SOME PROBLEMS ON ESTIMATION AFTER SELECTION

Ph. D. THESIS

by

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INDIAN INSTITUTE OF TECHNOLOGY ROORKEE
ROORKEE - 247 667 (INDIA)

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A THESIS

Submitted in partial fulfilment of the requirements for the award of the degree

of

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in

MATHEMATICS

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CANDIDATE'S DECLARATION

I hereby certify that the work which is being presented in the thesis entitled "**SOME PROBLEMS ON ESTIMATION AFTER SELECTION**" in partial fulfilment of the requirements for the award of the Degree of Doctor of Philosophy and submitted in the Department of Mathematics of the Indian Institute of Technology Roorkee, Roorkee is an authentic record of my own work carried out during a period from July, 2013 to March, 2018 under the supervision of Dr. Aditi Gangopadhyay, Associate Professor, Department of Mathematics, Indian Institute of Technology Roorkee, Roorkee.

The matter presented in this thesis has not been submitted by me for the award of any other degree of this or any other Institution.

(KALU RAM MEENA)

This is to certify that the above statement made by the candidate is correct to the best of my knowledge.

(Aditi Gangopadhyay)
Supervisor

The Ph.D. Viva-Voce Examination of **Mr. Kalu Ram Meena**, Research Scholar, has been held on…………………………

Chairman, SRC **Signature of External Examiner**

This is to certify that the student has made all the corrections in the thesis.

Signature of Supervisor **Head of the Department**

Dated:………………

Dedicated

to

My Parents

Abstract

In the practical world of everyday life, full of complexities, we face baffling problem of making the myriad of judgments. Every situation needs formulation of the problem under an appropriate theoretical model followed by sifting and a careful analysis of the evidence leading to a conclusion whose validity is based on reason. In the progression of scientific reasoning, statistical inference provides the methodology developed to meet these requirements. In many practical situations it may be of interest to select the best (or worst) of k (≥ 2) available populations (or options), where the quality of a population is defined in terms of unknown parameters associated with it. In the statistical literature, these type of problems are classified as "Ranking and Selection Problem". A problem of practical interest after selection of the best (or worst) population, using a given selection procedure, is estimation of the worth of the selected population. In the statistical literature, these problems are called "Estimation After Selection Problems". In this thesis, we study this problem of estimating parameters of the selected population(s) for certain distributions. An application of this theory is shown in this thesis. Most of the previous works are studied under the squared error loss function. In this thesis, some problems are studied under some other loss functions.

In this thesis, we study this problem of estimating parameters of the selected population(s) for certain distributions. A brief summary of the thesis is give below.

In **chapter 1**, a review of available work on the problem of estimation after selection is given. A summary of the results in the thesis is also given.

In **Chapter 2**, some basic definitions results and techniques are explained which are of use in this thesis.

In **Chapter 3**, two normal populations with different unknown means and same known variance are considered. The population with the smaller sample mean is selected. Various estimators are constructed for the mean of the selected normal population. Finally, we are compared with respect to the bias and Mean Squared Error (MSE) risks by the method of Monte-Carlo simulation and their performances are analyzed with the help of graphs.

In **Chapter 4**, we consider two competing pairs of random variables (X,Y_1) and (X,Y_2) satisfying linear regression models with equal intercepts. We describe the model which connects the

selection between two regression lines with the selection between two normal populations for estimating regression coefficients of the selected regression line. We apply this model to a problem in finance which involves selecting security with lower risk. We assume that an investor being risk averse always chooses the security with lower risk (or, volatility) while choosing one of two securities available to him for investment and further is interested in estimating the risk of the chosen security. We construct several estimators and apply the theory to real data sets. Finally, graphical representation of the results is given.

In **Chapter 5**, independent random samples are drawn from two normal populations with same unknown mean and different unknown variances. The population corresponding to the smallest sum of the squared deviations from the mean is selected as the best population. We consider estimation of quantiles of the selected population. Admissible class of estimators for the quantile of the selected population is found in certain subclasses of estimators. The biases and mean squared error risks of these estimators are compared numerically by Monte-Carlo simulation. Finally, the biases and risks of different estimators are represented by graphs.

In **Chapter 6**, we consider independent random samples $X_{i1},\ldots,X_{in}$ drawn from $k(k \geq 2)$ population $\Pi_i, i=1,...,k$. The observations from Π_i follows Pareto distribution with an unknown scale (θ_i) and common known shape parameters. In this chapter, estimation of an unknown scale parameter of the selected population from the given k Pareto population are discussed. The uniformly minimum risk unbiased (UMRU) estimator of scale parameter of the population corresponding to the largest and smallest θ_i, are determined under the Generalized Stein loss function. Sufficient condition for minimaxity of an estimator of θ_L(scale parameter of the population corresponding to the largest θ_i) and θ_S (scale parameter of the population corresponding to the smallest θ_i) are given, and we determine that the generalized Bayes estimator of θ_S is minimax for $k=2$. Also, found the class of linear admissible estimators of $\theta_L(\theta_S)$. Further, we demonstrate that the UMRU estimator of θ_S is inadmissible. Finally some results and discussions are reported.

In **Chapter 7**, we consider $\Pi_1,\ldots,\Pi_k$, k (≥ 2) independent populations, where Π_i follows the uniform distribution over the interval $(0,\theta_i)$ and $\theta_i > 0$ $(i=1,\ldots,k)$ is an unknown scale parameter. The population associated with the largest scale parameter is called the best population. The problem of estimating the scale parameter θ_L of the selected uniform population when sample sizes are unequal and the loss is measured by the squared log error (SLE) loss function is considered. We derive the uniformly minimum risk unbiased (UMRU) estimator of θ_L under the SLE loss function and two natural estimators of θ_L are also studied. For $k=2$, we derive a sufficient condition for inadmissibility of an estimator of θ_L. Using these conditions, we conclude that the UMRU estimator and natural estimator are inadmissible. Finally, the risk functions of various competing estimators of θ_L are compared through simulation.

In **Chapter 8**, $k(\geq 2)$ independent uniform populations, over the interval $(0, \theta_i)$ and $\theta_i > 0$ ($i = 1, \ldots, k$) be an unknown scale parameter, are considered . In this chapter, we consider the problem of estimating the scale parameter θ_L of the selected uniform population when sample sizes are unequal, and the loss is measured by the generalized stein loss (GSL) function. The uniformly minimum risk unbiased (UMRU) estimator of θ_L is derived, and two natural estimators of θ_L are also studied under the generalized stein loss (GSL) function. The natural estimator $\xi_{N,2}$ is proved to be the generalized Bayes estimator with respect to a noninformative prior. For $k = 2$, we give a sufficient condition for inadmissibility of an estimator of θ_L and show that the UMRU estimator and natural estimator are inadmissible. A simulation study is also carried out for the performance of the risk functions of various competing estimators. Finally some results and discussions are reported.

Finally, **Chapter 9** presents the summary and concluding remarks of this thesis and the possible directions of the future scope.

List of Research Papers

Published/Communicated

1. **Kalu Ram Meena**, Aditi Kar Gangopadhyay, Satrajit Mandal, "Estimation of the Mean of the Selected Population". World Academy of Science, Engineering, and Technology, International Journal of Mathematical, Computational, Physical, Electrical and Computer Engineering, Vol: 8(12), 1480-1485, 2014.

2. **K. R. Meena**, Aditi Kar Gangopadhyay, "Estimating Volatility of the Selected Security". American Journal of Mathematical and Management Sciences. Vol: 36(3), 177-187, 2017.

3. **K. R. Meena**, Mohd. Arshad and Aditi Kar Gangopadhyay, "Estimating the Parameter of Selected Uniform Population under the Squared log error Loss Function." Communications in Statistics-Theory and Methods, Vol: 47(7), 1679-1692, 2018.

4. Aditi Kar Gangopadhyay, **K. R. Meena**, S. Kumar, "Estimating Quantile of the Selected Normal Population" Submitted.

5. **K. R. Meena**, Aditi Kar Gangopadhyay, "Admissible and Minimax Estimators of the Parameter of the Selected Population under the Generalized Stein Loss Function" Submitted.

6. **K. R. Meena**, Aditi Kar Gangopadhyay, "Estimating the Parameter of Selected Uniform Population Under the Generalized Stein Loss Function" Submitted.

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With devotion and profound gratitude, I dedicate this thesis to my parents.

Roorkee (Kalu Ram Meena)

March , 2018

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Chapter 1

Introduction

1.1 Introduction

Let $\Pi_1, \ldots, \Pi_k$ be k populations with associated probability distribution being characterized by the parameters $\theta_1, \ldots, \theta_k$ respectively. Frequently, one is interested in selecting the best population or a subset of populations containing the best. The population is termed the best according to some characteristic such as the largest mean or the smallest variance etc. Some typical examples of practical interest are:

1. Out of different surgical strategies offered for the treatment of a particular disease, a surgeon would like to use that surgical methodology on patients that have the highest success rate. Here, the success rate of the strategy is measured by the time required for the patient to ensure the disease and be cured or on the proportion of the patients successfully recovered from the disease.

2. A farmer having many choices of fertilizers accessible to him would like to pick the one which can offer him the maximum yield.

3. An army chief will prefer to choose the most effective quality guns for his army, here the best may be decided on the basis of the proportion of successful hits or simple maneuverability of the gun carriage.

4. An investor would like to buy stocks of the companies which are expected to yield higher returns over next few years.

5. There are number of organic and chemical fertilizers that can be used for some crop. An agricultural farm proprietor wants to select a fertilizer for his /her crop that will provide maximum yield and also preserve the soil quality over a period of time.

In the statistical literature, these type of problems are commonly classified as Ranking and Selection Problems. The initial formulation of the ranking and selection problem was given by Bechhofer [23] and Gupta [48]. For a detailed discussion of the literature of these type of problems one may refer to Gibbons et al. [43] and Gupta and Panchapakesan [54]. A complete bibliography of the literature on these problems is given in Kulldorf [64] and Dudewicz and Koo [39]. Estimating parameters of the selected population (representing the best population Π_k) or to estimate a characteristic of the selected subset of the populations (the subset that involves the best population Π_k) is an important practical problem. For example:

1. The surgeon, after selecting the best surgical strategies for the treatment of a particular disease, would naturally be interested in having an estimate of the average success rate of the selected surgical procedure.

2. The farmer whereas applying the most effective chosen fertilizer to the forthcoming crop, would undoubtedly be keen to estimate the yield.

3. Similarly, the army chief would like to have an estimate of the effectiveness of the gun he has selected.

4. An investor will like to know the estimated returns from his/her investments are the best selected shares of the companies.

5. The farm proprietor will like to have an estimate of the expected yield if he/she is using the best fertilizer.

Therefore, for all such real-life applications, there's a requirement to develop estimators of parameters of the selected population from the populations under consideration. These type of problems are commonly referred to as "Estimation After Selection".

The problem of estimation after selection differs from the classical estimation problem in some basic sense. In the classical estimation problem, we are to estimate the parameter of the given population on the basis of a random sample from that population itself. The parameter to be estimated is a fixed quantity, and so for an unbiased estimator T of $g(\theta)$ we must have $E_\theta(T) = g(\theta)$. However, the parameter to be estimated in the estimation after selection problem is a random quantity say θ_J, where θ_J is parameter of the selected population. Therefore in the case of selection problem, for an unbiased estimator T of θ_J we must have $E(T - \theta_J) = 0$. Thus implying that the unbiased estimator of θ_J is nothing but the unbiased estimator of $E(\theta_J)$ and a UMVUE of θ_J is a UMVUE of $E(\theta_J)$. Moreover, In classical estimation problem, the mean squared error (MSE) of an unbiased estimator say, U_0 of θ_0 is its variance. That is,

$$
\begin{aligned}
MSE(U_0, \theta_0) &= E_{\theta_0}(U_0 - \theta_0)^2 \\
&= V_{\theta_0}(U_0)
\end{aligned}
$$

Therefore, if a UMVUE of θ_0 exists, then it is uniformly better than any other unbiased estimator. While in estimation after selection problem, the MSE of an unbiased estimator T of θ_J is not same as its variance. In fact

$$
\begin{aligned}
MSE(T, \theta_J) &= E(T - \theta_J)^2 \\
&= V_{\underline{\theta}}(T) + V_{\underline{\theta}}(\theta_J) - 2Cov(T, \theta_J),
\end{aligned}
$$

where $Cov(T, \theta_J)$ is the covariance between T and θ_J.

For uniformity in our presentation we will adopt the following notations throughout the thesis:

(i) $\Phi(.)$: Distribution function of normal distribution with mean 0 and variance 1.

(ii) $\phi(.)$: Standard normal density function.

(iii) $I(A)$: Indicator function of statement A, i.e.,

$$
I(A) = \begin{cases} 1 & \text{if } A \text{ is true} \\ 0 & \text{otherwise} \end{cases}
$$

(iv) $\mathbb{R}$: The real line, i.e., $(-\infty, \infty)$.

(v) $\mathbb{R}^k$: The k-dimensional Euclidean space.

(vi) $\mathbb{R}_+$: The positive real line, i.e., $(0, \infty)$.

(vii) ${\mathbb{R}_+}^k$: product space $\underbrace{\mathbb{R}_+ \times \cdots \times \mathbb{R}_+}_{k-\text{times}}$, i.e., $(0, \infty)^k$.

1.2 A Review of the Literature

In this section, we present the literature survey, in brief on the problems of estimation after selection that are related to our study. In this review, we have also included few papers which may not be directly related to our study. The problem of estimation after selection was initially formulated and investigated by Rubinstein [111, 112]. He studied the problem in the context of reliability estimation where the problem considered by him is in connection with a particular sequential scheme for selecting the components in a manufacturing process. The unbiased estimators were derived by him for the failure rates of the selected components. The generalized method given by him obtains unbiased estimators of selected Poisson parameters corresponding to a broad class of selection procedures.

1.2.1 Normal Population

The problem of estimating mean of the selected normal population was considered by Stein [122] with independent normal populations having unknown means and common known variance. One observation is taken from each of these k populations, and the population corresponding to the largest observation was selected. He considered estimating mean of the selected normal population with respect to the squared error loss function. He noted that $X_{(1)}$ is a generalized Bayes estimator when the generalized prior distribution is the Lebesgue measure. For $k = 2$, Stein showed that $X_{(1)} = \max(X_1, X_2)$ is admissible as well as minimax estimator, where X_i is $N(\theta_i, 1), i = 1, ...k$. However, Stein remarked for $k = 2$ and especially for $k > 2$, the estimator $X_{(1)}$ is positively biased and that its bias tends to infinity where the means of the populations are equal or close. The inadmissibility of $X_{(1)}$ was also conjectured in general for $k \geq 3$. However, this conjecture was later disproved by Brown in 1987. Brown proved that $X_{(1)}$ is admissible for any k.

Sarkadi [116] introduced the fixed-sample selection approach of one out of two populations and derived the unbiased estimator for parameter of the selected Poisson population. He also discussed the estimation of mean M of the selected normal population. For this, he took two normally distributed populations with common known variance and selected the one with the smallest sample mean. For estimating mean M of the selected population, he proposed an estimator t_c of the form

$$t_c = Y_2 + Y\Phi\left(\frac{cY}{\sigma}\right) - c\sigma\phi\left(\frac{cY}{\sigma}\right), \; c \geq 0,$$

where, c is an arbitrary constant, and Y_i is the mean of the sample drawn from the normal population $N(\theta_i, \tau^2)$, $i = 1, 2$. $Y = Y_1 - Y_2$, $\sigma^2 = V(Y) = \frac{2\tau^2}{n}$, and $\phi(x)$ and $\Phi(x)$ are the probability density function and cumulative distribution function of standard normal variate respectively. The expectation of the estimator t_c is given by

$$E(t_c) = \theta_2 + \theta\Phi\left(\frac{\theta c}{\sigma\sqrt{1+c^2}}\right),$$

where $\theta = \theta_1 - \theta_2$ is the mean of the variable Y. Also, he pointed out that $E(t_c)$ approaches to $E(M) = \theta_2 + \theta\Phi\left(\frac{\theta}{\sigma}\right)$ where c approaches to a large value. Hence showed that the bias of t_c as an estimator of M can be controlled by making c large. Hence, for large values of c, t_c becomes unbiased estimator of M. Sarkadi also provided a lower bound for the mean squared error (MSE) of the proposed estimators and stated that an exact expression is hard to obtain.

Putter and Rubinstein [105] considered estimating mean of the selected normal population studied out of k normal populations $\Pi_1, \Pi_2, \ldots, \Pi_k$. Let Y_i denote the mean of the random sample of size n from the population Π_i $(i = 1, 2, \ldots, k)$ and suppose the population corresponding to $Y_{max} = max(Y_1, Y_2, \ldots, Y_k)$ is selected. Putter and Rubinstein showed that Y_{max} is a positively biased estimator of M, the mean of the selected population. They also proved that the bias of Y_{max} for $k = 2$ is a decreasing function of $|\theta_1 - \theta_2|$ for the symmetric location parameter family. Further, for the case of two populations, they proved the non existence of unbiased estimator of M. The authors also proposed estimators of the form

$$\hat{M}_\lambda \;\;=\;\; Y_{max} - \lambda\sigma\phi\left(\frac{Y}{\sigma}\right), \;\; \lambda \geq 0,$$

which is just a decrement of $Y_{\max}$ by the λ multiple of its estimated bias. Also, it was shown that $Y_{\max}$ is the unique minimax estimator of M under the MSE criterion, whose MSE never exceeds σ^2.

Dahiya [36] also studied the problem of estimating mean of the selected normal population under the same model as was considered by Sarkadi [116]. He selected the population corresponding to the largest sample mean. He proposed some more estimators for the mean of the selected normal population. Dahiya noticed that the maximum likelihood estimator (MLE) of $E(M) = \theta_2 + \theta\,\Phi\left(\frac{\theta}{\sigma}\right)$ is $T = Y_2 + Y\Phi\left(\frac{Y}{\sigma}\right)$ for estimating mean of the selected population. Motivated by the concept of Putter and Rubinstein [105], used to obtain the estimator $\hat{M}_\lambda$ from the estimator Y_{max}. Dahiya studied a more general estimator T_λ:

$$T_\lambda = T - \lambda\left[\phi\left(\frac{Y}{\sqrt{2}\sigma}\right)\frac{\sigma}{\sqrt{2}} + Y\left\{\Phi\left(\frac{Y}{\sqrt{2}\sigma}\right) - \Phi\left(\frac{Y}{\sigma}\right)\right\}\right],$$

where $\lambda \geq 0$ is an arbitrary constant. The concept of estimator T_λ is similar to $\hat{M}_\lambda$. He obtained from the estimator T by subtracting a λ multiple of the estimated bias $B_{\underline{\theta}}(T)$ of T from itself.

Blumenthal and Cohen [26] proposed an estimator H_c to estimate $\theta_{max} = \max(\theta_1, \theta_2)$ and some times H_c is called the hybrid estimator. Dahiya [36] also studied the hybrid estimator H_c having the form

$$H_c = \begin{cases} \frac{Y_1 + Y_2}{2} & \text{if } |Y_1 - Y_2| < c\sigma \\ Y_{max} & \text{if } |Y_1 - Y_2| \geq c\sigma\ , \end{cases}$$

where $c \geq 0$ is an arbitrary constant. For $c = 0$, H_c reduces to Y_{max}, the same way as $\hat{M}_\lambda$ reduces to Y_{max} for $\lambda = 0$.

Dahiya also derived the exact expressions for the biases and MSE'S of the five estimators $t_c, \hat{M}_\lambda, T, T_\lambda$ and H_c and compared them numerically.

The above estimation problem was extended by Hsieh [58] for common but unknown variance τ^2. The author used the natural selection rule and selected the population yielding the largest sample mean and studied estimators similar to those in Dahiya [36]. Mainly, he modified the estimators of Dahiya by replacing the minimum variance unbiased estimator (UMVUE) of σ in place of σ. The modification of T_λ of Dahiya [36] was not included as there is no immediate justification for using this estimator when τ^2 is unknown. Derivations for the biases and MSE's of the estimators were obtained and compared numerically. His conclusions were close to those reported by Dahiya.

The problem of estimating mean of the selected normal population was extensively studied by Cohen and Sackrowitz [34] for general k normal populations with the common known variance σ^2. Authors mainly studied for the cases $k \geq 3$ and a family of estimators $d_k(\underline{x})$ has been produced, where $d_k(\underline{x})$ is of the form

$$d_k(\underline{x}) = \sum_{i=1}^{k} c_{(i),k}\, x_{(i)}, \tag{1.1}$$

with $c_{(1),k} \geq c_{(2),k} \geq \ldots \geq c_{(k),k}$ depending upon $\hat{r}_i$ which are non-negative functions of $x_{i+1} - x_i$ and $x_{(1)} \geq x_{(2)} \geq \ldots \geq x_{(k)}$ are the ordered statistics of $x_1, x_2, \ldots, x_k$. Authors also proved that the estimators $d_k(\underline{x})$ as expressed in (1.1) possess some desirable properties. Further, they are empirical Bayes estimators with respect to multivariate normal priors having the mean vector zero and the covariance matrices members of a set of k possibilities. For every possible covariance matrix is such that the means are correlated in a way to reflect the possibility that some or all of them are close together. In particular, they considered six types of $\hat{r}_i$ and numerically studied the corresponding six estimators $T_i\,(i = 1, 2, \ldots, 6)$ and $X_{(1)}$ with respect to the bias and MSE's using Monte Carlo Method for the values of $k = 3(2)7$. Based on numerical values, authors concluded that T_2 is performs better than any other estimator from several points of view and hence recommended it for practical use. It was also shown that the risk of $X_{(1)}$ is maximized when the means θ_i's are equal. For $k = 2$, the estimator $\hat{M}_\lambda$ (by Putter Rubinstein) and the estimator T_λ (by Dahiya) were proved to be inadmissible.

Further, a general problem of estimating mean of the selected normal population was introduced by Hwang [60]. Actually, he completely studied on this problem. Let $X_{(i)}$ be the ith order statistic and $\theta_{(i)}$ the mean associated with $X_{(i)}$. He considered estimation of $\theta_{(i)}$'s and introduced three natural selection criteria: the large sample consistency, the large population consistency and the boundedness of the risk for any fixed k. It is detected that the existing estimators by Dahiya [36], Hsieh [58] and Cohen and Sackrowitz [34] for $\theta_{(k)}$ do not satisfy a minimum of one amongst these three criteria. Hwang proposed an empirical Bayes estimators for $\theta_{(i)}$ for the case of common known variance σ^2. The empirical Bayes estimator is given as follows

$$\delta_{(i)}^{EB} = \bar{X} + \left[1 - \frac{(k-3)\,\sigma^2}{\Sigma(X_i - \bar{X})^2}\right] + \left(X_{(i)} - \bar{X}\right), \tag{1.2}$$

where for any number $a, a_+ = \max(a, 0)$. The author showed that this empirical Bayes estimator satisfies all the three criteria as described above. Further author proved that when $\theta_{(i)}$'s are equal, the estimator (1.2) has zero bias. That is, the estimator $\delta_{(i)}^{EB}$ is the unbiased estimator when the means are equal. Also, its performance is better than all the existing estimators for most of the parameter values that he considered.

Venter [133] introduced a new problem for estimating mean of the selected population. He considered k normal populations and ith population which is distributed $N(\theta_i, \sigma^2), i = 1, \ldots, k$ where σ^2 is known. He took a classical approach and proposed some new estimators. Let X_l denotes the largest sample mean, where $X_l = \max_i X_i$, where X_i denotes the sample mean of the ith population. Then author reduced the bias of X_l by estimating the bias and then subtracting it from itself. This leads to a new class of estimators of mean of the selected normal population and called bias reducing (BR) estimators which are defined as $X_l - b(X, a)$, where

$$b(X, a) = a \sum_{i=1}^{k} \int_{-\infty}^{\infty} z\phi(z) \Pi_{r \neq i} \Phi(z + a(X_i - X_r)) dz. \tag{1.3}$$

These estimators depend upon the term $b(X, a)$ which in turn depends upon the constant 'a' as is mentioned in (1.3). Therefore, the amount of bias reduction of X_l while using BR estimators is controlled through the constant 'a'. Constant 'a' also effects the mean squared error (MSE) of BR estimators. For $k = 2$, Venter selected some values of 'a' such that the MSE of $X_l - b(X, a)$ did not exceed by a preselected amount. The author also compared the MSE's of BR estimators $X_l - b(X, a)$ to that of cohen and sackrowitz's estimators. Further, he reported that the performances of both the classes of estimators are quite similar and no class dominates the other class.

Let $X_1, X_2, \ldots, X_k$ be k independent random variables with distributions of $X_j\,(j = 1, \ldots, k)$ belonging to one parameter families $f_{X_j}(x_j, \theta_j), j = 1, 2, \ldots, k$. Estimating mean of the selected population was introduced by Cohen and Sackrowitz [35] and derived some new estimators for the problem under consideration. They selected the population corresponding to the maximum X_i. That is, the population M is selected if $X_M = \max(X_1, \ldots, X_k)$. Estimators based on a two-stage sample are offered that are conditionally unbiased where the conditioning is on the ordering of the sample means $(X_1, \ldots, X_k)$ computed from the first stage of sampling. Conditionally unbiased estimators are also unconditionally unbiased. Cohen and Sackrowitz [35] proposed a two-stage sample where observations at stage two are taken from the selected population only. They obtained the uniformly minimum variance conditionally unbiased estimators for the normal (with known and unknown variance) and gamma cases. Further, in one parameter family cases, they observed that the uniformly minimum variance conditionally unbiased estimators depend only on $X_{(1)}$ and $X_{(2)}$.

The above estimation problem was continued to study by Sill and Sampson [118] for the bivariate normal distributions where mean vector $\boldsymbol{\mu_i^T} = (\mu_{1i}, \mu_{2i})$ and the covariance matrix is Σ/n_A. Let the vector $\boldsymbol{X_i^T} = (X_{1i}, X_{2i})$ denote the mean responses based upon n_A experimental units from the population where A is called the first stage of data collection τ_i. Let M be the population corresponding to the largest value of the sample means of the surrogate variables, $X_{1i}, i = 1\ldots,k$, which is used for population selection. The population τ_M is selected if it yields the largest response, i.e., $X_M = \max\{X_{11}, X_{12}, \ldots, X_{1k}\}$. After making this selection, n_B the collected additional independent observations from population M and B are called the second stage data collections. Let $\boldsymbol{Y_i^T} = (Y_1, Y_2)$ be the mean response from population τ_M in Stage B then that $\boldsymbol{Y}$ follows the bivariate normal distribution with mean vector $\boldsymbol{\mu}_M = (\mu_{1M}, \mu_{2M})^T$ and the covariance matrix Σ/n_B. They found the good unbiased estimators of the parameter μ_{2M} associated with the primary outcome variable, X_{2M}.

Further, estimating mean of the selected population from two normal populations with unknown means and common known variance has been addressed by Parsian and Farsipour [104] under the criterion of bias and LINEX loss function. They proposed seven different estimators for the mean of the selected population. They found expressions for the biases and risk functions of these estimators and compared numerically. Their results were further extended by Misra and van der Meulen [87]. They provided some admissibility results for a subclass of equivariant estimators, and a sufficient condition for the inadmissibility of equivariant estimators. It was proved that several estimators proposed by Parsian and Farsipour [104] were inadmissible and better estimators were obtained. Some further work on normal populations has been done by Qomi et al. [106] and Mohammadi and Towhidi [93].

1.2.2 Nonnormal Populations

The discussion so far was on normal populations only. The major work on estimating mean of the selected negative exponential population was developed by Sackrowitz and Samuel-Cahn [113]. Suppose there are k observations $X_i, i = 1,\ldots,k$ which are independent negative exponentially distributed with unknown expectations $\lambda_i, i = 1,\ldots,k$. They define the random variables $J(X_1,\ldots,X_k)$ and $M(X_1,\ldots,X_k)$ of maximal and minimal observations, respectively. Authors considered estimation of λ_J and λ_M, which denote the scale parameters associated with the largest and the smallest of the observations $X_1, \ldots, X_k$. Authors also studied the class of linear estimators of the form

$$\phi(\boldsymbol{X}) = \Sigma_{i=1}^{k} \alpha_i X_{(i)},$$

where $\alpha_1, \ldots, \alpha_k$ are fixed constants. They also found that the estimator $\phi_1(\boldsymbol{X}) = X_{(1)} - X_{(2)}$ is conditionally unbiased estimator as well as unique uniformly minimum variance unbiased estimator of λ_J, for $k = 2$. It is also shown that the estimator $\phi_2(\boldsymbol{X}) = kX_M$ is the unique uniformly minimum variance unbiased estimator for λ_J and proved that no conditionally unbiased estimator for λ_J exists. Further, authors have investigated some admissibility and minimaxity results in a certain class of estimators as well as discussed some other intuitively appealing estimators.

Sackrowitz and Samuel-Cahn [114] developed some general results for Bayes and minimax estimators of parameters of the selected population.

The problem of estimating mean of the selected uniform population has been investigated by Vellaisamy et al. [130]. Let Π_i be uniform population over the interval $(0, \theta_i), i = 1, \ldots, k$. Random sample of size n is drawn from each of the k populations and let Y_i denotes the largest observation of the sample drawn from the ith population. For selecting the best population, that is the one associated with the largest θ_i, the natural selection rule is to select the population corresponding to the largest Y_i. They estimated mean M of the selected population and found that the natural estimator $T_1(Y) = (n_1)Y_{(1)}/2n$ is positively biased and derived the UMVUE of M using the (U,V) method of Robbins [110] and also studied its asymptotic distribution. They obtained a generalized Bayes estimator of M and shown its minimaxity for the cases $k \leq 4$ and a class of admissible estimators. Further, the improvement over UMVUE was also found for the case $k = 2$. Song [121] extended the minimaxity and inadmissibility results of [130] for the case $k \leq 2$.

Nematollahi and Motamed-Shariati [102] estimated scale parameters of the selected uniform population under the entropy loss function. They selected the population corresponding to the largest as well as the smallest scale parameter. They generalized the (U,V) methods of Robbins [110] as well as derived the uniformly minimum risk unbiased (UMRU) estimators for both the cases. They characterized the admissible estimators and derived minimax estimator, as well as inadmissibility result for the scale-invariant estimator of θ_L (scale parameter of the selected population when the population corresponding to the largest scale parameter is selected) and the dominated estimator was obtained for the case $k = 2$. The risks of all proposed estimators were compared numerically.

The problem of estimating mean of the selected gamma population has been initiated by Vellaisamy and Sharma [131]. They considered two gamma populations Π_1 and Π_2 with unknown scale parameters α_1 and α_2 and common shape parameter p where p is taken to be a known positive integer. Let Y_i denotes the sample mean based on a random sample of size n from the ith population. Following natural selection rule, the population corresponding to the larger Y_i is selected. The problem considered is to estimate mean M of the selected population. Authors showed that $Y_{(1)}$ is positively biased and obtained the UMVUE of M. Improvements over the natural estimator and the UMVUE were obtained. Minimaxity and non-minimaxity of these estimators were also examined. They proved that UMVUE is not minimax estimator.

Later on, these results are further generalized for $k(> 2)$ gamma populations by Vellaisamy and Sharma [132]. Further dominating estimators have been obtained in Vellaisamy [125]. He used the method of differential inequalities to derive these estimators.

Vellaisamy [126] has also derived some general results concerning the UMVUE of the selected parameter. For the squared error loss, the conditions under which the UMVUE is also uniformly minimum mean squared error unbiased estimator (UMMSEUE), have been obtained. As an application, the UMVUE of θ_I, the reciprocal of the natural parameter of the population selected from k independent populations where densities were belonging to one parametric continuous exponential family, has been also derived. Vellaisamy has found that the UMVUE of θ_I is a UMMSEUE. He also gave some examples.

Misra et al. [88] considered the problem of estimating scale parameter of the selected gamma population out of the gamma populations with unknown scale parameters and common known shape parameter under the scale invariant squared error loss function. Authors characterized admissible estimators within certain subclass of equivariant estimators. Sufficient conditions for the inadmissibility of invariant estimators of selected scale parameters were obtained. As a consequence, various natural estimators were found to be inadmissible and the sufficient conditions were used to improve various natural estimators. Some problems have been continued the study by Misra et al. [89] under the squared error loss function and analyzed results of Misra et al. [88].

Motamed-Shariati and Nematollahi [95] obtained the minimax estimator for scale parameter of the selected gamma population under the scale-invariant squared error loss function , where the shape parameter $\alpha > 0$ is arbitrary. This is generalization of the result obtained by Vellaisamy and Sharma [131] for integer value α. Estimating scale parameters of the selected gamma populations were studied by Nematollahi and Motamed-Shariati [101] under the entropy loss function. The Uniform Minimum Risk Unbiased (UMRU) estimator for the selected parameter was derived. For k=2, a certain class of linear admissible estimators was investigated, and inadmissibility of UMRU estimator was also proved. Qomi et al. [106] also considered the problem of estimating scale parameter of the selected gamma population under the reflected normal loss function.

The problem of estimating scale parameter of the selected Pareto population has been investigated by Kumar and Kar Gangopadhyay [66]. They considered k Pareto distributed populations with Π_i having unknown scale parameter α_i and known shape parameter β_i; $i = 1, \ldots, k$ and selected the population with the largest X_i, where X_i is the smallest observation of the sample from the ith population. Assuming the shape parameters to be equal, they derived the uniformly minimum variance unbiased estimator(UMVUE) for the scale parameter α_J of the selected population. An admissible class of linear estimators was derived in the class $\delta_c = cX_J$ with respect to the squared error loss function. Further, they also proved a general inadmissibility result for the scale equivariant estimators.

Al-Mosawi and Khan [5] studied the case of Pareto populations with the same known shape parameter and different scale parameters. For the best population, they used the natural selection rule, which selects the population associated with the largest X_i, where X_i is the smallest observation of the ith sample from the population $\Pi_i, i = 1, \ldots, k$. They estimated the moments of the selected population with respect to asymmetric scale invariant loss function. They showed that natural estimators are consistent estimators and considered a class of linear estimators of the moments of the selected population and derived the admissibility of natural estimators. Further, they investigated the performance of the estimators through the simulation study and compared for the selected values of the order of moments and shape parameter.

Nematollahi [99] has considered k independent Pareto populations with θ_i's, unknown scale parameters and β common known shape parameter and estimated parameter of the selected population under the squared log error loss function. He found the uniformly minimum risk unbiased (UMRU) estimator of selected parameters. A sufficient condition for minimaxity of estimators was obtained for the case $k = 2$. It is also proved that the UMRU and natural estimators of θ_J are minimax estimators. He also studied a subclass of admissible linear estimators and derived the sufficient condition for θ_J and the UMRU estimator of θ_J to be inadmissible. The risks of the proposed estimators were compared numerically.

Tappin [123] has considered discrete distributions for the problem of estimation after selection. She estimated the parameter of the selected binomial population. The selection rule considered is to select the population with the greatest number of successes and, in the case of a tie, to follow one of the two schemes: either choose the population with the smallest index or randomize among the tied populations. Tappin employed the second stage of sampling scheme and took additional observations on the selected population. She obtained the UMVUE under the first tie break scheme and proved that no UMVUE exists under the second. Further, an unbiased estimator is found in the case where no UMVUE exists.

Vellaisamy and Jain [129] have estimated parameter of the population selected from the discrete exponential family under the ϑ-normalized squared error loss function. They considered the estimation of $\theta_{(1)}$ and $\theta_{(m+1)}$ for the special cases $\vartheta = 0$ and $\vartheta = 1$, which respectively correspond to squared error loss and normalized squared error loss. They proved that the natural estimators are inadmissible as well as dominating estimators were obtained by solving certain difference inequalities. For special cases, the improved estimators for the selected Poisson and negative binomial distributions were also produced.

Vellaisamy [128] established that the unbiased estimator of selected mean for normal population and some other distributions belonging to a one-parametric exponential family do not exist. He showed that whenever an unbiased estimator exists, it should be a function of order statistics. Further, Al-Mosawi and Vellaisamy [7] have discussed estimation of parameter of the selected binomial population. Based on single- stage sample scheme, it is proved that neither unbiased nor risk-unbiased estimator exit. They considered the situation where additional observations are available from the selected population, and an unbiased, risk-unbiased, and two-stage uniformly minimum variance conditional unbiased (UMVCU) estimators were obtained. They compared the bias and the risk of the derived estimators through the simulation study.

1.2.3 Estimation of Quantile

Sharma and Vellaisamy [117] first introduced the problem of estimating quantile of the selected population. They considered the problem of estimating quantile of the selected normal population. k normal populations with unknown means $\mu_i; i = 1,\ldots,k$ and common unknown variance σ^2 were considered. A quantile of the selected population is $\theta_J = \mu_J + \eta\sigma$, for $\eta \neq 0$. Sharma and Vellaisamy selected the population corresponding to the largest sample mean. They considered two estimators, one based on Y_J and S_J and another one based on Y_J and S. Y_J and S_J are the sample mean and the sum of squared deviations from the mean of the selected population respectively. $S_J = S_j$ if $Y_j \geq Y_i$, for $j \neq i$ and $S = \Sigma_{i=1}^{k} S_i$. The two estimators proposed by them are

$$T_1 = Y_J + \eta\, c_n\, S_J^{1/2}$$

and

$$T_2 = Y_J + \eta\, c_{\nu+1}\, S^{1/2}\ ,$$

with c_m defined as

$$c_m = \frac{\Gamma(m/2)}{\sqrt{2}\Gamma(m+1)/2}.$$

When $\eta < 0$, T_2 always improves T_1. However for $\eta > 0$, it is not the case. Sharma and Vellaisamy also gave a sufficient condition for an estimator of θ_J in a certain class to be inadmissible. As a consequence, the natural estimator T_2 of θ_J is proved to be inadmissible.

Kumar and Kar [67] estimated quantiles of the selected normal population when the underlying populations are normal with different unknown means and variances. The loss function is taken to be squared error.

Estimating a quantile of the selected exponential population has been studied by Kumar and Kar [68]. They considered k exponential populations with different scale parameters and a common location parameter. The population corresponding to the largest sample mean is selected. The problem is to estimate a quantile of the selected population. Kumar and Kar derived the uniformly minimum variance unbiased estimator (UMVUE) by using (U-V) method of Robbins [110] and Rao-Blackwellization. Further, using the Brewster and Zidek technique [27], they improved the UMVUE with respect to the squared error and the scale invariant loss functions. They also obtained a general inadmissibility result for affine equivariant estimators. Further, the bias and risk functions of proposed estimators were compared numerically.

Vellaisamy [127] has studied the problem of estimating quantile of the selected exponential population. He considered k exponential populations with Π_i having an exponential distribution with unknown location parameter ξ_i and common scale parameter σ. Let X_i denotes the minimum of a random sample of size n from Π_i and $X_J = max\{X_1, \dots, X_k\}$. The population corresponding to X_J is selected. The problem is to estimate quantile $\theta_J = \xi_J + b\,\sigma$, $b \geq 0$ of the selected population. It is shown that the best affine equivariant estimator (BAEE) does not exist. The method of differential inequalities has been used to derive a sufficient condition, for an estimator in the class of scale-equivariant estimators to be inadmissible. As a special case, he obtained improved estimators over the natural estimators of θ_J, for all values of $b \geq 0$. For more results on quantile estimation one can refer to Wang et al. [134].

1.2.4 Under Heteroscedasticity

Gupta and Miescke [52] have discussed the problem of selecting the normal population associated with the largest mean under the heteroscedasticity (unequal vaiances) when the population variances are known and unequal. Under the $0-1$ loss function, the risk of a selection rule d is measured by the probability of incorrect selection, i.e.,

$$R(\underline{\theta}, d) = 1 - P_{\underline{\theta}}(CS|d). \tag{1.4}$$

Under the risk function $R(\underline{\theta}, d)$ in (1.4), Gupta and Miescke showed that the natural selection rule d_N is minimax if and only if $\sigma_1^2 = \cdots = \sigma_k^2$. Moreover, the minimax value of the problem is $1 - \frac{1}{k}$. It is observed that the natural selection rule d_N is not minimax if the variances $\sigma_1^2, \dots, \sigma_k^2$ are unequal. However, in this situation no alternative of d_N is presented and Bayes rules with respect to various priors are studied.

The selection problem involving $k(\geq 2)$ binomial populations $\Pi_1, \dots, \Pi_k$ with unknown success probabilities $\theta_1, \dots, \theta_k$ and sample sizes $n_1, \dots, n_k$ was considered by Sobel and Huyett [120]. For $n_1 = \cdots = n_k = n$ (say), Sobel and Huyett [120] considered the goal of selecting the binomial population associated with $\theta_k = \max\{\theta_1, \dots, \theta_k\}$ under the indifference-zone approach of Bechhofer [23]. They have suggested selecting the population corresponding to the largest observed frequency of success, with ties broken at random. Hall [57] showed that Sobel and Huyett [120] selection rule is minimax for equal sample sizes.

Risko [107] investigated the problem of selecting the better of two binomial populations for unequal sample sizes n_1 and n_2, where the population associated with the larger probability of success $\max\{\theta_1, \theta_2\}$. Assume that the risk is measured by the probability of incorrect selection and the selection problem is formulated under the indifference zone approach of Bechhofer [23]. The author has seen that the intuitive selection rule performs very bad when one sample size is large in relation to the other. For this case, author procured a selection rule which is minimax in the limit as one sample size goes to infinity while the other's size is kept fixed. It is showed that this newly obtained selection rule also gives a suitable alternative to the intuitive selection rule for samples whose sizes differ slightly. For the cases where both the sample size are finite, a class $\mathscr{D}$ of the selection rules is developed. The minimax selection rule in the class $\mathscr{D}$, obtained here, is called restricted minimax selection rule. In certain small sample configurations, and when one of the sample size is very large, as demonstrated by them, this minimax selection rule is globally minimax.

The problem of finding a minimax selection rule for the selection of the better of two binomial populations with unequal sample sizes are constructed by Dhariyal et al. [37]. The authors found some necessary conditions for selection to be globally minimax and demonstrated that the restricted minimax selection rule developed by Risko [107] satisfies these conditions.

Abughalous and Miescke [3] considered the problem of selecting the best binomial population with unequal sample sizes $n_1,\ldots,n_k$ where the population associated with the largest probability of success is selected. The authors showed that under the $0-1$ loss, the necessary and sufficient condition for natural selection rule of selecting the population that corresponding to the largest proportion of successes (with ties broken at random) to be minimax is that $n_1 = \cdots = n_k$. This result is similar to the result obtained by Gupta and Miescke [52]. The authors also discussed some properties of Bayes selection rule for various prior distributions under the linear loss function and monotone permutation invariant loss function.

Misra and Dhariyal [83] generalized the results of Gupta and Miescke [52] and Abughalous and Miescke [3] to general probability distributions. Suppose that the observation X_i is from the population Π_i having a cumulative distribution function (c.d.f.) $F_{\alpha_i}(x|\theta_i)$, where θ_i being unknown parameter and $\alpha_i (i = 1,\ldots,k)$, known nuisance parameter. Misra and Dhariyal [83] considered the goal of selecting the population associated with $\theta_{[k]} = \max\{\theta_1,\ldots,\theta_k\}$. The risk function of a selection rule $d \in \mathscr{D}$ assumed to be measured by the probability of incorrect selection as

$$R(\underline{\theta},d) = 1 - P_{\underline{\theta}}(CS|d). \tag{1.5}$$

Under the risk function (1.5) authors proved that the minimax value is $1-\frac{1}{k}$. The random variables $X_1,\ldots,X_k$ are statistically independent and observations X_i is from Π_i with cumulative distribution function (c.d.f.) $F_{\alpha_i}(x|\theta_i)$ having stochastically increasing property. Under the risk function (1.5), authors proved that the natural selection rule d_N which selects the population corresponding to the largest observation is minimax if and only if $\alpha_1 = \cdots = \alpha_k$. For $k = 2$, the natural selection rule d_N is minimax showing that the underlying distributions are symmetric.

The problem of selecting the population with the smallest scale parameter from k independent populations from gamma, exponential and Weibull populations have been studied by Abughalous and Bansal [2]. The $0-1$ loss function is used (here loss is 0 if correct selection is made and is 1 otherwise). The authors showed that the natural selection rule d_N is minimax if and only if $n_1 = \cdots = n_k$. Moreover the minimax value is $1-\frac{1}{k}$. Abughalous and Bansal [2] also discussed the Bayes selection rule under the $0-1$ loss function and linear loss function.

Misra and Arshad [82] examined the problem of selecting the better of two gamma populations with unequal shape parameter $\alpha_i > 0$ and unknown scale parameter $\theta_i > 0$, $i = 1,2$. Under the indifference zone approach of Bechhofer [23], the authors considerd the goal of selecting the population associated with $\max\{\theta_1, \theta_2\}$, when the quality of a selection rule is assessed in regard of the infimum of the probability of correct selection over the preference-zone. This goal is equivalent to deriving the minimax selection rule when (θ_1, θ_2) lie in the preference-zone and $0-1$ loss function is used (here loss is 0 if the correct selection is made and 1 if the correct selection is not made) for decision-theoretic framework. The authors proposed a class of natural selection rules and derived restricted minimax selection rule. This restricted minimax selection rule was proved to be globally minimax, generalized Bayes and admissible. Various natural selection rules are outperformed by the minimax selection rule, as can be observed by numerical comparison. Similar problem for the case of two independent exponential populations have been considered by Arshad and Misra [10].

Arshad et al. [13] estimated the largest mean of the selected gamma population under the assumption that the k populations have unequal known shape parameters. Arshad and Misra [9] initiated the work on estimation of the largest (smallest) mean of the selected uniform population. They obtained the UMVU estimators and considerd the three natural estimators $\varphi_{N,1}, \varphi_{N,2}$ and $\varphi_{N,3}$ of selected parameters, based on the maximum likelihood estimators, UMVU estimators, and minimum risk equivariant estimators for component estimation problems. They proved that the natural estimator $\varphi_{N,3}$ was the generalized estimator with respect to a non informative prior. Further, a general result for improving scale-invariant estimator and some better estimators are discussed. The UMVU estimator as well as natural estimator $\varphi_{N,2}$ are shown to be inadmissible.

Arshad et al. [12] derived the UMRU estimator under the entropy loss function and also obtained some inadmissible results for largest scale parameter of selected uniform population.

Under the decision theoretic framework, although large literature by Bahadur and Goodman [16], Hall [56, 57], Eaton [40], Mieseke [78, 79], Risko [107], Mulekar [96], Gupta and Miesecke [52], Dhariyal et al. [37], Abughalous and bansal [1, 2], Misra and Dhariyal [84], Bansal et al. [20], Miesecke and Park [81], Gupta and Liang [51], Mulekar and Matejcik [98], Bansal and Miescke [18, 19], Miescke [80], Gupta and Li [50], Al-Mosawi and Shanubhogue [6], Misra and Gupta [90] and Arshad and Misra [11] on the treatment of ranking and selection problems is available, still not much work has been done involving heteroscedasticity (unequal sample size and /or unequal nuisance parameters). In this thesis we make an attempt in this direction. In Chapter 7, we have considered the problem of selecting the uniform population under the squared log error (SLE) loss function when sample sizes are unequal. In Chapter 8, we have considered the problem of selecting the uniform population under the Generalized Stein Loss (GSL) function with unequal sample sizes.

1.3 A Summary of the Results in the Thesis

In this section, we give chapter wise brief description of this thesis. In **Chapter 2**, some basic definitions, techniques and selection rule required for this thesis are presented.

In **Chapter 3**, suppose $X_{i1},\ldots,X_{in}$, $i=1,2$, be pair of random samples from populations which are normally distributed with mean α_i, and common known variance τ^2 . The selection procedure is that the population giving the smallest sample mean is selected. In this chapter, the aim is to examine different estimators for the mean of the selected population from two normal populations. With the help of Mote-Carlo simulation method, the bias and mean squared errors of the various estimators are computed as well as their performances are compared with the help of graphs and tables.

In **Chapter 4**, two competing pairs of random variables (X,Y_1) and (X,Y_2) satisfying linear regression models with equal intercepts are considered. The model which connects the selection between two regression lines from two normal populations for estimating regression coefficients of the selected regression line is described. This model is applied to a problem in finance which involves selecting security with lower risk. We assume that an investor being risk averse always chooses the security with lower risk (or, volatility) while choosing one of two securities available to him for investment and further is interested in estimating the risk of the chosen security. Several estimators are constructed and their developed theory is applied to real data sets. The bias and mean squared error (MSE) risk performances of the estimators of volatility of the selected security are numerically compared, and the graphs representing the bias and MSE risks of the estimators are drawn. The results has been discussed.

In **Chapter 5**, we take up the problem of estimating quantile of a selected normal population. Suppose independent random samples $(X_{11},\ldots,X_{1n_1}), n_1 \geq 2$ and $(X_{21},\ldots,X_{2n_2}), n_2 \geq 2$ are available from these two normal populations with same mean and different variances where both are unknown. The population corresponding to the smallest sum of the squared deviations from the mean is selected as the best population. We address estimation of quantiles of the selected population. Several estimators are proposed. The problem is formulated and the admissibility of a natural estimator within a class of linear estimators is proved. We consider a more general class of estimators and found a class of admissible estimators. The biases and mean squared error risks of the proposed estimators are compared numerically by Monte-Carlo simulation. Finally, the biases and risks of different estimators are represented by graph.

In **Chapter 6** suppose $X_{i1},\ldots,X_{in}$ be an independent random sample drawn from $k(k\geq 2)$ populations $\Pi_i, i=1,...,k$ having Pareto distributions with different and unknown scale and common known shape parameters. Let $X_i=\min\{X_{i1},\ldots,X_{in}\}, i=1,\ldots,k$ and $X_{(1)}\leq X_{(2)}\leq\cdots\leq X_{(k)}$ be the order statistics of $X_1,\ldots,X_k$. The population corresponding to the largest $X_{(k)}$(or the smallest $X_{(1)}$) is selected as the best population. In this chapter estimating scale parameter of a selected population is considered under Generalized Stein loss (GSL) function. The uniformly minimum risk unbiased (UMRU) estimator of scale parameter of the population corresponding to the largest and the smallest θ_i's, are obtained. Sufficient condition for minimaxity of an estimator of θ_L(scale parameter of the population corresponding to the largest θ_i) and θ_S (scale parameter of the population corresponding to the smallest θ_i) are obtained. We show that the generalized Bayes estimator of θ_S is minimax for $k=2$. Also, we have found the class of linear admissible estimators of $\theta_L(\theta_S)$, respectively. The technique of Brewster and Zidek [27] is employed to provide a sufficient condition for inadmissibility of some scale and permutation invariant estimators of θ_S and the UMRU estimator of θ_S is shown to be inadmissible and some better estimators are provided. Finally, the results have been discussed.

In **Chapter 7** suppose $\Pi_1,\ldots,\Pi_k$ be k (≥ 2) independent populations, where Π_i denotes the uniform distribution over the interval $(0,\theta_i)$ and $\theta_i>0$ $(i=1,\ldots,k)$ is an unknown scale parameter. The population associated with the largest scale parameter is called the best population. For selecting the best population, we use a selection rule based on the natural estimators of $\theta_i, i=1,\ldots,k$, for the case of unequal sample sizes. Consider the problem of estimating scale parameter θ_L of the selected uniform population when sample sizes are unequal and the loss is measured by the squared log error (SLE) loss function. We derive the uniformly minimum risk unbiased (UMRU) estimator under the SLE loss function. Two natural estimators $\Psi_{N,1}$ and $\Psi_{N,2}$, which are respectively the analogs of the maximum likelihood estimator (MLE) and the UMRU estimator of θ_i's for the component problem, are studied. For $k=2$, we derive a sufficient condition for inadmissibility of an estimator of θ_L. Using these conditions, we show that the UMRU estimator and natural estimator $\Psi_{N,1}$ are inadmissible. Some results for estimating scale parameter of the selected uniform population when the goal of selection is to select a population associated with the smallest scale parameter are provided. Finally, the risk functions of various competing estimators of θ_L are compared through simulation.

In **Chapter 8** we consider $k(\geq 2)$ independent uniform populations over the interval $(0,\theta_i)$ and $\theta_i>0$ $(i=1,\ldots,k)$ be an unknown scale parameter. For selecting the best population associated with the largest (or smallest) scale parameter, we have considered a class of selection rules based on the natural estimators of $\theta_i, i=1,...,k$. In this chapter, we have considered the problem of estimating scale parameter θ_L of the selected uniform population when sample sizes are unequal, and the loss is measured by the Generalized Stein Loss (GSL) function. The UMRU estimator of θ_L is derived.

Two natural estimators $\xi_{N,1}$ and $\xi_{N,2}$, which are respectively the analogs of the maximum likelihood estimator (MLE) and the UMRU estimators of θ_i's for the component estimation problem, are studied. The natural estimator $\xi_{N,2}$ is proved to be the generalized Bayes estimator with respect to a noninformative prior. For $k = 2$, we give a sufficient condition for inadmissibility of an estimator of θ_L and show that the UMRU estimator and natural estimator $\xi_{N,1}$ are inadmissible. A simulation study is also carried out for the performance of the risk functions of various competing estimators of θ_L based on minimax selection rule, and it is found satisfactory. Finally some results and discussions are reported.

In **Chapter 9** the work of this thesis is concluded and the possible directions of the future scope is provided.

Chapter 2

Some Basic Definitions and Techniques

2.1 Introduction

In this chapter, we discuss some of the basic definitions related to this study, results and techniques which will be used in the sequel for estimating parameter(s) of the selected population. For more study on these topics one may refer to Ferguson [41], Gupta and Panchapakesan [55], Lehmann [73], Berger [24], Casella and Berger [30], Gibbons et al. [43] and Bain and Engelhardt [17].

2.2 The Standard Estimation or the Component Problem

In the theory of estimation, the fundamental problem is to make a presumption about the values of a specific characteristic, called parameter, of a given population. Usually, presumption depends on a sample from that population itself. Let $\boldsymbol{X} = (X_1, \ldots, X_n)$ be a random sample from a population with distribution P_θ, which is a member of the family of probability distribution $\mathscr{P} = \{P_\theta : \theta \in \Omega\}$. The set Ω is referred as the parameter space and the set of all possible values of the random variable $\boldsymbol{X}$ is called the sample space and is denoted by χ, taken to be a finite dimensional Euclidean space. In general, we are interested in estimating some function $h(\theta)$ of θ. The statistic $\delta(x)$ used to estimate $h(\theta)$ is called an estimator of $h(\theta)$ and the particular value of $\delta(x)$ is known as an estimate of $h(\theta)$. The estimates lie in a space $\mathscr{A}$, which is usually taken as the convex closure of the set $\{g(\theta) : \theta \in \Omega\}$. $\mathscr{A}$ is the set of actions available to the statistician. The $L(\theta, \delta(x))$ denotes the loss incurred when $h(\theta)$ is estimated by $\delta(\underline{x})$, where $\underline{x} = (x_1, \ldots, x_n)$ is the observed value of the sample. The loss function is assumed to be nonnegative. Usually the loss function is considered to be a convex and increasing function of the Euclidean distance $|h(\theta) - \delta(x)|$. A loss function is said to be convex if it is a convex function in the argument δ. Some examples of loss functions are as follows:

1. The most widely used loss function is squared error

$$L_1(g(\theta),\delta) = (g(\theta)-\delta)^2 \tag{2.1}$$

2. Absolute error loss function

$$L_2(g(\theta),\delta) = |g(\theta)-\delta| \tag{2.2}$$

3. Generalized stein loss (GSL) function

$$L_3\left(g(\theta),\delta\right) = \left(\frac{\delta}{g(\theta)}\right)^p - p\ln\left(\frac{\delta}{g(\theta)}\right) - 1, \quad p \neq 0. \tag{2.3}$$

4. Squared log error (SLE) loss function

$$L_4\left(g(\theta),\delta\right) = (\ln(\delta) - \ln(g(\theta)))^2 \tag{2.4}$$

The expected value of $L(\theta,\delta(x))$ with respect to probability distribution P_θ is known as the risk function of the estimator $\delta(x)$ and its risk function is denoted by

$$R(\theta,\delta) = E_\theta(L(\theta,\delta(x))) \tag{2.5}$$

At a given θ, the risk function is the average loss function that will be incurred if the estimator δ is used. For squared error loss, the risk function (2.5) is called the mean squared error (MSE).

