EXISTENCE AND CONTROLLABILITY RESULTS TO SOME FUNCTIONAL DIFFERENTIAL SYSTEMS

Ph.D. THESIS

by

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CANDIDATE'S DECLARATION

I hereby certify that the work which is being presented in this thesis entitled **"EXISTENCE AND CONTROLLABILITY RESULTS TO SOME FUNCTIONAL DIFFERENTIAL SYSTEMS"** in partial fulfilment of the requirements for the award of the degree of "Doctor of Philosophy" and submitted in the Department of Mathematics, Indian Institute of Technology Roorkee, Roorkee is an authentic record of my own work carried out during a period from July, 2013 to July, 2018 under the supervision of Dr. D. N. Pandey, Associate Professor, Department of Mathematics, Indian Institute of Technology Roorkee, Roorkee.

The matter presented in this thesis has not been submitted by me for the award of any other degree of this or any other Institution.

(**VIKRAM SINGH**)

This is to certify that the above statement made by the candidate is correct to the best of my knowledge.

(D. N. Pandey)
Supervisor

Date:

The Ph.D. Viva-Voce Examination of **Mr. Vikram Singh**, Research Scholar, has been held on……………

Chairman, SRC | Signature of External Examiner

This is to certify that the student has made all the corrections in the thesis.

Signature of Supervisor | Head of the Department/ Chairman, ODC

Dedicated to my grandfather

Mr. Omkar Singh

and my parents

Mr. Naresh Chand *and* ***Mrs. Shree Mati Devi***

who supported me in all the circumstances.

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Vikram Singh

Abstract

The work presented in this thesis deals with the investigations of existence, uniqueness and some controllability results for mild and integral solutions to various types of fractional differential systems in abstract spaces. To deal with such problems some tools which we have used are the semigroup theory of linear operators, concepts of fractional calculus, functional analysis and some suitable fixed point theorems. We may divide our work into three major parts.

In the first part (Chapters 3, 4 and 5), the existence and uniqueness of mild solutions for deterministic and stochastic fractional differential systems are investigated. In order to obtain the desired results, monotonic iterative technique, condensing theorem and Picard type iterations are employed.

It is well-known that the concept of controllability is a valuable property of a control system, and it plays a very important role in several control problems in both finite and infinite dimensional spaces. In controllability of a system, we show the existence of a control function which steers to the mild solution of the system from its initial state to the desired final state, where the initial and final states may vary over the entire space. There are several concepts related to controllability, such as exact controllability, optimal controllability, trajectory controllability and approximate controllability. In the thesis, we study exact and approximate controllability

results for some fractional differential systems.

Motivated by the above discussion, in the second part (Chapter 6) of the thesis, the exact controllability results are established for some fractional impulsive delay differential systems using some basic tools of fractional calculus, measure of non-compactness and Mönch fixed point theorem.

In the third part (Chapter 7), some existence, uniqueness and approximate controllability of integral solutions for fractional differential systems involving Hilfer fractional derivative with non dense domain are discussed in a Banach space.
The chapter-wise organization of the thesis is as follows:

Chapter 1 contains a brief introduction to the problems which are discussed in later chapters, and provides a motivational background to study the problems discussed in this thesis. Further, it contains a review of relevant literature.

Chapter 2 contains some basic concepts of fractional calculus, functional analysis, semigroup theory and stochastic analysis that will be required in the subsequent chapters.

In Chapter 3, we obtain some existence and uniqueness results for mild solutions to Sobolev type fractional impulsive differential systems with fractional order nonlocal conditions by applying monotone iterative technique coupled with the method of lower and upper solutions. The sufficient conditions are obtained by measure of noncompactness and generalized Gronwall inequality. Finally, an application is given to illustrate the obtained results.

In Chapter 4, the existence and uniqueness results for mild solutions of a abstract multi-term time-fractional stochastic differential system are investigated. We use the tools of fractional calculus, generalized semigroup theory and stochastic analysis techniques to obtain the desired results. We come up with a new set of sufficient conditions using standard Picard's iterations on the coefficients in the equations satisfy some non-Lipschitz conditions. Finally, an application is given to illustrate the obtained results.

In Chapter 5, some existence and uniqueness results for mild solutions to the multi-term time-fractional differential systems with not-instantaneous impulses and finite delay are established. We use the tools of Banach fixed point theorem and condensing map along with generalization of the semigroup theory for linear operators and fractional calculus to come up with a new set of sufficient conditions for the existence and uniqueness of the mild solutions. An illustration is provided at the end of the chapter to demonstrate the established results

In Chapter 6, we obtain some exact controllability results for an abstract fractional impulsive quasilinear integro-differential system with state-dependent delay. We use the concepts of fractional calculus, measure of noncompactness and abstract phase space to come up with a new set of sufficient conditions for the exact controllability by using Mönch's fixed point theorem. At the end, an example is discussed to demonstrate the application of the obtained abstract results.

Chapter 7 is concerned with existence and approximate controllability of integral solutions to the systems determined by abstract fractional differential equations with nondense domain. We establish the existence and uniqueness results of integral solution by generalized Banach contraction principle. Moreover, our approximate controllability results are based on a sequencing technique in which the compactness of semigroup and uniformly boundedness of nonlinear functions are not required. Finally, an application is given to illustrate the obtained results.

The relevant references are appended at the end.

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Nomenclature

Let $\mathbb{X}$ and $\mathbb{Y}$ be two Banach spaces. Let $\mathbb{H}$ and $\mathbb{K}$ be two Hilbert spaces. We denote the space of bounded linear operators, from the space $\mathbb{X}$ into $\mathbb{Y}$, by $\mathcal{L}(\mathbb{X}, \mathbb{Y})$ endowed with uniform operator topology. The notation $\mathcal{L}(\mathbb{X}, \mathbb{Y})$ abbreviated by $\mathcal{L}(\mathbb{X})$ when $\mathbb{X} = \mathbb{Y}$. For a linear operator A on $\mathbb{X}$, $\mathcal{R}(A)$, $\mathcal{D}(A)$ and $\varrho(A)$ represent the range, domain and resolvent of A, respectively. For sake of convenience, we consider the following notations

Notation	**Description**
$\mathbb{R}$	Set of Real numbers
$\mathbb{N}$	Set of Natural numbers
$\mathbb{C}$	Set of Complex numbers
$B_r(x)$	Ball in $\mathbb{X}$ with center at x and radius r
$\mathcal{C}([a,b], \mathbb{X})$	Space of all continuous functions from $[a,b]$ into $\mathbb{X}$
$\mathcal{PC}([a,b], \mathbb{X})$	The space of all functions $y : [a,b] \to \mathbb{X}$, which are continuous everywhere except the point $t_j \in (a,b), j = 1, 2, ..., m$. At the points t_j, the right limit $y(t_j^+)$ and the left limit $y(t_j^-)$ of $y(t)$ exist and $y(t_j^-) = y(t_j)$. Moreover, the space $\mathcal{PC}([a,b], \mathbb{X})$ is Banach space equipped with the norm $\|y\| = \sup_{t\in[a,b]} \|y(t)\|$.
$W^{m,p}([a,b], \mathbb{X})$	Sobolev space

$\Gamma(n)$	Euler's continuous gamma fuctions
$J_{0^+}^{q}$	Riemann-Liouville (in short RL) fractional integral operator of order q
${}^{R}D_{0^+}^{q}$	Riemann-Liouville fractional differential operator of order q
${}^{c}D_{0^+}^{q}$	Caputo fractional differential operator of order q
$D_{0^+}^{p,q}$	Hilfer fractional differential operator of order q and type p.

Chapter 1

Introduction

1.1 General Introduction

A system is defined as a collection, set or arrangement of objects which are related by interactions and produce various outputs in response to different inputs. Moreover, a system is called dynamical system if it varies with respect to time. For example electromechanical machines such as motor car, aircraft or spaceships, biological systems such as human body, economic structures of countries or regions and population growth in a region are dynamical systems.

A differential equation in which the derivative $y'(t)$ of an unknown function y at some time t is related to the unknown function y as a function of some other function of time t is called functional differential equation. In other words the relation

$$y'(t) = f(t, y(t), y(h(t))), \quad t \in (0, b], b < \infty, \tag{1.1.1}$$

$$y(0) = y_0, \tag{1.1.2}$$

where f and h are some suitable functions, may be denoted as a functional differential equation with initial condition (1.1.2). Here the function h can be taken as deviated time-argument, delay or state dependent delay function i.e. in functional differential equations the state can no longer be represented by a vector $y(t)$ at discrete time t but may be represented as a history valued function y_t corresponding to previous time.

Functional differential equations were first studied by Bernoulli, Laplace and Condorcet in the late eighteenth century and considerable investigation of such equations through semigroup theory, dynamical system hypotheses and functional analysis, have been accomplished since then. Unlike ordinary differential equations(ODEs), functional differential equations are generally infinite dimensional and appears in numerous biological, chemical and physical systems which communicate through lossless channels.

Many real world phenomena such as the heat conduction in materials, vibrations in wires, propagation of small disturbances through a gas, or liquid motion of elastic bodies and many more physical problems can be modeled and studied by their governing partial differential equations. On basis of some common basic properties these partial differential equations are divided into several classes. In this work, we mainly concentrate on the heat equation and the wave equation with fractional order. The properties, which remain same for each member of the same class, are known as invariant properties of that class.

A functional analytic representation of the differential equation is called an abstract formulation for the problem and known as evolution equation. Evolution equations are the equations which may be interpreted as the differential law of the development in time of a system in Hilbert spaces or more generally in the Banach spaces of functions. It is always beneficial to consider the abstract formulation to study the invariant properties of certain class of problems. In such formulations, instead of investigating an individual problem we may concentrate on some invariant properties of the class of problems to which the problem belongs. Thus, in study of a abstract problem only invariant properties come into the picture and all unnecessary details of an individual problem get suppressed.

Now, we have an example to demonstrate the idea of abstract formulations of a partial differential equation.

Example 1.1.1. *Consider the following initial value problem for the semi-linear hyperbolic integro-differential equation in the n-dimensional Euclidean space $\mathbb{R}^n$,*

$$\begin{aligned}\frac{\partial^2}{\partial t^2} z(t,y) =&\Delta z(t,y) + F(t, z(t,y), \frac{\partial z}{\partial t}(t,y)),\\ &+ \int_{t_0}^{t} K(t-s) h\Big(s, z(s,y), \frac{\partial z}{\partial s}(s,y)\Big) ds, \quad t > t_0,\ y \in \mathbb{R}^n, \\ z(t_0, y) =& u_1(y), \quad \frac{\partial z}{\partial t}(t_0,y) = u_2(y), \quad y \in \mathbb{R}^n,\end{aligned} \tag{1.1.3}$$

where $t_0 < t_1 < t_2 < \cdots < t_n \leq t_0 + b$, $n \in \mathbb{N}$, $0 < b < \infty$, Δ denotes the n-dimensional Laplace operator, F, h are smooth nonlinear functions and K is a locally p-integrable function for $1 < p < \infty$. The system (1.1.3) is equivalent to the first-order system

$$\begin{aligned}\frac{\partial}{\partial t}\begin{pmatrix} z(t,y) \\ u(t,y)\end{pmatrix} =& \begin{pmatrix} 0 & I \\ \Delta & 0\end{pmatrix} \times \begin{pmatrix} z(t,y) \\ u(t,y)\end{pmatrix} + \begin{pmatrix} 0 \\ F(t, z(t,y), u(t,y))\end{pmatrix} \\ &+ \int_{t_0}^{t} K(t-s) \begin{pmatrix} 0 \\ h(s, z(s,y), u(s,y))\end{pmatrix} ds, \quad t > t_0,\ x \in \mathbb{R}^n, \\ \begin{pmatrix} z(t_0,y) \\ u(t_0,y)\end{pmatrix} =& \begin{pmatrix} u_1(y) \\ u_2(y)\end{pmatrix}, \quad y \in \mathbb{R}^n.\end{aligned} \tag{1.1.4}$$

Let

$$v(t) = \begin{pmatrix} z(t,\cdot) \\ u(t,\cdot)\end{pmatrix}$$

regarded as a function of y and take Y to be some space of functions on $\Omega \subset \mathbb{R}^n$. The derivatives $\frac{dv}{dt}$ and $\frac{\partial}{\partial t}\binom{z(t,\cdot)}{u(t,\cdot)}$ are both the limits of the difference quotient

$$\frac{\binom{z(t+h,\cdot)}{u(t+h,\cdot)} - \binom{z(t,\cdot)}{u(t,\cdot)}}{h}$$

first limit being in the sense of the norm of Y and the second limit being a pointwise one. We may formally recognize $\frac{\partial}{\partial t}\binom{z(t,\cdot)}{u(t,\cdot)}$ with $\frac{dv}{dt}$.

Therefore, the problem (1.1.3) can be rewritten with some proper choice of a Banach space Y as

$$\left.\begin{aligned} v'(t) &= Av(t) + \widetilde{F}(t, v(t)) + \int_{t_0}^{t} K(t-s)\widetilde{h}(s, v(s)) ds \\ v(t_0) &= v_0, \quad t \in [t_0, t_0 + b],\end{aligned}\right\} \tag{1.1.5}$$

where $A := \begin{pmatrix} 0 & I \\ \Delta & 0\end{pmatrix}$ is a linear operator with domain $\mathcal{D}(A) \subset Y$ which generates *the strongly continuous semigroup* $\mathcal{S}(\cdot)$*. The non-linear maps*

$$\widetilde{F} = \begin{pmatrix} 0 \\ F(t, z(t,y), u(t,y)), \end{pmatrix}$$
$$\widetilde{h} = \begin{pmatrix} 0 \\ h(s, z(s,y), u(s,y)) \end{pmatrix}$$

are defined as follows $\widetilde{F}, \widetilde{h} : [t_0, t_0 + b] \times Y \to Y$*,* $v_0 \in Y$ *and* K *(knows as Kernel function) is defined on* $[t_0, t_0 + b]$ *to* $\mathbb{R}$*. For (1.1.3), one may choose the space* Y *as* $Y = H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$*. The operator* A *defined by the matrix of operators* $\begin{pmatrix} 0 & I \\ \Delta & 0 \end{pmatrix}$*, generates a* C_0 *semigroup of bounded linear operators in* Y*.*

Thus, from the above example, the evolution equations may be considered as initial value problems (in short IVPs) for ordinary differential equations in an infinite dimensional space and are connected with partial differential equations which characterizing certain physical phenomena.

In various physical phenomena, more measurements are required at some instances in addition to standard initial data and, therefore, initial conditions may be replaced by nonlocal conditions. Nonlocal conditions, which are generalization of classical initial conditions, were firstly introduced by Byszewski [33]. He introduced the nonlocal conditions into IVPs and proved that many important physical phenomena may be modeled as certain partial differential equations with nonlocal conditions. In nonlocal conditions, more information is taken into account and that reduces the ill effects occurred due to a single initial measurement, therefore these conditions are usually more precise and useful for measurements in comparison to the classical ones.

Fractional calculus, a generalization of integer calculus, deals with the investigation and applications of derivatives and integrals of non-integer order. In particular, in this branch of mathematics, we study the notion and methods of solutions for differential equations involving fractional derivatives of the unknown function, which are called fractional differential equations (in short FDEs). The applications of fractional methodology are quite diverse and may be used in nonlinear dynamics, complex system dynamics, electrochemistry, viscoelasticity and image processing. The analysis of nonlinear oscillations of an earthquake, continuum and statistical mechanics, relaxation in fluid polymers and the modeling of visco-plasticity are some of the fields in which employment of differential equations involving fractional derivative provide more realistic analysis of the problem considered.

In last three decades, fractional differential equations have become of great importance as it describe the memory and hereditary properties of various materials and phenomena. Due to involvement of integral operator in the definition of fractional derivative, fractional differential operator is a non-local operator. That is, a fractional order derivative of a function at a certain point in space or time consists of information about the function at previous points in space or time, respectively. For example, viscoelastic materials and polymers which are related to systems with memory may be efficiently described with fractional differential equations.

It is observed that the present state of many physical phenomena depends on some previous (history) state. If we take this hypothesis into consideration while dealing with modeling problems, we end up with another class of differential equations called delay differential equations. Precisely, a differential equation in which the derivative of an unknown function at certain time is given by the value of function at previous time is called a delay differential equation. According to the situation the delay time can be finite, infinite or it may depend upon the state variable. Delay differential equations may be in almost all the areas of sciences, especially in biological sciences due to its numerous applications. For example in prey predator system, the predator decreases the average growth rate of the prey. In mathematical analysis, the mature rate for particular duration of time before predator is capable of decreasing the average growth rate of the particular species is assumed to be infinite. Thus it represents a system with infinite delay.

Various evolutionary processes such as population dynamics, orbital transfer of satellites and sampled-data systems are characterized by the abrupt changes in their state. These abrupt changes occur for a very short interval of time and can be approximated in terms of instantaneous changes of state, i.e., impulses. Such processes can be appropriately modeled by impulsive differential equations. In last few years, theory of impulsive differential systems has been developed as a beneficial tool, which may precisely define a mathematical model in various realistic situations, for example biological phenomena which involves thresholds, optimal control models in economics and bursting rhythm models in medicine. These equations are usually defined by a pair of equations, an ordinary differential equation to be satisfied during the continuous portion of evolution and a difference equation defining the discrete impulsive actions.

Generally, the impulses start abruptly for very short duration of time that can be negligible in comparison to the overall process. But in many practical situations instantaneous impulses failed to describe the certain dynamics of evolution processes. For example, pharmacotherapy, in which the hemodynamic equilibrium of a person is considered. The initiation of the drugs in the bloodstream and the resultant absorption for the body are gradual and continuous process. Therefore instantaneous impulses failed to describe such process. To characterize these type of situations Hernàndez and O′Regan [102] introduce a new case of impulsive action, which triggered abruptly at an arbitrary instant and remains active during a finite time interval. These type of impulsive conditions are called non-instantaneous impulsive conditions.

The differential equations that involve randomness in the mathematical description of a given phenomenon are known as stochastic differential equations. Due to randomness these differential equations may provide more accurate descriptions than the deterministic differential equations. In recent years, stochastic differential equations in both finite and infinite dimensions have attracted a lot of attention in many areas such as physics, population dynamics, electrical engineering, ecology, medicine biology and other areas of science and engineering, because of their practical applications in these areas. In fact, real phenomena in different fields, which involves stochastic excitations of a Gaussian white noise type have been extensively investigated both theoretically and experimentally over a long period of time. Remember that Gaussian white noise may be mathematically described as a formal derivative of a Brownian motion process, is a tolerable abstraction and is never a completely faithful representation of a physical noise source. The properties of the stochastic and the deterministic models differ significantly, so deep investigations of stochastic models are required. Therefore, it is beneficial to study the theory of stochastic differential equations as a well deserved discipline and it is also due to the increasing applications of stochastic differential equations in various fields.

For most of the differential equations, it is difficult to find exact solutions in closed forms. To overcome this difficulty, many numerical and analytical techniques have been designed for example, the homotopy analysis method and the Adomian decomposition method have been applied to integrate various systems of fractional order. However, in recent years, considerable work has been done using monotone iterative technique (in short MIT), which is a productive procedure for proving existence results in a closed set generated by upper and lower solutions. In MIT, by choosing upper and lower solutions as two initial iterations, one may construct two monotone sequences which converge monotonically from above and below, respectively, to a solution of the problem. The monotone behavior of the sequence of iterations is also useful in the treatment of numerical solutions of various boundary value and initial boundary value problems. A major advancement of this technique is the extension of the idea of upper-lower solutions to coupled systems of a finite number of parabolic and elliptic equations. Ladde et al. [128] has described a comprehensive introduction to the monotone iterative techniques in their book

"Monotone iterative techniques for nonlinear differential equations".

If a dynamical system is controlled by suitable inputs (controls) to obtain desired output (state) then it is called a control system. In other words, a control system is an interconnection of components forming a system configuration that provides a desired system response. The control system is an interdisciplinary field covering wide areas of engineering and sciences. It exists in everyday work of human life. For example, our body temperature and blood sugar level needs to be controlled at desired set points, insect and animal populations are controlled by very delicately balanced prey predator relationship. These control systems are provided to us by nature. There are several simple as well as complex man-made control systems which are used in our everyday life. Automatic water heater, washing machine, missiles etc. are some examples of man made control systems. However, whether a control system is natural or man-made, share a common aim, to control or regulate a particular variable within certain operating limits. Controllability is a mathematical problem, which analyzes the possibility of steering a system from an arbitrary initial state to an arbitrary final state using a set of admissible controls. It plays an important role in control problems such as stabilization of unstable systems by feedback control or in the study of optimal control. For this reason, it has been studied by several authors during the past few decades.

It is reveled that in ancient time some control techniques had been used by the Romans in order to control their aqueducts. Indeed, they used ingenious systems of regulating valves to make the water level constant in these constructions. Some historiographer says that some control systems were also used in the ancient Mesopotamia civilization (2000BC) such as the control system for irrigation system was well developed.

The modern mathematical control theory was introduced in the seventeenth century. It was started with the development of a pendulum clock in order to analysis the problem of speed control by a Dutch astronomer and mathematician and astronomer Krichristiaan Hugens. The invention of steam engine by James Watt in 1769 made control mechanisms very popular. In 1860s, James Clerk Maxwell published the first complete mathematical treatment of the steady state behavior of control systems. Characterizations of stability were independently obtained for linear systems by mathematicians Hurwitz and Routh. This theory was applied in various different areas such as the study of ship steering system. During 1930s, Bonde, Nyquist and others developed frequency domain approach and feedback control approach for linear systems. At the time of second world war and following years, the scientists and engineers improved their experiences on the control mechanisms for plane tracking, ballistic missiles and in designing of anti-aircraft missiles.

In 1950s, the control theory started with the powerful general techniques that were developed for treating time-varying nonlinear systems. The appreciable work of Kalman in filtering techniques and the algebraic approach to linear systems, Bellman in the context of dynamic programming and Pontryagin with the maximum principle for nonlinear optimal control problems, made a great contribution to the foundation of modern mathematical control theory.

1.2 Literature Survey

1.2.1 Existence of Solutions

The description of many physical phenomena such as nonlinear oscillations of earthquake, seepage flow in porus media, flow of fluid through fissured rocks [25], propagation of mechanical diffusive waves in viscoelastic media [147], relaxation phenomena in complex viscoelastic material [93] and biological models [2] may be described by FDEs. Fractional derivative was first mentioned in a letter correspondence between Leibniz and L'Hospital in 1695. Later on, many famous mathematicians e.g. Euler, Laplace, Fourier, Abel, Grünwald, Riemann, Liouville, Caputo etc. contributed a lot in this field. For introduction of theory of FDEs and its applications one may refer to the books by Kilbas and Trujillo [122], Miller and Ross [152], Podlubny [167] and Kilbas and Samko [185].

Many authors [63; 67; 77; 78] discussed the qualitative properties of the solution of the following FDE

$$^{R}D^{q}_{0+}y(t) = f(t, y(t)), \quad y(a) = b, \tag{1.2.1}$$

where $q \in (0,1)$, $a \in \mathbb{R}^{+}$, $b \in \mathbb{R}$ and f is a given continuous function. In [63], authors reduced equation (1.2.1) into an integral equation with weak singularity and applied basic techniques of nonlinear analysis. In [216], authors improved the existing results [63; 67; 77; 78] and obtained the results for (1.2.1) using Schauder fixed point theorem.

In [79], El-Sayed established the existence and uniqueness results for the following diffusion wave equation of fractional order

$$^{R}D^{q}_{0+}y(t) = Ay(t), \quad t > 0, \quad q \in (0,2], \tag{1.2.2}$$

$$y(0) = y_0 \qquad y'(0) = y_1 \tag{1.2.3}$$

where A generates an analytic semigroup $\mathcal{S}(t)$. Equation (1.2.2), represents the diffusion equation when $q \to 1$ and to the wave equation when $q \to 2$. Later on, Kaufmann and Mboumi [118] established the results for FDEs and provide sufficient conditions for the existence of at least one and at least three positive solutions to the nonlinear fractional boundary value problem. In [73], El-Borai introduced the definition of mild solution in terms of probability density function with Laplace transform to a Cauchy problem in a Banach space $\mathbb{X}$. In [74], El-Borai studied the existence and uniqueness of the solution of the FDE

$$^{c}D^{q}_{0+}y(t) = Ay(t) + F(t, B_1(t)y(t), \ldots, B_m(t)y(t)) \quad t > 0,\ q \in (0,1], \tag{1.2.4}$$

$$y(0) = y_0 \in \mathbb{X}, \tag{1.2.5}$$

where A generates an analytic semigroup, function F satisfies uniformly Hölder continuity in t and $\{B_j(t), j = 1, 2, \dots m\}$ is a family of closed densely defined linear operators on $\mathbb{X}$. In [222] and [224], Zhou obtained various results on solutions for fractional evolution equations. Later on, utilizing the concept of mild solution introduced by Zhou [222], many authors studied different type of fractional differential equations see [1; 4; 15; 41; 72; 103; 116; 124; 146; 155; 178; 212] and references therein.

In past few years, the theory of impulsive differential equations has grabbed a lot of attention of researchers as it provide an understanding of mathematical models to simulate the dynamics of processes in which sudden and discontinuous jumps occur. In order to solve impulsive fractional differential equations there are two main approaches. The first approach (also called multiple base point approach) was introduced by Benchohra and Slimani [30] in which they considered the following fractional initial value problem with impulsive effects

$$^cD^q_{0+}y(t) = F(t, y(t)),\ q \in (0,1],\ t \in [0,b],\ t \neq t_i, \tag{1.2.6}$$

$$\Delta y(t_i) = I_i(y(t_i^-)),\ i = 1, \cdots, m;\ m \in \mathbb{N}, \tag{1.2.7}$$

$$y(0) = y_0, \tag{1.2.8}$$

the functions F and $I_i,\ i = 1, \cdots, m$ are appropriate continuous functions. To find out the mild solution of (1.2.6), authors proposed the definition of classical Caputo fractional derivative and revised it in each subintervals $(t_i, t_{i+1}]$ for the equation (1.2.6), where the impulses start the lower bound from each impulsive points $t_i,\ i = 1, \cdots, m$. Moreover, authors gave the formula of solutions for (1.2.6) as

$$y(t) = \begin{cases} y_0 + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} F(s, y(s)) ds, \quad t \in [0, t_1], \\ y_0 + \frac{1}{\Gamma(q)} \sum_{j=1}^{i} \int_{t_{j-1}}^{t_j} (t_j - s)^{q-1} F(s, y(s)) ds \\ + \frac{1}{\Gamma(q)} \int_{t_i}^{t} (t-s)^{q-1} F(s, y(s)) ds + \sum_{j=1}^{i} I_j(y(t_j^-)), \quad t \in (t_i, t_{i+1}], \end{cases} \tag{1.2.9}$$

where $i = 1, \cdots, m$. The second approach (also called the single base point approach) was introduced by Fečkan et al. [85] in which they used the generalized Caputo fractional derivative with lower bound at zero. Since the generalized Caputo fractional derivative should be fixed at lower bound at zero, thus they did not change the lower bound in each subinterval in the definition of the Caputo derivative and suggested the following formula for the impulsive differential equation (1.2.6)-(1.2.8)

$$y(t) = \begin{cases} y_0 + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} F(s, y(s)) ds, & t \in [0, t_1], \\ y_0 + I_1(y(t_1^-)) + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} F(s, y(s)) ds, & t \in (t_1, t_2], \\ \vdots & \vdots \\ y_0 + \sum_{i=1}^m I_i(y(t_i^-)) + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} F(s, y(s)) ds, & t \in (t_m, b]. \end{cases} \tag{1.2.10}$$

Wang et al. [172] introduced the new concept of mild solution using Laplace transform and probability density function for the following FDE with impulsive conditions

$$^cD_{0+}^q y(t) = Ay(t) + F(t, y(t)), \quad q \in (0, 1], \, t \in (0, b], \; t \neq t_i, \tag{1.2.11}$$

$$y(0) = y_0, \tag{1.2.12}$$

$$y(t_i^+) = y(t_i^-) + y_i, \quad i = 1, \cdots, m; \; m \in \mathbb{N}, \tag{1.2.13}$$

where A generates a C_0-semigroup and F is a continuous function satisfying some appropriate conditions. Also, they extended the results for (1.2.11)-(1.2.13) to semilinear fractional evolution equations with nonlocal initial condition and impulsive conditions. For more details on impulsive differential equations we refer the reader to monographs [16; 29; 130; 186], papers [9; 30; 38; 39; 40; 45; 47; 109; 112; 134; 137; 138; 139; 146; 162; 180; 213] and references therein.

In [102], Hernández and O′Regan proposed a different case of impulsive action, which triggered abruptly at an arbitrary instant and remains active during a finite period of time. They study the existence of solutions for an impulsive equation

$$z'(t) = Ay(t) + f(t, y(t)), \quad t \in (s_j, t_{j+1}],\ j = 0, 1, \ldots, m, \tag{1.2.14}$$

$$y(t) = g_i(t, y(t)), \quad t \in (t_j, s_j],\ j = 1, 2, \ldots, m, \tag{1.2.15}$$

$$y(0) = y_0, \tag{1.2.16}$$

where A generates a C_0-semigroup of linear operators on a Banach space $\mathbb{X}$, points t_j and s_j are pre-fixed numbers, f and g are given suitable functions. Meanwhile, Pierri et al. [166] extended the results of [102] in a α-normed Banach space. In [54] and [161], Pandey et al. extended the results of [102] for second order differential equation and proved the existence results using measure of noncompactness and fixed point theorems. Later on, many authors extended the results of [102] for fractional differential equations see [3; 89; 92; 165; 191; 212].

In 1991, Byszewski [33] introduced the nonlocal Cauchy problem. He has done pioneering work on nonlocal condition problems [34; 35; 36] and generalized the Cauchy problem with initial condition to the Cauchy problem with nonlocal conditions in which authors allow the measurements at more than one point rather than at a single point so that these conditions become more precise for physical models than the classical ones. In [33], Byszewski proved the three main results on the existence and uniqueness of solutions of a Cauchy problem with nonlocal initial conditions, which generalizes known results given by Pazy [163] for the solutions of Cauchy problem. In [64], Deng shows that the diffusion of a small amount of gas in a transparent tube can be described using the nonlocal conditions efficiently than using local conditions. Later on, many authors study the differential equations with nonlocal conditions see [4; 12; 13; 22; 28; 41; 42; 57; 70; 72; 116; 159; 188; 209; 210] and references therein.

In the seventieth century, there were number of works devoted to stochastic partial differential equations. The pioneering papers of D. Dawson [56], N. V. Krylov and B. L. Rosovski [123], Ruth F Curtain [49], Akira Ichikawa [111], Yamada and

Watanabe [211] and Da prato et al. [168] setup a base for the study of solutions to stochastic differential equations.

In [208], Xu and Hu investigated the mild solution of a Cauchy problem for semilinear stochastic evolution equation in a Hilbert space by virtue of the C_0-semigroup theory. El-Borai et al.[198] investigated some classes of stochastic fractional integro-partial differential equations and proved the existence of a stochastic mild solution with the help of Leray-Schauder principle. Using fixed-point theorems, Balasubramaniam and Dauer [21] discussed the controllability of stochastic delay evolution equations. In [20], Balasubramaniam et al. discussed the Faedo-Galerkin approximate solutions for stochastic semilinear integrodifferential equations. In [48], Cui and Yan considered a fractional stochastic differential equations and derived the existence of a stochastic solution by means of Sadovskii's fixed point theorem and semigroup theory. For further details and applications of stochastic differential equations one may see monographs and papers [9; 14; 22; 46; 75; 80; 91; 109; 127; 129; 149; 150; 160; 164; 169; 183; 184; 197; 212; 213].

Moreover, Sobolev type fractional differential equations admit more adequate abstract representation to partial differential equations arising in numerous applications for example in control theory of dynamical systems, flow of fluid through fissured rocks [26], propagation of long waves of small amplitude, shear in second order fluids [110], thermodynamics [44] etc. In particular, Sobolev type fractional differential equations serve abstract formulation in the form of implicit operator differential equations when an operator coefficient multiplied by the highest derivative [60]. For more literature on Sobolev type differential equations, see [17; 61; 83; 120; 133; 173; 189] and references therein.

In 1982, Du and Lakshmikantham [69] proposed a monotone iterative technique (in short MIT) for ordinary differential equations

$$\begin{cases} y'(t) = f(t, y(t)), \quad t \in (0, b], \, b < \infty; \\ y(0) = y_0, \end{cases} \tag{1.2.17}$$

in an ordered Banach space $\mathbb{X}$. Here f satisfies certain monotonicity and measure of noncompactness conditions. They proved the existence of minimal and maximal mild solutions for (1.2.17) lying in between lower and upper solutions using MIT. In [107] and [108], Hristova and Bainov applied MIT to functional and impulsive differential equations. In [193], Sun improved the result of Du and Lakshmikantham [69] and removed the measure of noncompactness condition on function f. In [194], Sun and Zhao improved the result of [193] and further investigated the results for differential equations in a weakly sequentially complete Banach space $\mathbb{X}$ to find extremal solutions of

$$\begin{cases} y'(t) = f(t, y(t), Ty(t)), \quad t \in (0, b], \, b < \infty; \\ y(0) = y_0. \end{cases} \tag{1.2.18}$$

They removed the measure of noncompactness condition but extended the monotonicity condition on function f. In [96], Guo and Liu used MIT to find the existence of extremal solution of impulsive integro differential equation in a weakly sequentially complete Banach space $\mathbb{X}$. Later on, Li and Liu [134] used Bellman inequality to improve the result of Guo and Liu [96]. In [219], Zhang used the monotone iterative technique to evolution equations with the assumption that the semigroup $\mathcal{S}(t)$ generated by A is equicontinuous. But this assumption was very strong. To remove this assumption, Chen and Mu [47] apply monotone iterative technique to discuss the solutions of impulsive differential equation.

In [131], Lakshmikantham and Vatsala extended the monotone iterative technique to find global existence of solutions for Riemann-Liouville FDE given by

$$\begin{cases} {}^cD^q_{0+}y(t) = f(t, y), \qquad t \in (0, b], \, q \in (0, 1); \\ y(0) = y_0. \end{cases} \tag{1.2.19}$$

Later on, many authors [45; 113; 132] applied this technique coupled with method of lower and upper solution for initial value problem of Caputo or Riemann fractional differential equations or impulsive fractional differential equation using the monotonicity and measure of noncompactness condition. Recently Kamaljeet and Bahuguna [116] extended this technique for nonlocal FDE with finite delay and proved the results using monotonicity and measure of noncompactness conditions on nonlinearities. For more details on this technique one can see [3; 6; 134; 137; 151; 205; 206; 218] and references therein.

Zhang et al. [217] investigated existence and asymptotic stability for a class of fractional stochastic differential equations by virtue of some fixed point theorems. Recently, Rajivganthi et al. [175] established some existence results for mild solutions and optimal controls by applying successive approximation approach for a class of fractional neutral stochastic differential equations.

Recently, multi-term time-fractional differential equations generating great interest among the mathematicians and engineers. For instance, in the papers [121; 141; 8; 143] a deterministic two-term time-fractional differential equation is studied in the abstract context, which include a concrete case of fractional diffusion-wave problem. On the other hand, the multi-term time-fractional diffusion wave equation was recently considered in [53] and [142] with constant and variable coefficients, respectively. Moreover, for multi-term time-fractional diffusion equations in [114; 135] the authors studied analytic solutions and numerical solutions and Pardo at al. in [7] studied the existence of mild solutions with Caratheodory type conditions with measure of noncompactness techniques.

1.2.2 Controllability

Basic results of controllability for finite and infinite dimensional spaces are well established in [27; 225]. Consider the following first order semi-linear differential system

$$\begin{aligned} \frac{dy(t)}{dt} &= Ay(t) + Bu(t) + f(t, y(t)), \quad t \in (t_0, b], \\ y(t_0) &= y_0, \end{aligned} \tag{1.2.20}$$

where $A : \mathcal{D}(A) \subseteq \mathbb{X} \to \mathbb{X}$ is the infinitesimal generator of C_0-semigroup $\{\mathcal{S}(t)\}_{t\geq 0}$, the state y takes its value in a Banach space $\mathbb{X}$ and u is a control function having its values in Banach space $L^2([t_0, b], \mathbb{U})$. The operator $B : \mathbb{U} \to \mathbb{X}$ is a bounded linear operator and f is a nonlinear function.

A continuous function $y \in \mathcal{C}([t_0, b], \mathbb{X})$ defined by $y(t_0) = y_0$ and

$$y(t) = \mathcal{S}(t - t_0)y_0 + \int_{t_0}^{t} \mathcal{S}(t - s)[Bu(s) + f(s, y(s))]ds \tag{1.2.21}$$

is called a mild solution of the system (1.2.20).

We denote by $y(t) = y(t, y_0, u)$ the state value of system (1.2.20) corresponding to the control function u and initial values y_0 at the time t. The set $\mathfrak{R}(b, y_0, f) := \{y(b, y_0, u) : u(\cdot) \in L^2([0, b], \mathbb{U})\}$ is called the reachable set of the system (1.2.20) corresponding to the function $f(t, y(t))$, and its closure is denoted by $\overline{\mathfrak{R}(b, y_0, f)}$.

Definition 1.2.1. *On the time interval* $[0, b]$*, the system* (1.2.20) *is said to be*

- *approximate controllable if* $\overline{\mathfrak{R}(b, y_0, f)} = \mathbb{X}$*;*
- *exact controllable if* $\mathfrak{R}(b, y_0, f) = \mathbb{X}$*.*

In other words, if for every arbitrary final state $y_b \in \mathbb{X}$ and $\epsilon > 0$, there exists a control function $u \in L^2([0, b], \mathbb{U})$ such that

- $\|y_b - y(b, y_0, u)\| < \epsilon$, then the system (1.2.20) is approximate controllable;
- $y(t_0) = y_0$ and $y(b) = y_b$, then the system (1.2.20) is exact controllable.

In (1962-63), theory of controllability originated from the famous work [115] done by Kalman. In this work, Kalman investigated the controllability of a finite dimensional linear system and proved the controllability under a rank condition of the controllability matrix. [27].

In (1967), Tarnove [199] suggested a method to obtain the controllability of nonlinear systems, in which he studied the controllability by investigating the existence of a fixed point of a certain set valued mapping. He used the fixed point theorem due to Bohnenblust-Karlin to obtain sufficient conditions for A-controllability of a nonlinear system $x'(t) = f(t, x, u)$, where A is a non-empty, bounded, closed convex subset of continuous functions. Subsequently, this idea was used by Dauer (1972) [55] for systems of the form $x'(t) = f(t, x) + g(t, u)$ in finite dimensional spaces. Fattorini (1966) [81] considered a more general model of a system and studied the controllability for the case when A is densely defined closed linear operator which generates a strongly continuous semigroup $\{\mathcal{S}(t), t \geq 0\}$. Moreover, in (1967) [82], he derived some necessary and sufficient conditions for the approximate controllability for the case when A is self adjoint, semibounded above and defined on a Hilbert space and the dynamical system has only a finite number of scalar controls.

In (1975), Triggiani [201] extended the classical theory of controllability and observability of finite dimensional spaces to linear abstract systems defined on infinite dimensional Banach spaces, under the basic assumption that the operator acting on the state is bounded. In (1983), Zhou [221] established some controllability results for the approximate controllability of the semilinear control system by assuming that the linear control system is approximate controllable. The approximate controllability results were proved for the case when the range of the control operator B satisfies an inequality condition.

Controllability results were developed by Carmichael and Quinn [37] for the nonlinear control system in an infinite dimensional setting. They formulated the controllability problem as a fixed-point problem and used Nussbaum fixed-point theorem to establish conditions under which the nonlinear control system is exact controllable from the origin to some ball contained in an appropriate function space.

Using Schauder's degree theorem, Naito [157] analyzed the problem of approximate controllability of the semilinear control system under the uniform boundedness condition on the nonlinear operator f and a range condition on B and F. In (1989) [158], he replaced the uniform boundedness condition on f by the inequality condition given as $(1 - lMb\|P\| \exp(lMb)) > 0$ along with $f(0) = 0$, where l is the Lipschitz constant, M is such that $\sup\{\|\mathcal{S}(t)\| : t \in [0, b]\} = M$ and P is a projective type operator introduced by estimating the control efficiency of the operator B. This inequality condition implies that the system is approximate controllable, if lb is sufficiently small. In recent years, various kinds of nonlinear differential systems have been considered for the study of controllability results using different kinds of approaches in many recent publications, see [43; 71; 156; 173; 174; 187; 203] and the references therein.

Balachandran and Park [18] discussed the controllability of fractional semilinear integrodifferential system with nonlocal condition, using the Banach fixed point theorem with the tools of fractional calculus. Tai and Wang [196] addressed the controllability results for a impulsive fractional integrodifferential system with semigroup theory and fractional calculus in a Banach space via Krasnoselskii's fixed poit theorem. Wang and Zhou [207] studies some complete controllability results for a fractional differential system with the concept of Kuratowski's measure of noncompactness, Krasnoselskii's and Sadovskii's fixed point theorems. Feckan et al. [83] investigated the controllability results for sobolev type fractional functional system via character solution operators with Schauder fixed point theorem. Vijayakumar et al. [204] established controllability results for a class of fractional neutral integro-differential systems with infinite delay in the case when the corresponding linear system is controllable. Tai and Lun [195] establish controllability results for fractional impulsive neutral infinite delay evolution integrodifferential systems with Krasnoselskii fixed point theorem in a Banach space. Qin et al. [170] established some controllability results with measure of noncompactness by virtue of convex-power condensing operator for fractional integrodifferential systems. Arora and Sukavanam [11] establish some controllability results via Nussbaum fixed point theorem for a retarded semilinear fractional system with non-local conditions using the compactness condition on the nonlinear function. Some controllability results are also obtained using the concept of integral contractor by Arora and Sukavanam [10]. Heping and Biu [101] represented exact controllability results with the tools of resolvent operator and some analytic methods a fractional neutral integro-differential equations with state-dependent delay. Aissani et al. [5] obtained sufficient conditions for controllability results with state-dependent delay via Sadovskii's fixed point theorem for fractional integro-differential system. Du et al. [68] obtained controllability results for fractional neutral integro-differential systems with the measure of noncompactness and Mönch fixed point theorem.

Triggiani [202] proved that a dynamical system is not exactly controllable if the semigroup or control operator associated with the control system is compact. Thus, the exact controllability is a strong concept and, indeed admits limited applications in infinite dimensional spaces. On the other side, being a weaker concept, approximate controllability is almost adequate in applications. So, the approximate controllability of fractional differential system requires a more detailed study.

The approximate controllability results for fractional differential systems may be found in the noble work of Sakthivel et al. [182]. In this work, the sufficient conditions for approximate controllability are obtained by fractional power theory, semigroup theory and fixed point strategy under the assumption that the associated linear system is approximately controllable. Sukavanam and Surendra [192] established the approximate controllability results for a class of fractional order semilinear delay control systems with the approximate controllability of linear system. The sufficient conditions are also obtained for the existence and uniqueness of the mild solution. Surendra and Sukavanam [126] studied approximate controllability of a class of fractional order semilinear delay control systems using Schauder fixed point theorem and contraction principle. Sakthivel and Yong [176] obtained the sufficient conditions of approximate controllability by employing semigroup theory, fractional calculus and fixed point technique to a abstract fractional differential system with state dependent delay. For more details on approximate controllability, see the novel papers [144; 145; 181; 182].

In the above cited papers, the controllability results are investigated for the densely defined abstract differential systems i.e. $\overline{\mathcal{D}(A)} = \mathbb{X}$. However, as investigated by Prato and Sinestrari [50], there are some real life problems where we need to deal the problems with non-densely defined operators, such as in study of one-dimensional heat equation with Dirichlet conditions. Up to now there are very few literature dealing with the case that the linear parts are defined non-densely. For more remarks and examples concerning the non-densely defined operators, see [50]. Particularly, the existence of solutions with nondense domain, see [50; 95; 154]. Moreover, the control problems are also investigated in [87; 88; 119; 220] for non-densely defined abstract first order differential systems.

