FOURIER SERIES APPROXIMATION OF FUNCTIONS IN $L_p\left(p \geq 1\right)$-SPACES

Ph.D. THESIS

by

SOSHAL

DEPARTMENT OF MATHEMATICS
INDIAN INSTITUTE OF TECHNOLOGY ROORKEE
ROORKEE – 247 667 (INDIA)

JULY, 2018

FOURIER SERIES APPROXIMATION OF FUNCTIONS IN $L_p\left(p \geq 1\right)$-SPACES

A THESIS

Submitted in partial fulfilment of the requirements for the award of the degree

of

DOCTOR OF PHILOSOPHY

in

MATHEMATICS

by

SOSHAL

DEPARTMENT OF MATHEMATICS
INDIAN INSTITUTE OF TECHNOLOGY ROORKEE
ROORKEE – 247 667 (INDIA)
JULY, 2018

INDIAN INSTITUTE OF TECHNOLOGY ROORKEE
ROORKEE

CANDIDATE'S DECLARATION

I hereby certify that the work which is being presented in the thesis entitled "**FOURIER SERIES APPROXIMATION OF FUNCTIONS IN** $L_p\left(p\geq 1\right)$**-SPACES**" in partial fulfilment of the requirements for the award of the Degree of Doctor of Philosophy and submitted in the Department of Mathematics of the Indian Institute of Technology Roorkee, Roorkee is an authentic record of my own work carried out during a period from July, 2012 to July, 2018 under the supervision of Dr. U. Singh, Associate Professor, Department of Mathematics, Indian Institute of Technology Roorkee, Roorkee.

The matter presented in this thesis has not been submitted by me for the award of any other degree of this or any other Institution.

(SOSHAL)

This is to certify that the above statement made by the candidate is correct to the best of my knowledge.

(U. Singh)
Supervisor

Date:

Abstract

In the present thesis, we study the degree of approximation of functions belonging to certain Lipschitz classes and subclasses of $L_p (p \geq 1)$-space (written as L^p throughout the thesis) through trigonometric Fourier series. We also study the approximation properties of some functions by means of Walsh–Fourier series and Fourier–Laguerre series. This thesis is divided into six chapters.

Chapter 1 is introductory in nature and includes the introduction of different Fourier series, basic definitions, concepts and the literature review. The objective of the work done and layout of the thesis is also given in this chapter.

In **Chapter** 2, we estimate the pointwise approximation of periodic functions belonging to $L^p(\omega)_\beta$(or $L^p(\tilde{\omega})_\beta$)-class, where ω(or $\tilde{\omega}$) is an integral modulus of continuity type function associated with f, and its conjugate function using product summability generated by the product of two general linear operators. We also measure the degree of approximation in the weighted norm for a function f belonging to weighted Lipschitz class $W(L^p, \xi(t))$ and its conjugate $\tilde{f}$, respectively. We prove the following theorems in this chapter:

Theorem 2.3.1 Let $f \in L^p(\omega)_\beta$ with $0 < \beta < 1 - \frac{1}{p}$, $p > 1$, and the entries of the lower triangular matrices $A \equiv (a_{n,k})$ and $B \equiv (b_{n,k})$ satisfy the following conditions:

$$b_{n,n} \ll \frac{1}{n+1}, \; n \in \mathbb{N}_0, \tag{0.0.1}$$

$$|b_{n,m}a_{m,0} - b_{n,m+1}a_{m+1,1}| \ll \frac{b_{n,m}}{(m+1)^2} \text{ for } 0 \leq m \leq n-1 \tag{0.0.2}$$

and

$$\sum_{k=0}^{m-1} |(b_{n,m}a_{m,m-k} - b_{n,m+1}a_{m+1,m+1-k}) - (b_{n,m}a_{m,m-k-1} - b_{n,m+1}a_{m+1,m-k})|$$

$$\ll \frac{b_{n,m}}{(m+1)^2} \quad \text{for} \;\; 0 \le m \le n-1, \tag{0.0.3}$$

with $A_{n,n} = B_{n,n} = 1$ for $n = 0, 1, 2, \ldots$. Then the degree of approximation of f by BA means of its Fourier series is given by

$$\left|t_n^{BA}(f;x) - f(x)\right| = O_x\left(\sum_{m=0}^{n} \frac{b_{n,m}}{m+1}(n+1)^{\beta+1}\omega(1/(n+1))\right),$$

provided that the positive non-decreasing function ω satisfies the following conditions:

$$\omega(t)/t \;\; \text{is a non-increasing function}, \tag{0.0.4}$$

$$\left\{\int_0^{\pi/(n+1)} \left(\frac{|\phi(x,t)|\sin^{\beta}(t/2)}{\omega(t)}\right)^p dt\right\}^{1/p} = O_x\left((n+1)^{-1/p}\right), \tag{0.0.5}$$

$$\left\{\int_{\pi/(n+1)}^{\pi} \left(t^{-\gamma}\frac{|\phi(x,t)|\sin^{\beta}(t/2)}{\omega(t)}\right)^p dt\right\}^{1/p} = O_x\left((n+1)^{\gamma-1/p}\right), \tag{0.0.6}$$

where γ is an arbitrary number such that $1/p < \gamma < \beta + 1/p$ and $p^{-1} + q^{-1} = 1$.

Theorem 2.3.2 Let f be a 2π-periodic function belonging to the class $L^p(\tilde{\omega})_\beta$, $0 < \beta < 1/p$, $p > 1$ and the entries of the lower triangular matrices $A \equiv (a_{n,k})$ and $B \equiv (b_{n,k})$ satisfy the following conditions:

$$b_{n,n} \ll \frac{1}{n+1}, \;\; n \in \mathbb{N}_0, \tag{0.0.7}$$

$$|b_{n,m}a_{m,m-l} - b_{n,m+1}a_{m+1,m+1-l}| \ll \frac{b_{n,m}}{(m+1)^2} \quad \text{for} \;\; 0 \le l \le m \le n-1. \tag{0.0.8}$$

with $A_{n,n} = B_{n,n} = 1$ for $n = 0, 1, 2, \ldots$. Then the degree of approximation of $\tilde{f}$, conjugate of f, by BA means of its conjugate Fourier series is given by

$$\left|\tilde{t}_n^{BA}(f;x) - \tilde{f}(x)\right| = O_x\left(\sum_{m=0}^{n} \frac{b_{n,m}}{m+1}(n+1)^{\beta+1}\tilde{\omega}(1/(n+1))\right),$$

provided that the positive non-decreasing function $\tilde{\omega}$ satisfies the following conditions:

$$\tilde{\omega}(t)/t^{\beta+1-\sigma} \;\; \text{is non-decreasing for} \; \beta < \sigma < 1/p, \tag{0.0.9}$$

$$\left\{\int_0^{\pi/(n+1)} \left(\frac{t^{-\sigma}|\psi(x,t)|\sin^{\beta}(t/2)}{\tilde{\omega}(t)}\right)^p dt\right\}^{1/p} = O_x\left((n+1)^{\sigma-1/p}\right), \tag{0.0.10}$$

$$\left\{\int_{\pi/(n+1)}^{\pi} \left(\frac{t^{-\gamma}|\psi(x,t)|\sin^{\beta}(t/2)}{\tilde{\omega}(t)}\right)^p dt\right\}^{1/p} = O_x\left((n+1)^{\gamma-1/p}\right), \tag{0.0.11}$$

where γ is an arbitrary number such that $1/p < \gamma < \beta + 1/p$ and $p^{-1} + q^{-1} = 1$. We discussed the case $p = 1$ separetly and two more theorems are proved for $p = 1$ (**Theorem 2.6.1** and **Theorem 2.6.2**).

In weighted L^p-norm, we prove following theorems:

Theorem 2.10.1 Let f be a 2π-periodic function belonging to $W(L^p, \xi(t))$ with $p \geq 1$, $\beta \geq 0$ and let the entries of the lower triangular matrices $A \equiv (a_{n,k})$ and $B \equiv (b_{n,k})$ satisfy the conditions (0.0.1) - (0.0.3) of Theorem 2.3.1 with $A_{n,n} = B_{n,n} = 1$ for $n = 0, 1, 2, \ldots$. Then the degree of approximation of f by BA means of its Fourier series is given by

$$\left\| t_n^{BA}(f;x) - f(x) \right\|_{p,\beta} = O\left(\xi(\pi/(n+1)) + (n+1)^{1-\sigma} \sum_{m=0}^{n} \frac{b_{n,m}}{m+1} \right),$$

provided that the positive non-decreasing function $\xi(t)$ satisfies the condition:

$$\xi(t)/t^{\sigma} \text{ is non–decreasing function for some } 0 < \sigma < 1. \quad (0.0.12)$$

Theorem 2.10.2 Let f be a 2π-periodic function belonging to $W(L^p, \xi(t))$ with $p \geq 1$, $\beta \geq 0$ and let the entries of the lower triangular matrices $A \equiv (a_{n,k})$ and $B \equiv (b_{n,k})$ satisfy the conditions (0.0.1) – (0.0.3) of Theorem 2.3.1 with $A_{n,n} = B_{n,n} = 1$ for $n = 0, 1, 2, \ldots$. Then the degree of approximation of $\tilde{f}$, conjugate of f, by BA means of its conjugate Fourier series is given by

$$\left\| \tilde{t}_n^{BA}(f;x) - \tilde{f}(x) \right\|_{p,\beta} = O\left(\xi(\pi/(n+1)) + (n+1)^{1-\sigma} \sum_{m=0}^{n} \frac{b_{n,m}}{m+1} \right)$$

where $\xi(t)$ and σ are the same as in Theorem 2.10.1.

In **Chapter** 3, we determine the degree of trigonometric approximation of 2π-periodic functions and their conjugates, in terms of the moduli of continuity associated with them, by matrix means of corresponding Fourier series. We also discuss some analogous results with remarks and corollaries.

Theorem 3.3.1 Let f be a 2π-periodic function belonging to the class $L^p(\omega)_\beta$, $\beta \geq 0$ and let $T \equiv (a_{n,k})$ be a lower triangular regular matrix with non-negative and non-decreasing (with respect to $0 \leq k \leq n$) entries with $A_{n,n-\tau} = O\left(1/t(n+1)\right)$. Then the degree of approximation of f by matrix means of its Fourier series is given by

$$\left\| t_n(f;x) - f(x) \right\|_p = O\left(\frac{1}{n+1} \int_{1/(n+1)}^{\pi} \frac{\omega(t)}{t^{\beta+2}} dt \right),$$

provided that ω is a function of modulus of continuity type such that

$$\int_0^v \frac{\omega(t)}{t^{\beta+1}}dt = O\left(\frac{\omega(v)}{v^\beta}\right),\ 0 < v < \pi. \tag{0.0.13}$$

Theorem 3.3.2 Let f be a 2π-periodic function belonging to the class $L^p(\tilde{\omega})_\beta,\ \beta \geq 0$ and let $T \equiv (a_{n,k})$ be a lower triangular regular matrix with non-negative and non-decreasing (with respect to $0 \leq k \leq n$) entries with $A_{n,n-\tau} = O\left(1/t(n+1)\right)$. Then the degree of approximation of $\tilde{f}$, conjugate of f, by matrix means of conjugate Fourier series is given by

$$\left\|\tilde{t}_n(f;x) - \tilde{f}(x)\right\|_p = O\left(\frac{1}{n+1}\int_{1/(n+1)}^{\pi} \frac{\tilde{\omega}(t)}{t^{\beta+2}}dt\right),$$

provided that $\tilde{\omega}$ is a function of modulus of continuity type such that

$$\int_0^v \frac{\tilde{\omega}(t)}{t^{\beta+1}}dt = O\left(\frac{\tilde{\omega}(v)}{v^\beta}\right),\ 0 < v < \pi. \tag{0.0.14}$$

Theorem 3.10.1 Let f be a 2π-periodic function belonging to $Lip(\omega(t),p)$-class with $p \geq 1$ and let $T \equiv (a_{n,k})$ be a lower triangular regular matrix with non-negative and non-decreasing (with respect to $0 \leq k \leq n$) entries with $A_{n,n-\tau} = O\left(1/t(n+1)\right)$. Then the degree of approximation of f by matrix means of its Fourier series is given by

$$\|t_n(f;x) - f(x)\|_p = O\left(\frac{1}{n+1}\int_{1/(n+1)}^{\pi} \frac{\omega(t)}{t^{2+1/p}}\,dt\right),$$

provided $\omega(t)$ is a positive non-decreasing function satisfying the following condition:

$$\int_0^v \frac{\omega\left(t\right)}{t^{1+1/p}}\,dt = O\left(\frac{\omega\left(v\right)}{v^{1/p}}\right),\ 0 < v < \pi. \tag{0.0.15}$$

Theorem 3.10.2 Let f be a 2π-periodic function belonging to $Lip(\omega(t),p)$-class with $p \geq 1$ and let $T \equiv (a_{n,k})$ be a lower triangular regular matrix with non-negative and non-decreasing (with respect to $0 \leq k \leq n$) entries with $A_{n,n-\tau} = O\left(1/t(n+1)\right)$. Then the degree of approximation of $\tilde{f}$, conjugate of f, by matrix means of its conjugate Fourier series is given by

$$\left\|\tilde{t}_n(f;x) - \tilde{f}(x)\right\|_p = O\left(\frac{1}{n+1}\int_{1/(n+1)}^{\pi} \frac{\omega(t)}{t^{2+1/p}}\,dt\right),$$

provided $\omega(t)$ is a positive non-decreasing function satisfying the condition (0.0.15) of Theorem 3.10.1.

In **Chapter** 4, we generalize the definition of $Lip\,(\alpha, p, w)$ defined by Guven [36] to the weighted Lipschitz class $Lip(\xi(\delta), p, w)$, where $\xi(\delta)$ is a positive non-decreasing function, and determine the degree of approximation of $f \in Lip(\xi(\delta), p, w)$ through matrix means of its trigonometric Fourier series. We note that some earlier results are particular cases of our following result:

Theorem 4.2.1 Let $1 < p < \infty$, $w \in \mathcal{A}_p$, $f \in Lip(\xi(\delta), p, w)$ and $A = (a_{n,k})$ be a lower triangular regular matrix satisfying one of the following conditions:

(i) $\{a_{n,k}\} \in AMDS$ in k and $(n+1)a_{n,0} = O(1)$,

(ii) $\{a_{n,k}\} \in AMIS$ in k,

(iii) $\sum_{k=0}^{n} \left| \Delta_k \left(\frac{A_{n,0} - A_{n,k+1}}{k} \right) \right| = O\,(1/n)\,.$

Then

$$\|f(x) - t_n(f;x)\|_{p,w} = O(\xi(1/n)),$$

where $\xi(\delta)$ is a positive non-decreasing function satisfying

$$\xi(1/\delta)\delta^{\sigma} \text{ is an non-decreasing function for some } \sigma > 0. \tag{0.0.16}$$

Chapter 5 deals with the approximation by triangular matrix means of Walsh–Fourier series in $L^p[0,1)$-space, where $\{a_{n,k}\}$ is almost monotone sequence. We generalize some earlier results [91; 93; 105] under less conditions on $a_{n,k}$. We prove the following:

Theorem 5.2.1 Let $f \in L^p[0,1)$, $1 \le p \le \infty$. Let $T \equiv (a_{n,k})$ be a lower triangular regular matrix with non-negative entries, where $n = 2^m + k$ for $1 \le k \le 2^m$ and $m \ge 1$. Then

(i) if $\{a_{n,k}\} \in AMIS$ in k and $na_{n,n} = O(1)$, then

$$\|t_n(f;x) - f(x)\|_p = O\left(\sum_{j=0}^{m-1} 2^j a_{n,2^{j+1}-1} \dot{\omega}_p(f; 2^{-j}) + \dot{\omega}_p(f; 2^{-m}) \right),$$

(ii) if $\{a_{n,k}\} \in AMDS$ in k, then

$$\|t_n(f;x) - f(x)\|_p = O\left(\sum_{j=0}^{m-1} 2^j a_{n,2^j} \dot{\omega}_p(f; 2^{-j}) + \dot{\omega}_p(f; 2^{-m}) \right).$$

Theorem 5.2.2 Let $f \in Lip(\alpha, p)$, $\alpha > 0$ and $1 \le p \le \infty$. Let $T \equiv (a_{n,k})$ be a lower triangular regular matrix with non-negative entries, where $n = 2^m + k$ for

$1 \le k \le 2^m$ and $m \ge 1$. Then

(i) if $\{a_{n,k}\} \in AMIS$ in k and $na_{n,n} = O(1)$, then

$$\|t_n(f;x) - f(x)\|_p = \begin{cases} O(n^{-\alpha})\,, & \text{if } 0 < \alpha < 1 \\ O(n^{-1}\log n)\,, & \text{if } \alpha = 1 \\ O(n^{-1})\,, & \text{if } \alpha > 1, \end{cases}$$

(ii) if $\{a_{n,k}\} \in AMDS$ in k, then

$$\|t_n(f;x) - f(x)\|_p = O\left(\sum_{j=0}^{m-1} 2^{(1-\alpha)j} a_{n,2^j} + 2^{-m\alpha}\right).$$

Chapter 6 deals with the approximation properties of $f \in L[0,\infty)$ by Cesáro means of order $\lambda \ge 1$ of the Fourier-Laguerre series of f for any $x > 0$. We prove the result for $x = 0$ separately.

Theorem 6.2.1 Let f be a function belonging to $L[0,\infty)$. Then the degree of approximation of f by the Cesáro means of order $\lambda \ge 1$ of the Fourier-Laguerre series of f is given by

$$|C_n^\lambda(f;x) - f(x)| = o\left(\xi(n)\right),$$

where $\xi(t)$ is a positive non-decreasing function such that $\xi(t) \to \infty$ as $t \to \infty$ and satisfies the following conditions:

$$\Phi(t) = \int_t^\epsilon y^{\alpha/2-1/4}|\phi(x,y)|dy = o\left(\xi(1/t)\right), \quad t \to 0, \tag{0.0.17}$$

$$\int_t^\delta \frac{|\psi(x,u)|}{u} du = o\left(\xi(1/t)\right), \quad t \to 0, \tag{0.0.18}$$

and

$$\int_n^\infty e^{-y/2}\, y^{\alpha/2-13/12}|\phi(x,y)|dy = o\left(n^{-1/2}\xi(n)\right), \quad n \to \infty, \tag{0.0.19}$$

where δ is a fixed positive constant and $\alpha \ge \frac{-1}{2}$. This holds uniformly for every fixed positive interval $0 < \epsilon \le x \le \omega < \infty$.

For $x = 0$, we prove the following theorems:

Theorem 6.4.1 Let f be a function belonging to $L[0,\infty)$. Then the degree of approximation of f at $x = 0$ by the Cesáro means of order $\lambda \ge 1$ of the Fourier-Laguerre series of f is given by

$$|C_n^\lambda(f;0) - f(0)| = o\left(n^{\alpha/2+3/4}\xi(n)\right),$$

where $\xi(t)$ is a positive non-decreasing function such that $\xi(t) \to \infty$ as $t \to \infty$ and satisfies the conditions (0.0.17) and (0.0.19) of Theorem 6.2.1 for $x = 0,\ \epsilon > 0$ and $\alpha \in [-1/2,\ 1/2]$.

Theorem 6.7.1 The degree of approximation of $f \in L[0, \infty)$ at $x = 0$ by the Hausdorff means of the Fourier-Laguerre series generated by $H \in H_1$ is given by

$$|H_n(f; 0) - f(0)| = o(\xi(n)),$$

where $\xi(t)$ is a positive non-decreasing function such that $\xi(t) \to \infty$ as $t \to \infty$ and satisfies the following conditions

$$\Phi(y) = \int_0^t |\varphi(y)| dy = o\left(t^{\alpha+1}\xi(1/t)\right), \quad t \to 0, \tag{0.0.20}$$

$$\int_\epsilon^n e^{y/2}\, y^{-((2\alpha+3)/4)} |\varphi(y)| dy = o\left(n^{-((2\alpha+1)/4)}\xi(n)\right), \tag{0.0.21}$$

and

$$\int_n^\infty e^{y/2}\, y^{-1/3} |\varphi(y)| dy = o(\xi(n)), \quad n \to \infty, \tag{0.0.22}$$

where ϵ is a fixed positive constant and $\alpha > -1/2$.

We also discuss some particular cases of Theorem 6.7.1.

List of Publications

Journal Papers

1. U. Singh and Soshal, Approximation of periodic integrable functions in terms of modulus of continuity, Acta Comment. Univ. Tartu. Math. **20** (2016), no. 1, 23–34.(Scopus indexed)

2. S. Saini and U. Singh, Degree of approximation of functions belonging to $Lip(\omega(t), p)$-class by linear operators based on Fourier series, Boll. Unione Mat. Ital. **9** (2016), no. 4, 495–504. (Springer, Scopus indexed, IF 0.182)

3. U. Singh and S. Saini, Approximation of periodic functions in certain subclasses of $L^p[0, 2\pi]$, Asian-Eur. J. Math. **10** (2017), no. 3, 1750046, 12 pp. (World Scientific, Scopus indexed)

4. S. Saini and U. Singh, Approximation of $\tilde{f}$, conjugate function of f belonging to a subclass of L^p-space, by product means of conjugate Fourier series, The Journal of Analysis (2017), 1–13 (Springer, Google Scholar indexed)

Conference Papers

1. S. Saini and U. Singh, Degree of approximation of $f \in L[0, \infty)$ by means of Fourier-Laguerre series, Mathematical Analysis and its Applications (2015), 207–217, Springer Proc. Math. Stat., 143, Springer, New Delhi.

Communicated Papers

The following papers are communicated for possible publication.

1. Soshal and Uaday Singh, Degree of approximation of functions in a weighted L^p-norm by product means of Fourier series.

2. Uaday Singh and Soshal, Approximation in generalized Lipschitz class with Muckenhoupt weights.

3. Soshal and Uaday Singh, Approximation by matrix means of Walsh–Fourier series in L^p-norm.

4. Uaday Singh and Soshal, Uniform approximation in $L[0,\infty)$-space by Cesáro means of Fourier–Laguerre series.

Acknowledgements

This doctoral work would not have been possible without the support and encouragement of numerous people including my well wishers and my friends. I would like to express my gratitude to all those who have helped me to complete this thesis.

It is a great pleasure to express my gratitude and everlasting indebtedness to my research supervisor Dr. Uaday Singh, Associate Professor, Department of Mathematics, I.I.T. Roorkee, Roorkee, for his help, advice and continuous encouragement throughout my research work. Without his active guidance, it would not have been possible for me to complete this work.

I am grateful to my SRC committee members, Prof. N. Sukavanam, Prof. P.N. Agrawal, Department of Mathematics and Prof. S.C. Sharma, Department of Mechanical and Industrial Engineering, for critically examining my progress and providing valuable suggestions. My sincere thanks to all the faculty members of the Department of Mathematics, I.I.T. Roorkee, for their cooperation. I would also like to thank all the non-teaching staff for their constant support.

I am indebted to the The Council of Scientific and Industrial Research, India for providing me financial support to carry out my research work.

The inspiration, support, cooperation and patience which I have received from my friends Dr. Shailesh Srivastava, Dr. Sumana Ghosh, Arti Rathor, Birendra , Ruchi, Pooja Rai and Pooja Saini are beyond the scope of any acknowledgement, yet I would like to express my heartfelt gratitude to them.
To my family, gratitude from the core of my heart. My affectionate thanks to my in laws, my sisters and brother for their unconditional love and affection which constantly provides me a strong moral support.

I must express my very profound gratitude to my husband Jaivindra Tomar to encourage me and make me happy throughout the years of my study.

The acknowledgement cannot be complete without the mention of the epicenter of my existence, my strongest support, my mother, Mrs. Murti Devi who has always extended her support and encouragement. Her blessing has been the constant source of inspiration to me. I extend my gratitude towards my father Late Karna Singh, whose eternal blessings always provide me a strong inner strength to overcome all the hurdles of life.

Last but not the least, many thanks to God for all my strengths, willpower and strong determination in the process of researching and writing this thesis so that I could achieved this accomplishment.

Roorkee (Soshal)

July , 2018

Table of Contents

Chapter 1

Introduction

Approximation theory is a vast field of mathematical territory which works out to approximate a typically unknown function [i.e. we have limited information about the function in terms of its properties] by some simple function, e.g. algebraic or trigonometric polynomials. Intuitively, Fourier approximation is the study of error estimation of functions through their Fourier series (which may be trigonometric, Walsh, Laguerre etc.) using summability techniques.

As we know that the Fourier Series of a function need not to be convergent to the function everywhere in the domain, the summability methods play key role to find the sum of non-convergent series which in turn approximate the function under consideration.

1.1 Fourier Series

A Fourier series has been defined to be a series in terms of orthogonal elements in the space, and this orthogonality property is attractive due to its computational aspects. Corresponding to different orthogonal sets, we can define several Fourier series. Some of them are discussed here:

Trigonometric Fourier Series:

Let f be a 2π periodic function belonging to $L^p := L^p[0,2\pi](p \geq 1)$-space. The trigonometric Fourier series of f is defined as

$$f(x) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k \cos kx + b_k \sin kx). \tag{1.1.1}$$

The n^{th} partial sum of the Fourier series (1.1.1), i.e.,

$$s_n(f;x) := \frac{a_0}{2} + \sum_{\nu=1}^{n}(a_\nu \cos \nu x + b_\nu \sin \nu x),\ \forall\, n \in \mathbb{N} \text{ with } s_0(f;x) = \frac{a_0}{2}, \tag{1.1.2}$$

is also called trigonometric polynomial of degree (or order) $\leq n$.

Consider the power series $\sum_{\nu=0}^{\infty} c_\nu z^\nu,\ z \in C$, where

$$c_\nu = \begin{cases} a_\nu - ib_\nu, & \nu \geq 1 \\ a_0/2, & \nu = 0. \end{cases}$$

If $z = e^{ix}$, then

$$\sum_{\nu=0}^{\infty} c_\nu z^\nu = \sum_{\nu=0}^{\infty} c_\nu e^{i\nu x} = a_0/2 + \sum_{\nu=1}^{\infty}(a_\nu - ib_\nu)(\cos \nu x + i \sin \nu x)$$
$$= a_0/2 + \sum_{\nu=1}^{\infty}(a_\nu \cos \nu x + b_\nu \sin \nu x) + i\sum_{\nu=1}^{\infty}(a_\nu \sin \nu x - b_n \cos \nu x).$$

The imaginary part of $\sum_{\nu=0}^{\infty} c_\nu e^{i\nu x}$ is called **conjugate Fourier series** and its n^{th} partial sum is defined by

$$\tilde{s}_n(f;x) := \sum_{\nu=1}^{n}(a_\nu \sin \nu x - b_\nu \cos \nu x),\ \forall\, n \in \mathbb{N} \text{ with } \tilde{s}_0(f;x) = 0. \tag{1.1.3}$$

The conjugate of f, denoted by $\tilde{f}$, is defined as

$$\tilde{f}(x) = -\frac{1}{2\pi} \lim_{\epsilon \to 0} \int_{\epsilon}^{\pi} \psi(x,t) \cot(t/2) dt, \tag{1.1.4}$$

where $\psi(x,t) = f(x+t) - f(x-t)$. We also write $\phi(x,t) := f(x+t) + f(x-t) - 2f(x)$ [152].

It is known that the series conjugate to a Fourier series is not necessarily itself a Fourier series, e.g. the series

$$\sum_{\nu=0}^{\infty} \frac{\cos \nu\theta}{log(\nu+2)}$$

is a Fourier series but the corresponding sine series is not a Fourier series. Therefore, a separate study of conjugate Fourier series is required.

Walsh Fourier Series:

Consider a sequence of functions $\{r_n(x)\}$ on the interval $I := [0, 1)$ defined as:

$$r_0(x) = \begin{cases} 1, & 0 \leq x < 2^{-1} \\ -1, & 2^{-1} \leq x < 1. \end{cases}$$

Now, extend it to the real line by periodicity of 1, i.e., $r_0(x+1) = r_0(x)$ and set $r_n(x) = r_0(2^n x)$, $n \geq 1$, $x \in \mathbb{R}$. These functions are known as Rademacher's functions.

The Walsh system $\{w_n(x)\}$ is obtained by taking all possible products of Rademacher's functions. For this, we use the Paley enumeration of the Walsh system [99]. Set $w_0(x) = 1$ and if

$$n = \sum_{j=0}^{\infty} k_j 2^j, \quad k_j = 0 \,\text{or}\, 1$$

is the dyadic representation of $n \in \mathbb{N}$, then $w_n(x) = \prod_{j=0}^{\infty} [r_j(x)]^{k_j}$ for $n \in \mathbb{N}$.

The Walsh system is complete orthonormal system.

The dyadic addition of x, $y \in I$ with dyadic expansions $x = \sum_{k=0}^{\infty} x_k 2^{-k-1}$ and $y = \sum_{k=0}^{\infty} y_k 2^{-k-1}$ is defined as

$$x \oplus y = \sum_{k=0}^{\infty} \frac{|x_k - y_k|}{2^{k+1}}.$$

Let f be a 1-periodic Lebesgue integrable function on I. The Walsh-Fourier series of f is defined as

$$\sum_{k=0}^{\infty} \hat{f}(k) w_k(x), \tag{1.1.5}$$

where $\hat{f}(k)$ is known as the k^{th}-Walsh-Fourier coefficient of f and given by $\hat{f}(k) = \int_0^1 f(t)w_k(t)dt$.
The n^{th} partial sum of the Walsh-Fourier series (1.1.5), i.e.,

$$s_n(f;x) := \sum_{k=0}^{n-1} \hat{f}(k)w_k(x),\ \forall\, n \in \mathbb{N}, \tag{1.1.6}$$

is also called the Walsh polynomial of degree (or order) $\leq n$. The collection of all Walsh polynomials of degree $\leq n$ is denoted by $\mathcal{P}_n$. The collection $\mathcal{P}_{2^n}$ coincides with the collection of $\mathcal{A}_n$-measurable functions on I, where $\mathcal{A}_n$ is the finite σ-algebra generated by the collection of dyadic intervals of the form $I_m(k) := [k2^{-m}, (k+1)2^{-m}],\ m \geq 0,\ k = 0, 1, ..., 2^m - 1$.
The integral representation of $s_n(f;x)$ in (1.1.6) is given by

$$s_n(f;x) = \int_0^1 f(x \oplus t)D_n(t)dt, \tag{1.1.7}$$

where $D_n(t) = \sum_{k=0}^{n-1} w_k(t)$ is the Walsh-Dirichlet kernel having the property that

$$D_{2^n}(t) = \begin{cases} 2^n, & t \in [0, 2^{-n}) \\ 0, & t \in [2^{-n}, 1]. \end{cases} \tag{1.1.8}$$

For more details one can see [32].

Fourier-Laguerre Series:

Let f be a Lebesgue integrable function on $[0, \infty)$, i.e., $f \in L[0, \infty)$. The Fourier-Laguerre expansion of f is given by

$$f(x) \sim \sum_{n=0}^{\infty} a_n L_n^{(\alpha)}(x), \tag{1.1.9}$$

where

$$\Gamma(\alpha+1) \begin{pmatrix} n+\alpha \\ n \end{pmatrix} a_n = \int_0^{\infty} e^{-x}x^{\alpha} f(x)L_n^{(\alpha)}(x)dx, \tag{1.1.10}$$

the existence of the above integral is presumed and $L_n^{(\alpha)}(x)$ denotes the n^{th} Laguerre polynomial of order $\alpha > -1$, defined by the generating function

$$\sum_{n=0}^{\infty} L_n^{(\alpha)}(x)\omega^n = (1-\omega)^{-\alpha-1} \exp\left(\frac{-x\omega}{1-\omega}\right).$$

The n^{th} partial sum of the Fourier-Laguerre series (1.1.9), i.e.,

$$s_n(f;x) = \sum_{k=0}^{n} a_k L_k^{(\alpha)}(x), \tag{1.1.11}$$

is also called Fourier-Laguerre polynomial of degree (or order) $\leq n$.
When $x = 0$,

$$L_n^{(\alpha)}(0) = \begin{pmatrix} n+\alpha \\ n \end{pmatrix}$$

and

$$\sum_{r=0}^{n} L_r^{(\alpha)}(x) = L_n^{(\alpha+1)}(x).$$

Laguerre polynomials $L_n^{(\alpha)}(x)$ form orthogonal set with the weight function $e^{-x}x^{\alpha}$ on $[0,\infty)$. Fore more details one can see [136].

The study of other Fourier series like corrected Fourier series [107], Fourier–Bessel's series [141] and hexagonal Fourier series [148] also develops an extensive field of Fourier expansions and approximations.

1.2 Summability

Summability is a well developed field for the study of non-convergent series in which one attempts to attach a value to the non-convergent infinite series, i.e., summability extends the notion of the sum of a convergent series. Throughout the last century, the mathematicians such as Abel, Cesáro, Euler, Hausdorff, Hölder, Nörlund, Riesz and Lebesgue have formulated various process of summability of infinite series.

The most well known methods of summability are linear. Toeplitz [140] was the first to put such methods systematically and we call them T-methods. Let $\sum u_n$ be an infinite series with sequence of partial sums $\{s_n\}$ and let $T = (a_{n,k})$ be an infinite lower triangular matrix. Then the sequence -to-sequence transformation

$$t_n = \sum_{k=0}^{n} a_{n,k} s_k = \sum_{k=0}^{n} a_{n,n-k} s_{n-k}, \qquad n = 0,\ 1,\ 2, \dots,$$

are called linear means (determined by the matrix T) of the sequence $\{s_n\}$.

Definition 1.2.1. The infinite series $\sum u_n$ is said to be summable to s by T-method, if

$$\lim_{n\to\infty} t_n = s,$$

where s is a finite number.

Regularity of Summability Process:

Definition 1.2.2. The summability matrix T or the sequence-to-sequence transformation $\{t_n\}$ is said to be regular, if

$$\lim_{n\to\infty} s_n = s \Rightarrow \lim_{n\to\infty} t_n = s.$$

The necessary and sufficient conditions on $a_{n,k}$ which make the matrix T regular were found by Toeplitz [140] and Silverman [120] and given as follows:

1. $\lim_{n\to\infty} \sum_{k=0}^{\infty} a_{n,k} = 1$ (row sums tend to 1),
2. $\sum_{k=0}^{\infty} |a_{n,k}| < M$ for every $n \in \mathbb{N} \cup \{0\}$ (uniformly bounded summable rows),
3. $\lim_{n\to\infty} a_{n,k} = 0$ for every $k \in \mathbb{N} \cup \{0\}$ (terms of any column tends to 0),

where M is a constant independent of n.

1.2.1 Some Particular Cases of Matrix-T

1. **Cesàro Matrix of Order δ:** For a positive real number δ, if

$$a_{n,k} = \begin{cases} E_{n-k}^{\delta-1}/E_n^{\delta}, & 0 \le k \le n \\ 0, & k > n, \end{cases}$$

where

$$E_n^{\delta} = \frac{\Gamma(n+\delta+1)}{\Gamma(n+1)\Gamma(\delta+1)} = \sum_{k=0}^{n} E_k^{\delta-1},$$

then the linear means t_n become Cesàro means of order δ, i.e., (C,δ)-means. The series $\sum u_n$ is said to be summable to s by Cesàro means, if

$$t_n^{\delta} = \frac{1}{E_n^{\delta}} \sum_{k=0}^{n} E_{n-k}^{\delta-1} s_k \to s \text{ as } n \to \infty.$$

2. **Harmonic Matrix:** If

$$a_{n,k} = \begin{cases} \frac{1}{(n-k+1)\log n}, & 0 \leq k \leq n \\ 0, & k > n, \end{cases}$$

then the linear means t_n become Harmonic means, i.e., $(H, 1)$-means. The series $\sum u_n$ is said to be summable to s by Harmonic means, if

$$t_n = \frac{1}{\log n} \sum_{k=0}^{n} \frac{s_k}{n-k+1} \to s \text{ as } n \to \infty.$$

3. **Nörlund Matrix:** If

$$a_{n,k} = \begin{cases} p_{n-k}/P_n, & 0 \leq k \leq n \\ 0, & k > n, \end{cases}$$

where $P_n = \sum_{k=0}^{n} p_k \neq 0$, and $p_{-1} = 0 = P_{-1}$, $\{p_n\}$ is any sequence of real numbers, then the linear means t_n become Nörlund means, i.e., (N_p)- means. The series $\sum u_n$ is said to be summable to s by Nörlund means, if

$$t_n = \frac{1}{P_n} \sum_{k=0}^{n} p_{n-k} s_k \to s \text{ as } n \to \infty.$$

4. **Riesz Matrix:** If

$$a_{n,k} = \begin{cases} p_k/P_n, & 0 \leq k \leq n \\ 0, & k > n, \end{cases}$$

where $P_n = \sum_{k=0}^{n} p_k \neq 0$, and $p_{-1} = 0 = P_{-1}, \{p_n\}$ is any sequence of real numbers, then the linear means t_n become Riesz means, i.e., $(\bar{N}_p)$- means. The series $\sum u_n$ is said to be summable to s by Riesz means, if

$$t_n = \frac{1}{P_n} \sum_{k=0}^{n} p_k s_k \to s \text{ as } n \to \infty.$$

5. **Euler Matrix:** For a positive real number q, if

$$a_{n,k} = \begin{cases} \begin{pmatrix} n \\ k \end{pmatrix} q^{n-k}/(1+q)^n, & 0 \leq k \leq n \\ 0, & k > n, \end{cases}$$

then the linear means t_n become Euler means, i.e., (E,q)-means of order q. The series $\sum u_n$ is said to be summable to s by Euler means, if

$$t_n = \frac{1}{(1+q)^n}\sum_{k=0}^{n}\binom{n}{k} q^{n-k}s_k \to s \text{ as } n\to\infty.$$

6. **Hausdorff Matrix:** The Hausdorff means of $\sum u_n$ are defined by

$$H_n = \sum_{k=0}^{n} h_{n,k}s_k, \quad n = 0,1,2,...,$$

with

$$h_{n,k} = \begin{cases} \binom{n}{k}\Delta^{n-k}\mu_k, & 0\le k\le n \\ 0, & k>n, \end{cases}$$

where $\Delta\mu_n = \mu_n - \mu_{n+1}$ is forward difference operator and $\Delta^{k+1}\mu_n = \Delta^k(\Delta\mu_n)$. If $H = (h_{n,k})$ is regular, then $\{\mu_n\}$ known as moment sequence, has the representation

$$\mu_n = \int_0^1 u^n d\gamma(u),$$

where $\gamma(u)$, known as mass function, has the following properties:

(a) $\gamma(u)$ is continuous at $u = 0$,

(b) $\gamma(u) \in BV[0,1]$ such that $\gamma(0) = 0,\ \gamma(1) = 1$,

(c) $\gamma(u) = 2^{-1}[\gamma(u+0)+\gamma(u-0)]$ for $0 < u < 1$.

