MATHEMATICAL MODELING OF RENEWABLE RESOURCES

Ph. D. THESIS

by

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INDIAN INSTITUTE OF TECHNOLOGY ROORKEE
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A THESIS

Submitted in partial fulfilment of the requirements for the award of the degree

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in

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ROORKEE

CANDIDATE'S DECLARATION

I hereby certify that the work which is being presented in the thesis entitled **" MATHEAMTICAL MODELING OF RENEWABLE RESOURCES''** in partial fulfilment of the requirements for the award of the Degree of Doctor of Philosophy and submitted in the Department of Mathematics of the Indian Institute of Technology Roorkee, Roorkee is an authentic record of my own work carried out during a period from July, 2012 to December, 2017 under the supervision of Prof. Sunita Gakkhar, Department of Mathematics, Indian Institute of Technology Roorkee, Roorkee.

The matter presented in this thesis has not been submitted by me for the award of any other degree of this or any other Institution.

(REENU RANI)

This is to certify that the above statement made by the candidate is correct to the best of my knowledge.

(Sunita Gakkhar)
Supervisor

The Ph.D. Viva-Voce Examination of **Mrs. Reenu Rani,** Research Scholar, has been held on..

Chairman, SRC **Signature of External Examiner**

This is to certify that the student has made all the corrections in the thesis.

Signature of Supervisor **Head of the Department**
Dated:............................

Dedicated

to

My Parents and Family

Abstract

The main motive of the present thesis is to study the dynamical behavior of some bio-economical and ecological models in which to preserve the species from extinction. This thesis comprise of eight chapters.

The **Chapter 1** includes the literature survey, basic definitions and technical tools which will be used throughout the thesis. The chapter wise summary of the thesis is given below:

In **Chapter 2**, a nonlinear harvesting of a modified Leslie–Gower type predator-prey dynamical system is studied where harvesting effort is taken as a dynamic variable. The conditions of existence and local asymptotic stability of various equilibrium states have been obtained. The dynamical behavior of the interior state of the system shows that it is locally and globally asymptotically stable under certain condition. It is established that the coexistence of prey and predator population depend upon the proper harvesting strategies. Using analytical and numerical results, it is examined that for a fixed value of price per unit mass and other parameters of the system, as the value of cost per unit mass is increasing, the level of harvesting start increases. After some times, a level of cost is obtained where harvesting effort will tend to zero. Accordingly, for the coexistence of prey predator population along with effort, optimal level of cost is obtained.

In **Chapter 3**, model of chapter 2 is extended incorporating taxation as a control instrument. The existence of interior steady state of this system is strongly depends on range of taxation. The bionomic equilibrium of the system provides the range of harvesting rate which may be useful for a harvesting agency to get the profitable

yields. The sufficient condition for global stability of unique interior equilibrium point provides a domain for global solutions. The conditions of persistence for the system are derived. It is also investigated that coexistence of prey and predator population depends upon the proper harvesting strategies such as the risk of extinction of species can be avoided. The objective of this work includes both ecological and economic aspects. The economic objective is to maximize the net economic revenue and ecologically, want to keep the prey and predator population from extinction.

In **Chapter 4**, a two dimensional predator–prey dynamical system where predator is provided an additional food resource, is studied incorporating combined harvesting. In this chapter, prey and predator both are affected by some external toxicant substances which are harmful for both species. The steady states of the system and their stability analysis have been carried out for all possible feasible equilibrium points. The system undergoes some local bifurcations i.e., trans-critical, Hopf, saddle node bifurcations for a threshold level of parametric values. Also, some global bifurcations i.e., Bogdanov–Taken bifurcation (BT) and Generalized Hopf bifurcation (GH) are detected w.r.t. different parametric values. The sufficient condition for the bionomic equilibrium and optimal harvesting policy for the model is obtained.

Chapter 5 analyzed a mathematical model of a dynamical Stock-Effort system with nonlinear harvesting of species where taxation is used as a control instrument. This system is considered in two different fishing zones. The migration rates of fishing vessels between patches is assumed to be stock dependence. In this model, we have assumed that there are two time scales, a fast one for movement fish and boats between patches, and slow one corresponding to fish population growth and fishery dynamics. The aggregated method is used to simplify the mathematical analysis of the complete model. Qualitative analysis of this system reveals that nonlinear harvesting term plays an important role to determine the dynamics and bifurcation of system. Existence of bifurcations indicates that the high taxes will cause closed of fishery. However, the Maximum Sustainable yield (MSY) and Optimal Taxation Policy is discussed for the aggregated model.

In **Chapter 6**, a spatial prey predator mathematical model has been proposed and analyzed. This system is based upon the two time scales: fast one and slow one. Therefore, to simplify the mathematical analysis, an aggregation method is used. The effect of toxicity is considered in the system. The unique interior equilibrium point exists under certain condition and it is globally asymptotically stable. Bendixon–Dulac Criteria confirms that there does not exist any periodic solutions in the interior. Numerically it is also shown that the trajectories of aggregated model remain close to the trajectories of the complete model.

Chapter 7 investigates a predator–prey dynamic reaction model in a heterogeneous water body where only prey is subjected to harvesting. The consequences of prey refuge, availability of alternative food resource for predators and effects of harvesting effort on the dynamics of prey predator system are explored. The two time scales are considered in the dynamics of the model. The reduced aggregated model is analyzed analytically as well as numerically using dynamic of harvesting effort. Numerical simulation shows that introducing the dynamics of harvesting effort can destabilize prey predator system. This is confirmed by the bifurcation diagrams and dynamics of Lyapuonov exponent w.r.t. bifurcation parameter.

The **Chapter 8** is about the achievements and future scope and possible extension of the present work.

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Roorkee (Reenu Rani)
December , 2017

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Chapter 1

Introduction

1.1 General Introduction

Renewable natural resources such as fishery and forestry are essential for the survival and growth of human beings. These resources are to harvested to fulfill the needs and daily requirements. These resources are also harvested for many commercial purposes to attain the maximum profits. Due to rapid growth of human population and their requirements, the utilization of these resources has been increased extensively. The over exploitation of these resources have led to extinction. For sustainable development of these resources a proper Harvesting policy or management is required. This will control the over exploitations and fulfill the human demands. The main objective of bionomic modeling is to maximize utilization of resources without depleting the stocks to extinction.

In ecological systems, different levels of organizations are required: individual level, population level, community level and ecosystem level. Different time scales are associated with them. Aggregation methods are used when the system involves more than one time scale. By aggregating some variables ([1]-[7]), it is possible to obtain a reduced model governing few global variables which are varying at a slow time scale.

In this thesis, some mathematical models of fishery system are proposed and analyzed which deal with various types of interacting species using different harvesting strategies. The consideration of toxicity in ecological environment effects the growth of a fishery system. Such models are also studied in the presence of alternate food resources which helps the species from extinction. Further, some problems involving two time scales are analyzed using aggregated methods. A brief description of ecological background of various ecological systems and various harvesting strategies, relevant to the study of this research work are presented in this chapter. A detailed literature survey and mathematical models and tools are also presented.

1.2 Literature Survey

A brief literature survey is presented that includes the studies of those articles which are related to the research work of this thesis.

An excellent initiation to the optimal management of renewable resources is introduced by Clark [22]. He studied the optimal harvesting of a logistically growing species. The effect of combined harvesting in two ecologically independent species have been investigated by Clark [22] and Mesterton-Gibbons [89], [90]. Leung and Wang [77] proposed a mathematical model in which the phenomena of non-extinctive fishery resources is investigated. Brauer and Soudack [13] and Brauer et al. [16] have studied the dynamical behavior of predator–prey system with constant rate of prey harvesting. They have observed that the region of asymptotic stability can be reduced by using constant harvesting rate. Mesterton-Gibbons [91] proposed a Lotka–Volterra model of two independent populations and studied an optimal harvesting policy for the system. Fan and Wang [38] generalized the classical model given by Clark [22], [24] by modifying with time-dependent Logistic equation with periodic coefficients and they concluded that there exists a unique and positive periodic solution for the model and it is globally asymptotically stable for positive solutions. Martin and Ruan [82] investigated a predator–prey model using delay and constant rate of prey harvesting. They observed that maximum sustainable yield at equilibrium depends upon the carrying capacity of environment. Dubey et al. [32] also proposed and investigated an inshore-offshore fishery model where the fish population is being harvested in both areas. They have studied the stability analysis and optimal harvesting policy for the system where taxation is taken as a control instrument. Kar et al. [66] considered a mathematical model of one predator and two prey where both the preys grow logistically and harvested. Again Kar and Chattopadhay [67] proposed a single species model and this model has two stages: one is mature stage and second is an immature stage. They explored the existence of equilibrium points and their stability. They proved that the optimal harvesting policy of the system is much better to the MSY policy. The optimal paths of the system always take less time to reach the optimal steady state as compared to the suboptimal path.

Mena-Lorca et al. [88] investigated the dynamical behavior of Leslie-Gower type predator-prey model with proportional harvesting. Lin and Ho [79] proposed a modified Leslie-Gower type predator-prey model using Holling-type II functional response with time delay. They have studied the local and global dynamics of this system. Li and Xiao [78] also studied Leslie-Gower predator-prey model using Holling–type III functional response which gives a rich and a complex dynamics of the system. Song and Li [122] proposed and analyzed the periodic Modified Leslie-Gower type predator-prey model with Holling-type II scheme and they have investigated the impulsive effect for its dynamical behavior. Zhu and Lan [135] have investigated a Leslie-Gower predator prey model with constant harvesting rate in prey. They have observed that the interior state can be saddle, stable and unstable, saddle- node under certain parametric conditions. Zhang et al. [133] investigated Leslie-Gower tpye predator-prey model with proportional harvesting of both prey and predator and they studied the persistence and global stability of the system. The dynamics of Leslie- Gower type model subjected to Allee effect has been investigated by Rojas-Palma and Gonzalez [109] with proportional harvesting. Gupta and Banerjee [45] proposed a predator- prey model with non-linear harvesting of prey. The prey is considered growing logistically and predator is assumed to follow modified Leslie-Gower type predation. They found that the system exhibits complex dynamical behavior including several local and global bifurcations.

The effects of toxicants on different ecological communities have become a major problem in recent years. Several investigations are made to study the effects of toxicant on biological species using mathematical models. The Mathematical models dealing with such problems are studied by Hallam and Clark [46], Hallam et al. [47], Hallam and De Luna [48], De Luna and Hallam [28], Freedman and Shukla

[40] etc. Some other studies also include the works of Chattopadhyay [21], Shukla and Dubey [119], Dubey and Hussain [30], Shukla et al. [120], etc. Growing with the human needs, a huge amount of toxicant are emitted into environment (such as in marine water) from industries and household sources. These toxicants affects the growth of living organisms of that environment. In particular, Freedman and Shukla [40] investigated the toxicant effects in a single species and predator-prey system. On the other hand, Huaping and Ma [57] studied the effects of toxicant in two competing species system. Maynard Smith [81] investigated the effects of toxicant in a competitive two species Lotka-Volterra type system. He considered that each species produces toxicant substances to the other only in the presence of other species. Further, this idea was extended by Kar and Chaudhuri [63] to a two species competing fish species where both species are commercially exploited. Tapasi et al. [27] also incorporated the effects of toxicant in a predator-prey model.

In many ecological systems, it is observed that most of the predators do not feed on a single prey only. They also depend on some other alternate food resource (prey species). The role of alternative prey in sustaining predator population has been widely studied in literature (Baalen and Kivan [10], Rijn and Houten [107], Harwood and Obrycki [49]. Srinivasu et al. [124] proposed a predator-prey model and incorporated provision of an additional food to predator. In this system, he observed that for a suitable quality and quantity of the additional food, the asymptotic interior state of the system can be either an equilibrium state or a limit cycle. Sahoo [111] investigated a predator-prey model in presence of alternative food to predator. This model is incorporated with different growth rate functions and different functional responses. Furthermore, Sahoo [112] investigated that for the conservation of biological species, alternative food plays an important role in an ecosystem. Sahoo and Poria [113] have investigated a non-chemical consideration for controlling disease in prey population by providing alternative food sources to predator.

Various investigations are dealing with the effect of the refuge in a heterogeneous habitat of interconnected patches. This Spatial heterogeneity leads to the consideration of two types of dynamics: local interactions between species on one hand and their migration between different patches on the other. Aggregation methods have been extensively performed for continuous system of differential equations (Auger and Roussarie [1], Auger and Poggiale,([5], [3])) and for time discrete models (Bravo de la Parra et al. [86], Bravo de la Parra and Sanchez [18]). Auger and Poggiale [17], Auger and Chiorino [4] and Auger and Charles [6] investigated that it is possible to reduce the dimension of a system to obtain a reduced model using results provided by geometrical singular perturbation (GSP) theory that can be handled analytically. Using aggregated methods, Poggiale J.C. and Auger P. [105] have shown that the refuge has a stabilizing effect on the equilibrium for a simple Lotka-Volterra type model with density-independent migration and refuge.

1.3 Objectives of the Thesis

In this thesis, two types of ecological and bio-economical models (Predator–Prey model, Stock-Effort model) using different types of harvesting strategies are analyzed. The dynamical behavior of proposed models are investigated to see the effects of harvesting. Taxation is used as a control instrument. The main objective of present thesis is to a complete mathematical investigations for the proposed models and to describe significant results which are important from biological and economical point of view. The stability, bifurcations and complexity these systems has been carried out. Pontryagin's Maximum Principle is used to study the optimal control of the system. The proposed mathematical models and results are illustrated with the help of numerical simulations.

Some of the important and basic topics/tools used in this thesis are presented below to discuss the above mentioned problems in a proper prospective way.

1.4 The Mathematical Concepts

In this thesis, the attempts are made to study the dynamical models for resource management. In such models, the biological species are growing, interacting and being harvested. The mathematical models for growth, interactions and harvesting are discussed below:

1.4.1 Growth of Population

The most commonly used growth models for biological species growing in isolation are exponential growth model and logistic growth models. These are briefly described below:

Exponential Growth Model

In exponential growth, the populations growth rate increases in proportion to the size of the population. The rate of change of population density x with time t is given as

$$\frac{dx}{dt} = G(x) = rx \tag{1.4.1}$$

Here, r is the constant of proportionality. The parameter $r = b - d$ is interpreted as a difference between the birth rate b and the death rate d and it is called the intrinsic growth rate of population.

The solution of this model is given as follows:

$$x(t) = x_0 e^{rt} \tag{1.4.2}$$

where, x_0 is the initial population size. Accordingly, the population density x increases exponentially. It tends to infinity as $t \to \infty$. This means that density increases unboundedly without any restriction on resources which are required for growth. Such a population growth may be valid for a short time, but it cannot go on forever.

Logistic Growth Model

Due to the drawback of exponential growth, Pierre Verhulst (1838) developed a model, called logistic growth model and it is described as follows:

$$\frac{dx}{dt} = G(x) = rx\left(1 - \frac{x}{K}\right). \tag{1.4.3}$$

This model combines two ecological processes: reproduction and competition. Both processes depend on population density. Population density x increases with time when $0 < x < K$. However, it decreases when $x > K$. The constant K is defined as carrying capacity of environment.

Further, it is observed that $G(x)$ increases with x for $0 < x < K/2$, reaches maximum at $x = K/2$ and then it declines. It becomes zero for $x = K$.

The logistic equation gives the following solution:

$$x(t) = \frac{x_0 K}{x_0 + (K - x_0)e^{-rt}} \tag{1.4.4}$$

It can be observed that for initial population size $x_0 > 0$, the solution $x(t) \to K$ as $t \to \infty$.

1.4.2 Species Interactions in Ecosystem

The biological species do not exist in isolation. They depend on many other species for their vital activities. There are several types of interactions identified by ecologists. However, only predation interaction is considered in this work. Predation occurs when a predator (parasite) feeds on its prey (host).

In the following, some basic predator–prey models and different forms of their functional dependence are discussed.

Basic Predator–Prey Model

In general, the predator–prey dynamics is described by the the following system of two autonomous differential equations:

$$\begin{aligned} \frac{dx}{dt} &= xG(x) - ap(x)y \\ \frac{dy}{dt} &= -dy + bp(x)y \end{aligned} \tag{1.4.5}$$

The variables $x(t)$ and $y(t)$ represent the prey and predator densities, respectively. Let $G(x)$ denotes the growth rate of prey in absence of predator. However, the predator will decay exponentially in absence of its food (prey). This is represented by the first term in predator equation. The function $ap(x)y$ representing the interaction of prey with predator having negative effect on the dynamics of prey and this function is called functional response. On the other hand, the term $bp(x)y$ represents the per capita growth rate of predator due to prey consumption and this function is called numeric response.

Functional Response in ecology is the intake rate of a consumer as a function of food density. Various forms of response functions are available in the literature. They are classified into categories:
(i) Prey dependent functional response.
(ii) Ratio dependent functional response.

1. **Prey Dependent Functional Response:**

   Typically the response function $p(x)$ satisfy the following:

   (i). $p(0) = 0$.

   (ii). $p'(x) > 0$.

   This functional response is generally categorized into three types, Holling type-I, II, III.

   Holling Type-I (Linear) functional response is the response that assumes a linear increase in the intake rate of consumer with respect to food density.

$$p(x) = \begin{cases} \alpha x \quad for \quad 0 < x < \nu \\ \gamma \qquad for \quad x \geq \nu. \end{cases} \tag{1.4.6}$$

   Here ν is the value of resource at which predator is satiated at γ.

   Generally, this functional response is used in the Lotka-Volterra predator-prey model for large ν. The Lotka-Volterra model is the simplest and a basic model of predator–prey interactions which is of the following form:

$$\begin{aligned} \frac{dx}{dt} &= rx - axy, \\ \frac{dy}{dt} &= -dy + bxy. \end{aligned} \tag{1.4.7}$$

   The parameters a and b are real positive parameters describing the interactions between prey and predator.

   Holling Type-II (Cyrtoid) functional response describes that the attack rate of a predator increases with a decreasing rate with prey density. It increases until it becomes constant at saturation. This function will take the following form:

$$p(x) = \frac{\alpha x}{1+\alpha h x} = \frac{ax}{b+x}; \quad \alpha, h, a, b > 0 \tag{1.4.8}$$

The constant α is the search rate, h is the handling time, a is the maximum consumption rate and b is the half saturation rate. This is also called Michaelis-Menten type functional response.

Holling Type-III (Sigmoid) is the functional response in which the attack rate accelerates at first and then decelerates towards saturation.

$$p(x) = \frac{\alpha x^2}{1+\alpha h x^2} = \frac{ax^2}{b+x^2}; \quad \alpha, a, b > 0 \tag{1.4.9}$$

2. **Ratio Dependent Functional Response:**

The ratio dependent functional response is represented as $p\left(\frac{x}{y}\right)$. The earliest ratio dependent model was introduced by Leslie and it is discussed by Leslie-Gower and Pielou (1948, 1958). In Leslie–Gower model, the carrying capacity of the predator is proportional to the prey population. This model depends on the fact that both prey and predator population grow with some upper limits which are not identified in the Lotka-Volterra model. Accordingly, Leslie-Gower predator-prey model is given as follows:

$$\begin{aligned} \frac{dx}{dt} &= (r_1 - b_1 x)x - p(x)y, \\ \frac{dy}{dt} &= y\left(r_2 - \frac{a_2 y}{x}\right). \end{aligned} \tag{1.4.10}$$

The term $\frac{r_1}{b_1}$ gives the carrying capacity of the prey in the absence of predation. The functional response $p(x)$ represents the predator's consumption rate w.r.t. prey. The predator grows logistically with the growth rate r_2 and carrying capacity $\frac{r_2 x}{a_2}$ which is proportional to the density of prey population. The term $\frac{y}{x}$ is known as the Leslie-Gower term that measures the depletion in the predator population due to scarcity of its favorite food.

Further, it is observed that in the case of severe scarcity (i.e., $x \to 0$), predator can switch over to some alternate prey. Note that the model (1.4.10) is not well behaved mathematically as $x \to 0$. To solve this deficiency occurring in the system (1.4.10), Aziz-Alaoui and Daher [9] proposed and analyzed modified Leslie-Gower predator-prey model with Holling-type II schemes and it is given as follows:

$$\begin{aligned} \frac{dx}{dt} &= (r_1 - b_1 x)x - \frac{a_1 xy}{x + k_1}, && (1.4.11) \\ \frac{dy}{dt} &= y\left(r_2 - \frac{a_2 y}{x + k_2}\right). && (1.4.12) \end{aligned}$$

where r_1, b_1, r_2, a_2 have the same meaning as described in the above system (1.4.10). The parameter a_1 is the maximum reduction rate of x. k_1 and k_2 measure the extent to which environment provides protection to prey x and to predator y, respectively.

1.4.3 Harvesting

A Basic model of renewable resource harvesting [22] is described as

$$\begin{aligned} \frac{dx}{dt} &= G(x) - H(x,t) && (1.4.13) \\ x(0) &= x_0 \end{aligned}$$

where $x(t)$ denotes the density of the resource biomass at time t, $G(x)$ represents the net growth rate of the population biomass. The function $H(x,t)$ represents the rate of harvesting of resource stock at time t.

Basically, there are two types of harvesting policies: density independent and density dependent. In the following, a brief description of these functions are discussed.

a). **Density Independent Harvesting Function:**

In this form, a constant number of individuals are harvested per unit of time irrespective of the species density. Accordingly, the harvesting function $H(x,t)$ in the equation (1.4.13) will take the form:

$$H(x,t) = h \qquad (1.4.14)$$

Here, the value h represents the constant rate of harvesting per unit of time.

b). **Density Dependent Harvesting Function:**

In the density dependent harvesting functions, the following different forms of harvesting functions are discussed:

(i). **Proportional Harvesting**

The harvesting function H is considered to be proportional to the species densities. The constant of proportionality depends on the catch–ability q and the effort of harvesting E. Accordingly, harvesting function is presented as follows:

$$H(x, E) = qxE \tag{1.4.15}$$

where q is catchability coefficient, x is the resource biomass and E is the harvesting effort.

The following assumptions are made in proportional harvesting function:
(i). Random search for the resources.
(ii). Equal likelihood of being captured for every resources.

(ii). **Nonlinear Harvesting**

In the proportional harvesting, it can be seen that there is an unbounded linear increase of H with E for fixed x. Similarly, there is an unbounded linear increase of H with x for fixed E. To overcome this, Holling type-II harvesting function [26],[70] has been suggested and it is given as follows:

$$H(x, E) = \frac{qxE}{m_1 E + m_2 x} \tag{1.4.16}$$

where m_1 and m_2 are positive constant. This harvesting function is always saturated with respect to effort level and stock abundance. The parameter m_1 is proportional to the ratio of the stock-level to the catch rate at higher level of effort and m_2 is proportional to the ratio of the effort level to the catch rate at higher stock levels. In this harvesting function, the following can be observed:

a). $H(x,E) \to \dfrac{qE}{m_2}$ as $x \to \infty$ for fixed value of E.

b). $H(x,E) \to \dfrac{qx}{m_1}$ as $E \to \infty$ for fixed value of x.

c). $H(x,E)$ has singularity at $x = 0$ and $E = 0$.

In order to remove the singularity of $H(x,E)$ at $(0,0)$ in (1.4.16), the harvesting function $H(x,E)$ is modified as follows [37] :

$$H(x,E) = \frac{qxE}{1 + m_1 E + m_2 x} \tag{1.4.17}$$

1.4.4 Maximum Sustainable Yield

Maximum sustainable yield (or MSY) is the largest possible yield (or catch) that can be sustained over time. Generally, it is used in the context of fisheries, forestay and wildlife management.

Maximum Sustainable Yield (MSY) constitutes two fundamental goals: one is attaining the highest possible catch in consistent way and second is the population persistence in progression. This concept is based on biological growth models that assumes that a surplus stock exists that can be harvested at any given population level less than a certain level say carrying capacity K, without varying the stock level. Surplus stock level equals to the sustainable yield at each population level. This follows that MSY can be achieved at the population level where surplus stock level is maximum

1.4.5 Taxation

Many control Instruments such as taxation, license fees, seasonal harvesting, reserve area etc. are usually considered as possible instruments in resource regulation. Taxation is considered as a suitable and standard measure to manage and preserve the fishery system from over-exploitation. A regulatory agency should set taxes on effort or landings such that perceived bionomic equilibrium level for the fleet should be the optimal. Otherwise, for the resource biomass below the optimal stock, the revenue earned by the fishery will not exceed the cost used [22].

Clark ([22], [23]) studied a single-species fishery model using taxation as a control instrument. Based on this work, many researchers (Chaudhuri [19], Mesterton-Gibbons [90], Fan and Wang [38], Pradhan and Chudhuri [106], Dubey et al. ([30], [31], [33], [34], [36], [32]), Ji and Wu [62], Kar et al. [66], Misra and Dubey [95], Huo et al. [58]) have analyzed the utilization of numerous renewable resources using optimal management policy.

1.5 Aggregation Method

In ecological systems, different levels of organizations are required: individual level, population level, community level and ecosystem level. The co-evolution of these ecological levels are involved with the dynamics of an ecological system. Ecosystems and communities exhibits complex graph of interacting populations incorporating many state variables and parameters. Each population is divided into subpopulations that corresponds to their ages and individuals states etc. Individuals can migrate between spacial patches to feed or for many reasons. Environment fluctuations with seasons also effects the dynamics of population and community. Taking into account of all these aspects in the dynamics of a community, a complex or complicated model involving many variables can be considered. Different time scales are associated with these different ecological levels.

Aggregation methods are used when the system involves two different time scale: slow one and fast one. A system at slow time scale is associated at population or a community level which is called macro-system. The system at fast time scale is associated at an individual level and it is called micro-system. Taking the advantage of these two time scales, it is possible to obtain a reduced (aggregated) model. This reduced model is based on variables called global variables varying at a slow time scale. This reduced form describes the dynamics of a system at the community or population level. This aggregated method reduces the system in a simple way. Moreover, this reduced model is easier to handle analytically that the original one.

The following two types of aggregation methods are presented:

Perfect Aggregation corresponds to the exact replacement of the micro-system by a macro-system for an appropriate choice of global aggregated variables. This aggregation is the simplest case when reduced system can be associated to the original one. Each solution of the original (complete) system is associated to the solution of the aggregated one. This aggregation assumes very particular values of parameters. Generally, a perfect aggregation is not possible.

Approximate Aggregation corresponds to the replacement of the micro-system by a macro-system which is obtained by an approximation. It implies that some simplifications can be justified and some approximations can be realized. This is the case when some variables are fast with respect to others. Fast variables can rapidly reach an attractor and the approximation consists of replacing the fast variables with equilibrium values. This approximated version is not a simple copy of the micro-system, it is another system different from it, but having certain similarities to it at some level of observation. To perform approximate aggregation, a mathematical method is used which is based on Central Manifold Theorem.

1.5.1 Aggregation of System of ODE's with two time scales

A micro-system is constructed in N number of subsystems. This type of system is regarded as hierarchically organized and illustrated by a set of ordinary differential equations governing N number of micro-variables. Consider a population dynamics with micro variables ${x_i}^\alpha$ as densities of individuals of sub-populations i associating to population α. Considering τ is fast time scale and $t = \varepsilon\tau$ is slow time scale, where ε is a small dimensionless parameter, such a micro-system will take the following form:

$$\frac{d{x_i}^\alpha}{d\tau} = {f_i}^\alpha(x^1, x^2, ...x^N) + \varepsilon {F_i}^\alpha(x^1, x^2, ...x^N) \tag{1.5.1}$$

with $x^\alpha = ({x_1}^\alpha, {x_2}^\alpha, ...{x_N}^\alpha)$ and $\varepsilon << 1$, a very small parameter. Functions f_i and F_i correspond to the fast and slow part of the system (1.5.1).

While making transition from a micro-level to a macro-level, the macro variables are assumed to be invariant for the fast part of micro system. This invariance is necessary condition for introducing fast and slow time scales.

In the following, an aggregated method is presented in the case where the fast part of the system (1.5.1) can be obtained by putting $\varepsilon = 0$ and aggregated variables

$x^\alpha = \sum_i x_i{}^\alpha$. The aggregated (or reduced) system will take the following form:

$$\frac{dx_i{}^\alpha}{dt} = \sum_{i=1}^{N^\alpha} F_i{}^\alpha(v_1{}^{1*}x^1, v_2{}^{1*}x^2, ...v_1{}^{2*}x^1, v_2{}^{2*}x^2......v_1{}^{N*}x^N) \tag{1.5.2}$$

where $v_i{}^{\alpha*} = \dfrac{x_i{}^{\alpha*}}{x^\alpha}$ represents the equilibrium frequencies or proportions of each subpopulation at fast equilibrium. The dynamics of aggregated model (1.5.2) gives a good approximation of the dynamics of complete system (1.5.1). The aggregated method not only reduces the dimension and complexity of the complete (micro-system) but also provides some new and global properties emerging to the dynamics of system at macro level.

Coupling effects between the dynamics of slow system (aggregated system) and dynamics of fast system (complete system) can be performed. Equilibrium frequencies approach to a stable equilibrium at a fast time scale. There are two cases arises corresponding to the fast part of the micro system:

(i). When the fast part of the micro system is linear then the equilibrium frequencies $v_i{}^{\alpha*}$ tend to constant values.

(ii). When the fast part $f_i{}^\alpha$ of the micro system is non-linear, the equilibrium frequencies $v_i{}^{\alpha*}$ are not constant and they are the functions of slow variables $(x^1, x^2, ...x^N)$. The frequencies will take the form $v_i{}^{\alpha*}(x^1, x^2, ...x^N)$, Using these frequencies, the complete (micro) system (1.5.1) will take the following form:

$$\begin{aligned}\frac{dx_i{}^\alpha}{dt} &= \sum_{i=1}^{N^\alpha} F_i{}^\alpha(v_1{}^{1*}(x^1, x^2, ...x^N)x^1, v_2{}^{1*}(x^1, x^2, ...x^N)x^2, ...v_1{}^{2*}(x^1, x^2, ..\\ &\quad ....x^N)x^1, v_2{}^{2*}(x^1, x^2, ...x^N)x^2......v_1{}^{N*}(x^1, x^2, ...x^N)x^N)\end{aligned} \tag{1.5.3}$$

The reduced system (1.5.3) includes new terms w.r.t. slow part of the complete system because of density dependence in equilibrium frequencies. The fast system will approach to a different equilibrium for each set of slow variables. As a consequence, different fast part of the system coupled to the same slow system can lead to different global dynamics.

1.6 Mathematical Techniques/ Tools

The dynamical models in fisheries are represented by coupled system of non-linear differential equations To study the long term behavior of such systems, stability analysis play an important role. Further, these systems may have complex and rich dynamical behavior such as bifurcations, quasi-periodic or chaotic solutions. Variety of stability concepts are addressed in literature: local stability, global stability and stability of periodic solutions. Results related to stability analysis, bifurcations and optimal control for ordinary differential equations are presented in this section.