The concept of sequencing technique introduced by Zhou [221] to establish the sufficient conditions for approximate controllability to the abstract semilinear differential systems. For more details on this technique see [125; 190]. In this technique, it is not required to assume the compactness of semigroup and uniformly boundedness of nonlinear function associated with the system.

1.3 Organization of Thesis

In this thesis, we study some existence, uniqueness and controllability results of mild and integral solutions to differential systems of arbitrary order, integro-differential equations, delay differential equations, stochastic fractional differential equations involving nonlocal initial conditions and impulsive effects, by the means

of semigroup theory, fixed point techniques, monotone iterative technique and sequencing technique.
This thesis is divided into the following chapters.

Chapter 1: In this chapter, we provide a brief introduction to the problems considered in the subsequent chapters and prepare a motivational background to study the problems which are discussed in the thesis. Further, we give a literature review of related work done in respective areas.

Chapter 2: In this chapter, we provide some basic concepts of fractional calculus, functional analysis, semigroup theory and stochastic analysis required for the subsequent chapters.

Chapter 3: In this chapter, we consider a nonlinear Sobolev type fractional impulsive differential systems with fractional order nonlocal conditions. The sufficient conditions are obtained by measure of noncompactness and generalized Gronwall inequality. Finally, an application is given to illustrate the obtained results.

The contents of this chapter are published in **International Journal of Applied and Computational Mathematics** as Singh V., Pandey D. N.: A study of Sobolev type fractional impulsive differential systems with fractional nonlocal conditions, vol 4, issue 1, 2018, 1–12.

Chapter 4: In this chapter, we are concerned with the existence and uniqueness of mild solutions for a class of multi-term time-fractional stochastic differential equations in Hilbert spaces. In order to obtain the required results, fractional calculus, generalized semigroup theory and stochastic analysis techniques are employed. New results are obtain involving the coefficients in the equations satisfying some non-Lipschitz conditions and using standard Picard's iteration. Finally, an application is given to illustrate that our obtained results are valuable.

The contents of this chapter are accepted in **Differential Equations and Dynamical Systems** as Singh V., Pandey D. N.: Multi-term time-fractional stochastic differential systems with non-Lipschitz coefficients.

Chapter 5: In this chapter, we study the existence and uniqueness resultd for mild solutions of multi-term time-fractional differential systems with not-instantaneous impulses and finite delay. The main results are obtained with generalization of the semigroup theory for linear operators and fractional calculus. An illustration is provided to demonstrate the established results

The contents of this chapter are published in **Nonlinear Dynamics and System Theory** as Singh V., Pandey D. N.: Mild solutions for multi-term time-fractional impulsive differential systems, vol 18, issue 3, 2018, 307–318.

Chapter 6: In this chapter, we study some controllability results for an abstract fractional impulsive quasilinear integro-differential system with state-dependent delay. We will use the concepts of fractional calculus, measure of noncompactness and abstract phase space to come up with a new set of sufficient conditions for controllability by using Mönch's fixed point theorem. Finally, an application is given to illustrate the obtained results.

The contents of this chapter are published in **International Journal of Dynamics and Control** as Singh V., Pandey D. N.: Controllability of fractional impulsive quasilinear differential systems with state dependent delay, DOI 10.1007/s40435-018-0425-z.

Chapter 7: In this chapter, we are concerned with the existence and approximate controllability of integral solutions to the systems determined by abstract fractional differential equations with nondense domain. The main results are established using tools of generalized Banach contraction principle and a sequencing technique in which the compactness of semigroup and uniformly boundedness of nonlinear functions are not required. Finally, an application is given to illustrate the obtained results.

The contents of this chapter are submitted in **Collectanea Mathematica**.

Chapter 2

Preliminaries

In this chapter, we provide some basic concepts of functional analysis, fractional calculus, semigroup theory and stochastic analysis which serve as prerequisite for subsequent chapters.

2.1 Basic Concepts of Functional Analysis

We will use norm $\|f\|_{L^p}$ of f whenever $f \in L^p([a,b],\mathbb{X}), a < b < \infty$, for some $1 \leq p \leq \infty$, where $L^p([a,b],\mathbb{X})$ represents a Banach space of Bochner integrable functions equipped with the norm $\|f\|_{L^p}$.

$$\|f\|_{L^p} := \begin{cases} (\int_a^b \|f(s)\|^p ds)^{1/p}, & 1 \leq p < \infty, \\ \sup_{t\in[a,b]} \|f(t)\|, & p = \infty. \end{cases} \tag{2.1.1}$$

Now, we have some important inequalities:

- The **Hölder inequality**: Let $p \in [1,\infty)$ and q is such that $1/p + 1/q = 1$. Then

$$\|fg\|_{L^1} \leq \|f\|_{L^p}\|g\|_{L^q}, \tag{2.1.2}$$

where $f \in L^p([a,b],\mathbb{X})$, $g \in L^q([a,b],\mathbb{X})$.

- The **Young inequality**: Assume $f \in L^p([a,b],\mathbb{X})$, $g \in L^q([a,b],\mathbb{X})$ and $1/p + 1/q = 1/r + 1$ such that $1 \leq p,q,r \leq \infty$. Then

$$\|f * g\|_{L^r} \leq \|f\|_{L^p}\|g\|_{L^q}, \tag{2.1.3}$$

where " $*$ " denotes the convolution.

Lemma 2.1.1. ***(Generalized Gronwall's inequality [97])****: Let $a \geq 0$, $\beta > 0$, $c(t)$ and $u(t)$ be the nonnegative locally integrable functions on $0 \leq t < b < +\infty$, such that*

$$u(t) \leq c(t) + a\int_0^t (t-s)^{\beta-1}u(s)ds,$$

then

$$u(t) \leq c(t) + \int_0^t \left[\sum_{n=1}^{\infty} \frac{(a\Gamma(\beta))^n}{\Gamma(n\beta)}(t-s)^{n\beta-1}c(s)\right]ds, \quad 0 \leq t < b.$$

The notations $\mathcal{C}(J,\mathbb{X})$ and $\mathcal{C}^m(J,\mathbb{X})$ stand for the spaces of all continuous functions and m-times continuously differentiable functions, respectively. Set $J = [0,b]$, $b < \infty$. Then, $\mathcal{C}(J,\mathbb{X})$ and $\mathcal{C}^m(J,\mathbb{X})$ denote the Banach spaces equipped with the norm denoted by

$$\|f\|_{\mathcal{C}} := \sup_{t\in J}\|f(t)\|, \quad \|f\|_{\mathcal{C}^m} := \sup_{t\in J}\sum_{k=0}^{m}\|f^{(k)}(t)\|, \tag{2.1.4}$$

respectively.

Definition 2.1.1. *The* ***Laplace transform*** *of a function $f \in L^1(\mathbb{R}^+,\mathbb{X})$ is given by*

$$\widehat{f}(\lambda) := \int_0^\infty e^{-\lambda t}f(t)dt, \tag{2.1.5}$$

Definition 2.1.2. *Let $\mathbb{X}$ and $\mathbb{Y}$ be two Banach spaces. A function $f : \mathbb{X} \to \mathbb{Y}$ satisfying the following condition for a constant $L > 0$ such that*

$$\|f(z_1) - f(z_2)\|_{\mathbb{Y}} \leq L\|z_1 - z_2\|_{\mathbb{X}}, \ \textit{for all} \ z_1, \ z_2 \in \mathbb{X}, \tag{2.1.6}$$

is called a ***Lipschitz continuous*** *function.*

Definition 2.1.3. *A function $f : \mathbb{X} \to \mathbb{Y}$ is said to be a* ***Hölder continuous*** *if there exist nonnegative constants $C > 0$ and $\theta \in (0,1]$ such that*

$$\|f(z_1) - f(z_2)\|_{\mathbb{Y}} \leq C\|z_1 - z_2\|_{\mathbb{X}}^{\theta} \ \textit{for each} \ z_1, z_2 \in \mathbb{X}. \tag{2.1.7}$$

The number θ is called the Hölder exponent of the condition. In particular, the function is Lipschitz continuous if $\theta = 1$ and bounded if $\theta = 0$.

Definition 2.1.4. *A family F of functions defined on a set $\mathbb{E}$ in $\mathbb{X}$ is called* ***equicontinuous*** *on $\mathbb{E}$ if for given $\epsilon > 0$, we can find a $\delta > 0$ in a way that*

$$\|f(z_0) - f(z)\| < \epsilon \quad \textit{whenever} \ \|z_0 - z\| < \delta, \ z_0, \ z \in \mathbb{E} \ \textit{and} \ f \in F. \tag{2.1.8}$$

Definition 2.1.5. *Let $F : \mathbb{X} \to \mathbb{X}$ be a function. Then a solution of the equation*

$$F(z) = z, \quad z \in \mathbb{X} \tag{2.1.9}$$

is known as a fixed point of the function F.

Definition 2.1.6. *Let $\mathbb{X}$ be a normed linear space. A mapping $F : B \subset \mathbb{X} \to \mathbb{X}$ satisfying the following condition for a constant $0 < C < 1$ in a way that*

$$\|F(z_1) - F(z_2)\| \le C\|z_1 - z_2\| \quad \textit{for all } \; z_1, z_2 \in B. \tag{2.1.10}$$

Then, F is called a ***contraction mapping****.*

Definition 2.1.7. *Let $\mathbb{X}$ and $\mathbb{Y}$ be normed linear spaces. An operator $\mathbb{T} : \mathbb{X} \to \mathbb{Y}$ is called* ***compact*** *if it maps every bounded subset of $\mathbb{X}$ into a relatively compact subset of $\mathbb{Y}$.*

Theorem 2.1.2. *(**Arzela-Ascoli theorem**) Let N be a compact set in $\mathbb{R}^n$, $n \ge 1$. Then, a set $B \subset C(N)$ is relatively compact in $C(N)$ if and only if the functions in B are uniformly bounded and equicontinuous on N.*

Theorem 2.1.3. *Let $\mathbb{X}$ and $\mathbb{Y}$ be two normed linear spaces. A linear operator $\mathbb{T} : \mathbb{X} \to \mathbb{Y}$ is compact if and only if for every bounded sequence (z_n) in $\mathbb{X}$ there exists a sequence $(\mathbb{T}(z_n))$ in $\mathbb{Y}$ which has a convergent subsequence.*

2.2 Semigroup Theory

Definition 2.2.1. *[163]A one parameter family $\{\mathcal{S}(t)\}_{t\ge 0}$, of bounded linear operators from Banach space $\mathbb{X}$ into $\mathbb{X}$ is known as semigroup of bounded linear operators on $\mathbb{X}$ if the following properties hold:*

(1) *$\mathcal{S}(0) = I$, where I represents the identity operator on $\mathbb{X}$.*

(2) *$\mathcal{S}(t + s) = \mathcal{S}(t)\mathcal{S}(s)$ for every $t, s \ge 0$(the semigroup property).*

Definition 2.2.2. *[163]A semigroup $\{\mathcal{S}(t)\}_{t\ge 0}$ of bounded linear operators on $\mathbb{X}$ is called strongly continuous semigroup or C_0-semigroup if*

$$\lim_{t\downarrow 0} \mathcal{S}(t)z = z, \quad \textit{for every } \; z \in \mathbb{X}. \tag{2.2.1}$$

Definition 2.2.3. *[163]The semigroup $\{\mathcal{S}(t)\}_{t\ge 0}$ of bounded linear operators is called uniformly continuous semigroup if $\lim_{t\to 0} \| \mathcal{S}(t) - I\| = 0$.*

Definition 2.2.4. *[163]The infinitesimal generator of a semigroup of bounded linear operator, $\{\mathcal{S}(t)\}_{t\ge 0}$, on Banach space $\mathbb{X}$ is a linear operator A on $\mathbb{X}$ defined by*

$$Az = \lim_{t\downarrow 0} \frac{\| \mathcal{S}(t)z - z\|}{t}, \; \textit{for } z \in D(A), \tag{2.2.2}$$

whenever this limit exists. The domain of A denoted by $D(A)$ defined as

$$D(A) = \{z \in \mathbb{X} : \lim_{t\downarrow 0} \frac{\| \mathcal{S}(t)z - z\|}{t} \; \textit{exists}\}. \tag{2.2.3}$$

Remark 2.2.1. *A semigroup $\mathcal{S}(t)$ has a unique infinitesimal generator.*

Theorem 2.2.2. *[163]Let $\mathcal{S}(t)$ be the C_0-semigroup. Then we can find constants $\delta \in \mathbb{R}$ and $M \geq 1$ in a way that*

$$\|\mathcal{S}(t)\| \leq Me^{\delta t}, \quad \textit{for all } t \geq 0. \tag{2.2.4}$$

Remark 2.2.3. *If $\delta = 0$, then, $\mathcal{S}(t)$ is called uniformly bounded semigroup. Moreover, if $M = 1$, then $\mathcal{S}(t)$ is called C_0-semigroup of contractions.*

Theorem 2.2.4. *[163] Let $\mathcal{S}(t)$ be a C_0-semigroup of bounded linear operators on $\mathbb{X}$ which is generated by A. Then,*

(1) *$\mathcal{S}(t)$ is bounded on every finite subinterval of $[0, \infty)$,*

(2) *for each $z \in \mathbb{X}$, $\lim_{h\to 0} \frac{1}{h} \int_t^{t+h} \mathcal{S}(s)zds = \mathcal{S}(t)z$,*

(3) *for all $z \in \mathbb{X}$, $\int_0^t \mathcal{S}(s)zds \in D(A)$ and*

$$A\bigg(\int_0^t \mathcal{S}(s)zds\bigg) = \mathcal{S}(t)z - z, \tag{2.2.5}$$

(4) *for $z \in D(A)$, $\mathcal{S}(t)z \in D(A)$ and*

$$\frac{d}{dt}\mathcal{S}(t)z = A\mathcal{S}(t)z = \mathcal{S}(t)Az, \tag{2.2.6}$$

(5) *for all $z \in D(A)$,*

$$\mathcal{S}(t)z - \mathcal{S}(s)z = \int_s^t \mathcal{S}(\tau)Azd\tau = \int_s^t A\mathcal{S}(\tau)zd\tau, \tag{2.2.7}$$

Corollary 2.2.5. *[163] Let A be the infinitesimal generator of a C_0-semigroup $\{\mathcal{S}(t)\}_{t\geq 0}$. Then, $D(A)$ is dense in $\mathbb{X}$ and A is a closed bounded linear operator.*

For a linear operator A (bounded or unbounded in $\mathbb{X}$) the resolvent set $\varrho(A)$ of A consists all $\lambda \in \mathbb{C}$ such that $(\lambda I - A)^{-1}$ is a bounded linear operator in $\mathbb{X}$. The resolvent of A is a family $R(\lambda, A) = (\lambda I - A)^{-1}$, $\lambda \in \varrho(A)$. For the resolvent operator $R(\lambda, A)$ of the generator A of a C_0-semigroup, we have the following result which shows that the resolvent operator is just the Laplace transform of the semigroup.

Lemma 2.2.6. *[163] Let $\mathcal{S}(t)$ be a C_0-semigroup with infinitesimal generator A and growth bound w_0. If $Re(\lambda) > w > w_0$, then $\lambda \in \varrho(A)$, and for all $y \in \mathbb{X}$ the following results hold:*

(a) $R(\lambda, A)y = (\lambda I - A)^{-1}y = \int_0^\infty e^{-\lambda t}\mathcal{S}(t)ydt$ *and* $\|R(\lambda, A)\| \leq \frac{M}{\mu - w}$; $\mu = Re(\lambda)$;

(b) *For all* $y \in \mathbb{X}$, $\lim_{\beta\to\infty} \beta(\beta I - A)^{-1}y = y$, *where* β *is constrained to be real.*

Theorem 2.2.7. *[163](**Hille-Yosida Theorem**)A necessary and sufficient condition for a closed linear densely defined operator* A *on a Banach space* $\mathbb{X}$ *to be the infinitesimal generator of a strongly continuous semigroup* $\mathcal{S}(t)$, $t \geq 0$ *on* $\mathbb{X}$ *is that there exist real numbers* M *and* δ *such that every real* $\lambda > \delta$ *belongs to* $\varrho(A)$ *and for such* λ

$$\|R(\lambda, A)^k\| \leq \frac{M}{(\lambda - \delta)^k}, \forall \ k \geq 1, \tag{2.2.8}$$

where $R(\lambda, A) = (\lambda I - A)^{-1}$ *denotes the resolvent operator of* A.

Theorem 2.2.8. *[163](**Hille-Yosida Theorem for Contraction Semigroups**) Let* $A : D(A) \subset \mathbb{X} \to \mathbb{X}$ *be a closed densely defined linear operator. Then* A *is the infinitesimal generator of a contraction semigroup if and only if every real* $\lambda \in \varrho(A)$ *and*

$$\|(\lambda I - A)^{-1}\| \leq \frac{1}{\lambda}. \tag{2.2.9}$$

Now, we state the following results which provide the representation of the semigroup generated by a bounded linear operator.

Theorem 2.2.9. *Let* U *be a bounded linear operator. If* $\|U\| \leq \upsilon$, *then*

$$e^{tU} = \frac{1}{2\pi i}\int_{\upsilon - i\infty}^{\upsilon + i\infty} e^{\lambda t}(\lambda I - U)^{-1}d\lambda. \tag{2.2.10}$$

The convergence in (2.2.10) is in the uniform operator topology and uniformly in t *on bounded intervals.*

Theorem 2.2.10. *Let* A *be a densely defined linear operator in* $\mathbb{X}$ *which satisfies the following two conditions:*

(1) $\sum_\mu = \{\lambda : |arg\lambda| < \frac{\pi}{2} + \mu\} \cup \{0\} \subset \varrho(A)$, *for some* $0 < \mu < \pi/2$;

(2) *there is a constant* $\mathcal{M}$ *in such a way that*

$$(\lambda I - A)^{-1} \leq \frac{\mathcal{M}}{|\lambda|}, \ \ for \ \lambda \in \sum_\mu \ \ and \ \lambda > 0.$$

Then, A *generates a* C_0*-semigroup* $\mathcal{S}(t)$ *fulfilling* $\|\mathcal{S}(t)\| \leq N$ *for constant* $N > 0$ *and*

$$\mathcal{S}(t) = \frac{1}{2\pi i}\int_\Gamma e^{\lambda t}(\lambda I - A)^{-1}d\lambda,$$

where Γ *is a smooth curve in* $\sum_\mu$ *starting from* $\infty e^{-i\theta}$ *to* $\infty e^{i\theta}$ *for some* $\pi/2 < \theta < \pi/2 + \mu$ *and for* $t > 0$, *the integral converges in the uniform operator topology.*

Definition 2.2.5. *A one parameter family of bounded linear operators* $\{C(t)\}_{t\in[0,b]}$ *is a strongly continuous cosine family, if the following conditions hold:*

(*i*) $C(0) = I$,(I *denotes the identity operator on* $\mathbb{X}$).

(*ii*) $C(t+s) + C(t-s) = 2C(t)C(s)$, *for all* $s, t \in [0, b]$.

(*iii*) *For each fixed* $y \in \mathbb{X}$, *and* $t \in [0, b]$, $C(t)y$ *is continuous.*

Moreover, sine family $\{S(t)\}_{t\in[0,b]}$ of bounded linear operators associated to the cosine family $\{C(t)\}_{t\in[0,b]}$ is defined by $S(t)y = \int_0^t C(s)yds$, $y \in \mathbb{X}$, $t \in [0, b]$. The infinitesimal generator $A : \mathbb{X} \to \mathbb{X}$ of cosine family $\{C(t)\}_{t\in[0,b]}$, is given by

$$Ay = \frac{d^2}{dt^2}C(t)y|_{t=0}, \quad y \in \mathcal{D}(A),$$

where $\mathcal{D}(A) := \{y \in \mathbb{X} : C(t)y \,\text{is twice continuously differentiable in}\; t \in [0, b]\}$.

2.3 Basic Concepts of Fractional Calculus

Nowadays, the functional differential equations involving fractional order derivatives are receiving increasing interest in the scientific community due to numerous applications in widespread areas of science and engineering such as in models of medicines, electrical engineering, biochemistry and, for more applications, one may see [25; 105]. It have been shown that the fractional differential equations are capable to describe the dynamical behavior of a real life phenomena more precisely.

In literature, there are more than fifteen definitions of fractional derivative. But the most commonly used definitions are the definitions given by Caputo and Riemaan-Liouville, as these two definitions coincide with the ordinary derivative at integers. Here we have the following definitions of fractional calculus.

Define $g_\eta(t)$ for $\eta > 0$ by

$$g_\eta(t) = \begin{cases} \frac{1}{\Gamma(\eta)}t^{\eta-1}, & t > 0; \\ 0, & t \leq 0, \end{cases}$$

The function g_η has the properties $(g_a * g_b)(t) = g_{a+b}(t)$, for $a, b > 0$ and $\widehat{g_\eta}(\lambda) = \frac{1}{\lambda^\eta}$ for $\eta > 0$ and $\text{Re } \lambda > 0$, where $\widehat{(\cdot)}$ and $*$ denote the Laplace transformation and convolution, respectively.

Definition 2.3.1. *The Riemann-Liouville fractional integral of a function $f \in L^1_{loc}([0,\infty),\mathbb{R})$ of order $\eta > 0$ with lower limit zero is defined as follows*

$$J^\eta_{0^+} f(t) = (g_\eta * f)(t) = \int_0^t g_\eta(t-s) f(s) ds, \quad t > 0,$$

and $J^0_{0^+} f(t) = f(t)$.

This fractional integral satisfies the properties $J^\eta_{0^+} \circ J^b_{0^+} = J^{\eta+b}_{0^+}$ for $b > 0$, $J^\eta_{0^+} f(t) = (g_\eta * f)(t)$ and $\widehat{J^\eta_{0^+} f}(t) = \frac{1}{\lambda^\eta} \widehat{f}(\lambda)$ for $\text{Re}\,\lambda > 0$.

Definition 2.3.2. *Let $\eta > 0$ be given and denote $m = \lceil \eta \rceil$. The Riemann-Liouville fractional derivative of order $\eta > 0$ for a function $f : \mathbb{R}^+ \to \mathbb{X}$ is defined by*

$${}^R D^\eta_{0^+} f(t) = D^m (g_{m-\eta} * f)(t) = \frac{1}{\Gamma(m-\eta)} \frac{d^m}{dt^m} \int_0^t (t-s)^{m-\eta-1} f(s) ds,$$

and ${}^R D^0_{0^+} f(t) = f(t)$, where $\lceil \cdot \rceil$ denotes the ceiling function and $D^m = \frac{d^m}{dt^m}$.

Definition 2.3.3. *Let $\eta > 0$ be given and denote $m = \lceil \eta \rceil$. The Caputo fractional derivative of order $\eta > 0$ of a function $f \in \mathcal{C}^m([0,\infty),\mathbb{R})$ with lower limit zero is given by*

$${}^c D^\eta_{0^+} f(t) = J^{m-\eta}_{0^+} D^m f(t) = \int_0^t g_{m-\eta}(t-s) D^m f(s) ds,$$

and ${}^c D^0_{0^+} f(t) = f(t)$. In addition, we have ${}^c D^\eta_{0^+} f(t) = (g_{m-\eta} * D^m f)(t)$ and the Laplace transformation of Caputo fractional derivative is given by

$$\widehat{{}^c D^\eta_{0^+} f}(t) = \lambda^\eta \widehat{f}(\lambda) - \sum_{d=0}^{m-1} f^{(d)}(0) \lambda^{\eta-1-d}, \quad \lambda > 0. \tag{2.3.1}$$

Remark 2.3.1. *Let $m - 1 < \eta \leq m$, then*

$$(J^\eta_{0^+} \circ {}^c D^\eta_{0^+}) f(t) = f(t) - \sum_{d=0}^{m-1} f^{(d)}(0) g_{d+1}(t), \quad t > 0. \tag{2.3.2}$$

If $f^{(d)}(0) = 0$, for $d = 1, 2, 3, ..., m-1$, then $(J^\eta_{0^+} \circ {}^c D^\eta_{0^+}) f(t) = f(t)$ and $\widehat{{}^c D^\eta_{0^+} f}(t) = \lambda^\eta \widehat{f}(\lambda)$.

This definition is more restrictive than Riemann-Lioville one because it requires the absolute integrability of the m^{th}-order derivative of the function $f(t)$. In fact, between the two definitions the following relation holds:

$$^{c}D_{0+}^{\eta}f(t) = {}^{R}D_{0+}^{\eta}\left[f(t) - \sum_{k=0}^{m-1}\frac{t^k}{k!}\frac{d^m}{ds^m}f(s)|_{s=0}\right], \tag{2.3.3}$$

where $m = \lceil \eta \rceil$.

The main advantage of the Caputo derivative is that the initial conditions for FDEs are of the same form as that of integer-order differential equations and Caputo fractional derivative of a constant is zero, which is nonzero according to the Riemann-Liouville definition.

Definition 2.3.4. *The left-sided Hilfer fractional derivative of order* $0 < \beta < 1$ *and* $0 \le \alpha \le 1$ *of a function* $f : [0, \infty) \to \mathbb{R}$ *with lower limit* 0 *is given by*

$$D_{0^+}^{\alpha,\beta}f(t) = J_{0^+}^{\alpha(1-\beta)}\frac{d}{dt}J_{0^+}^{(1-\alpha)(1-\beta)}f(t),$$

provided that the right side expression exists.

Remark 2.3.2. (i) *The Hilfer fractional derivative with* $\alpha = 0$, $0 < \beta < 1$ *and* $a = 0$ *implies classical Riemann-Liouville fractional derivative i.e.* $D_{0^+}^{0,\beta}f(t) = \frac{d}{dt}J_{0^+}^{(1-\beta)}f(t) = {}^{R}D_{0+}^{\beta}f(t)$.

(ii) *The Hilfer fractional derivative with* $\alpha = 1$, $0 < \beta < 1$ *and* $a = 0$ *implies classical Caputo fractional derivative i.e.* $D_{0^+}^{1,\beta}f(t) = J_{0^+}^{(1-\beta)}\frac{d}{dt}f(t) = {}^{c}D_{0+}^{\beta}f(t)$.

2.3.1 Solutions of Caputo Fractional Differential Equations

We consider the following homogenous problem

$$^{c}D_{0+}^{q}y(t) = 0, \quad t > 0, \quad q \in (n-1, n), \ n = \lceil q \rceil. \tag{2.3.4}$$

Then, solution of the above equation is given by

$$y(t) = d_0 + d_1 t + d_2 t^2 + \cdots + d_{n-1}t^{n-1}, \tag{2.3.5}$$

where $d_i \in \mathbb{R}$, $i = 1, \cdots, n-1$ and $\lceil q \rceil$ is ceiling function. For the nonhomogenous fractional differential equation

$$^{c}D_{0+}^{q}y(t) = F(t), \quad t > 0, \quad q \in (n-1, n), \quad t \in [0, b], \ b \in \mathbb{R}^{+}, \tag{2.3.6}$$

we get the following integral equation

$$y(t) = d_0 + d_1 t + d_2 t^2 + \cdots + d_{n-1} t^{n-1} + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} F(s) ds. \tag{2.3.7}$$

Thus, a function $y \in \mathcal{C}(\mathbb{R}^+, \mathbb{R})$ is said to be a solution of (2.3.6) if and only if y satisfies the integral equation (2.3.7). Now, we consider the infinite dimensional fractional order problem illustrated as

$$^cD^q_{0+} y(t) = Ay(t), \; t \in [0, T], T < \infty, \tag{2.3.8}$$

$$y(0) = y_0, \tag{2.3.9}$$

where $^cD^q_{0+}$ denotes the fractional derivative in the Caputo sense of order q, $0 < q < 1$, the state $y(\cdot)$ takes its values in $\mathbb{X}$, $A : \mathcal{D}(A) \subseteq \mathbb{X} \to \mathbb{X}$ is a closed densely defined linear operator which generates C_0-semigroup of bounded linear operator $\mathcal{S}(t), \; t \geq 0$.

The equation (2.3.8) is equivalent to the integral equation

$$y(t) = y_0 + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} Ay(s) ds. \tag{2.3.10}$$

The solution to (2.3.8) is associated with a function $y \in \mathcal{C}([0, T], \mathbb{X})$ that satisfies the following assumptions

(i) y is continuous on $[0, T]$ and $y(t) \in \mathcal{D}(A)$ for each $t \in [0, T]$,

(ii) $^cD^q y(t)$ exists and is continuous on $[0, T]$ with $0 < q < 1$,

(iii) y satisfies the equation (2.3.8) on $[0, T]$ and initial condition $y(0) = y_0$.

Taking the Laplace transformation, we get

$$\begin{aligned} L[y(t)] &= L[y_0] + L[\frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} Ay(s) ds] \\ &= L[y_0] + \frac{1}{\lambda^q} AL[y(t)] \\ &= \lambda^{q-1} (\lambda^q I - A)^{-1} y_0 = \lambda^{q-1} \int_0^\infty e^{-\lambda^q s} \mathcal{S}(s) y_0 ds, \end{aligned} \tag{2.3.11}$$

where I denotes the identity operator on $\mathbb{X}$.

Consider the following one-sided stable probability density [148]

$$\Phi_q(\zeta) = \frac{1}{\pi}\sum_{n=1}^{\infty}(-1)^{n-1}\zeta^{-nq-1}\frac{\Gamma(nq+1)}{n!}\sin(n\pi q),\ \zeta \in (0,\infty). \tag{2.3.12}$$

whose Laplace transform is given by

$$\int_0^\infty e^{-\lambda\zeta}\Phi_q(\zeta)d\zeta = e^{-\lambda^q}, \quad q \in (0,1). \tag{2.3.13}$$

Therefore, we get

$$\begin{aligned}
\lambda^{q-1}&\int_0^\infty e^{-\lambda^q s}\mathcal{S}(s)y_0 ds \\
&= \int_0^\infty q(\lambda t)^{q-1}e^{-(\lambda t)^q}\mathcal{S}(t^q)y_0 dt, \quad (\text{put } s = t^q) \\
&= \int_0^\infty \frac{-1}{\lambda}\frac{d}{dt}[e^{-(\lambda t)^q}]\mathcal{S}(t^q)y_0 dt \\
&= \int_0^\infty \int_0^\infty \zeta\Phi_q(\zeta)e^{-(\lambda t\zeta)}\mathcal{S}(t^q)y_0 d\zeta dt, \\
&= \int_0^\infty e^{-\lambda t}\Big[\int_0^\infty \Phi_q(\zeta)\mathcal{S}(t^q/\zeta^q)y_0 d\zeta\Big]dt.
\end{aligned} \tag{2.3.14}$$

From (2.3.11) and (2.3.14), we get

$$L[y(t)] = \int_0^\infty e^{-\lambda t}\Big[\int_0^\infty \Phi_q(\zeta)\mathcal{S}(t^q/\zeta^q)y_0 d\zeta\Big]dt. \tag{2.3.15}$$

Taking inverse Laplace transform of the above equation, we get

$$\begin{aligned}
y(t) &= \int_0^\infty \Phi_q(\zeta)\mathcal{S}(t^q/\zeta^q)y_0 d\zeta \\
&= \int_0^\infty \Psi_q(\zeta)\mathcal{S}(t^q\zeta)y_0 d\zeta \\
&= \mathcal{S}_q(t)y_0,
\end{aligned} \tag{2.3.16}$$

where $\Psi_q(\zeta) = \frac{1}{q}\zeta^{-1-\frac{1}{q}}\Phi_q(\zeta^{-1/q})$ satisfies the conditions of a probability density function defined on $(0,\infty)$, i.e. $\Psi_q(\zeta) \geq 0$, and $\int_0^\infty \Psi_q(\zeta)d\zeta = 1$. Therefore, the solution of (2.3.8) is given as

$$y(t) = \mathcal{S}_q(t)y_0, \tag{2.3.17}$$

where $\mathcal{S}_q(t),\ t \geq 0$ is defined by

$$\mathcal{S}_q(t)y = \int_0^\infty \Psi_q(\zeta)\mathcal{S}(t^q\zeta)yd\zeta, \quad y \in \mathcal{D}(A). \tag{2.3.18}$$

The Laplace transform of Ψ_q is given by

$$L[\Psi_q(t)] = \int_0^\infty e^{-\lambda t}\Psi_q(t)dt = F_q(\lambda) = \sum_{m=0}^{\infty} \frac{(-\lambda)^m}{\Gamma(qm+1)} = E_q(-\lambda), \tag{2.3.19}$$

for $0 < q < 1$.

Next, we consider the following fractional differential equation

$$^cD^q y(t) = Ay(t) + F(t), \quad t \in [0,T], \ \ 0 \leq T < \infty, \tag{2.3.20}$$

$$y(0) = y_0, \tag{2.3.21}$$

where $F \in L^1([0,T],\mathbb{X})$.

Taking Laplace transform on both sides, we have

$$\begin{aligned} L[y(t)] &= \lambda^{q-1}(\lambda^q I - A)^{-1}y_0 + (\lambda^q I - A)^{-1}L[F(t)] \\ &= \lambda^{q-1}\int_0^\infty e^{-\lambda^q s}\mathcal{S}(s)y_0 ds + \int_0^\infty e^{-\lambda^q s}\mathcal{S}(s)L[F(s)]ds. \end{aligned} \tag{2.3.22}$$

Now, we estimate

$\int_0^\infty e^{-\lambda^q s}\mathcal{S}(s)L[F(s)]ds$

$$\begin{aligned} &= \int_0^\infty \int_0^\infty qt^{q-1}e^{-(\lambda t)^q}\mathcal{S}(t^q)e^{-\lambda s}F(s)dsdt \\ &= \int_0^\infty \int_0^\infty \int_0^\infty q\Phi_q(\zeta)e^{-(\lambda t\theta)}\mathcal{S}(t^q)e^{-\lambda s}t^{q-1}F(s)d\zeta dsdt \\ &= \int_0^\infty \int_0^\infty \int_0^\infty q\Phi_q(\zeta)e^{-\lambda(t+s)}\mathcal{S}(t^q/\zeta^q)\frac{t^{q-1}}{\zeta^q}F(s)d\zeta dsdt \\ &= \int_0^\infty e^{-\lambda t}\Big[q\int_0^t\int_0^\infty \Phi_q(\zeta)\mathcal{S}\Big(\frac{(t-s)^{q-1}}{\zeta^q}\Big)F(s)\frac{(t-s)^q}{\zeta^q}d\zeta ds\Big]dt. \end{aligned}$$

Thus, we get

$$\begin{aligned} L[y(t)] = &\int_0^\infty e^{-\lambda t}\big[\int_0^\infty \Phi_q(\zeta)\mathcal{S}(t^q/\zeta^q)y_0 d\zeta\big]dt \\ &+ \int_0^\infty e^{-\lambda t}\Big[q\int_0^t\int_0^\infty \Phi_q(\zeta)\mathcal{S}\Big(\frac{(t-s)^{q-1}}{\zeta^q}\Big)F(s)\frac{(t-s)^q}{\zeta^q}d\zeta ds\Big]dt. \end{aligned}$$

Taking inverse Laplace transform of the above equation, we get

$$\begin{aligned} y(t) &= \int_0^\infty \Phi_q(\zeta)\mathcal{S}(t^q/\zeta^q)y_0 d\zeta + \\ &+ q\int_0^t \int_0^\infty \Phi_q(\zeta)\mathcal{S}\Big(\frac{(t-s)^{q-1}}{\zeta^q}\Big)F(s)\frac{(t-s)^q}{\zeta^q}d\zeta ds \\ &= \int_0^\infty \Psi_q(\zeta)\mathcal{S}(t^q\zeta)y_0 d\zeta + q\int_0^t\int_0^\infty \zeta(t-s)^{q-1}\Psi_q(\zeta)\mathcal{S}((t-s)^{q-1}\zeta)F(s)d\zeta ds \\ &= \mathcal{P}(t)y_0 + \int_0^t (t-s)^{q-1}\mathcal{R}(t-s)F(s)ds, \end{aligned} \tag{2.3.23}$$

where, the operator $\mathcal{P}(t)$ and $\mathcal{R}(t)$ are defined by

$$\mathcal{P}(t) = \int_0^\infty \Psi_q(\zeta)\mathcal{S}(t^q\zeta)d\zeta, \quad \mathcal{R}(t) = q\int_0^\infty \zeta\Psi_q(\zeta)\mathcal{S}(t^q\zeta)d\zeta. \tag{2.3.24}$$

Definition 2.3.5. *A continuous function $y \in \mathcal{C}([0,T],\mathbb{X})$ is said to be the mild solution of the problem (2.3.20)-(2.3.21) if the following integral equation*

$$y(t) = \mathcal{P}(t)y_0 + \int_0^t (t-s)^{q-1}\mathcal{R}(t-s)F(s)ds, \tag{2.3.25}$$

holds.

Lemma 2.3.3. *([222]) The operators $\{\mathcal{R}(t), t \geqslant 0\}$ and $\{\mathcal{P}(t), t \geqslant 0\}$ are bounded linear such that*

(i) $\|\mathcal{R}(t)z\| \leq \frac{qM}{\Gamma(1+q)}\|z\|$, *and* $\|\mathcal{P}(t)z\| \leq M\|z\|$, *for any* $z \in \mathbb{X}$.

(ii) *The families $\{\mathcal{R}(t) : t \geqslant 0\}$ and $\{\mathcal{P}(t) : t \geqslant 0\}$ are strongly continuous.*

(iii) *The families $\{\mathcal{R}(t) : t \geqslant 0\}$ and $\{\mathcal{P}(t) : t \geqslant 0\}$ are compact, if $\mathcal{S}(t)$ is compact for any $t > 0$.*

2.4 Basic Concepts of Measure of Noncompactness

Let $(\mathbb{X}, d)$ be a complete metric space with metric d and $\mathcal{B}_{\mathbb{X}}$ denote the class of all bounded subsets of $\mathbb{X}$. Now, we have some notations which will be used in the subsequent chapters. If U is a subset of a metric space $(\mathbb{X}, d)$, then $\mathrm{diam}(U) = \sup\{d(y,x) : y,\ x \in U\}$ is called the diameter of U.

Definition 2.4.1. *[200] A function $\gamma : \mathcal{B}_{\mathbb{X}} \to [0, \infty)$ is said to be a* ***measure of noncompactness*** *on a complete space $\mathbb{X}$ if the following conditions are true:*

(i) $\gamma(G) = 0$ *if and only if $G \in \mathcal{B}_{\mathbb{X}}$ is precompact [Regularity];*

(ii) $\gamma(G) = \gamma(\overline{G})$, *where $\overline{G}$ denotes the closure of $G \in \mathcal{B}_{\mathbb{X}}$; [Invariance under closure]*

(iii) $\gamma(G_1 \cup G_2) = \max\{\gamma(G_1), \gamma(G_2)\}$, $\forall\ G_1, G_2 \in \mathcal{B}_{\mathbb{X}}$*[Semi-additivity].*

Proposition 2.4.1. *[200] For bounded sets $G, G_1, G_2 \in \mathcal{B}_{\mathbb{X}}$, a measure of noncompactness function γ fulfills the following conditions*

(i) $\gamma(G_1) \leq \gamma(G_2)$, *when $G_1 \subset G_2$, [Monotonicity];*

(ii) $\gamma(G_1 \cap G_2) \leq \min\{\gamma(G_1), \gamma(G_2)\}$;

(iii) $\gamma(G) = 0$ *for each finite set G, [Non-singularity];*

(iv) Let $\{G_n\}$ be a decreasing sequence of nonempty, closed sets in $\mathcal{B}_{\mathbb{X}}$ such that $\lim_{n\to\infty} \gamma(G_n) = 0$. Then $G = \cap_{n=1}^{\infty} G_n \neq \emptyset$ is compact (Cantor's generalized intersection property).

Proposition 2.4.2. *For sets $G, G_1, G_2 \in \mathcal{B}_{\mathbb{X}}$, we have*

(i) G is relatively compact if and only if $\gamma(G) = 0$;

(ii) $\gamma(G_1 + G_2) \leq \gamma(G_1) + \gamma(G_2)$*(Algebraic semi-additivity);*

(iii) $\gamma(G_1) \leq \gamma(G_2)$ *when $G_1 \subset G_2$;*

(iv) $\gamma(\alpha \cdot G) \leq |\alpha| \cdot \gamma(G)$, *$\alpha$ is a number(Semi-homogeneity);*

(v) $\gamma(G + z) = \gamma(G)$ *for each $z \in \mathbb{X}$(Translation invariance);*

(vi) $\gamma(G) = \gamma(\overline{G}) = \gamma(conv(G))$, *where $\overline{G}$ and $conv(G)$ denotes the closure and convex hull of G respectively.*

(vii) If a map $F : \mathcal{D}(F) \subset \mathbb{X} \to \mathbb{X}$ is Lipschitz continuous with Lipschitz constant L, then $\gamma(F(G)) \leq L\gamma(G)$ for every bounded subset $G \subset \mathcal{D}(F)$.

Definition 2.4.2. *A continuous and bounded map $F : D \subseteq \mathbb{X} \to \mathbb{X}$ is called γ-contraction if we can find a constant $0 < \kappa < 1$ in a way that*

$$\gamma(F(G)) \leq \kappa\gamma(G),$$

for any noncompact bounded subset $G \subset D$.

Lemma 2.4.3. *[23] For any $G \subset \mathcal{C}([a, b], \mathbb{X})$, set $G(t) = \{w(t) : w \in G\}$. If G is bounded in $\mathcal{C}([a, b], \mathbb{X})$, then $G(t)$ is bounded in $\mathbb{X}$ and $\gamma(G) = \sup_{t\in[a,b]} \gamma(G(t))$.*

Lemma 2.4.4. *[100] If $\{w_n\}_{n=1}^{\infty} \subset L^1([a,b],\mathbb{X})$ and there exists an $m \in L^1([a,b],\mathbb{X})$ such that $\|w_n(t)\| \leq m(t)$, a.e. $t \in [a,b]$, then $\gamma(\{w_n(t)\}_{n=1}^{\infty})$ is integrable and*

$$\gamma\Bigg(\bigg\{\int_0^t w_n(s)ds\bigg\}_{n=1}^{\infty}\Bigg) \leq 2\int_0^t \gamma(\{w_n(s)\}_{n=1}^{\infty} ds.$$

Lemma 2.4.5. *[31] If G is bounded subset of $\mathbb{X}$, then there exists $\{w_n\}_{n=1}^{\infty} \subset G$, such that $\gamma(G) \leq 2\gamma(\{w_n\}_{n=1}^{\infty})$.*

Lemma 2.4.6. *[23] Let $G \subset \mathcal{PC}([a,b],\mathbb{X})$ be bounded and piecewise equicontinuous, then $\gamma(G(t))$ is piecewise continuous for $t \in [a,b]$, and $\gamma(G) = \sup\{\mu(G(t)) : t \in [a,b]\}$, where $G(t) = \{y(t) : y \in G\}$.*

Definition 2.4.3. *[200]Let $(\mathbb{X}, d)$ be a metric space. The **Kuratowski measure of noncompactness** $\zeta(U)$ of a set $U \subset \mathbb{X}$ is the greatest lower bound of those $\kappa > 0$, for which U admits a finite subdivision into sets, whose diameters are less than κ i.e.*

$$\zeta(U) := \inf\{\kappa > 0 : U \subset \cup_{k=1}^{n} U_k,\ U_k \subset \mathbb{X},\ diam(U_k) < \kappa,\ k = 1, 2, \cdots, n \in \mathbb{N}\}.$$

Clearly, the set U is completely bounded if and only if $\zeta(U) = 0$.

Definition 2.4.4. *The **Hausdorff measure of noncompactness** $\mu(U)$ of a set U in the metric space $(\mathbb{X}, d)$ is the greatest lower bounded of those $\kappa > 0$ for which the set U has a finite κ-net in the space $\mathbb{X}$ i.e.*

$$\mu(U) := \inf\{\kappa > 0 : U \subset \cup_{k=1}^{n} B_{r_k}(x_k),\ x_k \in \mathbb{X}, r_k < \kappa,\ k = 1, \cdots, n \in \mathbb{N}\},$$

where $B_{r_k}(x_k) = \{x \in \mathbb{X} : d(x, x_k) < r_k\}$ denotes the open ball of radius r_k with center at $x_k \in \mathbb{X}$.