2.3 Criteria for Selecting an Estimator

The merit of an estimator is measured by its risk function and the estimator having the minimum value of the risk is treated as the best estimator. That is why the statisticians like to find such an estimator δ for which $R(\theta,\delta)$ is minimum for all $\theta \in \Omega$. This would mean that, regardless of the all values of θ, the estimator δ will have a minimum expected loss. Unfortunately, this is not possible in most of the practical cases, since it depends on the unknown parameter θ itself. But it may be possible to determine such an estimator δ in a subclass of estimators. Let δ_0 and δ_1 be two estimators for $h(\theta)$ and if the efficiency of these estimators are to be compared, then they will be analyzed by comparing their risk functions. An estimator δ_0 is said to be better estimator than δ_1, if it satisfies the following conditions

$$R(\theta,\delta_0) \leq R(\theta,\delta_1) \text{ for all } \theta \in \Omega \tag{2.6}$$

and

$$R(\theta_*,\delta_0) < R(\theta_*,\delta_1) \text{ for some } \theta_* \in \Omega. \tag{2.7}$$

This can also be interpreted that δ_0 improves upon δ_1, or that δ_1 is dominated by δ_0.

The two estimators δ_0 and δ_1 are said to be equivalent if the their risk functions are equal. That is, if

$$R(\theta,\delta_0) = R(\theta,\delta_1) \text{ for all } \theta \in \Omega. \tag{2.8}$$

Further, an estimator δ_0 is said to be at lest as good as δ_1 if either δ_0 is better than δ_1 or it is equivalent to δ_1.

2.3.1 Admissible Estimators

Let $\mathscr{D}$ denotes the class of all estimators for an estimation problem and $\mathscr{C}$ be a subclass of $\mathscr{D}$. Consider an estimator δ such that there exists no other estimator δ_* in $\mathscr{C}$ which is better than δ. Then, an estimator δ is said to be inadmissible in the class $\mathscr{C}$ if there exists another estimator δ_* in $\mathscr{C}$ such that $R(\theta,\delta_*) \leq R(\theta,\delta)$ for all θ, with strict inequality holding for some θ. An estimator is said to be inadmissible if it is not admissible. Thus if δ is an inadmissible estimator, then there must exist at least one δ_* which is better than δ.

2.3.2 Complete and Essentially Complete classes

The subclass $\mathscr{C}$ of class $\mathscr{D}$ is said to be complete (essentially complete) class of estimators if for any estimator $\delta_1 \notin \mathscr{C}$, there exists an estimator $\delta_2 \in \mathscr{C}$ such that δ_2 is better than (as good as) δ_1. Considerably naturally, in the problems of estimation, it is desirable to obtain complete (essentially complete) classes of estimators, for we need not look outside these classes for obtaining good estimators for the estimation problem in hand.

Let $\mathscr{D}^*$ be the class of all nonrandomized estimators (one may refer Ferguson [41] for a definition of randomized estimators). When the loss function is convex, then $\mathscr{D}^*$ is essentially complete in $\mathscr{D}$. Further, let $T(\underline{X})$ be a sufficient statistic and $\mathscr{D}^*{}_T$ denote the class of all nonrandomized estimators based on T. Then $\mathscr{D}^*{}_T$ is also essentially complete.

2.3.3 Minimaxity Criterion

An estimator δ_M is said to be minimax if

$$\sup_{\theta\in\Omega} R(\theta,\delta_M) = \inf_{\delta\in\mathscr{D}} \sup_{\theta\in\Omega} R(\theta,\delta).$$

In other words an estimator δ_M is said to be minimax with respect to the risk function $R(\theta,\delta)$ if δ_M minimizes the maximum risk amongst all estimators in $\mathscr{D}$. If δ_M is a minimax estimator than the values of $\sup_{\theta \in \Omega} R(\theta, \delta_M)$ is said to have minimax value of the selection problem. The right side portion in the above expression as the minimax value or the upper value of the estimation problem.

2.3.4 Bayes Criterion

A measure τ on $(\Omega, B(\Omega))$ is said to be proper prior if $\tau(\Omega) < \infty$, where $B(\Omega)$ is a σ-filed of subsets of Ω. The measure τ is said to be improper prior if $\tau(\Omega) = \infty$ and

$$\int_{\Omega} p_\theta(x) d\tau(\theta) < \infty \ \text{ for almost all } \ x.$$

Where, $p_\theta(x)$ is the density of P_θ with respect to a σ- finite μ on the measurable space of X- values. According to Bayes criterion, the parameter θ itself is assumed to be a random variable with some known prior $\tau(\theta)$. The conditional posterior probability distribution of θ given x is defined as

$$\frac{p_\theta(x) d\tau(\theta)}{\int_\Omega p_\theta(x) d\tau(\theta)}$$

is known as the posterior or formal posterior distribution according as τ is proper or improper prior respectively. The posterior or formal posterior risk of an estimator δ is then described as

$$\frac{\int_\Omega L(\theta, \delta(x)) p_\theta(x) d\tau(\theta)}{\int_\Omega p_\theta d\tau(\theta)}.$$

An estimator δ_r which minimizes the above expression of risk is called the Bayes or formal Bayes (or generalized Bayes) estimator with respect to the prior τ. The Bayes risk of an estimator δ with respect to a prior τ is defined as

$$r(\tau, \delta) = E_r R(\theta, \delta).$$

If τ is defined as proper prior then δ_r also minimizes the Bayes risk

$$r(\tau, \delta) = \int_\Omega R(\theta, \delta) d\tau(\theta).$$

Here, it should be noted that when the loss function is proportional to squared error loss function then Bayes estimator is the mean of the posterior distribution and when the loss function is proportional to absolute error then the Bayes estimator is the median of the posterior distribution. Next, we address the property of invariance in estimation problems. It is one useful way of restricting the class of possible estimators so that one can view for the best estimator in this class.

2.3.5 Invariance

Let G denote a group of transformations from the sample space into itself. The operation on the group is considered to be the composition of functions. The family of distributions $\mathscr{P} = \{P_\theta : \theta \in \Omega\}$ is said to be invariant under the group G, if for every $g \in G$ and every $\theta \in \Omega$ there exits a unique $\theta^* \in \Omega$ such that the distribution of $g(\mathbf{X})$ is given by P_{θ^*} whenever the distribution of $\mathbf{X}$ is given byP_θ, where θ^* is uniquely determined by g and is denoted by $\overline{g}(\theta)$.

In estimation problem, if the family of distributions $\mathscr{P} = \{P_\theta : \theta \in \Omega\}$ is invariant under the group G, the loss function $L(\theta, a)$ is said to be invariant under the G, if for every $g \in G$ and every $a \in \mathscr{A}$, there exits a unique $a^* \in \mathscr{A}$ such that

$$L(\theta, a) = L(\overline{g}(\theta), a^*) \quad \text{for all } \ \theta \in \Omega.$$

In this condition, the action a^* is uniquely determined by g and is denoted by $\tilde{g}(a)$.

Here, it is observed that if $\mathscr{P}$ is invariant under the group G, then set $\overline{G} = \{\overline{g} : g \in G\}$ is a group of transformations on Ω and $\tilde{G} = \{\tilde{g} : g \in G\}$ is a group of transformations on the action space $\mathscr{A}$.

An estimation problem is said to be invariant under the group G if the underlying family of distributions and the loss function are invariant. For an invariant estimation problem, it is natural to use estimators which show symmetry. A nonrandomized estimator $\delta \in \mathscr{D}^*$ is said to be invariant if

$$\delta(g(x)) = \overline{g}(\delta(x)), \quad \text{for all } \ x \in \chi, g \in G.$$

Let two points θ_1 and θ_2 belongs to parameter space Ω are said to be equivalent if there exits a $\overline{g} \in \overline{G}$ such that $\theta_2 = \overline{g}(\theta_1)$. This is an equivalence relation and the partitions of the parameter space into equivalence classes are called orbits. An important property of an equivariant estimator is that its risk function is constant on orbits, that is,

$$R(\theta, \delta) = R(\overline{g}(\theta), \delta) \quad \text{for all } \ \theta \in \theta \ \text{ and } \ \overline{g} \in \overline{G}.$$

Thus if Ω has only one orbit, then the risk function of an equivariant estimator is independent of the parameter.

A function $T(x)$ is invariant with respect to the group G if

$$T(x) = T(g(x)) \quad \text{for all } \ x \ \text{ and } \ g \in G.$$

A function $T(x)$ is maximal invariant if it is invariant and $T(x_1) = T(x_2)$ implies $x_1 = g(x_2)$ for some $g \in G$. An important property of maximal invariants is that if $g(\theta)$ is maximal invariant under $\overline{G}$, then the distribution of $T(x)$ depends on θ only through $g(\theta)$.

2.4 The Brewster-Zidek Technique for Improving Estimators

An interesting and appropriate technique for improving on equivariant estimators have been presented by Brewster and Zidek [27]. However, their method as such can be employed in other situations also. The technique may be implemented on all the families of underlying distributions. The loss function must satisfy a natural conditions but may otherwise be arbitrary. We discuss two methods of using this technique.

Suppose we want to improve an estimator δ_0 of $g(\theta)$, $\theta \in \Omega$. Consider a class of estimators $C = \{\delta_c : c \text{ is real}\}$, where $c = c_0$ corresponds to the estimator δ_0. Minimize the risk function $R(\theta, \delta_c)$ with respect to c for each $\theta \in \Omega$. If the minimizing choice $\hat{c}$ is independent of θ and $\hat{c} \neq c_0$, then clearly the estimator $\delta_{\hat{c}}$ improve δ_0. For instance, consider translation equivariant estimators of the form $\overline{X} + c$ for estimating normal mean. Hear, clearly, $c = 0$ is the minimizing choice, say, when the loss is squared error.

However, in most reasonable situations $\hat{c}$ depends on θ, call it $\hat{c}(\theta)$. Improvement of δ_0 is still possible if $R(\theta, \delta_c)$ is strictly convex function of c and either $c_0 > \sup_{\theta \in \Omega} \hat{c}$ or $c_0 < \inf_{\theta \in \Omega} \hat{c}(\theta)$. Also the class of estimating $\{\delta_c : \inf \hat{c}(\theta) \leq c \leq \sup \hat{c}(\theta)\}$ is essentially complete in C.

This method can readily be generalized to cases, where we consider classes of estimators characterized by two or more constants. The risk function need to be strictly bowl-shaped (For a definition of bowl-shaped functions, refer to Brewster and Zidek [27]). A second approach involves reducing the risk of an equivariant estimator on the orbits of some invariant statistics W. Usually W is taken to be a maximum invariant estimator. Consider estimator of the form $\delta_{\phi(W)}$, where $\phi(W) = \phi_0(W)$ corresponds to the estimator δ_0. One can then apply the Brewster and Zidek technique as described in the preceding paragraph on the conditional risk function of $\delta_{\phi(W)}$ given $W = w$, that is, on

$$R(\theta, \delta_{\phi(w)}) = E\left[L\left(\theta, \delta_{\phi(W)}\right)|W = w\right].$$

Let $R(\theta, \delta_{\phi(w)})$ be a strictly convex function of $\phi(w)$ and $\hat{\phi}_\theta(w)$ be the choice of ϕ minimizing $R(\theta, \delta_{\phi(w)})$ for given w and θ. Define $\phi_*(w) = \inf_{\theta \in \Omega} \hat{\phi}_\theta(w)$ and $\phi^*(w) = \sup_{\theta \in \Omega} \hat{\phi}_\theta(w)$. Then if $\phi_0(w) < \phi_*(w)$, we have $R(\theta, \delta_{\phi_0(w)}) > R(\theta, \delta_{\phi_*(w)})$ and if $\phi_0(w) > \phi_*(w)$, $R(\theta, \delta_{\phi_0(w)}) > R(\theta, \delta_{\phi_*(w)})$. This leads us to estimators $\delta_{\tilde{\phi}(W)}$, where

$$\begin{aligned}
\tilde{\phi}(w) &= \phi_*(w), \text{ if } w \in A, \\
&= \phi^*(w), \text{ if } w \in B, \\
&= \phi_0(w), \text{ otherwise}
\end{aligned}$$

and A and B are given by

$$A = \{w : \phi_0(w) < \phi_*(w)\} \text{ and } B = \{w : \phi_0(w) > \phi^*(w)\}.$$

Then the estimator $\delta_{\tilde{\phi}(W)}$ will improve upon δ_0 if $P_\theta\,(A \bigcup B) > 0$ for some $\theta \in \Omega$.

2.5 The Problem of Selection

Let $\Pi_k, \ldots, \Pi_k$ be $k(\geq 2)$ independent populations such that the characteristic of the population Π_i described by a random variable X_i having the probability density function $f_{X_i}(.|\theta_i)$; where $\theta_i \in \Theta(i = 1, \ldots, k)$ is an unknown parameter. Saying that the populations $\Pi_k, \ldots, \Pi_k$ are independent we mean that the random observations from these populations are independent. Quite often, one is interested in selecting the best population or a subset of populations containing the best population from among the given k population. A population may be called the best according to some characteristic such as the largest mean, the smallest variance, the largest quantile etc. For the goal of selecting the best population, a most commonly used selection procedure is the "Natural Selection Rule", which is explained in the following subsequent section.

2.5.1 The Natural Selection Rule

Suppose we have k populations from $\Pi_1, \ldots, \Pi_k$ being characterized by the parameters $\theta_1, \ldots, \theta_k$. Let $\Theta = (\theta_1, \ldots, \theta_k)$ and $g(.)$ is a real valued function such that $g(\theta_i) \leq g(\theta_j)$ whenever $\theta_i \leq \theta_j, i \neq j$. Suppose the problem of interest is to select the population associated with $\max\limits_{1\leq i\leq k} g(\theta_i)$. Normally $T = (t_1, \ldots, t_k)$ is an appropriately chose sufficient statistics based on a random sample from population $\Pi_1, \ldots, \Pi_k$. Then the natural selection rule says to select the population Π_j if t_j is the unique largest among $t_1, \ldots, t_k$ and chose populations $\Pi_{i1}, \Pi_{i2}, \ldots, \Pi_{ir}$ each with probability $\frac{1}{r}$ if $t-$ values equal to $\max t_j$.

Bahadur [15] and Bahadur and Goodman [16] have analyzed the optimality properties of the natural selection rule. They have discussed that the natural selection rule minimizes the risk in the class of impartial decision rules or permutation invariant rule and proved that, for certain families of distributions, the natural selection procedure uniformly minimizes the risk among all symmetric procedures for a wide class of loss functions. Lehmann [72] provided alternative proof of this result by applying some invariance arguments. He also indicated several other optimum properties of the natural selection rule. Further, Eston [40] generalized the result by Lehmann [72] to cover the case of distributions having a generalization of the monotone likelihood ratio property of densities.

In this thesis, we use natural selection rule for selecting the best population when the underlying populations are Normal, Pareto and Uniform.

2.6 The Problem of Estimation After selection

Let $\Pi_1, \Pi_2, \ldots, \Pi_k$ be k independent populations with Π_i having associated the probability density function $f(x, \theta_i),\ i = 1, \ldots, k$. Suppose $\underline{X}$ be the sample observations from $\Pi_1, \ldots, \Pi_k$. Further suppose $I(\underline{X})$ denotes the selection procedure according to which a population is to be selected. The selection procedure $I(\underline{X})$ can be explained as follows:

$I(\underline{X})$ is a discrete valued random variable such that

$$I(\underline{X}) = j \text{ if } \underline{X} \in A_j.$$

where $A_1, A_2, \ldots, A_k$ is a partitions of the sample space of $\underline{X}$. The population Π_j is selected when $I(\underline{X}) = j$. The parameter corresponding to the selected population is denoted by θ_I. In the standard estimation problem presented in section 2.1, the parameter to be estimated is a fixed quantity, whereas θ_I is a random quantity which depends upon the outcome of the selection procedure. An estimator U($\underline{\text{X}}$) is said to be unbiased estimator of θ_I if it is unbiased for expectation of θ_I, that is if satisfy

$$E(U(\underline{X})) = E(\theta_I), \text{ for all } \underline{\theta} = (\theta_1, \ldots, \theta_k)$$

or equivalently if

$$E_{\underline{\theta}}(U(\underline{X}) - \theta_I), \text{ for all } \underline{\theta} = (\theta_1, \ldots, \theta_k).$$

Similarly, a uniformly minimum variance unbiased (UMVU)estimator of θ_I is actually a UMVU estimator of $E(\theta_I)$. Also it is interesting to note that, as in case of component problem the mean squared error (MSE) of the unbiased estimator is defined to be its variance, it is not the case in the problem of estimation after selection, if U is the unbiased estimator of θ_I. Then,

$$MSE(U) = E(U - \theta_I)^2 = V(U) + V(\theta_I) - 2Cov(U, \theta_I),$$

where $V(U)$ and $V(\theta_I)$ denotes the variance of U and θ_I, respectively.

For more discussion, on unbiased estimation following selection one may refer to Vellaisamy [126, 128].

Now we present a general technique to derived unbiased estimators for the problem of estimation after selection by Robbins [110]. We illustrate this technique underneath.

2.6.1 (U, V) Method of Robbins

This method is utilized to derive unbiased estimators of the random parameters of the selected population. For the component problem too, this method is used for detecting unbiased estimators of the random parameters. Let a random variable X with associated the probability density function $f(x,\theta),\ \ \theta \in \Omega$ where $f(x,\theta)$ is a parametric probability density functions with respect to some σ-finite measure μ, for a given function $v(x)$, one must find a function $u(x)$ such that

$$E_\theta\left[u(X)\right] = \theta E_\theta\left[v(X)\right],\ \ \forall\ \ \theta \in \Omega,$$

then $u(X)$ is an unbiased estimator of $\theta E_\theta\left(v\left(X\right)\right)$. For example, if we consider an exponential distribution with scale parameter θ and location parameter zero, that is

$$f(x,\theta) = \frac{1}{\theta}e^{-\frac{x}{\theta}},\ \ x > 0,\ \ \theta > 0. \tag{2.9}$$

We can easily see that for a given function $v(x)$, $u(x)$ can be defined as

$$u(x) = \int_0^x v(t)dt.$$

Next, we generalize for the k populations. Suppose we have k populations $\Pi_1,\ldots,\Pi_k$ with Π_i having the density function $f(x,\theta_i)$. We select the random sample X_i form $\Pi_i,\ i = 1,\ldots,k$. We may be interested in estimating

$$\theta_I = \sum_{i=1}^{k} v_i(\underline{x})\theta_i,$$

here $v_i(\underline{x}),\ i = 1,\ldots,k$ are the functions of $\underline{x} = (x_1,\ldots,x_k)$. If we obtain functions $u_i(\underline{x})$ such that $E_{\underline{\theta}}u_i(\underline{X}) = \theta_i E_{\underline{\theta}}v_i(\underline{X})$ for $i = 1,\ldots,k$, then $\sum_{i=1}^{k} u_i(\underline{X})$ is an unbiased estimator of $\theta_I(\underline{x})$. For example, Let $X_1,\ldots,X_k$ be exponential random variable with associate densities $f(x,\theta_1),\ldots,f(x,\theta_k)$, where $f(x,\theta)$ is defined in (2.9). Then for a given function $v_i(\underline{x})$, $u_i(\underline{x})$ defined as

$$u_i(\underline{x}) = \int_0^{x_1} v_i(x_1,\ldots,x_{i-1},t,x_{i+1},\ldots,x_k)dt.$$

satisfies the condition $Eu_i(\underline{X}) = \theta_i Ev_i(\underline{X})$, hence $\sum_{i=1}^{k} u_i(\underline{X})$ will be unbiased estimator of θ_I.

Chapter 3

Estimation of Mean of the selected Population

3.1 Introduction

Suppose we have two securities and we choose the security which is less risky. Then what can we say about the estimate of the risk of the selected security? For tackling this kind of situation we may consider the following model. Suppose we have two normal populations with means α_i, $i = 1,2$ respectively, and each having the same variance τ^2. A random sample of size n is drawn from each of the populations. Let $X_{11},\ldots,X_{1n}$ and $X_{21},\ldots,X_{2n}$ be the two random samples drawn from the first and second population respectively. Let Y_1 and Y_2 be the sample means of $\{X_{1j}\}$ and $\{X_{2j}\}$, $j = 1,\ldots,n$ respectively. Then the expectation of Y_i is α_i and the variance of Y_i is $\frac{\tau^2}{n}$. Now we are interested in selecting the normal population with the smaller mean. For this purpose we select the population with smaller sample mean. The problem corresponding to higher mean was studied in [36]. So, in the current problem the first population is selected if $Y_1 \leq Y_2$ and the second population is selected otherwise. Hence we define I_1 and I_2 as

$$I_1 = \begin{cases} 1 & \text{if } Y_1 \leq Y_2 \\ 0 & \text{otherwise} \end{cases}$$

and $I_2 = 1 - I_1$. Therefore, the mean of the selected population is

$$M = \alpha_1 I_1 + \alpha_2 I_2,$$

where $Y_{\min} = \min(Y_1, Y_2)$.

In this chapter, we derive four various estimators and the improved estimator of the mean of the selected population from two normal populations with unknown mean and common known variance under the squared error loss function. In section 3.2, we present these estimators. In section 3.3, we found the bias for the four estimators which are discussed in computable forms and also obtain MSE for $\hat{M}_\lambda$, and determine the improved estimator in section 3.4. In section 3.5, the comparison of bias and MSE risks of proposed estimators are analyzed through Monte- Carlo simulation technique.

3.2 Derivation of the estimators

In this section we consider the analogous estimators to the proposed estimators in [36]. The natural estimator of M is $Y_{\min}$.

The bias of $Y_{\min}$ as an estimator M is

$$\begin{aligned} B(Y_{min}) &= E[Y_{\min} - M], \\ &= E[Y_{\min}] - E[M], \end{aligned} \tag{3.1}$$

where

$$\begin{aligned} E[Y_{\min}] &= E[Y_1 I_1 + Y_2 I_2], \\ &= E[Y_1 I_1] + E[Y_2 I_2], \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{y_2} I_1 y_1 f(y_1,y_2)dy_1dy_2, + \int_{-\infty}^{\infty}\int_{-\infty}^{y_1} I_2 y_2 f(y_1,y_2)dy_1dy_2, \\ &= A + B. \end{aligned} \tag{3.2}$$

Here, we evaluate A as

$$\begin{aligned} A &= \int_{-\infty}^{\infty}\int_{-\infty}^{y_2} \frac{y_1}{\sqrt{2\pi}}\frac{\sqrt{n}}{\tau} exp\left(\frac{-(y_1-\alpha_1)^2}{\frac{2\tau^2}{n}}\right) \frac{1}{\sqrt{2\pi}}\frac{\sqrt{n}}{\tau} exp\left(\frac{-(y_2-\alpha_2)^2}{\frac{2\tau^2}{n}}\right) dy_1dy_2, \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{y_2} y_1 \frac{\sqrt{n}}{\tau}\phi\left(\frac{y_1-\alpha_1}{\frac{\tau}{\sqrt{n}}}\right)\frac{\sqrt{n}}{\tau}\phi\left(\frac{y_2-\alpha_2}{\frac{\tau}{\sqrt{n}}}\right) dy_1dy_2, \\ &= \frac{\sqrt{n}}{\tau}\int_{-\infty}^{\infty}\phi\left(\frac{y_2-\alpha_2}{\frac{\tau}{\sqrt{n}}}\right)\left[\frac{\sqrt{n}}{\tau}\int_{-\infty}^{y_2} y_1\phi\left(\frac{y_1-\alpha_1}{\frac{\tau}{\sqrt{n}}}\right)dy_1\right]dy_2. \end{aligned}$$

Using the transformation $\frac{y_1-\alpha_1}{\frac{\tau}{\sqrt{n}}} = u$, we obtain

$$\begin{aligned} A &= \frac{\sqrt{n}}{\tau}\int_{-\infty}^{\infty}\phi\left(\frac{y_2-\alpha_2}{\frac{\tau}{\sqrt{n}}}\right)\left[\int_{-\infty}^{\frac{y_2-\alpha_1}{\frac{\tau}{\sqrt{n}}}}\left(u\frac{\tau}{\sqrt{n}}+\alpha_1\right)\phi(u)du\right]dy_2, \\ &= \frac{\sqrt{n}}{\tau}\int_{-\infty}^{\infty}\phi\left(\frac{y_2-\alpha_2}{\frac{\tau}{\sqrt{n}}}\right)\left[\frac{\tau}{\sqrt{n}}\int_{-\infty}^{\frac{y_2-\alpha_1}{\frac{\tau}{\sqrt{n}}}} u\phi(u)du + \alpha_1\int_{-\infty}^{\frac{y_2-\alpha_1}{\frac{\tau}{\sqrt{n}}}}\phi(u)du\right]dy_2, \\ &= \frac{\sqrt{n}}{\tau}\int_{-\infty}^{\infty}\phi\left(\frac{y_2-\alpha_2}{\frac{\tau}{\sqrt{n}}}\right)\left[\frac{\tau}{\sqrt{n}}\int_{-\infty}^{\frac{y_2-\alpha_1}{\frac{\tau}{\sqrt{n}}}} u\left(\frac{1}{\sqrt{2\pi}}exp\left(-\frac{u^2}{2}\right)\right)du + \alpha_1\Phi\left(\frac{y_2-\alpha_1}{\frac{\tau}{\sqrt{n}}}\right)\right]dy_2. \end{aligned}$$

Further Putting $\frac{u^2}{2}=v$, we get

$$A=\frac{\sqrt{n}}{\tau}\int_{-\infty}^{\infty}\phi\left(\frac{y_2-\alpha_2}{\frac{\tau}{\sqrt{n}}}\right)\left[\frac{1}{\sqrt{2\pi}}\frac{\tau}{\sqrt{n}}\int_{\infty}^{\frac{1}{2}\left(\frac{y_2-\alpha_1}{\frac{\tau}{\sqrt{n}}}\right)^2} exp\left(-v\right)dv+\alpha_1\Phi\left(\frac{y_2-\alpha_1}{\frac{\tau}{\sqrt{n}}}\right)\right]dy_2,$$

$$=\frac{\sqrt{n}}{\tau}\int_{-\infty}^{\infty}\phi\left(\frac{y_2-\alpha_2}{\frac{\tau}{\sqrt{n}}}\right)\left[-\frac{1}{\sqrt{2\pi}}\frac{\tau}{\sqrt{n}}exp\left(-\frac{1}{2}\left(\frac{y_2-\alpha_1}{\frac{\tau}{\sqrt{n}}}\right)^2\right)+\alpha_1\Phi\left(\frac{y_2-\alpha_1}{\frac{\tau}{\sqrt{n}}}\right)\right]dy_2,$$

$$=\frac{\sqrt{n}}{\tau}\int_{-\infty}^{\infty}\phi\left(\frac{y_2-\alpha_2}{\frac{\tau}{\sqrt{n}}}\right)\left[-\frac{\tau}{\sqrt{n}}\phi\left(\frac{y_2-\alpha_1}{\frac{\tau}{\sqrt{n}}}\right)+\alpha_1\Phi\left(\frac{y_2-\alpha_1}{\frac{\tau}{\sqrt{n}}}\right)\right]dy_2.$$

Again letting $\frac{y_2-\alpha_2}{\frac{\tau}{\sqrt{n}}}=u$ and A simplifies to

$$A=-\frac{\tau}{\sqrt{n}}\int_{-\infty}^{\infty}\phi(u)\phi\left(u+\alpha\frac{\sqrt{n}}{\tau}\right)du+\alpha_1\int_{-\infty}^{\infty}\phi(u)\Phi\left(u+\alpha\frac{\sqrt{n}}{\tau}\right)du,$$

where $\alpha=\alpha_2-\alpha_1$.

For assessing the integral on the right side of A, we use the identity

$$\phi(ax+b)\phi(cx+d)=\phi\left(\frac{\left(a^2+c^2\right)x+ab+cd}{\sqrt{a^2+c^2}}\right)\phi\left(\frac{ad-bc}{\sqrt{a^2+c^2}}\right). \tag{3.3}$$

and the following integrals

$$\int_{-\infty}^{\infty}x\Phi(a+bx)\phi(x)dx=\frac{b}{\sqrt{1+b^2}}\phi\left(\frac{a}{\sqrt{1+b^2}}\right), \tag{3.4}$$

$$\int_{-\infty}^{\infty}\Phi(a+bx)\phi(x)dx=\Phi\left(\frac{a}{\sqrt{1+b^2}}\right), \tag{3.5}$$

and obtain A as

$$A=-\frac{\tau}{\sqrt{n}}\int_{-\infty}^{\infty}\phi\left(\sqrt{2}u+\frac{\alpha}{\tau\sqrt{\frac{2}{n}}}\right)\phi\left(\frac{\alpha}{\tau\sqrt{\frac{2}{n}}}\right)du+\alpha_1\Phi\left(\frac{\alpha}{\tau\sqrt{\frac{2}{n}}}\right),$$

$$=-\frac{1}{\sqrt{2}}\frac{\tau}{\sqrt{n}}\phi\left(\frac{\alpha}{\tau\sqrt{\frac{2}{n}}}\right)+\alpha_1\Phi\left(\frac{\alpha}{\tau\sqrt{\frac{2}{n}}}\right). \tag{3.6}$$

Similarly, B can be obtained as

$$B=\int_{-\infty}^{\infty}\int_{-\infty}^{y_1}y_2\frac{1}{\sqrt{2\pi}}\frac{\sqrt{n}}{\tau}exp\left(\frac{-\left(y_1-\alpha_1\right)^2}{\frac{2\tau^2}{n}}\right)\frac{1}{\sqrt{2\pi}}\frac{\sqrt{n}}{\tau}exp\left(\frac{-\left(y_2-\alpha_2\right)^2}{\frac{2\tau^2}{n}}\right)dy_1dy_2,$$

$$=-\frac{1}{\sqrt{2}}\frac{\tau}{\sqrt{n}}\phi\left(\frac{\alpha}{\tau\sqrt{\frac{2}{n}}}\right)+\alpha_2-\alpha_2\Phi\left(\frac{\alpha}{\tau\sqrt{\frac{2}{n}}}\right). \tag{3.7}$$

Addition of (3.6) and (3.7), because of (3.2), gives

$$E(Y_{\min}) = -\frac{1}{\sqrt{2}}\frac{\tau}{\sqrt{n}}\phi\left(\frac{\alpha}{\tau\sqrt{\frac{2}{n}}}\right) + \alpha_1\Phi\left(\frac{\alpha}{\tau\sqrt{\frac{2}{n}}}\right) + \alpha_2 - \frac{1}{\sqrt{2}}\frac{\tau}{\sqrt{n}}\phi\left(\frac{\alpha}{\tau\sqrt{\frac{2}{n}}}\right) - \alpha_2\Phi\left(\frac{\alpha}{\tau\sqrt{\frac{2}{n}}}\right). \quad (3.8)$$

The expected value of the mean of the selected population is defined as

$$\begin{aligned} E[M] &= E\left[\alpha_1 I_1 + \alpha_2 I_2\right], \\ &= E\left[\alpha_1 I_1\right] + E\left[\alpha_2 I_2\right], \\ &= \alpha_1 P\left(Y_1 < Y_2\right) + \alpha_2 P\left(Y_1 > Y_2\right). \end{aligned} \quad (3.9)$$

where

$$\begin{aligned} P(Y_1 < Y_2) &= \int_{-\infty}^{\infty}\int_{-\infty}^{y_2}\frac{1}{\sqrt{2\pi}}\frac{\sqrt{n}}{\tau}exp\left(\frac{-\left(y_1-\alpha_1\right)^2}{\frac{2\tau^2}{n}}\right)\frac{1}{\sqrt{2\pi}}\frac{\sqrt{n}}{\tau}exp\left(\frac{-\left(y_2-\alpha_2\right)^2}{\frac{2\tau^2}{n}}\right)dy_1dy_2, \\ &= \frac{1}{\sqrt{2\pi}}\frac{\sqrt{n}}{\tau}\int_{-\infty}^{\infty}exp\left(\frac{-\left(y_2-\alpha_2\right)^2}{\frac{2\tau^2}{n}}\right)\left[\frac{1}{\sqrt{2\pi}}\frac{\sqrt{n}}{\tau}\int_{-\infty}^{y_2}exp\left(\frac{-\left(y_1-\alpha_1\right)^2}{\frac{2\tau^2}{n}}\right)dy_1\right]dy_2, \\ &= \frac{\sqrt{n}}{\tau}\int_{-\infty}^{\infty}\phi\left(\frac{y_2-\alpha_2}{\frac{\tau}{\sqrt{n}}}\right)\left[\frac{\sqrt{n}}{\tau}\int_{-\infty}^{y_2}\phi\left(\frac{y_1-\alpha_1}{\frac{\tau}{\sqrt{n}}}\right)dy_1\right]dy_2. \end{aligned}$$

Let $u = \frac{y_1-\alpha_1}{\frac{\tau}{\sqrt{n}}}$, and $dy_1 = \frac{\tau}{\sqrt{n}}du$,

$$\begin{aligned} P(Y_1 < Y_2) &= \frac{\sqrt{n}}{\tau}\int_{-\infty}^{\infty}\phi\left(\frac{y_2-\alpha_2}{\frac{\tau}{\sqrt{n}}}\right)\left[\int_{-\infty}^{\frac{y_2-\alpha_1}{\frac{\tau}{\sqrt{n}}}}\phi(u)du\right]dy_2, \\ &= \frac{\sqrt{n}}{\tau}\int_{-\infty}^{\infty}\phi\left(\frac{y_2-\alpha_2}{\frac{\tau}{\sqrt{n}}}\right)\Phi\left(\frac{y_2-\alpha_1}{\frac{\tau}{\sqrt{n}}}\right)dy_2. \end{aligned}$$

Again let $u = \frac{y_2-\alpha_2}{\frac{\tau}{\sqrt{n}}}$, and $dy_2 = \frac{\tau}{\sqrt{n}}du$,

$$\begin{aligned} P\left(Y_1 < Y_2\right) &= \int_{-\infty}^{\infty}\phi(u)\Phi\left(u + \frac{\alpha\sqrt{n}}{\tau}\right)du, \\ &= \Phi\left(\frac{\alpha}{\tau\sqrt{\frac{2}{n}}}\right), \text{ where } \alpha = \alpha_2 - \alpha_1. \end{aligned} \quad (3.10)$$

Similarly, we obtain the expression for $P\left(Y_2 < Y_1\right)$ as

$$\begin{aligned} P\left(Y_2 < Y_1\right) &= \int_{-\infty}^{\infty}\int_{-\infty}^{y_1}\frac{1}{\sqrt{2\pi}}\frac{\sqrt{n}}{\tau}exp\left(\frac{-\left(y_1-\alpha_1\right)^2}{\frac{2\tau^2}{n}}\right)\frac{1}{\sqrt{2\pi}}\frac{\sqrt{n}}{\tau}exp\left(\frac{-\left(y_2-\alpha_2\right)^2}{\frac{2\tau^2}{n}}\right)dy_1dy_2, \\ &= 1 - \Phi\left(\frac{\alpha}{\tau\sqrt{\frac{2}{n}}}\right). \end{aligned} \quad (3.11)$$

Substituting the values of (3.10) and (3.11) in (3.9), we get the expected value of M as

$$E[M] = \alpha_1 \Phi\left(\frac{\alpha}{\tau\sqrt{\frac{2}{n}}}\right) + \alpha_2 - \alpha_2 \Phi\left(\frac{\alpha}{\tau\sqrt{\frac{2}{n}}}\right). \tag{3.12}$$

Using the equations (3.8) and (3.2), and (3.1) we get the bias of $Y_{\min}$ as

$$\begin{aligned} B(Y_{\min}) = & -\frac{1}{\sqrt{2}}\frac{\tau}{\sqrt{n}}\phi\left(\frac{\alpha}{\tau\sqrt{\frac{2}{n}}}\right) + \alpha_1\Phi\left(\frac{\alpha}{\tau\sqrt{\frac{2}{n}}}\right) - \frac{1}{\sqrt{2}}\frac{\tau}{\sqrt{n}}\phi\left(\frac{\alpha}{\tau\sqrt{\frac{2}{n}}}\right) + \alpha_2 \\ & - \alpha_2\Phi\left(\frac{\alpha}{\tau\sqrt{\frac{2}{n}}}\right) - \left[\alpha_1\Phi\left(\frac{\alpha}{\tau\sqrt{\frac{2}{n}}}\right) + \alpha_2 - \alpha_2\Phi\left(\frac{\alpha}{\tau\sqrt{\frac{2}{n}}}\right)\right], \\ = & -\frac{1}{\sqrt{2}}\frac{\tau}{\sqrt{n}}\phi\left(\frac{\alpha}{\tau\sqrt{\frac{2}{n}}}\right) - \frac{1}{\sqrt{2}}\frac{\tau}{\sqrt{n}}\phi\left(\frac{\alpha}{\tau\sqrt{\frac{2}{n}}}\right), \\ = & -\frac{2}{\sqrt{2}}\frac{\tau}{\sqrt{n}}\phi\left(\frac{\alpha}{\tau\sqrt{\frac{2}{n}}}\right), \\ = & -\sqrt{2}\frac{\tau}{\sqrt{n}}\phi\left(\frac{\alpha}{\tau\sqrt{\frac{2}{n}}}\right), \\ = & -\sigma\phi\left(\frac{\alpha}{\sigma}\right), \end{aligned}$$

where $\sigma = \tau\sqrt{\frac{2}{n}}$.

Now, we get the estimator $\hat{M}_\lambda$ of the random variable M as

$$\hat{M}_\lambda = Y_{\min} + \lambda\sigma\phi\left(\frac{Y}{\sigma}\right),$$

where $Y = Y_2 - Y_1$ and $\lambda \geq 0$ is an arbitrary constant. Note: Y $\sim N(\alpha, \sigma^2)$ and $E[Y_2 - Y_1] = \alpha_2 - \alpha_1 = \alpha, Var(Y) = \frac{2\tau^2}{n} = \sigma^2$.

Next, we propose an estimator t_c for the mean of the selected population according to Sarkadi [116], is given by

$$t_c = Y_2 - Y\Phi\left(\frac{cY}{\sigma}\right) + c\sigma\phi\left(\frac{cY}{\sigma}\right),$$

where $c > 0$ is an arbitrary constant and $\Phi(u)$ is the standard normal cdf. Now

$$\begin{aligned} E\left[t_c\right] &= E\left[Y_2 - Y\Phi\left(\frac{cY}{\sigma}\right) + c\sigma\phi\left(\frac{cY}{\sigma}\right)\right], \\ E\left[t_c\right] &= E\left[Y_2\right] - E\left[Y\Phi\left(\frac{cY}{\sigma}\right)\right] + E\left[c\sigma\phi\left(\frac{cY}{\sigma}\right)\right], \end{aligned} \tag{3.13}$$

where

$$E[Y_2] = \alpha_2, \tag{3.14}$$

and

$$\begin{aligned}
E\left[Y\Phi\left(\frac{cY}{\sigma}\right)\right] &= \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} exp\left(-\frac{1}{2}\left(\frac{y-\alpha}{\sigma}\right)^2\right) y\Phi\left(\frac{cy}{\sigma}\right) dy,\\
&= \frac{1}{\sigma}\int_{-\infty}^{\infty} \phi\left(\frac{y-\alpha}{\sigma}\right) y\Phi\left(\frac{cy}{\sigma}\right) dy.
\end{aligned}$$

$\Longrightarrow$ Let $\frac{y-\alpha}{\sigma} = u$, then $dy = \sigma du$,

$$\begin{aligned}
E\left[Y\Phi\left(\frac{cY}{\sigma}\right)\right] &= \int_{-\infty}^{\infty} (\sigma u+\alpha)\,\phi(u)\Phi\left(cu+\frac{c\alpha}{\sigma}\right) du,\\
&= \sigma\int_{-\infty}^{\infty} u\phi(u)\Phi\left(cu+\frac{c\alpha}{\sigma}\right) du + \alpha\int_{-\infty}^{\infty} \phi(u)\Phi\left(cu+\frac{c\alpha}{\sigma}\right),\\
&= \frac{\sigma c}{\sqrt{1+c^2}}\phi\left(\frac{c\alpha}{\sigma\sqrt{1+c^2}}\right) + \alpha\Phi\left(\frac{c\alpha}{\sigma\sqrt{1+c^2}}\right).
\end{aligned} \tag{3.15}$$

Further, we obtain the expression of $E\left[c\sigma\phi\left(\frac{cY}{\sigma}\right)\right]$ as

$$\begin{aligned}
E\left[c\sigma\phi\left(\frac{cY}{\sigma}\right)\right] &= \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} exp\left(-\frac{1}{2}\left(\frac{y-\alpha}{\sigma}\right)^2\right) c\sigma\phi\left(\frac{cy}{\sigma}\right) dy,\\
&= c\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} exp\left(-\frac{1}{2}\left(\frac{y-\alpha}{\sigma}\right)^2\right) \frac{1}{\sqrt{2\pi}} exp\left(-\frac{1}{2}\left(\frac{cy}{\sigma}\right)^2\right) dy,\\
&= \frac{c}{2\pi}\int_{-\infty}^{\infty} exp\left(-\frac{1}{2\sigma^2}\left(y^2\left(1+c^2\right)+\alpha^2-2y\alpha\right)\right) dy,\\
&= \frac{c}{2\pi}\int_{-\infty}^{\infty} exp\left[-\frac{1}{2\sigma^2}\left\{\left(y\sqrt{1+c^2}-\frac{\alpha}{\sqrt{1+c^2}}\right)^2+\frac{\alpha^2c^2}{1+c^2}\right\}\right] dy,\\
&= \frac{c}{2\pi} exp\left[-\frac{1}{2\sigma^2}\left(\frac{\alpha^2c^2}{1+c^2}\right)\right]\int_{-\infty}^{\infty} exp\left[-\frac{1}{2\sigma^2}\left(y\sqrt{1+c^2}-\frac{\alpha}{\sqrt{1+c^2}}\right)^2\right] dy.
\end{aligned}$$

Let $y\sqrt{1+c^2} = u$, and $dy = \frac{1}{\sqrt{1+c^2}}du$, we obtain

$$\begin{aligned}
E\left[c\sigma\phi\left(\frac{cY}{\sigma}\right)\right] &= \frac{c}{2\pi\sqrt{1+c^2}} exp\left[-\frac{1}{2\sigma^2}\left(\frac{\alpha^2c^2}{1+c^2}\right)\right]\int_{-\infty}^{\infty} exp\left[-\frac{1}{2\sigma^2}\left(u-\frac{\alpha}{\sqrt{1+c^2}}\right)^2\right] du,\\
&= \frac{c\sigma}{\sqrt{1+c^2}}\phi\left(\frac{\alpha c}{\sigma\sqrt{1+c^2}}\right)\left[\frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} exp\left[-\frac{1}{2\sigma^2}\left(u-\frac{\alpha}{\sqrt{1+c^2}}\right)^2\right] du\right],\\
&= \frac{c\sigma}{\sqrt{1+c^2}}\phi\left(\frac{\alpha c}{\sigma\sqrt{1+c^2}}\right).
\end{aligned} \tag{3.16}$$

From (3.13), (3.14), (3.15) and (3.16), we get expected value of t_c as

$$E[t_c] = \alpha_2 - \alpha\Phi\left(\frac{c\alpha}{\sigma\sqrt{1+c^2}}\right).$$

So, we can say that estimator t_c is an unbiased estimator of $\alpha_2 - \alpha\Phi\left(\frac{c\alpha}{\sigma\sqrt{1+c^2}}\right)$, which approaches to $E[M] = \alpha_2 - \alpha\Phi\left(\frac{\alpha}{\sigma}\right)$ for large c. Hence, if c is large then the bias of t_c as an estimator of M is controlled.

Here, we can consider an estimator T of $E[M]$ given by

$$T = Y_2 - Y\Phi\left(\frac{Y}{\sigma}\right) \tag{3.17}$$

and propose to use it to estimate M. The estimator T, actually, is a maximum likelihood estimator of $E[M]$ and its bias will be the same whether we can use it as an estimator of $E[M]$ or M. A more general estimator of this type is given by

$$T_\lambda = T - \lambda\left[Y\left\{\phi\left(\frac{Y}{\sigma}\right) - \Phi\left(\frac{Y}{\sqrt{2}\sigma}\right)\right\} - \frac{\sigma}{\sqrt{2}}\phi\left(\frac{Y}{\sqrt{2}\sigma}\right)\right] \tag{3.18}$$

where $\lambda \geq 0$ is an arbitrary constant. The estimator of T_λ is the same as that of $\hat{M}_\lambda$ and is obtained on subtracting a λ multiple of the estimated bias of T from itself.

Another estimator that we intend to investigate here is given by

$$H_c = \begin{cases} \frac{Y_1+Y_2}{2}, & \text{if} \quad |Y_1 - Y_2| < c\sigma \\ \min(Y_1, Y_2), & \text{if} \quad |Y_1 - Y_2| \geq c\sigma. \end{cases} \tag{3.19}$$

where $c \geq 0$ is an arbitrary constant. Note that for c=0 we get $Y_{\min}$ which is the same as $\hat{M}_\lambda$ for $\lambda = 0$. Then H_c is sometimes called an hybrid estimator. Finally, it may be mentioned that the Bayes estimator of M, for squared error loss with uniform prior distribution on α_1 and α_2, turns out to be $\min(Y_1, Y_2)$ which is the same as $\hat{M}_\lambda$ for $\lambda = 0$.

3.3 Suitable formulae for Biases and MSE $\hat{M}_\lambda$

Since M is random variable, the general definition of the bias of an estimator $\hat{M}$ of M as $E(\hat{M}) - M$ has to be converted in a rather obvious way i.e. the Bias of estimator $\hat{M}$ is defined as

$$B(\hat{M}) = E(\hat{M} - M)$$

and similarly, the MSE of estimator $\hat{M}$ is defined as

$$MSE(\hat{M}) = E(\hat{M} - M)^2.$$

In the following theorem, we present the bias of the proposed estimators.

Theorem 3.3.1. *The respective bias of $\hat{M}_\lambda$, t_c, T and H_c are as follows*

(a)

$$B(\hat{M}_\lambda) = -\sigma\phi(\gamma) + \frac{\sigma\lambda}{\sqrt{2}}\phi\left(\frac{\gamma}{\sqrt{2}}\right), \tag{3.20}$$

(b)

$$B(t_c) = \gamma\sigma\left[\Phi(\gamma) - \Phi\left(\frac{c\gamma}{\sqrt{1+c^2}}\right)\right], \tag{3.21}$$

(c)

$$B(T) = \sigma\left[\gamma\left\{\Phi(\gamma) - \Phi\left(\frac{\gamma}{\sqrt{2}}\right)\right\} - \frac{1}{\sqrt{2}}\phi\left(\frac{\gamma}{\sqrt{2}}\right)\right], \tag{3.22}$$

(d)

$$B(H_c) = \frac{\sigma}{2}\left[\gamma\{2\Phi(\gamma) - \Phi(\gamma - c) - \Phi(c+\gamma)\} - \phi(c-\gamma) - \phi(c+\gamma)\right], \tag{3.23}$$

where $\gamma = \frac{\alpha}{\sigma}$.

Proof. (a) **The bias of $\hat{M}_\lambda$:**

$$B\left[\hat{M}_\lambda\right] = E\left[\hat{M}_\lambda - M\right] = E\left[\hat{M}_\lambda\right] - E[M]. \tag{3.24}$$

Now, we define expectation $\hat{M}_\lambda$

$$E\left[\hat{M}_\lambda\right] = E[Y_{\min}] + E\left[\lambda\sigma\phi\left(\frac{Y}{\sigma}\right)\right]. \tag{3.25}$$

where

$$\begin{aligned}
E\left[\lambda\sigma\phi\left(\frac{Y}{\sigma}\right)\right] &= \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} exp\left[-\frac{1}{2}\left(\frac{y-\alpha}{\sigma}\right)^2\right]\lambda\sigma\phi\left(\frac{y}{\sigma}\right)dy, \\
&= \frac{\lambda}{\sqrt{2\pi}}\int_{-\infty}^{\infty} exp\left[-\frac{1}{2}\left(\frac{y-\alpha}{\sigma}\right)^2\right]\frac{1}{\sqrt{2\pi}}exp\left[-\frac{y^2}{2\sigma^2}\right]dy, \\
&= \frac{\lambda}{\sqrt{2\pi}\sqrt{2\pi}}\int_{-\infty}^{\infty} exp\left[-\frac{1}{2\sigma^2}\left(\sqrt{2}y - \frac{\alpha}{\sqrt{2}}\right)^2 - \frac{1}{2}\left(\frac{\alpha}{\sqrt{2}\sigma}\right)^2\right]dy.
\end{aligned}$$

Let $\sqrt{2}y = u$, and $dy = \frac{du}{\sqrt{2}}$, we obtain

$$\begin{aligned}
E\left[\lambda\sigma\phi\left(\frac{Y}{\sigma}\right)\right] &= \frac{\lambda}{\sqrt{2\pi}\sqrt{2\pi}}exp\left[-\frac{1}{2}\left(\frac{\alpha}{\sqrt{2}\sigma}\right)^2\right]\int_{-\infty}^{\infty} exp\left[-\frac{1}{2\sigma^2}\left(u - \frac{\alpha}{\sqrt{2}}\right)^2\right]\frac{du}{\sqrt{2}}, \\
&= \frac{1}{\sqrt{2}}\phi\left(\frac{\alpha}{\sqrt{2}\sigma}\right)\frac{\sigma\lambda}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} exp\left[-\frac{1}{2\sigma^2}\left(u - \frac{\alpha}{\sqrt{2}}\right)^2\right]du, \\
&= \frac{\sigma\lambda}{\sqrt{2}}\phi\left(\frac{\alpha}{\sqrt{2}\sigma}\right).
\end{aligned} \tag{3.26}$$

where

$$\int_{-\infty}^{\infty} exp\left[-\frac{1}{2\sigma^2}\left(u-\frac{\alpha}{\sqrt{2}}\right)^2\right]du = 1.$$

Using the equations (3.8) and (3.26) in the equation (3.25), we get the expectation of $\hat{M}_\lambda$ as

$$E\left[\hat{M}_\lambda\right] = -\frac{\tau}{\sqrt{2n}}\phi\left(\frac{\alpha_2-\alpha_1}{\tau\sqrt{\frac{2}{n}}}\right)+\alpha_1\Phi\left(\frac{\alpha_2-\alpha_1}{\tau\sqrt{\frac{2}{n}}}\right)-\frac{\tau}{\sqrt{2n}}\phi\left(\frac{\alpha}{\tau\sqrt{\frac{2}{n}}}\right)$$
$$+\alpha_2\Phi\left(\frac{\alpha}{\tau\sqrt{\frac{2}{n}}}\right)+\frac{\sigma\lambda}{\sqrt{2}}\phi\left(\frac{\alpha}{\sqrt{2}\sigma}\right). \tag{3.27}$$

Using the expressions of the equations (3.12) and (3.27) in (3.25), we get the bias of $\hat{M}_\lambda$ as

$$\begin{aligned} B\left[\hat{M}_\lambda\right] &= -\frac{2\tau}{\sqrt{2n}}\phi\left(\frac{\alpha}{\tau\sqrt{\frac{2}{n}}}\right)+\frac{\sigma\lambda}{\sqrt{2}}\phi\left(\frac{\alpha}{\sqrt{2}\sigma}\right), \\ &= -\sigma\phi\left(\frac{\alpha}{\sigma}\right)+\frac{\sigma\lambda}{\sqrt{2}}\phi\left(\frac{\alpha}{\sqrt{2}\sigma}\right), \quad \text{where } \sigma^2=\frac{2\tau^2}{n} \\ B\left[\hat{M}_\lambda\right] &= -\sigma\phi\left(\gamma\right)+\frac{\sigma\lambda}{\sqrt{2}}\phi\left(\frac{\gamma}{\sqrt{2}}\right), \quad \text{where } \gamma=\frac{\alpha}{\sigma}. \end{aligned}$$

(b) **The bias of t_c:**

$$\begin{aligned} B\left[t_c\right] &= E\left[t_c-M\right]=E\left[t_c\right]-E\left[M\right], \\ &= E\left[Y_1-Y\Phi\left(\frac{cY}{\sigma}\right)+c\sigma\phi\left(\frac{cY}{\sigma}\right)\right]-E\left[M\right], \\ B\left[t_c\right] &= E\left[Y_1\right]-E\left[Y\Phi\left(\frac{cY}{\sigma}\right)\right]+E\left[c\sigma\phi\left(\frac{cY}{\sigma}\right)\right]-E\left[M\right]. \end{aligned} \tag{3.28}$$

Use equations (3.12), (3.14), (3.15) and (3.16) in (3.28), we obtain the bias of t_c as

$$\begin{aligned} B\left[t_c\right] &= \alpha\left[\Phi\left(\frac{\alpha}{\sigma}\right)-\Phi\left(\frac{c\alpha}{\sigma\sqrt{1+c^2}}\right)\right], \\ B\left[t_c\right] &= \gamma\sigma\left[\Phi\left(\gamma\right)-\Phi\left(\frac{c\gamma}{\sqrt{1+c^2}}\right)\right], \quad \text{where } \gamma=\frac{\alpha}{\sigma}. \end{aligned} \tag{3.29}$$

(c) **The bias of T:**

$$\begin{aligned} B\left[T\right] &= E\left[T-M\right]=E\left[T\right]-E\left[M\right], \\ &= E\left[Y_1-Y\Phi\left(\frac{Y}{\sigma}\right)\right]-E\left[M\right], \\ B\left[T\right] &= E\left[Y_1\right]-E\left[Y\Phi\left(\frac{Y}{\sigma}\right)\right]-E\left[M\right]. \end{aligned} \tag{3.30}$$

Now, we find the expression for $E\left[Y\Phi\left(\frac{Y}{\sigma}\right)\right]$ as

$$E\left[Y\Phi\left(\frac{Y}{\sigma}\right)\right] = \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} exp\left[-\frac{1}{2}\left(\frac{y-\alpha}{\sigma}\right)^2\right] y\Phi\left(\frac{y}{\sigma}\right)dy,$$
$$= \frac{1}{\sigma}\int_{-\infty}^{\infty}\phi\left(\frac{y-\alpha}{\sigma}\right)y\Phi\left(\frac{y}{\sigma}\right)dy.$$

Let $u = \frac{y-\alpha}{\sigma}$ then we obtain

$$E\left[Y\Phi\left(\frac{Y}{\sigma}\right)\right] = \int_{-\infty}^{\infty}(\sigma u+\alpha)\,\phi(u)\,\Phi\left(u+\frac{\alpha}{\sigma}\right)du,$$
$$= \sigma\left[\int_{-\infty}^{\infty}u\phi(u)\,\Phi(u+\gamma)\,du + \gamma\int_{-\infty}^{\infty}\phi(u)\,\Phi(u+\gamma)\,du\right],$$

where $\gamma = \frac{\alpha}{\sigma}$.

Use of equations (3.4) and (3.5) we yields

$$E\left[Y\Phi\left(\frac{Y}{\sigma}\right)\right] = \sigma\left[\frac{1}{\sqrt{2}}\phi\left(\frac{\gamma}{\sqrt{2}}\right) + \gamma\Phi\left(\frac{\gamma}{\sqrt{2}}\right)\right]. \tag{3.31}$$

Using the equations (3.12), (3.14), (3.31) and (3.30), we lead to the bias of T as

$$B[T] = \alpha_1 - \sigma\left[\frac{1}{\sqrt{2}}\phi\left(\frac{\gamma}{\sqrt{2}}\right) + \gamma\Phi\left(\frac{\gamma}{\sqrt{2}}\right)\right] - \left[\alpha_1 - \alpha\Phi\left(\frac{\alpha}{\sigma}\right)\right],$$
$$B[T] = \sigma\left[\gamma\{\Phi(\gamma) - \Phi\left(\frac{\gamma}{\sqrt{2}}\right)\} - \frac{1}{\sqrt{2}}\phi\left(\frac{\gamma}{\sqrt{2}}\right)\right],\ \text{where}\ \gamma = \frac{\alpha}{\sigma}. \tag{3.32}$$

(d) **The bias of H_c:**

$$B[H_c] = E[H_c - M] = E[H_c] - E[M], \tag{3.33}$$

Here,

$$H_c = \begin{cases} \frac{Y_1+Y_2}{2}, & \text{if } |Y_1-Y_2| < c\sigma \\ \min(Y_1,Y_2), & \text{if } |Y_1-Y_2| \geq c\sigma. \end{cases}$$

where $\min(Y_1,Y_2) = \frac{Y_1+Y_2}{2} - \frac{|Y_1-Y_2|}{2}$.

Let $Y \sim Y_1 - Y_2$, and $Y \sim N(\alpha,\sigma^2)$, therefor $E\left[\frac{Y_1+Y_2}{2}\right] = \frac{\alpha_1+\alpha_2}{2}$.