1.6.1 Stability Theory

Consider the following nonlinear autonomous system:

$$\frac{dX}{dt} = F(X); \quad X = (x_1, x_2, ..., x_n)^T, F = (F_1, F_2, ...F_n)^T \tag{1.6.1}$$

subject to the initial condition $x(0) = x_0 > 0$ and $F : D \to \mathbb{R}^n$ is locally Lipschitz map from domain D into $\mathbb{R}^n$.

The steady state $P^*({x_1}^*, {x_2}^*, ...{x_n}^*)$ of the system (1.6.1) are obtained by solving

$$\frac{dx_i}{dt} = F_i({x_1}^*, {x_2}^*, ...{x_n}^*) = 0 \quad for \quad i = 1, 2, ...n \tag{1.6.2}$$

Definition 1.6.1. The steady state $x_i = {x_i}^*$ of the system (1.6.1) is said to be Stable if , for $\epsilon > 0$, there exists $\delta > 0$ such that

$$\|x_i(t) - {x_i}^*\| < \epsilon \quad for \quad \|x_i(0) - {x_i}^*\| < \delta \quad \forall t \geq 0, \tag{1.6.3}$$

Definition 1.6.2. The steady state $x_i = {x_i}^*$ is said to be unstable, if it is not stable.

Definition 1.6.3. The steady state $x_i = {x_i}^*$ is said to asymptotically stable if it is stable and for the value of $\delta > 0$ it can be obtained

$$\|x_i(0) - {x_i}^*\| < \delta \Rightarrow \lim_{t\to\infty} x_i(t) = {x_i}^*. \tag{1.6.4}$$

To determine the linear stability of the steady state, consider a small perturbation u_i from the steady state by assuming

$$x_i = x_i{}^* + u_i \quad for \quad i = 1, 2, ...n \tag{1.6.5}$$

Using Taylor Series expansion about $x_i = x_i{}^*$ by substituting the value (1.6.5) in the system (1.6.1) and neglecting the second and higher order terms, the following linearized system can be obtained.

$$\frac{du}{dt} = Ju \quad for \quad i = 1, 2, ...n \tag{1.6.6}$$

where $u = (u_1, u_2, ..., u_n)^T$ and $J = D_X F(X^*) = \left(\frac{\partial F_i}{\partial x_j}\right)_{X=X^*}$ is called the jacobian matrix of the system (1.6.1). The steady state $X = X^*$ is said to be locally stable if all of the eigenvalues of the Jacobian matrix have negative real parts [102]. On the other hand the system is said to be unstable if at least one of the eigenvalues have positive real part.

Next, asymptotic stability of a steady state of a non-linear autonomous system can be established by using the Hartman-Grobman theorem. The theorem states that the dynamical behavior of a non-linear system (1.6.1) near a hyperbolic equilibrium point (i.e. no eigenvalue of the linearized matrix has real part equal to zero) in a given domain is qualitatively same as the behavior of its linearized system (1.6.6) near equilibrium point.

Theorem 1.6.1. ***(Hartman-Grobman Theorem)*** *[69]*
Consider the following nonlinear autonomous system:

$$\frac{dX}{dt} = F(X) \tag{1.6.7}$$

for some smooth mapping $F : \mathbb{R}^n \to \mathbb{R}^n$.

If $X = X^*$ *is a hyperbolic equilibrium point, then there is homeomorphism from* $\mathbb{R}^n$ *to* $\mathbb{R}^n$ *defined in the neighborhood of* $X = X^*$ *that maps from the trajectories of a nonlinear system to the trajectories of a linearized system* (1.6.6)*. This means that these two system have same qualitative behavior in the neighborhood of* $X = X^*$.

Routh-Hurwitz Criterion: ([100], [102]) This Criterion gives the necessary and sufficient conditions under which all roots λ_i (i.e., eigenvalues) of a characteristic polynomial lie in the left half of the complex plane. The characteristic polynomial corresponding to the system (1.6.6) is obtained as

$$\lambda^n + A_1\lambda^{n-1} + A_2\lambda^{n-2} + ... + A_n = 0 \tag{1.6.8}$$

For $n = 2$, the necessary and sufficient conditions for all roots of above polynomial equation lie in the left half of the complex plane are

$$A_1 > 0 \quad and \quad A_2 > 0 \tag{1.6.9}$$

For $n = 3$, the necessary and sufficient conditions will take the following form:

$$A_1 > 0, A_2 > 0 \quad and \quad A_1A_2 - A_3 > 0. \tag{1.6.10}$$

1.6.2 Bendixon-Dulac Theorem

This criteria ensures that there does not contain any periodic solution in a given region. Basically, this criteria is useful to prove that a locally asymptotically stable equilibrium point is globally stable or not.

Consider D be a simply connected region and a planer system defined in D is given as follows:

$$\begin{aligned} \frac{dx}{dt} &= f(x,y) \\ \frac{dy}{dt} &= g(x,y). \end{aligned} \tag{1.6.11}$$

Consider a function $B(x,y) \in C^1$ in a simply connected region $D \subset \mathbb{R}^2$. If the expression $\dfrac{\partial(Bf)}{\partial x} + \dfrac{\partial(Bg)}{\partial y}$ doesn't change sign in connected region D of plane and it is not identically equal to zero. Then the system does not have nonconstant periodic solutions lying entirely in the region D. [128]

1.6.3 Lyapunov's Direct Method

To analyze a nonlinear dynamical system, Lyapunov stability theory plays a vital role. Lyapunov stability theory generally includes Lyapunov's first and second methods. The Lyapunov's first method is based on lowest order approximation around a given point. This result is applicable in a small neighborhood of the point under consideration. The stability theory based on Lyapunov's second method for the dynamical system (1.6.1) having an equilibrium at $X = 0$, is described as below:

Let $\Psi \subset D$ be a sub–region containing the origin in its interior. Consider a function $V(X) : \mathbb{R}^n \to \mathbb{R}$ such that

(i). $V(X) = 0$ if and only if $X = 0$.

(ii). $V(X) > 0$ if and only if $X \neq 0$.

(iii). $\dot{V}(X) = \dfrac{dV(X)}{dt} \leq 0$.

Then $V(X)$ is called a Lyapunov function and the zero solution of given system is stable. If $\dfrac{dV(X)}{dt} < 0, X \in \Psi$, then the solution is asymptotically stable. However, if $\dfrac{dV(X)}{dt} > 0, X \in \Psi$ then the solution is unstable. If $\dfrac{dV(X)}{dt} = 0, X \in \Psi$, then origin is a center, i.e., all solutions are periodic solutions.

Moreover, the method does not provide any general technique for construction of Lyapunov function.

1.6.4 Bifurcation

Bifurcation of a dynamical system is a qualitative change in its dynamics produced by varying parameters.

Definition 1.6.4. Consider an autonomous system of ordinary differential equations (ODEs)

$$\frac{dX}{dt} = f(X, \mu); \quad f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n \tag{1.6.12}$$

where f is smooth. A bifurcation occurs at parameter $\mu = \mu_0$ if there is a parametric value μ_1 arbitrarily close to μ_0 with dynamics topologically inequivalent from those at μ_0. The number or stability of equilibria or periodic orbits of function f may change with perturbations of μ from μ_0.

Local Bifurcations

A local bifurcation occurs when a change in parametric value causes the change in stability of an equilibrium (or fixed point). In continuous systems, this corresponds to the real part of an eigenvalue of an equilibrium passing through zero. The equilibrium point will become non-hyperbolic at the bifurcation point. In other words, A local bifurcation occurs at (X_0, μ_0) if the Jacobian matrix $df \mid_{(X_0,\mu_0)}$ has an eigenvalue with zero real part.

In this thesis, some local bifurcations e.g. transcritical, saddle-node and Hopf bifurcation are studied which are discussed as follows:

Transcritical Bifurcation

In transcritical bifurcation, an equilibrium point of a dynamical system always exists w.r.t. all values of a parameter and is never destroyed. This point interchanges its stability with some another equilibrium point as the parametric value is varied. This gives one stable and other unstable equilibrium point before and after the bifurcation. In other words, the stability of equilibrium points exchanges when they collide. So the unstable fixed point becomes stable and visa versa.

Saddle-Node Bifurcation

A saddle-node bifurcation occurs when two equilibria of a dynamical system collides and then disappear. At this bifurcation point, the equilibrium has one zero eigenvalue. This phenomenon is also called fold or limit point bifurcation.

Theorem 1.6.2. ***(Sotomayer's Theorem)*** *[102] Consider the following nonlinear autonomous system:*

$$\frac{dX}{dt} = F(X, \mu); \quad F = (F^{(1)}, F^{(2)}......F^{(n)})^T, \quad X \in \mathbb{R}^n, \quad \mu \in \mathbb{R}. \qquad (1.6.13)$$

Suppose that $F(X^*, \mu^*) = 0$ and that the $n \times n$ matrix $J = D_X F(X^*, \mu^*) \equiv DF(X^*, \mu^*)$ has a simple eigenvalue $\mu = 0$ with eigenvector and that J^T has an eigenvector W corresponding to eigenvalue $\mu = 0$. Furthermore, suppose that J has k eigenvalues with negative real part and $(n - k - 1)$ eigenvalues with positive real *part and that the following condition are satisfied*

$$W^T F_\mu(X^*,\mu^*) \neq 0 \quad \text{and} \quad W^T\big[D^2F_\mu(X^*,\mu^*)(V,V)\big] \neq 0. \tag{1.6.14}$$

Then there is a smooth curve of equilibrium points of system (1.6.12) *in* $\mathbb{R}^n \times \mathbb{R}$ *passing through* (X^*,μ^*) *and tangent to hyperplane* $\mathbb{R}^n \times \mu^*$. *Depending on the signs of the expressions in* (1.6.14), *there are no equilibrium points of* (1.6.12) *near* X^* *when* $\mu < \mu^*$ *(or* $\mu > \mu^*$*) and there are two equilibrium points of* (1.6.12) *near* X^* *when* $\mu > \mu^*$ *(or* $\mu < \mu^*$*). The two equilibrium points of* (1.6.12) *near* X^* *are hyperbolic and have stable manifolds of dimension* k *and* $k+1$, *respectively, i.e., the system* (1.6.12) *experiences a saddle-node bifurcation at the equilibrium point* X^* *as the parameter* μ *passes through the bifurcation value* $\mu = \mu^*$

If the condition (1.6.14) *are changed to*

$$\begin{aligned} W^T F_\mu(X^*,\mu^*) &= 0, \\ W^T\big[DF_\mu(X^*,\mu^*)V\big] &\neq 0 \quad \text{and} \\ W^T\big[D^2F_\mu(X^*,\mu^*)(V,V)\big] &\neq 0. \end{aligned} \tag{1.6.15}$$

with

$$DF = (\nabla F^{(1)}, \nabla F^{(2)}........, \nabla F^{(n)})^T = \begin{pmatrix} \frac{\partial F^{(1)}}{\partial x_1} & \frac{\partial F^{(2)}}{\partial x_1} & \cdots & \cdots & \frac{\partial F^{(n)}}{\partial x_1} \\ \frac{\partial F^{(1)}}{\partial x_2} & \frac{\partial F^{(2)}}{\partial x_2} & \cdots & \cdots & \frac{\partial F^{(n)}}{\partial x_2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial F^{(1)}}{\partial x_n} & \frac{\partial F^{(2)}}{\partial x_n} & \cdots & \cdots & \frac{\partial F^{(n)}}{\partial x_n} \end{pmatrix}^T$$

$$D^2F = (\nabla^2 F^{(1)}, \nabla^2 F^{(2)}........, \nabla^2 F^{(n)})^T = \begin{pmatrix} \nabla\frac{\partial F^{(1)}}{\partial x_1} & \nabla\frac{\partial F^{(2)}}{\partial x_1} & \cdots & \cdots & \nabla\frac{\partial F^{(n)}}{\partial x_1} \\ \nabla\frac{\partial F^{(1)}}{\partial x_2} & \nabla\frac{\partial F^{(2)}}{\partial x_2} & \cdots & \cdots & \nabla\frac{\partial F^{(n)}}{\partial x_2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \nabla\frac{\partial F^{(1)}}{\partial x_n} & \nabla\frac{\partial F^{(2)}}{\partial x_n} & \cdots & \cdots & \nabla\frac{\partial F^{(n)}}{\partial x_n} \end{pmatrix}^T$$

where

$$\nabla\frac{\partial F^{(i)}}{\partial x_1} = \left(\frac{\partial^2 F^{(i)}}{\partial {x_1}^2}, \frac{\partial^2 F^{(i)}}{\partial x_1 \partial x_2}, ..., \frac{\partial^2 F^{(i)}}{\partial x_1 \partial x_n}\right)^T, ..., \nabla\frac{\partial F^{(i)}}{\partial x_n} = \left(\frac{\partial^2 F^{(i)}}{\partial x_n \partial x_1}, ..., \frac{\partial^2 F^{(i)}}{\partial {x_n}^2}\right)^T$$

$for \quad i = 1, 2, ...n$

Then the system (1.6.12) *undergoes a transcritical bifurcation around the equilibrium point* X^* *for the bifurcation parameter* $\mu = \mu^*$.

Hopf Bifurcation

In a two-dimensional dynamical system, Hopf bifurcation occurs when the stability of an equilibrium point changes via a pair of purely imaginary eigenvalues λ_i, $(i = 1, 2)$ and this bifurcation gives the birth of a limit cycle from the equilibrium as a parameter crosses a critical value. This implies that a Hopf bifurcation can occur in a system dimension two or higher. The bifurcation can be supercritical or subcritical, resulting in stable or unstable limit cycle, respectively.

Theorem 1.6.3. *Suppose that the equilibrium point at* (X^*, μ^*) *of the system* (1.6.12) *with the eigenvalues* λ_i, $(i = 1, 2)$ *at which the following properties are satisfied:*

(*A*1). $D_X f(X^*, \mu^*)$ *has a simple pair of purely imaginary eigenvalues and no other eigenvalues with zero real parts. This implies that there is a smooth curve of equilibrium points* $(X(\mu), \mu)$ *with* $X(\mu^*) = x^*$. *The eigenvalues* $\lambda(\mu)$, $\bar{\lambda}(\mu)$ *of* $D_X F(X(\mu), \mu^*)$ *which are imaginary at* $\mu = \mu^*$

(*A*2). $Re\left[\frac{d\lambda(\mu)}{d\mu}\right]_{\mu=\mu^*} \neq 0$

then there exists a unique branch of periodic solutions of the system (1.6.12) *near* (X^*, μ^*).

Liu [80] derived a criteria of Hopf bifurcation without using the eigenvalues of the Jacobian matrix evaluated at an equilibrium point of a three dimensional system.

Liu's Criteria [11]: The characteristics equation of the Jacobian matrix evaluated at an equilibrium point of a three dimensional system is given by

$$\lambda^3 + A_1(\mu)\lambda^2 + A_2(\mu)\lambda + A_3(\mu) = 0 \qquad (1.6.16)$$

where $A_1(\mu)$, $A_2(\mu)$, $A_3(\mu)$ and $\Delta(\mu) = A_1(\mu)A_2(\mu) - A_3(\mu)$ are smooth function of μ in an open interval of $\mu^* \in (0, \mu)$ such that

$$(i) \quad A_1(\mu^*) > 0, \quad A_3(\mu^*) > 0$$

$$(ii) \quad A_1(\mu^*)A_2(\mu^*) - A_3(\mu^*) = 0$$

$$(iii) \quad Re\left[\frac{d\Delta(\mu)}{d\mu}\right]_{\mu=\mu^*} \neq 0.$$

Then a simple Hopf bifurcation occurs at $\mu = \mu^*$.

As it is known that the bifurcating periodic solutions through Hopf bifurcation may be stable or unstable. Therefore, in order to establish the stability of these periodic solutions we compute the sign of the first lyapunov number. Hence in the following, the methodology of its computation is presented.

Stability and direction of Periodic Solutions

Consider a general planer system as follows:

$$\begin{aligned} \frac{dx}{dt} &= ax + by + p(x, y) \\ \frac{dy}{dt} &= cx + dy + q(x, y) \end{aligned} \tag{1.6.17}$$

with $\Delta = ad - bc > 0$, $a + d = 0$ and

$$p(x, y) = \sum_{i+j\geq 2} a_{ij}x^i y^j, \quad q(x, y) = \sum_{i+j\geq 2} b_{ij}x^i y^j \tag{1.6.18}$$

The Liapunouv coefficient σ for the planer system [102] is given by

$$\begin{aligned} \sigma = \quad & \frac{-3\pi}{2a_{01}\Delta^{\frac{3}{2}}}\Big[[ac({a_{11}}^2 + a_{11}b_{02} + a_{02}b_{11}) + ab({b_{11}}^2 + b_{11}a_{02} + b_{02}a_{11}) + \\ & c^2(a_{11}a_{02} + 2a_{02}b_{02}) - 2ac({b_{02}}^2 - a_{02}a_{20}) - 2ab({a_{20}}^2 - b_{02}b_{20}) \\ & -b^2(2a_{20}b_{20} + b_{11}b_{20}) + (bc - 2a^2)(b_{11}b_{02} - a_{11}a_{20})] - (a^2 + \\ & bc)[3(cb_{03} - ba_{30}) + 2a(a_{21} + b_{12}) + (ca_{12} - bb_{21})]\Big] \end{aligned}$$

There are stable periodic solutions (i.e., super-critical bifurcation) if the Liapunouv coefficient $\sigma < 0$ and it is unstable (i.e., sub-critical bifurcation) when $\sigma > 0$.

1.7 Chaos

Chaos is irregular, a-periodic long-term behavior (means that there are trajectories which do not settle down to fixed points, periodic orbits, or quasi-periodic orbits as time tends to infinity) of a nonlinear deterministic system that exhibits sensitive dependence on initial conditions.

The strange attractor is an attractor that exhibits highly sensitive dependence on initial conditions. The geometric structure of the chaotic attractor in the Poincare map appears as a totally disconnected and uncountable set of points. The Liapunov exponent is the important tool for chaotic solutions. The solution of dynamical system is chaotic if the Liapunov exponent is positive. The solution of dynamical system has stable fixed points and cycles if all the Liapunov exponents are negative.

According to Poincare–Bendixton theorem, a two dimensional continuous system has limited behavior. However, a three dimensional nonlinear continuous systems can have a rich dynamical behavior including quasi-periodic and chaotic behavior. Three patterns of the coexistence of species are possible (i)-coexistence at a globally (or locally) stable equilibrium point; (ii) - coexistence in a stable periodic motion (a limit cycle); (iii) - coexistence in a chaotic motion.

1.7.1 Pontryagin's Maximum Principle

Pontryagin's Maximum Principle [104] deals with a general control problem of maximizing an objective function

$$J = \int_{t_0}^{t_1} g(X(t), u(t), t)\, dt, \tag{1.7.1}$$

The state equations of a system are described as

$$\begin{aligned} \dot{X}(t) &= \frac{dX}{dt} = g(X(t), u(t), t) \\ x_i(0) &= x_{i0} > 0 \end{aligned} \tag{1.7.2}$$

with the state variables as follows:

$$X(t) = (x_1(t), x_2(t), x_3(t), ...x_n(t))^T \in \mathbb{R}^n, \quad t_0 < t < t_1$$

and the control variable is given by

$$u(t) = (u_1(t), u_2(t), u_3(t), ...u_n(t))^T \in \zeta(X(t), t) \subset \mathbb{R}^p, \quad t_0 < t < t_1$$

which is piecewise continuous and lies in the control region $\zeta(X(t), t)$.

The associated Hamiltonian function $\mathcal{H}$ is defined as follows:

$$\mathcal{H}[X(t), u(t), t; \lambda(t)] = g[X(t), u(t), t] + \sum_{i=1}^{n} \lambda_i g_i[X(t), u(t), t] \tag{1.7.3}$$

where, $\lambda(t) = (\lambda_1(t), \lambda_2(t), \lambda_3(t), ...\lambda_n(t))^T$ are called *adjoint variables*. This maximum principle states that if $u(t)$ is the optimal control and $X(t)$ is corresponding response then there exists an adjoint variable $\lambda_i(t)$ such that

$$\frac{d\lambda_i}{dt} = -\frac{\partial \mathcal{H}}{\partial x_i}. \tag{1.7.4}$$

The optimal control maximizes the Hamiltonian function $\mathcal{H}$ given in (1.7.3) such that

$$\mathcal{H}[X(t), u(t), t; \lambda(t)] = \max_{u \in \zeta} \mathcal{H}[X(t), u(t), t; \lambda(t)] \tag{1.7.5}$$

It is useful to observe that if the optimal control $u(t)$ happens to lie in the interior of the control interval, (i.e., if the control constraints are not binding), then for $\mathcal{H}$ to be maximum,

$$\frac{\partial \mathcal{H}}{\partial u_j} = 0; \quad j = 1, 2, 3, ..., p \tag{1.7.6}$$

Accordingly, the functions $X(t)$, $u(t)$ and $\lambda(t)$ can be determined with the help of the equations (1.7.4) and (1.7.6).

Numerical Simulations

The proposed models and results are illustrated with the help of numerical simulations using softwares: Matlab, Matcont.

1.8 Chapter-Wise Summary

The aim of the present thesis is to study the dynamical behavior of some bio-economical and ecological models in the presence of different type of harvesting functions. The main aim of the present work is to give a detailed study of mathematical analysis for the models presented in this thesis. The numerical simulations are carried out to validate the analytic findings.

In **Chapter 2**, nonlinear harvesting of a predator-prey dynamical system is studied. The harvesting effort is considered as a dynamic variable. Prey population is subjected to grow logistically. The Holling type-II functional response is assumed for prey and predator is to follow Modified Leslie-Gower type dynamics. The conditions of existence and local asymptotic stability of various equilibrium states have been obtained. The dynamical behavior of the system at interior shows that it is locally as well as globally asymptotically stable under certain condition. It is established that the coexistence of both populations depend upon the proper harvesting strategies. The proper harvesting strategies help to avoid the risk of extinction or over exploitation of species. Using analytical and numerical results, it is examined that for a fixed value of price per unit mass and other parameters of the system, as the value of cost per unit mass is increasing, the level of harvesting start increases. After some times, a level of cost is obtained where harvesting effort will tend to zero. Accordingly, for the coexistence of prey predator population along with effort, optimal level of cost is obtained.

In **Chapter 3**, model of chapter 2 is extended incorporating taxation as a control instrument. The conditions for existence of all possible steady states and their stability analysis have been examined. The existence of interior steady state of this system is strongly dependent on range of taxation. This range of tax is useful for the regulatory agency for formulating a tax structure. The bionomic equilibrium of the system has been obtained and it provides the range of harvesting rate (or catch-ability) that can be profitable for a harvesting agency to get maximum yields. The sufficient condition for global stability of unique interior equilibrium point provides a domain for global solutions. The conditions of persistence for this system are also derived. It is also investigated that coexistence of both prey and predator populations depending upon the proper harvesting schemes such as the risk of extinction of species can be avoided. The Optimal Taxation Policy for the problem has been investigated by using Pontrygin's Maximum Principle. The optimal solution and the optimal path have been derived. The impact of taxation levels on the dynamic of system shows that the densities of prey and predator increases as the tax rate increases whereas the density (level) of harvesting effort decreases as the tax rate increases. This observation gives the idea to achieve the optimal level of taxation corresponding to optimal equilibrium level of prey, predator populations and harvesting effort. Therefore, the aim of this work includes both ecological and economic aspects. The economical objective is to maximize the net economic revenue and ecologically, want to keep away the prey and predator populations from extinction.

In **Chapter 4**, a two dimensional pre predator dynamical system is studied incorporating combined harvesting where predator is provided an additional food resource. In this chapter, prey and predator both are affected by some external toxicant substances which are harmful for both species. Here, additional food is playing an important role in predator prey system which preserves predator from extinction. The steady states of the system and their stability analysis have been carried out for all possible feasible equilibrium points. The system undergoes some local bifurcations i.e., trans-critical, Hopf, saddle node bifurcations for a threshold level of parametric values. Also, some global bifurcations i.e., Bogdanov- Taken bifurcation and Generalized Hopf bifurcation are detected in the continuation of Hopf bifurcation point (or limit point), using software MATCONT w.r.t. different parametric values. The sufficient condition for the bionomic equilibrium has been derived. The optimal harvesting policy is explored by using Pontrygin's Maximum Principle such that the risk of extinction of the species can be avoided.

In **Chapter 5**, a mathematical model of a dynamical stock Effort system with nonlinear harvesting of species has been proposed and analyzed. This system is considered in two different fishing zones where fishing vessels also move between the patches to increase their revenue. The migration rate of fishing vessels between patches is assumed to be stock dependence. Further, some special cases are also discussed including the constant migration rate of fishing vessels between the patches. In this work, system comprises two time scales: fast one for movement fish and boats between patches, and slow one corresponds to growth of fish population and fishery dynamics. The aggregated method is used to simplify the mathematical analysis of the complete model. Taxation policy can be used as an effective control instrument. Qualitative analysis reveals that nonlinear harvesting term plays an important role to determine the dynamics and bifurcation of system. Existence of bifurcations indicates that the high taxes will cause closed of fishery. However, the Maximum Sustainable yield (MSY) and Optimal Taxation Policy is discussed for the aggregated model. Some numerical results are also illustrated to verify analytical results.

In **Chapter 6**, a spatial prey predator mathematical model has been proposed and analyzed. This system is based upon the two time scales: fast one for the movement of prey species between the patches and slow one corresponds to the growth of prey-predator and their interactions. Therefore, to simplify the mathematical analysis, an aggregation method is used. The effect of toxicity is considered in the system. Using the Routh-Hurwitz criteria, it has been shown that the unique interior equilibrium state exists under certain condition and it is globally asymptotically stable. There does not exist any periodic solutions in the interior and it is also confirmed through Bendixon-Dulac Criteria. Numerically it is also shown that the trajectories of aggregated model remain close to the trajectories of the complete model.

The **Chapter 7** investigates a predator-prey dynamic reaction model in a heterogeneous water body. In this model, only prey population is subjected to harvesting. The surface layer provides food for the two species. The prey migrates to deeper layers to take refuge from predator. Although, the prey is the preferred food for predator, but the predator also takes alternate food available in abundance. The consequences of prey refuge, availability of alternative food resource for predators and effects of harvesting effort on the dynamics of prey and predator populations are explored. The two time scales are considered to describe the dynamics of the model. The aggregated model is analyzed analytically as well as numerically using dynamic of harvesting effort. Numerical simulation shows that introducing the dynamics of harvesting effort can destabilize prey predator system. This is confirmed by the bifurcation diagrams and dynamics of Lyapuonov exponent w.r.t. bifurcation parameter.

The **Chapter 8** is about the achievements and future scope of this work. A list of references is appended at the end of thesis.

Chapter 2

A Harvesting Model with Non-linear Effort dynamics and Modified Leslie–Gower type Predator–Prey Model

2.1 Introduction

Predator-prey systems are most commonly used to describe the interaction of species. The simplest predator–prey dynamical model is the Lotka–Volterra model and this model is modified in various ways by many researchers. Meno-Lorca [88], Li and Xiao [78] and Zhu and Lan [73] have investigated the Leslie–Gower predator–prey model incorporating different harvesting functions. They have studied the local and global dynamics of systems with multiple bifurcations. Zhang et al. [133] also investigated the Leslie–Gower predator–prey model with proportional harvesting in both species to study the persistence and global stability of the system. Further, Huang and Gong [56] studied the same model with constant yield prey harvesting to study the multiple bifurcations. Again, Gupta and Benergee [45] studied a modified Leslie–Gower model with non–linear harvesting of prey. They found the several local and global bifurcation in the system.

Moreover, many investigations are made to study the Modified Leslie–Gower predator–prey model considering different effects like, Allee effect, time delay, and reaction–diffusion [Rojas-Palma and Gonzalez [109], Yuan and Jiang [132] and Zhuang and Zhao [134]]

In this chapter, a Modified Leslie–Gower predator–prey model is studied incorporating non–linear harvesting of prey. In this model, harvesting effort is considered as a dynamic variable. Although many researchers have discussed Modified Leslie–Gower predator–prey model incorporating non–linear harvesting of prey, the dynamics of harvesting function has not been investigated yet. This chapter deals with a three–dimensional model (of predator–prey model with effort dynamics), where the level of fishing effort can be expanded or contracted according as the perceived rent (i.e., the net economic revenue to the fishermen) is positive or negative.

2.2 The Mathematical Model

Let $x(t)$ denotes the population density of a logistically growing prey with Holling type-II functional response and $y(t)$ be the density of predator assuming Modified Leslie-Gower type predation. Let the prey species be harvested with effort E and $H(x, E)$ denotes the harvesting function. The dynamics of system with $H(x, E)$ is governed by following set of equations:

$$\frac{dx}{dt} = rx\left(1 - \frac{x}{k}\right) - \frac{\alpha xy}{a + x} - H(x, E) \tag{2.2.1}$$

$$\frac{dy}{dt} = sy\left(1 - \frac{\beta y}{a + x}\right) \tag{2.2.2}$$

The parameter r is the intrinsic growth rate and k is the environmental carrying capacity for the prey. For the predator, s is the growth rate, α is its encounter rate with the prey and β is maximum rate of the reduction of predator population. All these parameters are assuming only positive values.

The following more realistic non-linear harvesting function [45] is considered instead of constant and proportional harvesting.

$$H(x, E) = \frac{qEx}{m_1 E + m_2 x}$$

The net economic revenue of fishermen from harvesting of prey species is given by

Net Revenue $= T.R. - T.C. = E\left(\dfrac{qpx}{m_1E + m_2x} - c\right)$

Harvesting effort $E(t)$ is taken as a dynamic variable at time t. Therefore, assuming p and c are price and cost per unit mass, q is catch-ability and η is the stiffness parameter, the effort dynamics is determined as follows [103]:

$$\frac{dE}{dt} = \eta E\left(\frac{qpx}{m_1E + m_2x} - c\right). \tag{2.2.3}$$

The coupled dynamical equations (2.2.1)-(2.2.3) constitute the model for the harvesting of prey. The model with associated initial conditions is presented as follows:

$$\begin{aligned}
\frac{dx}{dt} &= rx\left(1 - \frac{x}{k}\right) - \frac{\alpha xy}{a + x} - \frac{qEx}{m_1E + m_2x} = xf(x, y, E) \equiv F(x, y, E),\\
\frac{dy}{dt} &= sy\left(1 - \frac{\beta y}{a + x}\right) = yg(x, y) \equiv G(x, y, E),\\
\frac{dE}{dt} &= \eta E\left(\frac{qpx}{m_1E + m_2x} - c\right) = Eh(x, E) \equiv H(x, y, E).
\end{aligned} \tag{2.2.4}$$

$$x(0) = x_0, y(0) = y_0, E(0) = E_0; \quad (x_0, y_0, E_0) \in \mathbb{R}^3_+.$$

It is observed that $\lim_{(x,y,E)\to(0,0,0)} F(x, y, E) = G(x, y, E) = H(x, y, E) = 0$. Further, assume that $F(0, 0, 0) = G(0, 0, 0) = H(0, 0, 0) = 0$. With this assumption, it is concluded that all the functions $F(x, y, E)$, $G(x, y, E)$ and $H(x, y, E)$ are continuous in the positive quadrant $\mathbb{R}^3_+$, where $\mathbb{R}^3_+ = \{(x, y, E) : x > 0, y > 0, E > 0\}$. [53]

2.3 The Model Analysis

Lemma 2.3.1. *All the solutions* $(x(t), y(t), E(t))$ *of the system* (2.2.4) *with positive initial condition remain positive for all* $t > 0$.