Remark 2.4.7. *The measure of noncompactness functions μ and ζ satisfy the following inequality*

$$\mu(U) \leq \zeta(U) \leq 2\mu(U), \quad \textit{for all}\ U \in \mathcal{B}_{\mathbb{X}}.$$

2.5 Basic Concepts of Stochastic Analysis

We first recall some general terms of probability theory.

Definition 2.5.1. *If Ω is a given set. Then, a σ-algebra $\mathcal{F}$ on Ω is a family of subsets of Ω with the following properties:*
(i) $\emptyset \in \mathcal{F}$;
(ii) $F \in \mathcal{F} \Rightarrow F^C \in \mathcal{F}$, where $F^C = \Omega - F$ is the complement of F in Ω;
(iii) $F_1, F_2, \cdots \in \mathcal{F} \Rightarrow F := \cup_{j=1}^{\infty} F_j \in \mathcal{F}$.
The pair $(\Omega, \mathcal{F})$ is said to be a measurable space.

If $\mathcal{F}_1$ and $\mathcal{F}_2$ are two σ-algebras of subsets of Ω, by $\mathcal{F}_1 \vee \mathcal{F}_2$ we denote the smallest σ-algebra of subsets of Ω which contains the σ-algebras $\mathcal{F}_1$ and $\mathcal{F}_2$.

By $\mathcal{B}(\mathbb{R}^n)$, we denote the σ-algebra of Borel subsets of $\mathbb{R}^n$, i.e. the smallest σ-algebra containing all open subsets of $\mathbb{R}^n$.

For a family $\mathcal{C}$ of subsets of Ω, $\sigma(\mathcal{C})$ denote the smallest σ-algebra of subsets of Ω containing $\mathcal{C}$, $\sigma(\mathcal{C})$ is termed as the σ-algebra generated by $\mathcal{C}$.

Definition 2.5.2. *(a) Let $(\Omega, \mathcal{F})$ be a measurable space. A function $\zeta : \mathcal{F} \to [0, \infty)$ is called a measure if:*
(i) $\zeta(\emptyset) = 0$;
(ii) if $A_n \in \mathcal{F}$, $n \geq 1$ and $A_i \cap A_j = \emptyset$ for $i \neq j$, then

$$\zeta(\cup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \zeta(A_n).$$

(b) A triplet $(\Omega, \mathcal{F}, \zeta)$ is said to be a space with measure ζ.

A measure ζ is said to be σ-finite if there exists a sequence A_n, $n \geq 1$, $A_n \in \mathcal{F}$ with $A_i \cap A_j = \emptyset$ for $i \neq j$ and $\Omega = \cup_{n=1}^{\infty} A_n$ and $\zeta(A_n) < \infty$ for every n.

Definition 2.5.3. *Let $(\Omega, \mathcal{F})$ be a measurable space. A function $f : \Omega \to \mathbb{R}$ is said to be a measurable function if for every $N \in \mathcal{B}(\Omega)$, we have $f^{-1}(N) \in \mathcal{F}$, where*

$$f^{-1}(N) = \{\omega \in \Omega : \; f(\omega) \in N\}.$$

We shall write a.a., a.e. and a.s. for almost all, almost everywhere and almost surely respectively; $f = g$ a.e. means $\zeta(f \neq g) = 0$.

If $\zeta(\Omega) = 1$ we say that ζ is a probability measure on $\mathcal{F}$. In this case the triplet $(\Omega, \mathcal{F}, \zeta)$ is termed a probability measure space.

Definition 2.5.4. *A probability measure $\boldsymbol{P}$ on a measurable space $(\Omega, \mathcal{F})$ is a function $\boldsymbol{P} : \mathcal{F} \to [0, 1]$ such that*
(1) $\boldsymbol{P}(\emptyset) = 0$, $\boldsymbol{P}(\Omega) = 1$.
(2) If $F_1, F_2, \ldots \in \mathcal{F}$ are disjoint, then

$$\boldsymbol{P}(\bigcup_{j=1}^{\infty} F_j) = \sum_{j=1}^{\infty} \boldsymbol{P}(F_j).$$

(3) If $F_1, F_2, \ldots, F_n, \ldots \in \mathcal{F}$, then $\boldsymbol{P}(\bigcup_{j=1}^{\infty} F_j) \leq \sum_{j=1}^{\infty} \boldsymbol{P}(F_j)$.

Definition 2.5.5. *The* $(\Omega, \mathcal{F}, \boldsymbol{P})$ *is said to be a probability space.*

Probability space is called a complete probability space if $\mathcal{F}$ contains all the subsets B of Ω with $\mathbf{P}$-outer measure zero, i.e.

$$\mathbf{P}^*(B) = \inf\{\mathbf{P}(F); F \in \mathcal{F}, B \subset F\} = 0,$$

where $\mathbf{P}^*$ represents the outer measure of B.

For any family $\mathcal{U}$ of subsets of Ω, there is a smallest σ-algebra $H_{\mathcal{U}}$ containing $\mathcal{U}$

$$H_{\mathcal{U}} = \bigcap\{H : \ H \text{ is a } \sigma\text{-algebra of } \Omega, \ \ \mathcal{U} \subset H\}.$$

We say that $H_{\mathcal{U}}$ is the σ-algebra generated by $\mathcal{U}$.

Definition 2.5.6. *Let* $(\Omega, \mathcal{F}, \boldsymbol{P})$ *be a probability space. Then, a function* $f : \Omega \to \mathbb{R}^n$ *is called* $\mathcal{F}$*-measurable if*

$$f^{-1}(U) := \{\omega \in \Omega : \ f(\omega) \in U\} \in \mathcal{F},$$

for all open sets $U \in \mathbb{R}^n$*.*

Definition 2.5.7. *Let* $(\Omega, \mathcal{F}, \boldsymbol{P})$ *be a probability space. A mapping* $\boldsymbol{X} : \Omega \to \mathbb{R}^n$ *is said to be an n-dimensional random variable if for each* $F \in \mathcal{F}$*, we have*

$$\boldsymbol{X}^{-1}(F) \in \mathcal{F}.$$

The random variable $\boldsymbol{X}$ *is also* $\mathcal{F}$*-measurable.*

Definition 2.5.8. *If* $\int_{\Omega} |\boldsymbol{X}| d\boldsymbol{P} < \infty$*, then the number*

$$E[\boldsymbol{X}] = \int_{\Omega} |\boldsymbol{X}| d\boldsymbol{P},$$

is called the expectation of $\boldsymbol{X}$ *(with respect to* $\boldsymbol{P}$*).*

Now, we define random variables depending upon time.

Definition 2.5.9. *A stochastic process is a parameterized collection of random variables* $\{\boldsymbol{X}(t)| \ t \geq 0\}$ *defined on a probability space* $(\Omega, \mathcal{F}, \boldsymbol{P})$ *and assuming values in* $\mathbb{R}^n$*.*

Note that for each fix $t \geq 0$, we have a random variable

$$\omega \to \mathbf{X}(t, \omega); \quad \omega \in \Omega.$$

while, by fixing $\omega \in \Omega$, we can consider

$$t \to \mathbf{X}(t, \omega); \quad t \geq 0,$$

which is called a path of $\mathbf{X}(t)$.

Usually we denote a stochastic process by $\{\mathbf{X}(t),\ t \in J \subset \mathbb{R}\}$, $X = \{\mathbf{X}(t)\}_{t\in J}$ or $\mathbf{X}(t),\ t \in J$, the dependence upon the second argument may be omitted.

Definition 2.5.10. *Let $(\Omega, \mathcal{F}, \boldsymbol{P})$ be a probability space. An event $E \in \mathcal{F}$ happens almost surely if the probability of E not occurring is zero i.e. $\boldsymbol{P}(E^c) = 0$.*

Let $J \subset \mathbb{R}$ be an interval. Now, we state following result which is used to study the stochastic process.

Definition 2.5.11. *(i) The process $X = \boldsymbol{X}(t),\ t \in J$ is continuous if for a.a. ω, the functions $\boldsymbol{X}(\cdot, \omega)$ are continuous on J.*
(ii) The process X is called to be right continuous if for a.a. ω, the functions $\boldsymbol{X}(\cdot, \omega)$ are right continuous on J.
(iii) The process $X = \{\boldsymbol{X}(t) : t \in J\}$ is continuous in probability if $t_n \to t_0$ with $t_n, t_0 \in J$ implies $\boldsymbol{X}(t_n) \xrightarrow{\boldsymbol{P}} \boldsymbol{X}(t_0)$.
(iv) The process X is said to be a measurable process if it is measurable on the product space with respect to the σ-algebra $\mathcal{B}(J) \otimes \mathcal{F}$, $\mathcal{B}(J)$ is a σ-algebra of Borel sets in J.

Remark 2.5.1. *Every right continuous stochastic process is a measurable process.*

Definition 2.5.12. *Let $(\Omega, \mathcal{F}, \boldsymbol{P})$ be a probability space. A filtration $\{\mathcal{F}_t |\ t \in J\}$ is a weakly increasing collection of σ-algebras on Ω and bounded above by $\mathcal{F}$, i.e. for $s, t \in J$ with $s < t$,*

$$\mathcal{F}_s \subseteq \mathcal{F}_t \subseteq \mathcal{F}.$$

A stochastic process X is said to be adapted to the filtration if, for every $t \in J$, $\boldsymbol{X}(t)$ is $\mathcal{F}_t$-measurable.

Definition 2.5.13. *The filtration is said to be normal if*
(i) $\mathcal{F}_0$ contains all $B \in \mathcal{F}$ such that $\boldsymbol{P}(B) = 0$,
(ii) $\mathcal{F}_t = \mathcal{F}_{t^+}$, $t \in J$, where $\mathcal{F}_{t^+}$ denotes the intersection of all $\mathcal{F}_s$ for $s > t$.

Let us consider a family $\mathbf{F} = \{\mathcal{F}_t : t \in J\}$ of σ-algebras $\mathcal{F}_t \subset \mathcal{F}$ with the property that $t_1 < t_2$ gives $\mathcal{F}_{t_1} \subset \mathcal{F}_{t_2}$.

Definition 2.5.14. *A continuous stochastic process $w(t),\ t \geq 0$ is called a standard Brownian motion or a standard Wiener process if:*
(i) $w(0) = 0$,
(ii) $w(t)$ is an almost surely continuous stochastic process with independent increments,
(iii) $\boldsymbol{E}(w(t)) = 0,\ t \geq 0$, and $\boldsymbol{E}(|w(t) - w(s)|^2) = |t - s|$ for $t \geq s \geq 0$.

Definition 2.5.15. *An n-dimensional stochastic process $\boldsymbol{X}(t) = (\boldsymbol{X}^1(t), \cdots, \boldsymbol{X}^n(t))$, $t \geq 0$ is called an n-dimensional standard Wiener process if each process $w^i(t)$ is a standard Brownian motion and the σ-algebras $\sigma(w^i(t) :\ t \geq 0),\ 1 \leq i \leq n$ are independent.*

Definition 2.5.16. *Let $(\mathbb{X}, d)$ be a metric space, and let $E \subseteq \mathbb{R}$. A function $f : E \to \mathbb{X}$ is called a càdlàg function, if f is right-continuous function with left limit exists.*

Remark: All continuous functions are càdlàg functions.

Definition 2.5.17. *A linear bounded operator $\mathbb{T} : \mathbb{X} \to \mathbb{Y}$ is said to be a nuclear operator if there exist two sequences $\{a_j\} \subset \mathbb{Y}$, $\{\varphi_j\} \subset \mathbb{X}^*$ such that*

$$\sum_{j=1}^{\infty} \|a_j\| \cdot \|\varphi_j\| < +\infty$$

and $\mathbb{T}$ has the representation $\mathbb{T}x = \sum_{j=1}^{\infty} a_j \varphi_j(x),\ \ x \in \mathbb{X}$.

The space of all nuclear operators from $\mathbb{X}$ into $\mathbb{Y}$, with the norm

$$\|\mathbb{T}\|_1 \ = \ \inf\left\{ \sum_{j=1}^{\infty} \|a_j\| \cdot \|\varphi_j\| : \mathbb{T}x = \sum_{j=1}^{\infty} a_j \varphi_j(x) \right\}, \tag{2.5.1}$$

is a Banach space and denoted by $\mathcal{L}_1(\mathbb{X}, \mathbb{Y})$.

Let $\mathbb{H}$ be a separable Hilbert space and let $\{e_k\}$ be a complete orthonormal system in $\mathbb{H}$. If $\mathbb{T} \in \mathcal{L}_1(\mathbb{H}, \mathbb{H})$, then we define trace of $\mathbb{T}$,

$$\mathrm{Tr}(\mathbb{T}) = \sum_{j=1}^{\infty} < \mathbb{T}e_j, e_j > .$$

Definition 2.5.18. *Let $\mathbb{X}$ be a separable Hilbert spaces with complete orthonormal basis $\{e_k\} \subset \mathbb{H}$. A linear bounded operator $\mathbb{T} : \mathbb{H} \to \mathbb{H}$ is said to be Hilbert-Schmidt operator if $\sum_{k=1}^{\infty} \|\mathbb{T}e_k\|^2 < \infty$.*

2.6 Some Fixed Point Theorems

In this section, we have listed some fixed point theorems, which play an important role in proving the results in the subsequent chapters.

Theorem 2.6.1. *[32, Banach contraction principle] Let $\Phi : \mathbb{X} \to \mathbb{X}$ be a contraction mapping on a complete space $\mathbb{X}$. Then, Φ has a unique fixed point in $\mathbb{X}$.*

Theorem 2.6.2. *[32, Generalized Banach contraction principle] Let $\Phi : \mathbb{X} \to \mathbb{X}$ be a mapping on a Banach space $\mathbb{X}$ such that Φ^{n_0} is contraction for some large enough natural number n_0. Then, Φ has a unique fixed point.*

Theorem 2.6.3. *[153, Mönch fixed point theorem] Let $\mathbb{D}$ be a closed and convex subset of $\mathbb{X}$ and $0 \in \mathbb{D}$. Then a continuous mappings $\Phi : \mathbb{D} \to \mathbb{X}$ satisfying Mönch's condition (i.e. $\mathbb{B}_1 \subseteq \mathbb{D}$ is countable and $\mathbb{B}_1 \subseteq \overline{conv}(\{0\} \cup F(\mathbb{B}_1)) \Rightarrow \overline{\mathbb{B}}_1$ is compact) has a fixed point in $\mathbb{D}$.*

Theorem 2.6.4. *[65, Condensing theorem] Let $\mathbb{D}$ be a closed, bounded and convex subset of Banach space $\mathbb{X}$ and let $\Phi : \mathbb{D} \to \mathbb{D}$ be a condensing map. Then Φ admits a fixed point in $\mathbb{D}$.*

Chapter 3

A Study of Sobolev Type Fractional Impulsive Differential System via Monotone Iterative Technique

3.1 Introduction

Balachandran et al. [17] investigated some existence results using Krasnoselskii fixed point theorem for the following Sobolev type impulsive fractional integro-differential system

$$\begin{cases} {}^cD^{\beta}_{0^+}[By(t)] + Ay(t) = f(t, y(t)) + \int_0^t h(t, s, y(s))ds, \quad t \in (0, b],\ t \neq t_j; \\ \Delta y|_{t=t_j} = I_j(y(t_j)), \quad j = 1, 2, \dots m;\ m \in \mathbb{N}, \\ y(0) = y_0, \end{cases} \tag{3.1.1}$$

where $0 < \beta < 1$, B and A are linear operator with domain contained in $\mathbb{X}$. The nonlinear functions f, h and I_j, $j = 1, 2, 3, ..., m$ are given and satisfy some suitable conditions.

Kerboua et al. [120] established the approximate controllability results by

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generalized Banach contraction principle for the following Sobolev type fractional stochastic differential system

$$\begin{cases} {}^{c}D_{0^{+}}^{\beta}[Ly(t)] = My(t) + Bu(t) + f(t, y(t)) + g(t, y(t))\dfrac{d\omega_1(t)}{dt}, \quad t \in (0, b]; \\ {}^{R}D_{0^{+}}^{1-\beta}y(t)|_{t=0} = h(t, y(t))\frac{d\omega_1(t)}{dt}, \end{cases} \tag{3.1.2}$$

where $0 < \beta < 1$, L and M are linear operator with domain contained in Hilbert space $\mathbb{H}$. In the system (3.1.2), u is a control functional and B is a bounded linear operator. The nonlinear functions f, g and h are given and satisfy some suitable conditions. In [66], Dhage and Imdad discussed the asymptotic behaviour of non-linear quadratic functional integral equations using Carathéodory conditions.

Motivated by the systems (3.1.1) and (3.1.2), in this chapter, we will study the existence and uniqueness results for a abstract Sobolev type fractional impulsive differential system by applying monotone iterative technique using some basic tools of measure of noncompactness.

3.2 Problem Formulation

Let $(\mathbb{X}, \|.\|)$ be a Banach space with zero element θ. A cone $\mathcal{P}$ defines a partial ordering in $\mathbb{X}$ by $u \leq v$ if and only if $v - u \in \mathcal{P}$. We symbolize $u < v$ to indicate $u \leq v$ but $u \neq v$. We call a cone $\mathcal{P}$ as a normal cone if there exists a constant $N > 0$(called normal constant) such that $\theta \leq x \leq y$ implies $\|x\| \leq N\|y\|$. A cone $\mathcal{P} \subset \mathbb{X}$ is said to be regular cone if every increasing, bounded above sequence is convergent i.e. if $\{w_n\}$ be a sequence such that

$$w_1 \leq w_2 \leq \cdots \leq w_n \leq \cdots \leq z,$$

for some $z \in \mathbb{X}$, where $\theta \leq w_1$. Then there exists a $w \in \mathbb{X}$ such that $\|w_n - w\| \to 0$ as $n \to \infty$. Equivalently, a cone $\mathcal{P} \subset \mathbb{X}$ is said to be regular if every bounded below and decreasing sequence is convergent. It should be noted that every regular cone is a normal cone. For more details regarding to the cone $\mathcal{P}$, one may see [62; 179].

In this chapter, we study the following Sobolev type fractional impulsive differential system with fractional order nonlocal conditions

$$\begin{cases} {}^cD^{\beta}_{0+}[By(t)] = Ay(t) + f\Big(t, y(t), \int_0^t K(t,s,y(s))ds\Big), & t \in (0,b],\, t \neq t_j; \\ \Delta y|_{t=t_j} = I_j(y(t_j)), \quad j = 1,2,\dots m,\, m \in \mathbb{N}, \\ {}^RD^{1-\beta}_{0+}[Ty(t)]|_{t=0} = u_0 + g(y(t)), \end{cases} \tag{3.2.1}$$

where ${}^cD^q_{0+}$ and ${}^RD^q_{0+}$ denote Caputo and Riemann-Liouville fractional order derivatives of order $q \geq 0$, respectively. The nonlinear functions $f : [0,b] \times \mathbb{X} \times \mathbb{X} \to \mathbb{X}$, $K : \Delta \times \mathbb{X} \to \mathbb{X}$, $g : \mathcal{C}([0,b], \mathbb{X}) \to \mathbb{X}$ and $I_j : \mathbb{X} \to \mathbb{X}, j = 1,2,3,..,m$ are given which satisfy some appropriate assumptions. Here $\Delta := \{(t,s) \in \mathbb{R}^2 : 0 \leq s \leq t \leq b\}$. $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = b$ are prefixed points and $u_0 \in \mathbb{X}$. The jumps of $y(t)$ at $t = t_j$ are characterized by $\Delta y|_{t=t_j}$ such that $\Delta y|_{t=t_j} = y(t_j^+) - y(t_j^-)$. We symbolize $[0,b]' := [0,b] - \{t_1, t_2, \dots, t_m\}$.

The operators $B : \mathcal{D}(B) \subset \mathbb{X} \to \mathbb{X}$, $T : \mathcal{D}(T) \subset \mathbb{X} \to \mathbb{X}$ are linear and the operator $A : \mathcal{D}(A) \subset \mathbb{X} \to \mathbb{X}$ is a closed and linear operator such that the following conditions hold:

$(H1)$ $\mathcal{D}(T) \subset \mathcal{D}(B) \subset \mathcal{D}(A)$ and B, T are bijective operators.

$(H2)$ $B^{-1} : \mathbb{X} \to \mathcal{D}(B) \subset \mathbb{X}$ and $T^{-1} : \mathbb{X} \to \mathcal{D}(T) \subset \mathbb{X}$ are bounded and linear operators.

From $(H2)$, we have that B^{-1} is closed and injective therefore its inverse i.e. B is closed. Using closed graph theorem with hypotheses $(H1) - (H2)$, we obtain that the linear operator $AB^{-1} : \mathbb{X} \to \mathbb{X}$ is bounded. Consequently, AB^{-1} generates a semigroup $\{\mathcal{S}(t)\}(t \geq 0)$. Throughout this chapter, we assume that the semigroup $\{\mathcal{S}(t)\}(t \geq 0)$ generated by AB^{-1} is a strongly continuous semigroup of bounded linear operators on $\mathbb{X}$ and there exists a constant $M \geq 1$ such that $\|\mathcal{S}(t)\| \leq M$.

For sake of convenience, we denote $\|B\| = M_1$, $\|B^{-1}\| = M_2$ and $\|T^{-1}\| = M_3$.

Using the definition of Caputo fractional derivative, the problem (3.2.1) can be written in the following integral equation

$$By(t) = By(0) + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} \Big[Ay(s) + f\Big(s, y(s), \int_0^s K(s,\xi,y(\xi))d\xi\Big)\Big] ds \tag{3.2.2}$$

provided that the integral in (3.2.2) exists for $t \in [0, t_1]$. Moreover, from equation (3.2.1), we conclude:

(a) Nonlocal type condition given in (3.2.1) is well defined, i.e., it will reduce to the usual nonlocal condition if $\beta = 1$ and T is a identity operator.

(b) The value of $y(0)$ is given by

$$y(0) = T^{-1}\tilde{u}_0 + \frac{1}{\Gamma(1-\beta)} \int_0^t \frac{T^{-1}[u_0 + g(y(s))]}{(t-s)^{\beta}} ds, \tag{3.2.3}$$

where $Ty(t)|_{t=0} = \tilde{u}_0$.

(c) The integral given in (3.2.2) exists and taken in Bochner sense.

Motivated by [84; 86], using the arguments used in [51; 223], we define a mild solution to the problem (3.2.1)as follows:

Definition 3.2.1. *A function $y \in \mathcal{PC}([0,b],\mathbb{X})$ is called a mild solution of the problem (3.2.1) if $\Delta y|_{t=t_j} = I_j(y(t_j))$, $j = 1, 2, \ldots m$, $y(0)$ is given by* (3.2.3) *and satisfies the following equation*

$$y(t) = \begin{cases} P_\beta(t)BT^{-1}\Big[\tilde{u}_0 + \frac{1}{\Gamma(1-\beta)}\int_0^t (t-s)^{-\beta}(u_0 + g(y(s)))ds\Big] \\ + \int_0^t (t-s)^{\beta-1} Q_\beta(t-s) f\Big(s, y(s), \int_0^s K(s,\xi,y(\xi))d\xi\Big)ds, \quad t \in [0, t_1]; \\ P_\beta(t)BT^{-1}\Big[\tilde{u}_0 + \frac{1}{\Gamma(1-\beta)}\int_0^t (t-s)^{-\beta}(u_0 + g(y(s)))ds\Big] \\ + \int_0^t (t-s)^{\beta-1} Q_\beta(t-s) f\Big(s, y(s), \int_0^s K(s,\xi,y(\xi))d\xi\Big)ds, \\ + P_\beta(t-t_1)I_1(y(t_1)), \quad t \in (t_1, t_2]; \\ \vdots \\ P_\beta(t)BT^{-1}\Big[\tilde{u}_0 + \frac{1}{\Gamma(1-\beta)}\int_0^t (t-s)^{-\beta}(u_0 + g(y(s)))ds\Big] \\ + \sum_{j=1}^m P_\beta(t-t_j)I_j(y(t_j)) \\ + \int_0^t (t-s)^{\beta-1} Q_\beta(t-s) f\Big(s, y(s), \int_0^s K(s,\xi,y(\xi))d\xi\Big)ds, \quad t \in (t_m, b]. \end{cases} \tag{3.2.4}$$

where

$$P_\beta(t) = \int_0^\infty B^{-1}\zeta_\beta(\theta)\mathcal{S}(t^\beta\theta)d\theta, \quad \textit{and} \quad Q_\beta(t) = \beta\int_0^\infty B^{-1}\theta\zeta_\beta(\theta)\mathcal{S}(t^\beta\theta)d\theta,$$

and $\zeta_\beta(\theta) := \frac{1}{\beta}\theta^{1-\frac{1}{\beta}} \times \psi_\beta(\theta^{-\frac{1}{\beta}})$ *is a probability density function defined on* $(0,\infty)$ *i.e.,* $\zeta_\beta(\theta) \geq 0$, $\int_0^\infty \zeta_\beta(\theta)d\theta = 1$ *and*

$$\psi_\beta(\theta) = \frac{1}{\pi}\sum_{n=1}^{\infty}(-1)^{n-1}\theta^{-n\beta-1}\frac{\Gamma(n\beta+1)}{n!}\sin(n\pi\beta), 0 < \theta < \infty.$$

For more details see [76].

Lemma 3.2.1. *The operators* $\{P_\beta(t), t \geq 0\}$ *and* $\{Q_\beta(t), t \geq 0\}$ *are bounded linear operators such that*

(i) $\|P_\beta(t)u\| \leq M\|u\|, \|Q_\beta(t)u\| \leq \frac{M}{\Gamma(\beta)}\|u\|$ *for any* $u \in \mathbb{X}$.

(ii) *The operators* $\{P_\beta(t) : t \geq 0\}$ *and* $\{Q_\beta(t) : t \geq 0\}$ *are strongly continuous.*

Let $\mathcal{PC}^\beta([0,b],\mathbb{X}) = \{y \in \mathcal{PC}([0,b],\mathbb{X}) : {}^cD^\beta y(t)$ exists and continuous on $[0,b]$ and $y(t) \in \mathcal{D}(A)$ for all $t \in [0,b]'\}$. An abstract function $y \in \mathcal{PC}^\beta([0,b],\mathbb{X})$ is said to be a solution of (3.2.1) if $y(t)$ satisfies the system (3.2.1).

Definition 3.2.2. *A function* $u^{(0)} \in \mathcal{PC}^\beta([0,b],\mathbb{X})$ *is called a lower solution of* (3.2.1) *if it satisfies the following inequality*

$$\begin{cases} {}^cD^\beta_{0+}[Bu^{(0)}(t)] \leq Au^{(0)}(t) + f(t, u^{(0)}(t), \int_0^t K(t,s,u^{(0)}(s))ds), & t \in [0,b]\, t \neq t_j; \\ \Delta u^{(0)}|_{t=t_j} \leq I_j(u^{(0)}(t_j)), \quad j = 1,2,\ldots m; \\ {}^RD^{1-\beta}_{0+}[Tu^{(0)}(0)] \leq u_0^{(0)} + g(u^{(0)}(t)). \end{cases} \tag{3.2.5}$$

If all the inequalities in above system are reversed, then the function $u^{(0)}$ *is called upper solution of* (3.2.1) *and denoted by* $v^{(0)}$.

Definition 3.2.3. *The semigroup* $\{\mathcal{S}(t)\}(t \geq 0)$ *in* $\mathbb{X}$ *is called a positive semigroup, if for all* $u > \theta$ *and* $t \geq 0$ *the inequality* $\mathcal{S}(t)u > \theta$ *holds.*

3.3 Existence and Uniqueness Results

In order to obtain the main results for the system (3.2.1), we consider the following assumptions:

(A1) The function $f : [0,b]\times\mathbb{X}\times\mathbb{X} \to \mathbb{X}$ and $K : \Delta\times\mathbb{X} \to \mathbb{X}$ satisfies Carathéodory conditions i.e.

1. $K(t,s,\cdot) : \mathbb{X} \to \mathbb{X}$ is continuous a.e. for $(t,s) \in \Delta$ and for each $v \in \mathbb{X}$, the function $K(\cdot,\cdot,v) : \Delta \to \mathbb{X}$ is strongly measurable.

2. $f(\cdot,v,w) : [0,b] \to \mathbb{X}$ is strongly measurable and $f(t,\cdot,\cdot) : \mathbb{X} \times \mathbb{X} \to \mathbb{X}$ is continuous a.e. for $t \in [0,b]$ and for all $(v,w) \in \mathbb{X} \times \mathbb{X}$.

(A2) For any lower and upper solutions $u^{(0)}, v^{(0)} \in \mathcal{PC}^{\beta}([0,b],\mathbb{X})$ of the system (3.2.1) such that $u^{(0)} \leq v^{(0)}$, for all $t \in [0,b]$, we have

1. The function $K(t,s,\cdot) : \mathbb{X} \to \mathbb{X}$ satisfies $K(t,s,v_1) \leq K(t,s,v_2)$, for any $(t,s) \in \Delta$, $v_1, v_2 \in \mathbb{X}$ with $u^{(0)} \leq v_1 \leq v_2 \leq v^{(0)}$.

2. The function $f(t,\cdot,\cdot) : \mathbb{X} \times \mathbb{X} \to \mathbb{X}$ satisfies $f(t,v_1,w_1) \leq f(t,v_2,w_2)$, for all $v_1, v_2 \in \mathbb{X}$ with $u^{(0)} \leq v_1 \leq v_2 \leq v^{(0)}$ and $w_1, w_2 \in \mathbb{X}$ with $\int_0^t K(t,s,u^{(0)}(s))ds \leq w_1 \leq w_2 \leq \int_0^t K(t,s,v^{(0)}(s))ds$.

3. The impulsive function $I_j : \mathbb{X} \to \mathbb{X}$ satisfies $I_j(u_1) \leq I_j(u_2)$, for all $j = 1, 2, \ldots m$, where $u^{(0)} \leq u_1 \leq u_2 \leq v^{(0)}$.

(A3)

1. For each bounded set $V \subset \mathbb{X}$, there exists an integrable function $\omega : \Delta \to [0,\infty)$ such that

$$\zeta(\{K(t,s,V)\}) \leq \omega(t,s)\zeta(V),$$

a.e. for $(t,s) \in \Delta$. For simplification, put $w^* = \max \int_0^t \omega(t,s)ds$.

2. There exists a constant $L_f \geq 0$ such that

$$\zeta(f(t,V_1,V_2)) \leq L_f \Big[\zeta(V_1) + \zeta(V_2)\Big],$$

for a.e. $t \in [0,b]$ and $V_1, V_2 \subset \mathbb{X}$.

(A4) The function $g : \mathcal{C}([0,b],\mathbb{X}) \to \mathbb{X}$ is compact, continuous and increasing.

Theorem 3.3.1. *Let $\mathcal{P}$ be a normal cone with normal constant N in ordered Banach space $\mathbb{X}$. Allow that the system (3.2.1) has lower and upper solutions $u^{(0)}, v^{(0)} \in \mathcal{PC}^{\beta}([0,b],\mathbb{X})$ such that $u^{(0)} \leq v^{(0)}$, and $\mathcal{S}(t)(t \geq 0)$ is positive semigroup and the assumptions (A1)-(A4) are fulfilled. Then, minimal and maximal mild solutions exist for the system (3.2.1) in between $u^{(0)}$ and $v^{(0)}$.*

Proof. Let $G = [u^{(0)}, v^{(0)}] = \{u \in \mathcal{PC}([0,b],\mathbb{X}) : u^{(0)} \leq u \leq v^{(0)}\}$. Define a map $\Phi : G \to \mathcal{PC}([0,b],\mathbb{X})$ by

$$\Phi y(t) = \begin{cases} P_\beta(t)BT^{-1}\Big[\tilde{u}_0 + \frac{1}{\Gamma(1-\beta)}\int_0^t (t-s)^{-\beta}(u_0 + g(y(s)))ds\Big] \\ + \int_0^t (t-s)^{\beta-1}Q_\beta(t-s)f\Big(s, y(s), \int_0^s K(s,\xi,y(\xi))d\xi\Big)ds,\ t \in [0,t_1]; \\ P_\beta(t)BT^{-1}\Big[\tilde{u}_0 + \frac{1}{\Gamma(1-\beta)}\int_0^t (t-s)^{-\beta}(u_0 + g(y(s)))ds\Big] \\ + \int_0^t (t-s)^{\beta-1}Q_\beta(t-s)f\Big(s, y(s), \int_0^s K(s,\xi,y(\xi))d\xi\Big)ds, \\ +P_\beta(t-t_1)I_1(y(t_1)) \quad t \in (t_1,t_2]; \\ \cdots, \\ P_\beta(t)BT^{-1}\Big[\tilde{u}_0 + \frac{1}{\Gamma(1-\beta)}\int_0^t (t-s)^{-\beta}(u_0 + g(y(s)))ds\Big] \\ + \sum_{j=1}^m P_\beta(t-t_j)I_j(y(t_j)) \\ + \int_0^t (t-s)^{\beta-1}Q_\beta(t-s)f\Big(s, y(s), \int_0^s K(s,\xi,y(\xi))d\xi\Big)ds,\ t \in (t_m,b]. \end{cases} \tag{3.3.1}$$

Using Lebesgue dominated convergence theorem accompanying with the assumptions $(A1)$, $(A4)$, we can easily see that the map $\Phi : G \to \mathcal{PC}([0,b],\mathbb{X})$ is continuous. Using $(A2)$, for any $u \in G$, we have

$$\begin{aligned} f\Big(t, u^{(0)}(t), \int_0^t K(t,\xi,u^{(0)}(\xi))d\xi\Big) &\leq f\Big(t, y(t), \int_0^t K(t,\xi,y(\xi))d\xi\Big) \\ &\leq f\Big(t, v^{(0)}(t), \int_0^t K(t,\xi,v^{(0)}(\xi))d\xi\Big). \end{aligned}$$

In view of normality property of the positive cone N, there exists a constant $C > 0$ such that

$$\|f\Big(t, y(t), \int_0^t K(t,\xi,y(\xi))d\xi\Big)\| \leq C, \quad y \in G.$$

For convenience, we divide the proof in the following steps.
Step 1: We show that Φ is equicontinuous.

For any $y \in G$ and l_1, $l_2 \in (t_p, t_{p+1}]$ for any $1 \le p \le m$ such that $l_1 < l_2$, we have

$$
\begin{aligned}
&\|\Phi y(l_2) - \Phi y(l_1)\| \\
\le& \|P_\beta(l_2)BT^{-1}\tilde{u}_0 - P_\beta(l_1)BT^{-1}\tilde{u}_0\| + \frac{\|P_\beta(l_2)BT^{-1}\|}{\Gamma(1-\beta)} \\
&\times \left\| \int_0^{l_2} (l_2 - s)^{-\beta}(u_0 + g(y(s)))ds - \int_0^{l_1} (l_1 - s)^{-\beta}(u_0 + g(y(s)))ds \right\| \\
&+ \frac{\|P_\beta(l_2) - P_\beta(l_1)\|\|BT^{-1}\|}{\Gamma(1-\beta)} \left\| \int_0^{l_1} (l_1 - s)^{-\beta}(u_0 + g(y(s)))ds \right\| \\
&+ \sum_{j=1}^{p} \|P_\beta(l_1 - t_j) - P_\beta(l_1 - t_j)\|\|I_j(y(t_j))\| \\
&+ \int_0^{l_1} (l_2 - s)^{\beta-1}\|Q_\beta(l_2 - s) - Q_\beta(l_1 - s)\|\|f\left(s, y(s), \int_0^s K(s,\xi,y(\xi))d\xi\right)\|ds \\
&+ \int_0^{l_1} |(l_2 - s)^{\beta-1} - (l_1 - s)^{\beta-1}|\|Q_\beta(l_1 - s)\|\|f\left(s, y(s), \int_0^s K(s,\xi,y(\xi))d\xi\right)\|ds \\
&+ \int_{l_1}^{l_2} (l_2 - s)^{\beta-1}\|Q_\beta(l_2 - s)\|\|f\left(s, y(s), \int_0^s K(s,\xi,y(\xi))d\xi\right)\|ds.
\end{aligned}
$$

From the Lemma 3.2.1, we can easily deduce that $\|\Phi y(l_2) - \Phi y(l_1)\| \to 0$ as $l_2 \to l_1$ independently of $y \in G$. Therefore $\Phi(G)$ is equicontinuous on $[0, b]$.
Step 2: We show that Φ is an increasing monotonic operator.
For time $t \in [0, b]'$, let u and v be two elements of G such that $u^{(0)} \le u \le v \le v^{(0)}$. From the assumptions $(A2)$ and $(A4)$ and the positive property of the operators $P_\beta(t)$ and $Q_\beta(t)$ for $t \ge 0$, it follows that

$$\Phi u \le \Phi v. \tag{3.3.2}$$

To show $u^{(0)}(t) \le \Phi u^{(0)}(t)$ and $\Phi v^{(0)}(t) \le v^{(0)}(t)$ for all $t \in [0, b]'$, let $h(t) = {}^cD^\beta[Bu^{(0)}(t)] - Au^{(0)}(t)$. Then, we have $h(t) \le f(t, u^{(0)}(t), \int_0^t K(t,\xi,u^{(0)}(\xi))d\xi)$ and $h \in \mathcal{PC}([0, b], \mathbb{X})$. Thus for any $t \in [0, t_1]$, by Definition (3.2.4), (3.2.2) and positivity of the operators $P_\beta(t)$ and $Q_\beta(t)$, we get

$$
\begin{aligned}
u^{(0)}(t) =& P_\beta(t)BT^{-1}u^{(0)}(t)|_{t=0} + \int_0^t (t-s)^{\beta-1}Q_\beta(t-s)h(s)ds \\
\le& P_\beta(t)BT^{-1}\left[\tilde{u}_0^{(0)} + \frac{1}{\Gamma(1-\beta)}\int_0^t (t-s)^{-\beta}(u_0^{(0)} + g(u^{(0)}(s)))ds\right] \\
&+ \int_0^t (t-s)^{\beta-1}Q_\beta(t-s)f\left(s, u^{(0)}(s), \int_0^s K(s,\xi,u^{(0)}(\xi))d\xi\right)ds \\
\le& \Phi u^{(0)}(t).
\end{aligned}
$$

For any $t \in (t_1, t_2]$, by Definition (3.2.4), (3.2.2) and the positive property of the operators $P_\beta(t)$ and $Q_\beta(t)$, we get

$$\begin{aligned}
u^{(0)}(t) =&P_\beta(t)BT^{-1}u^{(0)}(t)|_{t=0} + P_\beta(t-t_1)\Delta u^{(0)}|_{t=t_1} + \int_0^t (t-s)^{\beta-1}Q_\beta(t-s)h(s)ds\\
\leq &P_\beta(t)BT^{-1}\left[\tilde{u}_0^{(0)} + \frac{1}{\Gamma(1-\beta)}\int_0^t (t-s)^{-\beta}(u_0^{(0)} + g(u^{(0)}(s)))ds\right]\\
&+ \int_0^t (t-s)^{\beta-1}Q_\beta(t-s)f\left(s, u^{(0)}(s), \int_0^s K(s,\xi,u^{(0)}(\xi))d\xi\right)ds\\
&+ P_\beta(t-t_1)I_1(u^{(0)}(t_1)) \leq \Phi u^{(0)}(t).
\end{aligned}$$

Similarly, we can show that $u^{(0)}(t) \leq \Phi u^{(0)}(t)$ for any $t \in (t_j, t_{j+1}]$, $j = 2, 3, \ldots, m$. Thus we have $u^{(0)}(t) \leq \Phi u^{(0)}(t)$ for all $t \in [0, b]'$. Using the same argument as above we can show that $\Phi v^{(0)}(t) \leq v^{(0)}(t)$ for all $t \in [0, b]'$. Hence Φ is an increasing monotonic operator.
Now, we define two sequences $\{u^{(n)}\}$ and $\{v^{(n)}\}$ in G by the iterative scheme

$$u^{(n)} = \Phi u^{(n-1)} \quad \text{and} \quad v^{(n)} = \Phi v^{(n-1)}, \quad n = 1, 2, \ldots \tag{3.3.3}$$

and from (3.3.2), we have

$$u^{(0)} \leq u^{(1)} \leq \cdots \leq v^{(n)} \leq v^{(n-1)} \leq \cdots \leq v^{(0)}. \tag{3.3.4}$$

Step 3: Now, we demonstrate that the sequences $\{u^{(n)}\}$ and $\{v^{(n)}\}$ are uniformly convergent on $[0, b]$. Let $S = \{u^{(n)} : n \in \mathbb{N}\}$ and $S_0 = \{u^{(n-1)} : n \in \mathbb{N}\}$. Then $S_0 = S \cup \{u^{(0)}\}$ which follows that $\zeta(S(t)) = \zeta(S_0(t))$ for $t \in [0, b]$. Let

$$\psi(t) := \zeta(S(t)) = \zeta(S_0(t)).$$

Since $S = \Phi(S_0)$, we have

$$\zeta(S(t)) = \zeta(\Phi(S_0(t))).$$

For $t \in [0, t_1]$, using assumptions (A3), (A4) and (3.3.1), we get

$$\begin{aligned}
\psi(t) =&\zeta\Bigg(P_\beta(t)BT^{-1}\left[\tilde{u}_0^{(n-1)} + \frac{1}{\Gamma(1-\beta)}\int_0^t (t-s)^{-\beta}[u_0^{(n-1)} + g(u^{(n-1)}(s))]ds\right]\\
&+ \int_0^t (t-s)^{\beta-1}Q_\beta(t-s)f\left(s, u^{(n-1)}(s), \int_0^s K(s,\xi,u^{(n-1)}(\xi))d\xi\right)ds\Bigg)
\end{aligned}$$

$$
\begin{aligned}
\leq & \zeta\Big(P_\beta(t)BT^{-1}\tilde{u}_0^{(n-1)}\Big)+\zeta\Big(\frac{P_\beta(t)BT^{-1}}{\Gamma(1-\beta)}\int_0^t(t-s)^{-\beta}[u_0^{(n-1)}+g(u^{(n-1)}(s))]ds\Big)\\
&+\zeta\Big(\int_0^t(t-s)^{\beta-1}Q_\beta(t-s)f\Big(s,u^{(n-1)}(s),\int_0^s K(s,\xi,u^{(n-1)}(\xi))d\xi\Big)ds\Big)\\
\leq & \frac{2M}{\Gamma\beta}\int_0^t(t-s)^{\beta-1}\zeta\Big(f\Big(s,u^{(n-1)}(s),\int_0^s K(s,\xi,u^{(n-1)}(\xi))d\xi\Big)\Big)ds\\
\leq & \frac{2M}{\Gamma\beta}\int_0^t(t-s)^{\beta-1}\Big(\zeta(u^{(n-1)}(s))+\zeta\Big(\int_0^s K(s,\xi,u^{(n-1)}(\xi))d\xi\Big)\Big)ds\\
\leq & \frac{2M}{\Gamma\beta}\int_0^t(t-s)^{\beta-1}\Big(\zeta(u^{(n-1)}(s))+2\int_0^s \zeta(s,\xi)\zeta(\{u^{(n-1)}(\xi)\})d\xi\Big)ds\\
\leq & \frac{2M}{\Gamma\beta}(1+2w^*)\int_0^t(t-s)^{\beta-1}\zeta(u^{(n-1)}(s))ds\\
\leq & \frac{2M}{\Gamma\beta}(1+2w^*)\int_0^t(t-s)^{\beta-1}\psi(s)ds.
\end{aligned}
$$

Using Lemma 2.1.1, we get $\psi(t)=0$ for $t\in[0,t_1]$ i.e. $\{u^{(n)}(t):n\in\mathbb{N}\}$ is precompact in $\mathbb{X}$ for $t\in[0,t_1]$. Precisely, $\psi(t_1)=\zeta(S(t_1))=\zeta(S_0(t_1))=0$, which shows that $S(t_1)$ and $S_0(t_1)$ are precompact in $\mathbb{X}$. Thus $I_1(S_0(t_1))$ is precompact in $\mathbb{X}$ i.e. $\zeta(I_1(S_0(t_1)))=0$.
For $t\in(t_1,t_2]$, using the same argument as above, we get

$$
\begin{aligned}
\psi(t)=&\zeta\Big(P_\beta(t)BT^{-1}\Big[\tilde{u}_0^{(n-1)}+\frac{1}{\Gamma(1-\beta)}\int_0^t(t-s)^{-\beta}[u_0^{(n-1)}+g(u^{(n-1)}(s))]ds\Big]\\
&+P_\beta(t-t_1)I_1(u^{(n-1)}(t_1))\\
&+\int_0^t(t-s)^{\beta-1}Q_\beta(t-s)f(s,u^{(n-1)}(s),\int_0^s K(s,\xi,u^{(n-1)}(\xi))d\xi)ds\Big)\\
\leq&\frac{2M}{\Gamma\beta}(1+2w^*)\int_0^t(t-s)^{\beta-1}\psi(s)ds.
\end{aligned}
$$

Again using Lemma (2.1.1), we get $\psi(t)=0$ for $t\in(t_1,t_2]$, from which we obtain that $\psi(t_2)=\zeta(S(t_2))=\zeta(S_0(t_2))=0$, which shows that $S(t_1)$ and $S_0(t_1)$ are precompact in $\mathbb{X}$. Thus $I_2(S_0(t_2))$ is precompact in $\mathbb{X}$ i.e. $\zeta(I_2(S_0(t_2)))=0$.