Now, we find the expectation,

$$E[H_c] = E\left[\frac{Y_1+Y_2}{2}\right] - \int\int_{|Y_1-Y_2|\geq c\sigma}\frac{|y_1-y_2|}{2}f(y_1,y_2)dy_1dy_2,$$
$$= \frac{\alpha_1+\alpha_2}{2} - \int_{|Y|\geq c\sigma}\frac{|\,y\,|}{2}f_Y(y)dy,$$
$$= \frac{\alpha_1+\alpha_2}{2} - \frac{1}{2}\left[\int_{c\sigma}^{-\infty} yf_Y(y)dy - \int_{-\infty}^{-c\sigma} yf_Y(y)dy\right],$$
$$= \frac{\alpha_1+\alpha_2}{2} - \frac{1}{2}\left[\int_{c\sigma}^{-\infty}\frac{y}{\sigma}\phi\left(\frac{y-\alpha}{\sigma}\right)dy - \frac{1}{\sigma} - \int_{-\infty}^{-c\sigma} y\phi\left(\frac{y-\alpha}{\sigma}\right)dy\right].$$

Next, let $u=\frac{y-\alpha}{\sigma}$,and $dy=\sigma du$, and we simplify

$$E\left[H_c\right]=\frac{\alpha_1+\alpha_2}{2}-\frac{1}{2}\left[\int_{\left(\frac{c\sigma-\alpha}{\sigma}\right)}^{\infty}(u\sigma+\alpha)\,\phi(u)du-\int_{-\infty}^{-\left(\frac{c\sigma+\alpha}{\sigma}\right)}(u\sigma+\alpha)\,\phi(u)du\right],$$

$$=\frac{\alpha_1+\alpha_2}{2}-\frac{1}{2}\left[\sigma\int_{\left(\frac{c\sigma-\alpha}{\sigma}\right)}^{\infty}u\phi(u)du+\alpha\int_{\left(\frac{c\sigma-\alpha}{\sigma}\right)}^{\infty}\phi(u)du-\sigma\int_{-\infty}^{-\left(\frac{c\sigma+\alpha}{\sigma}\right)}u\phi(u)du\right.$$
$$\left.-\alpha\int_{-\infty}^{-\left(\frac{c\sigma+\alpha}{\sigma}\right)}\phi(u)du\right].$$

Since, we solve the following integrals

$$\int_{\left(\frac{c\sigma-\alpha}{\sigma}\right)}^{\infty}u\phi(u)du=\frac{1}{\sqrt{2\pi}}\int_{\left(\frac{c\sigma-\alpha}{\sigma}\right)}^{\infty}ue^{-\frac{u^2}{2}}du,$$

Let, $z=\frac{u^2}{2}$, $udu=dz$,

$$\frac{1}{\sqrt{2\pi}}\int_{\left(\frac{c\sigma-\alpha}{\sigma}\right)}^{\infty}ue^{-\frac{u^2}{2}}du,=\frac{1}{\sqrt{2\pi}}\int_{\frac{1}{2}\left(\frac{c\sigma-\alpha}{\sigma}\right)^2}^{\infty}e^{-z}dz,=\phi\left(\frac{c\sigma-\alpha}{\sigma}\right).$$

Similarly

$$\int_{-\infty}^{-\left(\frac{c\sigma+\alpha}{\sigma}\right)}u\phi(u)du=-\phi(\frac{c\sigma+\alpha}{\sigma}).$$

Using these integrals, we obtain

$$E\left[H_c\right]=\frac{\alpha_1+\alpha_2}{2}-\frac{1}{2}\left[\sigma\phi\left(\frac{c\sigma-\alpha}{\sigma}\right)+\alpha\left\{\Phi(u)\right\}\Big|_{\left(\frac{c\sigma-\alpha}{\sigma}\right)}^{\infty}-\sigma\left\{-\phi\left(\frac{c\sigma+\alpha}{\sigma}\right)\right\}\right.$$
$$\left.-\alpha\left\{\Phi(u)\right\}\Big|_{-\infty}^{-\left(\frac{c\sigma+\alpha}{\sigma}\right)}\right].$$

As, $\Phi(-u)=1-\Phi(u), \Phi(\infty)=1$ and $\Phi(-\infty)$, we obtain

$$E\left[H_c\right]=\frac{\alpha_1+\alpha_2}{2}-\frac{\sigma}{2}\left[\phi\left(c-\gamma\right)+\phi\left(c+\gamma\right)-\gamma\{\Phi\left(c-\gamma\right)-\Phi\left(c+\gamma\right)\}\right], \tag{3.34}$$

where $\gamma=\frac{\alpha}{\sigma}$.

Using the equations (3.12) and (3.34) in (3.33), we obtain the bias of H_c,

$$B\left(H_c\right)=\frac{\sigma}{2}\left[\gamma\{2\Phi\left(\gamma\right)-\Phi\left(\gamma-c\right)-\Phi\left(c+\gamma\right)\}-\phi\left(c-\gamma\right)-\phi\left(c+\gamma\right)\right].$$

The proof of this theorem is complete. □

For finding the MSE expression of the proposed estimator $\hat{M}_\lambda$ we give the following theorem and its proof.

Theorem 3.3.2. *The MSE of $\hat{M}_\lambda$ is,*

$$MSE\left[\hat{M}_\lambda\right] = \sigma^2\left[\frac{1}{2}+\frac{\lambda^2}{\sqrt{6\pi}}\phi\left(\frac{\gamma\sqrt{2}}{\sqrt{3}}\right)-\lambda\phi\left(\frac{\gamma}{\sqrt{2}}\right)\left\{\phi\left(\frac{\gamma}{\sqrt{2}}\right)+\frac{3\gamma}{\sqrt{2}}\left(\frac{1}{2}-\Phi\left(\frac{\gamma}{\sqrt{2}}\right)\right)\right\}\right].$$

Proof. First, we consider $Y_{\min} = \overline{Y} - \frac{YsgnY}{2}$ and $M = \overline{\alpha} - \frac{\alpha sgnY}{2}$, where $\overline{Y} = \frac{y_1+y_2}{2}$, $Y = y_1 - y_2$, $\overline{\alpha} = \frac{\alpha_1+\alpha_2}{2}$ and

$$sgn(x) = \begin{cases} -1 & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ 1 & \text{if } x > 0. \end{cases}$$

Therefor $\hat{M}_\lambda = Y_{\min} + \lambda\sigma\phi\left(\frac{Y}{\sigma}\right) = \overline{Y} - \frac{YsgnY}{2} + \lambda\sigma\phi\left(\frac{Y}{\sigma}\right)$, we have

$$\begin{aligned}
MSE\left[\hat{M}_\lambda\right] =& E\left[\hat{M}_\lambda - M\right]^2, \\
=& E\left[\overline{Y} - \frac{YsgnY}{2} + \lambda\sigma\phi\left(\frac{Y}{\sigma}\right) - \left(\overline{\alpha} - \frac{\alpha sgnY}{2}\right)\right]^2, \\
=& E\left[\left(\overline{Y} - \frac{YsgnY}{2} - \overline{\alpha} + \frac{\alpha sgnY}{2}\right) + \lambda\sigma\phi\left(\frac{Y}{\sigma}\right)\right]^2, \\
=& E\left[\left\{\left(\overline{Y} - \overline{\alpha}\right) - (Y-\alpha)\frac{sgnY}{2}\right\} + \lambda\sigma\phi\left(\frac{Y}{\sigma}\right)\right]^2, \\
=& E\left[\left(\overline{Y} - \overline{\alpha}\right) - (Y-\alpha)\frac{sgnY}{2}\right]^2 + \lambda^2\sigma^2 E\left[\phi\left(\frac{Y}{\sigma}\right)\right]^2 \\
&+ 2\lambda\sigma E\left[\left\{\left(\overline{Y} - \overline{\alpha}\right) - (Y-\alpha)\frac{sgnY}{2}\right\}\phi\left(\frac{Y}{\sigma}\right)\right], \\
=& E\left[\left(\overline{Y} - \overline{\alpha}\right)^2\right] + E\left[\left\{(Y-\alpha)\frac{sgnY}{2}\right\}^2\right] - 2E\left[\left(\overline{Y} - \overline{\alpha}\right)(Y-\alpha)\frac{sgnY}{2}\right] \\
&+ \lambda^2\sigma^2 E\left[\phi\left(\frac{Y}{\sigma}\right)\right]^2 + \lambda\sigma E\left[\left\{\left(\overline{Y} - \overline{\alpha}\right) - (Y-\alpha)\frac{sgnY}{2}\right\}\phi\left(\frac{Y}{\sigma}\right)\right].
\end{aligned}$$

Since $\overline{Y}$ and Y are independent, so $E\left[\left(\overline{Y} - \overline{\alpha}\right)\phi\left(\frac{Y}{\sigma}\right)\right] = 0$, then we obtain

$$\begin{aligned}
MSE\left[\hat{M}_\lambda\right] =& E\left[\left(\overline{Y} - \overline{\alpha}\right)^2\right] + E\left[\left\{(Y-\alpha)\frac{sgnY}{2}\right\}^2\right] + \lambda^2\sigma^2 E\left[\phi\left(\frac{Y}{\sigma}\right)\right]^2 \\
&- 2\lambda\sigma E\left[(Y-\alpha)\frac{sgnY}{2}\phi\left(\frac{Y}{\sigma}\right)\right].
\end{aligned} \tag{3.35}$$

we let $\frac{Y}{\sigma} = v$ and $v \sim N(\gamma, 1)$. Therefore, we evaluate the following expression

$$\begin{aligned} E\left[\phi\left(\frac{Y}{\sigma}\right)\right]^2 &= E\left[\phi(v)\right]^2 = E\left[\frac{\phi\left(v\sqrt{2}\right)}{\sqrt{2\pi}}\right], \\ &= \frac{1}{\sqrt{2\pi}}\int \phi\left(v\sqrt{2}\right)\phi\left(v-\gamma\right)dv, \\ &= \frac{1}{\sqrt{6\pi}}\phi\left(\frac{\gamma\sqrt{2}}{\sqrt{3}}\right), \end{aligned} \tag{3.36}$$

and

$$E\left[\overline{Y} - \overline{\alpha}\right]^2 = \frac{\sigma^2}{4}, \tag{3.37}$$

$$\begin{aligned} E\left[(Y-\alpha)\frac{sgnY}{2}\right]^2 &= E\left[(Y-\alpha)\right]^2, \\ &= \frac{\sigma^2}{4}. \end{aligned} \tag{3.38}$$

$$\begin{aligned} E\left[(Y-\alpha)\phi\left(\frac{Y}{\sigma}\right)\frac{sgnY}{2}\right] =& E\left[(\sigma v-\alpha)\phi(v)\frac{sgn(\sigma v)}{2}\right], \\ =& \int_{-\infty}^{\infty}(\sigma v-\alpha)\phi(v)\phi(v-\gamma)\frac{sgn(\sigma v)}{2}dv \;\; \text{(we use the identity (3.3))}, \\ =& \int_{-\infty}^{\infty}(\sigma v-\alpha)\phi\left(\sqrt{2}v-\frac{\gamma}{\sqrt{2}}\right)\phi\left(\frac{\gamma}{\sqrt{2}}\right)\frac{sgn(\sigma v)}{2}, \\ =& \frac{\sigma}{2}\left[\int_{-\infty}^{\infty}v\phi\left(\sqrt{2}v-\frac{\gamma}{\sqrt{2}}\right)\phi\left(\frac{\gamma}{\sqrt{2}}\right)sgn(\sigma v)dv\right] \\ & -\frac{\alpha}{2}\left[\int_{-\infty}^{\infty}\phi\left(\sqrt{2}v-\frac{\gamma}{\sqrt{2}}\right)\phi\left(\frac{\gamma}{\sqrt{2}}\right)sgn(\sigma v)dv\right], \\ =& \frac{\sigma}{2}\phi\left(\frac{\gamma}{\sqrt{2}}\right)\left[-\int_{-\infty}^{0}v\phi\left(\sqrt{2}v-\frac{\gamma}{\sqrt{2}}\right)dv+\int_{0}^{\infty}v\phi\left(\sqrt{2}v-\frac{\gamma}{\sqrt{2}}\right)dv\right] \\ & -\frac{\alpha}{2}\phi\left(\frac{\gamma}{\sqrt{2}}\right)\left[-\int_{-\infty}^{0}\phi\left(\sqrt{2}v-\frac{\gamma}{\sqrt{2}}\right)dv+\int_{0}^{\infty}\phi\left(\sqrt{2}v-\frac{\gamma}{\sqrt{2}}\right)dv\right]. \end{aligned}$$

Let $\sqrt{2}v - \frac{\gamma}{\sqrt{2}} = u$ and $dv = \frac{1}{\sqrt{2}}du$, then

$$\begin{aligned} =& \frac{\sigma}{2}\phi\left(\frac{\gamma}{\sqrt{2}}\right)\left[-\int_{-\infty}^{0}v\phi\left(\sqrt{2}v-\frac{\gamma}{\sqrt{2}}\right)dv+\int_{0}^{\infty}v\phi\left(\sqrt{2}v-\frac{\gamma}{\sqrt{2}}\right)dv\right] \\ & -\frac{\alpha}{2\sqrt{2}}\phi\left(\frac{\gamma}{\sqrt{2}}\right)\left[-\int_{-\infty}^{-\frac{\gamma}{\sqrt{2}}}\phi(u)du+\int_{-\frac{\gamma}{\sqrt{2}}}^{\infty}\phi(u)du\right], \\ =& \frac{\sigma}{2}\phi\left(\frac{\gamma}{\sqrt{2}}\right)\left[\phi\left(\frac{\gamma}{\sqrt{2}}\right)-\frac{\gamma}{\sqrt{2}}\Phi\left(\frac{\gamma}{\sqrt{2}}\right)+\frac{\gamma}{2\sqrt{2}}\right]-\frac{\sigma\gamma}{2\sqrt{2}}\phi\left(\frac{\gamma}{\sqrt{2}}\right)\left[2\Phi\left(\frac{\gamma}{\sqrt{2}}\right)-1\right], \\ =& \frac{\sigma}{2}\phi\left(\frac{\gamma}{\sqrt{2}}\right)\left[\phi\left(\frac{\gamma}{\sqrt{2}}\right)+\frac{3\gamma}{\sqrt{2}}\left(\frac{1}{2}-\Phi\left(\frac{\gamma}{\sqrt{2}}\right)\right)\right]. \end{aligned} \tag{3.39}$$

Putting the expressions given in the equations (3.36), (3.37), (3.38) and (3.39) in the equation (3.35), and simplifying it, then we get the following result

$$MSE\left[\hat{M}_\lambda\right] = \sigma^2\left[\frac{1}{2}+\frac{\lambda^2}{\sqrt{6\pi}}\phi\left(\frac{\gamma\sqrt{2}}{\sqrt{3}}\right)-\lambda\phi\left(\frac{\gamma}{\sqrt{2}}\right)\left\{\phi\left(\frac{\gamma}{\sqrt{2}}\right)+\frac{3\gamma}{\sqrt{2}}\left(\frac{1}{2}-\Phi\left(\frac{\gamma}{\sqrt{2}}\right)\right)\right\}\right].$$

□

3.4 Improvement upon $\hat{M}_\lambda$

Consider the group $G = g(x)_c = x+c, c \in R$ of affine transformations. Under this transformation :$X_{ij} \to X_{ij}+c, Y_i \to Y_i+c$ and $M \to M+c$, if we take the the squared error loss $L(d,M) = (d-M)^2$ then estimation problem is invariant.

Theorem 3.4.1. *Let us consider the class of estimators of the form* $\hat{M}_\lambda = Y_{min}+\lambda\sigma\phi\left(\frac{Y}{\sigma}\right)$. *Further, define the estimator* $\hat{M}_{\lambda *}$ *by*

$$\begin{aligned}\hat{M}_{\lambda *} &= \hat{M}_\lambda, if \quad \lambda \le \gamma * \\ &= \hat{M}_{\gamma *}, if \quad \lambda > \gamma *,\end{aligned}$$

where $\gamma * = \frac{\sqrt{3}}{2}$. *Then* $\hat{M}_{\lambda *}$ *improves* $\hat{M}_\lambda$ *with respect to the squared error loss if* $\lambda > \gamma * > 0$ *for some* $\eta = \{\alpha_1, \alpha_2\}$.

Proof. The mean squared error risk of $\hat{M}_\lambda$ is given by

$$\begin{aligned}\frac{MSE\left[\hat{M}_\lambda\right]}{\sigma^2} &= \frac{E\left[\left(Y_{min}+\lambda\sigma\phi\left(\frac{Y}{\sigma}\right)-M\right)^2\right]}{\sigma^2} \\ &= \left[\frac{1}{2}+\frac{\lambda^2}{\sqrt{6\pi}}\phi\left(\frac{\gamma\sqrt{2}}{\sqrt{3}}\right)-\lambda\phi\left(\frac{\gamma}{\sqrt{2}}\right)\left\{\phi\left(\frac{\gamma}{\sqrt{2}}\right)-\frac{3\gamma}{\sqrt{2}}\left(\Phi\left(\frac{\gamma}{\sqrt{2}}\right)-\frac{1}{2}\right)\right\}\right] \\ &= \psi(\lambda)\end{aligned}$$

where $\gamma = \frac{\alpha}{\sigma}$.
Now $\psi'(\lambda) = \left[\frac{2\lambda}{\sqrt{6\pi}}\phi\left(\frac{\gamma\sqrt{2}}{\sqrt{3}}\right)-\phi\left(\frac{\gamma}{\sqrt{2}}\right)\left\{\phi\left(\frac{\gamma}{\sqrt{2}}\right)-\frac{3\gamma}{\sqrt{2}}\left(\Phi\left(\frac{\gamma}{\sqrt{2}}\right)-\frac{1}{2}\right)\right\}\right] = 0$ gives

$$\begin{aligned}\lambda &= \frac{\phi\left(\frac{\gamma}{\sqrt{2}}\right)\left\{\phi\left(\frac{\gamma}{\sqrt{2}}\right)-\frac{3\gamma}{\sqrt{2}}\left(\Phi\left(\frac{\gamma}{\sqrt{2}}\right)-\frac{1}{2}\right)\right\}}{\frac{2}{\sqrt{6\pi}}\phi\left(\frac{\gamma\sqrt{2}}{\sqrt{3}}\right)} \\ &= \frac{\sqrt{3}}{2}e^{-\gamma^2+\frac{2\gamma^2}{3}}\left\{1-\frac{\frac{3\gamma}{\sqrt{2}}\left(\Phi\left(\frac{\gamma}{\sqrt{2}}\right)-\frac{1}{2}\right)}{\phi\left(\frac{\gamma}{\sqrt{2}}\right)}\right\} \\ &= f(\gamma)\end{aligned}$$

which is the minima of $\psi(\lambda)$. We know $\Phi(0) = \frac{1}{2}$. When γ is positive, $\Phi(\frac{\gamma}{\sqrt{2}})$ is greater than $\frac{1}{2}$, that is $\Phi\left(\frac{\gamma}{\sqrt{2}}\right) - \frac{1}{2} > 0$. Thus, $\frac{3\gamma}{\sqrt{2}}\{\Phi\left(\frac{\gamma}{\sqrt{2}}\right) - \frac{1}{2}\}$ is positive. If γ is negative, $\frac{3\gamma}{\sqrt{2}}\{\Phi\left(\frac{\gamma}{\sqrt{2}}\right) - \frac{1}{2}\}$ is positive. Therefore, $f(\gamma)$ has the maximum value at $\gamma = 0$ and we obtain

$$f(\gamma) \leq \frac{\sqrt{3}}{2} e^{-\gamma^2/3} \leq \frac{\sqrt{3}}{2}.$$

Thus, $\text{Sup}_{\gamma \in R} f(\gamma) = \frac{\sqrt{3}}{2} = \gamma*$. According to Brewster-Zidek technique [27] we can say that if $\lambda > \gamma*$ then $\hat{M}_\lambda$ can be improved by $\hat{M}_{\gamma*}$ as in that case the risk of $\hat{M}_{\gamma*}$ will be less than that of $\hat{M}_\lambda$ and this completes the proof of the theorem. □

3.5 Comparison of different estimators based on bias and MSE

The biases and mean squared error risks are calculated for the above estimators by the method of Monte-Carlo simulation for different values of $|\gamma|$, where $\gamma = \frac{\alpha_1 - \alpha_2}{\sigma}$. The bias and MSE risks of $\hat{M}_\lambda$, T_λ, t_c and H_c are calculated for several values of λ and c. Among these values the values of λ and c which give better results are chosen to be presented in the thesis. The graphs of these biases and MSE risks are drawn for the above estimators. In these graphs the X axis represents $|\gamma|$ and Y axis represents the bias or MSE risks. The bias performances of the above estimators are given in Table 3.1. From Table 3.1 we observe that as $|\gamma|$ increases the value of absolute biases of all the estimators decrease excluding the estimator T_λ. The MSE risks performances are given in Table 3.2. From Table 3.2, it is observed that the risk performances of the estimators become better as the value of $|\gamma|$ increases. It should be noted that $\hat{M}_\lambda$ reduces to $Y_{\min}$ for $\lambda = 0$ and the other suitable choices of λ are $\sqrt{3}/2$, $4 - 2\sqrt{2}$ and $\sqrt{2}$. It is also observed that in most of the cases the risk of $\hat{M}_\lambda$, for $\lambda = \sqrt{3}/2$ is lower than the other estimators. The bias and risk of T_λ are considered for the values of $\lambda = \sqrt{3}/2, 1.0, 1.25, 1.50$. From Table 3.1, it can be seen that for all values of $|\gamma|$, the estimator T_λ has the best bias performance than $\hat{M}_\lambda$ most of the cases when $\lambda = \sqrt{3}/2$. Also observed from the Table 3.1 and Fig. 3.2, the estimator T_λ perform best for smaller values of $|\gamma|$, whereas estimator T perform the best for moderate values of $|\gamma|$. From the graphs also it is clear that as the value of $|\gamma|$ tends to ∞ the bias and risk values tend to 0, hence, all the estimators are consistent. From the Fig. 3.6 and Table 3.2 it is observed that the risk performance of the estimator T is better than T_λ. In fact, as the value of λ increases the bias and risk values increase. Bias and risk computations were made for estimator t_c for $1/3, 1/2, 1, \sqrt{3}$, and for hybrid estimator H_c for $c = 0.5, 1, \sqrt{2}, 2$. From Table 3.1 and Fig. 3.3, it can be seen that for all values of $|\gamma|$, bias of the estimator t_c decreases as c increases. Other hand, From Table 3.2 and Fig 3.7, we see that the MSE of t_c is increases for small values of $|\gamma|$, whereas, for moderate and larger values of $|\gamma|$, the MSE of t_c is decreases. Estimator t_c with $c = \sqrt{3}$

seem to be a good choice if bias of an estimator is the main criterion. On the other hand, if MSE risk is the main criterion, $H_c(c = 0.5)$ seem to be reasonable good choice. It is observed that the values of $|\gamma| \leq 1.0$ and $c = 0.5$ and $c = 1.0$, the bias performances of the estimator t_c is better than estimator H_c, whereas H_c performs better for moderate and large values of $|\gamma|$. When the values of $|\gamma|$ takes below 1.0, the MSE risk of the estimator H_c is decreases as c increases, whereas the MSE risk of the estimator H_c is increases as c increases for all moderate and larger values of $|\gamma|$.

We present below the tables and graphs of biases and mean squared error risks of different estimators.

Table 3.1: Bias performances of various estimators of M

		$\hat{M}_\lambda$				T_λ				t_c				H_c			
$\lvert\gamma\rvert$	T	$\lambda = 0$	$\lambda = \sqrt{3}/2$	$\lambda = 4-2\sqrt{2}$	$\lambda = \sqrt{2}$	$\lambda = \sqrt{3}/2$	$\lambda = 1$	$\lambda = 1.25$	$\lambda = 1.5$	$c = 1/3$	$c = 1/2$	$c = 1$	c=$\sqrt{3}$	$c = 0.5$	$c = 1$	c=$\sqrt{2}$	$c = 2$
0	-0.3622	-0.5077	-0.2048	-0.0979	-0.0130	-0.1710	-0.1414	-0.0862	-0.0310	-0.0059	-0.0078	-0.0124	-0.0163	0.7629	0.6950	0.6082	0.4810
0.5	-0.2230	-0.3369	-0.1048	-0.0229	0.0421	-0.0783	-0.0559	-0.0142	0.0276	0.0762	0.0656	0.0451	0.0336	0.4826	0.4643	0.4372	0.3843
1.0	-0.1190	-0.2008	-0.0499	0.0033	0.0455	-0.0313	-0.0177	0.0076	0.0329	0.1676	0.1274	0.0552	0.0183	0.3258	0.3384	0.3557	0.3671
1.5	-0.0391	-0.0991	-0.00069	0.0340	0.0616	0.0119	0.0198	0.0346	0.0493	0.2535	0.1844	0.0746	0.0255	0.2516	0.2753	0.3185	0.3742
2.0	0.0118	-0.0251	0.0285	0.0474	0.0625	0.0333	0.0366	0.0428	0.0490	0.2967	0.2026	0.0737	0.0295	0.1832	0.2037	0.2398	0.3186
2.5	0.0098	-0.0058	0.0148	0.0221	0.0279	0.0147	0.0154	0.0168	0.0182	0.2096	0.1269	0.0337	0.0109	0.0798	0.0848	0.0983	0.1364
3.0	0.0027	-0.0014	0.0035	0.0053	0.0067	0.0024	0.0024	0.0023	0.0022	0.1016	0.0524	0.0084	0.0014	0.0199	0.0206	0.0224	0.0295
3.5	-0.00005	-0.00056	0.00002	0.00022	0.00038	-0.00029	-0.00033	-0.0004	-0.00047	0.0229	0.0097	0.00062	-0.00034	0.0013	0.0013	0.0013	0.0016
4.0	-0.00002	-0.0001	-0.00001	0.00002	0.00004	-0.00012	-0.00013	-0.00016	-0.00019	0.0081	0.0028	0.00008	-0.00008	0.00019	0.00019	0.0002	0.00022
4.5	0.00001	0.00000	0.00001	0.00001	0.00001	-0.00001	-0.00001	-0.00002	-0.00002	0.0017	0.00049	0.00002	0.00000	0.00001	0.00001	0.00001	0.00001
5.0	-0.00002	-0.00002	-0.00002	-0.00002	-0.00002	-0.00003	-0.00003	-0.00003	-0.00003	0.00054	0.0001	-0.00002	-0.00002	0.00000	0.00000	0.00000	0.00001
6.0	0.00000	0.00001	0.00001	0.00001	0.00001	0.00001	0.00001	0.00001	0.00001	0.0001	0.00002	0.00001	0.00001	0.00000	0.00000	0.00000	0.00000

Table 3.2: Risk performances of various estimators of M

		$\hat{M}_\lambda$				T_λ				t_c				H_c			
$\lvert\gamma\rvert$	T	$\lambda = 0$	$\lambda = \sqrt{3}/2$	$\lambda = 4-2\sqrt{2}$	$\lambda = \sqrt{2}$	$\lambda = \sqrt{3}/2$	$\lambda = 1$	$\lambda = 1.25$	$\lambda = 1.5$	$c = 1/3$	$c = 1/2$	$c = 1$	c=$\sqrt{3}$	$c = 0.5$	$c = 1$	c=$\sqrt{2}$	$c = 2$
0	0.6572	0.7767	0.6659	0.6776	0.7058	0.6432	0.6494	0.6671	0.6926	0.4249	0.4765	0.6523	0.8718	0.7629	0.6950	0.6082	0.4810
0.5	0.4331	0.4866	0.4479	0.4650	0.4901	0.4428	0.4494	0.4653	0.4859	0.3326	0.3627	0.4640	0.5818	0.4826	0.4643	0.4372	0.3843
1.0	0.3136	0.3239	0.3301	0.3472	0.3663	0.3364	0.3424	0.3551	0.3701	0.2932	0.3091	0.3597	0.4012	0.3258	0.3384	0.3557	0.3671
1.5	0.2594	0.2485	0.2759	0.2933	0.3100	0.2870	0.2925	0.3035	0.3156	0.2950	0.2985	0.3132	0.3139	0.2516	0.2753	0.3185	0.3742
2.0	0.1971	0.1804	0.2094	0.2228	0.2345	0.2187	0.2224	0.2298	0.2375	0.2665	0.2538	0.2403	0.2250	0.1832	0.2037	0.2398	0.3186
2.5	0.0858	0.0791	0.0888	0.0929	0.0964	0.0916	0.0926	0.0945	0.0965	0.1291	0.1154	0.0992	0.0902	0.0798	0.0848	0.0983	0.1364
3.0	0.0210	0.0198	0.0213	0.0219	0.0224	0.0216	0.0217	0.0219	0.0221	0.0334	0.0283	0.0229	0.0211	0.0199	0.0206	0.0224	0.0295
3.5	0.0013	0.0013	0.0013	0.0013	0.0014	0.0013	0.0013	0.0013	0.0013	0.0021	0.0017	0.0014	0.0013	0.0013	0.0013	0.0013	0.0016
4.0	0.0002	0.00019	0.0002	0.0002	0.0002	0.0002	0.0002	0.0002	0.00019	0.00031	0.00025	0.0002	0.00019	0.00019	0.00019	0.0002	0.00022
4.5	0.00001	0.00001	0.00001	0.00001	0.00001	0.00001	0.00001	0.00001	0.00001	0.00002	0.00001	0.00001	0.00001	0.00001	0.00001	0.00001	0.00001
5.0	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
6.0	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000

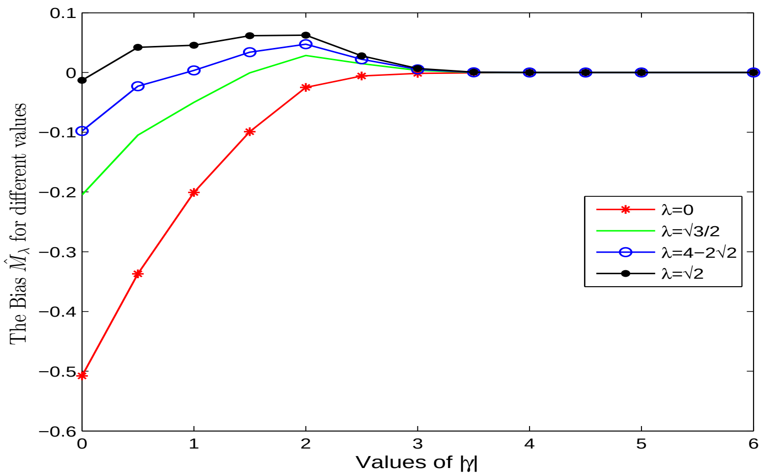

Figure 3.1: Bias of $\hat{M}_\lambda$.

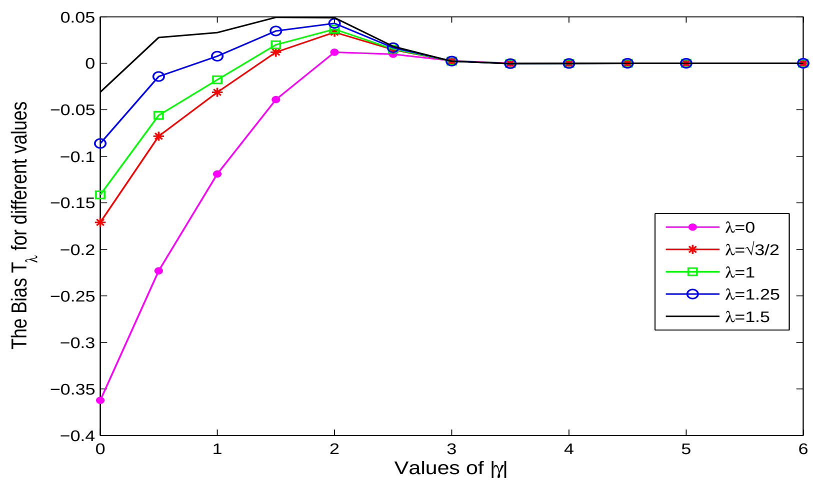

Figure 3.2: Bias of T_λ.

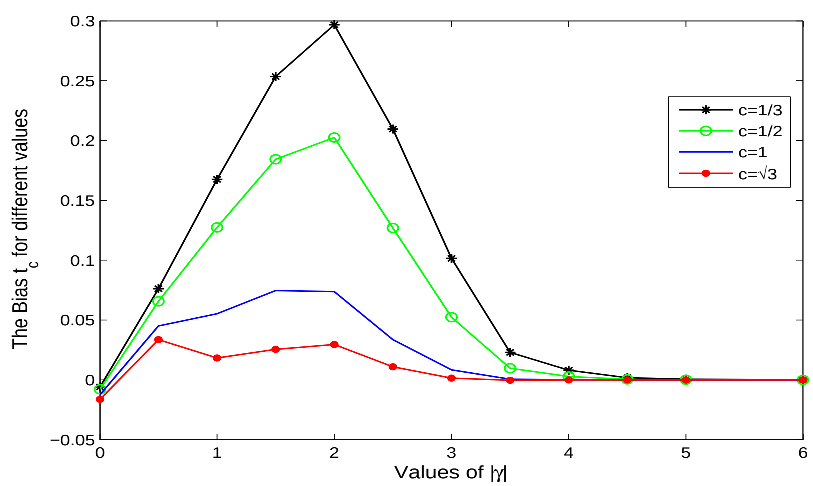

Figure 3.3: Bias of T_c.

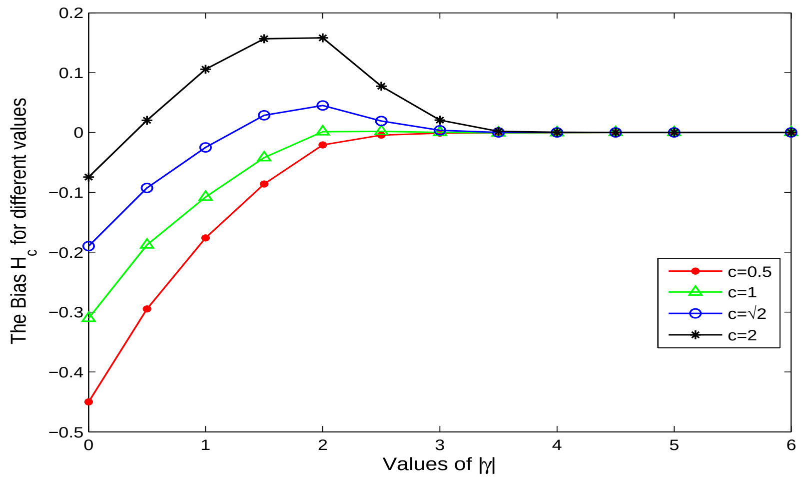

Figure 3.4: Bias of H_c.

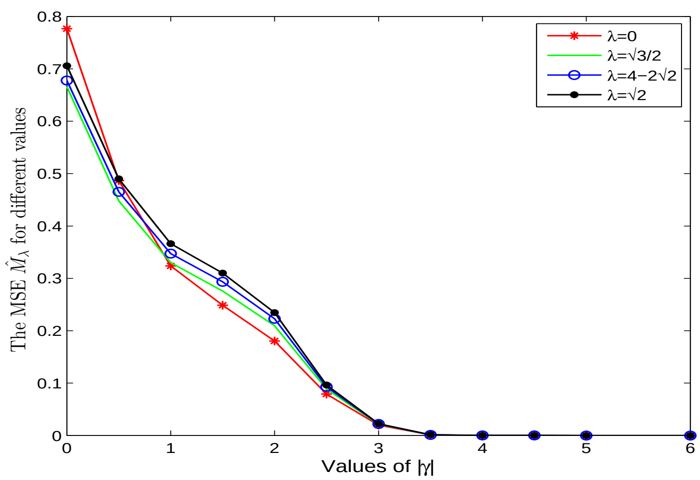

Figure 3.5: MSE of $\hat{M}_\lambda$.

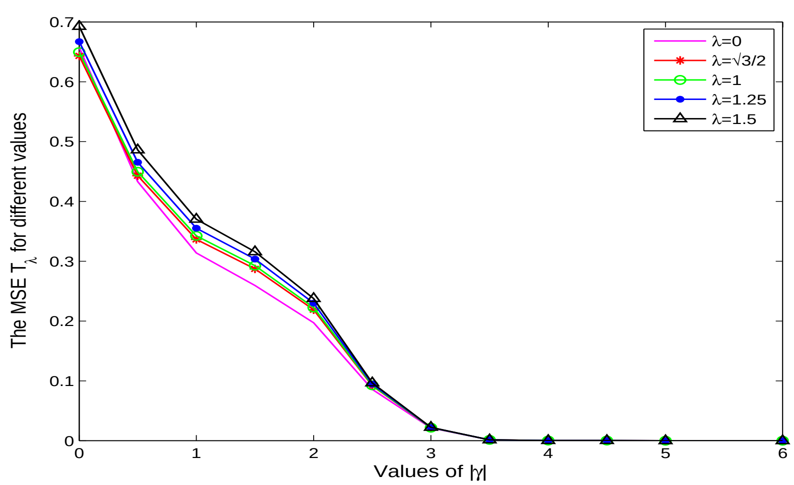

Figure 3.6: MSE of T_λ.

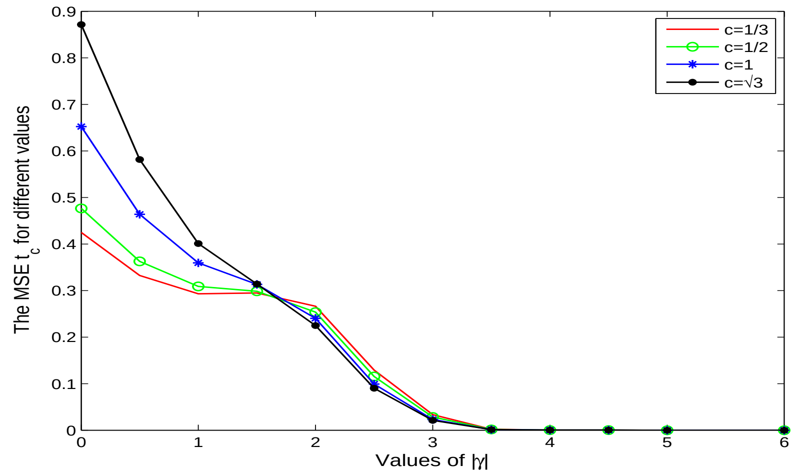

Figure 3.7: MSE of T_c.

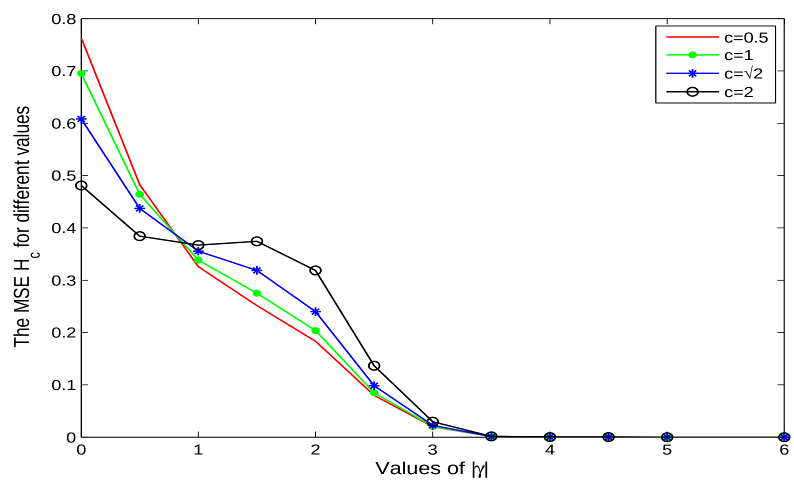

Figure 3.8: MSE of H_c.

Chapter 4

Estimating Volatility of the Selected Security

4.1 Introduction

All through this chapter we assume that investors avoid risks while taking investment related decisions and therefore when compelled to choose between two securities one with the lower volatility, or risk, is chosen. Typically a selection rule is used for this purpose. Given two populations (e.g., the two securities) and a selection rule, it is natural for the decision maker (in this case, the investor) to look for an estimate of the parameters (e.g., volatility) of the population which is eventually selected by using the specified selection rule. It is noteworthy that this estimate is computed before any actual selection is made, by using the samples from both populations and the selection rule under consideration. As an outcome of this exercise the investor gets an idea of the expected risk involved with the selection rule employed on the populations under consideration.

In this chapter we use the model developed in Gangopadhyay et al. [42] a model is developed for selecting the regression line with higher slope to estimate the risk of the security selected with the goal of risk-minimization. For this purpose we use Capital Asset Pricing Model (CAPM). In this model the rate of return (y) of a security relates to regression coefficient β of the security as given below:

$$y - r_f = \beta(x - r_f) + \varepsilon$$

where r_f is the risk-free rate of return, for example, the return on short-term bonds, β is the market risk of the security, x is the rate of return on the market portfolio and ε is the random error. Some works related to regression models are available in Chaturvedi and Shalabh [31], Chaturvedi and Wan [32], Singh et al. [119], Bastien et al. [21], Magnanensi et al. [75], Meyer et al. [77], Ageeva and Kharin [4], Kharin [62], Maevskii and Kharin [74], Kumar et al. [65] and Misra et al. [91].

In this chapter, the model is described in Section 4.2. Here we have discussed how the problem of selecting the regression line with lower slope is converted to the problem of selecting the normal population with lower mean. In Section 4.3 this model is applied to the decision problem of an investor in stocks considered in this chapter. Constructions of the estimators are described in Section 4.4. Here, we construct the estimators for volatility of the selected security when the security corresponding to lower volatility is selected. Finally in Section 4.5 the developed theory is applied to real data. The bias and mean squared error risk performances of the estimators of volatility of the selected security are numerically compared and the graphs representing the bias and MSE risks of the estimators are drawn. Section 4.6 is the discussions of results.

4.2 Description of the model

Let X, Y_1 and Y_2 be three random variables. Suppose we choose n values $x_{[1]} < x_{[2]} < \ldots < x_{[n]}$ of X. The density function of Y_i, where $i \in \{1,2\}$, given $X = x_{[j]}$, where $j \in \{1,2,\ldots,n\}$, is denoted by $f_{i|j}$ and the corresponding random variable by $Y_{i|j}$. For each pair (i,j), $f_{i|j}$ is assumed to be normal with mean $\mu_{i|j}$ and known standard deviation σ. We assume that the dependence of each Y_i on X satisfies the linear regression model with the regression line $Y_i = \alpha_i + \beta_i X$, for each $i \in \{1,2\}$. Consequently, $\mu_{i|j} = \alpha_i + \beta_i x_{[j]}$, for all $j \in \{1,2,\ldots,n\}$ and $i \in \{1,2\}$. It is realistic to assume that $Y_{i|j}$'s are mutually independent.

Consider the problem of selecting the population having regression line with the lower slope between two available populations. Therefore, our problem is to select the population corresponding to lower β_i. Further, we assume that the intercepts $\alpha_1 = \alpha_2 = \alpha$. This assumption is acceptable because in Capital Asset Pricing Model (CAPM) of the securities, which is our application domain, the intercepts are the risk free rates of returns which are assumed to be the same for all the securities under consideration. Suppose that we have independent random samples $Y_{i1|j}, Y_{i2|j}, \ldots, Y_{im|j}$ of size m where $i \in \{1,2\}$ and $j \in \{1,2,\ldots,n\}$ from the two populations. Since each $Y_{i|j}$ is a normal variate with mean $\mu_{i|j}$ and known standard deviation σ, the random variable $Z_i = \Sigma_{t=1}^{m} \Sigma_{j=1}^{n} Y_{it|j}/m$ is normal with mean $\sum_{j=1}^{n} \mu_{i|j}$ and variance $n\sigma^2/m$ for $i \in \{1,2\}$. So, we can say that selecting the regression line with lower slope is same as selecting the population with lower $\Sigma_{j=1}^{n} \mu_{i|j}$ where $i \in \{1,2\}$. That means we have to select the normal population with lower mean for which we can use the results obtained by Dahiya [36]. The connection between the problem of selecting the regression line corresponding to higher regression coefficient and the problem considered in Dahiya [36] was established by Gangopadhyay et al. [42]. The optimality properties of the natural selection rule are discussed in Bahadur and Goodman [16], Lehmann[72] and Eaton [40]. The natural selection rule says that for choosing the population corresponding to lower mean we have to select the population corresponding to lower sample mean because sample mean is the sufficient and complete statistic for the population mean. Let us denote the population with mean $\Sigma_{j=1}^{n}\mu_{i|j}$ as Π_i where $i \in \{1,2\}$. So, according to the natural selection rule we select the population Π_1 if $Z_1 \leq Z_2$ and select the population Π_2 otherwise. Further, we have to estimate the regression coefficient of the selected population. So our parameter of interest is

$$\beta_J, \text{ where } J = 1 \text{ if } Z_1 \leq Z_2 \text{ and } J = 2 \text{ otherwise.}$$

In the next section we apply this model to the problem of estimating risk of the selected security.

4.3 Application to finance

Suppose that an investor decides to purchase common stocks or securities whose annual rates of return can be related to the annual rate of return from a market portfolio (e.g., S&P BSE Sensex Index[India], S&P 100 composite stock index etc.). Let $Y_{i|t}$ be the rate of return (in percentage) on the ith security or common stock at time t and X_t be the rate of return (in percentage) on the market portfolio at time t. In modern portfolio theory, the "characteristic line" relating these rates of return for the ith security, $i \in \{1,2\}$, is of the form

$$Y_{i|t} = \alpha_i + \beta_i X_t + \varepsilon_{i|t}, t = 1, \ldots, T, \tag{4.1}$$

where the regression coefficients β_i and α_i are the slope coefficient and the intercept coefficient respectively and $\varepsilon_{i|t}$ is the random error. The regression coefficient β_i is also referred to as the *beta coefficient* of the ith security which is a measure of the market risk of a security.

Without loss of generality, we set the intercept term to zero. At this point, let us consider the Capital Asset Pricing Model (CAPM) of modern portfolio theory which represents the relationship between $Y_{i|t}$ and regression coefficient β_i of security i as given below:

$$(Y_{i|t} - r_f) = \beta_i (X_t - r_f) + \varepsilon_{i|t}, t = 1, ..., T, i \in \{1,2\} \tag{4.2}$$

where r_f is the risk-free rate of return, for example, the return on short-term bonds, which is not random. The intercept coefficient $\alpha_i = 0$ for $i = 1,2$ for the CAPM. For detailed discussion we refer to [46][Pages 165–166]. The beta coefficient, β_i is a measure of the systematic risk during time t which cannot be eliminated or minimized by diversification. It also provides an idea of the strength of the relationship between the security under consideration and the market portfolio. If the value of beta is more than one, then the corresponding security is volatile, or aggressive, because in that case, a one percent transformation in the market rate of return leads to more than one percent transformation in the ith security's rate of return. If the value of beta is lower than one, then the corresponding security is conservative and if the value is equal to one, then the corresponding security is neutral. An investor interested in choosing the security with the lower volatility chooses the one with the least value of the beta coefficient according to the selection rule. The estimate of the beta coefficient of the selected security is obtained by relating the technique developed in Gangopadhyay et al. [42] to CAPM. This is reasonable, since CAPM is a linear regression model. Suppose that $(rm_j, r_{i|j})$ are ordered pairs of observations with the rate of return on the market portfolio as the first coordinate and the rate of return on ith security as the second coordinate at time $j = 1, \ldots, n$, for $i = 1, 2$. Without loss of generality, we can assume that all rm_j's are distinct, and our $x_{[j]} = (rm_j - r_f)$, $j = 1, \ldots, n$, where r_f= risk-free rate of return (eg. return on short-term bonds). Our $Y_{i|j} = (r_{i|j} - r_f)$, which is assumed to have the density function $f_{i|j}$ for each pair (i, j) where $f_{i|j}$ is normal with mean $\mu_{i|j}$ and known standard deviation σ. Then according to equation (4.2), we can write

$$\mu_{i|j} = \beta_i x_{[j]}$$

where $i \in \{1,2\}$ and $j \in \{1,...,n\}$. We assume that $r_{i|j}$'s are mutually independent. So, $Y_{i|j}$'s are also mutually independent, because r_f is a constant. From the above discussions it is clear that selecting the security with lower regression coefficient is the same as selecting the security with lower $\mu_{i|j}$. $Y_{i|jt}, i = 1,2; j = 1, \ldots, n; t = 1, \ldots, m$, an independent random sample of size m is drawn from each of these two security returns. Then the "natural selection rule" in Bahadur and Goodman [16] for selecting a security will be as follows:

"*Select the security 1 if* $Z_1 < Z_2$ *and security 2, otherwise*".

Our objective is to estimate β_J, where $J = 1$, if $Z_1 \leq Z_2$ and $J = 2$, otherwise. Since, each $Y_{i|j}$ is normally distributed with mean $\mu_{i|j}$ and known standard deviation σ, the probability distribution of the random variable Z_i is normal with mean $\theta_i = \sum_{j=1}^{n} \mu_{i|j} = \beta_i \Sigma_{j=1}^{n} x_{[j]} = c\beta_i$ and variance $\tau^2 = n\sigma^2/m$, for $i = 1, 2$.

4.4 Construction of the estimators

For finding the estimators of parameters of the selected population we always start with the component problem, i.e. in case, we have only one population then we find the maximum likelihood estimator, take the analogue of that estimator and call it a natural estimator. This approach is followed by several previous investigations, such as Dahiya [36], Hsieh [58], Vellaisamy et al. [130], Vellaisamy and Sharma [131], Vellaisamy [125], and Kumar and Gangopadhyay [66]. We consider the component problem and obtain the maximum likelihood estimator of β_i as follows

$$b_i = \frac{\sum_{j=1}^{n} \left(Y_{i|j} - \frac{1}{n}\sum_{k=1}^{n} Y_{i|k}\right)\left(x_{[j]} - \frac{1}{n}\sum_{k=1}^{n} x_{[k]}\right)}{\sum_{j=1}^{n} \left(x_{[j]} - \frac{1}{n}\sum_{k=1}^{n} x_{[k]}\right)^2}$$

Let, $c_j = x_{[j]} - \sum_{k=1}^{n} x_{[k]}/n$ and $d = 1/\sum_{j=1}^{n}\left(x_{[j]} - \frac{1}{n}\sum_{k=1}^{n} x_{[k]}\right)^2$. Here, it is to be noted that $\sum_{j=1}^{n} c_j = 0$. Therefore we obtain

$$\begin{aligned} b_i &= d\left[\sum_{j=1}^{n} c_j Y_{i|j} - \frac{1}{n}\left(\sum_{j=1}^{n} c_j\right)\left(\sum_{k=1}^{n} Y_{i|k}\right)\right] \\ &= d\left(\sum_{j=1}^{n} c_j Y_{i|j}\right) \\ &= \sum_{j=1}^{n} w_j Y_{i|j}, \text{ where } w_j = dc_j \end{aligned}$$

Therefore, we get a natural estimator of $c\beta_J$ as $\sum_{t=1}^{m}\sum_{j=1}^{n} w_j Y_{Jt|j}/m$. Below we list the other estimators of $c\beta_J$ based on the derived estimators in Chapter 3.

1. $$\hat{M}_\lambda = Z_J + \lambda\sigma_z\phi\left(\frac{Z}{\sigma_z}\right)$$

   where $\lambda \geq 0$ is an arbitrary constant, $Z = Z_2 - Z_1$ and $Z_J = Z_1$ if $Z_1 < Z_2$ and $Z_J = Z_2$ otherwise, $\sigma_z^2 = Var(Z) = 2\tau^2$, and $\phi(x)$ is the standard normal density function given by $\phi(x) = \mathrm{e}^{-\frac{x^2}{2}}/\sqrt{2\pi}$, where $-\infty < x < \infty$ and the bias of Z_J is $E(Z_J) - E(\beta_J) = -\sigma_z\phi(\theta/\sigma_z)$, where $\theta = \theta_2 - \theta_1$. The estimator $\hat{M}_\lambda$ is obtained by subtracting a λ multiple of the estimated bias of Z_J from itself.

2. $t_c = Z_2 - Z\Phi(cZ/\sigma_z) + c\sigma_z\phi(cZ/\sigma_z)$, where $c > 0$ is an arbitrary constant and $\Phi(x)$ is the standard normal cdf. The estimator t_c is an unbiased estimator of $\theta_2 - \theta\Phi\left(\frac{c\theta}{\sigma_z\sqrt{1+c^2}}\right)$, which approaches $E[\theta_J] = E[\theta_1 I_1 + \theta_2 I_2] = E[\theta_1 P(Z_1 \leq Z_2) + \theta_2 P(Z_1 > Z_2)]$ where $\theta = \theta_2 - \theta_1$ for larger c. Hence the bias of t_c as an estimator of θ_J can be controlled by making c large.

3. The estimator T of $E[\theta_J]$ given by

   $$T = Z_2 - Z\Phi\left(\frac{Z}{\sigma_z}\right)$$

   and propose to use it to estimate θ_J. The estimator T, actually, is natural estimator of $E[\theta_J]$ and its bias will be the same whether we use it as an estimator of θ_J or $E[\theta_J]$.

4. Another estimator of this type is defined as

   $$T_\lambda = T - \lambda\left[Z\left\{\phi\left(\frac{Z}{\sigma_z}\right) - \Phi\left(\frac{Z}{\sqrt{2}\sigma_z}\right)\right\} - \frac{\sigma_z}{\sqrt{2}}\phi\left(\frac{Z}{\sqrt{2}\sigma_z}\right)\right]$$

   where $\lambda \geq 0$ is an arbitrary constant. The estimator T_λ is the same as that of $\hat{M}_\lambda$ and is obtained on subtracting a λ multiple of the estimated bias of T from itself.

5. An hybrid estimator is considered as

$$H_c = \begin{cases} \frac{Z_1+Z_2}{2}, & \text{if } |Z_1 - Z_2| < c\sigma_z \\ \min(Z_1, Z_2), & \text{if } |Z_1 - Z_2| \geq c\sigma_z. \end{cases}$$

where $c \geq 0$ is an arbitrary constant. Note that for c=0 we obtain Z_J which is the same as $\hat{M}_\lambda$ for $\lambda = 0$.

6. Finally we have an improved Estimator: Let us consider the class of estimators of the form $\hat{M}_\lambda = Z_J + \lambda \sigma_z \phi\left(\frac{Z}{\sigma_z}\right)$. Further, define the estimator $\hat{M}_{\lambda*}$ by

$$\hat{M}_{\lambda*} = \begin{cases} \hat{M}_\lambda, & \text{if } \lambda \leq \gamma* \\ \hat{M}_{\gamma*}, & \text{if } \lambda > \gamma*, \end{cases}$$

where $\gamma* = \sqrt{3}/2$. Then $\hat{M}_{\lambda*}$ improves $\hat{M}_\lambda$ with respect to the squared error loss if $\lambda > \gamma* > 0$ for some $\eta = \{\theta_1, \theta_2\}$.

Consider the group $G = g(x)_c = x + c, c \in R$ of affine transformations. Under this transformation $X_{ij} \mapsto X_{ij} + c, Z_i \mapsto Z_i + c$ and $\theta_J \mapsto \theta_J + c$, if we take the squared error loss $L(d, \theta_J) = (d - \theta_J)^2$, then the estimation problem is invariant.

Theorem 4.4.1. *Let us consider the class of estimators of the form* $\hat{M}_\lambda = Z_J + \lambda \sigma_z \phi\left(\frac{Z}{\sigma_z}\right)$. *Further, define the estimator* $\hat{M}_{\lambda*}$ *by*

$$\hat{M}_{\lambda*} = \begin{cases} \hat{M}_\lambda, & \text{if } \lambda \leq \gamma* \\ \hat{M}_{\gamma*}, & \text{if } \lambda > \gamma*, \end{cases}$$

where $\gamma* = \sqrt{3}/2$. *Then* $\hat{M}_{\lambda*}$ *improves* $\hat{M}_\lambda$ *with respect to the squared error loss if* $\lambda > \gamma* > 0$ *for some* $\eta = \{\alpha_1, \alpha_2\}$.