Proof. The solution of the system (2.2.4) is obtained as follows:

$$\begin{aligned}
x(t) &= x(0)exp\left(\int_0^t f(x(p), y(p), E(p))dp\right) > 0,\\
y(t) &= y(0)exp\left(\int_0^t g(x(p), y(p), E(p))dp\right) > 0,\\
E(t) &= E(0)exp\left(\int_0^t h(x(p), y(p), E(p))dp\right) > 0.
\end{aligned}$$

This shows that the solution of the system (2.2.4) is positive for all $t > 0$.

□

Lemma 2.3.2. *All the solutions of the system* (2.2.4) *which start in the region* $\mathbb{R}^3_+$ *are uniformly bounded.*

Proof. let us consider a function $\psi(t)$ such that

$$\begin{aligned}
\psi(t) &= x(t) + y(t) + \frac{1}{\eta p}E(t) \\
\frac{d\psi(t)}{dt} &= x'(t) + y'(t) + \frac{1}{\eta p}E'(t), \\
&= rx\left(1 - \frac{x}{k}\right) - \frac{\alpha xy}{a+x} - \frac{qEx}{m_1E + m_2x} + sy - \frac{\beta sy^2}{a+x} + \frac{qEx}{m_1E + m_2x} - \frac{cE}{\eta p}, \\
\frac{d\psi(t)}{dt} &\leq \left(rx - \frac{rx^2}{k}\right) + sy - \frac{\beta sy^2}{a+k} - \frac{cE}{\eta p}.
\end{aligned}$$

Introducing a positive constant N and rewrite the above equation as below:

$$\frac{d\psi(t)}{dt} + N\psi(t) \leq \left((r+N)x - \frac{r}{k}x^2\right) + \left((s+N)y - \frac{s\beta}{a+k}y^2\right) - \frac{(c-N)}{\eta p}E,$$

Assuming $c > N$, the above equation will take the following form:

$$\begin{aligned}
\frac{d\psi(t)}{dt} + N\psi(t) &\leq -\frac{r}{k}\left(x - \frac{k(r+N)}{2r}\right)^2 - \frac{s\beta}{a+k}\left(y - \frac{(s+N)(k+a)}{2s\beta}\right)^2 + M, \\
\frac{d\psi(t)}{dt} + N\psi(t) &\leq M; \quad M = \left(\frac{k^2(r+N)^2}{4r^2} + \frac{(s+N)^2(k+a)^2}{4s^2\beta^2}\right) \qquad (2.3.1)
\end{aligned}$$

Solving the above differential inequality (2.3.1), the following can be obtained:

$$\begin{aligned}
\psi(t) &\leq \frac{M}{N}\left(1 - e^{-Nt}\right) + \psi(0)e^{-Nt}, \\
0 < \lim_{t\to\infty} \psi(t) &\leq \frac{M}{N}.
\end{aligned}$$

Hence, all the solutions of system (2.2.4) initiating from $\mathbb{R}^3_+$ are confined in the region

$$R = \left\{(x, y, E) \in \mathbf{R}; \quad 0 < x(t) + y(t) + \frac{1}{\eta p}E(t) \leq \frac{M}{N} + \phi \text{ for any } \phi > 0\right\}$$

This proves the result. □

2.4 Existence of Various Equilibrium Points

For the model (2.2.4), there exists six non-negative equilibrium points which are given below:

1. $P_0(0,0,0)$ is the trivial equilibrium point and always exists. Moreover, the axial equilibrium points $P_1(k,0,0)$ on x-axis and $P_2\left(0,\frac{a}{\beta},0\right)$ on y-axis always exist.

2. $P_3(\overline{x},\overline{y},0)$ is the boundary equilibrium point in $xy-$plane in the absence of harvesting. The values of $\overline{x}$ and $\overline{y}$ are the positive solutions of the following equations:

$$\begin{aligned} r\left(1-\frac{\overline{x}}{k}\right)-\frac{\alpha\overline{y}}{a+\overline{x}} &= 0, \\ s\left(1-\frac{\beta\overline{y}}{a+\overline{x}}\right) &= 0. \end{aligned}$$

   The point P_3 is obtained as $P_3(\overline{x},\overline{y},0)=\left(k\left(1-\frac{\alpha}{r\beta}\right),\frac{1}{\beta}(a+\overline{x}),0\right)$ and it is positive for $r\beta>\alpha$.

3. $P_4(\hat{x},0,\hat{E})$ is boundary equilibrium point in $xE-$plane in the absence of predator where $\hat{x}$ and $\hat{E}$ are the positive solutions of the following equations:

$$r\left(1-\frac{\hat{x}}{k}\right)-\frac{q\hat{E}}{m_1\hat{E}+m_2\hat{x}} = 0, \tag{2.4.1}$$

$$\eta\left(\frac{qp\hat{x}}{m_1\hat{E}+m_2\hat{x}}-c\right) = 0. \tag{2.4.2}$$

   Solving above set of equations (2.4.1)-(2.4.2), the point P_4 is obtained as

$$(\hat{x},0,\hat{E})=\left(k\left(1-\frac{qL}{r(m_1L+m_2)}\right),0,L\hat{x}\right);\quad L=\frac{pq-cm_2}{cm_1}$$

   The point P_4 exists provided:

$$r(m_1L+m_2)>qL \quad and \quad c<\frac{pq}{m_2} \tag{2.4.3}$$

4. $P^*(x^*, y^*, E^*)$ is the unique interior equilibrium point of the following equations:

$$r\left(1-\frac{x^*}{k}\right)-\frac{\alpha y^*}{a+x^*}-\frac{qE^*}{m_1E^*+m_2x^*}=0$$
$$s\left(1-\frac{\beta y^*}{a+x^*}\right)=0$$
$$\eta\left(\frac{qpx^*}{m_1E^*+m_2x^*}-c\right)=0$$

After solving, the interior equilibrium point is obtained as follows:

$$(x^*, y^*, E^*)=\left(k\left(1-\frac{\alpha}{r\beta}-\frac{qL}{r(m_1L+m_2)}\right), \frac{a+x^*}{\beta}, Lx^*\right).$$

This point is positive for

$$\frac{\alpha}{r\beta}+\frac{qL}{r(m_1L+m_2)}<1 \quad and \quad c<\frac{pq}{m_2} \tag{2.4.4}$$

2.5 Stability of Equilibrium Points

The local stability of all feasible equilibrium points is discussed in this section. It can be observed that the system (2.2.4) cannot be linearized at the equilibrium points $(0,0,0)$ and $\left(0, \frac{a}{\beta}, 0\right)$.

Routh–Hurwitz Criteria is is used to study the local stability conditions for feasible equilibrium points of the system (2.2.4) except $(0,0,0)$ and $\left(0, \frac{a}{\beta}, 0\right)$. This criteria determines the nature of eigenvalues of the Jacobian matrix evaluated at the corresponding equilibrium points [102]. The Jacobian matrix of the system (2.2.4) at (x, y, E) is given by

$$J(x,y,E)=\begin{bmatrix} x\frac{df}{dx}+f & x\frac{df}{dy} & x\frac{df}{dE} \\ y\frac{dg}{dx} & y\frac{dg}{dy}+g & y\frac{dg}{dE} \\ E\frac{dh}{dx} & E\frac{dh}{dy} & E\frac{dh}{dE}+h \end{bmatrix}$$

$$J(x,y,E)=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

with

$$a_{11} = x\left(-\frac{r}{k} + \frac{\alpha y}{(a+x)^2} + \frac{qEm_2}{(m_1E+m_2x)^2} \right) + f, \quad a_{12} = \frac{-\alpha x}{a+x},$$

$$a_{13} = x\left(\frac{m_2 q x}{(m_1E+m_2x)^2} \right), \quad a_{21} = y\left(\frac{s\beta y}{(a+x)^2} \right), \quad a_{22} = y\left(\frac{-\beta s}{a+x} \right) + g,$$

$$a_{31} = \eta E\left(\frac{qpm_1E}{(m_1E+m_2x)^2} \right), \quad a_{32} = a_{23} = 0, \quad a_{33} = \eta E\left(\frac{-qpm_1x}{(m_1E+m_2x)^2} \right) + h.$$

In the following, some theorems are stated for the stability of various equilibrium states.

Theorem 2.5.1. *The axial equilibrium state $P_1(k,0,0)$ is always a saddle point with unstable manifold in y-direction and stable manifold in x-direction. It has a stable manifold in E-direction provided*

$$c > \frac{pq}{m_2} \tag{2.5.1}$$

Proof. The Jacobian matrix evaluated at $P_1(k,0,0)$ is given by

$$J_1(k,0,0) = \begin{bmatrix} -r & \dfrac{-\alpha k}{a+k} & \dfrac{q}{m_2} \\ 0 & s & 0 \\ 0 & 0 & \eta\left(\dfrac{pq}{m_2} - c\right) \end{bmatrix}$$

The eigenvalues of J_1 are given as below:

$$\lambda_1 = -r < 0, \quad \lambda_2 = s > 0 \quad and \quad \lambda_3 = \eta\left(\frac{pq}{m_2} - c\right)$$

The equilibrium point $(k,0,0)$ is always a saddle point as $\lambda_2 = s > 0$ with unstable manifold in y-direction. It has a stable manifold in x-direction as $\lambda_1 = -r < 0$. There is a stable manifold in E-direction for the condition (2.5.1). However, there is unstable manifold in E-direction, if the condition (2.5.1) is violated.

Remark 2.5.2. *For $c < \frac{pq}{m_2}$, the solution trajectories will never enter into $E = 0$ plane.*

□

Theorem 2.5.3. *The planar equilibrium state $P_3(\overline{x}, \overline{y}, 0)$ is locally asymptotically stable for the following conditions:*

$$\frac{s\beta\overline{xy}}{a+\overline{x}} < \left(r+s-\frac{\alpha}{\beta}\right) \quad \text{and} \tag{2.5.2}$$

$$\frac{pq}{m_2} < c \tag{2.5.3}$$

Proof. The Jacobian matrix evaluated at $P_3(\overline{x}, \overline{y}, 0)$ is given by

$$J_3(\overline{x}, \overline{y}, 0) = \begin{bmatrix} \overline{x}\left(-\frac{r}{k}+\frac{\alpha\overline{y}}{(a+\overline{x})^2}\right) & \frac{-\alpha\overline{x}}{a+\overline{x}} & \frac{q}{m_2} \\ \frac{s\beta(\overline{y})^2}{(a+\overline{x})^2} & \frac{-s\beta\overline{y}}{a+\overline{x}} & 0 \\ 0 & 0 & \eta\left(\frac{pq}{m_2}-c\right) \end{bmatrix}$$

The characteristic equation associated to the matrix $J_3(\overline{x}, \overline{y}, 0)$ is given by

$$(\lambda-\lambda_1)(\lambda^2-T\lambda+D)=0$$

where,

$$\lambda_1=\eta\left(\frac{pq}{m_2}-c\right), \quad T=\left(\frac{\alpha}{\beta}-r-s\right)+\frac{\alpha\overline{x}}{\beta(a+\overline{x})}, \quad D=\frac{s\beta\overline{xy}}{a+\overline{x}}$$

The point P_3 is locally asymptotically stable for $\lambda_3 < 0$ and $T < 0$ which give the conditions (2.5.2) and (2.5.3). □

Remark 2.5.4. *The point P_3 is a saddle point if one of the conditions* (2.5.2) *or* (2.5.3) *is violated. The point P_3 becomes unstable when both the conditions are violated and it may enter into $xyE-$octant. For, $T=0$, there is a pair of purely imaginary roots. Therefore, there is possibility of occurrence of periodic solutions around the equilibrium point P_3.*

Theorem 2.5.5. *The planar equilibrium state $P_4(\hat{x}, 0, \hat{E})$ is a saddle point provided*

$$\frac{qL(m_2-\eta m_1 p)}{(m_1L+m_2)^2} < r-\frac{qL}{m_1L+m_2} \tag{2.5.4}$$

Proof. The Jacobian matrix evaluated at $P_4(\hat{x}, 0, \hat{E})$ is given by

$$J_4(\hat{x},0,\hat{E}) = \begin{bmatrix} \hat{x}\Big(-\frac{r}{k}+\frac{q\hat{E}m_2}{(m_1\hat{E}+m_2\hat{x})^2}\Big) & \frac{-\alpha\hat{x}}{a+\hat{x}} & \frac{qm_2\hat{x}^2}{(m_1\hat{E}+m_2\hat{x})^2} \\ 0 & s & 0 \\ \frac{\eta\hat{E}^2qpm_1}{(m_1\hat{E}+m_2\hat{x})^2} & 0 & \frac{-\eta\hat{E}\hat{x}qpm_1}{(m_1\hat{E}+m_2\hat{x})^2} \end{bmatrix}$$

The one of eigenvalue of the Jacobian matrix J_4 is $\lambda = s > 0$. The other two eigenvalues $\lambda_{\pm}$ can be obtained from the following 2×2 matrix:

$$J_4^*(\hat{x},0,\hat{E}) = \begin{bmatrix} \hat{x}\Big(-\frac{r}{k}+\frac{q\hat{E}m_2}{(m_1\hat{E}+m_2\hat{x})^2}\Big) & \frac{qm_2\hat{x}^2}{(m_1\hat{E}+m_2\hat{x})^2} \\ \frac{\eta\hat{E}^2qpm_1}{(m_1\hat{E}+m_2\hat{x})^2} & \frac{-\eta\hat{E}\hat{x}qpm_1}{(m_1\hat{E}+m_2\hat{x})^2} \end{bmatrix}$$

From the above matrix, it can be observed that

$$det(J_4^*) = \frac{r\hat{x}}{k} > 0$$

and

$$tr(J_4^*) = \Big(\frac{qL}{m_1L+m_2} - r\Big) + \frac{qL(m_2-\eta m_1 p)}{(m_1L+m_2)^2}$$

. Accordingly, P_4 is a saddle point when $tr(J_4^*) < 0$ which gives the condition (2.5.4).

□

Remark 2.5.6. *The equilibrium point P_4 becomes unstable when the condition* (2.5.4) *is violated.*

If $tr(J_4^*) = 0$, then there is a pair of purely imaginary roots. Therefore, there is possibility of occurrence of periodic solutions in $xE-$plane.

Theorem 2.5.7. *The interior equilibrium state $P^*(x^*, y^*, E^*)$ is locally asymptotically stable for the following sufficient condition:*

$$\frac{\alpha y^*}{(a+x^*)^2} + \frac{qE^*m_2}{(m_1E^*+m_2x^*)^2} < \frac{r}{k} \tag{2.5.5}$$

Proof. The Jacobian matrix evaluated at $P_5(x^*, y^*, E^*)$ is given by

$$J_5 = \begin{bmatrix} x^*\Big(-\frac{r}{k}+\frac{\alpha y^*}{(a+x^*)^2}+\frac{qE^*m_2}{(m_1E^*+m_2x^*)^2}\Big) & \frac{-\alpha x^*}{a+x^*} & \frac{m_2q(x^*)^2}{(m_1E^*+m_2x^*)^2} \\ \frac{\beta s y^*}{(a+x^*)^2} & \frac{-\beta s y^*}{a+x^*} & 0 \\ \frac{\eta qpm_1(E^*)^2}{(m_1E^*+m_2x^*)^2} & 0 & \frac{\eta qpm_1x^*E^*}{(m_1E^*+m_2x^*)^2} \end{bmatrix}$$

The characteristics equation of the above Jacobian matrix about the interior equilibrium point (x^*, y^*, E^*) is given as follows:

$$\lambda^3 + A_1\lambda^2 + A_2\lambda + A_3 = 0$$

with
$A_1 = -(a_{11} + a_{22} + a_{33})$
$A_2 = a_{22}a_{33} + (a_{11}a_{33} - a_{13}a_{31}) + (a_{11}a_{22} - a_{21}a_{21})$
$A_3 = a_{11}a_{22}a_{33} - a_{12}a_{21}a_{33} - a_{13}a_{31}a_{22}$
Using Routh–Harwitz Creteria, it is concluded that

$$A_1 > 0 \quad iff \quad a_{ii} < 0 \quad for \quad i = 1, 2 \quad and \quad 3$$

It can be observed that

$$a_{22} < 0, \quad a_{33} < 0.$$

The value of $a_{11} < 0$ for the condition (2.5.5).

Also, $A_2 > 0, A_3 > 0$ and $A_1A_2 - A_3 > 0$ for the condition (2.5.5)
Accordingly, the interior equilibrium point (x^*, y^*, E^*) is locally asymptotically stable provided the condition (2.5.5) is satisfied. □

Remark 2.5.8. *The interior equilibrium point* (x^*, y^*, E^*) *may be unstable or saddle if the condition* (2.5.5) *is violated.*

2.6 Global Stability

Theorem 2.6.1. *The locally asymptotically stable interior equilibrium point* (x^*, y^*, E^*) *of the system* (2.2.4) *is globally asymptotically stable in the domain* $D = \{(x, y, E) : m_1E + m_2x > M, (x, y, E) \in \mathbb{R}^3_+\}$, *where* $M = \dfrac{Lqka\beta}{(m_1L + m_2)(ra\beta - \alpha k)}$.

Proof. Consider a Lyapunov function $V(x, y, E)$ such that

$$\begin{aligned} V(x, y, E) \quad = \quad & d_0\left[(x - x^*) - x^* \log \frac{x}{x^*}\right] + d_1\left[(y - y^*) - y^* \log \frac{y}{y^*}\right] + d_2\left[(E - E^*)\right. \\ & \left. - E^* \log \frac{E}{E^*}\right] \end{aligned}$$

The function $V(x,y,E)$ is zero at the equilibrium point (x^*,y^*,E^*) i.e $V(x^*,y^*,E^*) = 0$ and is positive for all values (x,y,E) other than (x^*,y^*,E^*). Now differentiate V w.r.t. time t.

$$\begin{aligned}\frac{dV}{dt} &= d_0(x-x^*)\frac{x^*}{x} + d_1(y-y^*)\frac{y^*}{y} + d_2(E-E^*)\frac{E^*}{E}\\ &= d_0(x-x^*)\left(r\left(1-\frac{x}{k}\right) - \frac{\alpha y}{a+x} - \frac{qE}{m_1E+m_2x}\right) + d_1(y-y^*)\left(s - \frac{\beta sy}{a+x}\right)\\ &\quad + d_2(E-E^*)\eta\left(\frac{qpx}{m_1E+m_2x} - c\right).\end{aligned}$$

After solving, the following can be obtained:

$$\begin{aligned}\frac{dV}{dt} =&- d_0(x-x^*)^2\left[\frac{r}{k} - \frac{\alpha y^*}{(a+x)(a+x^*)} - \frac{qE^*}{(m_1E+m_2x)(m_1E^*+m_2x^*)}\right] +\\ &\frac{(x-x^*)(y-y^*)}{a+x}\left[-d_0\alpha x^* + \frac{d_1 s\beta E^*}{a+x^*}\right] - \frac{d_1 s\beta(y-y^*)^2}{a+x}\\ &+ \frac{(-d_0qx^* + d_2qpE^*)(x-x^*)(E-E^*)}{(m_1E+m_2x)(m_1E^*+m_2x^*)} - \frac{d_2qpx^*(E-E^*)^2}{(m_1E+m_2x)(m_1E^*+m_2x^*)}\end{aligned}$$

Choosing $d_0 = 1, \quad d_1 = \dfrac{\alpha x^*(a+x^*)}{s\beta E^*}$ and $d_2 = \dfrac{qx^*}{qpE^*}$, the following can be derived:

$$\begin{aligned}\frac{dV}{dt} \le&- d_0(x-x^*)^2\left[\frac{r}{k} - \frac{\alpha y^*}{(a+x)(a+x^*)} - \frac{qE^*}{(m_1E+m_2x)(m_1E^*+m_2x^*)}\right]\\ &- \frac{d_1 s\beta(y-y^*)^2}{a+x} - \frac{d_2qpx^*(E-E^*)^2}{(m_1E+m_2x)(m_1E^*+m_2x^*)}\end{aligned}$$

$$\begin{aligned}\frac{dV}{dt} \le& -(x-x^*)^2\left[\frac{r}{k} - \frac{\alpha y^*}{a(a+x^*)} - \frac{qE^*}{(m_1E+m_2x)(m_1E^*+m_2x^*)}\right] -\\ &\frac{\alpha_1(a+x^*)x^*(y-y^*)^2}{(a+x)E^*} - \frac{qx^{2*}(E-E^*)^2}{(m_1E+m_2x)(m_1E^*+m_2x^*)},\end{aligned}$$

.

$$\frac{dV}{dt} < 0 \quad if \quad \frac{r}{k} - \frac{\alpha y^*}{a(a+x^*)} - \frac{qE^*}{(m_1E+m_2x)(m_1E^*+m_2x^*)} > 0$$

.

Substituting the values of y^* and E^* in the above expression, the following plane can be obtained:

$$(m_1E+m_2x) > \frac{Lqka\beta}{(m_1L+m_2)(ra\beta-\alpha k)} = M(say). \tag{2.6.1}$$

This implies that $\frac{dV}{dt}$ is negative definite .

Accordingly, the interior equilibrium (x^*, y^*, E^*) is globally asymptotically stable for the sufficient condition (2.6.1). □

2.7 Numerical Simulations

In this section, numerical results are illustrated for following choice of parameters to investigate the dynamic behavior and to validate the analytic results of the system, keeping all the parameters fixed except c.

$$r = 0.5, k = 100, \alpha = 0.005, m_1 = 0.5, m_2 = 0.5,$$
$$q = 0.15, a = 3, s = 1, \beta = 0.15, \eta = 1, p = 5. \tag{2.7.1}$$

In the figure-2.1, diagrams (A), (B) and (C) give the long run behavior of trajectories of prey x, predator y and harvesting effort E w.r.t time t. This figure shows that for the initial condition $(40, 350, 10)$, all the trajectories in the interior $\mathbb{R}^3_+$ converges to its interior equilibrium point for different values of cost c and keeping other parameters fixed. This shows that interior equilibrium is locally asymptotically stable. In the figure-2.1, it can be observed that the harvesting effort level E decreases with the increase of cost level c. Moreover, harvesting effort level E will tend to zero for $c > 1.5$.

Figure-2.2 (A) represents phase plane trajectories of species x, y and effort E for different initial levels. This shows that the interior equilibrium point $(x^*, y^*, E^*) = (81.3332, 562.2056, 20.333)$ is globally asymptotically stable for $c = 1.2$ $(0 < c < 1.5)$. The figure-2.2 (B) represents phase plane trajectories of different biomass with the different initial levels in the interior for $c = 1.8$ which converges to the point $(93.3340, 642.2877, 0.0000)$ in $xy-$plane. This concludes that harvesting of the prey population is not possible for $c > 1.5$ as it will not remain profitable to continue harvesting.

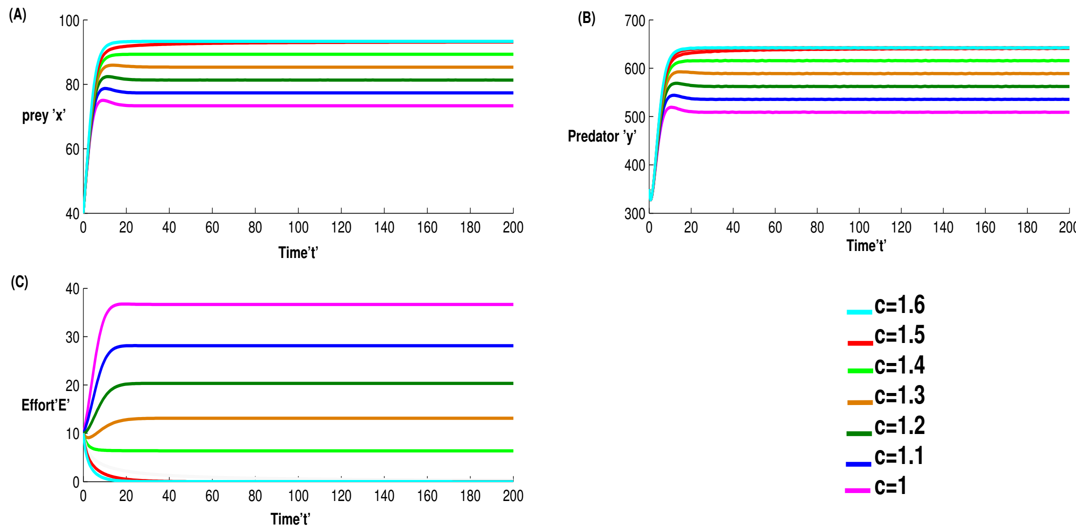

Figure 2.1: Time series analysis of prey x, predator y and effort E for different value of cost c with initial condition $(40, 350, 10)$.

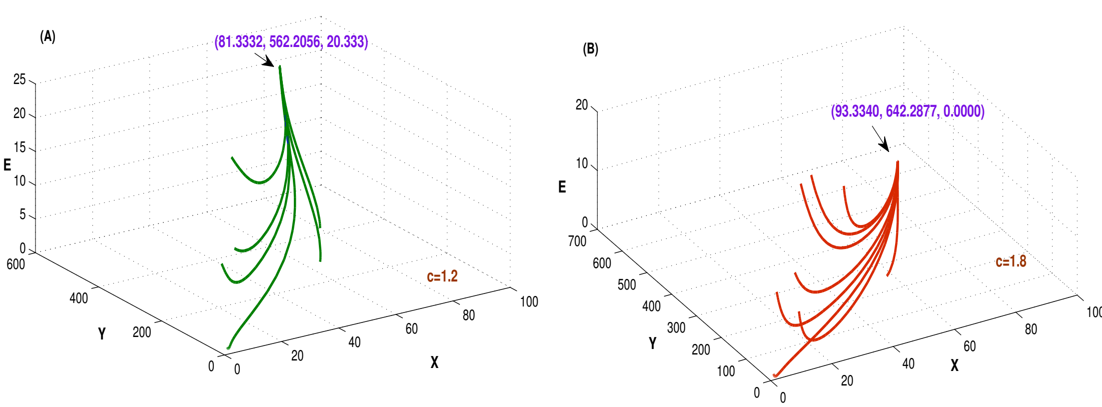

Figure 2.2: Figures (A) and (B) represent phase plane trajectories of prey x, predator y and effort E using different initial conditions for the fixed values of $c = 1.2$ and $c = 1.8$.

2.8 Conclusion

This chapter is concerned with the dynamical study of nonlinear harvesting of prey in Modified Leslie–Gower type predator-prey system. The harvesting effort is taken as a dynamic variable. The conditions for existence and local asymptotic stability of various equilibrium points have been examined. It can be concluded that the interior state of the system is locally and globally asymptotically stable under certain conditions. Using analytical and numerical results, it is observed that the level of harvesting effort decreases with the increasing cost while other parameters are fixed. The harvesting will not remain profitable for high value of cost and harvesting effort will tend to zero in this case. Accordingly, for the coexistence of prey and predator population with harvesting effort, the optimal cost is obtained.

Chapter 3

A Harvesting Model with Non-linear Effort dynamics and Modified Leslie–Gower type Predator–Prey Model using Taxation as a Control Instrument

3.1 Introduction

Regulation of renewable resources is an essential and important part in the optimal management of renewable resources. The over exploitation of these resources is controlled by imposing taxation or license fees. In fishery resource management, many investigations have been carried out using taxation as a control instrument. In this Chapter, model of chapter 2 is extended incorporating taxation as a control instrument.

Harvesting problems with taxation is introduced by Clark [22]. He studied a single-species fishery model using taxation as a control instrument. Based on this work, many researchers (Chaudhuri [19], Mesterton-Gibbons [90], Fan and Wang [38], Pradhan and Chudhuri [106], Dubey et al. ([30], [31], [33], [34], [36], [32]), Ji and Wu [62], Kar et al. [66], Misra and Dubey [95], Huo et al. [58]) have analyzed the utilization of numerous renewable resources using optimal management policy. Dubey et al. [31] analyzed a non-linear mathematical model to study a resource dependent fishery model with optimal harvesting policy by considering taxation as a control instrument. They also proved that the fishery resources can be protected from over-exploitation by increasing the tax and discounted rate. Pradhan and Chaudhuri [106] also proposed and analyzed a dynamical reaction model of two species fishery with taxation as a control variable and then discussed its optimal harvesting policy. Recently, Huo et al. [58] discussed a dynamic model for fishery resource with reserve area and taxation as a control parameter.

In this chapter, a Modified Leslie–Gower predator–prey model is investigated incorporating the non-linear harvesting of prey with effort dynamics. The taxation is used as a control instrument. This model is analyzed for the different level of tax and its global dynamics and singular optimal control. The range of tax may be useful for the regulatory agency for formulating a tax structure. This imposition of tax helps to control over harvesting of prey species and it helps the predator population to grow.