The series $\sum u_n$ is said to be summable to s by the Hausdorff means, if $H_n \to s \textit{ as } n\to\infty$.

Tylor means [17], Bochner–Riesz means [18] and almost Riesz means [22] are also very useful summability techniques.

1.2.2 Composition of Two Summability Methods

Let $A \equiv (a_{n,m})$ and $B \equiv (b_{n,m})$ be two infinite lower triangular matrices of real numbers. When we superimpose the $B-$summability on $A-$summability, we get

BA means of $\sum u_n$ defined by

$$t_n^{BA} \;=\; \sum_{m=0}^{n} b_{n,m}\left(\sum_{k=0}^{m} a_{m,k}s_k\right) = \sum_{m=0}^{n}\sum_{k=0}^{m} b_{n,m}a_{m,k}s_k.$$

The composite summability method is more effective than the individual summability method. It has been discussed very well by Mishra et al. [81, pp.261–262] that the product summability C^1N_p is stronger than the single $(C,1)$ summability method.

1.3 Some Basic Definitions and Results

1. **L^p-Norm:** The L^p norm of $f \in L^p[0,2\pi]$ is defined by

$$\|f\|_p = \begin{cases} \left(\frac{1}{2\pi}\int_0^{2\pi} |f(x)|^p dx\right)^{1/p}, & 1 \le p < \infty \\ \text{ess}\sup\limits_{x\in[0,2\pi]} |f(x)|, & p = \infty. \end{cases}$$

2. **Weighted L^p-Norm:** The weighted norm of $f \in L^p[0,2\pi]$ with the weighted function $w(x)$ is defined by

$$\|f\|_{p,w} = \left(\frac{1}{2\pi}\int_0^{2\pi} |f(x)|^p w(x)dx\right)^{1/p}, \; 1 \le p < \infty. \tag{1.3.1}$$

3. **Modulus of Continuity:** Let $f(x)$ be a function in the interval $[a,b]$. Then the modulus of continuity $\omega(\delta)$ of the function $f(x)$ is defined as

$$\omega(\delta) := \omega(f;\delta) = \sup_{|x-y|\le\delta} |f(x) - f(y)|, \quad a \le x, \; y \le b.$$

   Some basic properties of the modulus of continuity $\omega(\delta)$ [56; 95] are as follows:

   (a) $\omega(\delta) \ge 0$, and $\omega(\delta) = 0$, if $\delta = 0$.

   (b) $\omega(\delta_1) \le \omega(\delta_2)$ for $0 < \delta_1 < \delta_2$.

   (c) f is uniformly continuous on $[a,b]$ if and only if

$$\lim_{\delta\to 0} \omega(\delta) = 0.$$

   (d) $\omega(n\delta) \le n\,\omega(\delta)$ for $n \in \mathbb{N}$ and $\delta > 0$.

Also ω satisfies the following:

$$\delta_2^{-1}\omega(\delta_2) \le 2\delta_1^{-1}\omega(\delta_1) \quad \text{for } \delta_2 \ge \delta_1 > 0. \tag{1.3.2}$$

4. **Integral Modulus of Continuity:** The integral modulus of continuity of a 2π periodic function f in $L^p[0, 2\pi](1 \le p < \infty)$-space is defined by

$$\omega_p(f; h) = \sup_{0<|t|\le h} \| f(x+t) - f(x) \|_p .$$

5. **Generalized Modulus of Continuity:** The generalized moduli of continuity of f in $L^p[0, 2\pi]-$space are defined by

$$\omega_\beta f(\delta)_{L^p} := \sup_{0\le|t|\le\delta} \left\{ \left|\sin\frac{t}{2}\right|^{\beta p} \int_0^{2\pi} |\phi(x,t)|^p dx \right\}^{1/p}, \quad \beta \ge 0$$

and

$$\tilde{\omega}_\beta f(\delta)_{L^p} := \sup_{0\le|t|\le\delta} \left\{ \left|\sin\frac{t}{2}\right|^{\beta p} \int_0^{2\pi} |\psi(x,t)|^p dx \right\}^{1/p}, \quad \beta \ge 0.$$

6. **The Hölder Inequality:** Let x and y be scalar-valued Lebesgue-measurable functions on the Lebesgue-measurable set E such that $\int_E | x(t) |^p \, dt < \infty$ and $\int_E | y(t) |^q \, dt < \infty$, where $p > 1$ and $p^{-1} + q^{-1} = 1$. Then $\int_E | x(t)y(t) | \, dt < \infty$ and

$$\int_E | x(t)y(t) | \, dt \le \left(\int_E | x(t) |^p \, dt \right)^{1/p} \left(\int_E | y(t) |^q \, dt \right)^{1/q}.$$

For $p = 1$ and $q = \infty$, we have

$$\int_E | x(t)y(t) | \, dt \le \left(\int_E | x(t) | \, dt \right) (\operatorname{ess\,sup}_{t\in E} | y(t) |).$$

7. **Generalized Minkowski Inequality:** Let $g(x,t) \in L^p([a,b] \times [c,d])$ for $p \ge 1$. Then

$$\left\{ \int_a^b \left| \int_c^d g(x,t) dt \right|^p dx \right\}^{1/p} \le \int_c^d \left\{ \int_a^b |g(x,t)|^p dx \right\}^{1/p} dt.$$

8. **Abel's Transformation**: Let $\{a_n\}_{n=0}^\infty$ and $\{b_n\}_{n=0}^\infty$ be two sequences of real numbers, then

$$\sum_{k=m}^{n} a_k b_k = \sum_{k=m}^{n-1} A_k \triangle b_k + A_n b_n - A_{m-1} b_m, \tag{1.3.3}$$

where $A_k = \sum_{r=0}^{k} a_r$ and $\triangle b_k \equiv b_k - b_{k+1}$.
For $m = 0$, (1.3.3) reduces to

$$\sum_{k=0}^{n} a_k b_k = \sum_{k=0}^{n-1} A_k \triangle b_k + b_n A_n.$$

1.4 Important Functions Spaces

A function $f \in Lip\alpha$, if $|f(x+t) - f(x)| = O(t^\alpha)$ for $0 < \alpha \le 1$,
$f \in Lip(\alpha, p)$, if $\left(\int_0^{2\pi} |f(x+t) - f(x)|^p dx\right)^{1/p} = O(t^\alpha)$ for $0 < \alpha \le 1,\ p \ge 1$,
$f \in Lip(\xi(t), p)$, if $\left(\int_0^{2\pi} |f(x+t) - f(x)|^p dx\right)^{1/p} = O(\xi(t))$
and $f \in W(L^p, \xi(t))$, if

$$\left(\int_0^{2\pi} |(f(x+t) - f(x)) \sin^\beta(x/2)|^p dx\right)^{1/p} = O(\xi(t)),\ \beta \ge 0,\, p \ge 1,$$

where $\xi(t)$ is positive non-decreasing modulus of continuity type function.

It is important to note that the function $\xi(t)$, in the definition of $W(L^p, \xi(t))$-class is not the same as the $\xi(t)$ in the definition of $Lip(\xi(t), p)$-class. The $\xi(t)$ in the $Lip(\xi(t), p)$-class depends on t only, whereas the $\xi(t)$ in the $W(L^p, \xi(t))$-class depends on both t and β [52]. If we take $\xi(t) = t^\beta \psi(t)$ for $\beta \ge 0$ and some positive non-decreasing function $\psi(t)$, then the $W(L^p, \xi(t))$-class defined above reduces to the $W(L^p, \psi(t))$-class defined by Khan [52].
If $\beta = 0$, then $W(L^p, \xi(t)) \equiv Lip(\xi(t), p)$ and for $\xi(t) = t^\alpha (0 < \alpha \le 1)$, $Lip(\xi(t), p) \equiv Lip(\alpha, p)$. $Lip(\alpha, p) \to Lip\alpha$ as $p \to \infty$. Thus

$$Lip\alpha \subseteq Lip(\alpha, p) \subseteq Lip(\xi(t), p) \subseteq W(L^p, \xi(t)).$$

In the thesis, we also used some subclasses of L^p-space in terms of generalized integral modulus of continuity [58; 77].
Let ω be a modulus of continuity type function on $[0, 2\pi]$. Then two subclasses of L^p-space are defined by

$$L^p(\omega)_\beta = \{f \in L^p : \omega_\beta f(\delta)_{L^p} \le \omega(\delta)\},$$

$$L^p(\tilde{\omega})_\beta = \{f \in L^p : \tilde{\omega}_\beta f(\delta)_{L^p} \le \tilde{\omega}(\delta)\}.$$

If $\beta = 0$, $L^p(\omega)_\beta$ (or $L^p(\tilde{\omega}_\beta)$) coincides with $Lip(\xi(\delta), p)$ for $\omega(\delta) = \delta^\alpha (0 < \alpha \leq 1)$, $L^p(\omega)_\beta \equiv Lip(\alpha, p)$ and $Lip(\alpha, p) \to Lip\alpha$ as $p \to \infty$.

There are many more function spaces such as Basove spaces [3], Lebesgue space with Muckenhoupt weights [38], grand Lorentz space [47], sobolev space [100; 142] and generalized sobolev space [103] etc.

1.5 Fourier Approximation

If a function f in L^p-space is approximated by a polynomial $T_n(x)$ of degree $\leq n$ (which are either partial sums or some summability means of the Fourier series of f), then the error of approximation $E_n(f)$, in terms of n, is given by

$$E_n(f) = \min_{T_n} \| f(x) - T_n(x) \|_p .$$

The polynomial $T_n(x)$ is known as the Fourier-approximant of f, and this method of approximation is called Fourier approximation.
If $T_n^*(x)$ is the polynomial of best approximation, then

$$E_n(f) = \min_{T_n} \| f(x) - T_n(x) \|_p = \| f(x) - T_n^*(x) \|_p .$$

1.6 Literature Review

Approximation by Fourier series lies at the heart of signal analysis and in digital signal processing [106] including audio, videos, images, speech, radio transmission etc. Nowadays wavelet transform and Fourier transform also have been of growing interest for researchers [101; 104] to provide additional and remarkable tools in the area of filtering and signal analysis but Fourier series is the best choice for processing periodic signals.

Among the results studied by various researchers over the last decades, perhaps the best known result was due to Weierstrass [146]. This result has been extended further for trigonometric polynomials. Lebesgue [72] obtained the result for approximation of $f \in Lip\alpha (0 < \alpha \leq 1)$ by partial sums of its Fourier series and obtained $|s_n(f; x) - f(x)| = O\left(n^{-\alpha} \log n\right)$. Bernstien [7] used Cesàro summability of order 1, i.e. $(C, 1)$ means, for $f \in Lip\alpha(0 < \alpha \leq 1)$ to prove that $E_n(f) = O(n^{-\alpha})$ for

$0 < \alpha < 1$ and $E_n(f) = O\left(n^{-1}\log n\right)$ for $\alpha = 1$. Jackson [45] also gave theorems dealing with the errors in the approximation of $f \in Lip\alpha(0 < \alpha \leq 1)$ by the partial sums $s_n(f;x)$ and $(C,1)$ means of its Fourier series. Alextis [1] extended the results of Bernstien [7] for Cesàro summability of order δ for $f \in Lip\alpha(0 < \alpha \leq 1)$. Results of Jackson [45] has been studied further by Quade [108] in L^p-norm. He proved $\|s_n(f;x) - f(x)\|_p = \begin{cases} O\left(n^{-\alpha}\right), & p > 1 \\ O\left(n^{-\alpha}\log n\right), & p = 1. \end{cases}$ He also gave the similar results for $(C,1)$ summability means. Sahney and Goel [117] studied Nörlund means of Fourier series in $f \in Lip\alpha(0 < \alpha \leq 1)$-class. Holland et.al. [43], Chandra [13; 14] has extended the problem further using Nörlund and Riesz means with monotonic weights p_n. Holland [42] has gone through all the results on trigonometric approximation of continuous functions and published a survey paper in 1981 for these results.

Chandra [13; 14] studied the problem further and proved the approximation results for $f \in Lip\alpha$ choosing Nörlund means with monotonicity condition on p_n and a lower triangular matrix $T \equiv (a_{n,k})$ such that $a_{n,k} \leq a_{n,k+1}$. Chandra and Mohapatra [16] proved the results for absolute Nörlund summability of Fourier series. Chandra [15], Leinder [73] and Mittal et al. [87; 88] extended the results of [13; 14; 108] for $f \in Lip(\alpha, p)$ by relaxing the condition of monotonicity on $a_{n,k}$ to obtain $E_n(f)$ of order $n^{-\alpha}$. A nice discussion has been done by Bojanić and Mazhar [10] on the error when a function of bounded variation has been approximated by various summability means of its Fourier series. Wei and Yu [145] also gave nice results and discussed the applications of their results to more general classes of sequences.

Mittal [83] proved the results for F_1-effectiveness of C^1T method. The author [83] also discussed the F_1-effectiveness of C^1N_p and N_p methods on which the approximation results already have been proved. Mittal and Bhardwaj [84] studied the results of [13; 14; 15; 108] further for a linear matrix operator. Mittal and Rhoades [85] have obtained the error estimation of f through a matrix which does not have monotone rows. A new class $Lip(\psi(t), p)(p > 1)$ of 2π periodic functions was defined by Khan and Ram [54] as $Lip(\psi(t), p) = \{f : |f(x+t) - f(x)| \leq M\left(\psi(t)\, t^{-1/p}\right)\}$ $0 < t < \pi$ and M is a positive number independent of x and t. The authors in [54] determined the degree of approximation for $f \in Lip(\psi(t), p)$ using Euler means. Nigam [96] and Nigam and Sharma [98] obtained $E_n(f) =$

$O\left((n+1)^{1/p}\xi(1/(n+1))\right)$ for $f \in Lip(\xi(t),p)$ using $(E,q)(C,1)$ and $(C,1)(E,q)$ summability methods, respectively. Although, $\xi(t)$ does not depend on p but these results depend on p. Lal and Yadav [71] obtained the degree of approximation of $f \in W(L^p,\xi(t))$ using a triangular matrix involving the product $(C,1)(E,1)$. Rhoades [113; 116] extended these results to a wider class of Hausdorff matrices and proved that for $f \in W(L^p,\xi(t))$, $\|f(x) - H_n(f;x)\|_p = O\left(n^{\beta+1/p}\xi(1/n)\right)$ assuming two additional conditions on $\xi(t)$. He also derived a result for $f \in Lip\alpha$. Lal [66] had initiate for the study of C^1N_p summability of $f \in W(L^p,\xi(t))$ which was further improved by Singh et al. [123]. These results depend on p. Singh and Sonker [125] pointed out some remarks in the results of [66; 71; 113] and studied this problem further using Hausdorff summability means. Singh and Srivastava [135] studied all these results to obtain the degree of approximation free from p. The authors [135] obtained the results for $f \in W(L^p,\omega(t),\beta)$-class using C^1T operator with weaker conditions on $\omega(t)$. Obviously, the results are independent of p and hence more sharper than the earlier. Krasniqi [59], Łenski [75] and Łenski and Szal [78] have given results in terms of modulus of continuity for more general summability means, generated by product of two triangular matrices. Mohapatra and Szal [90] also obtained similar results in L^p-space and $Lip(\alpha,p)$-class. A number of results have been obtained by Kranz et al. [58], Krasniqi [61] and Łenski and Szal [76; 77] in the space $L^p(\omega)_\beta$.

Following Ky [62], recently Guven [36; 37] extended the results of Chandra [15] to a weighted Lipschitz class. Singh and Srivastava [126] continued this work and extended the theorem of Guven [36] to the matrix means under the relaxed condition of monotonicity and replaced the results of [37] by a single result. Das et al. [19] obtained the approximation results in Banach space with Hölder metric using $(E\ C)$ means of Fourier series. Mohapatra and Chandra [89] also obtained results in Hölder metric using lower triangular matrix means. Das et al. [20; 21] introduced a regular trigonometric summation method and applied it to determined $E_n(f)$ in generalized Hölder metric space $H_p^{(w)}$. Leindler [74] and Singh and Sonker [124] obtained the degree of approximation in generalized Hölder metric. The authors [124] generalized results of Leindler [74] using matrix means which has almost monotone rows. Very recently, Jafrov [46] approximated f through the Nörlund submethod of summability with monotonicity on p_n. Guven and Yurt [39] have given the direct and converse trigonometric approximation results in the Lebesgue space of function of severable variable with the help of moduli of smoothness of fractional order.

As conjugate Fourier series needs the separate study, many researchers investigated about the conjugate function of f in different classes, namely, $Lip\alpha$, $Lip(\alpha,p)$, $Lip(\xi(t),p)$ and $W(L^p,\xi(t))$, through various summability methods. Quarshi [109; 110], Lal and Kushwaha [67], Lal and Mishra [68] and Kęska [50] have proved some useful results on $E_n(\tilde{f})$ for $f \in Lip\alpha(0<\alpha<1)$ using almost Riesz, triangular matrix and Euler–Hausdorff means of conjugate Fourier series. Lal [65] proved the results for conjugate of almost Lipschitz functions by $(C,1)(E,1)$ means. Qureshi [111], Lal and Singh [70] and Sonker and Singh [134] proved $E_n(\tilde{f}) = O\left(n^{-\alpha+1/p}\right)$ using Nörlund, $(N,p,q)(E,1)$ and $(C,1)(E,q)$ means, respectively in $Lip(\alpha,p)(p \geq 1)$-class.

Being motivated by these results in the $Lip\alpha$, $Lip(\alpha,p)$-classes, some authors studied the problem further for $Lip(\xi(t),p)$ and $W(L^p,\xi(t))$-classes. Mittal et al. [86] approximated $\tilde{f}$ by linear opertors in $W(L^p,\xi(t))$-class. Rhoades [115] and Lal and Mishra [68] proved their results for the class $Lip(\xi(t),p)$ using the Hausdorff means and product means $E^q\Delta H$, respectively. The authors gave their results free from p in terms of $\xi(t)$ with a single condition on $\xi(t)$. Mishra et al. [82] used C^1N_p summability means of the conjugate Fourier series for the $W(L^p,\xi(t))$-class and proved $E_n(\tilde{f}) = O\left(n^{\beta/2+p/2}\xi(1/\sqrt{n})\right)$ with additional assumption conditions on $\xi(t)$ and monotonic weight p_n. Further, Mishra et al. [81] studied this result by dropping the monotonicity on p_n which in turn generalized the results of Lal [66]. Singh and Srivastava [127] approximated conjugate of functions belonging to $W(L^p,\xi(t))$ by Hausdorff means of conjugate Fourier series and proved $E_n(\tilde{f}) = O\left((n+1)^{\beta+1/p}\xi(1/(n+1))\right)$. The authors [128] studied this problem again to obtain the results free from p by using the C^1T means to proved that $E_n(\tilde{f}) = O\left((n+1)^{\beta}\xi(1/(n+1))\right)$.

In analogy with the theory of trigonometric Fourier series, an extensive parallel theory of Wash-Fourier series has been developed. In the past few decades, the system of Walsh functions has been the subject of growing interest. A list of papers on Walsh functions have been published in [5] stressing their application in computer science and electric engineering. Billard [8] and Gosselin [33] proved that the Walsh series of L^2 functions converges almost everywhere. For $f \in L^p,\ 1 < p < \infty$ Paley [99] proved that $W_n(f) \to f$ as $n \to \infty$. Results of Ladhawala [64] and Butzer and Wagner [11] conclude that Walsh Fourier series is absolutely convergent for $f \in Lip(\alpha, p)$. Móricz and Schipp [92] studied the integrability and convergence of Walsh-Fourier series in $L[0,1]$ with coefficients of bounded variation. The investigators like Butzer and Wagner [12], Schipp [119], Skvorcov [131] and Powell [102] examined the conditions for Walsh series to represent a dydically differentiable function f. A good amount of work has been done by Arutjunjan and Talaljan [4], Skvorcov [130] on the uniqueness of approximation by Walsh series. Wade [144] published a paper related to the research done on Walsh series during the period $1971-1981$. Fine [24], Yano [149], Jastrebova [49], Schipp [118], Baǐarstanova [6] and Skvorcov [132] have done good work on Cesáro summability of Walsh–Fourier series. Móricz and Siddiqi [93] considered the Nörlund means of Walsh–Fourier series of a function in $L^p(1 \leq p \leq \infty)$-space and gave the results on the rate of convergence by Nörlund means. The authors [93] observed that the earlier results on Cesàro means are special cases of their results. The authors also suggested an extension of their work [93, pp.386-388] which was carried out by Fridli [26]. Tevzade [138] and Goginava [30] studied the problem for Cesàro means of negative order.

Móricz and Rhoades [91] considered the weighted means and obtained the results for the Walsh–Fourier series of functions in $L^p[0,1)$ and $Lip(\alpha, p)(\alpha > 0,\ 1 \leq p \leq \infty)$-classes. Later Priti et al. [105] obtained the results using a lower triangular matrix means $T \equiv (a_{n,k})$ such that $\sum_{k=0}^{n-1} a_{n,n-k}^{\gamma} = O(n^{1-\gamma})$ for some $1 < \gamma \leq 2$. The analogous of [91; 93; 105] has been discussed by Blahota and Nagy [9] very recently. Rhoades [114] proved the results for a regular triangular matrix means with coefficients of general bounded variation. Weisz [147] introduced dyadic martingale Hardy spaces H_p and obtained that the maximal operator of the (C, α)-means is bounded from the space H_p to $L^p(1/(\alpha+1) < p < \infty)$ and uniformly continuous on H_p. Goginava [31] considered the maximal operator of Fejer-Walsh means. Fridli et al. [27] also extended the results of Yano [149], Jastrebova [49] and Skvorcov [132] to

H_p-space and homogeneous Banach spaces. Tephnadze [137] has derived the necessary and sufficient conditions for modulus of continuity so that Fejer-Walsh means is convergent in H_p-space.
Some authors such as Konyagin [57], Episkoposian and Müller [23] and Gàt et al. [29] studied the convergence and divergence properties of Walsh–Fourier series. The authors [23] proved some universality properties that there are countable sets E such that $s_n(f;x)$ is not uniformly universal on E, $\forall\ f \in C_w$. Hankel matrix transformation of Walsh–Fourier series also has been of growing interest for researchers (see [2] and references therein).

A smooth function can be represented as a series of orthogonal polynomials. Laguerre polynomials $L_n(x)$ and generalized Laguerre polynomials $L_n^{(\alpha)}(x)$ form complete orthogonal set of functions on the interval $[0,\infty)$. Laguerre series is useful to get good results in data compression task and it is also useful in the spectral theory for nonlinear differential equations [51]. Hille [40] proved that Laguerre series of a certain class of functions are Able summable and used the results in convergence of Laguerre series [41]. The equiconvergence theorems and summability theorem at $x = 0$ has been given in [136]. The author [136] proved that Fourier–Laguerre series of a Lebesgue measurable function on $[0,\infty)$ and continuous at $x = 0$ is (C,K) summable at $x = 0$ with the sum $f(0)$ provided $K > \alpha + 1/2$. Gupta [35] also proved that $\sigma_n^K(0) = o(\log n)$, $K > \alpha + 1/2$, where $\sigma_n^K(0)$ is n^{th} (C,K) means of Fourier-Laugerre series. Similar result has been proved by Singh [122] that $\sigma_n^K(0) = f(0) + O(n^{-1/2}) + O(\psi(1/n))$, $K > \alpha + 1/2$, $\alpha > -1$ with some assumption conditions on the function $\psi(t)$. The author [121] also studied the absolute $(C,1)$ summability factors at the point $x \neq 0$. Khan and Khan [53] established theorems on $E_n(f)$ of a function by Cesáro means of Fourier-Laguerre series, the series obtained by product of two Laugerre polynomials. The authors [60; 69; 97; 133; 139] has obtained the results at $x = 0$ by Nörlund, (N,p,q), $(E,1)$, and $(C,1)(E,q)$ summability means of Fourier-Laugerre series for $\alpha \in (-1,-1/2)$ with a set of assumption conditions. The authors in [55] used Harmonic-Euler product sumability means and proved the similar results. Greene [34] obtained the results on the oscillatory region of Fourier–Laguerre series and proved that outside this region $|L_n^{(\alpha)}(x)|$ increases monotonically bounded.

1.7 Objective of the Study

The objective of present study is to fill the gap in the literature by making some advancement in the field of Fourier approximation. The pointwise objectives are as follows:

- To study the trigonometric approximation of functions and their conjugates belonging to certain subclasses of L^p-space such as $L^p(\omega)_\beta,\ L^p(\tilde{\omega})_\beta$ and $W(L^p, \xi(t))$ using product summability methods (Chapter 2).

- To obtain the degree of approximation of functions in the classes $L^p(\omega)_\beta,\ L^p(\tilde{\omega})_\beta$ and $Lip(\omega(t), p)$ by linear operators (Chapter 3).

- To introduced generalized Lipschitz class with Muckenhoupt weights $Lip(\xi(\delta), p, w)$ and to determine the degree of approximation of functions by matrix means of trigonometric Fourier series (Chapter 4).

- To determine $E_n(f)$ for functions in $L^p[0, 1)$-space by means of its Walsh-Fourier series (Chapter 5).

- To study the approximation of $f \in L[0, \infty)$ by summability means of its Fourier-Laguerre series (Chapter 6).

Chapter 2

Approximation in Certain Subclasses of L^p-space Using Product Summability

2.1 Introduction

A good amount of research work has been done in the field of Fourier series approximation in the subclasses of L^p-space through the summability means (as filters). Product summability means are more general methods as the product transformation has a key role in signal theory in the form of double digital filter. To enhance the quality of digital filter, the summability matrices without monotone rows are more useful than the matrices with monotone rows.

Let $A \equiv (a_{n,m})$ and $B \equiv (b_{n,m})$ be two infinite lower triangular matrices of real numbers such that

$$A(\text{or } B) = \begin{cases} a_{n,m}(\text{or } b_{n,m}) \geq 0, & 0 \leq m \leq n \\ a_{n,m}(\text{or } b_{n,m}) = 0, & m > n, \end{cases}$$

$$\sum_{m=0}^{n} a_{n,m} = 1 \text{ and } \sum_{m=0}^{n} b_{n,m} = 1, \text{ where } n = 0, 1, 2, ...,$$

and let

$$A_{n,r} = \sum_{m=0}^{r} a_{n,m} \text{ and } B_{n,r} = \sum_{m=0}^{r} b_{n,m}.$$

When we superimpose the $B-$summability on $A-$summability, we have BA means of $\{s_k(f;x)\}$ defined by (see [59; 78; 79])

$$t_n^{BA}(f;x) = \sum_{m=0}^{n}\sum_{k=0}^{m} b_{n,m}a_{m,k}s_k(f;x), \quad n = 0,1,2,\dots . \tag{2.1.1}$$

Similarly, BA means of $\{\tilde{s}_k(f;x)\}$ defined by

$$\tilde{t}_n^{BA}(f;x) = \sum_{m=0}^{n}\sum_{k=0}^{m} b_{n,m}a_{m,k}\tilde{s}_k(f;x), \quad n = 0,1,2,\dots . \tag{2.1.2}$$

We write $(BA)_n(t)$ as

$$(BA)_n(t) = \frac{1}{2\pi}\sum_{m=0}^{n}\sum_{k=0}^{m} b_{n,m}a_{m,k}\frac{\sin{(k+1/2)t}}{\sin{(t/2)}}$$

and $(\widetilde{BA})_n(t)$ as

$$(\widetilde{BA})_n(t) = \frac{1}{2\pi}\sum_{m=0}^{n}\sum_{k=0}^{m} b_{n,m}a_{m,k}\frac{\cos{(k+1/2)t}}{\sin{(t/2)}}.$$

We also write $K_1 \ll K_2$ if $\exists$ a positive constant C (it may depend on some parameters) such that $K_1 \leq CK_2$.

2.2 Degree of Approximation in $L^p(\omega)_\beta$ and $L^p(\tilde{\omega})_\beta$-Classes

The product summability means of Fourier series have been considered in various directions, for example, Mittal [83, Theorem 1, p.437] has estimated the deviation $t_n^{BA}(f;.) - f(.)$ pointwise with lower triangular infinite matrix B defined by:

$$b_{n,m} = \begin{cases} \frac{1}{n+1}, & 0 \leq m \leq n \\ 0, & m > n. \end{cases}$$

This matrix corresponds to the Cesàro summability of order 1 and is denoted by C^1. He also discussed the (F_1)-effectiveness of C^1A method. Lenski and Szal [78, Theorem 2.1, p.1121] have extended the results of Mittal [83] to more general means

BA and proved their results in terms of modulus of continuity. They proved the following result:

$$\left|t_n^{BA}(f;x) - f(x)\right| \ll \sum_{m=0}^{n} b_{n,m} \left[\frac{1}{m+1} \sum_{k=0}^{m} \omega_x f(\pi/k+1)\right],$$

for every natural number n and all real x, where $\omega_x f(\delta) = \sup\limits_{0\leq t\leq\delta} \left|\frac{1}{t}\int_0^t \phi(x,u)du\right|$, known as the integral modulus of continuity of f.

Lenski and Szal [77] defined the class $L^p(\omega)_\beta$ and proved their results by using the sequence $\alpha_n = (a_{n,k})_{k=0}^n$ of Rest Bounded Variation ($RBVS$) or Head Bounded Variation ($HBVS$). They estimated the pointwise deviation as follows [77, Theorem 3, p.16] :

$$|T_{n,A}(f;x) - f(x)| = O_x\left((n+1)^{\beta+\frac{1}{p}+1}\, a_n\, \omega\left(\pi/n+1\right)\right),$$

where $a_n = \begin{cases} a_{n,0} & \text{for} \quad \{a_{n,k}\} \in RBVS \\ a_{n,n} & \text{for} \quad \{a_{n,k}\} \in HBVS. \end{cases}$

The authors have also obtained these type of deviations for $f \in L^p(\tilde{\omega})_\beta$-class [77, Theorems 1 and 2, p.16] as follows:

$$|\tilde{T}_{n,A}(f;x) - \tilde{f}(x)| = O_x\left((n+1)^{\beta+1+1/p} a_n \tilde{\omega}(\pi/(n+1))\right),$$

under assumption conditions on $\tilde{\omega}$. Kranz et al. [58] have also obtained such type of results. Later, Krasniqi [59] replaced the summability means by the product means of Fourier series and the result is as follows:

Theorem A [59, Theorem 3.1, p.4]: *Let $a_{n,p}$, $b_{p,k}$ satisfy the following conditions:*

$$a_{n,m} \geq 0, b_{m,k} \geq 0 \quad \text{and} \quad \sum_{m=0}^{n}\sum_{k=0}^{m} a_{n,m} b_{m,k} = 1,$$

$$\sum_{k=0}^{r-1} |b_{m,k} b_{m,k+1}| \leq K b_{m,r}, \quad 0 \leq r \leq m \quad \forall m.$$

Suppose $\omega(t)$ is such that

$$\int_u^\pi t^{-2}\omega(t)dt = O(H(u)) \quad (u \to +0),$$

where $H(u) \geq 0$ and

$$\int_0^t H(u)du = O(tH(t)) \quad (t \to +0).$$

Then

$$\|T_{n,AB}(f;x)-f(x)\|=O\left(\sum_{m=0}^{n}a_{n,m}b_{m,m}\times H\left(\sum_{m=0}^{n}a_{n,m}b_{m,m}\right)\right).$$

Recently, Krasniqi [61, Theorem 10, p.97] used the lower triangular infinite matrix $A\equiv(a_{n,k})$ with $a_{n,m}\leq\sum_{k=m}^{n}|\triangle a_{n,k}|$ and proved his result for the same class $L^p(\omega)_\beta$ as follows:

$$|T_{n,A}(f;x)-f(x)|=O_x\left((n+1)^{\beta+\frac{1}{p}+1}\sum_{k=0}^{n}|\Delta a_{n,k}|\,\omega\left(\pi/n+1\right)\right).$$

Similar results are also proved for $f\in L^p(\tilde{\omega})_\beta$. Clearly in these results, the error of approximation depends on p.

In the sequence of product means of Fourier series, Lenski and Szal [79] estimated the pointwise deviation of $\tilde{f}$, conjugate function of f, by general product means of its conjugate Fourier series in terms of modulus of continuity. Further, very recently Singh and Srivastava [129, Theorem 2.2, p.4] obtained the degree of approximation of functions belonging to weighted Lipschitz class by C^1A means of its Fourier series and the result is given as

$$\|t_n^{C^1A}(f;x)-f(x)\|_p=O((n+1)^\beta\omega(1/(n+1))),$$

where $\omega(t)$ is a positive non-decreasing function.

We note that deviation in this result is free from p.

Being motivated from these results, we study the earlier results further for f and its conjugate function $\tilde{f}$ with less assumption conditions on the product matrix. In our theorems, we obtain the deviations free from p and more sharper than the earlier results.

2.3 Main Results

Theorem 2.3.1. *Let $f\in L^p(\omega)_\beta$ with $0<\beta<1-\frac{1}{p}$, $p>1$, and the entries of the lower triangular matrices $A\equiv(a_{n,k})$ and $B\equiv(b_{n,k})$ satisfy the following conditions:*

$$b_{n,n}\ll\frac{1}{n+1},\ n\in\mathbb{N}_0, \tag{2.3.1}$$

$$|b_{n,m}a_{m,0} - b_{n,m+1}a_{m+1,1}| \ll \frac{b_{n,m}}{(m+1)^2} \ \ \textit{for} \ \ 0 \le m \le n-1 \tag{2.3.2}$$

and

$$\sum_{k=0}^{m-1} |(b_{n,m}a_{m,m-k} - b_{n,m+1}a_{m+1,m+1-k}) - (b_{n,m}a_{m,m-k-1} - b_{n,m+1}a_{m+1,m-k})|$$

$$\ll \frac{b_{n,m}}{(m+1)^2} \ \ \textit{for} \ \ 0 \le m \le n-1, \tag{2.3.3}$$

with $A_{n,n} = B_{n,n} = 1$ *for* $n = 0, 1, 2, \dots$. *Then the degree of approximation of* f *by* BA *means of its Fourier series is given by*

$$\left|t_n^{BA}(f;x) - f(x)\right| = O_x\left(\sum_{m=0}^{n} \frac{b_{n,m}}{m+1}(n+1)^{\beta+1}\omega(1/(n+1))\right),$$

provided that the positive non-decreasing function ω *satisfies the following conditions:*

$$\omega(t)/t \ \ \textit{is non-increasing function}, \tag{2.3.4}$$

$$\left\{\int_0^{\pi/(n+1)} \left(\frac{|\phi(x,t)|\sin^\beta(t/2)}{\omega(t)}\right)^p dt\right\}^{1/p} = O_x\left((n+1)^{-1/p}\right), \tag{2.3.5}$$

$$\left\{\int_{\pi/(n+1)}^{\pi} \left(t^{-\gamma}\frac{|\phi(x,t)|\sin^\beta(t/2)}{\omega(t)}\right)^p dt\right\}^{1/p} = O_x\left((n+1)^{\gamma-1/p}\right), \tag{2.3.6}$$

where γ *is an arbitrary number such that* $1/p < \gamma < \beta + 1/p$ *and* $p^{-1} + q^{-1} = 1$.

