejection of x is zero, the DM is not ready to accept fully. Then the membership function μ_{U_k} and the non-membership function ν_{U_k} of the fuzzy goal assigned to the kth objective function $\phi_k(.;c'_k)$

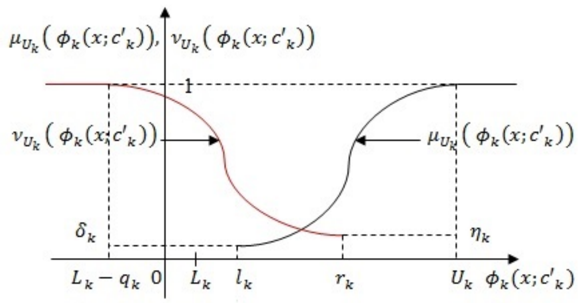

Figure 4.5: Membership and non-membership functions for maximizing objective in optimistic approach.

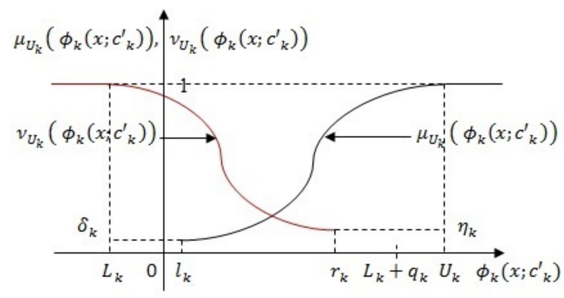

Figure 4.6: Membership and non-membership functions for maximizing objective in pessimistic approach.

in pessimistic approach for maximization problem are of the following forms (see Figure 4.6)

$$\mu_{U_k}(\phi_k(x;c'_k)) = \begin{cases} 0, & \phi_k(x;c'_k) < l_k, \\ L\Big(\frac{U_k-\phi_k(x;c'_k)}{\alpha_k}\Big), & l_k \leq \phi_k(x;c'_k) \leq U_k, \\ 1, & \phi_k(x;c'_k) \geq U_k, \end{cases} \tag{4.24A}$$

and

$$\nu_{U_k}(\phi_k(x;c'_k)) = \begin{cases} 1, & \phi_k(x;c'_k) \leq L_k, \\ R\Big(\frac{\phi_k(x;c'_k)-L_k}{\beta_k}\Big), & L_k \leq \phi_k(x;c'_k) \leq r_k, \\ 0, & \phi_k(x;c'_k) > r_k, \end{cases} \tag{4.24B}$$

where L and R are the reference functions. The values of l_k and r_k are the real numbers such that $L_k \leq l_k < r_k \leq L_k + q_k$. We assume that $\mu_{U_k}(l_k) = \delta_k$ and $\nu_{U_k}(r_k) = \eta_k$ (see Figure 8), where δ_k and η_k are the real numbers such that $0 \leq \delta_k < 1$, $0 \leq \eta_k < 1$ and $0 \leq \delta_k + \eta_k < 1$. Therefore, α_k and β_k must fulfil the following relations:

$$\alpha_k = \frac{U_k - l_k}{L^{-1}(\delta_k)} \text{ and } \beta_k = \frac{r_k - L_k}{R^{-1}(\eta_k)}, \tag{4.24C}$$

where L^{-1} and R^{-1} are the inverse functions of L and R, respectively, in the proper interval. The possible shapes of μ_{U_k} and ν_{U_k} are shown in Figure 4.6. From it, it is observed that there is an interval (r_k, U_k) in which the membership degree of achieving the goal is not one but the non-membership degree is zero.

The mixed approach:

In this approach, the DM is not flexible to reject and is not capable for extra acceptance. Then the membership function μ_{U_k} and the non-membership function ν_{U_k} of the fuzzy goal assigned

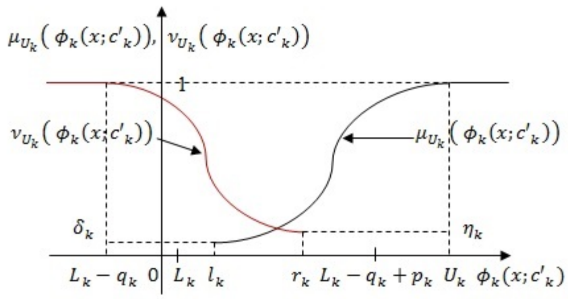

Figure 4.7: Membership and non-membership functions for maximizing objective in mixed approach.

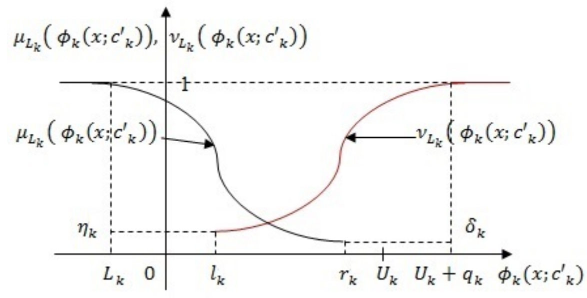

Figure 4.8: Membership and non-membership functions for minimizing objective in optimistic approach.

to the k-th objective function $\phi_k(.;c'_k)$ in mixed approach for maximization problem are of the following forms (see Figure 4.7)

$$\mu_{U_k}(\phi_k(x;c'_k)) = \begin{cases} 0, & \phi_k(x;c'_k) < l_k, \\ L\left(\frac{U_k - \phi_k(x;c'_k)}{\alpha_k}\right), & l_k \le \phi_k(x;c'_k) \le U_k, \\ 1, & \phi_k(x;c'_k) \ge U_k, \end{cases} \tag{4.25A}$$

and

$$\nu_{U_k}(\phi_k(x;c'_k)) = \begin{cases} 1, & \phi_k(x;c'_k) \le L_k - q_k, \\ R\left(\frac{\phi_k(x;c'_k) - (L_k - q_k)}{\beta_k}\right), & L_k - q_k \le \phi_k(x;c'_k) \le r_k, \\ 0, & \phi_k(x;c'_k) > r_k, \end{cases} \tag{4.25B}$$

where L and R are the reference functions. The values of l_k and r_k are the real numbers such that $L_k \le l_k < r_k \le L_k - q_k + p_k$. We assume that $\mu_{U_k}(l_k) = \delta_k$ and $\nu_{U_k}(r_k) = \eta_k$ (see Figure 9), where δ_k and η_k are the numbers such that $0 \le \delta_k < 1,\ 0 \le \eta_k < 1$ and $0 \le \delta_k + \eta_k < 1$. Therefore, α_k and β_k must fulfil the following relations:

$$\alpha_k = \frac{U_k - l_k}{L^{-1}(\delta_k)} \text{ and } \beta_k = \frac{r_k - (L_k - q_k)}{R^{-1}(\eta_k)}, \tag{4.25C}$$

where L^{-1} and R^{-1} are the inverse functions of L and R, respectively, in the proper interval. The possible shapes of μ_{U_k} and ν_{U_k} are shown in Figure 4.7.

Case 2: Min objectives Now, we consider the objectives $\phi_k(.;c'_k)$; $k = K_1 + 1, \dots, K$ to be minimized. The degree of DM's satisfaction increases as each objective value approaches its respective lower bound L_k and he/she is fully satisfied if all the objectives reach their lower bounds. But it is quite common that, practically, attaining of the exact values of these upper bounds is uncertain. Let $p_k > 0, q_k > 0$ be the respective tolerance for $\phi_k(.;c'_k)$ such that

$q_k < p_k$. For this, based on the decision and judgement of the DM, the degree of attainability and non-attainability of L_k, respectively, have been interpreted in three different ways- the optimistic, pessimistic and mixed approaches.

The optimistic approach:

In this approach, the DM takes a flexible way about rejection. Specially, if the degree of acceptance of x is zero, the DM do not reject fully. Then the membership function μ_{L_k} and the non-membership function ν_{L_k} of the fuzzy goal assigned to the kth objective function $\phi_k(.;c'_k)$ in optimistic approach for minimization problem are of the following forms (see Figure 4.8)

$$\mu_{L_k}(\phi_k(x;c'_k)) = \begin{cases} 1, & \phi_k(x;c'_k) \leq L_k, \\ R\left(\frac{\phi_k(x;c'_k)-L_k}{\beta_k}\right), & L_k \leq \phi_k(x;c'_k) \leq r_k, \\ 0, & \phi_k(x;c'_k) > r_k, \end{cases} \tag{4.26A}$$

and

$$\nu_{L_k}(\phi_k(x;c'_k)) = \begin{cases} 0, & \phi_k(x;c'_k) < l_k, \\ L\left(\frac{U_k+q_k-\phi_k(x;c'_k)}{\alpha_k}\right), & l_k \leq \phi_k(x;c'_k) \leq U_k+q_k, \\ 1, & \phi_k(x;c'_k) \geq U_k+q_k, \end{cases} \tag{4.26B}$$

where L and R are the reference functions. The values of l_k and r_k are the real numbers such that $L_k \leq l_k < r_k \leq U_k$. We assume that $\mu_{L_k}(l_k) = \delta_k$ and $\nu_{L_k}(r_k) = \eta_k$ (see Figure 10), where δ_k and η_k are the real numbers such that $0 \leq \delta_k < 1,\ 0 \leq \eta_k < 1$ and $0 \leq \delta_k + \eta_k < 1$. Therefore, α_k and β_k must fulfil the following relations:

$$\alpha_k = \frac{U_k+q_k-l_k}{L^{-1}(\delta_k)} \quad \text{and} \quad \beta_k = \frac{r_k-L_k}{R^{-1}(\eta_k)}, \tag{4.26C}$$

where L^{-1} and R^{-1} are the inverse functions of L and R, respectively, in the proper interval. The possible shapes of μ_{L_k} and ν_{L_k} are shown in Figure 4.8. From it, it is observed that there is an interval (r_k, U_k+q_k) in which the membership degree of achieving the goal is zero but the non-membership degree is not one.

The pessimistic approach:

In this approach, the DM is possibly ready for extra acceptance, i.e., if the degree of rejection of x is zero, the DM is not ready to accept fully. Then the membership function μ_{L_k} and the non-membership function ν_{L_k} of the fuzzy goal assigned to the kth objective function $\phi_k(.;c'_k)$

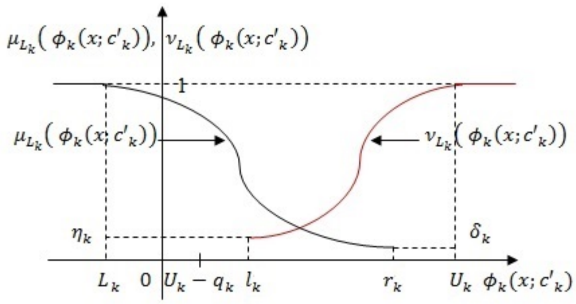

Figure 4.9: Membership and non-membership functions for minimizing objective in pessimistic approach.

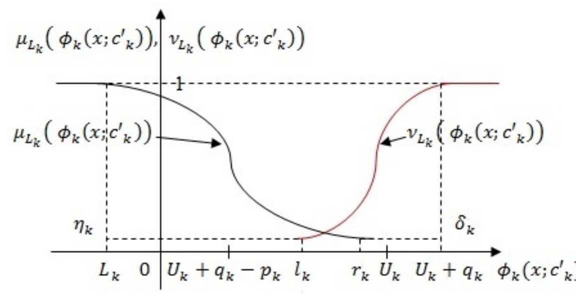

Figure 4.10: Membership and non-membership functions for minimizing objective in mixed approach.

in pessimistic approach for minimization problem are of the following forms (see Figure 4.9)

$$\mu_{L_k}(\phi_k(x;c'_k)) = \begin{cases} 1, & \phi_k(x;c'_k) \le L_k, \\ R\Big(\frac{\phi_k(x;c'_k)-L_k}{\beta_k}\Big), & L_k \le \phi_k(x;c'_k) \le r_k, \\ 0, & \phi_k(x;c'_k) > r_k, \end{cases} \tag{4.27A}$$

and

$$\nu_{L_k}(\phi_k(x;c'_k)) = \begin{cases} 0, & \phi_k(x;c'_k) < l_k, \\ L\Big(\frac{U_k-\phi_k(x;c'_k)}{\alpha_k}\Big), & l_k \le \phi_k(x;c'_k) \le U_k, \\ 1, & \phi_k(x;c'_k) \ge U_k, \end{cases} \tag{4.27B}$$

where L and R are the reference functions. The values of l_k and r_k are the real numbers such that $U_k - q_k \le l_k < r_k \le U_k$. We assume that $\mu_{L_k}(l_k) = \delta_k$ and $\nu_{L_k}(r_k) = \eta_k$ (see Figure 11), where δ_k and η_k are the real numbers such that $0 \le \delta_k < 1$, $0 \le \eta_k < 1$ and $0 \le \delta_k + \eta_k < 1$. Therefore, α_k and β_k must fulfil the following relations:

$$\alpha_k = \frac{U_k - l_k}{L^{-1}(\delta_k)} \text{ and } \beta_k = \frac{r_k - L_k}{R^{-1}(\eta_k)}, \tag{4.27C}$$

where L^{-1} and R^{-1} are the inverse functions of L and R, respectively, in the proper interval. The possible shapes of μ_{L_k} and ν_{L_k} are shown in Figure 4.9. From it, it is observed that there is an interval (L_k, l_k) in which the membership degree of achieving the goal is not one but the non-membership degree is zero.

The mixed approach:

In this approach, the DM is not flexible to reject and is not capable for extra acceptance. Then the membership function μ_{L_k} and the non-membership function ν_{L_k} of the fuzzy goal assigned to the kth objective function $\phi_k(.;c'_k)$ in mixed approach for minimization problem are of the following forms (see Figure 4.10)

$$\mu_{L_k}(\phi_k(x;c'_k)) = \begin{cases} 1, & \phi_k(x;c'_k) \leq L_k, \\ R\left(\frac{\phi_k(x;c'_k)-L_k}{\beta_k}\right), & L_k \leq \phi_k(x;c'_k) \leq r_k, \\ 0, & \phi_k(x;c'_k) > r_k, \end{cases} \tag{4.28A}$$

and

$$\nu_{L_k}(\phi_k(x;c'_k)) = \begin{cases} 0, & \phi_k(x;c'_k) < l_k, \\ L\left(\frac{U_k+q_k-\phi_k(x;c'_k)}{\alpha_k}\right), & l_k \leq \phi_k(x;c'_k) \leq U_k+q_k, \\ 1, & \phi_k(x;c'_k) \geq U_k+q_k, \end{cases} \tag{4.28B}$$

where L and R are the reference functions. The values of l_k and r_k are the real numbers such that $U_k + q_k - p_k \leq l_k < r_k \leq U_k$. We assume that $\mu_{L_k}(l_k) = \delta_k$ and $\nu_{L_k}(r_k) = \eta_k$ (see Figure 12), where δ_k and η_k are the real numbers such that $0 \leq \delta_k < 1,\ 0 \leq \eta_k < 1$ and $0 \leq \delta_k + \eta_k < 1$. Therefore, α_k and β_k must fulfil the following relations:

$$\alpha_k = \frac{U_k + q_k - l_k}{L^{-1}(\delta_k)} \quad \text{and} \quad \beta_k = \frac{r_k - L_k}{R^{-1}(\eta_k)}, \tag{4.28C}$$

where L^{-1} and R^{-1} are the inverse functions of L and R, respectively, in the proper interval. The possible shapes of μ_{L_k} and ν_{L_k} are shown in Figure 4.10.

Theorem 4.5.1. *Let $f : \mathbb{S} \to \mathbb{R}$ be a real valued function, where $\mathbb{S} \subseteq \mathbb{R}^n$ is a convex set.*

1. *If f is a convex function over $\mathbb{S}$, then $\{x \in \mathbb{S} : f(x) \leq c, c \in \mathbb{R}\}$ is a convex set.*

2. *If f is a concave function over $\mathbb{S}$, then $\{x \in \mathbb{S} : f(x) \geq c, c \in \mathbb{R}\}$ is a convex set.*

3. *f is a quasi convex function over $\mathbb{S}$ iff the lower level set $\{x \in \mathbb{S} : f(x) \leq c, c \in \mathbb{R}\}$ is a convex set.*

4. *f is a quasi concave function over $\mathbb{S}$ iff the upper level set $\{x \in \mathbb{S} : f(x) \geq c, c \in \mathbb{R}\}$ is a convex set.*

Remark 4.5.2. *Obviously, if f is a convex function over $\mathbb{S}$, then $\{x \in \mathbb{S} : f(x) \geq c, c \in \mathbb{R}\}$ is not a convex set and if f is a concave function over $\mathbb{S}$, then $\{x \in \mathbb{S} : f(x) \leq c, c \in \mathbb{R}\}$ is not a convex set; if f is a quasi convex function over $\mathbb{S}$, then $\{x \in \mathbb{S} : f(x) \geq c, c \in \mathbb{R}\}$ is not a convex set and if f is a quasi concave function over $\mathbb{S}$, then $\{x \in \mathbb{S} : f(x) \leq c, c \in \mathbb{R}\}$ is not a convex set.*

Lemma 4.5.3. *The function* $f(\lambda,\lambda') = \lambda + \frac{\alpha(1-\lambda-\lambda')}{2-\lambda-\lambda'}$ *is a concave function for every* $\lambda,\lambda' \in [0,1]$, $\lambda+\lambda' \leq 1$, *and* $\alpha \in (0,1]$.

Proof. For the proof of concavity of function $f(\lambda,\lambda')$, we have to show that the Hessian matrix H of $f(\lambda,\lambda')$ is negative semi-definite. The Hessian matrix H of $f(\lambda,\lambda')$ is given by

$$H_f(\lambda,\lambda') = \frac{-2\alpha}{(2-\lambda-\lambda')^3}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

which is negative semi-definite for every $\lambda,\lambda' \in [0,1]$, $\lambda+\lambda' \leq 1$, and $\alpha \in (0,1]$. Hence, the function $f(\lambda,\lambda') = \lambda + \frac{\alpha(1-\lambda-\lambda')}{2-\lambda-\lambda'}$ is a concave function for every $\lambda,\lambda' \in [0,1]$, $\lambda+\lambda' \leq 1$, and $\alpha \in (0,1]$. □

Corollary 4.5.4. *The sets* $\{(\lambda,\lambda') : \lambda + \frac{\alpha(1-\lambda-\lambda')}{2-\lambda-\lambda'} \geq \eta, \lambda+\lambda' \leq 1; \lambda,\lambda' \geq 0\}$, $\{\phi_k(x;c'_k) : \mu_{U_k}(\phi_k(x;c'_k)) \geq \lambda\}$, $\{\phi_k(x;c'_k) : \mu_{L_k}(\phi_k(x;c'_k)) \geq \lambda\}$, $\{\phi_k(x;c'_k) : \nu_{U_k}(\phi_k(x;c'_k)) \geq \lambda'\}$ *and* $\{\phi_k(x;c'_k) : \nu_{L_k}(\phi_k(x;c'_k)) \geq \lambda'\}$ *are convex sets.*

4.5.1 Auxiliary optimization problem

For solving Problem (4.7), we consider the following auxiliary optimization problem of Problem (4.7):

$$\begin{aligned}
\max \quad & \eta, \\
\text{subject to} \quad & \lambda + \frac{\alpha(1-\lambda-\lambda')}{2-\lambda-\lambda'} \geq \eta, \\
& \mu_{U_k}(\phi_k(x;c'_k)) \geq \lambda,\ k = 1,2,\ldots,K_1, \\
& \nu_{U_k}(\phi_k(x;c'_k)) \leq \lambda',\ k = 1,2,\ldots,K_1, \\
& \mu_{L_k}(\phi_k(x;c'_k)) \geq \lambda,\ k = K_1+1, K_1+2,\ldots,K, \\
& \nu_{L_k}(\phi_k(x;c'_k)) \leq \lambda',\ k = K_1+1, K_1+2,\ldots,K, \\
& \psi_l(x;a'_l) \leq b'_l,\ l = 1,2,\ldots,L_1, \\
& \psi_l(x;a'_l) \geq b'_l,\ l = L_1+1, L_1+2,\ldots,L_2, \\
& \psi_l(x;a'_l) = b'_l,\ l = L_2+1, L_2+2,\ldots,L, \\
& 0 \leq \lambda' \leq \lambda,\ \lambda+\lambda' \leq 1, 0 \leq \eta \leq 1 \\
& x \geq 0,
\end{aligned} \tag{4.29}$$

Here, the meaning of the symbols in Problem (4.29) is given in Problem (4.7).

Since every crisp convex MOOP has unique efficient solution, Problem (4.29) has unique efficient solution. Let us assume that x^* is an efficient for Problem (4.7), i.e., $\nexists$ any $x =$

$[x_1, x_2, \ldots, x_n]^T$ s.t. $\phi_k(x; c'_k) \geq \phi_k(x^*; c'_k), k = 1, 2, \ldots, K_1$ and $\phi_k(x; c'_k) > \phi_k(x^*; c'_k)$ for at least one $k \in \{1, 2, \ldots, K_1\}$; and $\phi_k(x; c'_k) \leq \phi_k(x^*; c'_k), k = K_1+1, K_1+2, \ldots, K$ and $\phi_k(x; c'_k) < \phi_k(x^*; c'_k)$ for at least one $k \in \{K_1+1, K_1+2, \ldots, K\}$.

Theorem 4.5.5. *An optimal solution x^* of Problem (4.29) is also an efficient solution for Problem (4.7).*

Proof. Let us assume that x^* is not an efficient for Problem (4.7). Then $\exists$ an $x = [x_1, x_2, \ldots, x_n]^T$ s.t. $\phi_k(x; c'_k) \geq \phi_k(x^*; c'_k), k = 1, 2, \ldots, K_1$ and $\phi_k(x; c'_k) > \phi_k(x^*; c'_k)$ for at least one $k \in \{1, 2, \ldots, K_1\}$; and $\phi_k(x; c'_k) \leq \phi_k(x^*; c'_k), k = K_1+1, K_1+2, \ldots, K$ and $\phi_k(x; c'_k) < \phi_k(x^*; c'_k)$ for at least one $k \in \{K_1+1, K_1+2, \ldots, K\}$.

Since the μ_{U_k} and ν_{U_k} are non-decreasing and non-increasing functions respectively with the non-decreasing values of the corresponding objective $\phi_k(x; c'_k)$; and μ_{L_k} and ν_{U_k} are non-increasing and non-decreasing functions respectively with the non-decreasing values of the corresponding objective $\phi_k(x; c'_k)$, we have $\mu_{U_k}(\phi_k(x; c'_k)) \geq \mu_{U_k}(\phi_k(x^*; c'_k))$ and $\nu_{U_k}(\phi_k(x; c'_k)) \leq \nu_{U_k}(\phi_k(x^*; c'_k))$ $k = 1, 2, \ldots, K_1$; and $\mu_{L_k}(\phi_k(x; c'_k)) \leq \mu_{L_k}(\phi_k(x^*; c'_k))$ and $\nu_{L_k}(\phi_k(x; c'_k)) \geq \nu_{L_k}(\phi_k(x^*; c'_k))$ $k = K_1+1, K_2+2, \ldots, K$. Hence $\lambda = \min\{\mu_{U_k}(\phi_k(x; c'_k)) : k = 1, 2, \ldots, K_1\} \geq \min\{\mu_{U_k}(\phi_k(x^*; c'_k)) : k = 1, 2, \ldots, K_1\} = \lambda^*$ (say) and $\lambda' = \max\{\nu_{U_k}(\phi_k(x; c'_k)) : k = 1, 2, \ldots, K_1\} \leq \max\{\nu_{U_k}(\phi_k(x^*; c'_k)) : k = 1, 2, \ldots, K_1\} = \lambda^*$ (say); and $\lambda = \min\{\mu_{L_k}(\phi_k(x; c'_k)) : k = K_1+1, K_2+2, \ldots, K\} \leq \min\{\mu_{L_k}(\phi_k(x^*; c'_k)) : k = K_1+1, K_2+2, \ldots, K\} = \lambda^*$ (say) and $\lambda' = \max\{\nu_{L_k}(\phi_k(x; c'_k)) : k = K_1+1, K_2+2, \ldots, K\} \geq \max\{\nu_{L_k}(\phi_k(x^*; c'_k)) : k = K_1+1, K_2+2, \ldots, K\} = \lambda'^*$ (say). This imply that $\eta \geq \eta^*$ (Max) and $\eta \leq \eta^*$ (Min), which is contradict to the fact that x^* is an optimal solution of Problem (4.29). □

The overall solution can be summarized as follows:

Step 1. Model the IFMOOP (4.5) as the crisp MOOP (4.7) by using the accuracy index and value function.

Step 2. Solve the single objective programming problem (SOPP) by considering one objective function at a time and ignoring all others. Repeat the for all objective functions. Let the optimal solutions obtained be $X_1, \ldots, X_{K_1}, \ldots, X_K$ respectively. Let $\mathbb{X} = \{X_k, k = 1, \ldots, K_1, \ldots, K\}$.

Step 3. Find the values of the objective functions $\phi_k(., c'_k), k = 1, \ldots, K_1, \ldots, K$ at each point obtained in Step 2.

Step 4. Find minimum and maximum values of each objective function over $\mathbb{X}$. Let L_k be the minimum and U_k be the maximum value of $\phi_k(., c'_k)$ over $\mathbb{X}$ i.e., $L_k = \min\{\phi_k(x, c'_k), x \in \mathbb{X}\}$ and $U_k = \max\{\phi_k(x, c'_k), x \in \mathbb{X}\}$.

Step 5. Construct the membership and non-membership functions for each objective functions by using the techniques explained in (4.23) to (4.28) as per cases.

Step 6. Construct the auxiliary optimization problem as given in (4.29), and solve it for finding the values of the decision variables and the levels of acceptance.

Step 7. The algorithm stops if the DM is satisfied with the obtained solution. Otherwise, the key parameters, i.e., preferences of each objective function, the tolerances for each objective etc. can be altered to meet the choice. The process is repeated until the DM is satisfied with the obtained solutions.

4.6 Numerical example

A manufacturing factory produces three types of products A, B and C during a period (say one month). Three types of resources R_1, R_2 and R_3 are required to produce these products. One unit of type A product needs around 3 units of R_1, 2 units of R_2 and 3 units of R_3; One unit of type B product needs around 4 units of R_1, 3 units of R_2 and 2 units of R_3 and One unit of type C product needs around 2 units of R_1, 3 units of R_2 and 3 units of R_3. The planned availabilities resource of R_1 and R_2 are around 320 and 350 units respectively with the additional amount around 25 and 20 units in safety store for the emergency purpose which is administrated by the General manager (GM). For better quality of the products at least amount 360 units approximately of resource R_3 must be utilized with some allowed tolerance by the managerial board. To reach the goals, assuming x_1, x_2 and x_3 units are the planned production quantities of A, B and C. The profit of selling each unit of products A, B and C are around 7, 10 and 8 rupees respectively and the estimated time requirements in producing each unit of products A, B and C are around 3, 4 and 5 hours respectively. The GM wants to maximize total profit and minimize total time requirement.

For better dealing with the uncertainties as well as hesitation of the problem, let us assume that all the parameters of the problem are TIFNs. Then this problem can be formulated as follows:

$$\begin{aligned}
\max \quad & \tilde{\phi}_1^I(x) = \tilde{7}^I x_1 + \tilde{10}^I x_2 + \tilde{8}^I x_3, \\
\min \quad & \tilde{\phi}_2^I(x) = \tilde{3}^I x_1 + \tilde{4}^I x_2 + \tilde{5}^I x_3, \\
\text{subject to} \quad & \tilde{3}^I x_1 + \tilde{2}^I x_2 + \tilde{3}^I x_3 \leq \tilde{320}^I, \\
& \tilde{4}^I x_1 + \tilde{3}^I x_2 + \tilde{2}^I x_3 \leq \tilde{350}^I, \\
& \tilde{2}^I x_1 + \tilde{3}^I x_2 + \tilde{3}^I x_3 \geq \tilde{360}^I, \\
& x = (x_1, x_2, x_3) \geq 0,
\end{aligned} \tag{4.30}$$

We assume that all the estimated parameters taken by the manager are TIFNs. Let $\tilde{320}^I = (320, 320, 350;\ 320, 320, 355)$, $\tilde{350}^I = (350, 350, 370; 350, 350, 375)$, $\tilde{360}^I = (340,\ 360,\ 360;\ 340,\ 360,\ 360)$, $\tilde{7}^I = (7, 7, 9; 6, 7,\ 10)$, $\tilde{10}^I = (9, 10, 11; 9, 10, 12)$, $\tilde{8}^I = (7.5, 8, 8.5; 7, 8, 9)$, $\tilde{2}^I = (1,\ 2,\ 3;\ 0.5,\ 2,\ 4)$, $\tilde{3}^I = (2, 3, 4; 1.5, 3, 4)$, $\tilde{4}^I = (3, 4,\ 5; 2, 4, 5)$, $\tilde{5}^I = (4.5, 5, 6; 4, 5, 6.5)$.

Using accuracy index, Problem (4.30) is transformed into the following crisp MOOP.

$$\begin{aligned}
\max \quad & \phi_1(x) = 7.5x_1 + 10.125x_2 + 8x_3, \\
\min \quad & \phi_2(x) = 2.9375x_1 + 3.8750x_2 + 5.1250x_3, \\
\text{subject to} \quad & 2.9375x_1 + 2.0625x_2 + 2.9375x_3 \leq 328.125, \\
& 3.875x_1 + 2.9375x_2 + 2.0625x_3 \leq 355.625, \\
& 2.0625x_1 + 2.9375x_2 + 2.9375x_3 \geq 355, \\
& x_1, x_2, x_3 \geq 0,
\end{aligned} \tag{4.31}$$

Solving the the LPPs by taking the Ist and 2nd objectives on using Mathematica 9.0, we get the optimal solutions as $X_1 = (0, 84.09, 52.66)$, $X_2 = (0, 120.85, 0)$. The ideal and anti-ideal values for each of the objectives are found to be $L_1 = 1223.62$, $U_1 = 1272.69$ and $L_2 = 468.30$, $U_2 = 595.73$. The allowed tolerances given by the DM are $p_1 = 45, p_2 = 90$ and $q_1 = 35, q_2 = 70$. Since ϕ_1 is to be maximized and ϕ_2 is to be minimized simultaneously, the MOOP (4.31) is solved by using (4.29) with different approaches, viz., optimistic, pessimistic and mixed as follows:

The optimistic approach:

The MOOP (4.31) is solved using the reference functions $L(x) = R(x) = \max\{0, 1 - x^t\},\ t > 0$ and $L(x) = R(x) = \frac{e^{-tx} - e^{-t}}{1 - e^{-t}}$, $t > 0$ for different values of t with the help of Mathematica 9.0 software. The solutions for different values of t are given in Table 4.1. These solutions are graphically shown in Figures 4.11 and 4.12.

Table 4.1: Optimistic approach solutions

$L(x)=R(x)=1-x^t,\ t>0,\ x\in[0,1]$							$L(x)=R(x)=\frac{e^{-tx}-e^{-t}}{1-e^{-t}},\ t>0,\ x\in[0,1]$						
t	x_1	x_2	x_3	$\phi_1(x)$	$\phi_2(x)$	η	t	x_1	x_2	x_3	$\phi_1(x)$	$\phi_2(x)$	η
1	0	103.987	24.321	1247.440	527.595	0.862	1	0	103.073	25.6225	1248.59	530.723	0.8706
2	0	104.007	24.2941	1247.42	527.534	0.7	2	0	103.064	25.6307	1248.57	530.73	0.8461
3	0	104.284	23.9143	1247.19	526.661	0.7	3	0	103.061	25.6406	1248.62	530.769	0.8104
4	0	104.699	23.3315	1246.73	525.283	0.7	4	0	103.058	25.6452	1248.62	530.781	0.7774
5	0	106.149	21.269	1244.91	520.331	0.7	5	0	103.044	25.66	1248.60	530.803	0.7518

The pessimistic approach:

The MOOP (4.31) is solved using reference functions $L(x) = R(x) = \max\{0, 1-x^t\},\ t > 0$ and $L(x) = R(x) = \frac{e^{-tx}-e^{-t}}{1-e^{-t}},\ t > 0$ for different values of t with the help of Mathematica 9.0 software. The solutions for different values of t are given in Table 4.2. These solutions are graphically shown in Figures 4.13 and 4.14.

Table 4.2: Pessimistic approach solutions

$L(x)=R(x)=1-x^t,\ t>0,\ x\in[0,1]$							$L(x)=R(x)=\frac{e^{-tx}-e^{-t}}{1-e^{-t}},\ t>0,\ x\in[0,1]$						
t	x_1	x_2	x_3	$\phi_1(x)$	$\phi_2(x)$	η	t	x_1	x_2	x_3	$\phi_1(x)$	$\phi_2(x)$	η
1	0	100.55	29.2169	1251.8	539.368	0.9086	1	0	102.791	26.0246	1248.96	531.691	0.8831
2	0	100.55	29.2168	1251.8	539.367	0.8484	2	0	102.999	25.7288	1248.7	530.981	0.8527
3	0	100.55	29.2107	1251.75	539.336	0.7221	3	0	103.045	25.6629	1248.63	530.822	0.8144
4	0	100.678	29.063	1251.87	539.075	0.6999	4	0	103.254	25.4812	1249.3	530.7	0.7649
5	0	100.825	28.8828	1251.92	538.721	0.6999	5	0	103.4125	25.2453	1249.01	530.106	0.7397

The mixed approach:

The MOOP (4.31) is solved using the reference functions $L(x) = R(x) = \max\{0, 1-x^t\},\ t > 0$ and $L(x) = R(x) = \frac{e^{-tx}-e^{-t}}{1-e^{-t}},\ t > 0$ for t = 2 with the help of Mathematica 9.0 software. The solutions are $x_1 = 0,\ x_2 = 103.052,\ x_3 = 25.652,\ \phi_1 = 1248.770,\ \phi_2 = 530.655$ when $L(x) = R(x) = \max\{0, 1-x^t\},\ t = 2$, and $x_1 = 0,\ x_2 = 103.054,\ x_3 = 25.650,\ \phi_1 = 1248.622$, $\phi_2 = 530.791$, when $L(x) = R(x) = \frac{e^{-tx}-e^{-t}}{1-e^{-t}},\ t = 2$.

4.7 Comparative study

The above example is also solved with some other existing models like Zimmermann's approach, maximum additive operator and maximum product operator [203] by considering the nonlinear membership and nonmembership functions for each of the objectives and constraints. The comparison of the results obtained is given in Table 4.3.

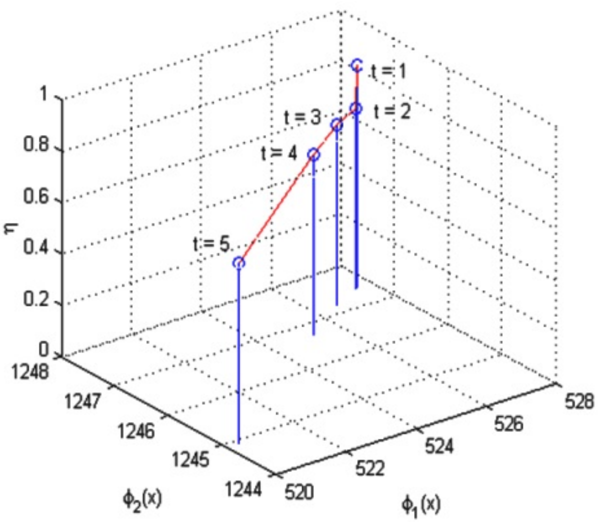

Figure 4.11: Graphical representation of the solutions by optimistic approach when $L(x) = R(x) = \max\{0, 1 - x^t\}$, $t = 1, 2, 3, 4, 5$.

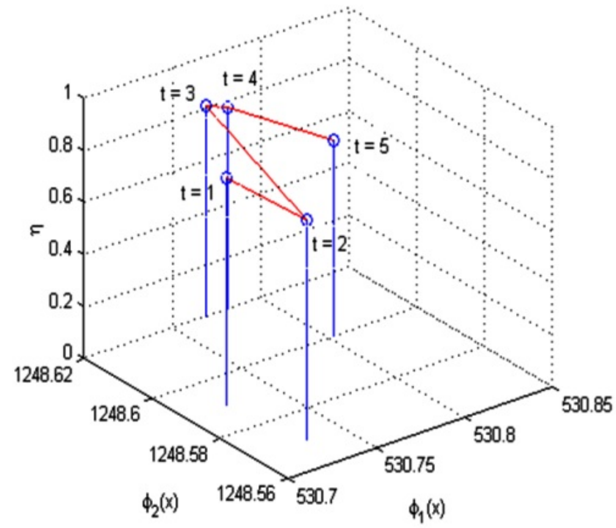

Figure 4.12: Graphical representation of the solutions by optimistic approach when $L(x) = R(x) = \frac{e^{-tx} - e^{-t}}{1 - e^{-t}}$, $t = 1, 2, 3, 4, 5$.

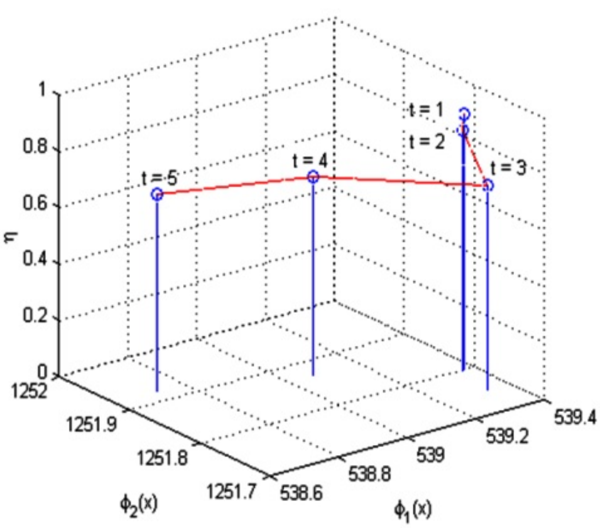

Figure 4.13: Graphical representation of the solutions by pessimistic approach when $L(x) = R(x) = \max\{0, 1 - x^t\}$, $t = 1, 2, 3, 4, 5$.

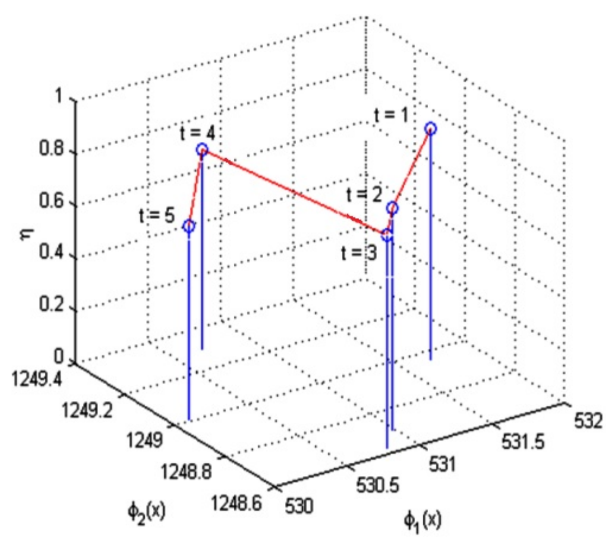

Figure 4.14: Graphical representation of the solutions by pessimistic approach when $L(x) = R(x) = \frac{e^{-tx} - e^{-t}}{1 - e^{-t}}$, $t = 1, 2, 3, 4, 5$.

Zimmemann's approach:

$$
\begin{aligned}
\max \quad & \lambda, \\
\text{subject to} \quad & \mu_{U_k}(\phi_k(x)) \geq \lambda,\ k = 1, 2, 3, \ldots K_1, \\
& \mu_{L_k}(\phi_k(x)) \geq \lambda,\ k = K_1+1, K_1+2, K_1+3, \ldots, K, \\
& \psi_l(x; a'_l) \leq b'_l,\ l = 1, 2, 3, \ldots, L_1, \\
& \psi_l(x; a'_l) \geq b'_l,\ l = L_1+1, L_1+2, L_1+3, \ldots, L_2, \\
& \psi_l(x; a'_l) = b'_l,\ l = L_2+1, L_2+2, L_2+3, \ldots, L, \\
& 0 \leq \lambda \leq 1, \\
& x \geq 0.
\end{aligned} \tag{4.32}
$$

Maximum additive operator:

$$
\begin{aligned}
\max \quad & \mu_{U_k}(\phi_k(x)) + \mu_{L_{k'}}(\phi_{k'}(x)),\ k = 1, 2, 3, \ldots K_1,\ k' = K_1+1, K_1+2, K_1+3, \ldots, K, \\
\text{subject to} \quad & 0 \leq \mu_{U_k}(\phi_k(x)) \leq 1,\ k = 1, 2, 3, \ldots K_1, \\
& 0 \leq \mu_{L_{k'}}(\phi_{k'}(x)) \leq 1,\ k' = K_1+1, K_1+2, K_1+3, \ldots, K, \\
& 0 \leq \mu_{U_k}(\phi_k(x)) + \mu_{L_{k'}}(\phi_{k'}(x)) \leq 1,\ k = 1, 2, 3, \ldots K_1,\ k' = K_1+1, K_1+2, \\
& K_1+3, \ldots, K, \\
& \psi_l(x; a'_l) \leq b'_l,\ l = 1, 2, 3, \ldots, L_1, \\
& \psi_l(x; a'_l) \geq b'_l,\ l = L_1+1, L_1+2, L_1+3, \ldots, L_2, \\
& \psi_l(x; a'_l) = b'_l,\ l = L_2+1, L_2+2, L_2+3, \ldots, L, \\
& 0 \leq \lambda \leq 1, \\
& x \geq 0.
\end{aligned} \tag{4.33}
$$

Maximum product operator:

$$
\begin{aligned}
\max \quad & (\mu_{U_k}(\phi_k(x))) * (\mu_{L_{k'}}(\phi_{k'}(x))),\ k = 1, 2, 3, \ldots K_1,\ k' = K_1+1, K_1+2, \\
& K_1+3, \ldots, K, \\
\text{subject to} \quad & 0 \leq \mu_{U_k}(\phi_k(x)) \leq 1,\ k = 1, 2, 3, \ldots K_1, \\
& 0 \leq \mu_{L_{k'}}(\phi_{k'}(x)) \leq 1,\ k' = K_1+1, K_1+2, K_1+3, \ldots, K,
\end{aligned} \tag{4.34}
$$

$$\psi_l(x; a'_l) \leq b'_l, \ l = 1, 2, 3, \ldots, L_1,$$
$$\psi_l(x; a'_l) \geq b'_l, \ l = L_1 + 1, L_1 + 2, L_1 + 3, \ldots, L_2,$$
$$\psi_l(x; a'_l) = b'_l, \ l = L_2 + 1, L_2 + 2, L_2 + 3, \ldots, L,$$
$$0 \leq \lambda \leq 1,$$
$$x \geq 0.$$

From Table 4.3, it is clear that the average of d_1 and d_2 values is minimum by the proposed

Table 4.3: Comparison table when $L(x) = R(x) = \max\{0, 1 - x^t\}, \ t = 2$

Method	ϕ_1	ϕ_2	Deviations from U_1 (d_1)	Deviations from L_2 (d_2)	Average of d_1 and d_2
Zimmermann's Technique	1248.63	530.796	24.06	62.496	43.278
Maximum additive operator	1234.97	549.844	37.72	81.544	59.632
Maximum product operator	1248.36	530.069	24.33	61.769	43.0495
Proposed method	1247.42	527.534	25.27	59.234	42.252

method. Therefore, objective values obtained by the proposed method are better than those obtained by the existing methods.

4.8 Advantages of the proposed method over the existing methods

The advantages of the proposed method over existing methods ([9], [189], [66], [139], [163], [153]) for solving IFMOOPs are summarized in Table 4.4.

4.9 Concluding remarks

In this chapter, we have developed an algorithm to solve the MOOP in IFE and illustrated the same by solving a numerical problem. The membership and non-membership functions play a vital role while designing a model in IFE. Most of the techniques in the existing literature [9, 66, 139, 153, 163] are based on constructing the membership and non-membership functions in which the lower and upper end points are fixed for the IF objectives/constraints. Moreover, the membership and non-membership functions in which the lower and upper end points are also fixed, do not deal with the mutual conflicting nature of the objectives and always do justice while modeling a real-life decision model. So, the general membership and non-membership functions governed by the reference functions are considered in this chapter from different viewpoints,

Table 4.4: Advantages of the proposed method

Existing models	Proposed model
1. Angelov [9] used a value function for solving IFMOOPs which has draw backs as pointed by Yager [189].	**1.** The proposed method proposes a new value function for solving IFMOOPs which is free from such draw backs.
2. Dubey et al. [66], Rani et al. [153] used the value function proposed by Yager [189] for solving IFMOOPs which has draw backs as pointed by us (see Section 4).	**2.** The proposed method proposes a new value function for solving IFMOOPs which is free from such draw backs.
3. Singh and Yadav [163], Rani et al. [153] have used only membership functions for solving IFMOOPs; the nonmembership functions are not used. But IF governed by membership and nonmembership functions.	**3.** The proposed method has uses both the membership and nonmembership functions for solving IFMOOPs.
4. The methods developed by Angelov [9], Dubey et al. [66], Nishad and Singh [139], Rani et al. [153], Singh and Yadav [163] are based on constructing the membership and non-membership functions in which the lower and upper end points are fixed for the IF objectives or constraints. Also, then do not deal with the mutual conflicting nature of the objectives.	**4.** The proposed method defines the membership and non-membership functions governed by reference functions in which the lower and upper end point are not fixed. Also, they deal with the mutual conflicting nature of the objectives.

viz., optimistic, pessimistic and mixed. The obtained results are found to be better and near the ideal and anti-ideal values of each of the objectives.

Chapter 5

Information measures in intuitionistic fuzzy environment: analysis and some relationships

This chapter considers some information measures, such as, normalized divergence measure, similarity measure, dissimilarity measure and normalized distance measure in IFE, which measure the uncertainty and hesitancy, and which can be applied to the selection of alternatives in group decision problems. We introduce and study the continuity of considered measures. Next, we prove some results that can be used to generate measures for FSs as well as for IFSs and we also prove some approaches to construct point measures from set measures in IFE. We define the weight set for one and many preference orders of alternatives. After, we investigate the properties and results related to the weight set. Based on the weight set, we develop the model for finding the uncertain weights corresponding to attribute. Also, we develop the model to finding positive certain weights corresponding to each attribute by using uncertain weights. Finally, an algorithm for choosing the best alternative according to the preference orders of alternatives in decision-making problems is proposed and its validity is shown with the help of a numerical example.

5.1 Introduction

Most of the information present in the real-life is uncertain in nature. Generally, decision-maker (DM) can not handle completely such complex information. Zadeh [194] introduced the concept of FS theory to model uncertainty by assigning degree of association called the membership degree. Several authors have given different types of measures to deal the uncertain information and have studied their theoretical properties and also established the interrelationship between them [38, 48, 53, 72, 93, 114, 195, 198]. There are several kinds of information which contain uncertainty as well as vagueness. Such information can not be modeled by FS theory. For such situation, Atannasov [11] gave the concept of IFS which handle the uncertainty as well as hesitation by assigning degrees known as membership degree and non-membership degree. If the sum of membership and non-membership degrees at each point of the universe is one, then the IFS becomes FS. If the sum of membership and non-membership degrees at each point of the universe lies in (0, 1), then the IFS is called pure IFS. Atanassov [13] has also given the operations on IFSs and their analysis. During last decades, IFS theory played an important role in modeling uncertain and vague systems, received much attention from the researchers and meaningful results were obtained in the field of decision-making problems [138], pattern recognition [54, 143] to name a few. Decision making is one of the popular branches of Operations Research in which the problem to choose the best alternative from the given set of feasible alternatives is considered. There exist several processes in literature but there are mainly four stages required to choose the best alternative: (i) Evaluate the set of feasible alternatives from given information. (ii) Determine the weight vector corresponding to alternatives or attributes which depend on DM. (iii) Aggregate alternatives by taking weight vector given by DM. (iv) Rank the alternatives in order of preference and select the best one.

There are several information measures in IFE, such as divergences measures, similarity measures, dissimilarity measures, and distance measures. They model uncertain and vague information. The inclusion between two IFSs can be measured by the concept of inclusion measure [79] and the commonality between two IFSs can be measured by the concept of similarity measure [95, 96]. Also, new similarity measures are constructed and used in pattern recognition [54]. Moreover, several authors established the relationship between point similarity measures and similarity measures [41, 97, 115]. The concept of distance measure in IFE and different types of distance measures are given in [167]. The concept of H-max distance measure of IFSs is given in [138] and it is used in decision-making problems. Grzegorzewski [77] gave distances and orderings in a family of IF numbers. The comparative analysis between similarity measures and distance measures in IFE are discussed from a pattern recognition point of view [143] and theoretical point of view [166]. The concept of divergence measures, local divergence measures and the relationship between distance, similarity and divergence measures are given in [133] and [134]. The monotonic similarity measures between IFSs and their relationship with entropy and inclusion measures given by Deng et al. [52]. Das et al. [50] gave information measures in the

IF framework and their relationships.

The motivation and our contributions are of three folds. In the first fold, we were inspired by the papers [45, 110] in which the authors have calculated the similarity measure between 2-vague values, n-vague values, 2-vague sets and n-vague sets, and also applied these concepts for behavior analysis in an organization. Based on the relations in [45] and [110], we have given the concept of continuity of measures in IFE, and we have constructed the point measures derived from measures of IFSs. Moreover, we have analyzed the continuity relationship between them. Further, several results concerning the point measures derived from the set of all IFSs measures and aggregation operator, and the relationship between IF-measures and fuzzy point measures are given in the form of theorems (see Sections 5.2 and 5.3). In the second fold, we were inspired by the papers [182, 183, 190, 191] in which the authors have given the concept of additive multi-attribute value models based on the weight-set to satisfy preference orders of alternatives and to determine the compromise weights for group decision-making. Based on the concepts of these papers, we have modeled the mathematical programming problem for finding uncertain attribute weights when one and many preferences of alternatives are given. Also, we have given the new concept of weight set corresponding to one and many preferences of alternatives, and have studied and analyzed the properties and results related to the weight set. After, we have modeled the mathematical programming problem to find the uncertain attribute weights (see Sections 5.4, 5.5, 5.6, 5.7). In the third fold, we were inspired by the papers [144, 145] in which the authors have given the concept of choosing the best alternative in decision making problems when the weights information are incomplete. Motivated by the facts in [144] and [145], we have modeled mathematical programming problem for finding the positive certain attribute weights with the help of uncertain attribute weights (Section 5.7). Further, an algorithm for solving multi-attribute decision making (MADM) problems is given. Also, a test example is given to demonstrate the practicality and effectiveness of the introduced measures and the proposed algorithm (see Section 5.8).

The rest of the chapter is organized as follows. In Section 5.2, we introduce some necessary basic definitions. Section 5.3 contains the notion of continuity of IF-measures and the relationship between point measures and the set of measures in IFE. The core of the chapter is presented in Sections 5.4, 5.5, 5.6 and 5.7 concerning the formulation of decision-making problem, the structure of weight set of one preference and many preferences simultaneously, determination of attribute weights for given preferences of alternatives. In Section 5.8, an algorithm for solving MADM problems and a test example are given. The chapter ends with Section 5.9 containing some concluding remarks.

5.2 Preliminaries

Let $\mathcal{I}$ denote the set of IFSs over $X = \{x_1, \ldots, x_j, \ldots, x_n\}$.

5.2.1 IF-difference

Definition 5.2.1. *[134] Let $\tilde{A}^I$, $\tilde{B}^I$ and $\tilde{C}^I \in \mathcal{I}$. Then an operator $- : \mathcal{I} \times \mathcal{I} \to \mathcal{I}$ is defined as a difference operator for IFSs, called IF-difference operator, if it satisfies the following properties:*

(D1) $\tilde{A}^I - \tilde{B}^I = \phi$ *if* $\tilde{A}^I \subseteq \tilde{B}^I$;

(D2) $\tilde{B}^I - \tilde{A}^I \subseteq \tilde{C}^I - \tilde{A}^I$ *if* $\tilde{B}^I \subseteq \tilde{C}^I$.

$\tilde{A}^I - \tilde{B}^I$ is called the IF difference of $\tilde{A}^I$ and $\tilde{B}^I$. The following are other interesting properties that IF-differences may satisfy:

(D'1) $(\tilde{A}^I \cap \tilde{C}^I) - (\tilde{B}^I \cap \tilde{C}^I) \subseteq \tilde{A}^I - \tilde{B}^I$.

(D'2) $(\tilde{A}^I \cup \tilde{C}^I) - (\tilde{B}^I \cup \tilde{C}^I) \subseteq \tilde{A}^I - \tilde{B}^I$.

(D'3) $\tilde{A}^I - \tilde{B}^I = \phi \;\Rightarrow\; \tilde{A}^I \subseteq \tilde{B}^I$.

Example 5.2.2. *[134] Consider the function $- : \mathcal{I} \times \mathcal{I} \to \mathcal{I}$ given by*

$$\tilde{A}^I - \tilde{B}^I = \{(x, \mu_{\tilde{A}^I - \tilde{B}^I}(x), \nu_{\tilde{A}^I - \tilde{B}^I}(x)) : x \in X\},$$

where $\mu_{\tilde{A}^I - \tilde{B}^I}(x) = \max(0, \mu_{\tilde{A}^I}(x) - \mu_{\tilde{B}^I}(x))$ and $\nu_{\tilde{A}^I - \tilde{B}^I}(x) = 1 - \mu_{\tilde{A}^I - \tilde{B}^I}(x)$ if $\nu_{\tilde{A}^I}(x) > \nu_{\tilde{B}^I}(x)$ or $\nu_{\tilde{A}^I - \tilde{B}^I}(x) = \min(1 + \nu_{\tilde{A}^I}(x) - \nu_{\tilde{B}^I}(x), 1 - \mu_{\tilde{A}^I - \tilde{B}^I}(x))$ if $\nu_{\tilde{A}^I}(x) \leq \nu_{\tilde{B}^I}(x)$. Then

(i) $\tilde{A}^I \subseteq \tilde{B}^I \;\Rightarrow\; \mu_{\tilde{A}^I}(x) \leq \mu_{\tilde{B}^I}(x)$ and $\nu_{\tilde{A}^I}(x) \geq \nu_{\tilde{B}^I}(x)$. This implies that $\mu_{\tilde{A}^I - \tilde{B}^I}(x) = 0$ and $\nu_{\tilde{A}^I - \tilde{B}^I}(x) = 1$. Therefore, $\tilde{A}^I - \tilde{B}^I = \phi$.

(ii) $\tilde{B}^I \subseteq \tilde{C}^I \;\Rightarrow\; \mu_{\tilde{B}^I}(x) \leq \mu_{\tilde{C}^I}(x)$ and $\nu_{\tilde{B}^I}(x) \geq \nu_{\tilde{C}^I}(x)$. This implies that $\mu_{\tilde{B}^I - \tilde{A}^I}(x) \leq \mu_{\tilde{C}^I - \tilde{A}^I}(x)$ and $\nu_{\tilde{B}^I - \tilde{A}^I}(x) \geq \nu_{\tilde{C}^I - \tilde{A}^I}(x)$. Therefore, $\tilde{B}^I - \tilde{A}^I \subseteq \tilde{C}^I - \tilde{A}^I$.

Thus, the function '-' is an IF-difference.

5.2.2 IF Normalized Distance Measure

Definition 5.2.3. *[166] An operator $N_d : \mathcal{I} \times \mathcal{I} \to [0, 1]$ is said to be an IF normalized distance measure if the following conditions are satisfied for every pair of IFSs $\tilde{A}^I$ and $\tilde{B}^I$:*

(d1) $N_d(\tilde{A}^I, \tilde{B}^I) = N_d(\tilde{B}^I, \tilde{A}^I)$,

(d2) $N_d(\tilde{A}^I, \tilde{B}^I) = 0$ *iff* $\tilde{A}^I = \tilde{B}^I$,

(d3) $N_d(\tilde{A}^I, \tilde{A}'^I) = 1$ *if* $\tilde{A}^I$ *is a crisp set,*

(d4) $\tilde{A}^I \subseteq \tilde{B}^I \subseteq \tilde{C}^I \;\Rightarrow\; N_d(\tilde{A}^I, \tilde{B}^I) \leq N_d(\tilde{A}^I, \tilde{C}^I),\; N_d(\tilde{A}^I, \tilde{C}^I) \geq N_d(\tilde{B}^I, \tilde{C}^I)$.

For example, the normalized Hamming and Euclidean distances as defined below are normalized distance measures:
The normalized Hamming and the normalized Euclidean distances between $\tilde{A}_1^I$ and $\tilde{A}_2^I$ are given as follows:

- The normalized Hamming distance [167]:

$$H(\tilde{A}_1^I, \tilde{A}_2^I) = \frac{1}{2n} \sum_{j=1}^{n} \left[|\mu_{\tilde{A}_1^I}(x_j) - \mu_{\tilde{A}_2^I}(x_j)| + |\nu_{\tilde{A}_1^I}(x_j) - \nu_{\tilde{A}_2^I}(x_j)| + |\pi_{\tilde{A}_1^I}(x_j) - \pi_{\tilde{A}_2^I}(x_j)|\right] \quad (5.1)$$

- The normalized Euclidean distance [167]:

$$E(\tilde{A}_1^I, \tilde{A}_2^I) = \Big[\frac{1}{2n} \sum_{j=1}^{n} (\mu_{\tilde{A}_1^I}(x_j) - \mu_{\tilde{A}_2^I}(x_j))^2 + (\nu_{\tilde{A}_1^I}(x_j) - \nu_{\tilde{A}_2^I}(x_j))^2 + (\pi_{\tilde{A}_1^I}(x_j) - \pi_{\tilde{A}_2^I}(x_j))^2\Big]^{1/2} \quad (5.2)$$

Definition 5.2.4. *The operator N_d defined on $\mathcal{L} \times \mathcal{L}$ satisfying conditions (d1)-(d4) is called IF normalized point distance measure.*

Remark 5.2.5. *An IF normalized distance measure N_d can be seen as a normalized point distance measure whenever X is a singleton, $X = \{x_1\}$.*

5.2.3 IF Similarity Measure

Definition 5.2.6. *[166] An operator $S_m : \mathcal{I} \times \mathcal{I} \to [0, 1]$ is said to be an IF similarity measure if the following conditions are satisfied for every pair of IFSs $\tilde{A}^I$ and $\tilde{B}^I$:*

(SM1) $S_m(\tilde{A}^I, \tilde{B}^I) = S_m(\tilde{B}^I, \tilde{A}^I)$,

(SM2) $S_m(\tilde{A}^I, \tilde{B}^I) = 1$ *iff* $\tilde{A}^I = \tilde{B}^I$,

(SM3) $S_m(\tilde{A}^I, \tilde{A}') = 0$ *if* $\tilde{A}^I$ *is a crisp set,*

(SM4) $\tilde{A}^I \subseteq \tilde{B}^I \subseteq \tilde{C}^I \;\Rightarrow\; S_m(\tilde{A}^I, \tilde{C}^I) \leq S_m(\tilde{A}^I, \tilde{B}^I),\; S_m(\tilde{A}^I, \tilde{C}^I) \leq S_m(\tilde{B}^I, \tilde{C}^I)$.

For example, the measures as defined below are similarity measures:
The similarity measures between $\tilde{A}_1^I$ and $\tilde{A}_2^I$ are given as follows:

- Similarity measure proposed by Hung and Yang [95]:

$$S^{HY}(\tilde{A}_1^I, \tilde{A}_2^I) = 1 - \frac{1}{n}\sum_{j=1}^{n}(|\mu_{\tilde{A}_1^I}(x_j) - \mu_{\tilde{A}_2^I}(x_j)| \vee |\nu_{\tilde{A}_1^I}(x_j) - \nu_{\tilde{A}_2^I}(x_j)|) \qquad (5.3)$$

- Similarity measure proposed by Chen [41]:

$$S^{C}(\tilde{A}_1^I, \tilde{A}_2^I) = 1 - \frac{1}{2n}\sum_{j=1}^{n}(|(\mu_{\tilde{A}_1^I}(x_j) - \mu_{\tilde{A}_2^I}(x_j)) - (\nu_{\tilde{A}_1^I}(x_j) - \nu_{\tilde{A}_2^I}(x_j))|) \qquad (5.4)$$

- Similarity measure proposed by Hong and Kim [97]:

$$S^{HK}(\tilde{A}_1^I, \tilde{A}_2^I) = 1 - \frac{1}{2n}\sum_{j=1}^{n}(|(\mu_{\tilde{A}_1^I}(x_j) - \mu_{\tilde{A}_2^I}(x_j))| + |(\nu_{\tilde{A}_1^I}(x_j) - \nu_{\tilde{A}_2^I}(x_j))|) \qquad (5.5)$$

Definition 5.2.7. *The operator S_m defined on $\mathcal{L} \times \mathcal{L}$ satisfying conditions (SM1)-(SM4) is called IF point similarity measure.*

5.2.4 IF Inclusion Measure

In IFS theory, the degree to which IFS $\tilde{A}^I$ is included in IFS $\tilde{B}^I$, denoted as $I_{nc}(\tilde{A}^I, \tilde{B}^I)$, is called an IF inclusion measure. Mathematically, it can be defined as follows:

Definition 5.2.8. *[79] An operator $I_{nc} : \mathcal{I} \times \mathcal{I} \to [0, 1]$ is said to be an IF inclusion measure if the following conditions are satisfied for every IFSs $\tilde{A}^I$, $\tilde{B}^I$ and $\tilde{C}^I$:*

(IM1) $I_{nc}(X, \phi) = 0,$

(IM2) $\tilde{A}^I \subseteq \tilde{B}^I \;\Rightarrow\; I_{nc}(\tilde{A}^I, \tilde{B}^I) = 1,$

(IM3) $\tilde{A}^I \subseteq \tilde{B}^I \subseteq \tilde{C}^I \;\Rightarrow\; I_{nc}(\tilde{C}^I, \tilde{A}^I) \leq \min(I_{nc}(\tilde{B}^I, \tilde{A}^I), I_{nc}(\tilde{C}^I, \tilde{B}^I)).$

Definition 5.2.9. *The operator I_{nc} defined on $\mathcal{L} \times \mathcal{L}$ satisfying conditions (IM1)-(IM3) is called IF point inclusion measure.*

5.2.5 IF Normalized Divergence Measure

Definition 5.2.10. *[134] An operator $N_D : \mathcal{I} \times \mathcal{I} \to [0, 1]$ is said to be an IF normalized divergence measure if the following conditions are satisfied for every pair of IFSs $\tilde{A}^I$ and $\tilde{B}^I$:*

(DM1) $N_D(\tilde{A}^I, \tilde{B}^I) = N_D(\tilde{B}^I, \tilde{A}^I),$

(DM2) $N_D(\tilde{A}^I, \tilde{A}^I) = 0,$

(DM3) $N_D(\tilde{A}^I, \tilde{A}') = 1$ *if* $\tilde{A}^I$ *is a crisp set,*

(DM4) $N_D(\tilde{A}^I \cap \tilde{C}^I, \tilde{B}^I \cap \tilde{C}^I) \leq N_D(\tilde{A}^I, \tilde{B}^I) \; \forall \, \tilde{C}^I \in \mathcal{I}$,

(DM5) $N_D(\tilde{A}^I \cup \tilde{C}^I, \tilde{B}^I \cup \tilde{C}^I) \leq N_D(\tilde{A}^I, \tilde{B}^I) \; \forall \, \tilde{C}^I \in \mathcal{I}$.

Definition 5.2.11. *The operator* N_D *defined on* $\mathcal{L} \times \mathcal{L}$ *satisfying conditions (DM1)-(DM5) is called IF normalized point divergence measure.*

5.2.6 IF Dissimilarity Measure

Definition 5.2.12. *[134] An operator* $d_s : \mathcal{I} \times \mathcal{I} \to [0, \infty)$ *is said to be an IF dissimilarity measure if the following conditions are satisfied for every pair of IFSs* $\tilde{A}^I$ *and* $\tilde{B}^I$*:*

(diss1) $d_s(\tilde{A}^I, \tilde{B}^I) = d_s(\tilde{B}^I, \tilde{A}^I)$,

(diss2) $d_s(\tilde{A}^I, \tilde{A}^I) = 0$,

(diss3) $\tilde{A}^I \subseteq \tilde{B}^I \subseteq \tilde{C}^I \;\Rightarrow\; d_s(\tilde{A}^I, \tilde{C}^I) \geq \max(d_s(\tilde{A}^I,\, \tilde{B}^I),\, d_s(\tilde{B}^I,\, \tilde{C}^I))$.

Definition 5.2.13. *The operator* d_s *defined on* $\mathcal{L} \times \mathcal{L}$ *satisfying conditions (diss1)-(diss3) is called IF point dissimilarity measure.*

Example 5.2.14 (IF-dissimilarity measures that are also IF normalized divergences [97])**.** *The IF dissimilarity measures between* $\tilde{A}^I_1$ *and* $\tilde{A}^I_2$ *are given by*

- $$d_{sH}(\tilde{A}^I_1, \tilde{A}^I_2) = \frac{1}{2n}\sum_{j=1}^{n}(|\mu_{\tilde{A}^I_1}(x_j) - \mu_{\tilde{A}^I_2}(x_j)| + |\nu_{\tilde{A}^I_1}(x_j) - \nu_{\tilde{A}^I_2}(x_j)|) \tag{5.6}$$

- $$d_{sL}(\tilde{A}^I_1, \tilde{A}^I_2) = \frac{1}{4n}\sum_{j=1}^{n}(|(S_{\tilde{A}^I_1}(x_j) - S_{\tilde{A}^I_2}(x_j))| + |(S_{\tilde{A}^I_1}(x_j) + S_{\tilde{A}^I_2}(x_j))|) \tag{5.7}$$

  where $S_{\tilde{A}^I_1}(x_j) = |\mu_{\tilde{A}^I_1}(x_j) - \nu_{\tilde{A}^I_1}(x_j)|$ *and* $S_{\tilde{A}^I_2}(x_j) = |\mu_{\tilde{A}^I_2}(x_j)\text{-}\nu_{\tilde{A}^I_2}(x_j)|$*. These IF-dissimilarity measures are also IF normalized divergence measures.*

5.3 Continuity property for the measures of information

In this section, we define the continuity for the information measure in IFE and investigate their properties.