Proof. We refer proof of Theorem 3.4.1 in Section 3.4. □

4.5 Application to real data and comparison of estimators

We have considered the data for the returns of a market portfolio (rm_j) (BSE SENSEX) and two securities (Reliance Industrial Infrastructure Ltd (RIIL.BO) -BSE $(r_j^{(1)})$ and Reliance Industries Ltd (RELIANCE.NS)) $(r_j^{(2)})$ from 07 July 2014 to 24 November 2014 for $n = 20$ weeks (data retrieved from https://in.finance.yahoo.com). We take that the risk-free rate of return (r_f) is 8.50% according to the current interest rate of short term bonds. We calculate that the value of β_1 = 1.290 and β_2

=1.177 through the CAPM model for the considered data using the least squared error estimates. We compare the bias and risk values of the estimators of β_J. For this purpose, we have generated random samples of size $m = 20$ from both the normal populations with means $\beta_1 x_{[j]}$ and $\beta_2 x_{[j]}$, respectively, for different known variances, where $j = 1,...,20$. These samples are denoted by $Y_{jt}^{(i)}$ for $j = 1,...,20; t = 1,...,20; i = 1,2$. We have calculated $Z_i = \Sigma_{t=1}^{20}\Sigma_{j=1}^{20} Y_{jt}^{(i)}/20, i = 1,2$. We repeat this process 5000 times. The biases and mean squared error risks are calculated for all the above estimators using Monte-Carlo simulation for different values of τ^2, where $\tau^2 = n\sigma^2/m$. The values of λ and c are chosen in such a way that they give better results. The biases and mean square error risks performances for the various estimators are given in Tables $4.1 - 4.2$. The graphs representing biases and mean squared error risks of the estimators mentioned above are displayed in Figure $4.1 - 4.8$. In these graphs the X axis represents τ , which is the standard deviation of $Z_i, i = 1,2$ and Y axis represents the bias or MSE risks.

Tables and Graphs of Biases of different estimators:

From Figure $4.1 - 4.4$, the X axis represents τ, standard deviation of $Z_i, i = 1,2$ and the Y axis represents the bias values of the estimators. In the graphs of biases,we observe that with respect to bias $\hat{M}_\lambda$ for $\lambda = \sqrt{2}$, T_λ for $\lambda = 1.5$, H_c for $c = 2$ are better than other cases.

From Table 4.1, it can be observed that for all values of τ, the estimator T_λ is better than $\hat{M}_\lambda$ for $\lambda = \sqrt{3}/2$. The estimator $\hat{M}_\lambda$ is dominated by T for $\lambda = 0$. From Table 4.1, it can be seen that for all value of τ, the estimator H_c is dominated by estimator t_c for values $c = 0.5, 1$. Also observed from Table 4.1, the bias performance of the estimator t_c is better than all other estimators for all values of τ. As we decrease the value of standard deviation then bias performances become better for all the estimators. We see that the estimator t_c is overall a good estimator based on the biases performances.

Graphs of Mean squared error risks of different estimators:

In Figure $4.5 - 4.8$, the X axis represents τ, standard deviation of $Z_i, i = 1,2$ and the Y axis represents mean squared error risks of different estimators. In the graphs of mean squared error (MSE) risks, we observe that with respect to risk $\hat{M}_\lambda$ for $\lambda = \sqrt{3}/2$, T_λ for $\lambda = \sqrt{3}/2$, T_c for $c = 1/3$, H_c for $c = 2$ perform better among all cases.

From Table 4.2, it can seen that for all values of τ, the estimator T_λ is better than the estimator $\hat{M}_\lambda$ when value of $\lambda = \sqrt{3}/2$. From Table 4.2, we observe that for all values of τ, the estimator t_c is better than the estimator H_c for value $c = 1/2$ and 1. For all values of τ and for $\lambda = 0$, the estimator $\hat{M}_\lambda$ is dominated by T. From Table 4.2 it can be seen that for all values of λ, the estimator T_λ has the smallest risk for all values of τ. From Table 4.2 we observe that the estimator T_c and H_c have smaller risk for c=1/3 and c=2, respectively. As we decrease the value of standard deviation then risk performances become better for all estimators.

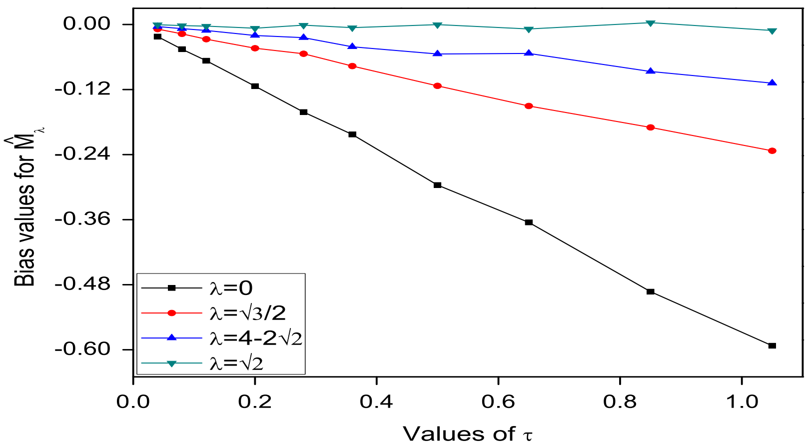

Figure 4.1: Bias of $\hat{M}_\lambda$ for different values of λ.

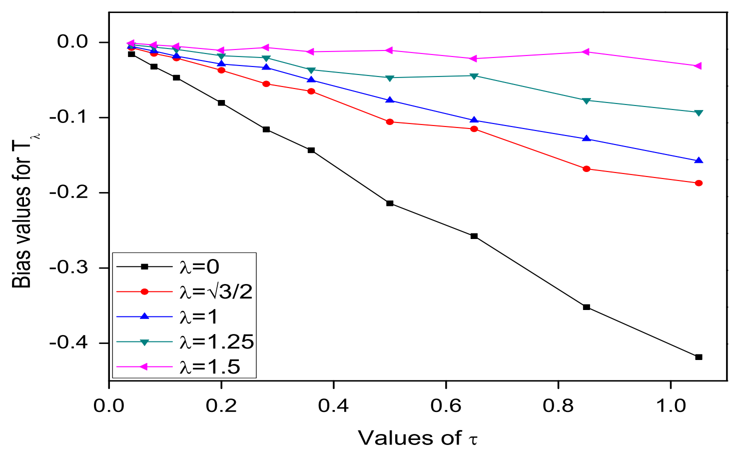

Figure 4.2: Bias of T_λ for different values of λ.

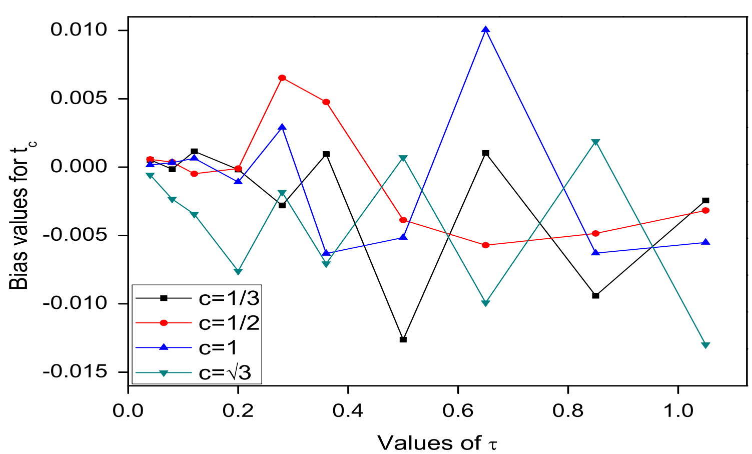

Figure 4.3: Bias of t_c for different values of c.

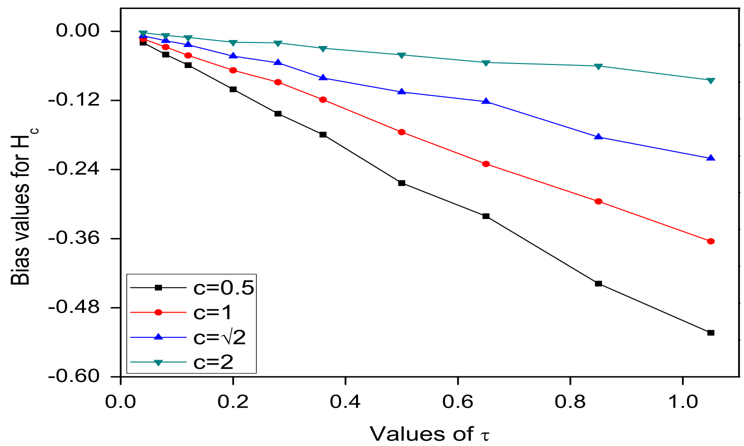

Figure 4.4: Bias of H_c for different values of c.

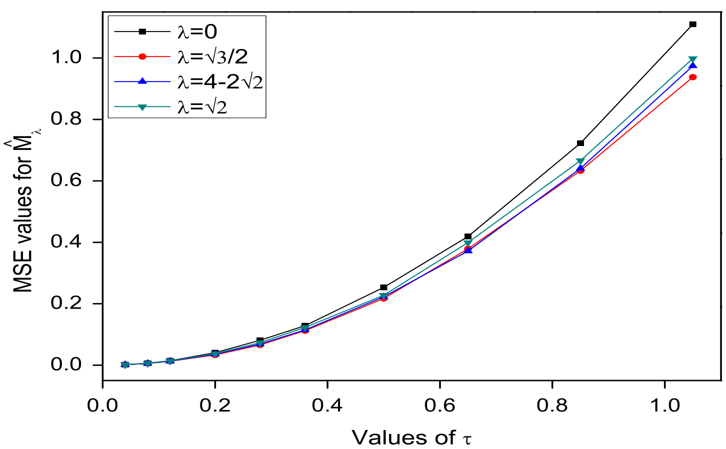

Figure 4.5: MSE of $\hat{M}_\lambda$ for different values of λ.

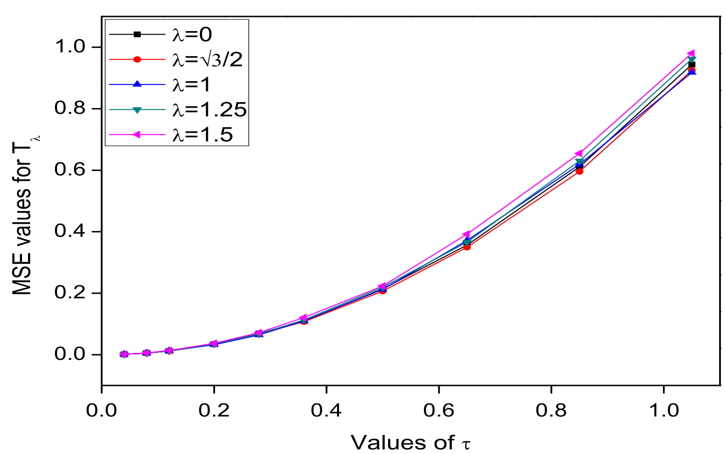

Figure 4.6: MSE of T_λ for different values of λ.

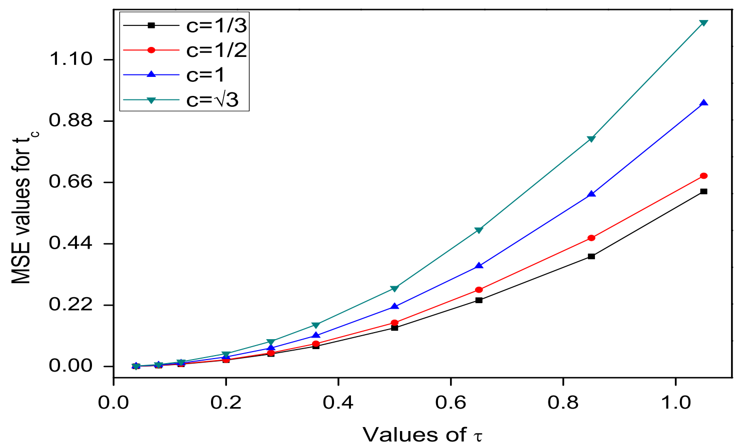

Figure 4.7: MSE of t_c for different values of c.

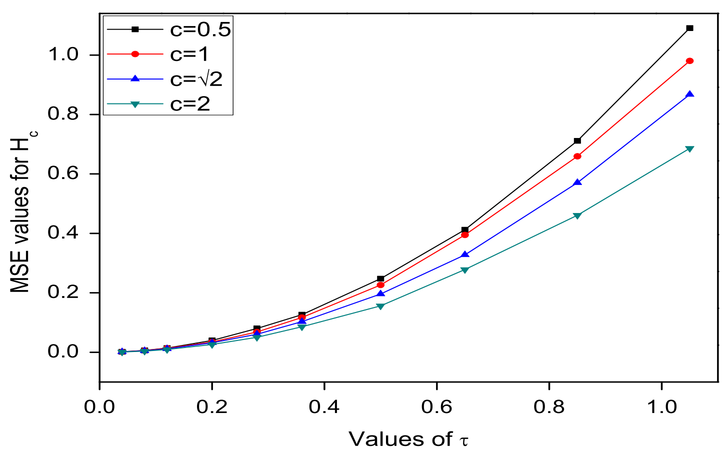

Figure 4.8: MSE of H_c for different values of c.

Table 4.1: Bias of various estimators

		$\hat{M}_\lambda$				T_λ				t_c				H_c			
τ	T	$\lambda=0$	$\lambda=\sqrt{3}/2$	$\lambda=4-2\sqrt{2}$	$\lambda=\sqrt{2}$	$\lambda=\sqrt{3}/2$	$\lambda=1$	$\lambda=1.25$	$\lambda=1.5$	$c=1/3$	$c=1/2$	$c=1$	c=$\sqrt{3}$	$c=0.5$	$c=1$	c=$\sqrt{2}$	$c=2$
0.04	-0.01565	-0.02226	-0.00824	-0.00368	-0.00024	-0.00704	-0.00539	-0.00318	-0.00104	0.00054	0.00057	0.00016	-0.00056	-0.01960	-0.01319	-0.00798	-0.00246
0.08	-0.03223	-0.04542	-0.01722	-0.00747	-0.00193	-0.01483	-0.01156	-0.00633	-0.00343	-0.00015	0.00037	0.00033	-0.00233	-0.04032	-0.02729	-0.01628	-0.00701
0.12	-0.04695	-0.06667	-0.02692	-0.01103	-0.00291	-0.02085	-0.01833	-0.00939	-0.0053	0.00116	-0.00048	0.00065	-0.00344	-0.05859	-0.04180	-0.02354	-0.01060
0.20	-0.08050	-0.11374	-0.04356	-0.02017	-0.00667	-0.03708	-0.02882	-0.01774	-0.01058	-0.00017	-0.0001	-0.00109	-0.00761	-0.10082	-0.06758	-0.04303	-0.01893
0.28	-0.11570	-0.16146	-0.05387	-0.02423	-0.00106	-0.05517	-0.03351	-0.02045	-0.00687	-0.0028	0.00654	0.0029	-0.00183	-0.14307	-0.08818	-0.05461	-0.02000
0.36	-0.14334	-0.20277	-0.07659	-0.04113	-0.00571	-0.06503	-0.05016	-0.03637	-0.01248	0.00095	0.00477	-0.00632	-0.00706	-0.17939	-0.11871	-0.08112	-0.02938
0.50	-0.21392	-0.29624	-0.11316	-0.05436	-0.00018	-0.10563	-0.07718	-0.04694	-0.01061	-0.01262	-0.00386	-0.00515	0.00071	-0.26342	-0.17496	-0.10542	-0.04088
0.65	-0.25752	-0.36506	-0.15019	-0.05329	-0.00805	-0.11498	-0.10359	-0.04416	-0.02165	0.00104	-0.00571	0.01004	-0.00992	-0.32102	-0.23029	-0.12207	-0.05400
0.85	-0.35190	-0.49287	-0.18983	-0.08668	0.00357	-0.16809	-0.12831	-0.07698	-0.01259	-0.00941	-0.00485	-0.00629	0.00189	-0.43813	-0.29539	-0.18358	-0.06006
1.05	-0.41838	-0.59261	-0.23285	-0.10804	-0.01094	-0.18709	-0.15763	-0.09289	-0.03129	-0.00244	-0.00317	-0.00551	-0.01299	-0.52343	-0.36461	-0.22064	-0.08459

Table 4.2: Mean Squared Error of various estimators

		$\hat{M}_\lambda$				T_λ				t_c				H_c			
τ	T	$\lambda=0$	$\lambda=\sqrt{3}/2$	$\lambda=4-2\sqrt{2}$	$\lambda=\sqrt{2}$	$\lambda=\sqrt{3}/2$	$\lambda=1$	$\lambda=1.25$	$\lambda=1.5$	$c=1/3$	$c=1/2$	$c=1$	c=$\sqrt{3}$	$c=0.5$	$c=1$	c=$\sqrt{2}$	$c=2$
0.04	0.00137	0.00160	0.00141	0.00146	0.00147	0.00135	0.00139	0.00144	0.00144	0.00092	0.00105	0.00142	0.00181	0.00158	0.00147	0.00132	0.00101
0.08	0.00543	0.00638	0.00547	0.00553	0.00583	0.00531	0.00537	0.00546	0.00573	0.00360	0.00402	0.00536	0.00721	0.00626	0.00574	0.00498	0.00399
0.12	0.01225	0.01434	0.01291	0.01278	0.01372	0.01210	0.01268	0.01261	0.01352	0.00819	0.00945	0.01239	0.01679	0.01416	0.01351	0.01159	0.00961
0.20	0.03452	0.04056	0.03325	0.03578	0.03765	0.03378	0.03251	0.03521	0.03723	0.02302	0.02423	0.03451	0.04609	0.03979	0.03474	0.03202	0.02613
0.28	0.06934	0.08117	0.06583	0.06851	0.07313	0.06838	0.06447	0.06754	0.07181	0.04500	0.04901	0.06637	0.09012	0.08004	0.06854	0.06051	0.05055
0.36	0.10924	0.12853	0.11227	0.11483	0.12275	0.10710	0.11005	0.11332	0.1209	0.07256	0.08193	0.11075	0.15017	0.12640	0.11687	0.10327	0.08578
0.50	0.21431	0.25306	0.21722	0.22258	0.22782	0.20658	0.21250	0.21932	0.22349	0.13857	0.15697	0.21458	0.28121	0.24766	0.22651	0.19611	0.15623
0.65	0.35659	0.41905	0.37881	0.37151	0.3992	0.34961	0.37152	0.36629	0.39196	0.23783	0.27503	0.36025	0.49084	0.41223	0.39467	0.32801	0.27855
0.85	0.61074	0.72300	0.63315	0.64077	0.66612	0.59561	0.61908	0.63040	0.65484	0.39492	0.46097	0.61739	0.81786	0.71149	0.65952	0.57034	0.46107
1.05	0.94372	1.10985	0.93801	0.97437	0.99851	0.92515	0.91811	0.96231	0.98031	0.62789	0.68408	0.94425	1.23441	1.09101	0.98044	0.86743	0.68601

4.6 Results and Discussions

From all the figures and tables, we observe that as τ increases the value of absolute biases and risks of all the estimators increase. From Figure 4.5 we observe the risk performance of $\hat{M}_\lambda$ for $\lambda = \sqrt{3}/2$ is lower than the other estimators in this class. As we decrease the value of standard deviation then bias and risk performances become better for all the estimators. From Figure 4.7, we observe that as we reduce the value of c the MSE risk performance of T_c becomes better. Figure 4.8 shows the improvement of the risk performance of H_c as the value of c increases. Figure 4.1 – 4.4 show that the bias performance of $\hat{M}_\lambda$ for $\lambda = \sqrt{2}$ is better than all the other estimators. Risk performance of T_c for $c = 1/3$ and H_c for $c = 2$ are better than other estimators.

Chapter 5

Estimating Quantile of the Selected Normal Population

5.1 Introduction

Let Π_1 and Π_2 be two normal populations with unknown mean μ and unknown variances σ_1^2 and σ_2^2 respectively. Suppose independent random samples $(X_{11},..,X_{1n_1}), n_1 \geq 2$ and $(X_{21},..,X_{2n_2}), n_2 \geq 2$ are available from these two normal populations Π_1 and Π_2 respectively. Further, let the mean and the sum of squared deviations from the mean for the two samples are denoted by $X_1 = \frac{1}{n_1}\Sigma_{j=1}^{n_1} X_{1j}$ and $S_1^2 = \Sigma_{j=1}^{n_1}(X_{1j} - X_1)^2$ and $X_2 = \frac{1}{n_2}\Sigma_{j=1}^{n_2} X_{2j}$ and $S_2^2 = \Sigma_{j=1}^{n_2}(X_{2j} - X_2)^2$ respectively.

For the first population, (X_1, S_1^2) is the complete and sufficient statistic for (μ, σ_1^2). Similarly, for the 2nd population, (X_2, S_2^2) is the complete and sufficient statistic for (μ, σ_2^2). X_1 and X_2 have $N(\mu, \sigma_1^2/n_1)$ and $N(\mu, \sigma_2^2/n_2)$ distributions respectively. S_1^2 and S_2^2 have $\sigma_1^2\chi_{n_1-1}^2$ and $\sigma_2^2\chi_{n_2-1}^2$ distributions respectively. We would like to select the population with smaller variance. For this, we select the population corresponding to the smallest S_i^2, $i = 1, 2$, that is, Π_1 is selected if $S_1^2 \leq S_2^2$ and Π_2 is selected if $S_2^2 < S_1^2$. In this chapter, we consider the problem of estimating a quantile of the selected normal population.

$$\theta_J = \mu + \eta\ \sigma_J, \tag{5.1}$$

where $J = 1$ if $S_1^2 \leq S_2^2$ and $J = 2$, otherwise and η is any constant . For $\eta \neq 0$, the only works available so far are Sharma and Vellaisamy [117], who have considered the case of different means and equal variances and Kumar and Kar [67], who have considered the case of different means and different variances. Some inadmissibility results of affine equivariant estimators of quantiles of normal populations have been discussed by Kumar and Tripathy [69] and Tripathy and Kumar [124].

In this chapter, we consider the problem of estimating quantile of a selected normal population when mean is same and unknown but variances are unknown and different. In Section 5.2, several estimators are proposed. In Section 5.3, we formulate the problem and prove the admissibility of a natural estimator within a class of linear estimators. In Section 5.4, we consider a more general class of estimators and found a class of admissible estimators. In Section 5.5, the numerical comparisons of the estimators are obtained.

5.2 Some Improved Estimators

In this section, we propose several estimators for $\theta_J = \mu + \eta\ \sigma_J$, where J = 1 if $S_1^2 < S_2^2$ and $J = 2$, otherwise and η is a constant.

It is to be noted here that the minimal sufficient statistic is not complete statistic. We consider the problem of estimating a quantile $\theta_1 = \mu + \eta\sigma_1$ of the first population with respect to the scale invariant loss function. The form of an affine equivariant estimator of θ_1 is $X_1 + \eta b S_1$ where b is real. The choice of b minimizing the risk of $X_1 + \eta b S_1$ is b_{n_1}, where

$$b_{n_1} = \frac{\Gamma(\frac{n_1}{2})}{\sqrt{2}\Gamma(\frac{n_1+1}{2})}.$$

The analogue of the best affine equivariant estimator is $\alpha_1 = X_1 + b_{n_J} S_J$, where $n_J = n_1$ if $S_J = S_1$ and $n_J = n_2$ if $S_J = S_2$.

In this kind of situation, a natural estimator of μ is $\hat{\mu} = \frac{S_2^2 X_1 + S_1^2 X_2}{S_2^2 + S_1^2}$, obtained and studied by Graybill and Deal [45], Khatri and Shah [63], Brown and Cohen [29], Cohen and Sackrowitz [33], and Moore and Krishnamoorthy [94]. We propose some estimators of θ_J having smaller risk than α_1 by replacing X_1 by $\hat{\mu}$ and these estimators are as follows:

$$\alpha_2 = \frac{n_1(n_1-1)S_2^2 X_1 + n_2(n_2-1)S_1^2 X_2}{n_1(n_1-1)S_2^2 + n_2(n_2-1)S_1^2} + \eta b_{n_J} S_J,$$

$$\alpha_3 = \frac{n_1(n_1-3)S_2^2 X_1 + n_2(n_2-3)S_1^2 X_2}{n_1(n_1-3)S_2^2 + n_2(n_2-3)S_1^2} + \eta b_{n_J} S_J,$$

$$\alpha_4 = X_1 + (X_2 - X_1)\left[\frac{b_1 S_1^2/n_1(n_1-1)}{S_1^2/n_1(n_1-1) + S_2^2/(n_2+2) + (\overline{Y}-\overline{X})^2/(n_2+2)}\right] + \eta b_{n_J} S_J,$$

$$\alpha_5 = X_1 + (X_2 - X_1)\left[\frac{b_2 n_2(n_2-1)S_1^2}{n_2(n_2-1)S_1^2 + n_1(n_1-1)S_2^2}\right] + \eta b_{n_J} S_J,$$

where $0 < b_1 < b_{\max}(n_1, n_2)$, $0 < b_2 < b_{\max}(n_1, n_2 - 3)$ and $b_{\max}(n_1, n_2) = \frac{2(n_2+2)}{n_2 E\{\max(V^{-1}, V^{-2})\}}$, where it is seen that V has an F distribution with (n_2+2) and (n_1-1) degrees of freedom.

$$\alpha_6 = (1 - \delta_{n_2} H(y))X_1 + \delta_{n_2} H(y) X_2 + \eta b_{n_J} S_J,$$

Where

$$\delta_{n_2} = \frac{(n_2-5)^2}{(n_2-1)^2}, \text{ if } n_2 \text{ is odd}$$
$$= \frac{(n_2-4)(n_2-6)}{n_2(n_2-2)}, \text{ if } n_2 \text{ is even}$$

and

$$H(y) = F\left(1, 1-\frac{n_2}{2}; \frac{n_2-1}{2}, y\right), \text{ if } 0 \leq y < 1$$
$$= \frac{1}{2}, \text{ if } y = 1$$
$$= \frac{(n_2-3)}{(n_2-1)y} F\left(1, \frac{3-n_2}{2}; \frac{n_2}{2}, \frac{1}{2}\right), \text{ if } y > 1$$

where F is the hypergeometric function(Lebedev, [70]) and denoted by ${}_2F_1$, and $y = \frac{S_2^2}{S_1^2}$.

$$\alpha_7 = \frac{\sqrt{n_1(n_1-1)}S_2X_1 + \sqrt{n_2(n_2-1)}S_1X_2}{\sqrt{n_1(n_1-1)}S_2 + \sqrt{n_2(n_2-1)}S_1} + \eta b_{n_J} S_J,$$

$$\alpha_8 = \frac{\sqrt{n_1}b_{n_2-1}S_2X_1 + \sqrt{n_2}b_{n_1-1}S_1X_2}{\sqrt{n_1}b_{n_2-1}S_2 + \sqrt{n_2}b_{n_1-1}S_1} + \eta b_{n_J} S_J,$$

and the estimator α_9 depends on the grand mean,

$$\alpha_9 = \frac{n_1X_1 + n_2X_2}{n_1+n_2} + \eta b_{n_J} S_J.$$

5.3 An Inadmissible Class of Estimators

Here, we consider the group $G = \{g_c : g_c(x) = x + c, -\infty < c < \infty\}$ of location transformations. Under this transformation, $X_1 \to X_1 + c$, $X_2 \to X_2 + c$, $S_1^2 \to S_1^2$, $S_2^2 \to S_2^2$, $\mu \to \mu + c$, $\sigma_J \to \sigma_J$ and $\theta_J \to \theta_J + c$. The estimation problem will be invariant if we consider the following loss function

$$L(\delta, \theta_J) = \left(\frac{\delta - \theta_J}{\sigma_J}\right)^2. \tag{5.2}$$

Under this loss function the equivariant estimator is given in this form

$$\delta_\phi = X_1 + \phi(\underline{Z}), \tag{5.3}$$

where, $\underline{Z} = (U, S_1^2, S_2^2)$, $U = X_2 - X_1$. We define the following functions for a function $\phi(\underline{z})$, $\underline{z} = (u, s_1^2, s_2^2)$

$$\begin{aligned} \phi_0(\underline{z}) &= \min\{u,0\}, \text{ if } \phi(\underline{z}) < \min\{u,0\}, \\ &= \phi(\underline{z}), \text{ if } \min\{u,0\} \leq \phi(\underline{z}) \leq \max\{u,0\}, \\ &= \max\{u,0\}, \text{ if } \phi(\underline{z}) > \max\{u,0\}, \end{aligned} \tag{5.4}$$

$$\phi_1(\underline{z}) \quad = \quad \max\{\min\{u,0\},\phi(\underline{z})\}, \tag{5.5}$$

and

$$\phi_2(\underline{z}) = \min\{\max\{u,0\},\phi(\underline{z})\}. \tag{5.6}$$

Then we get the following inadmissibility result of estimators which are equivariant under location groups of transformation.

Theorem 5.3.1. *Let δ_ϕ be a location equivariant estimator of θ_J. Let us define the functions ϕ_0, ϕ_1 and ϕ_2 as in* (5.4), (5.5) *and* (5.6) *respectively. Then with respect to the loss function* (5.2) *or squared error loss function we can show the following results.*

(i) For $\eta = 0$, δ_ϕ is improved by δ_{ϕ_0} if $P_{\underline{\beta}}(\phi(\underline{z}) \neq \phi_0(\underline{z})) > 0$ for some values of $\underline{\beta}$.

(ii) For $\eta > 0$, δ_ϕ is improved by δ_{ϕ_1} if $P_{\underline{\beta}}(\phi(\underline{z}) < \min\{U,0\}) > 0$ for some values of $\underline{\beta}$.

(iii) For $\eta < 0$, δ_ϕ is improved by δ_{ϕ_2} if $P_{\underline{\beta}}(\phi(\underline{z}) > \max\{U,0\}) > 0$ for some values of $\underline{\beta}$, where, $\underline{\beta} = \{\mu, \sigma_1, \sigma_2\}$.

Proof. Let us consider the conditional risk function of the estimator δ_ϕ given $\underline{Z} = \underline{z}$,

$$R(\underline{\beta}, \delta_\phi|\underline{z}) = E\{(X_1 + \phi(\underline{Z}) - \theta_J)^2|\underline{Z} = \underline{z}\}. \tag{5.7}$$

It is easily seen that, the right hand side of the above equation is a convex function of $\phi(\underline{Z})$. Therefore, minimum choice of $\phi(\underline{Z})$ is attained at

$$\phi(\underline{Z}) = -E[(X_1 - \theta_J)|\underline{Z} = \underline{z}]. \tag{5.8}$$

Since (X_1, X_2) and (S_1^2, S_2^2) are independently distributed, the required conditional distribution of X_1 given $\underline{Z} = \underline{z}$ is same as that of X_1 given $U = u$, which can be shown after some simplification to be normal distribution with mean $\mu - \frac{u}{1+\lambda}$ and variance $\frac{\sigma_2^2}{n_2(1+\lambda)}$, where $\lambda = \frac{n_1\sigma_2^2}{n_2\sigma_1^2}$. This gives,

$$\begin{aligned} \phi(\underline{z}, \underline{\beta}) \quad &= \quad \eta\sigma_1 + \frac{u}{1+\lambda}, \text{if } S_1^2 \leq S_2^2 \\ &= \quad \eta\sigma_2 + \frac{u}{1+\lambda}, \text{ otherwise} \end{aligned} \tag{5.9}$$

Now, we find the infimum and supremum values of $\phi(\underline{z}, \underline{\beta})$ with respect to $\underline{\beta}$ for all choices of η and $\underline{z}$. After some calculations from equation (5.9), we get the following consequences.
Case-I: when $\eta = 0$,

$$\inf_{\underline{\beta}} \phi(\underline{z}, \underline{\beta}) = \min\{u, 0\} \text{ and } \sup_{\underline{\beta}} \phi(\underline{z}, \underline{\beta}) = \max\{u, 0\}. \tag{5.10}$$

Case-II: when $\eta > 0$,

$$\begin{aligned} \sup_{\underline{\beta}} \phi(\underline{z}, \underline{\beta}) &= +\infty, \\ \text{and } \inf_{\underline{\beta}} \phi(\underline{z}, \underline{\beta}) &= 0, \text{ if } u \geq 0, \\ &= u, \text{ if } u < 0 \end{aligned} \tag{5.11}$$

Case-III: when $\eta < 0$,

$$\begin{aligned} \inf_{\underline{\beta}} \phi(\underline{z}, \underline{\beta}) &= -\infty, \\ \text{and } \sup_{\underline{\beta}} \phi(\underline{z}, \underline{\beta}) &= u, \text{ if } u \geq 0, \\ &= 0, \text{ if } u < 0 \end{aligned} \tag{5.12}$$

In Case-I to Case-III, we apply the orbit-by-orbit improvement technique of Brewster and Zidek technique [27]. It proves the theorem. □

We conclude this section with the following remark.

Remark 5.3.2. *It is noted from theorem* 5.3.1*, a location equivariant estimator* δ_ϕ *as defined in* (5.3) *lying outside the interval* $(\min\{X_1, X_2\}, \max\{X_1, X_2\})$ *with a positive probability is inadmissible.*

5.4 A general inadmissibility result for affine equivariant estimators

In this section, we first discuss the idea of invariance to the problem of estimating quantiles under the loss function (5.2) with the common location parameter μ. Also we show some inadmissibility results for estimators which are equivariant under the location and affine group of transformations.

Let us define the affine group of transformations $G_{a,b} = \{g_{a,b} : g_{a,b}(x) = ax + b, a > 0, b \in R\}$. Under $g_{a,b}$, $(X_1, X_2, S_1^2, S_2^2) \to (aX_1 + b, aX_2 + b, a^2 S_1^2, a^2 S_2^2), (\mu, \sigma_1^2, \sigma_2^2) \to (a\mu + b, a^2\sigma_1^2, a^2\sigma_2^2), \theta_J \to a\theta_J + b$. The loss function (5.2) remains invariant under $g_{a,b}$ if $\delta \to a\delta + b$. The estimation problem is invariant if we choose the loss function (5.2). Therefore, an affine equivariant estimator satisfies this relation,

$$d(aX_1 + b, aX_2 + b, a^2 S_1^2, a^2 S_2^2) = ad(X_1, X_2, S_1^2, S_2^2) + b.$$

The choice of $b = -aX_1$ and then $a = \frac{1}{S_1}$ give us the form of an affine equivariant estimator for θ_J as,

$$\begin{aligned} d(X_1, X_2, S_1^2, S_2^2) &= X_1 + S_1 \varphi(\underline{U}), \\ &= d_\varphi \qquad (say), \end{aligned} \tag{5.13}$$

where $\underline{U} = (U_1, U_2)$, $U_1 = \frac{X_2 - X_1}{S_1}$ and $U_2 = \frac{S_2^2}{S_1^2}$.

Next, we apply the orbit-by-orbit improvement technique of Brewster and Zidek [27] to prove a general result which provides a sufficient condition for the inadmissibility of an affine equivariant estimator (5.13) under the loss function (5.2).

Theorem 5.4.1. *Let $d_\varphi = X_1 + S_1 \varphi(\underline{U})$ be an affine equivariant estimator of θ_J as in* (5.13) *for estimating θ_J and the affine invariant loss as defined in* (5.2) *is considered. Let us define the following functions*

$$\begin{aligned} \varphi_0(\underline{u}) &= \min\{u_1, 0\}, \text{ if } \varphi(\underline{u}) < \min\{u_1, 0\}, \\ &= \varphi(\underline{u}), \text{ if } \min\{u_1, 0\} \leq \varphi(\underline{u}) \leq \max\{u_1, 0\} \\ &= \max\{u_1, 0\}, \text{ if } \varphi(\underline{u}) > \max\{u_1, 0\}, \end{aligned} \tag{5.14}$$

$$\varphi_1(\underline{u}) = \max\{\min\{u_1, 0\} + \eta b_{n_1+n_2}, \varphi(\underline{u})\}, \tag{5.15}$$

and

$$\varphi_2(\underline{u}) = \min\{\max\{u_1, 0\} + \eta b_{n_1+n_2}, \varphi(\underline{u})\}. \tag{5.16}$$

Then the following results can be shown:

(i) For $\eta = 0$, the estimator d_φ is improved by d_{φ_0} if $P_{\underline{\beta}}(\varphi(\underline{U}) \neq \varphi_0(\underline{U})) > 0$ for some values of $\underline{\beta}$.

(ii) For $\eta > 0$, the estimator d_φ is improved by d_{φ_1} if $P_{\underline{\beta}}(\varphi(\underline{U}) < \min\{U_1, 0\} + \eta b_{n_1+n_2}) > 0$ for some values of $\underline{\beta}$.

(iii) For $\eta < 0$, the estimator d_φ is improved by d_{φ_2} if $P_{\underline{\beta}}(\varphi(\underline{U}) > \max\{U_1, 0\} + \eta b_{n_1+n_2}) > 0$ for some values of $\underline{\beta}$.

Proof. Let us consider the conditional risk function of d_φ given $\underline{U} = \underline{u}$, given by

$$\begin{aligned} R(\underline{\beta}, d_\varphi | \underline{U}) &= E\{d_\varphi - \theta_J)^2 | \underline{U} = \underline{u}\}. \\ &= E\{X_1 + S_1 \varphi(\underline{U}) - \theta_J)^2 | \underline{U} = \underline{u}\}. \end{aligned} \tag{5.17}$$

Being a convex function of $\varphi(\underline{t})$ the above risk function attains its minimum value at

$$\varphi(\underline{u}, \underline{\beta}) = \frac{E\{(\theta_J - X_1)S_1 | \underline{U} = \underline{u}\}}{E\{S_1^2 | \underline{U} = \underline{u}\}}. \tag{5.18}$$

It can be written in this form,

$$\varphi * (\underline{u}, \lambda) = -\frac{E(P_1 Q_1^{1/2} | \underline{U} = \underline{u})}{\sqrt{n}_1 E(Q_1 | \underline{U} = \underline{u})} + \eta \frac{E(\frac{\sigma_J}{\sigma_1} Q_1^{1/2} | \underline{U} = \underline{u})}{E(Q_1 | \underline{U} = \underline{t})}, \tag{5.19}$$

where, $P_1 = \frac{\sqrt{n_1}(X_1-\mu)}{\sigma_1}$, $P_2 = \frac{\sqrt{n_1}(X_2-\mu)}{\sigma_1}$, $Q_1 = \frac{S_1^2}{\sigma_1^2}$, $Q_2 = \frac{S_1^2}{\sigma_1^2}$, $\lambda = \frac{n_1\sigma_2^2}{n_2\sigma_1^2}$.

Here, we evaluate the conditional expectations in the above expression. For that, we are required the conditional density functions of (P_1, Q_1) given $\underline{U} = \underline{u}$ and Q_1 given $\underline{U} = \underline{u}$. It can be easily seen that P_1 follows the standard normal distribution, P_2 follows the normal distribution with mean zero and variance λ and Q_1, Q_2 follows the chi-square distribution with (n_1-1) and (n_2-1) degrees of freedom, respectively. Let P_1, P_2, Q_1 and Q_2 are statistically independent random variables. So, the joint density function of P_1, P_2, Q_1, Q_2 is given by

$$f(p_1,p_2,q_1,q_2) = Ke^{-\frac{1}{2}(p_1^2+\frac{p_2^2}{\lambda}+q_1+q_2)}q_1^{\frac{n_1-3}{2}}q_2^{\frac{n_2-3}{2}}, -\infty < p_1 < \infty, -\infty < p_2 < \infty,$$
$$q_1 > 0, q_2 > 0,$$

where

$$K = \frac{1}{\sqrt{\lambda}\pi 2^{\frac{n_1+n_2}{2}}\Gamma(\frac{n_1-1}{2})\Gamma(\frac{n_2-1}{2}))}$$

.

Let us consider the transformation $U_1 = \frac{P_2-P_1}{\sqrt{n_1}Q_1^{\frac{1}{2}}}$, $U_2 = \frac{n_2\lambda Q_2}{n_1 Q_1}$, $U_3 = Q_1$ and $U_4 = P_1$. Then making the inverse transformation is $P_1 = U_4$, $P_2 = \sqrt{(}n_1)U_1U_3^{\frac{1}{2}} + U_4$, $Q_1 = U_3$, and $Q_2 = \frac{n_1U_2U_3}{n_2\lambda}$. The Jacobin of the transformation is $J = \frac{(n_1u_3)^{\frac{3}{2}}}{n_2\lambda}$. Therefore, the joint probability density function of (U_1, U_2, U_3, U_4) is given by

$$f(u_1,u_2,u_3,u_4) = K_1\sqrt{\frac{1+\lambda}{2\pi\lambda}}e^{-\frac{1}{2}(\frac{1+\lambda}{\lambda})(u_4+\frac{\sqrt{n_1}}{1+\lambda}u_1u_3^{\frac{1}{2}})^2}e^{-\frac{u_3\nu}{2}}u_2^{\frac{n_2-3}{2}}u_3^{\frac{n_1+n_2-3}{2}}, -\infty < t_1 < \infty,$$
$$-\infty < u_2 < \infty, u_2 > 0, u_3 > 0, \quad (5.20)$$

where $K_1 = \frac{Kn_1^{\frac{n}{2}}}{(n_2\lambda)^{\frac{n_2-1}{2}}}\sqrt{\frac{2\pi\lambda}{1+\lambda}}$ and $\nu = \frac{n_1u_1^2}{1+\lambda} + \frac{n_1t_2}{n_2\lambda} + 1$.

Thus, we obtain marginal distributions of (U_1, U_2, U_3) and (U_1, U_2) from (5.20), respectively as

$$f(u_1,u_2,u_3) = K_1e^{-\frac{u_3\nu}{2}}u_2^{\frac{n_2-3}{2}}u_3^{\frac{n_1+n_2-3}{2}}, \quad -\infty < u_1 < \infty, u_2 > 0, u_3 > 0,$$

and

$$f(u_1,u_2) = K_1 2^{\frac{n_1+n_2-1}{2}}\Gamma(\frac{n_1+n_2-1}{2})\nu^{-(\frac{n_1+n_2-1}{2})}, -\infty < u_1 < \infty, u_2 > 0.$$

Therefore, we can easily verify that the conditional distribution of U_4 given $(U_1, U_2, U_3) = (u_1, u_2, u_3)$ and U_3 given $(U_1, U_2) = (u_1, u_2)$ as

$$f_{U_4|(U_1,U_2,U_3)}(u_4|(u_1,u_2,u_3)) = \sqrt{\frac{1+\lambda}{2\pi\lambda}}e^{-\frac{1}{2}(\frac{1+\lambda}{\lambda})(u_4+\frac{\sqrt{n_1}}{1+\lambda}u_1u_3^{\frac{1}{2}})^2}, \quad (5.21)$$
$$-\infty < u_1 < \infty, -\infty < u_4 < \infty, u_2 > 0, u_3 > 0,$$

and

$$f_{U_3|(U_1,U_2)}(u_3|(u_1,u_2)) = (\frac{\nu}{2})^{\frac{n_1+n_2-1}{2}} \frac{1}{\Gamma(\frac{n_1+n_2-1}{2})} u_3^{\frac{n_1+n_2-3}{2}} e^{-\frac{u_3\nu}{2}}, \quad -\infty < u_1 < \infty, u_2 > 0, u_3 > 0. \tag{5.22}$$

respectively. Here the conditional density function of $U_4|(U_1,U_2,U_3)$ follows normal distribution function with mean $-\frac{\sqrt{n_1}u_1u_3^{\frac{1}{2}}}{1+\lambda}$ and variance $\frac{\lambda}{1+\lambda}$. Also the conditional distribution of $U_3|((U_1,U_2) = (u_1,u_2))$ is gamma distribution with scale parameter $\frac{2}{\nu}$ and shape parameter $\frac{n_1+n_2-1}{2}$. This gives

$$E\left(Q_1|\underline{U} = \underline{u}\right) = E\left\{U_3|\left(U_1,U_2\right) = \left(u_1,u_2\right)\right\} = \frac{n_1+n_2-1}{\nu}, \tag{5.23}$$

$$\begin{aligned} E\left\{\left(\frac{\sigma_J}{\sigma_1}\right)Q_1^{\frac{1}{2}}|\underline{U} = \underline{u}\right\} &= E\left\{\left(\frac{\sigma_J}{\sigma_1}\right)U_3^{\frac{1}{2}}|(U_1,U_2) = (u_1,u_2)\right\}, \\ &= \frac{n_1+n_2-1}{\sqrt{\nu}}b_{n_1+n_2} \text{ if } S_1^2 \leq S_2^2, \\ &= \frac{\sigma_2}{\sigma_1}\frac{n_1+n_2-1}{\sqrt{\nu}}b_{n_1+n_2} \text{ otherwise} \end{aligned} \tag{5.24}$$

$$\begin{aligned} E(P_1Q_1^{\frac{1}{2}}|\underline{U} = \underline{u}) &= E(U_4U_3^{\frac{1}{2}}|\underline{U} = \underline{u}), \\ &= E_{U_3|\underline{U}}\left[E\left\{U_4U_3^{\frac{1}{2}}\left|\left(U_1,U_2,U_3 = (u_1,u_2,u_3)\right)\right.\right\}\right], \\ &= -E\left\{\frac{\sqrt{n_1}u_1}{1+\tau}U_3|\underline{U} = \underline{u}\right\}, \\ &= -\frac{\sqrt{n_1}(n_1+n_2-1)u_1}{\nu(1+\lambda)}. \end{aligned} \tag{5.25}$$

Substituting the expressions from equations (5.23) , (5.24) and (5.25) in equation (5.19) , and simplifying we have the minimizing choice as

$$\begin{aligned} \varphi*(\underline{u},\lambda) &= \frac{u_1}{1+\lambda} + \eta\sqrt{\nu}b_{n_1+n_2} \text{ if } S_1^2 \leq S_2^2 \\ &= \frac{u_1}{1+\lambda} + \eta\frac{\sigma_2}{\sigma_1}\sqrt{\nu}b_{n_1+n_2} \text{ otherwise} \end{aligned} \tag{5.26}$$

where, $\nu = \frac{n_1u_1^2}{1+\lambda} + \frac{n_1u_2}{n_2\lambda} + 1$. So, $\varphi*$ can be rewritten as,

$$\begin{aligned} \varphi*(\underline{u},\lambda) &= \frac{u_1}{1+\lambda} + \eta\sqrt{\nu}b_{n_1+n_2} \text{ if } S_1^2 \leq S_2^2 \\ &= \frac{u_1}{1+\lambda} + \eta\sqrt{\nu}\sqrt{\frac{n_2\lambda}{n_1}}b_{n_1+n_2} \text{ otherwise} \end{aligned} \tag{5.27}$$

Now, we have to find the minimum and maximum values of $\varphi*$ with respect to λ for all values of η and $\underline{u}$. The optimum choices for $\varphi*$ give us the following cases:

Case-I: When $\eta = 0$,

$\inf_\lambda \varphi * (\underline{u}, \lambda) = \min\{u_1, 0\}$ and $\sup_\lambda \varphi * (\underline{u}, \lambda) = \max\{u_1, 0\}$.

Case-II: When $\eta > 0$, $u_1 \geq 0$, $\varphi * (\underline{u}, \lambda)$ is a decreasing function of λ and so

$\inf_\lambda \varphi * (\underline{u}, \lambda) = \lim_{\lambda \to \infty} \varphi * (\underline{u}, \lambda) = \eta b_{n_1+n_2}$ and $\sup_\lambda \varphi * (\underline{u}, \lambda) = \lim_{\lambda \to 0} \varphi * (\underline{u}, \lambda) = +\infty$.

Case-III: When $\eta < 0$, $u_1 < 0$, $\inf_\lambda \varphi * (\underline{u}, \lambda) \geq u_1$, (using lower bounds for both terms),

and $\sup_\lambda \varphi * (\underline{u}, \lambda) \geq u_1 = +\infty$.

Case-IV: When $\eta < 0$, $u_1 > 0$,

$\sup_\lambda \varphi * (\underline{u}, \lambda) \leq u_1 + \eta b_{n_1+n_2}$, (using upper bounds for both terms),

and $\inf_\lambda \varphi * (\underline{u}, \lambda) = -\infty$.

Case-V: When $\eta < 0$, $u_1 \leq 0$, $\varphi * (\underline{u}, \lambda)$ is an increasing function of λ and therefore,

$\inf_\lambda \varphi * (\underline{u}, \lambda) = -\infty$ and $\sup_\lambda \varphi * (\underline{u}, \lambda) \leq u_1 + \eta b_{n_1+n_2}$.

Using the above cases the orbit-by-orbit improvement technique of Brewster and Zidek [27] proves the theorem. □

Remark 5.4.2. *Above results say that an affine equivariant estimators of the form* (5.13) *lying outside the interval* $(\min\{X_1, X_2\} + \eta b_{n_1+n_2} S_1, \max\{X_1, X_2\} + \eta b_{n_1+n_2} S_1)$ *with a non zero probability is inadmissible.*

5.5 Numerical Comparisons

In this section, we numerically compare the bias and mean squared error risk values of the various estimators α_i (in section 2), $i = 1, 2, ..., 9$. It is easily seen that the bias and mean squared error risk functions of estimators $\alpha_i, i = 1, ..., 9$ depends on $\rho = \frac{\sigma_2}{\sigma_1}, n_1, n_2$, and $|\eta|$. For simulation, we consider various values of η and ρ, but results for selected values are reported here. Brown and Cohen [29] have given maximum acceptable limits of b_1 and b_2 for the estimators α_4 and α_5. The estimators α_4, α_5 of the maximum values b_1 and b_2 are improvements over X_1 of $b_{\max}(n_1, n_2)$ and $b_{\max}(n_1, n_2 - 3)$, respectively. If we take the values of b_1 and b_2 close to zero, then estimators α_4 and α_5 tend to α_1. If we can take the values of b_2 close to one, the estimator α_5 tends to α_2. For numerical comparisons, we select b_1 and b_2 as $\frac{1}{2} b_{\max}(n_1, n_2)$ and $\frac{1}{2} b_{\max}(n_1, n_2 - 3)$, respectively. The numerical values have been calculated using simulations based on 10,000 random samples of size n_1 and n_2, from two normal populations $N(\mu, \sigma_1^2)$, and $N(\mu, \sigma_2^2)$ respectively. In Tables 5.1 and 5.4 the bias and risk values have been presented when $n_1 = n_2$. We notice that for equal sample size, $\alpha_2 = \alpha_3$ and $\alpha_7 = \alpha_8$. In Tables 5.2-5.3 and 5.4-5.6, we report the biases and risk performances of estimators when sample sizes are unequal, i.e $n_1 \neq n_2$. If $n_1 \neq n_2$, then estimator α_6 is not defined, so, its bias and risk values are not reported for $n_1 \neq n_2$. Furthermore, if $n_1 = n_2 = 6$ the coefficient δ_{n_2} equals to zero, therefore the estimator α_6 is the same as α_1. In Tables 5.1 to 5.6, bias and risk performances of the estimators $\alpha_i, i = 1, ..., 9$ are presented. Three values in each cell represent bias and risk values corresponding to three selection of choices of (n_1, n_2) mentioned above each table. From tables we observe that all the absolute biases decrease as n_1 and n_2 increase. The absolute bias values of the estimators $\alpha_i, i = 1, ..., 9$ increase as the value of ρ increases. From the tables and graphs it is also observed that all risk values decrease as both n_1 and n_2 increase. Risk values of the estimators $\alpha_i, i = 1, ..., 9$ increase as ρ increases. From Tables 5.1-5.6 and the graphs the following conclusions can be made.

(a) (i) Absolute bias comparisons when sample sizes are equal $(n_1 = n_2)$

From Table 5.1 and corresponding Figure $5.1 - 5.3$, it is observed that for small $\rho (0 < \rho \leq 1.5)$ the absolute biases of the estimators are almost same. The bias curves are not visible separately. But when the value of $\rho > 1.5$, the absolute bias performances are visible separately and it is observed that for small $\rho \quad (0 < \rho \leq 1)$ and sample sizes say $n_1 > 6$, the absolute bias of the estimator α_2 is better than other estimators.

But when the value of $\rho > 1$, the absolute bias performance of α_4 becomes better for $n_1 > 6$. As the values of ρ and n_1 increase the performance of α_7 becomes better and in fact for $\rho > 1$ it performs better than other estimators according to absolute bias. Another observation is that for $1 < \rho \leq 2$ and $n_1 = 12$ the performance of α_9 is better than all the other estimators.

(ii) Absolute bias comparisons when sample sizes are unequal $(n_1 < n_2)$

From the Table 5.2 and Figure $5.4 - 5.6$, we observe that for small ρ the bias curves of all the estimators coincide and are not visible separately. But as ρ increases they become visible separately. It is also observe that for higher values of sample sizes and ρ is small then the performances of α_2 and α_3 are better than other estimators. It is also observed that the estimator α_9 behaves better than other estimators in few cases as the value of ρ increases. When n_1 and n_2 increase the variability of the bias performances of all the estimators increase. They become zig-zag. For $n_1 = 18, n_2 = 30$ when $1 < \rho < 2.25$ the bias of α_1 is smaller than other estimators. Further, we noticed for $(n_1, n_2) = (12, 20)$ as ρ increases from 3.5 the bias performance of α_1 become better than other estimators.

(iii) Absolute biases comparisons when sample sizes are unequal $(n_1 > n_2)$

From Table 5.3 and Figure $5.7 - 5.9$, we see that when $0 < \rho < 3$ the bias performances of all the estimators are more or less same and so the curves are not visible separately in this range of all the cases of (n_1, n_2). It is also observe that as the value of ρ increases the performances of the estimators $\alpha_2, \ldots, \alpha_9$ become better and in fact, for $\rho > 1$ the estimators α_2, α_3, α_7, α_8 and α_9 perform better than other estimators. For $(n_1, n_2) = (10, 6)$, α_1 and α_6 are the same performance for all the value of ρ.

(b) (i) Risk comparisons when sample sizes are equal $(n_1 = n_2)$

From Table 5.4 and Figure $5.10 - 5.12$, we observe that for $0 < \rho < 2.5$ the performances of all the estimators are more or less same and the curves coincide. We also observe that for small values of ρ the estimator α_2 performs better than other estimators. As the value of ρ increases the performances of the estimators α_7 and α_9 become better and in fact, for $\rho < 1.50$ the estimator α_7 has smaller risk than other estimators in most of the cases. For $2 < \rho \leq 4$ the estimator α_5 performs better than other estimators. This trend is observed for all three cases of (n_1, n_2).

(ii) Risk comparisons when sample sizes are unequal $(n_1 < n_2)$

From Table 5.5 and Figure $5.13 - 5.15$, it is observed that for $0 < \rho < 2$ the risk performances of all the estimators are almost same and the curves coincide. It is also observed that for all values of ρ the performances of α_2 and α_3 are good and in fact for small ρ they perform better than the other estimators. But as ρ increases the performances of α_7 and α_8 become better and they behave better than other estimators for $1 \leq \rho \leq 2$. For $\rho > 2.5$ the risk values of α_7, α_8 and α_9 become larger than other estimators. In this range the risk performance of all the estimators excluding α_7, α_8 and α_9 are more or less same. As ρ increase the risk value of α_9 becomes vary high. This trend is observed for all three cases of (n_1, n_2).

(iii) Risk comparisons when sample sizes are unequal $(n_1 > n_2)$.

From Table 5.6 and Figure $5.16 - 5.18$, we find that for any ρ the estimators α_2 and α_3 perform well and in fact for small ρ they perform better than other estimators. In general if we see, for any ρ the estimators $\alpha_2, \ldots, \alpha_6$ perform better than other estimators. For $\rho < 1.5$ the performances of α_7, α_8 and α_9 are better than other estimators. As ρ increase the risk value of α_9 becomes vary high. This tendency is seen for all three cases of (n_1, n_2).

(c) Similar Risk and bias behavior are observed during simulation for various other values of n_1, n_2, ρ and η. We omitted the tables and comments for the sake of brevity. The absolute biases and risks of all estimators are plotted for equal and unequal sample sizes. It is noticed from the figures $5.1 - 5.18$ that as the values of sample sizes increase the risks and absolute bias values decrease.