Theorem 2.3.2. *Let* f *be a* 2π*-periodic function belonging to the class* $L^p(\tilde{\omega})_\beta$, $0 < \beta < 1/p$, $p > 1$ *and the entries of the lower triangular matrices* $A \equiv (a_{n,k})$ *and* $B \equiv (b_{n,k})$ *satisfy the following conditions:*

$$b_{n,n} \ll \frac{1}{n+1}, \ n \in \mathbb{N}_0, \tag{2.3.7}$$

$$|b_{n,m}a_{m,m-l} - b_{n,m+1}a_{m+1,m+1-l}| \ll \frac{b_{n,m}}{(m+1)^2}, \ \ \textit{for} \ \ 0 \le l \le m \le n-1. \tag{2.3.8}$$

with $A_{n,n} = B_{n,n} = 1$ *for* $n = 0, 1, 2, \dots$. *Then the degree of approximation of* $\tilde{f}$, *conjugate of* f, *by* BA *means of its conjugate Fourier series is given by*

$$\left|\tilde{t}_n^{BA}(f;x) - \tilde{f}(x)\right| = O_x\left(\sum_{m=0}^{n} \frac{b_{n,m}}{m+1}(n+1)^{\beta+1}\tilde{\omega}(1/(n+1))\right),$$

provided that the positive non-decreasing function $\tilde{\omega}$ *satisfies the following conditions:*

$$\tilde{\omega}(t)/t^{\beta+1-\sigma} \text{ is non-decreasing for } \beta < \sigma < 1/p, \tag{2.3.9}$$

$$\left\{\int_0^{\pi/(n+1)} \left(\frac{t^{-\sigma}|\psi(x,t)|\sin^\beta(t/2)}{\tilde{\omega}(t)}\right)^p dt\right\}^{1/p} = O_x\left((n+1)^{\sigma-1/p}\right), \tag{2.3.10}$$

$$\left\{\int_{\pi/(n+1)}^{\pi} \left(\frac{t^{-\gamma}|\psi(x,t)|\sin^\beta(t/2)}{\tilde{\omega}(t)}\right)^p dt\right\}^{1/p} = O_x\left((n+1)^{\gamma-1/p}\right), \tag{2.3.11}$$

where γ *is an arbitrary number such that* $1/p < \gamma < \beta + 1/p$ *and* $p^{-1} + q^{-1} = 1$.

Note 1. Condition (2.3.4) implies that

$$\frac{\omega(\pi/(n+1))}{\pi/(n+1)} \le \frac{\omega(1/(n+1))}{1/(n+1)}, \text{ i.e., } \omega\left(\frac{\pi}{n+1}\right) = O\left(\omega\left(\frac{1}{n+1}\right)\right).$$

Similarly, using the condition (1.3.2), we have

$$\frac{\tilde{\omega}(\pi/(n+1))}{\pi/(n+1)} \le 2\frac{\tilde{\omega}(1/(n+1))}{1/(n+1)}, \text{ i.e., } \tilde{\omega}\left(\frac{\pi}{n+1}\right) = O\left(\tilde{\omega}\left(\frac{1}{n+1}\right)\right).$$

2.4 Lemmas

To prove our theorems, we need the following lemmas:

Lemma 2.4.1. *If the conditions* (2.3.2) *and* (2.3.3) *hold, then*

$$|b_{n,r}a_{r,r-l} - b_{n,r+1}a_{r+1,r+1-l}| \ll \frac{b_{n,r}}{(r+1)^2} \text{ for } 0 \le l \le r-1 \le n-2.$$

For the proof one can see [78, Lemma 3.2, p.1123].

Lemma 2.4.2. *If the matrices* A *and* B *satisfy the conditions* $A_{n,n} = B_{n,n} = 1$, *then*

$$|(BA)_n(t)| = O(n+1) \text{ for } 0 < t \le \pi/(n+1).$$

Proof. Using $1/\sin(t/2) = O(\pi/t)$ and $0 \le sin(nt) \le nt$ for $0 < t \le \pi/(n+1)$, we have

$$
\begin{aligned}
|(BA)_n(t)| &= \left|\frac{1}{2\pi}\sum_{m=0}^{n}\sum_{k=0}^{m} b_{n,m}a_{m,k}\frac{\sin{(k+1/2)t}}{\sin(t/2)}\right| \\
&\leq \frac{1}{2\pi}\sum_{m=0}^{n}\sum_{k=0}^{m} b_{n,m}a_{m,k}\left|\frac{\sin{(k+1/2)t}}{\sin(t/2)}\right| \\
&= O\Bigg(\sum_{m=0}^{n}\sum_{k=0}^{m} b_{n,m}a_{m,k}\frac{(k+1)t}{t}\Bigg) \\
&= O\left((n+1)\sum_{m=0}^{n} b_{n,m}\left(\sum_{k=0}^{m} a_{m,k}\right)\right) \\
&= O\left((n+1)\sum_{m=0}^{n} b_{n,m}A_{m,m}\right) \\
&= O\left((n+1)B_{n,n}\right) = O(n+1),
\end{aligned}
$$

in view of $A_{n,n} = B_{n,n} = 1$. □

Lemma 2.4.3. *If the matrices A and B satisfy the conditions* (2.3.1) – (2.3.3) *of Theorem* 2.3.1*, then*

$$
|(BA)_n(t)| = O\left(\frac{1}{t^2}\left(\sum_{m=0}^{n}\frac{b_{n,m}}{m+1} + \frac{1}{n+1}\right)\right) \text{ for } \pi/(n+1) < t \leq \pi.
$$

Proof. Using $1/\sin(t/2) = O\left(\pi/t\right)$ for $\pi/(n+1) < t \leq \pi$,

$$
\begin{aligned}
|(BA)_n(t)| &= \left|\frac{1}{2\pi}\sum_{m=0}^{n}\sum_{k=0}^{m} b_{n,m}a_{m,k}\frac{\sin{(k+1/2)t}}{\sin(t/2)}\right| \\
&= O\left(\frac{1}{t}\right)\left|\sum_{m=0}^{n}\sum_{k=0}^{m} b_{n,m}a_{m,k}\sin{(k+1/2)t}\right|.
\end{aligned}
$$

Now, using Abel's transformation after changing the order of summation, we have

$$
\begin{aligned}
&\left|\sum_{m=0}^{n}\sum_{k=0}^{m} b_{n,m}a_{m,k}\sin(k+1/2)t\right| \\
&= \left|\sum_{m=0}^{n}\sum_{k=0}^{m} b_{n,m}a_{m,m-k}\sin\left(m-k+1/2\right)t\right| \\
&= \left|\sum_{k=0}^{n}\left[\sum_{m=k}^{n-1}(b_{n,m}a_{m,m-k}-b_{n,m+1}a_{m+1,m+1-k})\sum_{l=m}^{k}\sin\left(l-k+1/2\right)t\right.\right. \\
&\qquad \left.\left.+b_{n,n}a_{n,n-k}\sum_{l=k}^{n}\sin\left(l-k+1/2\right)t\right]\right| \\
&= O\left(\frac{1}{t}\right)\left(\sum_{m=0}^{n-1}\left[\sum_{k=0}^{m}\left|b_{n,m}a_{m,m-k}-b_{n,m+1}a_{m+1,m+1-k}\right|\right]+\sum_{k=0}^{n} b_{n,n}a_{n,n-k}\right) \\
&= O\left(\frac{1}{t}\right)\left[\sum_{m=0}^{n-1}\sum_{k=0}^{m-1}\left|b_{n,m}a_{m,m-k}-b_{n,m+1}a_{m+1,m+1-k}\right|+b_{n,n}\right. \\
&\qquad \left.+\sum_{m=0}^{n-1}\left|b_{n,m}a_{m,0}-b_{n,m+1}a_{m+1,1}\right|\right] \\
&= O\left(\frac{1}{t}\right)\left[\sum_{m=0}^{n-1} m.\frac{b_{n,m}}{(m+1)^2}+b_{n,n}+\sum_{m=0}^{n-1}\frac{b_{n,m}}{(m+1)^2}\right] \\
&= O\left(\frac{1}{t}\right)\left[\sum_{m=0}^{n}\frac{b_{n,m}}{(m+1)}+\frac{1}{(n+1)}\right],
\end{aligned}
$$

in view of Lemma 2.4.1, conditions (2.3.1) and (2.3.2); and $A_{n,n}=1$.
Hence

$$
|(BA)_n(t)| = O\left(\frac{1}{t^2}\left(\sum_{m=0}^{n}\frac{b_{n,m}}{m+1}+\frac{1}{n+1}\right)\right).
$$

□

Lemma 2.4.4. *If the matrices A and B satisfy the conditions $A_{n,n}=1$ and $B_{n,n}=1$, then*

$$
|(\widetilde{BA})_n(t)| = O\left(1/t\right) \;\; \textit{for} \;\; 0<t\leq \pi/(n+1).
$$

Proof. Using $1/\sin(t/2) = O(\pi/t)$ for $0 < t \leq \pi/(n+1)$, we have

$$
\begin{aligned}
|(\widetilde{BA})_n(t)| &= \left| \frac{1}{2\pi} \sum_{m=0}^{n} \sum_{k=0}^{m} b_{n,m} a_{m,k} \frac{\cos(k+1/2)t}{\sin(t/2)} \right| \\
&\leq \frac{1}{2\pi} \sum_{m=0}^{n} \sum_{k=0}^{m} b_{n,m} a_{m,k} \left| \frac{\cos(k+1/2)t}{\sin(t/2)} \right| \\
&= O\left(\frac{1}{t} \sum_{m=0}^{n} b_{n,m} A_{m,m} \right) \\
&= O\left(\frac{1}{t} B_{n,n} \right) = O(1/t),
\end{aligned}
$$

in view of $A_{n,n} = B_{n,n} = 1$. □

Lemma 2.4.5. *If the matrices A and B satisfy the conditions* (2.3.7) *and* (2.3.8) *of Theorem* 2.3.2*, then*

$$
|(\widetilde{BA})_n(t)| = O\left(\frac{1}{t^2} \left(\sum_{m=0}^{n} \frac{b_{n,m}}{m+1} + \frac{1}{n+1} \right) \right) \quad \textit{for } \pi/(n+1) < t \leq \pi.
$$

Proof. Using $1/\sin(t/2) = O(\pi/t)$ for $\pi/(n+1) < t \leq \pi$,

$$
\begin{aligned}
|(\widetilde{BA})_n(t)| &= \left| \frac{1}{2\pi} \sum_{m=0}^{n} \sum_{k=0}^{m} b_{n,m} a_{m,k} \frac{\cos(k+1/2)t}{\sin(t/2)} \right| \\
&= O\left(\frac{1}{t} \right) \left| \sum_{m=0}^{n} \sum_{k=0}^{m} b_{n,m} a_{m,k} \cos(k+1/2)t \right|.
\end{aligned}
$$

Now, using Abel's transformation after changing the order of summation, we have

$$\begin{aligned}
&\left|\sum_{m=0}^{n}\sum_{k=0}^{m} b_{n,m}a_{m,k}\cos(k+1/2)t\right| \\
&= \left|\sum_{m=0}^{n}\sum_{k=0}^{m} b_{n,m}a_{m,m-k}\cos\left(m-k+1/2\right)t\right| \\
&= \left|\sum_{k=0}^{n}\left[\sum_{m=k}^{n-1}(b_{n,m}a_{m,m-k}-b_{n,m+1}a_{m+1,m+1-k})\sum_{l=m}^{k}\cos\left(l-k+1/2\right)t\right.\right. \\
&\quad \left.\left.+b_{n,n}a_{n,n-k}\sum_{l=k}^{n}\cos\left(l-k+1/2\right)t\right]\right| \\
&= O\left(\frac{1}{t}\right)\left(\sum_{m=0}^{n-1}\left[\sum_{k=0}^{m}|b_{n,m}a_{m,m-k}-b_{n,m+1}a_{m+1,m+1-k}|\right]+\sum_{k=0}^{n} b_{n,n}a_{n,n-k}\right) \\
&= O\left(\frac{1}{t}\right)\left[\sum_{m=0}^{n-1}\sum_{k=0}^{m-1}|b_{n,m}a_{m,m-k}-b_{n,m+1}a_{m+1,m+1-k}|+b_{n,n}\right. \\
&\quad \left.+\sum_{m=0}^{n-1}|b_{n,m}a_{m,0}-b_{n,m+1}a_{m+1,1}|\right] \\
&= O\left(\frac{1}{t}\right)\left[\sum_{m=0}^{n-1} m\frac{b_{n,m}}{(m+1)^2}+b_{n,n}+\sum_{m=0}^{n-1}\frac{b_{n,m}}{(m+1)^2}\right] \\
&= O\left(\frac{1}{t}\right)\left[\sum_{m=0}^{n}\frac{b_{n,m}}{(m+1)}+\frac{1}{(n+1)}\right],
\end{aligned}$$

in view of conditions (2.3.7) and (2.3.8); and $A_{n,n}=1$.
Hence

$$|(\widetilde{BA})_n(t)| \;=\; O\left(\frac{1}{t^2}\left(\sum_{m=0}^{n}\frac{b_{n,m}}{m+1}+\frac{1}{n+1}\right)\right).$$

□

Note 2. Conditions (2.3.2) and (2.3.3) can of Theorem 2.3.1 be replaced by a single condition (2.3.8) of Theorem 2.3.2 in the light of Lemma 2.4.1.

2.5 Proof of Theorem 2.3.1

By using the integral representation of $s_k(f;x)$, we have

$$s_k(f;x)-f(x)=\frac{1}{2\pi}\int_0^{\pi}\phi(x,t)\frac{\sin\left(k+1/2\right)t}{\sin(t/2)}dt.$$

From (2.1.1), we have

$$
\begin{aligned}
t_n^{BA}(f;x)-f(x) &= \sum_{m=0}^{n}\sum_{k=0}^{m} b_{n,m}a_{m,k}(s_k(f;x)-f(x))\\
&= \sum_{m=0}^{n}\sum_{k=0}^{m} b_{n,m}a_{m,k}\left(\frac{1}{2\pi}\int_0^{\pi}\phi(x,t)\frac{\sin{(k+1/2)t}}{\sin(t/2)}dt\right)\\
&= \frac{1}{2\pi}\int_0^{\pi}\phi(x,t)\sum_{m=0}^{n}\sum_{k=0}^{m} b_{n,m}a_{m,k}\frac{\sin{(k+1/2)t}}{\sin(t/2)}dt\\
&= \int_0^{\pi}\phi(x,t)(BA)_n(t)dt\\
&= \int_0^{\pi/(n+1)}\phi(x,t)(BA)_n(t)dt+\int_{\pi/(n+1)}^{\pi}\phi(x,t)(BA)_n(t)dt\\
&= I_1+I_2. \qquad (2.5.1)
\end{aligned}
$$

Now, using Lemma 2.4.2, $1/\sin(t/2)=O\left(\pi/t\right)$ for $0<t\leq\pi/(n+1)$ and Hölder's inequality, we have

$$
\begin{aligned}
|I_1| &\leq \int_0^{\pi/(n+1)}|\phi(x,t)(BA)_n(t)|\,dt=\lim_{\varepsilon\to 0}\int_{\varepsilon}^{\pi/(n+1)}|\phi(x,t)||(BA)_n(t)|dt\\
&= O\left(\lim_{\varepsilon\to 0}\int_{\varepsilon}^{\pi/(n+1)}\frac{|\phi(x,t)|\sin^{\beta}(t/2)}{\omega(t)}\frac{(n+1)\omega(t)}{\sin^{\beta}(t/2)}dt\right)\\
&= O\Bigg[(n+1)\left\{\int_0^{\pi/(n+1)}\left(\frac{|\phi(x,t)|\sin^{\beta}(t/2)}{\omega(t)}\right)^p dt\right\}^{1/p}\\
&\qquad \times\left\{\lim_{\varepsilon\to 0}\int_{\varepsilon}^{\pi/(n+1)}\left(\frac{\omega(t)}{\sin^{\beta}(t/2)}\right)^q dt\right\}^{1/q}\Bigg]\\
&= O_x\left[(n+1)^{1-1/p}\omega(\pi/(n+1))\left\{\lim_{\varepsilon\to 0}\int_{\varepsilon}^{\pi/(n+1)}t^{-q\beta}dt\right\}^{1/q}\right]\\
&= O_x\left[(n+1)^{1-1/p}\omega(\pi/(n+1))(n+1)^{\beta-1/q}\right]\\
&= O_x\big(\omega(\pi/(n+1))(n+1)^{\beta+1-1/p-1/q}\big)\\
&= O_x\left(\omega(\pi/(n+1))(n+1)^{\beta}\right), \qquad (2.5.2)
\end{aligned}
$$

in view of condition (2.3.5), mean value theorem for integrals, $0<\beta<1-1/p$ and $p^{-1}+q^{-1}=1$.

Again, using Lemma 2.4.3, $1/\sin(t/2)=O\left(\pi/t\right)$ for $\pi/(n+1)<t\leq\pi$ and Hölder's

inequality, we have

$$\begin{aligned}|I_2| &\leq \int_{\pi/(n+1)}^{\pi} |\phi(x,t)(BA)_n(t)|\,dt \\ &= \int_{\pi/(n+1)}^{\pi} \frac{|\phi(x,t)|}{t^2(n+1)}dt + \int_{\pi/(n+1)}^{\pi} \frac{|\phi(x,t)|}{t^2}\sum_{m=0}^{n-1}\frac{b_{n,m}}{(m+1)}dt \\ &= I_{21} + I_{22}. \end{aligned} \tag{2.5.3}$$

$$\begin{aligned}I_{21} &= O\left[\int_{\pi/(n+1)}^{\pi} \frac{|\phi(x,t)|}{t^2(n+1)}dt\right] \\ &= O\left[(n+1)^{-1}\int_{\pi/(n+1)}^{\pi} \frac{t^{-\gamma}|\phi(x,t)|\sin^\beta(t/2)}{\omega(t)}\frac{\omega(t)}{t^{-\gamma+2}\sin^\beta(t/2)}dt\right] \\ &= O\Bigg[(n+1)^{-1}\left\{\int_{\pi/(n+1)}^{\pi}\left(\frac{t^{-\gamma}|\phi(x,t)|\sin^\beta(t/2)}{\omega(t)}\right)^p dt\right\}^{1/p} \\ &\quad \times\left\{\int_{\pi/(n+1)}^{\pi}\left(\frac{\omega(t)}{t^{-\gamma+2}\sin^\beta(t/2)}\right)^q dt\right\}^{1/q}\Bigg] \\ &= O_x\left[(n+1)^{\gamma-1-1/p}\left\{\int_{\pi/(n+1)}^{\pi}\left(\frac{\omega(t)}{t}t^{-(-\gamma+\beta+1)}\right)^q dt\right\}^{1/q}\right] \\ &= O_x\left[(n+1)^{\gamma-1-1/p}\omega(\pi/(n+1))(n+1)\left\{\int_{\pi/(n+1)}^{\pi} t^{-q(-\gamma+\beta+1)}dt\right\}^{1/q}\right] \\ &= O_x\left[(n+1)^{\gamma-1/p}\omega(\pi/(n+1))(n+1)^{-\gamma+\beta+1-1/q}\right] \\ &= O_x\left(\omega(\pi/(n+1))(n+1)^\beta\right), \end{aligned} \tag{2.5.4}$$

in view of conditions (2.3.4) and (2.3.6), mean value theorem for integrals, $1/p < \gamma < \beta + 1/p$ and $p^{-1} + q^{-1} = 1$.

$$\begin{aligned}I_{22} &= O\Bigg[\sum_{m=0}^{n}\frac{b_{n,m}}{(m+1)}\left\{\int_{\pi/(n+1)}^{\pi}\left(\frac{t^{-\gamma}|\phi(x,t)|\sin^\beta(t/2)}{\omega(t)}\right)^p dt\right\}^{1/p} \\ &\quad \times\left\{\int_{\pi/(n+1)}^{\pi}\left(\frac{\omega(t)}{t^{-\gamma+2}\sin^\beta(t/2)}\right)^q dt\right\}^{1/q}\Bigg] \\ &= O_x\left[\sum_{m=0}^{n}\frac{b_{n,m}}{(m+1)}(n+1)^{\gamma-1/p}\left\{\int_{\pi/(n+1)}^{\pi}\left(\frac{\omega(t)}{t}t^{-(-\gamma+\beta+1)}\right)^q dt\right\}^{1/q}\right] \\ &= O_x\left[\sum_{m=0}^{n}\frac{b_{n,m}}{(m+1)}(n+1)^{\gamma-1/p}\omega(\pi/(n+1))(n+1)(n+1)^{-\gamma+\beta+1-1/q}\right] \\ &= O_x\left(\sum_{m=0}^{n}\frac{b_{n,m}}{(m+1)}\omega(\pi/(n+1))(n+1)^{\beta+1}\right), \end{aligned} \tag{2.5.5}$$

in view of conditions (2.3.4) and (2.3.6), mean value theorem for integrals, $1/p < \gamma < \beta + 1/p$ and $p^{-1} + q^{-1} = 1$.

Further, we have

$$(n+1)^{\beta}\omega(\pi/(n+1)) + \sum_{m=0}^{n} \frac{b_{n,m}}{m+1}(n+1)^{\beta+1}\omega(\pi/(n+1)$$

$$= \omega(\pi/(n+1))(n+1)^{\beta}\left[1 + \sum_{m=0}^{n} \frac{b_{n,m}}{m+1}(n+1)\right]$$
$$\geq 2\omega(\pi/(n+1))(n+1)^{\beta},$$

that is

$$(n+1)^{\beta}\omega(\pi/(n+1)) = O\left(\sum_{m=0}^{n} \frac{b_{n,m}}{m+1}(n+1)^{\beta+1}\omega(\pi/(n+1)\right). \qquad (2.5.6)$$

Collecting (2.5.1) - (2.5.6), we have

$$|t_n^{BA}(f;x) - f(x)| = O_x\left(\sum_{m=0}^{n} \frac{b_{n,m}}{m+1}(n+1)^{\beta+1}\omega(1/(n+1))\right), \qquad (2.5.7)$$

in view of Note 1.

Hence the proof of Theorem 2.3.1 is completed.

2.6 Proof of Theorem 2.3.2

By using the integral representation of $\tilde{s}_n(f;x)$, we have

$$\tilde{s}_n(f;x) - \tilde{f}(x) = \frac{1}{2\pi}\int_0^{\pi} \psi(x,t)\frac{\cos(n+1/2)t}{\sin(t/2)}dt.$$

Therefore from (2.1.2), we have

$$\begin{aligned}\tilde{t}_n^{BA}(f;x) - \tilde{f}(x) &= \int_0^{\pi} \psi(x,t)(\widetilde{BA})_n(t)dt \\ &= \left(\int_0^{\pi/(n+1)} + \int_{\pi/(n+1)}^{\pi}\right)\left(\psi(x,t)(\widetilde{BA})_n(t)dt\right) \\ &= I_1^{'} + I_2^{'}. \end{aligned} \qquad (2.6.1)$$

Now, using Lemma 2.4.4, $1/\sin(t/2) = O\left(\pi/t\right)$ for $0 < t \leq \pi/(n+1)$ and Hölder's inequality, we have

$$\begin{aligned}
|I_1'| &\leq \int_0^{\pi/(n+1)} \left|\psi(x,t)(\widetilde{BA})_n(t)\right| dt = \lim_{\varepsilon\to 0}\int_\varepsilon^{\pi/(n+1)} |\psi(x,t)||(\widetilde{BA})_n(t)|dt \\
&= O\left(\lim_{\varepsilon\to 0}\int_\varepsilon^{\pi/(n+1)} \frac{t^{-\sigma}|\psi(x,t)|\sin^\beta(t/2)}{\tilde{\omega}(t)}\frac{\tilde{\omega}(t)}{t^{1-\sigma}\sin^\beta(t/2)}dt\right) \\
&= O\Bigg[\left\{\int_0^{\pi/(n+1)}\left(\frac{t^{-\sigma}|\psi(x,t)|\sin^\beta(t/2)}{\tilde{\omega}(t)}\right)^p dt\right\}^{1/p} \\
&\quad \times\left\{\lim_{\varepsilon\to 0}\int_\varepsilon^{\pi/(n+1)}\left(\frac{\tilde{\omega}(t)}{t^{1-\sigma}\sin^\beta(t/2)}\right)^q dt\right\}^{1/q}\Bigg] \\
&= O_x\left[(n+1)^{\sigma-1/p}\times\left\{\lim_{\varepsilon\to 0}\int_\varepsilon^{\pi/(n+1)}\left(\frac{\tilde{\omega}(t)}{t^{\beta+1-\sigma}}\right)^q dt\right\}^{1/q}\right] \\
&= O_x\left[(n+1)^{\sigma-1/p}\tilde{\omega}(\pi/(n+1))(n+1)^{\beta+1-1/q-\sigma}\right] \\
&= O_x\left(\tilde{\omega}(\pi/(n+1))(n+1)^\beta\right), \qquad (2.6.2)
\end{aligned}$$

in view of conditions (2.3.9) and (2.3.10), mean value theorem for integrals and $p^{-1}+q^{-1}=1$.

Again, using Lemma 2.4.5, $1/\sin(t/2) = O\left(\pi/t\right)$ for $\pi/(n+1) < t \leq \pi$ and Hölder's inequality, we have

$$\begin{aligned}
|I_2'| &\leq \int_{\pi/(n+1)}^{\pi} \left|\psi(x,t)(\widetilde{BA})_n(t)\right| dt \\
&= \int_{\pi/(n+1)}^{\pi}\frac{|\psi(x,t)|}{t^2(n+1)}dt + \int_{\pi/(n+1)}^{\pi}\frac{|\psi(x,t)|}{t^2}\sum_{m=0}^{n-1}\frac{b_{n,m}}{(m+1)}dt \\
&= I_{21}' + I_{22}'. \qquad (2.6.3)
\end{aligned}$$

Now,

$$\begin{aligned}
I_{21}' &= O\left[\int_{\pi/(n+1)}^{\pi}\frac{|\psi(x,t)|}{t^2(n+1)}dt\right] \\
&= O\left[(n+1)^{-1}\int_{\pi/(n+1)}^{\pi}\frac{t^{-\gamma}|\psi(x,t)|\sin^\beta(t/2)}{\tilde{\omega}(t)}\frac{\tilde{\omega}(t)}{t^{-\gamma+2}\sin^\beta(t/2)}dt\right] \\
&= O\Bigg[(n+1)^{-1}\left\{\int_{\pi/(n+1)}^{\pi}\left(\frac{t^{-\gamma}|\psi(x,t)|\sin^\beta(t/2)}{\tilde{\omega}(t)}\right)^p dt\right\}^{1/p} \\
&\quad \times\left\{\int_{\pi/(n+1)}^{\pi}\left(\frac{\tilde{\omega}(t)}{t^{-\gamma+2}\sin^\beta(t/2)}\right)^q dt\right\}^{1/q}\Bigg]
\end{aligned}$$

$$\begin{aligned}
&= O_x\left[(n+1)^{\gamma-1-1/p}\left\{\int_{\pi/(n+1)}^{\pi}\left(\frac{\tilde{\omega}(t)}{t}t^{-(-\gamma+\beta+1)}\right)^q dt\right\}^{1/q}\right]\\
&= O_x\left[(n+1)^{\gamma-1-1/p}\tilde{\omega}(\pi/(n+1))(n+1)\left\{\int_{\pi/(n+1)}^{\pi} t^{-q(-\gamma+\beta+1)}dt\right\}^{1/q}\right]\\
&= O_x\left[(n+1)^{\gamma-1/p}\tilde{\omega}(\pi/(n+1))(n+1)^{-\gamma+\beta+1-1/q}\right]\\
&= O_x\left(\tilde{\omega}(\pi/(n+1))(n+1)^{\beta}\right), \qquad (2.6.4)
\end{aligned}$$

in view of conditions (1.3.2) and (2.3.11), mean value theorem for integrals, $1/p < \gamma < \beta + 1/p$ and $p^{-1} + q^{-1} = 1$.

Similarly,

$$\begin{aligned}
I'_{22} &= O\left[\sum_{m=0}^{n}\frac{b_{n,m}}{(m+1)}\left\{\int_{\pi/(n+1)}^{\pi}\left(\frac{t^{-\gamma}|\psi(x,t)|\sin^{\beta}(t/2)}{\tilde{\omega}(t)}\right)^p dt\right\}^{1/p}\right.\\
&\quad\left.\times\left\{\int_{\pi/(n+1)}^{\pi}\left(\frac{\tilde{\omega}(t)}{t^{-\gamma+2}\sin^{\beta}(t/2)}\right)^q dt\right\}^{1/q}\right]\\
&= O_x\left[\sum_{m=0}^{n}\frac{b_{n,m}}{(m+1)}(n+1)^{\gamma-1/p}\left\{\int_{\pi/(n+1)}^{\pi}\left(\frac{\tilde{\omega}(t)}{t}t^{-(-\gamma+\beta+1)}\right)^q dt\right\}^{1/q}\right]\\
&= O_x\left[\sum_{m=0}^{n}\frac{b_{n,m}}{(m+1)}(n+1)^{\gamma-1/p}\tilde{\omega}(\pi/(n+1))(n+1)(n+1)^{-\gamma+\beta+1-1/q}\right]\\
&= O_x\left(\sum_{m=0}^{n}\frac{b_{n,m}}{(m+1)}\tilde{\omega}(\pi/(n+1))(n+1)^{\beta+1}\right), \qquad (2.6.5)
\end{aligned}$$

in view of conditions (1.3.2) and (2.3.11), mean value theorem for integrals, $1/p < \gamma < \beta + 1/p$ and $p^{-1} + q^{-1} = 1$.

Further, we have

$$\begin{aligned}
&(n+1)^{\beta}\tilde{\omega}(\pi/(n+1)) + \sum_{m=0}^{n}\frac{b_{n,m}}{m+1}(n+1)^{\beta+1}\tilde{\omega}(\pi/(n+1)\\
&\qquad = \tilde{\omega}(\pi/(n+1))(n+1)^{\beta}\left[1+\sum_{m=0}^{n}\frac{b_{n,m}}{m+1}(n+1)\right]\\
&\qquad \geq 2\tilde{\omega}(\pi/(n+1))(n+1)^{\beta},
\end{aligned}$$

that is

$$(n+1)^{\beta}\tilde{\omega}(\pi/(n+1)) = O\left(\sum_{m=0}^{n}\frac{b_{n,m}}{m+1}(n+1)^{\beta+1}\tilde{\omega}(\pi/(n+1)\right). \qquad (2.6.6)$$

Collecting (2.6.1) - (2.6.6), we have

$$|\tilde{t}_n^{BA}(f;x)-\tilde{f}(x)| = O_x\left(\sum_{m=0}^{n}\frac{b_{n,m}}{m+1}(n+1)^{\beta+1}\tilde{\omega}(1/(n+1))\right), \tag{2.6.7}$$

in view of Note 1.

Hence the proof of Theorem 2.3.2 is completed.

Remark 2.6.1. In the proof of above theorems, we have used Hölder's inequality for $p>1$. Therefore, the proof is not applicable for $p=1$. Thus, for $p=1$, we have the following theorems:

Theorem 2.6.1. *Let $f \in L^1(\omega)_\beta$ with $0<\beta<1$ and the entries of the lower triangular matrices A and B satisfy the conditions* (2.3.1) − (2.3.3) *of Theorem* 2.3.1 *with $A_{n,n}=B_{n,n}=1$ for $n=0,1,2,\ldots$. Then the degree of approximation of f by BA means of its Fourier series is given by*

$$\left|t_n^{BA}(f;x)-f(x)\right| = O_x\left(\sum_{m=0}^{n}\frac{b_{n,m}}{m+1}(n+1)^{\beta+1}\omega(1/(n+1))\right),$$

provided that the positive non-decreasing function ω satisfies the following condition:

$$\omega(t)/t^\beta \;\; \textit{is non-decreasing function} \tag{2.6.8}$$

with the conditions (2.3.4) − (2.3.6) *of Theorem* 2.3.1 *for $p=1$ and $1<\gamma<\beta+1$.*

Theorem 2.6.2. *Let f be a 2π-periodic function belonging to the class $L^1(\tilde{\omega})_\beta$ and the entries of the lower triangular matrices $A \equiv (a_{n,k})$ and $B \equiv (b_{n,k})$ satisfy the conditions* (2.3.7) *and* (2.3.8) *of Theorem* 2.3.2 *with $A_{n,n}=B_{n,n}=1$ for $n=0,1,2,\ldots$. Then the degree of approximation of $\tilde{f}$, conjugate of f, by BA means of its conjugate Fourier series is given by*

$$\left|\tilde{t}_n^{BA}(f;x)-\tilde{f}(x)\right| = O_x\left(\sum_{m=0}^{n}\frac{b_{n,m}}{m+1}(n+1)^{\beta+1}\tilde{\omega}(1/(n+1))\right),$$

provided that the positive non-decreasing function $\tilde{\omega}$, satisfies the conditions (2.3.9) − (2.3.11) *of Theorem* 2.3.2 *for $p=1$, $\beta<\sigma<1$ and $1<\gamma<\beta+1$.*

The work of Theorems 2.3.1 and 2.6.1 has been published in Acta Comment. Univ. Tartu. Math. **20**(2016), no. 1, 23 − 34.

The work of Theorems 2.3.2 and 2.6.2 has been published in The Journal of Analysis(Springer Publication) (2017), 1 − 13.

2.7 Proof of Theorem 2.6.1

Following the proof of Theorem 2.3.1 and Hölder's inequality for $p = 1$, $q = \infty$, we have

$$\begin{aligned} |I_1| &= O\Big[(n+1)\int_0^{\pi/(n+1)} \frac{|\phi(x,t)|\sin^\beta(t/2)}{\omega(t)}dt \\ &\quad \times \operatorname{ess}\sup_{0<t\le\pi/(n+1)}\left|\frac{\omega(t)}{\sin^\beta(t/2)}\right|\Big] \\ &= O_x\Big[(n+1)(n+1)^{-1}\operatorname{ess}\sup_{0<t\le\pi/(n+1)}\left|\frac{\omega(t)}{t^\beta}\right|\Big] \\ &= O_x\left(\omega(\pi/(n+1))(n+1)^\beta\right), \end{aligned} \tag{2.7.1}$$

in view of conditions (2.6.8) and (2.3.5) for $p = 1$.

Similarly,

$$\begin{aligned} I_{21} &= O\Big[(n+1)^{-1}\int_{\pi/(n+1)}^{\pi} \frac{t^{-\gamma}|\phi(x,t)|\sin^\beta(t/2)}{\omega(t)}dt \\ &\quad \times \operatorname{ess}\sup_{\pi/(n+1)<t\le\pi}\left|\frac{\omega(t)}{t^{-\gamma+2}\sin^\beta(t/2)}\right|\Big] \\ &= O_x\left[(n+1)^{\gamma-1-1}\omega\left(\frac{\pi}{n+1}\right)\left(\frac{(n+1)^{2+\beta-\gamma}}{\pi^{2+\beta-\gamma}}\right)\right] \\ &= O_x\left(\omega(\pi/(n+1))(n+1)^\beta\right), \end{aligned} \tag{2.7.2}$$

in view of decreasing nature of $\omega(t)/t^{\beta+2-\gamma}$ and condition (2.3.6) for $p = 1$.

$$\begin{aligned} I_{22} &= O\Big[\sum_{m=0}^{n}\frac{b_{n,m}}{(m+1)}\int_{\pi/(n+1)}^{\pi} \frac{t^{-\gamma}|\phi(x,t)|\sin^\beta(t/2)}{\omega(t)}dt \\ &\quad \times \operatorname{ess}\sup_{\pi/(n+1)<t\le\pi}\left|\frac{\omega(t)}{t^{-\gamma+2}\sin^\beta(t/2)}\right|\Big] \\ &= O_x\left[\sum_{m=0}^{n}\frac{b_{n,m}}{(m+1)}(n+1)^{\gamma-1}\omega\left(\frac{\pi}{n+1}\right)\left(\frac{(n+1)^{2+\beta-\gamma}}{\pi^{2+\beta-\gamma}}\right)\right] \\ &= O_x\left(\sum_{m=0}^{n}\frac{b_{n,m}}{(m+1)}\omega(\pi/(n+1))(n+1)^{\beta+1}\right), \end{aligned} \tag{2.7.3}$$

in view of decreasing nature of $\omega(t)/t^{\beta+2-\gamma}$ and condition (2.3.6) for $p = 1$.

Collecting (2.7.1) - (2.7.3), we have

$$\left|t_n^{BA}(f;x) - f(x)\right| = O_x\left(\sum_{m=0}^{n}\frac{b_{n,m}}{m+1}(n+1)^{\beta+1}\omega(1/(n+1))\right),$$

in view of Note 1.

Hence the proof of Theorem 2.6.1 is completed.

2.8 Proof of Theorem 2.6.2

Following the proof of Theorem 2.3.2, by Hölder's inequality for $p = 1,\ q = \infty$, we have

$$\begin{aligned}
|I_1^{'}| &= O\Bigg[\left\{\int_0^{\pi/(n+1)} \left(\frac{t^{-\sigma}|\psi(x,t)|\sin^{\beta}(t/2)}{\tilde{\omega}(t)}\right) dt\right\} \\
&\quad \times \text{ess} \sup_{0<t\leq \pi/(n+1)} \left|\frac{\tilde{\omega}(t)}{t^{1-\sigma}\sin^{\beta}(t/2)}\right|\Bigg] \\
&= O_x\left[(n+1)^{\sigma-1}\tilde{\omega}(\pi/(n+1))(n+1)^{\beta+1-\sigma}\right] \\
&= O_x\left(\tilde{\omega}(\pi/(n+1))(n+1)^{\beta}\right), \qquad (2.8.1)
\end{aligned}$$

in view of conditions (2.3.9) and (2.3.10) for $p = 1$.

$$\begin{aligned}
I_{21}^{'} &= O\Bigg[(n+1)^{-1}\left\{\int_{\pi/(n+1)}^{\pi} \left(\frac{t^{-\gamma}|\psi(x,t)|\sin^{\beta}(t/2)}{\tilde{\omega}(t)}\right) dt\right\} \\
&\quad \times \text{ ess} \sup_{\pi/(n+1)<t\leq \pi} \left|\frac{\tilde{\omega}(t)}{t^{-\gamma+2}\sin^{\beta}(t/2)}\right|\Bigg] \\
&= O_x\left[(n+1)^{\gamma-2}\tilde{\omega}(\pi/(n+1))(n+1)^{-\gamma+\beta+2}\right] \\
&= O_x\left(\tilde{\omega}(\pi/(n+1))(n+1)^{\beta}\right), \qquad (2.8.2)
\end{aligned}$$

in view of condition (1.3.2) and (2.3.11) for $p = 1$, mean value theorem for integrals and $1 < \gamma < \beta + 1$.