3.2 The Mathematical Model

Let $x(t)$, $y(t)$ and $E(t)$ are the densities of prey, predator population and the harvesting effort at a time t. The Holling type–II functional response and non-linear harvesting is considered for a logistically growing prey species and predator is assumed to be Modified Leslie–Gower type. In order to control over exploitation of the species, the regulatory agencies impose a tax on harvested species. Let $\tau \in [\tau_{min}, \tau_{max}]$ be the imposed tax per unit harvested prey species. For a tax $\tau > 0$, the revenue of the fishermen will be reduced by $(p - \tau)$, assuming $p > \tau$. The mathematical model for the dynamics of the system is governed by the following system of differential equations:

$$\begin{aligned}
\frac{dx}{dt} &= rx\left(1-\frac{x}{k}\right)-\frac{\alpha xy}{a+x}-\frac{qEx}{m_1E+m_2x}=xf(x,y,E)=F(x,y,E),\\
\frac{dy}{dt} &= sy\left(1-\frac{\beta y}{a+x}\right)=yg(x,y)=G(x,y,E), \qquad (3.2.1)\\
\frac{dE}{dt} &= \eta E\left(\frac{q(p-\tau)x}{m_1E+m_2x}-c\right)=Eh(x,E)=H(x,y,E).
\end{aligned}$$

$$x(0)=x_0, y(0)=y_0, E(0)=E_0; \quad (x_0,y_0,E_0)\in\mathbb{R}^3_+.$$

It is observed that $\lim_{(x,y,E)\to(0,0,0)} F(x,y,E) = G(x,y,E) = H(x,y,E) = 0$. Further, assume that $F(0,0,0) = G(0,0,0) = H(0,0,0) = 0$. With this assumption, it is concluded that all the functions $F(x,y,E)$, $G(x,y,E)$ and $H(x,y,E)$ are continuous in the positive octant $\mathbb{R}^3_+$, where $\mathbb{R}^3_+ = \{(x,y,E) : x > 0, y > 0, E > 0\}$. [53]

3.3 The Model Analysis

Lemma 3.3.1. *All the solutions* $(x(t), y(t), E(t))$ *of the system* (3.2.1) *with positive initial condition remain positive for all* $t > 0$.

Proof. The positivity of solutions of the system (3.2.1) can be easily proved as in Lemma 2.3.1.

□

Lemma 3.3.2. *The system* (3.2.1) *has uniformly bounded solution.*

Proof. Consider a function $\psi(t)$ such that

$$\begin{aligned}
\psi(t) &= x(t)+y(t)+\frac{1}{\eta(p-\tau)}E(t),\\
\frac{d\psi(t)}{dt} &= x'(t)+y'(t)+\frac{1}{\eta(p-\tau)}E'(t),\\
&= rx\left(1-\frac{x}{k}\right)-\frac{\alpha xy}{a+x}-\frac{qEx}{m_1E+m_2x}+sy-\frac{\beta sy}{a+x}+\frac{qEx}{m_1E+m_2x}-\frac{cE}{\eta(p-\tau)},\\
\frac{d\psi(t)}{dt} &\le \left(rx-\frac{rx^2}{k}\right)+sy-\frac{\beta sy}{a+k}-\frac{cE}{\eta(p-\tau)}.
\end{aligned}$$

Introduce a positive constant N and rewrite the above equation as follows:

$$\frac{d\psi(t)}{dt} + N\psi(t) \leq \left((r+N)x - \frac{r}{k}x^2\right) + \left((s+N)y - \frac{s\beta}{a+k}y^2\right) - \frac{(c-N)}{\eta(p-\tau)}E.$$

For $c > N$, further simplification yields,

$$\frac{d\psi(t)}{dt} + N\psi(t) \leq -\frac{r}{k}\left(x - \frac{k(r+N)}{2r}\right)^2 - \frac{s\beta}{a+k}\left(y - \frac{(s+N)(k+a)}{2s\beta}\right)^2 + M,$$
$$\frac{d\psi(t)}{dt} + N\psi(t) \leq M; \quad M = \left(\frac{k^2(r+N)^2}{4r^2} + \frac{(s+N)^2(k+a)^2}{4s^2\beta^2}\right).$$

Solution of above differential inequality gives,

$$\psi(t) \leq \frac{M}{N}\left(1 - e^{-Nt}\right) + \psi(0)e^{-Nt},$$
$$0 < \lim_{t\to\infty} \psi(t) \leq \frac{M}{N}.$$

Accordingly, all the solutions of (3.2.1) initiating from $\mathbb{R}^3_+$ are confined in the region

$$R = \left\{(x, y, E) \in \mathbf{R}; 0 < x(t) + y(t) + \frac{1}{\eta(p-\tau)}E(t) \leq \frac{M}{N} + \phi \text{ for any } \phi > 0\right\}$$

This proves the result. □

3.4 Existence of Equilibrium States

The system has six feasible non-negative equilibrium states, namely

(*i*) $P_0(0, 0, 0)$ is a trivial equilibrium point.

(*ii*) $P_1(k, 0, 0)$ is the axial equilibrium point on $x-$axis.

(*iii*) $P_2(0, \frac{a}{\beta}, 0)$ is the axial equilibrium point on $y-$axis.

(*iv*) $P_3(\overline{x}, \overline{y}, 0)$ is the boundary equilibrium point in xy-plane. The equilibrium level densities $\overline{x}$ and $\overline{y}$ are the positive solution of the following equations:

$$r\left(1 - \frac{\overline{x}}{k}\right) - \frac{\alpha\overline{y}}{a+\overline{x}} = 0,$$

$$1-\frac{\beta\overline{y}}{a+\overline{x}}=0.$$

The positive solution is obtained as

$$\overline{x}=k\left(1-\frac{\alpha}{r\beta}\right)\quad and\quad \overline{y}=\frac{1}{\beta}\left(a+\overline{x}\right)\quad with\quad r\beta>\alpha. \tag{3.4.1}$$

(v) $P_4(\hat{x},0,\hat{E})$ is boundary equilibrium point in $xE-$plane. Here $\hat{x}$ and $\hat{E}$ are the positive solution of the following equations:

$$r\left(1-\frac{\hat{x}}{k}\right)-\frac{q\hat{E}}{m_1\hat{E}+m_2\hat{x}}=0,$$

$$\frac{q(p-\tau)\hat{x}}{m_1\hat{E}+m_2\hat{x}}-c=0.$$

This gives

$$\hat{x}=k\left(\frac{(rm_1-q)L+rm_2}{r(m_1L+m_2)}\right)\quad and\quad \hat{E}=L_1\hat{x};\quad L_1=\frac{(p-\tau)q-cm_2}{cm_1}.$$

Accordingly, $\hat{x}$ is positive provided one of the following conditions is satisfied as follows:

$$m_1\geq\frac{q}{r}\quad and\quad \tau<p-\frac{cm_2}{q} \tag{3.4.2}$$

$$m_1<\frac{q}{r}\quad and\quad 0<\frac{q}{r}-m_1<\frac{rm_2}{L_1}\Rightarrow\tau>p-\left(\frac{cm_2}{q}+\frac{r^2cm_1m_2}{q(q-m_1r)}\right) \tag{3.4.3}$$

(vi) $P^*(x^*,y^*,E^*)$ is the unique interior equilibrium point of the system (3.2.1) and is obtained by solving the following equations:

$$r\left(1-\frac{x^*}{k}\right)-\frac{\alpha y^*}{a+x^*}-\frac{qE^*}{m_1E^*+m_2x^*}=0,$$

$$1-\frac{\beta y^*}{a+x^*}=0,$$

$$\frac{q(p-\tau)x^*}{m_1E^*+m_2x^*}-c=0.$$

These yield:

$$x^*=k\left(1-\frac{\alpha}{r\beta}-\frac{qL_1}{r(m_1L_1+m_2)}\right),y^*=\frac{a+x^*}{\beta}\quad and\quad E^*=L_1x^* \tag{3.4.4}$$

The interior equilibrium point (x^*, y^*, E^*) is positive for the condition

$$p - \left(\frac{cm_2}{q} + \frac{(r - \frac{\alpha}{\beta})cm_1m_2}{q(q - m_1(r - \frac{\alpha}{\beta}))}\right) < \tau < p - \frac{cm_2}{q}. \tag{3.4.5}$$

The condition (3.4.5) gives the range of tax for the existence of interior equilibrium and this range of tax can be useful for regulatory agency at the time of formulation of tax structure per unit biomass for controlling the fishery system.

3.5 Stability of Equilibrium States

In this section, the local stability of all feasible equilibrium points is discussed except $(0,0,0)$ and $\left(0, \frac{a}{\beta}, 0\right)$ as the system cannot be linearized at these equilibrium points.

For the local stability, the Jacobian matrix of the system (3.2.1) at (x, y, E) is given by

$$J = \begin{bmatrix} f + x\left(-\frac{r}{k} + \frac{\alpha y}{(a+x)^2} + \frac{qEm_2}{(m_1E + m_2x)^2}\right) & -\frac{\alpha x}{a+x} & -m_2q\left(\frac{x}{m_1E + m_2x}\right)^2 \\ \frac{\beta s y^2}{(a+x)^2} & g - \frac{\beta s y}{a+x} & 0 \\ \frac{\eta q(p-\tau)m_1E^2}{(m_1E + m_2x)^2} & 0 & h - \frac{\eta q(p-\tau)m_1xE}{(m_1E + m_2x)^2} \end{bmatrix}$$

Following some theorems are stated for the stability of various equilibrium states.

Theorem 3.5.1. *The axial equilibrium state $P_1(k, 0, 0)$ is always a saddle point with unstable manifold in y-direction and stable manifold in x-direction. It has a stable manifold in E-direction provided*

$$\tau > p - \frac{cm_2}{q}. \tag{3.5.1}$$

Proof. The Jacobian matrix of the system (3.2.1) evaluated at $P_1(k, 0, 0)$ is given by

$$J_1(k,0,0) = \begin{bmatrix} -r & \frac{-\alpha k}{a+k} & \frac{q}{m_2} \\ 0 & s & 0 \\ 0 & 0 & \eta\left(\frac{(p-\tau)q}{m_2} - c\right) \end{bmatrix}$$

The eigenvalues corresponding to the equilibrium point $P_1(k,0,0)$ are given by

$$\lambda_1 = -r < 0, \quad \lambda_2 = s > 0 \quad and \quad \lambda_3 = \eta\left(\frac{(p-\tau)q}{m_2} - c\right)$$

The point $(k,0,0)$ is a saddle point with unstable manifold in y-direction and stable manifold in x-direction. The system has a stable manifold in E-direction if the condition (3.5.1) is satisfied. Moreover, the system has an unstable manifold in E-direction if the condition (3.5.1) is violated.

□

Remark 3.5.2. *The equilibrium point $P_1(k,0,0)$ becomes non–hyperbolic and bifurcation may occur when*

$$\frac{(p-\tau)q}{m_2} = c. \tag{3.5.2}$$

Theorem 3.5.3. *The planar equilibrium state $P_3(\overline{x},\overline{y},0)$ is locally asymptotically stable for the following conditions:*

$$\frac{\alpha\overline{x}}{\beta(a+\overline{x})} < r + s - \frac{\alpha}{\beta} \quad and \tag{3.5.3}$$

$$\frac{(p-\tau)q}{m_2} < c. \tag{3.5.4}$$

Proof. The Jacobian matrix at the equilibrium point $P_3(\overline{x},\overline{y},0)$ is given by

$$J_3(\overline{x},\overline{y},0) = \begin{bmatrix} \overline{x}\left(-\frac{r}{k} + \frac{\alpha\overline{y}}{(a+\overline{x})^2}\right) & \frac{-\alpha\overline{x}}{a+\overline{x}} & \frac{q}{m_2} \\ \frac{s\beta(\overline{y})^2}{(a+\overline{x})^2} & \frac{-s\beta\overline{y}}{a+\overline{x}} & 0 \\ 0 & 0 & \eta\left(\frac{(p-\tau)q}{m_2} - c\right) \end{bmatrix}$$

The characteristic equation corresponding to the equilibrium point $P_3(\overline{x},\overline{y},0)$ yields the eigenvalues:

$$\lambda_{1,2} = \frac{1}{2}\left[\left(\frac{\alpha}{\beta} - r - s\right) + \frac{\alpha\overline{x}}{\beta(a+\overline{x})} \pm \sqrt{\left(\left(\frac{\alpha}{\beta} - r - s\right) + \frac{\alpha\overline{x}}{\beta(a+\overline{x})}\right)^2 - 4\frac{rs\overline{x}}{k}}\right]$$

and

$$\lambda_3 = \eta\left(\frac{(p-\tau)q}{m_2} - c\right).$$

Accordingly, The equilibrium point P_3 is locally asymptotically stable for the conditions (3.5.3) and (3.5.4).

□

Remark 3.5.4. *The point P_3 is a saddle point if one of the conditions* (3.5.3) *or* (3.5.4) *is violated. If both the conditions are violated then the point P_3 is unstable.*

The bifurcation is possible when

$$\frac{(p-\tau)q}{m_2} = c. \tag{3.5.5}$$

If $\left(\frac{\alpha}{\beta} - r - s\right) + \frac{\alpha\overline{x}}{\beta(a+\overline{x})} = 0$, then a pair of purely imaginary eigenvalues exists. The transversality condition for Hopf bifurcation at the equilibrium point P_3 is given by

$$\frac{d}{ds}\left[\left(\frac{\alpha}{\beta} - r - s\right) + \frac{\alpha\overline{x}}{\beta(a+\overline{x})}\right] \neq 0 \quad for \quad s = \frac{(r\beta-\alpha)[k(\alpha - r\beta - \alpha\beta) - ar\beta]}{\beta(ar\beta + k(r\beta - \alpha))}$$

Accordingly, the existence of periodic solutions around the equilibrium point P_3 are possible. This will attract all small perturbations in the neighborhood of xy-plane when $\frac{(p-\tau)q}{m_2} < c$.

Theorem 3.5.5. *The planar equilibrium state $P_4(\hat{x}, 0, \hat{E})$ is a saddle point with unstable manifold in y-direction for the following condition:*

$$\left(\frac{qL_1}{m_1L_1 + m_2} - r\right) + \frac{qL_1(m_2 - \eta m_1(p-\tau))}{(m_1L_1+m_2)^2} > 0 \tag{3.5.6}$$

Proof. The Jacobian matrix evaluated at the equilibrium point $P_4(\hat{x}, 0, \hat{E})$ is given by

$$J_4(\hat{x}, 0, \hat{E}) = \begin{bmatrix} \hat{x}\left(-\frac{r}{k} + \frac{q\hat{E}m_2}{(m_1\hat{E} + m_2\hat{x})^2}\right) & \frac{-\alpha\hat{x}}{a+\hat{x}} & \frac{qm_2\hat{x}^2}{(m_1\hat{E}+m_2\hat{x})^2} \\ 0 & s & 0 \\ \frac{\eta\hat{E}^2q(p-\tau)m_1}{(m_1\hat{E}+m_2\hat{x})^2} & 0 & \frac{-\eta\hat{E}\hat{x}q(p-\tau)m_1}{(m_1\hat{E}+m_2\hat{x})^2} \end{bmatrix}$$

Corresponding to the equilibrium point $P_4(\hat{x}, 0, \hat{E})$, one of the eigenvalue is $\lambda = s > o$, and the other two are obtained as eigenvalues of the following 2×2 matrix

$$J_4^*(\hat{x}, 0, \hat{E}) = \begin{bmatrix} \hat{x}\left(-\frac{r}{k} + \frac{q\hat{E}m_2}{(m_1\hat{E} + m_2\hat{x})^2}\right) & -\frac{qm_2\hat{x}^2}{(m_1\hat{E}+m_2\hat{x})^2} \\ \frac{\eta\hat{E}^2q(p-\tau)m_1}{(m_1\hat{E}+m_2\hat{x})^2} & -\frac{\eta\hat{E}\hat{x}q(p-\tau)m_1}{(m_1\hat{E}+m_2\hat{x})^2} \end{bmatrix}$$

It can be observed that

$$det(J_4^*) = \frac{r\hat{x}}{k} > 0 \quad and$$

$$\begin{aligned} tr(J_4^*) &= \hat{x}\bigg(-\frac{r}{k} + \frac{q\hat{E}m_2}{(m_1\hat{E} + m_2\hat{x})^2}\bigg) + \frac{-\eta\hat{E}\hat{x}q(p-\tau)m_1}{(m_1\hat{E} + m_2\hat{x})^2} \\ &= \bigg(\frac{qL_1}{m_1L_1 + m_2} - r\bigg) + \frac{qL_1(m_2 - \eta m_1(p-\tau))}{(m_1L_1 + m_2)^2}. \end{aligned}$$

The equilibrium point P_4 is a saddle point when $tr(J_4^*) < 0$ which satisfy the condition (3.5.6). □

Remark 3.5.6. *If the condition* (3.5.6) *is violated i.e., $tr(J_4^*) > 0$, the equilibrium point P_4 becomes an unstable point.*

If $tr(J_4^*) = 0$, then it has pair of purely imaginary roots. The transversality condition for Hopf bifurcation at the equilibrium point P_3 is given by

$$\frac{dtr(J_4^*}{dr} = -1 \neq 0 \quad for \quad r = \frac{qL_1}{m_1L_1 + m_2} + \frac{qL_1(m_2 - \eta m_1(p-\tau))}{(m_1L_1 + m_2)^2} \tag{3.5.7}$$

Therefore, there exists a family of an attracting periodic solutions through Hopf bifurcation from P_4 in the neighborhood of r, keeping other parameters fixed.

Theorem 3.5.7. *The positive interior equilibrium point $P^*(x^*, y^*, E^*)$ is asymptotically locally stable provided:*

$$M_1 = \frac{r}{k} - \frac{\alpha y^*}{(a + x^*)^2} - \frac{qE^*m_2}{(m_1E^* + m_2x^*)^2} > 0. \tag{3.5.8}$$

Proof. Let the Jacobian matrix of the system (3.2.1) evaluated at the equilibrium point P^* be $J^*(x^*, y^*, E^*) = (a_{ij})_{3\times3}$.

$$a_{11} = x^*\bigg(-\frac{r}{k} + \frac{\alpha y^*}{(a + x^*)^2} + \frac{qE^*m_2}{(m_1E^* + m_2x^*)^2}\bigg), \quad a_{12} = \frac{-\alpha x^*}{a + x^*},$$

$$a_{13} = \frac{m_2q(x^*)^2}{(m_1E^* + m_2x^*)^2}, \quad a_{21} = \frac{\beta sy^*}{(a + x^*)^2}, \quad a_{22} = \frac{-\beta sy^*}{a + x^*} \quad a_{23} = a_{32} = 0,$$

$$a_{31} = \frac{\eta q(p-\tau)m_1(E^*)^2}{(m_1E^* + m_2x^*)^2}, \quad a_{33} = -\frac{\eta q(p-\tau)m_1x^*E^*}{(m_1E^* + m_2x^*)^2}, \quad a_{33} = -a_{31}\frac{x^*}{E^*}.$$

Therefore, the characteristics equation of the jacobian matrix at $P^*(x^*, y^*, E^*)$ is obtained as

$$\lambda^3 + A_1\lambda^2 + A_2\lambda + A_3 = 0,$$

with

$$\begin{aligned} A_1 &= -(a_{11} + a_{22} + a_{33}), \\ A_2 &= a_{22}a_{33} + (a_{11}a_{33} - a_{13}a_{31}) + (a_{11}a_{22} - a_{21}a_{21}), \\ A_3 &= a_{11}a_{22}a_{33} - a_{12}a_{21}a_{33} - a_{13}a_{31}a_{22}. \end{aligned}$$

Using Routh-Hurwitz criteria, the condition for local stability of the equilibrium point $P^*(x^*, y^*, E^*)$ is

$$A_1 > 0, \quad A_2 > 0 \quad and \quad A_1A_2 - A_3 > 0.$$

Note that $A_1 > 0$ provided (3.5.8).

Also, $A_2 > 0$ and $A_1A_2 - A_3 > 0$ for the condition (3.5.8).
Accordingly, the interior equilibrium point (x^*, y^*, E^*) is locally asymptotically stable provided $M_1 > 0$.

□

Remark 3.5.8. *The interior equilibrium point* (x^*, y^*, E^*) *may be unstable or saddle for* $M_1 < 0$.

3.6 Global Stability

Theorem 3.6.1. *The interior equilibrium point* (x^*, y^*, E^*) *of the system* (3.2.1) *is globally asymptotically stable in the domain* $D = \{(x, y, E) : m_1E + m_2x > M_2, (x, y, E) \in \mathbb{R}^3_+\}$, *where* $M_2 = \dfrac{L_1qka\beta}{(m_1L_1 + m_2)(ra\beta - \alpha k)}$.

Proof. Consider a Lyapunov function $V(x, y, E)$ for arbitrary chosen positive constants d_0, d_1 and d_2 such that:

$$\begin{aligned} V(x,y,E) &= d_0\left[(x - x^*) - x^* \log\frac{x}{x^*}\right] + d_1\left[(y - y^*) - y^* \log\frac{y}{y^*}\right] + \\ & \quad d_2\left[(E - E^*) - E^* \log\frac{E}{E^*}\right]; \qquad V(x^*, y^*, E^*) = 0. \end{aligned}$$

Now differentiate V w.r.t. time t,

$$\begin{aligned}\frac{dV}{dt} &= d_0(x-x^*)\frac{\dot{x}}{x}+d_1(y-y^*)\frac{\dot{y}}{y}+d_2(E-E^*)\frac{\dot{E}}{E},\\ &= d_0(x-x^*)\left(r\left(1-\frac{x}{k}\right)-\frac{\alpha y}{a+x}-\frac{qE}{m_1E+m_2x}\right)+d_1(y-y^*)\left(s-\frac{\beta sy}{a+x}\right)\\ &+ d_2(E-E^*)\eta\left(\frac{q(p-\tau)x}{m_1E+m_2x}-c\right).\end{aligned}$$

After solving,

$$\begin{aligned}\frac{dV}{dt} &= -\ d_0(x-x^*)^2\left[\frac{r}{k}-\frac{\alpha y^*}{(a+x)(a+x^*)}-\frac{qE^*}{(m_1E+m_2x)(m_1E^*+m_2x^*)}\right]\\ &\left[-d_0\alpha x^*+\frac{d_1s\beta E^*}{a+x^*}\right]+\frac{-d_0qx^*+d_2q(p-\tau)E^*}{(m_1E+m_2x)(m_1E^*+m_2x^*)}(x-x^*)(E-E^*)\\ &+ \frac{(x-x^*)(y-y^*)}{a+x}-\frac{d_1s\beta(y-y^*)^2}{a+x}-\frac{d_2q(p-\tau)x^*(E-E^*)^2}{(m_1E+m_2x)(m_1E^*+m_2x^*)}.\end{aligned}$$

Choosing $d_0=1$, $d_1=\dfrac{\alpha x^*(a+x^*)}{s\beta E^*}$ and $d_2=\dfrac{x^*}{(p-\tau)E^*}>0$ for $p>\tau$, the above equation becomes,

$$\begin{aligned}\frac{dV}{dt} &= -d_0(x-x^*)^2\left[\frac{r}{k}-\frac{\alpha y^*}{(a+x)(a+x^*)}-\frac{qE^*}{(m_1E+m_2x)(m_1E^*+m_2x^*)}\right]\\ &- \frac{d_1s\beta}{a+x}(y-y^*)^2-\frac{d_2q(p-\tau)x^*}{(m_1E+m_2x)(m_1E^*+m_2x^*)}(E-E^*)^2,\\ \frac{dV}{dt} &\le -(x-x^*)^2\left[\frac{r}{k}-\frac{\alpha y^*}{a(a+x^*)}-\frac{qE^*}{(m_1E+m_2x)(m_1E^*+m_2x^*)}\right]-\\ &\frac{\alpha_1(a+x^*)x^*(y-y^*)^2}{(a+x)E^*}-\frac{qx^{2*}(E-E^*)^2}{(m_1E+m_2x)(m_1E^*+m_2x^*)},\end{aligned}$$

$$\frac{dV}{dt}<0 \quad \text{if} \quad \frac{r}{k}-\frac{\alpha y^*}{a(a+x^*)}-\frac{qE^*}{(m_1E+m_2x)(m_1E^*+m_2x^*)}>0.$$

Substituting the values of y^* and E^* in the above expression, a plane is obtained as follows:

$$(m_1E+m_2x)>\frac{L_1qka\beta}{(m_1L_1+m_2)(ra\beta-\alpha k)}=M_2(say). \qquad (3.6.1)$$

If $ra\beta<\alpha k$, the inequality (3.6.1) is trivially true and for $ra\beta>\alpha k$, a bound of plane (m_1E+m_2x) is obtained in positive octant. This shows that $\frac{dV}{dt}$ is negative definite for the condition (3.6.1).

Accordingly, the interior equilibrium point (x^*,y^*,E^*) is globally asymptotically stable for the sufficient condition (3.6.1). □

3.7 Bifurcations

Theorem 3.7.1. *The system* (3.2.1) *exhibits a transcritical bifurcation around the axial equilibrium point* $P_1(k,0,0)$ *if*

$$\tau_c = p - \frac{cm_2}{q} \tag{3.7.1}$$

Proof. The Jacobian of system (3.2.1) at equilibrium point $P_1(k,0,0)$ has a zero eigenvalue for the condition $\tau = p - \frac{cm_2}{q}$ and therefore, the equilibrium point $(k,0,0)$ becomes non-hyperbolic. So there is a chance of bifurcation around this equilibrium point. The threshold value of the bifurcation is $\tau_c = p - \frac{cm_2}{q}$.
The eigenvectors of $J(k,0,0)$ and $(J(k,0,0))^T$ corresponding to zero eigenvalue are obtained as

$$V = \left(1, 0, \frac{-rm_2}{q}\right)^T \quad \textit{and} \quad W = (0,0,1)^T, \quad \textit{respectively.} \tag{3.7.2}$$

Compute Δ_1, Δ_2 and Δ_3 as follows:

$$\Delta_1 = W^T F_\tau(P_1, \tau_c) = 0, \quad F = (F^1, F^2, F^3)^T = (xf, yg, Eh)^T.$$

$$\Delta_2 = W^T \left[DF_\tau(P_1, \tau_c)V\right] = r\eta \neq 0,$$

where

$$DF_\tau(P_1, \tau^{tc}) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{-\eta q}{m_2} \end{pmatrix}$$

.

$$\Delta_3 = W^T \left[D^2 F_\tau(P_1, \tau_c)(V,V)\right] = \frac{\eta m_1 r^2}{qk} \neq 0$$

. Since, $\Delta_1 = 0$, there is no chance of saddle–node bifurcation.
Accordingly, by the Sotomayors theorem [102], the system (3.2.1) undergoes a transcritical bifurcation around the axial equilibrium point $(k,0,0)$ for the condition (3.7.1).

□

Remark 3.7.2. *Similarly, the system* (3.2.1) *has a transcritical bifurcation around the boundary equilibrium point* $(\overline{x}, \overline{y}, 0)$ *for*

$$\tau = p - \frac{cm_2}{q}. \tag{3.7.3}$$

3.8 Persistence

Persistence ensures the long term co-existence of all species. The system is investigated near the boundaries of the positive octant. According to the approach of Freedman and Waltman [39] [section-4], consider the system (3.2.1) along with the following assumptions:

(B1) $f_y = \dfrac{-\alpha}{a+x} < 0, \qquad f_E = \dfrac{-m_2 q x}{(m_1 E + m_2 x)^2} < 0,$

$$g_x = \frac{s\beta y}{(a+x)^2} > 0,$$
$$h_x = \frac{\eta(p-\tau)m_1 E}{(m_1 E + m_2 x)^2} > 0,$$
$$g(0,y,E) = 1 - \frac{\beta}{a}y < 0 \quad \text{if} \quad y > \frac{a}{\beta},$$
$$h(0,0,E) = -\eta c < 0.$$

(B2) The prey species x grows to the carrying capacity in the absence of predator i.e.,

$$f(0,0,0) = r > 0 \quad \textit{and} \quad f(k,0,0) = 0.$$

While, due to the intra-specific competition within prey species, it is observed

$$\frac{\partial f}{\partial x}(x,0,0) = -\frac{r}{k} < 0.$$

(B3) There is no equilibrium point on yE-plane.

(B4) In the absence of harvesting ($E = 0$) and predator ($y = 0$), there exist equilibrium points $(\overline{x}, \overline{y}, 0)$ and $(\hat{x}, 0, \hat{E})$ respectively, such that

$$f(\overline{x}, \overline{y}, 0) = g(\overline{x}, \overline{y}, 0) = 0,$$
$$f(\hat{x}, 0, \hat{E}) = h(\hat{x}, 0, \hat{E}) = 0.$$

Therefore, the following results represent the conditions for persistence of the system (3.2.1).

Theorem 3.8.1. *Let the hypotheses [B1]-[B4] hold. The system* (3.2.1) *persists in the absence of periodic solutions in the boundary planes provided*

$$h(\hat{x}, 0, \hat{E}) = \frac{(p-\tau)q}{m_2} - c > 0. \tag{3.8.1}$$
$$g(\overline{x}, \overline{y}, 0) = s > 0. \tag{3.8.2}$$

Proof. For the boundary equilibrium point $P_3(\overline{x}, \overline{y}, 0)$ in xy-plane, the eigen value in E-direction is obtained as $\lambda_3 = \dfrac{(p-\tau)q}{m_2} - c$. The point $P_3(\overline{x}, \overline{y}, 0)$ is unstable provided (3.8.1) holds.
Similarly, $\lambda_2 = s > 0$ is eigenvalue in y-direction corresponding to the boundary equilibrium point $(\hat{x}, 0, \hat{E})$ in xE-plane, which is unstable provided (3.8.2) holds. Also, the points $(0, 0, 0)$ and $(k, 0, 0)$ are unstable. This shows that all trajectories are bounded away from all boundaries of the system. Hence, if there are no limit cycles on the boundary planes and the conditions (3.8.1) and (3.8.2) are satisfied , then the system (3.2.1) persists. □

Theorem 3.8.2. *Let there be a finite number of periodic solution in xy and xE-planes. Then, for each limit cycle $(u(t), v(t))$ in the xy-plane and $(\omega_1(t), \omega_2(t))$ in xE-plane, the persistence conditions for the system would take the form:*

$$\int_0^{\xi} h(u(t), v(t), 0)\, dt > 0 \quad and \quad \int_0^{\omega} g(\hat{u}(t), 0, \hat{v}(t))\, dt > 0,$$

where ξ and ω are the limit periods of the limit cycle.

Proof. Assume that there exists a limit cycle in the in the xy-plane , then the variational matrix about the limit cycle $x(t) = u(t), y(t) = v(t), z(t) = 0$ take the form

$$V(u(t), v(t), 0) = \begin{bmatrix} u(t)\Big(-\dfrac{r}{k} + \dfrac{\alpha v(t)}{(a+u(t))^2}\Big) & \dfrac{-\alpha u(t)}{a+u(t)} & -\dfrac{q}{m_2} \\ \dfrac{s\beta(v(t))^2}{(a+u(t))^2} & \dfrac{-s\beta v(t)}{a+u(t)} & 0 \\ 0 & 0 & \eta\Big(\dfrac{(p-\tau)q}{m_2} - c\Big) \end{bmatrix}$$

Consider the solution of given system with the initial condition (t, a_1, a_2, a_3) sufficiently close to the limit cycle. From the above variational matrix, it can be obtained that

$$\begin{aligned} \frac{dE}{dt} &= \eta\Big(\frac{(p-\tau)q}{m_2} - c\Big) \quad with \quad E(0) = a_3, \\ E &= a_3 exp\Big[\int_0^{\xi} \eta\Big(\frac{(p-\tau)q}{m_2} - c\Big)\, dt\Big], \\ \frac{\partial E}{\partial a_3} &= exp\Big[\int_0^{\xi} \eta\Big(\frac{(p-\tau)q}{m_2} - c\Big)\, dt\Big]. \end{aligned}$$

Using Taylor's expansion theorem:
$E(t, a_1, a_2, a_3) - E(t, a_1, a_2, 0) \backsimeq a_3 \dfrac{\partial E}{\partial a_3} \backsimeq a_3 exp\left[\int_0^{\xi} \eta \left(\dfrac{(p-\tau)q}{m_2} - c \right) dt \right]$
$\backsimeq a_3 exp\left[\int_0^{\xi} h(u(t), v(t), 0)\, dt \right].$

Therefore, E increases or decreases according to $\int_0^{\xi} h(u(t), v(t), 0)\, dt$ is positive or negative. Hence, the trajectories go away from the xy-plane under the assumptions of the theorem.