In $\mathbb{R}^2$, the well-known metrics, like, the Euclidean distance and the Hamming distance are defined as follows.

- The Euclidean distance between $u = (u_1, u_2)$ and $v = (v_1, v_2)$ in $\mathbb{R}^2$ is given by

$$d^E(u, v) = \sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2}.$$

- The Hamming distance between $u = (u_1, u_2)$ and $v = (v_1, v_2)$ in $\mathbb{R}^2$ is given by

$$d^H(u, v) = |u_1 - v_1| + |u_2 - v_2|.$$

If we restrict these distances to $\mathcal{L}$, then we obtain the metric space $(\mathcal{L}, d^E)$, where d^E is the Euclidean distance on $\mathcal{L}$, and the metric space $(\mathcal{L}, d^H)$, where d^H is the Hamming distance on $\mathcal{L}$. Denote, for any $u \in \mathcal{L}$, $u_\pi = 1 - u_1 - u_2$. Szmidt and Kacprzyk [167] have defined two distances on $\mathcal{L}$ based on the Euclidean and the Hamming distances, where also u_π is used.

- The $d^E_{\mathcal{L}}$ between $u = (u_1, u_2)$ and $v = (v_1, v_2)$ in $\mathcal{L}$ is given by

$$d^E(u, v) = \sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2 + (u_\pi - v_\pi)^2}.$$

- The $d^H_{\mathcal{L}}$ between $u = (u_1, u_2)$ and $v = (v_1, v_2)$ in $\mathcal{L}$ is given by

$$d^H(u, v) = |u_1 - v_1| + |u_2 - v_2| + |u_\pi - v_\pi|.$$

Deschrijver et. al [59] proved that these distances are topologically equivalent.

Definition 5.3.1. *Let $d : \mathcal{L} \times \mathcal{L} \to [0, 1]$ be the Euclidean distance or the Hamming distance. A function $\Gamma : \mathcal{I} \times \mathcal{I} \to [0, 1]$ is continuous if for every $\epsilon > 0 \; \exists \, \delta > 0$ such that for every $\tilde{A}^I_1, \; \tilde{A}^I_2 \in \mathcal{I}$, $|\Gamma(\tilde{A}^I_1, \tilde{B}^I) - \Gamma(\tilde{A}^I_2, \tilde{B}^I)| < \epsilon$ for every $\tilde{B}^I \in \mathcal{I}$ whenever $\max\limits_{i=1,2,\ldots,n} d((\mu_{\tilde{A}^I_1}(x_i), \; \nu_{\tilde{A}^I_1}(x_i)), \; (\mu_{\tilde{A}^I_2}(x_i), \; \nu_{\tilde{A}^I_2}(x_i))) < \delta$.*

Theorem 5.3.2. *Let $p = (p_1, p_2) \in \mathcal{L}$ and $\tilde{A}^I_p = \{(x_i, \mu_{\tilde{A}^I}(x_i) = p_1, \nu_{\tilde{A}^I}(x_i) = p_2) : x_i \in X\} \in \mathcal{I}$. Let $\Upsilon : \mathcal{I} \times \mathcal{I} \to [0, 1]$ be an IF normalized divergence measure, IF inclusion measure, IF similarity measure, IF dissimilarity measure or IF normalized distance measure. Then the function $\gamma : \mathcal{L} \times \mathcal{L} \to [0, 1]$ defined by*

$$\gamma(p, q) = \Upsilon(\tilde{A}^I_p, \tilde{A}^I_q) \; \forall \; p, q \in \mathcal{L} \text{ and } \tilde{A}^I_p, \tilde{A}^I_q \in \mathcal{I} \tag{5.8}$$

is an IF normalized point divergence measure, IF point inclusion measure, IF point similarity measure, IF point dissimilarity measure or IF normalized point distance measure respectively.

Proof. Let us suppose that $\Upsilon : \mathcal{I} \times \mathcal{I} \to [0, 1]$ is the IF normalized divergence measure. Then

(i) $\Upsilon(\tilde{A}^I_p, \tilde{A}^I_q) = \Upsilon(\tilde{A}^I_q, \tilde{A}^I_p) \;\Rightarrow\; \gamma(p, q) = \gamma(q, p)$,

(ii) $\gamma(p,q)=0 \Leftrightarrow \Upsilon(\tilde{A}^I_p,\tilde{A}^I_q)=0 \Leftrightarrow \tilde{A}^I_p=\tilde{A}^I_q \Leftrightarrow p=q$,

(iii) Suppose $\tilde{A}^I_p$ is crisp set, i.e., $\tilde{A}^I_p=\emptyset$ or X. Then $\Upsilon(\emptyset,X)=1 \Leftrightarrow \gamma(0_{\mathcal{L}},1_{\mathcal{L}})=1$,

(iv) $\gamma(p\wedge r,q\wedge r)=\Upsilon(\tilde{A}^I_{=p\wedge r},\tilde{A}^I_{=q\wedge r})\leq \Upsilon(\tilde{A}^I_p,\tilde{A}^I_q)=\gamma(p,q)$,

(v) $\gamma(p\vee r,q\vee r)=\Upsilon(\tilde{A}^I_{=p\vee r},\tilde{A}^I_{=q\vee r})\leq \Upsilon(\tilde{A}^I_p,\tilde{A}^I_q)=\gamma(p,q)$.

Thus, $\gamma:\mathcal{L}\times\mathcal{L}\to[0,1]$ is an IF normalized point divergence measure.
Similarly, if $\Upsilon:\mathcal{I}\times\mathcal{I}\to[0,1]$ is an IF inclusion measure, IF similarity measure, IF dissimilarity measure or IF normalized distance measure, then we can easily prove that $\gamma:\mathcal{L}\times\mathcal{L}\to[0,1]$ is an IF point inclusion measure, IF point similarity measure, IF point dissimilarity measure or IF normalized point distance measure respectively. □

Theorem 5.3.3. *Let γ be an IF normalized point divergence measure, IF point similarity measure, IF point dissimilarity measure or IF normalized point distance measure respectively obtained from the corresponding IF measure as given in Theorem 5.3.2. Then Υ is continuous iff γ is continuous.*

Proof. ($\Rightarrow$) Suppose Υ is continuous. Then for every $\frac{\epsilon}{2}>0\ \exists\ \delta>0$ such that for every $\tilde{A}^I_1,\tilde{A}^I_2\in\mathcal{I}$, $|\Upsilon(\tilde{A}^I_1,\tilde{B}^I)-\Upsilon(\tilde{A}^I_2,\tilde{B}^I)|<\frac{\epsilon}{2}$ for every $\tilde{B}^I\in\mathcal{I}$ whenever $\max\limits_{i=1,2,\ldots,n} d((\mu_{\tilde{A}^I_1}(x_i),\nu_{\tilde{A}^I_1}(x_i)),(\mu_{\tilde{A}^I_2}(x_i),\nu_{\tilde{A}^I_2}(x_i)))<\delta$. Therefore, for δ thus obtained, let p',q' be such that $\max\{d(p,p'),d(q,q')\}<\delta$. Let us take $\tilde{A}^I_1=\tilde{A}^I_q$, $\tilde{A}^I_2=\tilde{A}^I_{q'}$ and $\tilde{B}^I=\tilde{A}^I_{p'}$. Then

$$\max_{i=1,2,\ldots,n} d((\mu_{\tilde{A}^I_1}(x_i),\nu_{\tilde{A}^I_1}(x_i)),(\mu_{\tilde{A}^I_2}(x_i),\nu_{\tilde{A}^I_2}(x_i)))=d(q,q')<\delta$$

and therefore, from the continuity of Υ, it follows that

$$|\gamma(p',q')-\gamma(p',q)|=|\Upsilon(\tilde{B}^I,\tilde{A}^I_1)-\Upsilon(\tilde{B}^I,\tilde{A}^I_2)|<\frac{\epsilon}{2}$$

Similarly,

$$|\gamma(p',q)-\gamma(p,q)|<\frac{\epsilon}{2}$$

Therefore,

$$|\gamma(p',q')-\gamma(p,q)|\leq|\gamma(p',q')-\gamma(p',q)|+|\gamma(p',q)-\gamma(p,q)|<\epsilon.$$

Thus, γ is continuous.
($\Leftarrow$) Since γ is continuous, for every $\epsilon>0\ \exists\ \delta>0$ such that $|\gamma(q,p')-\gamma(q',p')|<\epsilon$ whenever $d(q,q')<\delta$. Let us put $\tilde{A}^I_1=\tilde{A}^I_q$, $\tilde{A}^I_2=\tilde{A}^I_{q'}$ and $\tilde{B}^I=\tilde{A}^I_{p'}$. Then for every $\tilde{A}^I_1,\tilde{A}^I_2\in\mathcal{I}$, $|\Upsilon(\tilde{A}^I_1,\tilde{B}^I)-\Upsilon(\tilde{A}^I_2,\tilde{B}^I)|=|\gamma(q,p')-\gamma(q',p')|<\epsilon$ for every $\tilde{B}^I\in\mathcal{I}$ whenever $\max\limits_{i=1,2,\ldots,n} d((\mu_{\tilde{A}^I_1}(x_i),\nu_{\tilde{A}^I_1}(x_i)),(\mu_{\tilde{A}^I_2}(x_i),\nu_{\tilde{A}^I_2}(x_i)))<\delta$. □

Definition 5.3.4. **(i)** *((α,β)-cut of IFS) The (α,β)-cut of an IFS $\tilde{A}^I$ is denoted by $A_{(\alpha,\beta)}$ and is defined by*

$$A_{(\alpha,\beta)} = \{x \in X : \mu_{\tilde{A}^I}(x) \geq \alpha, \nu_{\tilde{A}^I}(x) \leq \beta\},$$

where $(\alpha,\beta) \in \mathcal{L}$.

(ii) *The (α,β)-cut of the complement of IFS $\tilde{A}^I$ is denoted by $A'_{(\alpha,\beta)}$ and is defined by*

$$A'_{(\alpha,\beta)} = \{x \in X : \nu_{\tilde{A}^I}(x) \geq \alpha, \mu_{\tilde{A}^I}(x) \leq \beta\},$$

where $(\alpha,\beta) \in \mathcal{L}$.

(iii) *A new type of IFS, derived from (α,β)-cut of a $\tilde{A}'^I$, is denoted by ${}_{(\alpha,\beta)}A'_{(\alpha,\beta)}$ and is defined by*

$${}_{(\alpha,\beta)}A'_{(\alpha,\beta)}(x) = \begin{cases} (\alpha,\beta), & x \in A'_{(\alpha,\beta)}; \\ 0_{\mathcal{L}}, & \textit{otherwise}. \end{cases}$$

For simplicity, we denote $(\alpha,\beta) := \hat{\alpha} \in \mathcal{L}$.

Decomposition Theorem: For every IFS $\tilde{A}^I$,

$$\tilde{A}^I = \underset{\hat{\alpha}\in\mathcal{L}}{\cup} {}_{\hat{\alpha}}A'_{\hat{\alpha}},$$

where ${}_{\hat{\alpha}}A'_{\hat{\alpha}}$ is defined by Definition 5.3.4 and $\cup$ denotes the standard IF union.

Proof. For each point $x \in X$, $\tilde{A}^I(x) = (\mu_{\tilde{A}^I}(x), \nu_{\tilde{A}^I}(x))$ is an element of $\mathcal{L}$. Then

$$\begin{aligned}\left(\underset{\hat{\alpha}\in\mathcal{L}}{\cup} {}_{\hat{\alpha}}A'_{\hat{\alpha}}\right)(x) = \inf_{\alpha\in[0,1]}\left(\sup_{\beta\in[0,1-\alpha]} {}_{\hat{\alpha}}A'_{\hat{\alpha}}(x)\right) = \max\left(\inf_{\alpha\in[0,1]}\left(\sup_{\beta\in[0,\mu_{\tilde{A}^I}(x))} {}_{\hat{\alpha}}A'_{\hat{\alpha}}(x)\right),\right. \\ \left.\inf_{\alpha\in[0,1]}\left(\sup_{\beta\in[\mu_{\tilde{A}^I}(x),1-\alpha]} {}_{\hat{\alpha}}A'_{\hat{\alpha}}(x)\right)\right) = \inf_{\alpha\in[0,1]}\left(\sup_{\beta\in[\mu_{\tilde{A}^I}(x),1-\alpha]} {}_{\hat{\alpha}}A'_{\hat{\alpha}}(x)\right) \\ = \max\left(\inf_{\alpha\in[0,\nu_{\tilde{A}^I}(x)]}\left(\sup_{\beta\in[\mu_{\tilde{A}^I}(x),1-\alpha]} {}_{\hat{\alpha}}A'_{\hat{\alpha}}(x)\right), \inf_{\alpha\in(\nu_{\tilde{A}^I}(x),1]}\left(\sup_{\beta\in[\mu_{\tilde{A}^I}(x),1-\alpha]} {}_{\hat{\alpha}}A'_{\hat{\alpha}}(x)\right)\right) \\ = \inf_{\alpha\in[0,\nu_{\tilde{A}^I}(x)]}\left(\sup_{\beta\in[\mu_{\tilde{A}^I}(x),1-\alpha]} {}_{\hat{\alpha}}A'_{\hat{\alpha}}(x)\right) = (\mu_{\tilde{A}^I}(x), \nu_{\tilde{A}^I}(x)) \\ = \tilde{A}^I(x).\end{aligned}$$

Since the same argument is valid for each $x \in X$, the theorem is proved. □

Theorem 5.3.5. *Let $\Lambda : \mathcal{I} \times \mathcal{I} \to [0,1]$ be an IF normalized divergence measure, IF inclusion measure, IF similarity measure, IF dissimilarity measure or IF normalized distance measure. Then the function $\lambda : \mathcal{L} \times \mathcal{L} \to [0,1]$ defined by*

$$\lambda(\hat{\alpha}, \hat{\hat{\alpha}}) = \Lambda({}_{\hat{\alpha}}A'_{\hat{\alpha}}, {}_{\hat{\hat{\alpha}}}A'_{\hat{\hat{\alpha}}}) \ \forall\ \hat{\alpha}, \hat{\hat{\alpha}} \in \mathcal{L} \text{ and } {}_{\hat{\alpha}}A'_{\hat{\alpha}}, {}_{\hat{\hat{\alpha}}}A'_{\hat{\hat{\alpha}}} \in \mathcal{I} \tag{5.9}$$

is an IF normalized point divergence measure, IF point inclusion measure, IF point similarity measure, IF point dissimilarity measure or IF normalized point distance measure respectively.

Proof. Suppose $\Lambda : \mathcal{I} \times \mathcal{I} \to [0,1]$ is an IF normalized divergence measure. Then

(i) $\Lambda({}_{\hat{\alpha}}A'_{\hat{\alpha}}, {}_{\hat{\hat{\alpha}}}A'_{\hat{\hat{\alpha}}}) = \Lambda({}_{\hat{\hat{\alpha}}}A'_{\hat{\hat{\alpha}}}, {}_{\hat{\alpha}}A'_{\hat{\alpha}}) \Rightarrow \lambda(\hat{\alpha}, \hat{\hat{\alpha}}) = \lambda(\hat{\hat{\alpha}}, \hat{\alpha})$,

(ii) $\lambda(\hat{\alpha}, \hat{\hat{\alpha}}) = 0 \Leftrightarrow \Lambda({}_{\hat{\alpha}}A'_{\hat{\alpha}}, {}_{\hat{\hat{\alpha}}}A'_{\hat{\hat{\alpha}}}) = 0 \Leftrightarrow {}_{\hat{\alpha}}A'_{\hat{\alpha}} = {}_{\hat{\hat{\alpha}}}A'_{\hat{\hat{\alpha}}} \Leftrightarrow \hat{\alpha} = \hat{\hat{\alpha}}$,

(iii) Suppose ${}_{\hat{\alpha}}A'_{\hat{\alpha}}$ is crisp set, i.e., ${}_{\hat{\alpha}}A'_{\hat{\alpha}} = \emptyset$ or X. Then $\Lambda(\emptyset, X) = 1 \Leftrightarrow \lambda(0_{\mathcal{L}}, 1_{\mathcal{L}}) = 1$,

(iv) $\lambda(\hat{\alpha} \wedge \hat{\hat{\alpha}}, \hat{\hat{\alpha}} \wedge \hat{\hat{\alpha}}) = \Lambda({}_{\hat{\alpha}\wedge\hat{\hat{\alpha}}}A'_{\hat{\alpha}\wedge\hat{\hat{\alpha}}}, {}_{\hat{\hat{\alpha}}\wedge\hat{\hat{\alpha}}}A'_{\hat{\hat{\alpha}}\wedge\hat{\hat{\alpha}}}) \leq \Lambda({}_{\hat{\alpha}}A'_{\hat{\alpha}}, {}_{\hat{\hat{\alpha}}}A'_{\hat{\hat{\alpha}}}) = \lambda(\hat{\alpha}, \hat{\hat{\alpha}})$,

(v) $\lambda(\hat{\alpha} \vee \hat{\hat{\alpha}}, \hat{\hat{\alpha}} \vee \hat{\hat{\alpha}}) = \Lambda({}_{\hat{\alpha}\vee\hat{\hat{\alpha}}}A'_{\hat{\alpha}\vee\hat{\hat{\alpha}}}, {}_{\hat{\hat{\alpha}}\vee\hat{\hat{\alpha}}}A'_{\hat{\hat{\alpha}}\vee\hat{\hat{\alpha}}}) \leq \Lambda({}_{\hat{\alpha}}A'_{\hat{\alpha}}, {}_{\hat{\hat{\alpha}}}A'_{\hat{\hat{\alpha}}}) = \lambda(\hat{\alpha}, \hat{\hat{\alpha}})$.

Thus, $\lambda : \mathcal{L} \times \mathcal{L} \to [0,1]$ is an IF normalized point divergence measure.
Similarly, if $\Lambda : \mathcal{I} \times \mathcal{I} \to [0,1]$ is an IF-inclusion measure, IF-similarity measure, IF-dissimilarity measure or IF normalized distance measure, then we can easily prove that $\lambda : \mathcal{L} \times \mathcal{L} \to [0,1]$ is an IF point inclusion measure, IF point similarity measure, IF point dissimilarity measure or IF normalized point distance measure respectively. □

Theorem 5.3.6. *Let λ be an IF normalized point divergence measure, IF point similarity measure, IF point dissimilarity measure or IF normalized point distance measure respectively obtained from the corresponding IF measure as given in Theorem 5.3.5. Then Λ is continuous iff λ is continuous.*

Proof. ($\Rightarrow$) Suppose Λ is continuous. Then for every $\frac{\epsilon}{2} > 0 \ \exists\, \delta > 0$ such that for every $\tilde{A}_1^I, \tilde{A}_2^I \in \mathcal{I}$, $|\Lambda(\tilde{A}_1^I, \tilde{B}^I) - \Lambda(\tilde{A}_2^I, \tilde{B}^I)| < \frac{\epsilon}{2}$ for every $\tilde{B}^I \in \mathcal{I}$ whenever $\max_{i=1,2,\ldots,n} d((\mu_{\tilde{A}_1^I}(x_i), \nu_{\tilde{A}_1^I}(x_i)), (\mu_{\tilde{A}_2^I}(x_i), \nu_{\tilde{A}_2^I}(x_i))) < \delta$. Therefore, for δ thus obtained, let $\hat{\alpha}', \hat{\hat{\alpha}}'$ be such that $\max\{d(\hat{\alpha}, \hat{\alpha}'), d(\hat{\hat{\alpha}}, \hat{\hat{\alpha}}')\} < \delta$.
Let us take $\tilde{A}_1^I = {}_{\hat{\hat{\alpha}}}A'_{\hat{\hat{\alpha}}}$, $\tilde{A}_2^I = {}_{\hat{\hat{\alpha}}'}A'_{\hat{\hat{\alpha}}'}$ and $\tilde{B}^I = {}_{\hat{\alpha}'}A'_{\hat{\alpha}'}$. Then

$$\max_{i=1,2,\ldots,n} d((\mu_{\tilde{A}_1^I}(x_i), \nu_{\tilde{A}_1^I}(x_i)), (\mu_{\tilde{A}_2^I}(x_i), \nu_{\tilde{A}_2^I}(x_i))) = d(\hat{\hat{\alpha}}, \hat{\hat{\alpha}}') < \delta$$

and therefore, from the continuity of Υ, it follows that

$$|\lambda(\hat{\alpha}', \hat{\hat{\alpha}}') - \lambda(\hat{\alpha}', \hat{\hat{\alpha}})| = |\Lambda(\tilde{B}^I, \tilde{A}_1^I) - \Lambda(\tilde{B}^I, \tilde{A}_2^I)| < \frac{\epsilon}{2}$$

similarly,

$$|\lambda(\hat{\alpha}', \hat{\hat{\alpha}}) - \lambda(\hat{\alpha}, \hat{\hat{\alpha}})| < \frac{\epsilon}{2}$$

Therefore,

$$|\lambda(\hat{\alpha}',\hat{\hat{\alpha}}') - \lambda(\hat{\alpha},\hat{\hat{\alpha}})| \leq |\lambda(\hat{\alpha}',\hat{\hat{\alpha}}') - \lambda(\hat{\alpha}',\hat{\hat{\alpha}})| + |\lambda(\hat{\alpha}',\hat{\hat{\alpha}}) - \lambda(\hat{\alpha},\hat{\hat{\alpha}})| < \epsilon$$

Thus, λ is continuous.

($\Leftarrow$) Since λ is continuous, for every $\epsilon > 0$ $\exists$ $\delta > 0$ such that $|\lambda(\hat{\hat{\alpha}},\hat{\alpha}') - \lambda(\hat{\hat{\alpha}}',\hat{\alpha}')| < \epsilon$ whenever $d(\hat{\hat{\alpha}},\hat{\hat{\alpha}}') < \delta$. Let us put $\tilde{A}^I_1 = {}_{\hat{\alpha}}A'_{\hat{\hat{\alpha}}}$, $\tilde{A}^I_2 = {}_{\hat{\alpha}'}A'_{\hat{\hat{\alpha}}'}$ and $\tilde{B}^I = {}_{\hat{\alpha}'}A'_{\hat{\alpha}'}$. Then for every $\tilde{A}^I_1, \tilde{A}^I_2 \in \mathcal{I}$, $|\Lambda(\tilde{A}^I_1, \tilde{B}^I) - \Lambda(\tilde{A}^I_2, \tilde{B}^I)| \leq |\lambda(\hat{\hat{\alpha}},\hat{\alpha}')$- $\lambda(\hat{\hat{\alpha}}',\hat{\alpha}')| < \epsilon$ for every $\tilde{B}^I \in \mathcal{I}$ whenever $\max\limits_{i=1,2,\ldots,n} d((\mu_{\tilde{A}^I_1}(x_i)$, $\nu_{\tilde{A}^I_1}(x_i))$, $(\mu_{\tilde{A}^I_2}(x_i)$, $\nu_{\tilde{A}^I_2}(x_i))) < \delta$. □

Definition 5.3.7. *A function $M : [0,1]^n \to [0,1]$ is called an aggregation operator if it satisfies the following conditions:*

(i) *M(0, 0, ..., 0) = 0;*

(ii) *M(1, 1, ..., 1) = 1;*

(iii) *M is monotonic non-decreasing in each arguments.*

An aggregation operator $M : [0,1]^2 \to [0,1]$ is called a binary aggregation operator.

Example 5.3.8. *[76] Let $x_1, x_2, \ldots, x_n \in [0,1]$. Then*

(i) *the weighted arithmetic mean of $x_1, x_2, \ldots, x_n$ is defined as*
$\sum_{i=1}^n w_i x_i$, *where* $0 \leq w_i \leq 1$, $\sum_{i=1}^n w_i = 1$.

(ii) *the weighted geometric mean of $x_1, x_2, \ldots, x_n$ is defined as*
$\prod_{i=1}^n x_i^{w_i}$, *where* $0 \leq w_i \leq 1$, $\sum_{i=1}^n w_i = 1$.

(iii) *the gamma operator of $x_1, x_2, \ldots, x_n$ is defined as*

$$\big(\prod_{i=1}^n x_i\big)^{1-\gamma}\big(1 - \prod_{i=1}^n (1 - x_i)\big)^{\gamma},\ \gamma \in [0,1].$$

Remark 5.3.9. *([48], [53], [114], [198]) Different considered information measures introduced in this chapter for IFS were originally introduced for FSs, using the same axiomatic.*

Proposition 5.3.10. *Let ψ_1, ψ_2 be two normalized point divergence measures, point inclusion measures, point similarity measures, point dissimilarity measures or normalized point divergence measures for FSs on X and let a map $M : [0,1] \times [0,1] \to [0,1]$ be a binary aggregation operator. Then the function $\Psi : \mathcal{I} \times \mathcal{I} \to [0,1]$ defined by*

$$\Psi(\tilde{A}^I, \tilde{B}^I) = M(\psi_1(\mu_{\tilde{A}^I}(x), \mu_{\tilde{B}^I}(x)), \psi_2(\nu_{\tilde{B}^I}(x), \nu_{\tilde{A}^I}(x))) \tag{5.10}$$

$\forall\ \tilde{A}^I,\ \tilde{B}^I\ \in \mathcal{I},\ x \in X$, is an IF normalized divergence measure, IF-inclusion measure, IF-similarity measure, IF-dissimilarity measure or IF normalized distance measure respectively.

Proof. Suppose $\psi_1,\ \psi_2$ are two normalized point fuzzy divergence measures and $M : [0,1] \times [0,1] \to [0,1]$ is a binary aggregation operator. Then

(i) if ψ_1, ψ_2 are symmetric (i.e., all mentioned inform measures but not inclusion measures), then obviously it holds $\Psi(\tilde{A}^I, \tilde{B}^I) = \Psi(\tilde{B}^I, \tilde{A}^I)$,

(ii) $\Psi(\tilde{A}^I, \tilde{A}^I) = M(0,0) = 0$,

(iii) Suppose $\tilde{A}^I$ is crisp set, i.e., $\tilde{A}^I = \emptyset$ or X. Then $\Psi(\tilde{A}^I, \tilde{A}'^I) = M(1,1) = 1$,

(iv) $\Psi(\tilde{A}^I \cap \tilde{C}^I, \tilde{B}^I \cap \tilde{C}^I) = M(\psi_1(\mu_{\tilde{A}^I}(x) \wedge \mu_{\tilde{C}^I}(x),\ \mu_{\tilde{B}^I}(x) \wedge \mu_{\tilde{C}^I}(x)),\ \psi_2(\nu_{\tilde{B}^I}(x) \vee \nu_{\tilde{C}^I}(x),\ \nu_{\tilde{A}^I}(x) \vee \nu_{\tilde{C}^I}(x)) \leq M(\psi_1(\mu_{\tilde{A}^I}(x),\ \mu_{\tilde{B}^I}(x)),\ \psi_2(\nu_{\tilde{B}^I}(x),\ \nu_{\tilde{A}^I}(x))) = \Psi(\tilde{A}^I, \tilde{B}^I)$, (iv) $\Psi(\tilde{A}^I \cup \tilde{C}^I, \tilde{B}^I \cup \tilde{C}^I) = M(\psi_1(\mu_{\tilde{A}^I}(x) \vee \mu_{\tilde{C}^I}(x),\ \mu_{\tilde{B}^I}(x) \vee \mu_{\tilde{C}^I}(x)),\ \psi_2(\nu_{\tilde{B}^I}(x) \wedge \nu_{\tilde{C}^I}(x),\ \nu_{\tilde{A}^I}(x) \wedge \nu_{\tilde{C}^I}(x))) \leq M(\psi_1(\mu_{\tilde{A}^I}(x),\ \mu_{\tilde{B}^I}(x)),\ \psi_2(\nu_{\tilde{B}^I}(x),\ \nu_{\tilde{A}^I}(x))) = \Psi(\tilde{A}^I, \tilde{B}^I)$.

Thus, $\Psi : \mathcal{I} \times \mathcal{I} \to [0,1]$ defined by (5.10) is an IF normalized divergence measure.
Similarly, if $\psi_1, \psi_2 : \mathcal{I} \times \mathcal{I} \to [0,1]$ are the fuzzy point inclusion measures, fuzzy point similarity measures, fuzzy point dissimilarity measures and fuzzy normalized point divergence measures and $M : [0,1]^2 \to [0,1]$ is a binary aggregation operator. Then the function $\Psi : \mathcal{I} \times \mathcal{I} \to [0,1]$ defined by (5.10) is an IF inclusion measure, IF similarity measure, IF dissimilarity measure and IF normalized distance measure respectively. □

Theorem 5.3.11. *Let $\Gamma_1, \Gamma_2, \ldots, \Gamma_n : \mathcal{I} \times \mathcal{I} \to [0,1]$ be the IF normalized divergence measures, IF inclusion measures, IF similarity measures, IF dissimilarity measures or IF normalized distance measures, and $M : [0,1]^n \to [0,1]$ be an aggregation operator. Then the function $\Gamma : \mathcal{I} \times \mathcal{I} \to [0,1]$ defined by*

$$\Gamma(\tilde{A}^I, \tilde{B}^I) = M(\Gamma_1(\tilde{A}^I, \tilde{B}^I), \Gamma_2(\tilde{A}^I, \tilde{B}^I), \ldots, \Gamma_n(\tilde{A}^I, \tilde{B}^I)) \tag{5.11}$$

for all $\tilde{A}^I, \tilde{B}^I \in \mathcal{I}$, is an IF normalized divergence measure, IF inclusion measure, IF similarity measure, IF dissimilarity measure or IF normalized distance measure respectively.

Proof. Suppose $\Gamma_1, \Gamma_2, \ldots, \Gamma_n : \mathcal{I} \times \mathcal{I} \to [0,1]$ are the IF normalized divergence measures and $M : [0,1]^n \to [0,1]$ is an aggregation operator. Then

(i) the symmetricity of $\Gamma_1, \Gamma_2, \ldots, \Gamma_n$ follows that $\Gamma(\tilde{A}^I, \tilde{B}^I) = \Gamma(\tilde{B}^I, \tilde{A}^I)$,

(ii) $\Gamma(\tilde{A}^I, \tilde{A}^I) = M(0, 0, \ldots, 0) = 0$,

(iii) Suppose $\tilde{A}^I$ is crisp set, i.e., $\tilde{A}^I = \emptyset$ or X. Then $\Gamma(\tilde{A}^I, \tilde{A}'^I) = M(1, 1, \ldots, 1) = 1$,

(iv) the each Γ_i is an IF normalized divergence measure and M is a non-decreasing corresponding to each argument follow that $\Gamma(\tilde{A}^I \cap \tilde{C}^I, \tilde{B}^I \cap \tilde{C}^I) \leq \Gamma(\tilde{A}^I, \tilde{B}^I)$ for every $\tilde{C}^I \in \mathcal{I}$,

(v) the non-decreasing corresponding to each argument of M and each IF normalized divergence measure Γ_i follow that $\Gamma(\tilde{A}^I \cup \tilde{C}^I, \tilde{B}^I \cup \tilde{C}^I) \leq \Gamma(\tilde{A}^I, \tilde{B}^I)$ for every $\tilde{C}^I \in \mathcal{I}$.

Thus, $\Gamma : \mathcal{I} \times \mathcal{I} \to [0, 1]$ defined by (5.11) is an IF normalized divergence measure.
Similarly, if $\Gamma_1, \Gamma_2, \ldots, \Gamma_n : \mathcal{I} \times \mathcal{I} \to [0, 1]$ are the IF inclusion measures, IF similarity measures, IF dissimilarity measures and IF normalized distance measures, and $M : [0, 1]^n \to [0, 1]$ is an aggregation operator. Then the function $\Gamma : \mathcal{I} \times \mathcal{I} \to [0, 1]$ defined by (5.11) is an IF inclusion measure, IF similarity measure, IF dissimilarity measure and IF normalized distance measure respectively. □

Theorem 5.3.12. *Let $\Gamma_1, \Gamma_2, \ldots, \Gamma_n : \mathcal{I} \times \mathcal{I} \to [0, 1]$ be continuous, and $\Gamma : \mathcal{I} \times \mathcal{I} \to [0, 1]$ be given by (5.11), where all symbols and notation are same as Theorem 5.3.11. Then Γ is continuous iff M is continuous.*

Proof. ($\Rightarrow$) The continuity of $\Gamma_1, \Gamma_2, \ldots, \Gamma_n, M$ and (5.11) follow that Γ is continuous.
($\Leftarrow$) Let's see the converse. Since Γ is continuous, for every $\epsilon > 0 \ \exists \ \delta > 0$ such that for every $\tilde{A}^I_1, \ \tilde{A}^I_2 \in \mathcal{I}$,

$$|\Gamma(\tilde{A}^I_1, \tilde{B}^I) - \Gamma(\tilde{A}^I_1, \tilde{B}^I)| = |M(\Gamma_1(\tilde{A}^I_1, \tilde{B}^I), \Gamma_2(\tilde{A}^I_1, \tilde{B}^I), \ldots, \Gamma_n(\tilde{A}^I_1, \tilde{B}^I)) - M(\Gamma_1(\tilde{A}^I_2, \tilde{B}^I), \Gamma_2(\tilde{A}^I_2, \tilde{B}^I), \ldots, \Gamma_n(\tilde{A}^I_2, \tilde{B}^I))| < \epsilon$$

for every $\tilde{B}^I \in \mathcal{I}$ whenever $\max_{i=1,2,\ldots,n} d((\mu_{\tilde{A}^I_1}(x_i), \ \nu_{\tilde{A}^I_1}(x_i)), \ (\mu_{\tilde{A}^I_2}(x_i), \ \nu_{\tilde{A}^I_2}(x_i))) < \delta$.
Putting $\Gamma_i(\tilde{A}^I_1, \ \tilde{B}^I) = a_i$ and $\Gamma_i(\tilde{A}^I_2, \ \tilde{B}^I) = b_i$ for every $\tilde{B}^I \in \mathcal{I}$. Then

$$|M(\Gamma_1(\tilde{A}^I_1, \tilde{B}^I), \Gamma_2(\tilde{A}^I_1, \tilde{B}^I), \ldots, \Gamma_n(\tilde{A}^I_1, \tilde{B}^I)) - M(\Gamma_1(\tilde{A}^I_2, \tilde{B}^I), \Gamma_2(\tilde{A}^I_2, \tilde{B}^I), \ldots, \Gamma_n(\tilde{A}^I_2, \tilde{B}^I))| = |M(a_1, a_2, \ldots, a_n) - M(b_1, b_2, \ldots, b_n)| < \epsilon$$

for every $\tilde{B}^I \in \mathcal{I}$ whenever $\max_{i=1,2,\ldots,n} d((\mu_{\tilde{A}^I_1}(x_i), \ \nu_{\tilde{A}^I_1}(x_i)), \ (\mu_{\tilde{A}^I_2}(x_i), \ \nu_{\tilde{A}^I_2}(x_i))) < \delta$.
But from the continuity of each Γ_i, for every $\delta_0 > 0 \ \exists \ \delta > 0$ such that for every $\tilde{A}^I_1, \ \tilde{A}^I_2 \in \mathcal{I}$,

$$|\Gamma_i(\tilde{A}^I_1, \tilde{B}^I) - \Gamma_i(\tilde{A}^I_1, \tilde{B}^I)| = |a_i - b_i| < \delta_0$$

for every $\tilde{B}^I \in \mathcal{I}$ whenever $\max_{i=1,2,\ldots,n} d((\mu_{\tilde{A}^I_1}(x_i), \ \nu_{\tilde{A}^I_1}(x_i)), \ (\mu_{\tilde{A}^I_2}(x_i), \ \nu_{\tilde{A}^I_2}(x_i))) < \delta$. The result follows. □

5.4 Formulation of Decision Making Problem

We check whether the newly constructed information measures, namely, IF normalized divergence measure, IF normalized divergence measure, IF-dissimilarity measure or similar measures, produce reasonable and reliable results, when they are applied in solving MADM/MCDM problems. Let $I_m = \{1, 2, 3, \ldots, i, \ldots, m\}$ and $J_n = \{1, 2, 3, \ldots, j, \ldots, n\}$. Here, we take the MADM problems with different preferences with IF values and alternatives weights. Mathematically, let $\tilde{R} = (\tilde{r}_{ij})_{m\times n}$ be the IF values decision matrix, where each $\tilde{r}_{ij} = (a_{ij}, b_{ij})$ is the value of $\mathcal{L}$ corresponding to each alternative $A_i (i \in I_m)$ with respect to each attribute $C_j (j \in J_n)$ and let $w = (w_1, w_2, \ldots, w_n)^T$ be the weight vector of attributes, where $\sum_{k=1}^{n} w_k = 1$, $w_k \geq 0$, $k = 1, 2, \ldots, n$.. We calculate the IF normalized distance measure, IF normalized divergence measure, IF-dissimilarity measure or similar equivalent measures between $\tilde{r}_{ij}$ and $(1/2, 1/2)$ and naming its r_{ij}.
Let M be the weighted aggregation operator. For simplicity, here, we take M as additive or multiplicative weighted aggregation operator, i.e., $\sum_{j=1}^{n} w_j r_{ij}$ or $\prod_{j=1}^{n} r_{ij}^{w_j}$.

Definition 5.4.1. *The alternative A_k is preferred to the alternative A_l , we write $A_k \succ_w A_l$, if there are $w_1, w_2, \ldots, w_n$ such that*

$$M((w_1, r_{k1}), (w_2, r_{k2}), \ldots, (w_j, r_{kj}), \ldots, (w_n, r_{kn}))-$$
$$M((w_1, r_{l1}), (w_2, r_{l2}), \ldots, (w_j, r_{lj}), \ldots, (w_n, r_{ln})) \geq 0,$$

where $\sum_{t=1}^{n} w_t = 1$, $w_t \geq 0$, $t \in J_n$, and M is the additive or multiplicative weighted aggregation operator, i.e., $\sum_{j=1}^{n} w_j r_{ij}$ or $\prod_{j=1}^{n} r_{ij}^{w_j}$.

5.5 Structure of the Weight-Set for one preference

The mathematical model for finding the lower and upper value of attribute weights, when the one preference of alternates are given, namely, $A_k \succ_w A_l$, is given by

$$(CP^L) = \begin{cases} \min\ w \\ s.t.\ M((w_1, r_{k1}), (w_2, r_{k2}), \ldots, (w_j, r_{kj}), \ldots, (w_n, r_{kn}))- \\ \qquad M((w_1, r_{l1}), (w_2, r_{l2}), \ldots, (w_j, r_{lj}), \ldots, (w_n, r_{ln})) \geq 0, \\ \sum_{t=1}^{n} w_t = 1, \quad w_t \geq 0, \quad t \in J_n, \end{cases}$$

and

$$(CP^U) = \begin{cases} \max \ w \\ s.t. \ M((w_1, r_{k1}), (w_2, r_{k2}), \ldots, (w_j, r_{kj}), \ldots, (w_n, r_{kn})) - \\ \qquad M((w_1, r_{l1}), (w_2, r_{l2}), \ldots, (w_j, r_{lj}), \ldots, (w_n, r_{ln})) \geq 0, \\ \sum_{t=1}^{n} w_t = 1, \quad w_t \geq 0, \quad t \in J_n, \end{cases}$$

where M is the additive or multiplicative weighted aggregation operator, i.e., $\sum_{j=1}^{n} w_j r_{ij}$ or $\prod_{j=1}^{n} r_{ij}^{w_j}$.

Remark 5.5.1. *(i) Let M be the additive weighted aggregation operator, i.e., $M((w_1, r_{i1}), (w_2, r_{i2}), \ldots, (w_n, r_{in})) = \sum_{j=1}^{n} w_j r_{ij}$. Then $w(k,l) = \{w : rw = \sum_{t=1}^{n} r_t w_t \geq 0, \sum_{t=1}^{n} w_t = 1, w_t \geq 0, t \in J_n\}$, where $r = (r_1, r_2, \ldots, r_n)$, $r_t = r_{kt} - r_{lt}, t \in J_n$. $w(k,l)$ is called the weight-set for $A_k \succ_w A_l$ corresponding to the additive weighted aggregation operator.*

(ii) Let M be the multiplicative weighted aggregation operator, i.e., $M((w_1, r_{i1}), (w_2, r_{i2}), \ldots, (w_n, r_{in})) = \prod_{j=1}^{n} r_{ij}^{w_j}$. Then $w(k,l) = \{w : rw = \sum_{t=1}^{n} r_t w_t \geq 0, \sum_{t=1}^{n} w_t = 1, w_t \geq 0, t \in J_n\}$, where $r = (r_1, r_2, \ldots, r_n)$, $r_t = \ln r_{kt} - \ln r_{lt}, \ t \in J_n$. $w(k,l)$ is called the weight-set for $A_k \succ_w A_l$ corresponding to the multiplicative weighted aggregation operator.

Lemma 5.5.2. *The polyhedron $w(k,l)$, given in Remark 5.5.1, is bounded and convex.*

Proof. Since

$$w(k,l) \subset E_+^n := \{w : w_t \geq 0, t \in J_n\},$$

and E_+^n is bounded, $w(k,l)$ is bounded.

Let $w, w' \in w(k,l)$. Then

$$rw = \sum_{t=1}^{n} r_t w_t \geq 0, \sum_{t=1}^{n} w_t = 1, w_t \geq 0, t \in J_n,$$
$$rw' = \sum_{t=1}^{n} r_t w'_t \geq 0, \sum_{t=1}^{n} w'_t = 1, w'_t \geq 0, t \in J_n.$$

Now, ς

$$\begin{aligned} \varsigma rw + \varsigma' rw' &= \sum_{t=1}^{n} \varsigma r_t w_t + \varsigma' r_t w'_t \\ &= \sum_{t=1}^{n} r_t (\varsigma w_t + \varsigma' w'_t) \geq 0, \end{aligned}$$

where $\varsigma, \varsigma' \geq 0$ and $\varsigma + \varsigma' = 1$.

$$\begin{aligned}\sum_{t=1}^{n} \varsigma w_t + \varsigma' w'_t =& \varsigma \sum_{t=1}^{n} r_t w_t + \varsigma' \sum_{t=1}^{n} r_t w'_t \\ =& \varsigma + \varsigma' = 1.\end{aligned}$$

Therefore, $w(k,l)$ is convex.

Thus, $w(k,l)$ is bounded and convex. □

Lemma 5.5.3. *[155] If $w(k,l) \neq \emptyset$, then $w(k,l)$ has a finite number of extreme points.*

Our aim is to determine attribute weights $w_1, w_2, \ldots, w_n$.

Let $q^1, q^2, \ldots, q^s$ be the extreme points of $w(k,l)$. Then $w(k,l)$ can be written as a convex linear combination of the extreme points [155], we have

$$w(k,l) = \left\{ \sum_{h=1}^{s} q^h \xi_h : \sum_{h=1}^{s} \xi_h = 1, \xi_h \geq 0, h = 1, 2, \ldots, s \right\}$$

Consider the following constrained conditions:

$$\left.\begin{aligned} & w_1 + w_2 + \cdots + w_n = 1 \\ & r_1 w_1 + r_2 w_2 + \cdots + r_n w_n \geq 0 \\ & w_1 \geq 0, w_2 \geq 0, \ldots, w_n \geq 0 \end{aligned}\right\} \tag{5.12}$$

(5.12) can be written in the standard form as

$$\left.\begin{aligned} & w_1 + w_2 + \cdots + w_n = 1 \\ & r_1 w_1 + r_2 w_2 + \cdots + r_n w_n - z = 0 \\ & w_1 \geq 0, w_2 \geq 0, \ldots, w_n \geq 0, z \geq 0. \end{aligned}\right\} \tag{5.13}$$

We need the following results to determine $w_1, w_2, \ldots, w_n$.

Lemma 5.5.4. *[155] Every basic feasible solution of the system (5.13) is an extreme point of the convex set of the feasible solutions and conversely.*

Let $\begin{bmatrix} 1 & 1 \\ r_t & r_{t'} \end{bmatrix}^{-1}$, $\begin{bmatrix} 1 & 1 \\ r_t & r_{t'} \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \geq 0$; or $\begin{bmatrix} 1 & 0 \\ r_t & -1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \geq 0$, $t, t' \in J_n$, $t < t'$ exist. Then a basic feasible solution (i.e., an extreme point) of $w(k,l)$ can be obtained. Thus, we have the following Lemmas 5.5.5 and 5.5.6.

Lemma 5.5.5. *If $r_t \neq r_{t'}$ and $r_t r_{t'} < 0$ for $t < t'$, $t, t' \in J_n$, then $q = \left(0, 0, \ldots, 0, \frac{r_{t'}}{r_{t'} - r_t}, 0, \ldots, 0, \frac{-r_t}{r_{t'} - r_t}, 0, \ldots, 0\right)^T \in E^n$ is an extreme point of $w(k,l)$, where $\frac{r_{t'}}{r_{t'} - r_t}$ and $\frac{-r_t}{r_{t'} - r_t}$ are the t-th and t'-th components of q respectively.*

Proof. Let $R_E = \begin{bmatrix} 1 & 1 & \dots & 1 & \dots & 1 & \dots & 1 & 0 \\ r_1 & r_2 & \dots & r_t & \dots & r_{t'} & \dots & r_n & -1 \end{bmatrix}$,
$w_E = \begin{bmatrix} w_1 & w_2 & \dots & w_t & \dots & w_{t'} & \dots & w_n & z \end{bmatrix}$, where $w_1 \geq 0, w_2 \geq 0, \dots, w_n \geq 0, z \geq 0$. Then the $w(k,l)$ is written as

$$R_E w_E^T = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Since $r_t \neq r_{t'}$, $t, t' \in J_n$, $t < t'$,

$$\begin{bmatrix} 1 & 1 \\ r_t & r_{t'} \end{bmatrix}^{-1} \quad \text{exist.}$$

Since $\frac{r_{t'}}{r_{t'}-r_t} \geq 0$, $\frac{-r_t}{r_{t'}-r_t} \geq 0$, $t, t' \in J_n$, $t < t'$,

$$\begin{bmatrix} 1 & 1 \\ r_t & r_{t'} \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{r_{t'} - r_t} \begin{bmatrix} r_{t'} & -1 \\ -r_t & 1 \end{bmatrix} = \begin{bmatrix} \frac{r_{t'}}{r_{t'}-r_t} \\ \frac{-r_t}{r_{t'}-r_t} \end{bmatrix} \geq \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Thus, $q = \left(0, 0, \dots, 0, \frac{r_{t'}}{r_{t'}-r_t}, 0, \dots, 0, \frac{-r_t}{r_{t'}-r_t}, 0, \dots, 0\right)^T \in E^n$ is an extreme point of $w(k,l)$. □

Lemma 5.5.6. *If $r_t \geq 0$, $t \in J_n$, then*

$$q = e_t = (0, 0, \dots, 1, \dots, 0)^T \in E^n$$

is an extreme point of $w(k,l)$.

Proof. Since $r_t \geq 0$, $t \in J_n$,

$$\begin{bmatrix} 1 & 0 \\ r_t & -1 \end{bmatrix}^{-1} \quad \text{exist, and} \begin{bmatrix} 1 & 0 \\ r_t & -1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ r_t \end{bmatrix} \geq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

. Thus, $q = (0, 0, \dots, 1, \dots, 0)^T \in E^n$ is an extreme point of $w(k,l)$. □

Lemma 5.5.7. *If $r_t \neq r_{t'}, 1 \leq k < l \leq n$, then the necessary and sufficient condition for $\frac{r_{t'}}{r_{t'}-r_t} \geq 0$, $\frac{-r_t}{r_{t'}-r_t} \geq 0$ is $r_t r_{t'} \leq 0$.*

Proof. (⇒) Suppose that $\frac{r_{t'}}{r_{t'}-r_t} \geq 0$, $\frac{-r_t}{r_{t'}-r_t} \geq 0$. Then

$$r_t r_{t'} \leq 0.$$

(⇐) Suppose that $r_t r_{t'} \leq 0$. Then arises three cases:

Case 1: If $r_{t'} > 0$, $r_t < 0$, then $r_{t'} - r_t > 0$. Thus,

$$\frac{r_{t'}}{r_{t'} - r_t} > 0, \quad \frac{-r_t}{r_{t'} - r_t} > 0.$$

Case 2: If $r_{t'} < 0$, $r_t > 0$, then $r_{t'} - r_t < 0$. Thus,

$$\frac{r_{t'}}{r_{t'} - r_t} > 0, \quad \frac{-r_t}{r_{t'} - r_t} > 0.$$

Case 3: If $r_{t'} = 0$ (or $r_t = 0$), then $r_t \neq 0$ (or $r_{t'} \neq 0$). Thus,

$$\frac{r_{t'}}{r_{t'} - r_t} = 0 \ \left(\text{or } \frac{r_{t'}}{r_{t'} - r_t} = 1\right), \quad \frac{-r_t}{r_{t'} - r_t} = 1 \ \left(\text{or } \frac{-r_t}{r_{t'} - r_t} = 0\right).$$

□

In the following Theorem 5.5.8 and Corollary 5.5.9, we give the necessary and sufficient condition to judge whether the weight-set $w(k, l)$ is empty.

Theorem 5.5.8. *The necessary and sufficient condition for $w(k, l) \neq \emptyset$ is that either there exist t and t', where $t \in J_n$, $r_t \neq r_{t'}$, $r_t r_{t'} \leq 0$, or there exists t, where $t \in J_n$, $r_t \geq 0$.*

Proof. From Lemmas 5.5.6 and 5.5.7, we have

$$r_t \neq r_{t'}, \frac{r_{t'}}{r_{t'} - r_t} > 0, \frac{-r_t}{r_{t'} - r_t} > 0.$$

which are equivalent to the conditions

$$r_t \neq r_{t'}, r_t r_{t'} < 0$$

If $w(k, l) \neq \emptyset$, there exist extreme points, and the extreme points of $w(k, l)$ are given in the forms of Lemmas 5.5.5 and 5.5.6. □

Corollary 5.5.9. *The sufficient and necessary condition for $w(k, l) = \emptyset$ is that there are no t and t', where $t, t' \in J_n$ and $t < t'$, $r_t \neq r_{t'}$, $r_t r_{t'} \leq 0$, or there is no t, where $t \in J_n$, $r_t \geq 0$.*

A structure of the weight-set $w(k, l)$ is given in Theorem 5.5.10.

Theorem 5.5.10. *If $w(k, l) \neq \emptyset$, and $q^1, q^2, \ldots, q^s$ are the extreme points of $w(k, l)$ determined either by Lemma 5.5.5 or 5.5.6, then the weight-set for $A_k \succ_w A_l$ can be written as*

$$w(k, l) = \{\sum_{h=1}^{s} q^h \xi_h : \sum_{h=1}^{s} \xi_h = 1, \xi_h \geq 0, h = 1, 2, \ldots, s\}$$

Proof. Since $w(k,l) \neq \emptyset$, i.e. $w(k,l)$ is a bounded convex polyhedron, $w(k,l)$ can be written as a convex combination of the extreme points [155]. □

Let us consider the following example. Example 5.5.11(ii) is a situation where the condition of Theorem 5.5.8 is not satisfied, i.e., $w(k,l) = \emptyset$. Example 5.5.11(i) is a situation where $w(k,l) \neq \emptyset$.

Example 5.5.11. *The decision information of a MADM problem with four attributes ($n = 4$) and four alternatives ($m = 4$) is given in Table 5.1.*

(i) *We consider $A_2 \succ_w A_3$. Then from Table 5.1, we have* $\bar{r}_1 = (0.2, 0, -0.1, 0.3)$. *Since* $r_1 = 0.2 > 0$, $r_2 = 0$, $r_3 = -0.1 < 0$, $r_4 = 0.3 > 0$, *by Lemmas 5.5.5 and 5.5.6, we get*

$$q^1 = (1,0,0,0)^T, q^2 = (0,1,0,0)^T, q^3 = (0,0,0,1)^T, q^4 = (1/3,0,2/3,0)^T, q^5 = (0,0,3/4,1/4)^T \tag{5.14}$$

are the extreme points of $w(2,3)$. *Thus by theorem 5.5.10, we have*

$$w(2,3) = \{\sum_{h=1}^{s} q^h \xi_h : \sum_{h=1}^{5} \xi_h = 1, \xi_h \geq 0, h = 1,2,3,4,5\}$$

where q^1, q^2, q^3, q^4, q^5 *are given in (5.14).*

(ii) *We consider $A_1 \succ_w A_2$. Then from Table 5.1, we have* $\bar{r}_1 = (-0.1, -0.3, -0.1, -0.1)$. *Since* $r_1 = -0.1 < 0$, $r_2 = -0.3 < 0$, $r_3 = -0.1 < 0$, $r_4 = -0.1 < 0$, *by Corollary 5.5.9, we get* $w(2,3) = \emptyset$. *Thus, it is impossible to satisfy* $A_1 \succ_w A_2$.

Table 5.1: Decision matrix

	C_1	C_2	C_3	C_4
A_1	0.3	0	0.2	0.3
A_2	0.4	0.3	0.3	0.4
A_3	0.2	0.3	0.4	0.1
A_4	0.2	0.3	0.25	0.2

Table 5.2: Decision matrix in IFE

	C_1	C_2	C_3	C_4
A_1	(0.6, 0.2)	(0.5, 0.5)	(0.4, 0.3)	(0.7, 0.2)
A_2	(0.3, 0.3)	(0.2, 0.6)	(0.5, 0.2)	(0.1, 0.8)
A_3	(0.4, 0.4)	(0.6, 0.2)	(0.7, 0.1)	(0.4, 0.5)
A_4	(0.5, 0.3)	(0.8, 0.2)	(0.25, 0.75)	(0.3, 0.7)

5.6 Structure of the Weight-Set for many preferences simultaneously

The mathematical models for finding the lower and upper value of attribute weights, when the many preferences of alternatives are given, namely, $A_{i_h} \succ_w A_{i'_h}$, $i_h, i'_h \in I_m$, $i_h \neq i'_h$,

$h = 1, 2, \ldots, s$, are given by

$$(CP_A^L) = \begin{cases} \min\ w \\ s.t.\ M((w_1, r_{i_h1}), (w_2, r_{i_h2}), \ldots, (w_j, r_{i_hj}), \ldots, \\ \qquad (w_n, r_{i_hn})) - M((w_1, r_{i'_h1}), (w_2, r_{i'_h2}), \ldots, \\ \qquad (w_j, r_{i'_hj}), \ldots, (w_n, r_{i'_hn})) \geq 0,\ h = 1, 2, \ldots, s, \\ \sum_{t=1}^n w_t = 1,\ \ w_t \geq 0,\ \ t \in J_n, \end{cases}$$

and

$$(CP_A^U) = \begin{cases} \max\ w \\ s.t.\ M((w_1, r_{i_h1}), (w_2, r_{i_h2}), \ldots, (w_j, r_{i_hj}), \ldots, \\ \qquad (w_n, r_{i_hn})) - M((w_1, r_{i'_h1}), (w_2, r_{i'_h2}), \ldots, \\ \qquad (w_j, r_{i'_hj}), \ldots, (w_n, r_{i'_hn})) \geq 0,\ h = 1, 2, \ldots, s, \\ \sum_{t=1}^n w_t = 1,\ \ w_t \geq 0,\ \ t \in J_n. \end{cases}$$

We consider the weight-set for satisfying many preference orders of alternatives simultaneously. Note that $i_h, i'_h \in I_m$, $i_h \neq i'_h$, $h = 1, 2, \ldots, s$, and the weight-set for $A_{i_h} \succ_w A_{i'_h}$ is

$$w(i_h, i'_h) = \left\{ w : \sum_{t=1}^n r_{i_ht} w_t \geq \sum_{t=1}^n r_{i'_ht} w_t, \sum_{t=1}^n w_t = 1, w_t \geq 0, t \in J_n \right\}.$$

So the weight set for $A_{i_h} \succ_w A_{i'_h}$, $h = 1, 2, \ldots, s$ is

$$\begin{aligned} w(i_h, i'_h; h = 1, 2, \ldots, s) =& \{w : A_{i_1} \succ_w A_{i'_1}, A_{i_2} \succ_w A_{i'_2}, \ldots, A_{i_t} \succ_w A_{i'_t}\} \\ =& \left\{ w : \sum_{t=1}^n r_{i_ht} w_t \geq \sum_{t=1}^n r_{i'_ht} w_t, h = 1, 2, \ldots, s, \right. \\ & \left. \sum_{t=1}^n w_t = 1, w_t \geq 0, t = 1, 2, \ldots, n \right\}. \end{aligned}$$

Let

$$\begin{aligned} \bar{r}_h =& (r_{i_h^1}, r_{i_h^2}, \ldots, r_{i_h^n}) - (r_{i_h'^1}, r_{i_h'^2}, \ldots, r_{i_h'^n}) \\ =& (r_{i_h^1} - r_{i_h'^1}, r_{i_h^2} - r_{i_h'^2}, \ldots, r_{i_h^n} - r_{i_h'^n}), \quad h = 1, 2, \ldots, s. \end{aligned}$$

We have

$$w(i_h, i'_h; h = 1, 2, \ldots, s) = \{w : \bar{a}_h w \geq 0, h = 1, 2, \ldots, s, \sum_{t=1}^n w_t = 1, w_t \geq 0, t \in J_n\},$$

where $w = (w_1, w_2, \ldots, w_n)^T \in E^n$.

First we consider the weight-set

$$w(i_1, i_1') = \{w : \bar{r}_1 w \geq 0, \sum_{t=1}^{n} w_t = 1, w_t \geq 0, t \in J_n\},$$

By Theorem 5.5.10, we have

$$w(i_1, i_1') = \{Q^1\xi^1 : \sum_{t=1}^{t_1} \xi_t^1 = 1, \xi_t^1 \geq 0, t = 1, 2, \ldots, t_1\},$$

where $Q^1 = (q^{11}, q^{12}, \ldots, q^{1t_1})_{t_0 \times t_1}$, $t_0 = n$ and $q^{11}, q^{12}, \ldots, q^{1t_1}$ are the extreme points of $w(A_{i_1} \succ_w A_{i\prime_1})$.

Now we consider

$$\begin{aligned} w(i_h, i_h'; h = 1, 2) =& \{w : \bar{r}_1 w \geq 0, \bar{r}_2 w \geq 0, \sum_{t=1}^{n} w_t = 1, w \geq 0\} \\ =& \{w : w \in w(i_1, i_1'), \bar{r}_2 w \geq 0\} \\ =& \{w : w = Q^1\lambda^1 \bar{r}_2 w \geq 0, \sum_{t=1}^{t_1} \xi_t^1 = 1, \xi_t^1 \geq 0, t = 1, 2, \ldots, t_1\} \\ =& \{Q^1\xi^1 : (\bar{r}_2 Q^1)\xi^1 \geq 0, \sum_{t=1}^{t_1} \xi_t^1 = 1, \xi_t^1 \geq 0, t = 1, 2, \ldots, t_1\} \end{aligned}$$

Let the convex polyhedron be

$$\Xi^1 = \{\xi^1 : (\bar{r}_2 Q^1)\xi^1 \geq 0, \sum_{t=1}^{t_1} \xi_t^1 = 1, \xi_t^1 \geq 0, t = 1, 2, \ldots, t_1\}$$

We obtain Q^2 in a similar method (see Lemmas 5.5.5–5.5.7), where $Q^2 = (q^{21}, q^{22}, \ldots, q^{2t_2})_{t_1 \times t_2}$ and $q^{21}, q^{22}, \ldots, q^{2k_2}$ are the extreme points of Ξ^1. Here we have

$$\Xi^1 = \{Q^2\xi^2 : \sum_{t=1}^{t_2} \xi_t^2 = 1, \xi_t^2 \geq 0, t = 1, 2, \ldots, t_2\}$$

and

$$\begin{aligned} w(i_h, i_h'; h = 1, 2) =& \{Q^1\xi^1 : \xi^1 \in \Xi^1\} \\ =& \{Q^1 Q^2 \xi^2 : \sum_{t=1}^{t_2} \xi_t^2 = 1, \xi_t^2 \geq 0, t = 1, 2, \ldots, t_2\}. \end{aligned}$$

Similarly, we have

$$w(i_h, i_h'; h = 1, 2, 3) = \{Q^1 Q^2 Q^3 \xi^3 : \sum_{t=1}^{t_3} \xi_t^3 = 1, \xi_t^3 \geq 0, t = 1, 2, \ldots, t_3\},$$

where $Q^3 = (q^{31}, q^{32}, \dots, q^{3t_3})_{t_2\times t_3}$ and $q^{31}, q^{32}, \dots, q^{3t_3}$ are the extreme points of Ξ^2. Here we have

$$\begin{aligned}\Xi^2 =&\{\xi^2 : (\bar{r}_3 Q^1 Q^2)\xi^2 \geq 0, \sum_{t=1}^{t_2} \xi_t^2 = 1, \xi_t^2 \geq 0, t = 1, 2, \dots, t_2\}\\ =&\{Q^3\xi^3 : \sum_{t=1}^{t_3} \xi_t^3 = 1, \xi_t^3 \geq 0, t = 1, 2, \dots, t_3\}.\end{aligned}$$

Hence we obtain the weight-set for satisfying many preference orders of alternatives simultaneously $(A_{i_1} \succ_w A_{i'_1}, A_{i_2} \succ_w A_{i'_2}, \dots, A_{i_s} \succ_w A_{i'_s})$, i.e.,

$$w(i_h, i'_h; h = 1, 2, \dots, s) = \{Q^1 Q^2 \cdots Q^s \xi^s : \sum_{t=1}^{t_s} \xi_t^s = 1, \xi_t^s \geq 0, t = 1, 2, \dots, t_s\},$$

where $Q^t = (q^{s1}, q^{s2}, \dots, q^{st_s})_{t_{s-1}\times t_s}$ and $q^{s1}, q^{s2}, \dots, q^{st_s}$ are the extreme points of Ξ^{s-1}.

$$\begin{aligned}\Xi^{p-1} =&\{\xi^{p-1} : (\bar{r}_p Q^1 Q^2 \cdots Q^{p-1})\xi^{p-1} \geq 0, \sum_{t=1}^{t_{p-1}} \xi_t^{p-1} = 1, \xi_t^{p-1} \geq 0, t = 1, 2, \dots, t_{p-1}\}\\ =&\{Q^p\xi^p : \sum_{t=1}^{t_p} \xi_t^p = 1, \xi_t^p \geq 0, t = 1, 2, \dots, t_p\} \;\; p = 1, 2, \dots, s.\end{aligned}$$

Example 5.6.1. *The decision information of a MADM problem with four attribute attribute (n = 4) and four alternatives (m = 4) is given in Table 5.1. We consider $A_2 \succ_w A_3$ and $A_1 \succ_w A_4$. Then from Table 5.1, we have*
$\bar{r}_1 = (0.2, 0, -0.1, 0.3)$ *and* $\bar{r}_2 = (0.1, -0.3, -0.05, 0.1)$. *From Example 5.5.11, we have Since $r_1 = 0.2 > 0$, $r_2 = 0$, $r_3 = -0.1 < 0$, $r_4 = 0.3 > 0$, by Lemmas 5.5.5 and 5.5.6, we get*

$$\begin{aligned}w(2,3) =&\{w : \bar{r}_1 w \geq 0, w_1 + w_2 + w_3 + w_4 = 1, w_1, w_2, w_3, w_4 \geq 0\}\\ =&\{\sum_{h=1}^{s} q^{1h}\xi_h : \sum_{h=1}^{4} \xi_h = 1, \xi_h \geq 0, h = 1, 2, 3, 4, 5\}\\ =&\{Q^1\xi_h^1 : \sum_{h=1}^{4} \xi_h^1 = 1, \xi_h^1 \geq 0, h = 1, 2, 3, 4, 5\},\end{aligned}$$

where

$$q^{11} = (1, 0, 0, 0)^T, q^{12} = (0, 0, 0, 1)^T, q^{13} = (1/3, 0, 2/3, 0)^T, q^{14} = (0, 0, 3/4, 1/4)^T, q^{15} = (1, 0, 0, 0)^T$$

and

$$Q^1 = \begin{bmatrix} 1 & 0 & 1/3 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 2/3 & 3/4 & 0\\ 0 & 1 & 0 & 1/4 & 1\end{bmatrix}.$$

Note that the weight-set for $A_2 \succ_w A_3$ and $A_1 \succ_w A_4$ is

$$\begin{aligned} w(2,3;1,4) &= \{w : \bar{r}_1 w \geq 0, \bar{r}_2 w \geq 0, w_1 + w_2 + w_3 + w_4 = 1, w_1, w_2, w_3, w_4 \geq 0\} \\ &= \{Q^1 \xi_h^1 : (\bar{r}_2 Q^1)\xi_h^1 \geq 0, \sum_{h=1}^{4} \xi_h^1 = 1, \xi_h^1 \geq 0, h = 1,2,3,4,5\} \end{aligned}$$

Let the convex polyhedron be

$$\Xi_1 = \{\xi_h^1 : (\bar{r}_2 Q^1)\xi_h^1 \geq 0, \sum_{h=1}^{4} \xi_h^1 = 1, \xi_h^1 \geq 0, h = 1,2,3,4,5\}.$$

We have $\bar{r}_2 Q^1 = (0.1, -0.3, 0, -0.0125, 0.1)$. *Since* $0.1 > 0, -0.3 < 0, -0.0125 < 0$, *by Lemma 5.5.6, we get*

$$\begin{aligned} &q^{21} = (1,0,0,0,0)^T, q^{22} = (0,0,1,0,0)^T, q^{23} = (0,0,0,0,1)^T, q^{24} = (3/4, 1/4, 0, 0, 0)^T, \\ &q^{25} = (1/9, 0, 0, 8/9, 0)^T, q^{26} = (0, 1/4, 0, 0, 3/4)^T, q^{24} = (0, 0, 0, 8/9, 1/9)^T \end{aligned}$$

are the extreme points of convex polyhedron Ξ_1. *We have*

$$\Xi_1 = \{Q^2 \xi_h^2 : \sum_{h=1}^{4} \xi_h^2 = 1, \xi_h^2 \geq 0, h = 1,2,3,4,5,6,7\}$$

where

$$Q^2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Finally, we have

$$w(2,3;1,4) = \{Q^1 Q^2 \xi_h^2 : \sum_{h=1}^{4} \xi_h^2 = 1, \xi_h^2 \geq 0, h = 1,2,3,4\}$$

where Q^1, Q^2 *are given above in this problem.*

5.7 Determination of attribute weights for given preferences

In this section, we find the attribute weights in uncertain and certain environments.