Similarly,

$$\begin{aligned}
I_{22}^{'} &= O\Bigg[\sum_{m=0}^{n} \frac{b_{n,m}}{(m+1)} \int_{\pi/(n+1)}^{\pi} \left(\frac{t^{-\gamma}|\psi(x,t)|\sin^{\beta}(t/2)}{\tilde{\omega}(t)}\right) dt \\
&\quad \times \text{ ess} \sup_{\pi/(n+1)<t\leq \pi} \left|\frac{\tilde{\omega}(t)}{t^{-\gamma+2}\sin^{\beta}(t/2)}\right|\Bigg] \\
&= O_x\Bigg[\sum_{m=0}^{n} \frac{b_{n,m}}{(m+1)}(n+1)^{\gamma-1}\tilde{\omega}(\pi/(n+1))(n+1)(n+1)^{-\gamma+\beta+1}\Bigg] \\
&= O_x\left(\sum_{m=0}^{n} \frac{b_{n,m}}{(m+1)}\tilde{\omega}(\pi/(n+1))(n+1)^{\beta+1}\right), \qquad (2.8.3)
\end{aligned}$$

in view of conditions (1.3.2) and (2.3.11) for $p = 1$, mean value theorem for integrals and $1 < \gamma < \beta + 1$.

Collecting (2.8.1) - (2.8.3), we have

$$|\tilde{t}_n^{BA}(f;x) - \tilde{f}(x)| = O_x\left(\sum_{m=0}^{n} \frac{b_{n,m}}{m+1}(n+1)^{\beta+1}\tilde{\omega}(1/(n+1))\right). \tag{2.8.4}$$

Hence the proof of Theorem 2.6.2 is completed.

2.9 Degree of Approximation in a Weighted L^p-Norm

Various investigators have studied the problem of determining the degree of approximation of a 2π-periodic function f belonging to a weighted Lipschitz class $W(L^p, \xi(t))$ and its conjugate $\tilde{f}$ through different summability means of the Fourier series and its conjugate series, respectively. Lal [66], Singh, et al. [123] and Mishra et al. [81] have considered the C^1N_p means in various directions. Lal [66] have obtained the degree of approximation of $f \in W(L^p, \xi(t))$-class using the C^1N_p means of the Fourier series of f as follows:

Theorem A [66, Theorem 2] *Let f be a 2π periodic function and Lebesgue integrable on $[0, 2\pi]$ and is belonging to $W(L^p, \xi(t))$-class then its degree of approximation by C^1N_p means of its Fourier series is given by*

$$\left\|t_n^{CN}(f;x) - f(x)\right\|_p = O\left((n+1)^{\beta+1/p}\xi(1/(n+1))\right),$$

provided $\xi(t)$ satisfies the following conditions:

$$\xi(t)/t \ \ be\ a\ non-increasing\ sequence, \tag{2.9.1}$$

$$\left\{\int_0^{1/(n+1)} \left(\frac{t|\phi(x,t)|}{\xi(t)}\right) \sin^{\beta p}(t)dt\right\}^{1/p} = O((n+1)^{-1}), and \tag{2.9.2}$$

$$\left\{\int_{1/(n+1)}^{\pi} \left(\frac{t^{-\delta}|\phi(x,t)|}{\xi(t)}\right)^p dt\right\}^{1/p} = O\left((n+1)^{\delta}\right), \tag{2.9.3}$$

where δ is an arbitrary number such that $q(1-\delta)-1 > 0,\ p^{-1}+q^{-1} = 1,\ 1 \le p \le \infty$, conditions (2.9.2) and (2.9.3) hold uniformly in x.

Singh et al. [123] studied the results of Lal [66] further and pointed out some errors [123, Remark 2.4, p.4]. The authors [123] improved the earlier results of Lal [66] by replacing the monotonicity on $\{p_n\}$ by the condition $(n+1)p_n = O(P_n)$ [123, Theorem 3.2, pp.4–5].

In the sequel, recently Mishra et al. [81] have obtained the subsequent results for conjugate Fourier series using the C^1N_p means as follows:

Theorem B [81, Theorem 3.1] *Consider the regular Nörlund transform N_p generated by the non-negative $\{p_n\}$ such that*

$$(n+1)p_n = O(P_n). \tag{2.9.4}$$

Let f be a 2π periodic function and Lebesgue integrable on $[0, 2\pi]$, then the degree of approximation of $\tilde{f}(x)$, conjugate to $f \in W(L^p, \xi(t))(p \geq 1)$-class with $0 \leq \beta \leq 1 - 1/p$ by C^1N_p means of conjugate series of its Fourier series is given by

$$\left\| \tilde{t}_n^{CN}(f;x) - \tilde{f}(x) \right\|_p = O\left((n+1)^{\beta+1/p} \xi(1/(n+1)) \right),$$

provided positive integer function $\xi(t)$ satisfies the following conditions:

$$\xi(t)/t \ \ is\ non-increasing\ in\ t, \tag{2.9.5}$$

$$\left\{ \int_0^{\pi/(n+1)} \left(\frac{|\psi_x(t)|}{\xi(t)} \right)^p \sin^{\beta p}(t/2) dt \right\}^{1/p} = O(1), \tag{2.9.6}$$

and

$$\left\{ \int_{\pi/(n+1)}^{\pi} \left(\frac{t^{-\delta}|\psi_x(t)|}{\xi(t)} \right)^p dt \right\}^{1/p} = O\left((n+1)^{\delta} \right), \tag{2.9.7}$$

where δ is an arbitrary number such that $q(\beta-\delta)-1 > 0$, $p^{-1}+q^{-1} = 1$, $1 \leq p \leq \infty$, conditions (2.9.6) and (2.9.7) hold uniformly in x.

Recently, Zhang [150] has pointed out that results of Mishra et al. [81] hold only for the function which is constant almost everywhere under their assumption conditions. Zhang [150, p.1140] explained that the assumption conditions $(\beta-\delta)q - 1 > 0$ with $p^{-1} + q^{-1} = 1$ and $0 \leq \beta \leq 1 - 1/p$ gives that $\delta < 0$. From the condition (2.9.7), as $n \to \infty$, the function becomes constant almost everywhere. The same observations are also true for the results of Singh et al. [123].

We reformulate the problems further with less assumption conditions on $\xi(t)$ and resolve the issue raised by Zhang [150]. We note that several authors define the function class $W(L^p, \xi(t))$ with weight function $\sin^{\beta p}(x/2)$ [54; 123, and references therein] or $\sin^{\beta p}(x)$ [66, and references therein] but the degree of approximation is measured in ordinary L^p-norm. Also, the function class $W(L^p, \xi(t))$ is a subclass of the weighted $L^p[0, 2\pi]$-space with the weight function $\sin^{\beta p}(x/2)$, so it is relevant to measure the degree of approximation in the weighted norm defined in (1.3.1).

2.10 Main Results

We study the above problems further to get better degree of approximation under relaxed conditions on the positive non-decreasing function $\xi(t)$ using the more general summability means, the product means, defined in (2.1.1) and (2.1.2) for Fourier series and its conjugate series. More precisely, we prove the following:

Theorem 2.10.1. *Let f be a 2π-periodic function belonging to $W(L^p, \xi(t))$ with $p \geq 1$, $\beta \geq 0$ and let the entries of the lower triangular matrices $A \equiv (a_{n,k})$ and $B \equiv (b_{n,k})$ satisfy the conditions* (2.3.1) - (2.3.3) *of Theorem* 2.3.1 *with $A_{n,n} = B_{n,n} = 1$ for $n = 0, 1, 2, \ldots$. Then the degree of approximation of f by BA means of its Fourier series is given by*

$$\left\| t_n^{BA}(f; x) - f(x) \right\|_{p,\beta} = O\left(\xi(\pi/(n+1)) + (n+1)^{1-\sigma} \sum_{m=0}^{n} \frac{b_{n,m}}{m+1} \right)$$

provided that the positive non-decreasing function $\xi(t)$ satisfies the condition:

$$\xi(t)/t^\sigma \text{ is non-decreasing function for some } 0 < \sigma < 1. \tag{2.10.1}$$

Theorem 2.10.2. *Let f be a 2π-periodic function belonging to $W(L^p, \xi(t))$ with $p \geq 1$, $\beta \geq 0$ and let the entries of the lower triangular matrices $A \equiv (a_{n,k})$ and $B \equiv (b_{n,k})$ satisfy the conditions* (2.3.1) - (2.3.3) *of Theorem* 2.3.1 *with $A_{n,n} = B_{n,n} = 1$ for $n = 0, 1, 2, \ldots$. Then the degree of approximation of $\tilde{f}$, conjugate of f, by BA means of its conjugate Fourier series is given by*

$$\left\| \tilde{t}_n^{BA}(f; x) - \tilde{f}(x) \right\|_{p,\beta} = O\left(\xi(\pi/(n+1)) + (n+1)^{1-\sigma} \sum_{m=0}^{n} \frac{b_{n,m}}{m+1} \right)$$

where $\xi(t)$ and σ are the same as in Theorem 2.10.1.

Remark 2.10.1. If the entries of matrix B satisfy one more condition, that is, $\sum_{m=0}^{n}\frac{b_{n,m}}{m+1}=O(1/(n+1))$, then the degree of approximation in our results reduces to $O\big(\xi(\pi/(n+1)) + (n+1)^{-\sigma}\big)$ which is a better approximation.

2.11 Proof of Theorem 2.10.1

Following the proof of Theorem 2.3.1, we have

$$\begin{aligned} t_n^{BA}(f;x)-f(x) &= \frac{1}{2\pi}\int_0^{\pi}\phi(x,t)\sum_{m=0}^{n}\sum_{k=0}^{m} b_{n,m}a_{m,k}\frac{\sin{(k+1/2)t}}{\sin(t/2)}dt \\ &= \int_0^{\pi}\phi(x,t)(BA)_n(t)dt. \end{aligned}$$

Using generalized Minkowski inequality, we have

$$\begin{aligned} \left\|t_n^{BA}(f;x)-f(x)\right\|_{p,\beta} &= \left\{\frac{1}{2\pi}\int_0^{2\pi}\left|\int_0^{\pi}\phi(x,t)(BA)_n(t)dt\right|^p\sin^{\beta p}(x/2)dx\right\}^{1/p} \\ &\leq \int_0^{\pi}\left\{\frac{1}{2\pi}\int_0^{2\pi}|\phi(x,t)|^p\sin^{\beta p}(x/2)dx\right\}^{1/p}(BA)_n(t)dt \\ &= \int_0^{\pi}O(\xi(t))(BA)_n(t)dt \\ &= O\left(\int_0^{\pi/(n+1)}\xi(t)(BA)_n(t)dt+\int_{\pi/(n+1)}^{\pi}\xi(t)(BA)_n(t)dt\right) \\ &= I_1+I_2. \qquad (2.11.1) \end{aligned}$$

Now, using Lemma 2.4.2 and mean value theorem for integrals, we have

$$\begin{aligned} I_1 &= O\left(\int_0^{\pi/(n+1)}\xi(t)(BA)_n(t)dt\right) \\ &= O\left((n+1)\int_0^{\pi/(n+1)}\xi(t)dt\right) \\ &= O(\xi(\pi/(n+1))). \qquad (2.11.2) \end{aligned}$$

The work of Theorems 2.10.1 and 2.10.2 has been communicated for possible publication.

Further, using Lemma 2.4.3, we have

$$\begin{aligned} I_2 &= O\left(\int_{\pi/(n+1)}^{\pi} \frac{\xi(t)}{t^2}\left(\sum_{m=0}^{n} \frac{b_{n,m}}{m+1} + \frac{1}{n+1}\right) dt\right) \\ &= O\left(\left(\sum_{m=0}^{n} \frac{b_{n,m}}{m+1}\right) \int_{\pi/(n+1)}^{\pi} \frac{\xi(t)}{t^2} dt + \frac{1}{n+1} \int_{\pi/(n+1)}^{\pi} \frac{\xi(t)}{t^2} dt\right) \\ &= I_{21} + I_{22}. \end{aligned} \tag{2.11.3}$$

Now,

$$\begin{aligned} I_{21} &= O\left(\sum_{m=0}^{n} \frac{b_{n,m}}{m+1} \int_{\pi/(n+1)}^{\pi} \frac{\xi(t)}{t^\sigma} . \frac{1}{t^{2-\sigma}} dt\right) \\ &= O\left(\sum_{m=0}^{n} \frac{b_{n,m}}{m+1} \frac{\xi(\pi)}{\pi^\sigma} \left[t^{\sigma-1}\right]_{\pi/(n+1)}^{\pi}\right) \\ &= O\left((n+1)^{1-\sigma} \sum_{m=0}^{n} \frac{b_{n,m}}{m+1}\right), \end{aligned} \tag{2.11.4}$$

in view of condition (2.10.1) and mean value theorem for integrals.
Similarly,

$$\begin{aligned} I_{22} &= O\left(\frac{1}{(n+1)} \int_{\pi/(n+1)}^{\pi} \frac{\xi(t)}{t^\sigma} . \frac{1}{t^{2-\sigma}} dt\right) \\ &= O\left(\frac{1}{(n+1)} \frac{\xi(\pi)}{\pi^\sigma} \left[t^{\sigma-1}\right]_{\pi/(n+1)}^{\pi}\right) \\ &= O\left((n+1)^{-\sigma}\right), \end{aligned} \tag{2.11.5}$$

in view of condition (2.10.1) and mean value theorem for integrals.
Further,

$$\begin{aligned} (n+1)^{-\sigma} + (n+1)^{1-\sigma} \sum_{m=0}^{n} \frac{b_{n,m}}{m+1} &\geq (n+1)^{-\sigma} + (n+1)^{-\sigma} \sum_{m=0}^{n} b_{n,m} \\ &\geq 2(n+1)^{-\sigma}, \end{aligned}$$

so that,

$$(n+1)^{-\sigma} = O\left((n+1)^{1-\sigma} \sum_{m=0}^{n} \frac{b_{n,m}}{m+1}\right). \tag{2.11.6}$$

Therefore

$$I_2 = O\left((n+1)^{1-\sigma} \sum_{m=0}^{n} \frac{b_{n,m}}{m+1}\right). \tag{2.11.7}$$

Collecting (2.11.1) - (2.11.7), we have

$$\left\|t_n^{BA}(f;x) - f(x)\right\|_{p,\beta} = O\left(\xi(\pi/(n+1)) + (n+1)^{1-\sigma}\sum_{m=0}^{n}\frac{b_{n,m}}{m+1}\right).$$

Hence the proof of Theorem 2.10.1 is completed.

2.12 Proof of Theorem 2.10.2

Following the proof of Theorem 2.3.2

$$\tilde{t}_n^{BA}(f;x) - \tilde{f}(x) = \frac{1}{2\pi}\int_0^\pi \psi(x,t)(\widetilde{BA})_n(t)dt.$$

Using generalized Minkowski inequality, we have

$$\begin{aligned}
\left\|\tilde{t}_n^{BA}(f;x) - \tilde{f}(x)\right\|_{p,\beta} &= \left(\frac{1}{2\pi}\int_0^{2\pi}\left|\int_0^\pi \psi(x,t)(\widetilde{BA})_n(t)dt\right|^p \sin^{\beta p}(x/2)dx\right)^{1/p} \\
&\leq \int_0^\pi \left(\frac{1}{2\pi}\int_0^{2\pi}|\psi(x,t)|^p \sin^{\beta p}(x/2)dx\right)^{1/p} |(\widetilde{BA})_n(t)|dt \\
&= \int_0^\pi O(\xi(t))|(\widetilde{BA})_n(t)|dt \\
&= O\left(\int_0^{\pi/(n+1)} + \int_{\pi/(n+1)}^{\pi}\right)\left(\xi(t)|(\widetilde{BA})_n(t)|dt\right) \\
&= I_1' + I_2'. \qquad (2.12.1)
\end{aligned}$$

Now, using Lemma 2.4.4, we have

$$\begin{aligned}
I_1' &= O\left(\int_0^{\pi/(n+1)} \xi(t)(\widetilde{BA})_n(t)dt\right) \\
&= O\left(\int_0^{\pi/(n+1)} \frac{\xi(t)}{t^\sigma}t^{\sigma-1}dt\right) \\
&= O(\xi(\pi/(n+1))), \qquad (2.12.2)
\end{aligned}$$

in view of condition (2.10.1) and mean value theorem for integrals.

Further, using Lemma 2.4.5, we have

$$\begin{aligned}
I_2' &= O\left(\int_{\pi/(n+1)}^{\pi} \frac{\xi(t)}{t^2}\left(\sum_{m=0}^{n}\frac{b_{n,m}}{m+1} + \frac{1}{n+1}\right)dt\right) \\
&= O\left(\left(\sum_{m=0}^{n}\frac{b_{n,m}}{m+1}\right)\int_{\pi/(n+1)}^{\pi}\frac{\xi(t)}{t^2}dt + \frac{1}{n+1}\int_{\pi/(n+1)}^{\pi}\frac{\xi(t)}{t^2}dt\right) \\
&= I_{21}' + I_{22}'. \qquad (2.12.3)
\end{aligned}$$

Proceeding in the same manner as the proof of Theorem 2.10.1, we have

$$I_2^{'} = O\left((n+1)^{1-\sigma}\sum_{m=0}^{n}\frac{b_{n,m}}{m+1}\right). \tag{2.12.4}$$

Collecting (2.12.1) - (2.12.4), we have

$$\left\|\tilde{t}_n^{BA}(f;x)-\tilde{f}(x)\right\|_{p,\beta} = O\left(\xi(\pi/(n+1)) + (n+1)^{1-\sigma}\sum_{m=0}^{n}\frac{b_{n,m}}{m+1}\right).$$

Hence the proof of Theorem 2.10.2 is completed.

2.13 Particular Cases

1. If we replace the matrix $B \equiv (b_{n,k})$ by C^1 matrix i.e., the matrix corresponding to Cesàro means of order 1, then

$$b_{n,m} = \begin{cases} \frac{1}{n+1}, & 0 \le m \le n \\ 0, & m > n. \end{cases}$$

Thus, we get C^1A-version of Theorem 2.10.1 and 2.10.2.

2. Further, if we replace the right hand side of conditions (2.3.2) and (2.3.3) by $\frac{b_{n,m}}{(n+m+1)^2}$, then, for the C^1A-means, the conditions (2.3.2) and (2.3.3) reduces to

$$|a_{m,0}-a_{m+1,1}| \ll \frac{1}{(n+m+1)^2} \quad \text{for } 0 \le m \le n-1 \tag{2.13.1}$$

and

$$\sum_{k=0}^{m-1}|(a_{m,m-k}-a_{m+1,m+1-k})-(a_{m,m-k-1}-a_{m+1,m-k})| \ll \frac{1}{(n+m+1)^2} \quad \text{for } 0 \le m \le n-1. \tag{2.13.2}$$

Using the above conditions (2.13.1) and (2.13.2), the Lemma 2.4.3 becomes as following:

$$\begin{aligned} |(C^1A)_n(t)| &= O\left(\frac{1}{(n+1)t^2}\left(\sum_{m=0}^{n}\frac{1}{n+m+1}+\frac{1}{n+1}\right)\right) \\ &= O\left(\frac{t^{-2}}{n+1}\right), \end{aligned}$$

in view of $\sum_{m=0}^{n} \frac{1}{(n+m+1)} = O(1)$.
And for C^1A-means, the results of Theorem 2.10.1 becomes

$$\left\| t_n^{C^1A}(f;x) - f(x) \right\|_{p,\beta} = O\left(\xi(\pi/(n+1)) + (n+1)^{-\sigma}\right).$$

Similarly, Lemma 2.4.5 changes as

$$|(\widetilde{C^1A})_n(t)| \;=\; O\left(\frac{t^{-2}}{n+1}\right),$$

and for C^1A-means, the results of Theorem 2.10.2 becomes

$$\left\| \tilde{t}_n^{C^1A}(f;x) - \tilde{f}(x) \right\|_{p,\beta} = O\left(\xi(\pi/(n+1)) + (n+1)^{-\sigma}\right).$$

Chapter 3

Approximation in Certain Subclasses of L^p-space by Linear Operators

In this chapter, we study the degree of trigonometric approximation of 2π-periodic functions and their conjugates, in terms of the moduli of continuity associated with them, by matrix means of corresponding Fourier series. We also discuss some analogous results with remarks and corollaries.

3.1 Introduction

Let $T = (a_{n,k})$ be a lower triangular matrix with non-negative entries such that $a_{n,-1} = 0$ and $A_{n,k} = \sum_{r=k}^{n} a_{n,r}$.

The sequence to sequence transformations

$$t_n(f;x) = \sum_{k=0}^{n} a_{n,k} s_k(f;x), \quad n = 0,\ 1,\ 2\ldots \tag{3.1.1}$$

and

$$\tilde{t}_n(f;x) = \sum_{k=0}^{n} a_{n,k} \tilde{s}_k(f;x),\ n = 0,\ 1,\ 2\ldots, \tag{3.1.2}$$

define matrix means of the sequences $\{s_n(f;x)\}$ and $\{\tilde{s}_n(f;x)\}$, respectively. The Fourier series (1.1.1) of f is said to be summable to s by T-means if $\lim_{n\to\infty} t_n(f;x) = s$,

where s is a finite number. Similarly, the conjugate series of Fourier series of f is said to be summable to $\bar{s}$ by T-means if $\lim_{n\to\infty} \tilde{t}_n(f;x) = \bar{s}$, where $\bar{s}$ is a finite number.
We write
$K_n(t) = \frac{1}{2\pi}\sum_{k=0}^{n} a_{n,n-k}\frac{\sin(n-k+1/2)t}{\sin t/2},\ \widetilde{K}_n(t) = \frac{1}{2\pi}\sum_{k=0}^{n} a_{n,n-k}\frac{\cos(n-k+1/2)t}{\sin(t/2)}$

and $\tau = [1/t]$, the integer part of $1/t$.

3.2 Degree of Approximation in $L^p(\omega)_\beta$ and $L^p(\tilde{\omega})_\beta$-Classes

A number of papers have been written dealing with the degree of approximation through Fourier series representation of a function, or a conjugate function, by various summability means represented by lower triangular matrix (see e.g.,[61; 68; 77; 113; 115] and references therein). In the sequel, Lenski and Szal [77], Kranz et al. [58] and Krasniqi [61] have proved many interesting results on the point-wise approximation of $f \in L^p(\omega)_\beta$ (or $L^p(\tilde{\omega})_\beta$) and its conjugate $\tilde{f}$. In all these papers, the authors have used the following conditions or conditions similar to them:

$$\left(\int_0^{\pi/(n+1)} \left(\frac{t|\psi(x,t)|}{\tilde{\omega}(t)}\right)^p \sin^{\beta p}(t/2)dt\right)^{1/p} = O_x((n+1)^{-1}), \tag{3.2.1}$$

$$\left(\int_{\pi/(n+1)}^{\pi} \left(\frac{t^{-\gamma}|\psi(x,t)|}{\tilde{\omega}(t)}\right)^p \sin^{\beta p}(t/2)dt\right)^{1/p} = O_x((n+1)^{\gamma}), \tag{3.2.2}$$

$$\left(\int_0^{\pi/(n+1)} \left(\frac{|\psi(x,t)|}{\tilde{\omega}(t)}\right)^p \sin^{\beta p}(t/2)dt\right)^{1/p} = O_x((n+1)^{-1/p}), \tag{3.2.3}$$

$$\left(\int_0^{\pi/(n+1)} \left(\frac{\tilde{\omega}(t)}{t\sin^{\beta}(t/2)}\right)^q dt\right)^{1/q} = O_x\left((n+1)^{\beta+1/p}\,\tilde{\omega}\left(\frac{\pi}{n+1}\right)\right), \tag{3.2.4}$$

where $1 < p < \infty$ and $0 < \gamma < \beta + 1/p$.
For more details one can see Krasniqi [61, pp.91-92]. From the recent results of Lal and Mishra [68] and Rhoades [115], we observe that the conditions (3.2.1)-(3.2.4) can be replaced by a single condition. These observations motivated us to study the problem further.

3.3 Main Results

We extend the above results in two different directions namely (i) we replace the summability means used by the authors in [58; 61; 77] by matrix means and relax the conditions (3.2.1)-(3.2.4); (ii) we extend the results of Lal and Mishra [68]; and Rhoades [115] to more general classes $L^p(\omega)_\beta$ and $L^p(\tilde{\omega})_\beta$. More precisely, we prove the following:

Theorem 3.3.1. *Let f be a 2π-periodic function belonging to the class $L^p(\omega)_\beta$, $\beta \geq 0$ and let $T \equiv (a_{n,k})$ be a lower triangular regular matrix with non-negative and non-decreasing (with respect to $0 \leq k \leq n$) entries with $A_{n,n-\tau} = O\left(1/t(n+1)\right)$. Then the degree of approximation of f by matrix means of its Fourier series is given by*

$$\|t_n(f;x) - f(x)\|_p = O\left(\frac{1}{n+1}\int_{1/(n+1)}^{\pi} \frac{\omega(t)}{t^{\beta+2}}dt\right),$$

provided that ω is a function of modulus of continuity type such that

$$\int_0^v \frac{\omega(t)}{t^{\beta+1}}dt = O\left(\frac{\omega(v)}{v^\beta}\right),\ 0 < v < \pi. \tag{3.3.1}$$

Theorem 3.3.2. *Let f be a 2π-periodic function belonging to the class $L^p(\tilde{\omega})_\beta$, $\beta \geq 0$ and let $T \equiv (a_{n,k})$ be a lower triangular regular matrix with non-negative and non-decreasing (with respect to $0 \leq k \leq n$) entries with $A_{n,n-\tau} = O\left(1/t(n+1)\right)$. Then the degree of approximation of $\tilde{f}$, conjugate of f, by matrix means of conjugate Fourier series is given by*

$$\left\|\tilde{t}_n(f;x) - \tilde{f}(x)\right\|_p = O\left(\frac{1}{n+1}\int_{1/(n+1)}^{\pi} \frac{\tilde{\omega}(t)}{t^{\beta+2}}dt\right),$$

provided that $\tilde{\omega}$ is a function of modulus of continuity type such that

$$\int_0^v \frac{\tilde{\omega}(t)}{t^{\beta+1}}dt = O\left(\frac{\tilde{\omega}(v)}{v^\beta}\right),\ 0 < v < \pi. \tag{3.3.2}$$

Remark 3.3.1. $A_{n,n-\tau} = O\left(1/t(n+1)\right) \Rightarrow A_{n,0} = O(1)$ as for $\tau = n,\ 1/t < \tau + 1 = n + 1$.

The work of Theorems 3.3.1 and 3.3.2 has been published in Asian-Eur. J. Math.(World Scientific Publication) **10** (2017), no. 3, 12 pp.

Remark 3.3.2. (i) The degree of approximation in our results does not depend on p. (ii) For $\beta = 0$, class $L^p(\omega)_\beta$ or $L^p(\tilde{\omega})_\beta$ reduces to $Lip(\xi(t), p)$-class and condition (3.3.1) or (3.3.2) reduces to condition (6) i.e., $\int_0^v \frac{\omega(t)}{t} dt = O(\omega(v))$ of Rhoades [115, Theorem 2, p. 393; Theorem 5, p. 395] . Thus our theorems extend Theorem 2, 3, 5 and 6 of Rhoades [115] and Theorem 3.1, 3.2 of Lal and Mishra [68] to their matrix analogous.

3.4 Lemmas

To prove our theorems, we need the following lemmas:

Lemma 3.4.1. *If p_n is non-negative and non-increasing sequence, then for $0 \le a < b \le \infty,\ 0 < t \le \pi$ and for any n*

$$\left|\sum_{k=a}^{b} p_k e^{i(n-k)t}\right| = \begin{cases} O\left(P(t^{-1})\right), & \text{for any } a \\ O\left(t^{-1}p_a\right), & a \ge t^{-1}, \end{cases}$$

where $P[t^{-1}] = P_\tau = \sum_{k=0}^{\tau} p_k$.

Proof. For the proof one can see [80, Lemma 5.11, p.8].

Lemma 3.4.2. *Let $T \equiv (a_{n,k})$ be a lower triangular regular matrix with non-negative and non-decreasing (with respect to $0 \le k \le n$) entries with $A_{n,n-\tau} = O\left(1/t(n+1)\right)$. Then*

$$|K_n(t)| = \begin{cases} O\left(n+1\right), & \text{for } 0 < t \le 1/(n+1) \\ O\left(1/t^2(n+1)\right), & \text{for } 1/(n+1) < t \le \pi. \end{cases}$$

Proof. **Case I**: For $0 < t \le 1/(n+1)$, using $1/\sin(t/2) = O(\pi/t)$ and $\sin(nt) \le nt$, we have

$$\begin{aligned} |K_n(t)| &= \left|\frac{1}{2\pi}\sum_{k=0}^{n} a_{n,n-k}\frac{\sin(n-k+1/2)t}{\sin(t/2)}\right| \\ &\le \frac{1}{2\pi}\sum_{k=0}^{n} a_{n,n-k}\left|\frac{\sin(n-k+1/2)t}{\sin(t/2)}\right| \\ &= O\left(\sum_{k=0}^{n} a_{n,n-k}\frac{(n-k+1/2)t}{t}\right) \\ &= O(n+1)A_{n,0} = O\left(n+1\right), \qquad (3.4.1) \end{aligned}$$

in view of $A_{n,0} = O(1)$.

Case II: For $1/(n+1) < t \le \pi$, using $1/\sin(t/2) = O(\pi/t)$ and $\tau = [1/t]$, we have

$$\begin{aligned}
|K_n(t)| &= \left|\frac{1}{2\pi}\sum_{k=0}^{n} a_{n,n-k}\frac{\sin(n-k+1/2)t}{\sin(t/2)}\right| \\
&= O(1/t)\left|\sum_{k=0}^{n} a_{n,n-k}\sin(n-k+1/2)t\right| \\
&= O(1/t)\left|\mathrm{Im}\left(\sum_{k=0}^{n} a_{n,n-k}e^{i(n-k+1/2)t}\right)\right| \\
&= O(1/t)\left|\sum_{k=0}^{n} a_{n,n-k}e^{i(n-k)t}\right|. \qquad (3.4.2)
\end{aligned}$$

Following Lemma 3.4.1, we have

$$\begin{aligned}
\left|\sum_{k=0}^{n} a_{n,n-k}e^{i(n-k)t}\right| &= \left|e^{int}\sum_{k=0}^{n} a_{n,n-k}e^{-ikt}\right| \\
&\le \left|\sum_{k=0}^{\tau-1} a_{n,n-k}e^{-ikt}\right| + \left|\sum_{k=\tau}^{n} a_{n,n-k}e^{-ikt}\right| \\
&\le \sum_{k=0}^{\tau-1} a_{n,n-k} + 2a_{n,n-\tau}\max_{\tau\le k\le n}\left|\frac{1-e^{-i(k+1)t}}{1-e^{-it}}\right| \\
&\le A_{n,n-\tau+1} + 2a_{n,n-\tau}(1/\sin(t/2)) \\
&\le A_{n,n-\tau} + 2(\tau+1)a_{n,n-\tau} = O(A_{n,n-\tau}) \\
&= O(1/t(n+1)). \qquad (3.4.3)
\end{aligned}$$

Hence,

$$|K_n(t)| = O\left(1/t^2(n+1)\right). \qquad (3.4.4)$$

Collecting (3.4.1)-(3.4.4), Lemma 3.4.2 is completed. □

Lemma 3.4.3. *Let $T \equiv (a_{n,k})$ be a lower triangular regular matrix with non-negative and non-decreasing (with respect to $0 \le k \le n$) entries with $A_{n,n-\tau} = O(1/t(n+1))$. Then*

$$|\widetilde{K}_n(t)| = \begin{cases} O(1/t), & \text{for } 0 < t \le 1/(n+1) \\ O(1/(n+1)t^2), & \text{for } 1/(n+1) < t \le \pi. \end{cases}$$

Proof. **Case I**: For $0 < t \leq 1/(n+1)$, using $1/\sin(t/2) = O\left(\pi/t\right)$, we have

$$\begin{aligned} |\widetilde{K}_n(t)| &= \left| \frac{1}{2\pi} \sum_{k=0}^{n} a_{n,n-k} \frac{\cos\left(n-k+1/2\right)t}{\sin(t/2)} \right| \\ &\leq \frac{1}{2\pi} \sum_{k=0}^{n} a_{n,n-k} \left| \frac{\cos\left(n-k+1/2\right)t}{\sin\left(t/2\right)} \right| \\ &= O(1/t) \sum_{k=0}^{n} a_{n,n-k} = O(1/t) A_{n,0} = O\left(1/t\right), \end{aligned} \tag{3.4.5}$$

in view of $A_{n,0} = O(1)$.

Case II: For $1/(n+1) < t \leq \pi$, using $1/\sin(t/2) = O\left(\pi/t\right)$ and $\tau = [1/t]$, we have

$$\begin{aligned} |\widetilde{K}_n(t)| &= \left| \frac{1}{2\pi} \sum_{k=0}^{n} a_{n,n-k} \frac{\cos\left(n-k+1/2\right)t}{\sin\left(t/2\right)} \right| \\ &= O\left(\frac{1}{t}\right) \left| \sum_{k=0}^{n} a_{n,n-k} \cos\left(n-k+1/2\right)t \right| \\ &= O\left(\frac{1}{t}\right) \left| \mathrm{Re}\left(\sum_{k=0}^{n} a_{n,n-k} e^{i(n-k+1/2)t} \right) \right| \\ &= O\left(\frac{1}{t}\right) \left| \sum_{k=0}^{n} a_{n,n-k} e^{i(n-k)t} \right|. \end{aligned} \tag{3.4.6}$$

Following Lemma 3.4.2, we have

$$|\widetilde{K}_n(t)| = O\left(1/(n+1)t^2\right). \tag{3.4.7}$$

Collecting (3.4.5)-(3.4.7), Lemma 3.4.3 is completed. □

3.5 Proof of Theorem 3.3.1

Following the section 2.5, we have

$$t_n(f;x) - f(x) = \int_0^{\pi} \phi(x,t) K_n(t) dt.$$

Using generalized Minkowski inequality and $1/\sin(t/2) = O(\pi/t)$ for $0 < t \leq \pi$, we have

$$\begin{aligned}
\|t_n(f;x) - f(x)\|_p &= \left(\frac{1}{2\pi}\int_0^{2\pi}\left|\int_0^{\pi}\phi(x,t)K_n(t)dt\right|^p dx\right)^{1/p} \\
&\leq \int_0^{\pi}\left(\frac{1}{2\pi}\int_0^{2\pi}|\phi(x,t)|^p|\sin^{\beta p}(t/2)|dx\right)^{1/p}\frac{|K_n(t)|}{|\sin^{\beta}(t/2)|}dt \\
&= \int_0^{\pi}\frac{\omega(t)}{t^{\beta}}|K_n(t)|dt \\
&= \left(\int_0^{1/(n+1)} + \int_{1/(n+1)}^{\pi}\right)\left(\frac{\omega(t)}{t^{\beta}}|K_n(t)|dt\right) \\
&= I_1 + I_2. \qquad (3.5.1)
\end{aligned}$$

Now, using Lemma 3.4.2 for $0 < t \leq 1/(n+1)$ and mean value theorem for integrals, we have

$$\begin{aligned}
I_1 &= \int_0^{1/(n+1)}\frac{\omega(t)}{t^{\beta}}|K_n(t)|dt = O\left((n+1)\int_0^{1/(n+1)} t\frac{\omega(t)}{t^{\beta+1}}dt\right) \\
&= O\left((n+1)(1/(n+1))\int_0^{1/(n+1)}\frac{\omega(t)}{t^{\beta+1}}dt\right) \\
&= O\left(\omega\left(1/(n+1)\right)(n+1)^{\beta}\right), \qquad (3.5.2)
\end{aligned}$$

in view of condition (3.3.1).

Using Lemma 3.4.2 for $1/(n+1) < t \leq \pi$, we have

$$\begin{aligned}
I_2 &= \int_{1/(n+1)}^{\pi}\frac{\omega(t)}{t^{\beta}}|K_n(t)|dt \\
&= O\left(\frac{1}{n+1}\int_{1/(n+1)}^{\pi}\frac{\omega(t)}{t^{\beta+2}}dt\right).
\end{aligned}$$

Now,

$$\begin{aligned}
\frac{1}{n+1}\int_{1/(n+1)}^{\pi}\frac{\omega(t)}{t^{\beta+2}}dt &\geq \frac{1}{(n+1)}\omega\left(1/(n+1)\right)\int_{1/(n+1)}^{\pi}\frac{1}{t^{\beta+2}}dt \\
&= \frac{1}{n+1}\omega\left(1/(n+1)\right)\left[(n+1)^{\beta+1} - \frac{1}{\pi^{\beta+1}}\right] \\
&\geq \frac{1}{2}\omega\left(1/(n+1)\right)(n+1)^{\beta},
\end{aligned}$$

that is

$$\omega\left(1/(n+1)\right)(n+1)^{\beta} = O\left(\frac{1}{n+1}\int_{1/(n+1)}^{\pi}\frac{\omega(t)}{t^{\beta+2}}dt\right). \qquad (3.5.3)$$

Collecting (3.5.1)-(3.5.3), we have

$$\|t_n(f;x) - f(x)\|_p = O\left(\frac{1}{n+1}\int_{1/(n+1)}^{\pi} \frac{\omega(t)}{t^{\beta+2}}dt\right).$$

Hence the proof of Theorem 3.3.1 is completed.