Repeating this process interval by interval up to $(t_m,b]$, we get $\psi(t)=0$ on every interval $t\in(t_{j-1},t_j]$, $j=1,2,\dots,m+1$. Thus $\{u^{(n)}(t)\}$ is precompact for every $t\in[0,b]$ and has a convergent subsequence. Combining this with the monotonicity (3.3.4), we get $\{u^{(n)}\}$ is a convergent sequence. Similarly we can show that $\{v^{(n)}\}$ is also a convergent sequence. Let

$$\lim_{n\to\infty}u^{(n)}=u^*,\quad \lim_{n\to\infty}v^{(n)}=v^*.$$

Clearly, $\{u^{(n)}\}$, $\{v^{(n)}\}\subset\mathcal{PC}([0,b],\mathbb{X})$ therefore u^*, v^* are bounded integrable in every $t\in(t_{j-1},t_j]$, $j=1,2,\dots,m+1$. Since for any $t\in(t_{j-1},t_j]$, we have

$$u^{(n)}(t)\;=\;\Phi u^{(n-1)}(t)$$

as $n \to \infty$, using Lebesgue dominated convergence theorem, for all $t \in (t_{j-1}, t_j]$, $j = 1, 2, \ldots, m+1$, we get

$$u^*(t) = \begin{cases} P_\beta(t)BT^{-1}\Big[\tilde{u}_0^* + \frac{1}{\Gamma(1-\beta)}\int_0^t (t-s)^{-\beta}(u_0^* + g(u^*(s)))ds\Big] \\ + \int_0^t (t-s)^{\beta-1}Q_\beta(t-s)f\Big(s, u^*(s), \int_0^s K(s,\xi,u^*(\xi))d\xi\Big)ds, \quad t \in [0, t_1]; \\ P_\beta(t)BT^{-1}\Big[\tilde{u}_0^* + \frac{1}{\Gamma(1-\beta)}\int_0^t (t-s)^{-\beta}(u_0^* + g(u^*(s)))ds\Big] \\ +P_\beta(t-t_1)I_1(u^*(t_1)) \\ + \int_0^t (t-s)^{\beta-1}Q_\beta(t-s)f\Big(s, u^*(s), \int_0^s K(s,\xi,u^*(\xi))d\xi\Big)ds, \quad t \in (t_1, t_2]; \\ \cdots, \\ P_\beta(t)BT^{-1}\Big[\tilde{u}_0^* + \frac{1}{\Gamma(1-\beta)}\int_0^t (t-s)^{-\beta}(u_0^* + g(u^*(s)))ds\Big] \\ + \sum_{j=1}^m P_\beta(t-t_j)I_j(u^*(t_j)) \\ + \int_0^t (t-s)^{\beta-1}Q_\beta(t-s)f\Big(s, u^*(s), \int_0^s K(s,\xi,u^*(\xi))d\xi\Big)ds, \quad t \in (t_m, b]. \end{cases}$$

Therefore $u^* \in \mathcal{PC}([0,b],\mathbb{X})$ and $u^*(t) = \Phi u^*(t)$. Similarly, $v^* \in \mathcal{PC}([0,b],\mathbb{X})$ and $v^*(t) = \Phi v^*(t)$. Using this fact along with the monotonicity condition (3.3.4), we obtain $u^{(0)} \leq u^* \leq v^* \leq v^{(0)}$. Hence u^* and v^* are the minimal and maximal mild solutions of (3.2.1) on G respectively. □

Theorem 3.3.2. *Let $\mathcal{P}$ be a regular positive cone in ordered Banach space $\mathbb{X}$. Allow that the system* (3.2.1) *acquired lower and upper solutions $u^{(0)}, v^{(0)} \in \mathcal{PC}^\beta([0,b],\mathbb{X})$ such that $u^{(0)} \leq v^{(0)}$, $\mathcal{S}(t)(t \geq 0)$ is positive semigroup and the assumptions* $(A1)$, $(A2)$ *and* $(A4)$ *are fulfilled. Then minimal and maximal solutions exist for the system* (3.2.1) *in between $u^{(0)}$ and $v^{(0)}$.*

Proof. Since the assumptions $(A1)$, $(A2)$ and $(A4)$ holds, therefore equation (3.3.4) is satisfied. Let $\{u^{(n)}\}$ and $\{v^{(n)}\}$ be two increasing or decreasing sequences in G. Then using Definition of regular cone and assumption $(A2)$, $\{K(t,s,u^{(n)})\}$ is convergent. Therefore $\zeta(\{K(t,s,u^{(n)})\}) = \zeta(\{u^{(n)}\}) = 0$. Similarly we can get, $\zeta(\{f(t,u^{(n)},v^{(n)})\}) = \zeta(\{u^{(n)}\}) + \zeta(\{v^{(n)}\}) = 0$. Then assumption $(A3)$ holds. Hence from Theorem 3.3.1, the proof is complete. □

Theorem 3.3.3. *Let $\mathcal{P}$ be a normal positive cone in ordered and weakly sequentially complete Banach space $\mathbb{X}$. Allow that the system (3.2.1) acquired lower and upper solutions $u^{(0)}, v^{(0)} \in \mathcal{PC}^\beta([0,b],\mathbb{X})$ such that $u^{(0)} \leq v^{(0)}$, $\mathcal{S}(t)(t \geq 0)$ is positive semigroup and the assumptions* $(A1)$, $(A2)$ *and* $(A4)$ *are fulfilled. Then minimal and maximal solutions exist for the system* (3.2.1) *in between $u^{(0)}$ and $v^{(0)}$.*

Proof. Following the proof of Corollary 3.3.2, and using the fact that the normal cone $\mathcal{P}$ is regular cone in an ordered and weakly sequentially complete Banach space, we obtain the required results. □

We consider the following assumptions to show the uniqueness of the solution of the system (3.2.1):

(A5) The function $K : \Delta \times \mathbb{X} \to \mathbb{X}$, $f : [0,b] \times \mathbb{X} \times \mathbb{X} \to \mathbb{X}$ and $g : \mathcal{C}([0,b],\mathbb{X}) \to \mathbb{X}$ satisfies the following conditions:

1. The function $K : \Delta \times \mathbb{X} \to \mathbb{X}$ is continuous and there exists an integrable function $\lambda : \Delta \to [0,\infty)$ such that

$$K(t,s,u_2) - K(t,s,u_1) \leq \lambda(t,s)[u_2 - u_1],$$

for any $(t,s) \in \Delta$ and $u^{(0)} \leq u_1 \leq u_2 \leq v^{(0)}$. For simplification let $\lambda^* = \int_0^t \lambda(t,s)$.

2. The function $f : [0,b] \times \mathbb{X} \times \mathbb{X} \to \mathbb{X}$ is continuous and there exists $\gamma \geq 0$ such that

$$f(t,u_2,v_2) - f(t,u_1,v_1) \leq \gamma[(u_2 - u_1) + (v_2 - v_1)],$$

for all $t \in [0,b]$, $u_1, u_2 \in \mathbb{X}$ with $u^{(0)} \leq u_1 \leq u_2 \leq v^{(0)}$ and $v_1, v_2 \in \mathbb{X}$ such that $\int_0^t K(t,s,u^{(0)}(s))ds \leq v_1 \leq v_2 \leq \int_0^t K(t,s,v^{(0)}(s))ds$.

3. The function $g : \mathcal{C}([0,b],\mathbb{X}) \to \mathbb{X}$ is continuous and there exists a positive constant α such that

$$g(u_2) - g(u_1) \leq \alpha(u_2 - u_1),$$

for $u_1, u_2 \in \mathbb{X}$.

Theorem 3.3.4. *Let $\mathcal{P}$ be a normal positive cone in ordered Banach space $\mathbb{X}$ with normal constant N. Allow that the system* (3.2.1) *acquired lower and upper solutions $u^{(0)}, v^{(0)} \in \mathcal{PC}^{\beta}([0,b],\mathbb{X})$ such that $u^{(0)} \leq v^{(0)}$, $\mathcal{S}(t)(t \geq 0)$ is positive semigroup and the assumptions (A2), (A4) and (A5) are fulfilled. Then the unique mild solution exists for the system* (3.2.1) *in between $u^{(0)}$ and $v^{(0)}$.*

Proof. Let $\{u^n\}$ and $\{v^n\} \subset [u^{(0)}, v^{(0)}]$ be two monotonic increasing sequences. Take $n, p = 1, 2, \ldots$ such that $n \geq p$, using $(A2)$ and $(A5)$, we have

$$\theta \leq K(t,s,u^n) - K(t,s,u^p) \leq \lambda(t,s)[u^n - u^p],$$

and

$$\theta \leq f(t,u^n,v^n) - f(t,u^p,v^p) \leq \gamma[(u^n-u^p)+(v^n-v^p)].$$

Using normality of the positive cone P, we obtain

$$\|K(t,s,u^n) - K(t,s,u^p)\| \leq N\lambda(t,s)\|u^n-u^p\|,$$

and

$$\|f(t,u^n,v^n) - f(t,u^p,v^p)\| \leq N\gamma[\|(u^n-u^p)+(v^n-v^p)\|].$$

Using the above inequalities and the definition of measure of noncompactness, we have

$$\zeta(\{K(t,s,u^n)\}) \leq \omega(t,s)\zeta(\{u^n\}),$$

and

$$\zeta(\{f(t,u^n,v^n)\}) \quad \leq L_f\Big[\zeta(\{u^n\})+\zeta(\{v^n\})\Big],$$

where $L_f = N\gamma$ and $\omega(t,s) = N\lambda(t,s)$. If $\{u^n\}$ and $\{v^n\}$ be two decreasing sequences then also the above inequality is true. Thus assumption $(A3)$ is satisfied. Therefore from Theorem 3.3.1 there exists u^* and v^* which are the minimal and maximal mild solutions of (3.2.4) between $u^{(0)}$ and $v^{(0)}$ in G respectively.
Now, we will show $u^*(t) = v^*(t)$ for every $t \in (t_{j-1}, t_j]$, $j = 1,2,\ldots,m+1$.
For $t \in [0,t_1]$, using assumption $(A5)$, we have

$$\begin{aligned}
\|u^*(t)-v^*(t)\| =&\|\Phi u^*(t) - \Phi v^*(t)\| \\
\leq&\left\|\frac{P_\beta(t)BT^{-1}}{\Gamma(1-\beta)}\int_0^t \frac{[g(u^*(s))-g(v^*(s))]}{(t-s)^\beta}ds\right\| \\
&+\left\|\int_0^t (t-s)^{\beta-1}Q_\beta(t-s)\Big(f(s,u^*(s),\int_0^s K(s,\xi,u^*(\xi))d\xi)\right. \\
&\qquad \left. - f(s,v^*(s),\int_0^s K(s,\xi,v^*(\xi))d\xi)\Big)ds\right\| \\
\leq&\frac{MNM_1M_3\alpha}{\Gamma(1-\beta)}\int_0^t (t-s)^{-\beta}\|u^*(s)-v^*(s)\|ds + \frac{NM\gamma}{\Gamma(\beta)}\times \\
&\int_0^t (t-s)^{\beta-1}\Big\{\|u^*(s)-v^*(s)\| + \int_0^s \lambda(s,\xi)\|u^*(\xi)-v^*(\xi)\|d\xi\Big\}ds \\
\leq&\frac{MNM_1M_3\alpha}{\Gamma(1-\beta)}\int_0^t (t-s)^{-\beta}\|u^*(s)-v^*(s)\|ds \\
&+\frac{NM\gamma}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}\Big\{\|u^*(s)-v^*(s)\| + \lambda^*\|u^*(s)-v^*(s)\|\Big\}ds \\
\leq&\frac{MNM_1M_3\alpha}{\Gamma(1-\beta)}\int_0^t (t-s)^{-\beta}\|u^*(s)-v^*(s)\|ds \\
&+\frac{NM\gamma}{\Gamma(\beta)}(1+\lambda^*)\int_0^t (t-s)^{\beta-1}\|u^*(s)-v^*(s)\|ds.
\end{aligned}$$

Therefore, using Lemma 2.1.1, we obtain $u^*(t) \equiv v^*(t)$ for $t \in [0, t_1]$. Particularly, $u^*(t_1) \equiv v^*(t_1)$ so $I_1(u^*(t_1)) = I_1(v^*(t_1))$.
For $t \in (t_1, t_2]$, using assumption $(A5)$ and similar argument as above, we have

$$\begin{aligned}\|u^*(t) - v^*(t)\| \leq & \frac{MNM_1M_3\alpha}{\Gamma(1-\beta)} \int_0^t (t-s)^{-\beta}\|u^*(s) - v^*(s)\| ds \\ & + \frac{NM\gamma}{\Gamma(\beta)}(1+\lambda^*) \int_0^t (t-s)^{\beta-1}\|u^*(s) - v^*(s)\| ds \\ \leq & \frac{MNM_1M_3\alpha}{\Gamma(1-\beta)} \int_0^{t_1} (t-s)^{-\beta}\|u^*(s) - v^*(s)\| ds \\ & + \frac{NM\gamma}{\Gamma(\beta)}(1+\lambda^*) \int_0^{t_1} (t-s)^{\beta-1}\|u^*(s) - v^*(s)\| ds.\end{aligned}$$

Again, using Lemma 2.1.1, we get $u^*(t) \equiv v^*(t)$ for $t \in (t_1, t_2]$. Particularly, $u^*(t_2) \equiv v^*(t_2)$ so $I_1(u^*(t_2)) = I_1(v^*(t_2))$.

Repeating this procedure interval by interval up to $(t_m, b]$, we obtain $u^*(t) \equiv v^*(t)$ over the whole $[0, b]$. Hence $\widetilde{u} := u^*(t) \equiv v^*(t)$ is the unique mild solution of the problem (3.2.1) in G, which is acquired by the monotone iterative procedure beginning from $u^{(0)}$ and $v^{(0)}$. □

3.4 Example

Let $\mathbb{X} = L^2[0, \pi]$. Consider the following Sobolev type fractional impulsive differential system with fractional nonlocal conditions:

$$\begin{cases} {}^cD^\beta[u(t,v) - u_{vv}(t,v)] = -\frac{\partial^2}{\partial v^2}u(t,v) + L\left(\frac{u(t)}{1+u(t)} + \int_0^t \frac{u(t,s)}{\sqrt{s(t-s)}}ds\right), \ t \in (0,\pi]'; \\ \Delta u|_{t=\frac{\pi}{2}} = e^{u(\frac{\pi}{2})^-}, \\ u(0,v) = \frac{\partial^2}{\partial v^2}\left[\tilde{u}_0(v) + \frac{1}{\Gamma(1-\beta)}\int_0^t (t-s)^{-\beta}[u_0(v) + \rho(s)\frac{(u(s))^2}{(1+u(s))^2}]ds\right], \\ u(t,0) = u(t,\pi) = 0, \end{cases} \tag{3.4.1}$$

where $(0,\pi]' = (0,\pi] - \frac{\pi}{2}$, $\beta \in (0,1]$, $L \geq 0$, $\rho(s)$ is a continuous operator. The linear operators B, A and T with their domains and ranges contained in $L^2[0,\pi]$ are define by $Bu = u - u_{vv}$, $A = -u_{vv}$ and $Tu = u_{vv}$ where the domains $\mathcal{D}(B)$, $\mathcal{D}(A)$ and $\mathcal{D}(T)$ are given by

$$\{u \in \mathbb{X} : u, \ u_v \text{ is absolutely continuous}, \ u_{vv} \in \mathbb{X}, \ u(0) = u(\pi) = 0\}.$$

Then the operators B and A are given by

$$Bu = \sum_{m=1}^{\infty}(1+m^2)\langle u, u_m\rangle u_m \quad \text{and} \quad Au = \sum_{m=1}^{\infty} m^2\langle u, u_m\rangle u_m,$$

where $u_m(v) = (\sqrt{2/\pi})\sin mv$, $m = 1, 2, 3, \ldots$ is the orthogonal set of eigenfunctions of A. Moreover, for any $u \in \mathbb{X}$, we have

$$B^{-1}u = \sum_{m=1}^{\infty}\frac{1}{1+m^2}\langle u, u_m\rangle u_m, \quad AB^{-1}u = \sum_{m=1}^{\infty}\frac{m^2}{1+m^2}\langle u, u_m\rangle u_m,$$

and

$$\mathcal{S}(t)u = \sum_{m=1}^{\infty}\exp\left(\frac{m^2 t}{1+m^2}\right)\langle u, u_m\rangle u_m.$$

Here B^{-1} is a bounded operator with $\|B^{-1}\| \leq 1$ and $B^{-1}A$ generates the strongly continuous semigroup $\mathcal{S}(t)$ on $L^2[0,\pi]$ with $\|\mathcal{S}(t)\| \leq 1$.

Let $\mathcal{P} = \{u \in \mathbb{X} : u(v) \geq 0 \ a.e. \ v \in [0,\pi]\}$ be the normal cone in Banach space $\mathbb{X}$ with normal constant $N = 1$. Define

$$\begin{aligned}
u(t) &= u(t,v),\\
K(t,s,u(s)) &= \frac{u(t,s)}{\sqrt{s(t-s)}},\\
f(t,u(t),\int_0^t K(t,s,u(s))ds) &= L\bigg(\frac{u(t)}{1+u(t)} + \int_0^t \frac{u(t,s)}{\sqrt{s(t-s)}}ds\bigg),\\
g(u(t)) &= \rho(t)\frac{(u(t))^2}{(1+u(t))^2},\\
I(u(\frac{\pi}{2})) &= e^{u(\frac{\pi}{2})^-}.
\end{aligned}$$

Therefore the fractional integro-differential system (3.4.1) can be written as in the form of (3.2.1) i.e.

$$\begin{cases}
{}^cD^{\beta}[Bu(t)] = Au(t) + f(t,u(t),\int_0^t K(t,s,u(s))ds), \quad t \in (0,\frac{\pi}{2}) \cup (\frac{\pi}{2},\pi);\\
\Delta u|_{t=\frac{\pi}{2}} = I(u(\frac{\pi}{2})),\\
{}^LD^{1-\beta}[Tu(0)] = u_0 + g(u(t)).
\end{cases} \tag{3.4.2}$$

Now, let $f(t,0,0)\geq 0$, $I(0)\geq 0$, $g(0)\geq 0$ and there exists a function $\xi(t)\geq 0$ such that

$$\begin{cases} {}^cD^{\beta}[B\xi(t)] \geq A\xi(t) + f(t,\xi(t),\int_0^t K(t,s,\xi(s))ds), \quad t\in(0,\frac{\pi}{2})\cup(\frac{\pi}{2},\pi); \\ \Delta\xi|_{t=\frac{\pi}{2}} \geq I(u(\frac{\pi}{2})), \\ {}^LD^{1-\beta}[T\xi(0)] \geq \xi_0 + g(\xi(t)). \end{cases}$$

Thus $u^{(0)}=0$ and $v^{(0)}=\xi(t)$ become the lower and upper solutions of the system (3.4.1) respectively. We can easily check that the functions f, K, I and g satisfies the assumptions $(A1)$, $(A2)$ and $(A4)$. To satisfy $(A3)$, for $u_1,u_2\in\mathbb{X}$ and $v_1,v_2\in\mathbb{X}$, we have

$$\begin{aligned} \|K(t,s,v_1(t))-K(t,s,v_2(t))\| \leq& \frac{N}{\sqrt{s(t-s)}}\|v_1-v_2\|, \\ \|f(t,u_1(t),v_1(t))-f(t,u_2(t),v_2(t))\| \leq& LN[\|u_1-u_2\|+\|v_1-v_2\|], \end{aligned}$$

and

$$\|g(u_1)-g(u_2)\|\leq N\alpha\|u_1-u_2\|.$$

Using the above inequalities and the definition of measure of noncompactness, for any increasing or decreasing sequences $\{u^n\}$ and $\{v^n\}$, we have

$$\zeta(\{K(t,s,u^n)\})\leq\omega(t,s)\zeta(\{u^n\}),$$

$$\zeta(\{f(t,u^n,v^n)\})\leq L_f\Big[\zeta(\{u^n\})+\zeta(\{v^n\})\Big],$$

where $\omega(t,s)=\frac{N}{\sqrt{s(t-s)}}$ and $L_f=LN$. Thus assumptions $(A3)$ is also satisfied. Hence using Theorem 3.3.4, the system (3.4.1) has a unique mild solution between 0 and ξ.

Chapter 4

Multi-Term Time-Fractional Stochastic Differential Systems with Non-Lipschitz Coefficients

4.1 Introduction

Nowadays, the multi-term time-fractional differential equations generating great interest among the mathematicians and engineers. For instance, in the papers [8; 121; 141; 143] the following deterministic two-term time-fractional differential system is studied in abstract form

$$\begin{cases} {}^cD_{0^+}^{1+\beta}y(t) + a\,{}^cD_{0^+}^{\gamma}y(t) = Ay(t) + {}^cD_{0^+}^{\beta-1}f(t, y(t)), \quad t \in (0, b], \\ y(0) = 0, \qquad y'(0) = g(y), \end{cases} \tag{4.1.1}$$

where $0 < \beta \leq \gamma \leq 1$, $a \geq 0$ and A is a sectorial operator. The nonlinear functions f and g are given vector valued functions.

On the other hand, the multi-term time-fractional diffusion wave equation was recently considered in [53] and [142] with constant and variable coefficients, respectively. Moreover, for multi-term time-fractional systems in [114; 135] the authors

The contents of this chapter are accepted in **Differential Equations and Dynamical Systems** as Singh V., Pandey D. N.: Multi-term time-fractional stochastic differential systems with non-Lipschitz coefficients.

studied analytic and numerical solutions. Recently, Pardo at al. [7] studied the existence of mild solution to the following system with measure of noncompactness techniques

$$\begin{cases} {}^cD_{0^+}^{1+\beta}y(t) + \sum_{j=1}^{n} c_j {}^cD_{0^+}^{\gamma_j}y(t) = Ay(t) + {}^cD_{0^+}^{\beta-1}f(t, y(t)), \quad t \in (0, 1], \\ y(0) = 0, \qquad y'(0) = g(y), \end{cases} \tag{4.1.2}$$

where $0 < \beta \leq \gamma_n \leq \cdots \leq \gamma_1 \leq 1$, $c_j \geq 0$, $j = 1, 2, 3, ..., n$ and A is the generator of strongly continuous cosine family. The nonlinear functions f and g are given suitable functions.

Anticipating a wide interest in the problems modeled as a multi-term time-fractional stochastic differential system, this chapter contributes in study of some existence and uniqueness results for mild solution of (4.2.2) using non-Lipschitz conditions by applying successive approximation approach.

4.2 Problem Formulation

Let $w(t)$ be a Q-Wiener process on a complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbf{P})$ with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions(i.e right continuous and $\{\mathcal{F}_0\}$ containing all $\mathbf{P}$-null sets) with the linear bounded covariance operator $Q \in \mathcal{L}(\mathbb{K}, \mathbb{K}) = \mathcal{L}(\mathbb{K})$ such that $trQ < \infty$, where "tr" denotes the trace of the operator. Further, we assume that there exist a complete orthonormal system $\{e_n\}_{n\geq 1}$ in $\mathbb{K}$, a sequence of non-negative real numbers $\{\lambda_n\}_{n\geq 1}$ such that $Qe_n = \lambda_n e_n$, $n = 1, 2, 3, \ldots$ and a sequence $\{\zeta_n\}_{n\geq 1}$ of independent Brownian motions such that

$$\langle w(t), e\rangle = \sum_{n=1}^{\infty} \sqrt{\lambda_n}\langle e_n, e\rangle_{\mathbb{K}}\zeta_n(t), \quad e \in \mathbb{K},\, t \in [0, b] \tag{4.2.1}$$

and $\mathcal{F}_t = \mathcal{F}_t^w$, where $\mathcal{F}_t^w$ is the σ-algebra generated by $\{w(s) : 0 \leq s \leq t\}$ and $\mathcal{F}_b = \mathcal{F}$. Further, assume that $\mathcal{L}_2^0 = \mathcal{L}_2(Q^{\frac{1}{2}}\mathbb{K}, \mathbb{H})$ represents the space of all Hilbert Schmidt operators from $Q^{\frac{1}{2}}\mathbb{K}$ to $\mathbb{H}$ with norm $\|\phi\|_{\mathcal{L}_2^0} = tr[\phi Q\phi^*] < \infty$, $\phi \in \mathcal{L}(\mathbb{K}, \mathbb{H})$. Let $\mathcal{L}_2(\mathcal{F}_b, \mathbb{H})$ be the space of all $\mathcal{F}_b$ measurable $\mathbb{H}$ valued square integrable random variables. Moreover, let $\mathcal{L}_2^{\mathcal{F}}([0,b],\mathbb{H})$ be the Hilbert space of all square integrable and $\mathcal{F}_t$ adapted processes with value in $\mathbb{H}$. We denote by $\mathbb{B}_b$ the Banach space of all $\mathbb{H}$-valued $\mathcal{F}_t$ adapted processes $y(t,\omega):[0,b]\times\Omega\to\mathbb{H}$ which are continuous in t for a.e. fixed $\omega\in\Omega$ and satisfy

$$\|y\|_{\mathbb{B}_b}=\mathbf{E}\Big(\sup_{t\in[0,b]}\|y(t,\omega)\|^p\Big)^{\frac{1}{p}}<\infty,\quad p\geq 2.$$

In this chapter, we study the existence and uniqueness of mild solution to the following multi-term time-fractional stochastic differential system

$$\begin{cases} {}^cD_{0+}^{1+\beta}y(t)+\displaystyle\sum_{j=1}^{n}\alpha_j\,{}^cD_{0+}^{\gamma_j}y(t)=Ay(t)+F(t,y(t)) \\ \qquad\qquad\qquad\qquad\qquad +G(t,y(t))\frac{dw(t)}{dt},\quad t\in(0,b],\, b<\infty, \\ y(0)=\varphi,\qquad y'(0)=\chi, \end{cases} \tag{4.2.2}$$

where ${}^cD_{0+}^{\eta}$ is the Caputo fractional derivative of order $\eta>0$; $A:\mathcal{D}(A)\subset\mathbb{H}\to\mathbb{H}$ is a closed linear operator on $\mathbb{H}$, and $\alpha_j\geq 0$ for all $j=1,2,3,...,n$. Here, $\gamma_j, j=1,2,...,n$ are positive real numbers such that $0<\beta\leq\gamma_n\leq\cdots\leq\gamma_1\leq 1$. The functions F and G are suitable functions to be defined later. The initial given data φ,χ are in $\mathcal{F}_0$-measurable $\mathbb{H}$-valued random variable independent of w with finite p moments.

To give a appropriate representation for mild solution in terms of certain family of bounded and linear operators, we define following family of operators.

Definition 4.2.1. *[7] Let A be a closed linear operator on a Hilbert space $\mathbb{H}$ with the domain $\mathcal{D}(A)$ and let $\beta>0,\gamma_j,\alpha_j,\ j=1,2,3,...,n$ be the real positive numbers. Then A is called the generator of a $(\beta,\gamma_j)-$ resolvent family if there exists $\omega>0$ and a strongly continuous function $\mathcal{S}_{\beta,\gamma_j}:\mathbb{R}^+\to\mathcal{L}(\mathbb{H})$ such that $\{\lambda^{\beta+1}+\sum_{j=1}^{n}\alpha_j\lambda^{\gamma_j}: Re\,\lambda>\omega\}\subset\varrho(A)$ and*

$$\lambda^{\beta}\Big(\lambda^{\beta+1}+\sum_{j=1}^{n}\alpha_j\lambda^{\gamma_j}-A\Big)^{-1}y=\int_0^{\infty}e^{-\lambda t}\mathcal{S}_{\beta,\gamma_j}(t)y\,dt,\quad Re\,\lambda>\omega, y\in\mathbb{H}. \tag{4.2.3}$$

The following result guarantee for the existence of $(\beta,\gamma_j)-$ resolvent family under some suitable conditions.

Theorem 4.2.1. *[7] Let $0 < \beta \leq \gamma_n \leq \cdots \leq \gamma_1 \leq 1$ and $\alpha_j \geq 0$ be given and let A be a generator of a bounded and strongly continuous cosine family $\{C(t)\}_{t\in\mathbb{R}}$. Then, A generates a bounded $(\beta,\gamma_j)-$ resolvent family $\{\mathcal{S}_{\beta,\gamma_j}(t)\}_{t\geq 0}$.*

Now, consider the initial value problem

$$
\begin{aligned}
{}^cD_{0^+}^{1+\beta}y(t) + \sum_{j=1}^{n}\alpha_j {}^cD_{0^+}^{\gamma_j}y(t) &= Ay(t) + f(t), \quad t \in (0,b], && (4.2.4)\\
y(0) =& p, \quad y'(0) = q, && (4.2.5)
\end{aligned}
$$

where A is the generator of a bounded and strongly continuous cosine family, $p,q \in \mathbb{X}$, $0 < \beta \leq \gamma_n \leq \cdots \leq \gamma_1 \leq 1$, $\alpha_j \geq 0$, $j = 1,2,...,n$, and f is a Hölder continuous function.

With the aim to construct mild solution representation for the problem (4.2.4)−(4.2.5) in the term of the family $\{\mathcal{S}_{\beta,\gamma_j}(t)\}_{t\geq 0}$, we apply the Laplace transform on the both sides of (4.2.4), we obtain

$$
\lambda^{1+\beta}\widehat{y}(\lambda) - \sum_{i=0}^{\lceil 1+\beta\rceil -1} y^{(i)}(0)\lambda^{\beta-i} + \sum_{j=1}^{n}\alpha_j\left[\lambda^{\gamma_j}\widehat{y}(\lambda) - \sum_{i=0}^{\lceil\gamma_j\rceil-1} y^{(i)}(0)\lambda^{\gamma_j-1-i}\right]
$$

$$
= A\widehat{y}(\lambda) + \widehat{f}(\lambda).
$$

Using initial data given by (4.2.5), we have

$$
\lambda^{1+\beta}\widehat{y}(\lambda) - \lambda^{\beta}p - \lambda^{\beta-1}q + \sum_{j=1}^{n}\alpha_j\lambda^{\gamma_j}\widehat{y}(\lambda) - \sum_{j=1}^{n}\alpha_j\lambda^{\gamma_j-1}p = A\widehat{y}(\lambda) + \widehat{f}(\lambda).
$$

This is equivalent to

$$
\left(\lambda^{1+\beta} + \sum_{j=1}^{n}\alpha_j\lambda^{\gamma_j} - A\right)\widehat{y}(\lambda) = \lambda^{\beta}p + \lambda^{\beta-1}q + \sum_{j=1}^{n}\alpha_j\lambda^{\gamma_j-1}p + \widehat{f}(\lambda).
$$

Now, by Theorem 4.2.1 assuming the existence of $(\beta,\gamma_j)-$ resolvent family $\{\mathcal{S}_{\beta,\gamma_j}(t)\}_{t\geq 0}$, we have

$$
\begin{aligned}
\widehat{y}(\lambda) =& \lambda^{\beta}\left(\lambda^{1+\beta} + \sum_{j=1}^{n}\alpha_j\lambda^{\gamma_j} - A\right)^{-1}p + \lambda^{\beta-1}\left(\lambda^{1+\beta} + \sum_{j=1}^{n}\alpha_j\lambda^{\gamma_j} - A\right)^{-1}q\\
&+ \sum_{j=1}^{n}\alpha_j\lambda^{\gamma_j-1}\left(\lambda^{1+\beta} + \sum_{j=1}^{n}\alpha_j\lambda^{\gamma_j} - A\right)^{-1}p + \left(\lambda^{1+\beta} + \sum_{j=1}^{n}\alpha_j\lambda^{\gamma_j} - A\right)^{-1}\widehat{f}(\lambda).
\end{aligned}
$$

By inverse Laplace transformation, we obtain

$$y(t) = \mathcal{S}_{\beta,\gamma_j}(t)p + (g_1 * \mathcal{S}_{\beta,\gamma_j})(t)q + \sum_{j=1}^{n} \alpha_j \int_0^t \frac{(t-s)^{\beta-\gamma_j}}{\Gamma(1+\beta-\gamma_j)} \mathcal{S}_{\beta,\gamma_j}(s)pds \tag{4.2.6}$$

$$+ \int_0^t (g_\beta * \mathcal{S}_{\beta,\gamma_j})(t-s)f(s)ds. \tag{4.2.7}$$

The above representation, allow us to define the mild solution for the system (4.2.2).

Definition 4.2.2. *An $\mathbb{H}$-valued stochastic process $\{y(t)\}_{t\in[0,b]}$ is said to be mild solution of* (4.2.2) *if*

(i) $y(t)$ is measurable and $\mathcal{F}_t$ adapted, for each $t \in [0,b]$,

(ii) $y(t)$ satisfies the following equation

$$\begin{aligned} y(t) &= \mathcal{S}_{\beta,\gamma_j}(t)\varphi + (g_1 * \mathcal{S}_{\beta,\gamma_j})(t)\chi + \sum_{j=1}^{n} \alpha_j \int_0^t \frac{(t-s)^{\beta-\gamma_j}}{\Gamma(1+\beta-\gamma_j)} \mathcal{S}_{\beta,\gamma_j}(s)\varphi ds \\ &+ \int_0^t \mathcal{T}_{\beta,\gamma_j}(t-s)F(s,y(s))ds + \int_0^t \mathcal{T}_{\beta,\gamma_j}(t-s)G(s,y(s))dw(s), \end{aligned} \tag{4.2.8}$$

$\boldsymbol{P}$-a.s. for all $t \in [0,b]$, where $\mathcal{T}_{\beta,\gamma_j}(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} \mathcal{S}_{\beta,\gamma_j}(s)ds$.

Lemma 4.2.2. *[136], For any $p \geq 2$ and let h be $\mathcal{L}_0^2$-valued predictable process such that $\boldsymbol{E}\Big(\int_0^b \|h(s)\|^p_{\mathcal{L}_0^2} ds \Big) < +\infty$, then we have*

$$\begin{aligned} \boldsymbol{E}\bigg(\sup_{s\in[0,t]} \bigg\| \int_0^s h(r)dw(r) \bigg\|^p \bigg) &\leq c_p \sup_{s\in[0,t]} \boldsymbol{E}\bigg(\bigg\| \int_0^s h(r)dw(r) \bigg\|^p \bigg) \\ &\leq C_p \boldsymbol{E}\bigg(\int_0^t \|h(r)\|^p_{\mathcal{L}_0^2} dr \bigg), \quad t \in [0,b], \end{aligned}$$

where $c_p = \Big(\frac{p}{p-1}\Big)^p$ and $C_p = \Big(\frac{p}{2}(p-1)\Big)^{\frac{p}{2}} \Big(\frac{p}{p-1}\Big)^{\frac{p^2}{2}}$.

4.3 Existence and Uniqueness Results

In this section, we establish the existence and uniqueness results of mild solution for the system (4.2.2). Throughout in this section we denote $S_0 = \sup_{t\in[0,b]} \|\mathcal{S}_{\beta,\gamma_j}(t)\|$. We consider the following assumptions

(A_1) The functions $F : [0,b] \times \mathbb{H} \to \mathbb{H}, G : [0,b] \times \mathbb{H} \to \mathcal{L}_2^0$ are measurable and continuous in y for each $t \in [0,b]$ and there exists a function $U : [0,b]\times[0,\infty) \to [0,\infty)$ such that

$$\mathbf{E}(\|F(t,y)\|^p) + \mathbf{E}(\|G(t,y)\|^p_{\mathcal{L}_2^0}) \leq U(t, \mathbf{E}(\|y\|)^p) \tag{4.3.1}$$

for all $y \in L^p(\Omega, \mathcal{F}_b, \mathbb{H})$ and all $t \in [0,b]$.

(A_2) For each fixed $x \in [0,\infty)$, $U(t,x)$ is locally integrable in t and non-decreasing continuous in x for each fixed $t \in [0,b]$ and for all $\theta > 0, x_0 \geq 0$, the integral equation $x(t) = x_0 + \theta \int_0^t U(s,x(s))ds$ admits a global solution on $[0,b]$.

(A_3) There exist a function $V : [0,b] \times [0,\infty) \to [0,\infty)$ such that

$$\mathbf{E}(\|F(t,x) - F(t,y)\|^p) + \mathbf{E}(\|G(t,x) - G(t,y)\|^p_{\mathcal{L}_2^0}) \leq V(t, \mathbf{E}(\|x-y\|)^p) \tag{4.3.2}$$

for all $x, y \in L^p(\Omega, \mathcal{F}_b, \mathbb{H})$ and all $t \in [0,b]$.

(A_4) For each fixed $x \in [0,\infty)$, $V(t,x)$ is locally integrable in t and non-decreasing continuous in x for each fixed $t \in [0,b]$. Moreover, $V(t,0) = 0$ and if a non-negative continuous function $z(t)$, $t \in [0,b]$ satisfies $z(t) \leq \sigma \int_0^t V(s,z(s))ds$ for $t \in [0,b]$ subject to $z(0) = 0$ for some $\sigma > 0$, then $z(t) = 0$ for all $t \in [0,b]$.

Remark 4.3.1. *(i) For all $x \geq 0$, define $V(t,x) = Vx$, where $V > 0$ is a constant, then (A_3) implies global Lipschitz condition.*

(ii) If $V(t,x)$ is concave with respect to $x > 0$ for each fixed $t \geq 0$ and

$$\|F(t,x) - F(t,y)\|^p + \|G(t,x) - G(t,y)\|^p_{\mathcal{L}_2^0} \leq V(t, \|x-y\|^p),$$

for all $x, y \in \mathbb{H}$, and $t \geq 0$. Then by Jensen's inequality (4.3.2) is satisfied.

(iii) Let $V(t,x) = \xi(t)\vartheta(x)$, $t \in [0,b]$, $x \geq 0$, where $\vartheta : [0,\infty) \to [0,\infty)$ is monotone non-decreasing, continuous and concave function with $\vartheta(0) = 0$, $\vartheta(x) > 0$ for all $x > 0$ and $\int_{0+} 1/\vartheta(x)dx = \infty$ and $\xi(t) \geq 0$ is locally integrable. It can be observed that ϑ satisfies (4.3.2) [177].

For $\epsilon \in (0,1)$ sufficiently small, we define [177]

$$\vartheta_1(x) = \begin{cases} x\log(x^{-1}), & 0 \le x \le \epsilon; \\ \epsilon\log(\epsilon^{-1}) + \vartheta_1'(\epsilon^-)(x-\epsilon), & x > \epsilon. \end{cases} \tag{4.3.3}$$

$$\vartheta_2(x) = \begin{cases} x\log(x^{-1})\log\log(x^{-1}), & 0 \le x \le \epsilon; \\ \epsilon\log(\epsilon^{-1})\log\log(\epsilon^{-1}) + \vartheta_2'(\epsilon^-)(x-\epsilon), & x > \epsilon. \end{cases} \tag{4.3.4}$$

where ϑ_1' and ϑ_2' stand for left derivatives of ϑ_1 and ϑ_2 at the point ϵ. All the functions satisfy $\int_{0^+} 1/\vartheta_i(x)dx = \infty$, $i = 1,2$ and concave and nondecreasing. It should be noted that the proposed conditions are more general than the Lipschitz conditions.

Taking into account the aforementioned definitions and lemmas, we give the following existence and uniqueness results of mild solutions for the system (4.2.2).

Theorem 4.3.2. *Assume that the assumptions $(A_1) - (A_4)$ hold, then the system (4.2.2) admits a unique mild solution in $\mathbb{B}_b$.*

First, we prove the existence part of Theorem 4.3.2 based on the Picard type approximation technique. Let us construct a sequence of stochastic processes $\{y_n\}_{n\in\mathbb{N}\cup\{0\}}$ defined by

$$\begin{cases} y_0(t) = \mathcal{S}_{\beta,\gamma_j}(t)\varphi + (g_1 * \mathcal{S}_{\beta,\gamma_j})(t)\chi + \sum_{j=1}^{n}\alpha_j\int_0^t \frac{(t-s)^{\beta-\gamma_j}}{\Gamma(1+\beta-\gamma_j)}\mathcal{S}_{\beta,\gamma_j}(s)\varphi ds \\ y_{n+1}(t) = \mathcal{S}_{\beta,\gamma_j}(t)\varphi + (g_1 * \mathcal{S}_{\beta,\gamma_j})(t)\chi + \sum_{j=1}^{n}\alpha_j\int_0^t \frac{(t-s)^{\beta-\gamma_j}}{\Gamma(1+\beta-\gamma_j)}\mathcal{S}_{\beta,\gamma_j}(s)\varphi ds \\ \qquad\qquad +B_1(y_n)(t) + B_2(y_n)(t), \end{cases} \tag{4.3.5}$$

where

$$B_1(y_n)(t) = \int_0^t \mathcal{T}_{\beta,\gamma_j}(t-s)F(s,y_n(s))ds, \tag{4.3.6}$$

$$\text{and} \qquad B_2(y_n)(t) = \int_0^t \mathcal{T}_{\beta,\gamma_j}(t-s)G(s,y_n(s))dw(s). \tag{4.3.7}$$

In order to establish existence results of the Theorem 4.3.2, we require the following lemma.

Lemma 4.3.3. *Under the assumptions* $(A_1)-(A_4)$*, the sequence* $\{y_n\}_{n\in\mathbb{N}\cup\{0\}}$ *is well defined. Moreover, it is bounded in* $\mathbb{B}_b$ *i.e.* $\sup_{n\in\mathbb{N}\cup\{0\}}\|y_n\|_{\mathbb{B}_b}\leq C$*, where* $C>0$ *is constant.*

Proof. From (4.3.5), we have

$$\begin{aligned}
&\mathbf{E}\|y_{n+1}(t)\|^p\\
&\leq 5^{p-1}\mathbf{E}\|\mathcal{S}_{\beta,\gamma_j}(t)\varphi\|^p\\
&\quad+5^{p-1}\mathbf{E}\|(g_1*\mathcal{S}_{\beta,\gamma_j})(t)\chi\|^p+5^{p-1}\mathbf{E}\left\|\sum_{j=1}^{n}\alpha_j\int_0^t\frac{(t-s)^{\beta-\gamma_j}}{\Gamma(1+\beta-\gamma_j)}\mathcal{S}_{\beta,\gamma_j}(s)\varphi ds\right\|^p\\
&\quad+5^{p-1}\mathbf{E}\|B_1(y_n)(t)\|^p+5^{p-1}\mathbf{E}\|B_2(y_n)(t)\|^p.
\end{aligned}\tag{4.3.8}$$

Using Hölder inequality and monotonicity of U, from (4.3.5), we get

$$\begin{aligned}
\mathbf{E}\|B_1(y_n)(t)\|^p\leq&\frac{S_0^p}{(\Gamma(1+\beta))^p}\left(\frac{p-1}{\beta p+p-1}\right)^{p-1}b^{\beta p+p-1}\int_0^t\mathbf{E}(\|F(s,y_n(s))\|^p)ds\\
\leq&C_1\int_0^t U(s,\mathbf{E}\|y_n(s)\|^p)ds\\
\leq&C_1\int_0^t U(s,\|y_n\|^p_{\mathbb{B}_s})ds,
\end{aligned}$$

where $C_1=\frac{S_0^p}{(\Gamma(1+\beta))^p}\left(\frac{p-1}{\beta p+p-1}\right)^{p-1}b^{\beta p+p-1}$.