Table 5.1: Bias values for (n_1, n_2)= (6,6), (12,12), (18,18), $\eta = 1$

ρ	α_1	α_2	α_4	α_5	α_6	α_7	α_9
0.25	-0.008464	-0.008620	-0.008496	-0.008486	-0.008464	-0.008561	-0.008522
	-0.003694	-0.003439	-0.003683	-0.003562	-0.003591	-0.003505	-0.003524
	-0.001966	-0.001942	-0.001961	-0.001951	-0.001952	-0.001946	-0.001950
0.50	-0.036403	-0.034263	-0.036068	-0.036100	-0.036403	-0.034417	-0.034584
	-0.010745	-0.010064	-0.010587	-0.010393	-0.010472	-0.010187	-0.010338
	-0.006716	-0.006333	-0.006629	-0.006472	-0.006490	-0.006405	-0.006510
0.75	-0.07715	-0.078017	-0.077421	-0.077273	-0.077150	-0.077621	-0.077070
	-0.041462	-0.039372	-0.041191	-0.040381	-0.040652	-0.039511	-0.039662
	-0.027363	-0.026472	-0.027261	-0.026795	-0.026850	-0.026580	-0.026695
1.00	-0.129924	-0.129845	-0.129923	-0.129913	-0.129924	-0.130103	-0.130335
	-0.081513	-0.081521	-0.081471	-0.081517	-0.081501	-0.081575	-0.081633
	-0.061168	-0.062342	-0.061290	-0.061917	-0.061837	-0.062318	-0.062294
1.25	-0.162436	-0.160816	-0.162212	-0.162207	-0.162436	-0.160998	-0.161284
	-0.086936	-0.086802	-0.086914	-0.086866	-0.086887	-0.086781	-0.086726
	-0.058719	-0.059444	-0.058770	-0.059182	-0.059104	-0.059523	-0.059602
1.50	-0.167978	-0.166697	-0.167794	-0.167797	-0.167978	-0.166637	-0.166924
	-0.078275	-0.076475	-0.078132	-0.077344	-0.077654	-0.076226	-0.076144
	-0.043995	-0.042951	-0.043927	-0.043329	-0.043440	-0.042490	-0.041925
1.75	-0.174916	-0.173440	-0.174778	-0.174707	-0.174916	-0.171989	-0.169846
	-0.060646	-0.061128	-0.060720	-0.060896	-0.060838	-0.060815	-0.060355
	-0.035611	-0.034476	-0.035548	-0.034886	-0.035008	-0.033837	-0.033006
2.00	-0.175163	-0.178630	-0.175443	-0.175654	-0.175163	-0.180133	-0.182130
	-0.057495	-0.056359	-0.057424	-0.056907	-0.057116	-0.055666	-0.054909
	-0.027501	-0.027527	-0.027501	-0.027517	-0.027511	-0.027533	-0.027478
2.25	-0.172052	-0.171405	-0.172075	-0.171960	-0.172052	-0.170051	-0.168733
	-0.056142	-0.055510	-0.056078	-0.055815	-0.055921	-0.055547	-0.055879
	-0.028608	-0.028609	-0.028594	-0.028608	-0.028600	-0.029027	-0.029663
2.50	-0.170321	-0.169440	-0.170204	-0.170196	-0.170321	-0.171371	-0.176265
	-0.056161	-0.056027	-0.056155	-0.056091	-0.056116	-0.055447	-0.054163
	-0.025318	-0.026295	-0.025371	-0.025942	-0.025835	-0.027060	-0.028092
3.00	-0.149662	-0.151044	-0.149749	-0.149858	-0.149662	-0.152188	-0.151752
	-0.056717	-0.056565	-0.056697	-0.056639	-0.056663	-0.056927	-0.058129
	-0.037370	-0.039008	-0.037448	-0.038415	-0.038225	-0.041483	-0.045563
3.50	-0.162461	-0.157308	-0.161814	-0.161731	-0.162461	-0.153834	-0.149550
	-0.058882	-0.058272	-0.058837	-0.058566	-0.058674	-0.057475	-0.055502
	-0.033778	-0.032104	-0.033691	-0.032710	-0.032896	-0.030670	-0.028943
4.00	-0.157353	-0.158004	-0.157352	-0.157446	-0.157353	-0.158861	-0.157696
	-0.061539	-0.060918	-0.061486	-0.061217	-0.061321	-0.061185	-0.062013
	-0.042517	-0.044561	-0.042617	-0.043821	-0.043589	-0.046715	-0.048939

Table 5.2: Bias values for (n_1, n_2)= (6,10), (12,20), (18,30), $\eta = 1$

ρ	α_1	α_2	α_3	α_4	α_5	α_6	α_7	α_8	α_9
0.25	-0.003302	-0.004695	-0.004716	-0.003621	-0.003839	-0.003674	-0.004466	-0.004472	-0.004222
	-0.002319	-0.001811	-0.001807	-0.002185	-0.002002	-0.002050	-0.001894	-0.001893	-0.001967
	-0.002203	-0.001343	-0.001342	-0.001842	-0.001609	-0.001595	-0.001446	-0.001445	-0.001651
0.50	-0.026044	-0.024196	-0.024165	-0.025643	-0.025332	-0.025573	-0.024577	-0.024570	-0.024884
	-0.005773	-0.007001	-0.007016	-0.006063	-0.006539	-0.006387	-0.006815	-0.006818	-0.006683
	-0.003682	-0.003682	-0.003679	-0.003753	-0.003682	-0.003741	-0.003687	-0.003687	-0.003629
0.75	-0.068119	-0.065450	-0.065309	-0.067573	-0.067091	-0.067556	-0.065788	-0.065771	-0.065858
	-0.035911	-0.033534	-0.033463	-0.035677	-0.034428	-0.035058	-0.033808	-0.033799	-0.033645
	-0.019157	-0.019828	-0.019838	-0.019210	-0.019621	-0.019476	-0.019717	-0.019719	-0.019711
1.00	-0.110170	-0.111675	-0.111925	-0.110263	-0.110749	-0.110299	-0.111867	-0.111889	-0.112682
	-0.069570	-0.068917	-0.068910	-0.069457	-0.069162	-0.069251	-0.069039	-0.069038	-0.069094
	-0.053704	-0.053172	-0.053166	-0.053656	-0.053336	-0.053445	-0.053289	-0.053289	-0.053346
1.25	-0.133040	-0.134447	-0.134731	-0.133112	-0.133582	-0.133136	-0.134786	-0.134810	-0.135662
	-0.071287	-0.070900	-0.070913	-0.071203	-0.071046	-0.071060	-0.071197	-0.071198	-0.071586
	-0.049003	-0.050228	-0.050260	-0.049075	-0.049851	-0.049473	-0.050237	-0.050241	-0.050550
1.50	-0.131831	-0.132025	-0.131996	-0.131854	-0.131906	-0.131846	-0.131745	-0.131742	-0.131366
	-0.056531	-0.056010	-0.055968	-0.056510	-0.056206	-0.056425	-0.055671	-0.055665	-0.054949
	-0.028610	-0.029298	-0.029334	-0.028620	-0.029086	-0.028757	-0.029687	-0.029691	-0.030465
1.75	-0.119030	-0.121362	-0.121674	-0.119263	-0.119928	-0.119320	-0.121627	-0.121658	-0.122952
	-0.045879	-0.046238	-0.046269	-0.045877	-0.046103	-0.045930	-0.046363	-0.046366	-0.046629
	-0.022554	-0.023814	-0.023868	-0.022592	-0.023426	-0.022901	-0.024361	-0.024368	-0.025577
2.00	-0.123298	-0.119987	-0.119475	-0.123010	-0.122023	-0.122928	-0.119754	-0.119713	-0.118900
	-0.047283	-0.046332	-0.046283	-0.047214	-0.046690	-0.047011	-0.046206	-0.046200	-0.045788
	-0.023030	-0.024025	-0.024069	-0.023061	-0.023718	-0.023304	-0.024676	-0.024682	-0.026073
2.25	-0.126571	-0.123326	-0.123036	-0.126137	-0.125322	-0.126085	-0.123672	-0.123645	-0.123230
	-0.046341	-0.046483	-0.046455	-0.046382	-0.046430	-0.046472	-0.045897	-0.045893	-0.045066
	-0.032609	-0.030488	-0.030410	-0.032524	-0.031142	-0.031941	-0.029870	-0.029860	-0.028348
2.50	-0.125539	-0.123653	-0.123442	-0.125323	-0.124813	-0.125291	-0.123573	-0.123550	-0.122416
	-0.049896	-0.047534	-0.047378	-0.049775	-0.048422	-0.049330	-0.046477	-0.046457	-0.043912
	-0.034373	-0.034272	-0.034270	-0.034365	-0.034303	-0.034327	-0.034233	-0.034232	-0.034097
3.00	-0.122881	-0.123638	-0.123815	-0.122927	-0.123173	-0.122936	-0.122671	-0.122659	-0.118841
	-0.058227	-0.053376	-0.053051	-0.057982	-0.055201	-0.057068	-0.051123	-0.051080	-0.046071
	-0.033617	-0.034397	-0.034431	-0.033643	-0.034156	-0.033834	-0.035106	-0.035113	-0.036823
3.50	-0.125889	-0.123904	-0.123414	-0.125754	-0.125125	-0.125733	-0.122063	-0.122004	-0.117349
	-0.058542	-0.059140	-0.059184	-0.058568	-0.058915	-0.058674	-0.059467	-0.059473	-0.060615
	-0.038507	-0.036201	-0.036094	-0.038433	-0.036912	-0.037878	-0.033636	-0.033615	-0.027136
4.00	-0.127738	-0.127552	-0.127543	-0.127720	-0.127667	-0.127718	-0.127498	-0.127494	-0.123952
	-0.052955	-0.056041	-0.056276	-0.053095	-0.054880	-0.053641	-0.059765	-0.059813	-0.069960
	-0.043361	-0.043611	-0.043625	-0.043367	-0.043534	-0.043419	-0.044260	-0.044265	-0.045625

Table 5.3: Bias values for (n_1,n_2)= (10,6), (20,12), (30,18), $\eta = 1$

ρ	α_1	α_2	α_3	α_4	α_5	α_6	α_7	α_8	α_9
0.25	-0.007091	-0.007325	-0.007319	-0.007127	-0.007129	-0.007091	-0.007304	-0.007304	-0.007185
	-0.003136	-0.003677	-0.003673	-0.003326	-0.003435	-0.003362	-0.003600	-0.003599	-0.003353
	-0.001334	-0.002083	-0.002079	-0.001569	-0.001834	-0.001800	-0.001963	-0.001962	-0.001645
0.50	-0.022872	-0.023270	-0.023227	-0.022972	-0.022937	-0.022872	-0.023257	-0.023254	-0.023141
	-0.008830	-0.008436	-0.008441	-0.008739	-0.008612	-0.008657	-0.008516	-0.008517	-0.008671
	-0.004564	-0.004700	-0.004698	-0.004597	-0.004655	-0.004652	-0.004679	-0.004679	-0.004627
0.75	-0.064712	-0.063960	-0.064055	-0.064558	-0.064590	-0.064712	-0.063870	-0.063880	-0.063978
	-0.032263	-0.031337	-0.031365	-0.032111	-0.031751	-0.031782	-0.031433	-0.031437	-0.031730
	-0.021177	-0.020586	-0.020596	-0.021104	-0.020782	-0.020727	-0.020637	-0.020638	-0.020814
1.00	-0.111381	-0.111484	-0.111468	-0.111327	-0.111398	-0.111381	-0.111594	-0.111591	-0.111626
	-0.070306	-0.069636	-0.069677	-0.070254	-0.069935	-0.069840	-0.069481	-0.069486	-0.069575
	-0.053460	-0.053155	-0.053170	-0.053451	-0.053256	-0.053087	-0.052933	-0.052936	-0.052873
1.25	-0.138266	-0.137758	-0.137789	-0.138200	-0.138184	-0.138266	-0.137960	-0.137961	-0.138217
	-0.071055	-0.071143	-0.071147	-0.071073	-0.071103	-0.071051	-0.070979	-0.070980	-0.070869
	-0.050957	-0.051066	-0.051067	-0.050973	-0.051030	-0.051003	-0.050902	-0.050903	-0.050766
1.50	-0.137209	-0.139476	-0.139123	-0.137396	-0.137577	-0.137209	-0.139504	-0.139477	-0.138784
	-0.061067	-0.059947	-0.060021	-0.060975	-0.060447	-0.060271	-0.059209	-0.059221	-0.058956
	-0.028101	-0.028454	-0.028439	-0.028121	-0.028337	-0.028507	-0.028751	-0.028748	-0.028916
1.75	-0.135426	-0.135457	-0.135359	-0.135374	-0.135431	-0.135426	-0.136632	-0.136609	-0.137576
	-0.042238	-0.043873	-0.043774	-0.042382	-0.043143	-0.043390	-0.044747	-0.044732	-0.045119
	-0.022115	-0.021298	-0.021335	-0.022073	-0.021569	-0.021138	-0.020511	-0.020518	-0.020074
2.00	-0.125822	-0.124019	-0.124348	-0.125587	-0.125530	-0.125822	-0.121946	-0.122006	-0.120421
	-0.035953	-0.036249	-0.036236	-0.035985	-0.036116	-0.036121	-0.036223	-0.036222	-0.036076
	-0.019032	-0.018998	-0.018998	-0.019029	-0.019009	-0.019028	-0.019205	-0.019204	-0.019519
2.25	-0.130787	-0.132194	-0.131914	-0.130942	-0.131015	-0.130787	-0.133397	-0.133357	-0.133955
	-0.035683	-0.036058	-0.036027	-0.035705	-0.035891	-0.035982	-0.036745	-0.036737	-0.037396
	-0.014677	-0.013159	-0.013224	-0.014592	-0.013663	-0.012954	-0.012080	-0.012090	-0.011764
2.50	-0.109167	-0.110303	-0.110111	-0.109284	-0.109351	-0.109167	-0.111542	-0.111505	-0.112956
	-0.036525	-0.036074	-0.036104	-0.036492	-0.036275	-0.036160	-0.035133	-0.035144	-0.033635
	-0.017960	-0.019731	-0.019649	-0.018051	-0.019144	-0.020142	-0.021951	-0.021933	-0.023550
3.00	-0.095615	-0.094135	-0.094430	-0.095509	-0.095375	-0.095615	-0.092944	-0.092984	-0.092848
	-0.026985	-0.026303	-0.026343	-0.026928	-0.026608	-0.026530	-0.026054	-0.026059	-0.025964
	-0.023927	-0.023752	-0.023759	-0.023917	-0.023810	-0.023747	-0.023790	-0.023791	-0.024004
3.50	-0.095304	-0.097202	-0.096812	-0.095423	-0.095612	-0.095304	-0.100323	-0.100239	-0.104164
	-0.036275	-0.037663	-0.037557	-0.036375	-0.037043	-0.037421	-0.040036	-0.040008	-0.042621
	-0.020150	-0.020855	-0.020820	-0.020184	-0.020621	-0.021111	-0.022940	-0.022926	-0.026114
4.00	-0.090968	-0.088980	-0.089305	-0.090829	-0.090646	-0.090968	-0.089689	-0.089696	-0.094040
	-0.037401	-0.037206	-0.037225	-0.037391	-0.037293	-0.037180	-0.035226	-0.035245	-0.030129
	-0.022520	-0.024099	-0.024021	-0.022598	-0.023576	-0.024646	-0.028507	-0.028477	-0.035465

Table 5.4: Risk values for (n_1, n_2)= (6,6), (12,12), (18,18), $\eta = 1$

ρ	α_1	α_2	α_4	α_5	α_6	α_7	α_9
0.25	0.015795	0.001732	0.010373	0.012149	0.015795	0.002085	0.004726
	0.007571	0.000708	0.00378	0.002309	0.003168	0.000865	0.002182
	0.005154	0.000457	0.002524	0.001078	0.001273	0.000567	0.00148
0.50	0.026258	0.009974	0.020875	0.022026	0.026258	0.010082	0.011628
	0.011852	0.003877	0.008900	0.005746	0.006762	0.004025	0.004848
	0.007717	0.002361	0.005989	0.003084	0.003312	0.002496	0.003089
0.75	0.040138	0.024921	0.035421	0.035846	0.040138	0.023919	0.023744
	0.019016	0.010956	0.016732	0.012703	0.013860	0.010750	0.010895
	0.012590	0.007020	0.011364	0.007744	0.008035	0.006988	0.007160
1.00	0.070030	0.052461	0.064619	0.064606	0.070030	0.050290	0.049371
	0.033905	0.024320	0.031620	0.026221	0.027813	0.023749	0.023559
	0.021855	0.015491	0.020686	0.016076	0.016510	0.015205	0.015101
1.25	0.112517	0.093786	0.106749	0.106083	0.112517	0.090046	0.089236
	0.052338	0.041949	0.050078	0.043589	0.045542	0.041125	0.041358
	0.033270	0.026655	0.032187	0.027067	0.027594	0.026379	0.026662
1.50	0.157845	0.140465	0.151653	0.150549	0.157845	0.137214	0.141709
	0.071167	0.061571	0.069038	0.062333	0.064318	0.061510	0.064910
	0.045373	0.038314	0.044316	0.038654	0.039246	0.038386	0.040531
1.75	0.213531	0.203656	0.207712	0.206247	0.213531	0.201607	0.216380
	0.089411	0.079563	0.087227	0.079841	0.081993	0.081070	0.090803
	0.057703	0.050116	0.056640	0.050403	0.051039	0.051223	0.057665
2.00	0.277318	0.269186	0.271067	0.269268	0.277318	0.272184	0.312887
	0.115343	0.104669	0.113044	0.104740	0.107118	0.108641	0.130613
	0.072208	0.064409	0.071137	0.064534	0.065178	0.067673	0.083206
2.25	0.341344	0.335313	0.335399	0.333173	0.341344	0.343820	0.418437
	0.137320	0.129145	0.135244	0.127861	0.129834	0.138650	0.183719
	0.086572	0.079450	0.085559	0.079215	0.079782	0.086113	0.116046
2.50	0.401355	0.393206	0.394355	0.391812	0.401355	0.418073	0.570204
	0.162435	0.151180	0.160144	0.151128	0.153685	0.163622	0.234294
	0.103264	0.096742	0.102284	0.096120	0.096635	0.107504	0.158728
3.00	0.447051	0.446622	0.441195	0.438435	0.447051	0.490892	0.773488
	0.188371	0.182913	0.186653	0.180270	0.181829	0.213103	0.378538
	0.119065	0.113634	0.118247	0.112930	0.113352	0.133069	0.242797
3.50	0.515584	0.535883	0.510993	0.509292	0.515584	0.620946	1.178648
	0.214208	0.209898	0.212681	0.206890	0.208262	0.253788	0.542979
	0.142006	0.137212	0.141301	0.136535	0.136916	0.165588	0.361855
4.00	0.587802	0.604870	0.582242	0.580010	0.587802	0.735302	1.692821
	0.249218	0.246472	0.247963	0.243304	0.244328	0.308911	0.784411
	0.166128	0.161711	0.165484	0.161084	0.161432	0.201048	0.512820

Table 5.5: Risk values for (n_1, n_2)= (6,10), (12,20), (18,30), $\eta = 1$

ρ	α_1	α_2	α_3	α_4	α_5	α_6	α_7	α_8	α_9
0.25	0.015472	0.001020	0.000999	0.008450	0.006604	0.008743	0.001352	0.001336	0.002760
	0.007516	0.000421	0.000420	0.003259	0.001426	0.001686	0.000551	0.000548	0.001276
	0.005251	0.000283	0.000282	0.002377	0.000763	0.000761	0.000376	0.000374	0.000893
0.50	0.024630	0.007057	0.006950	0.017956	0.01391	0.017698	0.007524	0.007488	0.008158
	0.011213	0.002520	0.002513	0.008071	0.003803	0.004869	0.002783	0.002776	0.003168
	0.007286	0.001475	0.001473	0.005509	0.002018	0.002463	0.001627	0.001624	0.001877
0.75	0.037807	0.018799	0.018607	0.031963	0.026084	0.031451	0.018783	0.018735	0.018410
	0.018353	0.008360	0.008331	0.015918	0.009827	0.012034	0.008474	0.008463	0.008345
	0.011317	0.004885	0.004881	0.010083	0.005491	0.006569	0.004981	0.004977	0.004950
1.00	0.063526	0.039643	0.039510	0.057189	0.048370	0.056419	0.038727	0.038680	0.038178
	0.031040	0.018337	0.018307	0.028648	0.020133	0.023784	0.018171	0.018160	0.017879
	0.020485	0.011938	0.011932	0.019278	0.012715	0.014763	0.011892	0.011888	0.011777
1.25	0.097406	0.069820	0.069920	0.090680	0.078972	0.089595	0.067105	0.067086	0.067247
	0.045953	0.031324	0.031338	0.043592	0.033030	0.038000	0.030687	0.030685	0.031280
	0.030525	0.020627	0.020619	0.029421	0.021508	0.024437	0.020329	0.020327	0.020612
1.50	0.136209	0.108786	0.109418	0.129870	0.116744	0.128687	0.104857	0.104924	0.110424
	0.064572	0.048305	0.048349	0.062274	0.050043	0.056200	0.047255	0.047269	0.050337
	0.041307	0.030603	0.030628	0.040221	0.031314	0.034842	0.030253	0.030261	0.032757
1.75	0.175592	0.149679	0.151807	0.169141	0.154605	0.167783	0.145031	0.145288	0.163732
	0.083282	0.066727	0.066866	0.081104	0.068156	0.074938	0.065972	0.066020	0.075767
	0.055012	0.043292	0.043338	0.053928	0.044006	0.048176	0.043227	0.043249	0.049993
2.00	0.221332	0.194637	0.198070	0.214641	0.198257	0.213123	0.191439	0.191932	0.233111
	0.104804	0.087690	0.087982	0.102574	0.088649	0.096050	0.087889	0.087991	0.109731
	0.069227	0.056691	0.056752	0.068162	0.057417	0.062181	0.057228	0.057269	0.071727
2.25	0.274996	0.247009	0.252354	0.267690	0.249609	0.266050	0.248650	0.249532	0.335855
	0.129970	0.111204	0.111595	0.127675	0.112296	0.120662	0.114190	0.114368	0.158229
	0.083367	0.072522	0.072686	0.082411	0.072514	0.076901	0.075025	0.075107	0.104427
2.50	0.331730	0.307564	0.314990	0.324715	0.306321	0.323015	0.313384	0.314694	0.453290
	0.156888	0.139989	0.140570	0.154800	0.140074	0.148212	0.145344	0.145617	0.217730
	0.102586	0.089960	0.090111	0.101571	0.090292	0.095510	0.094433	0.094549	0.143818
3.00	0.387588	0.367343	0.376571	0.381570	0.364354	0.379911	0.391112	0.393292	0.693887
	0.178306	0.165346	0.166096	0.176679	0.164643	0.171373	0.179374	0.179828	0.328623
	0.120676	0.111796	0.112050	0.119919	0.111400	0.115293	0.123058	0.123261	0.225988
3.50	0.452634	0.444953	0.457122	0.448086	0.434718	0.446823	0.491261	0.494560	1.031217
	0.219954	0.205710	0.206412	0.218344	0.205625	0.212940	0.229508	0.230169	0.496192
	0.145883	0.137662	0.137948	0.145186	0.137089	0.140846	0.155476	0.155775	0.339231
4.00	0.513660	0.510280	0.524454	0.509535	0.497625	0.508369	0.587909	0.592594	1.493094
	0.248650	0.237772	0.238781	0.247224	0.236434	0.242461	0.277441	0.278406	0.720735
	0.165830	0.156613	0.156799	0.165173	0.156714	0.160917	0.182036	0.182429	0.479984

Table 5.6: Risk values for (n_1,n_2)= (10,6), (20,12), (30,18), $\eta = 1$

ρ	α_1	α_2	α_3	α_4	α_5	α_6	α_7	α_8	α_9
0.25	0.009409	0.001462	0.001499	0.006012	0.007042	0.009409	0.001660	0.001673	0.004113
	0.004685	0.000687	0.000689	0.002595	0.001489	0.001993	0.000794	0.000797	0.002048
	0.003118	0.000450	0.000450	0.001796	0.000740	0.000822	0.000519	0.000520	0.001356
0.50	0.016022	0.007543	0.007684	0.013028	0.013426	0.016022	0.007429	0.007448	0.009055
	0.007489	0.003244	0.003256	0.006082	0.004072	0.004285	0.003266	0.003270	0.004120
	0.004916	0.002203	0.002204	0.004180	0.002466	0.002361	0.002213	0.002214	0.002740
0.75	0.025768	0.018677	0.018726	0.023297	0.023267	0.025768	0.017895	0.017896	0.018097
	0.012343	0.008323	0.008332	0.011314	0.009018	0.008918	0.008161	0.008162	0.008459
	0.008267	0.005492	0.005494	0.007740	0.005752	0.005494	0.005417	0.005418	0.005629
1.00	0.047141	0.040409	0.040141	0.044565	0.044241	0.047141	0.039262	0.039205	0.038111
	0.023038	0.018777	0.018760	0.022081	0.019299	0.018920	0.018510	0.018500	0.018205
	0.014778	0.011857	0.011853	0.014311	0.012047	0.011703	0.011735	0.011731	0.011564
1.25	0.079046	0.073070	0.072382	0.076425	0.075835	0.079046	0.072378	0.072218	0.070475
	0.034794	0.031329	0.031228	0.033938	0.031289	0.030876	0.031632	0.031596	0.031092
	0.023433	0.020510	0.020487	0.022990	0.020572	0.020334	0.020747	0.020733	0.020457
1.50	0.115481	0.110384	0.109098	0.112527	0.111808	0.115481	0.111059	0.110731	0.109865
	0.049883	0.044879	0.044810	0.048901	0.045323	0.044672	0.045992	0.045930	0.046522
	0.029373	0.026404	0.026364	0.028945	0.026410	0.026360	0.027668	0.027638	0.028247
1.75	0.151629	0.147603	0.145740	0.148769	0.147875	0.151629	0.153188	0.152608	0.158176
	0.060392	0.057566	0.057324	0.059616	0.056888	0.056475	0.061619	0.061489	0.065403
	0.035400	0.032977	0.032909	0.035025	0.032751	0.032968	0.035600	0.035547	0.038172
2	0.195467	0.195539	0.192256	0.192592	0.191646	0.195467	0.208539	0.207519	0.224707
	0.070657	0.069202	0.068836	0.069988	0.067760	0.067502	0.076623	0.076418	0.086485
	0.041424	0.038705	0.038635	0.041035	0.038558	0.038913	0.043522	0.043441	0.050780
2.25	0.238609	0.242195	0.238033	0.236165	0.235049	0.238609	0.263541	0.262108	0.298954
	0.079166	0.076614	0.076267	0.078399	0.075598	0.075275	0.088243	0.087954	0.109602
	0.050036	0.046935	0.046874	0.049632	0.046884	0.047234	0.054091	0.053978	0.068826
2.50	0.281060	0.287455	0.282618	0.278291	0.277375	0.281060	0.318231	0.316269	0.383941
	0.093216	0.089623	0.089296	0.092341	0.088994	0.088642	0.106360	0.105973	0.144480
	0.061847	0.058546	0.058487	0.061436	0.058544	0.058994	0.068685	0.068531	0.094185
3.00	0.298901	0.301331	0.296859	0.296613	0.295152	0.298901	0.352237	0.349512	0.500422
	0.110558	0.106652	0.106374	0.109726	0.106333	0.105940	0.132436	0.131872	0.218686
	0.073995	0.071761	0.071673	0.073677	0.071523	0.072527	0.088675	0.088442	0.146123
3.50	0.345083	0.351117	0.344989	0.342627	0.341354	0.345083	0.434534	0.430469	0.723293
	0.133090	0.130844	0.130524	0.132484	0.130040	0.129907	0.166140	0.165402	0.315817
	0.086218	0.083606	0.083560	0.085914	0.083621	0.084248	0.105386	0.105089	0.207402
4.00	0.363403	0.367875	0.362398	0.361385	0.360035	0.363403	0.472602	0.467681	0.948421
	0.150150	0.147671	0.147393	0.149573	0.147143	0.147109	0.196573	0.195587	0.457667
	0.099292	0.097485	0.097409	0.099041	0.097294	0.098463	0.129095	0.128692	0.302410

Graphs of absolute Bias values for $n_1 = n_2$, $n_1 < n_2$ and $n_1 > n_2$ with $\eta = 1$.

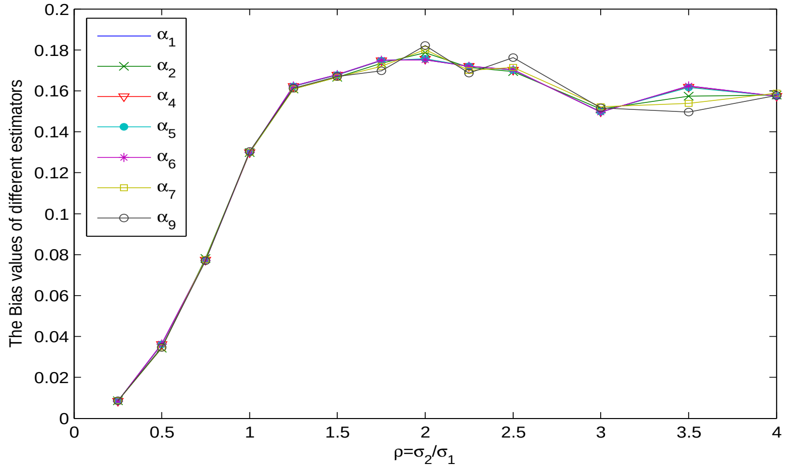

Figure 5.1: Comparison of Bias values of different estimators for $n_1 = 6, n_2 = 6$.

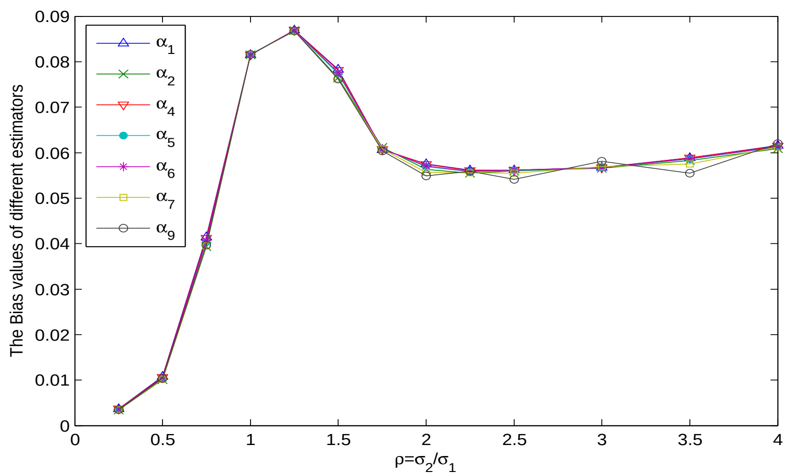

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Figure 5.2: Comparison of Bias values of different estimators for $n_1 = 12, n_2 = 12$.

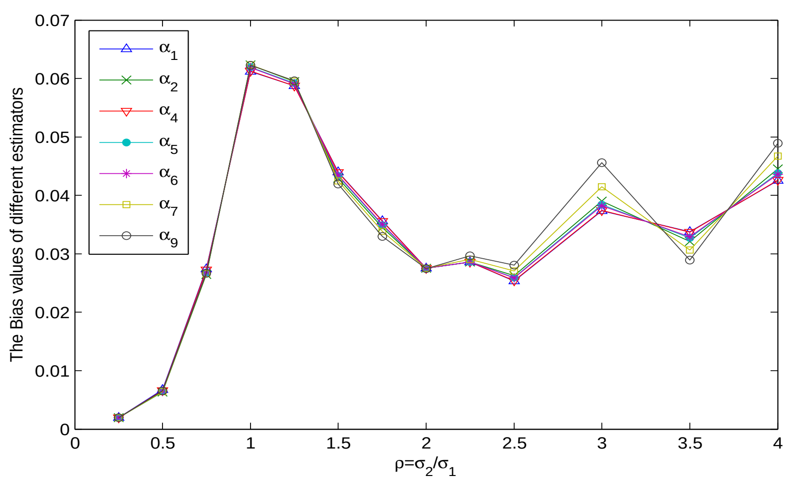

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Figure 5.3: Comparison of Bias values of different estimators for $n_1 = 18, n_2 = 18$.

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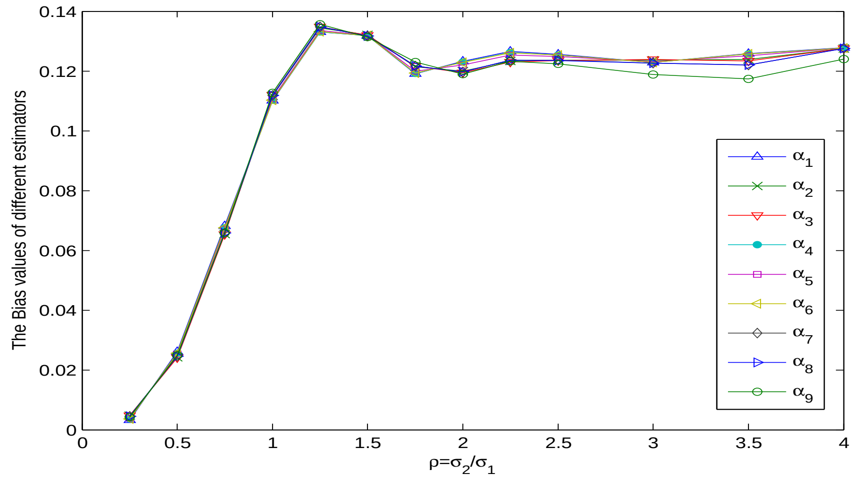

Figure 5.4: Comparison of Bias values of different estimators for $n_1 = 6, n_2 = 10$.

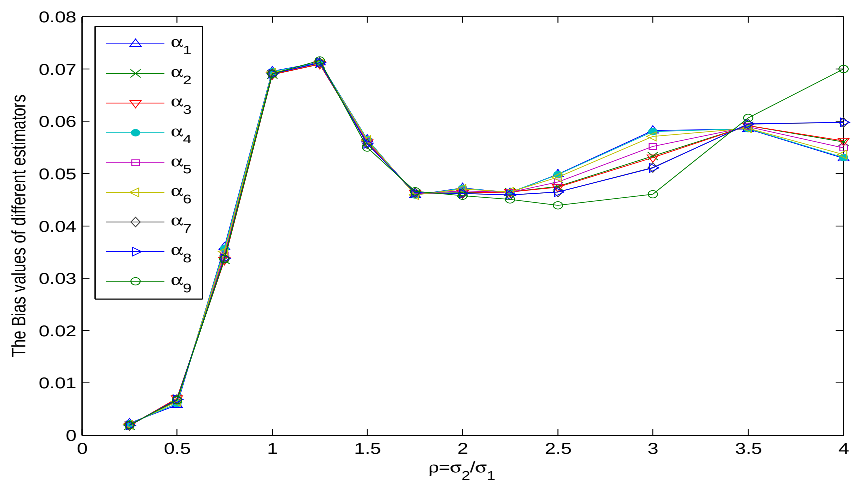

Figure 5.5: Comparison of Bias values of different estimators for $n_1 = 12, n_2 = 20$.

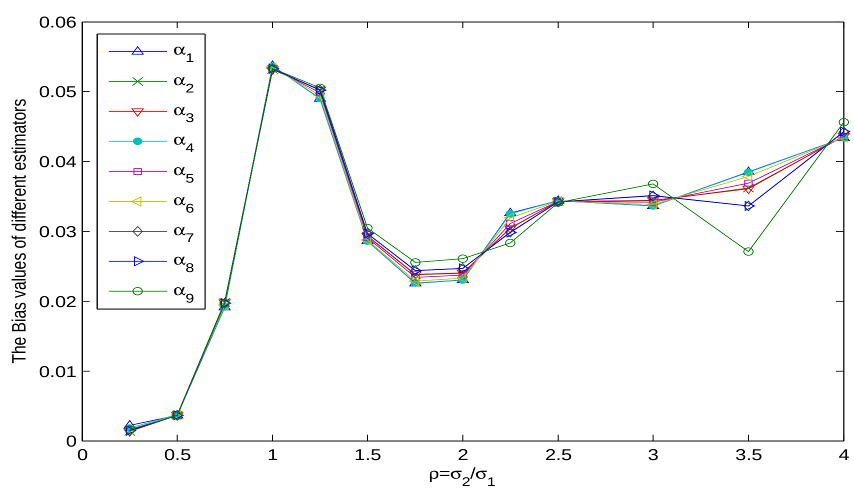

Figure 5.6: Comparison of Bias values of different estimators for $n_1 = 18, n_2 = 30$.

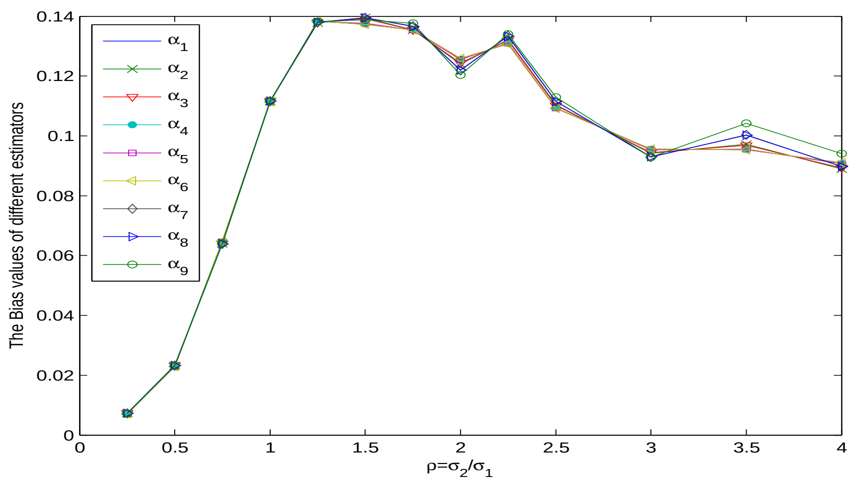

Figure 5.7: Comparison of Bias values of different estimators for $n_1 = 10, n_2 = 6$.

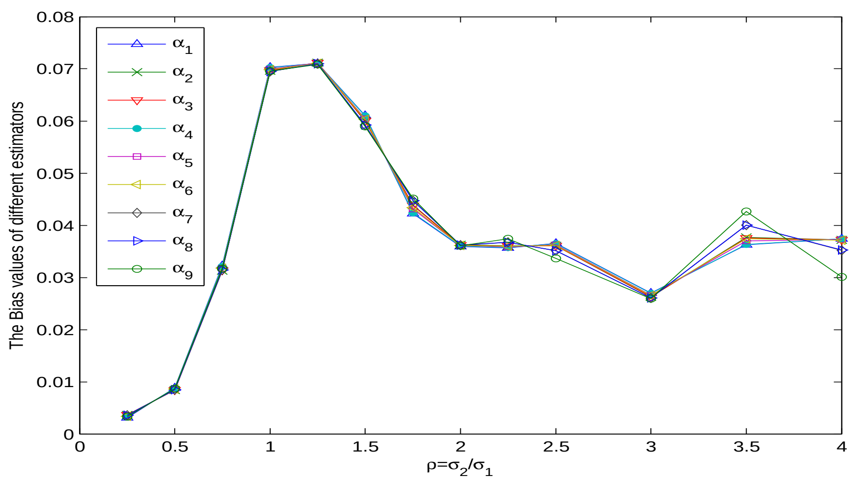

Figure 5.8: Comparison of Bias values of different estimators for $n_1 = 20, n_2 = 12$.

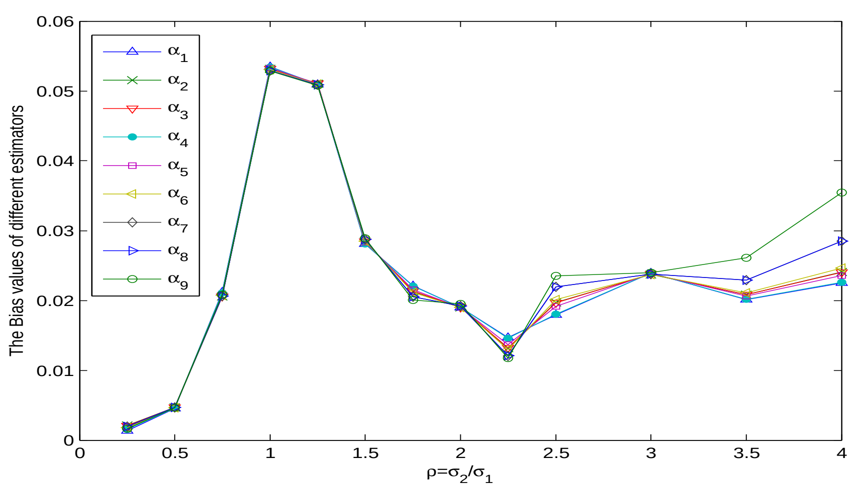

Figure 5.9: Comparison of Bias values of different estimators for $n_1 = 30, n_2 = 18$.

Graphs of Risk values for $n_1 = n_2$, $n_1 < n_2$ and $n_1 > n_2$ with $\eta = 1$.

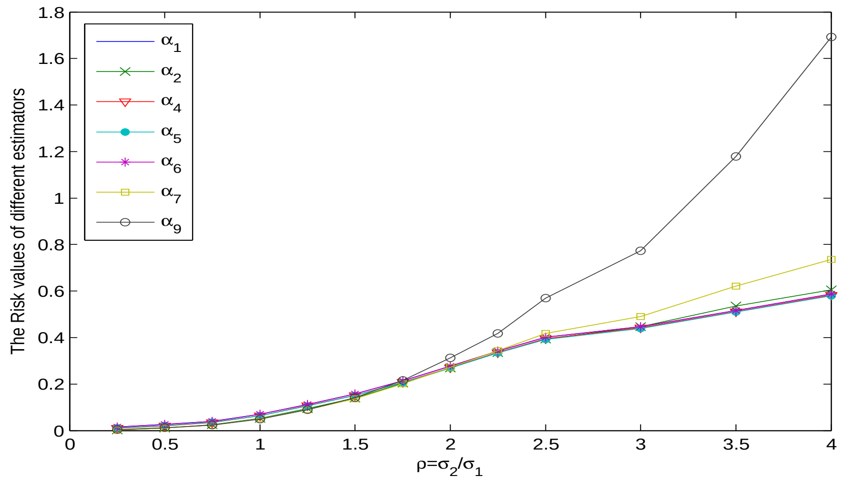

Figure 5.10: Comparison of Risk values of different estimators for $n_1 = 6$, $n_2 = 6$.

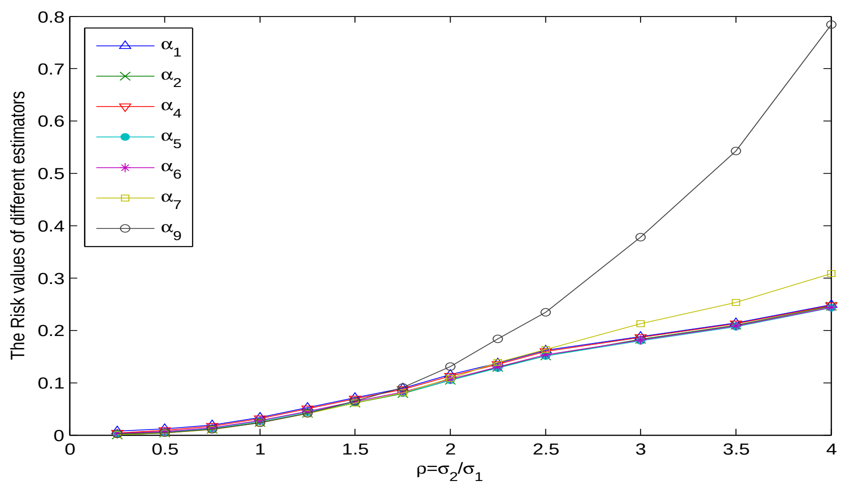

Figure 5.11: Comparison of Risk values of different estimators for $n_1 = 12$, $n_2 = 12$.

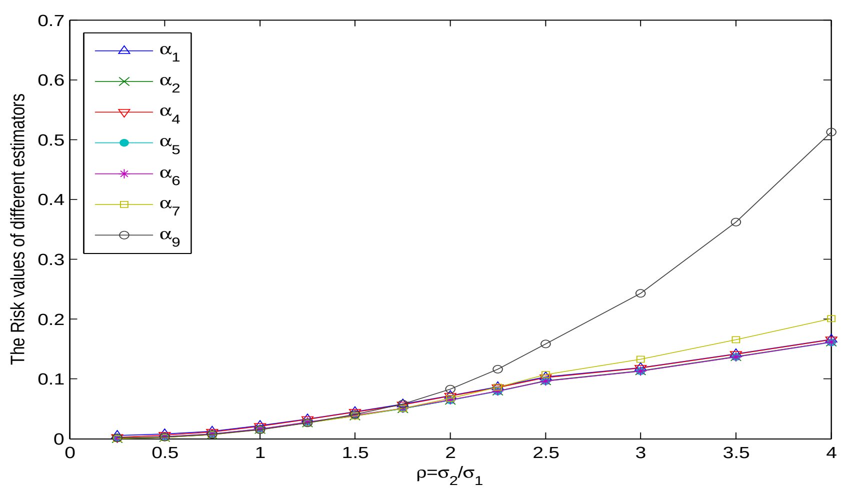

Figure 5.12: Comparison of Risk values of different estimators for $n_1 = 18$, $n_2 = 18$.

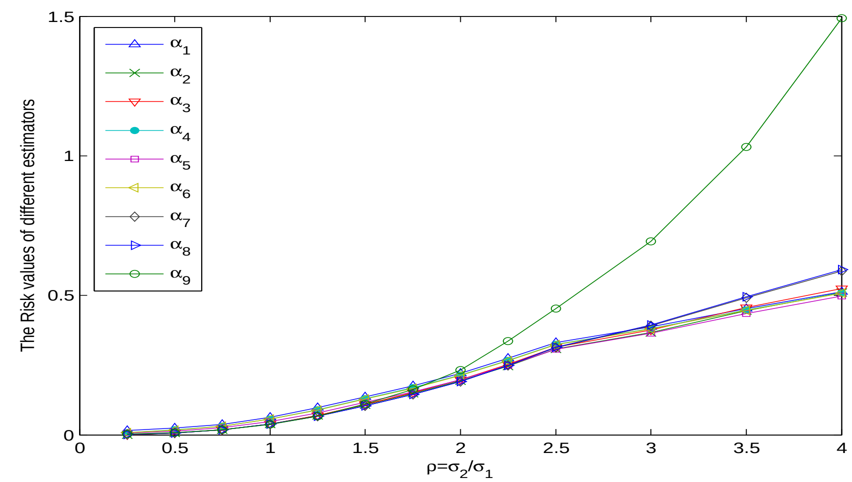

Figure 5.13: Comparison of Risk values of different estimators for $n_1 = 6$, $n_2 = 10$.

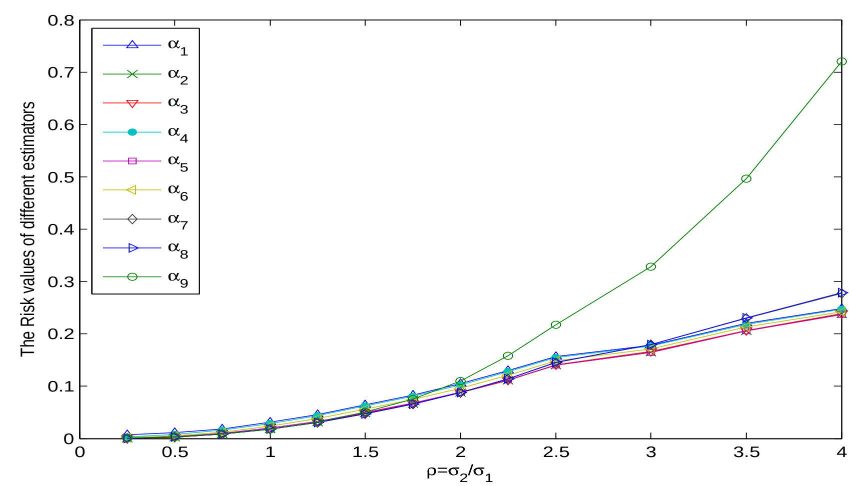

Figure 5.14: Comparison of Risk values of different estimators for $n_1 = 12$, $n_2 = 20$.

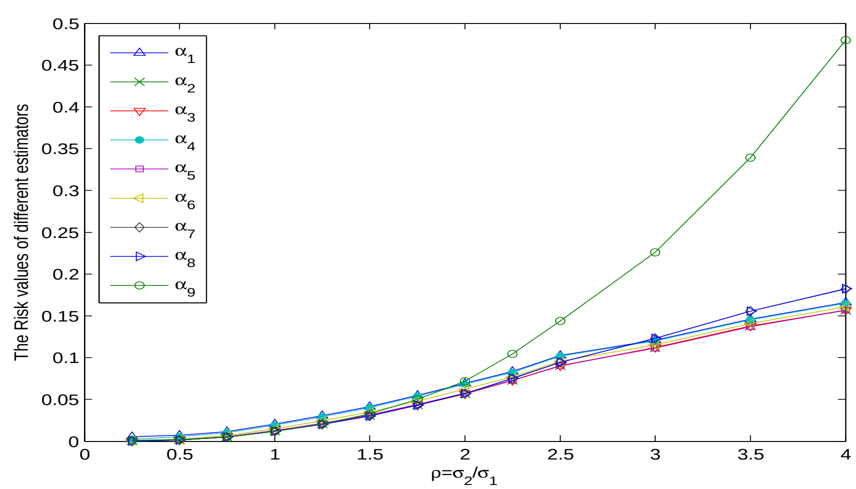

Figure 5.15: Comparison of Risk values of different estimators for $n_1 = 6$, $n_2 = 10$.

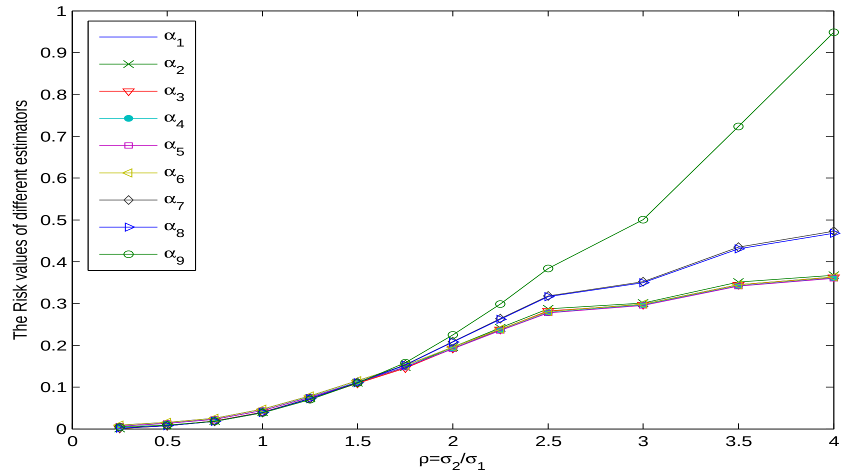

Figure 5.16: Comparison of Risk values of different estimators for $n_1 = 6$, $n_2 = 10$.

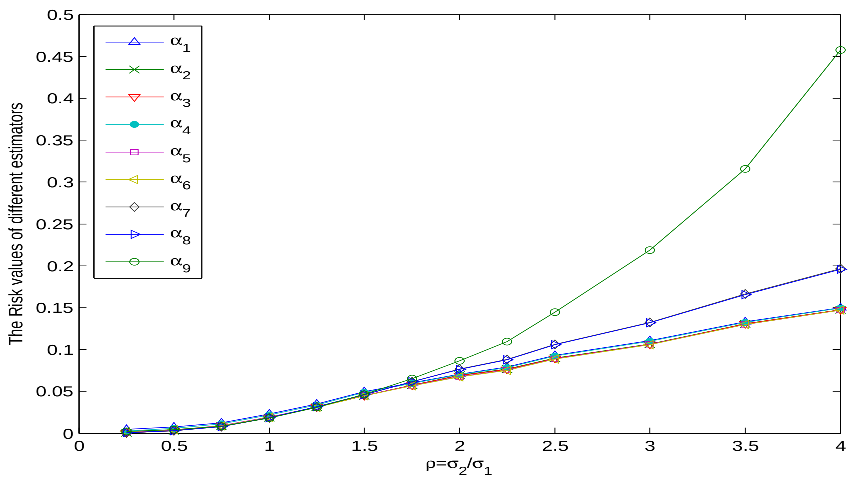

Figure 5.17: Comparison of Risk values of different estimators for $n_1 = 12$, $n_2 = 20$.

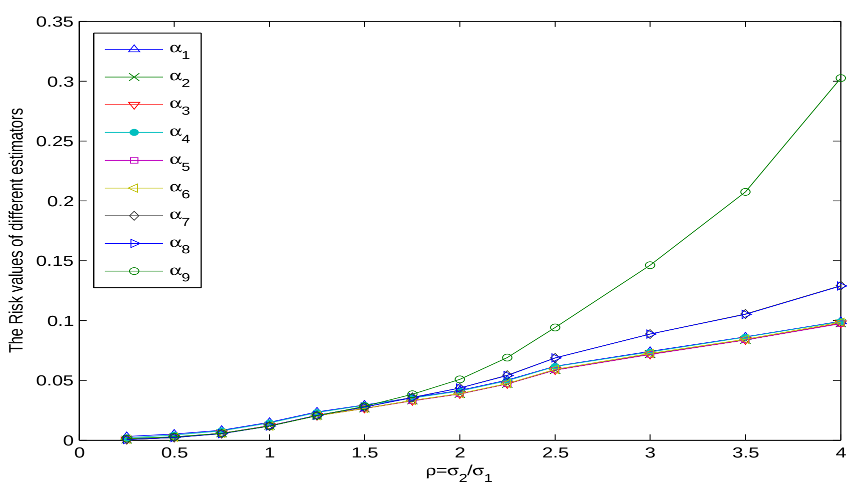

Figure 5.18: Comparison of Risk values of different estimators for $n_1 = 18$, $n_2 = 30$.

Chapter 6

Admissible and Minimax Estimators of Parameter of the Selected Pareto Population under the Generalized Stein Loss Function

6.1 Introduction

The Pareto distribution has been commonly used for approximating the right tails of distributions with positive skewness. This is also used to model the distribution of income, geophysical, migration, size of cities and firms, engineering field, property values, insurance risk, word frequencies, business mortality, etc. Such types of applications and estimation of parameters in the context of Pareto distribution have been discussed by many authors such as Malik[76] and Kern [61], Asrabadi [14], Amin [8], Dixit and Nooghabi [38], Bhattacharya et al. [25] and Mulekar and Fukasawa [97].

Let $\Pi_1,...,\Pi_k$ be $k(\geq 2)$ populations with Π_i having Pareto distribution, with associated probability density function(p.d.f.)

$$f(x|\theta_i,\beta) = \frac{\beta\theta_i^{\beta}}{x^{\beta+1}}, \quad \theta_i < x < \infty, \quad 0 < \beta, \quad i = 1,...,k, \tag{6.1}$$

where, θ_i's and β are unknown scale and the common known shape parameters respectively. Suppose we have random sample $X_{i1},...,X_{in}$ from the population $\Pi_i, i = 1,...,k$, and let $X_i = \min\{X_{i1},...,X_{in}\}$, $i = 1,...,k$. The statistic $\boldsymbol{X} = (X_1,\ldots,X_k)$ is sufficient and complete statistics and the density of X_i has the monotone likelihood ratio property in (θ_i, X_i). Then clearly X_i follows Pareto $(\theta_i, n\beta)$ distribution and the corresponding density function is

$$g(x|\theta_i,\beta) = \frac{n\beta\theta_i^{n\beta}}{x^{n\beta+1}}, \quad \theta_i < x < \infty, \quad 0 < \beta, \quad i = 1,...k. \tag{6.2}$$

Let $X_{(1)} \leq X_{(2)} \leq, ..., \leq X_{(k)}$ represent the order statistics of $\{X_1, X_2, ..., X_k\}$. In this problem, for selecting the best population, we use natural selection rule, according to which the population associated with the largest or smallest X_i is selected as the best population. Let $\theta_L(\theta_S)$ be the scale parameter of the selected population corresponding to the largest (smallest) X_i. We are interested in estimating θ_L and θ_S, which can be written as

$$\theta_L = \sum_{i=1}^{k} \theta_i I\left(X_i, \max_{j \neq i} X_j\right) \quad \text{and} \quad \theta_S = \sum_{i=1}^{k} \theta_i I\left(X_i, \min_{j \neq i} X_j\right). \tag{6.3}$$

Here, we use the indicator function $I(a,b)$ defined by

$$I(a,b) = \begin{cases} 1, & \text{if } \ a \geq b \\ 0, & \text{if } \ a < b. \end{cases}$$

For our convenience, throughout this chapter we denote the population corresponding to the largest X_i as the largest population and the population corresponding to the smallest X_i as the smallest population. Recently, some authors have carried out studies on estimating scale parameter of the selected Pareto population under various loss functions. The Uniformly Minimum Variance Unbiased (UMVU) estimator was obtained by Misra and van der Meulen [86] under squared error loss function. They also compared performances of three natural estimators and UMVU estimator under the mean squared error criterion. The UMVU estimator and an admissible class of linear estimators of selected scale parameter, as well as a general inadmissibility results from Pareto family for the scale-invariant estimators, were discussed by Kumar and Gangopadhyay [66] under squared error loss function. Nematollahi [99] considered estimation of scale parameters of selected Pareto population under the squared log error loss function. He obtained the UMVU estimator, and presented a sufficient condition of minimaxity of an estimator of scale parameters, as well as proved that natural estimators and UMRU estimator are minimax for smallest population. He has obtained the general class of linear admissible estimators of selected scale parameters and obtained a general inadmissibility result for the scale invariant estimators of the selected smallest population. Al-Mosawi and Khan [5] considered the case $k \geq 2$ Pareto populations and estimated the moments of the selected population. They proposed minimum risk equivariant (MRE) estimator of the scale parameter for the component problem when no selection is involved. They also constructed risk-unbiased estimators and studied consistency and admissibility of the natural estimators under the asymmetric scale-invariant loss function. For selected values of the order of moments and shape parameter, they further obtained the biases and risks of the natural estimators and compared the performance of the estimators through the simulation study as well.