3.6 Proof of Theorem 3.3.2

Following the section 2.6, we have

$$\tilde{t}_n(f;x) - \tilde{f}(x) = \int_0^{\pi} \psi(x,t)\widetilde{K}_n(t)dt.$$

Using generalized Minkowski inequality and $1/\sin(t/2) = O(\pi/t)$ for $0 < t \leq \pi$, we have

$$\begin{aligned}
\left\|\tilde{t}_n(f;x) - \tilde{f}(x)\right\|_p &= \left(\frac{1}{2\pi}\int_0^{2\pi}\left|\int_0^{\pi}\psi(x,t)\widetilde{K}_n(t)dt\right|^p dx\right)^{1/p} \\
&\leq \int_0^{\pi}\left(\frac{1}{2\pi}\int_0^{2\pi}|\psi(x,t)|^p|\sin^{\beta p}(t/2)|dx\right)^{1/p}\frac{|\widetilde{K}_n(t)|}{|\sin^{\beta}(t/2)|}dt \\
&= \int_0^{\pi}\frac{\tilde{\omega}(t)}{t^{\beta}}|\widetilde{K}_n(t)|dt \\
&= \left(\int_0^{1/(n+1)} + \int_{1/(n+1)}^{\pi}\right)\left(\frac{\tilde{\omega}(t)}{t^{\beta}}|\widetilde{K}_n(t)|dt\right) \\
&= I_1' + I_2'. \qquad (3.6.1)
\end{aligned}$$

Now, using Lemma 3.4.3 for $0 < t \leq 1/(n+1)$, we have

$$\begin{aligned}
I_1' &= \int_0^{1/(n+1)}\frac{\tilde{\omega}(t)}{t^{\beta}}|\widetilde{K}_n(t)|dt \\
&= O\left(\int_0^{1/(n+1)}\frac{\tilde{\omega}(t)}{t^{\beta+1}}dt\right) = O\left(\tilde{\omega}\left(1/(n+1)\right)(n+1)^{\beta}\right), \qquad (3.6.2)
\end{aligned}$$

in view of condition (3.3.2).

Again, using Lemma 3.4.3 for $1/(n+1) < t \leq \pi$, we have

$$\begin{aligned}
I_2' &= \int_{1/(n+1)}^{\pi}\frac{\tilde{\omega}(t)}{t^{\beta}}|\widetilde{K}_n(t)|dt \\
&= O\left(\frac{1}{n+1}\int_{1/(n+1)}^{\pi}\frac{\tilde{\omega}(t)}{t^{\beta+2}}dt\right).
\end{aligned}$$

Now,

$$\begin{aligned}
\frac{1}{n+1}\int_{1/(n+1)}^{\pi}\frac{\tilde{\omega}(t)}{t^{\beta+2}}dt &\geq \frac{1}{(n+1)}\tilde{\omega}\left(1/(n+1)\right)\int_{1/(n+1)}^{\pi}\frac{1}{t^{\beta+2}}dt \\
&= \frac{1}{n+1}\tilde{\omega}\left(1/(n+1)\right)\left[(n+1)^{\beta+1}-\frac{1}{\pi^{\beta+1}}\right] \\
&\geq \frac{1}{2}\tilde{\omega}\left(1/(n+1)\right)(n+1)^{\beta},
\end{aligned}$$

that is

$$\tilde{\omega}\left(1/(n+1)\right)(n+1)^{\beta} = O\left(\frac{1}{n+1}\int_{1/(n+1)}^{\pi}\frac{\tilde{\omega}(t)}{t^{\beta+2}}dt\right). \tag{3.6.3}$$

Collecting (3.6.1)-(3.6.3), we get

$$\left\|\tilde{t}_n(f;x)-\tilde{f}(x)\right\|_p = O\left(\frac{1}{n+1}\int_{1/(n+1)}^{\pi}\frac{\tilde{\omega}(t)}{t^{\beta+2}}dt\right).$$

Hence the proof of Theorem 3.3.2 is completed.

3.7 Particular Cases

1. If

$$a_{n,k} = \begin{cases} p_{n-k}/P_n, & 0 \leq k \leq n \\ 0, & k > n, \end{cases}$$

where $P_n = \sum_{k=0}^{n} p_k \neq 0 \to \infty$ as $n \to \infty$, then the matrix T reduces to the Nörlund matrix N_p. Also for non-decreasing sequence $\{a_{n,k}\}$ in k, $\{p_n\}$ is non-increasing. If we replace matrix $T = \{a_{n,k}\}$ with Nörlund matrix $N_p = \{p_n\}$, where p_n is non-increasing, we get N_p-version of Theorem 3.3.1 and Theorem 3.3.2.

2. If $\beta = 0$ in our theorems, we get analogous results for $Lip(\xi(t), p)$-class.

3. Further, if $\beta = 0$ and $\omega(t) = \tilde{\omega}(t) = t^{\alpha}(0 < \alpha \leq 1)$, then $f \in Lip(\alpha, p)$-class and conditions (3.3.1) and (3.3.2) reduce to

$$\int_0^v \frac{\omega(t)}{t}dt = \int_0^v t^{\alpha-1}dt = v^{\alpha} = O(\omega(v)),\ 0 < v < \pi$$

and

$$\int_0^v \frac{\tilde{\omega}(t)}{t}dt = \int_0^v t^{\alpha-1}dt = v^{\alpha} = O(\tilde{\omega}(v)),\ 0 < v < \pi.$$

Thus for $f \in Lip(\alpha, p)(0 < \alpha \leq 1)$, we have

$$\begin{aligned} \|t_n(f;x) - f(x)\|_p &= O\left(\frac{1}{n+1}\int_{1/(n+1)}^{\pi} t^{\alpha-2}dt\right) \\ &= \begin{cases} O\left((n+1)^{-\alpha}\right), & \text{for } 0 < \alpha < 1 \\ O\left(\frac{\log \pi(n+1)}{n+1}\right), & \text{for } \alpha = 1. \end{cases} \end{aligned}$$

Similarly,

$$\left\|\tilde{t}_n(f;x) - \tilde{f}(x)\right\|_p = \begin{cases} O\left((n+1)^{-\alpha}\right), & \text{for } 0 < \alpha < 1 \\ O\left(\frac{\log \pi(n+1)}{n+1}\right), & \text{for } \alpha = 1. \end{cases}$$

3.8 Remarks

1. We note that the auxiliary conditions mentioned in (3.2.1)-(3.2.4) have been replaced by a single condition (3.3.1) or (3.3.2).

   Further, using Hölder's inequality and condition (3.2.4), we have

$$\begin{aligned} \left\{\frac{(n+1)\pi^{\beta}}{\pi}\int_0^{\pi/(n+1)}\frac{\tilde{\omega}(t)}{t^{\beta+1}}dt\right\} &\leq \left\{\frac{(n+1)}{\pi}\int_0^{\pi/(n+1)}\frac{\tilde{\omega}(t)}{t\sin^{\beta}(t/2)}dt\right\} \\ &\leq (n+1)^{1-1/p}\left\{\int_0^{\pi/(n+1)}\left(\frac{\tilde{\omega}(t)}{t\sin^{\beta}(t/2)}\right)^q dt\right\}^{1/q} \\ &= O\left((n+1)^{\beta+1}\tilde{\omega}(\pi/(n+1))\right), \end{aligned}$$

   which is particular case of $\int_0^v \frac{\tilde{\omega}(t)}{t^{\beta+1}}dt = O\left(\frac{\tilde{\omega}(v)}{v^{\beta}}\right)$, but not otherwise. Thus the conditions (3.3.1) or (3.3.2) are more general than (3.2.4).

2. From condition (3.5.3), we have

$$\frac{1}{n+1}\int_{1/(n+1)}^{\pi}\frac{\omega(t)}{t^{\beta+2}}dt \geq \frac{1}{2}(n+1)^{\beta}\omega\left(1/(n+1)\right),$$

   and using $\delta_2^{-1}\omega(\delta_2) \leq 2\delta_1^{-1}\omega(\delta_1)$ for $\delta_2 \geq \delta_1 > 0$, we have

$$\begin{aligned} \frac{1}{n+1}\int_{1/(n+1)}^{\pi}\frac{\omega(t)}{t^{\beta+2}}dt &\leq \frac{1}{n+1}2\frac{\omega(1/(n+1))}{1/(n+1)}\int_{1/(n+1)}^{\pi}\frac{1}{t^{\beta+1}}dt \\ &\leq \frac{2}{\beta}(n+1)^{\beta}\omega(1/(n+1)). \end{aligned}$$

   Thus the degree of approximation in our results is the same like $O\big((n+1)^{\beta}$ $\omega(1/(n+1))\big)$. Also for a given $\omega(t)$ [Particular Case 3], the degree of approximation is independent of p.

3.9 Degree of Approximation in $Lip(\omega(t), p)-$Class

Recently, Srivastava and Singh [135] defined a function class:

$$Lip(\omega(t), p) = \left\{ f \in L^p[0, 2\pi] : \|f(x+t) - f(x)\|_p = O(t^{-1/p}\omega(t)) \right\},$$

where $t > 0,\ p \geq 1$ and $\omega(t)$ is a positive non-decreasing function. Note that the function class $Lip(\psi(t), p)$ defined by Khan and Ram [54, p.47] and classical Lipschitz classes $Lip(\xi(t), p),\ Lip(\alpha, p)$ and $Lip\alpha$ are included in $Lip(\omega(t), p)$ [135, p.224]. They [135] proved the following theorem:

Theorem A [135, Theorem 1] *Let $T \equiv (a_{n,k})$ be a lower triangular matrix with non-negative and non-decreasing $(with\, respect\, to\ k)$ entries. Then the degree of approximation of a 2π -periodic function $f \in Lip(\omega(t), p)$ with $p \geq 1$ by matrix means of its Fourier series is given by*

$$\|t_n(f; x) - f(x)\|_p = O\left((n+1)^{1/p}\omega(\pi/(n+1))\right),$$

provided $\omega\,(t)$ is a positive non-decreasing function and satisfies the following conditions:

$$\omega(t)/t^{\sigma}\ \ \text{is an non-decreasing function for}\ \ 0 < \sigma < 1, \tag{3.9.1}$$

$$|\phi(x,t)|/\omega(t)\, t^{-1/p}\ \ \text{is bounded function of}\ t, \tag{3.9.2}$$

$$\left(\int_{\pi/(n+1)}^{\pi} \left(\frac{\omega(t)}{t^{1+1/p}}\right)^p dt\right)^{1/p} = O\left((n+1)\,\omega\left(\pi/\left(n+1\right)\right)\right), \tag{3.9.3}$$

where $p^{-1} + q^{-1} = 1$. Also condition (3.9.2) holds uniformly in x.

Very recently, the above result of Srivastava and Singh [135] has been improved by Zhang [151] up to a limited extent. He [151] replaced the conditions (3.9.1)-(3.9.3) of Theorem A by some weaker conditions and gave the result as follows:

Theorem B [151, Theorem 1] *Let $T \equiv (a_{n,k})$ be a lower triangular matrix with non-negative and non-decreasing $(with\, respect\, to\, k)$ entries. Then the degree of approximation of a 2π -periodic function $f(x)$ by matrix means of its Fourier series is given by*

$$|t_n(f; x) - f(x)| = O\left((n+1)^{1/p}\omega(\pi/(n+1)) + (n+1)^{-\sigma}\right),\ \ p > 1,$$

provided a positive non-decreasing function $\omega(t)$ *satisfies the following conditions:*

$$\text{there exists a } \sigma\,(1/p < \sigma < 1) \text{ such that } \frac{\omega(t)}{t^{\sigma}} \text{ is an non-decreasing function,} \tag{3.9.4}$$

$$\frac{|\phi(x,t)|}{\omega(t)\,t^{-1/p}} \text{ is a bounded function of } t, \text{ also it bounds uniformly in } x. \tag{3.9.5}$$

3.10 Main Results

We study the problem further and extend it to conjugate functions also. We replace the conditions (3.9.1)-(3.9.5) by a single condition. More precisely, we prove the following:

Theorem 3.10.1. *Let* f *be a* 2π*-periodic function belonging to* $Lip(\omega(t), p)$*-class with* $p \geq 1$ *and let* $T \equiv (a_{n,k})$ *be a lower triangular regular matrix with non-negative and non-decreasing (with respect to* $0 \leq k \leq n$*) entries with* $A_{n,n-\tau} = O\left(1/t(n+1)\right)$*. Then the degree of approximation of* f *by matrix means of its Fourier series is given by*

$$\|t_n(f;x) - f(x)\|_p = O\left(\frac{1}{n+1}\int_{1/(n+1)}^{\pi} \frac{\omega(t)}{t^{2+1/p}}\,dt\right),$$

provided $\omega(t)$ *is a positive non-decreasing function satisfying the following condition:*

$$\int_0^v \frac{\omega\,(t)}{t^{1+1/p}}\,dt = O\left(\frac{\omega\,(v)}{v^{1/p}}\right),\ 0 < v < \pi. \tag{3.10.1}$$

Theorem 3.10.2. *Let* f *be a* 2π*-periodic function belonging to* $Lip(\omega(t), p)$*-class with* $p \geq 1$ *and let* $T \equiv (a_{n,k})$ *be a lower triangular regular matrix with non-negative and non-decreasing (with respect to* $0 \leq k \leq n$*) entries with* $A_{n,n-\tau} = O\left(1/t(n+1)\right)$*. Then the degree of approximation of* $\tilde{f}$*, conjugate of* f*, by matrix means of its conjugate Fourier series is given by*

$$\left\|\tilde{t}_n(f;x) - \tilde{f}(x)\right\|_p = O\left(\frac{1}{n+1}\int_{1/(n+1)}^{\pi} \frac{\omega(t)}{t^{2+1/p}}\,dt\right),$$

provided $\omega(t)$ *is a positive non-decreasing function satisfying the condition* (3.10.1) *of Theorem* 3.10.1.

Remark 3.10.1. We see that $\omega(t)$ satisfying (3.9.4) also satisfies the condition (3.10.1) for $1/p < \sigma < 1$, as:

$$\int_0^v \frac{\omega(t)}{t^{1+1/p}}dt = \int_0^v \frac{\omega(t)}{t^{\sigma}}t^{\sigma-1-1/p}dt = O\left(\frac{\omega(v)}{v^{\sigma}}\left[\frac{t^{\sigma-1/p}}{\sigma-1/p}\right]_0^v\right) = O\left(\frac{\omega(v)}{v^{1/p}}\right).$$

Thus condition (3.9.4) is replaced by (3.10.1) and condition (3.9.5) is removed as it is not required.

Remark 3.10.2. For $\omega(t) = t^{1/p}\xi(t)$, the $Lip(\omega(t),p)-$class coincides with $Lip(\xi(t),p)$ -class and condition (3.10.1) of Theorem 3.10.1 reduces to the condition (6) of Rhoades [115, Theorem 2, p.393]. Thus our theorems extend Theorems 2, 3, 5 and 6 of Rhoades [115] and Theorems 3.1, 3.2 of Lal and Mishra [68] to their matrix analogous.

3.11 Proof of Theorem 3.10.1

Following the section 2.5, we have

$$t_n(f;x) - f(x) = \int_0^{\pi} \phi(x,t)K_n(t)dt.$$

Using generalized Minkowski inequality, we have

$$\begin{aligned}
\|t_n(f;x) - f(x)\|_p &= \left(\frac{1}{2\pi}\int_0^{2\pi}\left|\int_0^{\pi}\phi(x,t)K_n(t)dt\right|^p dx\right)^{1/p} \\
&\leq \int_0^{\pi}\left(\frac{1}{2\pi}\int_0^{2\pi}|\phi(x,t)|^p dx\right)^{1/p}|K_n(t)|dt \\
&= \int_0^{\pi}\frac{\omega(t)}{t^{1/p}}|K_n(t)|dt \\
&= \left(\int_0^{1/(n+1)} + \int_{1/(n+1)}^{\pi}\right)\left(\frac{\omega(t)}{t^{1/p}}|K_n(t)|dt\right) \\
&= I_1 + I_2. \qquad (3.11.1)
\end{aligned}$$

Now, using Lemma 3.4.2 for $0 < t \leq 1/(n+1)$ and mean value theorem, we have

$$\begin{aligned}
I_1 &= \int_0^{1/(n+1)}\frac{\omega(t)}{t^{1/p}}|K_n(t)|dt = O\left((n+1)\int_0^{1/(n+1)} t.\frac{\omega(t)}{t^{1+1/p}}dt\right) \\
&= O\left((n+1)(n+1)^{-1}\omega(1/(n+1))(n+1)^{1/p}\right) \\
&= O\left(\omega(1/(n+1))(n+1)^{1/p}\right), \qquad (3.11.2)
\end{aligned}$$

The work of Theorems 3.10.1 and 3.10.2 has been published in Boll. Unione Mat. Ital.(Springer Publication) **9** (2016), no. 4, 495 – 504.

in view of condition (3.10.1).

Again, using Lemma 3.4.2 for $1/(n+1) < t \leq \pi$, we have

$$\begin{aligned} I_2 &= \int_{1/(n+1)}^{\pi} \frac{\omega(t)}{t^{1/p}} |K_n(t)| dt \\ &= O\left(\frac{1}{n+1} \int_{1/(n+1)}^{\pi} \frac{\omega(t)}{t^{2+1/p}} dt\right). \end{aligned} \tag{3.11.3}$$

Now, using mean value theorem, we have

$$\begin{aligned} \frac{1}{n+1} \int_{1/(n+1)}^{\pi} \frac{\omega(t)}{t^{2+1/p}} dt &\geq \frac{1}{(n+1)} \omega(1/(n+1)) \int_{1/(n+1)}^{\pi} \frac{1}{t^{2+1/p}} dt \\ &= \frac{1}{(n+1)} \omega(1/(n+1)) \left[(n+1)^{1+1/p} - \frac{1}{\pi^{1+1/p}}\right] \\ &= \frac{(n+1)^{1+1/p}}{(n+1)} \omega(1/(n+1)) \left[1 - \frac{1}{((n+1)\pi)^{1+1/p}}\right] \\ &\geq \frac{1}{2} \omega(1/(n+1)) (n+1)^{1/p}, \end{aligned}$$

that is

$$\omega(1/(n+1)) (n+1)^{1/p} = O\left(\frac{1}{n+1} \int_{1/(n+1)}^{\pi} \frac{\omega(t)}{t^{2+1/p}} dt\right). \tag{3.11.4}$$

Collecting (3.11.1)-(3.11.4), we have

$$\|t_n(f;x) - f(x)\|_p = O\left(\frac{1}{n+1} \int_{1/(n+1)}^{\pi} \frac{\omega(t)}{t^{2+1/p}} dt\right).$$

Hence the proof of Theorem 3.10.1 is completed.

3.12 Proof of Theorem 3.10.2

Following the section 2.6, we have

$$\begin{aligned} \tilde{t}_n(f;x) - \tilde{f}(x) &= \int_0^{\pi} \psi(x,t) \frac{1}{2\pi} \sum_{k=0}^{n} a_{n,n-k} \frac{\cos(n-k+1/2)t}{\sin(t/2)} dt \\ &= \int_0^{\pi} \psi(x,t) \widetilde{K}_n(t) dt. \end{aligned}$$

Using generalized Minkowski inequality, we have

$$\begin{aligned}
\left\|\tilde{t}_n(f;x)-\tilde{f}(x)\right\|_p &= \left(\frac{1}{2\pi}\int_0^{2\pi}\left|\int_0^{\pi}\psi(x,t)\widetilde{K}_n(t)dt\right|^p dx\right)^{1/p} \\
&\leq \int_0^{\pi}\left(\frac{1}{2\pi}\int_0^{2\pi}|\psi(x,t)|^p dx\right)^{1/p}|\widetilde{K}_n(t)|dt \\
&= \int_0^{\pi}\frac{\omega(t)}{t^{1/p}}|\widetilde{K}_n(t)|dt \\
&= \left(\int_0^{1/(n+1)}+\int_{1/(n+1)}^{\pi}\right)\left(\frac{\omega(t)}{t^{1/p}}|\widetilde{K}_n(t)|dt\right) \\
&= I_1^{'}+I_2^{'}. \qquad (3.12.1)
\end{aligned}$$

Now, using Lemma 3.4.3 for $0<t\leq 1/(n+1)$, we have

$$\begin{aligned}
I_1^{'} &= \int_0^{1/(n+1)}\frac{\omega\,(t)}{t^{1/p}}|\widetilde{K}_n(t)|dt \\
&= O\left(\int_0^{1/(n+1)}\frac{\omega\,(t)}{t^{1+1/p}}\,dt\right) \\
&= O\left(\omega(1/(n+1)(n+1)^{1/p}\right), \qquad (3.12.2)
\end{aligned}$$

in view of condition (3.10.1).

Again, using Lemma 3.4.3 for $1/(n+1)<t\leq\pi$, we have

$$\begin{aligned}
I_2^{'} &= \int_{1/(n+1)}^{\pi}\frac{\omega\,(t)}{t^{1/p}}|\widetilde{K}_n(t)|dt \\
&= O\left(\frac{1}{n+1}\int_{1/(n+1)}^{\pi}\frac{\omega\,(t)}{t^{2+1/p}}\,dt\right).
\end{aligned}$$

Now, following the proof of Theorem 3.10.1, we have

$$\omega\left(1/(n+1)\right)(n+1)^{1/p}=O\left(\frac{1}{n+1}\int_{1/(n+1)}^{\pi}\frac{\omega(t)}{t^{2+1/p}}\,dt\right). \qquad (3.12.3)$$

Collecting (3.12.1)-(3.12.3), we get

$$\left\|\tilde{t}_n(f;x)-\tilde{f}(x)\right\|_p=O\left(\frac{1}{n+1}\int_{1/(n+1)}^{\pi}\frac{\omega(t)}{t^{2+1/p}}\,dt\right).$$

Hence the proof of Theorem 3.10.2 is completed.

3.13 Corollaries

The following corollaries can be derived from our theorems:

1. If $\omega(t) = t^{1/p}\xi(t)$, then $f \in Lip(\xi(t), p)$-class and condition (3.10.1) reduces to the condition
$$\int_0^v \frac{\xi(t)}{t}\, dt = O(\xi(v)), \quad 0 < v < \pi.$$
Thus for $f \in Lip(\xi(t), p)$, we have
$$\|t_n(f;x) - f(x)\|_p = O\left(\frac{1}{n+1}\int_{1/(n+1)}^{\pi} \frac{\xi(t)}{t^2}\, dt\right)$$
and
$$\left\|\tilde{t}_n(f;x) - \tilde{f}(x)\right\|_p = O\left(\frac{1}{n+1}\int_{1/(n+1)}^{\pi} \frac{\xi(t)}{t^2}\, dt\right).$$

2. If $\omega(t) = t^{\alpha+1/p}$, then $f \in Lip(\alpha, p)$-class and condition (3.10.1) reduces to the condition
$$\int_0^v t^{\alpha-1} dt = O(v^\alpha), \quad 0 < v < \pi.$$
Thus for $f \in Lip(\alpha, p)$, we have
$$\begin{aligned}\left\|\tilde{t}_n(f;x) - \tilde{f}(x)\right\|_p &= O\left(\frac{1}{n+1}\int_{1/(n+1)}^{\pi} t^{\alpha-2} dt\right) \\ &= \begin{cases} O\left((n+1)^{-\alpha}\right), & \text{for } 0 < \alpha < 1 \\ O\left(\frac{\log \pi(n+1)}{n+1}\right), & \text{for } \alpha = 1. \end{cases}\end{aligned}$$

   Corollaries 1 and 2 are analogous to the results given by Rhoades [115, Theorems 2, 3, 5 and 6, pp.393-395] and Lal and Mishra [68, Theorem 3.1, Theorem 3.2, pp.4-5].

3. If we take
$$a_{n,k} = \begin{cases} p_{n-k}/P_n, & 0 \le k \le n \\ 0, & k > n, \end{cases}$$
where $P_n = \sum_{k=0}^{n} p_k \neq 0 \to \infty$ as $n \to \infty$, then the matrix T reduces to Nörlund matrix N_p. Also for non-decreasing sequence $\{a_{n,k}\}$ in k, $\{p_n\}$ is non-increasing. If we replace matrix $T = \{a_{n,k}\}$ with Nörlund matrix $N_p = \{p_n\}$ where p_n is non-increasing, we get Nörlund-version of Theorem 3.10.1 and Theorem 3.10.2.

Chapter 4

Approximation in Generalized Lipschitz Class with Muckenhoupt Weights

4.1 Introduction

In this chapter, we generalize the definition of $Lip\,(\alpha, p, w)$ to the weighted Lipschitz class $Lip(\xi(\delta), p, w)$, where $\xi(\delta)$ is a positive non-decreasing function and determine the degree of approximation of $f \in Lip(\xi(\delta), p, w)$ through matrix means of its trigonometric Fourier series. Our results generalize some earlier results.

A measurable 2π periodic function $w : \mathbb{R} \to [0, \infty]$ is said to be a weight function if the set $w^{-1}(\{0, \infty\})$ has Lebesgue measure zero. For $1 < p < \infty$, a weight function w belongs to the Muckenhoupt class $\mathcal{A}_p$ if

$$\sup_I \left(\frac{1}{|I|} \int_I w(x)dx \right) \left(\frac{1}{|I|} \int_I [w(x)]^{-1/(p-1)} dx \right)^{p-1} < \infty,$$

where the supremum is taken over all intervals I with length $|I| \leq 2\pi$. The weighted Lebesgue space L^p_w, space of all 2π-periodic measurable functions f on $[0, 2\pi]$ is defined as:

$$f \in L^p_w \ \text{ if } \ \|f\|_{p,w} = \left(\int_0^{2\pi} |f(x)|^p w(x)dx \right)^{1/p} < \infty.$$

For $1 < p < \infty$, let $w \in \mathcal{A}_p$ and $f \in L^p_w$. Then the modulus of continuity of the function f is defined by

$$\Omega(f;\delta)_{p,w} = \sup_{|h|\leq\delta} \|\Delta_h(f)\|_{p,w},\ \delta > 0,$$

where $\Delta_h(f;x) = \frac{1}{h}\int_0^h |f(x+t) - f(x)|dt$. The modulus of continuity $\Omega(f;\delta)_{p,w}$ is non-negative, non-decreasing, continuous function such that $\lim\limits_{\delta\to 0}\Omega(f;\delta)_{p,w} = 0$ and $\Omega(f_1 + f_2;\delta)_{p,w} \leq \Omega(f_1;\delta)_{p,w} + \Omega(f_2;\delta)_{p,w}$. For more details one can see [63; 94]. The weighted Lipschitz class $Lip(\alpha, p, w)(0 < \alpha \leq 1)$ [36; 37; 126] is defined by

$$Lip(\alpha, p, w) = \{f \in L^p_w : \Omega(f;\delta)_{p,w} = O(\delta^\alpha),\ \delta > 0\}$$

We extend this class to the generalized Lipschitz class $Lip(\xi(\delta), p, w)$ defined by

$$Lip(\xi(\delta), p, w) = \{f \in L^p_w : \Omega(f;\delta)_{p,w} = O(\xi(\delta)),\ \delta > 0\},$$

where $\xi(\delta)$ is a positive non-decreasing function of δ.
For $\xi(\delta) = \delta^\alpha$, the $Lip(\xi(\delta), p, w)$-class reduces to the $Lip(\alpha, p, w)$-class. For $w(x) = 1\ \forall x \in [0, 2\pi]$ and $\xi(\delta) = \delta^\alpha(< \alpha \leq 1)$, the $Lip(\xi(\delta), p, w)$-class reduces to the well known $Lip(\alpha, p)$-class.

A positive sequence $\mathbf{a} = \{a_{n,k}\}$ is called almost monotonically decreasing($AMDS$) or almost monotonically increasing($AMIS$) with respect to k, if there exist a constant $K = K(\mathbf{a})$, depending on the sequence $\mathbf{a}$ only, such that $a_{n,p} \leq Ka_{n,m}$ or $a_{n,p} \geq Ka_{n,m}$ for all $p \geq m$. We write $\Delta_k a_{n,k} = a_{n,k} - a_{n,k+1}$ and $[x]$, the greatest integer contained in x.

Many authors such as Chandra [15], Mittal et al. [87] and Leindler [73] have studied the approximation properties of the means $N_n(f;x)$, $R_n(f;x)$ and matrix means $t_n(f;x)$ with almost monotone sequence in the Lebesgue space L^p. Chandra [15] has proved $\|N_n(f;x) - f(x)\|_p = \|R_n(f;x) - f(x)\|_p = O(n^{-\alpha})$, $n \in \mathbb{N}$, in $Lip(\alpha, p)(0 < \alpha \leq 1)$ with monotonicity conditions on the summability means generated by the sequence $\{p_n\}$. Mittal et al. [87] generalized the paper of Chandra [15] partially, and extended its results to matrix means with $\left|\sum\limits_{k=0}^{n} a_{n,k} - 1\right| = O(n^{-\alpha})$. On the other hand, Leindler [73] has relaxed the condition of monotonicity on $\{p_n\}$

and proved some of the results of Chandra [15] for almost monotone weights $\{p_n\}$. Later, Guven [36] has extended the results of Chandra [15] for the weighted Lipschitz class $Lip(\alpha, p, w)(0 < \alpha \leq 1,\ 1 < p < \infty)$ and proved the following results:

Theorem A [36, Theorem 1, p.101]

Let $1 < p < \infty$, $w \in \mathcal{A}_p$, $0 < \alpha \leq 1$ and let $\{p_n\}$ be a monotonic sequence of positive real numbers such that $(n+1)p_n = O(P_n)$. Then, for every $f \in Lip(\alpha, p, w)$ the estimate

$$\|N_n(f;x) - f(x)\|_{p,w} = O(n^{-\alpha}),\ \ n = 1, 2, ...$$

holds.

Theorem B [36, Theorem 2, p.101]

Let $1 < p < \infty$, $w \in \mathcal{A}_p$, $0 < \alpha \leq 1$ and let $\{p_n\}$ be a sequence of positive real numbers satisfying the relation $\sum_{m=0}^{n-1} \left| \frac{P_m}{m+1} - \frac{P_{m+1}}{m+2} \right| = O\left(\frac{P_n}{n+1}\right)$. Then, for every $f \in Lip(\alpha, p, w)$ the estimate

$$\|R_n(f;x) - f(x)\|_{p,w} = O(n^{-\alpha}),\ \ n = 1, 2, ...$$

satisfied.

Also Guven [37] has proved the following results for $a_{n,k} \in AMIS$ or $AMDS$:

Theorem C [37, Theorem 1, p.14]

Let $1 < p < \infty$, $w \in \mathcal{A}_p$, $0 < \alpha \leq 1$, $f \in Lip(\alpha, p, w)$ and $A = (a_{n,k})$ be a lower triangular regular matrix with $|s_n^{(A)} - 1| = O(n^{-\alpha})$. If one of the conditions

(i) A has almost monotone decreasing rows and $(n+1)a_{n,0} = O(1)$,

(ii) A has almost monotone increasing rows and $(n+1)a_{n,r} = O(1)$,

where $r = [n/2]$, holds, then

$$\|f - T_n^{(A)}(f)\|_{p,w} = O(n^{-\alpha}).$$

Recently, Singh and Srivastava [126] extended the results of Guven [36] to the matrix means $\tau_n(f;x)$ under the relaxed conditions of monotonicity in the same class $Lip(\alpha, p, w)$. We study these results further for more general class $Lip(\xi(\delta), p, w)$.

4.2 Main Result

Theorem 4.2.1. *Let* $1 < p < \infty$, $w \in \mathcal{A}_p$, $f \in Lip(\xi(\delta), p, w)$ *and* $A = (a_{n,k})$ *be a lower triangular regular matrix with satisfying one of the following conditions:*
(i) $\{a_{n,k}\} \in AMDS$ *in* k *and* $(n+1)a_{n,0} = O(1)$,
(ii) $\{a_{n,k}\} \in AMIS$ *in* k,
(iii) $\sum_{k=0}^{n} \left| \Delta_k \left(\frac{A_{n,0} - A_{n,k+1}}{k} \right) \right| = O\left(1/n\right).$
Then

$$\|f(x) - t_n(f;x)\|_{p,w} = O(\xi(1/n)),$$

where $\xi(\delta)$ *is a positive non-decreasing function satisfying*

$$\xi(1/\delta)\delta^{\sigma} \ \text{ is an non-decreasing function for some } \sigma > 0. \tag{4.2.1}$$

Note 3. For the case (iii), we must have $0 < \sigma < 1$.

4.3 Lemmas

To prove our theorem, we need the following lemmas:

Lemma 4.3.1. *Let* $1 < p < \infty$, $w \in \mathcal{A}_p$, *and* $f \in Lip(\xi(\delta), p, w)$. *Then*

$$\|f - s_n(f;x)\|_{p,w} = O(\xi(1/n)), \ n \in \mathbb{N}. \tag{4.3.1}$$

Proof. Let T_n^* $(n \in \mathbb{N} \cup \{0\})$ be the trigonometric polynomial of best approximation of f. Then

$$E_n(f)_{p,w} = \| f(x) - T_n^*(x) \|_{p,w} .$$

From Theorem 2 of [63], we have

$$\begin{aligned} E_n(f)_{p,w} &= O\left(\Omega(f, 1/n)_{p,w}\right) \\ &= O\left(\xi(1/n)\right). \end{aligned}$$

The work of this chapter has been communicated for possible publication.

By the uniform boundedness of $s_n(f;x)$ in the space L^p_w (for more details see [44]), we have

$$\begin{aligned}\|f-s_n(f;x)\|_{p,w} &\leq \|f-T^*_n(x)\|_{p,w}+\|T^*_n(x)-s_n(f;x)\|_{p,w}\\ &= \|f-T^*_n(x)\|_{p,w}+\|s_n(f(x)-T^*_n(x))\|_{p,w}\\ &= O\left(\|f-T^*_n(x)\|_{p,w}\right)\\ &= O\left(\xi(1/n)\right).\end{aligned}$$

□

Lemma 4.3.2. *Let either* $\{a_{n,k}\}\in AMDS$ *in* k *and* $(n+1)a_{n,0}=O(1)$ *or* $\{a_{n,k}\}\in AMIS$. *Then*

$$\sum_{k=0}^{n} a_{n,k}\xi(1/k)=O(\xi(1/n)).$$

Proof. Let $r=[n/2]$ and $\{a_{n,k}\}\in AMIS$. Then

$$\begin{aligned}\sum_{k=0}^{n} a_{n,k}\xi(1/k) &\leq \sum_{k=0}^{r} a_{n,k}\xi(1/k)+\sum_{k=r+1}^{n} a_{n,k}\xi(1/k)\\ &\leq Ka_{n,r}\sum_{k=0}^{r}\frac{\xi(1/k)k^\sigma}{k^\sigma}+\xi(1/(r+1))\sum_{k=r+1}^{n} a_{n,k}\\ &\leq Ka_{n,r}\xi(1/(r))r^\sigma\sum_{k=0}^{n} k^{-\sigma}+\xi(1/(r+1))\sum_{k=0}^{n} a_{n,k}\\ &= O(\xi(1/r)(r/n)^{1-\sigma})+\xi(1/n)\\ &= O(\xi(1/n)),\end{aligned}$$

in view of $ra_{n,r}\leq(n-r)a_{n,r}\leq K(a_{n,r+1}+...+a_{n,n})\leq A_{n,0}$ and condition (4.2.1). If $\{a_{n,k}\}\in AMDS$ and $(n+1)a_{n,0}=O(1)$, then

$$\begin{aligned}\sum_{k=0}^{n} a_{n,k}\xi(1/k) &\leq Ka_{n,0}\sum_{k=0}^{n}\frac{\xi(1/k)k^\sigma}{k^\sigma}\\ &= O(Ka_{n,0}\xi(1/n)n^\sigma n^{1-\sigma})\\ &= O(\xi(1/n)),\end{aligned}$$

in view of condition (4.2.1). □

4.4 Proof of Theorem 4.2.1

We prove both the cases (i) and (ii) together. From 3.1.1, we have

$$t_n(f;x) - f(x) = \sum_{k=0}^{n} a_{n,k}\{s_k(f;x) - f(x)\},$$

Using Lemma 4.3.1 and 4.3.2, we have

$$\begin{aligned} \|t_n(f;x) - f(x)\|_{p,w} &= \sum_{k=0}^{n} a_{n,k}\|s_k(f;x) - f(x)\|_{p,w} \\ &= O\left(\sum_{k=0}^{n} a_{n,k}\xi(1/k)\right) \\ &= O\left(\xi(1/n)\right), \end{aligned} \tag{4.4.1}$$

Now, we prove the case (iii). Using Abel's transform, we have

$$\begin{aligned} \|t_n(f;x) - f(x)\|_{p,w} &= O\left(\sum_{k=0}^{n} a_{n,k}\xi(1/k)\right) \\ &= \sum_{k=0}^{n-1} \Delta_k\xi(1/k) \sum_{i=0}^{k} a_{n,i} + \xi(1/n) \sum_{i=0}^{n} a_{n,i} \\ &= \sum_{k=0}^{n-1} (A_{n,0} - A_{n,k+1})\left(\xi(1/k) - \xi(1/k+1)\right) + \xi(1/n)A_{n,0} \\ &= \sum_{k=0}^{n-1} (A_{n,0} - A_{n,k+1})\left(\frac{k^\sigma\xi(1/k)}{k^\sigma} - \frac{(k+1)^\sigma\xi(1/(k+1))}{(k+1)^\sigma}\right) + \xi(1/n) \\ &= O(n^\sigma\xi(1/n)) \sum_{k=0}^{n-1} (A_{n,0} - A_{n,k+1})\left(\frac{1}{k^\sigma} - \frac{1}{(k+1)^\sigma}\right) + \xi(1/n) \\ &= O(n^\sigma\xi(1/n)) \sum_{k=0}^{n} \frac{k^{-\sigma}(A_{n,0} - A_{n,k+1})}{k} + \xi(1/n), \end{aligned} \tag{4.4.2}$$

in view of $A_{n,0} = 1$.