Similarly, result can be proved for xE-plane.

This completes the result. □

3.9 Bionomic Equilibrium

The net economic revenue to the society is represented as the sum of net economic revenue to the fishermen and net economic revenue to the regulatory agency, i.e.,

$$P(t, x, y, E, \tau) = \left(\frac{q(p-\tau)x}{m_1 E + m_2 x} - c \right) E + \frac{q\tau x E}{m_1 E + m_2 x} = \left(\frac{qpx}{m_1 E + m_2 x} - c \right) E.$$

Clark [22] defined the bionomic equilibrium point as the point of intersection of the interior equilibrium of the system (3.2.1) along with zero net economic revenue. The bionomic equilibrium $P_{BE}(x_{BE}, y_{BE}, E_{BE})$ is obtained as the positive solution of the system

$$\frac{dx}{dt} = \frac{dy}{dt} = \frac{dE}{dt} = P = 0.$$

It gives

$$x_{BE} = \frac{k}{m_1 r}\left[\left(r - \frac{\alpha}{\beta} \right) m_1 - q + \frac{cm_2}{p} \right], \quad y_{BE} = \frac{a + x_{BE}}{\beta} \quad and \quad E_{BE} = \frac{(pq - cm_2)}{cm_1} x_{BE},$$

for

$$\frac{cm_2}{p} < q < \left(r - \frac{\alpha}{\beta} \right) m_1 + \frac{cm_2}{p}. \tag{3.9.1}$$

3.10 Optimal Taxation Policy

In this section, an optimal harvesting policy for the system (3.2.1) is investigated to maximize the total discounted net revenue using taxation as a control instrument.

The optimal control problem over an infinite time horizon is given by

$$\max_{\tau_{min}<\tau(t)<\tau_{max}} I = \int_0^{\infty} e^{-\delta t}\left(\frac{qpx}{m_1E+m_2x} - c\right) dt. \tag{3.10.1}$$

The constant δ is the instantaneous annual rate of discount decided by harvesting agencies. Let $X = (x, y, E)$ and $X^* = (x^*, y^*, E^*)$ are the positions such that there exist a tax policy $\tau(t)$. The system (3.2.1) with $X(t_1) = X^*$ has a positive solution for $t > t_1$ under the policy $\tau(t)$.
Therefore, the taxation policy is assumed as follows:

$$\tau(t) = \begin{cases} \overline{\tau}(t) & for \quad t \in [0, t_1] \\ \tau^* & for \quad t > t_1. \end{cases}$$

The objective is to determine an optimal taxation policy $\tau = \tau(t)$ to maximize (3.10.1) subject to the state equations in the system (3.2.1) and the control constraints $\tau_{min} < \tau(t) < \tau_{max}$. Pontryagin's Maximum Principle is used to obtain the optimal level of the solution of the problem (3.10.1). Let $\lambda_1(t), \lambda_2(t)$ and $\lambda_3(t)$ are adjoint variables w.r.t. the time t corresponding to the variables x, y and E, respectively. The associated Hamiltonian function $\mathcal{H}$ is given by

$$\begin{aligned} \mathcal{H}(t, x, y, E, \tau) \quad = \quad & e^{-\delta t}\left(\frac{qpx}{m_1E+m_2x} - c\right) + \lambda_1\left[rx\left(1-\frac{x}{k}\right) - \frac{\alpha xy}{a+x} - \frac{qEx}{m_1E+m_2x}\right] \\ & +\lambda_2\left[sy\left(1-\frac{\beta y}{a+x}\right)\right] + \lambda_3\left[\eta E\left(\frac{q(p-\tau)x}{m_1E+m_2x} - c\right)\right] \end{aligned} \tag{3.10.2}$$

Notice that Hamiltonian is linear in control variable τ. The optimal control problem involves singular and bang-bang controls. Also, the optimal control must satisfy the following conditions to maximize $\mathcal{H}$:

$$\overline{\tau} = \begin{cases} \tau_{max} & \forall t \in [0, t_1] \quad with \quad \dfrac{d\mathcal{H}}{d\tau} > 0 \\ \tau_{min} & \forall t \in [0, t_1] \quad with \quad \dfrac{d\mathcal{H}}{d\tau} < 0. \end{cases}$$

The Hamiltonian in (3.10.2) must be maximized for $\tau \in [\tau_{min}, \tau_{max}]$. Assume that the control constraints are not binding (i.e., the optimal solution does not occur at τ_{min} or τ_{max}). Thus, the considered control problem admits a singular solution on the control set (τ_{min}, τ_{max}) if

$$\begin{aligned} \frac{\partial \mathcal{H}}{\partial \tau} \quad &= \quad 0, \\ i.e., \frac{-qx\lambda_3}{m_1E+m_2x} \quad &= \quad 0 \quad \Rightarrow \quad \lambda_3(t) = 0. \end{aligned} \tag{3.10.3}$$

In order to find a singular control, Pontryagin's Maximum Principle [104] is utilized and the adjoint variables must satisfy the adjoint equations given by

$$\frac{d\lambda_1}{dt} = -\frac{\partial \mathcal{H}}{\partial x}, \quad \frac{d\lambda_2}{dt} = -\frac{\partial \mathcal{H}}{\partial y}, \quad \frac{d\lambda_3}{dt} = -\frac{\partial \mathcal{H}}{\partial E}. \tag{3.10.4}$$

The optimal equilibrium point is the equilibrium point corresponding to the optimal tax. Such a path is called the optimal path and is a solution of the system (3.2.1). Now, the adjoint equations are

$$\begin{aligned}
\frac{d\lambda_1}{dt} &= -\frac{\partial \mathcal{H}}{\partial x} = -\Bigg[e^{-\delta t}\bigg(\frac{pqm_1E^2}{(m_1E+m_2x)^2}\bigg) + \lambda_1\bigg(-\frac{rx}{k} + \frac{\alpha xy}{(a+x)^2} - \frac{qExm_2}{(m_1E+m_2x)^2}\bigg) \\
&+ \lambda_2\bigg(\frac{s\beta y^2}{(a+x)^2}\bigg)\Bigg], \qquad (3.10.5)\\
\frac{d\lambda_2}{dt} &= -\frac{\partial \mathcal{H}}{\partial y} = -\Bigg[-\lambda_1\bigg(\frac{\alpha x}{a+x}\bigg) - \lambda_2\bigg(\frac{s\beta y}{a+x}\bigg)\Bigg], \qquad (3.10.6)\\
\frac{d\lambda_3}{dt} &= -\frac{\partial \mathcal{H}}{\partial E} = -\Bigg[e^{-\delta t}\bigg(\frac{pqm_2x^2}{c(m_1E+m_2x)^2}\bigg) + \bigg(\frac{-qm_2x^2}{c(m_1E+m_2x)^2}\bigg)\Bigg] \\
&= -\Bigg[e^{-\delta t}\bigg(p - \frac{c(m_1E+m_2x)^2}{qm_2x^2}\bigg) - \lambda_1\Bigg]. \qquad (3.10.7)
\end{aligned}$$

Also, the considered control problem admits a singular solution on the control set $[0, E_{max}]$ if $\frac{\partial \mathcal{H}}{\partial E} = 0$,

$$\Rightarrow \lambda_1(t) = e^{-\delta t}\bigg(p - \frac{c(m_1E+m_2x)^2}{qm_2x^2}\bigg). \tag{3.10.8}$$

Let $\lambda_i(t) = \mu_i(t)e^{-\delta t}$, where $\mu_i(t) = \lambda_i(t))e^{\delta t}$ $(i = 1, 2, 3)$ are known as the shadow prices and they should remain constant over time. Solving (3.10.6), a linear differential equation is obtained in λ_2 and in the interior equilibrium (x^*, y^*, E^*) such that

$$\frac{d\lambda_2}{dt} - A_1\lambda_2 = -e^{-\delta t}A_2, \tag{3.10.9}$$

where

$$A_1 = \frac{s\beta y^*}{a+x^*} \quad and \quad A_2 = \frac{\alpha x^*}{a+x^*}\bigg(p - \frac{c(m_1E^*+m_2x^*)^2}{qm_2x^{*2}}\bigg)$$

Solving equation (3.10.9),

$$\lambda_2(t) = \frac{A_1}{A_2+\delta}e^{-\delta t} \tag{3.10.10}$$

To solve (3.10.5), put the value of $\lambda_2(t)$ using (3.10.10) in (3.10.5),

$$\begin{aligned}\frac{d\lambda_1}{dt} &= -e^{-\delta t}\left(\frac{pqm_1E^{*2}}{(m_1E^*+m_2x^*)^2}\right)+\lambda_1\left(\frac{rx^*}{k}-\frac{\alpha x^*y^*}{(a+x^*)^2}+\frac{qE^*x^*m_2}{(m_1E^*+m_2x^*)^2}\right)\\ &- \frac{A_1}{A_2+\delta}e^{-\delta t}\left(\frac{s\beta y^{*2}}{(a+x^*)^2}\right).\end{aligned}$$

$$\frac{d\lambda_1}{dt}-B_1\lambda_1=-e^{-\delta t}B_2, \tag{3.10.11}$$

where

$$B_1=\frac{rx^*}{k}-\frac{\alpha x^*y^*}{(a+x^*)^2}+\frac{qE^*x^*m_2}{(m_1E^*+m_2x^*)^2}$$

and

$$B_2=\frac{pqm_1E^{*2}}{(m_1E^*+m_2x^*)^2}+\frac{A_1}{(A_2+\delta)}\frac{s\beta y^{*2}}{(a+x^*)^2}.$$

Solving equation (3.10.11),

$$\lambda_1(t)=\frac{B_1}{B_2+\delta}e^{-\delta t} \tag{3.10.12}$$

Using (3.10.8) and (3.10.12),

$$p-\frac{c(m_1E^*+m_2x^*)^2}{qm_2x^{*2}}=\frac{B_1}{B_2+\delta} \tag{3.10.13}$$

Therefore, the expression (3.10.13) gives desired singular path. Next, Arrow Sufficiency condition for infinite time horizon [42] is applied, for the optimal level of this singular solution. It is observed that

$$\begin{aligned}\frac{\partial^2\mathcal{H}}{\partial x^2} &= \frac{-r\lambda_1}{k}-\frac{m_1E^{*2}}{(m_1E^*+m_2x^*)x^{*2}}\left[\frac{\partial\mathcal{H}}{\partial E}+ce^{-\delta t}\right]-\frac{m_2^2qE^*x^*}{(m_1E^*+m_2x^*)^3}\\ &- \frac{s\beta\lambda_2y^{*2}}{(a+x^*)^3}-\frac{\alpha y^*(a-x^*)\lambda_1}{(a+x^*)^3}.\end{aligned}$$

For $\frac{\partial^2\mathcal{H}}{\partial x^2}<0$, it is observed that $\lambda_1>0, \lambda_2>0$ and $x^*<a$ with the singular control i.e., $\frac{\partial\mathcal{H}}{\partial E}=0$ and $\frac{\partial^2\mathcal{H}}{\partial y^2}=\frac{-\lambda_2 s\beta}{a+x^*}<0$.

Therefore, $\frac{\partial^2\mathcal{H}}{\partial x^2}<0$ and $\frac{\partial^2\mathcal{H}}{\partial y^2}<0$ for all $t\in[0,\infty)$ This shows that the Hamiltonian $\mathcal{H}$ is concave in both x and y for all $t\in[0,\infty)$ provided the required conditions are satisfied. Hence, the Arrow Sufficiency condition for infinite time horizon shows that the singular solution is the part of optimal solution.

3.11 Numerical Simulations

In this section, numerical simulations are carried out for suitable choice of parameters to investigate the dynamical behavior of the system, keeping all the parameters fixed except τ. Hence, τ is known as bifurcation parameter. Consider the following choice of data in appropriate units:

$$\begin{aligned} r = 0.3, k = 100, \alpha = 0.005, m_1 = 0.5, m_2 = 0.5, \\ q = 0.15, a = 3, s = 1, \beta = 0.15, \eta = 1, p = 5, c = 1 \end{aligned} \tag{3.11.1}$$

For the above data set, it can be observed that the axial equilibrium point $P_1(100, 0, 0)$ is saddle point and $P_2(0, 20, 0)$ is locally asymptotically stable. The boundary equilibrium points $P_3(10, 86.667, 0)$ and $P_4(0.952, 0, 0.048)$ are saddle points.

As a regulatory is always interested in the interior states. So, for the above set of parameters, examine the condition of existence and the stability of the steady state $P^*(x^*, y^*, E^*)$. To ensure the existence of the non- trivial steady states P^* , the value of taxation τ can be obtained as $-0.7329 < \tau < 1.667$. For $\tau_{min} < 0$, there is a case of subsidies provided by government to the fishermen at the time of fishing. If there is no case of subsidy, then we will take $\tau_{min} = 0$ and $\tau_{max} = 1.6$(say). For the $\tau_{min} = 0$ and $\tau_{max} = 1.6$, the steady states can be obtained as $(55.557, 390.4073, 27.7777)$ and $(86.7306, 598.3166, 1.8993)$. It can be observed that when a fisherman have to pay no tax, he uses maximum amount of efforts to obtained the maximum benefits from fishery as compared in the case of taxation. The parameter values also satisfy the condition (3.5.8), which shows that steady state $P^*(x^*, y^*, E^*)$ is locally asymptotically stable.

In the figure-3.1, diagrams (a), (b) and (c) give long term behavior of trajectories of prey and predator population and effort E w.r.t. time t for the different low values of tax τ. This shows that for the fixed initial level $(70, 550, 5)$, all the trajectories converges to its interior equilibrium point in the positive octant. Also, it can be observed that as the value of taxation τ increases, the harvesting effort decreases. In resulted, prey population increases which helps predator population to grow. Figure-3.1(d) represents phase plane trajectories of species x, y and harvesting effort E with the different initial levels and it represents that the interior point (x^*, y^*, E^*)= $(84.1232, 580.8527, 4.2091)$ is globally stable corresponding to $\tau = 1.5$ for different initial levels in positive octant.

In figure-3.2, diagrams (a), (b) and (c) gives long term behavior of trajectories of prey and predator population and effort E w.r.t. time t for the different high values of taxation τ. This shows that the population densities for the prey x and predator y increase as the tax rates increase, where as the density of harvesting effort E decreases as the tax rates increase. Therefore, a level of taxation i.e., $\tau = 1.667$ is obtained where effort level will tends to zero. The figure-3.3, represents phase plane trajectories of different biomass with the different initial levels at the interior, which converge to the point $(88.8892, 612.6444, 0.0000)$ on the boundary plane i.e., xy-plane corresponding to $\tau = 1.9$, keeping other parameters fixed. Therefore, for the condition $\tau > 1.667$, it shows that, for the every different initial levels on the xyE-octant converge to a point on xy-plane which means that if a threshold level of taxation i.e.,$\tau = 1.667$ is crossed, then there is no beneficial harvesting of the prey population for this condition as it is not profitable to continue harvesting of prey species.

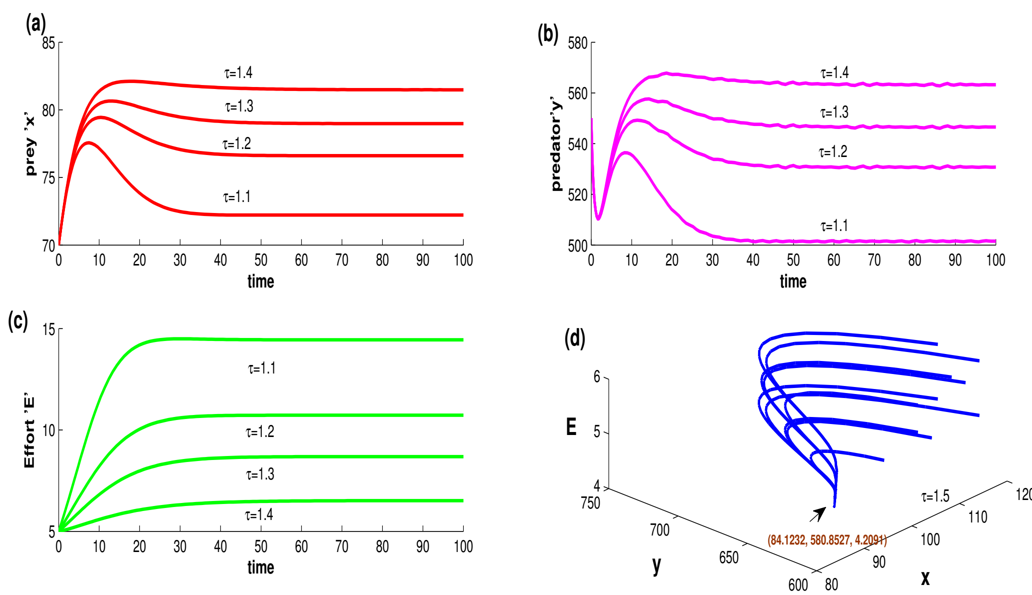

Figure 3.1: Figures (a), (b) and (c) represent solution curves of the x, y and E w.r.t. time t for different low values of tax τ, for a fixed initial level $(70, 550, 5)$ and (d) represents phase plane trajectories of species x, y and Effort E using different initial levels for fixed value of $\tau = 1.5$

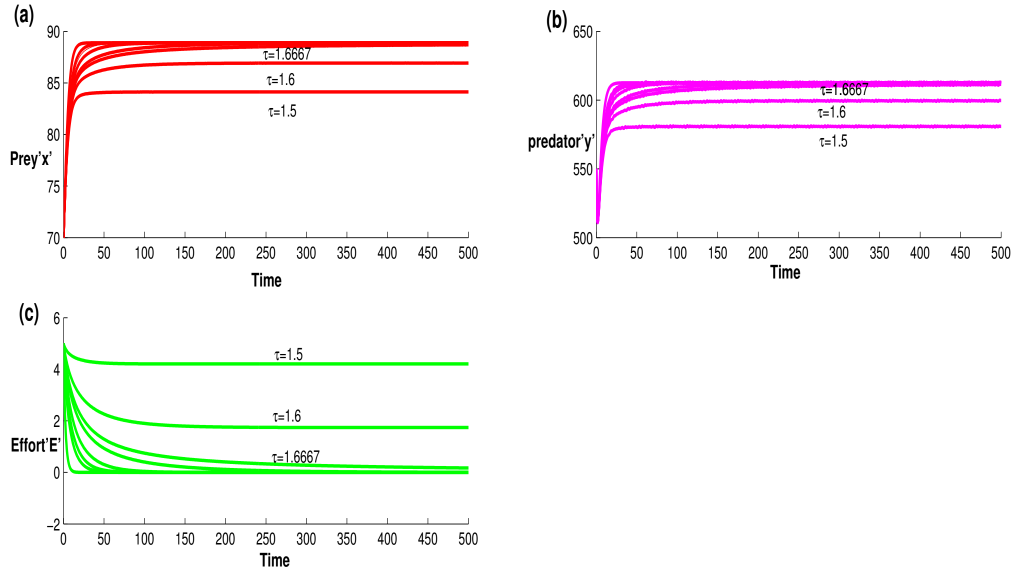

Figure 3.2: Figures (a), (b) and (c) represent solution curves of x, y and E as a function of time t for different high values of tax τ.

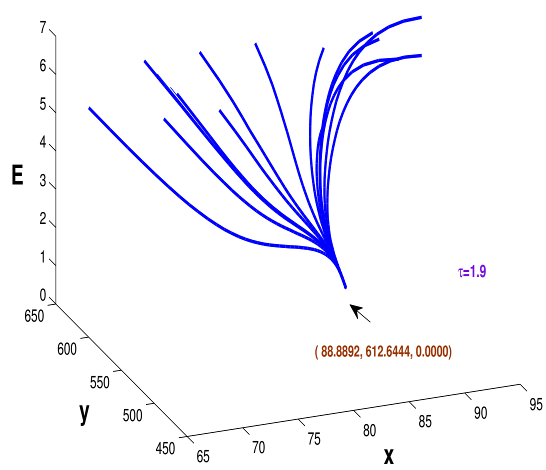

Figure 3.3: Phase plane trajectories of x, y and E with the different initial levels corresponding to $\tau = 1.9$.

3.12 Conclusion

This chapter is concerned with the study of a Modified Leslie–Gower type predator in a predator–prey system with nonlinear harvesting of prey population.The harvesting effort is taken as a dynamic variable and taxation as a control instrument. The conditions for existence of steady states and their stability behavior have been examined by using Routh–Hurwitz Criteria and Lyapunov method. The existence of interior steady state strongly depends on range of tax. This range of tax may be useful for the regulatory agency for formulating a tax structure. The bionomic equilibrium of the system has been derived and it provides the range of harvesting co-efficient (or catch ability of harvest) that can be useful for a harvesting agency to get the profitable yields. The sufficient condition for global stability of unique interior equilibrium point provides a domain for global solution. The conditions of persistence for the system is derived. It is also investigated that the coexistence of prey and predator population depends upon the proper harvesting strategies such that the risk of extinction (or over exploitation) of the species can be avoided. The optimal taxation policy for the control problem has been studied by using Pontryagin's Maximum Principle. The optimum solution and optimum path has been derived. The impact of taxation on the system shows that the density of harvesting effort decreases as the tax rates increases. This increases in the densities of the prey and predator populations. It can be concluded that the equilibrium level of predator-prey system can be increased by increasing tax level. This observations gives the idea to obtain optimal level of taxation corresponding the optimal equilibrium level of prey, predator population and effort dynamics.

Chapter 4

The Impact of Provision of Additional Food to Predator in Predator Prey Model with Combined Harvesting in the Presence of Toxicity

4.1 Introduction

Growing with human needs, the industries are producing huge amount of toxicants which are released in marine water also. The toxicants adversely affects the marine ecology. Mathematical models dealing with the effects of toxicants on ecological communities started with the work of Hallam and Clark [46], Hallam et al. [47], Hallam and De Luna [48], De Luna and Hallam [28] and others. Some more mathematical studies on this topic include the works done by Freedman and Shukla [40], Shukla and Dubey, Dubey and Hussain [30], Shukla et al. [120] and Chattopadhyay [21]. Maynard Smith [81] incorporated the effects of toxic substances in a two species Lotka-Volterra competitive system by considering that each species produces a substance toxic to the other only when the other is present. The idea of Maynard Smith was extended further by Kar and Chaudhuri [63] to a two species competing fish species which are commercially exploited. Tapasi et al. [27] considered a prey-predator fishery in which the growth of both species is affected differently by some toxicants.

In recent years, many authors have concentrated on studying the consequences of provision of additional food to predators in a predator-prey system [49], [107]. Further, some investigations on the controllability with provision of additional food are investigated by Srinivasu et al. [124], Sahoo [111], Sahoo and Poria [113] etc.

In this chapter, a combined harvesting of a predator–prey fishery system with toxicant effect is investigated. Here, an additional food is provided to the predator so that predator can survive in the absence of prey. The global dynamics of this system is provided in this chapter.

4.2 The Mathematical Model

Let N and P represent biomass of prey and predator in the Lotka-Volterra predator-prey system, respectively. Modifying this model by supplying additional food to predators is studied by Srinivasu et al. [124]. Some assumptions are made for this type of model: i) Constant biomass (A) of additional food is provided to predator and it is distributed among them uniformly. This constant supply of additional food (A) is supported either by some external sources or by nature. ii) The number of encounters per predator with the additional food is proportional to the density of the additional food. iii) The handling time of both predators per unit quantity of additional food are assumed to be same.

The predator-prey model in presence of additional food to predators is governed by the following set of differential equations [124]:

$$\begin{aligned} \frac{dN}{dT} &= rN\left(1 - \frac{N}{k_1}\right) - \frac{cNP}{a + \alpha\eta A + N} \\ \frac{dP}{dT} &= \frac{b(N + \eta A)P}{a + \alpha\eta A + N} - dP \end{aligned} \tag{4.2.1}$$

The parameter a represents the half saturation value of the predator in the absence of additional food, b represents the maximum birth rate of the predator and c is the maximum capture rate of prey by predator, respectively. The system assumes

Holling type–II predator functional response for its prey and that the number of encounters per predator with the additional food is proportional to additional food biomass. Let us consider the presence of toxic substance, for example industrial waste, in the habitat affecting growth of both the species. The toxicants affect the quality of food for both the species. There is an accelerated growth in the toxicants as the density of prey density increases. The prey species is affected more severely than predator. Accordingly, the decay due to toxicity in prey and predator species are assumed as uN^3 and vP^2 with $0 < v < u < 1$, where u and v are the coefficient of toxicity to the prey and predator species [27]. Since $\frac{d(uN^3)}{dN} = 3uN^2 > 0$ and $\frac{d^2(uN^3)}{dN^2} = 6uN > 0$, there is an accelerated growth in the production of the toxic substance to the density of the N species as more and more of the species consume the toxic infected foods. Here, prey species are directly infected by toxicant through some external sources where predators are also infected indirectly through the infected prey. Considering the combined harvesting of prey and predator i.e., $H_1 = q_1EN$ and $H_1 = q_1EN$, the non linear dynamical model can be written as:

$$\begin{aligned} \frac{dN}{dT} &= rN\left(1 - \frac{N}{k_1}\right) - \frac{cNP}{a + \alpha\eta A + N} - uN^3 - q_1EN, \\ \frac{dP}{dT} &= \frac{b(N + \eta A)P}{a + \alpha\eta A + N} - dP - vP^2 - q_2EP, \end{aligned} \tag{4.2.2}$$

$$N(0) = N_0, P(0) = P_0; \quad (N_0, P_0) \in \mathbb{R}^2_+.$$

The model is non-dimensionalized using the transformations $x = \dfrac{N}{a}, y = \dfrac{cP}{ar}, t = rT$ with the following dimensionless parameters:

$$k = \frac{k_1}{a}, \beta = \frac{b}{r}, \xi = \frac{\eta A}{a}, \delta = \frac{m}{r}, \alpha_1 = \frac{ua^2}{r}, \alpha_2 = \frac{va}{c}, h_1 = \frac{q_1E}{r}, h_2 = \frac{q_2E}{r}.$$

The coupled dynamical equations constitute the model with positive initial conditions and $(0 < \alpha_2 < \alpha_1 < 1)$ as follows:

$$\begin{aligned} \frac{dx}{dt} &= x\left(1 - \frac{x}{k}\right) - \frac{xy}{1 + \alpha\xi + x} - \alpha_1x^3 - h_1x = xf(x, y) \\ \frac{dy}{dt} &= \frac{\beta(x + \xi)y}{1 + \alpha\xi + x} - \delta y - \alpha_2y^2 - h_2y = yg(x, y) \end{aligned} \tag{4.2.3}$$

$$x(0) = x_0, \quad y(0) = y_0; \quad (x_0, y_0) \in \mathbb{R}^2_+$$

4.3 The Model Analysis

4.3.1 Positive Invariance

For $X(0) = X_0 \in \mathbb{R}^2_+$, where $X = (x, y) \in \mathbb{R}^2_+$, the system (4.2.3) can be formulated in a matrix form as $\dot{X} = F(X)$ given by

$$F(X) = \begin{bmatrix} x\left(1 - \dfrac{x}{k}\right) - \dfrac{xy}{1 + \alpha\xi + x} - \alpha_1 x^3 - h_1 x \\ \dfrac{\beta(x + \xi)y}{1 + \alpha\xi + x} - \delta y - \alpha_2 y^2 - h_2 y \end{bmatrix}$$

where $F : C_+ \to \mathbb{R}^2$ and $F \in C^\infty(\mathbb{R}^2)$.

This can be observed that $F_i(X) \mid_{X_i=0} \geq 0$ (for $i = 1, 2$) whenever $X(0) \in \mathbb{R}^2_+$ such that $X_i = 0$. Thus, the system (4.2.3) admits positive solution and it is positively invariant in $\mathbb{R}^2_+$ for all $t > 0$ [Nagumo [101]].

4.3.2 Boundedness

Theorem 4.3.1. *The system* (4.2.3) *has uniformly bounded solution.*

Proof. From the system (4.2.3), the following is derived

$$\frac{dx}{dt} < x\left(1 - \frac{x}{k}\right). \tag{4.3.1}$$

This shows that all the solutions of the system (4.3.1) must satisfy $x(t) \leq k, \quad \forall t > 0$.