Estimation after selection has been studied under various loss functions, either symmetric or asymmetric. Quadratic , squared error, Scale invariant squared error loss functions have been used in the literature. These loss functions are symmetric about the parameter value. Symmetric loss (Squared error loss) function assigns the same penalties to over-estimation and under-estimation of the same magnitude (Gupta and Miescke [53]; Parsion and Sanjari Farsipour [104] etc.). The utilization of the symmetric loss functions is not appropriate for estimation of the selected parameters, because it is not a scale-invariant loss function. So, an asymmetric loss function is needed to be applied (this kind of loss function has been implemented by some researchers such as Zellner[135], Sadooghi-Alvandi [115], Basu and Ebrahimi [22], etc.). This loss is useful in situations where underestimation appears to have greater significance than over-estimation. Zellner [135] expressed that in the dam construction, an underestimation of the peak water level is commonly much more serious than an overestimation. One famous asymmetric loss function that has been considered for handling such situations is the generalized Stein loss (GSL) function given by

$$L(g(\theta),\Psi)=\left(\frac{\Psi}{g(\theta)}\right)^{p}-p\ln\left(\frac{\Psi}{g(\theta)}\right)-1,\qquad p\neq 0. \tag{6.4}$$

This loss function is not symmetric but convex in $\Delta=\frac{\Psi}{g(\theta)}$ when p=1 and quasi-convex otherwise, and has a unique minimum at $\Delta=1$, GSL is scale invariant and also is useful in situations where under-estimation and over-estimation do not have the same penalty. The GSL function with negative p values penalizes over-estimation more than under-estimation and with positive p values it acts vice-versa. It is worth mentioning that near $\Delta=1$,

$$\left(\frac{\Psi}{g(\theta)}\right)^{p}-p\ln\left(\frac{\Psi}{g(\theta)}\right)-1\approx\frac{p^2}{2}\left(\frac{\Psi}{g(\theta)}-1\right)^{2},$$

and for small $|p|$ values,

$$\left(\frac{\Psi}{g(\theta)}\right)^{p}-p\ln\left(\frac{\Psi}{g(\theta)}\right)-1\approx\frac{p^2}{2}\left(\ln\Psi-\ln g(\theta)\right)^{2}.$$

The remainder of the chapter is organized as follows. In Section 6.2, the uniform minimum risk unbiased (UMRU) estimator of θ_L and θ_S are derived based on the results of Nematollahi and Jafari Jozani [100]. In Section 6.3, we have proved a sufficient condition for minimaxity of estimators of θ_L and θ_S, and shown that the generalized Bayes estimators of θ_S are minimax for $k=2$. We have obtained a class of admissible linear estimators of θ_L and θ_S in Section 6.4. The technique of Brewster and Zidek [27] is employed to provide a sufficient condition for inadmissibility of some scale and permutation invariant estimators of θ_S and the UMRU estimator of θ_S is shown to be inadmissible and provided some dominated or better estimators. Finally, some results and discussions are given in section 6.5.

6.2 UMRU Estimation

In this section, we introduce the concept of the general form of uniformly minimum risk unbiased (UMRU) estimator of θ_L and obtain the conditions for the risk-unbiased estimator under the GSL function (6.4). Then we derive the UMRU estimators of θ_L and θ_S given in (6.3). The following definition of risk-unbiased estimator is adopted from Lehmann [71].

Definition 6.2.1. [Lehmann [71]] An estimator $\Psi(\boldsymbol{X})$ of the parameter $g(\boldsymbol{\theta})$ is said to be risk-unbiased if it satisfies the inequality

$$E_{\boldsymbol{\theta}}\left[L(g(\boldsymbol{\theta}),\Psi(\boldsymbol{X}))\right] \leq E_{\boldsymbol{\theta}}[L(g(\boldsymbol{\theta}^{'}),\Psi(\boldsymbol{X}))], \qquad \text{for all } \boldsymbol{\theta} \neq \boldsymbol{\theta}^{'}.$$

A decision-theoretic approach of unbiasedness that mainly depends on the type of loss functions is applied. Using the definition 6.2.1 and the GSL function (6.4), an estimator $\Psi(\boldsymbol{X})$ is a risk-unbiased estimator of the parameter $g(\boldsymbol{\theta})$, if it satisfies the following condition

$$E_{\boldsymbol{\theta}}[\Psi^p(\boldsymbol{X})] = g^p(\boldsymbol{\theta}), \text{ for all } \boldsymbol{\theta}. \tag{6.5}$$

So, the condition for being risk-unbiased estimator of θ_L is given by

$$E_{\boldsymbol{\theta}}\left[\Psi^p(\boldsymbol{x})\right] = E_{\boldsymbol{\theta}}\left[\theta_L^p\right], \qquad \text{for all } \boldsymbol{\theta}.$$

Nematollahi and Jafari Jozani [100] introduced the concept of UMRU estimation for the random parameter $g(\theta)(=\theta_L \text{or } \ \theta_S)$ under the general ξ-loss function

$$L(g(\theta),\Psi) = (\xi(\Psi) - \xi(g(\theta)))^2,$$

while it has risk-unbiased condition $E_\theta[\xi(\Psi(\boldsymbol{X}))] = E_\theta[\xi(g(\theta))]$. They have shown that for Pareto distribution under the $\xi-$ loss function, the UMRU estimators of θ_L and θ_S are given by

$$\Psi_L^U(\boldsymbol{X}) = \xi^{-1}\left[\xi(X_{(k)}) - \frac{1}{n\beta}\sum_{i=1}^{k}\xi^{'}(X_{(i)})X_{(i)}\left(\frac{X_{(i)}}{X_{(k)}}\right)^{n\beta}\right] \tag{6.6}$$

and

$$\Psi_S^U(\boldsymbol{X}) = \xi^{-1}\left[\xi(X_{(1)}) - \xi^{'}(X_{(1)})\frac{X_{(1)}}{n\beta}\left(1-\left(\frac{X_{(1)}}{X_{(2)}}\right)^{n\beta}\right)\right] \tag{6.7}$$

respectively. Since the unbiased condition under GSL function, is equivalent to unbiased condition under ξ-loss function with $\xi(x) = x^p$, then from (6.6) and (6.7) the UMRU estimators of θ_L and θ_S for Pareto population under the GSL function (6.4) are given by

$$\Psi_L^U(\boldsymbol{X}) = X_{(k)}\left[1 - \frac{p}{n\beta}\sum_{i=1}^{k}\left(\frac{X_{(i)}}{X_{(k)}}\right)^{n\beta+p}\right]^{1/p} \tag{6.8}$$

and

$$\Psi_S^U(\boldsymbol{X}) = X_{(1)} \left[1 - \frac{p}{n\beta} + \frac{p}{n\beta} \left(\frac{X_{(1)}}{X_{(2)}} \right)^{n\beta} \right]^{1/p} \tag{6.9}$$

respectively.

Remark 6.2.1. *Let $X_{[1]} \leq \cdots \leq X_{[k]}$ denote the ordered values of random variable $X_1, X_2, ..., X_k$. For $p = -1$, it should be mentioned from* (6.8) *and* (6.9)*, that the UMRU estimators of scale parameters θ_L and θ_S under the entropy loss function are given by*

$$\Psi_L^U(\boldsymbol{X}) = \left[\frac{n\beta X_{(k)}}{n\beta + \sum_{i=1}^{k} \left(\frac{X_{(i)}}{X_{(k)}} \right)^{n\beta - 1}} \right] \tag{6.10}$$

and

$$\Psi_S^U(\boldsymbol{X}) = \left[\frac{n\beta X_{(1)}}{(n\beta + 1) - \left(\frac{X_{(1)}}{X_{(2)}} \right)^{n\beta}} \right] \tag{6.11}$$

respectively.

6.3 Minimax estimation

In this section, we deal with minimax estimation of selected scale parameters when we have two independent populations Π_1 and Π_2, i.e. $k = 2$. We are able to provide a sufficient condition for minimaxity of estimators of θ_L and θ_S under the GSL function as defined in (6.4), and next we prove that generalized Bayes estimators of θ_S is minimax.

6.3.1 Sufficient condition for minimaxity

We use some results of Sackrowitz and Samuel-Cahn [114] for finding the sufficient condition of minimax estimation. We state the following theorem for the desired sufficient condition.

Theorem 6.3.1. *Suppose $X_{i1}, \ldots, X_{in}$ denotes a random sample of size n from Pareto population and having the density* (6.1)*. Let $X_{(i)} = \min(X_{i1}, ..., X_{in}), i = 1, 2$, and $X_{(1)} \leq X_{(2)}$ are the ordered statistics of X_1, X_2. An estimator $\Psi^*(X_1, X_2)$ is minimax for $g(\theta) = \theta_L$ or θ_S under the GSL function as defined in* (6.4) *if its risk satisfies*

$$R(g(\theta), \Psi^*) \leq \ln \left(\frac{n\beta}{n\beta - p} \right) - \frac{p}{n\beta}. \tag{6.12}$$

Proof. To establish the inequality (6.12), the consequences of Sackrowitz and Samuel-Cahn [114] are applied to determine the minimax estimator in the component problem for $\theta_i, i = 1,2$. Nematollahi [99] considered the following prior for $\theta_i, i = 1,2$,

$$\Pi_i^m(\theta_i) = \frac{\theta_i^{\frac{1}{m}-1}}{ma^{\frac{1}{m}}}, \quad 0 < \theta_i < a. \tag{6.13}$$

Since $X_i|\theta_i$ has pareto pdf (6.2), the posterior density can be effortlessly obtained, that's defined as

$$\Pi_i^m(\theta_i|x_i) = \frac{(n\beta + \frac{1}{m})\theta_i^{n\beta+\frac{1}{m}-1}}{(a_i^*)^{n\beta+\frac{1}{m}}}, \quad 0 < \theta_i < \min(x_i, a) = a_i^*. \tag{6.14}$$

It is straightforward to see that the Bayes estimator of θ_i under the GSL function is identical to

$$\Psi_{\Pi_i^m}(x_i) = \left[E\left(\frac{1}{\theta_i^p}|x_i\right)\right]^{-\frac{1}{p}} = a_i^*\left[\frac{n\beta - p + \frac{1}{m}}{n\beta + \frac{1}{m}}\right]^{\frac{1}{p}} \tag{6.15}$$

and the posterior risk of $\Psi_{\Pi_i^m}(X_i)$ is given by

$$\begin{aligned} r\left(x_i, \Psi_{\Pi_i^m}(x_i)\right) &= E\left[\left(\frac{\Psi_{\Pi_i^m}(x_i)}{\theta_i}\right)^p - p\ln\left(\frac{\Psi_{\Pi_i^m}(x_i)}{\theta_i}\right) - 1|x_i\right] \\ &= \ln\left[E\left(\frac{1}{\theta_i^p}|x_i\right)\right] + pE\left(\ln\theta_i|x_i\right) \\ &= \ln\left[\frac{n\beta + \frac{1}{m}}{n\beta - p + \frac{1}{m}}\right] - \frac{p}{n\beta + \frac{1}{m}} \end{aligned}$$

Since the posterior risk does not depend on x_i, therefore the Bayes risk of $\Psi_{\Pi_i^m}(x_i)$ is also

$$r^*\left(\Pi_i^m, \Psi_{\Pi_i^m}\right) = \ln\left[\frac{n\beta + \frac{1}{m}}{(n\beta - p + \frac{1}{m})}\right] - \frac{p}{n\beta + \frac{1}{m}}, i = 1,2. \tag{6.16}$$

Now consider the Bayes estimator of θ_L and θ_S under the GSL function (6.4). Suppose θ_1 and θ_2 are independently and identically distributed (i.i.d.) random variables with prior, whose density is (6.13) with $a = m$. Then using the result of Sackrowitz and Samuel-Cahn [114] (from lemma 3.2) and equation (6.15), the unique Bayes estimator of θ_L and θ_S under the GSL function, and the prior $\Pi^m = (\Pi_1^*, \Pi_2^*)$ are given by

$$\Psi_1^m(X_1, X_2) = \min(X_{(2)}, m)\left[\frac{n\beta - p + \frac{1}{m}}{n\beta + \frac{1}{m}}\right]^{\frac{1}{p}} \tag{6.17}$$

and

$$\Psi_2^m(X_1, X_2) = \min(X_{(1)}, m)\left[\frac{n\beta - p + \frac{1}{m}}{n\beta + \frac{1}{m}}\right]^{\frac{1}{p}} \tag{6.18}$$

respectively. Since the posterior risk of the component problem is independent of $\boldsymbol{x} = (x_1, x_2)$, therefore by using the result of Sackrowitz and Samuel-Cahn [114] (from Theorem 3.1), the Bayes risk $r_i^*(\Pi^m, \Psi_i^m)$ of $\Psi_i^m(X_1, X_2), i = 1, 2$, are the same as the one given in (6.16), i.e.,

$$\begin{aligned} r_1^*(\Pi^m, \Psi_1^m) &= r_2^*(\Pi^m, \Psi_2^m) = r^*(\Pi_i^m, \Psi_i^m) \\ &= \ln\left[\frac{n\beta + \frac{1}{m}}{(n\beta - p + \frac{1}{m})}\right] - \frac{p}{n\beta + \frac{1}{m}}, i = 1, 2, \end{aligned}$$

and hence

$$\lim_{m\to\infty} r_i^*(\Pi^m, \Psi_i^m) = \ln\left[\frac{n\beta}{(n\beta - p)}\right] - \frac{p}{n\beta}, \quad i = 1, 2. \tag{6.19}$$

Now, we apply result of Sackrowitz and Samuel-Cahn [114] (from Theorem 3.2), the estimator $\Psi_L(X_1, X_2)$ and $\Psi_S(X_1, X_2)$ are minimax for θ_L and θ_S, respectively, if

$$R(\theta_L, \Psi_L) \le \lim_{m\to\infty} r_1^*(\Pi^m, \Psi_1^m) = \ln\left[\frac{n\beta}{(n\beta - p)}\right] - \frac{p}{n\beta} \tag{6.20}$$

and

$$R(\theta_S, \Psi_S) \le \lim_{m\to\infty} r_2^*(\Pi^m, \Psi_2^m) = \ln\left[\frac{n\beta}{(n\beta - p)}\right] - \frac{p}{n\beta}, \tag{6.21}$$

where $R(\theta_L, \Psi_L)$ and $R(\theta_S, \Psi_S)$ are the risk functions of Ψ_L and Ψ_S under the GSL function, respectively. We should note that the limiting Bayes estimators $\Psi_1^\infty(X_1, X_2) = X_{(2)}\left(\frac{n\beta - p}{n\beta}\right)^{\frac{1}{p}}$ and $\Psi_2^\infty(X_1, X_2) = X_{(1)}\left(\frac{n\beta - p}{n\beta}\right)^{\frac{1}{p}}$ of θ_L and θ_S are Generalized Bayes estimators of θ_L and θ_S under the noninformative prior $\Pi_i(\theta) = \frac{1}{\theta_i}, \theta_i > 0$, $i = 1, 2$. □

6.3.2 Minimax estimators of Selected Parameters

Consider the assumption of Theorem 6.3.1 and with respect to the group of scale transformation $G = \{g_c : g_c(x_1, x_2) = (cx_1, cx_2), c > 0\}$. It is easy to verify that the best invariant estimator of θ_i is $\Psi(X_i) = X_i\left(\frac{n\beta - p}{n\beta}\right)^{\frac{1}{p}}, i = 1, 2$. Therefore, the natural (generalized Bayes (GB)) estimators of θ_L and θ_S are given as

$$\Psi_L^{GB}(\boldsymbol{X}) = X_{(2)}\left(\frac{n\beta - p}{n\beta}\right)^{\frac{1}{p}} \quad \text{and} \quad \Psi_S^{GB}(\boldsymbol{X}) = X_{(1)}\left(\frac{n\beta - p}{n\beta}\right)^{\frac{1}{p}} \tag{6.22}$$

respectively. We further try to find that the the GB estimators of θ_S is minimax. We need the following lemma, which will be helpful in acquiring the results.

Lemma 6.3.2. *Under the assumptions of Theorem* 6.3.1*, let* $\lambda = \min(\theta_1, \theta_2)/\max(\theta_1, \theta_2)$ *and* $n\beta > p$ *then* *(a)* $E\left[\left(\frac{X_{(2)}}{\theta_L}\right)^p\right] = \frac{(n\beta)^2\lambda^{n\beta-p}}{(n\beta-p)(2n\beta-p)} + \frac{n\beta}{(n\beta-p)} - \frac{n\beta\lambda^{n\beta}}{(2n\beta-p)}$

(b) $E\left[\ln\left(\frac{X_{(2)}}{\theta_L}\right)\right] = \frac{\lambda^{n\beta}}{2}\left(\frac{1}{n\beta} - \ln(\lambda)\right) + \frac{1}{n\beta}$

(c) $E\left[\left(\frac{X_{(1)}}{\theta_S}\right)^p\right] = \frac{n\beta\lambda^{n\beta}}{(2n\beta-p)} + \frac{n\beta}{(n\beta-p)} - \frac{(n\beta)^2\lambda^{n\beta-p}}{(n\beta-p)(2n\beta-p)}$

(d) $E\left[\ln\left(\frac{X_{(1)}}{\theta_S}\right)\right] = \frac{\lambda^{n\beta}}{2}\left(\ln(\lambda) - \frac{1}{n\beta}\right) + \frac{1}{n\beta}$

Proof. Let $Y_1 = \frac{\theta_1}{X_1}$, $Y_2 = \frac{\theta_2}{X_2}$, $W = \frac{X_1}{X_2}$ and $W^* = \frac{Y_1}{Y_2}$. Then Y_1 and Y_2 are independent with pdf $f_{Y_i}(y_i) = n\beta y_i^{n\beta-1}, 0 < y_i < 1, i = 1,2$. Let $I(A)$ be the indicator function of the set A.

(a) If $\theta_1 < \theta_2$,i.e. $\lambda = \frac{\theta_1}{\theta_2} < 1$, then

$$\begin{aligned}
E\left[\left(\frac{X_{(2)}}{\theta_L}\right)^p\right] =& E\left[\left(\frac{X_2}{\theta_2}\right)^p I(W \le 1)\right] + E\left[\left(\frac{X_1}{\theta_1}\right)^p I(W > 1)\right] \\
=& E\left[\frac{1}{Y_2^p} I(\lambda \le W^*)\right] + E\left[\frac{1}{Y_1^p} I(\lambda > W^*)\right] \\
=& \int_0^1 n\beta y_2^{n\beta-p-1}\left[\int_{\lambda y_2}^1 n\beta y_1^{n\beta-1} dy_1\right] dy_2 \\
&+ \int_0^1 n\beta y_2^{n\beta-1}\left[\int_0^{\lambda y_2} n\beta y_1^{n\beta-p-1} dy_1\right] dy_2 \\
=& \int_0^1 n\beta y_2^{n\beta-p-1}\left[n\beta\left(\frac{y_1^{n\beta}}{n\beta}\right)_{\lambda y_2}^1\right] dy_2 + \int_0^1 n\beta y_2^{n\beta-1}\left[n\beta\left(\frac{y_1^{n\beta-p}}{n\beta-p}\right)_0^{\lambda y_2}\right] dy_2 \\
=& \int_0^1 n\beta y_2^{n\beta-p-1}\left(1 - (\lambda y_2)^{n\beta}\right) dy_2 + \frac{(n\beta)^2\lambda^{n\beta-p}}{n\beta-p}\int_0^1 y_2^{2n\beta-p-1} dy_2.
\end{aligned}$$

Integrating the terms and simplifying further we get

$$E\left[\left(\frac{X_{(2)}}{\theta_L}\right)^p\right] = \frac{(n\beta)^2\lambda^{n\beta-p}}{(n\beta-p)(2n\beta-p)} + \frac{n\beta}{(n\beta-p)} - \frac{n\beta\lambda^{n\beta}}{(2n\beta-p)}.$$

Similarly, for $\theta_1 > \theta_2$,i.e. $\lambda = \frac{\theta_2}{\theta_1} > 1$, we have

$$E\left[\left(\frac{X_{(2)}}{\theta_L}\right)^p\right] = \frac{(n\beta)^2\lambda^{n\beta-p}}{(n\beta-p)(2n\beta-p)} + \frac{n\beta}{(n\beta-p)} - \frac{n\beta\lambda^{n\beta}}{(2n\beta-p)}.$$

(b) If $\theta_1 < \theta_2$, i.e., $\frac{\theta_1}{\theta_2} < 1$.

$$
\begin{aligned}
E\left[\ln\left(\frac{X_{(2)}}{\theta_L}\right)\right] =& E\left[\ln\left(\frac{X_2}{\theta_2}\right)I(W \le 1)\right] + E\left[\ln\left(\frac{X_1}{\theta_1}\right)I(W > 1)\right] \\
=& E\left[\ln\left(\frac{1}{Y_2}\right)I(\lambda \le W^*)\right] + E\left[\ln\left(\frac{1}{Y_1}\right)I(\lambda > W^*)\right] \\
=& -\int_0^1 n\beta y_2^{n\beta-1}\ln(y_2)\left[\int_{\lambda y_2}^1 n\beta y_1^{n\beta-1}dy_1\right]dy_2 \\
& -\int_0^1 n\beta y_2^{n\beta-1}\left[\int_0^{\lambda y_2} n\beta y_1^{n\beta-1}\ln(y_1)dy_1\right]dy_2 \\
=& -\int_0^1 n\beta y_2^{n\beta-1}\ln(y_2)\left[n\beta\left(\frac{y_1^{n\beta}}{n\beta}\right)_{\lambda y_2}^1\right]dy_2 \\
& -\int_0^1 n\beta y_2^{n\beta-1}\left[n\beta\left(\ln(y_1)\frac{y_1^{n\beta}}{n\beta} - \frac{y_1^{n\beta}}{(n\beta)^2}\right)_0^{\lambda y_2}\right]dy_2 \\
=& -\int_0^1 n\beta y_2^{n\beta-1}\ln(y_2)\left[1-(\lambda y_2)^{n\beta}\right]dy_2 \\
& -\int_0^1 n\beta y_2^{n\beta-1}\left[(\lambda y_2)^{n\beta}\ln(\lambda y_2) - \frac{(\lambda y_2)^{n\beta}}{n\beta}\right]dy_2.
\end{aligned}
$$

Integrating the terms and simplifying further we get

$$
E\left[\ln\left(\frac{X_{(2)}}{\theta_L}\right)\right] = \frac{\lambda^{n\beta}}{2}\left(\frac{1}{n\beta} - \ln(\lambda)\right) + \frac{1}{n\beta}.
$$

Similarly, for $\theta_1 > \theta_2$, i.e, $\lambda = \frac{\theta_2}{\theta_1} < 1$, we have

$$
E\left[\ln\left(\frac{X_{(2)}}{\theta_L}\right)\right] = \frac{\lambda^{n\beta}}{2}\left(\frac{1}{n\beta} - \ln(\lambda)\right) + \frac{1}{n\beta}.
$$

(c) If $\theta_1 < \theta_2$, i.e. $\lambda = \frac{\theta_1}{\theta_2} < 1$, we have

$$
E\left[\left(\frac{X_{(1)}}{\theta_S}\right)^p\right] = E\left[\left(\frac{X_1}{\theta_1}\right)^p I(W \le 1)\right] + E\left[\left(\frac{X_2}{\theta_2}\right)^p I(W > 1)\right]
$$

$$
\begin{aligned}
&=E\left[\frac{1}{Y_1^p} I(\lambda \le W^*)\right] + E\left[\frac{1}{Y_2^p} I(\lambda > W^*)\right] \\
&= \int_0^1 n\beta y_2^{n\beta-1}\left[\int_{\lambda y_2}^1 n\beta y_1^{n\beta-p-1} dy_1\right] dy_2 + \int_0^1 n\beta y_2^{n\beta-p-1}\left[\int_0^{\lambda y_2} n\beta y_1^{n\beta-1} dy_1\right] dy_2 \\
&= \int_0^1 n\beta y_2^{n\beta-1}\left[n\beta\left(\frac{y_1^{n\beta-p}}{n\beta-p}\right)_{\lambda y_2}^1\right] dy_2 + \int_0^1 n\beta y_2^{n\beta-p-1}\left[n\beta\left(\frac{y_1^{n\beta}}{n\beta}\right)_0^{\lambda y_2}\right] dy_2 \\
&=\frac{(n\beta)^2}{(n\beta-p)}\int_0^1 y_2^{n\beta-1} dy_2 - \frac{n\beta\lambda^{n\beta-p}}{(n\beta-p)}\int_0^1 y_2^{2n\beta-p-1} dy_2 + n\beta\lambda^{n\beta}\int_0^1 y_2^{2n\beta-p-1} dy_2 \\
&=\frac{(n\beta)^2}{(n\beta-p)}\left(\frac{y_2^{n\beta}}{n\beta}\right)_0^1 - \frac{n\beta\lambda^{n\beta-p}}{(n\beta-p)}\left(\frac{y_2^{2n\beta-p}}{2n\beta-p}\right)_0^1 + n\beta\lambda^{n\beta}\left(\frac{y_2^{2n\beta-p}}{2n\beta-p}\right)_0^1.
\end{aligned}
$$

Therefore, after the simplification, we give

$$
E\left[\left(\frac{X_{(1)}}{\theta_S}\right)^p\right] = \frac{n\beta\lambda^{n\beta}}{(2n\beta-p)} + \frac{n\beta}{(n\beta-p)} - \frac{(n\beta)^2\lambda^{n\beta-p}}{(n\beta-p)(2n\beta-p)}.
$$

Similarly, if $\theta_1 > \theta_2$, $\lambda = \frac{\theta_2}{\theta_1} < 1$, we have obtained

$$
E\left[\left(\frac{X_{(1)}}{\theta_S}\right)^p\right] = \frac{n\beta\lambda^{n\beta}}{(2n\beta-p)} + \frac{n\beta}{(n\beta-p)} - \frac{(n\beta)^2\lambda^{n\beta-p}}{(n\beta-p)(2n\beta-p)}.
$$

(d) If $\theta_1 < \theta_2$, i.e., $\lambda = \frac{\theta_1}{\theta_2} < 1$,

$$
\begin{aligned}
E\left[\ln\left(\frac{X_{(1)}}{\theta_S}\right)\right] &= E\left[\ln\left(\frac{X_1}{\theta_1}\right) I(W \le 1)\right] + E\left[\ln\left(\frac{X_2}{\theta_2}\right) I(W > 1)\right] \\
&= E\left[\ln\left(\frac{1}{Y_1}\right) I(\lambda \le W^*)\right] + E\left[\ln\left(\frac{1}{Y_2}\right) I(\lambda > W^*)\right] \\
&= -\int_0^1 n\beta y_2^{n\beta-1}\left[\int_{\lambda y_2}^1 n\beta y_1^{n\beta-1}\ln(y_1) dy_1\right] dy_2 \\
&\quad -\int_0^1 n\beta y_2^{n\beta-1}\ln(y_2)\left[\int_0^{\lambda y_2} n\beta y_1^{n\beta-1} dy_1\right] dy_2 \\
&= -\int_0^1 n\beta y_2^{n\beta-1}\left[n\beta\left(\frac{y_1^{n\beta}\ln(y_1)}{n\beta} - \frac{y_1^{n\beta}}{(n\beta)^2}\right)_{\lambda y_2}^1\right] dy_2 \\
&\quad -\int_0^1 n\beta y_2^{n\beta-1}\ln(y_2)\left[n\beta\left(\frac{y_1^{n\beta}}{n\beta}\right)_0^{\lambda y_2}\right] dy_2 \\
&= \int_0^1 y_2^{n\beta-1} dy_2 + n\beta\lambda_{n\beta}\ln(\lambda)\int_0^1 y_2^{2n\beta-1} dy_2 - \lambda^{n\beta}\int_0^1 y_2^{2n\beta-1} dy_2 \\
&= \left(\frac{y_2^{n\beta}}{n\beta}\right)_0^1 + n\beta\lambda^{n\beta}\ln(\lambda)\left(\frac{y_2^{2n\beta-1}}{2n\beta}\right)_0^1 - \lambda^{n\beta}\left(\frac{y_2^{2n\beta}}{2n\beta}\right)_0^1.
\end{aligned}
$$

Therefore, after the simplification, we obtain

$$E\left[\ln\left(\frac{X_{(1)}}{\theta_S}\right)\right]=\frac{\lambda^{n\beta}}{2}\left[\ln(\lambda)-\frac{1}{n\beta}\right]+\frac{1}{n\beta}.$$

Similarly, if $\theta_1>\theta_2$, $\lambda=\frac{\theta_2}{\theta_1}<1$, we have obtain

$$E\left[\ln\left(\frac{X_{(1)}}{\theta_S}\right)\right]=\frac{\lambda^{n\beta}}{2}\left[\ln(\lambda)-\frac{1}{n\beta}\right]+\frac{1}{n\beta}.$$

□

Theorem 6.3.3. *Let X_1 and X_2 be two independent random variables where X_i having a Pareto distribution defined in* (6.2)*. Then, GB estimator $\Psi_S^{GB}(\boldsymbol{X})$ defined in* (6.22) *is minimax estimator of θ_S under the GSL function* (6.4)*, when $n\beta>p$.*

Proof.

$$\begin{aligned}
R\left(\theta_S,\Psi_S^{GB}\right)=&E\left[\left(\frac{\Psi_S^{GB}}{\theta_S}\right)^p-p\ln\left(\frac{\Psi_S^{GB}}{\theta_S}\right)-1\right]\\
=&E\left[\left(\frac{X_{(1)}\left(\frac{n\beta-p}{n\beta}\right)^{\frac{1}{p}}}{\theta_S}\right)^p-p\ln\left(\frac{X_{(1)}\left(\frac{n\beta-p}{n\beta}\right)^{\frac{1}{p}}}{\theta_S}\right)-1\right]\\
=&\left(\frac{n\beta-p}{n\beta}\right)E\left(\frac{X_{(1)}}{\theta_S}\right)^p-pE\left[\ln\left(\frac{X_{(1)}}{\theta_S}\right)\right]-\ln\left(\frac{n\beta-p}{n\beta}\right)-1\\
=&\left(\frac{n\beta-p}{n\beta}\right)\left[\frac{n\beta\lambda^{n\beta}}{(2n\beta-p)}+\frac{n\beta}{(n\beta-p)}-\frac{(n\beta)^2\lambda^{n\beta-p}}{(n\beta-p)(2n\beta-p)}\right]\\
&-p\left[\frac{\lambda^{n\beta}}{2}\left(\ln(\lambda)-\frac{1}{n\beta}\right)+\frac{1}{n\beta}\right]+\ln\left(\frac{n\beta}{n\beta-p}\right)-1\\
=&\ln\left(\frac{n\beta}{n\beta-p}\right)-\frac{p}{n\beta}+\lambda^{n\beta}\left[\frac{(n\beta-p)}{(2n\beta-p)}-\frac{n\beta}{\lambda^p(2n\beta-p)}-\frac{p\ln(\lambda)}{2}+\frac{p}{2n\beta}\right]\\
=&\ln\left(\frac{n\beta}{n\beta-p}\right)-\frac{p}{n\beta}+\lambda^{n\beta}g(\lambda).
\end{aligned}$$

We observed that $g^{'}(\lambda)=\frac{p}{\lambda}\left(\frac{n\beta}{\lambda^p(2n\beta-p)}-\frac{1}{2}\right)>0$ for $0<\lambda\leq 1, 2n\beta>p$, therefore $g(\lambda)$ is a strictly increasing function of λ and $g(\lambda)\leq g(1)=\frac{-p^2}{2n\beta(2n\beta-p)}<0$. Thus it satisfies the following inequality

$$R\left(\theta_S,\Psi_S^{GB}\right)<\ln\left(\frac{n\beta}{n\beta-p}\right)-\frac{p}{n\beta}.$$

This, completes the proof of theorem. □

6.4 Admissibility of Estimators

Let random samples $X_{i1},...,X_{in}$, be available from the ith population $\Pi_i, i=1,2$, and having pdf (6.1) as well as $X_i = \min(X_{i1},...,X_{in}), i=1,2, n\geq 2$ and $X_{(1)} \leq X_{(2)}$ be the order statistics of X_1, X_2. In this section we study the class of linear admissible estimators of the form $cX_{(2)}$ and $cX_{(1)}$ for θ_L and θ_S, respectively. Also, we derive a sufficient condition for inadmissibility of a scale invariant estimator of scale parameter θ_S under the GSL function (6.4).

6.4.1 Characterization of linear admissible estimators

Let $G_A = \{g_A : g_A(x_1,x_2) = (cx_1, cx_2), c>0\}$ be a scale group of transformations. Under this transformations $(X_1,X_2) \to (cX_1, cX_2), c>0$. The given problem is invariant if we take the scale invariant loss function (6.4). Then , we define the following subclasses

$$D_L = \left\{\Psi_{1c} : \Psi_{1c}(X_1,X_2) = cX_{(2)}, c>0\right\} \tag{6.23}$$

and

$$D_S = \left\{\Psi_{2c} : \Psi_{2c}(X_1,X_2) = cX_{(1)}, c>0\right\} \tag{6.24}$$

of invariant estimators for θ_L and θ_S, respectively. The following theorems, we characterize the admissible estimators of θ_L and θ_S, that belongs to the class D_L and D_S, respectively.

Theorem 6.4.1. *Under the assumptions of Theorem* (6.3.1) *and let* $d_1^* = \left(\frac{(n\beta-p)(2n\beta-p)}{2(n\beta)^2}\right)^{\frac{1}{p}}$, $d_2^* = \left(\frac{(n\beta-p)}{n\beta}\right)^{\frac{1}{p}}$, $d_3^* = \left(1-\frac{p}{2n\beta}\right)^{\frac{1}{p}}$ *and* $d_4^* = \left(\frac{(n\beta-p)}{n\beta}\right)^{\frac{1}{p}}$. *Then under the GSL function* (6.4) *hold following results,*

(a) When $c \in [d_1^*, d_2^*]$ *and* $2n\beta > p$, *then the estimators* $\Psi_{1c}(X_1,X_2) = cX_{(2)}$ *are admissible within the class* D_L *of invariant estimators of* θ_L.

(b) When $c \in \left[d_3^*, d_4^*\right]$ *and* $2n\beta > p$, *then the estimators* $\Psi_{2c}(X_1,X_2) = cX_{(1)}$ *are admissible within the class* D_S *of invariant estimators of* θ_S.

Proof. (a) Let us consider the risk function of $\Psi_{1c} = cX_{(2)}$ which is defined as

$$\begin{aligned} R(\theta_L, \Psi_{1c}) =& E_\theta\left[\left(\frac{\Psi_{1c}}{\theta_L}\right)^p - p\ln\left(\frac{\Psi_{1c}}{\theta_L}\right) - 1\right] \\ =& E_\theta\left[\left(\frac{cX_{(2)}}{\theta_L}\right)^p - p\ln\left(\frac{cX_{(2)}}{\theta_L}\right) - 1\right] \\ =& c^p E_\theta\left[\left(\frac{X_{(2)}}{\theta_L}\right)^p\right] - pE_\theta\left[\ln\left(\frac{X_{(2)}}{\theta_L}\right)\right] - p\ln(c) - 1. \end{aligned}$$

For fixed λ, this risk function is a convex function of c, and takes its minimum at $c = K(\lambda)$, where

$$K(\lambda) = \left[E\left(\frac{X_{(2)}}{\theta_L}\right)^p \right]^{-\frac{1}{p}}.$$

Using the Lemma (6.3.2), we have obtained

$$K(\lambda) = \left[\frac{(n\beta)^2 \lambda^{n\beta-p}}{(n\beta-p)(2n\beta-p)} + \frac{n\beta}{(n\beta-p)} - \frac{n\beta\lambda^{n\beta}}{(2n\beta-p)} \right]^{-\frac{1}{p}}.$$

It is to observe that $K(\lambda)$ is a continuous function of λ and is a strictly decreasing on (0,1]. Therefore, we have

$$\inf_{0<\lambda\leq 1} K(\lambda) = K(1) = \left(\frac{(n\beta-p)(2n\beta-p)}{2(n\beta)^2} \right)^{\frac{1}{p}} = d_1^*,$$

and

$$\sup_{0<\lambda\leq 1} K(\lambda) = \lim_{\lambda\to 0^+} K(\lambda) = \left(\frac{(n\beta-p)}{n\beta} \right)^{\frac{1}{p}} = d_2^*.$$

It is clear that $K(\lambda)$ is continuous function of λ, it follows that $K(\lambda)$ assumes all values in the interval $[d_1^*, d_2^*)$. Therefore, any value of $c \in [d_1^*, d_2^*)$ minimizes the risks function $R(\theta_L, \Psi_{1c})$, for some values of $0 \leq \lambda < 1$, and hence such c corresponds to an admissible estimator. This shows that the estimators Ψ_{1c} are admissible within the subclass D_L, for any $c \in [d_1^*, d_2^*)$. The admissibility of the estimator $\Psi_{1d_2^*}$ follows from the continuity of the risks function.

Notice that, for every fixed $0 \leq \lambda < 1$, the risk function $R(\theta_L, \Psi_{1c})$ is an increasing function of c if $c > K(\lambda)$ and it is a decreasing function of c if $c < K(\lambda)$. Since $d_1^* \leq K(\lambda) \leq d_2^*$, $\forall$ $0 < \lambda \leq 1$, and can drew the conclusion that the estimators $\Psi_{1c} = cX_{(2)}$, within the subclass D_L, for $c \in (0, d_1^*) \bigcup (d_2^*, \infty)$ are inadmissible in estimating θ_L, hence complete the proof.

(b) The proof is similar to the proof (a) and therefore is omitted here.

□

Remark 6.4.2. *It follows from Theorem* (6.4.1)*, and under the entropy loss function, i.e.,* $p = -1$ *and the estimators* $\Psi_{1c}(X_1, X_2) = cX_{(2)}$*, for* $\left(\frac{2(n\beta)^2}{n\beta(2n\beta+3)+1}\right) \leq c \leq \left(\frac{n\beta}{n\beta+1}\right)$*, are admissible in the class of linear invariant estimators of* θ_L*, and the estimators* $\Psi_{2c}(X_1, X_2) = cX_{(1)}$*, for* $\left(\frac{2n\beta}{2n\beta+1}\right) \leq c \leq \left(\frac{n\beta}{n\beta+1}\right)$ *are admissible in the class of linear invariant estimators of* θ_S*.*

Remark 6.4.3. *The above theorem basically tells that the natural estimators (GB estimators)* $\Psi_L^{GB}(X_1, X_2)$ $= X_{(2)} \left(\frac{n\beta-p}{n\beta}\right)^{\frac{1}{p}}$ *and* $\Psi_S^{GB}(X_1, X_2) = X_{(1)} \left(\frac{n\beta-p}{n\beta}\right)^{\frac{1}{p}}$ *of* θ_L *and* θ_S *are admissible within the class of linear and invariant estimators* D_L *and* D_S *of* θ_L *and* θ_S*, respectively.*

6.4.2 Sufficient condition for inadmissibility

In this subsection, for $k=2$, we will provide sufficient condition for inadmissibility of selected scale parameter θ_S. Consider a general class of scale and permutation invariant estimators for θ_S, defined as

$$D_U = \left\{\Psi_\phi : \Psi_\phi\left(X_1, X_2\right) = X_{(1)}\phi(Y)\right\},$$

where $Y = \frac{X_{(2)}}{X_{(1)}}$ and $\phi(.)$ is some real valued function defined on $[1,\infty)$. Let $T = \frac{X_2}{X_1}$, then

$$\Psi_\phi(X_1,X_2) = X_{(1)}\phi(Y) = \begin{cases} X_1\phi\left(\frac{X_2}{X_1}\right), & \text{if} \quad X_1 < X_2, \\ X_2\phi\left(\frac{X_1}{X_2}\right), & \text{if} \quad X_1 \geq X_2, \end{cases} = \begin{cases} X_1\phi\left(T\right), & \text{if} \quad T < 1, \\ X_1 T\phi\left(\frac{1}{T}\right), & \text{if} \quad T \geq 1. \end{cases}$$

Therefore, $\Psi_\phi(X_1,X_2) = X_1\psi(T)$, where

$$\psi(T) = \begin{cases} \phi\left(T\right), & \text{if} \quad T < 1, \\ T\phi\left(\frac{1}{T}\right), & \text{if} \quad T \geq 1. \end{cases}$$

Next, we use the idea of Brewster and Zidek [27] to obtain estimators of the from $\Psi_{\phi_*} = X_{(1)}\phi_*(Y) = X_1\psi_*(T)$, which are dominate the estimators in class D_U and $\Psi_\phi = X_{(1)}\phi(Y) = X_1\psi(T)$. The following theorem provides a sufficent condition for inadmissiblity of the estimators $\Psi_\phi \in D_U$.

Theorem 6.4.4. *Let X_1 and X_2 be two independent random variables where X_i has a pareto distribution as defined in* (6.2)*. Let $\Psi_\phi(X_1,X_2) = X_1\psi(T) \in D_U$ be an invariant of θ_S. Define*

$$\psi_1(T) = \begin{cases} T\left(\frac{2n\beta-p}{2n\beta}\right)^{\frac{1}{p}}, & \text{if } 0 < T \leq 1, \\ \left(\frac{2n\beta-p}{2n\beta}\right)^{\frac{1}{p}}, & \text{if } T \geq 1. \end{cases}$$

and $P_\theta\left(\psi(T) > \psi_1(T)\right) > 0, \forall\theta = (\theta_1,\theta_2) \in \mathbb{R}_+^2 = (0,\infty)^2$, then under GSL function the invariant estimator $\Psi_\phi(X_1,X_2) = X_1\psi(T)$ is inadmissible for estimating θ_S and is dominated by $\Psi_{\phi_*}(X_1,X_2) = X_1\psi_*(T)$, where $\psi_*(T) = \min(\psi(T),\psi_1(T))$.

Proof. Consider the risk difference of the Ψ_ϕ and Ψ_{ϕ_*} is

$$\begin{aligned}\Delta =& R(\theta_S,\Psi_\phi) - R(\theta_S,\Psi_{\phi_*}) \\ =& E_\theta\left[\left(\frac{X_1\psi(T)}{\theta_S}\right)^p - p\ln\left(\frac{X_1\psi(T)}{\theta_S}\right) - 1\right] - E_\theta\left[\left(\frac{X_1\psi_*(T)}{\theta_S}\right)^p - p\ln\left(\frac{X_1\psi_*(T)}{\theta_S}\right) - 1\right] \\ =& E_\theta\left[\left(\psi^p(T) - \psi_*^p(T)\right)\left(\frac{X_1}{\theta_S}\right)^p - p\ln\left(\frac{\psi(T)}{\psi_*(T)}\right)\right] \\ =& E_\theta\left[D_\theta(T)\right],\end{aligned}$$

where

$$D_\theta(T) = \left(\psi^p(T) - \psi_*^p(T)\right) E_\theta\left(\left(\frac{X_1}{\theta_S}\right)^p \big| T\right) - p\ln\left(\frac{\psi(T)}{\psi_*(T)}\right)$$

It is easy to find the conditional pdf of X_1 given $T = \frac{X_2}{X_1} = t$ is given by

$$f_{X_1|T}(x_1|t) = \begin{cases} \frac{2n\beta\theta_2^{2n\beta}}{x_1^{2n\beta+1}t^{2n\beta}}, & \text{if} \quad x_1 > \frac{\theta_2}{t}, \quad 0 < t < \frac{\theta_2}{\theta_1} \\ \frac{2n\beta\theta_1^{2n\beta}}{x_1^{2n\beta+1}}, & \text{if} \quad x_1 > \theta_1, \frac{\theta_2}{\theta_1} < t. \end{cases} \tag{6.25}$$

Note that,

$$E_\theta\left[\left(\frac{X_1}{\theta_S}\right)^p \big| T = t\right] = \begin{cases} \frac{1}{\theta_2^p} E\left(X_1^p \big| T = t\right), & \text{if} \quad t \le 1 \\ \frac{1}{\theta_1^p} E\left(X_1^p \big| T = t\right), & \text{if} \quad t > 1. \end{cases} \tag{6.26}$$

From (6.25), we have

$$\begin{aligned} E\left(X_1^p \big| T = t\right) &= \begin{cases} \int_{\frac{\theta_2}{t}}^{\infty} \frac{2n\beta\theta_2^{2n\beta}}{x_1^{2n\beta-p+1}t^{2n\beta}} dx_1, & \text{if} \quad 0 < t < \frac{\theta_2}{\theta_1} \\ \int_{\theta_1}^{\infty} \frac{2n\beta\theta_1^{2n\beta}}{x_1^{2n\beta-p+1}} dx_1, & \text{if} \quad t > \frac{\theta_2}{\theta_1}. \end{cases} \\ &= \begin{cases} \left(\frac{2n\beta}{2n\beta-p}\right)\left(\frac{\theta_2}{t}\right)^p, & \text{if} \quad 0 < t < \frac{\theta_2}{\theta_1} \\ \left(\frac{2n\beta}{2n\beta-p}\right)\theta_1^p, & \text{if} \quad t > \frac{\theta_2}{\theta_1}. \end{cases} \end{aligned} \tag{6.27}$$

For $\theta_1 > \theta_2 \left(\lambda = \frac{\theta_2}{\theta_1} < 1\right)$, from (6.26) and (6.27), we conclude that

$$E_\theta\left[\left(\frac{X_1}{\theta_S}\right)^p \big| T = t\right] = \begin{cases} \frac{2n\beta}{(2n\beta-p)}\frac{1}{t^p}, & \text{if} \quad 0 < t \le \lambda \\ \frac{2n\beta}{(2n\beta-p)}\left(\frac{1}{\lambda}\right)^p, & \text{if} \quad \lambda < t \le 1 \\ \frac{2n\beta}{(2n\beta-p)}, & \text{if} \quad t > 1. \end{cases} \tag{6.28}$$

Also note that, for $\theta_1 < \theta_2 \left(\lambda = \frac{\theta_1}{\theta_2} < 1\right)$, from (6.26) and (6.27), we conclude that

$$E_\theta\left[\left(\frac{X_1}{\theta_S}\right)^p \big| T = t\right] = \begin{cases} \frac{2n\beta}{(2n\beta-p)}\frac{1}{t^p}, & \text{if} \quad 0 < t \le 1 \\ \frac{2n\beta}{(2n\beta-p)}\left(\frac{1}{t\lambda}\right)^p, & \text{if} \quad 1 < t \le \frac{1}{\lambda} \\ \frac{2n\beta}{(2n\beta-p)}, & \text{if} \quad t > \frac{1}{\lambda}. \end{cases} \tag{6.29}$$

In either cases, for $p < 0$, using (6.28) and (6.29), we get

$$\sup_{0<\lambda\leq 1} E\left(\left(\frac{X_1}{\theta_S}\right)^p \Big| T=t\right) = \begin{cases} \frac{2n\beta}{(2n\beta-p)}\frac{1}{t^p}, & \text{if} \quad 0<t\leq 1 \\ \frac{2n\beta}{(2n\beta-p)}, & \text{if} \quad t>1 \end{cases}$$

$$= \frac{1}{\psi_1^p(t)}. \tag{6.30}$$

and for $p > 0$, we get

$$\inf_{0<\lambda\leq 1} E\left(\left(\frac{X_1}{\theta_S}\right)^p \Big| T=t\right) = \begin{cases} \frac{2n\beta}{(2n\beta-p)}\frac{1}{t^p}, & \text{if} \quad 0<t\leq 1 \\ \frac{2n\beta}{(2n\beta-p)}, & \text{if} \quad t>1 \end{cases}$$

$$= \frac{1}{\psi_1^p(t)}. \tag{6.31}$$

It follows from (6.30) and (6.31) that, if $\psi_1(T) < \psi(T)$ then

$$\begin{aligned} D_\theta(T) &= \left(\psi^p(T) - \psi_*^p(T)\right) E_\theta\left[\left(\frac{X_1}{\theta_S}\right)^p \Big| T=t\right] - p\ln\left(\frac{\psi(T)}{\psi_*(T)}\right) \\ D_\theta(T) &\geq \left(\psi^p(T) - \psi_1^p(T)\right)\frac{1}{\psi_1^p(T)} - p\ln\left(\frac{\psi(T)}{\psi_1(T)}\right). \\ &= \left(\frac{\psi(T)}{\psi_1(T)}\right)^p - p\ln\left(\frac{\psi(T)}{\psi_1(T)}\right) - 1 \\ &\geq 0, \end{aligned}$$

with strict inequality holding for some $\theta = (\theta_1, \theta_2) \in \mathbb{R}_+^2$.
If $\psi_1(T) \geq \psi(T)$, then $D_\theta(T)$=0. Therefore

$$R(\theta_S, \Psi_\phi) \geq R(\theta_S, \Psi_{\phi_*}), \quad \text{for all}, \quad \theta = (\theta_1, \theta_2) \in \mathbb{R}_+^2,$$

where strict inequality holds for some θ. Hence, this completes the proof. □

Corollary 6.4.5. *For $k = 2$, the UMRU estimator $\Psi_S^U(X)$, under the GSL function, is inadmissible and dominated by*

$$\Psi_S^D(X) = \min\left(X_{(1)}\left(\frac{2n\beta - p}{2n\beta}\right)^{\frac{1}{p}}, \Psi_S^U(\boldsymbol{X})\right)$$

Proof. Let

$$\psi(T)=\begin{cases} T\left(\left(1-\frac{p}{n\beta}\right)+\frac{p}{n\beta}T^{n\beta}\right)^{\frac{1}{p}}, & \text{if } \ 0<t\leq 1 \\ \left(\left(1-\frac{p}{n\beta}\right)+\frac{p}{n\beta}\frac{1}{T^{n\beta}}\right)^{\frac{1}{p}}, & \text{if } \ t>1. \end{cases} \tag{6.32}$$

Then $\Psi_S^U(\boldsymbol{X})=X_{(1)}\left(\left(1-\frac{p}{n\beta}\right)+\frac{p}{n\beta}\left(\frac{X_{(1)}}{X_{(2)}}\right)^{n\beta}\right)^{\frac{1}{p}}=X_{(1)}\psi(T)=\Psi_\phi(X)$. Now, if $(\frac{1}{2})^{\frac{1}{n\beta}}<T<1$ then $T\left(\left(1-\frac{p}{n\beta}\right)+\frac{p}{n\beta}T^{n\beta}\right)^{\frac{1}{p}}>T\left(\frac{2n\beta-p}{2n\beta}\right)^{\frac{1}{p}}$ and if $1<T<2^{\frac{1}{n\beta}}$ then $\left(\left(1-\frac{p}{n\beta}\right)+\frac{p}{n\beta}\frac{1}{T^{n\beta}}\right)^{\frac{1}{p}}>\left(\frac{2n\beta-p}{2n\beta}\right)^{\frac{1}{p}}$. Therefore, from (6.30) and (6.31) and (6.32), $P\left(\psi(T)>\psi_1(T)\right)>0$, and from theorem 6.4.4, $\Psi_S^U(\boldsymbol{X})=\Psi_\phi(X)$ is inadmissible estimator and is dominated by

$$\begin{aligned} \Psi_S^D(\boldsymbol{X}) =& X_1\min\left(\psi(T),\psi_1(T)\right) \\ =& \begin{cases} X_1\min\left(T\left(\frac{2n\beta-p}{2n\beta}\right)^{\frac{1}{p}},T\left(\left(1-\frac{p}{n\beta}\right)+\frac{p}{n\beta}T^{n\beta}\right)^{\frac{1}{p}}\right), & \text{if } \ 0<t\leq 1 \\ X_1\min\left(\left(\frac{2n\beta-p}{2n\beta}\right)^{\frac{1}{p}},\left(\left(1-\frac{p}{n\beta}\right)+\frac{p}{n\beta}\frac{1}{T^{n\beta}}\right)^{\frac{1}{p}}\right), & \text{if } \ t>1. \end{cases} \\ =& X_{(1)}\min\left(\left(\frac{2n\beta-p}{2n\beta}\right)^{\frac{1}{p}},\left(\left(1-\frac{p}{n\beta}\right)+\frac{p}{n\beta}\left(\frac{X_{(1)}}{X_{(2)}}\right)^{n\beta}\right)^{\frac{1}{p}}\right) \\ =& \min\left(X_{(1)}\left(\frac{2n\beta-p}{2n\beta}\right)^{\frac{1}{p}},\Psi_S^U(\boldsymbol{X})\right). \end{aligned}$$

Thus the proof is completed. □

6.5 Results and Discussions

In this chapter, we have considered the problem of estimating the scale parameter of the selected population from Pareto population with respect to Generalized Stein loss function. Firstly, we have studied Uniform minimum risk unbiased(UMRU) estimator, Bayes estimator, generalized Bayes estimator and limiting generalized estimator and also obtained sufficient conditions for minimaxity of selected scale parameter from the Pareto population. It is shown that generalized Bayes estimator is minimax of scale parameter θ_S for $k=2$. we have also obtained some admissible all class of linear estimators. Under the Generalized Satin loss function, the sufficient condition for inadmissibility of some scale and permutation invariant estimator of selected scale parameter θ_S is obtained by Brewster and Zidek [27] technique. It is also found that the UMRU estimator is inadmissible and is a dominated estimator of θ_S. It should be noted that for $p=1$, the GSL function converts to Stein loss function, and in this situation, we have reported the following results

(a)

$$\Psi_L^U(\boldsymbol{X}) = X_{(k)}\left[1-\frac{1}{n\beta}\sum_{i=1}^{k}\left(\frac{X_{(i)}}{X_{(k)}}\right)^{n\beta+1}\right]$$

is the UMRU estimator of θ_L.

(b)

$$\Psi_S^U(\boldsymbol{X}) = X_{(1)}\left[\left(1-\frac{1}{n\beta}\right)+\frac{1}{n\beta}\left(\frac{X_{(1)}}{X_{(2)}}\right)^{n\beta}\right]$$

is the UMRU estimator of θ_S.

(c) The natural estimators (generalized Bayes estimators) of θ_L and θ_S are given by

$$\Psi_L^{GB}(\boldsymbol{X}) = X_{(2)}\left(\frac{n\beta-1}{n\beta}\right) \quad \text{and} \quad \Psi_S^{GB}(\boldsymbol{X}) = X_{(1)}\left(\frac{n\beta-1}{n\beta}\right)$$

respectively, and also noted that Generalized estimator of θ_S is minimax.

(d) The estimators $\Psi_{1c}(X_1,X_2) = cX_{(2)}$, for $\frac{(n\beta-1)(2n\beta-1)}{(2n\beta)^2} \leq c \leq \left(\frac{n\beta-1}{n\beta}\right)$, are admissible in the class of linear invariant estimators of θ_L.

(e) The estimators $\Psi_{1c}(X_1,X_2) = cX_{(1)}$, for $\left(1-\frac{1}{2n\beta}\right) \leq c \leq \left(1-\frac{1}{n\beta}\right)$, are admissible in the class of linear invariant estimators of θ_S.

(f) For $k=2$, the UMRU estimator of θ_S is inadmissible and dominated by

$$\Psi_S^D(\boldsymbol{X}) = \min\left(X_{(1)}\left(\frac{2n\beta-1}{2n\beta}\right), \Psi_S^U(\boldsymbol{X})\right).$$

For the future research, one can consider another loss function and study all the results afresh.