5.7.1 Determination of attribute weights in uncertain environment

Based on the structure of the weight-set, the uncertain weight for every attribute can be determined and used for decision analysis. Suppose that the weight-set for $A_i \succ_w A_j$ is

$$w(k,l) = \{\sum_{k=1}^{t} q^h \xi_t : \sum_{k=1}^{t} \xi_t = 1, \xi_t \geq 0, h = 1, 2, \ldots, s\},$$

where $q^1, q^2, \ldots, q^t$ are the extreme points of $w(k,l)$.
Let $q^h = (q_1^k, q_2^k, \ldots, q_n^k)^T$, $h = 1, 2, \ldots, s$. Note w_l^L and w_l^U are the lower bound and upper bound of the $l - th$ element w_l in weight vector w such that $A_i \succ_w A_j$, respectively. w_l^L and w_l^U can be obtained by solving the following linear programming problems (P^L) and (P^U), respectively,

$$(P^L) = \begin{cases} \min \ \sum_{k=1}^{t} q_l^k \xi_t \\ s.t. \ \sum_{k=1}^{t} \xi_t = 1, \\ \xi_t \geq 0, \quad h = 1, 2, \ldots, s, \end{cases}$$

and

$$(P^U) = \begin{cases} \max \ \sum_{k=1}^{t} q_l^k \xi_t \\ s.t. \ \sum_{k=1}^{t} \xi_t = 1, \\ \xi_t \geq 0, \quad h = 1, 2, \ldots, s. \end{cases}$$

Let $\underline{\lambda}^l = (\underline{\lambda}_1^l, \underline{\lambda}_2^l, \ldots, \underline{\lambda}_t^l)^T$ and $\overline{\lambda}^l = (\overline{\lambda}_1^l, \overline{\lambda}_2^l, \ldots, \overline{\lambda}_t^l)^T$ be the optimal solutions of (P^L) and (P^U), respectively. Then w_l^L and w_l^U can be obtained by

$$w_l^L = \min\{\sum_{k=1}^{t} q_l^k \underline{\lambda}_k^l, \sum_{k=1}^{t} q_l^k \overline{\lambda}_k^l\}, \quad w_l^U = \max\{\sum_{k=1}^{t} q_l^k \underline{\lambda}_k^l, \sum_{k=1}^{t} q_l^k \overline{\lambda}_k^l\}.$$

Thus, the uncertain weight vector w for $A_i \succ_w A_j$ can be obtained as

$$[w_1^L, w_1^U], [w_2^L, w_2^U], \ldots, [w_n^L, w_n^U]$$

5.7.2 Determination of attribute weights in certain environments

With the help of obtained uncertain attribute weights in previous section and decision matrix, we model the programming problem for finding certain positive attribute weights, as given below:

$$(CW) = \begin{cases} \max\ w(M((w_1, r_{k1}), (w_2, r_{k2}), \ldots, (w_j, r_{kj}), \ldots, (w_n, r_{kn})), \\ \ldots, M((w_1, r_{m1}), (w_2, r_{m2}), \ldots, (w_j, r_{mj}), \ldots, (w_n, r_{mn}))) \\ s.t.\ \epsilon \le w_t^L \le w_1 \le w_t^U,\ \ t \in J_n, \\ \sum_{t=1}^n w_t = 1, \end{cases}$$

where $\epsilon > 0$ is sufficiently small and M is the additive or multiplicative weighted aggregation operator, i.e., $M((w_1, r_{i1}), (w_2, r_{i2}), \ldots, (w_n, r_{in})) = \sum_{j=1}^n w_j r_{ij}$ or $\prod_{j=1}^n r_{ij}^{w_j}$.

5.8 Selection of alternatives

The selection of alternatives, when one or more preferences of alternatives is given, is summarized in Algorithm 1.

For the reasonability of the proposed algorithm, we consider a numerical example adapted from Das et al. [50]. An investment company wants to invest money in the best possible option. The four possible alternatives to invest money are as follows: (i) A car company (A_1). (ii) A food company (A_2). (iii) A computer company (A_3). (iv) An arms company (A_4).

Algorithm 1 Selection of alternatives

Step 1. Calculate the IF normalized distance measure, IF normalized divergence measure, IF-dissimilarity measure or similar equivalent measures of each $\tilde{r}_{ij}^I$ from $< 1/2, 1/2 >_X$ and naming it as r_{ij}.
Step 2. Let w_j be the weighting value corresponding to the jth attribute.
Step 3. Model mathematical programming problems as problems (CP^L) and (CP^U) for given one preference of alternatives, and (CP_A^L) and (CP_A^U) for given many preferences of alternatives.
Step 4. Model mathematical programming problems as problems (P^L) and (P^U).
Step 5. Calculate the uncertain attribute weights of the problems (P^L) and (P^U).
Step 6. Calculate each w_j of the problem (CW).
Step 7. Aggregate r_{ij} for each alternative into collective overall values r_i by using Definition 5.3.7, Example 5.3.8 and positive attributes weight vector.
Step 8. Rank all the alternatives $A_i (i = 1, 2, ..., m)$ according to the collective overall values $r_i (i = 1, 2, ..., m)$.
Step 9. Select the best alternative from ranked alternatives (as done in Step 8).

The investment company must take a decision according to the following four attributes: The risk analysis (C_1), the growth analysis (C_2), the environmental impact analysis (C_3) and the social, political impact analysis (C_4). The four possible alternatives A_i (i = 1, 2, 3, 4) are to be evaluated using the IF values information by the decision maker under the above four attributes C_j (j = 1, 2, 3, 4) with preferences $A_2 \succ_w A_3$ and $A_1 \succ_w A_4$, as listed in the following matrix.

Then, to find the most desirable alternative(s) based on proposed Algorithm 1.

Applying Algorithm 1 for the preferences $A_2 \succ_w A_3$ and $A_1 \succ_w A_4$, and taking normalized hamming distance H (eq.(5.1)) and $\epsilon = 0.01$, it is observed that used normalized hamming distance H (eq.(5.1)) measure indicates that the alternatives ordering is $A_2 \succ A_1 \succ A_4 \succ A_3$. It is confirmed that A_2 is the best alternative from given set of alternatives $\{A_1, A_2, A_3, A_4\}$.

5.9 Concluding remarks

In this chapter, the definitions of normalized divergence, similarity, dissimilarity, inclusion and normalized distance measures in IFE are analyzed the existing axiomatic. We have established the following: (i) the IF point measures generated from the measures of the standard IFSs constructed by level sets and other special set $\tilde{A}_p^I$ (ii) the measures derived from point measures (iii) aggregated measures from the set of measures, and studied the continuity relation relationship between them. We have given the concept of weights for one and many preferences of alternatives. Also, we have modeled the mathematical programming problems for determining the positive certain attribute weights. Finally, an algorithm is given for the selection of the best alternative from the given set of feasible alternatives with given preferences. A numerical example is given to demonstrate the effectiveness of the proposed algorithm.

Chapter 6

Residual implications on $\mathcal{L}$ based on powers of continuous t-norm

Residual implications constitute a special class of implications on $\mathcal{L}$, which play important roles from theoretical to practical aspects. Many authors investigated various properties of different types of implications on $\mathcal{L}$ and established the interrelationships among them. In this chapter, the powers of a t-norm $\mathcal{T}$ with identical tuple elements on $\mathcal{L}$ are introduced and their properties are studied. More specifically, a new type of implication on $\mathcal{L}$, known as the residual implication is derived from powers of continuous t-norm $\mathcal{T}$, which is denoted by $\mathtt{I}_{\mathtt{I}\mathcal{T}}$ and satisfies certain properties of residual implications by imposing some extra conditions. Moreover, some additional important properties are studied and analyzed. These altogether reveal that they do not intersect the most well-known classes of fuzzy implications. Finally, we investigate the solutions of Boolean-like laws in $\mathtt{I}_{\mathtt{I}\mathcal{T}}$.

6.1 Introduction

The fuzzy implication is equally important from both the theoretical and practical points of view. From the theoretical point of view, the development of algebra is done and their properties are studied. From the practical point of view, the fuzzy implication is used to study approximate reasoning and network problems, etc. (see [19, 106]). Several authors worked on fuzzy connectivity [102], continuity on t-norms and residual implications [99, 101, 103], and fuzzy modeling through grouping, overlap functions and generalized bientropic functions [39]. The concept of Archimedean overlap functions, the ordinal sum of overlap functions and their limiting properties are given in [64]. The cancelation property is useful for t-norms and t-conorms and their brief studies are given in [124]. Development of the powers of t-norms (powers acquire positive real numbers) and their properties are studied for strict and nilpotent cases (see [177]). After that, fuzzy implications have been derived from powers of continuous t-norms and studied their properties in [128]. In [175], $\circledast$-composition of fuzzy implications are given and investigated the properties over fuzzy implications. A special class of fuzzy implication operators known as R-implications (residual implications) is derived from t-norm [11], overlap functions [65] and aggregation operators[141] and properties are studied in brief. The distributive laws of fuzzy implications over overlap and grouping functions are given in [148].

Nowadays, many different extensions of FSs are known e.g., L-FSs proposed by Goguen [80], interval-valued FS proposed by Gorzalczany [81] represents the degree of membership of an element by an interval rather than exact numerical value, intuitionistic fuzzy set (IFS) proposed by Atanassov [11] etc. IFS characterized by membership function and nonmembership function which model the non-determinacy occurs in the system because of the hesitation of decision makers etc. Approximate reasoning on IFSs is studied by triple I method [200] and relating De Morgan triples with intuitionistic De Morgan triples via automorphisms [46]. Mathematically, interval-valued FS and IFS both are equivalent.

It has become one of the most important operators in logic [174]. The arithmetic operators in interval-valued FS theory [55] and other theories, like, interval fuzzy negations [26], generalized interval-valued OWA operators with interval weights derived from interval-valued overlap functions [29], interval additive generators of interval overlap functions and interval grouping functions [147] are developed. The implications in interval valued FS with several properties are developed in [28]. Implications based on binary aggregation operators in interval-valued FS theory are given in [61]. Moreover, the algebraic structures of interval-valued fuzzy (S, N)-implications are developed in [111]. IF t-norms and t-conorms are studied in [59]. The expression, construction, classification and several properties with applications of intuitionistic and interval-valued fuzzy implications are given in [33] and [45].

The following are main motivating facts behind the present work:

(i) In [128], the authors have proposed a new type of residual implication operator on [0, 1], viz., the T-power based implication as follows:

Definition 6.1.1 ([128], Definition 4). *An operator $I^T : [0,1]^2 \to [0,1]$ is called T-power based implication if there is a continuous t-norm T on [0, 1] such that*

$$I^T(a,b) = \sup\{t \in [0,1] : b_T^{(t)} \geq a\} \; \forall \, a, b \in [0,1] \tag{6.1}$$

Following this, the authors have shown that the T-power based implication is fuzzy implication operator (see, Proposition 4 in [128]). But do not necessarily satisfy certain properties such as neutrality property, exchange principle, etc. satisfied by such residual implications, i.e., weaker version of the residual implications. Moreover, they have studied the invariant with respect to T -powers. Further, the authors have proved that they do not intersect the most well-known classes of fuzzy implication operators.

(ii) In [65], the authors have given the new type of R-implication operator on [0, 1] generated by an overlap function as follows:

Definition 6.1.2. *[65] Let O be an overlap function on [0, 1]. Then the operator* $I_O : [0,1]^2 \to [0,1]$ *called the implication operator derived from O is given by*

$$I_O(a,b) = \max\{c \in [0,1] : O(a,c) \leq b\} \; \forall \; a,b \in [0,1] \tag{6.2}$$

Moreover, I_O is a fuzzy implication (see [65]). But this implication is weaker version of residual implication and satisfied certain properties over the residual implications by introducing the some extra conditions.

(iii) In paper [100], the author has solved the long-standing problem related to the continuity of residual implications derived from t-norm, and also have given special type of fuzzy negation as follows:

Definition 6.1.3. *[100] For any fixed* $b_0 \in [0,1)$, *the non-increasing partial function* $I(.,b_0) : [b_0,1] \to [b_0,1]$, *is denoted by* $g^T_{b_0}$. *Observe that (i)* $g^T_{b_0}(b_0) = 1$. *(ii)* $g^T_{b_0}(1) = b_0$. *(iii)* $g^T_{b_0}$ *is non-increasing. I and T are the implication and t-norm on [0, 1] respectively. If* $b_0 = 0$, *then* g^T_0 *is the natural negation associated with the t-norm T.*

(iv) In papers [42] and [43], the authors have given the following Boolean-like laws:

$$b \leq I(a,b), \; I(a,I(b,a)) = 1, \; I(a,b) = I(a,I(a,b)),$$
$$I(a,I(b,c)) = I(I(a,b),I(a,c)) \; \forall \; a,b,c \in [0,1],$$

where I is the fuzzy implications and the solutions of these Boolean-like laws are obtained in (S, N)-, R-, QL-, D-, (T, N)- and h-implications in fuzzy environment. The above Boolean-like laws in IFE are as follows:

$$v \leq_{\mathcal{L}} \mathtt{I}_{\mathtt{I}}(u,v), \; \mathtt{I}_{\mathtt{I}}(u,\mathtt{I}_{\mathtt{I}}(v,u)) = 1_{\mathcal{L}}, \; \mathtt{I}_{\mathtt{I}}(u,v) = \mathtt{I}_{\mathtt{I}}(u,\mathtt{I}_{\mathtt{I}}(u,v)),$$
$$\mathtt{I}_{\mathtt{I}}(u,\mathtt{I}_{\mathtt{I}}(v,w)) = \mathtt{I}_{\mathtt{I}}(\mathtt{I}_{\mathtt{I}}(u,v),\mathtt{I}_{\mathtt{I}}(u,w)) \; \forall \; u,v,w \in \mathcal{L} \tag{6.3}$$

where $\mathtt{I}_{\mathtt{I}}$ is the IF implication.

The above facts have motivated us to take up the study of residual implications on $\mathcal{L}$ generated by powers of continuous t-norm $\mathcal{T}$, denoted by $\mathtt{I}_{\mathtt{I}\mathcal{T}}$, and special negation by $\mathcal{N}^{\alpha}_{\mathtt{I}_{\mathtt{I}\mathcal{T}}}$ and also find the solutions of (6.3) in $\mathtt{I}_{\mathtt{I}\mathcal{T}}$.

In this work, we introduce a definition of the powers of continuous t-norm $\mathcal{T}$ with identical tuple elements whose powers acquire positive real numbers or positive real numbers in [0, 1] or members of $\mathcal{L}$. Inspired by Definitions 6.1.1 and 6.1.2, we define and study $\mathcal{T}$-power-based implications.

The rest of the chapter is organized as follows. In Section 6.2, we introduce some basic definitions needed throughout the chapter. Section 7.4 is devoted to some new definitions of the powers with respect to continuous t-norm and their results; also some new inequalities, linear translation of elements and powers. The core of the chapter is represented by Sections 6.4 and 6.5, concerning the development of $\mathcal{T}$-power-based implication and the proofs of specific results of $\mathcal{T}$-power-based implications. Also, we investigate the solutions of Boolean-like laws in $\mathcal{T}$-power-based implication. Finally, the chapter ends with Section 6.6 containing concluding remarks.

6.2 Preliminaries

In $\mathbb{R}^2$, the well-known metrics, like, the Euclidean distance and the Hamming distance are defined as follows.

- The Euclidean distance between $u = (u_1, u_2)$ and $v = (v_1, v_2)$ in $\mathbb{R}^2$ is given by

$$d^E(u, v) = \sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2}.$$

- The Hamming distance between $u = (u_1, u_2)$ and $v = (v_1, v_2)$ in $\mathbb{R}^2$ is given by

$$d^H(u, v) = |u_1 - v_1| + |u_2 - v_2|.$$

If we restrict these distances to $\mathcal{L}$, then we obtain the metric spaces $(\mathcal{L}, d^E)$ and $(\mathcal{L}, d^H)$, where d^E and d^H are the Euclidean and Hamming distances on $\mathcal{L}$ respectively. Denote, for any $u \in \mathcal{L}$, $u_\pi = 1 - u_1 - u_2$. Szmidt and Kacprzyk [167] have defined two distances on $\mathcal{L}$ based on the Euclidean and the Hamming distances, where also u_π is used.

- The $d^E_{\mathcal{L}}$ between $u = (u_1, u_2)$ and $v = (v_1, v_2)$ in $\mathcal{L}$ is given by

$$d^E(u, v) = \sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2 + (u_\pi - v_\pi)^2}.$$

- The $d^H_{\mathcal{L}}$ between $u = (u_1, u_2)$ and $v = (v_1, v_2)$ in $\mathcal{L}$ is given by

$$d^H(u, v) = |u_1 - v_1| + |u_2 - v_2| + |u_\pi - v_\pi|.$$

Deschrijver et. al [59] proved that these four distances are topologically equivalent. So, the continuity with respect to one of these metric spaces is equivalent to the continuity with respect to any other metric space.

Let $G : \mathcal{L} \to \mathcal{L}$ be an arbitrary mapping. Then G is called IF continuous if $\forall\, \epsilon > 0\ \exists\, \delta > 0$ such that, $\forall\, u, v \in \mathcal{L}$,

$$d(G(u), G(v)) < \epsilon \quad \text{whenever} \quad d(u, v) < \delta \tag{6.4}$$

where d is any of the metric on $\mathcal{L} \times \mathcal{L}$.

This metric $d : \mathcal{L} \times \mathcal{L} \to \mathbb{R}$ is extended for n-ary as:

$${}^n d(({}^1u, {}^2u, \ldots, {}^nu), ({}^1v, {}^2v, \ldots, {}^nv)) = \sqrt{(d({}^1u, {}^1v))^2 + (d({}^2u, {}^2v))^2 + \cdots + (d({}^nu, {}^nv))^2} \tag{6.5}$$

for every $({}^1u, {}^2u, \ldots, {}^nu),\ ({}^1v, {}^2v, \ldots, {}^nv) \in \mathcal{L}^n$.

Let $H : \mathcal{L}^n \to \mathcal{L}$ be an arbitrary mapping. Then H is called continuous if $\forall\, \epsilon > 0\ \exists\, \delta > 0$ such that, $\forall\ ({}^1u, {}^2u, \ldots, {}^nu),\ ({}^1v, {}^2v, \ldots, {}^nv) \in \mathcal{L}^n$,

$$d(H(({}^1u, {}^2u, \ldots, {}^nu)), H(({}^1v, {}^2v, \ldots, {}^nv))) < \epsilon \quad \text{whenever} \quad {}^n d(({}^1u, {}^2u, \ldots, {}^nu), ({}^1v, {}^2v, \ldots, {}^nv)) < \delta \tag{6.6}$$

where d and ${}^n d$ are metrics on $\mathcal{L} \times \mathcal{L}$ and $\mathcal{L}^n \times \mathcal{L}^n$ respectively.

Theorem 6.2.1. *[[59], Theorem 2] Given a t-norm T and t-conorm S on [0, 1] satisfying $T(a, b) \le 1 - S(1 - a, 1 - b)\ \forall\, a, b \in [0, 1]$. Then the mappings $\mathcal{T}$ and $\mathcal{S}$ defined by*

$$\mathcal{T}(u, v) = (T(u_1, v_1), S(u_2, v_2)),\ \mathcal{S}(u, v) = (S(u_1, v_1), T(u_2, v_2)) \text{ for } u = (u_1, u_2), v = (v_1, v_2) \in \mathcal{L}$$

are a t-norm and a t-conorm on $\mathcal{L}$ respectively.

It is well known that associativity property of the t-norms allows to extend each t-norm in IFE in a unique way to a n-dimensional operation as follows:

$$\mathcal{T}({}^1u, \ldots, {}^nu) = \mathcal{T}({}^1u, \mathcal{T}({}^2u, \ldots, \mathcal{T}({}^{n-1}u, {}^nu) \cdots) \tag{6.7}$$

The unicity is in the sense that

$$\mathcal{T}({}^1u, \mathcal{T}({}^2u, \ldots, \mathcal{T}({}^{n-1}u, {}^nu) \cdots) = \mathcal{T}(\mathcal{T}(\cdots \mathcal{T}(\mathcal{T}({}^1u, {}^2u), \ldots, {}^{n-1}u), {}^nu).$$

6.3 Powers with respect to t-norm

This chapter is inspired by the papers [128, 158, 177] concerning the powers of t-norms, the fuzzy implication operators based on powers of continuous t-norms in IFE. Based on these papers, we introduce the powers of continuous t-norm in IFE and powers may be positive real numbers and member of $\mathcal{L}$.

From the associativity of any t-norm $\mathcal{T}$, the positive integer powers with respect to $\mathcal{T}$ can be defined as follows:

$$u_{\mathcal{T}}^{(n)} = \begin{cases} \mathcal{T}(u, u_{\mathcal{T}}^{(n-1)}), & n \geq 2,\ n \in \mathbb{Z}^{+}, \\ u, & n = 1, \\ 1_{\mathcal{L}}, & n = 0, \end{cases}$$

for every $u \in \mathcal{L}$.

Example 6.3.1. *Let us consider a t-norm $\mathcal{T} : \mathcal{L}^2 \to \mathcal{L}$, $\mathcal{T}(u, v) = (\max(0, u_1 + v_1 - 1),$ $\min(1, u_2 + 1 - v_1,\ v_2 + 1 - u_1))$ $\forall$ $u = (u_1, u_2)$, $v = (v_1, v_2) \in \mathcal{L}$. Now we calculate, for $u = (u_1, u_2) \in \mathcal{L}$ and $n \in \mathbb{Z}^{+}$, $u_{\mathcal{T}}^{(n)}$*

$$u_{\mathcal{T}}^{(n)} = \begin{cases} 0_{\mathcal{L}}, & 0 \leq u_1 \leq \frac{n-1}{n}, 1 \geq u_2 \geq \max(0, (n \\ & -1)u_1 - (n-2)), u_1 + u_2 \leq 1; \\ (0, u_2 + (n-1)(1-u_1)), & 0 \leq u_1 \leq \frac{n-1}{n}, 0 \leq u_2 < (n-1)u_1 \\ & -(n-2), u_1 + u_2 \leq 1; \\ (nu_1 - (n-1), u_2 + (n-1)(1-u_1)), & 1 \geq v_1 > \frac{n-1}{n}, 0 \leq u_2 < (n-1)u_1 \\ & -(n-2), u_1 + u_2 \leq 1. \end{cases}$$

Lemma 6.3.2. *Let $\mathcal{T} : \mathcal{L}^2 \to \mathcal{L}$ be a continuous t-norm and $m, n \in \mathbb{Z}^{+}$. Then $\mathcal{T}(u_{\mathcal{T}}^{(m)}, u_{\mathcal{T}}^{(n)}) = u_{\mathcal{T}}^{(m+n)}$ $\forall$ $u \in \mathcal{L}$.*

Proof. Since $\mathcal{T}$ satisfies associativity, $\mathcal{T}(u, \mathcal{T}(v, w)) = \mathcal{T}(\mathcal{T}(u, v), w)$ $\forall$ $u, v, w \in \mathcal{L}$.

By using the definition of the positive integer powers of u with respect to $\mathcal{T}$ and the associative property of $\mathcal{T}$, we have

$$\begin{aligned} \mathcal{T}(u_{\mathcal{T}}^{(m)}, u_{\mathcal{T}}^{(n)}) =& \mathcal{T}(\mathcal{T}(u_{\mathcal{T}}^{(m)}, u), u_{\mathcal{T}}^{(n-1)}) = \mathcal{T}(u_{\mathcal{T}}^{(m+1)}, u_{\mathcal{T}}^{(n-1)}) = \mathcal{T}(\mathcal{T}(u_{\mathcal{T}}^{(m+1)}, u), u_{\mathcal{T}}^{(n-2)}) \\ =& \mathcal{T}(u_{\mathcal{T}}^{(m+2)}, u_{\mathcal{T}}^{(n-2)}) = \mathcal{T}(\mathcal{T}(u_{\mathcal{T}}^{(m+2)}, u), u_{\mathcal{T}}^{(n-3)}) = \mathcal{T}(u_{\mathcal{T}}^{(m+3)}, u_{\mathcal{T}}^{(n-3)}) \\ =& \cdots = \mathcal{T}(\mathcal{T}(u_{\mathcal{T}}^{(m+n-2)}, u), u_{\mathcal{T}}^{(1)}) = \mathcal{T}(u_{\mathcal{T}}^{(m+n-1)}, u) \\ =& u_{\mathcal{T}}^{(m+n)}. \end{aligned}$$

Thus,

$$\mathcal{T}(u_{\mathcal{T}}^{(m)}, u_{\mathcal{T}}^{(n)}) = u_{\mathcal{T}}^{(m+n)} \ \forall \ u \in \mathcal{L}.$$

□

Definition 6.3.3. *The n-th root power of an element $u \in \mathcal{L}$ with respect to a t-norm $\mathcal{T}$ is denoted by $u_{\mathcal{T}}^{(\frac{1}{n})}$ and is defined by*

$$u_{\mathcal{T}}^{(\frac{1}{n})} = \sup\{w \in \mathcal{L} : w_{\mathcal{T}}^{(n)} \leq_{\mathcal{L}} u\} \ \ \forall \ n \in \mathbb{Z}^+ \tag{6.8}$$

Definition 6.3.4. *Let $\mathcal{T} : \mathcal{L}^2 \to \mathcal{L}$ be a continuous t-norm. Then we can defined $x^{(t)}$, for $t > 0$ (t is positive real constant), as follows:*

$$u_{\mathcal{T}}^{(t)} = \sup\{w \in \mathcal{L} : i, j \in \mathbb{N}_0, w_{\mathcal{T}}^{(j)} \leq_{\mathcal{L}} v, v = u_{\mathcal{T}}^{(i)} \text{ and } i/j \leq t\}.$$

Lemma 6.3.5. *Let $\mathcal{T} : \mathcal{L}^2 \to \mathcal{L}$ be a continuous t-norm and $m, n, k \in \mathbb{Z}^+$. Then $u_{\mathcal{T}}^{(\frac{km}{kn})} = u_{\mathcal{T}}^{(\frac{m}{n})} \ \forall \ u \in \mathcal{L}$.*

Proof. Trivial □

Lemma 6.3.6. *Let $\mathcal{T} : \mathcal{L}^2 \to \mathcal{L}$ be an Archimedean continuous t-norm. Then, $\forall \ u \in \mathcal{L} \backslash \{0_{\mathcal{L}}, 1_{\mathcal{L}}\}$, it holds that $\mathcal{T}(u, u) <_{\mathcal{L}} u$.*

Proposition 6.3.7. *Let $\mathcal{T} : \mathcal{L}^2 \to \mathcal{L}$ be a continuous t-norm. Then the following properties hold:*

(i) For given $t \in \mathbb{R}^+$, $u_{\mathcal{T}}^{(t)} \leq_{\mathcal{L}} v_{\mathcal{T}}^{(t)} \ \forall \ u, v \in \mathcal{L}$ such that $u \leq_{\mathcal{L}} v$.

(ii) For a given $u \in \mathcal{L}$, $u_{\mathcal{T}}^{(t)} \geq_{\mathcal{L}} u_{\mathcal{T}}^{(t')} \ \forall \ t, t' \in [0, 1]$ such that $t < t'$.

Proof. Trivial □

Corollary 6.3.8. *Let $\mathcal{T} : \mathcal{L}^2 \to \mathcal{L}$ be a continuous t-norm. If $u_{\mathcal{T}}^{(t)} \leq_{\mathcal{L}} v$, then $u \leq_{\mathcal{L}} v_{\mathcal{T}}^{(1/t)} \ \forall \ t \in \mathbb{R}^+, \ u, v \in \mathcal{L}$.*

Proof. For given $t \in \mathbb{R}^+$ and $u, v \in \mathcal{L}$,

$$\begin{aligned}
u_{\mathcal{T}}^{(t)} \leq_{\mathcal{L}} v \ &\Rightarrow u_{\mathcal{T}}^{(t)} \leq_{\mathcal{L}} v_{\mathcal{T}}^{(1)} \\
&\Rightarrow (u_{\mathcal{T}}^{(t)})_{\mathcal{T}}^{(1)} \leq_{\mathcal{L}} (v_{\mathcal{T}}^{(1)})_{\mathcal{T}}^{(1)} \\
&\Rightarrow (u_{\mathcal{T}}^{(t)})_{\mathcal{T}}^{(1/t)} \leq_{\mathcal{L}} (v_{\mathcal{T}}^{(1)})_{\mathcal{T}}^{(1/t)} && \text{(by Proposition 6.3.7(i))} \\
&\Rightarrow u_{\mathcal{T}}^{(t/t)} \leq_{\mathcal{L}} v_{\mathcal{T}}^{(1/t)} && \text{(by (6.8))} \\
&\Rightarrow u \leq_{\mathcal{L}} v_{\mathcal{T}}^{(1/t)}.
\end{aligned}$$

□

Definition 6.3.9. *Let $\eta = (\eta_1, \eta_2) \in \mathcal{L}$ and $\mathcal{T} : \mathcal{L}^2 \to \mathcal{L}$ be a continuous t-norm. Then, for every $u \in \mathcal{L}$, the power $u_{\mathcal{T}}^{(\eta)}$ is defined by*

$$u_{\mathcal{T}}^{(\eta)} = (pr_1(u_{\mathcal{T}}^{(1-\eta_2)}), pr_2(u_{\mathcal{T}}^{(\eta_1)})) \tag{6.9}$$

Theorem 6.3.10. *The operator $u_{\mathcal{T}}^{(\eta)}$ defined by (6.9) is an element of $\mathcal{L}$ for every $\eta, u \in \mathcal{L}$.*

Proof. To show that $u_{\mathcal{T}}^{(\eta)} = (pr_1(u_{\mathcal{T}}^{(1-\eta_2)}), pr_2(u_{\mathcal{T}}^{(\eta_1)}))$ belongs to $\mathcal{L}$ for every $\eta = (\eta_1, \eta_2), u = (u_1, u_2) \in \mathcal{L}$.

We know that, for every $v = (v_1, v_2), w = (w_1, w_2) \in \mathcal{L}$ such that

$$v \leq_{\mathcal{L}} w \quad \Leftrightarrow \quad v_1 \leq w_1,\ v_2 \geq w_2 \quad \Leftrightarrow \quad pr_1(v) \leq pr_1(w),\ pr_2(v) \geq pr_2(w) \tag{6.10}$$

Since $\eta = (\eta_1, \eta_2) \in \mathcal{L}$, $\eta_1 \leq 1 - \eta_2$.

Now, by Proposition 6.3.7(ii), we have

$$u_{\mathcal{T}}^{(1-\eta_2)} \leq_{\mathcal{L}} u_{\mathcal{T}}^{(\eta_1)} \quad \Rightarrow \quad pr_1(u_{\mathcal{T}}^{(1-\eta_2)}) \leq pr_1(u_{\mathcal{T}}^{(\eta_1)}),\ pr_2(u_{\mathcal{T}}^{(1-\eta_2)}) \geq pr_2(u_{\mathcal{T}}^{(\eta_1)}) \tag{6.11}$$

Further,

$$\begin{aligned} &u = u_{\mathcal{T}}^{(1)} \leq_{\mathcal{L}} u_{\mathcal{T}}^{(1-\eta_2)} \leq_{\mathcal{L}} u_{\mathcal{T}}^{(0)} = 1_{\mathcal{L}}, \quad u = u_{\mathcal{T}}^{(1)} \leq_{\mathcal{L}} u_{\mathcal{T}}^{(\eta_1)} \leq_{\mathcal{L}} u_{\mathcal{T}}^{(0)} = 1_{\mathcal{L}} \\ \Leftrightarrow\ &0 \leq u_1 \leq pr_1(u_{\mathcal{T}}^{(1-\eta_2)}) \leq 1, \quad 1 \geq u_2 \geq pr_2(u_{\mathcal{T}}^{(\eta_1)}) \geq 0 \end{aligned} \tag{6.12}$$

$$pr_1(u_{\mathcal{T}}^{(1-\eta_2)}) + pr_2(u_{\mathcal{T}}^{(\eta_1)}) \leq pr_1(u_{\mathcal{T}}^{(1-\eta_2)}) + pr_2(u_{\mathcal{T}}^{(1-\eta_2)})$$

Since $u_{\mathcal{T}}^{(1-\eta_2)} \in \mathcal{L}$,

$$pr_1(u_{\mathcal{T}}^{(1-\eta_2)}) + pr_2(u_{\mathcal{T}}^{(1-\eta_2)}) \leq 1 \tag{6.13}$$

From (6.12) and (6.13), we have

$$0 \leq pr_1(u_{\mathcal{T}}^{(1-\eta_2)}) \leq 1, \quad 0 \leq pr_2(u_{\mathcal{T}}^{(1-\eta_2)}) \leq 1, \quad pr_1(u_{\mathcal{T}}^{(1-\eta_2)}) + pr_2(u_{\mathcal{T}}^{(1-\eta_2)}) \leq 1.$$

Thus,

$$u_{\mathcal{T}}^{(\eta)} = (pr_1(u_{\mathcal{T}}^{(1-\eta_2)}), pr_2(u_{\mathcal{T}}^{(\eta_1)})) \in \mathcal{L}.$$

□

Lemma 6.3.11. *Let $\mathcal{T} : \mathcal{L}^2 \to \mathcal{L}$ be a t-norm satisfying $\mathcal{T}(u + \epsilon_I, v + \epsilon_I) \leq_{\mathcal{L}} \mathcal{T}(u, v) + \epsilon_I\ \forall\ \epsilon_I = (\epsilon_1, \epsilon_2) \in \mathcal{L} \setminus \{0_{\mathcal{L}}, 1_{\mathcal{L}}\}, u, v \in \mathcal{L}$. Then*

$$(v + \epsilon_I)_{\mathcal{T}}^{(n)} \leq_{\mathcal{L}} v_{\mathcal{T}}^{(n)} + \epsilon_I\ \forall\ v \in \mathcal{L}; n \in \mathbb{Z}^+. \tag{6.14}$$

Proof. By induction, we prove that $(v+\epsilon_I)^{(n)}_{\mathcal{T}} \leq_{\mathcal{L}} v^{(n)}_{\mathcal{T}} + \epsilon_I \ \forall \ n \in \mathbb{Z}^+$. First one has that $(v+\epsilon_I)^{(1)}_{\mathcal{T}} = v+\epsilon_I = v^{(1)}_{\mathcal{T}} + \epsilon_I$. This shows that (6.14) hold for n = 1. Now, suppose that $(v+\epsilon_I)^{(k)}_{\mathcal{T}} \leq_{\mathcal{L}} v^{(k)}_{\mathcal{T}} + \epsilon_I$ holds $\forall\, n = k$ and $k \in \mathbb{Z}^+$. For $n = k+1$,

$$(v+\epsilon_I)^{(k+1)}_{\mathcal{T}} = \mathcal{T}((v+\epsilon_I)^{(k)}_{\mathcal{T}}, v+\epsilon_I) \leq_{\mathcal{L}} \mathcal{T}(v^{(k)}_{\mathcal{T}} + \epsilon_I, v+\epsilon_I) \leq_{\mathcal{L}} \mathcal{T}(v^{(k)}_{\mathcal{T}}, v) + \epsilon_I = v^{(k+1)}_{\mathcal{T}} + \epsilon_I.$$

This shows that (6.14) hold for n = k+1. Thus $(v+\epsilon_I)^{(n)}_{\mathcal{T}} \leq_{\mathcal{L}} v^{(n)}_{\mathcal{T}} + \epsilon_I \ \forall \ v \in \mathcal{L}; n \in \mathbb{Z}^+$. □

Assumption 6.3.12. *Let us suppose that the following results hold for the t-norm $\mathcal{T} : \mathcal{L}^2 \to \mathcal{L}$:*

(i) $(v-\epsilon_I)^{(w)}_{\mathcal{T}} \leq_{\mathcal{L}} v^{(w)}_{\mathcal{T}} - \epsilon_I \ \forall \ v, w \in \mathcal{L}$,

(ii) $(v-\epsilon_I)^{(w-\epsilon_I)}_{\mathcal{T}} \leq_{\mathcal{L}} v^{(w)}_{\mathcal{T}} \ \forall \ v, w \in \mathcal{L}$,

where $\epsilon_I = (\epsilon_1, \epsilon_2) \in \mathcal{L} \setminus \{0_{\mathcal{L}}, 1_{\mathcal{L}}\}$.

6.4 $\mathcal{T}$-power based implications and their properties

After studying the papers [45, 59, 128] and previous section, we define the following $\mathcal{T}$-power based implication.

Definition 6.4.1. *A function $\mathtt{I}_{\mathtt{I}R\mathcal{T}} : \mathcal{L}^2 \to \mathcal{L}$ is called a $\mathcal{T}$-power based implication if there exists a continuous t-norm $\mathcal{T} : \mathcal{L}^2 \to \mathcal{L}$ such that*

$$\mathtt{I}_{\mathtt{I}R\mathcal{T}}(u, v) = \sup\{\gamma \in \mathcal{L} : v^{(\gamma)}_{\mathcal{T}} \geq_{\mathcal{L}} u\} \ \forall \ u, v \in \mathcal{L}. \tag{6.15}$$

If $\mathtt{I}_{\mathtt{I}R\mathcal{T}}$ is a $\mathcal{T}$-power based implication generated by a continuous t-norm $\mathcal{T}$, then we will often denote it by $\mathtt{I}_{\mathtt{I}\mathcal{T}}$.

Remark 6.4.2. *Note that the set $\{\gamma \in \mathcal{L} : v^{(\gamma)}_{\mathcal{T}} \geq_{\mathcal{L}} u\}$ is always non-empty because of any continuous t-norm $\mathcal{T}$, $v^{(0_{\mathcal{L}})}_{\mathcal{T}} = 1_{\mathcal{L}} \geq_{\mathcal{L}} u \ \forall \ u, v \in \mathcal{L}$.*

Theorem 6.4.3. *The operator $\mathtt{I}_{\mathtt{I}\mathcal{T}}$ defined by (6.15) is an IFI.*

Proof. The fact that $\mathtt{I}_{\mathtt{I}\mathcal{T}}$ defined by (6.15) is an IFI can be seen from the following:

- Let $u <_{\mathcal{L}} u'$. Then, we have $\{\gamma \in \mathcal{L} : v^{(\gamma)}_{\mathcal{T}} \geq_{\mathcal{L}} u'\} \subset \{\gamma \in \mathcal{L} : v^{(\gamma)}_{\mathcal{T}} \geq_{\mathcal{L}} u\}$, and hence $\sup\{\gamma \in \mathcal{L} : v^{(\gamma)}_{\mathcal{T}} \geq_{\mathcal{L}} u'\} \leq_{\mathcal{L}} \sup\{\gamma \in \mathcal{L} : v^{(\gamma)}_{\mathcal{T}} \geq_{\mathcal{L}} u\} \ \Rightarrow \ \mathtt{I}_{\mathtt{I}\mathcal{T}}(u', v) \leq_{\mathcal{L}} \mathtt{I}_{\mathtt{I}\mathcal{T}}(u, v)$, i.e., $\mathtt{I}_{\mathtt{I}\mathcal{T}}$ satisfies (I1).

- Once again, let $v <_{\mathcal{L}} v'$. Then, we have $\{\gamma \in \mathcal{L} : v^{(\gamma)}_{\mathcal{T}} \geq_{\mathcal{L}} u\} \subset \{\gamma \in \mathcal{L} : v'^{(\gamma)}_{\mathcal{T}} \geq_{\mathcal{L}} u\}$, and hence $\sup\{\gamma \in \mathcal{L} : v^{(\gamma)}_{\mathcal{T}} \geq_{\mathcal{L}} u\} \leq_{\mathcal{L}} \sup\{\gamma \in \mathcal{L} : v'^{(\gamma)}_{\mathcal{T}} \geq_{\mathcal{L}} u\} \ \Rightarrow \ \mathtt{I}_{\mathtt{I}\mathcal{T}}(u, v) \leq_{\mathcal{L}} \mathtt{I}_{\mathtt{I}\mathcal{T}}(u, v')$, i.e., $\mathtt{I}_{\mathtt{I}\mathcal{T}}$ satisfies (I2).

- $\mathtt{I}_{\mathtt{I}\mathcal{T}}(0_{\mathcal{L}}, 0_{\mathcal{L}}) = \sup\{\gamma \in \mathcal{L} : (0_{\mathcal{L}})_{\mathcal{T}}^{(\gamma)} \geq_{\mathcal{L}} 0_{\mathcal{L}}\} = \sup(\mathcal{L}) = 1_{\mathcal{L}}$;
  $\mathtt{I}_{\mathtt{I}\mathcal{T}}(1_{\mathcal{L}}, 1_{\mathcal{L}}) = \sup\{\gamma \in \mathcal{L} : (1_{\mathcal{L}})_{\mathcal{T}}^{(\gamma)} \geq_{\mathcal{L}} 1_{\mathcal{L}}\} = \sup(\mathcal{L}) = 1_{\mathcal{L}}$;
  $\mathtt{I}_{\mathtt{I}\mathcal{T}}(1_{\mathcal{L}}, 0_{\mathcal{L}}) = \sup\{\gamma \in \mathcal{L} : (0_{\mathcal{L}})_{\mathcal{T}}^{(\gamma)} \geq_{\mathcal{L}} 1_{\mathcal{L}}\} = \sup(0_{\mathcal{L}}) = 0_{\mathcal{L}}$,
  i.e., $\mathtt{I}_{\mathtt{I}\mathcal{T}}$ satisfies (I3).

Thus, the operator $\mathtt{I}_{\mathtt{I}\mathcal{T}}$ defined by (6.15) is an IFI. □

6.4.1 Residuation Principle

We say that power of element in $\mathcal{L}$ w.r.t. $\mathcal{T}$ satisfies the residuation principle if and only if, for every $u, v, w \in \mathcal{L}$

$$v_{\mathcal{T}}^{(w)} \geq_{\mathcal{L}} u \quad \Leftrightarrow \quad \mathtt{I}_{\mathtt{I}\mathcal{T}}(u, v) \geq_{\mathcal{L}} w \tag{6.16}$$

where $\mathtt{I}_{\mathtt{I}\mathcal{T}}$ denotes the $\mathcal{T}$-power based implication generated by a continuous t-norm $\mathcal{T}$.

Example 6.4.4. *[62] Let $\mathcal{T} : \mathcal{L}^2 \to \mathcal{L}$ be a mapping defined by*

$$\mathcal{T}(u, v) = (\max(0, u_1 + v_1 - 1), \min(1, u_2 + v_2)) \ \forall \ u = (u_1, u_2), v = (v_1, v_2) \in \mathcal{L}.$$

Then, it is easily verified that $\mathcal{T}$ is a continuous t-norm. Now we calculate, for $u = (u_1, u_2), v = (v_1, v_2), w = (w_1, w_2) \in \mathcal{L}$ and $n \in \mathbb{Z}^+$, $v_{\mathcal{T}}^{(n)}, v_{\mathcal{T}}^{(1/n)}$ and $v_{\mathcal{T}}^{(w)}$

$$v_{\mathcal{T}}^{(n)} = \begin{cases} 0_{\mathcal{L}}, & 0 \leq v_1 \leq \frac{n-1}{n}, 1 \geq v_2 \geq \frac{1}{n}, v_1 + v_2 \leq 1; \\ (0, nv_2), & 0 \leq v_1 \leq \frac{n-1}{n}, 0 \leq v_2 < \frac{1}{n}, v_1 + v_2 \leq 1; \\ (nv_1 - (n-1), nv_2), & 1 \geq v_1 > \frac{n-1}{n}, 0 \leq v_2 < \frac{1}{n}, v_1 + v_2 \leq 1. \end{cases} n \in \mathbb{Z}^+.$$

$$v_{\mathcal{T}}^{(1/n)} = \left(\frac{v_1 + (n-1)}{n}, \frac{v_2}{n} \right) \ \forall \ v \in \mathcal{L}, \ n \in \mathbb{Z}^+.$$

$$v_{\mathcal{T}}^{(w)} = (1 - (1 - w_2)(1 - v_1), w_1 v_2) \ \ \forall \ v, w \in \mathcal{L}.$$

Now, we have, for $u = (u_1, u_2), v = (v_1, v_2), w = (w_1, w_2)$ in $\mathcal{L}$

$$\begin{aligned} v_{\mathcal{T}}^{(w)} \geq_{\mathcal{L}} u &\Leftrightarrow (1 - (1 - w_2)(1 - v_1), w_1 v_2) \geq_{\mathcal{L}} (u_1, u_2) \\ &\Leftrightarrow 1 - (1 - w_2)(1 - v_1) \geq u_1 \text{ and } w_1 v_2 \leq u_2 \end{aligned}$$

Hence,

$$\mathtt{I}_{\mathtt{I}R\mathcal{T}}(u, v) = \begin{cases} 1_{\mathcal{L}}, & u_1 \leq v_1 < 1, u_2 \geq v_2 > 0; \\ \left(\frac{u_2}{v_2}, 1 - \frac{1-u_1}{1-v_1}\right), & 0 < v_1 \leq u_1, 1 > v_2 \geq u_2; \\ \left(\frac{u_2}{v_2}, 0\right), & u_1 \leq v_1 < 1, v_2 \geq u_2 > 0, \end{cases} u, v \in \mathcal{L}.$$

Clearly, for each $u, v \in \mathcal{L}$, $\mathtt{I}_{\mathtt{I}R\mathcal{T}}(u,v) \in \mathcal{L}$. *From the above calculations it follows immediately that* $v_{\mathcal{T}}^{(w)} \geq_{\mathcal{L}} u$ *if and only if* $\mathtt{I}_{\mathtt{I}\mathcal{T}}(u,v) \geq_{\mathcal{L}} w$, *and so the residuation principle holds for power of element in* $\mathcal{L}$ *w.r.t.* $\mathcal{T}$.

Example 6.4.5. *[62] Let* $\mathcal{T} : \mathcal{L}^2 \to \mathcal{L}$ *be a mapping defined by*

$$\mathcal{T}(u,v) = (\max(0, u_1 + v_1 - 1), \min(1, u_2 + 1 - v_1, v_2 + 1 - u_1))\ \forall\, u = (u_1, u_2), v = (v_1, v_2) \in \mathcal{L}.$$

Then it is easily verified that $\mathcal{T}$ *is a continuous t-norm. Now we calculate, for* $u = (u_1, u_2), v = (v_1, v_2), w = (w_1, w_2) \in \mathcal{L}$ *and* $n \in \mathbb{Z}^+$, $v_{\mathcal{T}}^{(n)}, v_{\mathcal{T}}^{(1/n)}$ *and* $v_{\mathcal{T}}^{(w)}$

$$v_{\mathcal{T}}^{(n)} = \begin{cases} 0_{\mathcal{L}}, & 0 \leq v_1 \leq \frac{n-1}{n}, 1 \geq v_2 \geq \max(0, (n \\ & -1)v_1 - (n-2)), v_1 + v_2 \leq 1; \\ (0, v_2 + (n-1)(1-v_1)), & 0 \leq v_1 \leq \frac{n-1}{n}, 0 \leq v_2 < (n-1)v_1 \\ & -(n-2), v_1 + v_2 \leq 1; \\ (nv_1 - (n-1), v_2 + (n-1)(1-v_1)), & 1 \geq v_1 > \frac{n-1}{n}, 0 \leq v_2 < (n-1)v_1 \\ & -(n-2), v_1 + v_2 \leq 1. \end{cases} \quad n \in \mathbb{Z}^+.$$

$$v_{\mathcal{T}}^{(1/n)} = \left(\frac{v_1 + (n-1)}{n}, v_2 - \frac{(n-1)(1-v_1)}{n}\right) \ \forall\, v \in \mathcal{L},\ n \in \mathbb{Z}^+.$$

$$v_{\mathcal{T}}^{(w)} = (1 - (1-w_2)(1-v_1), v_2 - (1-w_1)(1-v_1)) \ \forall\, v, w \in \mathcal{L}.$$

Now, we have, for $u = (u_1, u_2), v = (v_1, v_2), w = (w_1, w_2)$ *in* $\mathcal{L}$

$$\begin{aligned} v_{\mathcal{T}}^{(w)} \geq_{\mathcal{L}} u &\Leftrightarrow (1 - (1-w_2)(1-v_1), v_2 - (1-w_1)(1-v_1)) \geq_{\mathcal{L}} (u_1, u_2) \\ &\Leftrightarrow 1 - (1-w_2)(1-v_1) \geq u_1, \quad v_2 - (1-w_1)(1-v_1) \leq u_2 \end{aligned}$$

Hence, $\forall\, u, v \in \mathcal{L}$.

$$\mathtt{I}_{\mathtt{I}R\mathcal{T}}(u,v) = \begin{cases} 1_{\mathcal{L}}, & u_1 \leq v_1 < 1, u_2 \geq v_2; \\ \left(1 - \frac{v_2 - u_2}{1 - v_1}, 1 - \frac{1-u_1}{1-v_1}\right), & 0 < v_1 \leq u_1, v_2 \geq u_2; \\ \left(1 - \frac{v_2 - u_2}{1-v_1}, 0\right), & u_1 \leq v_1 < 1, v_2 \geq u_2. \end{cases}$$

Clearly, for each $u, v \in \mathcal{L}$, $\mathtt{I}_{\mathtt{I}R\mathcal{T}}(u,v) \in \mathcal{L}$. *From the above calculations it follows immediately that* $v_{\mathcal{T}}^{(w)} \geq_{\mathcal{L}} u$ *if and only if* $\mathtt{I}_{\mathtt{I}\mathcal{T}}(u,v) \geq_{\mathcal{L}} w$, *and so the residuation principle holds for power of element in* $\mathcal{L}$ *w.r.t.* $\mathcal{T}$.

Lemma 6.4.6. *Let* $\mathcal{T} : \mathcal{L}^2 \to \mathcal{L}$ *be a continuous t-norm satisfying the residuation principle. Then, for any* $u, v, w \in \mathcal{L}$ *such that* $v_{\mathcal{T}}^{(w)} = u$, $\exists\, w' \in \mathcal{L}$ *such that* $w' \geq_{\mathcal{L}} w$, *and*

$$v_{\mathcal{T}}^{(w')} = u \quad \text{and} \quad \mathtt{I}_{\mathtt{I}R\mathcal{T}}(u,v) = w' \tag{6.17}$$

Proof. Let u, v and w be elements of $\mathcal{L}$ for which $v_{\mathcal{T}}^{(w)} = u$. Then, by residuation principle, it follows that $w \leq_{\mathcal{L}} \mathtt{I}_{\mathtt{I}R\mathcal{T}}(u, v)$. Define w' as $w' = \mathtt{I}_{\mathtt{I}R\mathcal{T}}(u, v)$. Then $w \leq_{\mathcal{L}} w'$. Since $\mathcal{T}$ holds Proposition 6.3.7,

$$v_{\mathcal{T}}^{(w')} \leq_{\mathcal{L}} v_{\mathcal{T}}^{(w)} \quad \Rightarrow \quad v_{\mathcal{T}}^{(w')} \leq_{\mathcal{L}} u$$

On the other hand, since $w' = \mathtt{I}_{\mathtt{I}R\mathcal{T}}(u, v)$, by residuation principle, it follows that $v_{\mathcal{T}}^{(w')} \geq_{\mathcal{L}} u$. Hence, $v_{\mathcal{T}}^{(w')} = u$. Thus, (6.17) holds. □

Lemma 6.4.7. *Let $\mathcal{T} : \mathcal{L}^2 \to \mathcal{L}$ be a continuous t-norm satisfying the residuation principle, and u, u', v, w, w' be the arbitrary elements of $\mathcal{L}$. Assume w and w' satisfy*

$$\begin{aligned} v_{\mathcal{T}}^{(w)} = u \quad &\text{and} \quad \mathtt{I}_{\mathtt{I}R\mathcal{T}}(u, v) = w \\ v_{\mathcal{T}}^{(w')} = u' \quad &\text{and} \quad \mathtt{I}_{\mathtt{I}R\mathcal{T}}(u', v) = w'. \end{aligned} \tag{6.18}$$

Then $u' \leq_{\mathcal{L}} u$ if and only if $w \leq_{\mathcal{L}} w'$.

Proof. Assume that u, u', v, w, w' satisfy the condition (6.18). If $u' = v_{\mathcal{T}}^{(w')} \leq_{\mathcal{L}} v_{\mathcal{T}}^{(w)} = u$, then, since $\mathtt{I}_{\mathtt{I}R\mathcal{T}}$ is non-increasing corresponding to first component, $w = \mathtt{I}_{\mathtt{I}R\mathcal{T}}(u, v) \leq_{\mathcal{L}} \mathtt{I}_{\mathtt{I}R\mathcal{T}}(u', v)$ $= w'$, so $w \leq_{\mathcal{L}} w'$. Conversely, $u' \leq_{\mathcal{L}} u$ follows immediately from $w \leq_{\mathcal{L}} w'$, since $\mathcal{T}$ holds from Proposition 6.3.7. □

6.4.2 Properties of $\mathtt{I}_{\mathtt{I}\mathcal{T}}$

Proposition 6.4.8. *Let $\mathtt{I}_{\mathtt{I}\mathcal{T}}$ be a $\mathcal{T}$-power based implication operator. Then*

(i) $\mathtt{I}_{\mathtt{I}\mathcal{T}}$ satisfies (OP) and (IP).

(ii) $\mathtt{I}_{\mathtt{I}\mathcal{T}}$ does not satisfy (NP).

(iii) $\mathtt{I}_{\mathtt{I}\mathcal{T}}$ satisfies (SBC), (LBC) and (RBC).

Proof. (i) For $u, v \in \mathcal{L}$,

$$u \leq_{\mathcal{L}} v \ \Leftrightarrow \ \mathtt{I}_{\mathtt{I}\mathcal{T}}(u, v) = \sup\{\gamma \in \mathcal{L} : v_{\mathcal{T}}^{(\gamma)} \geq_{\mathcal{L}} u\} = 1_{\mathcal{L}}.$$

Thus, $\mathtt{I}_{\mathtt{I}\mathcal{T}}$ satisfies (OP). Obviously, $\mathtt{I}_{\mathtt{I}\mathcal{T}}$ also satisfies (IP).

(ii) Putting $u = 1_{\mathcal{L}}$ in (6.15), we have

$$\mathtt{I}_{\mathtt{I}\mathcal{T}}(1_{\mathcal{L}}, v) = \sup\{\gamma \in \mathcal{L} : v_{\mathcal{T}}^{(\gamma)} \geq_{\mathcal{L}} 1_{\mathcal{L}}\}.$$

There are two cases:

Case 1: If $v <_{\mathcal{L}} 1_{\mathcal{L}}$, then

$$\begin{aligned}\mathtt{I}_{\mathtt{I}\mathcal{T}}(1_{\mathcal{L}}, v) =& \sup\{\gamma \in \mathcal{L} : v_{\mathcal{T}}^{(\gamma)} \geq_{\mathcal{L}} 1_{\mathcal{L}}\}\\ =& 0_{\mathcal{L}}.\end{aligned}$$

Case 2: If $v = 1_{\mathcal{L}}$, then

$$\begin{aligned}\mathtt{I}_{\mathtt{I}\mathcal{T}}(1_{\mathcal{L}}, v) =& \sup\{\gamma \in \mathcal{L} : v_{\mathcal{T}}^{(\gamma)} \geq_{\mathcal{L}} 1_{\mathcal{L}}\}\\ =& 1_{\mathcal{L}}.\end{aligned}$$

From Case 1 and Case 2, we conclude that

$$\mathtt{I}_{\mathtt{I}\mathcal{T}}(1_{\mathcal{L}}, v) = \begin{cases} 0_{\mathcal{L}}, & v <_{\mathcal{L}} 1_{\mathcal{L}}, \\ 1_{\mathcal{L}}, & v = 1_{\mathcal{L}}. \end{cases}$$

This shows that $\mathtt{I}_{\mathtt{I}\mathcal{T}}$ does not satisfy (NP).

(iii) Consider $u \neq 0_{\mathcal{L}}$. Then it follows that $\mathtt{I}_{\mathtt{I}\mathcal{T}}(1_{\mathcal{L}}, v) = \sup\{\gamma \in \mathcal{L} : (0_{\mathcal{L}})_{\mathcal{T}}^{(\gamma)} \geq_{\mathcal{L}} u\} = 0_{\mathcal{L}}$. This implies that $\mathtt{I}_{\mathtt{I}\mathcal{T}}$ satisfies (SBC). Obviously $\mathtt{I}_{\mathtt{I}\mathcal{T}}$ also satisfies (LBC) and (RBC).

□

Example 6.4.9. *Consider a continuous t-norm $\mathcal{T}$ defined by $\mathcal{T}(u, v) = (u_1 v_1, 1 - (1 - u_2)(1 - v_2))\ \forall\, u = (u_1, u_2), v = (v_1, v_2) \in \mathcal{L}$. Then $\mathcal{T}$-power based implication $\mathtt{I}_{\mathtt{I}\mathcal{T}}$ is given by*

$$\begin{aligned}\mathtt{I}_{\mathtt{I}\mathcal{T}}(u, v) =& \sup\{\gamma \in \mathcal{L} : v_{\mathcal{T}}^{(\gamma)} \geq_{\mathcal{L}} u\}\\ =& \sup\{(\gamma_1, \gamma_2) \in \mathcal{L} : (v_1^{1-\gamma_2}, 1 - (1 - v_2)^{\gamma_1}) \geq_{\mathcal{L}} (u_1, u_2)\}\\ =& \sup\{(\gamma_1, \gamma_2) \in \mathcal{L} : v_1^{1-\gamma_2} \geq u_1,\ \ 1 - (1 - v_2)^{\gamma_1} \leq u_2\}\end{aligned} \tag{6.19}$$

Putting $u = 1_{\mathcal{L}}$ in (6.19), we have

$$\begin{aligned}\mathtt{I}_{\mathtt{I}\mathcal{T}}(1_{\mathcal{L}}, v) =& \sup\{\gamma \in \mathcal{L} : v_{\mathcal{T}}^{(\gamma)} \geq_{\mathcal{L}} 1_{\mathcal{L}}\}\\ =& \sup\{(\gamma_1, \gamma_2) \in \mathcal{L} : (v_1^{1-\gamma_2}, 1 - (1 - v_2)^{\gamma_1}) \geq_{\mathcal{L}} (1, 0)\}\\ =& \sup\{(\gamma_1, \gamma_2) \in \mathcal{L} : v_1^{1-\gamma_2} \geq 1,\ \ 1 \leq (1 - v_2)^{\gamma_1}\}\\ =& \begin{cases} 0_{\mathcal{L}}, & v <_{\mathcal{L}} 1_{\mathcal{L}}, \\ 1_{\mathcal{L}}, & v = 1_{\mathcal{L}}, \end{cases}\end{aligned}$$

i.e., $\mathtt{I}_{\mathtt{I}\mathcal{T}}$ does not satisfy (NP).

Proposition 6.4.10. *The $\mathcal{T}$-power based implication operator $\mathtt{I}_{\mathtt{I}\mathcal{T}}$ satisfies (NP) if $\forall\, u \in \mathcal{L}$ the following two conditions hold:*

(i) $u_{\mathcal{T}}^{(w)} <_{\mathcal{L}} 1_{\mathcal{L}}$ *for any* $w >_{\mathcal{L}} u$;

(ii) for every $\epsilon_I >_{\mathcal{L}} 0_{\mathcal{L}}$ *there is* $u - \epsilon_I <_{\mathcal{L}} w \leq_{\mathcal{L}} u$ *such that* $u_{\mathcal{T}}^{(w)} \geq_{\mathcal{L}} 1_{\mathcal{L}}$.

Proof. Suppose conditions (i) and (ii) hold. Then prove that $\mathtt{I}_{\mathtt{I}\mathcal{T}}$ satisfies (NP). Condition (i) implies $\mathtt{I}_{\mathtt{I}\mathcal{T}}(1_{\mathcal{L}}, u) \leq_{\mathcal{L}} u$ and condition (ii) implies $\mathtt{I}_{\mathtt{I}\mathcal{T}}(1_{\mathcal{L}}, u) \geq_{\mathcal{L}} w >_{\mathcal{L}} u - \epsilon_I$ from which we obtain $\mathtt{I}_{\mathtt{I}\mathcal{T}}(1_{\mathcal{L}}, u) \geq_{\mathcal{L}} u$ since ϵ_I is arbitrary. Thus $\mathtt{I}_{\mathtt{I}\mathcal{T}}(1_{\mathcal{L}}, u) = u$, i.e., $\mathtt{I}_{\mathtt{I}\mathcal{T}}$ satisfies (NP). □

Corollary 6.4.11. *The $\mathcal{T}$-power based implication operator $\mathtt{I}_{\mathtt{I}\mathcal{T}}$ satisfies (NP) if and only if* $u_{\mathcal{T}}^{(u)} = 1_{\mathcal{L}}$ *for every* $u \in \mathcal{L}$.

Proof. (⇒) Suppose $\mathtt{I}_{\mathtt{I}\mathcal{T}}$ satisfies (NP). Then to prove that $u_{\mathcal{T}}^{(u)} = 1_{\mathcal{L}}$ for every $u \in \mathcal{L}$. By the (NP) of $\mathtt{I}_{\mathtt{I}\mathcal{T}}$, we have $\mathtt{I}_{\mathtt{I}\mathcal{T}}(1_{\mathcal{L}}, u) = u \;\Rightarrow\; \sup\{\gamma \in \mathcal{L} : u_{\mathcal{T}}^{(\gamma)} = 1_{\mathcal{L}}\} = u$. Thus $u_{\mathcal{T}}^{(u)} = 1_{\mathcal{L}}$ for every $u \in \mathcal{L}$.
(⇐) Suppose $u_{\mathcal{T}}^{(u)} = 1_{\mathcal{L}}$ for every $u \in \mathcal{L}$ holds. Then prove that $\mathtt{I}_{\mathtt{I}\mathcal{T}}$ satisfies (NP). Since $u_{\mathcal{T}}^{(u)} = 1_{\mathcal{L}}$ for every $u \in \mathcal{L}$, $\mathtt{I}_{\mathtt{I}\mathcal{T}}(1_{\mathcal{L}}, u) = \sup\{\gamma \in \mathcal{L} : u_{\mathcal{T}}^{(\gamma)} = 1_{\mathcal{L}}\} = u$. Thus $\mathtt{I}_{\mathtt{I}\mathcal{T}}(1_{\mathcal{L}}, u) = u$, i.e., $\mathtt{I}_{\mathtt{I}\mathcal{T}}$ satisfies (NP). □

Proposition 6.4.12. *The $\mathcal{T}$-power based implication operator $\mathtt{I}_{\mathtt{I}\mathcal{T}}$ satisfies (CB) if and only if* $v_{\mathcal{T}}^{(w)} \geq_{\mathcal{L}} u$ *for every* $w <_{\mathcal{L}} v$ *and* $u, v, w \in \mathcal{L}$.

Proof. (⇒) If possible let $u_0, v_0, w_0 \in \mathcal{L}$ with $w_0 <_{\mathcal{L}} v_0$ such that $v_{0\mathcal{T}}^{(w_0)} <_{\mathcal{L}} u_0$. This implies that $\mathtt{I}_{\mathtt{I}\mathcal{T}}(u_0, v_0) \leq_{\mathcal{L}} w_0 <_{\mathcal{L}} v_0$, which is contradiction because of $\mathtt{I}_{\mathtt{I}\mathcal{T}}$ satisfies (CB). Hence $v_{\mathcal{T}}^{(w)} \geq_{\mathcal{L}} u$ for every $w <_{\mathcal{L}} v$.
(⇐) If $v_{\mathcal{T}}^{(w)} \geq_{\mathcal{L}} u$ for every $w <_{\mathcal{L}} v$, then $\{w \in \mathcal{L} : 0_{\mathcal{L}} \leq_{\mathcal{L}} w <_{\mathcal{L}} v\} \subseteq \{w \in \mathcal{L} : v_{\mathcal{T}}^{(w)} \geq_{\mathcal{L}} u\}$. This implies that $\sup\{w \in \mathcal{L} : 0_{\mathcal{L}} \leq_{\mathcal{L}} w <_{\mathcal{L}} u\} \leq_{\mathcal{L}} \sup\{w \in \mathcal{L} : v_{\mathcal{T}}^{(w)} \geq_{\mathcal{L}} u\}$. Thus $v \leq_{\mathcal{L}} \mathtt{I}_{\mathtt{I}\mathcal{T}}(u, v)$, i.e., $\mathtt{I}_{\mathtt{I}\mathcal{T}}$ satisfies (CB). □

Proposition 6.4.13. *Let $\mathtt{I}_{\mathtt{I}\mathcal{T}}$ be a $\mathcal{T}$-power based implication operator. Then $\mathtt{I}_{\mathtt{I}\mathcal{T}}$ satisfies (SIB) if and only if $\mathtt{I}_{\mathtt{I}\mathcal{T}}$ satisfies (CB).*

Proof. For every $u, v, w \in \mathcal{L}$, $\mathtt{I}_{\mathtt{I}\mathcal{T}}$ satisfies (CB), i.e.,

$$\begin{aligned}
\mathtt{I}_{\mathtt{I}\mathcal{T}}(u, v) \geq_{\mathcal{L}} v &\Leftrightarrow \sup\{w \in \mathcal{L} : v_{\mathcal{T}}^{(w)} \geq_{\mathcal{L}} u\} \geq_{\mathcal{L}} v \\
&\Leftrightarrow \{w \in \mathcal{L} : v_{\mathcal{T}}^{(w)} \geq_{\mathcal{L}} u\} \subseteq \{w \in \mathcal{L} : (\sup\{z \in \mathcal{L} : v_{\mathcal{T}}^{(z)} \geq_{\mathcal{L}} u\})_{\mathcal{T}}^{(w)} \geq_{\mathcal{L}} u\} \\
&\Leftrightarrow \sup\{w \in \mathcal{L} : v_{\mathcal{T}}^{(w)} \geq_{\mathcal{L}} u\} \leq_{\mathcal{L}} \sup\{w \in \mathcal{L} : (\mathtt{I}_{\mathtt{I}\mathcal{T}}(u, v))_{\mathcal{T}}^{(w)} \geq_{\mathcal{L}} u\} \\
&\Leftrightarrow \mathtt{I}_{\mathtt{I}\mathcal{T}}(u, v) \leq_{\mathcal{L}} \mathtt{I}_{\mathtt{I}\mathcal{T}}(u, \mathtt{I}_{\mathtt{I}\mathcal{T}}(u, v)),
\end{aligned}$$

i.e., $\mathtt{I}_{\mathtt{I}\mathcal{T}}$ satisfies (SIB). □

Corollary 6.4.14. *Let $\mathtt{I}_{\mathtt{I}\mathcal{T}}$ be a $\mathcal{T}$-power based implication operator. Then the following statements are equivalent:*

(i) $\mathtt{I}_{\mathtt{I}\mathcal{T}}$ *satisfies (CB).*

(ii) $v_{\mathcal{T}}^{(w)} \geq_{\mathcal{L}} u$ *for every* $w <_{\mathcal{L}} v$ *and* $u, v, w \in \mathcal{L}$.