Again, using Lemma 4.2.2, Hölder inequality and monotonicity of U, we get

$$\begin{aligned}
\mathbf{E}\|B_2(y_n)(t)\|^p\leq&C_p\mathbf{E}\left(\int_0^t\|\mathcal{T}_{\beta,\gamma_j}(t-s)\|^2\|G(s,y_n(s))\|^2_{\mathcal{L}_2^2}ds\right)^{\frac{p}{2}}\\
\leq&C_p\left(\frac{S_0}{\Gamma(1+\beta)}\right)^{\frac{p}{2}}\left(\frac{p-2}{2\beta p+p-2}\right)^{\frac{p-2}{2}}b^{2\beta p+p-2}\int_0^t\mathbf{E}(\|G(s,y_n(s))\|^p_{\mathcal{L}_2^2})ds\\
\leq&C_2\int_0^t U(s,\mathbf{E}\|y_n(s)\|^p)ds\\
\leq&C_2\int_0^t U(s,\|y_n\|^p_{\mathbb{B}_s})ds,
\end{aligned}$$

where $C_2=C_p\left(\frac{S_0}{\Gamma(1+\beta)}\right)^{\frac{p}{2}}\left(\frac{p-2}{2\beta p+p-2}\right)^{\frac{p-2}{2}}b^{2\beta p+p-2}$.

Now, using the above inequalities in (4.3.8), we acquire

$$\begin{aligned}\mathbf{E}\|y_{n+1}(t)\|^p \leq & 5^{p-1}S_0^p\mathbf{E}(\|\varphi\|^p) + 5^{p-1}S_0^p b^p\mathbf{E}(\|\chi\|^p) \\ & + 5^{p-1}\bigg(\sum_{j=1}^{n}\frac{S_0\alpha_j b^{1+\beta-\gamma_j}}{\Gamma(2+\beta-\gamma_j)}\bigg)^p\mathbf{E}(\|\varphi\|^p) \\ & + 5^{p-1}(C_1+C_2)\int_0^t U(s,\|y_n\|^p_{\mathbb{B}_s})ds \\ \leq & k_1 + k_2\int_0^t U(s,\|y_n\|^p_{\mathbb{B}_s})ds,\end{aligned}$$

where $k_1 = 5^{p-1}\bigg[S_0^p\mathbf{E}(\|\varphi\|^p) + S_0^p b^p\mathbf{E}(\|\chi\|^p) + \bigg(\sum_{j=1}^{n}\frac{S_0\alpha_j b^{1+\beta-\gamma_j}}{\Gamma(2+\beta-\gamma_j)}\bigg)^p\mathbf{E}(\|\varphi\|^p)\bigg]$ and $k_2 = 5^{p-1}(C_1+C_2)$.

Therefore,

$$\|y_{n+1}\|^p_{\mathbb{B}_t} \leq k_1 + k_2\int_0^t U(s,\|y_n\|^p_{\mathbb{B}_s})ds. \tag{4.3.9}$$

Now, we consider the following integral equation

$$z(t) = k_1 + k_2\int_0^t U(s,z(s))ds. \tag{4.3.10}$$

By the assumption (A_2), (4.3.10) admits a global solution $z(\cdot)$ on $[0,b]$.
Next, we show by applying induction argument that $\|y_n\|^p_{\mathbb{B}_t} \leq z(t)$, for all $t \in [0,b]$. For all $t \in [0,b]$, we have

$$\begin{aligned}\|y_0\|^p_{\mathbb{B}_t} \leq & 3^{p-1}S_0^p\mathbf{E}(\|\varphi\|^p) + 3^{p-1}S_0^p b^p\mathbf{E}(\|\chi\|^p) + 3^{p-1}\bigg(\sum_{j=1}^{n}\frac{S_0\alpha_j b^{1+\beta-\gamma_j}}{\Gamma(2+\beta-\gamma_j)}\bigg)^p\mathbf{E}(\|\varphi\|^p) \\ \leq & k_1 \leq z(t).\end{aligned}$$

Now, let us assume that $\|y_n(t)\|^p_{\mathbb{B}_t} \leq z(t)$ for all $t \in [0,b]$. Then by (4.3.9), (4.3.10) and non-decreasing property on U in second variable, we obtain

$$z(t) - \|y_{n+1}\|^p_{\mathbb{B}_t} \geq k_2\int_0^t (U(s,z(s)) - U(s,\|y_n\|^p_{\mathbb{B}_s}))ds \geq 0, \quad \forall t \in [0,b]. \tag{4.3.11}$$

In particular, $\sup_{n\in\mathbb{N}\cup\{0\}}\|y_{n+1}\|_{\mathbb{B}_b} \leq z(b)^{1/p}$ i.e. $\{y_n\}_{n\in\mathbb{N}\cup\{0\}}$ is well defined. □

Lemma 4.3.4. *Under the assumptions $(A_1)-(A_4)$, the sequence $\{y_n\}_{n\in\mathbb{N}\cup\{0\}}$ is a Cauchy sequence in $\mathbb{B}_b$.*

Proof. Let us define $\delta_n(t) = \sup_{n\leq m}\|y_m - y_n\|^p_{\mathbb{B}_t}$. For all $m,n \in \mathbb{N}\cup\{0\}$, we obtain

$$\begin{aligned}y_m(t) - y_n(t) \leq & \int_0^t \mathcal{T}_{\beta,\gamma_j}(t-s)(F(s,y_m(s)) - F(s,y_n(s)))ds \\ & + \int_0^t \mathcal{T}_{\beta,\gamma_j}(t-s)(G(s,y_m(s)) - G(s,y_n(s)))dw(s).\end{aligned}$$

Now, recalling the same argument as in Lemma 4.3.3, we obtain

$$\|y_m - y_n\|^p_{\mathbb{B}_t} \leq C_3 \int_0^t V(s, \|y_{m-1} - y_{n-1}\|^p_{\mathbb{B}_s}) ds \tag{4.3.12}$$

where $C_3 = 2^{p-1}\left[\frac{S_0^p}{(\Gamma(1+\beta))^p}\left(\frac{p-1}{\beta p+p-1}\right)^{p-1} b^{\beta p+p-1} + C_p\left(\frac{S_0}{\Gamma(1+\beta)}\right)^{\frac{p}{2}}\left(\frac{p-2}{2\beta p+p-2}\right)^{\frac{p-2}{2}} b^{2\beta p+p-2}\right]$.
This shows that

$$\delta_n(t) \leq C_3 \int_0^t V(s, \delta_{n-1}(s)) ds. \tag{4.3.13}$$

It is clear that the functions δ_n are well defined for all $n \geq 0$, categorically monotone non-decreasing and uniformly bounded due to Lemma 4.3.3. Since $\{\delta_n(t)\}_{n\in\mathbb{N}\cup\{0\}}$ is a monotonic non-increasing sequence for each $t \in [0, b]$, there exists a monotone non-decreasing function δ such that $\lim_{n\to\infty} \delta_n(t) \to \delta(t)$.
Now, by virtue of Lebesgue convergence theorem, we follow from the inequality (4.3.13) that

$$\delta(t) \leq C_3 \int_0^t V(s, \delta(t)) ds, \qquad \text{as} \quad n \to \infty. \tag{4.3.14}$$

By the assumption (A_4) and Lemma 2.2 in [24] that $\delta = 0$, $\forall t \in [0, b]$. Since $0 \leq \|y_m - y_n\|^p_{\mathbb{B}_b} \leq \delta_n(b)$ and $\lim_{n\to\infty} \delta_n(b) \to \delta(b)$, therefore as a result $\{y_n\}_{n\in\mathbb{N}\cup\{0\}}$ is a Cauchy sequence in $\mathbb{B}_b$. □

Proof of Theorem 4.3.2. Existence: Form Lemma 4.3.4, let us denote y as a limit of the sequence $\{y_n\}_{n\in\mathbb{N}\cup\{0\}}$. Now, similar as in the proof of Lemma 4.3.4, we can show that the right side of the sequence $\{y_n\}_{n\in\mathbb{N}\cup\{0\}}$ in inequality (4.3.5) tends to

$$\begin{aligned}&\mathcal{S}_{\beta,\gamma_j}(t)\varphi + (g_1 * \mathcal{S}_{\beta,\gamma_j})(t)\chi + \sum_{j=1}^n \alpha_j \int_0^t \frac{(t-s)^{\beta-\gamma_j}}{\Gamma(1+\beta-\gamma_j)} \mathcal{S}_{\beta,\gamma_j}(s)\varphi ds \\ &+ \int_0^t \mathcal{T}_{\beta,\gamma_j}(t-s) F(s, y(s)) ds + \int_0^t \mathcal{T}_{\beta,\gamma_j}(t-s) G(s, y(s)) dw(s), \qquad \text{as} \quad n \to \infty.\end{aligned}$$

Uniqueness: Let $x, y \in \mathbb{B}_b$ be two mild solutions of the system (4.2.2). Now repeating the proof of Lemma 4.3.4, similar as (4.3.12) we can obtain

$$\|x - y\|^p_{\mathbb{B}_t} \leq C_3 \int_0^t V(s, \|x - y\|^p_{\mathbb{B}_s}) ds. \tag{4.3.15}$$

By using the assumption (A_4), similar as in proof of Lemma 4.3.4, we get $\|x-y\|^p_{\mathbb{B}_b} \to 0$, which shows that $x = y$. This completes the proof.

4.4 Example

We provide a concrete example to illustrate the feasibility of the established results. Let $\beta, \gamma_j > 0,\ j = 1, 2, 3, \ldots, n$ be given such that $0 < \beta \leq \gamma_n \leq \cdots \leq \gamma_1 \leq 1$. Let $\mathbb{H} = L^2([0, \pi])$. We consider the following system

$$
\begin{aligned}
{}^cD^{1+\beta}z(t,x) + \sum_{j=1}^{n} \alpha_j {}^cD^{\gamma_j} z(t,x) =& \frac{\partial^2}{\partial x^2} z(t,x) + \widehat{F}(t, z(t,x)) \\
&+ \widehat{G}(t, z(t,x)) \frac{dw(t)}{dt}, \qquad (4.4.1) \\
z(t,0) =& z(t,\pi) = 0, \quad t \in [0,1], \qquad (4.4.2) \\
z(0,x) = z_0(x), \qquad & \frac{\partial z(t,x)}{\partial t}|_{t=0} = z_1(x),\ 0 \leq x \leq \pi, \qquad (4.4.3)
\end{aligned}
$$

where $w(t)$ denotes one dimensional $\mathbb{R}$-valued Brownian motion and $z_0(x), z_1(x) \in L^2([0, \pi])$ are $\mathcal{F}_0$ measurable and satisfy $\mathbf{E}\|z_0\|^2 \leq \infty, \mathbf{E}\|z_1\|^2 \leq \infty$. Here, we consider $p = 2$.

Define a operator $A : \mathcal{D}(A) \subset \mathbb{H} \to \mathbb{H}$ by

$$
Az = z_{xx}, \quad z \in \mathcal{D}(A),
$$

where $\mathcal{D}(A) := \{z \in \mathbb{H} : z, z_x \text{ are absolutely continuous}, z_{xx} \in \mathbb{H}, z(0) = z(\pi) = 0\}$. Then the operator A has spectral representation given by

$$
Az = \sum_{n=1}^{\infty} -n^2 \langle z, z_n \rangle z_n, \quad z \in \mathcal{D}(A),
$$

where $z_n(x) = (\sqrt{2/\pi}) \sin nx,\ n = 1, 2, \ldots,$ is the orthogonal set of eigenfunctions corresponding to the eigenvalues $\lambda_n = -n^2$ of A. Then A will be a generator of cosine family such that

$$
C(t)z = \sum_{n=1}^{\infty} \cos nt \langle z, z_n \rangle z_n,
$$

Thus A generates a strongly continuous cosine family. Then, for $\beta, \gamma_j > 0,\ j = 1, 2, 3, \ldots, n$ such that $0 < \beta \leq \gamma_n \leq \cdots \leq \gamma_1 \leq 1$, by Theorem 4.2.1, we conclude that A generates a bounded $(\beta, \gamma_j)-$ resolvent family

$$\mathcal{S}_{\beta,\gamma_j}(t)z = \int_0^\infty \frac{1}{t^{\frac{(1+\beta)}{2}}} \Phi_{\frac{(1+\beta)}{2}}(st^{-\frac{(1+\beta)}{2}})C(s)zds, \quad t \in [0,1],$$

where

$$\Phi_{\frac{(1+\beta)}{2}}(v) = \sum_{n=0}^{\infty} \frac{(-v)^n}{n!\Gamma(-(\beta(n+1)) - n)}, v \in \mathbb{C},$$

is the Wright functions. Let us denote $u(t)(x) = z(t,x)$ and $\varphi = z_0(x),\ \chi = z_1(x)$ for $t \in [0,1],\ x \in [0,\pi]$. Then, $Au(t) = \frac{\partial^2}{\partial x^2} z(t,x)$ and for the functions $F, G : [0,1] \times \mathbb{H} \to \mathbb{H}$, we have

$$F(t,u(t))(x) = \widehat{F}(t, z(t,x)),\ G(t,u(t))(x) = \widehat{G}(t, z(t,x)).$$

Then the system (4.4.1) – (4.4.3) has a abstract form of the system (4.2.2). Now, by the Theorem 4.3.2 we may conclude that if the functions F and G satisfy the assumptions $(A_1)-(A_4)$, then the system (4.4.1)–(4.4.3) has a unique mild solution.

Chapter 5

Mild Solutions For Multi-Term Time-Fractional Impulsive Differential Systems

5.1 Introduction

The phenomena involving the effects of instant forces may be formulated by instantaneous impulsive differential equations. However, if the forces are employed for a certain time interval, then the instantaneous impulsive differential equations fail to describe the phenomena. For example, pharmacotherapy [166], in which the hemodynamic equilibrium of a person is considered. The initiation of the drugs in the bloodstream and the resultant absorption for the body are gradual and continuous process. Therefore instantaneous impulses failed to describe such process. To characterize these type of situations Hern$\acute{a}$ndez and O′Regan [102] introduce a new case of impulsive actions, which triggered abruptly an arbitrary instant and their action remains for a finite time interval.

On the other hand, the multi-term time-fractional diffusion-wave equations were

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recently considered in [53] and [142] with constant and variable coefficients, respectively. Moreover, for multi-term time-fractional systems in [114; 135] the authors studied analytic and numerical solutions. Recently, Pardo at al. [7] studied the existence of mild solution to the system (4.1.2) with measure of noncompactness techniques. In the best of our knowledge multi-term time-fractional diffusion wave equations are not investigated with impulsive conditions so far.

Motivated by the above discussion, in this chapter, we investigate the existence and uniqueness results for mild solution to a multi-term time-fractional differential system involving not-instantaneous impulses and delay.

5.2 Problem Formulation

To facilitate the discussion due to delay term, we use the Banach space $\mathcal{PC}_0 := \mathcal{C}([-\tau, 0], \mathbb{X})$ formed by the continuous functions from $[-\tau, 0]$ to $\mathbb{X}$ equipped with the norm $\|y\|_{\mathcal{PC}_0} = \sup_{t\in[-\tau,0]}\{\|y(t)\| : y \in \mathcal{PC}_0\}$. To work with impulsive forces, we denote the Banach space $\mathcal{PC}_b = \mathcal{PC}([-\tau, b], \mathbb{X})$, $b < \infty$ with the norm $\|y\|_{\mathcal{PC}_b} = \sup_{t\in[-\tau,b]}\{\|y(t)\| : y \in \mathcal{PC}_b\}$, of all functions $y : [-\tau, b] \to \mathbb{X}$, which are continuous everywhere except the points $t_k \in (0, b), k = 1, 2, ..., m$, at which $y(t_k^+)$ and $y(t_k^-) = y(t_k)$ exist

In this chapter, we study the existence and uniqueness of mild solution to the following multi-term time-fractional differential system

$$\begin{cases} {}^cD_{s_k}^{1+\beta}y(t) + \displaystyle\sum_{j=1}^{n} \alpha_j {}^cD_{s_k}^{\gamma_j}y(t) \\ \qquad\qquad = Ay(t) + F\left(t, y_t, \int_0^t \mathfrak{K}(t,s)(y_s)ds\right), \quad t \in \cup_{k=0}^{m}(s_k, t_{k+1}], \\ y(t) = G_k(t, y_t), \quad y'(t) = H_k(t, y_t), \qquad t \in \cup_{k=1}^{m}(t_k, s_k], \\ y(t) + g_1(y) = \phi(t), \quad y'(t) + g_2(y) = \varphi(t), \quad t \in [-\tau, 0], \end{cases} \tag{5.2.1}$$

where $A : \mathcal{D}(A) \subset \mathbb{X} \to \mathbb{X}$ is a closed linear operator and $\alpha_j \geq 0$ for all $j = 1, 2, 3, ..., n$. ${}^cD_{s_k}^{\eta}$ stands for the Caputo derivative of order $\eta > 0$ and operational interval $J = [0, b] = \{0\} \cup_{k=0}^{m} (s_k, t_{k+1}] \cup_{k=1}^{m} (t_k, s_k]$ such that $0 = s_0 < t_1 \le s_1 \le t_2 < \cdots < t_m \le s_m \le t_{m+1} = b$ are prefix numbers. All γ_j, $j = 1, 2, 3...n$, are positive real numbers such that $0 < \beta \le \gamma_n \le \cdots \le \gamma_1 \le 1$. G_k and H_k are continuous functions from $\cup_{k=1}^{m} (t_k, s_k] \times \mathcal{PC}_0$ into $\mathbb{X}$ for all $k = 1, 2, ..., m$. $F : J \times \mathcal{PC}_0 \times \mathcal{PC}_0 \to \mathbb{X}$ is suitable function. The history function $y_t : [-\tau, 0] \to \mathbb{X}$ is the element of $\mathcal{PC}_0$ characterized by $y_t(\theta) = y(t + \theta), \theta \in [-\tau, 0]$; $\phi, \varphi \in \mathcal{PC}_0$ and y' is usual derivative of y with respect to t. $\mathfrak{K}$ is a positive and continuous operator on $\Omega := \{(t, s) \in \mathbb{R}^2 : 0 \le s \le t < b\}$ and $k^0 = \sup_{0 \le s \le t < b} \int_0^t \mathfrak{K}(t, s) ds < \infty$. Here by not-instantaneous, we mean that the impulses start abruptly at t_k and their effect will continue on the intervals $[t_k, s_k]$ for $k = 1, 2, 3, ..., m$.

To give appropriate representation of mild solution in terms of certain family of bounded and linear operators, we define the following family of operators.

Definition 5.2.1. *[7] Let A be a closed linear operator on a Banach space $\mathbb{X}$ with the domain $\mathcal{D}(A)$ and $\beta > 0, \gamma_j, \alpha_j$, $j = 1, 2, 3...n$ be the real positive numbers. Then A is called the generator of a $(\beta, \gamma_j)-$ resolvent family if there exists $\omega > 0$ and a strongly continuous function $\mathcal{S}_{\beta,\gamma_j} : \mathbb{R}^+ \to \mathcal{L}(\mathbb{X})$ such that $\{\lambda^{\beta+1} + \sum_{j=1}^{n} \alpha_j \lambda^{\gamma_j} : Re\,\lambda > \omega\} \subset \varrho(A)$ and*

$$\lambda^{\beta} \left(\lambda^{\beta+1} + \sum_{j=1}^{n} \alpha_j \lambda^{\gamma_j} - A \right)^{-1} y = \int_0^{\infty} e^{-\lambda t} \mathcal{S}_{\beta,\gamma_j}(t) y dt, \quad Re\,\lambda > \omega, y \in \mathbb{X}. \quad (5.2.2)$$

The following result guarantee for the existence of $(\beta, \gamma_j)-$ resolvent family under some suitable conditions.

Theorem 5.2.1. *[7] Let $0 < \beta \le \gamma_i \le, \cdots, \le \gamma_1 \le 1$ and $\alpha_j \ge 0$ be given and let A be a generator of a bounded and strongly continuous cosine family $\{C(t)\}_{t \in \mathbb{R}}$. Then A generates a bounded $(\beta, \gamma_j)-$ resolvent family $\{\mathcal{S}_{\beta,\gamma_j}(t)\}_{t \ge 0}$.*

Motivated by (4.2.6), we define a mild solution for the system (5.2.1).

Definition 5.2.2. *A function $y \in \mathcal{PC}_b$ is called a mild solution of the system* (5.2.1), *if $y(t) = \phi(t) - g_1(y), y'(t) = \varphi(t) - g_2(y)$ for $[-\tau, 0]$ and $y(t) = G_k(t, y_t), y'(t) =$*

$H_k(t, y_t)$ for $t \in \cup_{k=1}^{m}(t_k, s_k]$ and satisfies the following integral equations:

$$y(t) = \begin{cases} \mathcal{S}_{\beta,\gamma_j}(t)[\phi(0) - g_1(y)] + \int_0^t \mathcal{S}_{\beta,\gamma_j}(s)[\varphi(0) - g_2(y)]ds \\ + \sum_{j=1}^{n} \alpha_j \int_0^t \frac{(t-s)^{\beta-\gamma_j}}{\Gamma(1+\beta-\gamma_j)} \mathcal{S}_{\beta,\gamma_j}(s)[\phi(0) - g_1(y)]ds \\ + \int_0^t \mathcal{T}_{\beta,\gamma_j}(t-s)F(s, y_s, K(y_s))ds, \qquad t \in [0, t_1]; \\ \mathcal{S}_{\beta,\gamma_j}(t-s_k)G_k(s_k, y_{s_k}) + \int_{s_k}^t \mathcal{S}_{\beta,\gamma_j}(s-s_k)H_k(s_k, y_{s_k})ds \\ + \sum_{j=1}^{n} \alpha_j \int_{s_k}^t \frac{(t-s)^{\beta-\gamma_j}}{\Gamma(1+\beta-\gamma_j)} \mathcal{S}_{\beta,\gamma_j}(s-s_k)G_k(s_k, y_{s_k})ds \\ + \int_{s_k}^t \mathcal{T}_{\beta,\gamma_j}(t-s)F(s, y_s, K(y_s))ds, \qquad t \in \cup_{k=1}^{m}(s_k, t_{k+1}], \end{cases} \tag{5.2.3}$$

where $\mathcal{T}_{\beta,\gamma_j}(t) = (g_\beta * \mathcal{S}_{\beta,\gamma_j})(t)$ and $K(y_s) = \int_0^s \mathfrak{K}(s,\xi)(y_\xi)d\xi$.

5.3 Existence and Uniqueness Results

Throughout the section, we denote $S_0 = \sup_{t\in[0,b]} \|\mathcal{S}_{\beta,\gamma_j}(t)\|_{\mathcal{L}}$. In order to establish the existence and uniqueness result by Banach fixed point theorem for the system (5.2.1), we consider the following assumptions:

(A_1) There exist positive constants μ_F and μ_F^0 such that

$$\|F(t, \psi_1, \chi_1) - F(t, \psi_2, \chi_2)\| \le \mu_F \|\psi_1 - \psi_2\|_{\mathcal{PC}_0} + \mu_F^0 \|\chi_1 - \chi_2\|_{\mathcal{PC}_0},$$

where $\psi_i, \chi_i \in \mathcal{PC}_0$, $i = 1, 2$.

(A_2) There exist positive constants μ_G, μ_{g_i} and μ_H such that

$$\|G_k(t, \psi) - G_k(t, \chi)\| \le \mu_G \|\psi - \chi\|_{\mathcal{PC}_0}, \quad \|g_i(x) - g_i(y)\| \le \mu_{g_i}\|x - y\|,$$

$$\|H_k(t, \psi) - H_k(t, \chi)\| \le \mu_H \|\psi - \chi\|_{\mathcal{PC}_0},$$

for all $\psi, \chi \in \mathcal{PC}_0$, $x, y \in \mathbb{X}$, $i = 1, 2$ and $k = 1, 2, 3, \ldots, m$.

Theorem 5.3.1. *Assume that the assumptions $(A_1) - (A_2)$ are fulfilled, then the system (5.2.1) has a unique mild solution in J if $\Theta < 1$, where*

$$\Theta = \max\left[S_0 d + T_0 S_0 e + \sum_{j=1}^{n} \frac{\alpha_j S_0 d T_0^{1+\beta-\gamma_j}}{\Gamma(2+\beta-\gamma_j)} + \frac{S_0 T_0^{1+\beta} b(b+\tau)}{\Gamma(2+\beta)}[\mu_F + \mu_F^0 k^0], \mu_G\right],$$

where $d = \max\{\mu_{g_1}, \mu_G\}$, $e = \max\{\mu_{g_2}, \mu_H\}$ and $T_0 = \max_{0\le k\le m} |t_{k+1} - s_k|$.

Proof. To transform the problem into a fixed point problem, we define the operator $Q : \mathcal{PC}_b \to \mathcal{PC}_b$ by $Qy(t) = \phi(t)$ for $t \in [-\tau, 0]$, $Qy(t) = G_k(t, y_t)$ for all $t \in \cup_{k=1}^m (t_k, s_k]$, and

$$Qy(t) = \begin{cases} \mathcal{S}_{\beta,\gamma_j}(t)[\phi(0) - g_1(y)] + \int_0^t \mathcal{S}_{\beta,\gamma_j}(s)[\varphi(0) - g_2(y)]ds \\ +\sum_{j=1}^n \alpha_j \int_0^t \frac{(t-s)^{\beta-\gamma_j}}{\Gamma(1+\beta-\gamma_j)} \mathcal{S}_{\beta,\gamma_j}(s)[\phi(0) - g_1(y)]ds \\ + \int_0^t \mathcal{T}_{\beta,\gamma_j}(t-s)F(s, y_s, K(y_s))ds, \qquad t \in [0, t_1]; \\ \mathcal{S}_{\beta,\gamma_j}(t - s_k)G_k(s_k, y_{s_k}) \\ + \int_{s_k}^t \mathcal{S}_{\beta,\gamma_j}(s - s_k)H_k(s_k, y_{s_k})ds \\ +\sum_{j=1}^n \alpha_j \int_{s_k}^t \frac{(t-s)^{\beta-\gamma_j}}{\Gamma(1+\beta-\gamma_j)} \mathcal{S}_{\beta,\gamma_j}(s - s_k)G_k(s_k, y_{s_k})ds \\ + \int_{s_k}^t \mathcal{T}_{\beta,\gamma_j}(t-s)F(s, y_s, K(y_s))ds, \qquad t \in \cup_{k=1}^m (s_k, t_{k+1}]. \end{cases} \tag{5.3.1}$$

Let $x, y \in \mathcal{PC}_b$. For $t \in [0, t_1]$, we have

$$\begin{aligned} &\|Qx(t) - Qy(t)\| \\ &\le \|\mathcal{S}_{\beta,\gamma_j}(t)\|_{\mathcal{L}} \|g_1(x) - g_1(y)\| + \int_0^t \|\mathcal{S}_{\beta,\gamma_j}(s)\|_{\mathcal{L}} \|g_2(x) - g_2(y)\|ds \\ &\quad + \sum_{j=1}^n \alpha_j \int_0^t \frac{(t-s)^{\beta-\gamma_j}}{\Gamma(1+\beta-\gamma_j)} \|\mathcal{S}_{\beta,\gamma_j}(s)\|_{\mathcal{L}} \|g_1(x) - g_1(y)\|ds \\ &\quad + \int_0^t \|\mathcal{T}_{\beta,\gamma_j}(t-s)\|_{\mathcal{L}} \|F(s, x_s, K(x_s)) - F(s, y_s, K(y_s))\|ds \\ &\le \left[S_0\mu_{g_1} + T_0 S_0 \mu_{g_2} + \sum_{j=1}^n \frac{\alpha_j S_0 \mu_{g_1} T_0^{1+\beta-\gamma_j}}{\Gamma(2+\beta-\gamma_j)} + \frac{S_0 T_0^{1+\beta} b(b+\tau)}{\Gamma(2+\beta)} [\mu_F + \mu_F^0 k^0] \right] \|x - y\|_{\mathcal{PC}_b}, \end{aligned}$$

Since

$$\begin{aligned} \int_0^b \|y_s\| ds \le & \int_0^b \int_{-\tau}^0 \|y(s+r)\| dr ds \\ = & \int_0^b \int_{s-\tau}^s \|y(v)\| dv ds, \quad \text{where } s + r = v \\ \le & \int_0^b \int_{-\tau}^b \|y(v)\| dv ds \\ \le & b(b+\tau) \sup_{v \in [-\tau, b]} \|y(v)\| \\ \le & b(b+\tau) \|y\|_{\mathcal{PC}_b} \end{aligned}$$

For $t \in \cup_{k=1}^m (t_k, s_k]$, we get

$$\|Qx(t) - Qy(t)\| \le \|G_k(t, x_t) - G_k(t, y_t)\| \le \mu_G \|x - y\|_{\mathcal{PC}_b}, \quad k = 1, 2, 3, \ldots, m.$$

Similarly, for $t \in \cup_{k=1}^{m}(s_k, t_{k+1}]$ we get

$$
\begin{aligned}
&\|Qx(t) - Qy(t)\| \\
&\quad \leq \|\mathcal{S}_{\beta,\gamma_j}(t - s_k)\|_{\mathcal{L}} \|G_k(s_k, x_{s_k}) - G_k(s_k, y_{s_k})\| \\
&\qquad + \int_{s_k}^{t} \|\mathcal{S}_{\beta,\gamma_j}(s - s_k)\|_{\mathcal{L}} \|H_k(s_k, x_{s_k}) - H_k(s_k, y_{s_k})\| ds \\
&\qquad + \sum_{j=1}^{n} \alpha_j \int_{s_k}^{t} \frac{(t - s)^{\beta - \gamma_j}}{\Gamma(1 + \beta - \gamma_j)} \|\mathcal{S}_{\beta,\gamma_j}(s - s_k)\|_{\mathcal{L}} \|G_k(s_k, x_{s_k}) - G_k(s_k, y_{s_k})\| ds \\
&\qquad + \int_{s_k}^{t} \|\mathcal{T}_{\beta,\gamma_j}(t - s)\|_{\mathcal{L}} \|F(s, x_s, K(x_s)(s)) - F(s, y_s, K(y_s))\| ds \\
&\quad \leq \left[S_0 \mu_G + T_0 S_0 \mu_H + \sum_{j=1}^{n} \frac{\alpha_j S_0 \mu_G T_0^{1+\beta-\gamma_j}}{\Gamma(2 + \beta - \gamma_j)} + \frac{S_0 T_0^{1+\beta} b(b + \tau)}{\Gamma(2 + \beta)} [\mu_F + \mu_F^0 k^0] \right] \|x - y\|_{\mathcal{PC}_b}.
\end{aligned}
$$

Gathering the above results, we have $\|Qx - Qy\|_{\mathcal{PC}_b} \leq \Theta \|x - y\|_{\mathcal{PC}_b}$. Since $\Theta < 1$, so by the Banach contraction principle the system (5.2.1) has a unique mild solution. □

In order to establish the existence results by the virtue of condensing map, we consider the following assumptions:

(A_3) The functions G_k, H_k, g_1 and g_2 are continuous functions and F is compact and continuous, and there exist positive constants ν_F, ν_F^0, ν_G, ν_H, ν_{g_1}, ν_{g_2} such that

$$
\begin{aligned}
\|F(t, \psi, \chi)\| \leq \nu_F \|\psi\|_{\mathcal{PC}_0} + \nu_F^0 \|\chi\|_{\mathcal{PC}_0}, \quad \|g_i(x)\| \leq \nu_{g_i} \|x\|, \\
\|G_k(t, \psi)\| \leq \nu_G \|\psi\|_{\mathcal{PC}_0}, \quad \|H_k(t, \psi)\| \leq \nu_H \|\psi\|_{\mathcal{PC}_0},
\end{aligned}
$$

for all $x \in \mathbb{X}, \psi, \chi \in \mathcal{PC}_0$.

Theorem 5.3.2. *Assume that the assumptions* $(A_2) - (A_3)$ *are fulfilled, then the system* (5.2.1) *has a mild solution in* J *if* $\Delta < 1$, *where*

$$
\Delta = \max \left[S_0 d + T_0 S_0 e + \sum_{j=1}^{n} \frac{\alpha_j S_0 d T_0^{1+\beta-\gamma_j}}{\Gamma(2 + \beta - \gamma_j)}, \mu_G \right],
$$

where $d = \max\{\mu_{g_1}, \mu_G\}$, $e = \max\{\mu_{g_2}, \mu_H\}$.

Proof. Consider the operator $Q : \mathcal{PC}_b \to \mathcal{PC}_b$ defined in Theorem 5.3.1. We show that Q has a fixed point. It is easy to see that $Qy(t) \in \mathcal{PC}_b$. Let $\mathcal{B}_{r_0} := \{y \in \mathcal{PC}_b : \|y\|_{\mathcal{PC}_b} \leq r_0\}$, where

$$r_0 \geq \max\Bigg[S_0Y_1 + T_0S_0Z_1 + \sum_{j=1}^{n} \frac{\alpha_j S_0 Y_1 T_0^{1+\beta-\gamma_j}}{\Gamma(2+\beta-\gamma_j)},\ \nu_G r_0,\ S_0\nu_G r_0 + T_0 S_0 \nu_H r_0 + \sum_{j=1}^{n} \frac{\alpha_j S_0 \nu_G r_0 T_0^{1+\beta-\gamma_j}}{\Gamma(2+\beta-\gamma_j)}\Bigg] + \frac{S_0 T_0^{1+\beta} b(b+\tau)}{\Gamma(2+\beta)}[\nu_F + \nu_F^0 k^0] r_0, \tag{5.3.2}$$

where $Y_1 = \|\phi(0)\| + \nu_{g_1} r_0$, $Z_1 = \|\varphi(0)\| + \nu_{g_2} r_0$. It is clear that $\mathcal{B}_{r_0}$ is a closed, bounded and convex subset of $\mathcal{PC}_b$. Let $y \in \mathcal{B}_{r_0}$, then for $t \in [0, t_1]$, we have

$$\begin{aligned}
\|Qy(t)\| \leq& \|\mathcal{S}_{\beta,\gamma_j}(t)\|_{\mathcal{L}}(\|\phi(0)\| + \|g_1(y)\|) + \int_0^t \|\mathcal{S}_{\beta,\gamma_j}(s)\|_{\mathcal{L}}(\|\varphi(0)\| + \|g_2(y)\|)ds \\
&+ \sum_{j=1}^{n} \alpha_j \int_0^t \frac{(t-s)^{\beta-\gamma_j}}{\Gamma(1+\beta-\gamma_j)} \|\mathcal{S}_{\beta,\gamma_j}(s)\|_{\mathcal{L}}(\|\phi(0)\| + \|g(y)\|)ds \\
&+ \int_0^t \|\mathcal{T}_{\beta,\gamma_j}(t-s)\|_{\mathcal{L}} \|F(s, y_s, K(y_s))\|ds \\
\leq& S_0Y_1 + T_0S_0Z_1 + \sum_{j=1}^{n} \frac{\alpha_j S_0 Y_1 T_0^{1+\beta-\gamma_j}}{\Gamma(2+\beta-\gamma_j)} + \frac{S_0 T_0^{1+\beta} b(b+\tau)}{\Gamma(2+\beta)}[\nu_F + \nu_F^0 k^0] r_0.
\end{aligned}$$

For $t \in \cup_{k=1}^{m}(t_k, s_k]$, we get

$$\|Qy(t)\| \leq \|G_k(t, y_t)\| \leq \nu_G r_0, \quad k = 1, 2, 3, \ldots, m.$$

Similarly, for $t \in \cup_{k=1}^{m}(s_k, t_{k+1}]$, we get

$$\begin{aligned}
\|Qy(t)\| \leq& \|\mathcal{S}_{\beta,\gamma_j}(t-s_k)\|_{\mathcal{L}} \|G_k(s_k, y_{s_k})\| + \int_{s_k}^t \|\mathcal{S}_{\beta,\gamma_j}(s-s_k)\|_{\mathcal{L}} \|H_k(s_k, y_{s_k})\|ds \\
&+ \sum_{j=1}^{n} \alpha_j \int_{s_k}^t \frac{(t-s)^{\beta-\gamma_j}}{\Gamma(1+\beta-\gamma_j)} \|\mathcal{S}_{\beta,\gamma_j}(s-s_k)\|_{\mathcal{L}} \|G_k(s_k, y_{s_k})\|ds \\
&+ \int_{s_k}^t \|\mathcal{T}_{\beta,\gamma_j}(t-s)\|_{\mathcal{L}} \|F(s, y_s, K(y_s))\|ds \\
\leq& S_0\nu_G r_0 + T_0 S_0 \nu_H r_0 + \sum_{j=1}^{n} \frac{\alpha_j S_0 \nu_G r_0 T_0^{1+\beta-\gamma_j}}{\Gamma(2+\beta-\gamma_j)} + \frac{S_0 T_0^{1+\beta} b(b+\tau)}{\Gamma(2+\beta)}[\nu_F + \nu_F^0 k^0] r_0.
\end{aligned}$$

We conclude by (5.3.2) that $\|Qy\|_{\mathcal{PC}_b} \leq r_0$. Thus we conclude that $Q(\mathcal{B}_{r_0}) \subseteq \mathcal{B}_{r_0}$. Next, we show that Q is a condensing operator. Let us decompose Q by $Q = Q_1 + Q_2$, where $Q_1y(t) = G_k(t, y_t)$ for all $t \in \cup_{k=1}^{m}(t_k, s_k]$ and

$$Q_1y(t) = \begin{cases} \mathcal{S}_{\beta,\gamma_j}(t)[\phi(0) - g_1(y)] + \int_0^t \mathcal{S}_{\beta,\gamma_j}(s)[\varphi(0) - g_2(y)]ds \\ \quad + \sum_{j=1}^{n} \alpha_j \int_0^t \frac{(t-s)^{\beta-\gamma_j}}{\Gamma(1+\beta-\gamma_j)} \mathcal{S}_{\beta,\gamma_j}(s)[\phi(0) - g_1(y)]ds \qquad t \in [0, t_1]; \\ \mathcal{S}_{\beta,\gamma_j}(t - s_k)G_k(s_k, y_{s_k}) + \int_{s_k}^t \mathcal{S}_{\beta,\gamma_j}(s - s_k)H_k(s_k, y_{s_k})ds \\ \quad + \sum_{j=1}^{n} \alpha_j \int_{s_k}^t \frac{(t-s)^{\beta-\gamma_j}}{\Gamma(1+\beta-\gamma_j)} \mathcal{S}_{\beta,\gamma_j}(s - s_k)G_k(s_k, y_{s_k})ds, \quad t \in \cup_{k=1}^{m}(s_k, t_{k+1}] \end{cases} \tag{5.3.3}$$

and

$$Q_2y(t) = \begin{cases} \int_0^t \mathcal{T}_{\beta,\gamma_j}(t-s)F(s, y_s, K(y_s))ds, \qquad t \in [0, t_1]; \\ \int_{s_k}^t \mathcal{T}_{\beta,\gamma_j}(t-s)F(s, y_s, K(y_s))ds, \qquad t \in \cup_{k=1}^{m}(s_k, t_{k+1}]. \end{cases} \tag{5.3.4}$$

First, we show that Q_1 is continuous, so consider a sequence in $\mathcal{B}_{r_0}$ such that $y^n \to y \in \mathcal{B}_{r_0}$, then for $t \in [0, t_1]$, we get

$$\begin{aligned} \|Q_1y^n(t) - Q_1y(t)\| \leq & S_0\|g_1(y^n) - g_1(y)\| + S_0T_0\|g_2(y^n) - g_2(y)\| \\ & + \sum_{j=1}^{n} \frac{\alpha_j S_0 T_0^{1+\beta-\gamma_j}}{\Gamma(2+\beta-\gamma_j)} \|g_1(y^n) - g_1(y)\|. \end{aligned}$$

For $t \in \cup_{k=1}^{m}(s_k, t_{k+1}]$, we obtain

$$\begin{aligned} \|Q_1y^n(t) - Q_1y(t)\| \leq & S_0\|G_k(s_k, y^n_{s_k}) - G_k(s_k, y_{s_k})\| \\ & + S_0T_0\|H_k(s_k, y^n_{s_k}) - H_k(s_k, y_{s_k})\|_{\mathbb{X}} \\ & + \sum_{j=1}^{n} \frac{\alpha_j S_0 T_0^{1+\beta-\gamma_j}}{\Gamma(2+\beta-\gamma_j)} \|G_k(s_k, y^n_{s_k}) - G_k(s_k, y_{s_k})\|. \end{aligned}$$

By continuity of G_k, H_k, g_1 and g_2, we have $\|Q_1y^n - Q_1y\|_{\mathcal{PC}_b} \to 0$ as $n \to \infty$. Hence Q_1 is continuous. Let $x, y \in \mathcal{PC}_b$. As we have done in Theorem 5.3.1 for $t \in [0, t_1]$, we have

$$\|Q_1x(t) - Q_1y(t)\| \leq \left[S_0\mu_{g_1} + T_0S_0\mu_{g_2} + \sum_{j=1}^{n} \frac{\alpha_j S_0 \mu_{g_1} T_0^{1+\beta-\gamma_j}}{\Gamma(2+\beta-\gamma_j)} \right] \|x - y\|_{\mathcal{PC}_b}.$$

For $t \in \cup_{k=1}^{m}(t_k, s_k]$, we get

$$\|Q_1x(t) - Q_1y(t)\| \leq \|G_k(t, x_t) - G_k(t, y_t)\| \leq \mu_G\|x - y\|_{\mathcal{PC}_b}, \quad k = 1, 2, 3, \ldots, m,$$

and for $t \in \cup_{k=1}^{m}(s_k, t_{k+1}]$, obtain

$$\|Q_1x(t) - Q_1y(t)\| \leq \left[S_0\mu_G + T_0S_0\mu_H + \sum_{j=1}^{n} \frac{\alpha_j S_0 \mu_G T_0^{1+\beta-\gamma_j}}{\Gamma(2+\beta-\gamma_j)} \right] \|x - y\|_{\mathcal{PC}_b}.$$

Gathering the above results, we have $\|Q_1x - Q_1y\|_{\mathcal{PC}_b} \leq \Delta\|x-y\|_{\mathcal{PC}_b}$, which shows that Q_1 is a contraction mapping.

Next, we show that Q_2 is completely continuous. First, we verify that Q_2 is continuous, so we consider a sequence in $\mathcal{B}_{r_0}$ such that $y^n \to y \in \mathcal{B}_{r_0}$ as $n \to \infty$, then for $t \in [0, t_1]$, we get

$$\begin{aligned}\|Q_2y^n(t) - &Q_2y(t)\| \\ &\leq \int_0^t \|\mathcal{T}_{\beta,\gamma_j}(t-s)\|_{\mathcal{L}}\|F(s, y_s^n, K(y_s^n)) - F(s, y_s, K(y_s))\|ds,\end{aligned}$$

for $t \in \cup_{k=1}^m (s_k, t_{k+1}]$, we obtain

$$\begin{aligned}\|Q_2y^n(t) - &Q_2y(t)\| \\ &\leq \int_{s_k}^t \|\mathcal{T}_{\beta,\gamma_j}(t-s)\|_{\mathcal{L}}\|F(s, y_s^n, K(y_s^n)) - F(s, y_s, K(y_s))\|ds.\end{aligned}$$

Using continuity of F, we get $\|Q_2y^n - Q_2y\|_{\mathcal{PC}_b} \to 0$ as $n \to \infty$. Hence Q_2 is continuous.