Chapter 7

Estimating parameter of the selected uniform population under the squared log error loss function

7.1 Introduction

Let $\Pi_1,...,\Pi_k$ be $k(\geq 2)$ independently and identically distributed uniform populations such that the observation X_i from population Π_i has the probability density function (p.d.f.)

$$g(x|\theta_i) = \begin{cases} \frac{1}{\theta_i}, & \text{if} \quad 0 < x < \theta_i \\ 0, & \text{otherwise.} \end{cases} \tag{7.1}$$

where $\theta_i > 0, (i = 1,...,k)$ is an unknown scale parameter. The population Π_i is called the best population if $\theta_i > \theta_j$, for all $i, j,\ i \neq j$ i.e., the best population is a population corresponding to the largest scale parameter $\theta_{[k]} = \max\{\theta_1,...,\theta_k\}$. In case of tie, it is assumed that one of the populations is arbitrarily selected as the best population. Let $X_{i1},...,X_{in_i}$ denote a random sample of size n_i from the population $\Pi_i, i = \{1,...,k\}$. Let $X_i = \max\{X_{i1},\ldots,X_{in_i}\}$, therefore $\boldsymbol{X} = (X_1,...,X_k)$ is a complete and sufficient statistic for $\boldsymbol{\theta} = (\theta_1,...,\theta_k) \in \mathbb{R}^k_+$; here $\mathbb{R}^k_+ = \{(x_1,...,x_k) \in \mathbb{R}^k : x_i > 0\ \forall\ i = 1,2,...,k\}$ denotes a subset of $k-$ dimensional Euclidean space $\mathbb{R}^k$. Clearly, $X_1,...,X_k$ are independent random variables and X_i has the following probability density function

$$f_i(x|\theta_i) = \begin{cases} \frac{n_i x^{n_i-1}}{\theta_i^{n_i}}, & \text{if} \quad 0 < x < \theta_i \\ 0, & \text{otherwise.} \end{cases} \tag{7.2}$$

where $\theta_i > 0, (i = 1,...,k)$ is an unknown scale parameter. It is appropriate to use the complete and sufficient statistic $\boldsymbol{X}$ for selecting the best population, and for estimating parameter of the selected population. Now, we define a selection rule for selecting the best population. "A non-randomized selection rule $\boldsymbol{d} = (d_1, ..., d_k)$ is defined on the sample space χ to $\{0,1\}^k$ such that $\sum_{i=1}^{k} d_i(\boldsymbol{x}) = 1$, for all $\boldsymbol{x} \in \chi$". For a given observation $\boldsymbol{x}$, the selection rule $\boldsymbol{d} = (d_1, ..., d_k)$ selects the population Π_i as the best population if $d_i(\boldsymbol{x}) = 1$ and $d_j(\boldsymbol{x}) = 0$ for $j \in \{1, ..., k\} \setminus \{i\}$. For the goal of selecting the best population, the natural selection rule is $\boldsymbol{d}^N(\boldsymbol{x}) = (d_1^N, d_2^N, ..., d_k^N)$, where

$$d_i^N(\boldsymbol{x}) = \begin{cases} 1, & \text{if} \quad x_i > \max_{j \neq i} x_j \\ 0, & \quad \text{otherwise.} \end{cases}$$

It is known that for $n_1 = n_2 = \cdots = n_k$, the natural selection rule $\boldsymbol{d}^N(\boldsymbol{x})$ is minimax under the $0-1$ loss function (see Misra and Dhariyal [83]). Misra and Dhariyal [83] that have shown if the sample sizes are unequal, then the natural selection rule $\boldsymbol{d}^N(\boldsymbol{x})$ is no longer minimax under the 0-1 loss function, and has many undesirable properties. The problem of selecting the best population was studied by Bechhofer [23], employing the indifference zone approach by Robbins [108, 109] using empirical Bayes approach while by Gupta [47, 48] employing the subset selection approach. These methodologies have been developed by many statisticians, one may refer to Gupta and Hsu [49], Huang and Lai [59], Misra et al. [85] and Golparvar and Parsian [44]. Golparvar and Parsian[44] developed empirical Bayes procedure for identifying the best exponential population under Type-II progressive censored data. Recently, Arshad and Misra [13] have proposed a class $\mathbb{C}$ of selection rules for selecting the best population when sample sizes are unequal. The form of the selection rule is $\boldsymbol{d}^{\boldsymbol{a}}(\boldsymbol{X}) = (d_1^{\boldsymbol{a}}, ..., d_k^{\boldsymbol{a}})$, where

$$d_i^{\boldsymbol{a}}(\boldsymbol{X}) = \begin{cases} 1, & \text{if} \quad a_i X_i > \max_{j \neq i} a_j X_j \\ 0, & \quad \text{otherwise.} \end{cases} \tag{7.3}$$

and $\boldsymbol{a} = (a_1, ..., a_k) \in \mathbb{R}_+^k$. For $k = 2$ and $n_1 \neq n_2$, it follows from Arshad and Misra [11] that the selection rule $\boldsymbol{d}^{a^*} = (d_1^{a^*}, d_2^{a^*})$, where

$$d_1^{a^*}(\boldsymbol{X}) = \begin{cases} 1, & \text{if} \quad X_1 > a^* X_2 \\ 0, & \text{if} \quad X_1 \leq a^* X_2. \end{cases} \; ; \; d_2^{a^*}(\boldsymbol{X}) = \begin{cases} 1, & \text{if} \quad X_1 \leq a^* X_2 \\ 0, & \text{if} \quad X_1 > a^* X_2. \end{cases}$$

and

$$a^* \equiv a^*(n_1, n_2) = \begin{cases} \left(\frac{n_1+n_2}{2n_2}\right)^{\frac{1}{n_1}}, & \text{if} \quad n_1 \leq n_2 \\ \left(\frac{2n_1}{n_1+n_2}\right)^{\frac{1}{n_2}}, & \text{if} \quad n_1 > n_2, \end{cases}$$

is generalized Bayes rule, admissible and minimax under the $0-1$ loss function. For selecting the best population, we use a fixed selection rule $\boldsymbol{d}^{\boldsymbol{a}} \in \mathbb{C}$, defined in Eq. (7.3). Then, the scale parameter

θ_L of the selected population is given by

$$\theta_L = \sum_{i=1}^{k} \theta_i \boldsymbol{d}_i^{\boldsymbol{a}}(\boldsymbol{X}).$$

For $i = 1, ..., k$, let $A_i = \{\boldsymbol{x} \in \chi : a_i x_i > a_j x_j \ \ \forall j \neq i, j = 1, 2, ..., k\}$ and let $I_A(.)$ be an indicator function of the set A. The scale parameter θ_L can be written as

$$\theta_L = \sum_{i=1}^{k} \theta_i I_{A_i}(\boldsymbol{X}). \tag{7.4}$$

Most of the works have been done to construct a good estimator of scale parameters of selected uniform populations under the various loss functions. For example, Vellaisamy et al. [130] proved that the natural estimator is positively biased and inadmissible. They obtained the uniformly minimum variance unbiased (UMVU) estimator and a generalized Bayes estimator of mean of the selected population, when the sample sizes are equal. For $k = 2$, the UMVU estimator has been improved under the squared error and scale-invariant squared error loss functions. Song [121] extended their results to k uniform population. Nematollahi and Motamed-Shariati [102] obtained the uniformly minimum risk unbiased (UMRU) estimator under the entropy loss function. They shown that the UMRU estimator is inadmissible and generalized Bayes estimator is minimax. Arshad and Misra [13] extended the results of Vellaisamy et al. [130] and Song [121] by considering the problem of estimating the scale parameter of selected uniform population when sample sizes are unequal. They derived UMVU estimator and a general result for improving a scale invariant estimator of selected population under the scale invariant scale squared error loss function. They showed that a subclass of natural estimators is inadmissible under the scaled-squared error loss function. Arshad and Misra [12] obtained the uniformly minimum risk unbiased (UMRU) estimator under the entropy loss function and also derived some inadmissible results for scale parameter of the selected population. Mohammadi [92] obtained the UMRU estimator under the squared log error (SLE) loss function, which is proposed by Brown [28] and is given by

$$L(g(\theta), \Psi) = [\ln(\Psi) - \ln(g(\theta))]^2 = \left[\ln\left(\frac{\Psi}{g(\theta)}\right)\right]^2, \quad \boldsymbol{\theta} \in \Omega, \Psi \in \mathbb{C}, \tag{7.5}$$

where $\mathbb{C}$ denotes the class of all estimators of $g(\theta)$ which is some function of parameter θ. This loss function is convex when $\frac{\Psi}{g(\theta)} \leq e$ and concave otherwise, and has unique minimum at $g(\theta) = \Psi$. The SLE loss function is useful in situations where underestimation appears to have more significance than overestimation. Mohammadi [92] also studied a class of linear estimators of scale parameter of selected uniform population. Nematollahi [99] studied the problem of estimating the scale parameter of selected Pareto population under the SLE loss function. They derived the UMRU estimator under the SLE loss function. For $k = 2$, the minimaxity and inadmissibility of the UMRU estimator have been shown. It is worth to mention that the works of Mohammadi [92] and Nematollahi [99] have done under the SLE loss function and in case of equal sample sizes. In this chapter, we address the problem of estimating scale parameter of the selected uniform population, under the SLE loss function, in case of unequal sample sizes.

For the component problem, the maximum likelihood estimator (MLE) and the UMRU estimator under the SLE loss function of θ_i are X_i and $e^{\frac{1}{n_i}}X_i, (i=1,...,k)$, respectively. Consider two natural estimators of the θ_L based on the these estimators, (under the SLE loss function) are given by

$$\Psi_{N,1}(\boldsymbol{X}) = \sum_{i=1}^{k} X_i I_{A_i}(\boldsymbol{X}), \text{ and } \quad \Psi_{N,2}(\boldsymbol{X}) = \sum_{i=1}^{k} e^{\frac{1}{n_i}} X_i I_{A_i}(\boldsymbol{X}). \tag{7.6}$$

In Section 7.2, we determine the UMRU estimator of θ_L under the SLE loss function. In Section 7.3, we derive a sufficient condition for inadmissibility of scale parameter θ_L under the SLE loss function and also shown that the natural estimator $\Psi_{N,1}$ and the UMRU estimator are inadmissible for estimating θ_L. In Section 7.4, we provide some results for estimating scale parameter of the selected uniform population when the goal of selection is to select a population associated with the smallest scale parameter. A simulation study on performance of various competing estimators is provided in section 7.5.

7.2 UMRU Estimator

In this section, we study uniformly minimum risk unbiased estimator of θ_L under the SLE loss function (7.5). We first obtain the conditions for the risk-unbiased estimator under the SLE loss function. The definition 6.2.1 is used to find the condition of the risk-unbiased estimator.

Using the Definition 6.2.1 and the SLE loss function (7.5), an estimator $\Psi(\boldsymbol{X})$ is a risk-unbiased estimator of the parameter $g(\boldsymbol{\theta})$, if it satisfies the following condition

$$E_{\boldsymbol{\theta}}[\ln(\Psi(\boldsymbol{X}))] = \ln(g(\boldsymbol{\theta})), \text{ for all } \boldsymbol{\theta}. \tag{7.7}$$

Since θ_L depends on $X_1,...,X_k$, the modification to risk unbiased condition (7.7) is required. Following Nematollahi and Jafari Jozani [100], the condition for the risk-unbiased estimator of θ_L is given by

$$E_{\boldsymbol{\theta}}[\ln(\Psi(\boldsymbol{x}))] = E_{\boldsymbol{\theta}}[\ln(\theta_L)], \qquad \text{for all } \boldsymbol{\theta}.$$

To find the UMRU estimator of θ_L, we use the following lemma given in Nematollahi and Jafari Jozani [100].

Lemma 7.2.1. *Suppose $X_1,...,X_k$ be k independent random variables such that X_i has p.d.f.* (7.2). *Let $U_1(\boldsymbol{X}),...,U_k(\boldsymbol{X})$ are k real valued functions on $\mathbb{R}_+^k$ such that* *(i)* $E_{\boldsymbol{\theta}}\left[|\ln(X_i)U_i(\boldsymbol{X})|\right] < \infty$, *for all* $\boldsymbol{\theta} \in \Omega$, $i = 1,...,k$.

(ii) $\int_0^{x_i} \ln(x_i)U_i(x_1,...,x_{i-1},t,x_{i+1},...,x_k)t^{n_i-1}dt < \infty$, *for all* $\boldsymbol{x} \in \mathbb{R}_+^k$, $i = 1,...,k$.

(iii) $\lim_{x_i \to 0}\left[\ln(x_i)\int_0^{x_i} U_i(x_1,...,x_{i-1},t,x_{i+1},...,x_k)t^{n_i-1}dt\right] = 0$, *for all* $\boldsymbol{x} \in \mathbb{R}_+^k$, $j \neq i, i = 1,...,k$.
Then, define the function $V_i(\boldsymbol{X})$ *such that*

$$V_i(\boldsymbol{X}) = \ln(X_i)U_i(\boldsymbol{X}) + \frac{1}{x_i^{n_i}}\int_0^{x_i} U_i(x_1,...,x_{i-1},t,x_{i+1},...,x_k)t^{n_i-1}dt,$$

satisfy

$$E_{\boldsymbol{\theta}}\left[\sum_{i=1}^k V_i(\boldsymbol{X})\right] = E_{\boldsymbol{\theta}}\left[\sum_{i=1}^k \ln(\theta_i)U_i(\boldsymbol{X})\right].$$

Theorem 7.2.2. *Under the SLE loss given in* (7.5)*, the uniformly minimum risk unbiased estimator of the scale parameter* θ_L *of the selected population is given by*

$$\Psi_U^L(\boldsymbol{X}) = exp\left[\sum_i^k \left\{\ln(X_i) + \frac{1}{n_i}\left\{1 - \left(\frac{\max_{j\neq i} a_jX_j}{a_iX_i}\right)^{n_i}\right\}\right\} I_{A_i(\boldsymbol{X})}\right]. \tag{7.8}$$

Proof. For $i = 1,...,k$, let $V_i(\boldsymbol{X})$ be a function defined on the sample space χ such that $E\left[V_i(\boldsymbol{X})\right] = E\left[\ln(\theta_i)I_{A_i}(\boldsymbol{X})\right]$.
Using Lemma 7.2.1, for $i = 1,...,k$, we have

$$\begin{aligned}
V_i(\boldsymbol{X}) &= \ln(X_i)I_{A_i}(\boldsymbol{X}) + \frac{1}{X_i^{n_i}}\int_0^{X_i} I_{A_i}(x_1,...,x_{i-1},t,x_{i+1},...,x_k)t^{n_i-1}dt \\
&= \ln(X_i)I_{A_i}(\boldsymbol{X}) + \frac{1}{X_i^{n_i}}\int_{\max_{j\neq i}\frac{a_jX_j}{a_i}}^{X_i} t^{n_i-1}dt I_{A_i}(\boldsymbol{X}) \\
&= \ln(X_i)I_{A_i}(\boldsymbol{X}) + \frac{1}{n_i}\left[1 - \left(\frac{\max_{j\neq i} a_jX_j}{a_iX_i}\right)^{n_i}\right] I_{A_i}(\boldsymbol{X}) \\
&= \left[\ln(X_i) + \frac{1}{n_i}\left\{1 - \left(\frac{\max_{j\neq i} a_jX_j}{a_iX_i}\right)^{n_i}\right\}\right] I_{A_i}(\boldsymbol{X}).
\end{aligned}$$

Clearly,

$$\ln\left[\Psi_U^L(\boldsymbol{X})\right] = \sum_{i=1}^k V_i(\boldsymbol{X}).$$

It follows that

$$\begin{aligned}
E_{\boldsymbol{\theta}}\left[\ln\left(\Psi_U^L(\boldsymbol{X})\right)\right] &= E_{\boldsymbol{\theta}}\left[\sum_{i=1}^k V_i(\boldsymbol{X})\right] \\
&= \sum_{i=1}^k E_{\boldsymbol{\theta}}\left[\ln(\theta_i)I_{A_i}(\boldsymbol{X})\right] \\
&= E_{\boldsymbol{\theta}}\left[\ln(\theta_L)\right].
\end{aligned}$$

Hence, the estimator $\Psi_U^L(\boldsymbol{X})$ is a risk unbiased estimator of θ_L. □

Remark 7.2.3. *Let $X_{[1]} \leq \cdots \leq X_{[k]}$ denote the ordered values of random variable $X_1, X_2, ..., X_k$. For $a_1 = a_2 = \cdots = a_k = 1$, and $n_1 = n_2 = \cdots = n_k = n$ (say), it follows from Theorem 7.2.2 that the UMRU estimator of scale parameter θ_L is given by*

$$\Psi_U^L(\boldsymbol{X}) = X_{(k)} e^{\frac{1}{n}\left\{1-\left(\frac{X_{(k-1)}}{X_{(k)}}\right)^n\right\}}.$$

This UMRU estimator depends only on two largest order statistics. This result is due to Mohammadi [92] and also reported by Nematollahi [99].

7.3 Inadmissibility results for scale invariant estimators

In this section, for the case of $k = 2$ uniform populations, we will provide some sufficient conditions for inadmissability of a scale invariant estimator of scale parameter θ_L under the SLE loss function (7.5).

Definition 7.3.1. An estimator $\Psi(X_1, X_2)$ of the scale parameter θ_L of the selected population is scale-invariant if

$$\Psi(cX_1, cX_2) = c\Psi(X_1, X_2) \quad \text{for all} \quad c > 0.$$

Let $c = \frac{1}{X_1}$ and let $Y = \frac{X_2}{X_1}$. Then an invariant estimator $\Psi(X_1, X_2)$ of θ_L can be written as

$$\Psi(X_1, X_2) = X_1 \psi(Y),$$

where $\psi(.)$ is a real valued function defined on $\mathbb{R}_+$. Now, we consider a general class $D_L = \{\Psi\psi : \Psi\psi(X_1, X_2) = X_1\psi(Y)\}$ of scale invariant estimators of θ_L. The following theorem provides a sufficient condition for inadmissible of an estimator of θ_L under the SLE loss function for selected populations.

Theorem 7.3.1. *Suppose $\Psi_\psi(X_1, X_2) = X_1\psi(Y) \in D_L$ is a scale-invariant estimator of θ_L, where $Y = \frac{X_2}{X_1}$ and $\psi(.)$ is a function defined on $\mathbb{R}_+$. Define the function ψ_1 on $\mathbb{R}_+$ as*

$$\psi_1(Y) = \begin{cases} e^{\frac{1}{n_1+n_2}}, & \text{if } \ 0 < Y < a \\ Ye^{\frac{1}{n_1+n_2}}, & \text{if } \ Y \geq a. \end{cases}$$

where $a = \frac{a_1}{a_2}$. If $P_{\boldsymbol{\theta}}\left(\psi_1(Y) > \psi(Y)\right) \geq 0$ for all $\boldsymbol{\theta} = (\theta_1, \theta_2) \in \mathbb{R}_+^2$, with strict inequality hold for some $\boldsymbol{\theta} \in \mathbb{R}_+^2$. Then, under SLE loss function, the estimator Ψ_ψ is inadmissible for estimating θ_L

and is dominated by estimator $\Psi_{\psi_*}(X_1,X_2) = X_1\psi_*(Y)$*, where*

$$\psi_*(Y) = \begin{cases} \psi_1(Y), & \textit{if} \quad \psi(Y) \le \psi_1(Y) \\ \psi(Y), & \textit{if} \quad \psi(Y) > \psi_1(Y). \end{cases} \tag{7.9}$$

Proof. Consider the risk-difference of two estimators Ψ_ψ and Ψ_{ψ_*}

$$\begin{aligned} \Delta &= R(\boldsymbol{\theta},\Psi_\psi) - R(\boldsymbol{\theta},\Psi_{\psi_*}) \\ &= E_{\boldsymbol{\theta}}\left[\ln\left(\frac{X_1\psi(Y)}{\theta_L}\right)\right]^2 - E_{\boldsymbol{\theta}}\left[\ln\left(\frac{X_1\psi_*(Y)}{\theta_L}\right)\right]^2 \\ &= E_{\boldsymbol{\theta}}\left[D_{\boldsymbol{\theta}}(Y)\right], \end{aligned}$$

where, for $y \in \mathbb{R}_+$ and $\theta \in \mathbb{R}_+^2$,

$$D_{\boldsymbol{\theta}}(y) = \left[\ln\left(\frac{\psi(y)}{\psi_*(y)}\right)\right]\left[2E\left(\ln\left(\frac{X_1}{\theta_L}\right)\Big|Y=y\right) + \ln\left(\psi(y)\psi_*(y)\right)\right]. \tag{7.10}$$

The conditional p.d.f. of X_1, given $Y=y$, is given by

$$f_{X_1|Y}(x_1|y) = \begin{cases} \frac{n_1+n_2}{\theta_1^{n_1+n_2}}x_1^{n_1+n_2-1}, & \text{if} \quad 0<x_1<\theta_1,\ y<\frac{\theta_2}{\theta_1} \\ \frac{n_1+n_2}{\theta_2^{n_1+n_2}}y^{n_1+n_2}x_1^{n_1+n_2-1}, & \text{if} \quad 0<x_1<\frac{\theta_2}{y}, y \ge \frac{\theta_2}{\theta_1} \end{cases}$$

Let $\lambda = \frac{\theta_2}{\theta_1}$, and let $a = \frac{a_1}{a_2}$. In calculation of $E\left(\ln\left(\frac{X_1}{\theta_L}\right)\Big|Y=y\right)$, the following two cases arise:

Case-I: when $y < a$

$$E\left(\ln\left(\frac{X_1}{\theta_L}\right)\Big|Y=y\right) = \begin{cases} -\frac{1}{n_1+n_2}, & \text{if} \quad y<\lambda \\ \ln(\lambda)-\ln(y)-\frac{1}{n_1+n_2}, & \text{if} \quad y \ge \lambda. \end{cases}$$

Case-II: when $y \ge a$

$$E\left(\ln\left(\frac{X_1}{\theta_L}\right)\Big|Y=y\right) = \begin{cases} -\frac{1}{n_1+n_2}-\ln(\lambda), & \text{if} \quad y<\lambda \\ -\ln(y)-\frac{1}{n_1+n_2}, & \text{if} \quad y \ge \lambda. \end{cases}$$

It follows from Case-I and Case-II that, for $\lambda < a$

$$E\left(\ln\left(\frac{X_1}{\theta_L}\right)\Big|Y=y\right) = \begin{cases} -\frac{1}{n_1+n_2}, & \text{if} \quad 0<y<\lambda \\ \ln(\lambda)-\ln(y)-\frac{1}{n_1+n_2}, & \text{if} \quad \lambda \le y < a \\ -\ln(y)-\frac{1}{n_1+n_2}, & \text{if} \quad 0<a \le y, \end{cases} \tag{7.11}$$

and, for $\lambda \geq a$

$$E\left(\ln\left(\frac{X_1}{\theta_L}\right)\Big|Y=y\right)=\begin{cases}-\frac{1}{n_1+n_2}, & \text{if} \quad 0<y<a\\ -\ln(\lambda)-\frac{1}{n_1+n_2}, & \text{if} \quad a\leq y<\lambda\\ -\ln(y)-\frac{1}{n_1+n_2}, & \text{if} \quad 0<\lambda\leq y.\end{cases} \tag{7.12}$$

Now, using (7.11) and (7.12), we get

$$\begin{aligned}\sup_{\lambda\in(0,\infty)} E\left(\ln\left(\frac{X_1}{\theta_L}\right)\Big|Y=y\right) &= \begin{cases}-\frac{1}{n_1+n_2}, & \text{if} \quad 0<y<a\\ -\ln(y)-\frac{1}{n_1+n_2}, & \text{if} \quad a\leq y\end{cases}\\ &= -\ln(\psi_1(y)). \end{aligned}\tag{7.13}$$

It follows from (7.9) and (7.10) that, if $\psi_1(y)\geq\psi(y)$ then

$$\begin{aligned}D_{\boldsymbol{\theta}}(y) &= \left[\ln\left(\frac{\psi(y)}{\psi_1(y)}\right)\right]\left[2E_{\boldsymbol{\theta}}\left(\ln\left(\frac{X_1}{\theta_L}\right)\Big|Y=y\right)+\ln\left(\psi(Y)\psi_1(Y)\right)\right]\\ &\geq \left[\ln\left(\frac{\psi(y)}{\psi_1(y)}\right)\right]\left[-2\ln\left(\psi_1(y)\right)+\ln\left(\psi(y)\right)+\ln\left(\psi_1(y)\right)\right]\\ &= \left[\ln\left(\frac{\psi(y)}{\psi_1(y)}\right)\right]^2\\ &\geq 0,\end{aligned}$$

where strict inequality holds for some $\boldsymbol{\theta}\in\mathbb{R}_+^2$.
If $\psi_1(y)<\psi(y)$, then $D_{\boldsymbol{\theta}}(y)$=0. Therefore

$$R(\boldsymbol{\theta},\Psi_\psi)\geq R(\boldsymbol{\theta},\Psi_{\psi_*}), \text{ for all }, \ \boldsymbol{\theta}\in\mathbb{R}_+^2,$$

where strict inequality holds for some θ. Hence, this complete the proof. □

Corollary 7.3.2. *For $k=2$, under the SLE loss function* (7.5)*, the UMRU estimator $\Psi_L^U(X)$ is inadmissible for estimating scale parameter θ_L of selected population and is dominated by $\Psi_L^{IU}(X)=X_1\max\{\psi^U(y),\psi_1(y)\}$, where*

$$\psi^U(y)=\begin{cases}e^{\frac{1}{n_1}\left[1-\left(\frac{y}{a}\right)^{n_1}\right]}, & \textit{if} \quad 0<y<a\\ ye^{\frac{1}{n_2}\left[1-\left(\frac{a}{y}\right)^{n_2}\right]}, & \textit{if} \quad y\geq a.\end{cases}$$

and $\psi_1(y)$ is defined in Theorem 7.3.1.

Corollary 7.3.3. *For $k=2$, under the SLE loss function* (7.5)*, the natural estimator $\Psi_{N,1}(\boldsymbol{X})$, given in* (7.6)*, is inadmissible and is dominated by*

$$\Psi^I_{N,1}(\boldsymbol{X}) = e^{\frac{1}{n_1+n_2}}\Psi_{N,1}(\boldsymbol{X}).$$

Remark 7.3.4. *For $k=2, n_1=n_2=n$ and $a_1=a_2=1$, it follows from Corollary* 7.3.2 *that the UMRU estimator of θ_L is inadmissible under the SLE loss function* (7.5)*. This result is due to Nematollahi [99]. Thus, Corollary* 7.3.2 *generalizes their result.*

Remark 7.3.5. *For $k=2$, $n_1=n_2=n$ and $a_1=a_2=1$, it follows from Corollary* 7.3.3 *that the natural estimator $\Psi_{N,1}$, corresponding to the largest MLE, of θ_L is inadmissible under SLE loss function. This results is also reported in Nematollahi [99].*

Theorem 7.3.6. *Let c_1 and c_2 be two possible constants and let $\boldsymbol{c}=(c_1,c_2)$. Consider the natural estimators*

$$\Psi_{\boldsymbol{c}}(X_1,X_2) = \begin{cases} c_1X_1, & \text{if} \quad \boldsymbol{X}\in A_1 \\ c_2X_2, & \text{if} \quad \boldsymbol{X}\in A_2. \end{cases}$$

Assume that $c_i \in (0, e^{\frac{1}{n_1+n_2}})\bigcup(e^{\frac{1}{n_i}},\infty)$, for $i=1,2$. Then, the estimators Ψ_c are inadmissible under the SLE loss function.

Proof. It is easy to see that the sufficient condition for inadmissibility given in Theorem 7.3.1 is satisfied by the estimators Ψ_c, if $c_i \in (0, e^{\frac{1}{n_1+n_2}}), i=1,2$. Thus, it follows from Theorem 7.3.1 that the estimators Ψ_c are inadmissible and are dominated by

$$\Psi^*_{\boldsymbol{c}}(\boldsymbol{X}) = \begin{cases} e^{\frac{1}{n_1+n_2}}X_1, & \text{if} \quad \mathbf{X}\in A_1 \\ e^{\frac{1}{n_1+n_2}}X_2, & \text{if} \quad \mathbf{X}\in A_2. \end{cases}$$

Further, assume that $c_i \in (e^{\frac{1}{n_i}},\infty)$ for $i=1,2$. Note that the risk function of the estimator Ψ_c is a function of $\lambda=\frac{\theta_2}{\theta_1}\in(0,\infty)$. The risk function of Ψ_c is given by

$$\begin{aligned} R(\lambda,\Psi_c) &= E_{\boldsymbol{\theta}}\left[\ln\left(\frac{\Psi_c}{\theta_L}\right)\right]^2 \\ &= \sum_{j=1}^{2} R_j(\lambda,c_j) \quad \text{(say)}, \end{aligned}$$

where

$$R_j(\lambda,c_j)=E_{\boldsymbol{\theta}}\left[\left(\ln\left(\frac{c_jX_j}{\theta_j}\right)I_{A_j(X)}\right)^2\right], j=1,2.$$

For a fixed $\lambda\in(0,\infty)$ and fixed $\in\{1,2\}$, $R_j(\lambda,c_j)$ takes its minimum at $c_j^*(\lambda)=e^{-K_j(\lambda)}$, where

$$K_j(\lambda)=\frac{E\left(\ln(\frac{X_1}{\theta_j})I_{A_j}(\boldsymbol{X})\right)}{E\left(I_{A_j}(\boldsymbol{X})\right)},\qquad j=1,2.$$

Using the p.d.f. of X_j, given in (7.2), we get

$$K_1(\lambda)=\begin{cases}\frac{\left(\frac{\lambda}{a}\right)^{n_1}\left[\frac{1}{n_1}-\left(\frac{n_2}{n_1+n_2}\right)\ln\left(\frac{\lambda}{a}\right)-\frac{n_1}{(n_1+n_2)^2}\right]-\frac{1}{n_1}}{1-\left(\frac{n_2}{n_1+n_2}\right)\left(\frac{\lambda}{a}\right)^{n_1}}, & \text{if} \quad \lambda<a\\ -\frac{1}{(n_1+n_2)}, & \text{if} \quad \lambda\geq a.\end{cases}$$

and

$$K_2(\lambda)=\begin{cases}\frac{\left(\frac{a}{\lambda}\right)^{n_2}\left[\frac{1}{n_2}-\left(\frac{n_1}{n_1+n_2}\right)\ln\left(\frac{a}{\lambda}\right)-\frac{n_2}{(n_1+n_2)^2}\right]-\frac{1}{n_2}}{1-\left(\frac{n_1}{n_1+n_2}\right)\left(\frac{a}{\lambda}\right)^{n_2}}, & \text{if} \quad \lambda\geq a\\ -\frac{1}{(n_1+n_2)}, & \text{if} \quad \lambda<a.\end{cases}$$

It is easy to check that $K_1(\lambda)$ and $K_2(\lambda)$ are continuous and non-decreasing function of $\lambda\in(0,\infty)$. Therefore, $c_1^*(\lambda)$ and $c_2^*(\lambda)$ are non-increasing functions of λ, and $\sup\limits_{\lambda\in(0,\infty)} c_1^*(\lambda)=e^{\frac{1}{n_1}}$ and $\sup\limits_{\lambda\in(0,\infty)} c_2^*(\lambda)=e^{\frac{1}{n_2}}$. Note that, for fixed $\lambda\in(0,\infty)$ and fixed $j=\{1,2\}$, $R_j(\lambda,c)$ is a decreasing function of $c\in(0,c_j^*)$ and is an increasing function of $c\in[c_j^*,\infty)$ with $c_j^*\leq e^{\frac{1}{n_j}}$. Therefore, for $c_j\geq e^{\frac{1}{n_j}}$,

$$R_j(\lambda,c_j)>R_j(\lambda,e^{\frac{1}{n_j}})\quad\forall\lambda\in(0,\infty)$$

This implies that

$$\begin{aligned}R(\lambda,\Psi_c)&=\sum_{j=1}^{2}R_j(\lambda,c_j)\\&>\sum_{j=1}^{2}R_j(\lambda,e^{\frac{1}{n_j}})\\&=R(\lambda,\Psi_d)\ \forall\ \lambda\in(0,\infty),\end{aligned}$$

where

$$\Psi_{\boldsymbol{d}}(X_1,X_2)=\begin{cases}e^{\frac{1}{n_1}}X_1, & \text{if} \quad \mathbf{X}\in A_1\\ e^{\frac{1}{n_2}}X_2, & \text{if} \quad \mathbf{X}\in A_2.\end{cases}$$

Hence the results follows.

Note: The choices of c_i's are arbitrary, so any value of c_i in the interval $(0,\infty)$ correspond to an estimator. It follows from the Theorem 7.3.6 that $c_i \in \left[e^{\frac{1}{(n_1+n_2)}}, e^{\frac{1}{n_i}}\right]$ minimizes the risk function $R(\lambda, \Psi_c)$ for some values of $\lambda > 0$. Therefore, under the SLE loss function, $c_i \in \left[e^{\frac{1}{(n_1+n_2)}}, e^{\frac{1}{n_i}}\right]$ correspond to an admissible estimator within a class of linear estimators. □

7.4 Results for the worst uniform population

In this section, we consider the problem of estimating the scale parameter of the selected uniform population when the selection good is to select a population associated with the smallest scale parameter $\theta_{[1]} = \min\{\theta_1, ..., \theta_k\}$. We call the population associated with $\theta_{[1]}$, the worst population. For selecting the worst population, we consider a class $\mathbb{D} = \{\boldsymbol{d^a} : \boldsymbol{d^a} = (d_1^{\boldsymbol{a}}, ..., d_k^{\boldsymbol{a}}), \boldsymbol{a} \in \mathbb{R}_+^k\}$ of selection rules, where

$$d_i^a(X) = \begin{cases} 1, & \text{if} \quad b_i X_i < \min\limits_{j\neq i} b_j X_j \\ 0, & \text{if} \quad b_i X_i \geq \min\limits_{j\neq i} b_j X_j, \end{cases}$$

and $\boldsymbol{b} = (b_1, ..., b_k)$. We estimate the scale parameter associated with the population selected by the selection rule $\boldsymbol{d^a} \in \mathbb{D}$. Thus the scale parameter of the selected population is given by

$$\theta_S = \sum_{i=1}^{k} \theta_i I_{B_i}(\boldsymbol{X}),$$

where $B_i = \{\boldsymbol{x} \in \mathbb{R}_+^k : b_i x_i < b_j x_j, \forall j \neq i, j = 1, ..., k\}, i = 1, ..., k$.

Based on the MLE, a natural estimator of θ_S is given by

$$\Psi_{N,1}^S(\boldsymbol{X}) = \sum_{i=1}^{k} X_i I_{B_i}(\boldsymbol{X}). \tag{7.14}$$

Similarly, another natural estimator of θ_S based on the UMRU estimator, under the SLE loss function of θ_i, in component estimation problem, is given by

$$\Psi_{N,2}^S(\boldsymbol{X}) = \sum_{i=1}^{k} e^{\frac{1}{n_i}} X_i I_{B_i}(\boldsymbol{X}). \tag{7.15}$$

Now we will provide some results (without proofs) similar to the results derived in the above section 7.2, and section 7.3. The following theorem is an analog of Theorem 7.2.2.

Theorem 7.4.1. *Under the SLE loss function, the uniformly minimum risk unbiased estimator of the scale parameter θ_S of the selected population is given by*

$$\Psi_U^S(\boldsymbol{X}) = exp\left[\sum_{i=1}^{k}\left\{\ln(X_i) I_{B_i(\boldsymbol{X})} + \frac{1}{n_i}\left\{\min\left(1, \frac{\min\limits_{j\neq i} a_j X_j}{a_i X_i}\right)\right\}^{n_i}\right\}\right]. \tag{7.16}$$

Remark 7.4.2. *Let $X_{[1]} \leq \cdots \leq X_{[k]}$ denote the ordered values of random variables $X_1, X_2, ..., X_k$. For $a_1 = a_2 = \cdots = a_k = 1$, and $n_1 = n_2 = \cdots = n_k = n$, it follows from Theorem 7.4.1 that the UMRU estimator of scale parameter θ_S is given by*

$$\Psi_U^S(\boldsymbol{X}) = X_{(1)} e^{\frac{1}{n}\sum_{i=1}^{k}\left(\frac{X_{(1)}}{X_{(i)}}\right)^n}. \tag{7.17}$$

This result is due to Nematollahi [99]. Thus, Theorem 7.4.1 generalizes their results.

The following theorem is an analogs of Theorem 7.3.1.

Theorem 7.4.3. *Let $\Psi_\psi(X_1, X_2) = X_1 \psi(Y)$ be a given scale-invariant estimator of scale parameter of θ_S, where $Y = \frac{X_2}{X_1}$ and $\psi(.)$ is a real -valued function defined on $(0, \infty)$. Define*

$$\psi_2(Y) = \begin{cases} Y e^{\frac{1}{n_1+n_2}}, & \text{if} \quad 0 < Y < b \\ e^{\frac{1}{n_1+n_2}}, & \text{if} \quad Y \geq b, \end{cases}$$

where $b = \frac{b_1}{b_2}$. If $P_{\boldsymbol{\theta}}(\psi(Y) < \psi_2(Y)) \geq 0$ for all $\boldsymbol{\theta} = (\theta_1, \theta_2) \in \mathbb{R}_+^2$, with strictly inequality hold for same $\boldsymbol{\theta} \in \mathbb{R}_+^2$. Then, under SLE loss function the scale-invariant estimator Ψ_ψ is inadmissible for estimation θ_S and is dominated by estimator $\Psi_{\psi_*}(X_1, X_2) = X_1 \psi_*(Y)$ where

$$\psi_*(Y) = \begin{cases} \psi_2(Y), & \text{if} \quad \psi(Y) \leq \psi_2(Y) \\ \psi(Y), & \text{if} \quad \psi(Y) > \psi_2(Y). \end{cases} \tag{7.18}$$

Corollary 7.4.4. *For $k = 2$, under the SLE loss function (7.5), the natural estimator $\Psi_{N,1}^S(\boldsymbol{X})$, given in (7.14), is inadmissible and is dominated by*

$$\Psi_{N,1}^{IS}(\boldsymbol{X}) = e^{\frac{1}{n_1+n_2}} \Psi_{N,1}^S(\boldsymbol{X}). \tag{7.19}$$

Remark 7.4.5. *For $k = 2$, $a_1 = a_2 = 1$, and $n_1 = n_2 = n$, it follows from Corollary 7.4.4 that the natural estimator $\Psi_{N,1}$, corresponding to the smallest MLE, of θ_S is inadmissible under the SLE loss function.*

7.5 Numerical Comparison

In this section, for $k = 2$, we compare the performances of UMRU estimator $\Psi_L^U(\boldsymbol{X})$, the improved estimator $\Psi_L^{UI}(\boldsymbol{X})$ upon the UMRU estimator, natural estimators $\Psi_{N,1}$, $\Psi_{N,2}$ and the improved estimator $\Psi_{N,1}^I$ upon natural estimator $\Psi_{N,1}$ of scale parameter θ_L of selected uniform population. For the goal of selecting the best uniform population, we consider the minimax selection rule $\boldsymbol{d}^{a^*}$, where

$a^* = a^*(n_1, n_2)$ defined in Section 7.1. It is easy to see that the minimax selection rule $\boldsymbol{d}^{a^*}$ is not same for different configurations of (n_1, n_2). We compare the risk functions of the five competing estimators of θ_L for different values of $\lambda = \frac{\theta_2}{\theta_1}$ and different configurations of sample sizes. For notational convenience, let $R_1(\lambda) = R(\lambda, \Psi_L^U(\boldsymbol{X})), R_2(\lambda) = R(\lambda, \Psi_L^{UI}(\boldsymbol{X})), R_3(\lambda) = R(\lambda, \Psi_{N,1}), R_4(\lambda) = R(\lambda, \Psi_{N,1}^I)$ and $R_5(\lambda) = R(\lambda, \Psi_{N,2})$ represent the risk functions of the various estimators. The risk functions of these estimator are plotted for $(n_1, n_2) \in \{(2,3), (3,2), (4,5), (5,4)\}$. We observed from the figures 7.1-7.4 that the the natural estimator $\Psi_{N,1}$ is dominated by all other estimators except the natural estimator $\Psi_{N,2}$. The improved estimator $\Psi_{N,1}^I$ provide significant improvement over the natural estimator $\Psi_{N,1}$. The improved estimator Ψ_L^{UI} provide marginal improvement over the UMRU estimator Ψ_L^U. The performance of the improved estimator $\Psi_{N,1}^I$ is satisfactory and hence the improved estimator $\Psi_{N,1}^I$ is recommended for practical applications.

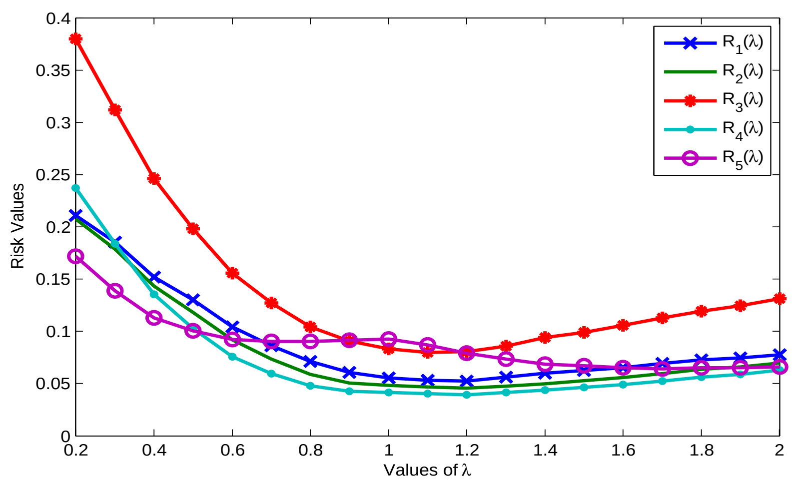

Figure 7.1: Risk performances of different estimators for $(n_1, n_2) = (2, 3)$.

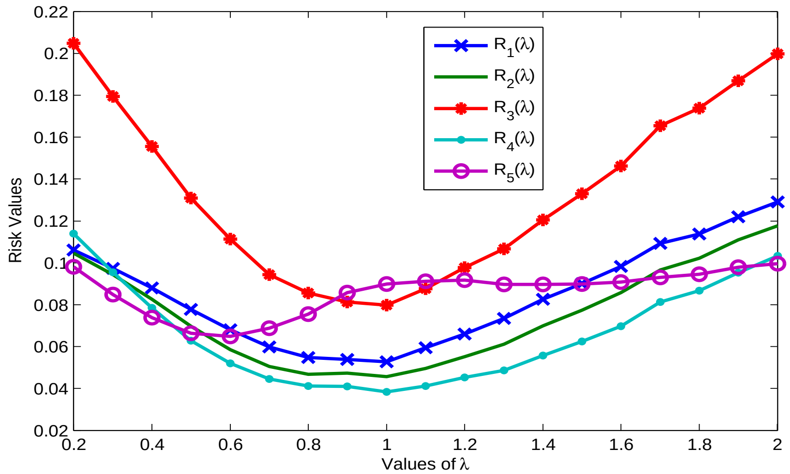

Figure 7.2: Risk performances of different estimators for $(n_1, n_2) = (3, 2)$.

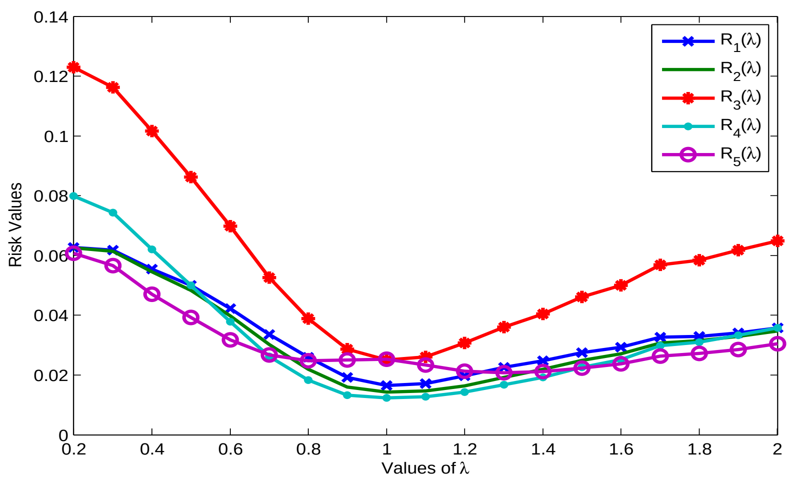

Figure 7.3: Risk performances of different estimators for $(n_1, n_2) = (4, 5)$.

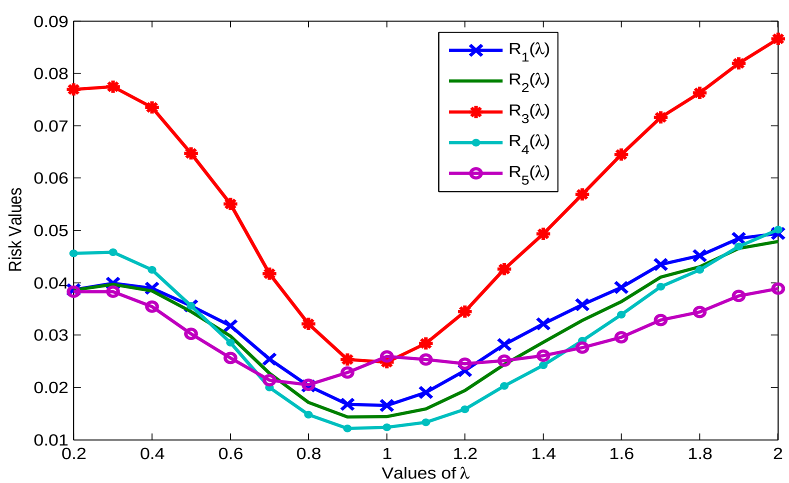

Figure 7.4: Risk performances of different estimators for $(n_1, n_2) = (5, 4)$.

Chapter 8

Estimating Parameter of the Selected Uniform Population Under the Generalized Stein Loss Function

8.1 Introduction

Let $X_{i1}, X_{i2}, ..., X_{in_i}$ be independent random sample of size n_i from the population Π_i $(i = 1, 2 \ldots, k)$ which are individually uniformly distributed over the interval $(0, \theta_i)$ with unknown scale parameter $\theta_i > 0$. Let $X_i = \max\{X_{i1}, ..., X_{in_i}\}$, therefore $\boldsymbol{X} = (X_1, ..., X_k)$ is a complete and sufficient statistic for $\boldsymbol{\theta} = (\theta_1, ..., \theta_k) \in \mathbb{R}_+^k$; here $\mathbb{R}_+^k = \{(x_1, ..., x_k) \in \mathbb{R}^k : x_i > 0 \ \forall \ i = 1, 2, ..., k\}$ denotes a subset of $k-$ dimensional Euclidean space $\mathbb{R}^k$. Let $X_1, ..., X_k$ denote independent random variables and X_i has the following probability density function

$$f_i(x|\theta_i) = \begin{cases} \frac{n_i x^{n_i - 1}}{\theta_i^{n_i}}, & \text{if} \quad 0 < x < \theta_i \\ 0, & \text{otherwise.} \end{cases} \tag{8.1}$$

where $\theta_i > 0, (i = 1, ..., k)$ are an unknown scale parameter. Then it may be interest to identify the best populations. The population Π_i is called the best population if $\theta_i > \theta_j$, for all i, j, $i \neq j$ i.e., the best population is a population associated with the largest scale parameter $\theta_{[k]} = \max\{\theta_1, ..., \theta_k\}$. If more than one of the θ_i are tied at the largest value, it is assumed that one of the populations is arbitrarily tagged as the best population.

According to Arshad and Misra [9] the natural selection rule for the goal of selecting the best population is $\boldsymbol{\delta}^N(\boldsymbol{x}) = (\delta_1^N, \delta_2^N, ..., \delta_k^N)$, where

$$\delta_i^N(\boldsymbol{x}) = \begin{cases} 1, & \text{if} \quad x_i > \max_{j \neq i} x_j \\ 0, & \text{otherwise.} \end{cases}$$

For samples of equal sizes i.e. $n_1 = n_2 = \cdots = n_k$, the natural selection rule $\boldsymbol{\delta}^N(\boldsymbol{x})$ is known to be minimax under the $0-1$ loss function (Misra and Dhariyal [83]). It follows from Misra and Dhariyal [83] that if the sample sizes are unequal, then the natural selection rule $\boldsymbol{\delta}^N(\boldsymbol{x})$ is no longer minimax under the 0-1 loss function, and has many undesirable properties. Recently, Arshad and Misra [9] proposed a class $\mathbb{C}$ of selection rules for selecting the best population when sample sizes are unequal. The from of the selection rule is $\boldsymbol{\delta}^{\boldsymbol{a}}(\boldsymbol{X}) = (\delta_1^{\boldsymbol{a}}, ..., \delta_k^{\boldsymbol{a}})$, where

$$\delta_i^{\boldsymbol{a}}(\boldsymbol{X}) = \begin{cases} 1, & \text{if} \quad a_i X_i > \max_{j \neq i} a_j X_j \\ 0, & \text{otherwise.} \end{cases} \tag{8.2}$$

and $\boldsymbol{a} = (a_1, ..., a_k) \in \mathbb{R}_+^k$. For $k = 2$ and $n_1 \neq n_2$, it follows from Arshad and Misra [11] that the selection rule $\boldsymbol{\delta}^{a^*} = (\delta_1^{a^*}, \delta_2^{a^*})$, where

$$\delta_1^{a^*}(\boldsymbol{X}) = \begin{cases} 1, & \text{if} \quad X_1 > a^* X_2 \\ 0, & \text{if} \quad X_1 \leq a^* X_2. \end{cases} ; \delta_2^{a^*}(\boldsymbol{X}) = \begin{cases} 1, & \text{if} \quad X_1 \leq a^* X_2 \\ 0, & \text{if} \quad X_1 > a^* X_2. \end{cases}$$

and

$$a^* \equiv a^*(n_1, n_2) = \begin{cases} \left(\frac{n_1+n_2}{2n_2}\right)^{\frac{1}{n_1}}, & \text{if} \quad n_1 \leq n_2 \\ \left(\frac{2n_1}{n_1+n_2}\right)^{\frac{1}{n_2}}, & \text{if} \quad n_1 > n_2, \end{cases}$$

and it is admissible and minimax under the $0-1$ loss function, and it is a generalized Bayes rule with respect to non-informative prior.

In this chapter, we consider the problem of estimating the scale parameter θ_L associated with the population selected by a selection rule $\boldsymbol{\delta}^{\boldsymbol{a}}$ given in (8.2). Let $A_i = \{\boldsymbol{x} \in \chi : a_i x_i > a_j x_j \ \ \forall j \neq i, j = 1, 2, ..., k\}$ and let $I_A(.)$ be the partition of sample space χ. Then scale parameter θ_L can be given by

$$\theta_L = \sum_{i=1}^{k} \theta_i I_{A_i}(\boldsymbol{X}). \tag{8.3}$$

where $I_A(.)$ denotes the indicator function of the set A. Arshad and Misra [12] obtained the uniformly minimum risk unbiased (UMRU) estimator under the entropy loss function and also derived some inadmissible results for scale parameter of the selected population. Pagheh and Nematollahi

[103] obtained the UMRU estimator and also derived some inadmissible estimates for scale parameter of selected papulation, under the Generalized Stein (GSL) Loss function, which is given by

$$L(g(\theta),\xi) = \left(\frac{\xi}{\theta}\right)^q - q\ln\left(\frac{\xi}{\theta}\right) - 1, \quad \boldsymbol{\theta} \in \Omega, \xi \in \mathbb{C}, q \neq 0, \tag{8.4}$$

where $\mathbb{C}$ denotes the class of all estimators of $g(\theta)$ and $g(\theta)$ is some function of parameter θ. This loss function is asymmetric and convex when $\Delta = \frac{\xi}{g(\theta)}$ and quasi concave otherwise, but its risk function has unique minimum at $\Delta = 1$. Being scale invariant, the GSL function is suitable for estimating a scale parameter. Further, it is useful where under estimation and over estimation are assigned unequal penalties. In upcoming discussion, we exploit this property of GSL to estimate the parameter of the selected uniform distribution from samples of unequal sizes.

For the component problem, we define two natural estimators of θ_L based on the maximum likelihood estimator (MLE) and the UMRU estimator, under the GSL function of θ_i as X_i and $\left(\frac{n_i+q}{n_i}\right)^{\frac{1}{q}} X_i, (i = 1,...,k)$, respectively. Therefore one may consider natural estimators of the θ_L of the selected population as:

$$\xi_{N,1}(\boldsymbol{X}) = \sum_{i=1}^{k} X_i I_{A_i}(\boldsymbol{X}), \text{and} \quad \xi_{N,2}(\boldsymbol{X}) = \sum_{i=1}^{k} \left(\frac{n_i+q}{n_i}\right)^{\frac{1}{q}} X_i I_{A_i}(\boldsymbol{X}). \tag{8.5}$$

The rest of this chapter is arranged as follows. In Section 8.2, we determine the UMRU and prove that the natural estimator $\xi_{N,2}(\boldsymbol{X})$ is the generalized Bayes estimator of θ_L under the GSL function. In Section 8.3, we derive a sufficient condition for inadmissibility of scale parameter θ_L under the GSL loss function and also show that the natural estimator $\xi_{N,1}$ and the UMRU estimator are inadmissible for estimating θ_L. In Section 8.4, we conducted a simulation study on performance of various competing estimators is provided. Finally some results and discussions are reported in section 8.5.

8.2 UMRU Estimator and Generalized Bayes Estimator

In this section, we discuss the general form of uniformly minimum risk unbiased estimator of θ_L, under the GSL function (8.4). We first introduce the concept of risk-unbiased estimator to our problem of estimating scale parameter θ_L under the GSL function. The definition given by Lehmann [71] and presented in 6.2.1 is a key in obtaining the condition of risk-unbiased estimator of θ_L.

Using this Definition and the GSL function (8.4), an estimator $\xi(\boldsymbol{X})$ is a risk-unbiased estimator of the parameter $g(\boldsymbol{\theta})$, if it satisfies the following condition

$$E_{\boldsymbol{\theta}}[\xi^q(\boldsymbol{X})] = g^q(\boldsymbol{\theta}), \text{ for all } \boldsymbol{\theta}. \tag{8.6}$$

Since θ_L depends on $X_1,...,X_k$, the modification to risk unbiased condition (8.6) is required. Following Nematollahi and Jafari Jozani [100], the condition for the risk-unbiased estimator of θ_L is given by

$$E_{\boldsymbol{\theta}}\left[\xi^q(\boldsymbol{x})\right] = E_{\boldsymbol{\theta}}\left[\theta_L^q\right] \qquad \text{for all } \boldsymbol{\theta}.$$

To find the UMRU estimator of θ_L, the following lemma is an application of the (u,v) method of Robbins(1988) which is given in Nematollahi and Jafari Jozani [100].

Lemma 8.2.1. *Suppose $X_1,...,X_k$ be k independent random variables, where X_i has a probability density function as given in* (8.2)*. Let $U_1(\boldsymbol{X}),...,U_k(\boldsymbol{X})$ be k real valued functions on $\mathbb{R}_+^k$ such that*

(i) $E_{\boldsymbol{\theta}}\left[|X_i^q U_i(\boldsymbol{X})|\right] < \infty$, *for all* $\boldsymbol{\theta} \in \Omega$, $i = 1,...,k$.

(ii) $\int_0^{x_i} x_i^q U_i(x_1,...,x_{i-1},t,x_{i+1},...,x_k)t^{n_i-1}dt < \infty$, *for all* $\boldsymbol{x} \in \mathbb{R}_+^k$, $i = 1,...,k$.

(iii) $\lim_{x_i \to 0}\left[x_i^q \int_0^{x_i} U_i(x_1,...,x_{i-1},t,x_{i+1},...,x_k)t^{n_i-1}dt\right] = 0$, *for all* $\boldsymbol{x} \in \mathbb{R}_+^k, j \neq i, i = 1,...,k$.

Then, the function $V_i(\boldsymbol{X})$ defined as

$$V_i(\boldsymbol{X}) = X_i^q U_i(\boldsymbol{X}) + q x_i^{q-n_i} \int_0^{x_i} U_i(x_1,...,x_{i-1},t,x_{i+1},...,x_k)t^{n_i-1}dt,$$

satisfies

$$E_{\boldsymbol{\theta}}\left[\sum_{i=1}^{k} V_i(\boldsymbol{X})\right] = E_{\boldsymbol{\theta}}\left[\sum_{i=1}^{k} \theta_i^q U_i(\boldsymbol{X})\right].$$

Now, we propose and derive the UMRU estimator of θ_S under the Generalized Stein loss function.

Theorem 8.2.2. *Consider the GSL function, as defined in* (8.4)*, then the uniformly minimum risk unbiased estimator of the scale parameter θ_L of the selected population is given by*

$$\xi_U(\boldsymbol{X}) = \sum_{i=1}^{k} X_i \left[1 + \frac{q}{n_i}\left\{1 - \left(\frac{\max_{j\neq i} a_j X_j}{a_i X_i}\right)^{n_i}\right\}\right]^{\frac{1}{q}} I_{A_i(\boldsymbol{X})}. \tag{8.7}$$

Proof. For $i = 1,...,k$, let $V_i(\boldsymbol{X})$ be a function defined on the sample space χ such that $E\left[V_i(\boldsymbol{X})\right] = E\left[\theta_i^q I_{A_i}(\boldsymbol{X})\right]$.