(iii) $\mathtt{I}_{\mathtt{I}\mathcal{T}}$ *satisfies (SIB).*

Proposition 6.4.15. *Let $\mathtt{I}_{\mathtt{I}\mathcal{T}}$ be a $\mathcal{T}$-power based implication operator satisfying the Assumption 6.3.12. Then the following properties hold:*

(i) $\mathtt{I}_{\mathtt{I}\mathcal{T}}(u + \epsilon_I, v) \geq_{\mathcal{L}} \mathtt{I}_{\mathtt{I}\mathcal{T}}(u, v - \epsilon_I) \;\forall\, u, v, w \in \mathcal{L}$,

(ii) $\mathtt{I}_{\mathtt{I}\mathcal{T}}(u, v) \geq_{\mathcal{L}} \mathtt{I}_{\mathtt{I}\mathcal{T}}(u, v - \epsilon_I) + \epsilon_I \;\forall\, u, v, w \in \mathcal{L}$,

(iii) $\mathtt{I}_{\mathtt{I}\mathcal{T}}(u + \epsilon_I, v + \epsilon_I) \geq_{\mathcal{L}} \mathtt{I}_{\mathtt{I}\mathcal{T}}(u, v) \;\forall\, u, v, w \in \mathcal{L}$.

Proof. Let $\mathcal{T} : \mathcal{L}^2 \to \mathcal{L}$ and $\mathtt{I}_{\mathtt{I}\mathcal{T}} : \mathcal{L}^2 \to \mathcal{L}$ be a continuous t-norm and a $\mathcal{T}$-power based implication operator respectively. In view of Assumption 6.3.12, $\mathcal{T}$ satisfies results (i) and (ii) of Assumption 6.3.12.

(i) For each $\epsilon_I \in \mathcal{L}$ with $\epsilon_I >_{\mathcal{L}} 0_{\mathcal{L}}$, $(v - \epsilon_I)_{\mathcal{T}}^{(w)} \leq_{\mathcal{L}} v_{\mathcal{T}}^{(w)} - \epsilon_I$, we have

$$\{w \in \mathcal{L} : v_{\mathcal{T}}^{(w)} \geq_{\mathcal{L}} u + \epsilon_I\} \supseteq \{w \in \mathcal{L} : (v - \epsilon_I)_{\mathcal{T}}^{(w)} \geq_{\mathcal{L}} u\},$$

which implies that

$$\sup\{w \in \mathcal{L} : v_{\mathcal{T}}^{(w)} \geq_{\mathcal{L}} u + \epsilon_I\} \geq_{\mathcal{L}} \sup\{w \in \mathcal{L} : (v - \epsilon_I)_{\mathcal{T}}^{(w)} \geq_{\mathcal{L}} u\}.$$

Thus, $\mathtt{I}_{\mathtt{I}\mathcal{T}}(u + \epsilon_I, v) \geq_{\mathcal{L}} \mathtt{I}_{\mathtt{I}\mathcal{T}}(u, v - \epsilon_I)$.

(ii) For each $\epsilon_I \in \mathcal{L}$ with $\epsilon_I >_{\mathcal{L}} 0_{\mathcal{L}}$, $(v - \epsilon_I)_{\mathcal{T}}^{(w-\epsilon_I)} \leq_{\mathcal{L}} v_{\mathcal{T}}^{(w)}$, we have

$$\{w \in \mathcal{L} : v_{\mathcal{T}}^{(w)} \geq_{\mathcal{L}} u\} \supseteq \{w \in \mathcal{L} : (v - \epsilon_I)_{\mathcal{T}}^{(w-\epsilon_I)} \geq_{\mathcal{L}} u\},$$

which implies that

$$\sup\{w \in \mathcal{L} : v_{\mathcal{T}}^{(w)} \geq_{\mathcal{L}} u\} \geq_{\mathcal{L}} \sup\{w \in \mathcal{L} : (v - \epsilon_I)_{\mathcal{T}}^{(w-\epsilon_I)} \geq_{\mathcal{L}} u\}, \text{or}$$
$$\sup\{w \in \mathcal{L} : v_{\mathcal{T}}^{(w)} \geq_{\mathcal{L}} u\} \geq_{\mathcal{L}} \sup\{w \in \mathcal{L} : (v - \epsilon_I)_{\mathcal{T}}^{(w)} \geq_{\mathcal{L}} u\} + \epsilon_I.$$

Thus, $\mathtt{I}_{\mathtt{I}\mathcal{T}}(u, v) \geq_{\mathcal{L}} \mathtt{I}_{\mathtt{I}\mathcal{T}}(u, v - \epsilon_I) + \epsilon_I$.

(iii) For each $\epsilon_I \in \mathcal{L}$ with $\epsilon_I >_{\mathcal{L}} 0_{\mathcal{L}}$, $v_{\mathcal{T}}^{(w)} + \epsilon_I \leq_{\mathcal{L}} (v + \epsilon_I)_{\mathcal{T}}^{(w)}$, we have

$$\{w \in \mathcal{L} : (v + \epsilon_I)_{\mathcal{T}}^{(w)} \geq_{\mathcal{L}} u + \epsilon_I\} \supseteq \{w \in \mathcal{L} : v_{\mathcal{T}}^{(w)} \geq_{\mathcal{L}} u\},$$

which implies that

$$\sup\{w \in \mathcal{L} : (v + \epsilon_I)_{\mathcal{T}}^{(w)} \geq_{\mathcal{L}} u + \epsilon_I\} \geq_{\mathcal{L}} \sup\{w \in \mathcal{L} : v_{\mathcal{T}}^{(w)} \geq_{\mathcal{L}} u\}.$$

Thus, $\mathtt{I}_{\mathtt{I}\mathcal{T}}(u + \epsilon_I, v + \epsilon_I) \geq_{\mathcal{L}} \mathtt{I}_{\mathtt{I}\mathcal{T}}(u, v)$.

□

Remark 6.4.16. *Note that, if $u >_{\mathcal{L}} v$, by Proposition 6.4.15(iii), for any $\epsilon_I >_{\mathcal{L}} 0_{\mathcal{L}}$ we have the following string of inequalities:*

$$\mathtt{I}_{\mathtt{I}\mathcal{T}}(u, v) \leq_{\mathcal{L}} \mathtt{I}_{\mathtt{I}\mathcal{T}}(u + \epsilon_I, v + \epsilon_I) \leq_{\mathcal{L}} \mathtt{I}_{\mathtt{I}\mathcal{T}}(u + 2\epsilon_I, v + 2\epsilon_I) \leq_{\mathcal{L}} \cdots \leq_{\mathcal{L}} \mathtt{I}_{\mathtt{I}\mathcal{T}}(1_{\mathcal{L}}, 1_{\mathcal{L}} - u + v).$$

Inspired by the paper [100], for any $\alpha \in \mathcal{L} \setminus \{1_{\mathcal{L}}\}$, the non-increasing partial function $\mathtt{I}_{\mathtt{I}\mathcal{T}}(., \alpha) : \mathcal{L} \to \mathcal{L}$, which will be denoted by $\mathcal{N}^{\alpha}_{\mathtt{I}_{\mathtt{I}\mathcal{T}}}$. Observe that

(i) $\mathcal{N}^{\alpha}_{\mathtt{I}_{\mathtt{I}\mathcal{T}}}(\alpha) = 1_{\mathcal{L}}$

(ii) $\mathcal{N}^{\alpha}_{\mathtt{I}_{\mathtt{I}\mathcal{T}}}(1_{\mathcal{L}}) = \alpha$ whenever two conditions hold:

- $\alpha_{\mathcal{T}}^{(w)} <_{\mathcal{L}} 1_{\mathcal{L}}$ for any $w >_{\mathcal{L}} \alpha$;
- for every $\epsilon_I >_{\mathcal{L}} 0_{\mathcal{L}}$ there is $\alpha - \epsilon_I <_{\mathcal{L}} w \leq_{\mathcal{L}} \alpha$ such that $\alpha_{\mathcal{T}}^{(w)} \geq_{\mathcal{L}} 1_{\mathcal{L}}$.

(iii) $\mathcal{N}^{\alpha}_{\mathtt{I}_{\mathtt{I}\mathcal{T}}}$ is non-increasing.

Remark 6.4.17. *If we put $\alpha = 0_{\mathcal{L}}$ in $\mathcal{N}^{\alpha}_{\mathtt{I}_{\mathtt{I}\mathcal{T}}}$, then $\mathcal{N}^{\alpha}_{\mathtt{I}_{\mathtt{I}\mathcal{T}}}$ is the natural negation of $\mathtt{I}_{\mathtt{I}\mathcal{T}}$.*

Remark 6.4.18. *(i) u is $\mathcal{T}-$idempotent, i.e., $\mathcal{T}(u, u) = u$ iff $\mathtt{I}_{\mathtt{I}\mathcal{T}}(u, v) = 0_{\mathcal{L}}\ \forall\ v <_{\mathcal{L}} u$.*

(ii) For every continuous t-norm $\mathcal{T}$, the natural negation of $\mathtt{I}_{\mathtt{I}\mathcal{T}}$ is the Gödel negation, i.e.,

$$\mathcal{N}_{\mathtt{I}_{\mathtt{I}\mathcal{T}}}(u) = \mathtt{I}_{\mathtt{I}\mathcal{T}}(u, 0_{\mathcal{L}}) = \sup\{\gamma \in \mathcal{L} : 0_{\mathcal{T}}^{(\gamma)} \geq_{\mathcal{L}} u\} = \begin{cases} 1_{\mathcal{L}}, & u = 0_{\mathcal{L}}, \\ 0_{\mathcal{L}}, & u >_{\mathcal{L}} 0_{\mathcal{L}}. \end{cases}$$

Proposition 6.4.19. *Let $\mathtt{I}_{\mathtt{I}\mathcal{T}}$ be a $\mathcal{T}$-power based implication operator. If $\mathtt{I}_{\mathtt{I}\mathcal{T}}$ satisfies contrapositive property w.r.t. negation $\mathcal{N}$, i.e., $\mathtt{I}_{\mathtt{I}\mathcal{T}}(u, v) = \mathtt{I}_{\mathtt{I}\mathcal{T}}(\mathcal{N}(v), \mathcal{N}(u))$, then the following hold:*

(i) The natural negation of $\mathtt{I}_{\mathtt{I}\mathcal{T}}$ is the Gödel negation.

(ii) $\mathcal{N}$ *is decreasing.*

(iii) If $\mathcal{N}$ *is continuous, then* u *is* $\mathcal{T}-$*idempotent iff* $\mathcal{N}(u)$ *is* $\mathcal{T}-$*idempotent*

Theorem 6.4.20. *For* $\alpha \in \mathcal{L} \setminus \{0_\mathcal{L}, 1_\mathcal{L}\}$, $\mathcal{N}^{\alpha}_{\mathtt{I}_{\mathtt{I}\mathcal{T}}}$ *is a continuous function.*

Proof. Let us suppose that $\mathcal{N}^{\alpha}_{\mathtt{I}_{\mathtt{I}\mathcal{T}}}$ is not continuous, i.e., there is $\alpha \in \mathcal{L} \setminus \{0_\mathcal{L}, 1_\mathcal{L}\}$ such that $\mathtt{I}_{\mathtt{I}\mathcal{T}}(., \alpha)$ is not continuous at some point $u_0 \in \mathcal{L}$. Thus there is a sequence $\{u_n\}$ in $\mathcal{L}$ such that $u_n \to u_0$ as $n \to \infty$ but $\mathtt{I}_{\mathtt{I}\mathcal{T}}(u_n, \alpha) \nrightarrow p_0 = \mathtt{I}_{\mathtt{I}\mathcal{T}}(u_0, \alpha)$, i.e., for some $\epsilon_I >_\mathcal{L} 0_\mathcal{L}$ and fixed N such that $p_0 + \epsilon_I <_\mathcal{L} \mathtt{I}_{\mathtt{I}\mathcal{T}}(u_n, \alpha)$ or $\mathtt{I}_{\mathtt{I}\mathcal{T}}(u_n, \alpha) <_\mathcal{L} p_0 - \epsilon_I$ for $n \geq N$ and $u_n \to u_0$. There are two cases:

Case 1: If $p_0 + \epsilon_I <_\mathcal{L} \mathtt{I}_{\mathtt{I}\mathcal{T}}(u_n, \alpha)$ for $n \geq N$ and $u_n \to u_0$, then

$$\begin{aligned} &\alpha_\mathcal{T}^{(p_0+\epsilon_I)} \geq_\mathcal{L} u_n \text{ for } n \geq N \\ \Rightarrow\ &\alpha_\mathcal{T}^{(p_0+\epsilon_I)} \geq_\mathcal{L} u_0 \\ \Rightarrow\ &\mathtt{I}_{\mathtt{I}\mathcal{T}}(u_0, \alpha) \geq_\mathcal{L} p_0 + \epsilon_I \ \Rightarrow\ p_0 \geq_\mathcal{L} p_0 + \epsilon_I \end{aligned}$$

which contradicts the fact $p_0 <_\mathcal{L} p_0 + \epsilon_I$.

Case 2: If $\mathtt{I}_{\mathtt{I}\mathcal{T}}(u_n, \alpha) <_\mathcal{L} p_0 - \epsilon_I$ for $n \geq N$ and $u_n \to u_0$, then

$$\begin{aligned} &\alpha_\mathcal{T}^{(p_0-\epsilon_I)} <_\mathcal{L} u_n \text{ for } n \geq N \\ \Rightarrow\ &\alpha_\mathcal{T}^{(p_0-\epsilon_I)} <_\mathcal{L} u_0 \\ \Rightarrow\ &\mathtt{I}_{\mathtt{I}\mathcal{T}}(u_0, \alpha) <_\mathcal{L} p_0 - \epsilon_I \ \Rightarrow\ p_0 <_\mathcal{L} p_0 - \epsilon_I \end{aligned}$$

which contradicts the fact $p_0 >_\mathcal{L} p_0 - \epsilon_I$.

From above two cases, it is clear that $\mathcal{N}^{\alpha}_{\mathtt{I}_{\mathtt{I}\mathcal{T}}}$ is a continuous function for any $\alpha \in \mathcal{L} \setminus \{0_\mathcal{L}, 1_\mathcal{L}\}$. □

Theorem 6.4.21. *For* $\alpha \in \mathcal{L} \setminus 0_\mathcal{L}$, $\mathcal{N}^{\alpha}_{\mathtt{I}_{\mathtt{I}\mathcal{T}}}$ *is decreasing.*

Proof. We know that $\mathcal{N}^{\alpha}_{\mathtt{I}_{\mathtt{I}\mathcal{T}}}$ is non-increasing for any fixed $\alpha \in \mathcal{L} \setminus 0_\mathcal{L}$. On the contrary, let us suppose that $\mathcal{N}^{\alpha}_{\mathtt{I}_{\mathtt{I}\mathcal{T}}}$ is constant on the u, $u_0 \leq_\mathcal{L} u \leq_\mathcal{L} v_0$, for some $\alpha <_\mathcal{L} u_0 <_\mathcal{L} v_0 <_\mathcal{L} 1_\mathcal{L}$, i.e., $\exists\, q$, $\alpha \leq_\mathcal{L} q \leq_\mathcal{L} 1_\mathcal{L}$, such that

$$\mathcal{N}^{\alpha}_{\mathtt{I}_{\mathtt{I}\mathcal{T}}}(u_0) = \mathcal{N}^{\alpha}_{\mathtt{I}_{\mathtt{I}\mathcal{T}}}(v_0) = q$$

For fix arbitrary w, $u_0 <_\mathcal{L} w <_\mathcal{L} v_0$, there are three cases:

Case 1: If $q = 1_\mathcal{L}$, then

$$\mathcal{N}^{\alpha}_{\mathtt{I}_{\mathtt{I}\mathcal{T}}}(w) = \sup\{\gamma \in \mathcal{L} : \alpha_\mathcal{T}^{(\gamma)} \geq_\mathcal{L} z\} = 1_\mathcal{L}.$$

Thus, $\alpha_\mathcal{T}^{(1_\mathcal{L}-\epsilon_I)} \geq_\mathcal{L} w$ for any $\epsilon_I \in \mathcal{L}\setminus\{0_\mathcal{L}, 1_\mathcal{L}\}$. Since $\mathcal{T}$ is continuous, $\lim_{\epsilon_I \to 0_\mathcal{L}^+} \alpha_\mathcal{T}^{(1_\mathcal{L}-\epsilon_I)} \geq_\mathcal{L} w$. This implies that $\alpha \geq_\mathcal{L} w$, which contradicts the fact that $\alpha <_\mathcal{L} w$.

Case 2: If $q = \alpha$, then

$$\mathcal{N}^{\alpha}_{\mathtt{I}_{\mathtt{I}\mathcal{T}}}(w) = \sup\{\gamma \in \mathcal{L} : \alpha_{\mathcal{T}}^{(\gamma)} \geq_{\mathcal{L}} z\} = \alpha.$$

Thus, $\alpha_{\mathcal{T}}^{(\alpha+\epsilon_I)} <_{\mathcal{L}} w$ for any ϵ_I such that $0_{\mathcal{L}} <_{\mathcal{L}} \epsilon_I <_{\mathcal{L}} 1_{\mathcal{L}} - \epsilon_I$. Since $\mathcal{T}$ is continuous, $\lim_{\epsilon_I \to 0^-_{\mathcal{L}}} \alpha_{\mathcal{T}}^{(\alpha+\epsilon_I)} <_{\mathcal{L}} w$. This implies that $\alpha_{\mathcal{T}}^{(\alpha)} <_{\mathcal{L}} w$, which is not uniformly true because of $\alpha <_{\mathcal{L}} w$. This can be shown with the help of example. Let us consider the continuous t-norm $\mathcal{T}$ given by

$$\mathcal{T}(u, v) = (u_1 v_1, 1 - (1 - u_2)(1 - v_2)) \; \forall u = (u_1, u_2), v = (v_1, v_2) \in \mathcal{L}.$$

Taking $\alpha = (0.1, 0.2)$ and $z = (0.2, 0.1)$. Clearly, $\alpha <_{\mathcal{L}} z$. Now,

$$\alpha_{\mathcal{T}}^{(\alpha)} = (0.1^{0.1}, 1 - 0.8^{0.2}) = (0.794, 0.044) >_{\mathcal{L}} (0.2, 0.1), \text{ i.e., } \alpha_{\mathcal{T}}^{(\alpha)} >_{\mathcal{L}} w \text{ while } \alpha <_{\mathcal{L}} w.$$

Thus, our assumption is wrong.

Case 3: If $\alpha <_{\mathcal{L}} q <_{\mathcal{L}} 1_{\mathcal{L}}$, then

$$\mathcal{N}^{\alpha}_{\mathtt{I}_{\mathtt{I}\mathcal{T}}}(w) = \sup\{\gamma \in \mathcal{L} : \alpha_{\mathcal{T}}^{(\gamma)} \geq_{\mathcal{L}} w\} = q.$$

Thus, $\alpha_{\mathcal{T}}^{(q-\epsilon_I)} \geq_{\mathcal{L}} w >_{\mathcal{L}} \alpha_{\mathcal{T}}^{(q+\epsilon_I)}$ for any $\epsilon_I >_{\mathcal{L}} 0_{\mathcal{L}}$ such that $q - \epsilon_I \geq_{\mathcal{L}} \alpha$ and $q + \epsilon_I \leq_{\mathcal{L}} 1_{\mathcal{L}}$. Since $\mathcal{T}$ is continuous, $\alpha_{\mathcal{T}}^{(q)} = w$. Now this happens for every q, $\alpha <_{\mathcal{L}} q <_{\mathcal{L}} 1_{\mathcal{L}}$, which contradicts the fact that $\mathcal{T}$ is a function itself.

Hence $\mathcal{N}^{\alpha}_{\mathtt{I}_{\mathtt{I}\mathcal{T}}}$ is decreasing. □

Theorem 6.4.22. *Let $\mathcal{T} : \mathcal{L}^2 \to \mathcal{L}$ be a continuous t-norm and $\mathtt{I}_{\mathtt{I}\mathcal{T}}$ be a $\mathcal{T}$-power based implication. Then $\mathtt{I}_{\mathtt{I}\mathcal{T}}$ is continuous iff $\mathcal{N}^{0_{\mathcal{L}}}_{\mathtt{I}_{\mathtt{I}\mathcal{T}}}$ is continuous.*

Proof. Suppose that $\mathcal{N}^{0_{\mathcal{L}}}_{\mathtt{I}_{\mathtt{I}\mathcal{T}}}$ is continuous. We have to prove that $\mathtt{I}_{\mathtt{I}\mathcal{T}}$ is continuous. By Theorem 6.4.20 and the continuity of $\mathcal{N}^{0_{\mathcal{L}}}_{\mathtt{I}_{\mathtt{I}\mathcal{T}}}(u)$, it clear that $\mathtt{I}_{\mathtt{I}\mathcal{T}}$ is continuous corresponding to first argument. We have only to show that $\mathtt{I}_{\mathtt{I}\mathcal{T}}$ is continuous in its second argument. If not, then there is $u_0 \in \mathcal{L}$ such that $\mathtt{I}_{\mathtt{I}\mathcal{T}}(u_0, .)$ is not continuous at some $v_0 \in \mathcal{L}$, i.e., there is a sequence $\{v_n\}$ in $\mathcal{L}$ such that $v_n \to v_0$ as $n \to \infty$ but $\mathtt{I}_{\mathtt{I}\mathcal{T}}(u_0, v_n) \nrightarrow q_0 = \mathtt{I}_{\mathtt{I}\mathcal{T}}(u_0, v_0)$, i.e., for some $\epsilon_I >_{\mathcal{L}} 0_{\mathcal{L}}$ and fixed N such that $q_0 + \epsilon_I <_{\mathcal{L}} \mathtt{I}_{\mathtt{I}\mathcal{T}}(u_0, v_n) <_{\mathcal{L}} q_0 - \epsilon_I$ for $n \geq N$ and $v_n \to v_0$. There are two cases:

Case 1: If $q_0 + \epsilon_I <_{\mathcal{L}} \mathtt{I}_{\mathtt{I}\mathcal{T}}(u_0, v_n)$ for $n \geq N$ and $v_n \to v_0$, then

$$\begin{aligned} & v_{n\mathcal{T}}^{(q_0+\epsilon_I)} \geq_{\mathcal{L}} u_0 \text{ for } n \geq N \\ \Rightarrow \; & v_{0\mathcal{T}}^{(q_0+\epsilon_I)} \geq_{\mathcal{L}} u_0 \\ \Rightarrow \; & \mathtt{I}_{\mathtt{I}\mathcal{T}}(u_0, v_0) \geq_{\mathcal{L}} q_0 + \epsilon_I \; \Rightarrow \; q_0 \geq_{\mathcal{L}} q_0 + \epsilon_I \end{aligned}$$

which contradicts the fact $q_0 <_{\mathcal{L}} q_0 + \epsilon_I$.

Case 2: If $\mathtt{I}_{\mathtt{I}\mathcal{T}}(u_0, v_n) <_{\mathcal{L}} q_0 - \epsilon_I$ for $n \geq N$ and $v_n \to v_0$, then

$$v_{n\mathcal{T}}^{(q_0-\epsilon_I)} <_{\mathcal{L}} u_0 \text{ for } n \geq N$$
$$\Rightarrow v_{0\mathcal{T}}^{(q_0-\epsilon_I)} <_{\mathcal{L}} u_0$$
$$\Rightarrow \mathtt{I}_{\mathtt{I}\mathcal{T}}(u_0, v_0) <_{\mathcal{L}} q_0 - \epsilon_I \Rightarrow q_0 <_{\mathcal{L}} q_0 - \epsilon_I$$

which contradicts the fact $q_0 >_{\mathcal{L}} q_0 - \epsilon_I$.

From above two cases, it is clear that $\mathtt{I}_{\mathtt{I}\mathcal{T}}$ is continuous corresponding to second argument. Thus $\mathtt{I}_{\mathtt{I}\mathcal{T}}$ is continuous in both arguments. Since $\mathtt{I}_{\mathtt{I}\mathcal{T}}$ is monotone, $\mathtt{I}_{\mathtt{I}\mathcal{T}}$ is continuous.
Conversely, if $\mathcal{N}_{\mathtt{I}_{\mathtt{I}\mathcal{T}}}^{0_{\mathcal{L}}}$ is not continuous, then $\mathtt{I}_{\mathtt{I}\mathcal{T}}(x, 0_{\mathcal{L}})$ is not continuous. □

6.5 Solutions of Boolean-like Laws

In this section, we investigate the solutions of Boolean-like laws in a $\mathcal{T}$-power based implication operator $\mathtt{I}_{\mathtt{I}\mathcal{T}}$.

6.5.1 Solution of $v \leq_{\mathcal{L}} \mathtt{I}_{\mathtt{I}}(u, v)$

In this subsection, we will discuss the solution of the Boolean-like law:

$$v \leq_{\mathcal{L}} \mathtt{I}_{\mathtt{I}}(u, v) \ \forall\, u, v \in \mathcal{L} \tag{6.20}$$

where $\mathtt{I}_{\mathtt{I}}$ is an IFI.

Lemma 6.5.1. *Every $\mathtt{I}_{\mathtt{I}\mathcal{T}}$ satisfies (6.20) if $\mathtt{I}_{\mathtt{I}\mathcal{T}}$ satisfies (NP), i.e., $\mathtt{I}_{\mathtt{I}\mathcal{T}}$ follows the Proposition 6.4.10.*

Proof. Since $u \leq_{\mathcal{L}} 1_{\mathcal{L}}$, by (I2) and (NP), we have $\mathtt{I}_{\mathtt{I}\mathcal{T}}(u, v) \geq_{\mathcal{L}} \mathtt{I}_{\mathtt{I}\mathcal{T}}(1_{\mathcal{L}}, v) = v$. □

(6.20) is (CB) property. For all other solutions of (6.20), please see Propositions 6.4.12 and 6.4.13.

6.5.2 Solution of $\mathtt{I}_{\mathtt{I}}(u, \mathtt{I}_{\mathtt{I}}(v, u)) = 1_{\mathcal{L}}$

In this subsection, we will study the solution of the Boolean-like law:

$$\mathtt{I}_{\mathtt{I}}(u, \mathtt{I}_{\mathtt{I}}(v, u)) = 1_{\mathcal{L}} \ \forall\, u, v \in \mathcal{L} \tag{6.21}$$

where $\mathtt{I}_{\mathtt{I}}$ is an IFI.

Theorem 6.5.2. *Every* $\mathtt{I}_{\mathtt{I}\mathcal{T}}$ *satisfies (6.21) iff* $u_{\mathcal{T}}^{(w)} \geq_{\mathcal{L}} v$ *for every* $w <_{\mathcal{L}} u$ *and* $u, v, w \in \mathcal{L}$.

Proof. Let $\mathcal{T}$ be a continuous t-norm and $\mathtt{I}_{\mathtt{I}\mathcal{T}}$ be a $\mathcal{T}-$ power based implication operator.
($\Rightarrow$) Let us suppose that $u_{\mathcal{T}}^{(w)} \geq_{\mathcal{L}} v$ for every $w <_{\mathcal{L}} u$ and $u, v, w \in \mathcal{L}$. Then, by Proposition 6.4.12, $\mathtt{I}_{\mathtt{I}\mathcal{T}}$ satisfies (CB) so $\mathtt{I}_{\mathtt{I}\mathcal{T}}(v, u) \geq_{\mathcal{L}} u$. Since $\mathtt{I}_{\mathtt{I}\mathcal{T}}$ is non-decreasing corresponding to second tuple, $\mathtt{I}_{\mathtt{I}\mathcal{T}}(u, \mathtt{I}_{\mathtt{I}\mathcal{T}}(v, u)) \geq_{\mathcal{L}} \mathtt{I}_{\mathtt{I}\mathcal{T}}(u, u) = 1_{\mathcal{L}}$. Thus $\mathtt{I}_{\mathtt{I}\mathcal{T}}(u, \mathtt{I}_{\mathtt{I}\mathcal{T}}(v, u)) = 1_{\mathcal{L}}$.
($\Leftarrow$) Suppose $\mathtt{I}_{\mathtt{I}\mathcal{T}}$ satisfies $\mathtt{I}_{\mathtt{I}\mathcal{T}}(u, \mathtt{I}_{\mathtt{I}\mathcal{T}}(v, u)) = 1_{\mathcal{L}}$. Since $\mathtt{I}_{\mathtt{I}\mathcal{T}}$ satisfies (OP), $\mathtt{I}_{\mathtt{I}\mathcal{T}}(u, \mathtt{I}_{\mathtt{I}\mathcal{T}}(v, u)) = 1_{\mathcal{L}} \Rightarrow u \leq_{\mathcal{L}} \mathtt{I}_{\mathtt{I}\mathcal{T}}(v, u)$. This implies that $\mathtt{I}_{\mathtt{I}\mathcal{T}}$ satisfies (CB). Then by Proposition 6.4.12, it is concluded that $u_{\mathcal{T}}^{(w)} \geq_{\mathcal{L}} v$ for every $w <_{\mathcal{L}} u$ and $u, v, w \in \mathcal{L}$. □

Corollary 6.5.3. *Every* $\mathtt{I}_{\mathtt{I}\mathcal{T}}$ *satisfies (6.21) iff* $\mathtt{I}_{\mathtt{I}\mathcal{T}}$ *satisfies (CB).*

6.5.3 Solution of $\mathtt{I}_{\mathtt{I}}(u, v) = \mathtt{I}_{\mathtt{I}}(u, \mathtt{I}_{\mathtt{I}}(u, v))$

In this subsection, we find the solution of the Boolean-like law:

$$\mathtt{I}_{\mathtt{I}}(u, v) = \mathtt{I}_{\mathtt{I}}(u, \mathtt{I}_{\mathtt{I}}(u, v)) \; \forall \, u, v \, \in \mathcal{L} \tag{6.22}$$

where $\mathtt{I}_{\mathtt{I}}$ is an IFI.

Theorem 6.5.4. *Every* $\mathtt{I}_{\mathtt{I}\mathcal{T}}$ *satisfies (6.22) iff* $\mathtt{I}_{\mathtt{I}\mathcal{T}}(u, v) = \begin{cases} 1_{\mathcal{L}}, & 0_{\mathcal{L}} \leq_{\mathcal{L}} u \leq_{\mathcal{L}} v, \\ 0_{\mathcal{L}}, & 0_{\mathcal{L}} \leq_{\mathcal{L}} v <_{\mathcal{L}} u, \end{cases}$ $\forall \, u, v, w \in \mathcal{L}$.

Proof. Let $\mathcal{T}$ be a continuous t-norm and $\mathtt{I}_{\mathtt{I}\mathcal{T}}$ be a $\mathcal{T}-$ power based implication operator.
($\Rightarrow$) Let us suppose that $\mathtt{I}_{\mathtt{I}\mathcal{T}}$ satisfies (6.22). Then we have to prove that $\mathtt{I}_{\mathtt{I}\mathcal{T}}(u, v) = \begin{cases} 1_{\mathcal{L}}, & 0_{\mathcal{L}} \leq_{\mathcal{L}} u \leq_{\mathcal{L}} v, \\ 0_{\mathcal{L}}, & 0_{\mathcal{L}} \leq_{\mathcal{L}} v <_{\mathcal{L}} u, \end{cases}$ $\forall \, u, v, w \in \mathcal{L}$. Firstly, we have to show that $u_{\mathcal{T}}^{(u)} = u$ for all $u \in \mathcal{L} \setminus 0_{\mathcal{L}}$.
Obviously, when t, $u \leq_{\mathcal{L}} t \leq_{\mathcal{L}} 1_{\mathcal{L}}$, $\mathtt{I}_{\mathtt{I}\mathcal{T}}(u, t) = 1_{\mathcal{L}} \geq_{\mathcal{L}} u$ holds by the (OP) of $\mathtt{I}_{\mathtt{I}\mathcal{T}}$. Let us suppose that there exist t_0, $0_{\mathcal{L}} \leq_{\mathcal{L}} t_0 <_{\mathcal{L}} u$ such that $\mathtt{I}_{\mathtt{I}\mathcal{T}}(u, t_0) \geq_{\mathcal{L}} u$. Then by (OP) of $\mathtt{I}_{\mathtt{I}\mathcal{T}}$, we have $\mathtt{I}_{\mathtt{I}\mathcal{T}}(u, \mathtt{I}_{\mathtt{I}\mathcal{T}}(u, t_0)) = 1_{\mathcal{L}}$. Using (6.22), we have $\mathtt{I}_{\mathtt{I}\mathcal{T}}(u, t_0) = \mathtt{I}_{\mathtt{I}\mathcal{T}}(u, \mathtt{I}_{\mathtt{I}\mathcal{T}}(u, t_0)) = 1_{\mathcal{L}}$. This implies that $u \leq_{\mathcal{L}} t_0$, which contradicts the fact $0_{\mathcal{L}} \leq_{\mathcal{L}} t_0 <_{\mathcal{L}} u$. Hence $\mathtt{I}_{\mathtt{I}\mathcal{T}}(u, t_0) <_{\mathcal{L}} u$ for all t_0, $0_{\mathcal{L}} \leq_{\mathcal{L}} t_0 <_{\mathcal{L}} u$.
Now, when $u \leq_{\mathcal{L}} t \leq_{\mathcal{L}} 1_{\mathcal{L}}$,

$$\mathtt{I}_{\mathtt{I}\mathcal{T}}(u, t) \geq_{\mathcal{L}} u \;\Rightarrow\; \sup\{\gamma \in \mathcal{L} : t_{\mathcal{T}}^{(\gamma)} \geq_{\mathcal{L}} u\} \geq_{\mathcal{L}} u \;\Rightarrow\; t_{\mathcal{T}}^{(u)} \geq_{\mathcal{L}} u \tag{6.23}$$

Moreover, when $0_{\mathcal{L}} \leq_{\mathcal{L}} t_0 <_{\mathcal{L}} u$,

$$\mathtt{I}_{\mathtt{I}\mathcal{T}}(u, t_0) <_{\mathcal{L}} u \;\Rightarrow\; \sup\{\gamma \in \mathcal{L} : (t_0)_{\mathcal{T}}^{(\gamma)} \geq_{\mathcal{L}} u\} <_{\mathcal{L}} u \;\Rightarrow\; (t_0)_{\mathcal{T}}^{(u)} <_{\mathcal{L}} u \tag{6.24}$$

Since $\mathcal{T}$ is the continuous, (6.23) and (6.24) are as

$$\lim_{t \to u} t_{\mathcal{T}}^{(u)} = u_{\mathcal{T}}^{(u)} \geq_{\mathcal{L}} u \tag{6.25}$$

$$\lim_{t_0 \to u} (t_0)_{\mathcal{T}}^{(u)} = u_{\mathcal{T}}^{(u)} <_{\mathcal{L}} u \tag{6.26}$$

From (6.25) and (6.26), we have

$$u_{\mathcal{T}}^{(u)} = u \tag{6.27}$$

Taking $0_{\mathcal{L}} \leq_{\mathcal{L}} u \leq_{\mathcal{L}} v$, we have

$$\mathtt{I}_{\mathtt{I}\mathcal{T}}(u,v) = \sup\{\gamma \in \mathcal{L} : v_{\mathcal{T}}^{(\gamma)} \geq_{\mathcal{L}} u\} = \sup\{1_{\mathcal{L}}\} = 1_{\mathcal{L}} \tag{6.28}$$

Taking $0_{\mathcal{L}} \leq_{\mathcal{L}} v <_{\mathcal{L}} u$, we have

$$\mathtt{I}_{\mathtt{I}\mathcal{T}}(u,v) = \sup\{\gamma \in \mathcal{L} : v_{\mathcal{T}}^{(\gamma)} \geq_{\mathcal{L}} u\} = \sup\{0_{\mathcal{L}}\} = 0_{\mathcal{L}} \tag{6.29}$$

Combining (6.28) and (6.29), we have

$$\mathtt{I}_{\mathtt{I}\mathcal{T}}(u,v) = \begin{cases} 1_{\mathcal{L}}, & v \geq_{\mathcal{L}} u \geq_{\mathcal{L}} 0_{\mathcal{L}}, \\ 0_{\mathcal{L}}, & u >_{\mathcal{L}} v \geq_{\mathcal{L}} 0_{\mathcal{L}}, \end{cases}$$

($\Leftarrow$) It is easy to see that the $\mathtt{I}_{\mathtt{I}\mathcal{T}}(u,v) = \begin{cases} 1_{\mathcal{L}}, & v \geq_{\mathcal{L}} u \geq_{\mathcal{L}} 0_{\mathcal{L}}, \\ 0_{\mathcal{L}}, & u >_{\mathcal{L}} v \geq_{\mathcal{L}} 0_{\mathcal{L}}, \end{cases}$

satisfies (6.22). □

Corollary 6.5.5. *Every* $\mathtt{I}_{\mathtt{I}\mathcal{T}}$ *satisfies (6.22) iff* $u_{\mathcal{T}}^{(u)} = u \; \forall \; u \in \mathcal{L} \setminus 0_{\mathcal{L}}$.

Proof. Let $\mathcal{T}$ be a continuous t-norm and $\mathtt{I}_{\mathtt{I}\mathcal{T}}$ be a $\mathcal{T}-$ power based implication operator.
($\Rightarrow$) Let us suppose that $\mathtt{I}_{\mathtt{I}\mathcal{T}}$ satisfies the (6.22). Then by using Theorem 6.5.4, we obtain $u_{\mathcal{T}}^{(u)} = u \; \forall \; u \in \mathcal{L} \setminus 0_{\mathcal{L}}$.
($\Leftarrow$) Suppose $u_{\mathcal{T}}^{(u)} = u$ holds $\forall \; u \in \mathcal{L} \setminus 0_{\mathcal{L}}$. Then by Theorem 6.5.4, we get

$$\mathtt{I}_{\mathtt{I}\mathcal{T}}(u,v) = \begin{cases} 1_{\mathcal{L}}, & v \geq_{\mathcal{L}} u \geq_{\mathcal{L}} 0_{\mathcal{L}}, \\ 0_{\mathcal{L}}, & u >_{\mathcal{L}} v \geq_{\mathcal{L}} 0_{\mathcal{L}}, \end{cases}$$

$\forall \; u, v, w \in \mathcal{L}$. It is easy to see that the $\mathtt{I}_{\mathtt{I}\mathcal{T}}$ satisfies (6.22). □

6.5.4 Solution of $\mathtt{I}_\mathtt{I}(u, \mathtt{I}_\mathtt{I}(v, w)) = \mathtt{I}_\mathtt{I}(\mathtt{I}_\mathtt{I}(u, v), \mathtt{I}_\mathtt{I}(u, w))$

In this subsection, we find the solution of the Boolean-like law:

$$\mathtt{I}_\mathtt{I}(u, \mathtt{I}_\mathtt{I}(v, w)) = \mathtt{I}_\mathtt{I}(\mathtt{I}_\mathtt{I}(u, v), \mathtt{I}_\mathtt{I}(u, w)) \ \forall\, u, v, w \in \mathcal{L} \tag{6.30}$$

where $\mathtt{I}_\mathtt{I}$ is an IFI.

Theorem 6.5.6. *Every $\mathtt{I}_{\mathtt{I}\mathcal{T}}$ satisfies (6.30) iff* $\mathtt{I}_{\mathtt{I}\mathcal{T}}(u, v) = \begin{cases} 1_\mathcal{L}, & v \geq_\mathcal{L} u \geq_\mathcal{L} 0_\mathcal{L}, \\ 0_\mathcal{L}, & u >_\mathcal{L} v \geq_\mathcal{L} 0_\mathcal{L} \end{cases}$ *$\forall\, u, v, w \in \mathcal{L}$.*

Proof. Let $\mathcal{T}$ be a continuous t-norm and $\mathtt{I}_{\mathtt{I}\mathcal{T}}$ be a $\mathcal{T}-$ power based implication operator.
($\Rightarrow$) Let us suppose that $\mathtt{I}_{\mathtt{I}\mathcal{T}}$ satisfies the (6.30). Then we have to prove that $\mathtt{I}_{\mathtt{I}\mathcal{T}}(u, v) = \begin{cases} 1_\mathcal{L}, & v \geq_\mathcal{L} u \geq_\mathcal{L} 0_\mathcal{L}, \\ 0_\mathcal{L}, & u >_\mathcal{L} v \geq_\mathcal{L} 0_\mathcal{L} \end{cases}$ $\forall\, u, v, w \in \mathcal{L}$. Firstly, we have to show that $u_\mathcal{T}^{(u)} = u \ \forall\, u \in \mathcal{L} \setminus 0_\mathcal{L}$. Obviously, when $u \leq_\mathcal{L} t \leq_\mathcal{L} 1_\mathcal{L}$, $\mathtt{I}_{\mathtt{I}\mathcal{T}}(u, t) = 1_\mathcal{L} \geq_\mathcal{L} u$ holds by the (OP) of $\mathtt{I}_{\mathtt{I}\mathcal{T}}$. Let us suppose that there exists t_0, $0_\mathcal{L} \leq_\mathcal{L} t_0 <_\mathcal{L} u$ such that $\mathtt{I}_{\mathtt{I}\mathcal{T}}(u, t_0) \geq_\mathcal{L} x$. Then by (OP) of $\mathtt{I}_{\mathtt{I}\mathcal{T}}$, we have $\mathtt{I}_{\mathtt{I}\mathcal{T}}(u, \mathtt{I}_{\mathtt{I}\mathcal{T}}(u, t_0)) = 1_\mathcal{L}$. Using the (6.30), we have $1_\mathcal{L} = \mathtt{I}_{\mathtt{I}\mathcal{T}}(u, \mathtt{I}_{\mathtt{I}\mathcal{T}}(u, t_0)) = \mathtt{I}_{\mathtt{I}\mathcal{T}}(\mathtt{I}_{\mathtt{I}\mathcal{T}}(u, u), \mathtt{I}_{\mathtt{I}\mathcal{T}}(u, t_0))$. Since $\mathtt{I}_{\mathtt{I}\mathcal{T}}$ satisfies (IP), $1_\mathcal{L} = \mathtt{I}_{\mathtt{I}\mathcal{T}}(\mathtt{I}_{\mathtt{I}\mathcal{T}}(u, u), \mathtt{I}_{\mathtt{I}\mathcal{T}}(u, t_0)) = \mathtt{I}_{\mathtt{I}\mathcal{T}}(1_\mathcal{L}, \mathtt{I}_{\mathtt{I}\mathcal{T}}(u, t_0)) \Rightarrow 1_\mathcal{L} \leq_\mathcal{L} \mathtt{I}_{\mathtt{I}\mathcal{T}}(u, t_0)$, by (OP) $\mathtt{I}_{\mathtt{I}\mathcal{T}}$, but $1_\mathcal{L} \geq_\mathcal{L} \mathtt{I}_{\mathtt{I}\mathcal{T}}(u, t_0)$. Thus $\mathtt{I}_{\mathtt{I}\mathcal{T}}(u, t_0) = 1_\mathcal{L}$. This implies that $u \leq_\mathcal{L} t_0$, which contradict the fact $0_\mathcal{L} \leq_\mathcal{L} t_0 <_\mathcal{L} u$. Hence $\mathtt{I}_{\mathtt{I}\mathcal{T}}(u, t_0) <_\mathcal{L} u$ for all t_0, $0_\mathcal{L} \leq_\mathcal{L} t_0 <_\mathcal{L} u$. Now, when $u \leq_\mathcal{L} t \leq_\mathcal{L} 1_\mathcal{L}$,

$$\mathtt{I}_{\mathtt{I}\mathcal{T}}(u, t) \geq_\mathcal{L} u \ \Rightarrow\ \sup\{\gamma \in \mathcal{L} : t_\mathcal{T}^{(\gamma)} \geq_\mathcal{L} u\} \geq_\mathcal{L} u \ \Rightarrow\ t_\mathcal{T}^{(u)} \geq_\mathcal{L} u \tag{6.31}$$

Moreover, when $0_\mathcal{L} \leq_\mathcal{L} t_0 <_\mathcal{L} u$,

$$\begin{aligned} \mathtt{I}_{\mathtt{I}\mathcal{T}}(u, t_0) <_\mathcal{L} u \ &\Rightarrow \sup\{\gamma \in \mathcal{L} : (t_0)_\mathcal{T}^{(\gamma)} \geq_\mathcal{L} u\} <_\mathcal{L} u \\ &\Rightarrow (t_0)_\mathcal{T}^{(u)} <_\mathcal{L} u \end{aligned} \tag{6.32}$$

Since $\mathcal{T}$ is the continuous, (6.31) and (6.32) imply that

$$\lim_{t \to u} t_\mathcal{T}^{(u)} = u_\mathcal{T}^{(u)} \geq_\mathcal{L} u \tag{6.33}$$

$$\lim_{t_0 \to u} (t_0)_\mathcal{T}^{(u)} = u_\mathcal{T}^{(u)} <_\mathcal{L} u \tag{6.34}$$

From (6.33) and (6.34), we have

$$u_\mathcal{T}^{(u)} = u \tag{6.35}$$

Taking $0_{\mathcal{L}} \leq_{\mathcal{L}} u \leq_{\mathcal{L}} v$, we obtain that

$$\mathtt{I}_{\mathtt{I}\mathcal{T}}(u,v) = \sup\{\gamma \in \mathcal{L} : v_{\mathcal{T}}^{(\gamma)} \geq_{\mathcal{L}} u\} = \sup\{1_{\mathcal{L}}\} = 1_{\mathcal{L}} \tag{6.36}$$

Taking $0_{\mathcal{L}} \leq_{\mathcal{L}} v <_{\mathcal{L}} u$, we have

$$\mathtt{I}_{\mathtt{I}\mathcal{T}}(u,v) = \sup\{\gamma \in \mathcal{L} : v_{\mathcal{T}}^{(\gamma)} \geq_{\mathcal{L}} u\} = \sup\{0_{\mathcal{L}}\} = 0_{\mathcal{L}} \tag{6.37}$$

Combining (6.36) and (6.37), we have

$$\mathtt{I}_{\mathtt{I}\mathcal{T}}(u,v) = \begin{cases} 1_{\mathcal{L}}, & v \geq_{\mathcal{L}} u \geq_{\mathcal{L}} 0_{\mathcal{L}}, \\ 0_{\mathcal{L}}, & u >_{\mathcal{L}} v \geq_{\mathcal{L}} 0_{\mathcal{L}}, \end{cases}$$

($\Leftarrow$) Easily, we see that the $\mathtt{I}_{\mathtt{I}\mathcal{T}}(u,v) = \begin{cases} 1_{\mathcal{L}}, & v \geq_{\mathcal{L}} u \geq_{\mathcal{L}} 0_{\mathcal{L}}, \\ 0_{\mathcal{L}}, & u >_{\mathcal{L}} v \geq_{\mathcal{L}} 0_{\mathcal{L}} \end{cases}$
satisfies (6.30). □

Corollary 6.5.7. *Every* $\mathtt{I}_{\mathtt{I}\mathcal{T}}$ *satisfies (6.30) iff* $u_{\mathcal{T}}^{(u)} = u \; \forall \; u \in \mathcal{L} \setminus 0_{\mathcal{L}}$.

6.6 Concluding remarks

In this chapter, $\mathcal{T}$-power-based implications as a new class of implication operators on $\mathcal{L}$ was introduced. We have studied the properties of these implications. We have observed that some of the properties of fuzzy implications acting on the real unit interval [0, 1] are not satisfied by related $\mathcal{T}$-power-based implications acting on $\mathcal{L}$. We have shown that the studied $\mathcal{T}$-power-based implications on $\mathcal{L}$ satisfy the discussed properties after addition of some extra conditions. After that, the string of inequality of $\mathtt{I}_{\mathtt{I}\mathcal{T}}$ has been established. We have also introduced a new type of negation $\mathcal{N}^{\alpha}_{\mathtt{I}_{\mathtt{I}\mathcal{T}}}$ based on $\mathtt{I}_{\mathtt{I}\mathcal{T}}$, continuity and strictly monotonicity of this negation has been analyzed. Finally, we have investigated the solutions of Boolean-like laws (6.3) in $\mathtt{I}_{\mathtt{I}\mathcal{T}}$.

Chapter 7

Distributivity of implication operators on $\mathcal{L}$ over t-representable t-norms: The case of strict and nilpotent t-norms

In this chapter, a new class of IFIs known as $(\mathtt{f_I}, \omega)$-implications is introduced which is a generalized form of Yager's f-implications in IFE. Basic properties of these implications are discussed in detail. It is shown that $(\mathtt{f_I}, \omega)$-implications are not only the generalizations of Yager's f-implications, but also the generalizations of $\mathcal{R}$-, $(\mathcal{S}, \mathtt{N})$- and $\mathcal{QL}$-implications in IFE. The distributive equations $\mathtt{I_I}(\mathcal{T}(u, v), w) = \mathcal{S}(\mathtt{I_I}(u, w), \mathtt{I_I}(v, w))$ and $\mathtt{I_I}(u, \mathcal{T}_1(v, w)) = \mathcal{T}_2(\mathtt{I_I}(u, v), \mathtt{I_I}(u, w))$ over t-representable t-norms and t-conorms generated from nilpotent and strict t-norms in IFS theory are discussed. Also, one of the open problems posed by Baczyński [17, 18] is solved.

7.1 Introduction

In fuzzy logic, FIs have become one of the most important operators [19]. These implications are derived from the new class of implications such as (g, min)-implications [116], generalized h-generators [117] and (g, u)-implications [196] etc. Fuzzy implications are also very useful in fuzzy connectivity ([44, 102, 172]). Nowadays, many different extensions of FSs are known as L-FSs [80], interval-valued FS [81] which represents the degree of membership of an element by an interval rather than exact numerical value and IFS [11] to model the non-determinacy which occurs in the system because of the the hesitation of decision maker etc. An IFS is described by a membership function and a non-membership function. It is proven to be a more suitable tool than an FS to describe imprecise or uncertain information.

Interval-valued FS theory has become very popular from both theoretical and practical aspects. One can find theoretical articles connected with different classes of interval-valued logical connectives like interval-valued fuzzy negations [26], interval-valued t-norms [56, 174], interval-valued fuzzy uninorms [57], interval-valued fuzzy implications [5, 27, 28, 111], interval-valued fuzzy implications based on binary aggregation operators [61, 118] and interval-valued fuzzy relations [55, 173]. Similarly, one can find many articles with practical applications of interval-valued FS theory to the robustness of interval-valued fuzzy inference and a representable of cardinality (see [58, 113]).

IFSs have become very popular that are connected with different classes of intuitionistic logical connectives like intuitionistic t-norms [59], IFIs [33, 45, 161] and intuitionistic fuzzy relations (IF relations) [34, 146]. Atanassov's intuitionistic De Morgan triple via automorphisms is introduced and used in approximate reasoning (see [46, 200]).

7.1.1 Motivation

Combs and Andrews [44] attempted to exploit the equivalence

$$(j \wedge k) \to l \equiv (j \to l) \vee (k \to l) \tag{7.1}$$

towards eliminating combinatorial rule explosion in fuzzy systems. It is one of the four equations given by Cox [49]. The remaining three equations are as follows:

$$(j \vee k) \to l \equiv (j \to l) \wedge (k \to l) \tag{7.2}$$

$$j \to (k \wedge l) \equiv (j \to k) \wedge (j \to l) \tag{7.3}$$

$$j \to (k \vee l) \equiv (j \to k) \vee (j \to l) \tag{7.4}$$

Obviously, the above equivalences are tautologies in classical logic and their generalizations in fuzzy logic lead to the distributivity of fuzzy implications over t-norms and t-conorms as given below:

$$I(T(a,b),c) = S(I(a,c),I(b,c)) \tag{7.5}$$

$$I(S(a,b),c) = T(I(a,c),I(b,c)) \tag{7.6}$$

$$I(a,T_1(b,c)) = T_2(I(a,b),I(a,c)) \tag{7.7}$$

$$I(a,S_1(b,c)) = S_2(I(a,b),I(a,c)) \tag{7.8}$$

for $a,b,c \in [0,1]$, where I is a fuzzy implication; T,T_1,T_2 are t-norms and S,S_1,S_2 are t-conorms in fuzzy environment. The above equations play an important role in lossless rule reduction in Fuzzy Rule Based Systems [21, 49]. Conditions under which eqs. (7.5)-(7.8) hold for (S, N)-, $R-, QL$-implications can be found in [19, 22, 172]. Balasubramaniam [21] studied the distributivity of Yager's f-implications over t-norms and t-conorms. Baczyński and Jayaram [20] investigated the implications that satisfy eqs. (7.5)-(7.8) for nilpotent or strict triangular conorms.

Atanassov and Gargov [15] proposed the elements of IF logic. Obviously, the equivalences (7.1)-(7.4) are tautologies in classical logic and their generalizations in IF logic lead to the distributivity of IFIs over t-norms and t-conorms in IFE as given below:

$$\mathtt{I}_{\mathtt{I}}(\mathcal{T}(u,v),w) = \mathcal{S}(\mathtt{I}_{\mathtt{I}}(u,w),\mathtt{I}_{\mathtt{I}}(v,w)) \tag{7.9}$$

$$\mathtt{I}_{\mathtt{I}}(\mathcal{S}(u,v),w) = \mathcal{T}(\mathtt{I}_{\mathtt{I}}(u,w),\mathtt{I}_{\mathtt{I}}(v,w)) \tag{7.10}$$

$$\mathtt{I}_{\mathtt{I}}(u,\mathcal{T}_1(v,w)) = \mathcal{T}_2(\mathtt{I}_{\mathtt{I}}(u,v),\mathtt{I}_{\mathtt{I}}(u,w)) \tag{7.11}$$

$$\mathtt{I}_{\mathtt{I}}(u,\mathcal{S}_1(v,w)) = \mathcal{S}_2(\mathtt{I}_{\mathtt{I}}(u,v),\mathtt{I}_{\mathtt{I}}(u,w)) \tag{7.12}$$

for $u, v, w \in \mathcal{L}$, where $\mathtt{I}_{\mathtt{I}}$ is an IFI; $\mathcal{T}, \mathcal{T}_1, \mathcal{T}_2$ are t-norms and $\mathcal{S}, \mathcal{S}_1, \mathcal{S}_2$ are t-conorms in IF environment. Now, we define Yager's class of implications in fuzzy and IF environment.

Definition 7.1.1. *[19] Let $f : [0,1] \to [0,\infty]$ be a decreasing and continuous function satisfying $f(1) = 0$. Then a function $I_f : [0,1]^2 \to [0,1]$ defined by*

$$I_f(a,b) = f^{(-1)}(af(b)) \; \forall \; a,b \in [0,1] \tag{7.13}$$

is called the Yager's class of fuzzy implications generated by f.

Definition 7.1.2. *Suppose that $\mathcal{L} = \{(u_1,u_2) \in [0,1]^2 : u_1 + u_2 \leq 1\}$. Let $\mathtt{f}_{\mathtt{I}} : \mathcal{L} \to [0,\infty]^2$ be a decreasing and continuous function satisfying $\mathtt{f}_{\mathtt{I}}(1_{\mathcal{L}}) = 0_{\mathcal{L}}$. Then a function $\mathtt{I}_{\mathtt{If}_{\mathtt{I}}} : \mathcal{L}^2 \to \mathcal{L}$ defined by*

$$\mathtt{I}_{\mathtt{If}_{\mathtt{I}}}(u,v) = \mathtt{f}_{\mathtt{I}}^{(-1)}(u\mathtt{f}_{\mathtt{I}}(v)) \; \forall \; u,v \in \mathcal{L}, \tag{7.14}$$

is called the Yager's class of IFIs generated by $\mathtt{f}_{\mathtt{I}}$.

Motivated by the Definition 7.1.2, we propose new implications known as $(\mathtt{f}_{\mathtt{I}}, \omega)$-implications which are generalized form of the Yager's class of IFIs. Baczyński [17] solved distributive eqs. (7.9) and (7.11) when t-norms are strict. The solutions of distributive eqs. (7.9) and (7.11) have been obtained with the help of all the solutions of the following functional equation:

$$h(y_1+z_1, y_2+z_2) = h(y_1,y_2) + h(z_1,z_2), \quad (y_1,y_2),(z_1,z_2) \in L^\infty \tag{7.15}$$

where $L^\infty = \{(y_1,y_2) \in [0,\infty]^2 : y_1 \geq y_2\}$ and $h : L^\infty \to [0,\infty]$ is an unknown function. This equation is related to the case with strict t-norms.

In each of the distributive eqs. (7.9) and (7.11), there are 15 possible solutions corresponding to the vertical section of $\mathtt{I}_{\mathtt{I}}(u,.)$ and 15 possible solutions corresponding to the horizontal section for a fixed $u \in \mathcal{L}$. Thus, 225 different possible solutions of each of distributive eqs. (7.9) and (7.11) are obtained. But all the possible solutions for the (7.9) and (7.11) are not correct solutions. Baczyński [17] has posed the following open problem.

Problem 1. *Characterize all of the correct solutions out of 225 different solutions of distributive eq. (7.11) when t-norms are strict (for a fixed $u \in \mathcal{L}$ or a fixed $v \in \mathcal{L}$).*

On other hand, Baczyński [18] solved distributive eqs. (7.9) and (7.11) when the t-norms are nilpotent. The solutions of distributive eqs. (7.9) and (7.11) are obtained with the help of all the obtained solutions of the following functional equation, for fix real numbers $a, b > 0$,

$$h(\min(y_1 + z_1, a), \min(y_2 + z_2, a)) = \min(h(y_1, y_2) + h(z_1, z_2), b), \quad (y_1, y_2), (z_1, z_2) \in L^a \quad (7.16)$$

where $L^a = \{(y_1, y_2) \in [0, a]^2 : y_1 \geq y_2\}$ and $h : L^a \to [0, a]$ is an unknown function. This equation is related to the case with nilpotent t-norms. In each of the distributive eqs. (7.9) and (7.11), there are 9 possible solutions corresponding to the vertical section of $\mathtt{I}_{\mathtt{I}}(u,.)$ and 9 possible solutions corresponding to the horizontal section for a fixed $u \in \mathcal{L}$. Thus, 81 different possible solutions of each of the distributive eqs. (7.9) and (7.11) are obtained. But all the possible solutions for the (7.9) and (7.11) are not correct solutions. Baczyński [18] has posed the following open problem.

Problem 2. *Characterize all of the correct solutions out of 81 different solutions of distributive eq. (7.11) when t-norms are nilpotent (for a fixed $u \in \mathcal{L}$ or a fixed $v \in \mathcal{L}$).*

After motivated by above facts, we solve the open problems posed by Baczyński [17, 18].

In this chapter, we have solved distributive eqs. (7.9) and (7.11) when the t-norms are nilpotent and distributive eqs. (7.10) and (7.12) when the t-norms.

The rest of the paper is organized as follows. In Section 7.2, we introduce some basic definitions needed throughout the whole paper. The core of the paper is represented by Sections 7.3 and 7.4 concerning to the development of the new class of IFI known as $(\mathtt{f}_{\mathtt{I}}, \omega)$-implication which is a generalization of Yager's f-implications in IFE, the discussion of some of their properties and the proofs of specific results, and the solution of open problems related to the distributivity of $(\mathtt{f}_{\mathtt{I}}, \omega)$-implication over t-representable t-norms in IFE. Finally, the paper ends with Section 7.5 containing conclusions.

7.2 Preliminaries

Definition 7.2.1. *[11]*

1). $\mathcal{L} = \{(u_1, u_2) : (u_1, u_2) \in [0,1]^2,\ u_1 + u_2 \leq 1\}$ *be an IF interpretation triangle and the operation* $\leq_{\mathcal{L}}$ *be defined on* $\mathcal{L}$ *by*

$$(u_1, u_2) \leq_{\mathcal{L}} (v_1, v_2) \Leftrightarrow u_1 \leq v_1,\ u_2 \geq v_2,\ \forall\ (u_1, u_2), (v_1, v_2) \in \mathcal{L}.$$

For each nonempty set $\mathcal{A} \subseteq \mathcal{L}$*, we have*

$$\begin{aligned} \sup \mathcal{A} =& (\sup\{u_1 : (u_1, u_2) \in \mathcal{A}\}, \inf\{u_2 : (u_1, u_2) \in \mathcal{A}\}), \\ \inf \mathcal{A} =& (\inf\{u_1 : (u_1, u_2) \in \mathcal{A}\}, \sup\{u_2 : (u_1, u_2) \in \mathcal{A}\}). \end{aligned}$$

Then $(\mathcal{L}, \leq_{\mathcal{L}})$ *is a complete lattice [60]. Equivalently, this lattice can also be defined as an algebraic structure* $(\mathcal{L}, \vee, \wedge)$ *where the join operator* $\vee$ *and the meet operator* $\wedge$ *are defined as follows:*
For $x, y \in \mathcal{L}$*,*

$$u \vee v = (\max(u_1, v_1), \min(u_2, v_2)),\ u \wedge v = (\min(u_1, v_1), \max(u_2, v_2)).$$

2). For an arbitrary number P *in* $[1, \infty)$*, we define the sets*

$$\begin{aligned} \mathcal{L}_{\leq P} =& \{(u_1, u_2) : (u_1, u_2) \in [0, P]^2,\ u_1 + u_2 \leq P\}, \\ \mathcal{L}_{=P} =& \{(u_1, u_2) : (u_1, u_2) \in [0, P]^2,\ u_1 + u_2 = P\}, \\ \mathcal{L}_{\geq P} =& \{(u_1, u_2) : (u_1, u_2) \in [0, \infty]^2,\ u_1 + u_2 \geq P\}. \end{aligned}$$

The operation $\leq_{\mathcal{L}_{\leq P}}$*, for fixed* P *on* $\mathcal{L}_{\leq P}$*, defined by*

$$(u_1, u_2) \leq_{\mathcal{L}_{\leq P}} (v_1, v_2) \Leftrightarrow u_1 \leq v_1,\ u_2 \geq v_2,\ \forall\ (u_1, u_2), (v_1, v_2) \in \mathcal{L}_{\leq P}.$$

For each nonempty set $\mathcal{A} \subseteq \mathcal{L}_{\leq P}$*, we have*

$$\begin{aligned} \sup \mathcal{A} =& (\sup\{u_1 : (u_1, u_2) \in \mathcal{A}\}, \inf\{u_2 : (u_1, u_2) \in \mathcal{A}\}), \\ \inf \mathcal{A} =& (\inf\{u_1 : (u_1, u_2) \in \mathcal{A}\}, \sup\{u_2 : (u_1, u_2) \in \mathcal{A}\}). \end{aligned}$$

Clearly $(\mathcal{L}_{\leq P}, \leq_{\mathcal{L}_{\leq P}})$ *is a complete lattice.*

3). For an arbitrary number P *in* $[1, \infty)$*, the operation* $\leq_{\mathcal{L}_{\leq \geq P}}$ *on* $\mathcal{L}_{\leq P} \cup \mathcal{L}_{\geq P}$ *defined by*

$$(u_1, u_2) \leq_{\mathcal{L}_{\leq \geq P}} (v_1, v_2) \Leftrightarrow u_1 \leq v_1,\ u_2 \geq v_2\ \ \forall\ (u_1, u_2), (v_1, v_2) \in \mathcal{L}_{\leq P} \cup \mathcal{L}_{\geq P}.$$

Then $(\mathcal{L}_{\leq P} \cup \mathcal{L}_{\geq P}, \leq_{\mathcal{L}_{\leq \geq P}})$ *is a lattice.*

4). The operations $\leq_\infty$ and $<_\infty$ on $[0,\infty]^2$ respectively defined by

$$(u_1,u_2) \leq_\infty (v_1,v_2) \Leftrightarrow u_1 \leq v_1,\ u_2 \geq v_2,\quad (u_1,u_2) <_\infty (v_1,v_2) \Leftrightarrow u_1 < v_1,\ u_2 > v_2 \quad \forall\,(u_1,u_2),(v_1,v_2) \in [0,\infty]^2.$$

Remark 7.2.2. *(i) Note that if for $(u_1,u_2),(v_1,v_2) \in \mathcal{L}$, $u_1 < v_1$* and *$u_2 < v_2$, then u and v are incomparable with respect to $\leq_\mathcal{L}$, written as $u\|_\mathcal{L} v$.*

(ii) Note that if for $(u_1,u_2),(v_1,v_2) \in \mathcal{L}_{\leq P}$, $u_1 < v_1$ and *$u_2 < v_2$, then u and v are incomparable with respect to $\leq_{\mathcal{L}_{\leq P}}$, written as $u\|_{\mathcal{L}_{\leq P}} v$.*

(iii) For $p = 1$, the set $\mathcal{L}_{\leq p}$ represents the set $\mathcal{L}$.

(iv) We denote the units $0_\mathcal{L} = (0,1)$, $1_\mathcal{L} = (1,0)$ for the set $\mathcal{L}$, and $0_{\mathcal{L}_{\leq P}} = (0,P)$, $1_{\mathcal{L}_{\leq P}} = (P,0)$ for the set $\mathcal{L}_{\leq P}$.

7.3 $(\mathtt{f}_\mathtt{I}, \omega)$-implications

Definition 7.3.1. *A function $\mathtt{f}_\mathtt{I} : \mathcal{L} \to [0,\infty]^2$ is said to be a decreasing generator on $\mathcal{L}$ if it satisfies the following conditions:*

(i) $\mathtt{f}_\mathtt{I}$ is decreasing,

(ii) $\mathtt{f}_\mathtt{I}$ is continuous,

(iii) $\mathtt{f}_\mathtt{I}(1_\mathcal{L}) = 0_\mathcal{L}$.