Further, we show that Q_2 is a family of equi-continuous functions. Let $l_2, l_1 \in [0, t_1]$ such that $0 \leq l_1 < l_2 \leq t_1$, we have

$$\begin{aligned}\|Q_2y(l_2) - &Q_2y(l_1)\| \\ \leq &\int_0^{l_1} \|\mathcal{T}_{\beta,\gamma_j}(l_2-s) - \mathcal{T}_{\beta,\gamma_j}(l_1-s)\|_{\mathcal{L}}\|F(s, y_s, K(y_s))\|ds \\ &+ \int_{l_1}^{l_2} \|\mathcal{T}_{\beta,\gamma_j}(l_2-s)\|_{\mathcal{L}}\|F(s, y_s, K(y_s))\|ds \\ \leq &S_0\left[\int_0^{l_1}\left(\frac{(l_2-s)^\beta}{\Gamma(1+\beta)} - \frac{(l_1-s)^\beta}{\Gamma(1+\beta)}\right)ds + \frac{(l_2-l_1)^{1+\beta}}{\Gamma(2+\beta)}\right][\nu_F + \nu_F^0 k^0]r_0 \\ \leq &\frac{S_0}{\Gamma(2+\beta)}\left[\left|(l_2^{1+\beta} - l_1^{1+\beta}) - (l_2-l_1)^{1+\beta}\right| + \frac{(l_2-l_1)^{1+\beta}}{\Gamma(2+\beta)}\right][\nu_F + \nu_F^0 k^0]r_0.\end{aligned}$$

For $l_2, l_1 \in \cup_{k=1}^m (s_k, t_{k+1}]$ such that $s_k \leq l_1 < l_2 \leq t_{k+1}$, we have

$$\begin{aligned}&\|Q_2y(l_2) - Q_2y(l_1)\| \\ &\leq \int_{s_k}^{l_1} \|\mathcal{T}_{\beta,\gamma_j}(l_2-s) - \mathcal{T}_{\beta,\gamma_j}(l_1-s)\|_{\mathcal{L}}\|F(s, y_s, K(y_s))\|ds \\ &\quad+ \int_{l_1}^{l_2} \|\mathcal{T}_{\beta,\gamma_j}(l_2-s)\|_{\mathcal{L}}\|F(s, y_s, K(y_s))\|ds \\ &\leq S_0\left[\int_{s_k}^{l_1}\left(\frac{(l_2-s)^\beta}{\Gamma(1+\beta)} - \frac{(l_1-s)^\beta}{\Gamma(1+\beta)}\right)ds + \frac{(l_2-l_1)^{1+\beta}}{\Gamma(2+\beta)}\right][\nu_F + \nu_F^0 k^0]r_0 \\ &\leq \frac{S_0}{\Gamma(2+\beta)}\left[\left|((l_2-s_k)^{1+\beta} - (l_1-s_k)^{1+\beta}) - (l_2-l_1)^{1+\beta}\right| + \frac{(l_2-l_1)^{1+\beta}}{\Gamma(2+\beta)}\right][\nu_F + \nu_F^0 k^0]r_0,\end{aligned}$$

from aforemention inequalities we conclude that $\|Q_2y(l_2) - Q_2y(l_1)\|_{\mathcal{PC}_b} \to 0$ as $l_2 \to l_1$ for $t \in [0, b]$. This shows that Q_2 is a family of equi-continuous functions.

Finally, we will show that $\mathbb{Y} = \{Q_2y(t) : y \in \mathbb{B}_{r_0}\}$ is precompact in $\mathcal{B}_{r_0}$. Let $t > 0$ be fixed and let $y^n \in \mathbb{B}_{r_0}$, $\{y^n\}$ be a bounded sequence in $\mathcal{PC}_b$. Then $\mathbb{Y} = \{Q_2y^n(t) : y^n \in \mathbb{B}_{r_0}\}$ is bounded sequence in $\mathbb{B}_{r_0}$. Hence, for any $t^* \in \cup_{k=0}^{m}(s_k, t_{k+1}]$, the sequence $\{y^n(t^*)\}$ is bounded in $\mathbb{B}_{r_0}$. Since F is compact, it has a convergent subsequence such that

$$F(t^*, y^n_{t^*}, K(y^n_{t^*})) \to F(t^*, y_{t^*}, K(y_{t^*})),$$

or

$$\|F(t^*, y^n_{t^*}, K(y^n_{t^*})) - F(t^*, y_{t^*}, K(y_{t^*}))\| \to 0 \text{ as } n \to \infty.$$

Using the bounded convergence theorem, we can conclude that

$$(Q_2y^n)(t) \to (Q_2y)(t), \text{ in } \mathbb{B}_{r_0}.$$

This proves that Q_2 is a compact operator. Therefore Q_1 is a continuous and contraction operator and Q_2 is a completely continuous operator, hence $Q = Q_1+Q_2$ is a condensing map on $\mathcal{B}_{r_0}$. Finally, by Theorem 2.6.4, we infer that there exists a mild solution of the system (5.2.1) in $\mathcal{B}_{r_0}$. □

5.4 Example

In this section, we provide an example to illustrate the feasibility of the established results. Set $\mathbb{X} = L^2(\mathbb{R}^n)$, then $\mathcal{PC}_0 := \mathcal{C}([-\tau, 0], L^2(\mathbb{R}^n))$. Let $\beta, \gamma_j > 0$ for $j = 1, 2, 3, \ldots, n$ be given, satisfying $0 < \beta \le \gamma_n \le \cdots \le \gamma_1 \le 1$ and $\tau \in \mathbb{R}$ such that $\tau > 0$. We consider the following system

$$\begin{aligned} \partial_t^{1+\beta}u(t,x) + \sum_{j=1}^{n} \alpha_j \partial_t^{\gamma_j} u(t,x) =& \Delta u(t,x) + \frac{u_t(\theta,x)}{50} \\ &+ \int_{-\tau}^{t} \cos(t-\xi)\frac{u_t(\theta,x)}{25} d\xi, \end{aligned} \tag{5.4.1}$$

for all $(t, x) \in \cup_{k=0}^{m}(s_k, t_{k+1}] \times [0, 1]$,

$$\begin{aligned} G_k(t, u_t(\theta,x)) &= \int_{-\tau}^{t} \frac{\sin(t-\xi)}{(k+1)} \frac{u_t(\theta,x)}{25} d\xi, \\ H_k(t, u_t(\theta,x)) &= \int_{-\tau}^{t} \frac{\cos(t-\xi)}{(k+1)} \frac{u_t(\theta,x)}{25} d\xi, \qquad t \in \cup_{k=1}^{m}(t_k, s_k], \end{aligned} \tag{5.4.2}$$

$$u(\theta,x) + \sum_{r=1}^{q} a_r y(t_r) = \phi(\theta,x), \quad u'(\theta,x) + \sum_{r=1}^{q} b_r y(t_r) = \varphi(\theta,x), \tag{5.4.3}$$

where $a_r, b_r \in \mathbb{R}, \theta \in [-\tau, 0]$. The points $0 = t_0 = s_0 < t_1 \leq s_1 \leq t_2 < \cdots < t_m \leq s_m \leq t_{m+1} = 1$ are prefix numbers, $\partial_t^{1+\beta}$ denotes the Caputo derivative of order $(1+\beta)$ and Δ is the Laplacian with a maximal domain $\{v \in \mathbb{X} : v \in H^2(\mathbb{R}^n)\}$. The history function $u_t(\theta, x) : [-\tau, 0] \to \mathbb{X}$ is the element of $\mathcal{PC}_0$ characterized by $u_t(\theta, x) = u(t+\theta, x), \theta \in [-\tau, 0]$. Setting $y(t)(x) = u(t, x)$, $g_1(x) = \sum_{r=1}^{p} a_r x(t_r)$, $g_2(x) = \sum_{r=1}^{p} b_r x(t_r)$, $\phi(\theta)(x) = \phi(\theta, x)$, $(\theta, x) \in [-\tau, 0] \times [0, 1]$. Now, we have $F(t, \psi, K(\psi)) = \frac{\psi}{50} + \int_{-\tau}^{t} \cos(t-\xi)\frac{\psi}{5^2} d\xi$, $G_k(t, \psi) = \int_{-\tau}^{t} \frac{\sin(t-\xi)}{(k+1)} \frac{\psi}{25} d\xi$, $H_k(t, \psi) = \int_{-\tau}^{t} \frac{\cos(t-\xi)}{(k+1)} \frac{\psi}{25} d\xi$. Now, we observe that the system (5.4.1) – (5.4.3) has a the abstract form of the system (5.2.1). Moreover, for $t \in [0, 1], \psi_i, \chi_i \in \mathcal{PC}_0, i = 1, 2$ and $x, y \in \mathbb{X}$, we have

$$\|F(t, \psi_1, K(\chi_1)) - F(t, \psi_2, K(\chi_2))\| \leq \frac{1}{50}\|\psi_1 - \psi_2\| + \frac{1}{25}\|\chi_1 - \chi_2\|,$$
$$\|G_k(t, \chi_1) - G_k(t, \chi_2)\| \leq \frac{2}{25}\|\chi_1 - \chi_2\|; \; \|H_k(t, \chi_1) - H_k(t, \chi_2)\| \leq \frac{1}{25}\|\chi_1 - \chi_2\|,$$
$$\|g_1(x) - g_1(y)\| \leq qa\|x - y\|; \; \|g_2(x) - g_2(y)\| \leq qb\|x - y\|,$$

where $a = \max_{1 \leq r \leq q} |a_r|$ and $b = \max_{1 \leq r \leq q} |b_r|$. Thus the assumptions (A_1) and (A_2) are satisfied. "On the other hand, it follows from the theory of cosine families that Δ generates a bounded cosine function $\{C(t)\}_{t \geq 0}$ on $L^2(\mathbb{R}^n)$. Moreover, by Theorem 5.2.1 the operator Δ in equation (5.4.1) generates a bounded $\{\mathcal{S}_{\beta,\gamma_j}(t)\}_{t\geq 0}$-resolvent family". Let $S_0 = \sup_{t\in[0,1]} \|\mathcal{S}_{\beta,\gamma_j}(t)\|_{\mathcal{L}}$. Now, by Theorem 5.3.1 if

$$\max\left[S_0 d + S_0 e + \sum_{j=1}^{n} \frac{\alpha_j S_0 d}{\Gamma(2+\beta-\gamma_j)} + \frac{3S_0}{50\Gamma(2+\beta)}, \frac{1}{25}\right] < 1,$$

where $d = \max\{qa, \frac{2}{25}\}$, $e = \max\{qb, \frac{2}{25}\}$, then the system (5.4.1) – (5.4.3) admits a unique mild solution.

Chapter 6

Exact Controllability of Fractional Impulsive Quasilinear Differential Systems with State Dependent Delay

6.1 Introduction

Balachandran et al. [19] studied the exact controllability results for the following first order quasilinear integrodifferential evolution system with the Lipschitz continuity of nonlinear functions

$$\begin{cases} \frac{dy(t)}{dt} + A(t, y(t))y(t) = Bu(t) + f\left(t, y(t), \int_0^t g(t, s, y(s)ds\right), & t \in (0, b]; \\ y(0) = y_0, \end{cases} \tag{6.1.1}$$

where the domain $\mathcal{D}(A)$ dense in $\mathbb{X}$, B is a bounded linear operator and $u(\cdot)$ is a control function. The nonlinear functions f and g are given and Lipschitz continuous.

Debbouche and Baleanu [59] investigated the exact controllability results for the

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following fractional order quasilinear impulsive evolution system with the Lipschitz continuity of nonlinear functions

$$\begin{cases} {}^cD^{\alpha}y(t) + A(t, y(t))y(t) = Bu(t) \\ \qquad\qquad +f\Big(t, g(t, y(\beta(t))), \int_0^t g(t, s, y(\gamma(s))ds\Big), \\ \Delta y_{|t=t_k} = y(t_k^+) - y(t_k^-) = I_k(y(t_k)),\ k = 1, 2, 3, \ldots, m, \\ y(t) + h(y) = y_0, \end{cases} \tag{6.1.2}$$

where $t \in [0, b]$, $-A(t, \cdot) : \mathcal{D}(A) \subseteq \mathbb{X} \to \mathbb{X}$ is a closed linear operator such that the domain $\mathcal{D}(A)$ is dense in $\mathbb{X}$ and independent of t. The given nonlinear functions f, g, β, γ and $I_k, k = 1, 2, 3, .., m$ are Lipschitz continuous.

Motivated by [19; 59; 99], in this chapter, we study the controllability results for the following abstract fractional impulsive quasilinear differential system with state-dependent delay

$$\begin{cases} {}^cD^{\alpha}_{0+}y(t) + A(t, y(t))y(t) = Bu(t) + \displaystyle\int_0^t H(t, s, y_{\rho(s,y_s)})ds \\ \qquad\qquad +F\Big(t, y_{\rho(t,y_t)}, \displaystyle\int_0^t K(t, s)y_{\rho(t,y_t)}ds\Big), t \in (0, b], t \neq t_k, \\ \Delta y_{|t=t_k} = y(t_k^+) - y(t_k^-) = G_k(y(t_k)), \qquad k = 1, 2, 3, \ldots, m, \\ y(t) = \varphi(t), \quad t \in (-\infty, 0], \quad \varphi \in \mathcal{B}_h, \end{cases} \tag{6.1.3}$$

where $0 = t_0 < t_1 < t_2 < \ldots < t_k < t_m < t_{m+1} = b$ are prefixed points. “The state $y(\cdot)$ takes values in $\mathbb{X}$ and the control function $u(\cdot)$ belongs to the space $L^2([0, b], \mathbb{U})$, a Banach space of admissible control functions with $\mathbb{U}$ as a Banach space, and $B : \mathbb{U} \to \mathbb{X}$ is a bounded linear operator. We assume that $-A(t, \cdot) : \mathcal{D}(A) \subset \mathbb{X} \to \mathbb{X}$ is a densely defined, closed linear operator such that the domain $\mathcal{D}(A)$ is independent of t and $-A(t, \cdot)$ generates an evolution family in the Banach space $\mathbb{X}$”. The history function $y_t : (-\infty, 0] \to \mathbb{X}$ characterized by $y_t(\theta) = y(t + \theta), \theta \in (-\infty, 0]$ is an element of a phase space $\mathcal{B}_h$ defined axiomatically. The functions $F : [0, b] \times \mathcal{B}_h \times \mathcal{B}_h \to \mathbb{X}$, $H : \Omega \times \mathcal{B}_h \to \mathbb{X}$, and $\rho : [0, b] \times \mathcal{B}_h \to (-\infty, b]$ are appropriate functions satisfying some assumptions. The operator $K \in \mathcal{C}(\Omega, \mathbb{R}^+)$ is a positive and continuous operator from $\Omega := \{(t, s) \in \mathbb{R}^2 : 0 \leq s \leq t < b\}$ to $\mathbb{R}^+$ such that $k^0 = \sup_{t,s\in[0,b]} \int_0^t K(t,s)ds < \infty$ and ${}^cD^{\alpha}_{0+}$ stands for the Caputo fractional derivative of order $\alpha \in (0,1)$.

6.2 Preliminaries

In case of infinite delay, we need to use the theoretical phase space $\mathcal{B}_h$ in a beneficial way i.e. fulfilling the elementary axioms given in [98]. In this chapter, we use the phase spaces $\mathcal{B}_h, \mathcal{B}'_h$ which are same as defined in [52]. So, we omit the details here.

If $y : (-\infty, b] \to \mathbb{X}, b > 0$, such that $y_0 \in \mathcal{B}_h$ and $y|_{[0,b]} \in \mathcal{PC}([0,b],\mathbb{X})$ and the following conditions hold:

(i) y_t is in $\mathcal{B}_h$.

(ii) $\|y(t)\| \le N\|y_t\|_{\mathcal{B}_h}$, where N is a constant.

(iii) $\|y_t\|_{\mathcal{B}_h} \le \xi_1(t)\sup\{\|y(s)\| : 0 \le s \le t\} + \xi_2(t)\|y_0\|_{\mathcal{B}_h}$, where $\xi_1 : [0,\infty) \to [0,\infty)$ is continuous, $\xi_2 : [0,\infty) \to [0,\infty)$ is locally bounded, and ξ_1, ξ_2 are independent of y.

(iv) The function $t \to \varphi_t$ is well defined and continuous from the set $\mathcal{R}(\rho^-) := \{\rho(s,\psi) : (s,\psi) \in [0,b] \times \mathcal{B}_h, \rho(s,\psi) \le 0\}$ into $\mathcal{B}_h$ and there exists a bounded and continuous function $h_\varphi : \mathcal{R}(\rho^-) \to (0,\infty)$ such that $\|y_t\|_{\mathcal{B}_h} \le h_\varphi(t)\|y\|_{\mathcal{B}_h}$.

Lemma 6.2.1. *[104] Let $y : (-\infty, b] \to \mathbb{X}$ be a function such that $y_0 = \varphi, y|_{[0,b]} \in \mathcal{PC}([0,b],\mathbb{X})$ and if (iv) satisfies, then*

$$\|y_s\|_{\mathcal{B}_h} \le (\xi_2^0 + h_\varphi^0)\|\varphi\|_{\mathcal{B}_h} + \xi_1^0 \sup\{\|y(\theta)\| : \theta \in [0, \max\{0,s\}]\}, \quad s \in \mathcal{R}(\rho^-) \cup [0,b],$$

where $h_\varphi^0 = \sup_{t\in\mathcal{R}(\rho^-)} h_\varphi(t)$, $\xi_1^0 = \sup_{s\in[0,b]} \xi_1(s)$ and $\xi_2^0 = \sup_{s\in[0,b]} \xi_2(s)$.

To give appropriate representation of mild solution in terms of certain family of bounded and linear operators, we define the following families of operators.

Definition 6.2.1. *A two parameter family of bounded linear operators $R(t,s)$, $0 \le s \le t \le b$ is said to be an evolution family if the following conditions hold:*

(i) $R(t,t) = I, \quad R(t,r)R(r,s) = R(t,s)$, *for* $0 \leq s \leq r \leq t \leq b$,

(ii) $(t,s) \to R(t,s)$ *is strongly continuous for* $0 \leq s \leq t \leq b$.

For more details on evolution family and quasilinear system of evolution see Chapter 5 and Section 6.4 in [163], respectively.

Definition 6.2.2. *[59] Let* $A(t,y)$ *be a closed linear operator on a Banach space* $\mathbb{X}$ *with domain* $\mathcal{D}(A)$ *and* $\alpha > 0$. *Then* $A(t,y)$ *is called the generator of an* $(\alpha, y)-$ *resolvent family if there exists* $\omega \geq 0$ *and a strongly continuous function* $\mathcal{S}_{(\alpha,y)} : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathcal{L}(\mathbb{X})$ *such that* $\{\lambda^\alpha : Re\,\lambda > \omega\} \subset \varrho(A)$, *for* $0 \leq s \leq t < \infty$, *and*

$$(\lambda^\alpha - A(t,y))^{-1}x = \int_0^\infty e^{-\lambda(t-s)}\mathcal{S}_{(\alpha,y)}(t,s)x\,dt, \quad Re\,\lambda > \omega, x, y \in \mathbb{X}. \tag{6.2.1}$$

Motivated by [19; 58; 59; 196], we have the following definition of mild solution for the system (6.1.3).

Definition 6.2.3. *Let* $A(t,y)$ *be a generator of a bounded* $(\alpha, y)-$ *resolvent family* $\{\mathcal{S}_{(\alpha,y)}(t,s)\}_{t\geq 0}$. *Then a function* $y : (-\infty, b] \to \mathbb{X}$ *is called a mild solution of the system* (6.1.3) *if* $\Delta y_{|t=t_k} = G_k(y(t_k))$ *and satisfies the integral equation*

$$y(t) = \begin{cases} \varphi(t), & t \in (-\infty, 0]; \\ \mathcal{S}_{(\alpha,y)}(t,0)\varphi(0) + \sum\limits_{0<t_k<t} \mathcal{S}_{(\alpha,y)}(t,t_k)G_k(y(t_k)) \\ + \int_0^t \mathcal{S}_{(\alpha,y)}(t,s)\int_0^s H(s,\tau,y_{\rho(\tau,y_\tau)})d\tau ds + \int_0^t \mathcal{S}_{(\alpha,y)}(t,s) \\ \times \left[Bu(s) + F\left(s, y_{\rho(s,y_s)}, \int_0^s K(s,\tau)y_{\rho(\tau,y_\tau)}d\tau\right)\right]ds & t \in [0,b]. \end{cases} \tag{6.2.2}$$

Lemma 6.2.2. *[117] Let* $(\Theta g)(t) := \int_0^t \mathcal{S}_{(\alpha,y)}(t,s)g(s)ds$ *and the sequence* $\{g_n\}_{n=1}^\infty \subset L^1([0,b],\mathbb{X})$ *be semicompact. Then the following conditions hold:*

(i) *The set* $\{\Theta g_n\}_{n=1}^\infty$ *is relatively compact in* $\mathcal{C}([0,b],\mathbb{X})$.

(ii) *Allow* $g_n \rightharpoonup g_0$, *then* $(\Theta g_n)(t) \to (\Theta g_0)(t)$, *as* $n \to \infty$, *for all* $t \in [0,b]$.

Definition 6.2.4. *[117] A countable set of functions* $\{g_n\}_{n=1}^\infty \subset L^1([0,b],\mathbb{X})$ *is called semicompact if there exists a function* $\gamma \in L^1([0,b],\mathbb{R}^+)$ *satisfying* $\sup\limits_{n\geq 1}\|g_n(t)\| \leq \gamma(t)$, *for a.e.* $t \in [0,b]$, *and the sequence* $\{g_n(t)\}_{n=1}^\infty$ *is relatively compact in* $\mathbb{X}$ *for a.e.* $t \in [0,b]$.

Definition 6.2.5. *The system* (6.1.3) *is said to be exact controllable on the time interval* $[0,b]$ *if for any arbitrary final state* $y_b \in \mathbb{X}$, *there exists a control function* $u \in L^2([0,b],\mathbb{U})$ *in a way that the mild solution* $y(t)$ *of system* (6.1.3) *corresponding to the control* u *satisfies* $y(0) = y_0$, *and* $y(b) = y_b$.

6.3 Exact Controllability Results

Let us denote $S_0 = \sup_{0\le s\le t\le b}\{\|\mathcal{S}_{(\alpha,y)}(t,s)\|_{\mathcal{L}} : y$ belongs to a bounded subset of $\mathbb{X}\}$. In order to obtain the controllability results for the system (6.1.3), we consider the following assumptions:

(A_1) The function $F : [0,b]\times\mathcal{B}_h\times\mathcal{B}_h \to \mathbb{X}$ satisfies:

(i) $F(t,\cdot,\cdot)$ is continuous for $t\in[0,b]$ a.e., and $F(\cdot,\varphi,\psi)$ is strongly measurable for $\varphi,\psi\in\mathcal{B}_h$.

(ii) There exists a function $m_F\in L^1([0,b],\mathbb{R}^+)$, and a nondecreasing continuous function $L_F:\mathbb{R}^+\to\mathbb{R}^+$ such that

$$\|F(t,\phi,\psi)\| \le m_F(t)L_F(\|\phi\|_{\mathcal{B}_h}+\|\psi\|_{\mathcal{B}_h}),$$

and $\lim_{r\to\infty}\frac{L_F(cr)}{r}=\Lambda<\infty$, for $c>0$ and $(t,\phi,\psi)\in[0,b]\times\mathcal{B}_h\times\mathcal{B}_h$.

(iii) There exists a function $J_F\in L^1([0,b],\mathbb{R}^+)$ for any bounded subsets $\mathbb{B}_1,\mathbb{B}_2\in\mathcal{B}_h$ such that

$$\mu(F(t,\mathbb{B}_1,\mathbb{B}_2)) \le J_F(t)[\sup_{-\infty<\theta\le 0}\mu(\mathbb{B}_1(\theta))+\sup_{-\infty<\theta\le 0}\mu(\mathbb{B}_2(\theta))].$$

(A_2) The function $H:\Omega\times\mathcal{B}_h\to\mathbb{X}$ satisfies:

(i) $H(t,s,\cdot)$ is continuous for $(t,s)\in\Omega$ a.e., and $H(\cdot,\cdot,\psi)$ is strongly measurable for $\psi\in\mathcal{B}_h$.

(ii) There exists a function $m_H\in L^1(\Omega,\mathbb{R}^+)$ and a nondecreasing continuous function $L_H:\mathbb{R}^+\to\mathbb{R}^+$ such that

$$\|H(t,s,\psi)\|\le m_H(t,s)L_H(\|\psi\|_{\mathcal{B}_h}),\quad (t,s,\psi)\in\Omega\times\mathcal{B}_h,$$

and $\lim_{r\to\infty}\frac{L_H(cr)}{r}=\gamma<\infty$, for $c>0$ and $\sigma=\sup_{t,s\in[0,b]}\int_0^t\int_0^s m_H(s,\tau)d\tau ds.$

(iii) There exists a function $J_H \in L^1(\Omega, \mathbb{R}^+)$ for every bounded subsets $\mathbb{B}_3 \in \mathcal{B}_h$ such that

$$\mu(H(t,s,\mathbb{B}_3)) \leq J_H(t,s)[\sup_{-\infty<\theta\leq 0} \mu(\mathbb{B}_3(\theta))].$$

For convenience let $\varsigma = \sup_{t,s\in[0,b]} \int_0^t \int_0^s J_H(s,\tau)d\tau ds$.

(A_3) The functions $G_k : \mathbb{X} \to \mathbb{X}$, for $k = 1, 2, 3, \ldots, m$, are continuous and satisfies:

(i) There exist non decreasing functions $L_{G_k} : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\|G_k(z)\| \leq L_{G_k}(\|z\|), \quad z \in \mathbb{X} \quad \text{and} \quad \lim_{r\to\infty} \frac{L_{G_k}(cr)}{r} = \lambda_k < \infty,$$

for $c > 0$ and $k = 1, 2, \ldots, m$ and.

(ii) For every bounded subset $\mathbb{B}_4 \subset \mathbb{X}$, there exists constants $J_{G_k} > 0$ such that

$$\mu(G_k(\mathbb{B}_4)) \leq J_{G_k}\mu(\mathbb{B}_4), \quad \text{for all } k = 1, 2, \ldots, m.$$

(A_4) The bounded linear operator $W : L^2([0,b], \mathbb{U}) \to \mathbb{X}$ characterized by $Wu = \int_0^T \mathcal{S}_{(\alpha,y)}(T,s)Bu(s)ds$ (for construction of W and W^{-1}, see [171]) satisfies:

(i) W has an induced inverse operator W^{-1} that takes values in $L^2([0,b], \mathbb{U})/KerW$, and there exist constants $b^0, w^0 > 0$ such that $\|B\| \leq b^0$ and $\|W^{-1}\| \leq w^0$.

(ii) There exists a function $J_W \in L^1([0,b], \mathbb{R}^+)$ such that, for every bounded subset $\mathbb{B}_5 \in \mathbb{X}$ we have

$$\mu(W^{-1}(\mathbb{B}_5)(t)) \leq J_W(t)\mu(\mathbb{B}_5).$$

Theorem 6.3.1. *Assume that the* $(\alpha, y)-$*resolvent family* $\mathcal{S}_{(\alpha,y)}(t,s)$ *generated by* $A(t,\cdot)$ *is equicontinuous and the assumptions* $(A_1) - (A_4)$ *are satisfied, then the system* (6.1.3) *is controllable on* $[0,b]$ *if* $\max(\Delta_1, \Delta_2) < 1$*, where*

$$\Delta_1 = \left(S_0 + S_0^2 b^0 b^{\frac{1}{2}} w^0\right)\left[\sum_{k=1}^m \lambda_k + \gamma\sigma + \|m_F\|_{L^1}\Lambda\right], \tag{6.3.1}$$

$$\Delta_2 = \left(S_0 + 2S_0^2 b^0 \|J_W\|_{L^1}\right)\left[\sum_{k=1}^m J_{G_k} + 4\varsigma + 2\|J_F\|_{L^1}(1 + k^0)\right]. \tag{6.3.2}$$

Proof. Using the assumption $(A_2)(i)$, we define the control $u^y(\cdot)$ for a arbitrary function $y \in \mathcal{PC}([0,b],\mathbb{X})$ by

$$\begin{aligned} u^y(t) =& W^{-1}\Bigg[y_b - \mathcal{S}_{(\alpha,y)}(b,0)\varphi(0) - \int_0^b \mathcal{S}_{(\alpha,y)}(b,s)\int_0^s H(s,\tau,y_{\rho(\tau,y_\tau)})d\tau ds \\ &- \int_0^b \mathcal{S}_{(\alpha,y)}(b,s)F\bigg(s, y_{\rho(s,y_s)}, \int_0^s K(s,\tau)y_{\rho(\tau,y_\tau)}d\tau\bigg)ds \\ &- \sum_{k=1}^m \mathcal{S}_{(\alpha,y)}(b,t_k)G_k(y(t_k))\Bigg](t). \end{aligned}$$

By applying the above control, we show that the operator $\Phi : \mathcal{B}'_h \to \mathcal{B}'_h$ defined by $\Phi y(t) = \varphi(t)$, $t \in (-\infty, 0]$ and

$$\begin{aligned} \Phi y(t) =& \mathcal{S}_{(\alpha,y)}(t,0)\varphi(0) + \sum_{0<t_k<t} \mathcal{S}_{(\alpha,y)}(t,t_k)G_k(y(t_k)) \\ &+ \int_0^t \mathcal{S}_{(\alpha,y)}(t,s)\int_0^s H(s,\tau,y_{\rho(\tau,y_\tau)})d\tau ds \\ &+ \int_0^t \mathcal{S}_{(\alpha,y)}(t,s)\bigg[Bu^y(s) + F\bigg(s, y_{\rho(s,y_s)}, \int_0^s K(s,\tau)y_{\rho(\tau,y_\tau)}d\tau\bigg)\bigg]ds, \end{aligned}$$

has a fixed point. Moreover, we obtain $\Phi y(b) = y_b$, which implies that the control $u^y(t)$ steers the system (6.1.3) from the initial state y_0 to the arbitrary final state y_b in the time interval $[0,b]$, provided that the nonlinear operator Φ admits a fixed point.
Let us define a function $x(\cdot) : (-\infty, b] \to \mathbb{X}$ by

$$x(t) = \begin{cases} \varphi(t), & t \in (-\infty, 0]; \\ \mathcal{S}_{(\alpha,y)}(t,0)\varphi(0), & t \in [0,b], \end{cases}$$

then $x_0 = \varphi$. For a function $z \in \mathcal{PC}([0,b],\mathbb{X})$ such that $z(0) = 0$, we define the function $\overline{z}$ by

$$\overline{z}(t) = \begin{cases} 0, & t \in (-\infty, 0]; \\ z(t), & t \in [0,b]. \end{cases}$$

If $y(\cdot)$ fulfills (6.2.2), we are able to split $y(\cdot)$ as $y(t) = x(t) + z(t)$, $t \in [0,b]$ which suggests that $y_t = x_t + z_t$, and $z(.)$ satisfies

$$\begin{aligned} z(t) =& \sum_{0<t_k<t} \mathcal{S}_{(\alpha,y)}(t,t_k)G_k(z(t_k) + x(t_k)) \\ &+ \int_0^t \mathcal{S}_{(\alpha,y)}(t,s)\int_0^s H(s,\tau,z_{\rho(\tau,z_\tau+x_\tau)} + x_{\rho(\tau,z_\tau+x_\tau)})d\tau ds + \int_0^t \mathcal{S}_{(\alpha,y)}(t,s)\bigg[Bu^{z+x}(s) \\ &+ F\bigg(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)}, \int_0^s K(s,\tau)(z_{\rho(\tau,z_\tau+x_\tau)} + x_{\rho(\tau,z_\tau+x_\tau)})d\tau\bigg)\bigg]ds, \end{aligned}$$

where

$$u^{z+x}(t) = W^{-1}\Big[y_b - \mathcal{S}_{(\alpha,y)}(b,0)\varphi(0) - \sum_{k=1}^{m} \mathcal{S}_{(\alpha,y)}(b,t_k)G_k(z(t_k)+x(t_k))$$
$$- \int_0^b \mathcal{S}_{(\alpha,y)}(b,s)\int_0^s H(s,\tau,z_{\rho(\tau,z_\tau+x_\tau)}+x_{\rho(\tau,z_\tau+x_\tau)})d\tau ds$$
$$- \int_0^b \mathcal{S}_{(\alpha,y)}(b,s)F\Big(s, z_{\rho(s,z_s+x_s)}+x_{\rho(s,z_s+x_s)}, \int_0^s K(s,\tau)(z_{\rho(\tau,z_\tau+x_\tau)}$$
$$+ x_{\rho(\tau,z_\tau+x_\tau)})d\tau\Big)ds\Big](t).$$

Let $\mathcal{B}''_h := \{z \in \mathcal{B}'_h : z_0 = 0\}$. Let $\|\cdot\|_{\mathcal{B}''_h}$ be the seminorm in $\mathcal{B}''_h$ characterized by

$$\|z\|_{\mathcal{B}''_h} := \sup_{t\in[0,b]} \|z(t)\| + \|z_0\|_{\mathcal{B}'_h} = \sup_{t\in[0,b]} \|z(t)\|, \quad z \in \mathcal{B}''_h,$$

as a results $(\mathcal{B}''_h, \|\cdot\|_{\mathcal{B}''_h})$ is a Banach space. For $r_0 > 0$, we define $\mathbb{D}_{r_0} = \{z \in \mathcal{B}''_h : \|z\|_{\mathcal{B}''_h} \le r_0\}$.
From the above assumptions and Lemma 6.2.1 we have the following estimates

$$\begin{aligned}
\|z_{\rho(s,z_s+x_s)}&+x_{\rho(s,z_s+x_s)}\|_{\mathcal{B}_h}\\
&\le \|z_{\rho(s,z_s+x_s)}\|_{\mathcal{B}_h} + \|x_{\rho(s,z_s+x_s)}\|_{\mathcal{B}_h}\\
&\le \xi_1^0 \sup_{0\le\tau\le s}\|z(\tau)\| + (\xi_2^0 + h_\varphi^0)\|z_0\|_{\mathcal{B}_h} + \xi_1^0 \sup_{0\le\tau\le s}|x(\tau)| + (\xi_2^0 + h_\varphi^0)\|x_0\|_{\mathcal{B}_h}\\
&\le \xi_1^0 \sup_{0\le\tau\le s}\|z(\tau)\| + \xi_1^0 \sup_{t\in[0,b]}\|\mathcal{S}_{(\alpha,y)}(t,s)\varphi(0)\|_{\mathcal{L}} + (\xi_2^0 + h_\varphi^0)\|\varphi\|_{\mathcal{B}_h}\\
&\le \xi_1^0 \sup_{0\le\tau\le s}\|z(\tau)\| + (\xi_1^0 S_0 N + \xi_2^0 + h_\varphi^0)\|\varphi\|_{\mathcal{B}_h}.
\end{aligned}$$

Here $\|z\| \le r_0$, then

$$\|z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)}\|_{\mathcal{B}_h} \le \xi_1^0 r_0 + C_0,$$

where $C_0 = (\xi_1^0 S_0 N + \xi_2^0 + h_\varphi^0)\|\varphi\|_{\mathcal{B}_h}$.

Similarly,
$$\begin{aligned}
\|z_t + x_t\|_{\mathcal{B}_h} &\le \|z_t\|_{\mathcal{B}_h} + \|x_t\|_{\mathcal{B}_h}\\
&\le \xi_1^0 \sup_{0\le\tau\le t}\|z(\tau)\| + \xi_2^0\|z_0\|_{\mathcal{B}_h} + \xi_1^0 \sup_{0\le\tau\le t}|x(\tau)| + \xi_2^0\|x_0\|_{\mathcal{B}_h}\\
&\le \xi_1^0 \sup_{0\le\tau\le t}\|z(\tau)\| + \xi_1^0 \sup_{t\in[0,b]}\|\mathcal{S}_{(\alpha,y)}(t,s)\varphi(0)\|_{\mathcal{L}} + \xi_2^0\|\varphi\|_{\mathcal{B}_h}\\
&\le \xi_1^0 \sup_{0\le\tau\le t}\|z(\tau)\| + (\xi_1^0 S_0 N + \xi_2^0)\|\varphi\|_{\mathcal{B}_h}.
\end{aligned}$$

Hence $\|z_t + x_t\|_{\mathcal{B}_h} \le \xi_1^0 r_0 + C'_0$, where $C'_0 = (\xi_1^0 S_0 N + \xi_2^0)\|\varphi\|_{\mathcal{B}_h}$. Now, we obtain

$$\|z(t_k) + x(t_k)\| \le N\|z_{t_k} + x_{t_k}\|_{\mathcal{B}_h} \le N(\xi_1^0 r_0 + C'_0).$$

Let us define the operator $\tilde{\Phi} : \mathcal{B}''_h \to \mathcal{B}''_h$ by

$$\begin{aligned}\tilde{\Phi}z(t) = &\sum_{0<t_k<t} \mathcal{S}_{(\alpha,y)}(t,t_k)G_k(z(t_k)+x(t_k))\\ &+ \int_0^t \mathcal{S}_{(\alpha,y)}(t,s)\int_0^s H(s,\tau,z_{\rho(\tau,z_\tau+x_\tau)} + x_{\rho(\tau,z_\tau+x_\tau)})d\tau ds\\ &+ \int_0^t \mathcal{S}_{(\alpha,y)}(t,s)\Bigg[Bu^{z+x}(s) + F\Bigg(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)},\\ &\int_0^s K(s,\tau)(z_{\rho(\tau,z_\tau+x_\tau)} + x_{\rho(\tau,z_\tau+x_\tau)})d\tau\Bigg)\Bigg]ds. \qquad (6.3.3)\end{aligned}$$

Clearly, the operator Φ has a fixed point if and only if $\tilde{\Phi}$ has a fixed point. So, let us demonstrate in the following steps that $\tilde{\Phi}$ has a fixed point.
Step 1 : There exists $r_0 > 0$ such that $\tilde{\Phi}(\mathbb{D}_{r_0}) \subseteq \mathbb{D}_{r_0}$.
Suppose the contrary, then for every $r_0 > 0$, there exists a function $z \in \mathbb{D}_{r_0}$ but $\tilde{\Phi}(z) \notin \mathbb{D}_{r_0}$ i.e. $\|\tilde{\Phi}z\|_{\mathcal{B}''_h} > r_0$ for some $t \in [0,b]$. We have

$$\begin{aligned}&\|\tilde{\Phi}z(t)\|\\ &\le S_0\sum_{k=1}^m \|G_k(z(t_k)+x(t_k))\| + S_0\int_0^t\int_0^s \|H(s,\tau,z_{\rho(\tau,z_\tau+x_\tau)} + x_{\rho(\tau,z_\tau+x_\tau)})\|d\tau ds\\ &\quad + S_0\int_0^t \|Bu^{z+x}(s)\|ds\\ &\quad + S_0\int_0^t \Bigg\|F\Bigg(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)}, \int_0^s K(s,\tau)(z_{\rho(\tau,z_\tau+x_\tau)} + x_{\rho(\tau,z_\tau+x_\tau)})d\tau\Bigg)\Bigg\|ds\\ &\le S_0\sum_{k=1}^m L_{G_k}(\|z(t_k)+x(t_k)\|)\\ &\quad + S_0\int_0^t\int_0^s m_H(s,\tau)L_H(\|z_{\rho(\tau,z_\tau+x_\tau)} + x_{\rho(\tau,z_\tau+x_\tau)}\|_{\mathcal{B}_h})d\tau ds\\ &\quad + S_0 b^0 b^{\frac{1}{2}}\|u^{z+x}(s)\|_{L^2} + S_0\int_0^t m_F(s)L_F\Bigg((1+k^0)\|z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)}\|_{\mathcal{B}_h}\Bigg)ds\\ &\le S_0\sum_{k=1}^m L_{G_k}(N(\xi_1^0 r_0 + C_0')) + S_0 L_H(\xi_1^0 r_0 + C_0)\sigma + S_0 b^0 b^{\frac{1}{2}}\|u^{z+x}\|_{L^2}\\ &\quad + S_0\|m_F\|_{L^1}L_F((1+k^0)(\xi_1^0 r_0 + C_0)), \qquad (6.3.4)\end{aligned}$$

where

$$\begin{aligned}\|u^{z+x}\|_{L^2} \le & w^0\Bigg[\|y_b\| + S_0\|\varphi(0)\| + S_0 L_H(\xi_1^0 r_0 + C_0)\sigma\\ &+ S_0\|m_F\|_{L^1}L_F((1+k^0)(\xi_1^0 r_0 + C_0)) + S_0\sum_{k=1}^m L_{G_k}(N(\xi_1^0 r_0 + C_0'))\Bigg].\end{aligned} \qquad (6.3.5)$$

Using (6.3.5) in (6.3.4) and taking supremum over t, we get

$$\|\tilde{\Phi}z\|_{\mathcal{B}''_h} \leq \left(1+S_0b^0b^{\frac{1}{2}}w^0\right)\left[S_0\sum_{k=1}^{m}L_{G_k}(N(\xi_1^0r_0+C_0')) + S_0L_H(\xi_1^0r_0+C_0)\sigma \right.$$
$$\left. + S_0\|m_F\|_{L^1}L_F((1+k^0)(\xi_1^0r_0+C_0))\right] + S_0b^0b^{\frac{1}{2}}w^0\left[\|y_b\|+S_0\|\varphi\|\right].$$

Dividing both sides by r_0, and as $r_0 \to \infty$, we get

$$1 \leq \left(1+S_0b^0b^{\frac{1}{2}}w^0\right)\left[S_0\sum_{k=1}^{m}\lambda_k + S_0\gamma\sigma + S_0\|m_F\|_{L^1}\Lambda\right],$$

which contradict to (6.3.1). Hence there exists a $r_0 > 0$ such that $\tilde{\Phi}(\mathbb{D}_{r_0}) \subseteq \mathbb{D}_{r_0}$.
Step 2 : $\tilde{\Phi}$ is continuous on $\mathbb{D}_{r_0}$.
To demonstrate the continuity of $\tilde{\Phi}$, we assume that there exists a sequence $z^n \to z$ in $\mathbb{D}_{r_0}$. Denote

$$F_n(s) = F\left(s, z^n_{\rho(s,z^n_s+x_s)} + x_{\rho(s,z^n_s+x_s)}, \int_0^s K(s,\tau)(z^n_{\rho(\tau,z^n_\tau+x_\tau)} + x_{\rho(\tau,z^n_\tau+x_\tau)})d\tau\right),$$
$$F(s) = F\left(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)}, \int_0^s K(s,\tau)(z_{\rho(\tau,z_\tau+x_\tau)} + x_{\rho(\tau,z_\tau+x_\tau)})d\tau\right),$$
$$H_n(s,\tau) = H(s,\tau, z^n_{\rho(\tau,z^n_\tau+x_\tau)} + x_{\rho(\tau,z^n_\tau+x_\tau)})$$
$$H(s,\tau) = H(s,\tau, z_{\rho(\tau,z_\tau+x_\tau)} + x_{\rho(\tau,z_\tau+x_\tau)}).$$

By Lebesgue Dominated convergence theorem accompanying with $(A_1)(i), (A_2)(i)$ and $(A_3)(i)$, we get

$$\begin{aligned}\|\tilde{\Phi}z^n(t) - &\tilde{\Phi}z(t)\| \\ &\leq S_0\sum_{k=1}^{m}\|G_k(z^n(t_k)+x(t_k)) - G_k(z(t_k)+x(t_k))\| \\ &\quad + S_0\int_0^t\int_0^s\|H_n(s,\tau)-H(s,\tau)\|d\tau ds \\ &\quad + S_0b^0b^{\frac{1}{2}}\|u^{z^n+x}-u^{z+x}\|_{L^2} + S_0\int_0^t\|F_n(s)-F(s)\|ds, \end{aligned} \tag{6.3.6}$$

where

$$\begin{aligned}\|u^{z^n+x} &- u^{z+x}\|_{L^2} \\ \leq & w^0\left[S_0\sum_{k=1}^{m}\|G_k(z^n(t_k)+x(t_k)) - G_k(z(t_k)+x(t_k))\|\right. \\ & \left. + S_0\int_0^b\int_0^s\|H_n(s,\tau)-H(s,\tau)\|d\tau ds + S_0\int_0^b\|F_n(s)-F(s)\|ds\right]. \end{aligned} \tag{6.3.7}$$

Now, we observe from (6.3.6) and (6.3.7) that $\|\tilde{\Phi}z^n - \tilde{\Phi}z\|_{\mathcal{B}_h''} \to 0$ as $n \to \infty$ and $t \in [0, b]$. Hence $\tilde{\Phi}$ is continuous on $\mathbb{D}_r$.
Step 3 : The Mönch's condition holds.
For this event, let us assume that $\mathbb{G}$ be a countable subset of $\mathbb{D}_{r_0}$ and $\mathbb{G} \subset \overline{conv}(\{0\} \cup \tilde{\Phi}(\mathbb{G}))$. Then, we demonstrate that $\mu(\mathbb{G}) = 0$, where μ is the Hausdroof measure of noncompactness.
For this purpose, without loss of generality, we may consider that $\mathbb{G} = \{z^n\}_{n=1}^{\infty}$. If we are able to show that $\{\tilde{\Phi}(z^n)\}_{n=1}^{\infty}$ is equicontinuous on $I_k = [t_k, t_{k+1}), k = 0, 1, 2, \ldots, m$, then $\mathbb{G} \subset \overline{conv}(\{0\} \cup \tilde{\Phi}(\mathbb{G}))$ is also equicontinuous on $I_k = [t_k, t_{k+1}), k = 0, 1, 2, \ldots, m$.