Consider a function $\psi(t)$ such that

$$\begin{aligned}
\psi(t) &= x(t) + \frac{1}{\beta}y(t), \\
\frac{d\psi(t)}{dt} &= x'(t) + \frac{1}{\beta}y'(t), \\
&= x\left(1 - \frac{x}{k}\right) - \alpha_1 x^3 - h_1 x - \frac{\delta y}{\beta} + \frac{\xi y}{1 + \alpha\xi + x} - \frac{\alpha_2 y^2}{\beta} - \frac{h_2 y}{\beta}, \\
&< x\left(1 - \frac{x}{k}\right) - h_1 x - \frac{\delta y}{\beta} + \frac{\xi y}{1 + \alpha\xi + x} - \frac{h_2 y}{\beta}.
\end{aligned}$$

Introducing a positive constant M and rewriting the above equation as follows:

$$\begin{aligned}
\frac{d\psi(t)}{dt} + M\psi(t) &\leq -\frac{1}{k}[x^2 - k(1 - h_1 + M)x] + y\frac{\xi}{(1 + \alpha\xi)} + y\left[\frac{M}{\beta} - \left(\frac{\delta}{\beta} + \frac{h_2}{\beta}\right)\right], \\
\frac{d\psi(t)}{dt} + M\psi(t) &\leq -\frac{1}{k}[x^2 - k(1 - h_1 + M)x] - y\left(\frac{(\delta + h_2 - M)}{\beta} - \frac{\xi}{(1 + \alpha\xi)}\right).
\end{aligned}$$

For $\delta + h_2 > M + \dfrac{\beta\xi}{(1+\alpha\xi)}$, further simplification yields,

$$\begin{aligned}
\frac{d\psi(t)}{dt} + M\psi(t) &\leq -\frac{1}{k}\left[x - \frac{k(1-h_1+M)}{2}\right]^2 + \frac{k^2(1-h_1+M)^2}{4}, \\
\frac{d\psi(t)}{dt} + M\psi(t) &\leq N, \quad N = \frac{k^2(1-h_1+M)^2}{4}.
\end{aligned}$$

Solution of above differential inequality gives,

$$\begin{aligned}
\psi(t) &\leq \frac{N}{M}\left(1 - e^{-Mt}\right) + \psi(0)e^{-Mt}, \\
0 < \lim_{t\to\infty} \psi(t) &\leq \frac{N}{M}.
\end{aligned}$$

Accordingly, all the solutions of (4.2.3) initiating from $\mathbb{R}^2_+$ are confined in the region

$$R = \left\{(x, y) \in \mathbf{R}; x(t) \leq k, \quad 0 < x(t) + \frac{1}{\beta}y(t) \leq \frac{N}{M} + \phi \text{ for any } \phi > 0\right\}.$$

This proves the result. □

4.3.3 Permanence

The system (4.2.3) is said to be permanent if there exist positive constants m_1, m_2 and M_1, M_2 such that each positive solution $(x(t, x_0, y_0), y(t, x_0, y_0))$ of system (4.2.3) with initial condition $(x_0, y_0) \in (\mathbb{R}^2_+)$ satisfies:

$$m_1 \leq \liminf_{t\to\infty} x(t, x_0, y_0) \leq \limsup_{t\to\infty} x(t, x_0, y_0) \leq M_1,$$

$$m_2 \leq \liminf_{t\to\infty} y(t, x_0, y_0) \leq \limsup_{t\to\infty} y(t, x_0, y_0) \leq M_2.$$

Theorem 4.3.2. *The system* (4.2.3) *with initial condition* (x_0, y_0) *is permanent if*

$$h_1 + \alpha_1 {M_1}^2 + M_2 < 1 \quad \textit{and} \quad h_2 < \frac{\beta\xi}{1+\alpha\xi+M_1} - \delta. \tag{4.3.2}$$

Proof. From the first equation of (4.2.3), can be obtained

$$\begin{aligned}
\frac{dx}{dt} &< x\left(1 - \frac{x}{k}\right), \\
\limsup_{t\to\infty} x(t) &\leq max\{x(0), k\} \equiv M_1.
\end{aligned} \tag{4.3.3}$$

Similarly, the second equation of (4.2.3), gives:

$$\frac{dy}{dt} \leq y\left(\frac{\beta(x+\xi)}{1+\alpha\xi} - (\delta+h_2) - \alpha_2 y\right),$$
$$\frac{dy}{dt} \leq \alpha_2 y\left(\frac{\beta(k+\xi)}{\alpha_2(1+\alpha\xi)} - y\right),$$

$$\limsup_{t\to\infty} y(t) \leq max\left\{y(0), \frac{\beta(k+\xi)}{\alpha_2(1+\alpha\xi)}\right\} \equiv M_2. \tag{4.3.4}$$

Also, from first equation, it is observed

$$\frac{dx}{dt} \geq x\left[\left(1-\frac{x}{k}\right) - 1.y - \alpha_1 x^2 - h_1\right],$$
$$\frac{dx}{dt} \geq x\left[k\left(1 - M_2 - \alpha_1 {M_1}^2 - h_1\right) - x\right],$$
$$\frac{dx}{dt} \geq x\left(L_1 - x\right), \tag{4.3.5}$$

for $\quad L_1 = k\left(1 - M_2 - \alpha_1 {M_1}^2 - h_1\right) > 0 \quad$ provided $\quad h_1 + \alpha_1 {M_1}^2 + M_2 < 1,$

$$\liminf_{t\to\infty} x(t) \geq min\{x(0), L_1\} \equiv m_1. \tag{4.3.6}$$

Again, from second equation, the following can be observed:

$$\frac{dy}{dt} \geq y\left(\frac{\beta(x+\xi)}{1+\alpha\xi+M_1} - (\delta+h_2) - \alpha_2 y\right),$$
$$\frac{dy}{dt} \geq \alpha_2 y\left[\frac{1}{\alpha_2}\left(\frac{\beta\xi}{1+\alpha\xi+M_1} - (\delta+h_2)\right) - y\right],$$
$$\frac{dy}{dt} \geq \alpha_2 y\left(L_2 - y\right), \tag{4.3.7}$$

say, $\quad L_2 = \frac{1}{\alpha_2}\left(\frac{\beta\xi}{1+\alpha\xi+M_1} - (\delta+h_2)\right) > 0 \quad$ provided $\quad h_2 < \frac{\beta\xi}{1+\alpha\xi+M_1} - \delta,$

$$\liminf_{t\to\infty} y(t) \geq min\{y(0), L_2\} \equiv m_2. \tag{4.3.8}$$

□

4.4 Existence of Equilibrium States

The following four non-negative equilibrium points are possible:

(*i*) $P_0(0,0)$ is a trivial equilibrium point which always exists.

(*ii*) $P_1(\bar{x},0)$; $\bar{x} = \dfrac{-1+\sqrt{1+4\alpha_1 k^2(1-h_1)}}{2\alpha_1 k}$, is predator-free equilibrium point and exists provided $h_1 < 1$. The point $\bar{x}$ decreases with the increasing h_1 and it will become zero when $h_1 = 1$. In the absence of harvesting, the equilibrium point $(\bar{x},0)$ is always exists.

(*iii*) $P_2(0,\hat{y})$; $\hat{y} = \dfrac{\beta\xi - (\delta+h_2)(1+\alpha\xi)}{\alpha_2(1+\alpha\xi)}$ is prey free equilibrium point and exists provided

$$\xi > \frac{\delta+h_2}{\beta-\alpha(\delta+h_2)} \quad or \quad h_2 < \frac{\beta\xi}{1+\alpha\xi} - \delta = \triangle_1. \tag{4.4.1}$$

In absence of prey, the survival of predator depends upon the alternate food. If sufficient amount of alternate food according to the condition (4.4.1) is provided, then predator may survive. Further, its survival also depends upon the level of harvesting of predator. If the harvesting of predator exceeds a critical level i.e., $h_2 > \triangle_1$, predator will not survive. However, in the absence of harvesting, the point $(0,\hat{y})$ is feasible for minimum level of alternative food and it given as follow:

$$\xi > \frac{\delta}{\beta-\alpha\delta} \tag{4.4.2}$$

Although, in the absence of toxicity, this equilibrium does not exists.

(*iv*) $P^*(x^*,y^*)$ is the interior equilibrium point of the system (4.2.3) and is obtained by solving the following equations:

$$y^* = (1+\alpha\xi+x^*)\left(1-\frac{x^*}{k}-\alpha_1 x^{*2}-h_1\right) \tag{4.4.3}$$

$$A_1x^{*4}+A_2x^{*3}+A_3x^{*2}+A_4x^*+A_5 = 0 \tag{4.4.4}$$

where,

$$\begin{aligned}
A_1 &= k\alpha_1\alpha_2, \\
A_2 &= \alpha_2[1+2k\alpha_1(1+\alpha\xi)], \\
A_3 &= \alpha_2[h_1k+k\alpha_1(1+\alpha\xi)^2+2(1+\alpha\xi)-k], \\
A_4 &= \alpha_2(1+\alpha\xi)^2+2k\alpha_1(1+\alpha\xi)(h_1-1)+k(\beta-(\delta+h_2)), \\
A_5 &= \alpha_2(1+\alpha\xi)^2k(h_1-1)+k(\beta\xi-(\delta+h_2)(1+\alpha\xi)).
\end{aligned} \tag{4.4.5}$$

Sufficient condition for the existence of atleast one real positive value of $x = x^*$, we must have $A_5 < 0$ which implies

$$h_2 > \frac{\beta\xi-(1-h_1)\alpha_2(1+\alpha\xi)^2}{1+\alpha\xi} - \delta = \triangle_2. \tag{4.4.6}$$

Further, the value of y^* will be positive provided

$$\frac{x^*}{k} + \alpha_1 x^{*2} + h_1 < 1 \tag{4.4.7}$$

Therefore, (4.4.6) and (4.4.7) are the sufficient conditions for the existence of interior equilibrium point (x^*, y^*).

In the absence of harvesting, the interior equilibrium point (x^*, y^*) exists and positive for the condition:

$$\beta < \frac{\alpha_2(1+\alpha\xi)^2+\delta(1+\alpha\xi)}{\xi} \quad and \quad \frac{x^*}{k} + \alpha_1 x^{*2} < 1 \tag{4.4.8}$$

However, in the absence of toxicity, (x^*, y^*) is positive provided

$$\alpha + \frac{1}{\xi} > 1 \quad and \quad x^* < k(1-h_1) \tag{4.4.9}$$

4.5 Stability and Bifurcations of Equilibrium States

For the local stability, the Jacobian matrix of the system (4.2.3) at any point (x, y) is given by

$$J(x,y) = \begin{bmatrix} x\left(-\frac{1}{k}+\frac{y}{(1+\alpha\xi+x)^2}-2\alpha_1 x\right)+f & -\frac{x}{1+\alpha\xi+x} \\ \frac{\beta y(1+\alpha\xi-\xi)}{(1+\alpha\xi+x)^2} & -\alpha_2 y+g \end{bmatrix}$$

Theorem 4.5.1. *The equilibrium point $P_0(0,0)$ is locally asymptotically stable for the following conditions:*

$$h_1 > 1 \tag{4.5.1}$$
$$h_2 > \frac{\beta\xi}{1+\alpha\xi} - \delta = \triangle_1. \tag{4.5.2}$$

Proof. The Jacobian matrix of the system (4.2.3) at $(0,0)$ is given by

$$J(0,0) = \begin{bmatrix} 1-h_1 & 0 \\ 0 & \dfrac{\beta\xi}{1+\alpha\xi} - (\delta + h_2) \end{bmatrix}$$

The eigenvalues of J_0 is given as: $\lambda_1 = 1 - h_1$ and $\lambda_2 = \dfrac{\beta\xi}{1+\alpha\xi} - (\delta + h_2)$. The equilibrium point $(0,0)$ is locally asymptotically stable for the conditions (4.5.1) and (4.5.2). □

Remark 4.5.2. *Further, it can be observed that if both the conditions* (4.5.1) *and*(4.5.2) *are violated, then the point* $(0,0)$ *becomes unstable. However, it is saddle if one of these conditions is violated. Thus, it can be seen that as the point* $(0,0)$ *crosses either* $h_1 = 1$ *or* $h_2 = \triangle_1$*, then there is a possibility of occurrence of transcritical bifurcation through these critical values of* h_1 *and* h_2*.*

Remark 4.5.3. *In the absence of harvesting, the equilibrium point* $(0,0)$ *is either saddle or unstable. However, toxicity does not effect the stability of the equilibrium point* $(0,0)$*.*

Theorem 4.5.4. *The axial equilibrium point* $(\bar{x},0)$ *is locally asymptotically stable provided*

$$h_2 > \triangle_3 = \frac{\beta(\bar{x}+\xi)}{1+\alpha\xi+\bar{x}} - \delta. \tag{4.5.3}$$

Proof. The Jacobian matrix evaluated at $(\bar{x},0)$ is given by

$$J(\bar{x},0) = \begin{bmatrix} \left(-\dfrac{\bar{x}}{k} - 2\alpha_1\bar{x}^2\right) & -\dfrac{\bar{x}}{1+\alpha\xi+\bar{x}} \\ 0 & \dfrac{\beta(\bar{x}+\xi)}{1+\alpha\xi+\bar{x}} - (\delta+h_2) \end{bmatrix}$$

The first eigenvalue $\lambda_1 = -\dfrac{\bar{x}}{k} - 2\alpha_1\bar{x}^2$ is always negative. The stability of $(\bar{x},0)$ depends on the second eigenvalue. Accordingly, $P_1(\bar{x},0)$ is locally asymptotically stable for the condition (4.5.3) $h_2 > \triangle_3$.

□

Remark 4.5.5. *The equilibrium point $P_1(\bar{x}, 0)$ becomes saddle when the condition* (4.5.3) *is violated. Thus, the equilibrium point $P_1(\bar{x}, 0)$ has a transcritical bifurcation at $h_2 = \triangle_3$ as it looses its stability through $h_2 = \triangle_3$ from stable to saddle.*

Remark 4.5.6. *In the absence of harvesting, the equilibrium point $(\bar{x}, 0)$ is locally asymptotically stable for*

$$\delta > \frac{\beta(x_1 + \xi)}{1 + \alpha\xi + x_1}. \tag{4.5.4}$$

When the condition (4.5.4) *is violated, then the point $P_1(\bar{x}, 0)$ is saddle point. Thus, it looses its stability through $\delta = \dfrac{\beta(x_1 + \xi)}{1 + \alpha\xi + x_1}$ in the absence of harvesting. However, toxicity does not effect the stability of $P_1(\bar{x}, 0)$.*

Theorem 4.5.7. *The axial equilibrium point $(0, \hat{y})$ is locally asymptotically stable provided*

$$\beta > \frac{\alpha_2(1 + \alpha\xi)^2 + (\delta + h_2)(1 + \alpha\xi)}{\xi}. \tag{4.5.5}$$

Proof. The Jacobian matrix of the system (4.2.3) at $(0, \hat{y})$ is given by

$$J(0, \hat{y}) = \begin{bmatrix} 1 - \dfrac{\beta\xi - (\delta + h_2)(1 + \alpha\xi)}{\alpha_2(1 + \alpha\xi)^2} - h_1 & 0 \\ \dfrac{\beta\hat{y}(1 + \alpha\xi - \xi)}{(1 + \alpha\xi)^2} & -\alpha_2\hat{y} \end{bmatrix}$$

As $\lambda_2 = -\alpha_2\hat{y} < 0$ and the equilibrium point $P_2(0, \hat{y})$ is locally asymptotically stable for the condition (4.5.5).

□

Remark 4.5.8. *When condition*(4.5.5) *is violated then it will become saddle point. Thus, the equilibrium point $P_2(0, \hat{y})$ undergoes a transcritical bifurcation through $\beta = \dfrac{\alpha_2(1 + \alpha\xi)^2 + (\delta + h_2)(1 + \alpha\xi)}{\xi}$*

Remark 4.5.9. *In the absence of harvesting, the equilibrium point $P_2(0, y_1)$ is locally asymptotically stable for the condition*

$$\beta > \frac{\alpha_2(1 + \alpha\xi)^2 + \delta(1 + \alpha\xi)}{\xi}. \tag{4.5.6}$$

If the condition (4.5.6) *is violated, then the point $P_2(0, y_1)$ becomes saddle. Thus, there is a transcritical bifurcation around the equilibrium point $P_2(0, y_1)$ for $\beta = \dfrac{\alpha_2(1 + \alpha\xi)^2 + \delta(1 + \alpha\xi)}{\xi}$.*

Theorem 4.5.10. *The equilibrium point* (x^*, y^*) *if exists, is locally asymptotically stable provided*

$$\frac{1}{k} + 2\alpha_1 x^* > \frac{y^*}{(1+\alpha\xi+x^*)^2} \quad \textit{and} \quad \alpha + \frac{1}{\xi} \geq 1. \tag{4.5.7}$$

Proof. For the local stability, the Jacobian matrix of the system (4.2.3) at (x^*, y^*) is given by

$$J(x^*, y^*) = \begin{bmatrix} x^*\left(-\frac{1}{k} + \frac{y^*}{(1+\alpha\xi+x^*)^2} - 2\alpha_1 x^*\right) & -\frac{x^*}{1+\alpha\xi+x^*} \\ \frac{\beta y^*(1+\alpha\xi-\xi)}{(1+\alpha\xi+x^*)^2} & -\alpha_2 y^* \end{bmatrix}$$

As the trace and determinant of the Jacobian matrix at the equilibrium point $P^*(x^*, y^*)$ are given as follow:

$$tracJ(x^*, y^*) = -\left(\frac{x^*}{k} - \frac{x^* y^*}{(1+\alpha\xi+x^*)^2} + 2\alpha_1 x^{*2} + \alpha_2 y^*\right),$$

$$detJ(x^*, y^*) = \alpha_2 y^*\left(\frac{x^*}{k} - \frac{x^* y^*}{(1+\alpha\xi+x^*)^2} + 2\alpha_1 x^{*2}\right) + \frac{\beta x^* y^*(1+\alpha\xi-\xi)}{(1+\alpha\xi+x^*)^3}.$$

It may be noted that the $trJ(x^*, y^*) < 0$ and $detJ(x^*, y^*) > 0$ when conditions (4.5.7) is satisfied. Hence, the equilibrium point (x^*, y^*) is locally asymptotically stable. □

Remark 4.5.11. *Further, it is observed that harvesting does not effect the stability of* (x^*, y^*)*. Although, the level of equilibrium point* (x^*, y^*) *may be shifted by harvesting. In the absence of toxicity, the necessary and sufficient condition for the local stability of the equilibrium point* (x^*, y^*) *is given as follows:*

$$\frac{1}{k} > \frac{y^*}{(1+\alpha\xi+x^*)^2} \quad \textit{and} \quad \alpha + \frac{1}{\xi} > 1. \tag{4.5.8}$$

4.6 Local Bifurcations

The Jacobian of system (4.2.3) at equilibrium point $P_1(\bar{x}, 0)$ has a zero eigenvalue for the condition $h_2 = \frac{\beta(\bar{x}+\xi)}{1+\alpha\xi+\bar{x}} - \delta$. Therefore, the equilibrium point $(\bar{x}, 0)$ becomes non-hyperbolic. So there is a chance of bifurcation around this equilibrium point.

Theorem 4.6.1. *The system* (4.2.3) *exhibits a transcritical bifurcation around the axial equilibrium point* $P_1(\bar{x}, 0)$ *if*

$$h_2^{tc} = \frac{\beta(\bar{x}+\xi)}{1+\alpha\xi+\bar{x}} - \delta. \tag{4.6.1}$$

Proof. The eigenvectors of $J(\bar{x}, 0)$ and $(J(\bar{x}, 0))^T$ corresponding to zero eigenvalue are obtained as

$$V = \left(1, -(1+\alpha\xi+\bar{x})\left(\frac{1}{k}+2\alpha_1\bar{x}\right)\right)^T \quad \textit{and} \quad W = (0,1)^T, \quad \textit{respectively.}$$

The value of Δ_1, Δ_2 and Δ_3 are computed as follows:

$$\Delta_1 = W^T F_{h_2}(P_1, h_2^{tc}) = 0, \quad F = (F^1, F^2, F^3)^T = (xf, yg, Eh)^T.$$

$$\Delta_2 = W^T\left[DF_{h_2}(P_1, h_2^{tc})V\right] = (1+\alpha\xi+\bar{x})\left(\frac{1}{k}+2\alpha_1\bar{x}\right) \neq 0,$$

where

$$DF_{h_2}(P_1, h_2^{tc}) = \begin{bmatrix} \left(-\dfrac{\bar{x}}{k} - 2\alpha_1\bar{x}^2\right) & -\dfrac{\bar{x}}{1+\alpha\xi+\bar{x}} \\ 0 & 0 \end{bmatrix}$$

$$\begin{aligned} \Delta_3 &= W^T\left[D^2F(P_1, h_2^{tc})(V,V)\right] \\ &= -\frac{\beta\left(\frac{1}{k}+2\alpha_1\bar{x}\right)(1+\alpha\xi-\xi)}{(1+\alpha\xi+\bar{x})} - \alpha_2(1+\alpha\xi+\bar{x})^2\left(\frac{1}{k}+2\alpha_1\bar{x}\right)^2 \neq 0. \end{aligned}$$

Since, $\Delta_1 = 0$, this gives that there is no chance of saddle-node bifurcation around the equilibrium point $P_1(\bar{x}, 0)$.
Thus, by the Sotomayors theorem [102], the system (4.2.3) undergoes a transcritical bifurcation around the axial equilibrium point $(\bar{x}, 0)$ for the condition (4.5.3). □

Again, the Jacobian of system (4.2.3) at equilibrium point $P_2(0, \hat{y})$ has a zero eigenvalue for the condition $\beta = \dfrac{\alpha_2(1+\alpha\xi)^2(h_1-1)+(\delta+h_2)(1+\alpha\xi)}{\xi}$ and therefore, the equilibrium point $(0, \hat{y})$ becomes non-hyperbolic. So there is a chance of bifurcation around this equilibrium point.

Theorem 4.6.2. *The system* (4.2.3) *exhibits a transcritical bifurcation around the axial equilibrium point* $P_2(0, \hat{y})$ *if*

$$\beta = \beta^{tc} = \frac{\alpha_2(1+\alpha\xi)^2(h_1-1)+(\delta+h_2)(1+\alpha\xi)}{\xi} \tag{4.6.2}$$

Proof. The eigenvectors of $J(0, \hat{y})$ and $(J(0, \hat{y}))^T$ corresponding to zero eigenvalue are obtained as

$$V = \left(1, \frac{\beta(1+\alpha\xi-\xi)}{\alpha_2(1+\alpha\xi)^2}\right)^T \quad \textit{and} \quad W = (1,0)^T, \quad \textit{respectively.}$$

Compute Δ_1, Δ_2 and Δ_3 as follows:

$$\Delta_1 = W^T F_\beta(P_2, \beta^{tc}) = 0, \quad F = (F^1, F^2, F^3)^T = (xf, yg, Eh)^T.$$

$$\Delta_2 = W^T \left[DF_\beta(P_2, \beta^{tc})V\right] = -1 \neq 0,$$

where

$$DF_\beta(P_2, \beta^{tc}) = \begin{bmatrix} 0 & 0 \\ \dfrac{\beta\hat{y}(1+\alpha\xi-\xi)}{(1+\alpha\xi)^2} & -\alpha_2\hat{y} \end{bmatrix}$$

$$\Delta_3 = W^T\left[D^2F(P_2, \beta^{tc})(V,V)\right] = -\left(\frac{2}{k} + 2\alpha_1 + \frac{2\hat{y})}{(1+\alpha\xi)^2}\right) + \frac{\beta(1+\alpha\xi-\xi)}{\alpha_2(1+\alpha\xi)^4} \neq 0$$

.

Since, $\Delta_1 = 0$, there is no chance of saddle-node bifurcation around the equilibrium point $P_2(0, \hat{y})$.

Thus, by the Sotomayors theorem [102], the system (4.2.3) undergoes a transcritical bifurcation around the axial equilibrium point $(0, \hat{y})$ for the condition (4.5.5).

□

4.6.1 Hopf Bifurcation

For $tr J(x^*, y^*) = 0$, there will be purely imaginary eigenvalues provided $det J(x^*, y^*) > 0$. This gives a Hopf bifurcation point at $\alpha_2 = \alpha_2^H$ around the equilibrium point (x^*, y^*). The critical value for the Hopf bifurcation parameter is $\alpha_2 = \alpha_2^H =$

$\dfrac{x^*}{(1+\alpha\xi+x^*)^2}-\dfrac{x^*}{y^*k}-\dfrac{2\alpha_1 x^{*2}}{y^*}$. The transversality condition for Hopf bifurcation at the equilibrium point P^* is given by

$$\frac{d(tr(J^*))}{d\alpha_2} = -y^* \neq 0 \quad for \quad \alpha_2 = \alpha_2^H. \tag{4.6.3}$$

This guarantees the existence of Hopf bifurcation (limit cycle) around (x^*, y^*). The stability of the limit cycle is discussed in following section:

Stability and Direction of Limit Cycle

In order to discuss the stability and direction of the limit cycle of the system, the Lyapunov coefficient σ at (x^*, y^*) is computed. For this, translate the equilibrium point (x^*, y^*) to the origin using the transformation $x - x^* = \hat{x}$, $y - y^* = \hat{y}$. Therefore, the system (4.2.3) in a neighborhood of origin can be derived as follows:

$$\frac{d\hat{x}}{dt} = a_{10}\hat{x} + a_{01}\hat{y} + a_{20}\hat{x}^2 + a_{11}\hat{x}\hat{y} + a_{02}\hat{y}^2 + a_{30}\hat{x}^3 + a_{21}\hat{x}^2\hat{y} + a_{12}\hat{x}\hat{y}^2 + a_{03}\hat{y}^3 + F_1(\hat{x},\hat{y}),$$
$$\frac{d\hat{y}}{dt} = b_{10}\hat{x} + b_{01}\hat{y} + b_{02}\hat{x}^2 + b_{11}\hat{x}\hat{y} + b_{02}\hat{y}^2 + b_{30}\hat{x}^3 + b_{21}\hat{x}^2\hat{y} + b_{12}\hat{x}\hat{y}^2 + b_{03}\hat{y}^3 + F_2(\hat{x},\hat{y}),$$

where,

$$a_{10} = f_x = x^*\left(-\frac{1}{k} + \frac{y^*}{(1+\alpha\xi+x^*)^2} - 2\alpha_1 x^*\right), \quad a_{01} = f_y = -\frac{x^*}{1+\alpha\xi+x^*},$$

$$a_{11} = f_{xy} = \frac{x^*}{(1+\alpha\xi+x^*)^2}, a_{20} = f_{xx} = \frac{-2}{k} + \frac{y^*}{(1+\alpha\xi+x^*)^2} - \frac{x^*y^*}{(1+\alpha\xi+x^*)^3} - 4\alpha_1 x^*,$$

$$a_{02} = f_{yy} = 0, \quad a_{30} = f_{xxx} = -\frac{y^*}{(1+\alpha\xi+x^*)^2} + \frac{3x^*y^*}{(1+\alpha\xi+x^*)^4} - 4\alpha_1,$$

$$a_{03} = 0, \quad a_{12} = 0, \quad a_{21} = f_{xxy} = \frac{1}{(1+\alpha\xi+x^*)^2} - \frac{x^*}{(1+\alpha\xi+x^*)^3}$$

$$b_{10} = g_x = \frac{\beta y^*(1+\alpha\xi-\xi)}{(1+\alpha\xi+x^*)^2}, \quad b_{01} = g_y = -\alpha_2 y^*, \quad b_{11} = g_{xy} = \frac{\beta(1+\alpha\xi-\xi)}{(1+\alpha\xi+x^*)^2},$$

$$b_{20} = g_{xx} = -\frac{2\beta y^*(1+\alpha\xi-\xi)}{(1+\alpha\xi+x^*)^3}, b_{03} = 0, \quad b_{12} = 0, \quad b_{02} = g_{yy} = -\alpha_2,$$

$$b_{21} = g_{xxy} = -\frac{2\beta(1+\alpha\xi-\xi)}{(1+\alpha\xi+x^*)^3}, \quad b_{30} = g_{xxx} = \frac{6\beta y^*(1+\alpha\xi-\xi)}{(1+\alpha\xi+x^*)^4}.$$

and $F_1(\hat{x},\hat{y})$ and $F_2(\hat{x},\hat{y})$ are power series in powers of $\hat{x}^i\hat{y}^j$ satisfying $i+j \geq 4$, i.e., with $F_1(\hat{x},\hat{y}) = \sum^{\infty}{}_{i+j=4} a_{ij}\hat{x}^i\hat{y}^j$ and $F_2(\hat{x},\hat{y}) = \sum^{\infty}{}_{i+j=4} b_{ij}\hat{x}^i\hat{y}^j$.

Hence, the first Lyapunov coefficient σ for the planer system [102] is given by

$$\sigma = \frac{-3\pi}{2a_{01}\Delta^{\frac{3}{2}}}\Big[[a_{10}b_{10}({a_{11}}^2 + a_{11}b_{02} + a_{02}b_{11}) + a_{01}a_{10}({b_{11}}^2 + a_{20}b_{11} + a_{11}b_{02}) + {b_{10}}^2(a_{11}a_{02} + 2a_{02}b_{02}) - 2a_{10}b_{10}({b_{02}}^2 - a_{02}a_{20}) - 2a_{01}a_{10}({a_{20}}^2 - b_{02}b_{20}) - {a_{01}}^2(2a_{20}b_{20} + b_{11}b_{20}) + (a_{01}b_{10} - 2{a_{10}}^2)(b_{11}b_{02} - a_{11}a_{20})] - ({a_{10}}^2 + a_{01}b_{10})[3(b_{10}b_{03} - a_{10}a_{30}) + 2a_{10}(a_{21} + b_{12}) + (b_{10}a_{12} - a_{10}b_{21})]\Big]$$

where

$$\Delta = \alpha_2 y^*\left(\frac{x^*}{k} - \frac{x^*y^*}{(1+\alpha\xi+x^*)^2} + 2\alpha_1 x^{*2}\right) + \frac{\beta x^*y^*(1+\alpha\xi-\xi)}{(1+\alpha\xi+x^*)^3}.$$

As the expression for Lyapunov coefficient σ is complex enough, so it is very difficult to decide about the sign of σ. Therefore, the some numerical example have drawn to verify its sign in numerical section with the Fig-4.5.

4.6.2 Saddle-Node Bifurcation

If $detJ(x^*, y^*) = 0$ and $trJ(x^*, y^*) < 0$, then the one of the eigenvalues of the Jacobian matrix $J(x^*, y^*)$ is zero so the point $P^* = (x^*, y^*)$ becomes non-hyperbolic. Thus, there is a chance of saddle-node bifurcation at $\alpha_1 = {\alpha_1}^{SN}$ around the equilibrium point (x^*, y^*).