Example 7.3.2. *Consider the following mappings from $\mathcal{L} \to \mathcal{L}$:*

(i) $\mathtt{f}_{1\mathtt{I}}(u) = (u_2, 1-u_2)$,

(ii) $\mathtt{f}_{2\mathtt{I}}(u) = (1-u_1+u_2, 1-u_2)$,

$\forall\ (u_1,u_2) \in \mathcal{L}$. It is easily verified that these are the decreasing generators on $\mathcal{L}$.

Definition 7.3.3. *Let $\mathtt{f}_\mathtt{I} : \mathcal{L} \to [0,\infty]^2$ be a decreasing generator on $\mathcal{L}$. Then the pseudo-inverse $\mathtt{f}_\mathtt{I}^{(-1)} : [0,\infty]^2 \to \mathcal{L}$ of $\mathtt{f}_\mathtt{I}$ is defined by*

$$\mathtt{f}_\mathtt{I}^{(-1)}(v) = \sup\{u : u \in \mathcal{L},\ \mathtt{f}_\mathtt{I}(u) >_\infty v\}\ \forall\, v \in [0,\infty]^2.$$

Generally, $\exists$ a $\mathtt{f}_\mathtt{I} : \mathcal{L} \to [0,\infty]^2$ mapping which is a decreasing generator on $\mathcal{L}$ but does not satisfy the property $\mathtt{f}_\mathtt{I}^{(-1)}(\mathtt{f}_\mathtt{I}(u)) = u$.

Example 7.3.4. *The pseudo-inverse of decreasing generators, given in Examples 7.3.2(i) and (ii), are as follows:*

(i)
$$\begin{aligned}\mathtt{f}_{\mathtt{I}}^{(-1)}(v) &= \sup\{u : u \in \mathcal{L},\ \mathtt{f}_{\mathtt{I}}(u) >_{\infty} v\},\ \forall\, v \in [0,\infty]^2 \\ &= \sup\{u : u \in \mathcal{L},\ (u_2, 1-u_2) >_{\infty} (v_1, v_2)\} \\ &= \sup\{u : u \in \mathcal{L},\ u_2 > v_1, 1-u_2 < v_2\} \\ &= (\min\{1, v_2, 1-v_1\}, \max\{v_1, 1-v_2\}),\ \forall\, v = (v_1, v_2) \in [0,\infty]^2.\end{aligned}$$

(ii)
$$\begin{aligned}\mathtt{f}_{\mathtt{I}}^{(-1)}(v) &= \sup\{u : u \in \mathcal{L},\ \mathtt{f}_{\mathtt{I}}(u) >_{\infty} v\},\ \forall\, v \in [0,\infty]^2 \\ &= \sup\{u : u \in \mathcal{L},\ (1-u_1+u_2, 1-u_2) >_{\infty} (v_1, v_2)\} \\ &= \sup\{u : u \in \mathcal{L},\ 1-u_1+u_2 > v_1, 1-u_2 < v_2\} \\ &= (\min\{1, v_2, \frac{2-v_1}{2}\}, \max\{0, 1-v_2\}),\ \forall\, v = (v_1, v_2) \in [0,\infty]^2.\end{aligned}$$

Definition 7.3.5. *A function $\omega_{\mathtt{f}} : \mathcal{L} \times ran(\mathtt{f}_{\mathtt{I}}) \to [0,\infty]^2$ is called a $\omega-$operator of a decreasing generator $\mathtt{f}_{\mathtt{I}}$ if it satisfies the following conditions:*

$$\omega_{\mathtt{f}} \text{ is non-decreasing in each argument} \tag{U1}$$

$$\omega_{\mathtt{f}}(0_{\mathcal{L}}, v) = 0_{\mathcal{L}}\ \forall\, v \in ran(\mathtt{f}_{\mathtt{I}}) \tag{U2}$$

$$\omega_{\mathtt{f}}(1_{\mathcal{L}}, v) = v\ \forall\, v \in ran(\mathtt{f}_{\mathtt{I}}) \tag{U3}$$

For simplicity, we use ω instead of $\omega_{\mathtt{f}}$.
The non-decreasing meaning of ω in each argument is as follows:

(i) for a fixed $v_0 \in ran(\mathtt{f}_{\mathtt{I}})$, $u <_{\mathcal{L}} u' \ \Rightarrow\ \omega(u, v_0) \leq_{\infty} \omega(u', v_0)$;

(ii) for a fixed $u_0 \in \mathcal{L}$, $v <_{\mathcal{L}_{\leq \geq \mathtt{f}_{\mathtt{I}}(0_{\mathcal{L}})}} v' \ \Rightarrow\ \omega(u_0, v) \leq_{\infty} \omega(u_0, v')$,

for $u, u' \in \mathcal{L}; v, v' \in ran(\mathtt{f}_{\mathtt{I}})$.

Example 7.3.6. *(i) If we take the $\mathtt{f}_{\mathtt{I}}$-operator $\mathtt{f}_{\mathtt{I}}(u_1, u_2) = (u_2, u_1)$, then the function $\omega : \mathcal{L} \times ran(\mathtt{f}_{\mathtt{I}}) \to [0,\infty]^2$ defined by*

$$\omega(u, v) = (u_1 v_1, 1-(1-u_2)(1-v_2))\ \forall\, u = (u_1, u_2) \in \mathcal{L},\ v = (v_1, v_2) \in ran(\mathtt{f}_{\mathtt{I}}).$$

satisfies the conditions $(U1)$, $(U2)$ and $(U3)$ of Definitions 7.3.5. Thus, ω is the $\omega-$operator of a decreasing generator $\mathtt{f}_{\mathtt{I}}$.

(ii) *Consider the decreasing generator* $\mathtt{f}_{\mathtt{I}}(u_1,u_2)=(u_2,u_1)$ *and the function* $\omega:\mathcal{L}\times ran(\mathtt{f}_{\mathtt{I}})\to[0,\infty]^2$ *defined by*

$$\omega(u,v)=\begin{cases}(v_1,v_2), & u=1_{\mathcal{L}},\ v=(v_1,v_2)\in ran(\mathtt{f}_{\mathtt{I}}),\\ (u_1,u_2), & u=(u_1,u_2)\in\mathcal{L}\setminus\{1_{\mathcal{L}}\},\ v=(v_1,v_2)\in ran(\mathtt{f}_{\mathtt{I}}).\end{cases}$$

It is easily to verified that ω *is the* $\omega-$*operator of a decreasing generator* $\mathtt{f}_{\mathtt{I}}$.

Definition 7.3.7. *A function* $\mathtt{I}_{\mathtt{I}(\mathtt{f}_{\mathtt{I}},\omega)}:\mathcal{L}^2\to\mathcal{L}$ *defined by*

$$\mathtt{I}_{\mathtt{I}(\mathtt{f}_{\mathtt{I}},\omega)}(u,v)=\mathtt{f}_{\mathtt{I}}^{(-1)}(\omega(u,\mathtt{f}_{\mathtt{I}}(v)))\ \forall\ u=(u_1,u_2),v=(v_1,v_2)\in\mathcal{L} \tag{7.17}$$

is called a $(\mathtt{f}_{\mathtt{I}},\omega)$*- implication generated by a decreasing generator* $\mathtt{f}_{\mathtt{I}}$ *in* $\mathcal{L}$ *and an* $\omega-$*operator. Here* $\mathtt{f}_{\mathtt{I}}^{(-1)}$ *is the pseudo-inverse of* $\mathtt{f}_{\mathtt{I}}$ *(see Definition 7.3.3).*

Example 7.3.8. (i) *Consider the decreasing generator* $\mathtt{f}_{\mathtt{I}}:\mathcal{L}\to\mathcal{L}$, $\mathtt{f}_{\mathtt{I}}(u)=(u_2,u_1)$ *and its* $\omega-$*operator given by*

$$\omega(u,v)=\begin{cases}\left(\frac{u_1^2+v_1^2}{2},\frac{u_2^2+v_2^2}{2}\right), & u=(u_1,u_2)\in\mathcal{L}\setminus\{0_{\mathcal{L}},1_{\mathcal{L}}\},\ v=(v_1,v_2)\in ran(\mathtt{f}_{\mathtt{I}}),\\ 0_{\mathcal{L}}, & u=0_{\mathcal{L}},v=(v_1,v_2)\in ran(\mathtt{f}_{\mathtt{I}}),\\ v, & u=1_{\mathcal{L}},v=(v_1,v_2)\in ran(\mathtt{f}_{\mathtt{I}}).\end{cases}$$

Then $(\mathtt{f}_{\mathtt{I}},\omega)$*-implication generated by* $\mathtt{f}_{\mathtt{I}}$ *and* ω *is given by*

$$\mathtt{I}_{\mathtt{I}(\mathtt{f}_{\mathtt{I}},\omega)}(u,v)=\begin{cases}(\frac{u_2^2+v_1^2}{2},\frac{u_1^2+v_2^2}{2}), & u=(u_1,u_2)\in\mathcal{L}\setminus\{0_{\mathcal{L}},1_{\mathcal{L}}\},\ v=(v_1,v_2)\in ran(\mathtt{f}_{\mathtt{I}}),\\ 1_{\mathcal{L}}, & u=0_{\mathcal{L}},v=(v_1,v_2)\in ran(\mathtt{f}_{\mathtt{I}}),\\ v, & u=1_{\mathcal{L}},v=(v_1,v_2)\in ran(\mathtt{f}_{\mathtt{I}}).\end{cases}$$

(ii) *Consider the decreasing generator* $\mathtt{f}_{\mathtt{I}}:\mathcal{L}\to\mathcal{L}$, $\mathtt{f}_{\mathtt{I}}(u)=(1-u_1+u_2,1-u_2)$ *and its* $\omega-$*operator given by*

$$\omega(u,v)=\begin{cases}(v_1,v_2), & u_1>v_1,v_2>u_2,\\ 0_{\mathcal{L}}, & u_1\le v_1,v_2\le u_2,\\ (1-v_2,v_2), & u_1\le v_1,v_2>u_2,\\ (v_1,0), & u_1>v_1,v_2\le u_2,\end{cases}$$

$\forall\ u=(u_1,u_2)\in\mathcal{L},\ v=(v_1,v_2)\in ran(\mathtt{f}_{\mathtt{I}})$. *Then* $(\mathtt{f}_{\mathtt{I}},\omega)$*-implication generated from* $\mathtt{f}_{\mathtt{I}}$ *and* ω*, with* $\mathtt{f}_{\mathtt{I}}^{(-1)}(v)=(\min\{1,v_2,\frac{2-v_1}{2}\},\max\{0,1-v_2\})$, $\forall\ v=(v_1,v_2)\in[0,\infty]^2$*, is given* by

$$\mathtt{I}_{\mathtt{I}(\mathtt{f_I},\omega)}(u,v)=\begin{cases}(\min\{1,1-v_2,\frac{1+v_1-v_2}{2}\},\max\{0,v_2\}), & u_1>1-v_1+v_2, 1-v_2>u_2,\\ 1_{\mathcal{L}}, & u_1\leq 1-v_1+v_2, 1-v_2\leq u_2,\\ (\min\{1,1-v_2\},\max\{0,v_2\}), & u_1\leq 1-v_1+v_2, 1-v_2>u_2,\\ 0_{\mathcal{L}}, & u_1>1-v_1+v_2, 1-v_2\leq u_2.\end{cases}$$

The following theorem shows that the $\mathtt{I}_{\mathtt{I}(\mathtt{f_I},\omega)}$ is an IFI in the sense of Definition 1.8.4.

Theorem 7.3.9. *The operator $\mathtt{I}_{\mathtt{I}(\mathtt{f_I},\omega)}$ defined by (7.17) is an IFI.*

Proof. The fact that $\mathtt{I}_{\mathtt{I}(\mathtt{f_I},\omega)}$ defined by (7.17) is an IFI can be seen from the following:

- Let $u <_{\mathcal{L}} u'$. Since $\mathtt{f_I}$ is a decreasing on $\mathcal{L}$, $\mathtt{f_I}^{(-1)}$ exists and ω is non-decreasing in each argument on $\mathcal{L}\times ran(\mathtt{f_I})$. Then, we have $\omega(u,\mathtt{f_I}(v)) \leq_{\mathcal{L}_{\leq\geq \mathtt{f_I}(0_{\mathcal{L}})}} \omega(u',\mathtt{f_I}(v))$ for any $v\in\mathcal{L}$, and hence

$$\mathtt{I}_{\mathtt{I}(\mathtt{f_I},\omega)}(u,v)=\mathtt{f_I}^{(-1)}(\omega(u,\mathtt{f_I}(v)))\geq_{\mathcal{L}}\mathtt{f_I}^{(-1)}(\omega(u',\mathtt{f_I}(v)))=\mathtt{I}_{\mathtt{I}(\mathtt{f_I},\omega)}(u',v), \tag{7.18}$$

  i.e., $\mathtt{I}_{\mathtt{I}(\mathtt{f_I},\omega)}$ satisfies (I1).

- Let $v <_{\mathcal{L}} v'$ for any $u\in\mathcal{L}$. Then

$$\begin{aligned}\mathtt{f_I}(y)\geq_{\mathcal{L}_{\leq\geq \mathtt{f_I}(0_{\mathcal{L}})}}\mathtt{f_I}(y') &\Rightarrow \omega(x,\mathtt{f_I}(y))\geq_{\mathcal{L}_{\leq\geq \mathtt{f_I}(0_{\mathcal{L}})}}\omega(x,\mathtt{f_I}(y'))\\ &\Rightarrow \mathtt{f_I}^{(-1)}(\omega(x,\mathtt{f_I}(y)))\leq_{\mathcal{L}}\mathtt{f_I}^{(-1)}(\omega(x,\mathtt{f_I}(y')))\\ &\Rightarrow \mathtt{I}_{\mathtt{I}(\mathtt{f_I},\omega)}(x,y)\leq_{\mathcal{L}}\mathtt{I}_{\mathtt{I}(\mathtt{f_I},\omega)}(x,y'),\end{aligned}$$

  i.e., $\mathtt{I}_{\mathtt{I}(\mathtt{f_I},\omega)}$ satisfies (I2).

- $\mathtt{I}_{\mathtt{I}(\mathtt{f_I},\omega)}(0_{\mathcal{L}},0_{\mathcal{L}})=\mathtt{f_I}^{(-1)}(\omega(0_{\mathcal{L}},\mathtt{f_I}(0_{\mathcal{L}})))=\mathtt{f_I}^{(-1)}(0_{\mathcal{L}})=1_{\mathcal{L}}$,
  $\mathtt{I}_{\mathtt{I}(\mathtt{f_I},\omega)}(1_{\mathcal{L}},1_{\mathcal{L}})=\mathtt{f_I}^{(-1)}(\omega(1_{\mathcal{L}},\mathtt{f_I}(1_{\mathcal{L}})))=\mathtt{f_I}^{(-1)}(\mathtt{f_I}(1_{\mathcal{L}}))=1_{\mathcal{L}}$,
  $\mathtt{I}_{\mathtt{I}(\mathtt{f_I},\omega)}(1_{\mathcal{L}},0_{\mathcal{L}})=\mathtt{f_I}^{(-1)}(\omega(1_{\mathcal{L}},\mathtt{f_I}(0_{\mathcal{L}})))=\mathtt{f_I}^{(-1)}(\mathtt{f_I}(0_{\mathcal{L}}))=0_{\mathcal{L}}$,
  i.e., $\mathtt{I}_{\mathtt{I}(\mathtt{f_I},\omega)}$ satisfies (I3).

□

Theorem 7.3.10. *Let $\mathtt{f_I},\mathtt{f_{I1}},\mathtt{f_{I2}}:\mathcal{L}\to[0,\infty]^2$ be the decreasing generators and $\mathtt{t_I}:\mathcal{L}\to\mathcal{L}$ be a decreasing function with $\mathtt{t_I}(0_{\mathcal{L}})=1_{\mathcal{L}}$ and $\mathtt{t_I}(1_{\mathcal{L}})=0_{\mathcal{L}}$. If the ω-operator generated by $\mathtt{t_I}$ is given by $\omega(u,v)=\mathtt{t_I}(u)v$, $\forall\ u\in\mathcal{L}$ and $\forall\ v\in ran(\mathtt{f_I})$, then the following statements are equivalent:*

(i) $\mathtt{I}_{\mathtt{I}(\mathtt{f}_{\mathtt{I}1},\omega)}(u,v) = \mathtt{I}_{\mathtt{I}(\mathtt{f}_{\mathtt{I}2},\omega)}(u,v)$.

(ii) $\exists$ *a constant* $c_{\mathcal{L}} \in [0,\infty]^2 \setminus \{0_{\mathcal{L}}, \infty_{\mathcal{L}}\}$ *such that* $\mathtt{f}_{\mathtt{I}2}(u) = c_{\mathcal{L}}\mathtt{f}_{\mathtt{I}1}(u),\ \forall\, u \in \mathcal{L}$.

Proof. *(i)* $\Rightarrow$ *(ii)* : Let $\mathtt{f}_{\mathtt{I}1}, \mathtt{f}_{\mathtt{I}2}$ be two decreasing generators on $\mathcal{L}$ such that $\mathtt{I}_{\mathtt{I}(\mathtt{f}_{\mathtt{I}1},\omega)}(u,v) = \mathtt{I}_{\mathtt{I}(\mathtt{f}_{\mathtt{I}2},\omega)}(u,v),\ \forall\, u,v \in \mathcal{L}$. For any $u \in \mathcal{L} \setminus \{0_{\mathcal{L}}\}$,

$$\begin{aligned}\mathtt{I}_{\mathtt{I}(\mathtt{f}_{\mathtt{I}1},\omega)}(u,v) = \mathtt{I}_{\mathtt{I}(\mathtt{f}_{\mathtt{I}2},\omega)}(u,v) &\Leftrightarrow \mathtt{f}_{\mathtt{I}1}^{-1}(\omega(u,\mathtt{f}_{\mathtt{I}1}(v))) = \mathtt{f}_{\mathtt{I}2}^{-1}(\omega(u,\mathtt{f}_{\mathtt{I}2}(v)))\\ &\Leftrightarrow \mathtt{f}_{\mathtt{I}1}^{-1}(\mathtt{t}_{\mathtt{I}}(u)\mathtt{f}_{\mathtt{I}1}(v)) = \mathtt{f}_{\mathtt{I}2}^{-1}(\mathtt{t}_{\mathtt{I}}(u)\mathtt{f}_{\mathtt{I}2}(v))\\ &\Leftrightarrow \mathtt{f}_{\mathtt{I}2} \circ \mathtt{f}_{\mathtt{I}1}^{-1}(\mathtt{t}_{\mathtt{I}}(u)\mathtt{f}_{\mathtt{I}1}(v)) = \mathtt{t}_{\mathtt{I}}(u)\mathtt{f}_{\mathtt{I}2} \circ \mathtt{f}_{\mathtt{I}1}^{-1}(\mathtt{f}_{\mathtt{I}1}(v)).\end{aligned}$$

By the substitutions $\mathtt{h}_{\mathtt{I}} = \mathtt{f}_{\mathtt{I}2} \circ \mathtt{f}_{\mathtt{I}1}^{-1}$ and $w = \mathtt{f}_{\mathtt{I}1}(v)$ for any $v \in \mathcal{L}$, we obtain

$$\mathtt{h}_{\mathtt{I}}(\mathtt{t}_{\mathtt{I}}(u)w) = \mathtt{t}_{\mathtt{I}}(u)\mathtt{h}_{\mathtt{I}}(w) \text{ for } u \in \mathcal{L} \setminus \{0_{\mathcal{L}}\} \text{ and } w \in [0,\infty]^2, \tag{7.19}$$

where $\mathtt{h}_{\mathtt{I}} : [0,\infty]^2 \to [0,\infty]^2$ is continuous, increasing and bijection with $\mathtt{h}_{\mathtt{I}}(0_{\mathcal{L}}) = 0_{\mathcal{L}}$. Taking $w = 1_{\mathcal{L}}$ in (7.19), we have

$$\mathtt{h}_{\mathtt{I}}(\mathtt{t}_{\mathtt{I}}(u)) = \mathtt{t}_{\mathtt{I}}(u)\mathtt{h}_{\mathtt{I}}(1_{\mathcal{L}}) \text{ for any } \mathtt{t}_{\mathtt{I}}(u) \in \mathcal{L} \setminus \{0_{\mathcal{L}}\} \text{ and } u \in \mathcal{L} \setminus \{0_{\mathcal{L}}\}.$$

Fix arbitrarily $w \in [0,\infty]^2$. Then $\exists\ \mathtt{t}_{\mathtt{I}}(u) \in \mathcal{L} \setminus \{0_{\mathcal{L}}\}$ such that $\mathtt{t}_{\mathtt{I}}(u)w \in \mathcal{L} \setminus \{0_{\mathcal{L}}\}$. Therefore,

$$\mathtt{h}_{\mathtt{I}}(\mathtt{t}_{\mathtt{I}}(u)w) = \mathtt{t}_{\mathtt{I}}(u)\mathtt{h}_{\mathtt{I}}(w) \ \Rightarrow\ \mathtt{h}_{\mathtt{I}}(w) = w.\mathtt{h}_{\mathtt{I}}(1_{\mathcal{L}}).$$

Now, we have

$$\mathtt{f}_{\mathtt{I}2} \circ \mathtt{f}_{\mathtt{I}1}^{-1}(w') = w'(\mathtt{f}_{\mathtt{I}2} \circ \mathtt{f}_{\mathtt{I}1}^{-1}(1_{\mathcal{L}})) \ \Rightarrow\ \mathtt{f}_{\mathtt{I}2}(u) = \mathtt{f}_{\mathtt{I}1}(u)(\mathtt{f}_{\mathtt{I}2} \circ \mathtt{f}_{\mathtt{I}1}^{-1}(1_{\mathcal{L}})).$$

Let $c_{\mathcal{L}} = \mathtt{f}_{\mathtt{I}2} \circ \mathtt{f}_{\mathtt{I}1}^{-1}(1_{\mathcal{L}}) >_{\mathcal{L}} 0_{\mathcal{L}}$. Then $\mathtt{f}_{\mathtt{I}2}(u) = c_{\mathcal{L}}\mathtt{f}_{\mathtt{I}1}(u)$ for $u \in \mathcal{L} \setminus \{0_{\mathcal{L}}\}$. Note that for $0_{\mathcal{L}}$, we also have $\mathtt{f}_{\mathtt{I}2}(u) = c_{\mathcal{L}}\mathtt{f}_{\mathtt{I}1}(u)$. Since $\mathtt{f}_{\mathtt{I}1}(0_{\mathcal{L}}) = \mathtt{f}_{\mathtt{I}2}(0_{\mathcal{L}}) = \infty_{\mathcal{L}}$, result (ii) is true for $u \in \mathcal{L}$.
(ii) $\Rightarrow$ *(i)* : Let $\mathtt{f}_{\mathtt{I}1}$ be a decreasing generator on $\mathcal{L}$ and $c_{\mathcal{L}} \in [0,\infty]^2 \setminus \{0_{\mathcal{L}}, \infty_{\mathcal{L}}\}$. Define $\mathtt{f}_{\mathtt{I}2}(u) = c_{\mathcal{L}}\mathtt{f}_{\mathtt{I}1}(u)$, for all $u \in \mathcal{L}$. Firstly, note that $\mathtt{f}_{\mathtt{I}2}$ is a well defined decreasing generator on $\mathcal{L}$. Moreover, $\mathtt{f}_{\mathtt{I}2}^{-1}(w) = \mathtt{f}_{\mathtt{I}1}^{-1}(\frac{w}{c_{\mathcal{L}}})$ for every $w \in ran(\mathtt{f}_{\mathtt{I}2})$. Now, for every $u,v \in \mathcal{L}$, we have

$$\begin{aligned}&u.c_{\mathcal{L}}\mathtt{f}_{\mathtt{I}1}(v) \leq_{\leq\geq \mathtt{f}_{\mathtt{I}}(0_{\mathcal{L}})} c_{\mathcal{L}}\mathtt{f}_{\mathtt{I}1}(v) = \mathtt{f}_{\mathtt{I}2}(v) \leq_{\leq\geq \mathtt{f}_{\mathtt{I}}(0_{\mathcal{L}})} \mathtt{f}_{\mathtt{I}2}(0_{\mathcal{L}}),\\ &\frac{u.c_{\mathcal{L}}\mathtt{f}_{\mathtt{I}1}(v)}{c_{\mathcal{L}}} \leq_{\leq\geq \mathtt{f}_{\mathtt{I}}(0_{\mathcal{L}})} u\mathtt{f}_{\mathtt{I}1}(v) = \mathtt{f}_{\mathtt{I}1}(v) \leq_{\leq\geq \mathtt{f}_{\mathtt{I}}(0_{\mathcal{L}})} \mathtt{f}_{\mathtt{I}1}(0_{\mathcal{L}}).\end{aligned}$$

Thus,

$$\begin{aligned}\mathtt{I}_{\mathtt{I}(\mathtt{f}_{\mathtt{I}2},\omega)}(u,v) =& \mathtt{f}_{\mathtt{I}2}^{-1}(\mathtt{t}_{\mathtt{I}}(u)\mathtt{f}_{\mathtt{I}2}(v)) = \mathtt{f}_{\mathtt{I}2}^{-1}(\mathtt{t}_{\mathtt{I}}(u)c_{\mathcal{L}}\mathtt{f}_{\mathtt{I}1}(v)) = \mathtt{f}_{\mathtt{I}1}^{-1}\Big(\frac{\mathtt{t}_{\mathtt{I}}(u)c_{\mathcal{L}}\mathtt{f}_{\mathtt{I}1}(v)}{c_{\mathcal{L}}}\Big) = \mathtt{f}_{\mathtt{I}1}^{-1}(\mathtt{t}_{\mathtt{I}}(u)\mathtt{f}_{\mathtt{I}1}(v))\\ =& \mathtt{I}_{\mathtt{I}(\mathtt{f}_{\mathtt{I}1},\omega)}(u,v),\ \forall\, u,v \in \mathcal{L}.\end{aligned}$$

$\square$

In the continuation, we will discuss some of the properties of $(\mathtt{f_I}, \omega)$-implications.

Theorem 7.3.11. *Let* $\mathtt{f_I}$ *and* ω *be a decreasing generator on* $\mathcal{L}$ *and an* ω*-operator on* $\mathcal{L} \times ran(\mathtt{f_I})$ *respectively. Then the* $(\mathtt{f_I}, \omega)$*-implication* $\mathtt{I}_{\mathtt{I}(\mathtt{f_I},\omega)}$ *defined by (7.17) satisfies the following properties:*

(i) $\mathtt{I}_{\mathtt{I}(\mathtt{f_I},\omega)}$ *satisfies (NP).*

(ii) $\mathtt{I}_{\mathtt{I}(\mathtt{f_I},\omega)}$ *satisfies (IP)* $\Leftrightarrow \omega(u, \mathtt{f_I}(u)) \leq_{\leq \geq \mathtt{f_I}(0_\mathcal{L})} \mathtt{f_I}(1_\mathcal{L}),\ \forall\, u \in \mathcal{L}$.

(iii) $\mathtt{I}_{\mathtt{I}(\mathtt{f_I},\omega)}$ *satisfies (OP)* $\Leftrightarrow \omega(u, \mathtt{f_I}(v)) \leq_\mathcal{L} \mathtt{f_I}(1_\mathcal{L}) \Leftrightarrow \mathtt{f_I}(u) \geq_{\leq \geq \mathtt{f_I}(0_\mathcal{L})} v,\ \forall\, u \in \mathcal{L},\ v \in ran(\mathtt{f_I})$.

(iv) $\mathtt{I}_{\mathtt{I}(\mathtt{f_I},\omega)}(u, v) \geq_{\leq \geq \mathtt{f_I}(0_\mathcal{L})} v$ *holds* $\forall\, u, v \in \mathcal{L}$.

Proof. Let $\mathtt{f_I}$ be a decreasing generator and ω be an ω-operator of $\mathtt{f_I}$. Then

$$\mathtt{I}_{\mathtt{I}(\mathtt{f},\omega)}(u, v) = \mathtt{f}_\mathtt{I}^{(-1)}(\omega(u, \mathtt{f_I}(v))),\ \forall\, u, v \in \mathcal{L}.$$

(i) For any $v \in \mathcal{L}$,

$$\mathtt{I}_{\mathtt{I}(\mathtt{f_I},\omega)}(1_\mathcal{L}, v) = \mathtt{f}_\mathtt{I}^{(-1)}(\omega(1_\mathcal{L}, \mathtt{f_I}(v))) = \mathtt{f}_\mathtt{I}^{(-1)}(\mathtt{f_I}(v)) = v, \text{ i.e., } \mathtt{I}_{\mathtt{I}(\mathtt{f_I},\omega)} \text{ satisfies (NP).}$$

(ii) Suppose that $\mathtt{I}_{\mathtt{I}(\mathtt{f_I},\omega)}$ satisfies (IP), i.e., for every $u \in \mathcal{L}$, we have $\mathtt{I}_{\mathtt{I}(\mathtt{f_I},\omega)}(u, u) = 1_\mathcal{L}$. By Definition 7.3.7, we obtain

$$\mathtt{I}_{\mathtt{I}(\mathtt{f_I},\omega)}(u, u) = 1_\mathcal{L} \ \Leftrightarrow\ \mathtt{f}_\mathtt{I}^{(-1)}(\omega(u, \mathtt{f_I}(u))) \ \Leftrightarrow\ \omega(u, \mathtt{f_I}(u)) \leq_{\leq \geq \mathtt{f_I}(0_\mathcal{L})} \mathtt{f_I}(1_\mathcal{L}),\ \forall\, u \in \mathcal{L}.$$

The converse part is straightforward.

(iii) Suppose that $\mathtt{I}_{\mathtt{I}(\mathtt{f_I},\omega)}$ satisfies (OP), i.e., $\forall\, \alpha, \beta \in \mathcal{L}$, $\mathtt{I}_{\mathtt{I}(\mathtt{f_I},\omega)}(\alpha, \beta) = 1_\mathcal{L}$ iff $\alpha \leq_\mathcal{L} \beta$, which means the following equivalences hold:

$$\mathtt{f}_\mathtt{I}^{(-1)}(\omega(\alpha, \mathtt{f_I}(\beta))) = 1_\mathcal{L} \ \Leftrightarrow\ \alpha \leq_\mathcal{L} \beta \text{ or } \omega(\alpha, \mathtt{f_I}(\beta)) \leq_{\leq \geq \mathtt{f_I}(0_\mathcal{L})} \mathtt{f_I}(1_\mathcal{L}) \ \Leftrightarrow\ \alpha \leq_\mathcal{L} \beta.$$

Putting $\alpha = u,\ \mathtt{f_I}(\beta) = v$, we get

$$\omega(u, v) \leq_{\leq \geq \mathtt{f_I}(0_\mathcal{L})} \mathtt{f_I}(1_\mathcal{L}) \ \Leftrightarrow\ u \leq_\mathcal{L} \mathtt{f}_\mathtt{I}^{(-1)}(v) \ \Leftrightarrow\ \mathtt{f_I}(v) \geq_{\leq \geq \mathtt{f_I}(0_\mathcal{L})} v,\ \forall\, u \in \mathcal{L},\ v \in ran(\mathtt{f_I}).$$

Conversely, assume that $\forall\ u \in \mathcal{L},\ v \in ran(\mathtt{f_I})$. Then $\omega(u, v) \leq_{\leq \geq \mathtt{f_I}(0_\mathcal{L})} \mathtt{f_I}(1_\mathcal{L}) \Leftrightarrow \mathtt{f_I}(u) \geq_{\leq \geq \mathtt{f_I}(0_\mathcal{L})} v$. For $\alpha, \beta \in \mathcal{L}$, suppose $u = \alpha,\ v = \mathtt{f_I}(\beta)$. Then $u \in \mathcal{L},\ v \in ran(\mathtt{f_I})$. Thus

$$\begin{aligned}
\mathtt{I}_{\mathtt{I}(\mathtt{f_I},\omega)}(\alpha, \beta) = 1_\mathcal{L} &\Leftrightarrow \mathtt{f}_\mathtt{I}^{(-1)}(\omega(\alpha, \mathtt{f_I}(\beta))) = 1_\mathcal{L} \\
&\Leftrightarrow \omega(\alpha, \mathtt{f_I}(\beta)) \leq_{\leq \geq \mathtt{f_I}(0_\mathcal{L})} \mathtt{f_I}(1_\mathcal{L}) \\
&\Leftrightarrow \omega(u, v) \leq_{\leq \geq \mathtt{f_I}(0_\mathcal{L})} \mathtt{f_I}(1_\mathcal{L}) \\
&\Leftrightarrow \mathtt{f_I}(u) \geq_{\leq \geq \mathtt{f_I}(0_\mathcal{L})} v \ \Leftrightarrow\ \alpha \leq_\mathcal{L} \beta.
\end{aligned}$$

(iv) For $u, v \in \mathcal{L}$,

$$\mathtt{I}_{\mathtt{I}(\mathtt{f}_\mathtt{I},\omega)}(u,v) = \mathtt{f}_\mathtt{I}^{(-1)}(\omega(u,\mathtt{f}_\mathtt{I}(v))) \geq_{\mathcal{L}} \mathtt{f}_\mathtt{I}^{(-1)}(\omega(1_\mathcal{L},\mathtt{f}_\mathtt{I}(v))) = \mathtt{f}_\mathtt{I}^{(-1)}(\mathtt{f}_\mathtt{I}(v)) = v.$$

□

The $(\mathtt{f}_\mathtt{I},\omega)$-implication does not necessarily satisfy (EP). This can be shown with the help of the following example:

Example 7.3.12. *Consider the decreasing generator $\mathtt{f}_\mathtt{I}$, $\mathtt{f}_\mathtt{I}(u) = (u_2, u_1)$ and its $\omega-$operator given by*

$$\omega(u,v) = \begin{cases} (\frac{u_1}{2}, \frac{u_2+v_2}{2}), & u = (u_1,u_2) \in \mathcal{L} \setminus \{0_\mathcal{L}, 1_\mathcal{L}\},\ v = (v_1,v_2) \in ran(\mathtt{f}), \\ 0_\mathcal{L}, & u = 0_\mathcal{L}, v = (v_1,v_2) \in ran(\mathtt{f}_\mathtt{I}), \\ v, & u = 1_\mathcal{L}, v = (v_1,v_2) \in ran(\mathtt{f}_\mathtt{I}). \end{cases}$$

Then $(\mathtt{f}_\mathtt{I},\omega)$-implication generated from $\mathtt{f}_\mathtt{I}$ and ω is given by

$$\mathtt{I}_{\mathtt{I}(\mathtt{f}_\mathtt{I},\omega)}(u,v) = \begin{cases} (\frac{u_2}{2}, \frac{u_1+v_2}{2}), & u = (u_1,u_2) \in \mathcal{L} \setminus \{0_\mathcal{L}, 1_\mathcal{L}\},\ v = (v_1,v_2) \in ran(\mathtt{f}_\mathtt{I}), \\ 1_\mathcal{L}, & u = 0_\mathcal{L}, v = (v_1,v_2) \in ran(\mathtt{f}_\mathtt{I}), \\ v, & u = 1_\mathcal{L}, v = (v_1,v_2) \in ran(\mathtt{f}_\mathtt{I}), \end{cases}$$

$$\mathtt{I}_{\mathtt{I}(\mathtt{f}_\mathtt{I},\omega)}(u,v) = \mathtt{f}_\mathtt{I}^{-1}(\omega(u,\mathtt{f}_\mathtt{I}(v))) = (1-(1-u_2)(1-v_1), u_1v_2),\ \forall u,v \in \mathcal{L}.$$

For $u = (0.4, 0.5)$, $v = (0.3, 0.6)$ and $w = (0.4, 0.6)$, we have

$$\mathtt{I}_{\mathtt{I}(\mathtt{f}_\mathtt{I},\omega)}(u, \mathtt{I}_{\mathtt{I}(\mathtt{f}_\mathtt{I},\omega)}(v,w)) = (0.25, 0.425),$$

while

$$\mathtt{I}_{\mathtt{I}(\mathtt{f}_\mathtt{I},\omega)}(v, \mathtt{I}_{\mathtt{I}(\mathtt{f}_\mathtt{I},\omega)}(u,w)) = (0.3, 0.4) \neq (0.25, 0.425).$$

This shows that $\mathtt{I}_{\mathtt{I}(\mathtt{f}_\mathtt{I},\omega)}$ does not satisfy (EP).

Theorem 7.3.13. *Let $\mathtt{t}_\mathtt{I} : \mathcal{L} \to \mathcal{L}$ be a strictly decreasing function with $\mathtt{t}_\mathtt{I}(0_\mathcal{L}) = 1_\mathcal{L}$ and $\mathtt{t}_\mathtt{I}(1_\mathcal{L}) = 0_\mathcal{L}$, and $\mathtt{f}_\mathtt{I}$ be a decreasing generator on $\mathcal{L}$. If the ω-operator generated by $\mathtt{t}_\mathtt{I}$ is given by $\omega(u,v) = \mathtt{t}_\mathtt{I}(u)v$, $\forall\, u \in \mathcal{L}$, $v \in ran(\mathtt{f}_\mathtt{I})$, then $\mathtt{I}_{\mathtt{I}(\mathtt{f}_\mathtt{I},\omega)}$, given by $\mathtt{I}_{\mathtt{I}(\mathtt{f}_\mathtt{I},\omega)}(u,v) = \mathtt{f}^{-1}(\mathtt{t}(u)\mathtt{f}(v))$, satisfies (EP).*

Proof. From Definition 7.3.7,

$$\mathtt{I}_{\mathtt{I}(\mathtt{f}_\mathtt{I},\omega)}(u,v) = \mathtt{f}_\mathtt{I}^{-1}(\omega(u,\mathtt{f}_\mathtt{I}(v))) = \mathtt{f}_\mathtt{I}^{-1}(\mathtt{t}_\mathtt{I}(u)\mathtt{f}_\mathtt{I}(v)),\ \forall\, u,v \in \mathcal{L}.$$

For $u, v, w \in \mathcal{L}$, by (7.17), we have

$$\begin{aligned}\mathtt{I}_{\mathtt{I}(\mathtt{f}_\mathtt{I},\omega)}(u, \mathtt{I}_{\mathtt{I}(\mathtt{f}_\mathtt{I},\omega)}(v,w)) =& \mathtt{f}_\mathtt{I}^{-1}(\mathtt{t}_\mathtt{I}(u).\mathtt{f}_\mathtt{I} \circ \mathtt{f}_\mathtt{I}^{-1}(\mathtt{t}_\mathtt{I}(v)\mathtt{f}_\mathtt{I}(w))) = \mathtt{f}_\mathtt{I}^{-1}(\mathtt{t}_\mathtt{I}(u).\mathtt{t}_\mathtt{I}(v)\mathtt{f}_\mathtt{I}(w))\\ =& \mathtt{f}_\mathtt{I}^{-1}(\mathtt{t}_\mathtt{I}(v).\mathtt{t}_\mathtt{I}(u)\mathtt{f}_\mathtt{I}(w)) = \mathtt{f}_\mathtt{I}^{-1}(\mathtt{t}_\mathtt{I}(v).\mathtt{f}_\mathtt{I} \circ \mathtt{f}_\mathtt{I}^{-1}(\mathtt{t}_\mathtt{I}(u)\mathtt{f}_\mathtt{I}(w)))\\ =& \mathtt{I}_{\mathtt{I}(\mathtt{f}_\mathtt{I},\omega)}(v, \mathtt{I}_{\mathtt{I}(\mathtt{f}_\mathtt{I},\omega)}(u,w)).\end{aligned}$$

Hence $\mathtt{I}_{\mathtt{I}(\mathtt{f}_\mathtt{I},\omega)}$ satisfies (EP). □

Theorem 7.3.14 ([59], Theorem 6.8)**.** *A function $\Phi_\mathtt{I} : \mathcal{L} \to \mathcal{L}$ is a continuous, increasing and bijection iff ∃ a continuous, non-decreasing and bijection $\varphi : [0,1] \to [0,1]$ such that*

$$\Phi_\mathtt{I}(u) = (\varphi(u_1), 1 - \varphi(1-u_2)),\ u = (u_1, u_2) \in \mathcal{L}.$$

Proof. See Theorem 6.8 in [59] for its proof. □

Let $\Phi_\mathtt{I}$ denote the family of all continuous, increasing and bijections from $\mathcal{L}$ to $\mathcal{L}$. We say that the function $F, G : \mathcal{L}^2 \to \mathcal{L}$ are $\Phi_\mathtt{I}$ conjugate, if there exists $\Phi_\mathtt{I} \in \Phi_\mathtt{I}$ such that $G = F_{\Phi_\mathtt{I}}$, where

$$F_{\Phi_\mathtt{I}}(u,v) = \Phi_\mathtt{I}^{-1}(F(\Phi_\mathtt{I}(u), \Phi_\mathtt{I}(v))),\ u, v \in \mathcal{L}.$$

Theorem 7.3.15. *If $\mathtt{I}_{\mathtt{I}(\mathtt{f}_\mathtt{I},\omega)}$ is a $(\mathtt{f}_\mathtt{I}, \omega)$-implication and $\Phi_\mathtt{I} : \mathcal{L} \to \mathcal{L}$ is a continuous, non-decreasing and bijection, then $(\mathtt{I}_{\mathtt{I}(\mathtt{f}_\mathtt{I},\omega)})_{\Phi_\mathtt{I}}(u,v) = \mathtt{I}_{\mathtt{I}(\mathtt{f}_\mathtt{I}\circ\Phi_\mathtt{I},\omega)}(\Phi_\mathtt{I}(u), v)$.*

Proof. Let $\Phi_\mathtt{I} \in \Phi_\mathtt{I}$, $\mathtt{f}_\mathtt{I}$ be an decreasing generator on $\mathcal{L}$ and ω be a ω-operator on $\mathcal{L} \times ran(\mathtt{f}_\mathtt{I})$. Then

$$\begin{aligned}(\mathtt{I}_{\mathtt{I}(\mathtt{f}_\mathtt{I},\omega)})_{\Phi_\mathtt{I}}(u,v) =& \Phi_\mathtt{I}^{-1}(\mathtt{I}_{\mathtt{I}(\mathtt{f}_\mathtt{I},\omega)}(\Phi_\mathtt{I}(u), \Phi_\mathtt{I}(v))) = \Phi_\mathtt{I}^{-1}(\mathtt{f}_\mathtt{I}^{-1}(\mathtt{u}(\Phi_\mathtt{I}(u), \mathtt{f}_\mathtt{I}(\Phi_\mathtt{I}(v)))))\\ =& (\mathtt{f}_\mathtt{I} \circ \Phi_\mathtt{I})^{-1}(\omega(\Phi_\mathtt{I}(u), \mathtt{f}_\mathtt{I} \circ \Phi_\mathtt{I}(v))) = \mathtt{I}_{\mathtt{I}(\mathtt{f}_\mathtt{I}\circ\Phi_\mathtt{I},\omega)}(\Phi_\mathtt{I}(u), v),\end{aligned}$$

for every $u, v \in \mathcal{L}$. □

Proposition 7.3.16. *Let $\mathtt{I}_\mathtt{I} : \mathcal{L}^2 \to \mathcal{L}$ be an IFI satisfying (NP). For a given $\mathtt{f}_\mathtt{I}$ generator, ∃ an ω-operator ω such that $\mathtt{I}_\mathtt{I}(u,v) = \mathtt{I}_{\mathtt{I}(\mathtt{f}_\mathtt{I},\omega)}(u,v)$, $\forall\, u, v \in \mathcal{L}$.*

Proof. Let $\mathtt{K} = \{(u,v) \in \mathcal{L}^2 : \mathtt{I}_\mathtt{I}(u,v) <_\mathcal{L} 1_\mathcal{L}\}$. Assume

$$\alpha = u \text{ and } \beta = \mathtt{f}_\mathtt{I}(v)$$

transform the region $\mathtt{K}$ to $\mathtt{K}'$ which is a subset of $\mathcal{L} \times ran(\mathtt{f}_\mathtt{I})$, i.e., for any $(u,v) \in \mathtt{K}$, ∃ a $(\alpha, \beta) \in \mathtt{K}'$ such that

$$\alpha = u \text{ and } \beta = \mathtt{f}_\mathtt{I}(v).$$

Define a function $\omega : \mathcal{L} \times ran(\mathtt{f_I}) \to \mathcal{L}$ by

$$\omega(\alpha,\beta) = \begin{cases} \mathtt{f_I}(\mathtt{I_I}(\alpha, \mathtt{f_I}^{-1}(\beta))), & (\alpha,\beta) \in \mathtt{K}', \\ 0_{\mathcal{L}}, & \alpha = 0_{\mathcal{L}}, \\ \mathtt{f_I}(1_{\mathcal{L}}), & \alpha = 1_{\mathcal{L}}, \beta = \mathtt{f_I}(1_{\mathcal{L}}), \\ c_{\mathcal{L}}, & otherwise, \end{cases}$$

where $c_{\mathcal{L}} \in [0,\infty]^2 \setminus ran(\mathtt{f_I})$ is a constant. Obviously, ω is non-decreasing corresponding to each argument. Further, we have $\omega(1_{\mathcal{L}}, \beta) = \beta$ since $\mathtt{I_I}$ satisfying (NP) and $\omega(0_{\mathcal{L}}, \beta) = 0_{\mathcal{L}}$. Hence, ω is a ω-operator of $\mathtt{f_I}$. In the following we will show that $\mathtt{I_I} = \mathtt{I}_{\mathtt{I}(\mathtt{f_I},\omega)}$.
For any $u, v \in \mathcal{L}$, if $\mathtt{I_I}(u,v) <_{\mathcal{L}} 1_{\mathcal{L}}$, then $(u, \mathtt{f_I}(v)) \in \mathtt{K}'$. Thus,

$$\mathtt{I}_{\mathtt{I}(\mathtt{f_I},\omega)}(u,v) = \mathtt{f_I}^{(-1)}(\omega(u, \mathtt{f_I}(v))) = \mathtt{f_I}^{-1}(\mathtt{f_I}(\mathtt{I_I}(u, \mathtt{f_I}^{-1}(\mathtt{f_I}(v))))) = \mathtt{I_I}(u,v).$$

If $\mathtt{I_I}(u,v) = 1_{\mathcal{L}}$, then $(u, \mathtt{f_I}(v)) \in \mathcal{L} \times ran(\mathtt{f_I}) \setminus \mathtt{K}'$. Thus,

$$\mathtt{I}_{\mathtt{I}(\mathtt{f_I},\omega)}(u,v) = \mathtt{f_I}^{(-1)}(\omega(u, \mathtt{f_I}(v))) = 1_{\mathcal{L}} = \mathtt{I_I}(u,v). \tag{7.20}$$

Since $c_{\mathcal{L}} \in [0,\infty]^2 \setminus ran(\mathtt{f_I})$, $\mathtt{f_I}^{-1}(c_{\mathcal{L}}) = 1_{\mathcal{L}}$. Thus $\mathtt{I}_{\mathtt{I}(\mathtt{f_I},\omega)}(u,v) = \mathtt{I_I}(u,v)$. □

Remark 7.3.17. *From the previous proof, we can see that for a given decreasing generator* $\mathtt{f_I}$ *with* $\mathtt{f_I}(0_{\mathcal{L}}) = \infty_{\mathcal{L}}$*, the* ω*-operator of* $\mathtt{f_I}$ *in the Proposition 7.3.16 will be uniquely determined by* $\mathtt{I_I}$*. However, in the case* $\mathtt{f_I}(0_{\mathcal{L}}) <_{\mathcal{L}} \infty_{\mathcal{L}}$*,* $\exists$ *an infinite number of* ω*-operators of* $\mathtt{f_I}$ *such that* $\mathtt{I_I} = \mathtt{I}_{\mathtt{I}(\mathtt{f_I},\omega)}$*, but in the region* $\mathtt{K}'$*,* ω *is uniquely determined by* $\mathtt{I_I}$*.*

Example 7.3.18. *Consider the Gaines-Rescher implication* $\mathtt{I}_{IGR}(u,v) =< 1 - sg(u_1 - v_1), v_2.sg(u_1 - v_1) >$*, i.e.,*

$$\mathtt{I}_{IGR}(u,v) = \begin{cases} (0, v_2), & u_1 > v_1, \\ 1_{\mathcal{L}}, & u_1 \le v_1, \end{cases}$$

and the $\mathtt{f_I}$*-operator* $\mathtt{f_I}(u_1,u_2) = (u_2,u_1)$*. For any* $\mathtt{f_I}$*-operator with* $\mathtt{f_I}(0_{\mathcal{L}}) <_{\mathcal{L}} \infty_{\mathcal{L}}$*, the coordinate transformation* $\alpha = u$ *and* $\beta = \mathtt{f_I}(v)$ *transform the region* $\mathtt{K} = \{(u,v) \in \mathcal{L}^2 : u_1 \le v_1\}$ *to* $\mathtt{K}' = \{(\alpha,\beta) \in \mathcal{L}^2 : pr_1(\alpha) \le pr_1(\mathtt{f_I}^{-1}(\beta))\}$*. Now*

$$\omega(u,v) = \begin{cases} \mathtt{f_I}(0, pr_2(\mathtt{f_I}^{-1}(\beta))), & pr_1(\alpha) \le pr_1(\mathtt{f_I}^{-1}(\beta)), \\ 0_{\mathcal{L}}, & otherwise, \end{cases}$$

then $\mathtt{I}_{IGR}(u,v) = \mathtt{I}_{\mathtt{I}(\mathtt{f_I},\omega)}$*.*

Definition 7.3.19. *Let* $\mathtt{I_I} : \mathcal{L}^2 \to \mathcal{L}$ *be an IFI. Then the function* $\mathcal{N}_{\mathtt{I_I}} : \mathcal{L} \to \mathcal{L}$*, defined by* $\mathcal{N}_{\mathtt{I_I}}(u) = \mathtt{I_I}(u, 0_{\mathcal{L}})$*, for all* $u \in \mathcal{L}$*, is said to be the natural negation of* $\mathtt{I_I}$*.*

Theorem 7.3.20. *Let* $\mathtt{f_I}$ *be the decreasing generator on* $\mathcal{L}$*, and* ω *be the* $\omega-$*operator associated with* $\mathtt{f_I}$*. Let* $\mathtt{I}_{\mathtt{I}(\mathtt{f_I},\omega)}$ *be the* $(\mathtt{f_I}, \omega)$*-IFI. Then*

$$\mathcal{N}_{\mathtt{I}_{\mathtt{I}(\mathtt{f_I},\omega)}}(u) = \begin{cases} 1_{\mathcal{L}}, & u = 0_{\mathcal{L}}, \\ \beta, & 0_{\mathcal{L}} <_{\mathcal{L}} u <_{\mathcal{L}} 1_{\mathcal{L}}, \\ 0_{\mathcal{L}}, & u = 1_{\mathcal{L}}, \end{cases}$$

where $0_{\mathcal{L}} \leq_{\mathcal{L}} \beta \leq_{\mathcal{L}} 1_{\mathcal{L}}$*.*

Proof. Clearly, $\mathcal{N}_{\mathtt{I}_{\mathtt{I}(\mathtt{f_I},\omega)}}(u) = \mathtt{I}_{\mathtt{I}(\mathtt{f_I},\omega)}(u, 0_{\mathcal{L}}) = \mathtt{f_I}^{(-1)}(\omega(u, \mathtt{f_I}(0_{\mathcal{L}})))$. If $u = 1_{\mathcal{L}}$, then $\mathcal{N}_{\mathtt{I}_{\mathtt{I}(\mathtt{f_I},\omega)}}(u) = \mathtt{f_I}^{-1}(\omega(1_{\mathcal{L}}, \mathtt{f_I}(0_{\mathcal{L}}))) = \mathtt{f_I}^{-1}(\mathtt{f_I}(0_{\mathcal{L}})) = 0_{\mathcal{L}}$. If $u = 0_{\mathcal{L}}$, then $\mathcal{N}_{\mathtt{I}_{\mathtt{I}(\mathtt{f_I},\omega)}}(u) = \mathtt{f_I}^{-1}(\omega(0_{\mathcal{L}}, \mathtt{f_I}(0_{\mathcal{L}}))) = \mathtt{f_I}^{-1}(0_{\mathcal{L}}) = 1_{\mathcal{L}}$. If $0_{\mathcal{L}} <_{\mathcal{L}} u <_{\mathcal{L}} 1_{\mathcal{L}}$, then $0_{\mathcal{L}} \leq_{\mathcal{L}} \mathcal{N}_{\mathtt{I}_{\mathtt{I}(\mathtt{f_I},\omega)}}(u) = \beta \leq_{\mathcal{L}} 1_{\mathcal{L}}$. □

Example 7.3.21. *(i) Let* $\mathtt{f_I}(u_1, u_2) = (u_2, u_1)$ *and*

$$\omega(u, v) = \begin{cases} (u_1, 1 - u_2), & u \in \mathcal{L} \setminus \{1_{\mathcal{L}}\}, \\ (v_1, v_2), & u = 1_{\mathcal{L}}. \end{cases}$$

Then $\mathcal{N}_{\mathtt{I}_{\mathtt{I}(\mathtt{f_I},\omega)}}(u) = (1 - u_1, u_1)$*.*

(ii) Let $\mathtt{f_I}(\frac{1}{u_1} - 1, 1 - u_2) = (u_2, u_1)$ *with* $\mathtt{f_I}(1_{\mathcal{L}}) = \infty_{\mathcal{L}}$ *and* $\omega(u, v) = (u_1 v_1, 1 - (1 - u_2)(1 - v_2))$*. Then*

$$\mathcal{N}_{\mathtt{I}_{\mathtt{I}(\mathtt{f_I},\omega)}}(u) = \begin{cases} 1_{\mathcal{L}}, & u = 0_{\mathcal{L}}, \\ (0, 1 - u_2), & u \neq 0_{\mathcal{L}}. \end{cases}$$

7.4 Distributivity of $(\mathtt{f_I}, \omega)$ implications over t-representable t-norms on $\mathcal{L}$

Proposition 7.4.1 ([17], Proposition 3.2)**.** *Let* $L^\infty = \{(x_1, x_2) \in [0, \infty]^2 : x_1 \geq x_2\}$ *and* $h : L^\infty \to [0, \infty]$ *be a function. Then the following statements are equivalent:*

(i) h satisfies the functional equation

$$h(x_1 + y_1, x_2 + y_2) = h(x_1, x_2) + h(y_1, y_2), \quad (x_1, x_2), (y_1, y_2) \in L^\infty. \tag{7.21}$$

(ii) Either $h = 0$, $h = \infty$,

$$h(x_1,x_2) = \begin{cases} 0, & x_2 = 0, \\ \infty, & x_2 > 0, \end{cases} \qquad h(x_1,x_2) = \begin{cases} 0, & x_2 < \infty, \\ \infty, & x_2 = \infty, \end{cases} \qquad h(x_1,x_2) = \begin{cases} 0, & x_1 = 0, \\ \infty, & x_1 > 0, \end{cases}$$

$$h(x_1,x_2) = \begin{cases} 0, & x_1 = x_2 < \infty, \\ \infty, & x_2 = \infty \\ & \text{or } x_1 > x_2, \end{cases} \qquad h(x_1,x_2) = \begin{cases} 0, & x_2 = 0, \\ & x_1 < \infty, \\ \infty, & x_2 > 0 \\ & \text{or } x_1 = \infty, \end{cases} \qquad h(x_1,x_2) = \begin{cases} 0, & x_1 < \infty, \\ \infty, & x_1 = \infty, \end{cases}$$

there exists unique $c \in (0,\infty)$ such that

$$h(x_1,x_2) = cx_2, \qquad h(x_1,x_2) = cx_1, \qquad h(x_1,x_2) = \begin{cases} cx_1, & x_1 = x_2, \\ \infty, & x_1 > x_2, \end{cases}$$

$$h(x_1,x_2) = \begin{cases} cx_2, & x_1 < \infty, \\ \infty, & x_1 = \infty, \end{cases} \quad h(x_1,x_2) = \begin{cases} cx_1, & x_2 = 0, \\ \infty, & x_2 > 0, \end{cases} \quad h(x_1,x_2) = \begin{cases} c(x_1 - x_2), & x_2 < \infty, \\ \infty, & x_2 = \infty, \end{cases}$$

or there exist unique $c_1, c_2 \in (0,\infty)$, $c_1 \neq c_2$ such that

$$h(x_1,x_2) = \begin{cases} c_1(x_1 - x_2) + c_2 x_2, & x_2 < \infty, \\ \infty, & x_2 = \infty, \end{cases}$$

for all $(x_1,x_2) \in L^\infty$.

Proposition 7.4.2 ([18], Proposition 5.2)**.** *Fix real $a, b > 0$. Let $L^a = \{(x_1,x_2) \in [0,a]^2 : x_1 \geq x_2\}$ and $h : L^\infty \to [0,a]$ be a function. Then the following statements are equivalent:*

(i) h satisfies the functional equation

$$h(\min(x_1+y_1,a), \min(x_2+y_2,a)) = \min(h(x_1,x_2) + h(y_1,y_2), b), \quad (x_1,x_2),(y_1,y_2) \in L^a. \tag{7.22}$$

(ii) Either $h = 0$, $h = b$,

$$h(x_1,x_2) = \begin{cases} 0, & x_2 = 0, \\ b, & x_2 > 0, \end{cases} \qquad h(x_1,x_2) = \begin{cases} 0, & x_1 = 0, \\ b, & x_1 > 0, \end{cases}$$

there exists unique $c \in [\frac{b}{a}, \infty)$ such that

$$h(x_1,x_2) = \min(cx_2, b), \qquad h(x_1,x_2) = \min(cx_1, b),$$

$$h(x_1,x_2) = \begin{cases} \min(cx_1, b), & x_1 = x_2, \\ b, & x_1 > x_2, \end{cases} \qquad h(x_1,x_2) = \begin{cases} \min(cx_1, b), & x_2 = 0, \\ b, & x_2 > 0, \end{cases}$$

or there exist unique $c_1, c_2 \in [b/a, \infty)$, $c_1 \neq c_2$ *such that*

$$h(x_1, x_2) = \begin{cases} \min(c_1(x_1 - x_2) + c_2 x_2, b), & x_1 < a, \\ b, & x_1 = a, \end{cases}$$

for all $(x_1, x_2) \in L^a$.

Lemma 7.4.3. *If* $t : [0,1] \to [0,\infty]$ *is an additive generator of a t-norm* $T : [0,1]^2 \to [0,1]$, *then the function* $s : [0,1] \to [0,\infty]$ *defined by*

$$s(a) = t(1 - a)$$

is an additive generator of the t-conorm $S : [0,1]^2 \to [0,1]$.

Conversely, if $s : [0,1] \to [0,\infty]$ *is an additive generator of a t-conorm* $S : [0,1]^2 \to [0,1]$, *then the function* $t : [0,1] \to [0,\infty]$ *defined by*

$$t(a) = s(1 - a)$$

is an additive generator of the t-norm $T : [0,1]^2 \to [0,1]$.

Remark 7.4.4. *Interval-valued fuzzy t-representable t-norm and intuitionistic fuzzy t-representable t-norm both are mathematically equivalent if t-norm* $T : [0,1]^2 \to [0,1]$ *and t-conorm* $S : [0,1]^2 \to [0,1]$ *are dual of each other.*

7.4.1 On the equation $\mathtt{I}_\mathtt{I}(\mathcal{T}(u,v),w) = \mathcal{S}(\mathtt{I}_\mathtt{I}(u,w), \mathtt{I}_\mathtt{I}(v,w))$

Theorem 7.4.5. *Let* $\mathtt{I}_\mathtt{I}$ *be the IFI satisfying (NP), and let t-norm* $\mathcal{T}$ *and t-conorm* $\mathcal{S}$ *on* $\mathcal{L}$ *be the t-representable, i.e.,* $\mathcal{T} = (T,S)$ *and* $\mathcal{S} = (S,T)$. *Then the triple* $(\mathtt{I}_\mathtt{I}, \mathcal{T}, \mathcal{S})$ *satisfies eq. (7.9) iff* $T = T_M$ *and* $S = S_M$.

Proof. (⇒) Distributive eq. (7.9) is given by

$$\mathtt{I}_\mathtt{I}(\mathcal{T}(u,v),w) = \mathcal{S}(\mathtt{I}_\mathtt{I}(u,w), \mathtt{I}_\mathtt{I}(v,w)),\ \forall\, u,v,w \in \mathcal{L},$$

where $\mathtt{I}_\mathtt{I}$ is an IFI, and t-norm $\mathcal{T}$ and t-conorm $\mathcal{S}$ on $\mathcal{L}$ are the t-representable, i.e., $\mathcal{T} = (T,S)$ and $\mathcal{S} = (S,T)$.

At this situation distributive eq. (7.9) has the following form

$$\begin{aligned} \mathtt{I}_\mathtt{I}((T(u_1,v_1), S(u_2,v_2)),(w_1,w_2)) = &(S(pr_1(\mathtt{I}_\mathtt{I}((u_1,u_2),(w_1,w_2))), pr_1(\mathtt{I}_\mathtt{I}((v_1,v_2),(w_1,w_2)))),\\ &T(pr_2(\mathtt{I}_\mathtt{I}((u_1,u_2),(w_1,w_2))), pr_2(\mathtt{I}_\mathtt{I}((v_1,v_2),(w_1,w_2))))), \end{aligned} \tag{7.23}$$

$\forall\, u=(u_1,u_2), v=(v_1,v_2), w=(w_1,w_2)\in \mathcal{L}$.

Letting $u=v=1_{\mathcal{L}}$ in (7.23), we have

$$\begin{aligned}\mathtt{I_I}((T(1,1),S(0,0)),(w_1,w_2)) =&(S(pr_1(\mathtt{I_I}((1,0),(w_1,w_2))),pr_1(\mathtt{I_I}((1,0),(w_1,w_2)))),\\ &T(pr_2(\mathtt{I_I}((1,0),(w_1,w_2))),pr_2(\mathtt{I_I}((1,0),(w_1,w_2))))).\end{aligned}$$

Since, $\mathtt{I_I}(1_{\mathcal{L}},w)=w$, $(w_1,w_2)=(S(w_1,w_1),T(w_2,w_2))$, i.e., $w_1=S(w_1,w_1)$ and $w_2=T(w_2,w_2)$. Hence $T=T_M$ and $S=S_M$ are the only idempotent t-norm and t-conorm respectively in fuzzy environment.

($\Leftarrow$) Suppose t-norm $\mathcal{T}$ and t-conorm $\mathcal{S}$ on $\mathcal{L}$ are the t-representable, i.e., $\mathcal{T}=(T_M,S_M)$ and $\mathcal{S}=(S_M,T_M)$. Then to prove that $\mathtt{I_I}$ satisfies (7.9), i.e., mainly satisfies (7.23). Let $u<_{\mathcal{L}} v$, i.e., $u_1<v_1,\ u_2>v_2$, for all $u=(u_1,u_2), v=(v_1,v_2)\in\mathcal{L}$. Then $\mathtt{I_I}(u,w)\geq_{\mathcal{L}}\mathtt{I_I}(v,w)$, i.e., $pr_1(\mathtt{I_I}(u,w))\geq pr_1(\mathtt{I_I}(v,w))$ and $pr_2(\mathtt{I_I}(u,w))\leq pr_2(\mathtt{I_I}(v,w))$, $\forall\, w\in\mathcal{L}$. It follows that the two sides of (7.23) are equal. On the other hand, let $u>_{\mathcal{L}} v$, i.e., $u_1>v_1,\ u_2<v_2$, $\forall\, u=(u_1,u_2), v=(v_1,v_2)\in\mathcal{L}$. Then $\mathtt{I_I}(u,w)\leq_{\mathcal{L}}\mathtt{I_I}(v,w)$, i.e., $pr_1(\mathtt{I_I}(u,w))\leq pr_1(\mathtt{I_I}(v,w))$ and $pr_2(\mathtt{I_I}(u,w))\geq pr_2(\mathtt{I_I}(v,w))$, for all $w\in\mathcal{L}$. It follows that the two sides of (7.23) are equal. We know that (7.23) is (7.9) when $\mathcal{T}$ and $\mathcal{S}$ are the t-representable. Thus $\mathtt{I_I}$ satisfies (7.9). □

7.4.2 General method for solving distributive eq. (7.9):

Distributive eq. (7.9) is given by

$$\mathtt{I_I}(\mathcal{T}(u,v),w)=\mathcal{S}(\mathtt{I_I}(u,w),\mathtt{I_I}(v,w)),\ \forall\, u,v,w\in\mathcal{L}, \tag{7.24}$$

where $\mathtt{I_I}:\mathcal{L}^2\to\mathcal{L}$ is the unknown function. The t-norm $\mathcal{T}$ and t-conorm $\mathcal{S}$ on $\mathcal{L}$ are the t-representable, i.e., $\mathcal{T}=(T,S)$ and $\mathcal{S}=(S,T)$.

At this situation distributive eq. (7.9) has the following form

$$\begin{aligned}\mathtt{I_I}((T(u_1,v_1),S(u_2,v_2)),(w_1,w_2)) =&(S(pr_1(\mathtt{I_I}((u_1,u_2),(w_1,w_2))),pr_1(\mathtt{I_I}((v_1,v_2),(w_1,w_2)))),\\ &T(pr_2(\mathtt{I_I}((u_1,u_2),(w_1,w_2))),pr_2(\mathtt{I_I}((v_1,v_2),(w_1,w_2))))),\end{aligned}$$

$\forall\, u=(u_1,u_2), v=(v_1,v_2), w=(w_1,w_2)\in\mathcal{L}$.

As a consequence we obtain the following two equations

$$pr_1(\mathtt{I_I}((T(u_1,v_1),S(u_2,v_2)),(w_1,w_2)))=S(pr_1(\mathtt{I_I}((u_1,u_2),(w_1,w_2))),pr_1(\mathtt{I_I}((v_1,v_2),(w_1,w_2)))), \tag{7.25}$$

$$pr_2(\mathtt{I_I}((T(u_1,v_1),S(u_2,v_2)),(w_1,w_2)))=T(pr_2(\mathtt{I_I}((u_1,u_2),(w_1,w_2))),pr_2(\mathtt{I_I}((v_1,v_2),(w_1,w_2)))), \tag{7.26}$$

$\forall\, u = (u_1, u_2), v = (v_1, v_2), w = (w_1, w_2) \in \mathcal{L}$.