Using Lemma 8.2.1, for $i = 1,...,k$, we have

$$\begin{aligned}
V_i(\boldsymbol{X}) &= X_i^q I_{A_i}(\boldsymbol{X}) + qX_i^{q-n_i}\int_0^{x_i} I_{A_i}(x_1,...,x_{i-1},t,x_{i+1},...,x_k)t^{n_i-1}dt \\
&= X_i^q I_{A_i}(\boldsymbol{X}) + qX_i^{q-n_i}\int_{\max\limits_{j\neq i}\frac{a_jX_j}{a_i}}^{X_i} t^{n_i-1}dt I_{A_i}(\boldsymbol{X}) \\
&= X_i^q I_{A_i}(\boldsymbol{X}) + \frac{qX_i^q}{n_i}\left[1-\left(\frac{\max\limits_{j\neq i} a_jX_j}{a_iX_i}\right)^{n_i}\right] I_{A_i}(\boldsymbol{X}) \\
&= X_i^q\left[1+\frac{q}{n_i}\left\{1-\left(\frac{\max\limits_{j\neq i} a_jX_j}{a_iX_i}\right)^{n_i}\right\}\right] I_{A_i}(\boldsymbol{X}).
\end{aligned}$$

Clearly,

$$\left[\xi_U^q(\boldsymbol{X})\right] = \sum_{i=1}^{k} V_i(\boldsymbol{X}).$$

It follows that

$$\begin{aligned}
E_{\boldsymbol{\theta}}\left[\left(\xi_U^q(\boldsymbol{X})\right)\right] &= E_{\boldsymbol{\theta}}\left[\sum_{i=1}^{k} V_i(\boldsymbol{X})\right] \\
&= \sum_{i=1}^{k} E_{\boldsymbol{\theta}}\left[\theta_i^q I_{A_i}(\boldsymbol{X})\right] \\
&= E_{\boldsymbol{\theta}}\left[\theta_L^q\right].
\end{aligned}$$

Since $\boldsymbol{X} = (X_1,...,X_k)$ is a complete and sufficient statistics. Hence, the estimator $\xi_U(\boldsymbol{X})$ is a risk unbiased estimator of θ_L. □

Remark 8.2.3. *Let $X_{[1]} \leq \cdots \leq X_{[k]}$ denote the ordered values of random variables $X_1, X_2, ..., X_k$. For $a_1 = a_2 = \cdots = a_k = 1$, and $n_1 = n_2 = \cdots = n_k = n$ (say), it follows from Theorem* 8.2.2 *that the UMRU estimator of scale parameter θ_L is given by*

$$\xi_U(\boldsymbol{X}) = X_{[k]}\left[1+\frac{q}{n}\left\{1-\left(\frac{X_{[k-1]}}{X_{[k]}}\right)^n\right\}\right]^{\frac{1}{q}}.$$

This UMRU estimator depends only on two largest order statistics. This result is due to Pagheh and Nematollahi [103].

Remark 8.2.4. *Arshad and Misra [12] obtained UMRU estimator of θ_L under the entropy loss function. Their result can be obtained from* (8.7) *by taking $q = -1$, i.e.,*

$$\xi_U(\boldsymbol{X}) = \sum_{i=1}^{k} \frac{n_iX_i}{\left[(n_i-1)+\left(\frac{\max\limits_{j\neq i} a_jX_j}{a_iX_i}\right)^{n_i}\right]} I_{A_i}(\boldsymbol{X}). \tag{8.8}$$

In the following proposed theorem by us, we obtain the generalized Bayes estimators of θ_L under the Generalized Stein loss function given in Eq. (8.4).

Theorem 8.2.5. *Consider the GSL function* (8.4)*, then the natural estimator* $\xi_{N,2}(\boldsymbol{X})$ *is the generalized Bayes estimator of* θ_L*, with respect to the noninformative prior distribution*

$$\pi_{\boldsymbol{\theta}}(\theta_1,...,\theta_k) = \begin{cases} \frac{1}{\theta_1,...,\theta_k}, & if \quad \boldsymbol{\theta} \in \Omega \\ 0, & otherwise. \end{cases} \tag{8.9}$$

Proof. Consider the noninformative prior distribution (8.9) for $\boldsymbol{\theta} = (\theta_1,...,\theta_k)$, then the posterior distribution of $\boldsymbol{\theta}$, given $\boldsymbol{X} = \boldsymbol{x}$ has the density function

$$\pi^p_{\boldsymbol{\theta}}(\theta_1,...,\theta_k|\boldsymbol{x}) = \begin{cases} \pi^k_{i=1}\frac{n_i x_i^{n_i}}{\theta_i^{n_i+1}}, & \text{if} \quad \theta_i > x_i, \; i = 1,...,k \\ 0, & otherwise. \end{cases} \tag{8.10}$$

The posterior risk of an estimator ξ under the GSL function (8.4) which can be written as

$$r^p(\xi,\boldsymbol{x}) = E_{\pi^p}\left[\left\{\left(\frac{\xi}{\theta_L}\right)^q - q\ln\left(\frac{\xi}{\theta_L}\right) - 1\right\}|\boldsymbol{X} = \boldsymbol{x}\right] \tag{8.11}$$

It is clear that the generalized Bayes estimator $\xi^{GB}(\boldsymbol{X})$, which minimizes the posterior risk (8.11), is as follows

$$\xi^{GB}(\boldsymbol{x}) = \sum_{i=1}^{k}\left[E_{\pi^p}\left(\frac{1}{\theta_i^q}|\boldsymbol{X} = \boldsymbol{x}\right)\right]^{-\frac{1}{q}} I_{A_i}(\boldsymbol{X})$$

So, the generalized Bayes estimator of θ_L with respect to the posterior density (8.9) is obtained as

$$\xi^{GB}(\boldsymbol{x}) = \sum_{i=1}^{n}\left[\frac{(q+n_i)x_i^q}{n_i}\right]^{\frac{1}{q}} I_{A_i}(\boldsymbol{X}) = \xi_{N,2}(\boldsymbol{X}).$$

Hence, the result follows. □

8.3 Inadmissibility results for scale invariant estimators

In this section, for the case of two uniform populations (i.e. for $k = 2$), we will provide some sufficient conditions for inadmissability of a scale invariant estimator of scale parameter θ_L under the GSL function (8.4). It also gives dominated estimators in these cases where the sufficient conditions of the results are satisfied. To do this, we employ the orbit-by-orbit improvement technique of Brewster and Zidek [27].

Using the definiation 7.3.1 and let $c = \frac{1}{X_2}$ and let $Y = \frac{X_1}{X_2}$. Then an invariant estimator $\xi(X_1, X_2)$ of θ_L can be written as

$$\xi(X_1, X_2) = X_2 \psi(Y),$$

where $\psi(.)$ is a real valued function defined on $\mathbb{R}_+$. Now, we consider a general form of the class of any scale invariant estimators of the scale parameter θ_L defined as

$$D_L = \{\xi\psi : \xi\psi(X_1, X_2) = X_2 \psi(Y)\}.$$

The following theorem is to study sufficient condition for inadmissible of an estimator of θ_L under the GSL function for selected populations.

Theorem 8.3.1. *Consider that $\xi_\psi(X_1, X_2) = X_2\psi(Y) \in D_L$ is a scale-invariant estimator of θ_L, where $Y = \frac{X_1}{X_2}$ and $\psi(.)$ is a function defined on $\mathbb{R}_+$. Define the function ψ_1 on $\mathbb{R}_+$ given by*

$$\psi_1(Y) = \begin{cases} \left(\frac{n_1+n_2+q}{n_1+n_2}\right)^{\frac{1}{q}}, & \text{if} \quad 0 < Y < a \\ Y\left(\frac{n_1+n_2+q}{n_1+n_2}\right)^{\frac{1}{q}}, & \text{if} \quad Y \geq a. \end{cases}$$

where $a = \frac{a_2}{a_1}$. If $P_{\boldsymbol{\theta}}(\psi_1(Y) > \psi(Y)) \geq 0$ for all $\boldsymbol{\theta} = (\theta_1, \theta_2) \in \mathbb{R}_+^2$, and strict inequality holds for some $\boldsymbol{\theta} \in \mathbb{R}_+^2$. Then, under GSL function, the estimator ξ_ψ is inadmissible for estimating θ_L, and is dominated by estimator $\xi_{\psi_*}(X_1, X_2) = X_2\psi_*(Y)$, where

$$\psi_*(Y) = \begin{cases} \psi_1(Y), & \text{if} \quad \psi(Y) \leq \psi_1(Y) \\ \psi(Y), & \text{if} \quad \psi(Y) > \psi_1(Y). \end{cases} \tag{8.12}$$

Proof. Consider the risk-difference of estimators ξ_ψ and ξ_{ψ_*}

$$\begin{aligned} \Delta &= R(\boldsymbol{\theta}, \xi_\psi) - R(\boldsymbol{\theta}, \xi_{\psi_*}) \\ &= E_{\boldsymbol{\theta}}\left[\left(\frac{X_2\psi(Y)}{\theta_L}\right)^q - \left(\frac{X_2\psi_*(Y)}{\theta_L}\right)^q - q\ln\left(\frac{\psi(Y)}{\psi_*(Y)}\right)\right] \\ &= E_{\boldsymbol{\theta}}\left[\left(\frac{X_2}{\theta_L}\right)^q\left(\psi^q(Y) - \psi_*^q(Y)\right) - q\ln\left(\frac{\psi(Y)}{\psi_*(Y)}\right)\right] \\ &= E_{\boldsymbol{\theta}}\left[D_{\boldsymbol{\theta}}(Y)\right], \end{aligned}$$

where, for $y \in \mathbb{R}_+$ and $\theta \in \mathbb{R}_+^2$,

$$D_{\boldsymbol{\theta}}(y) = \left(\psi^q(Y) - \psi_*^q\right) E_{\boldsymbol{\theta}}\left[\left(\frac{X_2}{\theta_L}\right)^q \middle| Y = y\right] - q\ln\left(\frac{\psi(Y)}{\psi_*(Y)}\right). \tag{8.13}$$

The conditional p.d.f. of X_2, given $Y = y$, is given by

$$f_{X_1|Y}(x_1|y) = \begin{cases} \frac{(n_1+n_2)x_2^{n_1+n_2-1}}{\theta_2^{n_1+n_2}}, & \text{if} \quad 0 < x_2 < \theta_2, \;\; y < \frac{\theta_1}{\theta_2} \\ \frac{(n_1+n_2)y^{n_1+n_2}x_2^{n_1+n_2-1}}{\theta_1^{n_1+n_2}}, & \text{if} \quad 0 < x_2 < \frac{\theta_1}{y}, y \geq \frac{\theta_1}{\theta_2} \end{cases}$$

Let $\lambda = \frac{\theta_1}{\theta_2}$, and let $a = \frac{a_2}{a_1}$. In calculation of $E_{\boldsymbol{\theta}}\left[\left(\frac{X_2}{\theta_L}\right)^q \middle| Y = y\right]$, the following two cases arise:
Case-I: when $y > \frac{a_2}{a_1}$

$$E\left(\left(\frac{X_2}{\theta_L}\right)^q \middle| Y = y\right) = \begin{cases} \frac{n_1+n_2}{n_1+n_2+q}\frac{1}{\lambda^q}, & \text{if} \quad y < \lambda \\ \frac{n_1+n_2}{n_1+n_2+q}\frac{1}{y^q}, & \text{if} \quad y \geq \lambda. \end{cases}$$

Case-II: when $y \leq a$

$$E\left(\left(\frac{X_2}{\theta_L}\right)^q \middle| Y = y\right) = \begin{cases} \frac{n_1+n_2}{n_1+n_2+q}, & \text{if} \quad y < \lambda \\ \frac{n_1+n_2}{n_1+n_2+q}\left(\frac{\lambda}{y}\right)^q, & \text{if} \quad y \geq \lambda. \end{cases}$$

It follows from Case-I and Case-II that, for $\lambda < a$

$$E\left(\left(\frac{X_2}{\theta_L}\right)^q \middle| Y = y\right) = \begin{cases} \frac{n_1+n_2}{n_1+n_2+q}, & \text{if} \quad 0 < y < \lambda \\ \frac{n_1+n_2}{n_1+n_2+q}\left(\frac{\lambda}{y}\right)^q, & \text{if} \quad \lambda \leq y < a \\ \frac{n_1+n_2}{n_1+n_2+q}\frac{1}{y^q}, & \text{if} \quad 0 < a \leq y, \end{cases} \tag{8.14}$$

and, for $\lambda \geq a$

$$E\left(\left(\frac{X_2}{\theta_L}\right) \middle| Y = y\right)^q = \begin{cases} \frac{n_1+n_2}{n_1+n_2+q}, & \text{if} \quad 0 < y < a \\ \frac{n_1+n_2}{n_1+n_2+q}\frac{1}{\lambda^q}, & \text{if} \quad a \leq y < \lambda \\ \frac{n_1+n_2}{n_1+n_2+q}\frac{1}{y^q}, & \text{if} \quad 0 < \lambda \leq y. \end{cases} \tag{8.15}$$

In either cases, for $q < 0$ using (8.14) and (8.14), we get

$$\begin{aligned} \inf_{\lambda \in (0,\infty)} E\left(\left(\frac{X_2}{\theta_L}\right)^q \middle| Y = y\right) &= \begin{cases} \frac{n_1+n_2}{n_1+n_2+q}, & \text{if} \quad 0 < y < a \\ \frac{n_1+n_2}{n_1+n_2+q}\frac{1}{y^q}, & \text{if} \quad a \leq y \end{cases} \\ &= \frac{1}{\psi_1^q(y)}. \end{aligned} \tag{8.16}$$

and for $q > 0$, we get

$$\sup_{\lambda\in(0,\infty)} E\left(\left(\frac{X_2}{\theta_L}\right)^q \Big| Y=y\right) = \begin{cases} \frac{n_1+n_2}{n_1+n_2+q}, & \text{if} \quad 0<y<a \\ \frac{n_1+n_2}{n_1+n_2+q}\frac{1}{y^q}, & \text{if} \quad a\le y \end{cases}$$
$$= \frac{1}{\psi_1^q(y)}. \tag{8.17}$$

It follows from (8.12), (8.13), (8.16) and (8.17) that, if $\psi_1(y) \geq \psi(y)$ then

$$\begin{aligned} D_{\boldsymbol{\theta}}(y) &= \left(\psi^q(Y)-\psi_*^q\right) E_{\boldsymbol{\theta}}\left[\left(\frac{X_2}{\theta_L}\right)^q \big| Y\right] - q\ln\left(\frac{\psi(Y)}{\psi_*(Y)}\right) \\ D_{\boldsymbol{\theta}}(y) &= \left(\psi^q(Y)-\psi_1^q(y)\right)\frac{1}{\psi_1^q(y)} - q\ln\left(\frac{\psi(Y)}{\psi_1(y)}\right). \\ &\geq \left(\frac{\psi(Y)}{\psi_1(Y)}\right)^q - q\ln\left(\frac{\psi(Y)}{\psi_1(y)}\right) - 1 \\ &\geq 0, \end{aligned}$$

and strict inequality holds for some $\boldsymbol{\theta} \in \mathbb{R}_+^2$.
If $\psi_1(y) < \psi(y)$, then $D_{\boldsymbol{\theta}}(y)$=0. Therefore

$$R(\boldsymbol{\theta},\xi_\psi) \geq R(\boldsymbol{\theta},\xi_{\psi_*}), \text{ for all }, \ \boldsymbol{\theta}\in\mathbb{R}_+^2,$$

and strict inequality holds for some θ. This completes the proof. □

Corollary 8.3.2. *For $k=2$, under the GSL function* (8.4)*, the UMRU estimator $\xi_U(X)$ is inadmissible for estimating scale parameter θ_L of selected population and is dominated by $\xi_U^D(X) = X_2\max\{\psi^U(y),\psi_1(y)\}$, where*

$$\psi^U(y) = \begin{cases} \left[1+\frac{q}{n_2}\left(1-\left(\frac{y}{a}\right)^{n_2}\right)\right]^{\frac{1}{q}}, & \textit{if} \quad 0<y<a \\ y\left[1+\frac{q}{n_1}\left(1-\left(\frac{a}{y}\right)^{n_1}\right)\right]^{\frac{1}{q}}, & \textit{if} \quad y\geq a. \end{cases}$$

and $\psi_1(y)$ is defined in Theorem 8.3.1.

Corollary 8.3.3. *For $k=2$, under the GSL function* (8.4)*, the natural estimator $\xi_{N,1}(\boldsymbol{X})$, given in* (8.5)*, is inadmissible and is dominated by*

$$\xi_{N,1}^{ID}(\boldsymbol{X}) = \left(\frac{n_1+n_2+q}{n_1+n_2}\right)^{\frac{1}{q}} \xi_{N,1}(\boldsymbol{X}).$$

Corollary 8.3.4. *$q<0$, For $k=2$, under the GSL function* (8.4)*, the natural estimator $\xi_{N,2}(\boldsymbol{X})$, given in* (8.5)*, is inadmissible and is dominated by*

$$\xi_{N,2}^{ID}(\boldsymbol{X}) = X_2 \max\{\xi_{N,2}(y), \psi_1(y)\}.$$

Remark 8.3.5. *For $k=2, n_1=n_2=n$ and $a_1=a_2=1$, it follows from Corollary* 8.3.2 *that the UMRU estimator of θ_L is inadmissible and is dominated under the GSL loss function* (5)*. This result is due to Pagheh and Nematollahi [103]. Thus, Corollary* 8.3.2 *generalizes their result.*

Remark 8.3.6. *For $k=2, n_1=n_2=n$ and $a_1=a_2=1$, it follows from Theorem* 8.3.1 *that the UMRU estimator of θ_L is improved and dominates under the entropy loss function. This result is due to Nematollahi and Motamed-Shariati [102]. Thus, their results can be derived from Theorem* 8.3.1 *by taking $q=-1$.*

Remark 8.3.7. *For $k=2$, $n_1=n_2=n$ and $a_1=a_2=1$, it follows from Corollary* 8.3.3 *that the natural estimator $\xi_{N,1}$, corresponding to the largest MLE, of θ_L is inadmissible under GSL function.*

Theorem 8.3.8. *Let $n_1+n_2+q>0$. Let c_1 and c_2 be two possible real constants and let $\boldsymbol{c}=(c_1,c_2)$. Suppose that $c_i \in \left(0, \left(\frac{n_1+n_2+q}{n_1+n_2}\right)^{\frac{1}{q}}\right) \bigcup \left(\left(\frac{n_i+q}{n_i}\right)^{\frac{1}{q}}, \infty\right)$, for $i=1,2$. Define the natural-type estimators*

$$\xi_{\boldsymbol{c}}(X_1,X_2) = \begin{cases} c_1X_1, & \text{if} \quad \boldsymbol{X} \in A_1 \\ c_2X_2, & \text{if} \quad \boldsymbol{X} \in A_2. \end{cases}$$

Then, under the GSL function (8.4)*, the natural-type estimators ξ_c are inadmissible for estimating θ_L.*

Proof. It is easy to see that the sufficient condition for inadmissibility given in Theorem 8.3.1 is satisfied by the estimators ξ_c, if $c_i \in \left(0, \left(\frac{n_1+n_2+q}{n_1+n_2}\right)^{\frac{1}{q}}\right), i=1,2$. Thus, it follows from Theorem 8.3.1 that the estimators ξ_c are inadmissible and are dominated by

$$\xi_{\boldsymbol{c}}^*(\boldsymbol{X}) = \begin{cases} \left(\frac{n_1+n_2+q}{n_1+n_2}\right)^{\frac{1}{q}} X_1, & \text{if} \quad \mathbf{X} \in A_1 \\ \left(\frac{n_1+n_2+q}{n_1+n_2}\right)^{\frac{1}{q}} X_2, & \text{if} \quad \mathbf{X} \in A_2. \end{cases}$$

Now, assume that $c_i \in \left(\left(\frac{n_i+q}{n_i}\right)^{\frac{1}{q}}, \infty\right)$ for $i=1,2$. Note that the risk function of the estimator ξ_c is a function of $\lambda = \frac{\theta_1}{\theta_2} \in (0,\infty)$. Therefore the risk function of ξ_c is given by

$$\begin{aligned} R(\lambda, \xi_c) &= E_{\boldsymbol{\theta}}\left[\left(\frac{\xi_c}{\theta_L}\right)^q - q\ln\left(\frac{\xi_c}{\theta_L}\right) - 1\right], q \neq 0 \\ &= \sum_{j=1}^{2} R_j(\lambda, c_j) \quad \text{(say)}, \end{aligned}$$

where

$$R_j(\lambda, c_j) = E_{\boldsymbol{\theta}}\left[\left\{\left(\frac{c_j X_j}{\theta_j}\right)^q - q\ln\left(\frac{c_j X_j}{\theta_j}\right) - 1\right\} I_{A_j(X)}\right]$$

Note that the risk function is a convex function of c, for a fixed $\lambda \in (0,\infty)$ and fixed $\in \{1,2\}$, $R_j(\lambda, c_j)$ achieves its minimum at $c_j^*(\lambda) = M_j(\lambda)$, where

$$M_j(\lambda) = \left[\frac{E\left(I_{A_j}(\boldsymbol{X})\right)}{E\left(\left(\frac{X_1}{\theta_j}\right)^q I_{A_j}(\boldsymbol{X})\right)}\right]^{\frac{1}{q}}, \qquad j = 1,2.$$

Now, using the p.d.f. of X_j, given in (2), we obtain,

$$M_1(\lambda) = \begin{cases} \left[\frac{1-\left(\frac{n_2}{n_1+n_2}\right)\left(\frac{a}{\lambda}\right)^{n_1}}{\left(\frac{n_1}{n_1+q}\right)\left\{1-\left(\frac{n_2}{n_1+n_2+q}\right)\left(\frac{a}{\lambda}\right)^{n_1+q}\right\}}\right]^{\frac{1}{q}}, & \text{if} \quad \lambda > a \\ \left(\frac{n_1+n_2+q}{n_1+n_2}\right)^{\frac{1}{q}}, & \text{if} \quad \lambda \le a. \end{cases}$$

and

$$M_2(\lambda) = \begin{cases} \left[\frac{1-\left(\frac{n_1}{n_1+n_2}\right)\left(\frac{\lambda}{a}\right)^{n_2}}{\left(\frac{n_2}{n_2+q}\right)\left\{1-\left(\frac{n_1}{n_1+n_2+q}\right)\left(\frac{\lambda}{a}\right)^{n_2+q}\right\}}\right]^{\frac{1}{q}}, & \text{if} \quad \lambda \le a \\ \left(\frac{n_1+n_2+q}{n_1+n_2}\right)^{\frac{1}{q}}, & \text{if} \quad \lambda > a. \end{cases}$$

It is easy to check that $M_1(\lambda)$ and $M_2(\lambda)$ are continuous and non-increasing function of $\lambda \in (0,\infty)$. Therefore, $c_1^*(\lambda)$ and $c_2^*(\lambda)$ are non-increasing functions of λ, and $\sup_{\lambda\in(0,\infty)} c_1^*(\lambda) = \left(\frac{n_1+q}{n_1}\right)^{\frac{1}{q}}$ and $\sup_{\lambda\in(0,\infty)} c_2^*(\lambda) = \left(\frac{n_2+q}{n_2}\right)^{\frac{1}{q}}$. Note that, for any fixed $\lambda \in (0,\infty)$ and fixed $j = \{1,2\}$, the risk function of $R_j(\lambda, c)$ is a decreasing function of $c \in (0, c_j^*)$ and is an increasing function of $c \in [c_j^*, \infty)$ with $c_j^* \le \left(\frac{n_j+q}{n_j}\right)^{\frac{1}{q}}$. Therefore, for $c_j \ge \left(\frac{n_j+q}{n_j}\right)^{\frac{1}{q}}$,

$$R_j(\lambda, c_j) > R_j\left(\lambda, \left(\frac{n_j+q}{n_j}\right)^{\frac{1}{q}}\right) \quad \forall \ \lambda \in (0,\infty)$$

This implies that

$$\begin{aligned} R(\lambda, \xi_c) &= \sum_{j=1}^{2} R_j(\lambda, c_j) \\ &> \sum_{j=1}^{2} R_j\left(\lambda, \left(\frac{n_j+q}{n_j}\right)^{\frac{1}{q}}\right) \\ &= R(\lambda, \xi_d) \ \ \forall \ \lambda \in (0,\infty), \end{aligned}$$

where

$$\xi_{\boldsymbol{d}}(X_1,X_2)=\begin{cases}\left(\frac{n_1+q}{n_1}\right)^{\frac{1}{q}}X_1, & \text{if} \quad \mathbf{X}\in A_1\\ \left(\frac{n_2+q}{n_2}\right)^{\frac{1}{q}}X_2, & \text{if} \quad \mathbf{X}\in A_2.\end{cases}$$

Hence, the proof of the theorem. □

8.4 Numerical Comparison

In this section, we present a numerical study to evaluate and compare the risk functions among various estimators under the GSL function. For $k=2$ and $\lambda=\frac{\theta_2}{\theta_1}$, it can be observed that the risk function of all the estimators depend on (θ_1,θ_2). Here the risk function of the UMRU estimator $\xi_U(\boldsymbol{X})$, the improved estimator $\xi_U^D(\boldsymbol{X})$ upon the UMRU estimator, natural estimator $\xi_{N,1}$, the improved estimator $\xi_{N,1}^{ID}$ upon natural estimator $\xi_{N,1}$, natural estimator $\xi_{N,2}$ and the improved estimator $\xi_{N,2}^{ID}$ upon natural estimator $\xi_{N,2}$ of scale parameter θ_L are compared. For selecting the best population, we consider the minimax selection rule $\boldsymbol{d}^{a^*}$, defined in Section 8.1. Recall that $a^*=a^*(n_1,n_2)$ is a function of n_1 and n_2. So a^* depends on the different sample sizes n_1 and n_2. It is clear that the minimax selection rule $\boldsymbol{d}^{a^*}$ is not same for different configurations of (n_1,n_2). We compare the risk functions of the five competing estimators of θ_L for different values of λ and different configurations of sample sizes. $R_1(\lambda)=R(\lambda,\xi_U(\boldsymbol{X}))$, $R_2(\lambda)=R(\lambda,\xi_U^D(\boldsymbol{X}))$,$R_3(\lambda)=R(\lambda,\xi_{N,1}(\boldsymbol{X}))$,$R_4(\lambda)=R(\lambda,\xi_{N,1}^{ID}(\boldsymbol{X}))$, and $R_5(\lambda)=R(\lambda,\xi_{N,2}(\boldsymbol{X}))$ denote the risk functions of the various estimators. The risk functions of these estimator are plotted for $(n_1,n_2)\in\{(2,3),(3,2),(4,5),(5,4)\}$. The following conclusions can be drawn from the figures $8.1-8.8$ as well as table from $8.1-8.8$.

(a) For $q=1$, the natural estimator $\xi_{N,1}$ is dominated by all the other estimators.

(b) For $q=-1$, the natural estimator $\xi_{N,1}$ is dominated by all the other estimators except $\xi_{N,2}$.

(c) The improved estimator ξ_U^D provides marginal improvement over the UMRU estimator ξ_U.

(d) The improved estimator $\xi_{N,1}^{ID}$ provides significant improvement over the natural estimator $\xi_{N,1}$.

(e) For $0<\lambda<0.8$, $1.4<\lambda$ and $q=1$, the estimator $\xi_{N,2}$ becomes better than all other estimators for all values of n_1,n_2.

(f) For $0<\lambda<0.6$, $1.6<\lambda$ and $q=-1$, the estimator $\xi_{N,2}$ becomes better than all other estimators when the values of n_1 and n_2 are $(3,4)$ and $(4,3)$.

(g) For $0<\lambda<0.8$, $1.4<\lambda$ and $q=-1$, the estimator $\xi_{N,2}$ performs better than other all estimators when (n_1,n_2) is $(5,8)$ and $(8,5)$.

(h) The estimators ξ_U, ξ_U^D and $\xi_{N,1}^{ID}$ perform better for moderate values of λ.

Here, it is noted from the over all performance of all the estimators that the performance of $\xi_{N,1}^{ID}$ is satisfactory. Therefor, estimator $\xi_{N,1}^{ID}$ recommended for use in paretical applications.

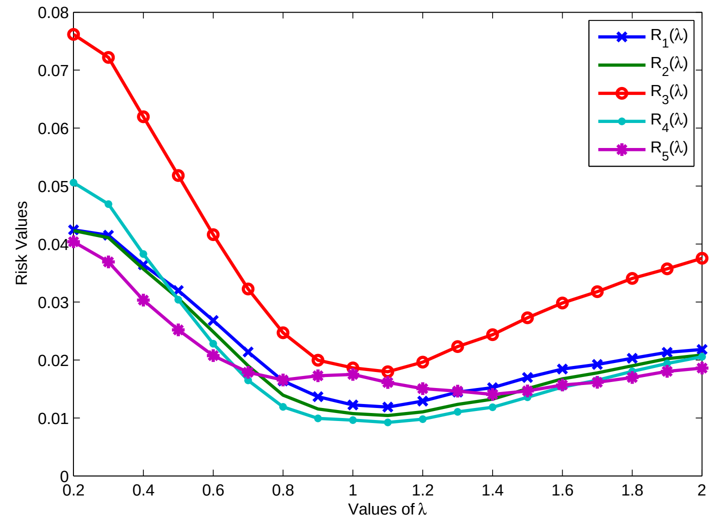

Figure 8.1: Risk performances of different estimators for $(n_1, n_2) = (3, 4)$ and $q = 1$.

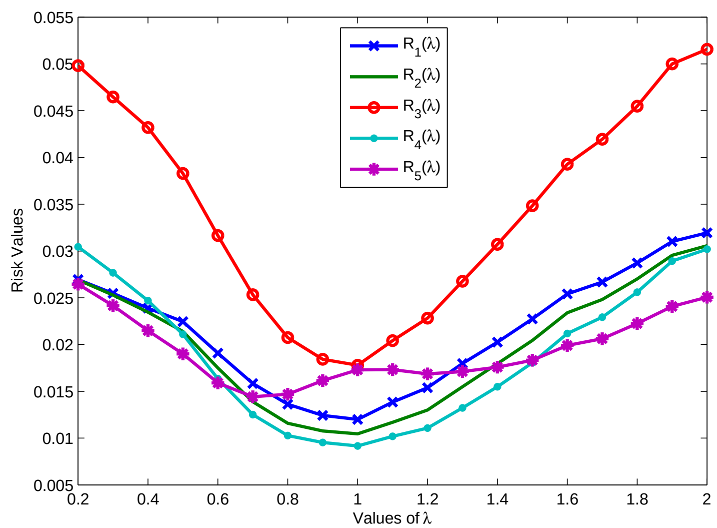

Figure 8.2: Risk performances of different estimators for $(n_1, n_2) = (4, 3)$ and $q = 1$.

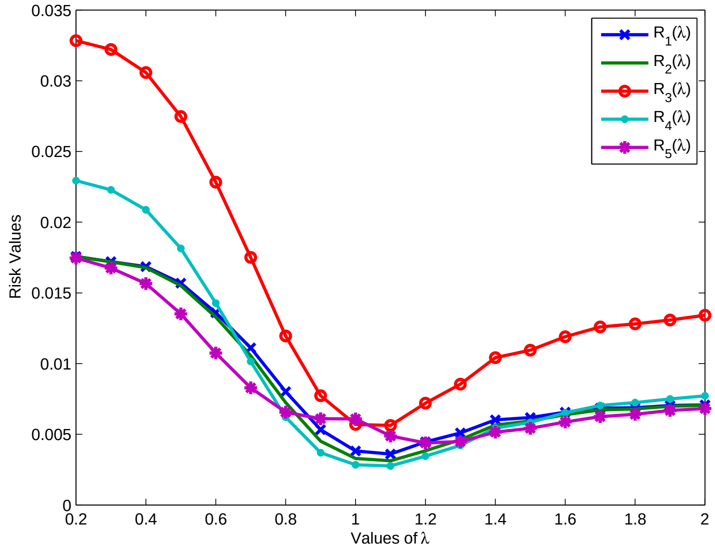

Figure 8.3: Risk performances of different estimators for $(n_1, n_2) = (5, 8)$ and $q = 1$.

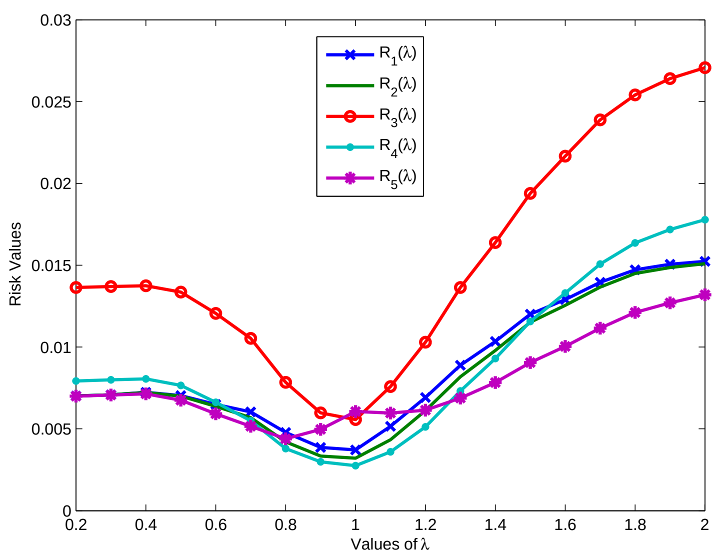

Figure 8.4: Risk performances of different estimators for $(n_1, n_2) = (8, 5)$ and $q = 1$.

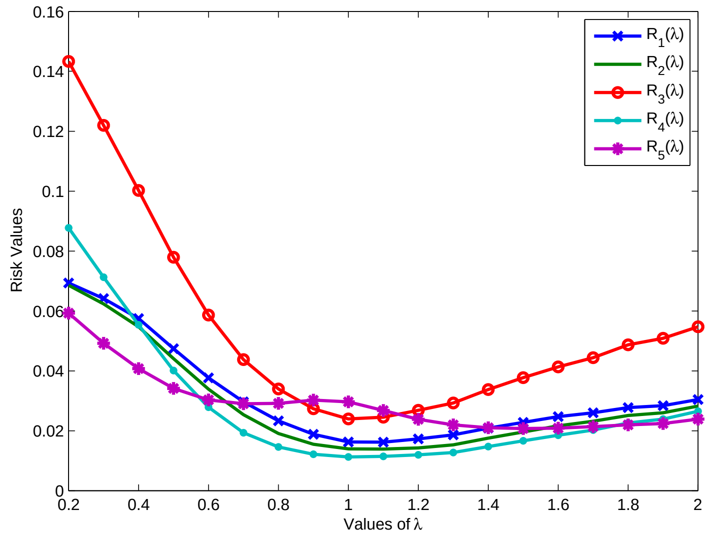

Figure 8.5: Risk performances of different estimators for $(n_1, n_2) = (3, 4)$ and $q = -1$.

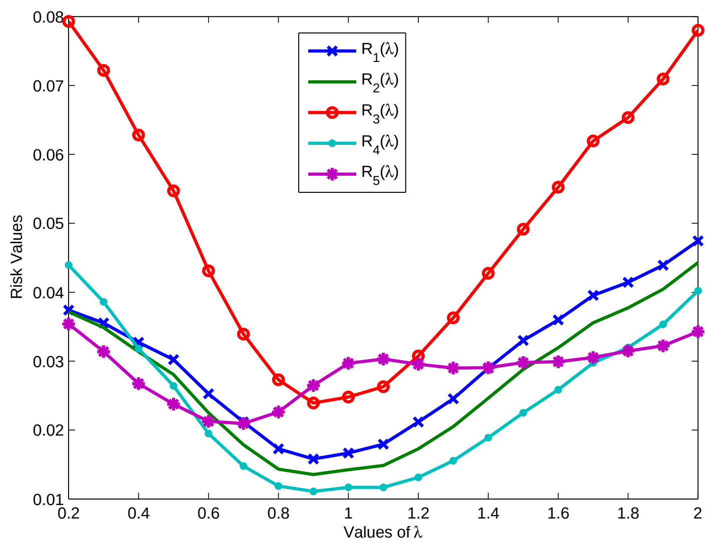

Figure 8.6: Risk performances of different estimators for $(n_1, n_2) = (4, 3)$ and $q = -1$.

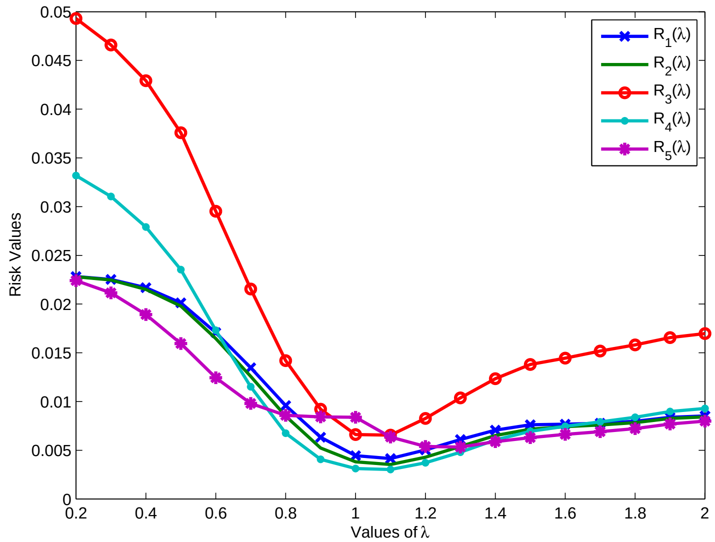

Figure 8.7: Risk performances of different estimators for $(n_1, n_2) = (5, 8)$ and $q = -1$.

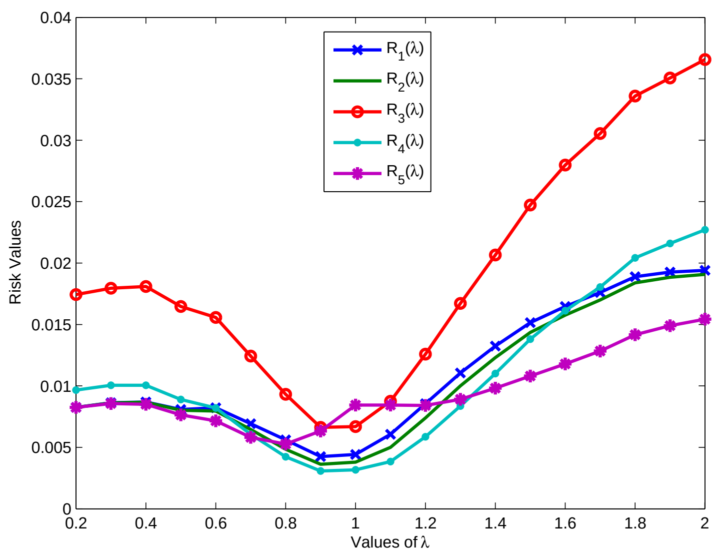

Figure 8.8: Risk performances of different estimators for $(n_1, n_2) = (8, 5)$ and $q = -1$.

Table 8.1: Risks of the UMRU estimator $\xi_U(\boldsymbol{X})$, the estimator $\xi_U^D(\boldsymbol{X})$ improved upon the UMVUE, and the natural estimators $\xi_{N,1}$, the estimator $\xi_{N,1}^{ID}$ improved upon the natural estimator $\xi_{N,1}$ and $\xi_{N,2}$ at different values of $\lambda = \frac{\lambda_2}{\lambda_1}$ and $q = 1$.

$(n_1, n_2) = (3,4); a^* = 0.9565$					
λ	$R(\lambda, \xi_U)$	$R(\lambda, \xi_U^D)$	$R(\lambda, \xi_{N,1})$	$R(\lambda, \xi_{N,1}^{ID})$	$R(\lambda, \xi_{N,2})$
0.2	0.04416	0.04399	0.07833	0.05248	0.04189
0.4	0.03598	0.03520	0.06086	0.03748	0.02994
0.6	0.02600	0.02419	0.04094	0.02228	0.02040
0.8	0.01690	0.01445	0.02487	0.01227	0.01711
1.0	0.01188	0.01041	0.01769	0.00908	0.01726
1.2	0.01318	0.01128	0.01999	0.01011	0.01534
1.4	0.01572	0.01363	0.02471	0.01219	0.01442
1.6	0.01796	0.01634	0.02970	0.01505	0.01526
1.8	0.02022	0.01895	0.03356	0.01785	0.01701
2.0	0.02230	0.02124	0.03798	0.02086	0.01881

Table 8.2: Risks of the UMRU estimator $\xi_U(\boldsymbol{X})$, the estimator $\xi_U^D(\boldsymbol{X})$ improved upon the UMVUE, and the natural estimators $\xi_{N,1}$, the estimator $\xi_{N,1}^{ID}$ improved upon the natural estimator $\xi_{N,1}$ and $\xi_{N,2}$ at different values of $\lambda = \frac{\lambda_2}{\lambda_1}$ and $q = 1$.

$(n_1, n_2) = (4,3); a^* = 1.0455$					
λ	$R(\lambda, \xi_U)$	$R(\lambda, \xi_U^D)$	$R(\lambda, \xi_{N,1})$	$R(\lambda, \xi_{N,1}^{ID})$	$R(\lambda, \xi_{N,2})$
0.2	0.02773	0.02769	0.05062	0.03123	0.02724
0.4	0.02479	0.02427	0.04360	0.02525	0.02212
0.6	0.01927	0.01774	0.03137	0.01639	0.01625
0.8	0.01347	0.01138	0.02086	0.01018	0.01447
1.0	0.01209	0.01063	0.01824	0.00939	0.01732
1.2	0.01585	0.01339	0.02359	0.01158	0.01703
1.4	0.02043	0.01810	0.03080	0.01555	0.01762
1.6	0.02426	0.02234	0.03792	0.02010	0.01923
1.8	0.02892	0.02724	0.04580	0.02583	0.02235
2.0	0.03185	0.03051	0.05172	0.03026	0.02500

Table 8.3: Risks of the UMRU estimator $\xi_U(\boldsymbol{X})$, the estimator $\xi_U^D(\boldsymbol{X})$ improved upon the UMVUE, and the natural estimators $\xi_{N,1}$, the estimator $\xi_{N,1}^{ID}$ improved upon the natural estimator $\xi_{N,1}$ and $\xi_{N,2}$ at different values of $\lambda = \frac{\lambda_2}{\lambda_1}$ and $q = 1$.

$(n_1,n_2) = (5,8); a^* = 0.9593$					
λ	$R(\lambda,\xi_U)$	$R(\lambda,\xi_U^D)$	$R(\lambda,\xi_{N,1})$	$R(\lambda,\xi_{N,1}^{ID})$	$R(\lambda,\xi_{N,2})$
0.2	0.01753	0.01753	0.03305	0.02306	0.01743
0.4	0.01696	0.01690	0.03102	0.02117	0.01575
0.6	0.01388	0.01352	0.02335	0.01466	0.01091
0.8	0.00781	0.00708	0.01174	0.00606	0.00649
1.0	0.00384	0.00332	0.00576	0.00287	0.00607
1.2	0.00441	0.00380	0.00713	0.00343	0.00440
1.4	0.00579	0.00539	0.00994	0.00510	0.00490
1.6	0.00665	0.00649	0.01209	0.00665	0.00595
1.8	0.00705	0.00697	0.01308	0.00745	0.00660
2.0	0.00692	0.00688	0.01312	0.00751	0.00665

Table 8.4: Risks of the UMRU estimator $\xi_U(\boldsymbol{X})$, the estimator $\xi_U^D(\boldsymbol{X})$ improved upon the UMVUE, and the natural estimators $\xi_{N,1}$, the estimator $\xi_{N,1}^{ID}$ improved upon the natural estimator $\xi_{N,1}$ and $\xi_{N,2}$ at different values of $\lambda = \frac{\lambda_2}{\lambda_1}$ and $q = 1$.

$(n_1,n_2) = (8,5); a^* = 1.0424$					
λ	$R(\lambda,\xi_U)$	$R(\lambda,\xi_U^D)$	$R(\lambda,\xi_{N,1})$	$R(\lambda,\xi_{N,1}^{ID})$	$R(\lambda,\xi_{N,2})$
0.2	0.00730	0.00730	0.01401	0.00825	0.00729
0.4	0.00710	0.00709	0.01365	0.00794	0.00702
0.6	0.00674	0.00659	0.01221	0.00680	0.00610
0.8	0.00475	0.00417	0.00773	0.00374	0.00436
1.0	0.00382	0.00331	0.00566	0.00285	0.00613
1.2	0.00704	0.00622	0.01038	0.00520	0.00624
1.4	0.01077	0.01015	0.01702	0.00974	0.00796
1.6	0.01333	0.01295	0.02217	0.01371	0.01030
1.8	0.01447	0.01422	0.02493	0.01600	0.01195
2.0	0.01549	0.01534	0.02761	0.01819	0.01340

Table 8.5: Risks of the UMRU estimator $\xi_U(\boldsymbol{X})$, the estimator $\xi_U^D(\boldsymbol{X})$ improved upon the UMVUE, and the natural estimators $\xi_{N,1}$, the estimator $\xi_{N,1}^{ID}$ improved upon the natural estimator $\xi_{N,1}$ and $\xi_{N,2}$ at different values of $\lambda = \frac{\lambda_2}{\lambda_1}$ and $q = -1$.

$(n_1,n_2) = (3,4); a^* = 0.9565$					
λ	$R(\lambda,\xi_U)$	$R(\lambda,\xi_U^D)$	$R(\lambda,\xi_{N,1})$	$R(\lambda,\xi_{N,1}^{ID})$	$R(\lambda,\xi_{N,2})$
0.2	0.06931	0.06828	0.14297	0.08734	0.05885
0.4	0.05672	0.05410	0.10120	0.05575	0.04033
0.6	0.03794	0.03388	0.05882	0.02787	0.03015
0.8	0.02378	0.01952	0.03437	0.01493	0.02961
1.0	0.01644	0.01412	0.02420	0.01141	0.02989
1.2	0.01752	0.01472	0.02680	0.01225	0.02425
1.4	0.02091	0.01738	0.03342	0.01448	0.02087
1.6	0.02451	0.02141	0.04130	0.01841	0.02069
1.8	0.02726	0.02486	0.04800	0.02236	0.02208
2.0	0.03001	0.02793	0.05467	0.02637	0.02373

Table 8.6: Risks of the UMRU estimator $\xi_U(\boldsymbol{X})$, the estimator $\xi_U^D(\boldsymbol{X})$ improved upon the UMVUE, and the natural estimators $\xi_{N,1}$, the estimator $\xi_{N,1}^{ID}$ improved upon the natural estimator $\xi_{N,1}$ and $\xi_{N,2}$ at different values of $\lambda = \frac{\lambda_2}{\lambda_1}$ and $q = -1$.

$(n_1,n_2) = (4,3); a^* = 1.0455$					
λ	$R(\lambda,\xi_U)$	$R(\lambda,\xi_U^D)$	$R(\lambda,\xi_{N,1})$	$R(\lambda,\xi_{N,1}^{ID})$	$R(\lambda,\xi_{N,2})$
0.2	0.03704	0.03691	0.07942	0.04393	0.03524
0.4	0.03339	0.03194	0.06345	0.03242	0.02735
0.6	0.02546	0.02262	0.04328	0.01961	0.02119
0.8	0.01794	0.01474	0.02800	0.01230	0.02259
1.0	0.01608	0.01390	0.02382	0.01122	0.02987
1.2	0.02157	0.01764	0.03102	0.01351	0.03004
1.4	0.02931	0.02505	0.04336	0.01938	0.02954
1.6	0.03642	0.03224	0.05578	0.02612	0.02993
1.8	0.04241	0.03875	0.06792	0.03344	0.03153
2.0	0.04534	0.04243	0.07561	0.03846	0.03338

Table 8.7: Risks of the UMRU estimator $\xi_U(\boldsymbol{X})$, the estimator $\xi_U^D(\boldsymbol{X})$ improved upon the UMVUE, and the natural estimators $\xi_{N,1}$, the estimator $\xi_{N,1}^{ID}$ improved upon the natural estimator $\xi_{N,1}$ and $\xi_{N,2}$ at different values of $\lambda = \frac{\lambda_2}{\lambda_1}$ and $q = -1$.

$(n_1, n_2) = (5, 8); a^* = 0.9593$					
λ	$R(\lambda, \xi_U)$	$R(\lambda, \xi_U^D)$	$R(\lambda, \xi_{N,1})$	$R(\lambda, \xi_{N,1}^{ID})$	$R(\lambda, \xi_{N,2})$
0.2	0.02231	0.02231	0.04886	0.03275	0.02201
0.4	0.02175	0.02157	0.04328	0.02814	0.01892
0.6	0.01716	0.01656	0.02927	0.01719	0.01255
0.8	0.00976	0.00868	0.01460	0.00706	0.00863
1.0	0.00435	0.00374	0.00649	0.00306	0.00844
1.2	0.00517	0.00436	0.00838	0.00380	0.00543
1.4	0.00687	0.00634	0.01217	0.00589	0.00573
1.6	0.00769	0.00740	0.01443	0.00742	0.00660
1.8	0.00802	0.00788	0.01586	0.00841	0.00726
2.0	0.00830	0.00823	0.01666	0.00906	0.00779

Table 8.8: Risks of the UMRU estimator $\xi_U(\boldsymbol{X})$, the estimator $\xi_U^D(\boldsymbol{X})$ improved upon the UMVUE, and the natural estimators $\xi_{N,1}$, the estimator $\xi_{N,1}^{ID}$ improved upon the natural estimator $\xi_{N,1}$ and $\xi_{N,2}$ at different values of $\lambda = \frac{\lambda_2}{\lambda_1}$ and $q = -1$.

$(n_1, n_2) = (8, 5); a^* = 1.0424$					
λ	$R(\lambda, \xi_U)$	$R(\lambda, \xi_U^D)$	$R(\lambda, \xi_{N,1})$	$R(\lambda, \xi_{N,1}^{ID})$	$R(\lambda, \xi_{N,2})$
0.2	0.00846	0.00846	0.01783	0.00993	0.00846
0.4	0.00868	0.00864	0.01777	0.00991	0.00846
0.6	0.00790	0.00766	0.01535	0.00796	0.00688
0.8	0.00555	0.00481	0.00930	0.00422	0.00524
1.0	0.00437	0.00377	0.00655	0.00310	0.00845
1.2	0.00830	0.00724	0.01231	0.00567	0.00841
1.4	0.01279	0.01193	0.02033	0.01072	0.00956
1.6	0.01669	0.01597	0.02824	0.01631	0.01184
1.8	0.01895	0.01840	0.03340	0.02030	0.01414
2.0	0.02062	0.02020	0.03794	0.02381	0.01609

8.5 Results and Discussions

In this chapter, we have addressed the problem of estimating the parameter of θ_L of the selected uniform population under the GSL function. We have derived the UMRU and generalized Bayes estimators for scale parameter of the selected uniform population under the GSL function. We have shown that the scale invariant estimators are inadmissible. Also, UMRU and natural estimators are inadmissible and dominated. Further, the risk of these estimators are compared numerically. In the literature, some other types of loss functions are used to estimate parameters of the selected population. These loss functions are considered for further research works. If we consider $q = 1$, then GSL function becomes Stein loss function and using similar technique in this chapter, we can conclude the following results

(i)

$$\xi_U(\boldsymbol{X}) = \sum_{i=1}^{k} X_i \left[1 + \frac{1}{n_i} \left\{ 1 - \left(\frac{\max\limits_{j \neq i} a_j X_j}{a_i X_i} \right)^{n_i} \right\} \right] I_{A_i(\boldsymbol{X})}$$

is the UMRU estimator of θ_L.

(ii)

$$\xi^{GB}(\boldsymbol{x}) = \sum_{i=1}^{n} \left[\frac{(1+n_i)x_i}{n_i} \right] I_{A_i}(\boldsymbol{X})$$

are the generalized Bayes estimator and natural estimator $\xi_{N,2}(\boldsymbol{X})$.

(iii) It should be noted here that we obtained the theorem 8.3.8, corollary 8.3.2 and corollary 8.3.3 in this case.

(iv) The natural estimators $\xi_c(X_1, X_2)$ which is defined in theorem 8.3.8, it is inadmissible for estimating θ_L, if and only if $\frac{n_1+n_2+1}{n_1+n_2} \leq c \leq \frac{n_i+1}{n_i}$, for $i = 1, 2$.

Remark 8.5.1. *Let $X_{[1]} \leq \cdots \leq X_{[k]}$ denote the ordered values of random variables $X_1, X_2, \ldots, X_k$. For $a_1 = a_2 = \cdots = a_k = 1$, and $n_1 = n_2 = \cdots = n_k = n$ (say), it follows from* (i) *that the UMRU estimator of scale parameter θ_L is given by*

$$\xi_U(\boldsymbol{X}) = \frac{X_{[k]}}{n} \left[n + 1 - \left(\frac{X_{[k-1]}}{X_{[k]}} \right)^n \right].$$

This UMRU estimator depends only on two largest order statistics. This result was derived by Misra and Mulen [86].

Chapter 9

Conclusions and Directions for Future Research

In this thesis, we have considered some problems on estimation after selection.

We have considered the problem of estimating mean of the selected population under the squared error loss function when the underlying distributions are normal with unknown mean and common known variance. The population yielding the smallest sample mean is selected. We have derived four different estimators and an improved estimator for the mean of the selected population. An application of this work is shown in finance, which is presented in the next chapter. We demonstrate that selecting the security with lower risk is the same as the selection of the population with the lower mean. We have obtained the estimators for the risk of the selected security and apply the theory to real data sets. Moreover, it is shown that the improved estimator performs better than the other estimators with respect to the bias and the mean squared error risk. It will be an interesting practical problem if this problem can be studied further when there are $k(\geq 2)$ securities.

We have considered the problem of estimating quantile of the selected normal population from two normal populations with same mean and different variances where both are unknown. We have proposed some estimators and obtained admissible classes estimators. A detailed simulation study has been carried out in order to numerically compare the bias and risk performances of all proposed estimators. Generalizing the above results for $k(\geq 2)$ populations is an interesting problem for further research.

The problem of estimating scale parameter of the selected Pareto population among $k(\geq 2)$ Pareto populations with common known shape parameter and different unknown scale parameters is considered. The population corresponding to the largest (smallest) scale parameter is selected and named as the largest (smallest) population. The uniformly minimum risk unbiased (UMRU) estimators of θ_L and θ_S, scale parameters of largest and smallest populations respectively, are derived under the Generalized Stein loss function. Sufficient condition for minimaxity of estimators of θ_L and θ_S are given, and it is shown that the generalized Bayes estimator of θ_S is minimax for $k=2$. Also, a class of linear admissible estimators of θ_L and θ_S are found. Further, it is shown that the UMRU estimator of θ_S is inadmissible. Studying this problem when the shape parameters are different and unknown is an open problem.

Next, we have considered $k(\geq 2)$ independent uniform populations with unequal sample sizes and an unknown scale parameter. For selecting the population associated with the largest scale parameter, we have considered a class $d^a(X)$ (see (7.3)) of natural selection rules. We have addressed the problem of estimating scale parameter θ_L of the selected population by a fixed selection rule in $d^{\boldsymbol{a}}(\boldsymbol{X})$ under the squared log error loss function. We obtain the uniformly minimum risk unbiased (UMRU) estimator of θ_L and two natural estimators of θ_L are also studied. We have shown that the UMRU estimator as well as natural estimator are inadmissible and better estimators are obtained. Furthermore, we produce related results for the problem of estimating scale parameter of the selected population when the selection goal is that of selecting the population corresponding to the smallest scale parameter. Finally, the risk functions of various competing estimators of θ_L are compared through simulation. Based on this simulation study the natural estimator $\Psi^{I}_{N,1}$ is recommended. One may study the above problem for other populations.

In the last chapter, we have addressed the problem of estimating scale parameter of the selected uniform population when sample sizes are unequal under the generalized Stein loss (GSL) function. The UMRU estimator of scale parameter is obtained and two natural estimators $\xi_{N,1}$ and $\xi_{N,2}$ (see Eq. (8.5)) are also studied. The natural estimator $\xi_{N,2}$ is shown to be the generalized Bayes estimator. The UMRU estimator as well as natural estimators are inadmissible and dominating estimators are also obtained. A numerical study on performance of various natural estimators is carried out. The above problem for some other population and under some other loss function can be studied further.

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