Again, using Abel's transform, we have

$$\sum_{k=0}^{n}\frac{k^{-\sigma}(A_{n,0}-A_{n,k+1})}{k}$$

$$\begin{aligned}
&= \sum_{k=0}^{n-1}\Delta_k\frac{(A_{n,0}-A_{n,k+1})}{k}\sum_{i=0}^{k}i^{-\sigma}+\frac{(A_{n,0}-A_{n,n+1})}{n}\sum_{i=0}^{n}i^{-\sigma}\\
&= \sum_{k=0}^{n-1}k^{1-\sigma}\Delta_k\frac{(A_{n,0}-A_{n,k+1})}{k}+\frac{(A_{n,0}-A_{n,n+1})}{n}n^{1-\sigma}\\
&= n^{1-\sigma}\sum_{k=0}^{n-1}\Delta_k\frac{(A_{n,0}-A_{n,k+1})}{k}+n^{-\sigma}\\
&= O(n^{-\sigma}), \qquad (4.4.3)
\end{aligned}$$

in view of $A_{n,n+1}=0$ and condition (iii) of Theorem 4.2.1 .
Collecting, (4.4.1) and (4.4.3), we have

$$\|t_n(f;x)-f(x)\|_{p,w}=O\left(\xi(1/n)\right).$$

Hence the proof of Theorem 4.2.1 is completed.

4.5 Particular Cases

1. If $\xi(\delta)=\delta^{\alpha}(0<\alpha\leq 1)$, then the $Lip(\xi(\delta),p,w)$-class reduces to the $Lip(\alpha,p,w)$ -class and Lemma 4.3.1 reduces to Lemma 4 of Guven [36, p.102]; and Theorem 4.2.1 is a partial extension of results of Singh and Srivastava [126, Theorem 1].

2. Further, if we take

$$a_{n,k}=\begin{cases} p_{n-k}/P_n, & 0\leq k\leq n\\ 0, & k>n,\end{cases}$$

then our matrix means reduce to Nörlund means. If $(a_{n,k})\in AMDS$, then the Nörlund matrix A has almost monotone increasing rows and if $(a_{n,k})\in AMIS$, then the Nörlund matrix A has almost monotone decreasing rows and $(n+1)a_{n,0}=(n+1)\frac{p_n}{P_n}=O(1)$. Thus, our Theorem 4.2.1 reduces to Theorem A of Guven [36, Theorem 1, p.101] for $\sigma>\alpha$.

3. Similarly, if $\xi(\delta) = \delta^\alpha (0 < \alpha \le 1)$ and

$$a_{n,k} = \begin{cases} p_k/P_n, & 0 \le k \le n \\ 0, & k > n, \end{cases}$$

then our matrix means reduce to $R_n(f;x)$ means. Also

$$\begin{aligned} A_{n,0} - A_{n,k+1} &= \sum_{i=0}^{n} a_{n,i} - \sum_{i=k+1}^{n} a_{n,i} \\ &= \left(\sum_{i=0}^{n} p_i - \sum_{i=k+1}^{n} p_i \right) / P_n \\ &= \sum_{i=0}^{k} p_i / P_n = P_k / P_n. \end{aligned}$$

So that,

$$\begin{aligned} \Delta_k \left(\frac{A_{n,0} - A_{n,k+1}}{k} \right) &= \frac{A_{n,0} - A_{n,k+2}}{k+1} - \frac{A_{n,0} - A_{n,k+1}}{k} \\ &= \frac{1}{P_n} \left(\frac{P_{k+1}}{k+1} - \frac{P_k}{k} \right), \end{aligned}$$

i.e., condition (iii) of Theorem 4.2.1 reduces to the condition on $\{p_n\}$ of Theorem B of Guven [36, Theorem 2, p.101] for $0 < \alpha < 1$. Thus, our Theorem 4.2.1 reduces to Theorem B of Guven [36, Theorem 2, p.101].

Chapter 5

Approximation by Matrix Means of Walsh–Fourier Series in L^p-Norm

5.1 Introduction

The study of approximation properties of the periodic functions in $L^p[0,1)(1 \leq p \leq \infty)$-spaces, in general, and in Lipschitz classes $Lip\alpha$ and $Lip(\alpha, p)$, in particular, through Walsh- Fourier series, has been a problem of growing interests over the decades. In this chapter, we determine the degree of approximation of 1-periodic functions in $L^p[0,1)$ and $Lip(\alpha, p)$ classes by the polynomials generated by matrix means of the Walsh-Fourier series associated with the functions under relaxed conditions on the matrix $T \equiv (a_{n,k})$. We then show that many of the theorems in the literature dealing with this area are special cases of our work.

Let $L^p(I)(1 \leq p < \infty)$ be the space of all p^{th} integrable functions defined on $I := [0,1)$. For $p = \infty$, L^p is interpreted to be $C_W(I)$. The space $C_W(I)$ is collection of all uniformly W-continuous functions on I, where a function $f : I \to \mathbb{R}$ is said to be uniformly W-continuous if it is uniformly continuous from the dyadic topology on I to the usual topology on $\mathbb{R}$. The dyadic topology is generated by the collection of all dyadic intervals $I_m(k)$. For more details one can see the litreture [30; 105; 144].

In this chapter, we study the approximation by means of Walsh-Fourier series in the norms of $L^p(I),\ 1 \le p < \infty$ and $C_W(I)$. The L^p norm of $f \in L^p(I)$ is defined by

$$\|f\|_p = \begin{cases} \left(\int_0^1 |f(x)|^p dx\right)^{1/p}, & 1 \le p < \infty \\ \operatorname*{ess\,sup}_{x\in I}|f(x)|, & p = \infty. \end{cases} \tag{5.1.1}$$

The dyadic modulus of continuity of f in L^p-space, denoted by $\dot{\omega}_p(f;\delta)$, is defined as

$$\dot{\omega}_p(f;\delta) = \sup_{0<t<\delta} \|f(x\oplus t) - f(x)\|_p, \tag{5.1.2}$$

where $1 \le p \le \infty,\ \delta > 0$. Accordingly, for $\alpha > 0$, the Lipschitz classes are defined by

$$Lip(\alpha, p) = \{f \in L^p(I) : \dot{\omega}_p(f;\delta) = O(\delta^\alpha)\}. \tag{5.1.3}$$

Let $T \equiv (a_{n,k})$ be a lower triangular matrix. Then the sequence to sequence transformation

$$t_n(f;x) = \sum_{k=1}^{n} a_{n,k} s_k(f;x),\ n \in \mathbb{N}_0 = \mathbb{N}\cup\{0\}, \tag{5.1.4}$$

defines the matrix means of $\{s_n(f;x)\}$. The Walsh-Fourier series (1.1.5) is said to be summable to s by T-means, if $\lim_{n\to\infty} t_n(f;x) = s$, where s is a finite number.
We also write

$$A_{n,k} := \sum_{r=k}^{n} a_{n,r},\ 0 \le k \le n,\ \text{where } A_{n,0} = 1\ \forall\, n;\ A_{n,k} \ge A_{n,k+1}\ \text{for } 0 \le k \le n,$$

$$L_n(t) := \sum_{k=1}^{n} a_{n,k} D_k(t),\ n \ge 1,\ \text{called the matrix mean kernel,}$$

$$K_n(t) := \frac{1}{n}\sum_{k=1}^{n} D_k(t),\ \ n \ge 1,\ \text{called the Walsh-Fejer kernel,}$$

and

$$R_{n,k}(t) = \sum_{i=1}^{k} a_{n,n-k+i} D_i(t)\quad \text{for}\ n = 2^m + k,\ 1 \le k \le 2^m,\ m \ge 1.$$

Recently, Priti et al. [105] extended the results of [91; 93] using a lower triangular matrix means $T \equiv (a_{n,k})$ as follows:

Theorem A: Let $f \in L^p(I)$, $1 \le p \le \infty,\ n = 2^m + k$, $1 \le k \le 2^m$, $m \ge 1$. Let $T \equiv (a_{n,k})$ be an infinite regular triangular matrix such that

$$\sum_{k=0}^{n-1} a^{\gamma}_{n,n-k} = O(n^{1-\gamma})\ \text{for some}\ 1 < \gamma \le 2.$$

(i) If $a_{n,k} \leq a_{n,k+1}$ $\forall$ $0 \leq k \leq n-1$, then

$$\|t_n(f;x) - f(x)\|_p \leq \frac{5}{2} \sum_{j=0}^{m-1} 2^j (A_{n,2^j} - A_{n,2^{j+1}}) \dot{\omega}_p(f; 2^{-j}) + \dot{\omega}_p(f; 2^{-m}),$$

(ii) if $a_{n,k} \geq a_{n,k+1}$ $\forall$ $0 \leq k \leq n-1$, then

$$\|t_n(f;x) - f(x)\|_p \leq \frac{5}{2} \sum_{j=0}^{m-1} 2^j a_{n,2^j} \dot{\omega}_p(f; 2^{-j}) + \dot{\omega}_p(f; 2^{-m}).$$

The analogous of [91; 93; 105] has been discussed by Blahota and Nagy [9] very recently. The authors [9] used Θ-means, generated by the sequence of lower triangular matrices $\Theta_n = (\theta_{k,n})$ with $\theta_{0,k} = 1$ $\forall$ $1 \leq k \leq n$.

5.2 Main Results

We consider matrix means of the Walsh–Fourier series generated by a lower triangular matrix $T \equiv (a_{n,k})$, where $\{a_{n,k}\}$ is almost monotone sequence with respect to k, and find the rate of approximation of $f \in L^p(I)$ and its subclass $Lip(\alpha, p)$ under the less conditions. More precisely, we prove the following:

Theorem 5.2.1. *Let $f \in L^p(I)$, $1 \leq p \leq \infty$. Let $T \equiv (a_{n,k})$ be a lower triangular regular matrix with non-negative entries, where $n = 2^m + k$ for $1 \leq k \leq 2^m$ and $m \geq 1$. Then*
(i) if $\{a_{n,k}\} \in AMIS$ in k and $na_{n,n} = O(1)$, then

$$\|t_n(f;x) - f(x)\|_p = O\left(\sum_{j=0}^{m-1} 2^j a_{n,2^{j+1}-1} \dot{\omega}_p(f; 2^{-j}) + \dot{\omega}_p(f; 2^{-m})\right),$$

(ii) if $\{a_{n,k}\} \in AMDS$ in k, then

$$\|t_n(f;x) - f(x)\|_p = O\left(\sum_{j=0}^{m-1} 2^j a_{n,2^j} \dot{\omega}_p(f; 2^{-j}) + \dot{\omega}_p(f; 2^{-m})\right).$$

Theorem 5.2.2. *Let $f \in Lip(\alpha, p)$, $\alpha > 0$ and $1 \leq p \leq \infty$. Let $T \equiv (a_{n,k})$ be a lower triangular regular matrix with non-negative entries, where $n = 2^m + k$ for $1 \leq k \leq 2^m$ and $m \geq 1$. Then* (*i*) *if* $\{a_{n,k}\} \in AMIS$ *in* k *and* $na_{n,n} = O(1)$, *then*

$$\|t_n(f;x) - f(x)\|_p = \begin{cases} O(n^{-\alpha})\,, & \text{if } 0 < \alpha < 1 \\ O(n^{-1}\log n)\,, & \text{if } \alpha = 1 \\ O(n^{-1})\,, & \text{if } \alpha > 1, \end{cases}$$

(*ii*) *if* $\{a_{n,k}\} \in AMDS$ *in* k, *then*

$$\|t_n(f;x) - f(x)\|_p = O\left(\sum_{j=0}^{m-1} 2^{(1-\alpha)j} a_{n,2^j} + 2^{-m\alpha}\right).$$

Note 4. For $\{a_{n,k}\} \in AMIS$ the condition $na_{n,n} = O(1)$ is sufficient to satisfy the condition (2.4) of Priti et al. [105, p. 35], i.e., $\sum_{k=0}^{n-1} a^{\gamma}_{n,n-k} = O(n^{1-\gamma})$. We have relaxed this condition for $\{a_{n,k}\} \in AMDS$ in our results.

5.3 Lemmas

To prove our results, we need the following lemmas.

Lemma 5.3.1. *Let* $n = 2^m + k$ *for* $1 \le k \le 2^m$ *and* $m \ge 1$. *Then*

$$\begin{aligned} L_n(t) &= -\sum_{j=0}^{m-1} r_j(t) w_{2^j-1}(t) \sum_{i=1}^{2^j-1} (a_{n,2^{j+1}-i} - a_{n,2^{j+1}-i-1}) i K_i(t) - \sum_{j=0}^{m-1} r_j(t) w_{2^j-1}(t) 2^j K_{2^j}(t) a_{n,2^j} \\ &\quad + \sum_{j=0}^{m-1} (A_{n,2^j} - A_{n,2^{j+1}}) D_{2^{j+1}}(t) + A_{n,n-k} D_{2^m}(t) + r_m(t) R_{n,k}(t). \end{aligned}$$

For the proof one can see [105, Lemma 2, p.37].

Lemma 5.3.2. *Let* $f \in L^p(I)(1 \le p \le \infty)$ *and* $g \in \mathcal{P}_{2^m}(m \ge 0)$. *Then*

$$\int_0^1 |g_j(t) r_j(t)| \dot{\omega}_p(f,t) dt \le 2^{-1} \dot{\omega}_p(f, 2^{-m}) \int_0^1 |g(t)| dt,$$

The work of this chapter has been communicated for possible publication.

Proof: Since $g \in \mathcal{P}_{2^m}$, it takes a constant value, say $g_m(k)$, on each dyadic interval $I_m(k)(0 \le k < 2^m)$. If $t \in I_m(k)$, then $t \oplus 2^{-m-1} \in I_m(k)$. For $1 \le p < \infty$, we have

$$\begin{aligned}
\int_0^1 |g_j(t) r_j(t)| \dot{\omega}_p(f,t)dt &= \left|\sum_{k=0}^{2^m-1} g_m(k)\right| \int_{I_{m+1}(2k)} \dot{\omega}_p(f;2^{-m-1})dt \\
&\le \sum_{k=0}^{2^m-1} |g_m(k)| \int_{I_{m+1}(2k)} \dot{\omega}_p(f;2^{-m-1})dt \\
&= \sum_{k=0}^{2^m-1} |g_m(k)| 2^{-m-1} \dot{\omega}_p(f;2^{-m}) \\
&= 2^{-1}\dot{\omega}_p(f;2^{-m}) \sum_{k=0}^{2^m-1} |g_m(k)| 2^{-m} \\
&= 2^{-1}\dot{\omega}_p(f;2^{-m}) \sum_{k=0}^{2^m-1} |g_m(k)| \int_{I_m(k)} dt \\
&= 2^{-1}\dot{\omega}_p(f;2^{-m}) \int_0^1 |g(t)|dt.
\end{aligned}$$

Similarly for $p = \infty$, we can prove that

$$\int_0^1 |g_j(t) r_j(t)| \dot{\omega}_\infty(f;t)dt \le 2^{-1}\dot{\omega}_\infty(f;2^{-m}) \int_0^1 |g(t)|dt.$$

Lemma 5.3.3.

$$K_{2^m}(t) \ge 0, \quad \int_0^1 |K_n(t)|dt \le 2, \quad \textit{and} \quad \int_0^1 K_{2^m}(t)dt = 1,$$

for each $t \in I$, *where* $m \ge 0$ *and* $n \ge 1$.

For more details one can see [93, p.380].

Lemma 5.3.4. *Let* $T \equiv (a_{n,k})$ *be a lower triangular regular matrix with non-negative entries satisfying one of the following conditions:*
(i) $\{a_{n,k}\} \in AMIS$ *in* k *and* $na_{n,n} = O(1)$,
(ii) $\{a_{n,k}\} \in AMDS$ *in* k.
Then there exist a constant C *such that*

$$I_{n,k} = \int_0^1 |R_{n,k}(t)|dt \le C \quad \textit{for } 0 \le k \le n \quad \textit{and } n \ge 1.$$

Proof: Using Lemma 2 [92, p.104], we have

$$\begin{aligned} I_{n,k} &= \int_0^1 |R_{n,k}|dt = \int_0^1 \left| \sum_{i=1}^{k} a_{n,n-k+i} D_i(t) \right| dt \\ &= \int_0^1 \left| \sum_{i=0}^{k-1} a_{n,n-i} D_{k-i}(t) \right| dt \leq 4(k-1) \max_{0\leq i\leq k-1} |a_{n,n-i}|. \end{aligned} \tag{5.3.1}$$

Now, if $\{a_{n,k}\} \in AMIS$ and $na_{n,n} = O(1)$, then

$$(k-1) \max_{0\leq i\leq k-1} |a_{n,n-i}| \leq K(k-1)a_{n,n} \leq na_{n,n} = O(1). \tag{5.3.2}$$

If $\{a_{n,k}\} \in AMDS$, then

$$(k-1) \max_{0\leq i\leq k-1} |a_{n,n-i}| \leq K(k-1)a_{n,n-k+1} = O(1). \tag{5.3.3}$$

Using (5.3.2) and (5.3.3) in (5.3.1), we have $I_{n,k} \leq 4 = C$.

5.4 Proof of Theorem 5.2.1

We will prove the results for $1 \leq p < \infty$. The proof is similar and even simpler for $p = \infty$.

From (1.1.7), we can write

$$s_n(f;x) - f(x) = \int_0^1 f(x \oplus t) D_n(t)dt - f(x),$$

and then from (5.1.4)

$$\begin{aligned} t_n(f;x) - f(x) &= \int_0^1 f(x \oplus t) \sum_{k=1}^{n} a_{n,k} D_k(t)dt - f(x) \\ &= \int_0^1 f(x \oplus t) L_n(t)dt - f(x) \\ &= \int_0^1 L_n(t)[f(x \oplus t) - f(x)]dt. \end{aligned}$$

Using generalized Minkowski's inequality, we have

$$\begin{aligned}
\|t_n(f;x)-f(x)\|_p &= \left(\int_0^1 \left|\int_0^1 L_n(t)[f(x\oplus t)-f(x)]dt\right|^p dx\right)^{1/p} \\
&\leq \int_0^1 \left(\int_0^1 |f(x\oplus t)-f(x)|^p dx\right)^{1/p} |L_n(t)|dt \\
&= \int_0^1 \|f(x\oplus t)-f(x)\|_p |L_n(t)|dt \\
&= O\left(\int_0^1 \dot{\omega}_p(f,t)|L_n(t)|dt\right).
\end{aligned}$$

Now, using Lemma 5.3.1,

$$\begin{aligned}
\|t_n(f;x)-f(x)\|_p &\leq \int_0^1 \left|\sum_{j=0}^{m-1} r_j(t)w_{2^j-1}(t)\sum_{i=1}^{2^j-1} i(a_{n,2^{j+1}-i}-a_{n,2^{j+1}-i-1})K_i(t)\right| \dot{\omega}_p(f;t)dt \\
&\quad + \int_0^1 \left|\sum_{j=0}^{m-1} r_j(t)w_{2^j-1}(t)2^j K_{2^j}(t)a_{n,2^j}\right| \dot{\omega}_p(f;t)dt \\
&\quad + \int_0^1 \left|\sum_{j=0}^{m-1}(A_{n,2^j}-A_{n,2^{j+1}})D_{2^{j+1}}(t)\right| \dot{\omega}_p(f;t)dt \\
&\quad + \int_0^1 |A_{n,n-k}D_{2^m}(t)|\dot{\omega}_p(f;t)dt + \int_0^1 |r_m(t)R_{n,k}(t)|\dot{\omega}_p(f;t)dt \\
&= I_1+I_2+I_3+I_4+I_5. \qquad (5.4.1)
\end{aligned}$$

Firstly, we estimate I_1 as follows:
Let $g_j(t) = w_{2^j-1}(t)\sum_{i=1}^{2^j-1} i(a_{n,2^{j+1}-i}-a_{n,2^{j+1}-i-1})K_i(t)$. Then using Lemma 5.3.2, we have

$$\begin{aligned}
I_1 &= \int_0^1 \left|\sum_{j=0}^{m-1} r_j(t)g_j(t)\right| \dot{\omega}_p(f;t)dt \leq \sum_{j=0}^{m-1}\int_0^1 |r_j(t)g_j(t)|\dot{\omega}_p(f;t)dt \qquad (5.4.2) \\
&\leq \sum_{j=0}^{m-1} 2^{-1}\dot{\omega}_p(f;2^{-j})\int_0^1 |g(t)|dt.
\end{aligned}$$

Further,

$$\begin{aligned}\int_0^1 |g_j(t)|dt &\le \int_0^1 |w_{2^j-1}(t)| \sum_{i=1}^{2^j-1} i|a_{n,2^{j+1}-i} - a_{n,2^{j+1}-i-1}||K_i(t)|dt \\ &\le \sum_{i=1}^{2^j-1} i|a_{n,2^{j+1}-i} - a_{n,2^{j+1}-i-1}| \int_0^1 |K_i(t)|dt \\ &\le 2\sum_{i=1}^{2^j-1} i|a_{n,2^{j+1}-i} - a_{n,2^{j+1}-i-1}|,\end{aligned}$$

in view of $\sup_{t\in I}|w_k(t)| = \sup_{t\in I}|\prod_{j=0}^{\infty}(r_j(t))^{k_j}| = 1$.
If $\{a_{n,k}\} \in AMIS$ and $na_{n,n} = O(1)$, then

$$\begin{aligned}2\sum_{i=1}^{2^j-1} i|a_{n,2^{j+1}-i} - a_{n,2^{j+1}-i-1}| &= 2\sum_{r=2^j+1}^{2^{j+1}-1} (2^{j+1}-r)|a_{n,r} - a_{n,r-1}| \\ &= 2^{j+2}\sum_{r=2^j+1}^{2^{j+1}-1} (a_{n,r} - a_{n,r-1}) - 2\sum_{r=2^j+1}^{2^{j+1}-1} r(a_{n,r} - a_{n,r-1}) \\ &= 2^{j+2}(a_{n,2^{j+1}-1} - a_{n,2^j}) - 2\left(-2^j a_{n,2^j} - \sum_{r=2^j+1}^{2^{j+1}-1} a_{n,r} + 2^{j+1}a_{n,2^{j+1}-1}\right) \\ &= 2\sum_{r=2^j}^{2^{j+1}-1} a_{n,r} - 2^j a_{n,2^j} \le 2\sum_{r=2^j}^{2^{j+1}-1} a_{n,r} \le K2^{j+1}a_{n,2^{j+1}-1}. \qquad (5.4.3)\end{aligned}$$

If $\{a_{n,k}\} \in AMDS$, then

$$\begin{aligned}2\sum_{i=1}^{2^j-1} i|a_{n,2^{j+1}-i} - a_{n,2^{j+1}-i-1}| &= 2\sum_{r=2^j+1}^{2^{j+1}-1} (2^{j+1}-r)|a_{n,r} - a_{n,r-1}| \\ &= 2^{j+2}\sum_{r=2^j+1}^{2^{j+1}-1} (a_{n,r-1} - a_{n,r}) + 2\sum_{r=2^j+1}^{2^{j+1}-1} r(a_{n,r} - a_{n,r-1}) \\ &\le K2^{j+1}a_{n,2^j}. \qquad (5.4.4)\end{aligned}$$

From (5.4.3) and (5.4.4), we have

$$\int_0^1 |g_j(t)|dt \le \begin{cases} K2^{j+1}a_{n,2^{j+1}-1}, & \text{if } \{a_{n,k}\} \in AMIS \text{ and } na_{n,n} = O(1) \\ K2^{j+1}a_{n,2^j}, & \text{if } \{a_{n,k}\} \in AMDS, \end{cases}$$

and

$$I_1 \leq \begin{cases} K\sum_{j=0}^{m-1} 2^j a_{n,2^{j+1}-1}\dot{\omega}_p(f;2^{-j}), & \text{if } \{a_{n,k}\} \in AMIS \text{ and } na_{n,n} = O(1) \\ K\sum_{j=0}^{m-1} 2^j a_{n,2^j}\dot{\omega}_p(f;2^{-j}), & \text{if } \{a_{n,k}\} \in AMDS. \end{cases}$$

Now, let $h_j(t) = 2^j a_{n,2^j} w_{2^j-1} K_{2^j}(t)$. Then

$$\begin{aligned} I_2 &= \int_0^1 \left|\sum_{j=0}^{m-1} r_j(t)h_j(t)\right| \dot{\omega}_p(f;t)dt \leq \sum_{j=0}^{m-1}\int_0^1 |r_j(t)h_j(t)|\dot{\omega}_p(f;t)dt \\ &\leq \sum_{j=0}^{m-1} 2^{-1}\dot{\omega}_p(f;2^{-j})\int_0^1 |h_j(t)|dt, \end{aligned}$$

and

$$\int_0^1 |h_j(t)|dt \leq 2^j a_{n,2^j}\int_0^1 K_{2^j}dt = 2^j a_{n,2^j},$$

in view of Lemma 5.3.3.

Hence,

$$I_2 \leq \sum_{j=0}^{m-1} 2^{j-1} a_{n,2^j}\dot{\omega}_p(f;2^{-j}). \tag{5.4.5}$$

Further,

$$\begin{aligned} I_3 &= \int_0^1 \left|\sum_{j=0}^{m-1}(A_{n,2^j} - A_{n,2^{j+1}})D_{2^{j+1}}(t)\right| \dot{\omega}_p(f;t)dt \\ &\leq \sum_{j=0}^{m-1}(A_{n,2^j} - A_{n,2^{j+1}})\int_0^1 |D_{2^{j+1}}(t)\dot{\omega}_p(f;t)|dt \\ &\leq \sum_{j=0}^{m-1}(A_{n,2^j} - A_{n,2^{j+1}})\dot{\omega}_p(f;2^{-j-1}), \end{aligned} \tag{5.4.6}$$

in view of (1.1.8).

If $\{a_{n,k}\} \in AMIS$ and $na_{n,n} = O(1)$, then

$$A_{n,2^j} - A_{n,2^{j+1}} = \sum_{k=2^j}^{n} a_{n,k} - \sum_{k=2^{j+1}}^{n} a_{n,k} = \sum_{k=2^j}^{2^{j+1}-1} a_{n,k} \leq K2^j a_{n,2^{j+1}}.$$

If $\{a_{n,k}\} \in AMDS$, then

$$A_{n,2^j} - A_{n,2^{j+1}} \leq K2^j a_{n,2^j}.$$

Hence,

$$I_3 \leq \begin{cases} \sum_{j=0}^{m-1} K2^j a_{n,2^{j+1}} \dot{\omega}_p(f;2^{-j-1}), & \text{if } \{a_{n,k}\} \in AMIS \text{ and } na_{n,n} = O(1) \\ \sum_{j=0}^{m-1} K2^j a_{n,2^j} \dot{\omega}_p(f;2^{-j-1}), & \text{if } \{a_{n,k}\} \in AMDS. \end{cases}$$

Similarly, using (1.1.8), we have

$$\begin{aligned} I_4 &= \int_0^1 |A_{n,n-k} D_{2^m}(t)| \dot{\omega}_p(f;t)dt \\ &= O(\dot{\omega}_p(f;2^{-m})). \end{aligned} \tag{5.4.7}$$

Using Lemma 5.3.4, we have

$$\begin{aligned} I_5 &= \int_0^1 |r_m(t) R_{n,k}(t)| \dot{\omega}_p(f;t)dt \\ &\leq 2^{-1} \dot{\omega}_p(f;2^{-m}) \int_0^1 |R_{n,k}(t)|dt \\ &= O\left(\dot{\omega}_p(f;2^{-m})\right). \end{aligned} \tag{5.4.8}$$

Collecting (5.4.1) - (5.4.8), we have

(i) if $\{a_{n,k}\} \in AMIS$ and $na_{n,n} = O(1)$, then

$$\begin{aligned} \|t_n(f;x) - f(x)\|_p &= O\Big(\sum_{j=0}^{m-1} 2^j a_{n,2^{j+1}-1} \dot{\omega}_p(f;2^{-j}) + \sum_{j=0}^{m-1} 2^{j-1} a_{n,2^j} \dot{\omega}_p(f;2^{-j}) \\ &\quad + \sum_{j=0}^{m-1} 2^j a_{n,2^{j+1}-1} \dot{\omega}_p(f;2^{-j}) \Big) + O(\dot{\omega}_p(f;2^{-m})) \\ &= O\left(\sum_{j=0}^{m-1} 2^j a_{n,2^{j+1}-1} \dot{\omega}_p(f;2^{-j}) + \dot{\omega}_p(f;2^{-m}) \right), \end{aligned} \tag{5.4.9}$$

(ii) if $\{a_{n,k}\} \in AMDS$, then

$$\begin{aligned} \|t_n(f;x) - f(x)\|_p &= O\Big(\sum_{j=0}^{m-1} 2^j a_{n,2^j} \dot{\omega}_p(f;2^{-j}) + \sum_{j=0}^{m-1} 2^{j-1} a_{n,2^j} \dot{\omega}_p(f;2^{-j}) \\ &\quad + \sum_{j=0}^{m-1} 2^j a_{n,2^j} \dot{\omega}_p(f;2^{-j}) \Big) + O(\dot{\omega}_p(f;2^{-m})) \\ &= O\left(\sum_{j=0}^{m-1} 2^j a_{n,2^j} \dot{\omega}_p(f;2^{-j}) + \dot{\omega}_p(f;2^{-m}) \right). \end{aligned} \tag{5.4.10}$$

Hence the proof of Theorem 5.2.1 is completed.

5.5 Proof of Theorem 5.2.2

We have $f \in Lip(\alpha, p) \Rightarrow \dot{\omega}_p(f;\delta) = O(\delta^\alpha)$.
(i) If $\{a_{n,k}\} \in AMIS$ with $na_{n,n} = O(1)$, then from (5.4.9), we have

$$\begin{aligned}
\|t_n(f;x) - f(x)\|_p &= O\left(\sum_{j=0}^{m-1} 2^{(1-\alpha)j} a_{n,2^{j+1}-1} + 2^{-m\alpha}\right)\\
&= O\left(a_{n,2^m-1}\sum_{j=0}^{m-1} 2^{(1-\alpha)j} + 2^{-m\alpha}\right)\\
&= O\left(2^{-m}\sum_{j=0}^{m-1} 2^{(1-\alpha)j} + 2^{-m\alpha}\right)\\
&= \begin{cases} O(n^{-\alpha}), & \text{if } 0<\alpha<1\\ O(n^{-1}\log n), & \text{if } \alpha = 1\\ O(n^{-1}), & \text{if } \alpha>1, \end{cases}
\end{aligned}$$

in view of $n^{-1} \leq 2^{-m}$ for $n = 2^m + k,\ 1 \leq k \leq 2^m$ and $m \geq 1$.
(ii) If $\{a_{n,k}\} \in AMDS$, then from (5.4.10), we have

$$\|t_n(f;x) - f(x)\|_p = O\left(\sum_{j=0}^{m-1} 2^{(1-\alpha)j} a_{n,2^j} + 2^{-m\alpha}\right).$$

Hence the proof of Theorem 5.2.2 is completed.

Note 5. If we take $\{a_{n,k}\} \in AMDS$ and $na_{n,0} = O(1)$ in case (ii) of Theorem 5.2.2, then for both the cases (i) and (ii)

$$\|t_n(f;x) - f(x)\|_p = \begin{cases} O(n^{-\alpha}), & \text{if } 0<\alpha<1\\ O(n^{-1}\log n), & \text{if } \alpha = 1\\ O(n^{-1}), & \text{if } \alpha>1. \end{cases}$$

5.6 Particular Cases

1. Since every monotone sequence is almost monotone, results of Priti et al. [105, Theorem 1 and 2, p.35] are particular cases our results in the light of Note 4.

2. If we take

$$a_{n,k} = \begin{cases} q_{n-k}/Q_n, & 0 \leq k \leq n\\ 0, & k>n, \end{cases}$$

where $Q_n = \sum_{k=0}^{n} q_k$, then our matrix reduces to Nörlund matrix. If $\{a_{n,k}\} \in AMDS$, then the Nörlund matrix has almost monotone increasing rows and if $(a_{n,k}) \in AMIS$, then the Nörlund matrix has almost monotone decreasing rows and

$$na_{n,n} = O(1) \Rightarrow \frac{nq_0}{Q_n} = O(1).$$

Then

$$\frac{n^{\gamma-1}}{Q_n^\gamma} \sum_{k=0}^{n-1} q_k^\gamma \leq \frac{n^{\gamma-1}}{Q_n^\gamma} q_0^\gamma.n = O(1)$$

which is the condition (2.6) of Móricz and Siddiqi [93, p.379]. Thus, our results reduce to the results of Móricz and Siddiqi [93, Theorem 1 and 2, pp.379–380].

3. Similarly, if

$$a_{n,k} = \begin{cases} p_k/P_n, & 0 \leq k \leq n \\ 0, & k > n, \end{cases}$$

where $P_n = \sum_{k=0}^{n} p_k$, then our matrix reduces to Riesz matrix of [91, p.2]. If $\{a_{n,k}\} \in AMIS$ with $na_{n,n} = O(1)$, then the Riesz matrix has almost monotone increasing rows with $\frac{np_n}{P_n} = O(1)$; and if $\{a_{n,k}\} \in AMDS$, then the Riesz matrix has almost monotone decreasing rows. Thus our results reduce to the results of Móricz and Rhoades [91, Theorem 1 and 2, p.3].

Chapter 6

Approximation of $f \in L[0,\infty)$ by Means of Fourier-Laguerre Series

The problem of determining the degree of approximation of $f \in L[0,\infty)$, the space of Lebesgue integrable functions defined on $[0,\infty)$, has been studied by various investigators through summability means of Fourier-Laguerre series of f at $x = 0$. We note that the problem is not studied for $x > 0$. In this chapter, we further study the problem and determine the degree of approximation of $f \in L[0,\infty)$ by Cesáro means of order $\lambda \geq 1$ of the Fourier-Laguerre series of f for any $x > 0$. We prove the result for $x = 0$ separately.

6.1 Introduction

The Cesáro means of order λ, of the Fourier-Laguerre series are given by

$$C_n^\lambda(f;x) = \frac{1}{\begin{pmatrix} n+\lambda \\ n \end{pmatrix}} \sum_{r=0}^{n} \begin{pmatrix} \lambda+n-r-1 \\ n-r \end{pmatrix} s_r(f;x). \tag{6.1.1}$$

The Fourier-Laguerre series is said to be summable to s by Cesáro means, if $C_n^\lambda(f;x) \to s$ as $n \to \infty$.

We also write

$$\phi(x,y) = f(y) - f(x),\ \ \psi(x,u) = f(x \pm u) - f(x)$$

and

$$\sum_{k=0}^{r}\begin{pmatrix} k+\alpha \\ k \end{pmatrix}^{-1} L_k^{(\alpha)}(x)L_k^{(\alpha)}(y) = \Gamma(\alpha+1)A_r^{\alpha}(x,y) \tag{6.1.2}$$

$$= \frac{(r+1)}{\begin{pmatrix} r+\alpha \\ r \end{pmatrix}} \frac{L_r^{(\alpha)}(x)L_{r+1}^{(\alpha)}(y) - L_{r+1}^{(\alpha)}(x)L_r^{(\alpha)}(y)}{x-y} \tag{6.1.3}$$

$$= \frac{(r+1)}{\begin{pmatrix} r+\alpha \\ r \end{pmatrix}} \frac{L_{r+1}^{(\alpha)}(x)L_{r+1}^{(\alpha-1)}(y) - L_{r+1}^{(\alpha-1)}(x)L_{r+1}^{(\alpha)}(y)}{x-y} \tag{6.1.4}$$

$$= \frac{(r+1)}{\begin{pmatrix} r+\alpha \\ r \end{pmatrix}} \left(L_{r+1}^{(\alpha)}(x)\frac{L_{r+1}^{(\alpha-1)}(y) - L_{r+1}^{(\alpha-1)}(x)}{x-y} - L_{r+1}^{(\alpha-1)}(x)\frac{L_{r+1}^{(\alpha)}(y) - L_{r+1}^{(\alpha)}(x)}{x-y} \right). \tag{6.1.5}$$

For more details one can see [136].