Now, let $w = (w_1, w_2) \in \mathcal{L}$ be arbitrary but fixed. Then we define two functions $g^1_{(w_1,w_2)}$, $g^2_{(w_1,w_2)} : [0,1] \to [0,1]$ by

$$g^1_{(w_1,w_2)}(.) := pr_1 \circ \mathtt{I_I}(., (w_1, w_2)), \;\; g^2_{(w_1,w_2)}(.) := pr_2 \circ \mathtt{I_I}(., (w_1, w_2)), \tag{7.27}$$

where $\circ$ represents standard composition of functions.

From (7.25), (7.26) and (7.27), we have

$$g^1_{(w_1,w_2)}((T(u_1, v_1), S(u_2, v_2))) = S(g^1_{(w_1,w_2)}(u_1, u_2), g^1_{(w_1,w_2)}(v_1, v_2)), \tag{7.28}$$

$$g^2_{(w_1,w_2)}((T(u_1, v_1), S(u_2, v_2))) = T(g^2_{(w_1,w_2)}(u_1, u_2), g^2_{(w_1,w_2)}(v_1, v_2)). \tag{7.29}$$

For simplicity, we put $g^1_{(w_1,w_2)} = g^1$ and $g^2_{(w_1,w_2)} = g^2$ in (7.28) and (7.29), we have

$$g^1((T(u_1, v_1), S(u_2, v_2))) = S(g^1(u_1, u_2), g^1(v_1, v_2)), \;\; g^2((T(u_1, v_1), S(u_2, v_2))) = T(g^2(u_1, u_2), g^2(v_1, v_2)). \tag{7.30}$$

Proposition 7.4.6. *Let $\mathcal{T} = (T, S)$, $\mathcal{S} = (S, T)$, where T and S are the strict t-norm and t-conorm respectively such that T and S are dual of each other. For a function $\mathtt{I_I} : \mathcal{L}^2 \to \mathcal{L}$, the following statements are equivalent:*

(i) The triple $(\mathcal{T}, \mathcal{S}, \mathtt{I_I})$ satisfies functional eq. (7.9), $\forall\, u, v, w \in \mathcal{L}$.

(ii) For every fixed $w \in \mathcal{L}$, $\mathtt{I_I}(., w)$ has one of the following forms:

$$\mathtt{I_I}(u, w) = 1_{\mathcal{L}}, \;\; \mathtt{I_I}(u, w) = 0_{\mathcal{L}}, \;\; \mathtt{I_I}(u, w) = \begin{cases} 0_{\mathcal{L}}, & u_2 < 1, \\ (0,0), & u_2 = 1, \end{cases} \;\; \mathtt{I_I}(u, w) = \begin{cases} 0_{\mathcal{L}}, & u_2 = 0, \\ (0,0), & u_2 > 0, \end{cases}$$

$$\mathtt{I_I}(u, w) = (0,0), \;\; \mathtt{I_I}(u, w) = \begin{cases} 0_{\mathcal{L}}, & u_1 = 1, \\ (0,0), & u_1 < 1, \end{cases} \;\; \mathtt{I_I}(u, w) = \begin{cases} 0_{\mathcal{L}}, & u_1 = 1 - u_2 > 0, \\ (0,0), & u_2 = 1 \textit{ or } \\ & u_1 < 1 - u_2, \end{cases}$$

$$\mathtt{I_I}(u, w) = \begin{cases} 0_{\mathcal{L}}, & u_2 = 0, \\ & u_1 > 0, \\ (0,0), & u_2 > 0 \textit{ or } \\ & u_1 = 0, \end{cases} \;\; \mathtt{I_I}(u, w) = \begin{cases} 0_{\mathcal{L}}, & u_1 > 0, \\ (0,0), & u_1 = 0, \end{cases} \;\; \mathtt{I_I}(u, w) = (0, t^{-1}(cs(u_2)))$$

$$\mathtt{I_I}(u, w) = \begin{cases} (0,0), & u_1 < 1 - u_2, \\ (0, t^{-1}(ct(1 - u_2))), & u_1 = 1 - u_2, \end{cases} \;\; \mathtt{I_I}(u, w) = \begin{cases} (0, t^{-1}(cs(u_2))), & u_1 > 0, \\ (0,0), & u_1 = 0, \end{cases}$$

$$\mathtt{I_I}(u,w)=\begin{cases}(0,t^{-1}(c(t(u_1)))), & u_2=0,\\(0,0), & u_2>0,\end{cases}\quad \mathtt{I_I}(u,w)=\begin{cases}(0,t^{-1}(c(t(u_1)-s(u_2)))), & u_2<1,\\(0,0), & u_2=1,\end{cases}$$

$$\mathtt{I_I}(u,w)=(0,t^{-1}(ct(u_1))),\quad \mathtt{I_I}(u,w)=\begin{cases}(0,0), & u_2=1,\\(0,t^{-1}(c_1(t(u_1)- & \\s(u_2))+c_2s(u_2))), & u_2<1,\end{cases}$$

$$\mathtt{I_I}(u,w)=\begin{cases}1_{\mathcal{L}}, & u_2>0,\\(0,0), & u_2=0,\end{cases}\quad \mathtt{I_I}(u,w)=\begin{cases}1_{\mathcal{L}}, & u_2>0,\\0_{\mathcal{L}}, & u_2=0,\end{cases}\quad \mathtt{I_I}(u,w)=\begin{cases}1_{\mathcal{L}}, & u_2>0,\ u_1<1,\\ & u_1+u_2\leq 1,\\0_{\mathcal{L}}, & u_2=0,\ u_1=1,\\(0,0), & u_2=0,\ u_1<1,\end{cases}$$

$$\mathtt{I_I}(u,w)=\begin{cases}1_{\mathcal{L}}, & u_2>0\ or\ (u_1=0,u_2>0),\\(0,0), & u_1=0,u_2=0,\\0_{\mathcal{L}}, & u_2=0,\ u_1>0,\end{cases}\quad \mathtt{I_I}(u,w)=\begin{cases}1_{\mathcal{L}}, & 0<u_2,\\(0,t^{-1}(ct(u_1))), & u_2=0,\end{cases}$$

$$\mathtt{I_I}(u,w)=\begin{cases}1_{\mathcal{L}}, & u_2=1,\\(0,0), & u_2<1,\end{cases}\quad \mathtt{I_I}(u,w)=\begin{cases}1_{\mathcal{L}}, & u_2=1,\\0_{\mathcal{L}}, & u_2=0,\\(0,0), & 0<u_2<1,\end{cases}$$

$$\mathtt{I_I}(u,w)=\begin{cases}1_{\mathcal{L}}, & u_2=1,\\0_{\mathcal{L}}, & u_2<1,\end{cases}\quad \mathtt{I_I}(u,w)=\begin{cases}1_{\mathcal{L}}, & u_1=0,\ u_2=1,\\0_{\mathcal{L}}, & u_1=1,\ u_2=0,\\(0,0), & u_1<1,\ u_2<1,\\ & u_1+u_2\leq 1,\end{cases}$$

$$\mathtt{I_I}(u,w)=\begin{cases}1_{\mathcal{L}}, & u_2=1,\\0_{\mathcal{L}}, & u_1=1-u_2>0,\ u_2<1,\\(0,0), & u_1<1-u_2,\ u_2<1,\end{cases}\quad \mathtt{I_I}(u,w)=\begin{cases}1_{\mathcal{L}}, & u_2=1,\\0_{\mathcal{L}}, & u_1>0,\ u_2=0,\\(0,0), & (u_1=0,\ u_2<1),\\ & ;or\ (0<u_2<1),\end{cases}$$

$$\mathtt{I_I}(u,w)=\begin{cases}1_{\mathcal{L}}, & u_2=1,\\0_{\mathcal{L}}, & u_1>0,\ u_2<1,\\ & u_1+u_2\leq 1,\\(0,0), & u_1=0,\ u_2<1,\end{cases}\quad \mathtt{I_I}(u,w)=\begin{cases}1_{\mathcal{L}}, & u_2=1,\\(0,t^{-1}(c & \\s(u_2))), & u_2<1,\end{cases}$$

$$\mathtt{I_I}(u,w)=\begin{cases}1_{\mathcal{L}}, & u_1=0,\ u_2=1,\\ (0,0), & u_2<1,\ u_1<1-u_2,\\ (0,t^{-1}(ct(1-u_2))), & u_2<1,\ u_1=1-u_2,\end{cases}\quad \mathtt{I_I}(u,w)=\begin{cases}1_{\mathcal{L}}, & u_1<1,\\ (0,0), & u_1=1,\end{cases}$$

$$\mathtt{I_I}(u,w)=\begin{cases}1_{\mathcal{L}}, & u_1=0,\ u_2=1,\\ (0,0), & u_1=0,\ u_2<1,\\ (0,t^{-1}(cs(u_2))), & u_1>0,\ u_2<1,\\ & u_1+u_2\leq 1,\end{cases}\quad \mathtt{I_I}(u,w)=\begin{cases}1_{\mathcal{L}}, & u_2=1,\\ (0,t^{-1}(c(t & \\ (u_1)-s(u_2))), & u_2<1,\end{cases}$$

$$\mathtt{I_I}(u,w)=\begin{cases}1_{\mathcal{L}}, & u_2=1,\\ (0,t^{-1}(ct(u_1))), & u_2<1,\end{cases}\quad \mathtt{I_I}(u,w)=\begin{cases}1_{\mathcal{L}}, & u_2=1,\\ (0,0), & 0<u_2<1,\\ (0,t^{-1}(c(t(u_1)))), & u_2=0,\end{cases}$$

$$\mathtt{I_I}(u,w)=\begin{cases}1_{\mathcal{L}}, & u_2=1,\\ (0,t^{-1}(c_1 & \\ (t(u_1)-s(u_2)) & \\ +c_2s(u_2)), & u_2<1,\end{cases}\quad \mathtt{I_I}(u,w)=\begin{cases}1_{\mathcal{L}}, & u_2=1\ or\\ & u_1<1-u_2,\\ (0,0), & u_1=1-u_2>0,\end{cases}$$

$$\mathtt{I_I}(u,w)=\begin{cases}1_{\mathcal{L}}, & (u_1=0,\ u_2=1)\ or\\ & (u_1<1-u_2,\ u_1<1),\\ 0_{\mathcal{L}}, & u_1=1,\ u_2=0,\\ (0,0), & u_1=1-u_2>0,\ 0<u_1<1,\end{cases}\quad \mathtt{I_I}(u,w)=\begin{cases}1_{\mathcal{L}}, & u_1<1,\\ 0_{\mathcal{L}}, & u_1=1,\end{cases}$$

$$\mathtt{I_I}(u,w)=\begin{cases}1_{\mathcal{L}}, & u_2=1\ or\\ & u_1<1-u_2,\\ 0_{\mathcal{L}}, & u_1=1-u_2>0,\end{cases}\quad \mathtt{I_I}(u,w)=\begin{cases}1_{\mathcal{L}}, & (u_1=0,\ u_2=1),\\ & or\ (u_1<1-u_2),\\ (0,t^{-1}(c(t(1-u_2)))), & u_1=1-u_2>0,\end{cases}$$

$$\mathtt{I_I}(u,w)=\begin{cases}1_{\mathcal{L}}, & u_1=0\ or\ u_2>0,\\ (0,0), & u_2=0,\ u_1>0,\end{cases}\quad \mathtt{I_I}(u,w)=\begin{cases}1_{\mathcal{L}}, & u_1=0,\\ (0,0), & u_1>0,\end{cases}$$

$$\mathtt{I_I}(u,w)=\begin{cases}1_{\mathcal{L}}, & (u_1=0),\ or\ (u_1<1,\\ & u_2>0,u_1+u_2\leq 1),\\ 0_{\mathcal{L}}, & u_1=1,\ u_2=0,\\ (0,0), & 0<u_1<1,\ u_2=0,\end{cases}\quad \mathtt{I_I}(u,w)=\begin{cases}0_{\mathcal{L}}, & u_1>0,\ u_2=0,\\ 1_{\mathcal{L}}, & u_1=0\ or\ u_2>0,\end{cases}$$

$$\mathtt{I_I}(u,w)=\begin{cases}1_{\mathcal{L}}, & u_1=0,\\ (0,0), & 0<u_1<1,\\ 0_{\mathcal{L}}, & u_1=1,\end{cases}\quad \mathtt{I_I}(u,w)=\begin{cases}1_{\mathcal{L}}, & (u_1=0),\ or\\ & (u_2>0),\\ (0,t^{-1}(c(t(u_1)))), & u_1>0,u_2=0,\end{cases}$$

$$\mathtt{I_I}(u,w)=\begin{cases}1_{\mathcal{L}}, & u_1=0,\\ 0_{\mathcal{L}}, & u_1>0,\end{cases}\quad \mathtt{I_I}(u,w)=\begin{cases}1_{\mathcal{L}}, & (u_1=0,\ u_2=1),\ or\\ & (u_1=0,u_1<1-u_2)\\ (0,t^{-1}(ct(1-u_2))), & u_1=1-u_2,\ u_1>0,\\ (0,0), & u_1<1-u_2,\ u_1>0,\end{cases}$$

$$\mathtt{I_I}(u,w)=\begin{cases}1_{\mathcal{L}}, & u_1=0,\\ (0,t^{-1}(ct(u_1))), & u_1>0,\end{cases}\quad \mathtt{I_I}(u,w)=\begin{cases}1_{\mathcal{L}}, & u_1=0,\\ (0,t^{-1}(cs(u_2))), & u_1>0,\end{cases}$$

$$\mathtt{I_I}(u,w)=\begin{cases}1_{\mathcal{L}}, & (u_1=0,u_2=1),\ or\\ & (u_1=0,u_1<1-u_2),\\ 0_{\mathcal{L}}, & u_1>0,u_1=1-u_2>0,\\ (0,0), & u_1=0,u_1<1-u_2,\end{cases}\quad \mathtt{I_I}(u,w)=\begin{cases}1_{\mathcal{L}}, & u_1=0,\\ 0_{\mathcal{L}}, & u_1>0,u_2=0,\\ (0,0), & u_1>0,u_2>0,\\ & u_1+u_2\leq 1,\end{cases}$$

$$\mathtt{I_I}(u,w)=\begin{cases}1_{\mathcal{L}}, & u_1=0,\\ (0,t^{-1}(ct(u_1))), & u_1>0,u_2=0,\\ (0,0), & u_1>0,u_2>0,\\ & u_1+u_2\leq 1,\end{cases}\quad \mathtt{I_I}(u,w)=\begin{cases}1_{\mathcal{L}}, & u_1=0,\\ (0,t^{-1}(c(t(u_1) & \\ -s(u_2)))), & u_1>0,u_2=0,\end{cases}$$

$$\mathtt{I_I}(u,w)=\begin{cases}1_{\mathcal{L}}, & u_1=0,\\ (0,t^{-1}(c_1(t(u_1)- & \\ s(u_2))+c_2s(u_2))), & u_1>0,u_2=0,\end{cases}\quad \mathtt{I_I}(u,w)=(s^{-1}(cs(u_2)),0),$$

$$\mathtt{I_I}(u,w)=\begin{cases}(s^{-1}(cs(u_2)),0), & u_2>0,\\ 0_{\mathcal{L}}, & u_2=0,\end{cases}\quad \mathtt{I_I}(u,w)=\begin{cases}(s^{-1}(cs(u_2)),0), & u_1<1,\\ 0_{\mathcal{L}}, & u_1=1,\end{cases}$$

$$\mathtt{I_I}(u,w)=\begin{cases}(s^{-1}(cs(u_2)),0), & u_1=0,\ or\ u_2>0,\\ 0_{\mathcal{L}}, & u_2=0,u_1>0,\end{cases}\quad \mathtt{I_I}(u,w)=(s^{-1}(cs(u_2)),t^{-1}(cs(u_2))),$$

$$\mathtt{I_I}(u,w)=\begin{cases}(s^{-1}(cs(u_2)),t^{-1}(ct(1-u_2))), & u_1=1-u_2,\\ (s^{-1}(cs(u_2)),0), & u_1<1-u_2,\end{cases}$$

$$\mathtt{I}_{\mathtt{I}}(u,w)=\begin{cases}(s^{-1}(cs(u_2)),t^{-1}(cs(u_2))), & u_1>0,\\ (s^{-1}(cs(u_2)),0), & u_1=0,\end{cases}$$

$$\mathtt{I}_{\mathtt{I}}(u,w)=\begin{cases}(s^{-1}(cs(u_2)),t^{-1}(ct(u_1))), & u_2=0,\\ (s^{-1}(cs(u_2)),0), & u_2>0,\end{cases}$$

$$\mathtt{I}_{\mathtt{I}}(u,w)=\begin{cases}(s^{-1}(cs(u_2)),t^{-1}(c(t(u_1) & \\ -s(u_2)))), & u_2<1,\\ 1_{\mathcal{L}}, & u_2=1,\end{cases}\quad \mathtt{I}_{\mathtt{I}}(u,w)=(s^{-1}(cs(u_2)),t^{-1}(ct(u_1))),$$

$$\begin{cases}(s^{-1}(cs(u_2)),t^{-1}(c_1(t(u_1) & \\ -s(u_2))+c_2s(u_2))), & u_2<1,\\ 1_{\mathcal{L}}, & u_2=1,\end{cases}\quad \mathtt{I}_{\mathtt{I}}(u,w)=\begin{cases}(s^{-1}(ct(1-u_2)),0), & u_1=1-u_2,\\ 1_{\mathcal{L}}, & u_1<1-u_2,\end{cases}$$

$$\mathtt{I}_{\mathtt{I}}(u,w)=\begin{cases}1_{\mathcal{L}}, & u_1<1-u_2,\ u_1<1,\\ (s^{-1}(ct(1-u_2)),0), & u_1=1-u_2,\ u_1<1,\\ 0_{\mathcal{L}}, & u_1=1,\ u_2=0,\end{cases}$$

$$\mathtt{I}_{\mathtt{I}}(u,w)=\begin{cases}(s^{-1}(ct(1-u_2)),t^{-1}(ct(1-u_2))), & u_1=1-u_2,\\ 1_{\mathcal{L}}, & u_1<1-u_2,\end{cases}$$

$$\mathtt{I}_{\mathtt{I}}(u,w)=\begin{cases}(s^{-1}(cs(u_2)),0), & u_1>0,\\ 1_{\mathcal{L}}, & u_1=1,\end{cases}\quad \mathtt{I}_{\mathtt{I}}(u,w)=\begin{cases}(s^{-1}(cs(u_2)),0), & 0<u_1<1,\\ 1_{\mathcal{L}}, & u_1=0,\\ 0_{\mathcal{L}}, & u_1=1,\end{cases}$$

$$\mathtt{I}_{\mathtt{I}}(u,w)=\begin{cases}1_{\mathcal{L}}, & u_1>0,u_2=0,\\ (s^{-1}(cs(u_2)),0), & u_1>0,u_2>0,\\ & u_1+u_2\le 1,\\ 1_{\mathcal{L}}, & u_1=0,\end{cases}$$

$$\mathtt{I}_{\mathtt{I}}(u,w)=\begin{cases}(s^{-1}(cs(u_2)),t^{-1}(ct(1-u_2))), & u_1=1-u_2,\ u_1>0,\\ (s^{-1}(cs(u_2)),0), & u_1<1-u_2,\ u_2>0,\\ 1_{\mathcal{L}}, & u_1=0,\ u_2\le 1,\end{cases}$$

$$\mathtt{I_I}(u,w) = \begin{cases} (s^{-1}(cs(u_2)), t^{-1}(cs(u_2))), & u_1 > 0, \\ 1_{\mathcal{L}}, & u_1 = 0, \end{cases} \quad \mathtt{I_I}(u,w) = \begin{cases} (0, t^{-1}(ct(u_1))), & u_1 > 0, u_2 = 0, \\ (s^{-1}(cs(u_2)), 0), & u_1 > 0, u_2 > 0, \\ & u_1 + u_2 \le 1, \\ 1_{\mathcal{L}}, & u_1 = 0, u_2 \ge 0, \end{cases}$$

$$\mathtt{I_I}(u,w) = \begin{cases} (s^{-1}(cs(u_2)), t^{-1}(ct((u_1) - s(u_2)))), & u_1 > 0, u_2 < 1, \\ & u_1 + u_2 \le 1, \\ 1_{\mathcal{L}}, & u_1 = 0, u_2 \le 1, \end{cases}$$

$$\mathtt{I_I}(u,w) = \begin{cases} (s^{-1}(cs(u_2)), t^{-1}(ct(u_1))), & u_1 > 0, \\ 1_{\mathcal{L}}, & u_1 = 0, \end{cases}$$

$$\mathtt{I_I}(u,w) = \begin{cases} (s^{-1}(cs(u_2)), t^{-1}(c_1(& \\ t(u_1) - s(u_2)) + c_2 s(u_2))), & u_1 > 0, u_2 < 1, u_1 + u_2 \le 1, \\ 1_{\mathcal{L}}, & u_1 = 0, u_2 \le 1, \end{cases}$$

$$\mathtt{I_I}(u,w) = \begin{cases} (s^{-1}(ct(u_1)), 0), & u_2 = 0, \\ 1_{\mathcal{L}}, & u_2 > 0, \end{cases}$$

$$\mathtt{I_I}(u,w) = \begin{cases} 1_{\mathcal{L}}, & u_1 < 1,\ u_2 > 0, \\ & u_1 + u_2 \le 1, \\ (s^{-1}(ct(u_1)), 0), & u_1 < 1,\ u_2 = 0, \\ 0_{\mathcal{L}}, & u_1 = 1,\ u_2 = 0, \end{cases}$$

$$\mathtt{I_I}(u,w) = \begin{cases} (s^{-1}(ct(u_1)), t^{-1}(cs(u_2))), & u_2 = 0, \\ 1_{\mathcal{L}}, & u_2 > 0, \end{cases}$$

$$\mathtt{I_I}(u,w) = \begin{cases} (s^{-1}(c(t(u_1) - s(u_2))), 0), & u_2 < 1, \\ (0,0), & u_2 = 1, \end{cases}$$

$$\mathtt{I_I}(u,w) = \begin{cases} (s^{-1}(c(t(u_1) - s(u_2))), 0), & u_1 < 1,\ u_2 < 1, \\ & u_1 + u_2 \le 1, \\ 1_{\mathcal{L}}, & u_1 = 0,\ u_2 = 1, \\ 0_{\mathcal{L}}, & u_1 = 1,\ u_2 = 0, \end{cases}$$

$$\mathtt{I}_{\mathtt{I}}(u,w)=\begin{cases}(s^{-1}(c(t(u_1)-s(u_2))),t^{-1}(cs(u_2))), & u_2<1,\\ 1_{\mathcal{L}}, & u_2=1,\end{cases}$$

$$\mathtt{I}_{\mathtt{I}}(u,w)=\begin{cases}(s^{-1}(c(t(u_1)-s(u_2))), & \\ t^{-1}(ct(1-u_2))), & u_1=1-u_2,\ u_2<1,\\ (s^{-1}(c(t(u_1)-s(u_2))),0), & u_1<1-u_2,\ u_2<1,\\ 1_{\mathcal{L}}, & u_1=0,\ u_2=1,\end{cases}$$

$$\mathtt{I}_{\mathtt{I}}(u,w)=\begin{cases}(s^{-1}(c(t(u_1)-s(u_2))), & \\ t^{-1}(ct((u_1)-s(u_2)))), & u_2<1,\\ 1_{\mathcal{L}}, & u_2=1,\end{cases}$$

$$\mathtt{I}_{\mathtt{I}}(u,w)=\begin{cases}(s^{-1}(c(t(u_1)-s(u_2))),t^{-1}(cs(u_2))), & u_1>0,u_2<1,\\ & u_1+u_2\le 1,\\ (s^{-1}(c(t(u_1)-s(u_2))),0), & u_1=0,u_2<1,\\ 1_{\mathcal{L}}, & u_1=0,u_2=1,\end{cases}$$

$$\mathtt{I}_{\mathtt{I}}(u,w)=\begin{cases}(s^{-1}(c(t(u_1)-s(u_2))),0), & 0<u_2<1,\\ (s^{-1}(c(t(u_1))),t^{-1}(c(t(u_1)))), & u_2=0,\\ 1_{\mathcal{L}}, & u_2=1,\end{cases}$$

$$\mathtt{I}_{\mathtt{I}}(u,w)=\begin{cases}(s^{-1}(c(t(u_1)-s(u_2))), & \\ t^{-1}(ct(u_1))), & u_2<1,\\ 1_{\mathcal{L}}, & u_2=1,\end{cases}$$

$$\mathtt{I}_{\mathtt{I}}(u,w)=\begin{cases}(s^{-1}(c(t(u_1)-s(u_2))),t^{-1}(c_1 & \\ (t(u_1)-s(u_2))+c_2s(u_2))), & u_2<1,\\ 1_{\mathcal{L}}, & u_2=1,\end{cases}\quad \mathtt{I}_{\mathtt{I}}(u,w)=(s^{-1}(ct(u_1)),0),$$

$$\mathtt{I}_{\mathtt{I}}(u,w)=\begin{cases}(s^{-1}(ct(u_1)),0), & u_1<1,\\ 0_{\mathcal{L}}, & u_1=1,\end{cases}\quad \mathtt{I}_{\mathtt{I}}(u,w)=(s^{-1}(ct(u_1)),t^{-1}(cs(u_2))),$$

$$\mathtt{I}_{\mathtt{I}}(u,w)=\begin{cases}(s^{-1}(ct(u_1)),t^{-1}(ct(1-u_2))), & u_1=1-u_2,\\ (s^{-1}(ct(u_1)),0), & u_1<1-u_2,\end{cases}$$

$$\mathtt{I_I}(u,w)=\begin{cases}(s^{-1}(ct(u_1)),t^{-1}(cs(u_2))), & u_1>0,\\ (s^{-1}(ct(u_1)),0), & u_1=0,\end{cases}$$

$$\mathtt{I_I}(u,w)=\begin{cases}(s^{-1}(ct(u_1)),t^{-1}(ct(u_1))), & u_2=0,\\ (s^{-1}(ct(u_1)),0), & u_2>0,\end{cases}$$

$$\mathtt{I_I}(u,w)=\begin{cases}(s^{-1}(ct(u_1)),t^{-1}(c(t(u_1)-s(u_2)))), & u_2<1,\\ (s^{-1}(ct(u_1)),0), & u_2=1,\end{cases}$$

$$\mathtt{I_I}(u,w)=(s^{-1}(ct(u_1)),t^{-1}(ct(u_1))),$$

$$\mathtt{I_I}(u,w)=\begin{cases}(s^{-1}(ct(u_1)),t^{-1}(c_1(t(u_1)-s(u_2))+c_2s(u_2))), & u_2<1,\\ 1_{\mathcal{L}}, & u_2=1,\end{cases}$$

$$\mathtt{I_I}(u,w)=\begin{cases}(s^{-1}(c_1(t(u_1)-s(u_2))+c_2s(u_2)),0), & u_2<1,\\ (0,0), & u_2=1,\end{cases}$$

$$\mathtt{I_I}(u,w)=\begin{cases}(s^{-1}(c_1(t(u_1)-s(u_2))+c_2s(u_2)),0), & u_1<1,\ u_2<1,\\ & u_1+u_2\leq 1,\\ 1_{\mathcal{L}}, & u_1=0,\ u_2=1,\\ 0_{\mathcal{L}}, & u_1=1,\ u_2=0,\end{cases}$$

$$\mathtt{I_I}(u,w)=\begin{cases}(s^{-1}(c_1(t(u_1)-s(u_2))+c_2s(u_2)),t^{-1}(cs(u_2))), & u_2<1,\\ 1_{\mathcal{L}}, & u_2=1,\end{cases}$$

$$\mathtt{I_I}(u,w)=\begin{cases}(s^{-1}(c_1(t(u_1)-s(u_2))+c_2s(u_2)),t^{-1}(ct(1-u_2))), & u_1=1-u_2,\ u_2<1,\\ (s^{-1}(c_1(t(u_1)-s(u_2))+c_2s(u_2)),0), & u_1<1-u_2,\ u_2<1,\\ 1_{\mathcal{L}}, & u_1=0,\ u_2=1,\end{cases}$$

$$\mathtt{I_I}(u,w)=\begin{cases}(s^{-1}(c_1(t(u_1)-s(u_2))+c_2s(u_2)),t^{-1}(cs(u_2))), & u_1>0,u_2<1,\\ & u_1+u_2\leq 1,\\ (s^{-1}(c_1(t(u_1)-s(u_2))+c_2s(u_2)),0), & u_1=0,u_2<1,\\ 1_{\mathcal{L}}, & u_1=0,u_2=1,\end{cases}$$

$$\mathtt{I_I}(u,w)=\begin{cases}(s^{-1}(c_1(t(u_1)-s(u_2))+c_2s(u_2)),0), & 0<u_2<1,\\ (s^{-1}(c_1(t(u_1))),t^{-1}(c(t(u_1)))), & u_2=0,\\ 1_{\mathcal{L}}, & u_2=1,\end{cases}$$

$$\mathtt{I}_{\mathtt{I}}(u,w)=\begin{cases}(s^{-1}(c_1(t(u_1)-s(u_2))+c_2s(u_2)),t^{-1}(ct((u_1)-s(u_2)))), & u_2<1,\\ 1_{\mathcal{L}}, & u_2=1,\end{cases}$$

$$\mathtt{I}_{\mathtt{I}}(u,w)=\begin{cases}(s^{-1}(c_1(t(u_1)-s(u_2))+c_2s(u_2)),t^{-1}(ct(u_1))), & u_2<1,\\ 1_{\mathcal{L}}, & u_2=1,\end{cases}$$

or

$$\mathtt{I}_{\mathtt{I}}(u,w)=\begin{cases}(s^{-1}(c_1(t(u_1)-s(u_2))+c_2s(u_2)),t^{-1}(c_1(t(u_1)-s(u_2))+c_2s(u_2))), & u_2<1,\\ 1_{\mathcal{L}}, & u_2=1.\end{cases}$$

Proof. Given that $\mathcal{T}$ and $\mathcal{S}$ are the t-representable on $\mathcal{L}$ such that $\mathcal{T}=(T,S)$ and $\mathcal{S}=(S,T)$, i.e., $\mathcal{T}(u,v)=(T(u_1,v_1),S(u_2,v_2))$ $\mathcal{S}(u,v)=(S(u_1,v_1),T(u_2,v_2))$, for all $u,v\in\mathcal{L}$. Given that t-norm T is strict, and T and S are dual of each other. It follows that t-conorm S is strict. Now, from Theorems 2.1.5 and 2.1.7 and Remarks 2.2.6 and 2.2.7, given in [19], there exist a decreasing continuous function $t:[0,1]\to[0,\infty]$ and a increasing continuous function $s:[0,1]\to[0,\infty]$ such that $t(0)=\infty$, $t(1)=0$, $s(0)=0$ and $s(1)=\infty$ which are uniquely determined a positive multiplicative constant such that, for all $a,b\in[0,1]$, $T(a,b)=t^{-1}(t(a)+t(b))$ and $S(a,b)=s^{-1}(s(a)+s(b))$.
Let us prove that $(ii)\Rightarrow(i)$.

(P1): Let $\mathtt{I}_{\mathtt{I}}$ have the form $\mathtt{I}_{\mathtt{I}}(u,w)=1_{\mathcal{L}}$. Then the LHS of (7.9) is equal to $1_{\mathcal{L}}$, and the RHS of (7.9) is equal to $1_{\mathcal{L}}$.

(P2): Let $\mathtt{I}_{\mathtt{I}}$ have the form $\mathtt{I}_{\mathtt{I}}(u,w)=(0,0)$. Then the LHS of (7.9) is equal to $(0,0)$, and the RHS of (7.9) is equal to $(0,0)$.

(P3): Let $\mathtt{I}_{\mathtt{I}}$ have the form $\mathtt{I}_{\mathtt{I}}(u,w)=0_{\mathcal{L}}$. Then the LHS of (7.9) is equal to $0_{\mathcal{L}}$, and the RHS of (7.9) is equal to $0_{\mathcal{L}}$.

(P4): Let $\mathtt{I}_{\mathtt{I}}$ have the form $\mathtt{I}_{\mathtt{I}}(u,w)=\begin{cases}0_{\mathcal{L}}, & u_2=0,\\ (0,0), & u_2>0.\end{cases}$

Then the LHS of (7.9) is equal to

$$\mathtt{I}_{\mathtt{I}}(\mathcal{T}(u,v),w)=\begin{cases}(0_{\mathcal{L}}, & S(u_2,v_2)=0,\\ (0,0), & S(u_2,v_2)>0.\end{cases}=\begin{cases}0_{\mathcal{L}}, & u_2=v_2=0,\\ (0,0), & u_2>0 \text{ or } v_2>0,\end{cases}$$

and the RHS of (7.9) is equal to

$$\mathcal{S}(\mathtt{I_I}(u,w),\mathtt{I_I}(v,w)) = \mathcal{S}\left(\begin{cases} 0_{\mathcal{L}}, & u_2 = 0, \\ (0,0), & u_2 > 0. \end{cases}, \begin{cases} 0_{\mathcal{L}}, & v_2 = 0, \\ (0,0), & v_2 > 0. \end{cases}\right)$$
$$= \begin{cases} 0_{\mathcal{L}}, & u_2 = v_2 = 0, \\ (0,0), & u_2 > 0 \text{ or } v_2 > 0. \end{cases}$$

(P5): Let $\mathtt{I_I}$ have the form $\mathtt{I_I}(u,w) = \begin{cases} 0_{\mathcal{L}}, & u_2 < 1, \\ (0,0), & u_2 = 1. \end{cases}$

Then the LHS of (7.9) is equal to

$$\mathtt{I_I}(\mathcal{T}(u,v),w) = \begin{cases} 0_{\mathcal{L}}, & S(u_2,v_2) < 1, \\ (0,0), & S(u_2,v_2) = 1. \end{cases} = \begin{cases} 0_{\mathcal{L}}, & u_2 < 1,\ v_2 < 1, \\ (0,0), & u_2 = 1 \text{ or } v_2 = 1, \end{cases}$$

and the RHS of (7.9) is equal to

$$\mathcal{S}(\mathtt{I_I}(u,w),\mathtt{I_I}(v,w)) = \mathcal{S}\left(\begin{cases} 0_{\mathcal{L}}, & u_2 < 1, \\ (0,0), & u_2 = 1. \end{cases}, \begin{cases} 0_{\mathcal{L}}, & v_2 < 1, \\ (0,0), & v_2 = 1. \end{cases}\right)$$
$$= \begin{cases} 0_{\mathcal{L}}, & u_2 < 1,\ v_2 < 1, \\ (0,0), & u_2 = 1 \text{ or } v_2 = 1, \end{cases}$$

(P6): Let $\mathtt{I_I}$ have the form $\mathtt{I_I}(u,w) = \begin{cases} 0_{\mathcal{L}}, & u_1 = 1, \\ (0,0), & u_1 < 1. \end{cases}$

Then the LHS of (7.9) is equal to

$$\mathtt{I_I}(\mathcal{T}(u,v),w) = \begin{cases} 0_{\mathcal{L}}, & T(u_1,v_1) = 1, \\ (0,0), & T(u_1,v_1) < 1. \end{cases} = \begin{cases} 0_{\mathcal{L}}, & u_1 = v_1 = 1, \\ (0,0), & u_1 < 1 \text{ or } v_1 < 1, \end{cases}$$

and the RHS of (7.9) is equal to

$$\mathcal{S}(\mathtt{I_I}(u,w),\mathtt{I_I}(v,w)) = \mathcal{S}\left(\begin{cases} 0_{\mathcal{L}}, & u_1 = 1, \\ (0,0), & u_1 < 1. \end{cases}, \begin{cases} 0_{\mathcal{L}}, & v_1 = 1, \\ (0,0), & v_1 < 1. \end{cases}\right)$$
$$= \begin{cases} 0_{\mathcal{L}}, & u_1 = v_1 = 1, \\ (0,0), & u_1 < 1 \text{ or } v_1 < 1. \end{cases}$$

(P7): Let $\mathtt{I_I}$ have the form $\mathtt{I_I}(u,w) = (0, 1 - s^{-1}(ct(1-u_2)))$.

Then the LHS of (7.9) is equal to

$$\mathtt{I}_{\mathtt{I}}(\mathcal{T}(u,v),w) = (0, 1 - s^{-1}(ct(1 - S(u_2, v_2)))) = (0, 1 - s^{-1}(ct(T(1 - u_2, 1 - v_2)))) \\ = (0, 1 - s^{-1}(c(t(1 - u_2) + t(1 - v_2)))),$$

and the RHS of (7.9) is equal to

$$\begin{aligned} \mathcal{S}(\mathtt{I}_{\mathtt{I}}(u,w), \mathtt{I}_{\mathtt{I}}(v,w)) =& \mathcal{S}((0, 1 - s^{-1}(ct(1 - u_2))), (0, 1 - s^{-1}(ct(1 - v_2)))) \\ =& (0, T(1 - s^{-1}(ct(1 - u_2)), 1 - s^{-1}(ct(1 - v_2)))) \\ =& (0, 1 - S(s^{-1}(ct(1 - u_2)), s^{-1}(ct(1 - v_2)))) \\ =& (0, 1 - s^{-1}(c(t(1 - u_2) + t(1 - v_2)))). \end{aligned}$$

Similarly we can verify the eq. (7.9) easily by all remaining forms of $\mathtt{I}_{\mathtt{I}}$.

Let us prove that $(i) \Rightarrow (ii)$.
From (7.30),

$$g^1(t^{-1}(t(u_1) + t(v_1)), s^{-1}(s(u_2) + s(v_2))) = s^{-1}(s(g^1(u_1, u_2)) + s(g^1(v_1, v_2))),$$
$$g^2(t^{-1}(t(u_1) + t(v_1)), s^{-1}(s(u_2) + s(v_2))) = t^{-1}(t(g^1(u_1, u_2)) + t(g^1(v_1, v_2))).$$

Hence

$$s \circ (g^1(t^{-1}(t(u_1) + t(v_1)), s^{-1}(s(u_2) + s(v_2)))) = s(g^1(u_1, u_2)) + s(g^1(v_1, v_2)), \tag{7.31}$$
$$t \circ (g^2(t^{-1}(t(u_1) + t(v_1)), s^{-1}(s(u_2) + s(v_2)))) = t(g^1(u_1, u_2)) + t(g^1(v_1, v_2)). \tag{7.32}$$

Let us put $t(u_1) = x_1, s(u_2) = x_2, t(v_1) = y_1$ and $s(v_2) = y_2$. Of course $x_1, x_2, y_1, y_2 \in [0, \infty]$, Moreover, $u = (u_1, u_2), v = (v_1, v_2) \in \mathcal{L}$, thus $u_1 \leq 1 - u_2$ and $v_1 \leq 1 - v_2$. Since t and s are decreasing generator and increasing generator respectively such that $t(a) = s(1-a), \forall a \in [0, 1]$, $x_1 \geq x_2$ and $y_1 \geq y_2$. This implies that $(x_1, x_2), (y_1, y_2) \in [0, \infty]^2$. If we put

$$f^1(x_1, x_2) := s \circ pr_1 \circ \mathtt{I}_{\mathtt{I}}((t^{-1}(x_1), s^{-1}(x_2)), (w_1, w_2)), \quad f^2(x_1, x_2) := t \circ pr_2 \circ \mathtt{I}_{\mathtt{I}}((t^{-1}(x_1), \\ s^{-1}(x_2)), (w_1, w_2)) \ \forall \ (x_1, x_2) \in [0, \infty]^2 \tag{7.33}$$

As a consequence we get the following two functional equations

$$f^1(x_1 + y_1, x_2 + y_2) = f^1(x_1, x_2) + f^1(y_1, y_2), \tag{7.34}$$
$$f^2(x_1 + y_1, x_2 + y_2) = f^2(x_1, x_2) + f^2(y_1, y_2), \tag{7.35}$$

where $(x_1, x_2), (y_1, y_2) \in [0, \infty]^2$.

Now, we find the possible solutions of (7.34) are as follows:

For $x_1, x_2 \in [0, \infty], x_1 \geq x_2$ and $(w_1, w_2) \in \mathcal{L}$, we have

(S1): $f^1 = 0 \;\Rightarrow\; s \circ pr_1 \circ \mathtt{I}_{\mathtt{I}}((t^{-1}(x_1), s^{-1}(x_2)), (w_1, w_2)) = 0 \;\Rightarrow\; pr_1 \circ \mathtt{I}_{\mathtt{I}}((u_1, u_2), (w_1, w_2)) = 0.$

(S2): $f^1 = \infty \;\Rightarrow\; s \circ pr_1 \circ \mathtt{I}_{\mathtt{I}}((t^{-1}(x_1), s^{-1}(x_2)), (w_1, w_2)) = \infty \;\Rightarrow\; pr_1 \circ \mathtt{I}_{\mathtt{I}}((u_1, u_2), (w_1, w_2)) = 1.$

(S3):
$$f^1(x_1, x_2) = \begin{cases} 0, & x_2 = 0, \\ \infty, & x_2 > 0, \end{cases}$$
$$\Rightarrow\; s \circ pr_1 \circ \mathtt{I}_{\mathtt{I}}((t^{-1}(x_1), s^{-1}(x_2)), (w_1, w_2)) = \begin{cases} 0, & x_2 = 0, \\ \infty, & x_2 > 0 \end{cases}$$
$$\Rightarrow\; pr_1 \circ \mathtt{I}_{\mathtt{I}}((u_1, u_2), (w_1, w_2)) = \begin{cases} 0, & u_2 = 0, \\ 1, & u_2 > 0. \end{cases}$$

(S4):
$$f^1(x_1, x_2) = \begin{cases} 0, & x_2 < \infty, \\ \infty, & x_2 = \infty, \end{cases}$$
$$\Rightarrow\; s \circ pr_1 \circ \mathtt{I}_{\mathtt{I}}((t^{-1}(x_1), s^{-1}(x_2)), (w_1, w_2)) = \begin{cases} 0, & x_2 < \infty, \\ \infty, & x_2 = \infty \end{cases}$$
$$\Rightarrow\; pr_1 \circ \mathtt{I}_{\mathtt{I}}((u_1, u_2), (w_1, w_2)) = \begin{cases} 0, & u_2 < 1, \\ 1, & u_2 = 1. \end{cases}$$

(S5):
$$f^1(x_1, x_2) = \begin{cases} 0, & x_1 = 0, \\ \infty, & x_1 > 0, \end{cases}$$
$$\Rightarrow\; s \circ pr_1 \circ \mathtt{I}_{\mathtt{I}}((t^{-1}(x_1), s^{-1}(x_2)), (w_1, w_2)) = \begin{cases} 0, & x_1 = 0, \\ \infty, & x_1 > 0. \end{cases}$$
$$\Rightarrow\; pr_1 \circ \mathtt{I}_{\mathtt{I}}((u_1, u_2), (w_1, w_2)) = \begin{cases} 0, & u_1 = 1, \\ 1, & u_1 < 1. \end{cases}$$

(S6):
$$f^1(x_1,x_2)=\begin{cases}0, & x_1=x_2<\infty,\\ \infty, & x_2=\infty \ or \ x_1>x_2,\end{cases}$$

$$\Rightarrow s\circ pr_1\circ \mathtt{I}_{\mathtt{I}}((t^{-1}(x_1),s^{-1}(x_2)),(w_1,w_2))=\begin{cases}0, & x_1=x_2<\infty,\\ \infty, & x_2=\infty \ or \ x_1>x_2.\end{cases}$$

$$\Rightarrow pr_1\circ \mathtt{I}_{\mathtt{I}}((u_1,u_2),(w_1,w_2))=\begin{cases}0, & u_1=1-u_2>0,\\ 1, & u_2=1 \ or \ u_1<1-u_2.\end{cases}$$

(S7):
$$f^1(x_1,x_2)=\begin{cases}0, & x_2=0,\ x_1<\infty,\\ \infty, & x_2>0 \ or \ x_1=\infty,\end{cases}$$

$$\Rightarrow s\circ pr_1\circ \mathtt{I}_{\mathtt{I}}((t^{-1}(x_1),s^{-1}(x_2)),(w_1,w_2))=\begin{cases}0, & x_2=0,\ x_1<\infty,\\ \infty, & x_2>0 \ or \ x_1=\infty.\end{cases}$$

$$\Rightarrow pr_1\circ \mathtt{I}_{\mathtt{I}}((u_1,u_2),(w_1,w_2))=\begin{cases}0, & u_2=0,\ u_1>0,\\ 1, & u_2>0 \ or \ u_1=0.\end{cases}$$

(S8):
$$f^1(x_1,x_2)=\begin{cases}0, & x_1<\infty,\\ \infty, & x_1=\infty,\end{cases}$$

$$\Rightarrow s\circ pr_1\circ \mathtt{I}_{\mathtt{I}}((t^{-1}(x_1),s^{-1}(x_2)),(w_1,w_2))=\begin{cases}0, & x_1<\infty,\\ \infty, & x_1=\infty.\end{cases}$$

$$\Rightarrow pr_1\circ \mathtt{I}_{\mathtt{I}}((u_1,u_2),(w_1,w_2))=\begin{cases}0, & u_1>0,\\ 1, & u_1=0.\end{cases}$$

(S9): $\exists\ c\in(0,\infty)$ such that

$$f^1(x_1,x_2)=cx_2 \ \Rightarrow\ s\circ pr_1\circ \mathtt{I}_{\mathtt{I}}((t^{-1}(x_1),s^{-1}(x_2)),(w_1,w_2))=cx_2$$
$$\Rightarrow\ pr_1\circ \mathtt{I}_{\mathtt{I}}((u_1,u_2),(w_1,w_2))=s^{-1}(cs(u_2)).$$

(S10): $$f^1(x_1,x_2) = \begin{cases} cx_1, & x_1 = x_2, \\ \infty, & x_1 > x_2, \end{cases}$$

$$\Rightarrow\ s \circ pr_1 \circ \mathtt{I_I}((t^{-1}(x_1), s^{-1}(x_2)), (w_1, w_2)) = \begin{cases} cx_1, & x_1 = x_2, \\ \infty, & x_1 > x_2. \end{cases}$$

$$\Rightarrow\ pr_1 \circ \mathtt{I_I}((u_1, u_2), (w_1, w_2)) = \begin{cases} s^{-1}(ct(1-u_2)), & u_1 = 1-u_2, \\ 1, & u_1 < 1-u_2. \end{cases}$$

(S11): $$f^1(x_1,x_2) = \begin{cases} cx_2, & x_1 < \infty, \\ \infty, & x_1 = \infty, \end{cases}$$

$$\Rightarrow\ s \circ pr_1 \circ \mathtt{I_I}((t^{-1}(x_1), s^{-1}(x_2)), (w_1, w_2)) = \begin{cases} cx_2, & x_1 < \infty, \\ \infty, & x_1 = \infty. \end{cases}$$

$$\Rightarrow\ pr_1 \circ \mathtt{I_I}((u_1, u_2), (w_1, w_2)) = \begin{cases} s^{-1}(cs(u_2)), & u_1 > 0, \\ 1, & u_1 = 0. \end{cases}$$

(S12): $$f^1(x_1,x_2) = \begin{cases} cx_1, & x_2 = 0, \\ \infty, & x_2 > 0, \end{cases}$$

$$\Rightarrow\ s \circ pr_1 \circ \mathtt{I_I}((t^{-1}(x_1), s^{-1}(x_2)), (w_1, w_2)) = \begin{cases} cx_1, & x_2 = 0, \\ \infty, & x_2 > 0. \end{cases}$$

$$\Rightarrow\ pr_1 \circ \mathtt{I_I}((u_1, u_2), (w_1, w_2)) = \begin{cases} s^{-1}(ct(u_1)), & u_2 = 0, \\ 1, & u_2 > 0. \end{cases}$$

(S13): $$f^1(x_1,x_2) = \begin{cases} c(x_1 - x_2), & x_2 < \infty, \\ \infty, & x_2 = \infty, \end{cases}$$

$$\Rightarrow\ s \circ pr_1 \circ \mathtt{I_I}((t^{-1}(x_1), s^{-1}(x_2)), (w_1, w_2)) = \begin{cases} c(x_1 - x_2), & x_2 < \infty, \\ \infty, & x_2 = \infty. \end{cases}$$

$$\Rightarrow\ pr_1 \circ \mathtt{I_I}((u_1, u_2), (w_1, w_2)) = \begin{cases} s^{-1}(c(t(u_1) - s(u_2))), & u_2 < 1, \\ 1, & u_2 = 1. \end{cases}$$

(S14): $\exists\, c \in (0, \infty)$ such that

$$f^1(x_1, x_2) = cx_1 \;\Rightarrow\; s \circ pr_1 \circ \mathtt{I}_\mathtt{I}((t^{-1}(x_1), s^{-1}(x_2)), (w_1, w_2)) = cx_1$$
$$\Rightarrow\; pr_1 \circ \mathtt{I}_\mathtt{I}((u_1, u_2), (w_1, w_2)) = s^{-1}(ct(u_1)).$$

(S15): $\exists\, c_1, c_2 \in (0, \infty),\ c_1 \neq c_2$ such that

$$f^1(x_1, x_2) = \begin{cases} c_1(x_1 - x_2) + c_2 x_2, & x_2 < \infty, \\ \infty, & x_2 = \infty, \end{cases}$$

$$\Rightarrow\; s \circ pr_1 \circ \mathtt{I}_\mathtt{I}((t^{-1}(x_1), s^{-1}(x_2)), (w_1, w_2)) = \begin{cases} c_1(x_1 - x_2) + c_2 x_2, & x_2 < \infty, \\ \infty, & x_2 = \infty. \end{cases}$$

$$\Rightarrow\; pr_1 \circ \mathtt{I}_\mathtt{I}((u_1, u_2), (w_1, w_2)) = \begin{cases} s^{-1}(c_1(t(u_1) - s(u_2)) + c_2(s(u_2))), & u_2 < 1, \\ 1, & u_2 = 1. \end{cases}$$

Similarly, we can find the possible solutions of (7.35) are as follows:

For $x_1, x_2 \in [0, \infty], x_1 \geq x_2$ and $(w_1, w_2) \in \mathcal{L}$, we have

(S'1): $pr_2 \circ \mathtt{I}_\mathtt{I}((u_1, u_2), (w_1, w_2)) = 1.$

(S'2): $pr_2 \circ \mathtt{I}_\mathtt{I}((u_1, u_2), (w_1, w_2)) = 0.$

(S'3): $pr_2 \circ \mathtt{I}_\mathtt{I}((u_1, u_2), (w_1, w_2)) = \begin{cases} 1, & u_2 = 0, \\ 0, & u_2 > 0. \end{cases}$

(S'4): $pr_2 \circ \mathtt{I}_\mathtt{I}((u_1, u_2), (w_1, w_2)) = \begin{cases} 1, & u_2 < 1, \\ 0, & u_2 = 1. \end{cases}$

(S'5): $pr_2 \circ \mathtt{I}_\mathtt{I}((u_1, u_2), (w_1, w_2)) = \begin{cases} 1, & u_1 = 1, \\ 0, & u_1 < 1. \end{cases}$

(S'6): $pr_2 \circ \mathtt{I}_\mathtt{I}((u_1, u_2), (w_1, w_2)) = \begin{cases} 1, & u_1 = 1 - u_2 > 0, \\ 0, & u_2 = 1 \text{ or } u_1 < 1 - u_2. \end{cases}$

(S'7): $pr_2 \circ \mathtt{I}_\mathtt{I}((u_1, u_2), (w_1, w_2)) = \begin{cases} 1, & u_2 = 0,\ u_1 > 0, \\ 0, & u_2 > 0 \text{ or } u_1 = 0. \end{cases}$

(S'8): $pr_2 \circ \mathtt{I_I}((u_1,u_2),(w_1,w_2)) = \begin{cases} 1, & u_1 > 0, \\ 0, & u_1 = 0. \end{cases}$

(S'9): $\exists\ c \in (0,\infty)$ such that

$$pr_2 \circ \mathtt{I_I}((u_1,u_2),(w_1,w_2)) = t^{-1}(cs(u_2)).$$

(S'10): $pr_2 \circ \mathtt{I_I}((u_1,u_2),(w_1,w_2)) = \begin{cases} t^{-1}(ct(1-u_2)), & u_1 = 1-u_2, \\ 0, & u_1 < 1-u_2. \end{cases}$

(S'11): $pr_2 \circ \mathtt{I_I}((u_1,u_2),(w_1,w_2)) = \begin{cases} t^{-1}(cs(u_2)), & u_1 > 0, \\ 0, & u_1 = 0. \end{cases}$

(S'12): $pr_2 \circ \mathtt{I_I}((u_1,u_2),(w_1,w_2)) = \begin{cases} t^{-1}(ct(u_1)), & u_2 = 0, \\ 0, & u_2 > 0. \end{cases}$

(S'13): $pr_1 \circ \mathtt{I_I}((u_1,u_2),(w_1,w_2)) = \begin{cases} t^{-1}(c(t(u_1) - s(u_2))), & u_2 < 1, \\ 0, & u_2 = 1. \end{cases}$

(S'14): $\exists\ c \in (0,\infty)$ such that
$pr_2 \circ \mathtt{I_I}((u_1,u_2),(w_1,w_2)) = t^{-1}(ct(u_1)).$

(S'15): $pr_2 \circ \mathtt{I_I}((u_1,u_2),(w_1,w_2)) = \begin{cases} s^{-1}(c_1(t(u_1) - s(u_2)) + c_2(s(u_2))), & u_2 < 1, \\ 0, & u_2 = 1. \end{cases}$

Of course not every combination of the above solutions give a correct value in the set $\mathcal{L}$. For example when $pr_1 \circ \mathtt{I_I}((u_1,u_2),(v_1,v_2)) = 0$ and $pr_2 \circ \mathtt{I_I}((u_1,u_2),(v_1,v_2)) = 1$, for every $(v_1,v_2) \in \mathcal{L}$, then our (constant) solution is correct: $\mathtt{I_I}((u_1,u_2),(v_1,v_2)) = (0,1) = 0_{\mathcal{L}}$.
Also when $pr_1 \circ \mathtt{I_I}((u_1,u_2),(w_1,w_2)) = \begin{cases} 0, & u_2 = 0, \\ 1, & u_2 > 0. \end{cases}$ and $pr_2 \circ \mathtt{I_I}((u_1,u_2),(w_1,w_2)) = \begin{cases} 1, & u_2 = 0, \\ 0, & u_2 > 0. \end{cases}$
for every $(v_1,v_2) \in \mathcal{L}$, then our (constant) solution is correct: $\mathtt{I_I}((u_1,u_2),(v_1,v_2)) = \begin{cases} 0_{\mathcal{L}}, & u_2 = 0, \\ 1_{\mathcal{L}}, & u_2 > 0. \end{cases}$

But if $pr_1 \circ \mathtt{I_I}((u_1,u_2),(w_1,w_2)) = \begin{cases} 0, & u_2 = 0, \\ 1, & u_2 > 0. \end{cases}$ and $pr_2 \circ \mathtt{I_I}((u_1,u_2),(w_1,w_2)) = \begin{cases} 1, & u_2 < 1, \\ 0, & u_2 = 1, \end{cases}$

for every $(v_1, v_2) \in \mathcal{L}$, then our solution is incorrect, since $\mathtt{I_I}((u_1,u_2),(v_1,v_2)) = \begin{cases} 0_{\mathcal{L}}, & u_2 = 0, \\ 1_{\mathcal{L}}, & u_2 = 1, \\ (1,1), & 0 < u_2 < 1, \end{cases}$

is not solution in $\mathcal{L}$ (since $(1,1) \notin \mathcal{L}$).

Similarly, we can find the possible combinations of the above solutions give a correct value in the set $\mathcal{L}$ is the required result (ii). □

Remark 7.4.7. *If we put $\alpha = u$, $\beta = \mathtt{f_I}(w)$ and $\mathtt{I_I} = \mathtt{I}_{\mathtt{I}(\mathtt{f_I},\omega)}$ in Proposition 7.4.6, then the above possible forms of $\mathtt{I_I}(u, w)$, for fixed $w \in \mathcal{L}$, convert into corresponding forms of $\omega(\alpha, \beta)$.*

Proposition 7.4.8. *Let $\mathcal{T} = (T, S)$, $\mathcal{S} = (S, T)$, where T and S are the nilpotent t-norm and t-conorm respectively such that T and S are dual of each other. For a function $\mathtt{I_I} : \mathcal{L}^2 \to \mathcal{L}$, the following statements are equivalent:*

(i) The triple $(\mathcal{T}, \mathcal{S}, \mathtt{I_I})$ satisfies the functional eq. (7.9) $\forall\, u, v, w \in \mathcal{L}$.

(ii) For every fixed $w \in \mathcal{L}$, $\mathtt{I_I}(., w)$ has one of the following forms:

$$\mathtt{I_I}(u,w) = 1_{\mathcal{L}}, \quad \mathtt{I_I}(u,w) = (0,0), \quad \mathtt{I_I}(u,w) = 0_{\mathcal{L}}, \quad \mathtt{I_I}(u,w) = \begin{cases} 0_{\mathcal{L}}, & u_2 = 0, \\ (0,0), & u_2 > 0, \end{cases}$$

$$\mathtt{I_I}(u,w) = \begin{cases} 0_{\mathcal{L}}, & u_1 = 1, \\ (0,0), & u_1 < 1, \end{cases} \quad \mathtt{I_I}(u,w) = (0, t^{-1}(\min(cs(u_2), b))),$$

$$\mathtt{I_I}(u,w) = \begin{cases} (0, t^{-1}(\min(\\ ct(u_1), b))), & u_1 = 1 - u_2, \\ (0,0), & u_1 < 1 - u_2, \end{cases} \quad \mathtt{I_I}(u,w) = \begin{cases} (0, t^{-1}(\min(ct(u_1), b))), & u_2 = 0, \\ (0,0), & u_2 > 0, \end{cases}$$

$$\mathtt{I_I}(u,w) = (0, t^{-1}(\min(ct(u_1), b))), \quad \mathtt{I_I}(u,w) = \begin{cases} (0, t^{-1}(\min(c_1(t(u_1) - \\ s(u_2)) + c_2 s(u_2), b))), & u_1 > 0, \\ (0,0), & u_1 = 0, \end{cases}$$

$$\mathtt{I_I}(u,v) = \begin{cases} (0,0), & u_2 = 0, \\ 1_{\mathcal{L}}, & u_2 > 0, \end{cases} \quad \mathtt{I_I}(u,w) = \begin{cases} 0_{\mathcal{L}}, & u_2 = 0, \\ 1_{\mathcal{L}}, & u_2 > 0, \end{cases}$$

$$\mathtt{I}_{\mathtt{I}}(u,w)=\begin{cases}0_{\mathcal{L}}, & u_1=1, u_2=0,\\ (0,0), & u_1<1, u_2=0,\\ 1_{\mathcal{L}}, & u_1<1, u_2>0\\ & \& \ u_1+u_2\leq 1.\end{cases} \quad \mathtt{I}_{\mathtt{I}}(u,w)=\begin{cases}(0,t^{-1}(\min(ct(u_1),b))), & u_2=0,\\ 1_{\mathcal{L}}, & u_2>0,\end{cases}$$

$$\mathtt{I}_{\mathtt{I}}(u,w)=\begin{cases}(0,0), & u_1=1,\\ 1_{\mathcal{L}}, & u_1<1,\end{cases} \quad \mathtt{I}_{\mathtt{I}}(u,w)=\begin{cases}0_{\mathcal{L}}, & u_1=1,\\ 1_{\mathcal{L}}, & u_1<1,\end{cases} \quad \mathtt{I}_{\mathtt{I}}(u,w)=(s^{-1}(\min(cs(u_2),b)),0),$$

$$\mathtt{I}_{\mathtt{I}}(u,w)=\begin{cases}0_{\mathcal{L}}, & u_2=0,\\ (s^{-1}(\min(cs(u_2),b)),0), & u_2>0,\end{cases} \quad \mathtt{I}_{\mathtt{I}}(u,w)=\begin{cases}0_{\mathcal{L}}, & u_1=1,\\ (s^{-1}(\min(cs(u_2),b)),0), & u_1<1,\end{cases}$$

$$\mathtt{I}_{\mathtt{I}}(u,w)=(s^{-1}(\min(cs(u_2),b)),t^{-1}(\min(cs(u_2),b))),$$

$$\mathtt{I}_{\mathtt{I}}(u,w)=\begin{cases}(s^{-1}(\min(cs(u_2),b)),\\ t^{-1}(\min(ct(u_1),b))), & u_1=1-u_2,\\ (s^{-1}(\min(cs(u_2),b)),0), & u_1<1-u_2,\end{cases}$$

$$\mathtt{I}_{\mathtt{I}}(u,w)=\begin{cases}(0,t^{-1}(\min(ct(u_1),b))), & u_2=0,\\ (s^{-1}(\min(cs(u_2),b)),0), & u_2>0,\end{cases}$$

$$\mathtt{I}_{\mathtt{I}}(u,w)=(s^{-1}(\min(cs(u_2),b)),t^{-1}(\min(ct(u_1),b))),$$

$$\mathtt{I}_{\mathtt{I}}(u,w)=\begin{cases}(s^{-1}(\min(cs(u_2),b)),t^{-1}(\min(c_1(t(u_1)-s(u_2))+c_2s(u_2),b))), & u_2=0,\\ (s^{-1}(\min(cs(u_2),b)),0), & u_1=0,\end{cases}$$

$$\mathtt{I}_{\mathtt{I}}(u,w)=\begin{cases}(s^{-1}(\min(ct(u_1),b)),0), & u_1=1-u_2,\\ 1_{\mathcal{L}}, & u_1<1-u_2,\end{cases}$$

$$\mathtt{I}_{\mathtt{I}}(u,w)=\begin{cases}0_{\mathcal{L}}, & u_1=1, u_2=0,\\ (s^{-1}(\min(ct(u_1),b)),0), & u_1=1-u_2, u_1<1,\\ 1_{\mathcal{L}}, & u_1<1-u_2, u_1<1,\end{cases}$$

$$\mathtt{I}_{\mathtt{I}}(u,w)=\begin{cases}(s^{-1}(\min(ct(u_1),b)),\\ t^{-1}(\min(c(t(u_1),b)))), & u_1=1-u_2,\\ 1_{\mathcal{L}}, & u_1<1-u_2,\end{cases}$$

$$\mathtt{I}_{\mathtt{I}}(u,w)=\begin{cases}(s^{-1}(\min(ct(u_1),b)),0), & u_2=0,\\ 1_{\mathcal{L}}, & u_2>0,\end{cases}$$

$$\mathtt{I}_{\mathtt{I}}(u,w)=\begin{cases}0_{\mathcal{L}}, & u_1=1,u_2=0,\\ (s^{-1}(\min(ct(u_1),b)),0), & u_1<1,u_2=0,\\ 1_{\mathcal{L}}, & u_1<1,u_2>0,u_1+u_2\leq 1,\end{cases}$$

$$\mathtt{I}_{\mathtt{I}}(u,w)=\begin{cases}(s^{-1}(\min(ct(u_1),b)),t^{-1}(\min(c(t(u_1),b)))), & u_2=0,\\ 1_{\mathcal{L}}, & u_2>0,\end{cases}$$

$$\mathtt{I}_{\mathtt{I}}(u,w)=(s^{-1}(\min(ct(u_1),b)),0),$$

$$\mathtt{I}_{\mathtt{I}}(u,w)=\begin{cases}0_{\mathcal{L}}, & u_1=1,\\ (s^{-1}(\min(ct(u_1),b)),0), & u_1<1,\end{cases}$$

$$\mathtt{I}_{\mathtt{I}}(u,w)=(s^{-1}(\min(ct(u_1),b)),t^{-1}(\min(c(s(u_2),b)))),$$

$$\mathtt{I}_{\mathtt{I}}(u,w)=\begin{cases}(s^{-1}(\min(ct(u_1),b)),\\ t^{-1}(\min(c(t(u_1),b)))), & u_1=1-u_2,\\ (s^{-1}(\min(ct(u_1),b)),0), & u_1<1-u_2,\end{cases}$$

$$\mathtt{I}_{\mathtt{I}}(u,w)=\begin{cases}(s^{-1}(\min(ct(u_1),b)),\\ t^{-1}(\min(c(t(u_1),b)))), & u_2=0,\\ (s^{-1}(\min(ct(u_1),b)),0), & u_2>0,\end{cases}$$

$$\mathtt{I}_{\mathtt{I}}(u,w)=(s^{-1}(\min(ct(u_1),b)),t^{-1}(\min(c(t(u_1),b)))),$$

$$\mathtt{I}_{\mathtt{I}}(u,w)=\begin{cases}(s^{-1}(\min(ct(u_1),b)),t^{-1}(\min(c_1\\ (t(u_1)-s(u_2))+c_2s(u_2),b))), & u_1>0,\\ (s^{-1}(\min(ct(u_1),b)),0), & u_1=0,\end{cases}$$

$$\mathtt{I}_{\mathtt{I}}(u,w)=\begin{cases}(s^{-1}(\min(c_1(t(u_1)-s(u_2))\\ +c_2s(u_2),b)),0), & u_1>0,\\ 1_{\mathcal{L}}, & u_1=0,\end{cases}$$

$$\mathtt{I}_{\mathtt{I}}(u,w)=\begin{cases}0_{\mathcal{L}}, & u_1=1,\\ 1_{\mathcal{L}}, & u_1=0,\\ (s^{-1}(\min(c_1(t(u_1)-s(u_2)) & \\ +c_2s(u_2),b)),0), & 0<u_1<1,\end{cases}$$

or

$$\mathtt{I}_{\mathtt{I}}(u,w)=\begin{cases}(s^{-1}(\min(c_1(t(u_1)-s(u_2))+c_2s(u_2),b)), & \\ t^{-1}(\min(c_1(t(u_1)-s(u_2))+c_2s(u_2),b))), & u_1>0,\\ 1_{\mathcal{L}}, & u_1=0,\end{cases}$$

Proof. Given that $\mathcal{T}$ and $\mathcal{S}$ are the t-representable on $\mathcal{L}$ such that $\mathcal{T}=(T,S)$ and $\mathcal{S}=(S,T)$, i.e., $\mathcal{T}(u,v)=(T(u_1,v_1),S(u_2,v_2))$ $\mathcal{S}(u,v)=(S(u_1,v_1),T(u_2,v_2))$, for all $u,v\in\mathcal{L}$. Given that t-norm T is nilpotent, and T and S are dual of each other. It follows that t-conorm S is nilpotent. Now, from Theorems 2.1.5 and 2.1.7 and Remarks 2.2.6 and 2.2.7 given in [19], there exists a decreasing continuous function $t:[0,1]\to[0,\infty]$ and a increasing continuous function $s:[0,1]\to[0,\infty]$ such that $t(0)<\infty$, $t(1)=0$, $s(0)=0$ and $s(1)<\infty$ which are uniquely determined a positive multiplicative constant such that, for all $a,b\in[0,1]$, $T(a,b)=t^{-1}(\min(t(a)+t(b),t(0)))$ and $S(a,b)=s^{-1}(\min(s(a)+s(b),s(1)))$.
Let us prove that $(ii)\Rightarrow(i)$.