Theorem 4.6.3. *The system* (4.2.3) *undergoes a saddle-node bifurcation around the equilibrium point* (x^*, y^*) *with respect to bifurcation parameter* $\alpha_1 = {\alpha_1}^{SN} = \frac{y^*}{2x^*(1+\alpha\xi+x^*)^2} - \frac{\beta(1+\alpha\xi-\xi)}{\alpha_2 x^*(1+\alpha\xi+x^*)^3} - \frac{1}{2kx^*}$ *if*

$$\frac{1}{k} + 2\alpha_1 x^* > \frac{y^*}{(1+\alpha\xi+x^*)^2} \quad \textit{and}$$

$$\alpha_2 y^*\left(\frac{x^*}{k} - \frac{x^*y^*}{(1+\alpha\xi+x^*)^2} + 2\alpha_1 x^{*2}\right) + \frac{\beta x^*y^*(1+\alpha\xi-\xi)}{(1+\alpha\xi+x^*)^3} = 0. \tag{4.6.4}$$

Proof. To prove that the system (4.2.3) undergoes a saddle-node bifurcation, Sotomayor's theorem [102] is used by considering α_1 as bifurcation parameter. According to Sotomayor's theorem, one of the eigenvalues of the Jacobian matrix $J(x^*, y^*)$ must be zero and the other eigenvalue must have negative real part . The eigenvectors of $J(x^*, y^*)$ and $(J(x^*, y^*))^T$ corresponding to zero eigenvalue are obtained as

$$V = \left(1, \frac{\beta(1+\alpha\xi-\xi)}{(1+\alpha\xi+x^*)^2}\right)^T \quad and \quad W = (-1,1)^T, \quad respectively.$$

Compute Δ_1 as follows:

$$\Delta_1 = W^T F_{h_2}(P_3, \alpha_1^{tc}) = 2x^{*2} \neq 0; \quad F = (F^1, F^2, F^3)^T = (xf, yg, Eh)^T$$

and

$$\begin{aligned}\Delta_3 &= W^T\big[D^2F(P_3,\alpha_1^{tc})(V,V)\big] = \frac{x^*y^*}{(1+\alpha\xi+x^*)^3} + \frac{y^*}{(1+\alpha\xi+x^*)^2} + \frac{\beta^2(1+\alpha\xi-\xi)^2}{\alpha_2(1+\alpha\xi+x^*)^4} \\ &\quad -\frac{2\beta(1+\alpha\xi-\xi)}{(1+\alpha\xi+x^*)^2}\left(\frac{x^*}{\alpha_2} + \frac{1}{\alpha_2(1+\alpha\xi+x^*)} + \frac{y^*}{1+\alpha\xi+x^*}\right) \neq 0\end{aligned}$$

Thus, from Sotomayor's theorem the system undergoes a saddle-node bifurcation around (x^*, y^*) at $\alpha_1 = {\alpha_1}^{SN}$. □

4.7 Bionomic Equilibrium

The net economic revenue to the society is represented as follows:

$$\pi(t, N, P, E) = (p_1q_1N + p_2q_2P - c)E \tag{4.7.1}$$

The parameters p_1 and p_2 represent the constant price per unit biomass of prey and predator. Clark [22] defined the bionomic equilibrium point as the point of intersection of the interior equilibrium of the system (4.2.2) along with zero net economic revenue. The bionomic equilibrium (N_∞, P_∞) is said to obtained when the total revenue (T.R.) obtained by harvester is equal to the total cost (T.C.) per unit biomass for effort E i.e.,

$$\pi(t, N, P, E) = 0, \tag{4.7.2}$$

The intersection of following curves referred as the biological equilibrium point .

$$\begin{aligned}\dot{N} &= rN\left(1-\frac{N}{k_1}\right) - \frac{cNP}{a+\alpha\eta A+N} - uN^3 - q_1EN = 0, \\ \dot{P} &= \frac{b(N+\eta A)P}{a+\alpha\eta A+N} - dP - vP^2 - q_2EP = 0.\end{aligned}$$

Now, $\dot{N}=0$ gives $E=\frac{1}{q_1}\left[r\left(1-\frac{N}{k_1}\right)-\frac{cP}{a+\alpha\eta A+N}-uN^2\right]$. Thus, E is positive for

$$r\left(1-\frac{N}{k_1}\right)>\frac{cP}{a+\alpha\eta A+N}+uN^2. \tag{4.7.3}$$

And $\dot{P}=0$ gives $E=\frac{1}{q_2}\left[\frac{b(N+\eta A)}{a+\alpha\eta A+N}-dP-vP\right]$. Here, E is positive for

$$\frac{b(N+\eta A)}{a+\alpha\eta A+N}>dP+vP. \tag{4.7.4}$$

As $\dot{N}=\dot{P}=0$, the non-trivial solution curve occur at a point on the following curve.

$$\frac{1}{q_1}\left[r\left(1-\frac{N}{k_1}\right)-\frac{cP}{a+\alpha\eta A+N}-uN^2\right]=\frac{1}{q_2}\left[\frac{b(N+\eta A)}{a+\alpha\eta A+N}-dP-vP\right] \tag{4.7.5}$$

Using the value of $P=\frac{c-p_1q_1N}{p_2q_2}$ from (4.7.2), the following equation for N is obtained:

$$N^3+A_1N^2+B_1N+C_1=0, \tag{4.7.6}$$

where

$A_1=(a+\alpha\eta)+\frac{r}{ku}-\frac{q_1v}{{q_2}^2up_2}$,

$B_1=\frac{r(a+\alpha\eta)}{uk}+\frac{bq_1}{q_2u}-\frac{r}{u}-\frac{q_1m}{q_2u}+\frac{q_1vc}{{q_2}^2up_2}-\frac{(a+\alpha\eta){q_1}^2vp_1}{{q_2}^2up_2}+\frac{cp_1q_1}{up_2q_2}$,

$C_1=\frac{(a+\alpha\eta)q_1vc}{{q_2}^2up_2}+\frac{c^2}{4p_2q_2}+bq_1\eta A-(a+\alpha\eta)(q_2r+q_1m)$.

For the at least one positive real root of the cubic equation (4.7.6),we must have $C_1<0$, which implies $N=N_\infty$ exists and is positive for the following condition.

$$\frac{(a+\alpha\eta)q_1vc}{{q_2}^2up_2}+\frac{c^2}{4p_2q_2}+bq_1\eta A<(a+\alpha\eta)(q_2r+q_1m)$$

i.e.,

$$A<\frac{1}{bq_1\eta}\left[(a+\alpha\eta)(q_2r+q_1m)-\left(\frac{(a+\alpha\eta)q_1vc}{{q_2}^2up_2}+\frac{c^2}{4p_2q_2}\right)\right] \tag{4.7.7}$$

and $P_\infty=\frac{c-p_1q_1N_\infty}{p_2q_2}>0$ provided

$$N_\infty<\frac{c}{p_1q_1}. \tag{4.7.8}$$

Thus, the bionomic equilibrium (N_∞, P_∞) exists for the condition (4.7.7) and (4.7.8) which gives the maximum level of additional food biomass and prey density for the existence of bionomic equilibrium solution.

4.8 Optimal Harvesting Policy

In this section, an optimal harvesting policy for the system (4.2.3) is investigated to maximize the total discounted net revenue. Therefore, the objective of the regulatory agency should be to maximize the total discounted net revenues that the society derives from the fishery. The present value J of a continuous time stream of revenues given by [22]

$$J = \int_0^\infty e^{-\delta t}(p_1q_1N + p_2q_2P - c)E\,dt, \tag{4.8.1}$$

where δ is the instantaneous annual rate of discount which is fixed amount decided by harvesting agencies. Here, the main objective is to determine an optimal harvesting policy that maximize (4.8.1) subject to the state equations. The control variable $E(t)$ is subjected to the constraints $0 \leq E(t) \leq E_{max}$. Pontryagin's Maximum Principle is used to obtain the optimal level of the solution of the problem (4.8.1). Let $\lambda_1(t)$ and $\lambda_2(t)$ are adjoint variables w.r.t. the time t corresponding to the variables N and P, respectively.

The associated Hamiltonian function is given by

$$\begin{aligned}\mathcal{H}(t,N,P,E) &= e^{-\delta t}(p_1q_1N + p_2q_2P - c)E + \lambda_1\left[rN\left(1-\frac{N}{k_1}\right) - \frac{cNP}{a+\alpha\eta A+N} - \right.\\ &\left. uN^3 - q_1EN\right] + \lambda_2\left[\frac{b(N+\eta A)P}{a+\alpha\eta A+N} - dP - vP^2 - q_2EP\right]. \end{aligned} \tag{4.8.2}$$

Here $\psi(t) = e^{-\delta t}(p_1q_1N + p_2q_2P - c) - \lambda_1q_1N - \lambda_2q_2P$ is called the switching function. Since Hamiltonian $\mathcal{H}$ is linear in the control variable $E(t)$, the optimal control will be a combination of extreme controls (bang-bang controls) and the singular control.

Thus, the optimal control $E(t)$ that maximizes $\mathcal{H}$ must satisfy the following conditions:

$$\overline{E} = \begin{cases} E_{max} & when \quad \psi(t) > 0, i.e., (\lambda_1q_1N + \lambda_2q_2P)e^{\delta t} < p_1q_1N + p_2q_2P - c \\ 0 & when \quad \psi(t) < 0, i.e., (\lambda_1q_1N + \lambda_2q_2P)e^{\delta t} > p_1q_1N + p_2q_2P - c. \end{cases}$$

The function $\lambda_i e^{\delta t}, (i=1,2)$ is called shadow price and $p_1q_1N + p_2q_2P - c$ is net economic revenue from harvesting. Economically, the condition $E = E_{max} \quad when \quad \psi(t) > 0$, shows that if the profit after paying all the expanses is positive then it is beneficial to harvest up to the limit of available efforts and for the condition $E =$

$0 \quad when \quad \psi(t) < 0$, the term $(\lambda_1 q_1 N + \lambda_2 q_2 P)e^{\delta t}$ exceeds the fisherman's net economic revenue on unit harvest, then it is no more profitable to harvest for a fisherman.

When $\psi(t) = 0$, then the Hamiltonian $\mathcal{H}$ becomes independent of the control variable $E(t)$, i.e., $\dfrac{d\mathcal{H}}{dE} = 0$. This is the necessary condition for the singular control $E^*(t)$ to be optimal over the control set $0 \leq E(t) \leq E_{max}$. Thus, the optimal harvesting policy is:

$$\overline{E} = \begin{cases} E_{max} & when \quad \psi(t) > 0, \\ 0 & when \quad \psi(t) < 0, \\ E^* & when \quad \psi(t) = 0. \end{cases}$$

When $\psi(t) = 0$, it follows as:

$$\begin{aligned} (\lambda_1 q_1 N + \lambda_2 q_2 P) = (p_1 q_1 N + p_2 q_2 P - c)e^{-\delta t} \\ (\lambda_1 q_1 N + \lambda_2 q_2 P) = \frac{\partial \pi}{\partial E} e^{-\delta t} \end{aligned} \tag{4.8.3}$$

Pontryagin's Maximum Principle is used in order to find a singular control and the adjoint variables must satisfy the adjoint equations given by

$$\frac{d\lambda_1}{dt} = -\frac{\partial \mathcal{H}}{\partial N}, \quad \frac{d\lambda_2}{dt} = -\frac{\partial \mathcal{H}}{\partial P}. \tag{4.8.4}$$

$$\begin{aligned} \frac{d\lambda_1}{dt} &= -\frac{\partial \mathcal{H}}{\partial N} = -p_1 q_1 E e^{-\delta t} - \lambda_1 \left(\left(r - \frac{2N}{k} \right) - \frac{cP(a + \alpha\eta A)}{(a + \alpha\eta A + N)^2} - 3uN^2 - q_1 E \right) \\ &\quad -\lambda_2 \left(\frac{bP(a + (\alpha - 1)\eta A}{(a + \alpha\eta A + N)^2} \right), \end{aligned} \tag{4.8.5}$$

$$\begin{aligned} \frac{d\lambda_2}{dt} &= -\frac{\partial \mathcal{H}}{\partial P} = -\left[[p_2 q_2 E e^{-\delta t} + \lambda_1 \left(\frac{-cN}{a + \alpha\eta A + N} \right) + \lambda_2 \left(\frac{b(N + \alpha\eta A)}{a + \alpha\eta A + N} - m \right. \right. \\ &\quad \left. \left. -2vP \right) \right] \end{aligned} \tag{4.8.6}$$

The considered control problem admits a singular solution on the control set $[0, E_{max}]$ if

$$\frac{\partial \mathcal{H}}{\partial E} = -(\lambda_1 q_1 N^* + \lambda_2 q_2 P^*) + (p_1 q_1 N^* + p_2 q_2 P^* - c)e^{-\delta t} = 0,$$

which implies

$$\lambda_1 = \frac{(p_1 q_1 N^* + p_2 q_2 P^* - c)e^{-\delta t} - \lambda_2 q_2 P^*}{q_1 N^*}. \tag{4.8.7}$$

Solving (4.8.6) using (4.8.7), a linear differential equation is obtained in λ_2 and in the interior equilibrium (N^*, P^*) such that

$$\frac{d\lambda_2}{dt} - M_1\lambda_2 = -e^{-\delta t}M_2 \tag{4.8.8}$$

where

$$M_1 = \frac{b(\eta A + N^*)}{a + \alpha\eta A + N^*} - m - 2vP^* - q_2P^* \quad and \quad M_2 = p_2q_2E - \frac{c(p_1q_1N^* + p_2q_2P^* - c)}{q_1(a + \alpha\eta A + N^*)}$$

Solving equation (4.8.8), the following is obtained,

$$\lambda_2(t) = \frac{M_2}{M_1 + \delta}e^{-\delta t} \tag{4.8.9}$$

To solve (4.8.5), put the value of $\lambda_2(t)$ by using (4.8.9) in (4.8.5),

$$\frac{d\lambda_1}{dt} - N_1\lambda_1 = -e^{-\delta t}N_2 \tag{4.8.10}$$

where

$$\begin{aligned} N_1 &= r - \frac{2N^*}{k} - \frac{cP^*(a + \alpha\eta A)}{(a + \alpha\eta A + N^*)^2} - 3uN^{*2} - q_1E \quad and \\ N_2 &= p_1q_1E + \left(\frac{bP^*(a + (\alpha - 1)\eta A}{(a + \alpha\eta A + N^*)^2}\right) \end{aligned}$$

Solving equation (4.8.10), it is obtained:

$$\lambda_1(t) = \frac{N_1}{N_2 + \delta}e^{-\delta t} \tag{4.8.11}$$

Substituting the value of λ_i $(i = 1, 2)$ along with the interior equilibrium point (x^*, y^*) in (4.8.3), the following desired singular path is obtained:

$$(\frac{N_2}{N_1 + \delta}q_1N^* + \frac{M_2}{M_1 + \delta}q_2P^*) = (p_1q_1N^* + p_2q_2P^* - c) \tag{4.8.12}$$

It is observed that $\lambda_i(t)e^{\delta t}$ $(i = 1, 2)$ is independent of time in an optimal equilibrium. Hence, they satisfy the transversality condition at ∞, they remain bounded at ∞. Also,

$$(p_1q_1N^* + p_2q_2P^* - c) = \left(\frac{N_2}{N_1 + \delta}q_1N^* + \frac{M_2}{M_1 + \delta}q_2P^*\right) \to 0 \quad as \quad \delta \to \infty. \tag{4.8.13}$$

(4.8.14)

The expression (4.8.14) implies that an infinite discount rate leads to the net economic revenue tending to zero. Hence fishery would remain closed. The equation (4.8.12) can be written as $F(N^*) = 0$ such that N_1, N_2, M_1, M_2 can be written as function of N^* only. Solve $F(N^*) = 0$ for the value of N^* such that there exists a unique positive root $N^* = N_\delta$ in the interval $0 < N_\delta < k$. Also using $N^* = N_\delta$, the values of P_δ and E_δ can be obtained. Thus, $(N_\delta, P_\delta, , E_\delta)$ is the desired optimal Singular solution for the system (4.2.3).

4.9 Numerical Simulations

In this section, numerical simulations are carried out for suitable choices of parameters to verify the analytical results and further investigate the dynamical behavior of the system. Consider the following parametric values for the system (4.2.3):

$$h_1 = 0.07, h_2 = 0.05, k = 70, \alpha = 2, \alpha_1 = 0.43,$$
$$\alpha_2 = 0.07, \beta = 0.2, \delta = 0.0007, \xi = 1. \tag{4.9.1}$$

For this data set, the trivial equilibrium point $(0, 0)$ is always unstable due to the violation of the conditions (4.5.1) and (4.5.2). The predator- free equilibrium point $(\bar{x}, 0) = (1.4541, 0)$ is a saddle point as the stability condition (4.5.4) of $(\bar{x}, 0)$ is not satisfied and prey free equilibrium point $(0, \hat{y}) = (0, 0.2281)$ is locally asymptotically stable due to the condition (4.5.6). The status of stability of the equilibrium point $(x^*, y^*) = (1.2980, 0.8034)$ of the system (4.2.3) is verified by the conditions (4.5.7) of Theorem (4.5.10) and solution curves are shown in Fig-4.1.

The dynamics of system is explored using the software package MATCONT [43], [29] by varying different parameters of the data set (4.9.1). Some general bifurcation, i.e., Hopf point, BP represents branching point and LP is the limiting point are identified by using this software. This package is a collection of numerical algorithms implemented as a MATLAB toolbox for the detection, continuation and identification of limit cycles (periodic orbits). According to Theorem (4.6.1), there is a Branch point (transcritical bifurcation) at $h_2 = h_2^{tc} = 0.109496$ around the equilibrium point $(\bar{x}, 0) \approx (1.454125, 0)$ and it is detected in the Fig-4.2. According to the stability condition (4.5.3) of $(\bar{x}, 0)$, the equilibrium point $(\bar{x}, 0)$ is saddle for

$h_2 < h_2^{tc}$ and as it crosses $h_2 = h_2^{tc}$, i.e., for $h_2 > h_2^{tc}$ it becomes locally asymptotically stable. Thus stability exchanges from saddle to stable via $h_2 = h_2^{tc}$. Similarly, according to the Theorem (4.6.2), a Branch point (transcritical bifurcation) occurs at $\beta = \beta^{tc} = 0.738000$ around the equilibrium point $(0, \hat{y}) \approx (0, 2.79000)$ shown in the Fig-4.3. Here, stability changes from saddle to stable via $\beta = \beta^{tc}$ as the value of β increases.

In Fig-4.4, one Hopf point (H) is detected with respect to parameter α_2, keeping others parameters fixed. The periodic solutions for $\alpha_2 = 0.007208$ are shown in Fig-4.5 with purely imaginary eigen value $Im(\lambda_{1,2}) = \pm i(0.0615085) \neq 0$ and $Re(\lambda_{1,2}) \approx 0$. For this Hopf point, the corresponding first Lyapunov coefficient is $(-1.133781e-001) < 0$, indicating a supercritical Hopf bifurcation. Thus, there should exist a stable limit cycle, bifurcating from the equilibrium.

In the Fig-4.6, one Limit point which is identified as the saddle node bifurcation is detected with respect to parameter $\alpha = 0.378060$ around $(x^*, y^*) \cong (0.329419, 1.500245)$. Its normal form coefficient is $-4.615967e - 001$. The phase portrait diagrams- 4.7 and 4.8 shows the existence of saddle node bifurcation point at $\alpha = 0.378060$.

Further, the numerical study of the system in the codim-2 parametric spaces is discussed as it complex enough to study its dynamics analytically. Here, in the continuation of Hopf point (H) for the parameter α_2, some global bifurcations i.e., Bogdanov Takens bifurcation (BT) and Generalized Hopf bifurcation (GH) are detected in different parametric space which are shown in the Fig-4.9-4.11. In Fig-4.9(A), one GH bifurcation occurs at $(\alpha_1, \alpha_2) = (0.010222, 0.018026)$ around the interior equilibrium point $(x^*, y^*) \cong (3.611219, 4.926058)$. The second lyapunov exponent is computed as $(-1.960298e - 003)$. The BogdanovTakens bifurcation(BT) is obtained at $(\alpha_1, \alpha_2) = (0.000958, 0.013127)$. The corresponding normal form coefficient is $(a, b) = (-3.584046e - 003, -7.595566e - 002)$. In the Fig-4.9(B), only one BT bifurcation point is detected at $(\alpha_2, h_2) = (0.000003, 0.065959)$ with the normal form $(a, b) = (-2.647543e - 006, -3.332983e - 001)$. In the Fig-4.10(A), one GH bifurcation point at $(\alpha, \beta) = (0.151123, 0.066412)$ is obtained and the second Lyapunov coefficient is $(-2.393379e + 000)$. Also, two BT bifurcation points at $(\alpha, \beta) = (0.007354, 0.059873)$ and $(\alpha, \beta) = (0.128352, 0.065742)$ are obtained and their normal forms are $(a, b) = (-3.610354e - 003, -8.133614e - 001)$ and $(a, b) = (1.372528e - 005, 5.732006e - 001)$ In the Fig-4.10(B), one GH bifurcation at $(\alpha, \delta) = (0.045917, 0.135117)$ and two BT bifurcation points at $(\alpha, \delta) = (0.002173, 0.140859)$ and $(\alpha, \delta) = (0.35634, 0.136177)$ are calculated. In the Fig-4.11(A), three GH bifurcation points at $(\xi, \alpha_1) = (1.209401, 0.001573), (\xi, \alpha_1) = (1.848611, 0.001356)$ and $(\xi, \alpha_1) = (3.945465, 0.000903)$ are calculated and their second Lyapunov coefficients are given as $-4.411241e - 004$, $-3.029045e - 004$ and $-6.050697e - 007$. The BT bifurcation at $(\xi, \alpha_1) = (3.057845, 0.000085)$ is obtained and its normal form is $(a, b) = (-1.160775e - 003, -2.798663e - 002)$. In Fig-4.11 (B), one GH bifurcation at $(\alpha, h_2) = (0.045933, 0.184414)$ ant two BT bifurcation at $(\alpha, h_2) = (0.002173, 0.190158)$ and $(\alpha, h_2) = (0.035634, 0.185476)$ are calculated. The corresponding second Lyapunov coefficient and normal form coefficients are given by $-2.553158e + 000$, $(a, b) = (-3.613471e - 003, -8.168234e - 001)$ and $(a, b) = (1.592518e - 005, 6.288138e - 001)$.

In the absence of toxicity, by taking $h_2 = 0.1$ and $\beta = 0.3$, other parameters being fixed, the dynamics of system is described as follows: There is a transcritical bifurcation (branch point) w.r.t. $h_2 = h_2^{tc} = 0.193426$ around $(\bar{x}, 0) \approx (65.1000, 0)$. The Hopf points (H_1, H_2, H_3) are detected w.r.t. the parameters $h_2 \cong 0.187553$, $\alpha \cong 21.828168$ and $d \cong 0.088253$. The periodic solutions w.r.t. these parameters h_2, α and d are shown in Fig-4.12, 4.13 and 4.14. The Hopf bifurcation w.r.t. $h_2 \cong 0.187553$ is detected with purely imaginary eigen value $\pm i(0.072186)$. The corresponding first Lyapunov coefficient is $-4.164581e - 004$, indicating supercritical Hopf bifurcation. For the Hopf point at $\alpha \cong 21.828168$, the eigen value and first Lyapunov coefficient are $\pm i(0.173155)$ and $-2.876108e - 004$. For the Hopf point at $d \cong 0.088253$, the eigen value and first Lyapunov coefficient are $\pm i(0.0527956)$ and $-4.178943e - 004$.

In the absence of harvesting effort, by taking $\beta = 0.3$ and other parameters being fixed, a transcritical bifurcations occur at $\delta = \delta^{tc} = 0.166917$ around $(\bar{x}, 0) \approx (1.508465, 0)$ and at $\beta = \beta^{tc} = 0.632100$ around $(0, \hat{y}) \approx (0, 3)$. A Hopf point is obtained at $\xi = 0.014425$. The first Lyapunov coefficient is $-4.615937e - 001$ and the purely imaginary eigen values are $\pm i(0.219119)$. The periodic solutions are drawn in Fig-4.15.

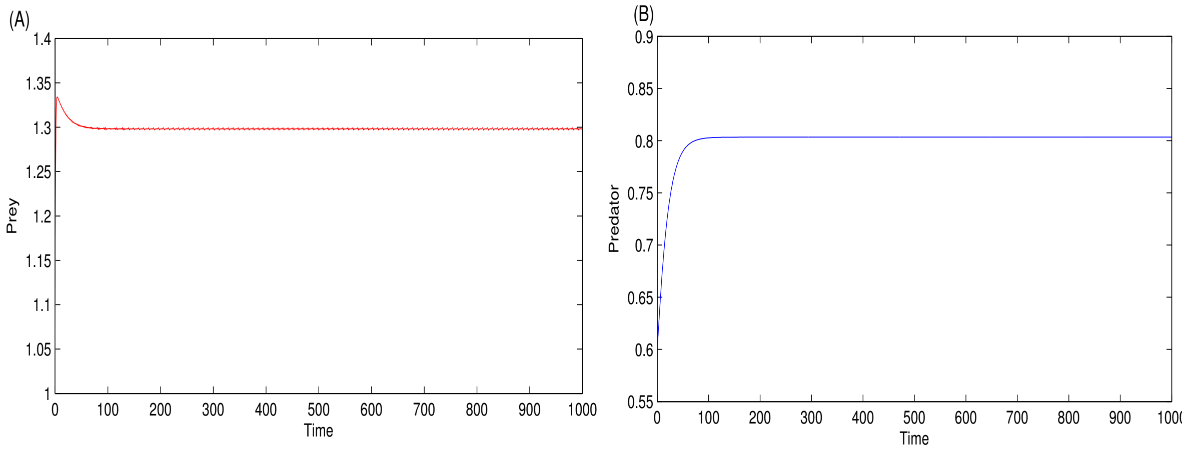

Figure 4.1: Solution curves of the prey and predator population w.r.t. time t for the set of parameters (4.9.1).

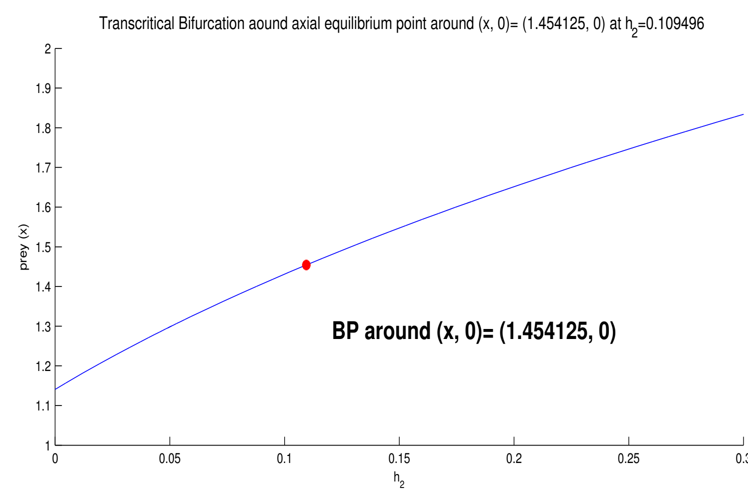

Figure 4.2: Bifurcation diagram of codimension-1 w.r.t. h_2 around the axial equilibrium point $(\bar{x}, 0)$ for the data set (4.9.1)

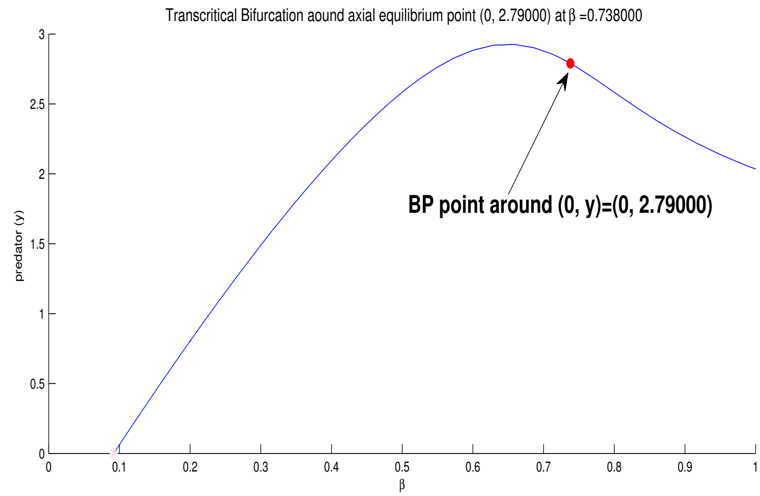

Figure 4.3: Bifurcation diagram of codimension-1 w.r.t. β around the axial equilibrium point $(0, \hat{y})$ for the data set (4.9.1)

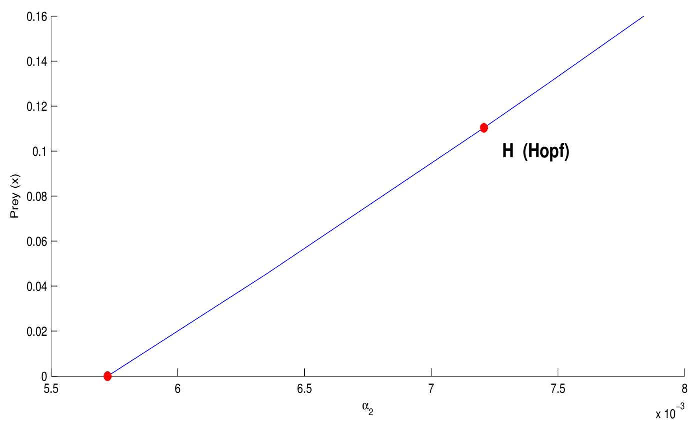

Figure 4.4: Bifurcation diagram of codimension-1 w.r.t. α_2 gives the Hopf bifurcation point around $(x^*, y^*) = (0.329419, 1.500245)$ at $\alpha_2 = 0.007208$ for the data set (4.9.1)

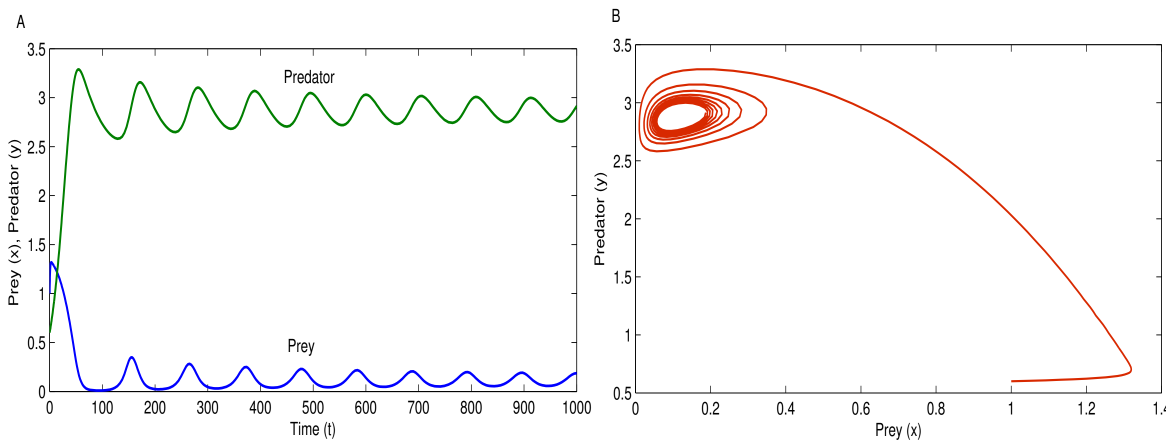

Figure 4.5: (A) represent time series of the prey population and predator population and (B) is the phase portrait of the solution curve which gives periodic solutions w.r.t. $\alpha_2 = 0.007208$ for the system (4.2.3) for the parameter set (4.9.1) in xy-plane.