The authors in [35; 55; 60; 97; 122; 139] have studied the problems of determining the degree of approximation of $f \in L[0,\infty)$ using different summability means such as Cesáro, Nörlund, Euler and product means of Fourier-Laguerre series of f at the point $x = 0$. Also these results are proved for different ranges of α, for instant, the authors in [35; 122] have fixed $\alpha \geq -1/2$, and the authors in [55; 60; 97; 139] have fixed $-1 \leq \alpha \leq -1/2$. Singh [121] has studied the absolute $(C,1)$-summability of the series

$$\sum_{n=1}^{\infty} \frac{a_n L_n^{(\alpha)}(x)}{(\log(n+1))^{1+\epsilon}},$$

where a_n's are Fourier-Laguerre coefficient of some $f \in L[0,\infty)$ with $x > 0$. However, the approximation of f by means of Fourier-Laguerre series has not been studied so far for $x > 0$, which motivated us to study the problem further.

6.2 Main Result

Theorem 6.2.1. *Let f be a function belonging to $L[0,\infty)$. Then the degree of approximation of f by the Cesáro means of order $\lambda \geq 1$ of the Fourier-Laguerre series* *of f is given by*

$$|C_n^\lambda(f;x) - f(x)| = o\left(\xi(n)\right)$$

where $\xi(t)$ is a positive non-decreasing function such that $\xi(t) \to \infty$ as $t \to \infty$ and satisfies the following conditions:

$$\Phi(t) = \int_t^\epsilon y^{\alpha/2-1/4}|\phi(x,y)|dy = o\left(\xi(1/t)\right), \quad t \to 0, \tag{6.2.1}$$

$$\int_t^\delta \frac{|\psi(x,u)|}{u} du = o\left(\xi(1/t)\right), \quad t \to 0, \tag{6.2.2}$$

and

$$\int_n^\infty e^{-y/2} y^{\alpha/2-13/12}|\phi(x,y)|dy = o\left(n^{-1/2}\xi(n)\right), \quad n \to \infty, \tag{6.2.3}$$

where δ is a fixed positive constant and $\alpha \geq \frac{-1}{2}$. This holds uniformly for every fixed positive interval $0 < \epsilon \leq x \leq \omega < \infty$.

6.3 Lemmas

To prove our theorem, we need the following lemmas:

Lemma 6.3.1. [136, Theorem 7.6.4 p.177]. Let α be an arbitrary real number, c and ϵ be fixed positive constants. Then

$$L_n^{(\alpha)}(x) = \begin{cases} O\left(n^\alpha\right), & 0 \leq x \leq \frac{c}{n} \\ O\left(x^{-(2\alpha+1)/4} n^{(2\alpha-1)/4}\right), & \frac{c}{n} \leq x \leq \epsilon, \end{cases}$$

as $n \to \infty$.

For $\alpha \geq -1/2$, both bounds hold in both intervals.

Lemma 6.3.2. [136, Theorem 8.22.1 p.198]. Let α be an arbitrary real number and $x \in [cn^{-1}, \omega]$. Then

$$L_n^\alpha(x) = k(x) n^{\frac{\alpha}{2}-\frac{1}{4}} \cos\left(2\sqrt{nx} + \gamma\right) + O\left(n^{\frac{\alpha}{2}-\frac{3}{4}}\right), \; x > 0,$$

where $k(x) = \pi^{-1/2} e^{x/2} x^{\frac{-\alpha}{2}-\frac{1}{4}}$ and $\gamma = -(\alpha+1/2)\pi/2$. The bound for the remainder holds uniformly in $[\epsilon, \omega]$.

Lemma 6.3.3. [136, Theorem 8.91.7, p.241]. Let α and ρ be arbitrary real numbers, $\omega > 0$ and $0 < \eta < 4$. Then

$$\max e^{-x/2} x^{\rho} |L_n^{(\alpha)}(x)| \sim n^Q,$$

where

$$Q = \begin{cases} \max\left(\rho - 1/2,\ \alpha/2 - 1/4\right), & \omega \le x \le (4-\eta)n \\ \max\left(\rho - 1/3,\ \alpha/2 - 1/4\right), & x > n. \end{cases}$$

Lemma 6.3.4. [136, p.237]. If both x and y belong to the interval $[cn^{-1},\ \omega]$, then

$$\begin{aligned} \frac{L_n^{(\alpha)}(y) - L_n^{(\alpha)}(x)}{\sqrt{y} - \sqrt{x}} &= k(y)\, n^{\frac{\alpha}{2}-\frac{1}{4}} \frac{\cos\left(2\sqrt{ny} + \gamma\right) - \cos\left(2\sqrt{nx} + \gamma\right)}{\sqrt{y} - \sqrt{x}} \\ &\quad + x^{\frac{-\alpha}{2}-\frac{3}{4}}\, O(n^{\frac{\alpha}{2}-\frac{1}{4}}) + y^{\frac{-\alpha}{2}-\frac{3}{4}}\, O(n^{\frac{\alpha}{2}-\frac{1}{4}}). \end{aligned}$$

Lemma 6.3.5. *If condition* (6.2.1) *holds, then*

$$\int_0^t y^{\alpha} |\phi(x,y)| dy = o\left(\xi\left(1/t\right) t^{\frac{\alpha}{2}+\frac{1}{4}}\right).$$

Proof: We have

$$\begin{aligned} \Phi(t) &= \int_t^{\delta} y^{\alpha/2-1/4} |\phi(x,y)| dy \\ \Phi^{'}(t) &= -t^{\alpha/2-1/4} |\phi(x,t)| \\ -\Phi^{'}(t) t^{\alpha/2+1/4} &= t^{\alpha} |\phi(x,t)| \\ -\int_0^t y^{\alpha/2+1/4} \Phi^{'}(y) dy &= \int_0^t y^{\alpha} |\phi(x,y)| dy \end{aligned}$$

on integrating L.H.S. by parts, we have

$$\int_0^t y^{\alpha} |\phi(x,y)| dy = o\left(t^{\frac{\alpha}{2}+\frac{1}{4}}\, \xi\left(1/t\right)\right).$$

Lemma 6.3.6. *If condition* (6.2.3) *holds, then*

$$\int_{\omega}^{n} e^{-y/2} y^{\alpha/2-3/4} |\phi(x,y)| dy = o\left(\xi(n)\right),$$

where ω is a fixed positive number and $n \to \infty$.

Proof: The proof is similar to the proof of Lemma 6.3.5.

6.4 Proof of Theorem 6.2.1

From (1.1.10) and (6.1.2), we have the following representation of $s_n(f;x)$ obtained in (1.1.11),

$$\begin{aligned} s_n(f;x) &= \sum_{k=0}^{n} \left(\Gamma(\alpha+1)\begin{pmatrix} k+\alpha \\ k \end{pmatrix}\right)^{-1} \left(\int_0^\infty e^{-y}y^\alpha f(y)L_k^{(\alpha)}(y)dy\right) L_k^{(\alpha)}(x) \\ &= (\Gamma(\alpha+1))^{-1}\int_0^\infty e^{-y}y^\alpha f(y)\sum_{k=0}^{n}\frac{L_k^{(\alpha)}(x)L_k^{(\alpha)}(y)}{\begin{pmatrix} k+\alpha \\ k \end{pmatrix}}dy \\ &= \int_0^\infty e^{-y}y^\alpha f(y)A_n^\alpha(x,y)dy. \end{aligned}$$

Therefore, from (6.1.1) we have

$$\begin{aligned} C_n^\lambda(f;x)-f(x) &= \frac{1}{\begin{pmatrix} n+\lambda \\ n \end{pmatrix}}\sum_{r=0}^{n}\begin{pmatrix} \lambda+n-r-1 \\ n-r \end{pmatrix}\int_0^\infty e^{-y}y^\alpha(f(y)-f(x))A_r^\alpha(x,y)dy \\ &= \frac{1}{\begin{pmatrix} n+\lambda \\ n \end{pmatrix}}\sum_{r=0}^{n}\begin{pmatrix} \lambda+n-r-1 \\ n-r \end{pmatrix}\int_0^\infty e^{-y}y^\alpha\phi(x,y)A_r^\alpha(x,y)dy. \end{aligned}$$

Now

$$\begin{aligned} \int_0^\infty e^{-y}y^\alpha\phi(x,y)A_r^\alpha(x,y)dy &= \left(\int_0^{1/n}+\int_{1/n}^{\epsilon}+\int_{\epsilon}^{x-\delta}+\int_{x-\delta}^{x+\delta}+\int_{x+\delta}^{\omega}\right. \\ &\quad \left.+\int_{\omega}^{n}+\int_{n}^{\infty}\right)\left(e^{-y}y^\alpha\phi(x,y)A_r^\alpha(x,y)dy\right) \\ &= I_1+I_2+I_3+I_4+I_5+I_6+I_7. \end{aligned} \tag{6.4.1}$$

For $0\le y<\epsilon$, using formula (6.1.4), Lemma 6.3.1 and Lemma 6.3.5, we have

$$\begin{aligned} |I_1| &\le \int_0^{1/n} y^\alpha|\phi(x,y)||A_r^\alpha(x,y)|dy \\ &= O\left(r^{1-\alpha}\int_0^{1/n} y^\alpha|\phi(x,y)|\left[r^{\alpha/2-1/4}r^{\alpha-1}+r^{\alpha/2-3/4}r^\alpha\right]dy\right) \\ &= O\left(r^{\alpha/2+1/4}\int_0^{1/n} y^\alpha|\phi(x,y)|dy\right) = o(\xi(n)). \end{aligned} \tag{6.4.2}$$

Similarly,

$$\begin{aligned}
|I_2| &\leq \int_{1/n}^{\epsilon} y^\alpha |\phi(x,y)||A_r^\alpha(x,y)|dy \\
&= O\Big(r^{1-\alpha}\int_{1/n}^{\epsilon} y^\alpha|\phi(x,y)|\Big[r^{\alpha/2-1/4}r^{\alpha/2-3/4}y^{-\alpha/2+1/4} \\
&\qquad + r^{\alpha/2-3/4}r^{\alpha/2-1/4}y^{-\alpha/2-1/4}\Big]dy\Big) \\
&= O\left(\int_{1/n}^{\epsilon} y^{\alpha/2-1/4}|\phi(x,y)|dy\right) = o(\xi(n)), \qquad (6.4.3)
\end{aligned}$$

in view of condition (6.2.1).

Further, using the formula (6.1.4), we have

$$\begin{aligned}
I_3 &= \int_{\epsilon}^{x-\delta} e^{-y}y^\alpha \phi(x,y)A_r^\alpha(x,y)dy \\
&= O\left(r^{1-\alpha}\int_{\epsilon}^{x-\delta} e^{-y}y^\alpha\phi(x,y)\frac{L_{r+1}^{(\alpha)}(x)L_{r+1}^{(\alpha-1)}(y) - L_{r+1}^{(\alpha-1)}(x)L_{r+1}^{(\alpha)}(y)}{x-y}dy\right) \\
&= I_{31} + I_{32}. \qquad (6.4.4)
\end{aligned}$$

Using Lemma 6.3.2, we have

$$\begin{aligned}
|I_{31}| &= O\Big(r^{1-\alpha}\int_{\epsilon}^{x-\delta}\frac{e^{-y}y^\alpha|\phi(x,y)|}{x-y}r^{\alpha/2-1/4}\Big[\pi^{-1/2}e^{y/2}y^{-\alpha/2+1/4}r^{\alpha/2-3/4} \\
&\qquad \cos(2\sqrt{ry} - (\alpha+1/2)\pi/2 + \pi/2) + r^{\frac{\alpha}{2}-\frac{5}{4}}\Big]dy\Big) \\
&= O\Big(\int_{\epsilon}^{x-\delta}\frac{e^{-y/2}y^{\alpha/2+1/4}|\phi(x,y)|}{x-y}\sin(2\sqrt{ry} - (\alpha+1/2)\pi/2)dy\Big) \\
&\qquad + O\Big(r^{-1/2}\int_{\epsilon}^{x-\delta}\frac{e^{-y}y^\alpha|\phi(x,y)|}{x-y}dy\Big) + o(1) \\
&= o(1), \qquad (6.4.5)
\end{aligned}$$

the first integral tends to 0 by Riemann-Lebesgue theorem and the second integral tends to 0 as $r \to \infty$.

Similarly,

$$|I_{32}| = o(1). \qquad (6.4.6)$$

Now we evaluate I_4, using Lemma 6.3.2 and Lemma 6.3.4 in formula (6.1.5). Since the variables are bounded by a fixed positive interval, so the remainder terms in the lammas depend only on n. Thus, we have (see [136, p.267])

$$\begin{aligned} A_r^\alpha(x,y) &= r^{1-\alpha} r^{\alpha/2-1/4} r^{(\alpha-1)/2-1/4} \frac{k(x)k(y)}{\sqrt{x}+\sqrt{y}} \\ &\left\{ \sqrt{y}\cos[2\sqrt{rx}+\gamma] \frac{\sin[2\sqrt{ry}+\gamma]-\sin[2\sqrt{rx}+\gamma]}{\sqrt{y}-\sqrt{x}} \right. \\ &\left. -\sqrt{x}\sin[2\sqrt{rx}+\gamma] \frac{\cos[2\sqrt{ry}+\gamma]-\cos 2[\sqrt{rx}+\gamma]}{\sqrt{y}-\sqrt{x}} + O(1) \right\}. \end{aligned}$$

Following the calculations of [136, p.267], we have

$$A_r^\alpha(x,y) = \frac{1}{2}\sqrt{x}\left(\pi^{-1/2} e^{x/2} x^{-\alpha/2-1/4}\right)^2 y^{-1/2} \frac{\sin\{2\sqrt{r}(\sqrt{y}-\sqrt{x})\}}{\sqrt{y}-\sqrt{x}} + O(1)$$

Thus,

$$\begin{aligned} I_4 &= \int_{x-\delta}^{x+\delta} e^{-y} y^\alpha \phi(x,y) A_r^\alpha(x,y) dy \\ &= \left(\int_{x-\delta}^{x-1/n} + \int_{x-1/n}^{x+1/n} + \int_{x+1/n}^{x+\delta} \right) e^{-y} y^\alpha \phi(x,y) A_r^\alpha(x,y) dy \\ &= I_{41} + I_{42} + I_{43}. \end{aligned}$$

$$\begin{aligned} |I_{41}| &= O\left(\int_{x-\delta}^{x-1/n} \frac{|\phi(x,y)|(\sqrt{x}+\sqrt{y})}{|x-y|} dy \right) \\ &= \int_{1/n}^{\delta} \frac{\psi(x,u)}{u} du = o(\xi(n)), \end{aligned}$$

in view of condition (6.2.2).

Similarly,

$$|I_{43}| = o(\xi(n)).$$

$$\begin{aligned} |I_{42}| &= O\left(\int_{x-1/n}^{x+1/n} |\phi(x,y)| \left| \frac{\sin\{2\sqrt{r}(\sqrt{y}-\sqrt{x})\}}{\sqrt{y}-\sqrt{x}} \right| dy \right) \\ &= O\left(r^{1/2} \int_0^{1/n} \psi(x,u) du \right) = O(r^{1/2} n^{-1} \xi(n)) = o(\xi(n)), \end{aligned}$$

in view of condition (6.2.2).

Hence,

$$|I_4| = o(\xi(n)). \tag{6.4.7}$$

Similar to I_3,

$$|I_5| = o(\xi(n)). \tag{6.4.8}$$

Again, using Lemma 6.3.3 for $\omega \le y \le n$ (taking $\eta = 3$) in formula (6.1.4), we have

$$\begin{aligned} |I_6| &\le O(r^{1-\alpha}) \int_{\omega}^{n} e^{-y} y^{\alpha-1} |\phi(x,y)||L_{r+1}^{(\alpha)}(x)||L_{r+1}^{(\alpha-1)}(y)|dy \\ &\quad + O(r^{1-\alpha}) \int_{\omega}^{n} e^{-y} y^{\alpha-1} |\phi(x,y)||L_{r+1}^{(\alpha-1)}(x)||L_{r+1}^{(\alpha)}(y)|dy \\ &= I_{61} + I_{62}. \end{aligned} \tag{6.4.9}$$

$$\begin{aligned} I_{61} &= O\left(r^{-\alpha/2+3/4} \int_{\omega}^{n} e^{-y/2} y^{\alpha/2-3/4} |\phi(x,y)| e^{-y/2} y^{\alpha/2-1/4} |L_{r+1}^{(\alpha-1)}(y)|dy \right) \\ &= \left(r^{-\alpha/2+3/4} r^{\alpha/2-3/4} \right) \int_{\omega}^{n} e^{-y/2} y^{\alpha/2-3/4} |\phi(x,y)|dy \\ &= o(\xi(n)), \end{aligned}$$

in view of Lemma 6.3.6.

Similarly,

$$I_{62} = o(\xi(n)).$$

Thus,

$$|I_6| = o(\xi(n)). \tag{6.4.10}$$

Further, using formula (6.1.3) and Lemma 6.3.3 for $y \ge n$, we have

$$\begin{aligned} |I_7| &= O(r^{1-\alpha}) r^{\alpha/2-1/4} \int_{n}^{\infty} e^{-y} y^{\alpha-1} |\phi(x,y)| \{ |L_r^{(\alpha)}(y)| + |L_{r+1}^{(\alpha)}(y)| \} dy \\ &= O(r^{1-\alpha}) r^{\alpha/2-1/4} r^{\alpha/2-1/4} \int_{\omega}^{n} e^{-y/2} y^{\alpha/2-13/12} |\phi(x,y)|dy \\ &= o(\xi(n)), \end{aligned} \tag{6.4.11}$$

in view of condition (6.2.3).

Collecting (6.4.1) - (6.4.11), we have

$$\begin{aligned} |C_n^{\lambda}(f;x) - f(x)| &= \frac{1}{\begin{pmatrix} n+\lambda \\ n \end{pmatrix}} \sum_{r=0}^{n} \begin{pmatrix} \lambda+n-r-1 \\ n-r \end{pmatrix} o(\xi(n)) \\ &= o\left(\xi(n) n^{-\lambda} \sum_{r=0}^{n} (n-r)^{\lambda-1} \right) = o(n\,\xi(n) n^{-\lambda} n^{\lambda-1}) \\ &= o(\xi(n)). \end{aligned}$$

Hence the proof of Theorem 6.2.1 is completed.

Note 6. As mentioned above, Theorem 6.2.1 is not true for $x = 0$. So, we prove the following theorem for $x = 0$:

Theorem 6.4.1. *Let f be a function belonging to $L[0,\infty)$. Then the degree of approximation of f at $x = 0$ by the Cesáro means of order $\lambda \geq 1$ of the Fourier-Laguerre series of f is given by*

$$|C_n^\lambda(f;0) - f(0)| = o\left(n^{\alpha/2+3/4}\xi(n)\right),$$

where $\xi(t)$ is a positive non-decreasing function such that $\xi(t) \to \infty$ as $t \to \infty$ and satisfies the conditions (6.2.1) *and* (6.2.3) *of Theorem* 6.2.1 *for $x = 0$, $\epsilon > 0$ and $\alpha \in [-1/2,\ 1/2]$.*

Note 7. For $x = 0$, $I_3 + I_4 + I_5 = \int_\epsilon^{-\delta} + \int_{-\delta}^{\delta} + \int_\delta^\omega = \int_\epsilon^\omega = I_3'$. So the condition (6.2.2) is not required for $x = 0$.

6.5 Proof of Theorem 6.4.1

For $x = 0$, from (6.1.2), we have

$$\begin{aligned} A_r^{(\alpha)}(0,y) &= A_r^{(\alpha)}(y) = \frac{1}{\Gamma(\alpha+1)} \sum_{k=0}^{r} \begin{pmatrix} k+\alpha \\ k \end{pmatrix}^{-1} L_k^{(\alpha)}(0) L_k^{(\alpha)}(y) \\ &= \frac{1}{\Gamma(\alpha+1)} \sum_{k=0}^{r} \begin{pmatrix} k+\alpha \\ k \end{pmatrix}^{-1} \begin{pmatrix} k+\alpha \\ k \end{pmatrix} L_k^{(\alpha)}(y) \\ &= \frac{1}{\Gamma(\alpha+1)} \sum_{k=0}^{r} L_k^{(\alpha)}(y) = \frac{1}{\Gamma(\alpha+1)} L_r^{(\alpha+1)}(y) \end{aligned} \tag{6.5.1}$$

and $\phi(y) := \phi(0,y) = f(y) - f(0)$.

$$C_n^\lambda(f;0) - f(0) = \frac{(\Gamma(\alpha+1))^{-1}}{\begin{pmatrix} n+\lambda \\ n \end{pmatrix}} \sum_{r=0}^{n} \begin{pmatrix} \lambda+n-r-1 \\ n-r \end{pmatrix} \int_0^\infty e^{-y} y^\alpha \phi(y) L_r^{(\alpha+1)}(y)\,dy.$$

The work of Theorems 6.2.1 and 6.4.1 has been communicated for possible publication.

Now

$$\int_0^\infty e^{-y}y^\alpha\phi(y)L_r^{(\alpha+1)}(y)dy = \left(\int_0^{1/n}+\int_{1/n}^{\epsilon}+\int_\epsilon^\omega+\int_\omega^n+\int_n^\infty\right)\left(e^{-y}y^\alpha\phi(y)L_r^{(\alpha+1)}(y)dy\right)$$
$$= I_1+I_2+I_3^{'}+I_4+I_5. \tag{6.5.2}$$

Using Lemma 6.3.1 for $0\le y\le \frac{1}{n}$, we have

$$|I_1| \le \int_0^{1/n} y^\alpha|\phi(y)|r^{\alpha+1}dy = o\left(\sup_{0\le y\le n^{-1}}|\phi(y)|r^{\alpha+1}n^{-\alpha-1}\right) = o(1). \tag{6.5.3}$$

Again using Lemma 6.3.1 for $\frac{1}{n}\le y\le\epsilon$, we have

$$|I_2| \le \int_{1/n}^{\epsilon} y^\alpha|\phi(y)|y^{-(\alpha+1)/2-1/4}r^{(\alpha+1)/2-1/4}dy = O\left(r^{\alpha/2+1/4}n^{1/2}\int_{1/n}^{\epsilon}y^{\alpha/2-1/4}|\phi(y)|dy\right) = o\left(r^{\alpha/2+1/4}n^{1/2}\xi(n)\right)=o\left(n^{\alpha/2+3/4}\xi(n)\right), \tag{6.5.4}$$

in view of condition (6.2.1).

Further, using Lemma 6.3.2, we have

$$|I_3^{'}| \le \int_\epsilon^\omega e^{-y}y^\alpha|\phi(y)|\big[\pi^{-1/2}e^{y/2}y^{-\alpha/2-3/4}r^{\alpha/2+1/4}\cos\left(2\sqrt{ry}-(\alpha+1/2)\pi/2-\pi/2\right)+r^{\alpha/2-1/4}\big]dy$$
$$= O\left(r^{\alpha/2+1/4}\int_\epsilon^\omega e^{-y/2}y^{\alpha/2-3/4}|\phi(y)|\sin(2\sqrt{ry}-(\alpha+1/2)\pi/2)dy\right) +O\left(r^{\alpha/2-1/4}\int_\epsilon^\omega e^{-y}y^\alpha|\phi(y)|dy\right)+o(1).$$
$$= o(1), \tag{6.5.5}$$

the first integral tends to 0 by Riemann-Lebesgue theorem and the second integral tends to 0 as $r\to\infty$ for $\alpha\in[-1/2,\ 1/2]$.

Using Lemma 6.3.3 for $\omega \le y \le n$ (taking $\alpha+1$ for α and $\eta = 3$), we have

$$\begin{aligned} |I_4| &\le \int_\omega^n e^{-y} y^\alpha |\phi(y)||L_r^{(\alpha+1)}(y)|dy \\ &= \int_\omega^n e^{-y/2} y^{\alpha/2-3/4} |\phi(y)| e^{-y/2} y^{\alpha/2+3/4} |L_r^{(\alpha+1)}(y)|dy \\ &= O\left(r^{\alpha/2+1/4} \int_\omega^n e^{-y/2} y^{\alpha/2-3/4} |\phi(y)|dy\right) = o\left(r^{\alpha/2+1/4}\xi(n)\right), \quad (6.5.6) \end{aligned}$$

in view of Lemma 6.3.6.
Similarly,

$$\begin{aligned} |I_5| &\le \int_n^\infty e^{-y} y^\alpha |\phi(y)||L_r^{(\alpha+1)}(y)|dy \\ &= \int_n^\infty e^{-y/2} y^{\alpha/2-13/12} |\phi(y)| e^{-y/2} y^{\alpha/2+13/12} |L_r^{(\alpha+1)}(y)|dy \\ &= o\left(r^{\alpha/2+3/4} n^{-1/2}\xi(n)\right) = o\left(r^{\alpha/2+1/4}\xi(n)\right), \quad (6.5.7) \end{aligned}$$

in view of condition 6.2.3.
Collecting (6.5.2) - (6.5.7), we have

$$|C_n^\lambda(f;0) - f(0)| = o\left(n^{\alpha/2+3/4}\xi(n)\right).$$

Hence the proof of Theorem 6.4.1 is completed.

Remark 6.5.1. When we approximate the function f satisfying the conditions (6.2.1)–(6.2.3) of Theorem 6.2.1 for $x > 0$, the Laguerre polynomials are of order $\alpha > -1/2$ while for $x = 0$, the Laguerre polynomials are of order $\alpha \in [-1/2, 1/2]$.

6.6 Approximation by Hausdorff Means of Fourier-Laguerre Series

Now, we determine the degree of approximation of functions belonging to $L[0,\infty)$ by the Hausdorff means of its Fourier-Laguerre series at $x = 0$. Our theorem extends some of the recent results of [60; 97; 133] in the sense that the summability methods used by these authors have been replaced by the Hausdorff matrices.

The Hausdorff means of the Fourier-Laguerre series are defined by

$$H_n(f;x) := \sum_{k=0}^n h_{n,k} s_k(f;x), \quad n = 0, 1, 2, \dots . \quad (6.6.1)$$

The Fourier-Laguerre series is said to be summable to s by the Hausdorff means, if $H_n(f;x) \to s\ as\ n \to \infty$. The class of all regular Hausdorff matrices with moment sequence $\{\mu_n\}$ associated with mass function $\gamma(u)$ having constant derivative, is denoted by H_1.

We also write

$$\varphi(y) = \frac{e^{-y}y^{\alpha}(f(y) - f(0))}{\Gamma(\alpha + 1)},$$

and

$$g(u,y) = \sum_{k=0}^{n} \left(\begin{array}{c} n \\ k \end{array} \right) u^k (1-u)^{n-k} L_k^{(\alpha+1)}(y).$$

Gupta [35] obtained the degree of approximation of $f \in L[0,\infty)$ by Cesàro means of order k of the Fourier-Laguerre series at the point $x = 0$, where $k > \alpha + 1/2$. Nigam and Sharma [97] have used $(E,1)$ means of the Fourier-Laguerre series for $-1 < \alpha < -1/2$ which is more appropriate range from the application point of view. The authors have proved the following result:

Theorem A. *If*

$$E_n^1 = \frac{1}{2^n} \sum_{k=0}^{n} \left(\begin{array}{c} n \\ k \end{array} \right) s_k \quad \to \infty \quad as\ n \to \infty,$$

then the degree of approximation of Fourier-Laguerre expansion at the point $x = 0$ *by* $(E,1)$ *means* E_n^1 *is given by*

$$E_n^1(0) - f(0) = o(\xi(n)),$$

provided that

$$\Phi(t) = \int_0^t |\varphi(y)| dy = o\left(t^{\alpha+1}\xi(1/t)\right), \quad t \to 0, \tag{6.6.2}$$

$$\int_{\delta}^{n} e^{y/2} y^{-((2\alpha+3)/4)} |\varphi(y)| dy = o\left(n^{-((2\alpha+1)/4)}\xi(n)\right), \tag{6.6.3}$$

$$\int_{n}^{\infty} e^{y/2} y^{-1/3} |\varphi(y)| dy = o(\xi(n)), \quad n \to \infty, \tag{6.6.4}$$

where δ *is a fixed positive constant and* $\alpha \in (-1, -1/2)$, *and* $\xi(t)$ *is a positive monotonic non-decreasing function of* t *such that* $\xi(n) \to \infty$ *as* $n \to \infty$.

Following, Nigam and Sharma [97], Krasniqi [60] has used the $(C,1)(E,q)$ means of the Fourier-Laguerre series to obtain the degree of approximation of $f \in L[0,\infty)$ at point $x = 0$ and has proved the following result:

Theorem B. *The degree of approximation of the Fourier-Laguerre expansion at the point $x = 0$ by the $[(C,1)(E,q)]_n$ means is given by*

$$[(C,1)(E,q)]_n(0) - f(0) = o(\xi(n)),$$

provided that the conditions (6.6.2) – (6.6.4) *given in Theorem A are satisfied.*

Remark 6.6.1. We observe that Krasniqi [60, p.37] has optimized $\sum_{k=0}^{v} \binom{v}{k} q^k k^{(2\alpha+1)/4}$ by its maximum value $(1+q)^v v^{(2\alpha+1)/4}$. This is possible only when $\alpha > -1/2$. But the author has used $-1 < \alpha < 1/2$ [60, Theorem 2.1, p.35]. Similar error can also be seen in [97; 133].

6.7 Main Result

Theorem 6.7.1. *The degree of approximation of $f \in L[0,\infty)$ at $x = 0$ by the Hausdorff means of the Fourier-Laguerre series generated by $H \in H_1$ is given by*

$$|H_n(f;0) - f(0)| = o(\xi(n)),$$

where $\xi(t)$ is a positive non-decreasing function such that $\xi(t) \to \infty$ as $t \to \infty$ and satisfies the following conditions

$$\Phi(y) = \int_0^t |\varphi(y)| dy = o\left(t^{\alpha+1}\xi(1/t)\right), \quad t \to 0, \tag{6.7.1}$$

$$\int_\epsilon^n e^{y/2}\, y^{-((2\alpha+3)/4)} |\varphi(y)| dy = o\left(n^{-((2\alpha+1)/4)}\xi(n)\right), \tag{6.7.2}$$

and

$$\int_n^\infty e^{y/2}\, y^{-1/3} |\varphi(y)| dy = o(\xi(n)), \quad n \to \infty, \tag{6.7.3}$$

where ϵ is a fixed positive constant and $\alpha > -1/2$.

Note 8. The functions f in Theorem 6.4.1 and Theorem 6.7.1 are not necessarily same.

The work of Theorem 6.7.1 has been published in Mathematical Analysis and its Applications, 207–217, Springer Proc. Math. Stat., 143, Springer, New Delhi, 2015.

6.8 Lemmas

Lemma 6.8.1. *For* $0 < u < 1$ *and* $0 \le y \le \epsilon$,

$$\left|\int_0^1 g(u,y)d\gamma(u)\right| = \begin{cases} O\left(n^{(\alpha+1)}\right), & \text{if } 0 \le y \le \frac{1}{n} \\ O\left(y^{-(2\alpha+3)/4} n^{(2\alpha+1)/4}\right), & \text{if } \frac{1}{n} \le y \le \epsilon, \end{cases}$$

as $n \to \infty$.

Proof.

$$\begin{aligned} \left|\int_0^1 g(u,y)d\gamma(u)\right| &= \left|\int_0^1 (1-u)^n \sum_{k=0}^n \binom{n}{k} \left(\frac{u}{1-u}\right)^k L_k^{(\alpha+1)}(y)d\gamma(u)\right| \\ &= \left|M \int_0^1 (1-u)^n \sum_{k=0}^n \binom{n}{k} \left(\frac{u}{1-u}\right)^k L_k^{(\alpha+1)}(y)du\right| \end{aligned}$$

Now, using Lemma 6.3.1 for $0 \le y \le \frac{1}{n}$ (taking $\alpha+1$ for α and $c=1$), we have

$$\begin{aligned} \left|\int_0^1 g(u,y)d\gamma(u)\right| &\le \int_0^1 (1-u)^n \sum_{k=0}^n \binom{n}{k} \left(\frac{u}{1-u}\right)^k O(k^{\alpha+1})du \\ &= O\left(n^{\alpha+1} \int_0^1 (1-u)^n \sum_{k=0}^n \binom{n}{k} \left(\frac{u}{1-u}\right)^k du\right) \\ &= O\left(n^{\alpha+1} \int_0^1 du\right) \\ &= O\left(n^{\alpha+1}\right). \end{aligned} \tag{6.8.1}$$

Again, using Lemma 6.3.1 for $\frac{1}{n} \le y \le \epsilon$, we have

$$\begin{aligned} \left|\int_0^1 g(u,y)d\gamma(u)\right| &= \int_0^1 (1-u)^n \sum_{k=0}^n \binom{n}{k} \left(\frac{u}{1-u}\right)^k O\left(y^{-(2\alpha+3)/4} k^{(2\alpha+1)/4}\right) du \\ &= O\left(y^{-(2\alpha+3)/4} n^{(2\alpha+1)/4} \int_0^1 (1-u)^n \sum_{k=0}^n \binom{n}{k} \left(\frac{u}{1-u}\right)^k du\right) \\ &= O\left(y^{-(2\alpha+3)/4} n^{(2\alpha+1)/4}\right). \end{aligned} \tag{6.8.2}$$

Collecting (6.8.1) and (6.8.2), the proof of Lemma 6.8.1 is completed. □

Lemma 6.8.2. *For* $0 < u < 1$,

$$\left|\int_0^1 g(u,y)d\gamma(u)\right| = \begin{cases} O\left(e^{y/2} y^{-(2\alpha+3)/4} n^{(2\alpha+1)/4}\right), & \text{if } \epsilon \le y \le n \\ O\left(e^{y/2} y^{-(3\alpha+5)/6} n^{(\alpha+1)/2}\right), & \text{if } y \ge n, \end{cases}$$

as $n \to \infty$.

Proof. following the Lemma 6.8.1, we have

$$\left|\int_0^1 g(u,y)d\gamma(u)\right| = \left|\int_0^1 (1-u)^n \sum_{k=0}^{n} \binom{n}{k} \left(\frac{u}{1-u}\right)^k L_k^{(\alpha+1)}(y)du\right|$$

Now, using Lemma 6.3.3 for $\epsilon \leq y \leq n$ (taking $\alpha+1$ for α and $\eta = 3$), we have

$$\begin{aligned}
\left|\int_0^1 g(u,y)d\gamma(u)\right| &= \left|\int_0^1 e^{(y/2)} y^{-(2\alpha+3)/4}(1-u)^n \sum_{k=0}^{n} \binom{n}{k} \left(\frac{u}{1-u}\right)^k \right. \\
&\qquad \left. e^{-(y/2)} y^{(2\alpha+3)/4} L_k^{(\alpha+1)}(y)du\right| \\
&= \int_0^1 e^{y/2} y^{-(2\alpha+3)/4}(1-u)^n \sum_{k=0}^{n} \binom{n}{k} \left(\frac{u}{1-u}\right)^k O\left(k^{(2\alpha+1)/4}\right) du \\
&= O\left(e^{y/2} y^{-(2\alpha+3)/4} n^{(2\alpha+1)/4}\right). \qquad (6.8.3)
\end{aligned}$$

Again, using Lemma 6.3.3 for $y \geq n$, we have

$$\begin{aligned}
\left|\int_0^1 g(u,y)d\gamma(u)\right| &= \left|\int_0^1 e^{(y/2)} y^{-(3\alpha+5)/6}(1-u)^n \sum_{k=0}^{n} \binom{n}{k} \left(\frac{u}{1-u}\right)^k \right. \\
&\qquad \left. e^{-(y/2)} y^{(3\alpha+5)/6} L_k^{(\alpha+1)}(y)du\right| \\
&= \int_0^1 e^{y/2} y^{-(3\alpha+5)/6}(1-u)^n \sum_{k=0}^{n} \binom{n}{k} \left(\frac{u}{1-u}\right)^k O\left(k^{(\alpha+1)/2}\right) du \\
&= O\left(e^{y/2} y^{-(3\alpha+5)/6} n^{(\alpha+1)/2}\right). \qquad (6.8.4)
\end{aligned}$$

Collecting (6.8.3) and (6.8.4), the proof of Lemma 6.8.2 is completed. □

6.9 Proof of Theorem 6.7.1

From 6.6.1, we have

$$\begin{aligned}
H_n(f;0) - f(0) &= \sum_{k=0}^{n} \binom{n}{k} \Delta^{n-k}\mu_k \left(\frac{1}{\Gamma(\alpha+1)} \int_0^\infty e^{-y} y^\alpha f(y) L_k^{(\alpha+1)}(y)dy - f(0) \right) \\
&= \sum_{k=0}^{n} \binom{n}{k} \Delta^{n-k}\mu_k \frac{1}{\Gamma(\alpha+1)} \int_0^\infty e^{-y} y^\alpha (f(y) - f(0)) L_k^{(\alpha+1)}(y)dy \\
&= \sum_{k=0}^{n} \binom{n}{k} \Delta^{n-k}\mu_k \int_0^\infty \varphi(y) L_k^{(\alpha+1)}(y)dy \\
&= \int_0^\infty \varphi(y) \left(\sum_{k=0}^{n} \binom{n}{k} \Delta^{n-k}\mu_k L_k^{(\alpha+1)}(y) \right) dy \\
&= \int_0^\infty \varphi(y) \left(\sum_{k=0}^{n} \binom{n}{k} \int_0^1 u^k (1-u)^{n-k} d\gamma(u) L_k^{(\alpha+1)}(y) \right) dy \\
&= \int_0^\infty \varphi(y) \left(\int_0^1 \sum_{k=0}^{n} \binom{n}{k} u^k (1-u)^{n-k} L_k^{(\alpha+1)}(y) d\gamma(u) \right) dy \\
&= \int_0^\infty \varphi(y) \left(\int_0^1 g(u,y) d\gamma(u) \right) dy \\
&= \left(\int_0^{1/n} + \int_{1/n}^{\epsilon} + \int_\epsilon^n + \int_n^\infty \right) \sum_{k=0}^{n} \binom{n}{k} \Delta^{n-k}\mu_k \varphi(y) L_k^{(\alpha+1)}(y)dy. \\
&= J_1 + J_2 + J_3 + J_4. \qquad (6.9.1)
\end{aligned}$$

Now, using Lemma 6.8.1 for $0 \leq y \leq \frac{1}{n}$, we have

$$\begin{aligned}
|J_1| &\leq \int_0^{1/n} |\varphi(y)| \left| \int_0^1 g(u,y) d\gamma(u) \right| dy \\
&= O\left(n^{\alpha+1}\right) \int_0^{1/n} |\varphi(y)| dy \\
&= O(n^{(\alpha+1)}) o\left(\left(\frac{1}{n}\right)^{\alpha+1} \xi(n) \right) \\
&= o(\xi(n)), \qquad (6.9.2)
\end{aligned}$$

in view of condition (6.7.1).