(P1): Let $\mathtt{I}_{\mathtt{I}}$ have the form $\mathtt{I}_{\mathtt{I}}(u,w)=1_{\mathcal{L}}$. Then the LHS of (7.9) is equal to $1_{\mathcal{L}}$, and the RHS of (7.9) is equal to $1_{\mathcal{L}}$.

(P2): Let $\mathtt{I}_{\mathtt{I}}$ have the form $\mathtt{I}_{\mathtt{I}}(u,w)=(0,0)$. Then the LHS of (7.9) is equal to $(0,0)$, and the RHS of (7.9) is equal to $(0,0)$.

(P3): Let $\mathtt{I}_{\mathtt{I}}$ have the form $\mathtt{I}_{\mathtt{I}}(u,w)=0_{\mathcal{L}}$. Then the LHS of (7.9) is equal to $0_{\mathcal{L}}$, and the RHS of (7.9) is equal to $0_{\mathcal{L}}$.

(P4): Let $\mathtt{I}_{\mathtt{I}}$ have the form $\mathtt{I}_{\mathtt{I}}(u,w)=\begin{cases}0_{\mathcal{L}}, & u_2=0,\\ (0,0), & u_2>0.\end{cases}$

Then the LHS of (7.9) is equal to

$$\mathtt{I}_{\mathtt{I}}(\mathcal{T}(u,v),w)=\begin{cases}0_{\mathcal{L}}, & S(u_2,y_2)=0,\\ (0,0), & S(u_2,v_2)>0.\end{cases}=\begin{cases}0_{\mathcal{L}}, & u_2=y_2=0,\\ (0,0), & u_2>0 \text{ or } v_2>0,\end{cases}$$

and the RHS of (7.9) is equal to

$$\mathcal{S}(\mathtt{I_I}(u,w),\mathtt{I_I}(v,w)) = \mathcal{S}\left(\begin{cases} 0_{\mathcal{L}}, & u_2=0, \\ (0,0), & u_2>0. \end{cases}, \begin{cases} 0_{\mathcal{L}}, & v_2=0, \\ (0,0), & v_2>0. \end{cases}\right)$$

$$= \begin{cases} 0_{\mathcal{L}}, & u_2=v_2=0, \\ (0,0), & u_2>0 \text{ or } v_2>0. \end{cases}$$

(P5): Let $\mathtt{I_I}$ have the form $\mathtt{I_I}(u,w) = \begin{cases} 0_{\mathcal{L}}, & v_1=1, \\ (0,0), & v_1<1. \end{cases}$

Then the LHS of (7.11) is equal to

$$\mathtt{I_I}(\mathcal{T}(u,v),w) = \begin{cases} 0_{\mathcal{L}}, & T(u_1,v_1)=1, \\ (0,0), & T(u_1,v_1)<1. \end{cases} = \begin{cases} 0_{\mathcal{L}}, & u_1=v_1=1, \\ (0,0), & u_1<1 \text{ or } v_1<1, \end{cases}$$

and the RHS of (7.11) is equal to

$$\mathcal{S}(\mathtt{I_I}(u,w),\mathtt{I_I}(v,w)) = \mathcal{S}\left(\begin{cases} 0_{\mathcal{L}}, & u_1=1, \\ (0,0), & u_1<1. \end{cases}, \begin{cases} 0_{\mathcal{L}}, & v_1=1, \\ (0,0), & v_1<1. \end{cases}\right)$$

$$= \begin{cases} 0_{\mathcal{L}}, & u_1=v_1=1, \\ (0,0), & u_1<1 \text{ or } v_1<1. \end{cases}$$

(P6): Let $\mathtt{I_I}$ have the form $\mathtt{I_I}(u,w) = \begin{cases} (0,0), & u_2=0, \\ 1_{\mathcal{L}}, & u_2>0. \end{cases}$

Then the LHS of (7.9) is equal to

$$\mathtt{I_I}(\mathcal{T}(u,v),w) = \begin{cases} (0,0), & S(u_2,v_2)=0, \\ 1_{\mathcal{L}}, & S(u_2,v_2)>0. \end{cases} = \begin{cases} (0,0), & u_2=v_2=0, \\ 1_{\mathcal{L}}, & u_2>0 \text{ or } v_2>0, \end{cases}$$

and the RHS of (7.9) is equal to

$$\mathcal{S}(\mathtt{I_I}(u,w),\mathtt{I_I}(v,w)) = \mathcal{S}\left(\begin{cases} (0,0), & u_2=0, \\ 1_{\mathcal{L}}, & u_2>0. \end{cases}, \begin{cases} (0,0), & v_2=0, \\ 1_{\mathcal{L}}, & v_2>0. \end{cases}\right)$$

$$= \begin{cases} (0,0), & u_2=v_2=0, \\ 1_{\mathcal{L}}, & u_2>0 \text{ or } v_2>0. \end{cases}$$

(P7): Let $\mathtt{I_I}$ have the form $\mathtt{I_I}(u,w) = \begin{cases} 0_{\mathcal{L}}, & u_2=0, \\ 1_{\mathcal{L}}, & u_2>0. \end{cases}$

Then the LHS of (7.9) is equal to

$$\mathtt{I}_{\mathtt{I}}(\mathcal{T}(u,v),w) = \begin{cases} 0_{\mathcal{L}}, & S(u_2,v_2)=0, \\ 1_{\mathcal{L}}, & S(u_2,v_2)>0. \end{cases} = \begin{cases} 0_{\mathcal{L}}, & u_2=v_2=0, \\ 1_{\mathcal{L}}, & u_2>0 \text{ or } v_2>0, \end{cases}$$

and the RHS of (7.9) is equal to

$$\mathcal{S}(\mathtt{I}_{\mathtt{I}}(u,w),\mathtt{I}_{\mathtt{I}}(v,w)) = \mathcal{S}\left(\begin{cases} 0_{\mathcal{L}}, & u_2=0, \\ 1_{\mathcal{L}}, & u_2>0. \end{cases}, \begin{cases} 0_{\mathcal{L}}, & v_2=0, \\ 1_{\mathcal{L}}, & v_2>0. \end{cases}\right) = \begin{cases} 0_{\mathcal{L}}, & u_2=v_2=0, \\ 1_{\mathcal{L}}, & u_2>0 \text{ or } v_2>0. \end{cases}$$

(P8): Let $\mathtt{I}_{\mathtt{I}}$ have the form $\mathtt{I}_{\mathtt{I}}(u,w) = \begin{cases} (0,0), & u_1=1, \\ 1_{\mathcal{L}}, & u_1<1. \end{cases}$

Then the LHS of (7.9) is equal to

$$\mathtt{I}_{\mathtt{I}}(\mathcal{T}(u,v),w) = \begin{cases} (0,0), & T(u_1,v_1)=1, \\ 1_{\mathcal{L}}, & T(u_1,v_1)<1. \end{cases} = \begin{cases} (0,0), & u_1=v_1=1, \\ 1_{\mathcal{L}}, & u_1<1 \text{ or } v_1<1, \end{cases}$$

and the RHS of (7.9) is equal to

$$\begin{aligned} \mathcal{S}(\mathtt{I}_{\mathtt{I}}(u,w),\mathtt{I}_{\mathtt{I}}(v,w)) =& \mathcal{S}\left(\begin{cases} (0,0), & u_1=1, \\ 1_{\mathcal{L}}, & u_1<1. \end{cases}, \begin{cases} (0,0), & v_1=1, \\ 1_{\mathcal{L}}, & v_1<1. \end{cases}\right) \\ =& \begin{cases} (0,0), & u_1=v_1=1, \\ 1_{\mathcal{L}}, & u_1<1 \text{ or } v_1<1. \end{cases} \end{aligned}$$

(P9): Let $\mathtt{I}_{\mathtt{I}}$ have the form $\mathtt{I}_{\mathtt{I}}(u,w) = \begin{cases} 0_{\mathcal{L}}, & u_1=1, \\ 1_{\mathcal{L}}, & u_1<1. \end{cases}$

Then the LHS of (7.9) is equal to

$$\mathtt{I}_{\mathtt{I}}(\mathcal{T}(u,v),w) = \begin{cases} 0_{\mathcal{L}}, & T(u_1,v_1)=1, \\ 1_{\mathcal{L}}, & T(u_1,v_1)<1. \end{cases} = \begin{cases} 0_{\mathcal{L}}, & u_1=v_1=1, \\ 1_{\mathcal{L}}, & u_1<1 \text{ or } v_1<1, \end{cases}$$

and the RHS of (7.9) is equal to

$$\mathcal{S}(\mathtt{I}_{\mathtt{I}}(u,w),\mathtt{I}_{\mathtt{I}}(v,w)) = \mathcal{S}\left(\begin{cases} 0_{\mathcal{L}}, & u_1=1, \\ 1_{\mathcal{L}}, & u_1<1. \end{cases}, \begin{cases} 0_{\mathcal{L}}, & v_1=1, \\ 1_{\mathcal{L}}, & v_1<1. \end{cases}\right) = \begin{cases} 0_{\mathcal{L}}, & u_1=v_1=1, \\ 1_{\mathcal{L}}, & u_1<1 \text{ or } v_1<1. \end{cases}$$

(P10): Let $\mathtt{I}_{\mathtt{I}}$ have the form $\mathtt{I}_{\mathtt{I}}(u,w) = \begin{cases} 0_{\mathcal{L}}, & u_1=1,\ u_2=0, \\ (0,0), & u_1<1,\ u_2=0, \\ 1_{\mathcal{L}}, & u_1<1,\ u_2>0,\ u_1+u_2\leq 1. \end{cases}$

Then the LHS of (7.9) is equal to

$$\mathtt{I_I}(\mathcal{T}(u,v),w) = \begin{cases} 0_{\mathcal{L}}, & T(u_1,v_1)=1,\ S(u_2,v_2)=0,\\ (0,0), & T(u_1,v_1)<1,\ S(u_2,v_2)=0,\\ 1_{\mathcal{L}}, & T(u_1,v_1)<1,\ S(u_2,v_2)>0,\\ & T(u_1,v_1)+S(u_2,v_2)\leq 1. \end{cases}$$

$$= \begin{cases} 0_{\mathcal{L}}, & u_1=v_1=1,\ u_2=v_2=0,\\ (0,0), & (u_1<1 \textit{ or } v_1<1),\ u_2=v_2=0,\\ 1_{\mathcal{L}}, & (u_1<1 \textit{ or } v_1<1),\ (u_2>0 \textit{ or}\\ & v_2>0),\ u_1+u_2\leq 1,\ v_1+v_2\leq 1, \end{cases}$$

and the RHS of (7.9) is equal to

$$\mathcal{S}(\mathtt{I_I}(u,w),\mathtt{I_I}(v,w)) = \mathcal{S}\left(\begin{cases} 0_{\mathcal{L}}, & u_1=1,\ u_2=0,\\ (0,0), & u_1<1,\ u_2=0,\\ 1_{\mathcal{L}}, & u_1<1,\ u_2>0,\\ & u_1+u_2\leq 1. \end{cases}, \begin{cases} 0_{\mathcal{L}}, & v_1=1,\ v_2=0,\\ (0,0), & v_1<1,\ v_2=0,\\ 1_{\mathcal{L}}, & v_1<1,\ v_2>0,\\ & v_1+v_2\leq 1. \end{cases}\right)$$

$$= \begin{cases} 0_{\mathcal{L}}, & u_1=v_1=1,\ u_2=v_2=0,\\ (0,0), & (u_1<1 \textit{ or } v_1<1),\ u_2=v_2=0,\\ 1_{\mathcal{L}}, & (u_1<1 \textit{ or } v_1<1),\ (u_2>0 \textit{ or } v_2>0)\\ & \&\ u_1+u_2\leq 1,\ v_1+v_2\leq 1. \end{cases}$$

Let us prove that $(i) \Rightarrow (ii)$.

From (7.30),

$$\begin{aligned} & g^1(t^{-1}(\min(t(u_1)+t(v_1),t(0))), s^{-1}(\min(s(u_2)+s(v_2),s(1))))\\ & = s^{-1}(\min(s(g^1(u_1,u_2))+s(g^1(v_1,v_2)),s(1))),\\ & g^2(t^{-1}(\min(t(u_1)+t(v_1),t(0))), s^{-1}(\min(s(u_2)+s(v_2),s(1))))\\ & = t^{-1}(\min(t(g^2(u_1,u_2))+t(g^2(v_1,v_2)),t(0))). \end{aligned}$$

Hence

$$s \circ (g^1(t^{-1}(\min(t(u_1)+t(v_1),t(0))), s^{-1}(\min(s(u_2)+s(v_2),s(1)))))$$
$$= \min(s(g^1(u_1,u_2)) + s(g^1(v_1,v_2)), s(1)), \quad (7.36)$$
$$t \circ (g^2(t^{-1}(\min(t(u_1)+t(v_1),t(0))), s^{-1}(\min(s(u_2)+s(v_2),s(1)))))$$
$$= \min(t(g^2(u_1,u_2)) + t(g^2(v_1,v_2)), t(0)). \quad (7.37)$$

Let us put $t(u_1) = x_1, s(u_2) = x_2, t(v_1) = y_1$, $s(v_2) = y_2$, $t(0) = a$ and $s(1) = b$. Of course $x_1, x_2, y_1, y_2 \in [0,\infty]$, Moreover, $u = (u_1,u_2), v = (v_1,v_2) \in \mathcal{L}$, thus $u_1 \leq 1-u_2$ and $v_1 \leq 1-v_2$. Since t and s are decreasing generator and increasing generator such that $t(1-a) = t(0)-t(a)$ and $s(a) = t(0)-t(a)$, $\forall\ a \in [0,1]$, $x_1 \geq x_2$ and $y_1 \geq y_2$. This implies that $(x_1,x_2),(y_1,y_2) \in [0,\infty]^2$. If we put

$$f^1(x_1,x_2) := s \circ pr_1 \circ \mathtt{I}_\mathtt{I}((t^{-1}(x_1), s^{-1}(x_2)),(w_1,w_2)), \quad f^2(x_1,x_2) := t \circ pr_2 \circ \mathtt{I}_\mathtt{I}((t^{-1}(x_1), \quad (7.38)$$
$$s^{-1}(x_2)),(w_1,w_2)) \ \forall\ (x_1,x_2) \in [0,\infty]^2 \quad (7.39)$$

As a consequence we get the following two functional equations

$$f^1(\min(x_1+y_1,a), \min(x_2+y_2,a)) = \min(f^1(x_1,x_2) + f^1(y_1,y_2), b), \quad (7.40)$$
$$f^2(\min(x_2+y_2,a), \min(x_1+y_1,a)) = \min(f^2(x_1,x_2) + f^2(y_1,y_2), b), \quad (7.41)$$

where $(x_1,x_2),(y_1,y_2) \in [0,\infty]^2$

For $x_1,x_2 \in [0,\infty], x_1 \geq x_2$ and $(w_1,w_2) \in \mathcal{L}$, we have

(S1): $f^1 = 0 \ \Rightarrow\ s \circ pr_1 \circ \mathtt{I}_\mathtt{I}((t^{-1}(x_1), s^{-1}(x_2)),(w_1,w_2)) = 0 \ \Rightarrow\ pr_1 \circ \mathtt{I}_\mathtt{I}((u_1,u_2),(w_1,w_2)) = 0.$

(S2): $f^1 = b \ \Rightarrow\ s \circ pr_1 \circ \mathtt{I}_\mathtt{I}((t^{-1}(x_1), s^{-1}(x_2)),(w_1,w_2)) = b \ \Rightarrow\ pr_1 \circ \mathtt{I}_\mathtt{I}((u_1,u_2),(w_1,w_2)) = 1.$

(S3): $$f^1(x_1,x_2) = \begin{cases} 0, & x_2 = 0, \\ b, & x_2 > 0, \end{cases}$$

$$\Rightarrow\ s \circ pr_1 \circ \mathtt{I}_\mathtt{I}((t^{-1}(x_1), s^{-1}(x_2)),(w_1,w_2)) = \begin{cases} 0, & x_2 = 0, \\ b, & x_2 > 0. \end{cases}$$

$$\Rightarrow\ pr_1 \circ \mathtt{I}_\mathtt{I}((u_1,u_2),(w_1,w_2)) = \begin{cases} 0, & u_2 = 0, \\ 1, & u_2 > 0. \end{cases}$$

(S4): $$f^1(x_1,x_2) = \begin{cases} 0, & x_1 = 0, \\ b, & x_1 > 0, \end{cases}$$

$$\Rightarrow\ s \circ pr_1 \circ \mathtt{I_I}((t^{-1}(x_1), s^{-1}(x_2)), (w_1, w_2)) = \begin{cases} 0, & x_1 = 0, \\ b, & x_1 > 0. \end{cases}$$

$$\Rightarrow\ pr_1 \circ \mathtt{I_I}((u_1, u_2), (w_1, w_2)) = \begin{cases} 0, & u_1 = 1, \\ 1, & u_1 < 1. \end{cases}$$

(S5): $\exists\ c \in [b/a, \infty)$ such that

$$\begin{aligned} f^1(x_1,x_2) = \min(cx_2, b) \ &\Rightarrow\ s \circ pr_1 \circ \mathtt{I_I}((t^{-1}(x_1), s^{-1}(x_2)), (w_1, w_2)) = \min(cx_2, b) \\ &\Rightarrow\ pr_1 \circ \mathtt{I_I}((u_1, u_2), (w_1, w_2)) = s^{-1}(\min(cs(u_2), b)). \end{aligned}$$

(S6): $$f^1(x_1,x_2) = \begin{cases} \min(cx_1, b), & x_1 = x_2, \\ b, & x_1 > x_2, \end{cases}$$

$$\Rightarrow\ s \circ pr_1 \circ \mathtt{I_I}((t^{-1}(x_1), s^{-1}(x_2)), (w_1, w_2)) = \begin{cases} \min(cx_1, b), & x_1 = x_2, \\ b, & x_1 > x_2. \end{cases}$$

$$\Rightarrow\ pr_1 \circ \mathtt{I_I}((u_1, u_2), (w_1, w_2)) = \begin{cases} s^{-1}(\min(ct(u_1), b)), & u_1 = 1 - u_2, \\ 1, & u_1 < 1 - u_2. \end{cases}$$

(S7): $$f^1(x_1,x_2) = \begin{cases} \min(cx_1, b), & x_2 = 0, \\ b, & x_2 > 0, \end{cases}$$

$$\Rightarrow\ s \circ pr_1 \circ \mathtt{I_I}((t^{-1}(x_1), s^{-1}(x_2)), (w_1, w_2)) = \begin{cases} \min(cx_1, b), & x_2 = 0, \\ b, & x_2 > 0. \end{cases}$$

$$\Rightarrow\ pr_1 \circ \mathtt{I_I}((u_1, u_2), (w_1, w_2)) = \begin{cases} s^{-1}(\min(ct(u_1), b)), & u_2 = 0, \\ 1, & u_2 > 0. \end{cases}$$

(S8): $f^1(x_1,x_2) = \min(cx_1, b) \Rightarrow\ pr_1 \circ \mathtt{I_I}((u_1, u_2), (w_1, w_2)) = s^{-1}(\min(ct(u_1), b)).$

(S9): $\exists\ c_1, c_2 \in [b/a, \infty),\ c_1 \neq c_2$ such that

$$f^1(x_1, x_2) = \begin{cases} \min(c_1(x_1 - x_2) + c_2 x_2, b), & x_1 < a, \\ b, & x_1 = a, \end{cases}$$

$$\Rightarrow\ s \circ pr_1 \circ \mathtt{I}_{\mathtt{I}}((t^{-1}(x_1), s^{-1}(x_2)), (w_1, w_2)) = \begin{cases} \min(c_1(x_1 - x_2) + c_2 x_2, b), & x_1 < a, \\ b, & x_1 = a. \end{cases}$$

$$\Rightarrow\ pr_1 \circ \mathtt{I}_{\mathtt{I}}((u_1, u_2), (w_1, w_2)) = \begin{cases} s^{-1}(\min(c_1(t(u_1) - s(u_2)) + c_2(s(u_2)), b)), & u_1 > 0, \\ 1, & u_1 = 0. \end{cases}$$

Similarly, we can find the possible solutions of (7.41) are as follows:

For $x_1, x_2 \in [0, \infty], x_1 \geq x_2$ and $(w_1, w_2) \in \mathcal{L}$, we have

(S'1): $pr_2 \circ \mathtt{I}_{\mathtt{I}}((u_1, u_2), (w_1, w_2)) = 1.$

(S'2): $pr_2 \circ \mathtt{I}_{\mathtt{I}}((u_1, u_2), (w_1, w_2)) = 0.$

(S'3): $pr_2 \circ \mathtt{I}_{\mathtt{I}}((u_1, u_2), (w_1, w_2)) = \begin{cases} 1, & u_2 = 0, \\ 0, & u_2 > 0. \end{cases}$

(S'4): $pr_2 \circ \mathtt{I}_{\mathtt{I}}((u_1, u_2), (w_1, w_2)) = \begin{cases} 1, & u_1 = 1, \\ 0, & u_1 < 1. \end{cases}$

(S'5): $\exists\ c \in [b/a, \infty)$ s.t.
$pr_2 \circ \mathtt{I}_{\mathtt{I}}((u_1, u_2), (w_1, w_2)) = t^{-1}(\min(cs(u_2), b)).$

(S'6): $pr_2 \circ \mathtt{I}_{\mathtt{I}}((u_1, u_2), (w_1, w_2)) = \begin{cases} t^{-1}(\min(ct(u_1), b)), & u_1 = 1 - u_2, \\ 0, & u_1 < 1 - u_2. \end{cases}$

(S'7): $pr_2 \circ \mathtt{I}_{\mathtt{I}}((u_1, u_2), (w_1, w_2)) = \begin{cases} t^{-1}(\min(ct(u_1), b)), & u_2 = 0, \\ 0, & u_2 > 0. \end{cases}$

(S'8): $pr_2 \circ \mathtt{I}_{\mathtt{I}}((u_1, u_2), (w_1, w_2)) = t^{-1}(\min(ct(u_1), b)).$

(S'9): $\exists\ c_1, c_2 \in [b/a, \infty),\ c_1 \neq c_2$ such that

$$pr_2 \circ \mathtt{I}_{\mathtt{I}}((u_1, u_2), (w_1, w_2)) = \begin{cases} t^{-1}(\min(c_1(t(u_1) - s(u_2)) + c_2(s(u_2)), b)), & u_1 > 0, \\ 0, & u_1 = 0. \end{cases}$$

Of course not every combination of the above solutions give a correct value in the set $\mathcal{L}$. For example when $pr_1 \circ \mathtt{I_I}((u_1,u_2),(v_1,v_2)) = 0$ and $pr_2 \circ \mathtt{I_I}((u_1,u_2),(v_1,v_2)) = 1$, for every $(v_1,v_2) \in \mathcal{L}$, then our (constant) solution is correct: $\mathtt{I_I}((u_1,u_2),(v_1,v_2)) = (0,1) = 0_{\mathcal{L}}$.

Also when $pr_1 \circ \mathtt{I_I}((u_1,u_2),(w_1,w_2)) = \begin{cases} 0, & u_2 = 0, \\ 1, & u_2 > 0. \end{cases}$ and $pr_2 \circ \mathtt{I_I}((u_1,u_2),(w_1,w_2)) = \begin{cases} 1, & u_2 = 0, \\ 0, & u_2 > 0. \end{cases}$

for every $(v_1,v_2) \in \mathcal{L}$, then our (constant) solution is correct: $\mathtt{I_I}((u_1,u_2),(v_1,v_2)) = \begin{cases} 0_{\mathcal{L}}, & u_2 = 0, \\ 1_{\mathcal{L}}, & u_2 > 0. \end{cases}$

But if $pr_1 \circ \mathtt{I_I}((u_1,u_2),(w_1,w_2)) = \begin{cases} 0, & u_2 = 0, \\ 1, & u_2 > 0. \end{cases}$ and $pr_2 \circ \mathtt{I_I}((u_1,u_2),(w_1,w_2)) = \begin{cases} 1, & u_2 < 1, \\ 0, & u_2 = 1, \end{cases}$

for every $(v_1,v_2) \in \mathcal{L}$, then our solution is incorrect, since $\mathtt{I_I}((u_1,u_2),(v_1,v_2)) = \begin{cases} 0_{\mathcal{L}}, & u_2 = 0, \\ 1_{\mathcal{L}}, & u_2 = 1, \\ (1,1), & 0 < u_2 < 1, \end{cases}$

is not solution in $\mathcal{L}$ (since $(1,1) \notin \mathcal{L}$).

Similarly, we can find the possible combinations of the above solutions give a correct value in the set $\mathcal{L}$ is the required result (ii). □

Remark 7.4.9. *If we put $\alpha = u$, $\beta = \mathtt{f_I}(w)$ and $\mathtt{I_I} = \mathtt{I}_{\mathtt{I}(\mathtt{f_I},\omega)}$ in Proposition 7.4.8, then the above possible forms of $\mathtt{I_I}(u,w)$, for fixed $w \in \mathcal{L}$, convert into corresponding forms of $\omega(\alpha,\beta)$.*

7.4.3 General method for solving distributive eq. (7.11):

Distributive eq. (7.11) is given by

$$\mathtt{I_I}(u, \mathcal{T}_1(v,w)) = \mathcal{T}_2(\mathtt{I_I}(u,v), \mathtt{I_I}(u,w)),\ \forall u,v,w \in \mathcal{L} \tag{7.42}$$

where $\mathtt{I_I}$ is the unknown function, and the t-norms $\mathcal{T}_1$ and $\mathcal{T}_2$ on $\mathcal{L}$ are the t-representable, i.e., $\mathcal{T}_1 = (T_1, T_2)$ and $\mathcal{T}_2 = (T_2, S_2)$.

At this situation distributive eq. (7.11) has the following form

$$\begin{aligned}\mathtt{I_I}((u_1,u_2),(T_1(v_1,w_1),S_1(v_2,w_2))) =& (T_2(pr_1(\mathtt{I_I}((u_1,u_2),(v_1,v_2))), pr_1(\mathtt{I_I}((u_1,u_2),(w_1,w_2)))),\\ & S_2(pr_2(\mathtt{I_I}((u_1,u_2),(v_1,v_2))), pr_2(\mathtt{I_I}((u_1,u_2),(w_1,w_2))))),\end{aligned}$$

$\forall\ u = (u_1,u_2), v = (v_1,v_2), w = (w_1,w_2) \in \mathcal{L}$.

As a consequence we obtain the following two equations

$$pr_1(\mathtt{I}_\mathtt{I}((u_1,u_2),(T_1(v_1,w_1),S_1(v_2,w_2)))) = T_2(pr_1(\mathtt{I}_\mathtt{I}((u_1,u_2),(v_1,v_2))),pr_1(\mathtt{I}_\mathtt{I}((u_1,u_2),(w_1,w_2)))), \tag{7.43}$$

$$pr_2(\mathtt{I}_\mathtt{I}((u_1,u_2),(T_1(v_1,w_1),S_1(v_2,w_2)))) = S_2(pr_2(\mathtt{I}_\mathtt{I}((u_1,u_2),(v_1,v_2))),pr_2(\mathtt{I}_\mathtt{I}((u_1,u_2),(w_1,w_2)))), \tag{7.44}$$

$\forall\, u = (u_1,u_2), v = (v_1,v_2), w = (w_1,w_2) \in \mathcal{L}$.

Now, let $u = (u_1,u_2) \in \mathcal{L}$ be arbitrary but fixed. Then we define two functions $g^1_{(u_1,u_2)}$, $g^2_{(u_1,u_2)} : [0,1] \to [0,1]$ by

$$g^1_{(u_1,u_2)}(.) := pr_1 \circ \mathtt{I}_\mathtt{I}((u_1,u_2),.), \quad g^2_{(u_1,u_2)}(.) := pr_2 \circ \mathtt{I}_\mathtt{I}((u_1,u_2),.), \tag{7.45}$$

where $\circ$ represents standard composition of functions.

From (7.43), (7.44) and (7.45), we have

$$g^1_{(u_1,2)}(T_1(v_1,w_1),S_1(v_2,w_2)) = T_2(g^1_{(u_1,u_2)}(v_1,v_2),g^1_{(u_1,u_2)}(w_1,w_2)), \tag{7.46}$$
$$g^2_{(u_1,u_2)}(T_1(v_1,w_1),S_1(v_2,w_2)) = S_2(g^2_{(u_1,u_2)}(v_1,v_2),g^2_{(u_1,u_2)}(w_1,w_2)). \tag{7.47}$$

For simplicity, we put $g^1_{(u_1,u_2)} = g^1$ and $g^2_{(u_1,u_2)} = g^2$ in (7.46) and (7.47), we have

$$g^1(T_1(v_1,w_1),S_1(v_2,w_2)) = T_2(g^1(v_1,v_2),g^1(w_1,w_2)), \quad g^2(T_1(v_1,w_1),S_1(v_2,w_2)) = S_2(g^2(v_1,v_2),g^2(w_1,w_2)). \tag{7.48}$$

Proposition 7.4.10. *Let $\mathcal{T} = (T,S)$, where T and S are the strict t-norm and t-conorm respectively such that T and S are dual of each other. For a function $\mathtt{I}_\mathtt{I} : \mathcal{L}^2 \to \mathcal{L}$, the following statements are equivalent:*

(i) The triple $(\mathcal{T}_1,\mathcal{T}_2,\mathtt{I}_\mathtt{I})$ satisfies the functional eq. (7.11) $\forall\, u,v,w \in \mathcal{L}$.

(ii) For every fixed $u \in \mathcal{L}$, $\mathtt{I}_\mathtt{I}(u,.)$ has one of the following form:

$$\mathtt{I}_\mathtt{I}(u,v) = 1_\mathcal{L}, \quad \mathtt{I}_\mathtt{I}(u,v) = (0,0), \quad \mathtt{I}_\mathtt{I}(u,v) = 0_\mathcal{L}, \quad \mathtt{I}_\mathtt{I}(u,v) = \begin{cases} (0,0), & v_2 = 0, \\ 0_\mathcal{L}, & v_2 > 0, \end{cases}$$

$$\mathtt{I}_\mathtt{I}(u,v) = \begin{cases} (0,0), & v_2 < 1, \\ 0_\mathcal{L}, & v_2 = 1, \end{cases} \quad \mathtt{I}_\mathtt{I}(u,v) = \begin{cases} (0,0), & v_1 = 1, \\ 0_\mathcal{L}, & v_1 < 1, \end{cases} \quad \mathtt{I}_\mathtt{I}(u,v) = \begin{cases} (0,0), & v_1 = 1 - v_2 > 0, \\ 0_\mathcal{L}, & v_2 = 1 \text{ or} \\ & v_1 < 1 - v_2, \end{cases}$$

$$\mathtt{I_I}(u,v)=\begin{cases}(0,0), & v_2=0, v_1>0,\\ 0_{\mathcal{L}}, & v_2>0 \ or \ v_1=0,\end{cases}\quad \mathtt{I_I}(u,v)=\begin{cases}(0,0), & v_1>0,\\ 0_{\mathcal{L}}, & v_1=0,\end{cases}$$

$$\mathtt{I_I}(u,v)=(0,s_2^{-1}(cs_1(v_2))),\quad \mathtt{I_I}(u,v)=\begin{cases}(0,s_2^{-1}(ct_1(v_1))), & v_1=1-v_2,\\ 0_{\mathcal{L}}, & v_1<1-v_2,\end{cases}$$

$$\mathtt{I_I}(u,v)=\begin{cases}(0,s_2^{-1}(cs_1(v_2))), & v_1>0,\\ 0_{\mathcal{L}}, & v_1=0,\end{cases}\quad \mathtt{I_I}(u,v)=\begin{cases}(0,s_2^{-1}(ct_1(v_1))), & v_2=0,\\ 0_{\mathcal{L}}, & v_2>0,\end{cases}$$

$$\mathtt{I_I}(u,v)=\begin{cases}(0,s_2^{-1}(c(t_1(v_1)-s_1(v_2)))), & v_2<1,\\ 0_{\mathcal{L}}, & v_2=1,\end{cases}\quad \mathtt{I_I}(u,v)=(0,s_2^{-1}(ct_1(v_1))),$$

$$\mathtt{I_I}(u,v)=\begin{cases}(0,s_2^{-1}(c_1(t_1(v_1)-s_1(v_2))+c_2s_1(v_2))), & v_2<1,\\ 0_{\mathcal{L}}, & v_2=1,\end{cases}\quad \mathtt{I_I}(u,v)=\begin{cases}1_{\mathcal{L}}, & v_2=0,\\ 0_{\mathcal{L}}, & v_2>0,\end{cases}$$

$$\mathtt{I_I}(u,v)=\begin{cases}1_{\mathcal{L}}, & v_2=0,\\ (0,0), & v_2>0,\end{cases}\quad \mathtt{I_I}(u,v)=\begin{cases}1_{\mathcal{L}}, & v_2<1,\\ (0,0), & v_2=1,\end{cases}\quad \mathtt{I_I}(u,v)=\begin{cases}1_{\mathcal{L}}, & v_1=1,\\ (0,0), & v_1<1,\end{cases}$$

$$\mathtt{I_I}(u,v)=\begin{cases}1_{\mathcal{L}}, & v_2<1,\\ 0_{\mathcal{L}}, & v_2=1,\end{cases}\quad \mathtt{I_I}(u,v)=\begin{cases}(0,0), & 0<v_2<1,\\ 1_{\mathcal{L}}, & v_2=0,\\ 0_{\mathcal{L}}, & v_2=1,\end{cases}$$

$$\mathtt{I_I}(u,v)=\begin{cases}1_{\mathcal{L}}, & v_2=0,\\ (0,s_2^{-1}(c(s_1 & \\ (v_2)))), & v_2>0,\end{cases}\quad \mathtt{I_I}(u,v)=\begin{cases}1_{\mathcal{L}}, & v_1=1,\\ 0_{\mathcal{L}}, & v_1<1,\end{cases}$$

$$\mathtt{I_I}(u,v)=\begin{cases}1_{\mathcal{L}}, & v_1=1,\ v_2=0,\\ (0,0), & v_1<1,\ v_2=0,\\ 0_{\mathcal{L}}, & v_1<1,\ v_2>0,\\ & v_1+v_2\le 1,\end{cases}\quad \mathtt{I_I}(u,v)=\begin{cases}1_{\mathcal{L}}, & v_1=1,\ v_2=0,\\ (0,0), & v_1<1,\ v_2<1,\\ & v_1+v_2\le 1,\\ 0_{\mathcal{L}}, & v_1=0,\ v_2=1,\end{cases}$$

$$\mathtt{I_I}(u,v)=\begin{cases}1_{\mathcal{L}}, & v_1=1,\ v_2=0,\\ (0,0), & v_1<1,\ v_1=1-v_2>0,\\ 0_{\mathcal{L}}, & v_1<1,\ v_1<1-v_2,\end{cases}\quad \mathtt{I_I}(u,v)=\begin{cases}1_{\mathcal{L}}, & v_1=1,\ v_2=0,\\ (0,0), & 0<v_1<1,\ v_2=0,\\ 0_{\mathcal{L}}, & (v_1=0,\ v_2>0),\\ & or/; v_1=0,\end{cases}$$

$$\mathtt{I}_{\mathtt{I}}(u,v)=\begin{cases}1_{\mathcal{L}}, & v_1=1-v_2>0,\\ (0,0), & v_2=1 \text{ } or \text{ } v_1<1-v_2,\end{cases}\quad \mathtt{I}_{\mathtt{I}}(u,v)=\begin{cases}1_{\mathcal{L}}, & v_1=0,\\ (0,0), & 0<v_1<1,\\ 0_{\mathcal{L}}, & v_1=1,\end{cases}$$

$$\mathtt{I}_{\mathtt{I}}(u,v)=\begin{cases}1_{\mathcal{L}}, & v_1=1,\\ (0,s_2^{-1}(cs_1 & \\ (v_2))), & v_1<1,\end{cases}\quad \mathtt{I}_{\mathtt{I}}(u,v)=\begin{cases}1_{\mathcal{L}}, & v_1=1,\ v_2=0,\\ (0,s_2^{-1}(ct_1(v_1))), & v_1<1,\ v_1=1-v_2,\\ 0_{\mathcal{L}}, & v_1<1,\ v_1<1-v_2,\end{cases}$$

$$\mathtt{I}_{\mathtt{I}}(u,v)=\begin{cases}1_{\mathcal{L}}, & v_1=1,\\ (0,s_2^{-1}(cs_1(v_2))), & 0<v_1<1,\\ 0_{\mathcal{L}}, & v_1=0,\end{cases}$$

$$\mathtt{I}_{\mathtt{I}}(u,v)=\begin{cases}1_{\mathcal{L}}, & v_1=1,v_2=0,\\ (0,s_2^{-1}(ct_1(v_1))), & v_1<1,v_2=0,\\ 0_{\mathcal{L}}, & v_1<1,v_2>0,\\ & v_1+v_2\le 1,\end{cases}$$

$$\mathtt{I}_{\mathtt{I}}(u,v)=\begin{cases}1_{\mathcal{L}}, & v_1=1,\ v_2=0,\\ 0_{\mathcal{L}}, & v_1=0,\ v_2=1,\\ (0,s_2^{-1}(c(t_1(v_1)-s_1(v_2)))), & v_1<1,\ v_2<1,\\ & v_1+v_2\le 1,\end{cases}$$

$$\mathtt{I}_{\mathtt{I}}(u,v)=\begin{cases}1_{\mathcal{L}}, & v_1=1,\ v_2=0,\\ 0_{\mathcal{L}}, & v_1=0,\ v_2=1,\\ (0,s_2^{-1}(c_1(t_1(v_1)-s_1(v_2))+c_2(s_1(v_2)))), & v_1<1,\ v_2<1,\\ & v_1+v_2\le 1,\end{cases}$$

$$\mathtt{I}_{\mathtt{I}}(u,v)=\begin{cases}1_{\mathcal{L}}, & v_1=1,\\ (0,s_2^{-1}(c(t_1(v_1)))), & v_1<1,\end{cases}\quad \mathtt{I}_{\mathtt{I}}(u,v)=\begin{cases}1_{\mathcal{L}}, & v_1=1-v_2>0,\ v_2<1,\\ 0_{\mathcal{L}}, & v_2=1,\\ (0,0), & v_1<1-v_2,v_2<1,\end{cases}$$

$$\mathtt{I}_{\mathtt{I}}(u,v)=\begin{cases}1_{\mathcal{L}}, & v_1=1-v_2>0,\\ 0_{\mathcal{L}}, & (v_2=1)\ or\ (v_1<1-v_2),\end{cases}\quad \mathtt{I}_{\mathtt{I}}(u,v)=\begin{cases}1_{\mathcal{L}}, & v_1=1-v_2>0,\ v_1>0,\\ (0,0), & (v_1>0,\ v_2=1),\\ & or\ (v_1<1-v_2,\ v_1>0)\\ 0_{\mathcal{L}}, & (v_1=0,v_2=1)\\ & or\ (v_1=0,v_2<1),\end{cases}$$

$$\mathtt{I}_{\mathtt{I}}(u,v)=\begin{cases}1_{\mathcal{L}}, & v_1>0,\ v_2=0,\\ 0_{\mathcal{L}}, & (v_1=0,v_2>0)\ or\ v_2>0,\end{cases}\quad \mathtt{I}_{\mathtt{I}}(u,v)=\begin{cases}1_{\mathcal{L}}, & v_1>0,\ v_2=0,\\ (0,0), & v_1=0\ or\ v_2>0,\end{cases}$$

$$\mathtt{I}_{\mathtt{I}}(u,v)=\begin{cases}1_{\mathcal{L}}, & v_1>0,\\ (0,0), & v_1=0,\end{cases}\quad \mathtt{I}_{\mathtt{I}}(u,v)=\begin{cases}1_{\mathcal{L}}, & v_1>0,\ v_2=0,\\ (0,0), & (0<v_2<1)\ or\\ & (v_1=0,\ v_2<1),\\ 0_{\mathcal{L}}, & (v_2=1)\ or\\ & (v_1=0,\ v_2=1),\end{cases}$$

$$\mathtt{I}_{\mathtt{I}}(u,v)=\begin{cases}1_{\mathcal{L}}, & v_1=1,\ v_2=0,\\ 0_{\mathcal{L}}, & (v_2=1)\ or\ (v_1=0,\ v_2\\ & =1)\ or\ (v_1<1-v_2,\ v_2\\ & >0)\ or\ (v_1=0,\ v_2<1),\end{cases}\quad \mathtt{I}_{\mathtt{I}}(u,v)=\begin{cases}1_{\mathcal{L}}, & v_1>0,\ v_2=0,\\ 0_{\mathcal{L}}, & v_1=0\ or\ v_2>0,\end{cases}$$

$$\mathtt{I}_{\mathtt{I}}(u,v)=\begin{cases}1_{\mathcal{L}}, & v_1>0,\ v_2=0,\\ 0_{\mathcal{L}}, & (v_1=0,\ v_2>0)\\ & or\ (v_1=0),\\ (0,0), & v_1>0,\ v_2>0,\\ & v_1+v_2\le 1,\end{cases}\quad \mathtt{I}_{\mathtt{I}}(u,v)=\begin{cases}1_{\mathcal{L}}, & v_1>0\ \ v_2=0,\\ (0,s_2^{-1}(cs_1(v_2))), & v_1=0\ or\ v_2>0,\end{cases}$$

$$\mathtt{I}_{\mathtt{I}}(u,v)=\begin{cases}1_{\mathcal{L}}, & v_1>0\ \ v_2=0,\\ (0,s_2^{-1}(cs_1(v_2))), & v_1>0,v_2>0,\\ & v_1+v_2\le 1,\\ 0_{\mathcal{L}}, & (v_2>0,v_1=0)\\ & or\ (v_1=0),\end{cases}\quad \mathtt{I}_{\mathtt{I}}(u,v)=\begin{cases}1_{\mathcal{L}}, & v_1>0,\ v_2<1,\\ & v_1+v_2\le 1\\ (0,0), & v_1=0,\ v_2<1,\\ 0_{\mathcal{L}}, & v_1=0,\ v_2=1,\end{cases}$$

$$\mathtt{I}_{\mathtt{I}}(u,v)=\begin{cases}1_{\mathcal{L}}, & v_1>0,\\ 0_{\mathcal{L}}, & v_1=0,\end{cases}\quad \mathtt{I}_{\mathtt{I}}(u,v)=(t_2^{-1}(cs_1(v_2)),0),$$

$$\mathtt{I}_{\mathtt{I}}(u,v)=\begin{cases}(t_2^{-1}(cs_1(v_2)),0), & v_2<1,\\ 0_{\mathcal{L}}, & v_2=1,\end{cases}\quad \mathtt{I}_{\mathtt{I}}(u,v)=(t_2^{-1}(cs_1(v_2)),s_2^{-1}(cs_1(v_2))),$$

$$\mathtt{I}_{\mathtt{I}}(u,v)=\begin{cases}(t_2^{-1}(cs_1(v_2)),s_2^{-1}(c(t_1 & \\ (v_1)-s_1(v_2)))), & v_2<1,\\ 0_{\mathcal{L}}, & v_2=1,\end{cases}\quad \mathtt{I}_{\mathtt{I}}(u,v)=(t_2^{-1}(cs_1(v_2)),s_2^{-1}(ct_1(v_1))),$$

$$\mathtt{I}_{\mathtt{I}}(u,v)=\begin{cases}(t_2^{-1}(cs_1(v_2)),s_2^{-1}(c_1(t_1(v_1) & \\ -s_1(v_2))+c_2s_1(v_2))), & v_2<1,\\ 0_{\mathcal{L}}, & v_2=1,\end{cases}\quad \mathtt{I}_{\mathtt{I}}(u,v)=\begin{cases}(t_2^{-1}(ct_1(v_1)),0), & v_1=1-v_2,\\ 0_{\mathcal{L}}, & v_1<1-v_2,\end{cases}$$

$$\mathtt{I}_{\mathtt{I}}(u,v)=\begin{cases}(t_2^{-1}(ct_1(v_1)),0), & v_1=1-v_2,\ v_2<1,\\ (0,0), & v_1<1-v_2,\ v_2<1,\\ 0_{\mathcal{L}}, & v_1=0,\ v_2=1,\end{cases}$$

$$\mathtt{I}_{\mathtt{I}}(u,v)=\begin{cases}(t^{-1}(ct(1-v_2)),0), & v_1=1-v_2>0,\\ 0_{\mathcal{L}}, & (v_1=0,\ v_2=1)\\ & \mathit{or}\ (v_1<1-v_2),\end{cases}$$

$$\mathtt{I}_{\mathtt{I}}(u,v)=\begin{cases}(t_2^{-1}(ct_1(v_1)),0), & v_1=1-v_2,\ v_1>0,\\ (0,0), & v_1<1-v_2,\ v_1>0,\\ 0_{\mathcal{L}}, & v_1=0,\ v_2<1,\\ 1_{\mathcal{L}}, & v_1=0,\ v_2=1,\end{cases}$$

$$\mathtt{I}_{\mathtt{I}}(u,v)=\begin{cases}(t_2^{-1}(ct_1(v_1)),s_2^{-1}(cs_1(v_2))), & v_1=1-v_2,\\ (0,s_2^{-1}(cs_1(v_2))), & v_1<1-v_2,\end{cases}$$

$$\mathtt{I}_{\mathtt{I}}(u,v)=\begin{cases}(t_2^{-1}(ct_1(v_1)),s_2^{-1}(ct_1(v_1))), & v_1=1-v_2,\\ 0_{\mathcal{L}}, & v_1<1-v_2,\end{cases}$$

$$\mathtt{I_I}(u,v)=\begin{cases}(t_2^{-1}(ct_1(v_1)),s_2^{-1}(cs_1(v_2))), & v_1=1-v_2, v_1>0,\\(0,s_2^{-1}(cs_1(v_2))), & v_1<1-v_2, v_1>0,\\0_{\mathcal{L}}, & (v_1\le 1-v_2, v_1=0),\\ & or\ (v_1=0,v_2=1),\end{cases}$$

$$\mathtt{I_I}(u,v)=\begin{cases}(t_2^{-1}(ct_1(v_1)),s_2^{-1}(c(t_1(v_1)-s_1(v_2)))), & v_1=1-v_2, v_2<1,\\(0,s_2^{-1}(c(t_1(v_1)-s_1(v_2)))), & v_1<1-v_2, v_2<1,\\0_{\mathcal{L}}, & v_1=0,v_2=1,\end{cases}$$

$$\mathtt{I_I}(u,v)=\begin{cases}(t_2^{-1}(ct_1(v_1)),s_2^{-1}(ct_1(v_1))), & v_1=1-v_2,\\(0,s_2^{-1}(ct_1(v_1))), & v_1<1-v_2,\end{cases}$$

$$\mathtt{I_I}(u,v)=\begin{cases}(t_2^{-1}(ct_1(v_1)),s_2^{-1}(c_1(t_1(v_1)-s_1(v_2))+c_2s_1(v_2))), & v_1=1-v_2, v_2<1,\\(0,s_2^{-1}(c_1(t_1(v_1)-s_1(v_2))+c_2s_1(v_2))), & v_1<1-v_2, v_2<1,\\0_{\mathcal{L}}, & v_1=0,v_2=1,\end{cases}$$

$$\mathtt{I_I}(u,v)=\begin{cases}(t_2^{-1}(cs_1(v_2)),0), & v_1>0,\\(0,0), & v_1=0,\end{cases}\quad \mathtt{I_I}(u,v)=\begin{cases}(t_2^{-1}(cs_1(v_2)),0), & v_1>0,\ v_2<1,\\ & v_1+v_2\le 1,\\(0,0), & v_1=0,\ v_2<1,\\0_{\mathcal{L}}, & v_1=0,\ v_2=1,\end{cases}$$

$$\mathtt{I_I}(u,v)=\begin{cases}(t_2^{-1}(cs_1(v_2)),0), & v_1>0,\\0_{\mathcal{L}}, & v_1=0,\end{cases}\quad \mathtt{I_I}(u,v)=\begin{cases}(t_2^{-1}(cs_1(v_2)), & \\ s_2^{-1}(cs_1(v_2))), & v_1>0,\\(0,s_2^{-1}(cs_1(v_2))), & v_1=0,\end{cases}$$

$$\mathtt{I_I}(u,v)=\begin{cases}(t_2^{-1}(cs_1(v_2)),s_2^{-1}(cs_1(v_2))), & v_1>0,\\0_{\mathcal{L}}, & v_1=0,\end{cases}$$

$$\mathtt{I_I}(u,v)=\begin{cases}(t_2^{-1}(cs_1(v_2)),s_2^{-1}(c(t_1(v_1)-s_1(v_2)))), & v_1>0,v_2<1,\\ & v_1+v_2\le 1,\\(0,s_2^{-1}(c(t_1(v_1)-s_1(v_2)))), & v_1=0,v_2<1,\\0_{\mathcal{L}}, & v_1=0,v_2=1,\end{cases}$$

$$\mathtt{I_I}(u,v)=\begin{cases}(t_2^{-1}(cs_1(v_2)),s_2^{-1}(ct_1(v_1))), & v_1>0,\\0_{\mathcal{L}}, & v_1=0,\end{cases}$$

$$\mathtt{I_I}(u,v) = \begin{cases} (t_2^{-1}(cs_1(v_2)), s_2^{-1}(c_1(t_1(v_1) - s_1(v_2)) + c_2 s_1(v_2))), & v_1 > 0, v_2 < 1, \\ & v_1 + v_2 \leq 1, \\ (0, s_2^{-1}(c_1(t_1(v_1) - s_1(v_2)) + c_2 s_1(v_2))), & v_1 = 0, v_2 < 1, \\ 0_{\mathcal{L}}, & v_1 = 0, v_2 = 1, \end{cases}$$

$$\mathtt{I_I}(u,v) = \begin{cases} (t_2^{-1}(ct_1(v_1)), 0), & v_2 = 0, \\ (0,0), & v_2 > 0, \end{cases} \quad \mathtt{I_I}(u,v) = \begin{cases} (t_2^{-1}(ct_1(v_1)), 0), & v_2 = 0, \\ 0_{\mathcal{L}}, & v_2 > 0, \end{cases}$$

$$\mathtt{I_I}(u,v) = \begin{cases} (t_2^{-1}(ct_1(v_1)), 0), & v_2 = 0, \\ 0_{\mathcal{L}}, & v_2 = 1, \\ (0,0), & 0 < v_2 < 1, \end{cases}$$

$$\mathtt{I_I}(u,v) = \begin{cases} (t_2^{-1}(ct_1(v_1)), 0), & v_2 = 0, v_1 > 0, \\ 0_{\mathcal{L}}, & (v_1 = 0, v_2 = 0),\ or\ (v_2 > 0) \\ & or\ (v_1 = 0, v_2 > 0), \end{cases}$$

$$\mathtt{I_I}(u,v) = \begin{cases} (t_2^{-1}(ct_1(v_1)), 0), & v_2 = 0, v_1 > 0, \\ 0_{\mathcal{L}}, & v_1 = 0, v_2 \geq 0, \\ (0,0) & v_1 > 0, v_2 > 0, v_1 + v_2 \leq 1, \end{cases}$$

$$\mathtt{I_I}(u,v) = \begin{cases} (t_2^{-1}(ct_1(v_1)), s_2^{-1}(cs_1(v_2))), & v_2 = 0, \\ (0, s_2^{-1}(cs_1(v_2))), & v_2 > 0, \end{cases}$$

$$\mathtt{I_I}(u,v) = \begin{cases} (t_2^{-1}(ct_1(v_1)), s_2^{-1}(cs_1(v_2))), & v_2 = 0, v_1 > 0, \\ 0_{\mathcal{L}}, & v_1 = 0, v_2 \geq 0, \\ (0, s_2^{-1}(cs_1(v_2))) & v_1 > 0, v_2 > 0, v_1 + v_2 \leq 1, \end{cases}$$

$$\mathtt{I_I}(u,v) = \begin{cases} (t_2^{-1}(ct_1(v_1)), s_2^{-1}(ct_1(v_1))), & v_2 = 0, \\ 0_{\mathcal{L}}, & v_2 > 0, \end{cases}$$

$$\mathtt{I_I}(u,v) = \begin{cases} (t_2^{-1}(ct_1(v_1)), s_2^{-1}(c(t_1(v_1) - s_1(v_2)))), & v_2 = 0, \\ 0_{\mathcal{L}}, & v_2 = 1, \\ (0, s_2^{-1}(c(t_1(v_1) - s_1(v_2)))) & 0 < v_2 < 1, \end{cases}$$

$$\mathtt{I_I}(u,v) = \begin{cases} (t_2^{-1}(ct_1(v_1)), s_2^{-1}(ct_1(v_1))), & v_2 = 0, \\ (0, s_2^{-1}(ct_1(v_1))), & v_2 > 0, \end{cases}$$

$$\mathtt{I_I}(u,v) = \begin{cases} (t_2^{-1}(ct_1(v_1)), s_2^{-1}(c_1(t_1(v_1) - s_1(v_2)) + c_2 s_1(v_2))), & v_2 = 0, \\ 0_{\mathcal{L}}, & v_2 = 1, \\ (0, s_2^{-1}(c_1(t_1(v_1) - s_1(v_2)) + c_2 s_1(v_2))) & 0 < v_2 < 1, \end{cases}$$

$$\mathtt{I_I}(u,v) = \begin{cases} (t_2^{-1}(c(t_1(v_1) - s_1(v_2))), 0), & v_2 < 1, \\ (0,0) & v_2 = 1, \end{cases}$$

$$\mathtt{I_I}(u,v) = \begin{cases} (t_2^{-1}(c(t_1(v_1) - s_1(v_2))), 0), & v_2 < 1, \\ 0_{\mathcal{L}} & v_2 = 1, \end{cases}$$

$$\mathtt{I_I}(u,v) = \begin{cases} (t_2^{-1}(c(t_1(v_1) - s_1(v_2))), 0), & v_2 < 1, v_1 > 0, \\ 0_{\mathcal{L}} & v_1 = 0, v_2 \le 1, \end{cases}$$

$$\mathtt{I_I}(u,v) = \begin{cases} (t_2^{-1}(c(t_1(v_1) - s_1(v_2))), s_2^{-1}(cs_1(v_2))), & v_2 < 1, \\ (0, s_2^{-1}(cs_1(v_2))) & v_2 = 1, \end{cases}$$

$$\mathtt{I_I}(u,v) = \begin{cases} (t_2^{-1}(c(t_1(v_1) - s_1(v_2))), s_2^{-1}(cs_1(v_2))), & v_1 > 0, v_2 < 1, \\ (0, s_2^{-1}(cs_1(v_2))) & v_1 = 0, v_2 \le 1, \end{cases}$$

$$\mathtt{I_I}(u,v) = \begin{cases} (t_2^{-1}(c(t_1(v_1) - s_1(v_2))), s_2^{-1}(c(t_1(v_1) - s_1(v_2)))), & v_2 < 1, \\ 0_{\mathcal{L}} & v_2 = 1, \end{cases}$$

$$\mathtt{I_I}(u,v) = \begin{cases} (t_2^{-1}(c(t_1(v_1) - s_1(v_2))), s_2^{-1}(c(t_1(v_1)))), & v_2 < 1, \\ (0, s_2^{-1}(c(t_1(v_1)))) & v_2 = 1, \end{cases}$$

$$\mathtt{I_I}(u,v) = \begin{cases} (t_2^{-1}(c(t_1(v_1) - s_1(v_2))), s_2^{-1}(c_1(t_1(v_1) - s_1(v_2)) + c_2 s_1(v_2))), & v_2 < 1, \\ 0_{\mathcal{L}} & v_2 = 1, \end{cases}$$

$$\mathtt{I_I}(u,v) = (t_2^{-1}(ct_1(v_1)), 0), \quad \mathtt{I_I}(u,v) = \begin{cases} (t_2^{-1}(ct_1(v_1)), 0), & v_2 < 1, \\ 0_{\mathcal{L}} & v_2 = 1, \end{cases}$$

$$\mathtt{I_I}(u,v) = (t_2^{-1}(ct_1(v_1)), s_2^{-1}(cs_1(v_2))),$$

$$\mathtt{I}_{\mathtt{I}}(u,v) = \begin{cases} (t_2^{-1}(ct_1(v_1)), 0), & v_1 > 0, \\ 0_{\mathcal{L}} & v_1 = 0, \end{cases} \quad \mathtt{I}_{\mathtt{I}}(u,v) = \begin{cases} (t_2^{-1}(ct_1(v_1)), \\ s_2^{-1}(cs_1(v_2))), & v_1 > 0, \\ 0_{\mathcal{L}} & v_1 = 0, \end{cases}$$

$$\mathtt{I}_{\mathtt{I}}(u,v) = \begin{cases} (t_2^{-1}(ct_1(v_1)), s_2^{-1}(c(t_1(v_1) - s_1(v_2)))), & v_2 < 1, \\ 0_{\mathcal{L}} & v_2 = 1, \end{cases}$$

$$\mathtt{I}_{\mathtt{I}}(u,v) = (t_2^{-1}(ct_1(v_1)), s_2^{-1}(ct_1(v_1)))$$

$$\mathtt{I}_{\mathtt{I}}(u,v) = \begin{cases} (t_2^{-1}(ct_1(v_1)), s_2^{-1}(c_1(t_1(v_1) - s_1(v_2)) + c_2 s_1(v_2))), & v_2 < 1, \\ 0_{\mathcal{L}} & v_2 = 1, \end{cases}$$

$$\mathtt{I}_{\mathtt{I}}(u,v) = \begin{cases} (t_2^{-1}(c_1(t_1(v_1) - s_1(v_2)) + c_2 s_1(v_2)), 0), & v_2 < 1, \\ (0, 0) & v_2 = 1, \end{cases}$$

$$\mathtt{I}_{\mathtt{I}}(u,v) = \begin{cases} (t_2^{-1}(c_1(t_1(v_1) - s_1(v_2)) + c_2 s_1(v_2)), 0), & v_2 < 1, \\ 0_{\mathcal{L}} & v_2 = 1, \end{cases}$$

$$\mathtt{I}_{\mathtt{I}}(u,v) = \begin{cases} (t_2^{-1}(c_1(t_1(v_1) - s_1(v_2)) + c_2 s_1(v_2)), 0), & v_1 > 0, v_2 < 1, \\ & v_1 + v_2 \leq 1, \\ 0_{\mathcal{L}} & v_1 = 0, v_2 \leq 1, \end{cases}$$

$$\mathtt{I}_{\mathtt{I}}(u,v) = \begin{cases} (t_2^{-1}(c_1(t_1(v_1) - s_1(v_2)) + \\ c_2 s_1(v_2)), s_2^{-1}(cs_1(v_2))), & v_2 < 1, \\ (0, s_2^{-1}(cs_1(v_2))) & v_2 = 1, \end{cases}$$

$$\mathtt{I}_{\mathtt{I}}(u,v) = \begin{cases} (t_2^{-1}(c_1(t_1(v_1) - s_1(v_2)) + \\ c_2 s_1(v_2)), s_2^{-1}(cs_1(v_2))), & v_1 > 0, v_2 < 1, \\ & v_1 + v_2 \leq 1, \\ 0_{\mathcal{L}} & v_1 = 0, v_2 \leq 1, \end{cases}$$

$$\mathtt{I}_{\mathtt{I}}(u,v) = \begin{cases} (t_2^{-1}(c_1(t_1(v_1) - s_1(v_2)) + c_2 s_1(v_2)), \\ s_2^{-1}(c(t_1(v_1) - s_1(v_2)))), & v_2 < 1, \\ 0_{\mathcal{L}} & v_2 = 1, \end{cases}$$

$$\mathtt{I}_{\mathtt{I}}(u,v)=\begin{cases}(t_2^{-1}(c_1(t_1(v_1)-s_1(v_2))+c_2s_1(v_2)), & \\ s_2^{-1}(ct_1(v_1))), & v_2<1,\\ (0,s_2^{-1}(ct_1(v_1))) & v_2=1,\end{cases}$$

$$or\ \ \mathtt{I}_{\mathtt{I}}(u,v)=\begin{cases}(t_2^{-1}(c_1(t_1(v_1)-s_1(v_2))+c_2s_1(v_2)), & \\ s_2^{-1}(c_1(t_1(v_1)-s_1(v_2))+c_2s_1(v_2))), & v_2<1,\\ 0_{\mathcal{L}} & v_2=1,\end{cases}$$

Proof. Given that $\mathcal{T}_1$ and $\mathcal{T}_2$ are the t-representable on $\mathcal{L}$, i.e., $\mathcal{T}_1=(T_1,S_1)$ and $\mathcal{T}_2=(S_2,T_2)$. Also, given that t-norm T is strict, and T and S are dual of each other. It follows that t-conorm S is strict. Moreover, $T_1=T_2$. Now , from Theorems 2.1.5 and 2.1.7 and Remarks 2.2.6 and 2.2.7, given in [19], there exists a decreasing continuous function $t:[0,1]\to[0,\infty]$ such that $t(0)=\infty$ and $t(1)=0$ which are uniquely determined a positive multiplicative constant such that $T(a,b)=t^{-1}(t(a)+t(b)),\ \forall\, a,b\in[0,1]$.

Let us prove that $(ii)\ \Rightarrow\ (i)$.

(P1): Let $\mathtt{I}_{\mathtt{I}}$ have the form $\mathtt{I}_{\mathtt{I}}(u,v)=1_{\mathcal{L}}$. Then the LHS of (7.11) is equal to $1_{\mathcal{L}}$, and the RHS of (7.11) is equal to $1_{\mathcal{L}}$.

(P2): Let $\mathtt{I}_{\mathtt{I}}$ have the form $\mathtt{I}_{\mathtt{I}}(u,v)=(0,0)$. Then the LHS of (7.11) is equal to $(0,0)$, and the RHS of (7.11) is equal to $(0,0)$.

(P3): Let $\mathtt{I}_{\mathtt{I}}$ have the form $\mathtt{I}_{\mathtt{I}}(u,v)=0_{\mathcal{L}}$. Then the LHS of (7.11) is equal to $0_{\mathcal{L}}$, and the RHS of (7.11) is equal to $0_{\mathcal{L}}$.

(P4): Let $\mathtt{I}_{\mathtt{I}}$ have the form $\mathtt{I}_{\mathtt{I}}(u,v)=\begin{cases}(0,0), & v_2=0,\\ 0_{\mathcal{L}}, & v_2>0.\end{cases}$

Then the LHS of (7.11) is equal to

$$\mathtt{I}_{\mathtt{I}}(u,\mathcal{T}_1(v,w))=\begin{cases}(0,0), & S_1(v_2,z_2)=0,\\ 0_{\mathcal{L}}, & S_1(v_2,w_2)>0.\end{cases}=\begin{cases}(0,0), & v_2=w_2=0,\\ 0_{\mathcal{L}}, & v_2>0 \text{ or } w_2>0,\end{cases}$$

and the RHS of (7.11) is equal to

$$\mathcal{T}_2(\mathtt{I}_{\mathtt{I}}(u,v),\mathtt{I}_{\mathtt{I}}(u,w))=\mathcal{T}_2\left(\begin{cases}(0,0), & v_2=0,\\ 0_{\mathcal{L}}, & v_2>0.\end{cases},\begin{cases}(0,0), & w_2=0,\\ 0_{\mathcal{L}}, & w_2>0.\end{cases}\right)$$

$$=\begin{cases}(0,0), & v_2=w_2=0,\\ 0_{\mathcal{L}}, & v_2>0 \text{ or } w_2>0.\end{cases}$$

(P5): Let $\mathtt{I}_{\mathtt{I}}$ have the form $\mathtt{I}_{\mathtt{I}}(u,v)=\begin{cases}(0,0), & v_2<1,\\ 0_{\mathcal{L}}, & v_2=1.\end{cases}$

Then LHS of (7.11) is equal to

$$\mathtt{I}_{\mathtt{I}}(u,\mathcal{T}_1(v,w))=\begin{cases}(0,0), & S_1(v_2,w_2)<1,\\ 0_{\mathcal{L}}, & S_1(v_2,w_2)=1.\end{cases}=\begin{cases}(0,0), & v_2<1,\ w_2<1,\\ 0_{\mathcal{L}}, & v_2=1 \text{ or } w_2=1,\end{cases}$$

and the RHS of (7.11) is equal to

$$\begin{aligned}\mathcal{T}_2(\mathtt{I}_{\mathtt{I}}(u,v),\mathtt{I}_{\mathtt{I}}(u,w))=&\mathcal{T}_2\left(\begin{cases}(0,0), & v_2<1,\\ 0_{\mathcal{L}}, & v_2=1.\end{cases},\begin{cases}(0,0), & w_2<1,\\ 0_{\mathcal{L}}, & w_2=1.\end{cases}\right)\\ =&\begin{cases}(0,0), & v_2<1,\ w_2<1,\\ 0_{\mathcal{L}}, & v_2=1 \text{ or } w_2=1.\end{cases}\end{aligned}$$

(P6): Let $\mathtt{I}_{\mathtt{I}}$ have the form $\mathtt{I}_{\mathtt{I}}(u,v)=\begin{cases}(0,0), & v_1=1,\\ 0_{\mathcal{L}}, & v_1<1.\end{cases}$

Then the LHS of (7.11) is equal to

$$\mathtt{I}_{\mathtt{I}}(u,\mathcal{T}_1(v,w))=\begin{cases}(0,0), & T_1(v_1,w_1)=1,\\ 0_{\mathcal{L}}, & T_1(v_1,w_1)<1.\end{cases}=\begin{cases}(0,0), & v_1=w_1=1,\\ 0_{\mathcal{L}}, & v_1<1 \text{ or } w_1<1,\end{cases}$$

and the RHS of (7.11) is equal to

$$\begin{aligned}\mathcal{T}_2(\mathtt{I}_{\mathtt{I}}(u,v),\mathtt{I}_{\mathtt{I}}(u,w))=&\mathcal{T}_2\left(\begin{cases}(0,0), & v_1=1,\\ 0_{\mathcal{L}}, & v_1<1.\end{cases},\begin{cases}(0,0), & w_1=1,\\ 0_{\mathcal{L}}, & w_1<1.\end{cases}\right)\\ =&\begin{cases}(0,0), & v_1=w_1=1,\\ 0_{\mathcal{L}}, & v_1<1 \text{ or } w_1<1.\end{cases}\end{aligned}$$

(P7): Let $\mathtt{I}_{\mathtt{I}}$ have the form $\mathtt{I}_{\mathtt{I}}(u,v)=(0,1-t^{-1}(ct(1-v_2)))$.