For this purpose, let $l_2, l_1 \in I_p$ such that $t_p \leq l_1 < l_2 \leq t_{p+1}$ for some $p \in \{0, 1, 2, ...m\}$ and $z \in \mathbb{D}_{r_0}$, we have

$$
\begin{aligned}
\|\tilde{\Phi}z^n(l_2) - &\tilde{\Phi}z^n(l_1)\| \\
\leq &\left\| \int_0^{l_2} \mathcal{S}_{(\alpha,y)}(l_2, s) \int_0^s H_n(s, \tau) d\tau ds - \int_0^{l_1} \mathcal{S}_{(\alpha,y)}(l_1, s) \int_0^s H_n(s, \tau) d\tau ds \right\| \\
&+ \left\| \int_0^{l_2} \mathcal{S}_{(\alpha,y)}(l_2, s)[Bu^{z^n+x}(s) + F_n(s)] ds \right. \\
&\left. - \int_0^{l_2} \mathcal{S}_{(\alpha,y)}(l_1, s)[Bu^{z^n+x}(s) + F_n(s)] ds \right\| \\
&+ \sum_{k=1}^{p} \|\mathcal{S}_{(\alpha,y)}(l_2, s) - \mathcal{S}_{(\alpha,y)}(l_1, s)\|_{\mathcal{L}} \|G_k(z^n(t_k) + x(t_k))\| \\
\leq &\int_0^{l_1} \|\mathcal{S}_{(\alpha,y)}(l_2, s) - \mathcal{S}_{(\alpha,y)}(l_1, s)\|_{\mathcal{L}} \left\| \int_0^s H_n(s, \tau) d\tau \right\| ds \\
&+ \int_{l_1}^{l_2} \|\mathcal{S}_{(\alpha,y)}(l_2, s)\|_{\mathcal{L}} \left\| \int_0^s H_n(s, \tau) d\tau \right\| ds \\
&+ \int_0^{l_1} \|\mathcal{S}_{(\alpha,y)}(l_2, s) - \mathcal{S}_{(\alpha,y)}(l_1, s)\|_{\mathcal{L}} \|Bu^{z^n+x}(s) + F_n(s)\| ds \\
&+ \int_{l_1}^{l_2} \|\mathcal{S}_{(\alpha,y)}(l_1, s)\|_{\mathcal{L}} \|Bu^{z^n+x}(s) + F_n(s)\| ds \\
&+ \sum_{k=1}^{p} \|\mathcal{S}_{(\alpha,y)}(l_2, s) - \mathcal{S}_{(\alpha,y)}(l_1, s)\|_{\mathcal{L}} \|G_k(z^n(t_k) + x(t_k))\|.
\end{aligned}
$$

By equicontinuity of $\mathcal{S}_{(\alpha,y)}(t, s)$ and absolute continuity of Lebesgue integral, we conclude that right side of the above inequality tends to zero as $l_2 \to l_1$ independently of z. Hence, $\tilde{\Phi}(\mathbb{G})$ is equicontinuous on I_k for all $k = 0, 1, 2, \ldots, m$. Now, by Theorem 2.4.4 and $(A_1)(iii), (A_2)(iii), (A_3)(ii)$ and $(A_4)(ii)$, we obtain

$$
\begin{aligned}
&\mu(\{u^{z^n+x}(\xi)\}_{n=1}^{\infty}) \\
\leq &J_W(\xi) \left[\mu\left(\left\{ \int_0^b \mathcal{S}_{(\alpha,y)}(b, s) \int_0^s H(s, \tau, z^n_{\rho(\tau, z^n_\tau + x_\tau)} + x_{\rho(\tau, z^n_\tau + x_\tau)}) d\tau ds \right\}_{n=1}^{\infty} \right) \right.
\end{aligned}
$$

$$
\begin{aligned}
&+\mu\Bigg(\Bigg\{\int_0^b \mathcal{S}_{(\alpha,y)}(b,s)F\Bigg(s, z^n_{\rho(s,z^n_s+x_s)}+x_{\rho(s,z^n_s+x_s)}, \int_0^s K(s,\tau)(z^n_{\rho(\tau,z^n_\tau+x_\tau)}\\
&\quad+x_{\rho(\tau,z^n_\tau+x_\tau)})d\tau\Bigg)ds\Bigg\}_{n=1}^{\infty}\Bigg)+\mu\Bigg(\Bigg\{\sum_{k=1}^m \mathcal{S}_{(\alpha,y)}(b,t_k)G_k(z^n(t_k)+x(t_k))\Bigg\}_{n=1}^{\infty}\Bigg)\Bigg]\\
\le& J_W(\xi)\Bigg[4S_0\int_0^b\int_0^s J_H(s,\tau)\sup_{-\infty<\theta\le 0}\mu(\{z^n(\tau+\theta)+x(\tau+\theta)\}_{n=1}^{\infty})d\tau ds\\
&+2S_0\int_0^b J_F(s)\Bigg[\sup_{-\infty<\theta\le 0}\mu(\{z^n(s+\theta)+x(s+\theta)\}_{n=1}^{\infty})\\
&+\int_0^s K(s,\tau)\sup_{-\infty<\theta\le 0}\mu(\{z^n(\tau+\theta)+x(\tau+\theta)\}_{n=1}^{\infty})d\tau\Bigg]ds\\
&+S_0\sum_{k=1}^m J_{G_k}\sup_{-\infty<\theta\le 0}\mu(\{z^n(t_k+\theta)+x(t_k+\theta)\}_{n=1}^{\infty})\Bigg]\\
\le& J_W(\xi)\Bigg[4S_0\int_0^b\int_0^s J_H(s,\tau)\sup_{0\le\eta\le\tau}\mu(\{z^n(\eta)\}_{n=1}^{\infty})d\tau ds\\
&+2S_0\int_0^b J_F(s)(1+k^0)\sup_{0\le\eta\le s}\mu(\{z^n(\eta)\}_{n=1}^{\infty})ds\\
&+S_0\sum_{k=1}^m J_{G_k}\sup_{0\le\eta_k\le t_k}\mu(\{z^n(\eta_k)\}_{n=1}^{\infty})\Bigg].
\end{aligned}\tag{6.3.8}
$$

Further, by Theorem 2.4.4, we have

$$
\begin{aligned}
&\mu(\tilde{\Phi}\{z^n(t)\}_{n=1}^{\infty})\\
\le&\mu\Bigg(\Bigg\{\sum_{k=1}^m \mathcal{S}_{(\alpha,y)}(b,t_k)G_k(z^n(t_k)+x(t_k))\Bigg\}_{n=1}^{\infty}\Bigg)\\
&+\mu\Bigg(\Bigg\{\int_0^t \mathcal{S}_{(\alpha,y)}(b,s)\int_0^s H(s,\tau,z^n_{\rho(\tau,z^n_\tau+x_\tau)}+x_{\rho(\tau,z^n_\tau+x_\tau)})d\tau ds\Bigg\}_{n=1}^{\infty}\Bigg)\\
&+\mu\Bigg(\Bigg\{\int_0^t \mathcal{S}_{(\alpha,y)}(b,s)\Bigg[Bu^{z^n+x}(s)\\
&+F\Bigg(s,z^n_{\rho(s,z^n_s+x_s)}+x_{\rho(s,z^n_s+x_s)},\int_0^s K(s,\tau)(z^n_{\rho(\tau,z^n_\tau+x_\tau)}+x_{\rho(\tau,z^n_\tau+x_\tau)})d\tau\Bigg)ds\Bigg]\Bigg\}_{n=1}^{\infty}\Bigg)\\
\le& S_0\sum_{k=1}^m J_{G_k}\sup_{0\le\eta_k\le t_k}\mu(\{z^n(\eta_k)\}_{n=1}^{\infty})+4S_0\int_0^b\int_0^s J_H(s,\tau)\sup_{0\le\eta\le\tau}\mu(\{z^n(\eta)\}_{n=1}^{\infty})d\tau ds\\
&+2S_0b^0\int_0^b\mu(\{u^{z^n+x}(s)\}_{n=1}^{\infty})ds+2S_0\int_0^b J_F(s)(1+k^0)\sup_{0\le\eta\le s}\mu(\{z^n(\eta)\}_{n=1}^{\infty})ds.
\end{aligned}\tag{6.3.9}
$$

From (6.3.8) and (6.3.9), we have

$$\begin{aligned}\mu(\tilde{\Phi}\{z^n(t)\}_{n=1}^{\infty}) \leq& \Big(S_0 + 2S_0^2 b^0 \int_0^b J_W(s)ds\Big)\Big[\sum_{k=1}^{m} J_{G_k} \sup_{0\leq\eta_k\leq t_k} \mu(\{z^n(\eta_k)\}_{n=1}^{\infty})\\ &+ 4\int_0^b\int_0^s J_H(s,\tau) \sup_{0\leq\eta\leq\tau} \mu(\{z^n(\eta)\}_{n=1}^{\infty})d\tau ds\\ &+ 2\int_0^b J_F(s)(1+k^0) \sup_{0\leq\eta\leq s} \mu(\{z^n(\eta)\}_{n=1}^{\infty})ds\Big]. \end{aligned} \tag{6.3.10}$$

Since $\mathbb{G}$ and $\tilde{\Phi}(\mathbb{G})$ are equicontinuous on every I_k, according to Lemma 2.4.6, the inequality (6.3.10), we obtain

$$\begin{aligned}\mu(\tilde{\Phi}\{z^n\}_{n=1}^{\infty}) \leq& \Big(S_0 + 2S_0^2 b^0 \|J_W\|_{L^1}\Big)\Big[\sum_{k=1}^{m} J_{G_k}\mu(\{z^n\}_{n=1}^{\infty}) + 4\varsigma\mu(\{z^n\}_{n=1}^{\infty})\\ &+ 2\|J_F\|_{L^1}(1+k^0)\mu(\{z^n\}_{n=1}^{\infty})\Big]\\ \leq& \Big(S_0 + 2S_0^2 b^0 \|J_W\|_{L^1}\Big)\Big[\sum_{k=1}^{m} J_{G_k} + 4\varsigma + 2\|J_F\|_{L^1}(1+k^0)\Big]\mu(\{z^n\}_{n=1}^{\infty}). \end{aligned} \tag{6.3.11}$$

That is $\mu(\tilde{\Phi}(\mathbb{G})) \leq \Delta_2\mu(\mathbb{G})$. Now, by Mönch's condition, we deduce

$$\mu(\mathbb{G}) \leq \mu(\overline{conv}(\{0\} \cup \tilde{\Phi}(\mathbb{G}))) = \mu(\tilde{\Phi}(\mathbb{G})) \leq \Delta_2\mu(\mathbb{G}),$$

which shows by the inequality (6.3.2) that $\mu(\mathbb{G}) = 0$.
Now, by applying Theorem 2.6.3, we observe that $\tilde{\Phi}$ has a fixed point z^* in $\mathbb{D}_{r_0}$. Then $y(t) = \overline{z}^*(t) + x(t)$ $t \in (-\infty, b]$ is a mild solution of the system (6.1.3) satisfying $y(b) = y_b$. Hence the system (6.1.3) is controllable on $[0, b]$. This completes the proof. □

Remark 6.3.2. *It should be noticed that if the functions F, H and G_k are Lipschitz continuous or compact, then $(A_1)(iii)$, $(A_2)(iii)$ and $(A_3)(ii)$ are automatically satisfied. Next, by choosing another measure of noncompact, we re-establish the controllability results for the system* (6.1.3) *in which equicontinuity of $\mathcal{S}_{(\alpha,y)}(t,s)$ and the inequality* (6.3.2) *are not required. So, our next results are going to be interesting and more general than the most of the previous available controllability results. In place of (A_3), we will consider the following assumption.*

$(A_3)'$ The functions $G_k : \mathbb{X} \to \mathbb{X}$, $k = 1, 2, 3, \ldots, m$, are compact, continuous, and there exist non decreasing functions $L_{G_k} : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\|G_k(z)\|_{\mathbb{X}} \leq L_{G_k}(\|z\|) \quad \text{and} \quad \lim_{r\to\infty} \frac{L_{G_k}(cr)}{r} = \lambda_k < \infty.$$

for $c > 0$ and $k = 1, 2, \ldots, m$ and $z \in \mathbb{X}$.

Theorem 6.3.3. *Assume that* $A(t,\cdot)$ *generates the* $(\alpha,y)-$*resolvent family* $\mathcal{S}_{(\alpha,y)}(t,s)$, *and the assumptions* (A_1), (A_2), $(A_3)'$ *and* (A_4) *are satisfied, then the fractional impulsive quasilinear differential system* (6.1.3) *is controllable on* $[0,b]$ *if*

$$\left(S_0+S_0^2b^0b^{\frac{1}{2}}w^0\right)\left[\sum_{k=1}^{m}\lambda_k+\gamma\sigma+\|m_F\|_{L^1}\Lambda\right]<1. \tag{6.3.12}$$

Proof. On account of Theorem 6.3.1, we should only show that the operator $\tilde{\Phi}:\mathcal{B}_h''\to\mathcal{B}_h''$ defined by (6.3.3) satisfies the Mönch's condition.

For this event, let us assume that $\mathbb{G}$ be a countable subset of $\mathbb{D}_{r_0}$ and $\mathbb{G}\subset\overline{conv}(\{0\}\cup\tilde{\Phi}(\mathbb{G}))$. In the sequel, we consider the measure of noncompact Υ in $\mathcal{B}_h''$ defined by (see [117])

$$\Upsilon(\mathbb{G})=\max_{\mathbb{E}\in\Delta(\mathbb{G})}(\nu(\mathbb{E}),mod_c(\mathbb{E})), \tag{6.3.13}$$

for all bounded subsets of $\mathbb{G}$ of $\mathcal{B}_h''$, where $\Delta(\mathbb{G})$ stands for the collection of all countable subsets of $\mathbb{G}$, ν is the real measure of noncompactness characterized by

$$\nu(\mathbb{E})=\sup_{t\in[0,b]}e^{-Lt}\mu(\mathbb{E}(t)), \tag{6.3.14}$$

where $\mathbb{E}(t)=\{y(t):y\in\mathbb{E}\}$, L is a suitable constant.

Here mod_c is the modulus of equicontinuity of the function set $\mathbb{E}$ characterized by

$$mod_c(\mathbb{E})=\lim_{\delta\to0}\sup_{y\in\mathbb{E}}\max_{0\le k\le m}\max_{l_1,l_2\in(t_k,t_{k+1}],\|l_2-l_1\|<\delta}\|y(l_2)-y(l_1)\|.$$

It is clear form [117] that Υ is well defined (i.e. there exists a $\mathbb{E}_0\in\Delta(\mathbb{G})$ which achieves the maximum in (6.3.13)), nonsingular, monotone and regular measure of noncompactness.

Let us choose an appropriate constant $L>0$, such that

$$\begin{aligned}q=&\left(S_0+2S_0^2b^0\|J_W\|_{L^1}\right)\left[4\int_0^b e^{-L(t-s)}\int_0^s J_H(s,\tau)d\tau ds\right.\\&\left.+2(1+k^0)\int_0^b J_F(s)e^{-L(t-s)}ds\right]<1,\end{aligned} \tag{6.3.15}$$

where J_F and J_H are the functions defined in $(A_1)(iii)$ and $(A_2)(iii)$, respectively. By using the regularity of Υ, we will demonstrate that $\mathbb{G}$ is relatively compact i.e. $\Upsilon(\mathbb{G})=0$. Since $\Upsilon(\tilde{\Phi}(\mathbb{G}))$ is a maximum, let $\{w^n\}_{n=1}^{\infty}\subseteq\tilde{\Phi}(\mathbb{G})$ be a denumerable set which attain its maximum. Then there exists a set $\{z^n\}_{n=1}^{\infty}\subseteq\mathbb{G}$ such that

$$w^n(t)=\tilde{\Phi}(z^n(t)). \tag{6.3.16}$$

By the compactness of G_k, we have $J_{G_k} = 0$, for all $k = 1, 2, ...m$. Then from (6.3.10), we deduce that

$$\begin{aligned}
&\mu(\tilde{\Phi}\{z^n(t)\}_{n=1}^{\infty}) \\
&\leq \sup_{t\in[0,b]} \left(S_0 + 2S_0^2 b^0 \|J_W\|_{L^1}\right)\left[4\int_0^t \int_0^s J_H(s,\tau) \sup_{0\leq\eta\leq\tau} \mu(\{z^n(\eta)\}_{n=1}^{\infty}) d\tau ds \right.\\
&\quad \left. + 2\int_0^t J_F(s)(1+k^0) \sup_{0\leq\eta\leq s} \mu(\{z^n(\eta)\}_{n=1}^{\infty}) ds\right] \\
&\leq \sup_{t\in[0,b]} \left(S_0 + 2S_0^2 b^0 \|J_W\|_{L^1}\right)\left[4\int_0^t \int_0^s J_H(s,\tau) e^{Ls} \sup_{\eta\in[0,b]} [e^{-L\eta}\mu(\{z^n(\eta)\}_{n=1}^{\infty})] d\tau ds \right.\\
&\quad \left. + 2\int_0^t J_F(s)(1+k^0) e^{Ls} \sup_{\eta\in[0,b]} (e^{-L\eta}\mu[\{z^n(\eta)\}_{n=1}^{\infty})] ds\right] \\
&\leq \sup_{t\in[0,b]} \left(S_0 + 2S_0^2 b^0 \|J_W\|_{L^1}\right)\nu(\{z^n\}_{n=1}^{\infty})\left[4\int_0^t \int_0^s J_H(s,\tau) e^{Ls} d\tau ds \right.\\
&\quad \left. + 2\int_0^t J_F(s)(1+k^0) e^{Ls} ds\right]. \qquad (6.3.17)
\end{aligned}$$

Further, from (6.3.14) and (6.3.17), we have

$$\begin{aligned}
\nu(\{w^n\}_{n=1}^{\infty}) = &\sup_{t\in[0,b]} e^{-Lt}\left(S_0 + 2S_0^2 b^0 \|J_W\|_{L^1}\right)\nu(\{z^n\}_{n=1}^{\infty})\left[4\int_0^t \int_0^s J_H(s,\tau) e^{Ls} d\tau ds \right.\\
&\left. + 2\int_0^t J_F(s)(1+k^0) e^{Ls} ds\right] \\
\leq &\left(S_0 + 2S_0^2 b^0 \|J_W\|_{L^1}\right)\nu(\{z^n\}_{n=1}^{\infty}) \sup_{t\in[0,b]} \left[4\int_0^t e^{-L(t-s)} \int_0^s J_H(s,\tau) d\tau ds \right.\\
&\left. + 2\int_0^t J_F(s)(1+k^0) e^{-L(t-s)} ds\right] \\
\leq &p\nu(\{z^n\}_{n=1}^{\infty}).
\end{aligned}$$

Thus, we conclude that

$$\nu(\{z^n\}_{n=1}^{\infty}) = \nu(\mathbb{G}) \leq \nu(\overline{conv}(\{0\} \cup \tilde{\Phi}(\mathbb{G}))) = \nu(\{w^n\}_{n=1}^{\infty}) \leq p\nu(\{z^n\}_{n=1}^{\infty}).$$

From (6.3.15), we obtain

$$\nu(\{z^n\}_{n=1}^{\infty}) = \nu(\mathbb{G}) = \nu(\{w^n\}_{n=1}^{\infty}) = 0.$$

In view of the definition of ν, we obtain

$$\mu(\{z^n(t)\}_{n=1}^{\infty}) = \mu(\{w^n(t)\}_{n=1}^{\infty})\}_{n=1}^{\infty} = 0, \qquad \text{for every } t \in [0, b].$$

From $(A_1)(iii), (A_2)(iii)$ and (6.3.8), we obtain

$$\mu\Bigg(\Bigg\{\int_0^t H(t,s,z^n_{\rho(s,z^n_s+x_s)}+x_{\rho(s,z^n_s+x_s)})ds + Bu^{z^n+x}(t) + F\Bigg(t, z^n_{\rho(t,z^n_t+x_t)}+x_{\rho(t,z^n_t+x_t)}, \int_0^t K(t,s)(z^n_{\rho(s,z^n_s+x_s)}+x_{\rho(s,z^n_s+x_s)})ds\Bigg)\Bigg\}_{n=1}^{\infty}\Bigg)$$

$$\begin{aligned}
\leq & 2\int_0^b J_H(t,s)\sup_{-\infty<\theta\leq 0}\mu(\{z^n(s+\theta)+x(s+\theta)\}_{n=1}^{\infty})ds \\
&+ J_W(t)b^0\Bigg[4S_0\int_0^b\int_0^s J_H(s,\tau)\sup_{-\infty<\theta\leq 0}\mu(\{z^n(\tau+\theta)+x(\tau+\theta)\}_{n=1}^{\infty})d\tau ds \\
&+ 2S_0\int_0^b J_F(s)(1+k^0)\sup_{-\infty<\theta\leq 0}\mu(\{z^n(s+\theta)+x(s+\theta)\}_{n=1}^{\infty})ds\Bigg] \\
&+ J_F(t)(1+k^0)\sup_{-\infty<\theta\leq 0}\mu(\{z^n(s+\theta)+x(s+\theta)\}_{n=1}^{\infty}) \\
\leq & 2\int_0^b J_H(t,s)\sup_{0\leq\eta\leq s}\mu(\{z^n(\eta)\}_{n=1}^{\infty})ds \\
&+ J_W(t)b^0\Bigg[4S_0\int_0^b\int_0^s J_H(s,\tau)\sup_{0\leq\eta\leq\tau}\mu(\{z^n(\eta)\}_{n=1}^{\infty})d\tau ds \\
&+ 2S_0\int_0^b J_F(s)(1+k^0)\sup_{0\leq\eta\leq s}\mu(\{z^n(\eta)\}_{n=1}^{\infty})ds\Bigg] \\
&+ J_F(t)(1+k^0)\sup_{0\leq\eta\leq t}\mu(\{z^n(\eta)\}_{n=1}^{\infty})ds = 0,
\end{aligned}$$

this implies that

$$\begin{aligned}
&\{(Qz^n)(t)\}_{n=1}^{\infty} \\
&\quad = \Bigg\{\int_0^t H(t,s,z^n_{\rho(s,z^n_s+x_s)}+x_{\rho(s,z^n_s+x_s)})ds + Bu^{z^n+x}(t) \\
&\qquad + F\Bigg(t, z^n_{\rho(t,z^n_t+x_t)}+x_{\rho(t,z^n_t+x_t)}, \int_0^t K(t,s)(z^n_{\rho(s,z^n_s+x_s)}+x_{\rho(s,z^n_s+x_s)})ds\Bigg)\Bigg\}_{n=1}^{\infty}
\end{aligned}$$

is relatively compact for almost all $t \in [0,b]$ in $\mathbb{X}$. Further, using the fact that $\{z^n\}_{n=1}^{\infty} \subseteq \mathbb{D}_{r_0}$, by $(A_1)(iii), (A_2)(iii)$ and (6.3.5), we conclude that $\{(Qz^n)(t)\}_{n=1}^{\infty}$ is uniformly integrable for a.e. $t \in [0,b]$. So, by the Definition 6.2.4, $\{(Qz^n)(\cdot)\}_{n=1}^{\infty}$ is semicompact. Moreover, by Lemma 6.2.2, $\{(Qz^n)\}_{n=1}^{\infty}$ is relatively compact in $\mathbb{D}_{r_0}$.

On the other hand, by compactness of G_k and strong continuity of $\mathcal{S}_{(\alpha,y)}(t,s)$, we conclude that $\sum\limits_{0<t_k<t}\mathcal{S}_{(\alpha,y)}(t,t_k)G_k(z(t_k)+x(t_k))$ is relatively compact. Then, by (6.3.16), $\{w^n\}_{n=1}^{\infty}$ is also relatively compact in $\mathbb{D}_{r_0}$. Now, by Mönch's condition, we deduce

$$\Upsilon(\mathbb{G}) \leq \Upsilon(\overline{conv}(\{0\}\cup\tilde{\Phi}(\mathbb{G}))) = \Upsilon(\{w^n\}_{n=1}^{\infty}) = 0,$$

which shows that $\mathbb{G}$ is relatively compact in $\mathbb{D}_{r_0}$. This completes the proof. □

6.4 Example

In this section, we provide a concrete example to a control problem described as fractional impulsive quasilinear linear differential system. In [93], authors have used a fractional order differential equation to represent charge transport in amorphous semiconductors and in [25], authors have utilized the concepts of fractional differential equations in the theory of the flow of fluid through fissured rocks.

Consider a control problem modeled as the following integro-differential equations with impulsive conditions that arise in theory of heat flow in materials with fading memory

$$\begin{aligned}\frac{\partial^\alpha z(t,x)}{\partial t^\alpha} &+ a(t,x,z(t,x))\frac{\partial^2}{\partial x^2}z(t,x) = Bw(t,x)\\ &+\int_0^t h_1(t-s)\int_{-\infty}^s e^{2(\tau-s)}z\Big(\tau-\rho_1(\tau)\rho_2\Big(\int_0^\pi b(\xi)|z(\tau,\xi)|^2d\xi\Big),x\Big)d\tau ds\\ &+\int_{-\infty}^t e^{2(t-s)}z\Big(s-\rho_1(s)\rho_2\Big(\int_0^\pi b(\xi)|z(s,\xi)|^2d\xi\Big),x\Big)ds\\ &+\int_0^t \sin(t-s)\int_{-\infty}^s e^{2(\tau-s)}z\Big(\tau-\rho_1(\tau)\rho_2\Big(\int_0^\pi b(\xi)|z(\tau,\xi)|^2d\xi\Big),x\Big)d\tau ds\end{aligned} \tag{6.4.1}$$

$$\begin{aligned} z(t,0) =& z(t,\pi)=0,\\ z(\theta,x) =& \varphi(\theta,x), \quad \theta\in(-\infty,0],\ 0\le x\le\pi,\\ \Delta z(t_k,x) =& \ z(t_k^+,x)-z(t_k^-,x)=\int_{-\infty}^{t_k} q_k(t_k-s)z(s,x)ds, \qquad k=1,2,3,\dots,m,\end{aligned}$$

where the function $a(t,x,\cdot)$ is continuous, $0<\alpha\le 1, 0<t_1<t_2<\ldots<t_{m+1}\le 1$ are prefixed numbers and $\varphi\in\mathcal{B}_h$. We consider the space $\mathbb{X}=L^2([0,\pi])$ equipped with norm $\|\cdot\|_{L^2}$. $\rho_i:[0,\infty)\to[0,\infty)$, $i=1,2$ are continuous, and the function $b(\cdot)$ is continuous and positive. The functions $q_k:\mathbb{R}\to\mathbb{R}, k=1,2,...,m$ are continuous and compact such that $d_k=\int_\infty^0 h(s)q_k^2(s)ds<\infty$, and $h_1(t-s)$ are continuous with $h_1(t-s)\ge 0$.

For the phase space $\mathcal{B}_h=\mathcal{C}_0\times L^2(g,\mathbb{X})$ (see [106] for details), we choose $h(s)=$

e^{2s}, $s < 0$, then $\int_{-\infty}^{0} e^{2s} ds = \frac{1}{2} < \infty$, for $t \le 0$. We determine

$$\|\varphi\|_{\mathcal{B}_h} = \int_{-\infty}^{0} h(s) \sup_{\tau \in [s,0]} \|\varphi(\tau)\|_{L^2} ds.$$

Moreover, for $(t, \varphi) \in [0, b] \times \mathcal{B}_h$, we denote $\varphi(\theta)x = \varphi(\theta, x)$, $\theta \in (-\infty, 0]$.

We define the operator $A(t, \cdot) : \mathbb{X} \to \mathbb{X}$ by $(A(t, \cdot)z)(x) = a(t, x, \cdot)z_{xx}$ such that

(i) The domain $\mathcal{D}(A(t, \cdot)) = \{z \in \mathbb{X} : z, z_x$ are absolute continuous $z_{xx} \in \mathbb{X}, z(0) = z(\pi) = 0\}$ is independent of t and dense in $\mathbb{X}$. Then

$$A(t, z)z = \sum_{n=1}^{\infty} n^2 \langle z, z_n \rangle z_n, \quad z \in \mathcal{D}(A(t, \cdot)),$$

where $\langle \cdot, \cdot \rangle$ represents inner product in $L^2([0, \pi])$, and $z_n = U_n \circ z$ is the set of orthogonal eigenvectors in $A(t, z)$, where $U_n(t - s) = \sqrt{\frac{2}{\pi}} \sin n(t - s)^{\alpha}$, $0 \le s \le t \le 1$, $n = 1, 2, 3, \ldots$.

(ii) For any λ such that $Re\, \lambda \le 0$, the operator $[\lambda^{\alpha} I + A(t, \cdot)]^{-1}$ exists in $\mathcal{L}(L^2([0, \pi]))$ and

$$[\lambda^{\alpha} I + A(t, \cdot)]^{-1} \le \frac{M_{\alpha}}{|\lambda| + 1}, \quad t \in [0, 1].$$

(iii) There exist constants M_{α} and $\beta \in (0, 1]$ such that

$$\|[A(t_1, \cdot) - A(t_2, \cdot)]A^{-1}(s, \cdot)\| \le M_{\alpha}|t_1 - t_2|^{\beta}, \quad t_2, t_1, s \in [0, 1].$$

Under the aforementioned conditions, the operator $-A^{-1}(s, \cdot)$, $s \in [0, 1]$ generates an evolution operator $\exp(-t^{\alpha} A^{-1}(s, \cdot))$, $t \ge 0$ and there exists a constant M_{α} such that

$$\|A^n(s, \cdot) \exp(-t^{\alpha} A^{-1}(s, \cdot))\| \le \frac{M_{\alpha}}{t^n}, \qquad n = 0, 1, \text{ and } t > 0, \ s \in [0, 1].$$

In particular, in fact the evolution operator is an $(\alpha, y)-$ resolvent family characterized by

$$\mathcal{S}_{(\alpha,y)}(t, s)z = \sum_{n=1}^{\infty} \exp\left[-n^2(t - s)^{\alpha}\right] \langle z, z_n \rangle z_n, \quad z \in \mathcal{D}(A(t, \cdot)).$$

Then, the linear operator W is given by

$$\begin{aligned}Ww(t,x) =& \int_0^1 \mathcal{S}_{(\alpha,y)}(t,s)Bw(s,x)ds\\ =&\sum_{n=1}^{\infty}\int_0^1 \exp\left[-n^2(t-s)^{\alpha}\right]\langle w(s,x), z_n\rangle z_n ds, \quad x\in[0,\pi].\end{aligned}$$

Assume that the operator W has induced inverse $W^{-1} \in L^2([0,1],U)/kerW$ and satisfies (A_4).

Put $y(t)(x) = z(t,x)$, $u(t)(x) = w(t,x)$ where $w(t,x) : [0,b] \times [0,\pi] \to [0,\pi]$ is continuous, then

$$\begin{aligned}\rho(s,\varphi) =&\rho_1(s)\rho_2\bigg(\int_0^{\pi} b(\xi)|\varphi(s,\xi)|^2 d\xi\bigg)\\ \int_0^t H(t,s,\varphi)(x)ds =& \int_0^t h_1(t-s)\int_{-\infty}^0 e^{2\tau}\varphi(\theta)(x)d\tau ds\\ F(t,\varphi,K\varphi)(x) =& \int_{-\infty}^0 e^{2\tau}\varphi(\theta)(x)d\tau + \int_0^t \sin(t-s)\int_{-\infty}^0 e^{2\tau}\varphi(\theta)(x)d\tau ds.\end{aligned}$$

Thus, the system (6.4.1) is an abstract form of the system (6.1.3). Clearly, F satisfies $(A_1)(i)$ and $(A_1)(ii)$. Further, for $(t,\varphi) \in [0,b] \times \mathcal{B}_h$, we get

$$\begin{aligned}&\|F(t,\varphi,K\varphi)(x)\|\\ &\leq\bigg[\int_0^{\pi}\bigg(\int_{-\infty}^0 e^{2(s)}\|\varphi(\theta)(x)\|ds + \int_0^t \|\sin(t-s)\|\int_{-\infty}^0 e^{2(\tau)}\|\varphi(\theta)(x)\|d\tau ds\bigg)^2 dx\bigg]^{\frac{1}{2}}\\ &\leq\bigg[\int_0^{\pi}\bigg(\int_{-\infty}^0 e^{2(s)}\sup\|\varphi(\theta)(x)\|ds + k^0\int_{-\infty}^0 e^{2(\tau)}\sup\|\varphi(\theta)(x)\|d\tau\bigg)^2\bigg]^{\frac{1}{2}}\\ &\leq\sqrt{\pi}\bigg(\|\varphi\|_{\mathcal{B}_h} + k^0\|\varphi\|_{\mathcal{B}_h}\bigg).\end{aligned}$$

where $k^0 = \sup_{t,s\in[0,1]}\int_0^t \sin(t-s)ds \leq 1$. Now, by the property of measure of noncompactness for $\mathbb{D}_1, \mathbb{D}_2 \subset \mathcal{B}_h$, we have

$$\mu(F(t,\mathbb{D}_1,\mathbb{D}_2)) \leq J_F(t)[\sup_{-\infty<\theta\leq 0}\mu(\mathbb{D}_1(\theta)) + k^0\sup_{-\infty<\theta\leq 0}\mu(\mathbb{D}_1(\theta))],$$

where $J_F(t) = \sqrt{\pi}$. Hence (A_1) holds.

Similarly, H clearly satisfies $(A_2)(i)$. Further, for $(t,\varphi)\in[0,b]\times\mathcal{B}_h$, we get

$$\begin{aligned}\|H(t,s,\varphi)(x)\| \leq& \bigg(\int_0^{\pi}\bigg(\int_0^t \|h_1(t-s)\|\int_{-\infty}^0 e^{2(s)}\|\varphi(\theta)(x)\|d\theta ds\bigg)^2 dx\bigg)^{\frac{1}{2}}\\ \leq&\bigg(\|h_1\|_{L^1}\int_0^{\pi}\bigg(\int_{-\infty}^0 e^{2(s)}\sup\|\varphi(\theta)(x)\|d\tau\bigg)^2 dx\bigg)^{\frac{1}{2}}\\ \leq&\sqrt{\pi}\bigg(\|h_1\|_{L^1}\|\varphi\|_{\mathcal{B}_h}\bigg),\end{aligned}$$

where $\|h_1\|_{L^1} = \sup_{t\in[0,1]}\int_0^t h_1(t-s)ds$. Now, by the property of measure of non-compactness for $\mathbb{D}_3\subset\mathcal{B}_h$, we have

$$\mu(H(t,s,\mathbb{D}_2)) \leq J_F(t,s)[\sup_{-\infty<\theta\leq 0}\mu(\mathbb{D}_1(\theta))],$$

where $J_F(t) = \sqrt{\pi}\|h_1\|_{L^1}$. Hence (A_2) holds. Further, we easily check that $(A_3)'$ holds with $L_{G_k} = d_k$. Hence, with these choices of F,H ρ, and $B=I$, the assumptions $(A_1)-(A_4)$ are fulfilled. Hence, by the Theorem 6.3.3, the system (6.4.1) is controllable on $[0,1]$.

Chapter 7

Approximate Controllability of Hilfer Fractional Differential System with Nondense Domain via Sequencing Technique

7.1 Introduction

Mostly, the existence and controllability results are investigated with dense domain i.e. $\overline{\mathcal{D}(A)} = \mathbb{X}$. However, some exact controllability results are also investigated for first order nondensely defined differential systems in [87; 88; 119].

Recently, Zhang and Liu [220] established the exact controllability results to the following control problem described as fractional differential system in a Banach space $\mathbb{X}$

$$\begin{cases} {}^{c}D^{\eta}_{0^+}y(t) = Ay(t) + Cu(t) + f(t, y_t), \quad t \in (0, b], \\ y(t)|_{t=0} = \phi(t), \quad t \in [-r, 0], \end{cases} \tag{7.1.1}$$

where $\eta \in (0, 1)$ and the operator A is a nondensely defined closed linear operator on $\mathbb{X}$ i.e. $\overline{\mathcal{D}(A)} \neq \mathbb{X}$. To the best of our knowledge, the approximate controllability

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results are not investigated to the fractional differential systems with nondense domain in the literature.

Motivated by the above facts, in this chapter, our aim is to study existence, uniqueness and approximate controllability for the following nondensely defined abstract fractional differential system by constructing a sequencing technique

$$\begin{cases} D_{0+}^{\alpha,\beta} y(t) = Ay(t) + Bu(t) + f(t, y(t)), \quad t \in (0, b], \\ J_{0+}^{(1-\alpha)(1-\beta)} y(t)|_{t=0} = y_0, \end{cases} \tag{7.1.2}$$

where $D_{0+}^{\alpha,\beta}$ represents the Hilfer fractional derivative of order $\beta \in (0, 1)$ and type $\alpha \in [0, 1]$. The state $y(\cdot)$ takes its values in the Banach space $\mathbb{X}$ and the control function $u(\cdot)$ belongs to the space $L^p([0, b], \mathbb{U})$, $p > \frac{1}{\beta}$, a Banach space of admissible control functions with $\mathbb{U}$ as a Banach space and $B : L^p([0, b], \mathbb{U}) \to L^p([0, b], \mathbb{X})$ is a bounded and linear operator. In (7.1.2), $A : \mathcal{D}(A) \subset \mathbb{X} \to \mathbb{X}$ is a nondensely closed linear operator on $\mathbb{X}$. The nonlinear function $f : [0, b] \times \mathbb{X} \to \mathbb{X}$ is a given function satisfying some appropriate assumptions to be defined later.

7.2 Preliminaries

In order to define mild solution to the system (7.1.2), we consider the following assumption:

(A_0) The operator $A : \mathcal{D}(A) \subset \mathbb{X} \to \mathbb{X}$ satisfies the Hille-Yosida condition, i.e. there exists constants $M_0 > 0$ and $\omega \in \mathbb{R}$ such that $(\omega, +\infty) \subseteq \varrho(A)$, and

$$\|(\lambda I - A)^{-k}\| \leq \frac{M_0}{(\lambda - \omega)^k}, \quad \text{for all } \lambda > \omega, \ k \geq 1.$$

Denote $\overline{\mathcal{D}(A)} = \mathbb{X}_0$. Let A_0 be the part of A in $\overline{\mathcal{D}(A)}$ defined by $A_0 y = Ay$, and the domain of A_0 is given by $\mathcal{D}(A_0) = \{y \in \mathcal{D}(A) : Ay \in \overline{\mathcal{D}(A)}\}$. Then, in view of [163], the part A_0 of the operator A generates a strongly continuous semigroup $\{\mathcal{S}(t)\}_{t\geq 0}$ on $\mathbb{X}_0$ with $\|\mathcal{S}(t)\| \leq M_0 e^{\omega t}$, where M_0 and ω are the constants introduced in Hille-Yosida condition. Let us denote $\sup_{t\in[0,b]} \|\mathcal{S}(t)\| \leq M$.

Let $B_\lambda = \lambda(\lambda I - A)^{-1}$, where I is the identity operator on $\mathbb{X}$, then for all $y \in \mathbb{X}_0$, we have $B_\lambda y = y$ as $\lambda \to \infty$. It may be concluded by Hille-Yosida condition that $\lim\limits_{\lambda\to\infty} \|B_\lambda y\| = M_0\|y\|$. For sake of convenience, let $\gamma = \alpha+\beta-\alpha\beta$, then $1-\gamma = (1-\alpha)(1-\beta)$. Define $Z = \{y \in \mathcal{C}((0,b],\mathbb{X}_0) : \lim_{t\to 0} t^{(1-\gamma)}y(t)$ exists and finite$\}$ equipped with the norm $\|y\|_Z = \sup_{t\in(0,b]}\{t^{1-\gamma}\|y(t)\| : \gamma = \alpha+\beta-\alpha\beta\}$. We may easily check that, Z is a Banach space. We note that $y(t) = t^{\gamma-1}z(t)$ for $t \in (0,b]$ and $y \in Z$ if and only if $z \in \mathcal{C}([0,b],\mathbb{X}_0)$ and $\|y\|_Z = \|z\|$.

To define an integral solution of the system (7.1.2), we introduce the Wright function $M_\beta(\theta)$ given by

$$M_\beta(\theta) = \sum_{n=1}^{\infty} \frac{(-\theta)^{(n-1)}}{(n-1)!\Gamma(1-\beta n)}, \quad 0 < \beta < 1,\ \theta \in \mathbb{C},$$

which satisfies the condition

$$\int_0^\infty \theta^\nu M_\beta(\theta)d\theta = \frac{\Gamma(1+\nu)}{\Gamma(1+\beta\nu)}, \quad \theta \geq 0.$$

Definition 7.2.1. *A continuous function* $y : [0,b] \to \mathbb{X}$ *is an integral solution of* (7.1.2) *if*

(i) $y : [0,b] \to \mathbb{X}$ *is a continuous,*

(ii) $J^\beta_{0+}y(t) \in \mathcal{D}(A)$ *for* $t \in [0,b]$ *and*

(iii) for $t \in (0,b]$, $y(t)$ *satisfies (see [90])*

$$\begin{aligned} y(t) =& \frac{y_0}{\Gamma(\alpha(1-\beta)+\beta)} t^{(\alpha-1)(1-\beta)} \\ &+ \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1}[Ay(s) + Bu(s) + f(s,y(s))]ds. \end{aligned} \tag{7.2.1}$$

Lemma 7.2.1. *If* y *is an integral solution of* (7.1.2)*, then for* $t \in [0,b]$*, we have* $y(t) \in \overline{\mathcal{D}(A)}$*. In particular* $y_0 \in \overline{\mathcal{D}(A)}$*.*

Proof. We may refer to [154] for the proof. □

The following Lemma provides another form of (7.2.1) with applications of Laplace transform.

Lemma 7.2.2. *If the function f take values in $\mathbb{X}$, then for $t \in (0,b]$ an integral solution* (7.2.1) *may be rewritten as*

$$y(t) = \mathcal{S}_{\alpha,\beta}(t)y_0 + \lim_{\lambda\to\infty}\int_0^t (t-s)^{\beta-1}\mathcal{P}_\beta(t-s)B_\lambda[Bu(s)+f(s,y(s))]ds, \qquad (7.2.2)$$

where $\mathcal{P}_\beta(t) = \int_0^\infty \beta\theta M_\beta(\theta)\mathcal{S}(t^\beta\theta)d\theta$, $\mathcal{S}_{\alpha,\beta}(t) = J_{0+}^{\alpha(1-\beta)}\mathcal{T}_\beta(t)$ *and* $\mathcal{T}_\beta(t) = t^{\beta-1}\mathcal{P}_\beta(t)$.