Further, using Lemma 6.8.1 for $\frac{1}{n} \leq y \leq \epsilon$, we have,

$$\begin{aligned} |J_2| &\leq \int_{1/n}^{\epsilon} |\varphi(y)| O\left(y^{-(2\alpha+3)/4} n^{(2\alpha+1)/4}\right) dy \\ &= O(n^{(2\alpha+1)/4}) \left(\int_{1/n}^{\epsilon} y^{-(2\alpha+3)/4} |\varphi(y)| dy\right). \end{aligned}$$

Following [97, p.6], we have

$$|J_2| = o(\xi(n)), \tag{6.9.3}$$

in view of condition (6.7.1).

Now, using Lemma 6.8.2 for $\epsilon \leq y \leq n$, we have

$$\begin{aligned} |J_3| &\leq \int_{\epsilon}^{n} |\varphi(y)| \left|\int_0^1 g(u,y) d\gamma(u)\right| dy \\ &= \int_{\epsilon}^{n} O\left(e^{y/2} y^{-((2\alpha+3)/4)} n^{(2\alpha+1)/4}\right) |\varphi(y)| dy \\ &= O\left(n^{(2\alpha+1)/4}\right) \left(\int_{\epsilon}^{n} e^{y/2} y^{-((2\alpha+3)/4)} |\varphi(y)| dy\right) \\ &= O\left(n^{(2\alpha+1)/4}\right) o\left((n^{-(2\alpha+1)/4}) \xi(n)\right) \\ &= o(\xi(n)), \end{aligned} \tag{6.9.4}$$

in view of condition (6.7.2).

Further, using Lemma 6.8.2 , we have

$$\begin{aligned} |J_4| &\leq \int_{n}^{\infty} |\varphi(y)| \left|\int_0^1 g(u,y) d\gamma(u)\right| dy \\ &= \int_{n}^{\infty} |\varphi(y)| O\left(e^{y/2} y^{-(3\alpha+5)/6} n^{(\alpha+1)/2}\right) dy \\ &= O\left(n^{(\alpha+1)/2}\right) \left(\int_{n}^{\infty} \frac{e^{y/2} y^{-1/3} |\varphi(y)|}{y^{(\alpha+1)/2}} dy\right) \\ &= o\left((\xi(n)) n^{(\alpha+1)/2} (n^{-(\alpha+1)/2})\right) \\ &= o(\xi(n)), \end{aligned} \tag{6.9.5}$$

in view of condition (6.7.3).

Collecting (6.9.1) - (6.9.5), we have

$$|H_n(f;0) - f(0)| = o(\xi(n)).$$

Hence the proof of Theorem 6.7.1 is completed.

6.10 Corollaries

The following corollaries can be derived from our Theorem 6.7.1.

1. As discussed in [113, Lemma 1, p.306] and [125, p.38], if we take the mass function $\gamma(u)$ given by

$$\gamma(u) = \begin{cases} 0, & 0 \leq u \leq a \\ 1, & a \leq u \leq 1, \end{cases}$$

where $a = 1/(1+q),\ q > 0$, the Hausdorff matrix H reduces to Euler matrix $(E, q),\ q > 0$ and defines the corresponding (E, q) means given by

$$E_q^n(f;x) = \frac{1}{(1+q)^n} \sum_{k=0}^{n} \binom{n}{k} q^{n-k} s_k(f;x), \quad q > 0.$$

Hence the Theorem 6.7.1 reduces to Theorem A (result proved by Nigam and Sharma [97, Theorem 2.1, p.3]).

2. As discussed in [28, p.400] and [112, p.2747], the Cesáro matrix of order λ, is also a Hausdorff matrix obtained by mass function $\gamma(u) = 1 - (1-u)^{\lambda}$ and the corresponding Cesáro means are given by

$$C_n^{\lambda}(f;x) = \frac{1}{\binom{n+\lambda}{n}} \sum_{k=0}^{n} \binom{\lambda+n-k-1}{n-k} s_k(f;x).$$

Further, Rhoades [113, p.308] and Rhoades et al. [116, p.6869] have mentioned that the product of two Hausdorff matrices is again a Hausdorff matrix. Hence the Theorem B and Theorem C (results proved by Krasniqi [60, Theorem 2.1, p.35] and Sonker [133, Theorem 1, p.126]) are also particular cases of our Theorem 6.7.1.

Conclusions and Future Scope

In the present thesis, we aimed to determine the degree of approximation of functions belonging to Lipschitz classes: $Lip(\omega(t), p)$ and $W(L^p, \xi(t))$, $p \geq 1$, and their conjugates using the lower triangular matrix means and product means. Functions belonging to certain subclasses of L^p-space; viz., $L^p(\omega)_\beta$ and $L^p(\tilde{\omega})_\beta$ and their conjugates are also considered to obtain degree of approximation by the summability means of trigonometric Fourier series and its conjugate Fourier series. Fourier–Laguerre series and Walsh–Fourier series are also used to estimate the degree of approximation of f belonging to $L[0, \infty)$ and $L[0, 1)$-spaces, respectively. Some corollaries and particular cases of the results are also discussed to justify that our results extend the earlier results, and contribute significantly to the literature.
During this study we observed that this work can be extended in multi directions. Some of the possible options are listed below:

- To obtain the approximation results in grand Lebesgue space with $B_{\phi,p}$ class of weights defined by Jain et al. [48].

- It will be possible to extend our work to obtain the degree of functions belonging to different function spaces as weighted grand Lebesgue space $L_w^{p)}(0, 1)$ with norm $\| \,.\, \|_{p),w}$ [25], W^p and $W^\Omega(C^n)$-spaces [101; 142].

- Our work can be extended to obtain the results on approximation properties of non-periodic functions by Fourier transform [104].

- The results on the degree of approximation of function by means of Fourier–Laguerre series can be extended to $L^p(p > 1)$-space and the Lipschitz classes associated with the periodic integrable functions. Also, we can study the degree of approximation of a function belonging to $H^1(0,\infty)$-space [143] by means of Fourier–Laguerre series.

Bibliography

[1] Alexits, G.: 1961, *Convergence problems of orthogonal series*, Translated from the German by I. Földer. International Series of Monographs in Pure and Applied Mathematics, Vol. 20, Pergamon Press, New York-Oxford-Paris.

[2] Alghamdi, M. A. and Mursaleen, M.: 2013, Hankel matrix transformation of the Walsh-Fourier series, *Appl. Math. Comput.* **224**, 278–282.

[3] Almira, J. M. and Luther, U.: 2004, Generalized approximation spaces and applications, *Math. Nachr.* **263/264**, 3–35.

[4] Arutjunjan, F. G. and Talaljan, A. A.: 1964, Uniqueness of series in Haar and Walsh systems, *Izv. Akad. Nauk SSSR Ser. Mat.* **28**, 1391–1408.

[5] Bass, C. A. (ed.): 1970, *Applications of Walsh functions*, 1970 Proceedings (31 March-3 April): Symposium and Workshop, held at Naval Research Laboratory, Washington, D.C. Edited by C. A. Bass, Naval Research Laboratory, Washington, D.C.; University of Maryland, College Park, Md.

[6] Baǐarstanova, A. S.: 1979, Summation of subsequences of partial sums of Fourier series according to the Walsh system, *Izv. Akad. Nauk Kazakh. SSR Ser. Fiz.-Mat.* (3), 59–62, 90.

[7] Bernstein, S. N. and Surl: 1912, Ordre de la meilleure approximation des fonctions par des polynomes de degre donne, *Mem. Cl. Acad. Roy. Belg.* **4**, 11–104.

[8] Billard, P.: 1967, Sur la convergence presque partout des séries de Fourier-Walsh des fonctions de l'espace $L^2\,(0,\,1)$, *Studia Math.* **28**, 363–388.

[9] Blahota, I. and Nagy, K.: 2018, Approximation by Θ-means of Walsh-Fourier series, *Anal. Math.* **44**(1), 57–71.

[10] Bojanić, R. and Mazhar, S. M.: 1988, An estimate of the rate of convergence of Cesàro means of Fourier series of functions of bounded variation, *Mathematical Analysis and its Applications (Kuwait, 1985)*, Vol. 3 of *KFAS Proc. Ser.*, Pergamon, Oxford, pp. 17–22.

[11] Butzer, P. L. and Wagner, H. J.: 1973, Walsh-Fourier series and the concept of a derivative, *Applicable Anal.* **3**, 29–46. Collection of articles dedicated to Eberhard Hopf on the occasion of his 70th birthday.

[12] Butzer, P. L. and Wagner, H. J.: 1975, On dyadic analysis based on the pointwise dyadic derivative, *Anal. Math.* **1**(3), 171–196.

[13] Chandra, P.: 1986, Approximation by Nörlund operators, *Mat. Vesnik* **38**(3), 263–269.

[14] Chandra, P.: 1988, On the degree of approximation of a class of functions by means of Fourier series, *Acta Math. Hungar.* **52**(3-4), 199–205.

[15] Chandra, P.: 2002, Trigonometric approximation of functions in L_p-norm, *J. Math. Anal. Appl.* **275**(1), 13–26.

[16] Chandra, P. and Mohapatra, R. N.: 1979, Absolute Nörlund summability factors of Fourier series, *Tamkang J. Math.* **10**(2), 245–252.

[17] Chandra, P., Thakur, S. S. and Singh, R.: 2013, Functions from L_p-spaces and Taylor means, *Mat. Vesnik* **65**(1), 35–45.

[18] Ciaurri, O. and Roncal, L.: 2005, The Bochner-Riesz means for Fourier-Bessel expansions, *J. Funct. Anal.* **228**(1), 89–113.

[19] Das, G., Ghosh, T. and Ray, B. K.: 1995, Degree of approximation of functions in the Hölder metric by (e, c) means, *Proc. Indian Acad. Sci. Math. Sci.* **105**(3), 315–327.

[20] Das, G., Ghosh, T. and Ray, B. K.: 1996, Degree of approximation of functions by their Fourier series in the generalized Hölder metric, *Proc. Indian Acad. Sci. Math. Sci.* **106**(2), 139–153.

[21] Das, G., Nath, A. and Ray, B. K.: 2002, A new trigonometric method of summation and its application to the degree of approximation, *Proc. Indian Acad. Sci. Math. Sci.* **112**(2), 299–319.

[22] Deepmala and Piscoran, L.-I.: 2016, Approximation of signals (functions) belonging to certain Lipschitz classes by almost Riesz means of its Fourier series, *J. Inequal. Appl.* (1), 163.

[23] Episkoposian, S. and Müller, J.: 2014, Universality properties of Walsh-Fourier series, *Monatsh. Math.* **175**(4), 511–518.

[24] Fine, N. J.: 1955, Cesàro summability of Walsh-Fourier series, *Proc. Nat. Acad. Sci. U.S.A.* **41**, 588–591.

[25] Fiorenza, A., Gupta, B. and Jain, P.: 2008, The maximal theorem for weighted grand Lebesgue spaces, *Studia Math.* **188**(2), 123–133.

[26] Fridli, S.: 1994, On the rate of convergence of Cesàro means of Walsh-Fourier series, *J. Approx. Theory* **76**(1), 31–53.

[27] Fridli, S., Manchanda, P. and Siddiqi, A. H.: 2008, Approximation by Walsh-Nörlund means, *Acta Sci. Math. (Szeged)* **74**(3-4), 593–608.

[28] Garabedian, H. L.: 1939, Hausdorff matrices, *Amer. Math. Monthly* **46**, 390–410.

[29] Gát, G., Goginava, U. and Karagulyan, G.: 2015, On everywhere divergence of the strong Φ-means of Walsh-Fourier series, *J. Math. Anal. Appl.* **421**(1), 206–214.

[30] Goginava, U.: 2002, On the approximation properties of Cesàro means of negative order of Walsh-Fourier series, *J. Approx. Theory* **115**(1), 9–20.

[31] Goginava, U.: 2008, Maximal operators of Fejér-Walsh means, *Acta Sci. Math. (Szeged)* **74**(3-4), 615–624.

[32] Golubov, B., Efimov, A. and Skvortsov, V.: 1991, *Walsh series and transforms*, Vol. 64 of *Mathematics and its Applications (Soviet Series)*, Kluwer Academic Publishers Group, Dordrecht. Theory and applications, Translated from the 1987 Russian original by W. R. Wade.

[33] Gosselin, J. A.: 1979, On the convergence of Walsh-Fourier series for $L^2(0, 1)$, *Studies in analysis*, Vol. 4 of *Adv. in Math. Suppl. Stud.*, Academic Press, New York-London, pp. 223–232.

[34] Greene, N.: 2008, Fourier series of orthogonal polynomials, *Proceedings of the American Conference on Applied Mathematics*, Cambridge, Massachusetts, pp. 247–251.

[35] Gupta, D. P.: 1971, Degree of approximation by Cesàro means of Fourier-Laguerre expansions, *Acta Sci. Math. (Szeged)* **32**, 255–259.

[36] Guven, A.: 2009, Trigonometric approximation of functions in weighted L^p spaces, *Sarajevo J. Math.* **5(17)**(1), 99–108.

[37] Guven, A.: 2012, Approximation in weighted L^p spaces, *Rev. Un. Mat. Argentina* **53**(1), 11–23.

[38] Guven, A. and Israfilov, D. M.: 2010, Trigonometric approximation in generalized Lebesgue spaces $L^{p(x)}$, *J. Math. Inequal.* **4**(2), 285–299.

[39] Guven, A. and Yurt, H.: 2016, Approximation theorems in L^p spaces of functions of several variables, *Ann. Univ. Sci. Budapest. Sect. Comput.* **45**, 5–22.

[40] Hille, E.: 1926a, On Laguerre's series first note, *Proceedings of the National Academy of Sciences* **12**(4), 261–265.

[41] Hille, E.: 1926b, On Laguerre's series second note, *Proceedings of the National Academy of Sciences* **12**(4), 265–269.

[42] Holland, A. S. B.: 1981, A survey of degree of approximation of continuous functions, *SIAM Rev.* **23**(3), 344–379.

[43] Holland, A. S. B., Sahney, B. N. and Tzimbalario, J.: 1976, On degree of approximation of a class of functions by means of Fourier series, *Acta Sci. Math. (Szeged)* **38**(1-2), 69–72.

[44] Hunt, R., Muckenhoupt, B. and Wheeden, R.: 1973, Weighted norm inequalities for the conjugate function and Hilbert transform, *Trans. Amer. Math. Soc.* **176**, 227–251.

[45] Jackson, D.: 1930, *The theory of approximation*, Vol. 11 of *American Mathematical Society Colloquium Publications*, American Mathematical Society, Providence, RI. Reprint of the 1930 original.

[46] Jafarov, S. Z.: 2017, Approximation by means of Fourier trigonometric series in weighted Lebesgue spaces, *Sarajevo J. Math.* **13(25)**(2), 217–226.

[47] Jain, P. and Kumari, S.: 2012, On grand Lorentz spaces and the maximal operator, *Georgian Math. J.* **19**(2), 235–246.

[48] Jain, P., Singh, M. and Singh, A. P.: 2016, Hardy type operators on grand Lebesgue spaces for non-increasing functions, *Trans. A. Razmadze Math. Inst.* **170**(1), 34–46.

[49] Jastrebova, M. A.: 1966, Approximation of functions satisfying a Lipschitz condition by arithmetic means of their Walsh-Fourier series, *Mat. Sb. (N.S.)* **71 (113)**, 214–226.

[50] Kęska, S.: 2016, The degree of approximation by Hausdorff means of a conjugate Fourier series, *Ann. Univ. Mariae Curie-Skłodowska Sect. A* **70**(2), 63–82.

[51] Khabibrakhmanov, I. and Summers, D.: 1998, The use of generalized Laguerre polynomials in spectral methods for nonlinear differential equations, *Computers & Mathematics with Applications* **36**(2), 65–70.

[52] Khan, H. H.: 1982, A note on a theorem of izumi, *Comm. Fac. Maths. Ankara* **31**, 123–127.

[53] Khan, H. H. and Khan, I. A.: 2012, Approximation by Cesaro means of product of two Laguerre polynomials, *Int. J. Math. Anal. (Ruse)* **6**(21-24), 1163–1172.

[54] Khan, H. H. and Ram, G.: 2003, On the degree of approximation, *Facta Univ. Ser. Math. Inform.* (18), 47–57.

[55] Khatri, K. and Mishra, V. N.: 2016, Approximation of functions belonging to $L[0,\infty)$ by product summability means of its Fourier-Laguerre series, *Cogent Math.* **3**, Art. ID 1250854, 7.

[56] Kolodii, I. and Khil'debrand, F.: 1971, Some properties of the modulus of continuity, *Mathematical notes of the Academy of Sciences of the USSR* **9**(5), 285–288.

[57] Konyagin, S. V.: 2006, Almost everywhere convergence and divergence of Fourier series, *International Congress of Mathematicians. Vol. II*, Eur. Math. Soc., Zürich, pp. 1393–1403.

[58] Kranz, R., Łenski, W. and Szal, B.: 2012, On the degrees of approximation of functions belonging to $L^p(\tilde{\omega})_\beta$ class by matrix means of conjugate Fourier series, *Math. Inequal. Appl.* **15**(3), 717–732.

[59] Krasniqi, X. Z.: 2010, On the degree of approximation of continuous functions that pertains to the sequence-to-sequence transformation, *Aust. J. Math. Anal. Appl.* **7**(2), Art. 13, 10.

[60] Krasniqi, X. Z.: 2013a, On the degree of approximation of a function by $(C,1)(E,q)$ means of its Fourier-Laguerre series, *International Journal of Analysis and Applications* **1**, 33–39.

[61] Krasniqi, X. Z.: 2013b, Slight extensions of some theorems on the rate of pointwise approximation of functions from some subclasses of L^p, *Acta Comment. Univ. Tartu. Math.* **17**(1), 89–101.

[62] Ky, N. X.: 1993, On approximation by trigonometric polynomials in L^p_u-spaces, *Studia Sci. Math. Hungar.* **28**(1-2), 183–188.

[63] Ky, N. X.: 1997, Moduli of mean smoothness and approximation with A_p-weights, *Ann. Univ. Sci. Budapest. Eötvös Sect. Math.* **40**, 37–48 (1998).

[64] Ladhawala, N. R.: 1976, Absolute summability of Walsh-Fourier series, *Pacific J. Math.* **65**(1), 103–108.

[65] Lal, S.: 1999, The degree of approximation of conjugates of almost Lipschitz functions by $(C, 1)(E, 1)$ means, *Bull. Calcutta Math. Soc.* **91**(2), 145–148.

[66] Lal, S.: 2009, Approximation of functions belonging to the generalized Lipschitz class by $C^1 \cdot N_p$ summability method of Fourier series, *Appl. Math. Comput.* **209**(2), 346–350.

[67] Lal, S. and Kushwaha, J.: 2009, Approximation of conjugate of functions belonging to the generalized Lipschitz class by lower triangular matrix means. int, *Journal of Math. Analysis (3)* **21**, 1031–1041.

[68] Lal, S. and Mishra, A.: 2013, Euler-Hausdorff matrix summability operator and trigonometric approximation of the conjugate of a function belonging to the generalized Lipschitz class, *J. Inequal. Appl.* pp. 2013:59, 14.

[69] Lal, S. and Nigam, H. K.: 2001, Degree of approximation by (N, p, q) summability means of the Fourier-Laguerre expansion, *Tamkang J. Math.* **32**(2), 143–150.

[70] Lal, S. and Singh, P.: 2002, Degree of approximation of conjugate of $\mathrm{Lip}(\alpha, p)$ function by $(C, 1)(E, 1)$ means of conjugate series of a Fourier series, *Tamkang J. Math.* **33**(3), 269–274.

[71] Lal, S. and Yadav, K. N. S.: 2001, On degree of approximation of function belonging to the Lipschitz class by $(C, 1)(E, 1)$ means of its Fourier series, *Bulletin of the Calcutta Mathematical Society* **93**(3), 191–196.

[72] Lebesgue, H.: 1910, Sur la représentation trigonométrique approchée des fonctions satisfaisant à une condition de Lipschitz, *Bull. Soc. Math. France* **38**, 184–210.

[73] Leindler, L.: 2005, Trigonometric approximation in L_p-norm, *J. Math. Anal. Appl.* **302**(1), 129–136.

[74] Leindler, L.: 2009, A relaxed estimate of the degree of approximation by Fourier series in generalized Hölder metric, *Anal. Math.* **35**(1), 51–60.

[75] Łenski, W.: 2001, On the rate of pointwise summability of Fourier series, *Appl. Math. E-Notes* **1**, 143–148.

[76] Łenski, W. and Szal, B.: 2008, Pointwise strong and very strong approximation by matrix means of Fourier series, *Comment. Math.* **48**(2), 177–198.

[77] Łenski, W. and Szal, B.: 2009, Approximation of functions belonging to the class $L^p(\omega)_\beta$ by linear operators, *Acta Comment. Univ. Tartu. Math.* (13), 11–24.

[78] Łenski, W. and Szal, B.: 2012, Approximation of integrable functions by general linear operators of their Fourier series, *Acta Math. Sin. (Engl. Ser.)* **28**(6), 1119–1134.

[79] Łenski, W. and Szal, B.: 2014, Approximation of conjugate functions by general linear operators of their Fourier series at the Lebesgue points, *Demonstr. Math.* **47**(4), 878–892.

[80] McFadden, L.: 1942, Absolute Nörlund summability, *Duke Math. J.* **9**, 168–207.

[81] Mishra, L. N., Mishra, V. N., Khatri, K. et al.: 2014, On the trigonometric approximation of signals belonging to generalized weighted Lipschitz $W(Lr, \xi(t))(r \geq 1)$-class by matrix (C1.Np) operator of conjugate series of its Fourier series, *Applied Mathematics and Computation* **237**, 252–263.

[82] Mishra, V. N., Sonavane, V. and Mishra, L. N.: 2013, L_r-approximation of signals (functions) belonging to weighted $W(L_r, \xi(t))$-class by $C^1 \cdot N_p$ summability method of conjugate series of its Fourier series, *J. Inequal. Appl.* pp. 2013:440, 15.

[83] Mittal, M. L.: 1998, A sufficient condition for (F^1)-effectiveness of the C^1T-method, *J. Math. Anal. Appl.* **220**(2), 434–450.

[84] Mittal, M. L. and Bhardwaj, N.: 1998, Approximation of functions by linear matrix operators, *Approximation theory IX, Vol. I. (Nashville, TN, 1998)*, Innov. Appl. Math., Vanderbilt Univ. Press, Nashville, TN, pp. 263–270.

[85] Mittal, M. L. and Rhoades, B. E.: 1999, On the degree of approximation of continuous functions by using linear operators on their Fourier series, *Int. J. Math. Game Theory Algebra* **9**(4), 259–267.

[86] Mittal, M. L., Rhoades, B. E. and Mishra, V. N.: 2006, Approximation of signals (functions) belonging to the weighted $W(L_p, \xi(t))$-class by linear operators, *Int. J. Math. Math. Sci.* pp. Art. ID 53538, 10.

[87] Mittal, M. L., Rhoades, B. E., Mishra, V. N. and Singh, U.: 2007, Using infinite matrices to approximate functions of class $\mathrm{Lip}(\alpha, p)$ using trigonometric polynomials, *J. Math. Anal. Appl.* **326**(1), 667–676.

[88] Mittal, M. L., Rhoades, B. E., Sonker, S. and Singh, U.: 2011, Approximation of signals of class $\mathrm{Lip}(\alpha, p)$ by linear operators, *Appl. Math. Comput.* **217**(9), 4483–4489.

[89] Mohapatra, R. N. and Chandra, P.: 1983, Degree of approximation of functions in the Hölder metric, *Acta Math. Hungar.* **41**(1-2), 67–76.

[90] Mohapatra, R. N. and Szal, B.: 2018, On trigonometric approximation of functions in the L^q norm, *Demonstr. Math.* **51**(1), 17–26.

[91] Móricz, F. and Rhoades, B. E.: 1996, Approximation by weighted means of Walsh-Fourier series, *Internat. J. Math. Math. Sci.* **19**(1), 1–8.

[92] Móricz, F. and Schipp, F.: 1990, On the integrability and L^1-convergence of Walsh series with coefficients of bounded variation, *J. Math. Anal. Appl.* **146**(1), 99–109.

[93] Móricz, F. and Siddiqi, A. H.: 1992, Approximation by Nörlund means of Walsh-Fourier series, *J. Approx. Theory* **70**(3), 375–389.

[94] Muckenhoupt, B.: 1972, Weighted norm inequalities for the Hardy maximal function, *Trans. Amer. Math. Soc.* **165**, 207–226.

[95] Natanson, I. and Obolensky, A. N.: 1964, *Constructive Function Theory: Vol. I: Uniform Approximation*, Frederick Ungar Publishing Company.

[96] Nigam, H. K.: 2010, On degree of approximation of a function belonging to $\mathrm{Lip}(\xi(t), r)$ class by $(E, q)(C, 1)$ product means of Fourier series, *Commun. Appl. Anal.* **14**(3-4), 607–614.

[97] Nigam, H. K. and Sharma, A.: 2010, A study on degree of approximation by $(E, 1)$ summability means of the Fourier-Laguerre expansion, *Int. J. Math. Math. Sci.* pp. Art. ID 351016, 7.

[98] Nigam, H. K. and Sharma, K.: 2011, Degree of approximation of a class of function by $(C, 1)(E, q)$ means of Fourier series, *IAENG Int. J. Appl. Math.* **41**(2), 128–132.

[99] Paley, R. E. A. C.: 1932, A Remarkable Series of Orthogonal Functions (I), *Proc. London Math. Soc. (2)* **34**(4), 241–264.

[100] Pathak, R. S. and Prasad, A.: 2006, A generalized pseudo-differential operator on Gel'fand-Shilov space and Sobolev space, *Indian J. Pure Appl. Math.* **37**(4), 223–235.

[101] Pathak, R. S. and Upadhyay, S. K.: 1994, W^p-spaces and Fourier transform, *Proc. Amer. Math. Soc.* **121**(3), 733–738.

[102] Powell, C. H. and Wade, W. R.: 1981, Term by term dyadic differentiation, *Canad. J. Math.* **33**(1), 247–256.

[103] Prasad, A. and Kumar, P.: 2016, The continuous fractional wavelet transform on a generalized Sobolev space, *Int. J. Wavelets Multiresolut. Inf. Process.* **14**(6), 1650046, 10.

[104] Prasad, A., Manna, S., Mahato, A. and Singh, V. K.: 2014, The generalized continuous wavelet transform associated with the fractional Fourier transform, *J. Comput. Appl. Math.* **259**(part B), 660–671.

[105] Priti, S., Mittal, S. S., Singh, U. and Kumar, V.: 2003, Approximation by matrix means of Walsh-Fourier series, *Advances in Mathematics Research. Vol. 5*, Nova Sci. Publ., Hauppauge, NY, pp. 31–45.

[106] Psarakis, E. Z. and Moustakides, G. V.: 1997, An L_2-based method for the design of 1-D zero phase FIR digital filters, *IEEE Trans. Circuits Systems I Fund. Theory Appl.* **44**(7), 591–601.

[107] Qing-Hua, Z., Shuiming, C. and Yuanyuan, Q.: 2005, Corrected Fourier series and its application to function approximation, *International Journal of Mathematics and Mathematical Sciences* **2005**(1), 33–42.

[108] Quade, E. S.: 1937, Trigonometric approximation in the mean, *Duke Math. J.* **3**(3), 529–543.

[109] Qureshi, K.: 1981, On the degree of approximation of functions belonging to the Lipschitz class by means of a conjugate series, *Indian J. Pure Appl. Math.* **12**(9), 1120–1123.

[110] Qureshi, K.: 1982a, On the degree of approximation of a periodic function f by almost Riesz means of its conjugate series, *Indian J. Pure Appl. Math.* **13**(10), 1136–1139.

[111] Qureshi, K.: 1982b, On the degree of approximation of functions belonging to the class Lip(α, p) by means of a conjugate series, *Indian J. Pure Appl. Math.* **13**(5), 560–563.

[112] Rhoades, B. E.: 1995, Commutants for some classes of Hausdorff matrices, *Proc. Amer. Math. Soc.* **123**(9), 2745–2755.

[113] Rhoades, B. E.: 2001, On the degree of approximation of functions belonging to the weighted $(L^p, \xi(t))$ class by Hausdorff means, *Tamkang J. Math.* **32**(4), 305–314.

[114] Rhoades, B. E.: 2008, Walsh-Fourier series with coefficients of generalized bounded variation, *Acta Sci. Math. (Szeged)* **74**(3-4), 609–614.

[115] Rhoades, B. E.: 2014, The degree of approximation of functions, and their conjugates, belonging to several general Lipschitz classes by Hausdorff matrix means of the Fourier series and conjugate series of a Fourier series, *Tamkang J. Math.* **45**(4), 389–395.

[116] Rhoades, B. E., Ozkoklu, K. and Albayrak, I.: 2011, On the degree of approximation of functions belonging to a Lipschitz class by Hausdorff means of its Fourier series, *Appl. Math. Comput.* **217**(16), 6868–6871.

[117] Sahney, B. N. and Goel, D. S.: 1973, On the degree of approximation of continuous functions, *Ranchi Univ. Math. J.* **4**, 50–53.

[118] Schipp, F.: 1975, über gewisse Maximaloperatoren, *Ann. Univ. Sci. Budapest. Eötvös Sect. Math.* **18**, 189–195 (1976).

[119] Schipp, F.: 1976, On term by term dyadic differentiability of Walsh series, *Anal. Math.* **2**(2), 149–154.

[120] Silverman, L. L.: 1913, *On the definition of the sum of a divergent series*, Vol. 1, University of Missouri.

[121] Singh, T.: 1976, On the absolute summability factors of Fourier-Laguerre expansion, *Indian J. Pure Appl. Math.* **7**(9), 961–968.

[122] Singh, T.: 1978, Degree of approximation by Cesàro means of Fourier-Laguerre series, *Indian J. Pure Appl. Math.* **9**(4), 394–399.

[123] Singh, U., Mittal, M. L. and Sonker, S.: 2012, Trigonometric approximation of signals (functions) belonging to $W(L^r, \xi(t))$ class by matrix $(C^1 \cdot N_p)$ operator, *Int. J. Math. Math. Sci.* pp. Art. ID 964101, 11.

[124] Singh, U. and Sonker, S.: 2012, Degree of approximation of function $f \in H_p^{(w)}$ class in generalized Hölder metric by matrix means, *Mathematical modelling and scientific computation*, Vol. 283 of *Commun. Comput. Inf. Sci.*, Springer, Heidelberg, pp. 1–10.

[125] Singh, U. and Sonker, S.: 2013, Trigonometric approximation of signals (functions) belonging to weighted $(L^p, \xi(t))$-class by Hausdorff means, *J. Appl. Funct. Anal.* **8**(1), 37–44.

[126] Singh, U. and Srivastava, S. K.: 2013, Degree of approximation of functions in Lipschitz class with Muckenhoupt weights by matrix means, *IAENG Int. J. Appl. Math.* **43**(4), 190–194.

[127] Singh, U. and Srivastava, S. K.: 2014a, Approximation of conjugate of functions belonging to weighted Lipschitz class $W(L^p, \xi(t))$ by Hausdorff means of conjugate Fourier series, *J. Comput. Appl. Math.* **259**(part B), 633–640.

[128] Singh, U. and Srivastava, S. K.: 2014b, On the degree of approximation of conjugate functions in weighted Lipschitz class, *IAENG Transactions on Engineering Sciences: Special Issue of the International MultiConference of Engineers and Computer Scientists 2013 and World Congress on Engineering 2013*, CRC Press, p. 81.

[129] Singh, U. and Srivastava, S. K.: 2014c, Trigonometric approximation of functions belonging to certain Lipschitz classes by C1.T operator, *Asian-European Journal of Mathematics* **7**(04), 1450064.

[130] Skvorcov, V. A.: 1977, Uniqueness condition for the representation of functions by Walsh series, *Mat. Zametki* **21**(2), 187–197.

[131] Skvorcov, V. A.: 1979, Null series with respect to a multiplicative system, *Vestnik Moskov. Univ. Ser. I Mat. Mekh.* (6), 63–67, 108.

[132] Skvorcov, V. A.: 1981, Some estimates of approximation of functions by Cesàro means of Fourier-Walsh series, *Mat. Zametki* **29**(4), 539–547, 633.

[133] Sonker, S.: 2014, Approximation of functions by $(C, 2)(E, q)$ means of its Fourier-Laguerre series., *Proceeding of ICMS-2014*, number 978-93-5107-261-4, International Conference on Mathematical Science, Elesvier, pp. 125–128.

[134] Sonker, S. and Singh, U.: 2012, Degree of approximation of the conjugate of

signals (functions) belonging to $Lip(\alpha, r)$-class by $(C, 1)(E, q)$ means of conjugate trigonometric Fourier series, *J. Inequal. Appl.* pp. 2012:278, 7.

[135] Srivastava, S. K. and Singh, U.: 2014, Trigonometric approximation of periodic functions belonging to $Lip(\omega(t), p)$-class, *J. Comput. Appl. Math.* **270**, 223–230.

[136] Szegö, G.: 1939, *Orthogonal polynomials.*, Vol. 23, American Mathematical Society Colloquium Publications.

[137] Tephnadze, G.: 2016, On the convergence of Fejér means of Walsh-Fourier series in the space H_p, *Izv. Nats. Akad. Nauk Armenii Mat.* **51**(2), 54–70.

[138] Tevzadze, V. I.: 1981, Uniform convergence of Cesàro means of negative order of Fourier-Walsh series, *Soobshch. Akad. Nauk Gruzin. SSR* **102**(1), 33–36.

[139] Tiwari, S. K. and Kachhara, D. K.: 2010, Degree of approximation by Nörlund summability means of Laguerre series, *J. Indian Math. Soc. (N.S.)* **77**(1-4), 207–213.

[140] Toeplitz, O.: 1911, Über allgemeine lineare mittelbildungen., *Prace Matematyczno-Fizyczne* **22**(1), 113–119.

[141] Tolstov, G. P.: 1976, *Fourier series*, Dover Publications, Inc., New York. Second English translation, Translated from the Russian and with a preface by Richard A. Silverman.

[142] Upadhyay, S. K.: 2010, Pseudo-differential operators on $W^{\Omega}(C^n)$-space, *J. Int. Acad. Phys. Sci.* **14**(1), 53–60.

[143] Upadhyay, S. K., Yadav, R. N. and Debnath, L.: 2012, Properties of the Hankel-Hausdorff operator on Hardy space $H^1(0, \infty)$, *Analysis (Munich)* **32**(3), 221–230.

[144] Wade, W. R.: 1982, Recent developments in the theory of Walsh series, *Internat. J. Math. Math. Sci.* **5**(4), 625–673.

[145] Wei, B. and Yu, D.: 2012, On the degree of approximation of continuous functions by means of Fourier series, *Math. Commun.* **17**(1), 211–219.

[146] Weierstrass, K.: 1885, Über die analytische darstellbarkeit sogenannter willkürlicher functionen einer reellen veränderlichen, *Sitzungsberichte der Königlich Preußischen Akademie der Wissenschaften zu Berlin* **2**, 633–639.

[147] Weisz, F.: 2001, (C, α) summability of Walsh-Fourier series, *Anal. Math.* **27**(2), 141–155.

[148] Xu, Y.: 2010, Fourier series and approximation on hexagonal and triangular domains, *Constr. Approx.* **31**(1), 115–138.

[149] Yano, S.: 1957, Cesàro summability of Walsh-Fourier series, *Tôhoku Math. J. (2)* **9**, 267–272.

[150] Zhang, R.-J.: 2014, On the trigonometric approximation of the generalized weighted Lipschitz class, *Appl. Math. Comput.* **247**, 1139–1140.

[151] Zhang, R.-J.: 2015, A note on the trigonometric approximation of $Lip(\omega(t), p)$-class, *Appl. Math. Comput.* **269**, 129–132.

[152] Zygmund, A.: 2002, *Trigonometric series. Vol. I, II*, Cambridge Mathematical Library, third edn, Cambridge University Press, Cambridge. With a foreword by Robert A. Fefferman.