Then the LHS of (7.11) is equal to

$$\begin{aligned}\mathtt{I}_{\mathtt{I}}(x,\mathcal{T}_1(y,z))=&(0,1-t^{-1}(ct(1-S_1(v_2,w_2))))=(0,1-t^{-1}(ct(T_1(1-v_2,1-w_2))))\\ =&(0,1-t^{-1}(ctt^{-1}(t(1-v_2)+t(1-w_2)))),\end{aligned}$$

and the RHS of (7.11) is equal to

$$\begin{aligned}\mathcal{T}_2(\mathtt{I}_{\mathtt{I}}(u,v),\mathtt{I}_{\mathtt{I}}(u,w))=&\mathcal{T}_2((0,1-t^{-1}(ct(1-v_2))),(0,1-t^{-1}(ct(1-w_2))))\\ =&(0,S_2(1-t^{-1}(ct(1-v_2)),1-t^{-1}(ct(1-w_2))))\\ =&(0,1-T_2(t^{-1}(ct(1-v_2)),t^{-1}(ct(1-w_2))))\\ =&(0,1-t^{-1}(ctt^{-1}(t(1-v_2)+t(1-w_2)))).\end{aligned}$$

Similarly we can verify eq. (7.11) easily by taking all remaining forms of $\mathtt{I}_{\mathtt{I}}$.

Let us prove that $(i) \Rightarrow (ii)$.
From (7.48),

$$g^1(t_1^{-1}(t_1(v_1)+t_1(w_1)), s_1^{-1}(s_1(v_2)+s_1(w_2))) = t_2^{-1}(t_2(g^1(v_1,v_2)) + t_2(g^1(w_1,w_2))),$$
$$g^2(t_1^{-1}(t_1(v_1)+t_1(w_1)), s_1^{-1}(s_1(v_2)+s_1(w_2))) = s_2^{-1}(s_2(g^2(v_1,v_2)) + s_2(g^2(w_1,w_2))).$$

Hence

$$t_2 \circ g^1(t_1^{-1}(t_1(v_1)+t_1(w_1)), s_1^{-1}(s_1(v_2)+s_1(w_2))) = t_2(g^1(v_1,v_2)) + t_2(g^1(w_1,w_2)),$$
$$s_2 \circ g^2(t_1^{-1}(t_1(v_1)+t_1(w_1)), s_1^{-1}(s_1(v_2)+s_1(w_2))) = s_2(g^2(v_1,v_2)) + s_2(g^2(w_1,w_2)).$$

Let us put $t_1(v_1) = y_1, s_1(v_2) = y_2, t_1(w_1) = z_1$ and $s_1(w_2) = z_2$. Of course $y_1, y_2, z_1, z_2 \in [0,\infty]$, Moreover $u = (u_1,u_2), w = (w_1,w_2) \in \mathcal{L}$, thus $v_1 \leq 1 - v_2$ and $w_1 \leq 1 - w_2$. Since t_1 and s_1 are decreasing generator and increasing generator respectively such that $t_1(a) = s_1(1-a)$, $\forall a \in [0,1]$, $y_1 \geq y_2$ and $z_1 \geq z_2$. This implies that $(y_1,y_2),(z_1,z_2) \in [0,\infty]^2$. If we put

$$f^1(y_1,y_2) := t_2 \circ pr_1 \circ \mathtt{I}_{\mathtt{I}}((u_1,u_2),(t_1^{-1}(y_1), s_1^{-1}(y_2))), \quad f^2(y_1,y_2) := s_2 \circ pr_2 \circ \mathtt{I}_{\mathtt{I}}((u_1,u_2), (t_1^{-1}(y_1), s_1^{-1}(y_2))) \ \forall\ (y_1,y_2) \in [0,\infty]^2 \tag{7.49}$$

As a consequence we get the following two functional equations

$$f^1(y_1+z_1, y_2+z_2) = f^1(y_1,y_2) + f^1(z_1,z_2), \tag{7.50}$$
$$f^2(y_1+z_1, y_2+z_2) = f^2(y_1,y_2) + f^2(z_1,z_2), \tag{7.51}$$

where $(y_1,y_2),(z_1,z_2) \in [0,\infty]^2$.

Now, we find the possible solutions of (7.50) are as follows:

For $y_1, y_2 \in [0,\infty], y_1 \geq y_2$ and $(v_1,v_2) \in \mathcal{L}$, we have

(S1): $f^1 = 0 \ \Rightarrow\ t_2 \circ pr_1 \circ \mathtt{I}_{\mathtt{I}}((u_1,u_2),(t_1^{-1}(y_1), t_1^{-1}(y_2))) = 0 \ \Rightarrow\ pr_1 \circ \mathtt{I}_{\mathtt{I}}((u_1,u_2),(v_1,v_2)) = 1.$

(S2): $f^1 = \infty \ \Rightarrow\ t_2 \circ pr_1 \circ \mathtt{I}_{\mathtt{I}}((u_1,u_2),(t_1^{-1}(y_1), t_1^{-1}(y_2))) = \infty \ \Rightarrow\ pr_1 \circ \mathtt{I}_{\mathtt{I}}((u_1,u_2),(v_1,v_2)) = 0.$

(S3): $f^1(y_1,y_2) = \begin{cases} 0, & y_2 = 0, \\ \infty, & y_2 > 0, \end{cases} \ \Rightarrow\ t_2 \circ pr_1 \circ \mathtt{I}_{\mathtt{I}}((u_1,u_2),(t_1^{-1}(y_1), t_1^{-1}(y_2))) = \begin{cases} 0, & y_2 = 0, \\ \infty, & y_2 > 0. \end{cases}$

$$\Rightarrow\ pr_1 \circ \mathtt{I}_{\mathtt{I}}((u_1,u_2),(v_1,v_2)) = \begin{cases} 1, & v_2 = 0, \\ 0, & v_2 > 0. \end{cases}$$

$$\text{(S4):}\quad f^1(y_1,y_2)=\begin{cases}0, & y_2<\infty,\\ \infty, & y_2=\infty,\end{cases}\quad \Rightarrow\; t_2\circ pr_1\circ \mathtt{I}_\mathtt{I}((u_1,u_2),(t_1^{-1}(y_1),t_1^{-1}(y_2)))=\begin{cases}0, & y_2<\infty,\\ \infty, & y_2=\infty.\end{cases}$$

$$\Rightarrow\; pr_1\circ \mathtt{I}_\mathtt{I}((u_1,u_2),(v_1,v_2))=\begin{cases}1, & v_2<1,\\ 0, & v_2=1.\end{cases}$$

$$\text{(S5):}\quad f^1(y_1,y_2)=\begin{cases}0, & y_1=0,\\ \infty, & y_1>0,\end{cases}\quad \Rightarrow\; t_2\circ pr_1\circ \mathtt{I}_\mathtt{I}((u_1,u_2),(t_1^{-1}(y_1),t_1^{-1}(y_2)))=\begin{cases}0, & y_1=0,\\ \infty, & y_1>0.\end{cases}$$

$$\Rightarrow\; pr_1\circ \mathtt{I}_\mathtt{I}((u_1,u_2),(v_1,v_2))=\begin{cases}1, & v_1=1,\\ 0, & v_1<1.\end{cases}$$

$$\text{(S6):}\qquad f^1(y_1,y_2)=\begin{cases}0, & y_1=y_2<\infty,\\ \infty, & y_2=\infty \text{ or } y_1>y_2,\end{cases}$$

$$\Rightarrow\; t_2\circ pr_1\circ \mathtt{I}_\mathtt{I}((u_1,u_2),(t_1^{-1}(y_1),t_1^{-1}(y_2)))=\begin{cases}0, & y_1=y_2<\infty,\\ \infty, & y_2=\infty \text{ or } y_1>y_2.\end{cases}$$

$$\Rightarrow\; pr_1\circ \mathtt{I}_\mathtt{I}((u_1,u_2),(v_1,v_2))=\begin{cases}1, & v_1=1-v_2>0,\\ 0, & v_2=1 \text{ or } v_1<1-v_2.\end{cases}$$

$$\text{(S7):}\qquad f^1(y_1,y_2)=\begin{cases}0, & y_2=0,\ y_1<\infty,\\ \infty, & y_2>0 \text{ or } y_1=\infty,\end{cases}$$

$$\Rightarrow\; t_2\circ pr_1\circ \mathtt{I}_\mathtt{I}((u_1,u_2),(t_1^{-1}(y_1),t_1^{-1}(y_2)))=\begin{cases}0, & y_2=0,\ y_1<\infty,\\ \infty, & y_2>0 \text{ or } y_1=\infty.\end{cases}$$

$$\Rightarrow\; pr_1\circ \mathtt{I}_\mathtt{I}((u_1,u_2),(v_1,v_2))=\begin{cases}1, & v_2=0,\ v_1>0,\\ 0, & v_2>0 \text{ or } v_1=0.\end{cases}$$

(S8):
$$f^1(y_1,y_2) = \begin{cases} 0, & y_1 < \infty, \\ \infty, & y_1 = \infty, \end{cases}$$

$$\Rightarrow\ t_2 \circ pr_1 \circ \mathtt{I}_{\mathtt{I}}((u_1,u_2),(t_1^{-1}(y_1),t_1^{-1}(y_2))) = \begin{cases} 0, & y_1 < \infty, \\ \infty, & y_1 = \infty. \end{cases}$$

$$\Rightarrow\ pr_1 \circ \mathtt{I}_{\mathtt{I}}((u_1,u_2),(v_1,v_2)) = \begin{cases} 1, & v_1 > 0, \\ 0, & v_1 = 0. \end{cases}$$

(S9): $\exists\ c \in (0,\infty)$ such that

$$\begin{aligned} f^1(y_1,y_2) = cy_2 \ &\Rightarrow\ t_2 \circ pr_1 \circ \mathtt{I}_{\mathtt{I}}((u_1,u_2),(t_1^{-1}(y_1),t_1^{-1}(y_2))) = cy_2 \\ &\Rightarrow\ pr_1 \circ \mathtt{I}_{\mathtt{I}}((u_1,u_2),(v_1,v_2)) = t_2^{-1}(cs_1(v_2)). \end{aligned}$$

(S10):
$$f^1(y_1,y_2) = \begin{cases} cy_1, & y_1 = y_2, \\ \infty, & y_1 > y_2, \end{cases} \Rightarrow\ t_2 \circ pr_1 \circ \mathtt{I}_{\mathtt{I}}((u_1,u_2),(t_1^{-1}(y_1),t_1^{-1}(y_2))) = \begin{cases} cy_1, & y_1 = y_2, \\ \infty, & y_1 > y_2. \end{cases}$$

$$\Rightarrow\ pr_1 \circ \mathtt{I}_{\mathtt{I}}((u_1,u_2),(v_1,v_2)) = \begin{cases} t_2^{-1}(ct_1(v_1)), & v_1 = 1 - v_2, \\ 0, & v_1 < 1 - v_2. \end{cases}$$

(S11):
$$f^1(y_1,y_2) = \begin{cases} cy_2, & y_1 < \infty, \\ \infty, & y_1 = \infty, \end{cases} \Rightarrow\ t_2 \circ pr_1 \circ \mathtt{I}_{\mathtt{I}}((u_1,u_2),(t_1^{-1}(y_1),t_1^{-1}(y_2))) = \begin{cases} cy_2, & y_1 < \infty, \\ \infty, & y_1 = \infty. \end{cases}$$

$$\Rightarrow\ pr_1 \circ \mathtt{I}_{\mathtt{I}}((u_1,u_2),(v_1,v_2)) = \begin{cases} t_2^{-1}(cs_1(v_2)), & v_1 > 0, \\ 0, & v_1 = 0. \end{cases}$$

(S12):
$$f^1(y_1,y_2) = \begin{cases} cy_1, & y_2 = 0, \\ \infty, & y_2 > 0, \end{cases} \Rightarrow\ t_2 \circ pr_1 \circ \mathtt{I}_{\mathtt{I}}((u_1,u_2),(t_1^{-1}(y_1),t_1^{-1}(y_2))) = \begin{cases} cy_1, & y_2 = 0, \\ \infty, & y_2 > 0. \end{cases}$$

$$\Rightarrow\ pr_1 \circ \mathtt{I}_{\mathtt{I}}((u_1,u_2),(v_1,v_2)) = \begin{cases} t_2^{-1}(ct_1(v_1)), & v_2 = 0, \\ 0, & v_2 > 0. \end{cases}$$

(S13): $$f^1(y_1,y_2) = \begin{cases} c(y_1-y_2), & y_2<\infty, \\ \infty, & y_2=\infty, \end{cases}$$

$$\Rightarrow\ t_2\circ pr_1\circ \mathtt{I}_{\mathtt{I}}((u_1,u_2),(t_1^{-1}(y_1),t_1^{-1}(y_2))) = \begin{cases} c(y_1-y_2), & y_2<\infty, \\ \infty, & y_2=\infty. \end{cases}$$

$$\Rightarrow\ pr_1\circ \mathtt{I}_{\mathtt{I}}((u_1,u_2),(v_1,v_2)) = \begin{cases} t_2^{-1}(c(t_1(v_1)-s_1(v_2))), & v_2<1, \\ 0, & v_2=1. \end{cases}$$

(S14): $\exists\ c\in(0,\infty)$ such that

$$f^1(y_1,y_2)=cy_1 \ \Rightarrow\ t_2\circ pr_1\circ \mathtt{I}_{\mathtt{I}}((u_1,u_2),(t_1^{-1}(y_1),t_1^{-1}(y_2))) = cy_1$$
$$\Rightarrow\ pr_1\circ \mathtt{I}_{\mathtt{I}}((u_1,u_2),(v_1,v_2)) = t_2^{-1}(ct_1(v_1)).$$

(S15): $\exists\ c_1,c_2\in(0,\infty),\ c_1\neq c_2$ such that

$$f^1(y_1,y_2) = \begin{cases} c_1(y_1-y_2)+c_2y_2, & y_2<\infty, \\ \infty, & y_2=\infty, \end{cases}$$

$$\Rightarrow\ t_2\circ pr_1\circ \mathtt{I}_{\mathtt{I}}((u_1,u_2),(t_1^{-1}(y_1),t_1^{-1}(y_2))) = \begin{cases} c_1(y_1-y_2)+c_2y_2, & y_2<\infty, \\ \infty, & y_2=\infty. \end{cases}$$

$$\Rightarrow\ pr_1\circ \mathtt{I}_{\mathtt{I}}((u_1,u_2),(v_1,v_2)) = \begin{cases} t_2^{-1}(c_1(t_1(v_1)-s_1(v_2))+c_2(s_1(v_2))), & v_2<1, \\ 0, & v_2=1. \end{cases}$$

Similarly, we can find the possible solutions of (7.51) are as follows:

For $y_1,y_2\in[0,\infty], y_1\geq y_2$ and $(v_1,v_2)\in\mathcal{L}$, we have

(S'1): $pr_2\circ \mathtt{I}_{\mathtt{I}}((u_1,u_2),(v_1,v_2))=0.$

(S'2): $pr_2\circ \mathtt{I}_{\mathtt{I}}((u_1,u_2),(v_1,v_2))=1.$

(S'3): $$pr_2\circ \mathtt{I}_{\mathtt{I}}((u_1,u_2),(v_1,v_2)) = \begin{cases} 0, & v_2=0, \\ 1, & v_2>0. \end{cases}$$

(S'4): $$pr_2\circ \mathtt{I}_{\mathtt{I}}((u_1,u_2),(v_1,v_2)) = \begin{cases} 0, & v_2<1, \\ 1, & v_2=1. \end{cases}$$

(S'5): $pr_2 \circ \mathtt{I}_\mathtt{I}((u_1,u_2),(v_1,v_2)) = \begin{cases} 0, & v_1 = 1, \\ 1, & v_1 < 1. \end{cases}$

(S'6): $pr_2 \circ \mathtt{I}_\mathtt{I}((u_1,u_2),(v_1,v_2)) = \begin{cases} 0, & v_1 = 1 - v_2 > 0, \\ 1, & v_2 = 1 \text{ } or \text{ } v_1 < 1 - v_2. \end{cases}$

(S'7): $pr_2 \circ \mathtt{I}_\mathtt{I}((u_1,u_2),(v_1,v_2)) = \begin{cases} 0, & v_2 = 0,\ v_1 > 0, \\ 1, & v_2 > 0 \text{ } or \text{ } v_1 = 0. \end{cases}$

(S'8): $pr_2 \circ \mathtt{I}_\mathtt{I}((u_1,u_2),(v_1,v_2)) = \begin{cases} 0, & v_1 > 0, \\ 1, & v_1 = 0. \end{cases}$

(S'9): $\exists\ c \in (0,\infty)$ such that

$$pr_2 \circ \mathtt{I}_\mathtt{I}((u_1,u_2),(v_1,v_2)) = s_2^{-1}(cs_1(v_2)).$$

(S'10): $pr_2 \circ \mathtt{I}_\mathtt{I}((u_1,u_2),(v_1,v_2)) = \begin{cases} s_2^{-1}(ct_1(v_1)), & v_1 = 1 - v_2, \\ 1, & v_1 < 1 - v_2. \end{cases}$

(S'11): $pr_2 \circ \mathtt{I}_\mathtt{I}((u_1,u_2),(v_1,v_2)) = \begin{cases} s_2^{-1}(cs_1(v_2)), & v_1 > 0, \\ 1, & v_1 = 0. \end{cases}$

(S'12): $pr_2 \circ \mathtt{I}_\mathtt{I}((u_1,u_2),(v_1,v_2)) = \begin{cases} s_2^{-1}(ct_1(v_1)), & v_2 = 0, \\ 1, & v_2 > 0. \end{cases}$

(S'13): $pr_2 \circ \mathtt{I}_\mathtt{I}((u_1,u_2),(v_1,v_2)) = \begin{cases} s_2^{-1}(c(t_1(v_1) - s_1(v_2))), & v_2 < 1, \\ 1, & v_2 = 1. \end{cases}$

(S'14): $\exists\ c \in (0,\infty)$ such that
$pr_2 \circ \mathtt{I}_\mathtt{I}((u_1,u_2),(v_1,v_2)) = s_2^{-1}(ct_1(v_1))$.

(S'15): $\exists\ c_1, c_2 \in (0,\infty),\ c_1 \neq c_2$ such that

$$pr_2 \circ \mathtt{I}_\mathtt{I}((u_1,u_2),(v_1,v_2)) = \begin{cases} s_2^{-1}(c_1(t_1(v_1) - s_1(v_2)) + c_2(s_1(v_2))), & v_2 < 1, \\ 1, & v_2 = 1. \end{cases}$$

Of course not every combination of the above solutions give a correct value in the set $\mathcal{L}$. For example when $pr_1 \circ \mathtt{I_I}((u_1,u_2),(v_1,v_2)) = \begin{cases} 1, & v_2 < 1, \\ 0, & v_2 = 1. \end{cases}$ and $pr_2 \circ \mathtt{I_I}((u_1,u_2),(v_1,v_2)) = \begin{cases} 0, & v_2 < 1, \\ 1, & v_2 = 1, \end{cases}$ for every $(w_1,w_2) \in \mathcal{L}$, then our (constant) solution is correct: $\mathtt{I_I}((u_1,u_2),(v_1,v_2)) = \begin{cases} 1_{\mathcal{L}}, & v_2 < 1, \\ 0_{\mathcal{L}}, & v_2 = 1, \end{cases}$.

Also when $pr_1 \circ \mathtt{I_I}((u_1,u_2),(v_1,v_2)) = \begin{cases} 1, & v_1 = 1, \\ 0, & v_1 < 1. \end{cases}$ and $pr_2 \circ \mathtt{I_I}((u_1,u_2),(v_1,v_2)) = \begin{cases} 0, & v_1 = 1, \\ 1, & v_1 < 1. \end{cases}$ for every $(w_1,w_2) \in \mathcal{L}$, then our (constant) solution is correct: $\mathtt{I_I}((u_1,u_2),(v_1,v_2)) = \begin{cases} 1_{\mathcal{L}}, & v_1 = 1, \\ 0_{\mathcal{L}}, & v_1 < 1. \end{cases}$

But if $pr_1 \circ \mathtt{I_I}((u_1,u_2),(v_1,v_2)) = \begin{cases} 1, & v_2 < 1, \\ 0, & v_2 = 1. \end{cases}$ and $pr_2 \circ \mathtt{I_I}((u_1,u_2),(v_1,v_2)) = \begin{cases} 0, & v_1 = 1, \\ 1, & v_1 < 1. \end{cases}$ for every $(v_1,v_2) \in \mathcal{L}$, then our solution is incorrect, since $\mathtt{I_I}((u_1,u_2),(v_1,v_2)) = \begin{cases} 1_{\mathcal{L}}, & v_1 = 1, v_2 = 0, \\ 1_{\mathcal{L}}, & v_1 = 0, v_2 = 1, \\ (1,1), & 0 < v_1 < 1, 0 < v_2 < 1, v_1 + v_2 \le 1, \end{cases}$ is not solution in $\mathcal{L}$ (since $(1,1) \notin \mathcal{L}$). Similarly, we can find the possible combinations of the above solutions give a correct value in the set $\mathcal{L}$ is the required result (ii). □

Remark 7.4.11. *If we put $\alpha = u$, $\beta = \mathtt{f_I}(v)$ and $\mathtt{I_I} = \mathtt{I_{I(f_I,\omega)}}$ in Proposition 7.4.10, then the above possible forms of $\mathtt{I_I}(u,v)$, for fixed $u \in \mathcal{L}$, convert into corresponding forms of $\omega(\alpha,\beta)$.*

Proposition 7.4.12. *Let $\mathcal{T} = (T,S)$, where T and S are the nilpotent t-norm and t-conorm respectively such that T and S are dual of each other. For a function $\mathtt{I_I} : \mathcal{L}^2 \to \mathcal{L}$, the following statements are equivalent:*

(i) The triple $(\mathcal{T}_1, \mathcal{T}_2, \mathtt{I_I})$ satisfies the functional eq. (7.11) $\forall\, u,v,w \in \mathcal{L}$.

(ii) For every fixed $u \in \mathcal{L}$, $\mathtt{I_I}(u,.)$ has one of the following forms:

$$\mathtt{I_I}(u,v) = 1_{\mathcal{L}}, \quad \mathtt{I_I}(u,v) = (0,0), \quad \mathtt{I_I}(u,v) = 0_{\mathcal{L}}, \quad \mathtt{I_I}(u,v) = \begin{cases} 1_{\mathcal{L}}, & v_1 = 1, \\ 0_{\mathcal{L}}, & v_1 < 1, \end{cases}$$

$$\mathtt{I_I}(u,v) = \begin{cases} (0,0), & v_2 = 0, \\ 0_{\mathcal{L}}, & v_2 > 0, \end{cases} \quad \mathtt{I_I}(u,v) = \begin{cases} (0,0), & v_1 = 1, \\ 0_{\mathcal{L}}, & v_1 < 1, \end{cases} \quad \mathtt{I_I}(u,v) = \begin{cases} 1_{\mathcal{L}}, & v_2 = 0, \\ (0,0), & v_2 > 0, \end{cases}$$

$$\mathtt{I_I}(u,v) = \begin{cases} 1_{\mathcal{L}}, & v_2 = 0, \\ 0_{\mathcal{L}}, & v_2 > 0, \end{cases} \quad \mathtt{I_I}(u,v) = \begin{cases} 1_{\mathcal{L}}, & v_2 = 0, \\ (0,0), & v_2 > 0, \end{cases} \quad \mathtt{I_I}(u,v) = \begin{cases} 1_{\mathcal{L}}, & v_1 = 1, v_2 = 0, \\ 0_{\mathcal{L}}, & v_1 < 1, v_2 > 0, \\ & v_1 + v_2 \leq 1, \\ (0,0), & v_1 < 1, v_2 = 0, \end{cases}$$

$$\mathtt{I_I}(u,v) = (0, s_2^{-1}(\min(cs_1(v_2), b))), \quad \mathtt{I_I}(u,v) = \begin{cases} (0, s_2^{-1}(\min(cs_1(v_2), b))), & v_1 = 1 - v_2, \\ 0_{\mathcal{L}}, & v_1 < 1 - v_2, \end{cases}$$

$$\mathtt{I_I}(u,v) = \begin{cases} (0, s_2^{-1}(\min(cs_1(v_2), b))), & v_2 = 0, \\ 0_{\mathcal{L}}, & v_2 > 0, \end{cases} \quad \mathtt{I_I}(u,v) = (0, s_2^{-1}(\min(ct_1(v_1), b))),$$

$$\mathtt{I_I}(u,v) = \begin{cases} (0, s_2^{-1}(\min(c_1(t_1(v_1) - & \\ s_1(v_2)) + c_2 - s_1(v_2), b))), & v_1 > 0, \\ 0_{\mathcal{L}}, & v_1 = 0, \end{cases} \quad \mathtt{I_I}(u,v) = \begin{cases} 1_{\mathcal{L}}, & v_2 = 0, \\ (0, s_2^{-1}(\min(cs_1(v_2), b))), & v_2 > 0, \end{cases}$$

$$\mathtt{I_I}(u,v) = \begin{cases} 1_{\mathcal{L}}, & v_1 = 1, \\ (0, s_2^{-1}(\min(cs_1(v_2), b))), & v_1 < 1, \end{cases}$$

$$\mathtt{I_I}(u,v) = \begin{cases} 1_{\mathcal{L}}, & v_1 = 1, v_2 = 0, \\ (0, s_2^{-1}(\min(cs_1(v_2), b))), & v_1 = 1 - v_2, v_1 < 1, \\ 0_{\mathcal{L}}, & v_1 < 1 - v_2, v_1 < 1, \end{cases}$$

$$\mathtt{I_I}(u,v) = \begin{cases} 1_{\mathcal{L}}, & v_1 = 1, v_2 = 0, \\ (0, s_2^{-1}(\min(ct_1(v_1), b))), & v_1 < 1, v_2 = 0, \\ 0_{\mathcal{L}}, & v_1 < 1, v_2 > 0, v_1 + v_2 \leq 1, \end{cases}$$

$$\mathtt{I_I}(u,v) = \begin{cases} 1_{\mathcal{L}}, & v_1 = 1, \\ (0, s_2^{-1}(\min(ct_1(v_1), b))), & v_1 < 1, \end{cases}$$

$$\mathtt{I_I}(u,v) = \begin{cases} 1_{\mathcal{L}}, & v_1 = 1, \\ (0, s_2^{-1}(\min(c_1(t_1(v_1) - & \\ s_1(v_2)) + c_2 - s_1(v_2), b))), & 0 < v_1 < 1, \\ 0_{\mathcal{L}}, & v_1 = 0, \end{cases} \quad \mathtt{I_I}(u,v) = (t_2^{-1}(\min(cs_1(v_2), b)), 0),$$

$$\mathtt{I}_{\mathtt{I}}(u,v) = (t_2^{-1}(\min(cs_1(v_2),b)), s_2^{-1}(\min(cs_1(v_2),b))),$$

$$\mathtt{I}_{\mathtt{I}}(u,v) = (t_2^{-1}(\min(cs_1(v_2),b)), s_2^{-1}(\min(ct_1(v_1),b))),$$

$$\mathtt{I}_{\mathtt{I}}(u,v) = \begin{cases} (t_2^{-1}(\min(ct_1(v_1),b)),0), & v_1 = 1 - v_2, \\ (0,0) & v_1 < 1 - v_2, \end{cases}$$

$$\mathtt{I}_{\mathtt{I}}(u,v) = \begin{cases} (t_2^{-1}(\min(ct_1(v_1),b)), s_2^{-1}(\min(cs_1(v_2),b))), & v_1 = 1 - v_2, \\ (0, s_2^{-1}(\min(cs_1(v_2),b))) & v_1 < 1 - v_2, \end{cases}$$

$$\mathtt{I}_{\mathtt{I}}(u,v) = \begin{cases} (t_2^{-1}(\min(ct_1(v_1),b)), & \\ s_2^{-1}(\min(ct_1(v_1),b))), & v_1 = 1 - v_2, \\ 0_{\mathcal{L}} & v_1 < 1 - v_2, \end{cases}$$

$$\mathtt{I}_{\mathtt{I}}(u,v) = \begin{cases} (t_2^{-1}(\min(ct_1(v_1),b)), s_2^{-1}(\min(ct_1(v_1),b))), & v_1 = 1 - v_2, \\ (0, s_2^{-1}(\min(ct_1(v_1),b))) & v_1 < 1 - v_2, \end{cases}$$

$$\mathtt{I}_{\mathtt{I}}(u,v) = \begin{cases} (t_2^{-1}(\min(ct_1(v_1),b)), s_2^{-1}(\min(c_1(t_1(v_1) - s_1(v_2)) & \\ +c_2 - s_1(v_2), b)))), & v_1 = 1 - v_2, v_1 > 0, \\ (0, s_2^{-1}(\min(c_1(t_1(v_1) - s_1(v_2)) + c_2 - s_1(v_2), b)))), & v_1 < 1 - v_2, v_1 > 0, \\ 0_{\mathcal{L}}, & (v_1 = 0, v_2 = 1) \ or \\ & (v_1 = 0, v_1 < 1 - v_2), \end{cases}$$

$$\mathtt{I}_{\mathtt{I}}(u,v) = \begin{cases} (t_2^{-1}(\min(ct_1(v_1),b)),0), & v_2 = 0, \\ (0,0) & v_2 > 0, \end{cases}$$

$$\mathtt{I}_{\mathtt{I}}(u,v) = \begin{cases} (t_2^{-1}(\min(ct_1(v_1),b)),0), & v_2 = 0, \\ 0_{\mathcal{L}} & v_2 > 0, \end{cases}$$

$$\mathtt{I}_{\mathtt{I}}(u,v) = \begin{cases} (t_2^{-1}(\min(ct_1(v_1),b)),0), & v_2 = 0, \\ (0, s_2^{-1}(\min(cs_1(v_2),b))) & v_2 > 0, \end{cases}$$

$$\mathtt{I}_{\mathtt{I}}(u,v) = \begin{cases} (t_2^{-1}(\min(ct_1(v_1),b)), s_2^{-1}(\min(ct_1(v_1),b))), & v_2 = 0, \\ 0_{\mathcal{L}}, & v_2 > 0, \end{cases}$$

$$\mathtt{I}_{\mathtt{I}}(u,v) = \begin{cases} (t_2^{-1}(\min(ct_1(v_1),b)), s_2^{-1}(\min(c_1t_1(v_1),b))), & v_1 > 0, v_2 = 0, \\ (0, s_2^{-1}(\min(c_1(t_1(v_1) - s_1(v_2)) + c_2 - s_1(v_2), b))), & v_1 > 0, v_2 > 0, v_1 + v_2 \le 1, \\ 0_{\mathcal{L}}, & v_1 = 0, v_2 \ge 0, \end{cases}$$

$$\mathtt{I_I}(u,v) = (t_2^{-1}(\min(ct_1(v_1),b)),0),$$

$$\mathtt{I_I}(u,v) = (t_2^{-1}(\min(ct_1(v_1),b)), s_2^{-1}(\min(cs_1(v_2),b))),$$

$$\mathtt{I_I}(u,v) = (t_2^{-1}(\min(ct_1(v_1),b)), s_2^{-1}(\min(ct_1(v_1),b))),$$

$$\mathtt{I_I}(u,v) = \begin{cases} (t_2^{-1}(\min(ct_1(v_1),b)), s_2^{-1}(\min(c_1(t_1(v_1)-s_1(v_2))+c_2-s_1(v_2),b))), & v_1 > 0, \\ 0_{\mathcal{L}}, & v_1 = 0, \end{cases}$$

$$\mathtt{I_I}(u,v) = \begin{cases} (t_2^{-1}(\min(ct_1(v_1),b)), s_2^{-1}(\min(c_1(t_1(v_1)-s_1(v_2))+c_2-s_1(v_2),b))), & v_1 > 0, \\ (0,0), & v_1 = 0, \end{cases}$$

$$\mathtt{I_I}(u,v) = \begin{cases} (t_2^{-1}(\min(c_1(t_1(v_1)-s_1(v_2))+c_2-s_1(v_2),b)), s_2^{-1}(\min(cs_1(v_2,b)))), & v_1 > 0, \\ (0, s_2^{-1}(\min(cs_1(v_2,b)))), & v_1 = 0, \end{cases}$$

$$\mathtt{I_I}(u,v) = \begin{cases} (t_2^{-1}(\min(c_1(t_1(v_1)-s_1(v_2))+c_2-s_1(v_2),b)), s_2^{-1}(\min(ct_1(v_1,b)))), & v_1 > 0, \\ 0_{\mathcal{L}}, & v_1 = 0, \end{cases}$$

or

$$\mathtt{I_I}(u,v) = \begin{cases} (t_2^{-1}(\min(c_1(t_1(v_1)-s_1(v_2))+c_2-s_1(v_2),b)), & \\ s_2^{-1}(\min(c_1(t_1(v_1)-s_1(v_2))+c_2-s_1(v_2),b))), & v_1 > 0, \\ 0_{\mathcal{L}}, & v_1 = 0, \end{cases}$$

Proof. Given that $\mathcal{T}_1$ and $\mathcal{T}_2$ are the t-representable on $\mathcal{L}$, i.e., $\mathcal{T}_1 = (T_1, S_1)$ and $\mathcal{T}_2 = (S_2, T_2)$. Also, given that t-norm T is nilpotent, and T and S are dual of each other. It follows that t-conorm S is nilpotent. Now, from Theorems 2.1.5 and 2.1.7 and Remarks 2.2.6 and 2.2.7 given in [19], there exist two decreasing continuous functions $t_1, t_2 : [0,1] \to [0,\infty]$ such that $t_1(0), t_2(0) < \infty$ and $t_1(1) = t_2(1) = 0$ which are uniquely determined a positive multiplicative constant such that $T_1(a,b) = t_1^{-1}(\min(t_1(a)+t_1(b), t_1(0)))$ and $T_2(a,b) = t_2^{-1}(\min(t_2(a)+t_2(b), t_2(0)))$, $\forall\, a,b \in [0,1]$.

Let us prove that $(ii) \Rightarrow (i)$.

(P1): Let $\mathtt{I_I}$ have the form $\mathtt{I_I}(u,v) = 1_{\mathcal{L}}$. Then LHS of (7.11) is equal to $1_{\mathcal{L}}$, and the RHS of (7.11) is equal to $1_{\mathcal{L}}$.

(P2): Let $\mathtt{I_I}$ have the form $\mathtt{I_I}(u,v) = (0,0)$. Then LHS of (7.11) is equal to $(0,0)$, and the RHS of (7.11) is equal to $(0,0)$.

(P3): Let $\mathtt{I_I}$ have the form $\mathtt{I_I}(u,v) = 0_{\mathcal{L}}$. Then LHS of (7.11) is equal to $0_{\mathcal{L}}$, and the RHS of (7.11) is equal to $0_{\mathcal{L}}$.

(P4): Let $\mathtt{I}_{\mathtt{I}}$ have the form $\mathtt{I}_{\mathtt{I}}(u,v) = \begin{cases} (0,0), & v_2 = 0, \\ 0_{\mathcal{L}}, & v_2 > 0. \end{cases}$

Then LHS of (7.11) is equal to

$$\mathtt{I}_{\mathtt{I}}(u, \mathcal{T}_1(v,w)) = \begin{cases} (0,0), & S_1(v_2, z_2) = 0, \\ 0_{\mathcal{L}}, & S_1(v_2, w_2) > 0. \end{cases} = \begin{cases} (0,0), & v_2 = w_2 = 0, \\ 0_{\mathcal{L}}, & v_2 > 0 \text{ or } w_2 > 0, \end{cases}$$

and the RHS of (7.11) is equal to

$$\begin{aligned} \mathcal{T}_2(\mathtt{I}_{\mathtt{I}}(u,v), \mathtt{I}_{\mathtt{I}}(u,w)) =& \mathcal{T}_2\left(\begin{cases} (0,0), & v_2 = 0, \\ 0_{\mathcal{L}}, & v_2 > 0. \end{cases}, \begin{cases} (0,0), & w_2 = 0, \\ 0_{\mathcal{L}}, & w_2 > 0. \end{cases} \right) \\ =& \begin{cases} (0,0), & v_2 = w_2 = 0, \\ 0_{\mathcal{L}}, & v_2 > 0 \text{ or } w_2 > 0. \end{cases} \end{aligned}$$

(P5): Let $\mathtt{I}_{\mathtt{I}}$ have the form $\mathtt{I}_{\mathtt{I}}(u,v) = \begin{cases} (0,0), & v_1 = 1, \\ 0_{\mathcal{L}}, & v_1 < 1. \end{cases}$

Then LHS of (7.11) is equal to

$$\mathtt{I}_{\mathtt{I}}(u, \mathcal{T}_1(v,w)) = \begin{cases} (0,0), & T_1(v_1, w_1) = 1, \\ 0_{\mathcal{L}}, & T_1(v_1, w_1) < 1. \end{cases} = \begin{cases} (0,0), & v_1 = w_1 = 1, \\ 0_{\mathcal{L}}, & v_1 < 1 \text{ or } w_1 < 1, \end{cases}$$

and the RHS of (7.11) is equal to

$$\begin{aligned} \mathcal{T}_2(\mathtt{I}_{\mathtt{I}}(u,v), \mathtt{I}_{\mathtt{I}}(u,w)) =& \mathcal{T}_2\left(\begin{cases} (0,0), & v_1 = 1, \\ 0_{\mathcal{L}}, & v_1 < 1. \end{cases}, \begin{cases} (0,0), & w_1 = 1, \\ 0_{\mathcal{L}}, & w_1 < 1. \end{cases} \right) \\ =& \begin{cases} (0,0), & v_1 = w_1 = 1, \\ 0_{\mathcal{L}}, & v_1 < 1 \text{ or } w_1 < 1. \end{cases} \end{aligned}$$

(P6): Let $\mathtt{I}_{\mathtt{I}}$ have the form $\mathtt{I}_{\mathtt{I}}(u,v) = \begin{cases} 1_{\mathcal{L}}, & v_2 = 0, \\ (0,0), & v_2 > 0. \end{cases}$

Then LHS of (7.11) is equal to

$$\mathtt{I}_{\mathtt{I}}(u, \mathcal{T}_1(v,w)) = \begin{cases} 1_{\mathcal{L}}, & S_1(v_2, w_2) = 0, \\ (0,0), & S_1(v_2, w_2) > 0. \end{cases} = \begin{cases} 1_{\mathcal{L}}, & v_2 = w_2 = 0, \\ (0,0), & v_2 > 0 \text{ or } w_2 > 0, \end{cases}$$

and the RHS of (7.11) is equal to

$$\mathcal{T}_2(\mathtt{I}_\mathtt{I}(u,v),\mathtt{I}_\mathtt{I}(u,w)) = \mathcal{T}_2\left(\begin{cases}1_\mathcal{L}, & v_2=0,\\ (0,0), & v_2>0.\end{cases}, \begin{cases}1_\mathcal{L}, & w_2=0,\\ (0,0), & w_2>0.\end{cases}\right)$$
$$= \begin{cases}1_\mathcal{L}, & v_2=w_2=0,\\ (0,0), & v_2>0 \text{ or } w_2>0.\end{cases}$$

(P7): Let $\mathtt{I}_\mathtt{I}$ have the form $\mathtt{I}_\mathtt{I}(u,v) = \begin{cases}1_\mathcal{L}, & v_2=0,\\ 0_\mathcal{L}, & v_2>0.\end{cases}$

Then LHS of (7.11) is equal to

$$\mathtt{I}_\mathtt{I}(u,\mathcal{T}_1(v,w)) = \begin{cases}1_\mathcal{L}, & S_1(v_2,w_2)=0,\\ 0_\mathcal{L}, & S_1(v_2,w_2)>0.\end{cases} = \begin{cases}1_\mathcal{L}, & v_2=w_2=0,\\ 0_\mathcal{L}, & v_2>0 \text{ or } w_2>0,\end{cases}$$

and the RHS of (7.11) is equal to

$$\mathcal{T}_2(\mathtt{I}_\mathtt{I}(u,v),\mathtt{I}_\mathtt{I}(u,w)) = \mathcal{T}_2\left(\begin{cases}1_\mathcal{L}, & v_2=0,\\ 0_\mathcal{L}, & v_2>0.\end{cases}, \begin{cases}1_\mathcal{L}, & w_2=0,\\ 0_\mathcal{L}, & w_2>0.\end{cases}\right) = \begin{cases}1_\mathcal{L}, & v_2=w_2=0,\\ 0_\mathcal{L}, & v_2>0 \text{ or } w_2>0.\end{cases}$$

(P8): Let $\mathtt{I}_\mathtt{I}$ have the form $\mathtt{I}_\mathtt{I}(u,v) = \begin{cases}1_\mathcal{L}, & v_1=1,\\ (0,0), & v_1<1.\end{cases}$

Then LHS of (7.11) is equal to

$$\mathtt{I}_\mathtt{I}(u,\mathcal{T}_1(v,w)) = \begin{cases}1_\mathcal{L}, & T_1(v_1,w_1)=1,\\ (0,0), & T_1(v_1,w_1)<1.\end{cases} = \begin{cases}1_\mathcal{L}, & v_1=w_1=1,\\ (0,0), & v_1<1 \text{ or } w_1<1,\end{cases}$$

and the RHS of (7.11) is equal to

$$\mathcal{T}_2(\mathtt{I}_\mathtt{I}(u,v),\mathtt{I}_\mathtt{I}(u,w)) = \mathcal{T}_2\left(\begin{cases}1_\mathcal{L}, & v_1=1,\\ (0,0), & v_1<1.\end{cases}, \begin{cases}1_\mathcal{L}, & w_1=1,\\ (0,0), & w_1<1.\end{cases}\right)$$
$$= \begin{cases}1_\mathcal{L}, & v_1=w_1=1,\\ (0,0), & v_1<1 \text{ or } w_1<1.\end{cases}$$

(P9): Let $\mathtt{I}_\mathtt{I}$ have the form $\mathtt{I}_\mathtt{I}(u,v) = \begin{cases}1_\mathcal{L}, & v_1=1,\\ 0_\mathcal{L}, & v_1<1.\end{cases}$

Then LHS of (7.11) is equal to

$$\mathtt{I}_\mathtt{I}(u,\mathcal{T}_1(v,w)) = \begin{cases}1_\mathcal{L}, & T_1(v_1,w_1)=1,\\ 0_\mathcal{L}, & T_1(v_1,w_1)<1.\end{cases} = \begin{cases}1_\mathcal{L}, & v_1=w_1=1,\\ 0_\mathcal{L}, & v_1<1 \text{ or } w_1<1,\end{cases}$$

and the RHS of (7.11) is equal to

$$\mathcal{T}_2(\mathtt{I_I}(u,v),\mathtt{I_I}(u,w)) = \mathcal{T}_2\left(\begin{cases}1_{\mathcal{L}}, & v_1 = 1,\\ 0_{\mathcal{L}}, & v_1 < 1.\end{cases}, \begin{cases}1_{\mathcal{L}}, & w_1 = 1,\\ 0_{\mathcal{L}}, & w_1 < 1.\end{cases}\right) = \begin{cases}1_{\mathcal{L}}, & v_1 = w_1 = 1,\\ 0_{\mathcal{L}}, & v_1 < 1 \text{ or } w_1 < 1.\end{cases}$$

(P10): Let $\mathtt{I_I}$ have the form $\mathtt{I_I}(u,v) = \begin{cases}1_{\mathcal{L}}, & v_1 = 1,\ v_2 = 0,\\ 0_{\mathcal{L}}, & v_1 < 1,\ v_2 > 0,\ v_1 + v_2 \leq 1,\\ (0,0), & v_1 < 1,\ v_2 = 0.\end{cases}$

Then LHS of (7.11) is equal to

$$\mathtt{I_I}(u,\mathcal{T}_1(v,w)) = \begin{cases}1_{\mathcal{L}}, & T_1(v_1,w_1) = 1,\ S_1(v_2,w_2) = 0,\\ 0_{\mathcal{L}}, & T_1(v_1,w_1) < 1,\ S_1(v_2,w_2) > 0,\\ & T_1(v_1,w_1) + S_1(v_2,w_2) \leq 1,\\ (0,0), & T_1(v_1,w_1) < 1,\ S_1(v_2,w_2) = 0,\\ . \end{cases}$$

$$= \begin{cases}1_{\mathcal{L}}, & v_1 = w_1 = 1,\ v_2 = w_2 = 0,\\ 0_{\mathcal{L}}, & (v_1 < 1 \textit{ or } w_1 < 1),\ (v_2 > 0 \textit{ or}\\ & w_2 > 0),\ v_1 + v_2 \leq 1,\ w_1 + w_2 \leq 1,\\ (0,0), & (v_1 < 1 \textit{ or } w_1 < 1),\ v_2 = w_2 = 0.\end{cases}$$

and the RHS of (7.11) is equal to

$$\mathcal{T}_2(\mathtt{I_I}(u,v),\mathtt{I_I}(u,w)) = \mathcal{T}_2\left(\begin{cases}1_{\mathcal{L}}, & v_1 = 1,\ v_2 = 0,\\ 0_{\mathcal{L}}, & v_1 < 1,\ v_2 > 0,\\ & v_1 + v_2 \leq 1.\\ (0,0), & v_1 < 1\ \ v_2 = 0.\end{cases}, \begin{cases}1_{\mathcal{L}}, & w_1 = 1,\ w_2 = 0,\\ 0_{\mathcal{L}}, & w_1 < 1,\ w_2 > 0,\\ & w_1 + w_2 \leq 1,\\ (0,0), & w_1 < 1,\ w_2 = 0.\end{cases}\right)$$

$$= \begin{cases}1_{\mathcal{L}}, & v_1 = w_1 = 1,\ v_2 = w_2 = 0,\\ 0_{\mathcal{L}}, & (v_1 < 1 \textit{ or } w_1 < 1),\ (v_2 > 0 \textit{ or } w_2 > 0)\\ & \&\ v_1 + v_2 \leq 1,\ w_1 + w_2 \leq 1,\\ (0,0), & (v_1 < 1 \textit{ or } w_1 < 1),\ v_2 = w_2 = 0.\end{cases}$$

Let us prove that $(ii) \Rightarrow (i)$.

Now from (7.48), we have

From (7.48),

$$\begin{aligned}
&g^1(t_1^{-1}(\min(t_1(v_1)+t_1(w_1),t_1(0))),s_1^{-1}(\min(s_1(v_2)+s_1(w_2),s_1(1))))\\
&=t_2^{-1}(\min(t_2(g^1(v_1,v_2))+t_2(g^1(w_1,w_2)),t_2(0))),\\
&g^2(t_1^{-1}(\min(t_1(v_1)+t_1(w_1),t_1(0))),s_1^{-1}(\min(s_1(v_2)+s_1(w_2),s_1(1))))\\
&=s_2^{-1}(\min(s_2(g^2(v_1,v_2))+s_2(g^2(w_1,w_2)),s_2(1))).
\end{aligned}$$

Hence

$$\begin{aligned}
&t_2\circ g^1(t_1^{-1}(\min(t_1(v_1)+t_1(w_1),t_1(0))),s_1^{-1}(\min(s_1(v_2)+s_1(w_2),s_1(1))))\\
&=\min(t_2(g^1(v_1,v_2))+t_2(g^1(w_1,w_2)),t_2(0)),\\
&s_2\circ g^2(t_1^{-1}(\min(t_1(v_1)+t_1(w_1),t_1(0))),s_1^{-1}(\min(s_1(v_2)+s_1(w_2),s_1(1))))\\
&=\min(s_2(g^2(v_1,v_2))+s_2(g^2(w_1,w_2)),s_2(1)).
\end{aligned}$$

Let us put $t_1(v_1)=y_1, s_1(v_2)=y_2, t_1(w_1)=z_1,\ s_1(w_2)=z_2,\ t_1(0)=a$ and $t_2(0)=b$. Of course $y_1,y_2,z_1,z_2\in[0,\infty]$, Moreover $u=(u_1,u_2), w=(w_1,w_2)\in\mathcal{L}$, thus $v_1\le 1-v_2$ and $w_1\le 1-w_2$. Since t and s are decreasing generator and increasing generator such that $t(1-a)=t(0)-t(a)$ and $s(a)=t(0)-t(a)$, $\forall a\in[0,1]$, $y_1\ge y_2$ and $z_1\ge z_2$. This implies that $(y_1,y_2),(z_1,z_2)\in[0,\infty]^2$. If we put

$$f^1(y_1,y_2):=t_2\circ pr_1\circ \mathtt{I}_{\mathtt{I}}((u_1,u_2),(t_1^{-1}(y_1),s_1^{-1}(y_2))),\quad f^2(y_1,y_2):=s_2\circ pr_2\circ \mathtt{I}_{\mathtt{I}}((u_1,u_2),(t_1^{-1}(y_1),s_1^{-1}(y_2)))\ \forall\,(x_1,x_2)\in[0,\infty]^2 \tag{7.52}$$

As a consequence we get the following two functional equations

$$f^1(\min(y_1+z_1,a),\min(y_2+z_2,a))=\min(f^1(y_1,y_2)+f^1(z_1,z_2),b), \tag{7.53}$$

$$f^2(\min(y_1+z_1,a),\min(y_2+z_2,a))=\min(f^2(y_1,y_2)+f^2(z_1,z_2),b), \tag{7.54}$$

where $(y_1,y_2),(z_1,z_2)\in[0,\infty]^2$.

Now, we find the possible solutions of (7.53) are as follows:

For $y_1,y_2\in[0,\infty], y_1\ge y_2$ and $(v_1,v_2)\in\mathcal{L}$, we have

(S1): $f^1=0\ \Rightarrow\ t_2\circ pr_1\circ\mathtt{I}_{\mathtt{I}}((u_1,u_2),(t_1^{-1}(y_1),s_1^{-1}(y_2)))=0\ \Rightarrow\ pr_1\circ\mathtt{I}_{\mathtt{I}}((u_1,u_2),(v_1,v_2))=1.$

(S2): $f^1=b\ \Rightarrow\ t_2\circ pr_1\circ\mathtt{I}_{\mathtt{I}}((u_1,u_2),(t_1^{-1}(y_1),s_1^{-1}(y_2)))=b\ \Rightarrow\ pr_1\circ\mathtt{I}_{\mathtt{I}}((u_1,u_2),(v_1,v_2))=0.$

(S3): $$f^1(y_1,y_2)=\begin{cases}0, & y_2=0,\\ b, & y_2>0\end{cases} \Rightarrow\ t_2\circ pr_1\circ \mathtt{I}_{\mathtt{I}}((u_1,u_2),(t_1^{-1}(y_1),s_1^{-1}(y_2)))=\begin{cases}0, & y_2=0,\\ b, & y_2>0\end{cases}$$

$$\Rightarrow\ pr_1\circ \mathtt{I}_{\mathtt{I}}((u_1,u_2),(v_1,v_2))=\begin{cases}1, & v_2=0,\\ 0, & v_2>0.\end{cases}$$

(S4): $$f^1(y_1,y_2)=\begin{cases}0, & y_1=0,\\ b, & y_1>0\end{cases} \Rightarrow\ t_2\circ pr_1\circ \mathtt{I}_{\mathtt{I}}((u_1,u_2),(t_1^{-1}(y_1),s_1^{-1}(y_2)))=\begin{cases}0, & y_1=0,\\ b, & y_1>0\end{cases}$$

$$\Rightarrow\ pr_1\circ \mathtt{I}_{\mathtt{I}}((u_1,u_2),(v_1,v_2))=\begin{cases}1, & v_1=1,\\ 0, & v_1<1.\end{cases}$$

(S5): $\exists\ c\in[b/a,\infty)$ such that

$$f^1(y_1,y_2)=\min(cv_2,b)\ \ \Rightarrow\ t_2\circ pr_1\circ \mathtt{I}_{\mathtt{I}}((u_1,u_2),(t_1^{-1}(y_1),s_1^{-1}(y_2)))=\min(cy_2,b)$$
$$\Rightarrow\ pr_1\circ \mathtt{I}_{\mathtt{I}}((u_1,u_2),(v_1,v_2))=t_2^{-1}(\min(cs_1(v_2),b)).$$

(S6): $$f^1(y_1,y_2)=\begin{cases}\min(cy_1,b), & y_1=y_2,\\ b, & y_1>y_2\end{cases}$$

$$\Rightarrow\ t_2\circ pr_1\circ \mathtt{I}_{\mathtt{I}}((u_1,u_2),(t_1^{-1}(y_1),s_1^{-1}(y_2)))=\begin{cases}\min(cy_1,b), & y_1=y_2,\\ b, & y_1>y_2\end{cases}$$

$$\Rightarrow\ pr_1\circ \mathtt{I}_{\mathtt{I}}((u_1,u_2),(v_1,v_2))=\begin{cases}t_2^{-1}(\min(ct_1(v_1),b)), & v_1=1-v_2,\\ 0, & v_1<1-v_2.\end{cases}$$

(S7): $$f^1(y_1,y_2)=\begin{cases}\min(cy_1,b), & y_2=0,\\ b, & y_2>0,\end{cases}$$

$$\Rightarrow\ t_2\circ pr_1\circ \mathtt{I}_{\mathtt{I}}((u_1,u_2),(t_1^{-1}(y_1),s_1^{-1}(y_2)))=\begin{cases}\min(cy_1,b), & y_2=0,\\ b, & y_2>0.\end{cases}$$

$$\Rightarrow\ pr_1\circ \mathtt{I}_{\mathtt{I}}((u_1,u_2),(v_1,v_2))=\begin{cases}t_2^{-1}(\min(ct_1(v_1),b)), & v_2=0,\\ 0, & v_2>0.\end{cases}$$

(S8): $f^1(y_1,y_2)=\min(cy_1,b)\Rightarrow\ \ pr_1\circ \mathtt{I}_{\mathtt{I}}((u_1,u_2),(v_1,v_2))=t_2^{-1}(\min(ct_1(v_1),b)).$

(S9): $\exists\, c_1, c_2 \in [b/a, \infty),\ c_1 \neq c_2$ such that

$$f^1(y_1, y_2) = \begin{cases} \min(c_1(y_1 - y_2) + c_2 y_2, b), & y_1 < a, \\ a, & y_1 = a \end{cases}$$

$$\Rightarrow\ t_2 \circ pr_1 \circ \mathtt{I}_\mathtt{I}((u_1, u_2), (t_1^{-1}(y_1), s_1^{-1}(y_2))) = \begin{cases} \min(c_1(y_1 - y_2) + c_2 y_2, b), & y_1 < a, \\ b, & y_1 = a. \end{cases}$$

$$\Rightarrow\ pr_1 \circ \mathtt{I}_\mathtt{I}((u_1, u_2), (v_1, v_2)) = \begin{cases} t_2^{-1}(\min(c_1(t_1(v_1) - s_1(v_2)) + c_2(s_1(v_2)), b)), & v_1 > 0, \\ 0, & v_1 = 0. \end{cases}$$

Similarly, we can find the possible solutions of (7.51) are as follows:

For $y_1, y_2 \in [0, \infty], y_1 \geq y_2$ and $(v_1, v_2) \in \mathcal{L}$, we have

(S1): $pr_2 \circ \mathtt{I}_\mathtt{I}((u_1, u_2), (v_1, v_2)) = 0.$

(S2): $pr_2 \circ \mathtt{I}_\mathtt{I}((u_1, u_2), (v_1, v_2)) = 1.$

(S3): $pr_2 \circ \mathtt{I}_\mathtt{I}((u_1, u_2), (v_1, v_2)) = \begin{cases} 0, & v_2 = 0, \\ 1, & v_2 > 0. \end{cases}$

(S4): $pr_2 \circ \mathtt{I}_\mathtt{I}((u_1, u_2), (v_1, v_2)) = \begin{cases} 0, & v_1 = 1, \\ 1, & v_1 < 1. \end{cases}$

(S5): $\exists\, c \in [b/a, \infty)$ such that

$$pr_2 \circ \mathtt{I}_\mathtt{I}((u_1, u_2), (v_1, v_2)) = s_2^{-1}(\min(c s_1(v_2), b)).$$

(S6): $pr_2 \circ \mathtt{I}_\mathtt{I}((u_1, u_2), (v_1, v_2)) = \begin{cases} s_2^{-1}(\min(c t_1(v_1), b)), & v_1 = 1 - v_2, \\ 1, & v_1 < 1 - v_2. \end{cases}$

(S7): $pr_2 \circ \mathtt{I}_\mathtt{I}((u_1, u_2), (v_1, v_2)) = \begin{cases} s_2^{-1}(\min(c t_1(v_1), b)), & v_2 = 0, \\ 1, & v_2 > 0. \end{cases}$

(S8): $pr_2 \circ \mathtt{I}_\mathtt{I}((u_1, u_2), (v_1, v_2)) = s_2^{-1}(\min(c t_1(v_1), b)).$

(S9): $\exists\, c_1, c_2 \in [b/a, \infty),\ c_1 \neq c_2$ such that

$$pr_2 \circ \mathtt{I}_\mathtt{I}((u_1, u_2), (v_1, v_2)) = \begin{cases} s_2^{-1}(\min(c_1(t_1(v_1) - s_1(v_2)) + c_2(s_1(v_2)), b)), & v_1 > 0, \\ 1, & v_1 = 0. \end{cases}$$

Of course not every combination of the above solutions give a correct value in the set $\mathcal{L}$. For example when $pr_1 \circ \mathtt{I_I}((u_1, u_2), (v_1, v_2)) = \begin{cases} 1, & v_2 < 1, \\ 0, & v_2 = 1. \end{cases}$ and $pr_2 \circ \mathtt{I_I}((u_1, u_2), (v_1, v_2)) = \begin{cases} 0, & v_2 < 1, \\ 1, & v_2 = 1, \end{cases}$ for every $(u_1, u_2) \in \mathcal{L}$, then our (constant) solution is correct: $\mathtt{I_I}((u_1, u_2), (v_1, v_2)) = \begin{cases} 1_{\mathcal{L}}, & v_2 < 1, \\ 0_{\mathcal{L}}, & v_2 = 1. \end{cases}$

Also when $pr_1 \circ \mathtt{I_I}((u_1, u_2), (v_1, v_2)) = \begin{cases} 1, & v_1 = 1, \\ 0, & v_1 < 1. \end{cases}$ and $pr_2 \circ \mathtt{I_I}((u_1, u_2), (v_1, v_2)) = \begin{cases} 0, & v_1 = 1, \\ 1, & v_1 < 1, \end{cases}$ for every $(u_1, u_2) \in \mathcal{L}$, then our (constant) solution is correct: $\mathtt{I_I}((u_1, u_2), (v_1, v_2)) = \begin{cases} 1_{\mathcal{L}}, & v_1 = 1, \\ 0_{\mathcal{L}}, & v_1 < 1. \end{cases}$

But if $pr_1 \circ \mathtt{I_I}((u_1, u_2), (v_1, v_2)) = \begin{cases} 1, & v_2 < 1, \\ 0, & v_2 = 1. \end{cases}$ and $pr_2 \circ \mathtt{I_I}((u_1, u_2), (v_1, v_2)) = \begin{cases} 0, & v_1 = 1, \\ 1, & v_1 < 1. \end{cases}$ for every $(v_1, v_2) \in \mathcal{L}$, then our solution is incorrect, since $\mathtt{I_I}((u_1, u_2), (v_1, v_2)) = \begin{cases} 1_{\mathcal{L}}, & v_1 = 1, v_2 = 0, \\ 1_{\mathcal{L}}, & v_1 = 0, v_2 = 1, \\ (1,1), & 0 < v_1 < 1, 0 < v_2 < 1, v_1 + v_2 \leq 1, \end{cases}$ is not solution in $\mathcal{L}$ (since $(1,1) \notin \mathcal{L}$). Similarly, we can find the possible combinations of the above solutions give a correct value in the set $\mathcal{L}$ is the required result (ii). □

Remark 7.4.13. *If we put $\alpha = u$, $\beta = \mathtt{f_I}(v)$ and $\mathtt{I_I} = \mathtt{I_{I(f_I,\omega)}}$ in Proposition 7.4.12, then the above possible forms of $\mathtt{I_I}(u, v)$, for fixed $u \in \mathcal{L}$, convert into corresponding forms of $\omega(\alpha, \beta)$.*

7.5 Concluding remarks

In this chapter, a new type of IFI known as $(\mathtt{f_I}, \omega)$-implication is introduced which is a generalized form of Yagers f-implications in IFE, and it is different from $(\mathcal{S}, \mathcal{N})$-, $\mathcal{R}$-, $\mathcal{QL}$-implications. We have also discussed some properties of the $(\mathtt{f_I}, \omega)$-implication. It is showed that the implications that satisfy (NP) and the $(\mathtt{f_I}, \omega)$-implication are equivalent. For flexibility and applications' point of views, this implication is interesting as well as important. After that, the distributivity of IFSs over t-representable t-norms and t-conorms for the cases of nilpotent and strict t-norms are discussed corresponding to IFIs as well as $(\mathtt{f_I}, \omega)$-implication. Also, we have solved the open problems posed by Baczyński [17, 18]. This work is useful for many application areas, like, fuzzy control, approximate reasoning etc.

Chapter 8

Conclusions and future research scope

This chapter configures the concluding part of the Thesis and also proposes some suggestions towards which the present work can be further continued. It consists of two sections; Section 8.1 brings out the overall conclusions of the research work carried out in this thesis and in Section 8.2 suggestions regarding the future research directions and possible extensions of the work presented in the thesis are made.

8.1 Conclusion

The aim of the work is to develop new methodologies for solving various optimization problems in IFE and to analyze the algebraic study of implication operators in IFE.

The overall conclusions of the thesis are as below:

- The product of unrestricted LR-type IFNs are proposed with the help of $\alpha-$cut, $\beta-$cut and $(\alpha, \beta)-$cut.

- A new method is proposed for solving unrestricted LR-type FIFPPs with the help of the proposed product of unrestricted LR-type IFNs, score index, and accuracy index.

- There exist several FIFLPPs which can not be solved by the existing methods but can be solved by the proposed method. Hence, the proposed method is better than the existing methods [9, 137, 165, 189] for solving FIFLPPs.

- The primal-dual problems, discussed in [24, 85], are extended in IFE by taking membership and non-membership functions governed by reference functions in different approaches, viz., pessimistic, optimistic and mixed.

- Developed an algorithm to model and solve MOLPPs using accuracy index and value function from different viewpoints, viz., optimistic, pessimistic and mixed, and compared it with the Zimmermann's technique, γ-connective and minimum bounded sum operator. Such conflicting optimization problems arise very usually in manufacturing, planning and scheduling systems.

- The definitions of normalized divergence, similarity, dissimilarity, inclusion, and normalized distance measures in IFE are analyzed.

- We have established the following:

    (i) the IF point measures generated from the measures of the standard IFSs constructed by level sets and other special set $\tilde{A}_p^I$,

    (ii) the measures derived from point measures,

    (iii) aggregated measures from the set of measures and studied the continuity relationship between them.

- We have given the concept of weights for one and many preferences of alternatives.

- We have modeled the mathematical programming problems for determining the positive certain attribute weights.

- An algorithm is given for the selection of the best alternative from the given set of feasible alternatives with given preferences.

- $\mathcal{T}$-power-based implications as a new class of implication operators on $\mathcal{L}$ is introduced and studied properties of these implications.

- We have observed that some of the properties of fuzzy implications acting on the real unit interval [0, 1] are not satisfied by related $\mathcal{T}$-power-based implications acting on $\mathcal{L}$.

- We have shown that the studied $\mathcal{T}$-power-based implications on $\mathcal{L}$ satisfy the discussed properties by addition of some extra conditions. Also, the string of inequality of $\mathtt{I}_{\mathtt{I}\mathcal{T}}$ has been established.

- We have also introduced a new type of negation $\mathcal{N}^{\alpha}_{\mathtt{I}_{\mathtt{I}\mathcal{T}}}$ based on $\mathtt{I}_{\mathtt{I}\mathcal{T}}$. Continuity and strict monotonicity of this negation are analyzed.

- We have investigated the solutions of Boolean-like laws in $\mathtt{I}_{\mathtt{I}\mathcal{T}}$.

- A new type of IFI known as $(\mathtt{f_I}, \omega)$-implication is introduced which is a generalized form of Yager's f-implications in IFE, and shown that it is different from $(\mathcal{S}, \mathcal{N})$-, $\mathcal{R}$-, $\mathcal{QL}$-implications.

- We have also discussed some properties of the $(\mathtt{f_I}, \omega)$-implication. It is shown that the implications that satisfy (NP) and the $(\mathtt{f_I}, \omega)$-implication are equivalent.

- From flexibility and applications point of views, this implication is interesting as well as important.

- We have solved distributive eqs. (7.9) - (7.12) over t-representable t-norms and t-conorms for the cases of nilpotent and strict t-norms corresponding to $(\mathtt{f_I}, \omega)$-implication.

8.2 Future scope

There are several interesting directions for further research and development based on the work in this thesis. Some of the suggestions for future work are as follows:

- Work on unrestricted LR-type fully IF matrix equations based on the product of unrestricted LR-type IFNs.

- Work on group decision making problems in IFE based on the membership and non-membership functions governed by the reference functions as well as point operator F_α.

- Investigation of all IF-measures based on implication operators on $\mathcal{L}$ and development of interrelationships between them.

- Development of preference relations in IFE and to apply in decision making theory.

- The study of the implication operators from analytic and algebraic point of views, and their applications to preference analysis.

- Application of implication operators to optimization problems in IFE.

- The study of the aggregation operator in IFE on the basis of decision making theory.

- Development of the implication operators based on aggregation operators in IFE and the study of these relations from theoretical and practical point of views.

- Development of the powers of aggregation operators, the implications based on powers of aggregation operator and their uses for MCDM / MADM problems in IFE.

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