Proof. Let $p > 0$. Denote the Laplace transforms

$$\chi(p) = \int_0^\infty e^{-ps}B_\lambda y(s)ds, \quad \text{and} \quad \varphi(p) = \int_0^\infty e^{-ps}B_\lambda[Bu(s)+f(s,y(s))]ds. \qquad (7.2.3)$$

Note that for $t \in (0,b]$, $B_\lambda y(t)$, $B_\lambda[Bu(t)+f(t,y(t))] \in \mathcal{D}(A)$, we have $\chi(p), \varphi(p) \in \overline{\mathcal{D}(A)}$. Applying the Laplace transform on (7.2.1), and using (7.2.3) we have

$$\begin{aligned}\chi(p) =& p^{(1-\alpha)(1-\beta)-1}B_\lambda y_0 + \frac{1}{p^\beta}A\chi(p) + \frac{1}{p^\beta}\varphi(p)\\ =& p^{\alpha(\beta-1)}(p^\beta I - A)^{-1}B_\lambda y_0 + (p^\beta I - A)^{-1}\varphi(p)\\ =& p^{\alpha(\beta-1)}\int_0^\infty e^{-p^\beta s}\mathcal{S}(s)B_\lambda y_0 ds + \int_0^\infty e^{-p^\beta s}\mathcal{S}(s)\varphi(p)ds, \end{aligned} \qquad (7.2.4)$$

provided that the integrals in (7.2.4) exist. Let $\psi_\beta(\theta) = \frac{\beta}{\theta^{\beta+1}}M_\beta(\theta^{-\beta})$ whose Laplace transform is given by

$$\int_0^\infty e^{-p\theta}\psi_\beta(\theta)d\theta = e^{-p^\beta}, \quad \text{where} \quad \beta \in (0,1). \qquad (7.2.5)$$

Using (7.2.5), we have

$$\begin{aligned}\int_0^\infty e^{-p^\beta s}\mathcal{S}(s)B_\lambda y_0 ds =& \int_0^\infty \beta t^{\beta-1}e^{-(pt)^\beta}\mathcal{S}(t^\beta)B_\lambda y_0 dt\\ =& \int_0^\infty\int_0^\infty \beta t^{\beta-1}\psi_\beta(\theta)e^{-(pt\theta)}\mathcal{S}(t^\beta)B_\lambda y_0 d\theta dt\\ =& \int_0^\infty e^{-pt}\left[\int_0^\infty \beta\frac{t^{\beta-1}}{\theta^\beta}\psi_\beta(\theta)\mathcal{S}\left(\frac{t^\beta}{\theta^\beta}\right)B_\lambda y_0 d\theta\right]dt\\ =& \int_0^\infty e^{-pt}[t^{\beta-1}\mathcal{P}_\beta(t)B_\lambda y_0]dt \end{aligned} \qquad (7.2.6)$$

and

$$\begin{aligned}
&\int_0^\infty e^{-p^\beta s}\mathcal{S}(s)\varphi(p)ds\\
&=\int_0^\infty\int_0^\infty \beta t^{\beta-1}e^{-(pt)^\beta}\mathcal{S}(t^\beta)e^{-ps}B_\lambda[Bu(s)+f(s,y(s))]dsdt\\
&=\int_0^\infty\int_0^\infty\int_0^\infty \beta t^{\beta-1}\psi_\beta(\theta)e^{-(pt\theta)}\mathcal{S}(t^\beta)e^{-ps}B_\lambda[Bu(s)+f(s,y(s))]d\theta dsdt\\
&=\int_0^\infty\int_0^\infty\int_0^\infty \beta\frac{t^{\beta-1}}{\theta^\beta}\psi_\beta(\theta)e^{-p(t+s)}\mathcal{S}\Big(\frac{t^\beta}{\theta^\beta}\Big)B_\lambda[Bu(s)+f(s,y(s))]d\theta dsdt\\
&=\int_0^\infty e^{-pt}\int_0^t\int_0^\infty \frac{\beta(t-s)^{\beta-1}}{\theta^\beta}\psi_\beta(\theta)\mathcal{S}\Big(\frac{(t-s)^\beta}{\theta^\beta}\Big)B_\lambda[Bu(s)+f(s,y(s))]d\theta dsdt\\
&=\int_0^\infty e^{-pt}\Big[\int_0^t[(t-s)^{\beta-1}\mathcal{P}_\beta(t-s)B_\lambda[Bu(s)+f(s,y(s))]ds\Big]dt. \qquad (7.2.7)
\end{aligned}$$

By the inverse Laplace transform of $p^{\alpha(\beta-1)}$, we get

$$L^{-1}(p^{\alpha(\beta-1)})=\begin{cases}\frac{t^{\alpha(1-\beta)-1}}{\Gamma(\alpha(1-\beta))}, & 0<\alpha\le 1;\\ \delta(t), & \alpha=0,\end{cases}$$

where $\delta(t)$ stands for Dirac delta function.
Thus, by (7.2.4), (7.2.6) and (7.2.7), for $t\in(0,b]$ and using the inverse Laplace transform, we have

$$\begin{aligned}
B_\lambda y(t)=&(L^{-1}(p^{\alpha(\beta-1)})*\mathcal{T}_\beta)(t)B_\lambda y_0+\int_0^t(t-s)^{\beta-1}\mathcal{P}_\beta(t-s)B_\lambda[Bu(s)+f(s,y(s))]ds\\
=&J_{0^+}^{\alpha(\beta-1)}\mathcal{T}_\beta(t)B_\lambda y_0+\int_0^t(t-s)^{\beta-1}\mathcal{P}_\beta(t-s)B_\lambda[Bu(s)+f(s,y(s))]ds\\
=&\mathcal{S}_{\alpha,\beta}(t)B_\lambda y_0+\int_0^t(t-s)^{\beta-1}\mathcal{P}_\beta(t-s)B_\lambda[Bu(s)+f(s,y(s))]ds.
\end{aligned}$$

In view of $\lim_{\lambda\to\infty}B_\lambda y=y$, for $y\in\mathbb{X}_0$, and Lemma 7.2.1, for $t\in(0,b]$, we have

$$y(t)=\mathcal{S}_{\alpha,\beta}(t)y_0+\lim_{\lambda\to\infty}\int_0^t(t-s)^{\beta-1}\mathcal{P}_\beta(t-s)B_\lambda[Bu(s)+f(s,y(s))]ds. \qquad (7.2.8)$$

This completes the proof. □

Remark 7.2.3. (*i*) *If* $\alpha=0$ *and* $\overline{\mathcal{D}(A)}=\mathbb{X}$, *then the fractional system* (7.1.2) *transforms in classical Riemann Liouville fractional differential equation studied in [140].*

(*ii*) *If* $\alpha=1$, *then the fractional system* (7.1.2) *transforms in classical Caputo fractional differential equation studied in [220].*

Lemma 7.2.4. *[94; 163; 214] The following properties hold :*

(i) The operator $\mathcal{P}_\beta(t)$ is continuous in the uniform operator topology for $t > 0$.

(ii) For $t > 0$, $\{\mathcal{S}_{\alpha,\beta}(t)\}$ and $\{\mathcal{T}_\beta(t)\}$ are strongly continuous.

(iii) For any fixed $t > 0$, $\{\mathcal{S}_{\alpha,\beta}(t)\}$ and $\{\mathcal{T}_\beta(t)\}$ are linear operator, and for any $y \in \mathbb{X}_0$, we have

$$\|\mathcal{T}_\beta(t)y\| \leq \frac{Mt^{\beta-1}}{\Gamma(\beta)}\|y\|, \quad \textit{and} \quad \|\mathcal{S}_{\alpha,\beta}(t)y\| \leq \frac{Mt^{\gamma-1}}{\Gamma(\alpha(1-\beta)+\beta)}\|y\|.$$

We denote by $y(t) = y(t, y_0, u)$ the state value of system (7.1.2) corresponding to the control function u and initial values y_0 at the time t. The set $\mathfrak{R}_b(u, y_0, f) := \{y(b, y_0, u) : u(\cdot) \in L^p([0, b], \mathbb{U})\}$ is called the reachable set of the system (7.1.2) corresponding to the function $f(t, y(t))$, and its closure is denoted by $\overline{\mathfrak{R}_b(u, y_0, f)}$. Thus the set $\mathfrak{R}_b(u, y_0, f)$ consists of all possible final states.

Definition 7.2.2. *The system (7.1.2) is said to be approximate controllable on the time interval $[0, b]$ if and only if $\overline{\mathfrak{R}_b(u, y_0, f)} = \mathbb{X}_0$.*

In other words, the system (7.1.2) is approximate controllable on the time interval $[0, b]$, if for every arbitrary final state $y_b \in \mathbb{X}_0$ and $\epsilon > 0$, there exists a control function $u \in L^p([0, b], \mathbb{U})$ such that $\|y_b - y(b, y_0, u)\| < \epsilon$.

7.3 Existence and Uniqueness Results

In this section, we establish some existence and uniqueness results for integral solution to the system (7.1.2). We consider the following assumptions:

(A_1) There exists a nonnegative function $\psi(\cdot) \in L^p([0, b], \mathbb{R}^+)$ and a constant $k > 0$ such that

$$\|f(t, y)\| \leq \psi(t) + kt^{1-\gamma}\|y\|, \quad \text{for all } \ y \in \mathbb{X}, \ \text{and } t \in [0, b] \ a.e.$$

(A_2) There exists a positive constant L_f such that

$$\|f(t, y) - f(t, z)\| \leq L_f\|y - z\|, \quad \text{for all } \ y, z \in \mathbb{X}, \ \text{and } t \in [0, b] \ a.e.$$

Theorem 7.3.1. *Let $(A_0), (A_1)$ and (A_2) be true. Then, for each control function $u \in L^p([0,b],\mathbb{U})$, the Hilfer fractional control system (7.1.2) has a unique integral solution on Z.*

Proof. Consider an operator $\Phi : Z \to Z$ defined by

$$(\Phi y)(t) = \mathcal{S}_{\alpha,\beta}(t)y_0 + \lim_{\lambda\to\infty}\int_0^t (t-s)^{\beta-1}\mathcal{P}_\beta(t-s)B_\lambda[Bu(s)+f(s,y(s))]ds, \quad (7.3.1)$$

By the definitions of $\mathcal{S}_{\alpha,\beta}(t)$ and $\mathcal{P}_\beta(t)$ with (A_1), we have

$$\begin{aligned}\lim_{t\to 0^+} t^{(1-\beta)(1-\alpha)}\mathcal{S}_{\alpha,\beta}(t)y_0 =& \lim_{t\to 0^+} t^{(1-\beta)(1-\alpha)}\frac{y_0}{\Gamma(\alpha(1-\beta))}\int_0^t (t-s)^{\alpha(1-\beta)-1}s^{\beta-1}\mathcal{P}_\beta(s)ds\\ =&\frac{y_0}{\Gamma(\alpha(1-\beta)+\beta)},\end{aligned}$$

$$\begin{aligned}\text{and}\quad &\lim_{t\to 0^+}\left\| t^{(1-\beta)(1-\alpha)}\lim_{\lambda\to\infty}\int_0^t (t-s)^{\beta-1}\mathcal{P}_\beta(t-s)B_\lambda[Bu(s)+f(s,y(s))]ds\right\|\\ &\le \lim_{t\to 0^+} t^{(1-\beta)(1-\alpha)}\frac{MM_0}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}\|Bu(s)+f(s,y(s))\|ds \to 0.\end{aligned}$$

Moreover, by the strong continuity of $\mathcal{S}_{\alpha,\beta}(t)$ and $\mathcal{P}_\beta(t)$ with the assumption (A_1), it is not difficult to show $(\Phi y)(\cdot)$ is continuous. Thus Φ maps Z into itself and

$$t^{(1-\gamma)}(\Phi y)(t) = \begin{cases} t^{(1-\gamma)}\mathcal{S}_{\alpha,\beta}(t)y_0 + \lim_{\lambda\to\infty} t^{(1-\gamma)}\int_0^t (t-s)^{\beta-1}\mathcal{P}_\beta(t-s) & \\ \times B_\lambda[Bu(s)+f(s,y(s))]ds, & t\in(0,b];\\ \frac{y_0}{\Gamma(\alpha(1-\beta)+\beta)}, & t=0.\end{cases}$$

Next, we show that the operator Φ^{n_0} is a contraction operator for large enough natural number n_0 on the space Z. In fact, for $y, z \in Z$ and $t\in(0,b]$, we have

$$\begin{aligned} t^{1-\gamma}&\|(\Phi y)(t)-(\Phi z)(t)\|\\ &=t^{1-\gamma}\left\|\lim_{\lambda\to\infty}\int_0^t (t-s)^{\beta-1}\mathcal{P}_\beta(t-s)B_\lambda[f(s,y(s))-f(s,z(s))]ds\right\|\\ &\le\frac{MM_0L_f t^{1-\gamma}}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}\|y(s)-z(s)\|ds\\ &\le\frac{MM_0L_f t^{1-\gamma}}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}s^{\gamma-1}[s^{1-\gamma}\|y(s)-z(s)\|]ds\\ &\le\left[\frac{\Gamma(\gamma)MM_0L_f t^{\beta}}{\Gamma(\gamma+\beta)}\right]\|y-z\|_Z.\end{aligned}$$

Similarly, for $y, z \in Z$ and $t \in (0, b]$, we have

$$
\begin{aligned}
t^{1-\gamma}\|(\Phi^2 y)(t) - (\Phi^2 z)(t)\| & \\
&\leq \frac{MM_0 L_f t^{1-\gamma}}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} s^{\gamma-1} [s^{1-\gamma}\|(\Phi y)(s) - (\Phi z)(s)\|] ds \\
&\leq \frac{MM_0 L_f t^{1-\gamma}}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} s^{\gamma-1} \left[\left[\frac{\Gamma(\gamma) MM_0 L_f s^{\beta}}{\Gamma(\gamma+\beta)}\right] \|y - z\|_Z\right] ds \\
&\leq \left[\frac{\Gamma(\gamma)(MM_0 L_f t^{\beta})^2}{\Gamma(\gamma+2\beta)}\right] \|y - z\|_Z.
\end{aligned}
$$

Using the above iteration up to a natural number n, we have

$$t^{1-\gamma}\|(\Phi^n y)(t) - (\Phi^n z)(t)\| \leq \left[\frac{\Gamma(\gamma)(MM_0 L_f t^{\beta})^n}{\Gamma(\gamma+n\beta)}\right] \|y - z\|_Z.$$

Taking supremum, we get

$$\|(\Phi^n y) - (\Phi^n z)\|_Z \leq \left[\frac{\Gamma(\gamma)(MM_0 L_f b^{\beta})^n}{\Gamma(\gamma+n\beta)}\right] \|y - z\|_Z.$$

Note that $\left[\frac{(MM_0 L_f b^{\beta})^k}{\Gamma(\gamma+k\beta)}\right]$ is a general term of the Mittag-Leffler series $E_{\beta,\gamma}(MM_0 L_f b^{\beta})$ $= \sum_{k=0}^{\infty} \frac{(MM_0 L_f b^{\beta})^k}{\Gamma(\gamma+k\beta)}$ and this series is uniformly convergent on $[0, b]$. Therefore, we can obtain a large enough natural number n_0 such that $\left[\frac{\Gamma(\gamma)(MM_0 L_f b^{\beta})^{n_0}}{\Gamma(\gamma+n_0\beta)}\right] < 1$. Hence, the operator Φ^{n_0} is contraction operator on Z. Now, as a consequence of generalized Banach contraction principle, we conclude that the operator Φ has a unique fixed point say $y(\cdot)$ on Z, which is the required integral solution of the system (7.1.2). The proof is complete. □

7.4 Approximate Controllability Results

In this section, the sufficient conditions for approximate controllability to the Hilfer fractional control system (7.1.2) are established by sequencing technique. For this purpose, we define a continuous linear mapping $\mathcal{Q} : L^p([0, b], \mathbb{X}) \to \mathbb{X}_0$ given by

$$\mathcal{Q}(\varphi) = \lim_{\lambda\to\infty} \int_0^b (b-s)^{\beta-1} \mathcal{P}_{\beta}(b-s) B_{\lambda} \varphi(s) ds, \quad \varphi \in L^p([0, b], \mathbb{X}).$$

We define by the pair (y, u) the integral solution of the system (7.1.2) corresponding to the control function $u \in L^p([0, b], \mathbb{U})$. We denote the terminal state of the integral solution y by $y_b = y(b)$ given by

$$y_b = y(b, y_0, u) = \mathcal{S}_{\alpha,\beta}(b)y_0 + \lim_{\lambda\to\infty}\int_0^b (b-s)^{\beta-1}\mathcal{P}_\beta(b-s)B_\lambda[Bu(s) + f(s, y(s))]ds.$$

So, the reachable set $\mathfrak{R}_b(u, y_0, f)$ of the control system (7.1.2) is given by

$$\mathfrak{R}_b(u, y_0, f) := \{y(b) = y(b, y_0, u) : u(\cdot) \in L^p([0, b], \mathbb{U})\}.$$

From the Definition 7.2.2, we notice that if $y_0 \in \mathbb{X}_0$, $u \in L^p([0, b], \mathbb{U})$ and $\overline{\mathfrak{R}_b(u, y_0, f)} = \mathbb{X}_0$, then the system (7.1.2) is approximately controllable on $[0, b]$. Equivalently, if for arbitrary final state $y_b \in \mathbb{X}_0$ and $\epsilon > 0$, there exists a control function $u_\epsilon \in L^p([0, b], \mathbb{U})$ such that the integral solution of the system (7.1.2) satisfies the condition

$$\|y_b - \mathcal{S}_{\alpha,\beta}(b)y_0 - \mathcal{Q}(F(s, y_\epsilon(s))) - \mathcal{Q}(Bu_\epsilon)\| < \epsilon,$$

where $y_\epsilon(s) = y(s, y_0, u_\epsilon)$ satisfies

$$y_\epsilon(s) = \mathcal{S}_{\alpha,\beta}(s)y_0 + \lim_{\lambda\to\infty}\int_0^s (s-\xi)^{\beta-1}\mathcal{P}_\beta(s-\xi)B_\lambda[F(\xi, y_\epsilon(\xi)) + Bu_\epsilon(\xi)]d\xi.$$

In what follows, to derive the approximate controllability results for the control system (7.1.2), we consider the following assumptions:

(A_2') For the nonlinear function $f : [0, b] \times \mathbb{X} \to \mathbb{X}$ there exists a constant $L_f' > 0$ such that

$$\|f(t, y) - f(t, z)\| \le L_f' t^{1-\gamma}\|y - z\|, \quad \text{for all } y, z \in \mathbb{X}, \text{ and } t \in [0, b].$$

(A_3) For $\varphi \in L^p([0, b], \mathbb{X})$ and given $\epsilon > 0$, there exists a control function $u \in L^p([0, b], \mathbb{U})$ such that

$$\|\mathcal{Q}(\varphi) - \mathcal{Q}(Bu)\| < \epsilon, \quad \text{and} \quad \|Bu\|_{L^p} \le \sigma\|\varphi\|_{L^p},$$

where $\sigma > 0$ is a constant independent of φ.

(A_4) The following inequality holds

$$\frac{\sigma L'_f M M_0 b^{1+\beta-\gamma-\frac{1}{p}}}{\Gamma(\beta)}\left(\frac{p-1}{p\beta-1}\right)^{1-\frac{1}{p}} E_{\beta,1}(MM_0L'_f b^{1+\beta-\gamma}) < 1. \tag{7.4.1}$$

It is clear that the condition (A_2) is implied by (A'_2). Therefore, for each control function $u \in L^p([0,b],\mathbb{U})$ the result of Theorem 7.3.1 still holds i.e. the control system (7.1.2) has a unique integral solution in Z.

Lemma 7.4.1. *Assume that the assumptions* $(A_0),(A_1)$ *and* (A_3) *hold. Then, the linear system corresponding to the control system* (7.1.2) (*putting* $f(t,y(t))=0$ *in* (7.1.2)) *is approximate controllable.*

Proof. We know that domain $\overline{\mathcal{D}(A_0)}$ of the operator A_0 is dense in $\mathbb{X}_0$, it is sufficient to show that $\mathcal{D}(A_0)\subseteq\overline{\mathfrak{R}_b(u,y_0,0)}$, that is, for each $y_b\in\mathcal{D}(A_0)$ and given $\epsilon>0$ there exists a control function $u\in L^p([0,b],\mathbb{U})$ such that

$$\|y_b-\mathcal{S}_{\alpha,\beta}(b)y_0-\mathcal{Q}(Bu)\|<\epsilon. \tag{7.4.2}$$

Let $y_b\in\mathcal{D}(A_0)$, then $y_b-\mathcal{S}_{\alpha,\beta}(b)y_0\in\mathcal{D}(A_0)$. Now, by definition of operator $\mathcal{Q}$ it may be seen that there exists some $\varphi\in L^p([0,b],\mathbb{X})$ such that

$$\zeta=\lim_{\lambda\to\infty}\int_0^b(b-s)^{\beta-1}\mathcal{P}_\beta(b-s)B_\lambda\varphi(s)ds,$$

where $\zeta=y_b-\mathcal{S}_{\alpha,\beta}(b)y_0$. By the assumption (A_3), we conclude that for any $\epsilon>0$ there exists a control function $u\in L^p([0,b],\mathbb{U})$ such that

$$\|\zeta-\mathcal{Q}(Bu)\|<\epsilon.$$

Thus, for $\zeta=y_b-\mathcal{S}_{\alpha,\beta}(b)y_0$ the inequality (7.4.2) holds and which implies that $\mathcal{D}(A_0)\subseteq\overline{\mathfrak{R}_b(u,y_0,0)}$. Further, the denseness of the domain $\mathcal{D}(A_0)$ in $\mathbb{X}_0$ implies that $\overline{\mathfrak{R}_b(u,y_0,0)}=\mathbb{X}_0$ i.e. the linear system corresponding to the control system (7.1.2) is approximately controllable. □

Lemma 7.4.2. *Let* (y_1,u_1) *and* (y_2,u_2) *be the integral solutions corresponding to the control functions* $u_1,u_2\in L^p([0,b],\mathbb{U})$. *Then, under the assumptions* (A_1) *and* (A'_2) *the following inequality holds:*

$$\begin{aligned}\|y\|_Z\le&C_1E_{\beta,1}(MM_0kb^{1+\beta-\gamma}),\\ \|y_1-y_2\|_Z\le&C_2E_{\beta,1}(MM_0L'_fb^{1+\beta-\gamma})\|Bu_1-Bu_2\|_{L^p},\end{aligned}$$

where $C_1=\frac{M}{\Gamma(\alpha(1-\beta)+\beta)}\|y_0\|+\frac{MM_0b^{1+\beta-\gamma-\frac{1}{p}}}{\Gamma(\beta)}\left(\frac{p-1}{p\beta-1}\right)^{1-\frac{1}{p}}[\|Bu\|_{L^p}+\|\psi\|_{L^p}]$ *and* $C_2=\frac{MM_0b^{1+\beta-\gamma-\frac{1}{p}}}{\Gamma(\beta)}\left(\frac{p-1}{p\beta-1}\right)^{1-\frac{1}{p}}$.

Proof. The integral solution $y(t) = y(t, y_0, u)$ of the system (7.1.2) in Z is given by

$$y(t) = \mathcal{S}_{\alpha,\beta}(t)y_0 + \lim_{\lambda\to\infty}\int_0^t (t-s)^{\beta-1}\mathcal{P}_\beta(t-s)B_\lambda[Bu(s) + f(s, y(s))]ds, \quad 0 < t \leq b.$$

Thus, for $y \in Z$, and $0 < t \leq b$, we have

$$\begin{aligned}
t^{1-\gamma}\|y(t)\| \leq & t^{1-\gamma}\|\mathcal{S}_{\alpha,\beta}(t)y_0\| + \left\| t^{1-\gamma}\lim_{\lambda\to\infty}\int_0^t (t-s)^{\beta-1}\mathcal{P}_\beta(t-s)B_\lambda Bu(s)ds\right\| \\
& + \left\| t^{1-\gamma}\lim_{\lambda\to\infty}\int_0^t (t-s)^{\beta-1}\mathcal{P}_\beta(t-s)B_\lambda f(s, y(s))ds\right\| \\
\leq & \frac{M}{\Gamma(\alpha(1-\beta)+\beta)}\|y_0\| + \frac{MM_0t^{1-\gamma}}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}\|Bu(s)\|ds \\
& + \frac{MM_0t^{1-\gamma}}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}[\psi(s) + ks^{1-\gamma}\|y(s)\|]ds \\
\leq & \frac{M\|y_0\|}{\Gamma(\alpha(1-\beta)+\beta)} + \frac{MM_0t^{1-\gamma}}{\Gamma(\beta)}\left(\int_0^t (t-s)^{q(\beta-1)}ds\right)^{\frac{1}{q}}\left(\int_0^t \|Bu(s)\|^p ds\right)^{\frac{1}{p}} \\
& + \frac{MM_0t^{1-\gamma}}{\Gamma(\beta)}\left(\int_0^t (t-s)^{q(\beta-1)}ds\right)^{\frac{1}{q}}\left(\int_0^t \|\psi(s)\|^p ds\right)^{\frac{1}{p}} \\
& + \frac{MM_0kt^{1-\gamma}}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}[s^{1-\gamma}\|y(s)\|]ds \\
\leq & \frac{M\|y_0\|}{\Gamma(\alpha(1-\beta)+\beta)} + \frac{MM_0b^{1+\beta-\gamma-\frac{1}{p}}}{\Gamma(\beta)}\left(\frac{p-1}{p\beta-1}\right)^{1-\frac{1}{p}}[\|Bu\|_{L^p} + \|\psi\|_{L^p}] \\
& + \frac{MM_0kb^{1-\gamma}}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}[s^{1-\gamma}\|y(s)\|]ds.
\end{aligned}$$

Set $t^{1-\gamma}\|y(t)\| = w(t)$, $t \in [0, b]$, then

$$w(t) \leq C_1 + \frac{MM_0kb^{1-\gamma}}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}w(s)ds,$$

Now, by Gronwall inequality (see Corollary 2 in [215] for convolution type integral equation, we get

$$w(t) \leq C_1E_{\beta,1}(MM_0kb^{1+\beta-\gamma}),$$

where $E_{\beta,1}$ is the Mittag-Leffler function. It follows that

$$\|y\|_Z = \sup_{t\in[0,b]}[t^{1-\gamma}\|y(t)\|] \leq C_1E_{\beta,1}(MM_0kb^{1+\beta-\gamma}).$$

Further, let y_1 and y_2 be the integral solutions of the system (7.1.2) corresponding to the control functions $u_1, u_2 \in L^p([0, b], \mathbb{U})$, respectively. Then, following the above steps, we may conclude that

$$\|y_1 - y_2\|_Z \leq C_2E_{\beta,1}(MM_0L'_fb^{1+\beta-\gamma})\|Bu_1 - Bu_2\|_{L^p}.$$

This completes the proof. □

Theorem 7.4.3. *Under the assumptions* (A_0), (A_1), (A_2'), (A_3) *and* (A_4) *the fractional differential system* (7.1.2) *with Hilfer derivative is approximate controllable on* $[0, b]$.

Proof. Since by the Lemma 7.4.1, we have $\overline{\mathfrak{R}_b(u, y_0, 0)} = \mathbb{X}_0$. Further, to show the approximate controllability of fractional order system (7.1.2), it is sufficient to show that $\overline{\mathfrak{R}_b(u, y_0, 0)} \subseteq \overline{\mathfrak{R}_b(u, y_0, f)}$. Thus, for given $\epsilon > 0$, and $\widetilde{y}_b \in \overline{\mathfrak{R}_b(u, y_0, 0)}$, there exists a control function $u \in L^p([0, b], \mathbb{U})$ such that

$$\|\widetilde{y}_b - \mathcal{S}_{\alpha,\beta}(b)y_0 - \mathcal{Q}(Bu)\| < \frac{\epsilon}{2^3}. \tag{7.4.3}$$

Now, we construct a sequence recursively as follows:

Let $u_1 \in L^p([0, b], \mathbb{U})$ be arbitrary control function. Then, by assumption (A_3), there exists a control function $u_2 \in L^p([0, b], \mathbb{U})$ such that

$$\|\mathcal{Q}\{(Bu) - f(s, y_1(s))\} - \mathcal{Q}(Bu_2)\| < \frac{\epsilon}{2^3}, \tag{7.4.4}$$

where $y_1(s) = y(s, y_0, u_1)$ satisfies

$$y_1(s) = \mathcal{S}_{\alpha,\beta}(s)y_0 + \lim_{\lambda\to\infty} \int_0^s (s-\xi)^{\beta-1}\mathcal{P}_\beta(s-\xi)B_\lambda[Bu_1(\xi) + f(\xi, y_1(\xi))]d\xi.$$

From (7.4.3) and (7.4.4), we have

$$\|\widetilde{y}_b - \mathcal{S}_{\alpha,\beta}(b)y_0 - \mathcal{Q}\{f(s, y_1(s))\} - \mathcal{Q}(Bu_2)\| < \frac{\epsilon}{2^2}. \tag{7.4.5}$$

For $u_2 \in L^p([0, b], \mathbb{U})$, by the assumption (A_3), we may determine $w_2 \in L^p([0, b], \mathbb{U})$ such that

$$\|\mathcal{Q}\{f(s, y_2(s)) - f(s, y_1(s))\} - \mathcal{Q}(Bw_2)\| < \frac{\epsilon}{2^3}, \tag{7.4.6}$$

also, by Lemma 7.4.2 and assumption (A_3), we have

$$\begin{aligned}\|Bw_2\|_{L^p} \leq& \sigma\|f(t, y_2) - f(t, y_1)\| \\ \leq& \sigma L_f' t^{1-\gamma}\|y_1 - y_2\| \\ \leq& \frac{\sigma L_f' M M_0 b^{1+\beta-\gamma-\frac{1}{p}}}{\Gamma(\beta)}\left(\frac{p-1}{p\beta-1}\right)^{1-\frac{1}{p}} E_{\beta,1}(MM_0L_f'b^{1+\beta-\gamma})\|Bu_1 - Bu_2\|_{L^p}.\end{aligned}$$

Now, define $u_3(\cdot) = u_2(\cdot) - w_2(\cdot)$ in $L^p([0, b], \mathbb{U})$, we obtain the following property

$$\begin{aligned}&\|\widetilde{y}_b - \mathcal{S}_{\alpha,\beta}(b)y_0 - \mathcal{Q}\{f(s, y_2(s))\} - \mathcal{Q}(Bu_3)\| \\ &\quad\leq \|\widetilde{y}_b - \mathcal{S}_{\alpha,\beta}(b)y_0 - \mathcal{Q}\{f(s, y_1(s))\} - \mathcal{Q}(Bu_2) \\ &\qquad + \mathcal{Q}(Bw_2) - \mathcal{Q}\{f(s, y_2(s)) - f(s, y_1(s))\}\| \\ &\quad\leq \|\widetilde{y}_b - \mathcal{S}_{\alpha,\beta}(b)y_0 - \mathcal{Q}\{f(s, y_1(s))\} - \mathcal{Q}(Bu_2)\| \\ &\qquad + \|\mathcal{Q}(Bw_2) - \mathcal{Q}\{f(s, y_2(s)) - f(s, y_1(s))\}\|.\end{aligned}$$

Using (7.4.5) and (7.4.6), we get

$$\|\widetilde{y}_b - \mathcal{S}_{\alpha,\beta}(b)y_0 - \mathcal{Q}\{f(s, y_2(s))\} - \mathcal{Q}(Bu_3)\| < \left(\frac{1}{2^2} + \frac{1}{2^3}\right)\epsilon.$$

Thus, by induction, we get that there exists a sequence $\{u_n : n \geq 1\}$ in $L^p([0, b], \mathbb{U})$ such that

$$\|\widetilde{y}_b - \mathcal{S}_{\alpha,\beta}(b)y_0 - \mathcal{Q}\{f(s, y_n(s))\} - \mathcal{Q}(Bu_{n+1})\| < \left(\frac{1}{2^2} + \cdots + \frac{1}{2^{n+1}}\right)\epsilon, \qquad (7.4.7)$$

where $y_n(\cdot) = y(\cdot, y_0, u_n)$, $t \in [0, b]$ and

$$\begin{aligned}&\|Bu_{n+1} - Bu_n\|_{L^p}\\ &\quad\leq \frac{\sigma L'_f M M_0 b^{1+\beta-\gamma-\frac{1}{p}}}{\Gamma(\beta)} \left(\frac{p-1}{p\beta - 1}\right)^{1-\frac{1}{p}} E_{\beta,1}(M M_0 L'_f b^{1+\beta-\gamma}) \|Bu_n - Bu_{n-1}\|_{L^p}.\end{aligned}$$

It may be concluded by (7.4.1) that the sequence $\{Bu_n : n \geq 1\}$ is a Cauchy sequence in the Banach space $L^p([0, b], \mathbb{X})$ and thus we obtain a function $v \in L^p([0, b], \mathbb{X})$ such that

$$\lim_{n\to\infty} Bu_n = v, \qquad v \in L^p([0, b], \mathbb{X}).$$

Since the mapping $\mathcal{Q} : L^p([0, b], \mathbb{X}) \to \mathbb{X}_0$ is a continuous linear mapping. Therefore, for given $\epsilon > 0$, we may find some integer $n_0 > 0$ such that

$$\|\mathcal{Q}(Bu_{n_0+1}) - \mathcal{Q}(Bu_{n_0})\| < \frac{\epsilon}{2}. \qquad (7.4.8)$$

Using (7.4.7) and (7.4.8), we obtain

$$\begin{aligned}\|\widetilde{y}_b &- \mathcal{S}_{\alpha,\beta}(b)y_0 - \mathcal{Q}\{f(s, y_{n_0}(s))\} - \mathcal{Q}(Bu_{n_0})\|\\ &\leq \|y_b - \mathcal{S}_{\alpha,\beta}(b)y_0 - \mathcal{Q}\{f(s, y_{n_0}(s))\} - \mathcal{Q}(Bu_{n_0+1})\|\\ &\quad + \|\mathcal{Q}(Bu_{n_0+1}) - \mathcal{Q}(Bu_{n_0})\|\\ &< \left(\frac{1}{2^2} + \cdots + \frac{1}{2^{n_0+1}}\right)\epsilon + \frac{\epsilon}{2} < \epsilon.\end{aligned}$$

This implies that $\widetilde{y}_b \in \overline{\mathfrak{R}_b(u, y_0, f)}$. Thus, $\overline{\mathfrak{R}_b(u, y_0, 0)} \subseteq \overline{\mathfrak{R}_b(u, y_0, f)}$. Hence by denseness of $\mathfrak{R}_b(u, y_0, 0)$ in $\mathbb{X}_0$ the fractional order control system (7.1.2) involving Hilfer derivative is approximate controllable on $[0, b]$. This completes the proof. □

Theorem 7.4.4. *Let the range of the operator B i.e. $\mathcal{R}(B)$ be dense in $L^p([0, b], \mathbb{X}_0)$. Then, under the assumptions $(A_0), (A_1)$ and (A'_2), the fractional order system* (7.1.2) *is approximate controllable on* $[0, b]$.

Proof. Since the range of the operator B i.e. $\mathcal{R}(B)$ is dense in $L^p([0,b],\mathbb{X}_0)$, for any $\zeta \in L^p([0,b],\mathbb{X}_0)$ and $\delta > 0$, there exist some points $Bu(\cdot) \in \mathcal{R}(B)$, where $u \in L^p([0,b],\mathbb{U})$ such that

$$\|Bu - \zeta\|_{L^p} < \delta\|\zeta\|_{L^p}. \tag{7.4.9}$$

Now, we obtain

$$\begin{aligned}
\|\mathcal{Q}(\zeta) - \mathcal{Q}(Bu)\| \leq & \left\| \lim_{\lambda\to\infty} \int_0^b (b-s)^{\beta-1}\mathcal{P}_\beta(b-s)B_\lambda[\zeta(s) - Bu(s)]ds \right\| \\
\leq & \frac{MM_0}{\Gamma(\beta)} \left(\int_0^b (b-s)^{q(\beta-1)}ds \right)^{\frac{1}{q}} \left(\int_0^b \|\zeta(s) - Bu(s)\|^p ds \right)^{\frac{1}{p}} \\
\leq & \frac{MM_0}{\Gamma(\beta)} \left[\frac{p-1}{p\beta - 1} \right]^{1-\frac{1}{p}} b^{\beta-\frac{1}{p}} \|\zeta - Bu\|_{L^p} \\
< & \frac{MM_0}{\Gamma(\beta)} \left[\frac{p-1}{p\beta - 1} \right]^{1-\frac{1}{p}} b^{\beta-\frac{1}{p}} \delta\|\zeta\|_{L^p} < \epsilon.
\end{aligned}$$

Also from (7.4.9), we get

$$\begin{aligned}
\|Bu\|_{L^p} \leq & \|Bu - \zeta + \zeta\|_{L^p} \\
\leq & \|Bu - \zeta\|_{L^p} + \|\zeta\|_{L^p} \\
\leq & \delta\|\zeta\|_{L^p} + \|\zeta\|_{L^p} \\
\leq & (\delta + 1)\|\zeta\|_{L^p}.
\end{aligned}$$

Thus from the above inequalities, we may infer that the assumptions (A_3) is satisfied. If we choose $\delta > 0$ such that the assumption (A_4) holds, then the approximate controllability of the fractional order system (7.1.2) follows from Theorem 7.4.3. This completes the proof. □

7.5 Example

We provide a concrete example to illustrate the feasibility of the established results.

Consider the control system involving the following partial differential system with Hilfer fractional derivative

$$D_{0^+}^{\alpha,\frac{2}{3}} z(t,x) = \frac{\partial^2}{\partial x^2} z(t,x) + \mu(t,x) + \frac{t^{\frac{1}{3}(1-\alpha)}e^t}{1+e^t} \sin(1 + z(t,x)), \quad t \in (0,1], \tag{7.5.1}$$

$$J_{0^+}^{\frac{1}{3}(1-\alpha)} z(t,x)|_{t=0} = z_0(x), \ \ x \in [0,\pi], \tag{7.5.2}$$

$$z(t,0) = z(t,\pi) = 0, \quad t \in (0,1], \tag{7.5.3}$$

where $D_{0+}^{\alpha,\frac{2}{3}}$ represents the Hilfer fractional derivative of order $\frac{2}{3}$ of type $\alpha \in [0,1]$ and $J_{0+}^{\frac{1}{3}(1-\alpha)}$ stands for Riemann-Liouville fractional order integral of order $\frac{1}{3}(1-\alpha)$. The function $\mu(t,x) : [0,1] \times [0,\pi] \to \mathbb{R}$ is a continuous function. Consider $\mathbb{X} = \mathbb{U} = \mathcal{C}([0,\pi],\mathbb{R})$ equipped with the uniform topology and the operator $A : \mathbb{X} \to \mathbb{X}$ defined by

$$\mathcal{D}(A) := \{z(t,\cdot) \in \mathcal{C}^2([0,\pi],\mathbb{R}) : \, z(t,0) = z(t,\pi) = 0\}, \quad Az(t,x) = \frac{\partial^2}{\partial x^2} z(t,x).$$

Then, we have

$$\overline{\mathcal{D}(A)} = \{z(t,\cdot) \in \mathcal{C}([0,\pi],\mathbb{R}) : \, z(t,0) = z(t,\pi) = 0\} \neq \mathbb{X}.$$

As we know from [50] that A satisfies Hille-Yosida condition with $(0,+\infty) \in \varrho(A)$, $\|(\lambda I - A)^{-1}\| \leq \frac{1}{\lambda}$ for $\lambda > 0$. Thus (A_0) satisfies with $M_0 = 1$. Since it is well known that A generates a C_0-semigroup $\mathcal{S}(t)$ in $\overline{\mathcal{D}(A)}$ such that $\|\mathcal{S}(t)\| \leq e^{-t} \leq 1 = M$. Set $w(t)(x) = z(t,x)$, then $f(t, z(t,x)) = f(t, w(t))$. Let the control operator $Bu : [0,1] \to \mathbb{R}$ defined by

$$(Bu)(t)(x) = \mu(t,x), \quad x \in [0,\pi].$$

Then, the system (7.5.1) – (7.5.3) has an abstract form of the system (7.1.2) i.e.

$$\begin{aligned} D_{0+}^{\alpha,\frac{2}{3}} w(t) &= Aw(t) + Bu(t) + \frac{t^{\frac{1}{3}(1-\alpha)} e^t}{1+e^t} \sin(1 + w(t)), \quad t \in (0,1], \\ J_{0+}^{(1-\alpha)\frac{1}{3}} w(t)|_{t=0} &= z_0, \end{aligned} \tag{7.5.4}$$

The system (7.5.4) can be handled by using classical semigroup theory when the nonlinear function satisfies the condition

$$f(t,0) = 0, \qquad 0 \leq t \leq 1, \tag{7.5.5}$$

In this case, the function f takes the values in $\overline{\mathcal{D}(A)}$ and A generates a strongly continuous semigroup on $\overline{\mathcal{D}(A)}$. However, the setting in this chapter allows the range of f to be $\mathbb{X}$ without the condition (7.5.5). In addition, the function $f : [0,1]\times\mathbb{X} \to \mathbb{R}$ given by

$$f(t, w(t)) = \frac{t^{\frac{1}{3}(1-\alpha)}e^t}{1+e^t}\sin(1+w(t)),$$

then the function f satisfies $(A_1), (A_2)$ with $\psi(t) = 0$, $k = 1$ and $L_f = 1$. Therefore, by Theorem 7.3.1 the system (7.5.1) – (7.5.3) has a unique integral solution. Moreover, the condition (A_2') is also satisfies with $L_f' = 1$. Further, if $\frac{\sigma}{\Gamma(\frac{2}{3})}\sqrt{3}E_{\beta,1}(1) < 1$ and the assumption (A_3) holds, then by Theorem 7.4.3 the system (7.5.1) – (7.5.3) is approximate controllable on $[0, 1]$.

List of Publications

List of papers published/accepted in International Refereed Journals

1. **Singh, V. and Pandey, D. N.**; A study of Sobolev type fractional impulsive differential systems with fractional nonlocal conditions, *International Journal of Applied and Computational Mathematics*, vol. 4, no. 1, (2018), 1–12.

2. **Singh, V. and Pandey, D. N.**; Pseudo almost automorphic mild solutions to some fractional differential equations with Stepanov-like pseudo almost automorphic forcing term, *Nonlinear Dynamics and Systems Theory*, vol. 17, no. 3, (2018), 409–420.

3. **Singh, V. and Pandey, D. N.**; Controllability of fractional impulsive quasilinear differential systems with state dependent delay, *International Journal of Dynamics and Control*, (2018), 1-13.

4. **Singh, V. and Pandey, D. N.**; Controllability of multi-term time-fractional differential systems, *Journal of Control and Decision*, DOI 10.1080/23307706.2018.1495584.

5. **Singh, V. and Pandey, D. N.**; Mild solutions for multi-term time-fractional impulsive differential systems, *Nonlinear Dynamics and Systems Theory*, 18(3), 307–318.

6. **Singh, V. and Pandey, D. N.**; Approximate controllability of Hilfer fractional differential system via sequencing technique, Accepted in *Journal of Mathematical Sciences*.

7. **Singh, V. and Pandey, D. N.**; Multi-term time-fractional stochastic differential equations with non-Lipschitz coefficients, Accepted in *Differential Equations and Dynamical Systems.*

8. **Singh, V. and Pandey, D. N.**; Controllability of second order Sobolev type impulsive delay differential systems, *Mathematical Methods in Applied Sciences*, DOI:10.1002/mma.5427.

9. **Singh, V. and Pandey, D. N.**; Exact controllability of multi-term time-fractional differential system with sequencing techniques, Accepted in *Indian Journal of Pure and Applied Mathematics.*

List of papers communicated in International Refereed Journals

1. **Singh, V. and Pandey, D. N.**; PC-mild solutions to Sobolev-type fractional differential equations with non-instantaneous impulses, communicated in *Mediterranean Journal of Mathematics.*

2. **Singh, V. and Pandey, D. N.**; Approximate controllability of second order non-autonomous stochastic impulsive differential system with delay, communicated in *Stochastic Analysis and Applications.*

3. **Singh, V. and Pandey, D. N.**; Approximate controllability of Hilfer fractional differential system with nondense domain via sequencing technique, communicated in *Collectanea Mathematica.*

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