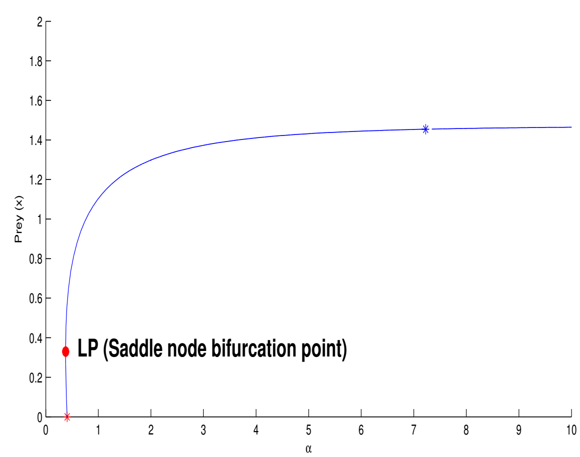

Figure 4.6: Bifurcation diagram of codimension-1 w.r.t. α gives the saddle node bifurcation point around $(x^*, y^*) = (0.329419, 1.500245)$ at $\alpha = 0.378060$ for the data set (4.9.1)

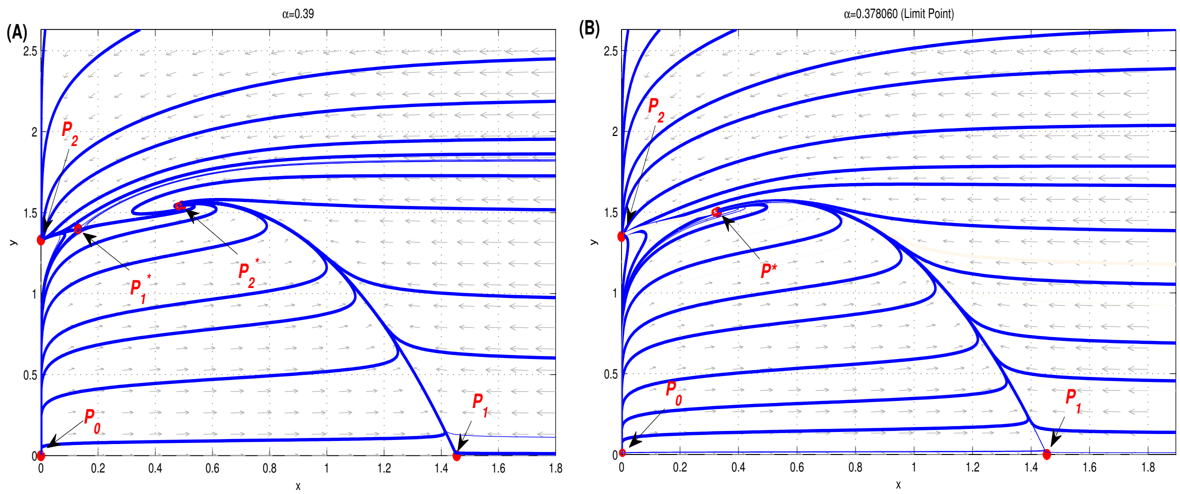

Figure 4.7: Solution curves of the prey predator population w.r.t. time t for changing value of α using the set of parameters (4.9.1).

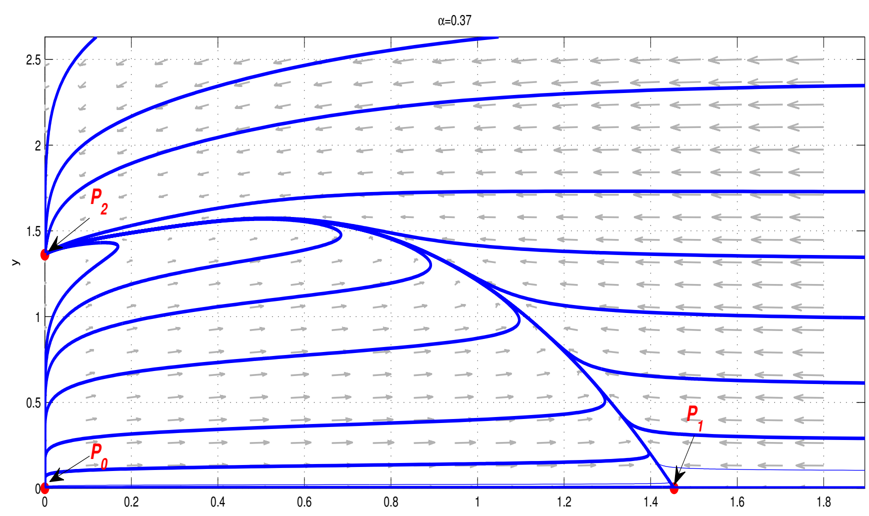

Figure 4.8: Solution curves of the prey and predator population w.r.t. time t for changing value of α using the set of parameters (4.9.1).

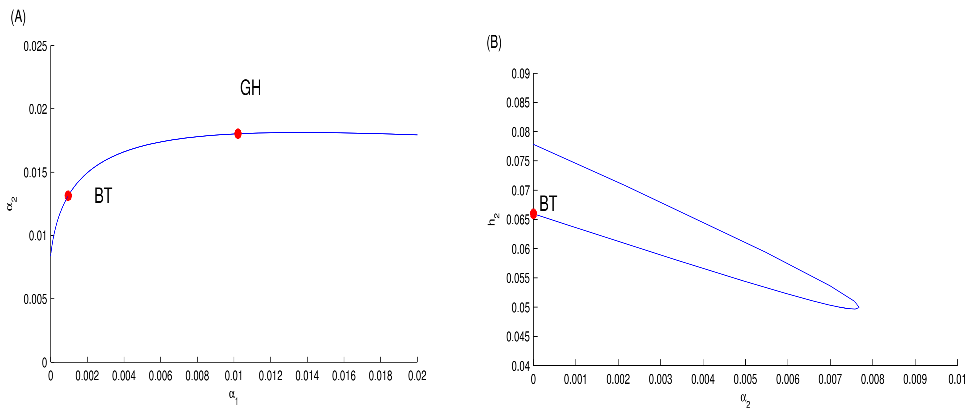

Figure 4.9: Bifurcation diagram of codimension-2 w.r.t.different parameter space for the parameter set (4.9.1) in the continuation of Hopf point.

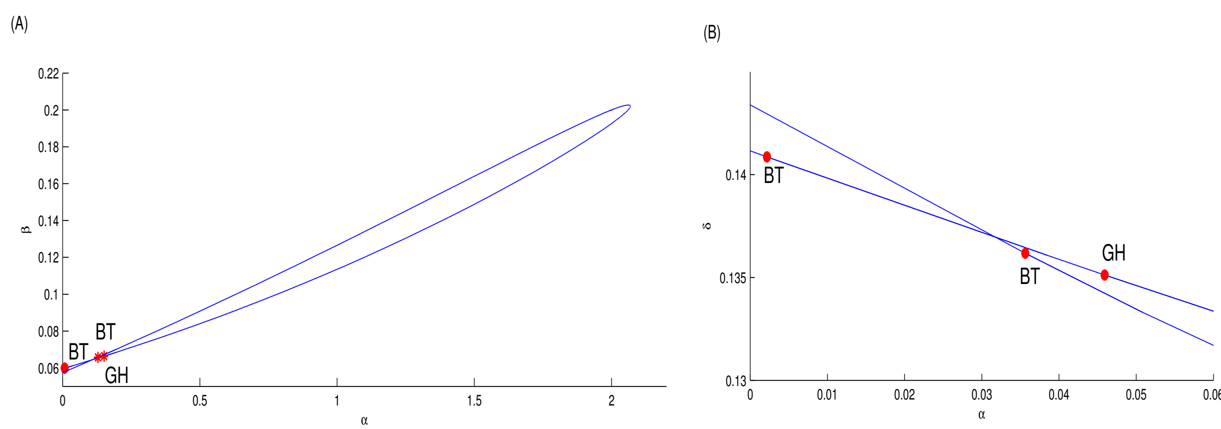

Figure 4.10: Bifurcation diagram of codimension-2 w.r.t.different parameter space for the parameter set (4.9.1) in the continuation of Hopf point.

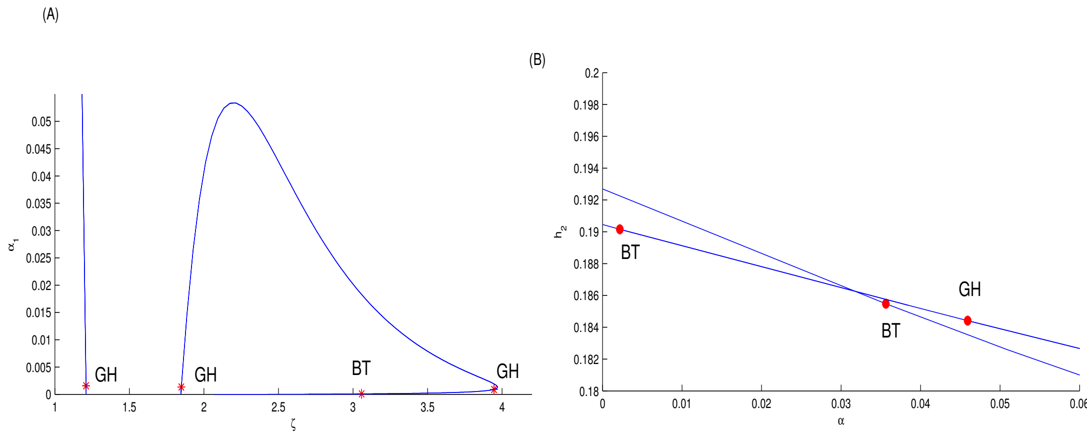

Figure 4.11: Bifurcation diagram of codimension-2 w.r.t.different parameter space for the parameter set (4.9.1) in the continuation of Hopf point.

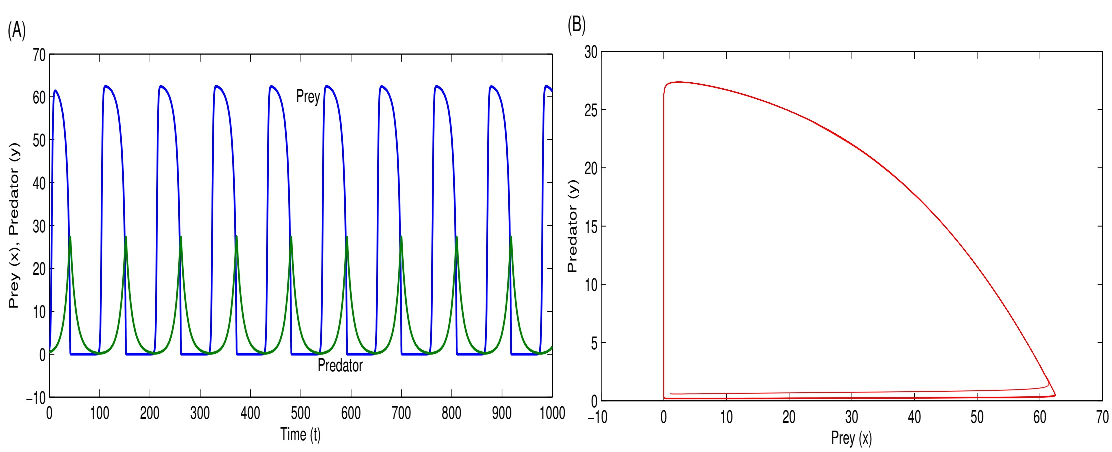

Figure 4.12: (A) represent time series of the prey population and predator population and (B) is the phase portrait of the solution curve which gives periodic solutions w.r.t. $h_2 = 0.187553$ in the absence of toxicity in xy-plane .

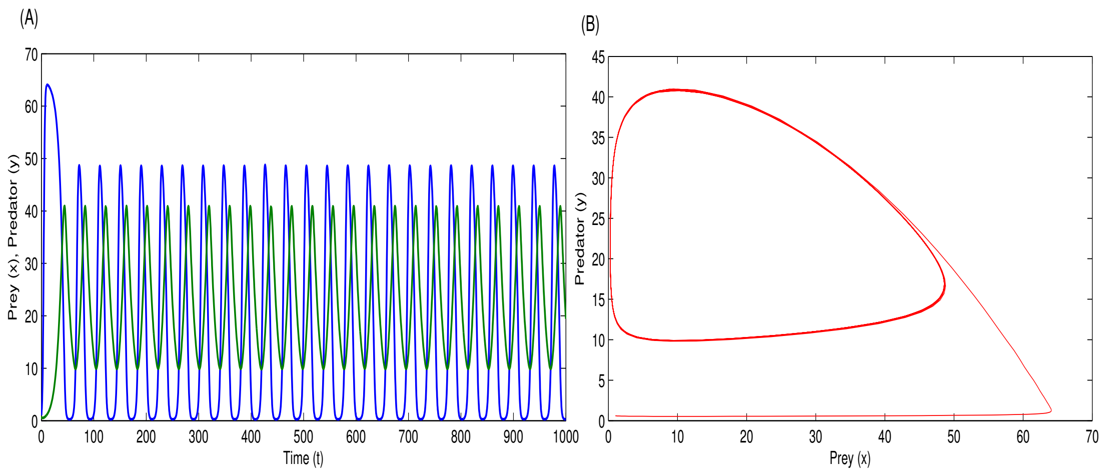

Figure 4.13: (A) represents time series of the prey population and predator population and (B) is the phase portrait of the solution curve which gives periodic solutions w.r.t. $\alpha = 21.828168$ in the absence of toxicity in xy- plane.

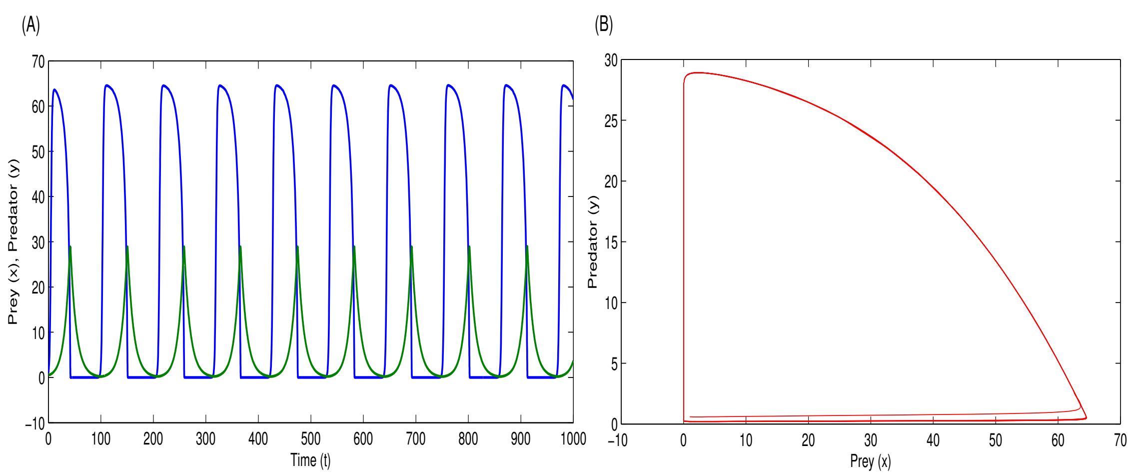

Figure 4.14: (A) represents time series of the prey population and predator population and (B) is the phase portrait of the solution curve which gives periodic solutions w.r.t. $d = 0.088253$ in the absence of toxicity in xy-plane .

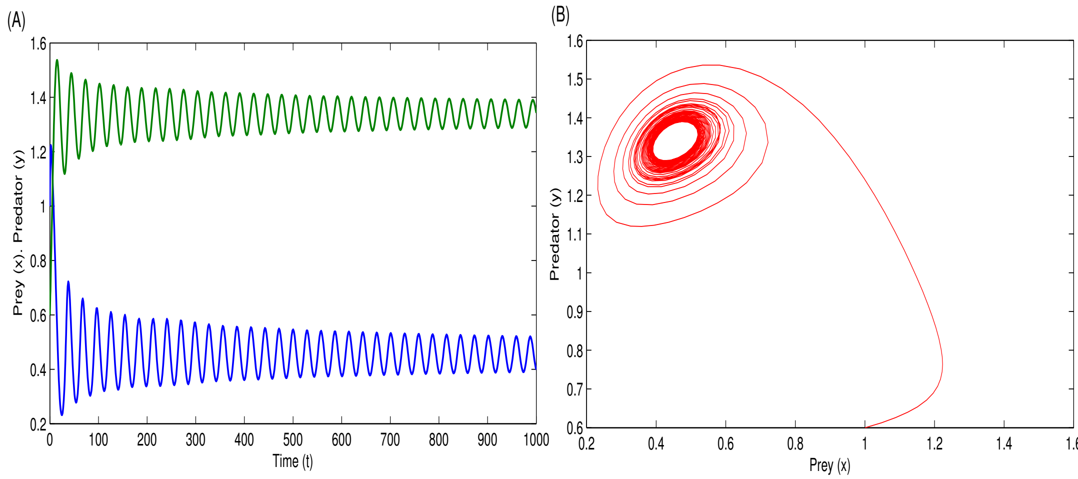

Figure 4.15: (A) represents time series of the prey population and predator population and (B) is the phase portrait of the solution curve which gives periodic solutions w.r.t. $\xi = 0.014425$ in the absence of harvesting in xy-plane .

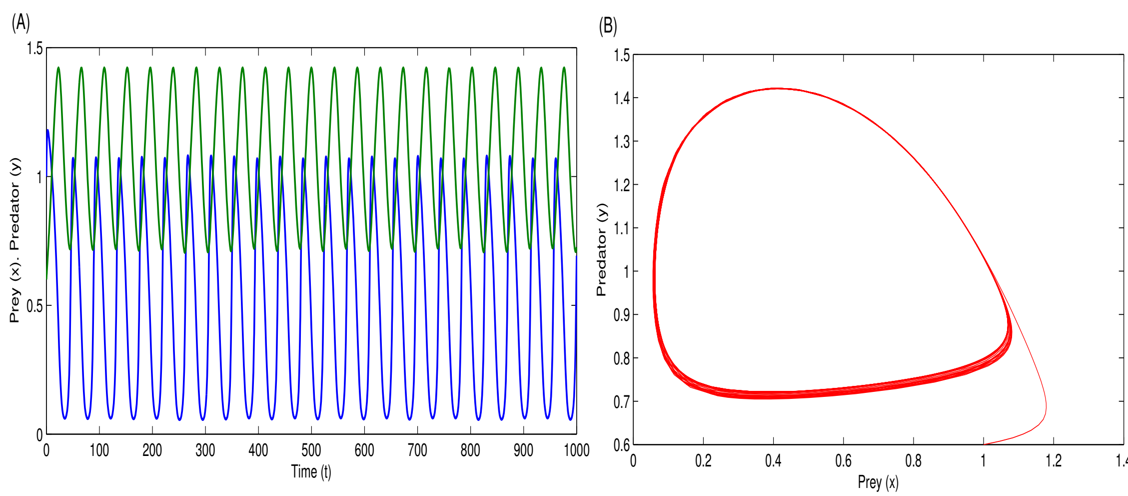

Figure 4.16: (A) represents time series of the prey population and predator population and (B) is the phase portrait of the solution curve which gives periodic solutions w.r.t. $\alpha_2 = 0.005471$ in the absence of additional food for the system (4.2.3), using the parameter set (4.9.1) in xy-plane.

4.10 Conclusion

This chapter is concerned with a two dimensional dynamical system incorporating combined harvesting for a predator-prey system where predator is provided an additional food. In this model, prey and predator both are affected by some external toxicant substances which is harmful for both the species. In this model, an addition food is playing an important role in predator-prey system which preserves predator population from extinction. The steady states of the system are obtained for suitable conditions. The stability analysis has been carried out for all possible feasible equilibrium points. The system undergoes local bifurcations i.e., transcritical, Hopf, saddle-node bifurcations for a threshold level of some parameters which are verified by numerical examples. Global bifurcations i.e., Bogdanov-Takens bifurcation (BT) and Generalized Hopf bifurcation (GH) are also detected in the continuation of Hopf bifurcation point (or limit point), using the software MATCONT w.r.t. different parameter values. The sufficient condition for the bionomic equilibrium have been derived. The optimal harvesting policy have been discussed by using Pontryagin's Maximum Principle.

Chapter 5

The Dynamics of a Fishery System in a Patchy Environment with Non–linear Harvesting

5.1 Introduction

An ecological system consists of different levels of organizations: individual level, population level, community level and ecosystem level. Different time scales are associated with them. Aggregation methods are used when the system involves more than one time scale. By aggregating some variables, it is possible to obtain a reduced model governing few global variables which are varying at a slow time scale. Aggregation methods have been used for continuous system of differential equations as well as for time discrete models by Auger and Roussarie [1], Auger and Poggiale [5], [3], Bravo de la Parra et al. [86] and Bravo de la Parra and Sanchez [18]. Auger and Poggiale [17], Auger and Chiorino [4] and Auger and Charles [6] investigated that it is possible to reduce the dimension of a system to obtain a reduced model using perturbation techniques and Center Manifold Theorem that can be handled analytically.

In the present chapter, a stock–effort dynamical model is investigated in a non-homogenous habitat with non-linear harvesting of stock. The habitat is divided into two fishing zones. The constant fish displacements and movements of fishing vessels between zones is assumed to be at a faster time scale as compared to the local growth / interaction.

5.2 The Mathematical Model

Let the heterogeneous habitat is divided in two zones with different carrying capacities. Let n_i $(i = 1, 2)$ be the fish size in ith zone such that $n(t) = n_1(t)+n_2(t)$. Similarly, E_i $(i = 1, 2)$ be the fishing effort in the ith zone such that $E(t) = E_1(t)+E_2(t)$. All fishing vessels are assumed to be identical and consequently, the fishing effort can be measured in terms of number of vessels. The fishing vessels are moving quickly between the zones to increase their net economic revenue. The fish moves at fast time scale between the two zones. At fast time scale, the total stock and the total number of vessels are assumed to be constant. Accordingly, the fast part of the model describes the movement of fish and vessels between the two zones. However, the total fish stock and the number of vessels are not constant at slow time scale. Considering τ is fast time scale and $t = \varepsilon\tau$ is slow time scale, where ε is a small dimensionless parameter, the complete Stock-Effort model reads as follows:

$$
\begin{aligned}
\frac{dn_1}{d\tau} &= (kn_2 - \hat{k}n_1) + \varepsilon\left(r_1 n_1\left(1 - \frac{n_1}{K_1}\right) - \frac{q_1 E_1 n_1}{b + a_1 n_1 + a_2 n_2}\right)\\
\frac{dn_2}{d\tau} &= (\hat{k}n_1 - kn_2) + \varepsilon\left(r_2 n_2\left(1 - \frac{n_2}{K_2}\right) - \frac{q_2 E_2 n_2}{b + a_1 n_1 + a_2 n_2}\right)\\
\frac{dE_1}{d\tau} &= \big(m(n_2)E_2 - \hat{m}(n_1)E_1\big) + \varepsilon E_1\left(\frac{q_1(p - T)n_1}{b + a_1 n_1 + a_2 n_2} - c\right)\\
\frac{dE_2}{d\tau} &= \big(\hat{m}(n_1)E_1 - m(n_2)E_2\big) + \varepsilon E_2\left(\frac{q_2(p - T)n_2}{b + a_1 n_1 + a_2 n_2} - c\right)
\end{aligned}
\tag{5.2.1}
$$

$$n_1(0) > 0, n_2(0) > 0, E_1(0) > 0, E_2(0) > 0.$$

The parameter r_i $(i = 1, 2)$ represents the intrinsic growth rate and K_i $(i = 1, 2)$ is the carrying capacity in zone-i. The parameter q_i is the catch-ability coefficient of the fleet in zone i $(i = 1, 2)$. Parameter p is the price of the catch and T is the tax imposed on per unit. The constant c is the cost of the fishing effort per unit. The constant b is the half saturation level and a_i $(i = 1, 2)$ is the handling time per harvesting vessel. The constant values k and $\hat{k}$ denote the fish per capita migration rates of fish from zone-2 to zone-1 and from zone-1 to zone-2, respectively. The migration rates ([7], [83]) for the fishing vessels, $\hat{m}(n_1)$ and $m(n_2)$, depend on the fish stock in the particular zone. These migration are assumed as follows:

$$\hat{m}(n_1) = \frac{1}{\alpha n_1 + \alpha_0} \quad and \quad m(n_2) = \frac{1}{\beta n_2 + \beta_0} \tag{5.2.2}$$

The migration functions $\hat{m}(n_1)$ and $m(n_2)$ of vessels from zone-1 to zone-2 and from zone-2 to zone-1 are assumed monotonically decreasing with population density. This shows that as the fish population n_i increases, the migration rates $m(n_i)$ decreases.

5.2.1 The Fast System

The fast system can be obtained by neglecting the slow part of the system i.e., taking $\varepsilon = 0$. Accordingly, the fast system will take the following form:

$$\begin{aligned} \frac{dn_1}{d\tau} &= (kn_2 - \hat{k}n_1) \\ \frac{dn_2}{d\tau} &= (\hat{k}n_1 - kn_2) \\ \frac{dE_1}{d\tau} &= \big(m(n_2)E_2 - \hat{m}(n_1)E_1\big) \\ \frac{dE_2}{d\tau} &= \big(\hat{m}(n_1)E_1 - m(n_2)E_2\big) \end{aligned} \tag{5.2.3}$$

Denote the total fish population as $n = n_1 + n_2$ and fishing effort as $E = E_1 + E_2$. These total population size are invariant at fast time scale. The following equilibria are obtained for the fast part of the system:

$$n_1^* = v_1^* n, \quad n_2^* = v_2^* n$$

$$E_1^* = \eta_1^*(n)E, \quad E_2^* = \eta_2^*(n)E$$

with

$$v_1^* = \frac{k}{k+\hat{k}}, \quad v_2^* = \frac{\hat{k}}{k+\hat{k}},$$

$$\eta_1^*(n) = \frac{m(n_2)}{m(n_2) + \hat{m}(n_1)} = \frac{\alpha v_1^* n + \alpha_0}{(\alpha v_1^* + \beta v_2^*)n + (\alpha_0 + \beta_0)},$$

$$\eta_2^*(n) = \frac{\hat{m}(n_1)}{m(n_2) + \hat{m}(n_1)} = \frac{\beta v_2^* n + \beta_0}{(\alpha v_1^* + \beta v_2^*)n + (\alpha_0 + \beta_0)}.$$

The constants v_1^* and v_2^* represent the proportions of stock on each patch at the fast equilibrium, whereas $\eta_1^*(n)$ and $\eta_2^*(n)$ represent the same interpretation for the fishing effort. To study the stability of fast equilibria of system in each zone-i, the following substitution can be made:

$$n_2 = n - n_1 \quad and \quad E_2 = E - E_1$$

Therefore, the system in the zone-1 for the fast part will take the form as follows:

$$\begin{aligned} \frac{dn_1}{d\tau} &= k(n - n_1) - \hat{k}n_1, \\ \frac{dE_1}{d\tau} &= m(n - n_1)(E - E_1) - \hat{m}(n_1)E_1 \end{aligned} \tag{5.2.4}$$

Fast equilibria must satisfy the following equations:

$$\begin{aligned} k(n - n_1^*) - \hat{k}n_1^* = 0 \\ m(n - n_1^*)(E - E_1^*) - \hat{m}(n_1^*)E_1^* = 0 \end{aligned}$$

As for each pair (n, E), a unique equilibrium (n_1^*, E_1^*) of system (5.2.4) exists. To study the stability of (n_1^*, E_1^*), Jacobian matrix is evaluated as follows:

$$J(n_1^*, E_1^*) = \begin{bmatrix} -k & -\hat{k} \\ -m'(n - n_1^*)(E - E_1^*) - \hat{m}'(n_1^*)E_1^* & -m(n - n_1^*) - \hat{m}(n_1^*) \end{bmatrix}$$

The functions $\hat{m}(n_1^*)$ and $m(n - n_1^*)$ are positive and decreasing. This means that their derivatives are negative. Therefore, it can be seen that

$$tr(J(n_1^*, E_1^*)) < 0 \quad and \quad det(J(n_1^*, E_1^*)) > 0$$

This implies that the fast equilibrium (n_1^*, E_1^*) is asymptotically stable. Similarly, it can be easily proved that the fast equilibrium (n_2^*, E_2^*) is also locally asymptotically stable. Accordingly, each pair (n_i^*, E_i^*), for $i = 1, 2$ for fast part is locally asymptotically stable.

5.3 The Aggregated Model

The following aggregated system is obtained by substituting the fast equilibria in (5.2.1) and by adding the fish stock and the fishing effort equations. The following system of two equations governing the total fish stock and fishing effort variables at the slow time scales is obtained, that is called the aggregated model ([1]-[7]):

$$\begin{aligned}\frac{dn}{dt} &= rn\left(1-\frac{n}{K}\right)-\frac{Q(n)nE}{b+an} = nF(n,E),\\ \frac{dE}{dt} &= E\left(\frac{Q(n)(p-T)n}{b+an}-c\right) = EG(n,E).\end{aligned} \tag{5.3.1}$$

$$n(0)=n_0, E(0)=E_0, \quad (n_0,E_0)\in\mathbb{R}_+^2$$

with

$$r=r_1v_1^*+r_2v_2^*, \quad \frac{r}{K}=\frac{r_1v_1^{*2}}{K_1}+\frac{r_2v_2^{*2}}{K_2}, \quad a=a_1v_1^*+r_2v_2^*,$$

$$Q(n)=q_1v_1^*\eta_1^*(n)+q_2v_2^*\eta_2^*(n)=\frac{(q_1\alpha v_1^{*2}+q_2\beta v_2^{*2})n+(q_1v_1\alpha_0+q_2v_2\beta_0)}{(\alpha v_1^*+\beta v_2^*)n+(\alpha_0+\beta_0)}$$

Introduce the constants Q_1, Q_0, Q_{11} and Q_{00} as

$$Q_1=q_1\alpha v_1^{*2}+q_2\beta v_2^{*2}, \quad Q_0=q_1v_1\alpha_0+q_2v_2\beta_0$$

$$Q_{11}=\alpha v_1^*+\beta v_2^*, \quad Q_{00}=\alpha_0+\beta_0$$

Accordingly,

$$Q(n)=\frac{Q_1n+Q_0}{Q_{11}n+Q_{00}}$$

The constants r and K are positive parameters w.r.t. the local parameters of the complete model (5.2.1).

Remark 5.3.1. *If $p<T$, the derivative of E is negative and E goes to extinction then we must assume that $p>T$.*

5.4 Positivity and Boundedness of Aggregated Model

Lemma 5.4.1. *All the solutions* $(n(t), E(t))$ *of the system* (5.3.1) *with positive initial conditions remain positive for all* $t > 0$*.*

Proof. The positivity of solutions of the system (5.3.1) can be easily proved as in Lemma 2.3.1. □

Lemma 5.4.2. *The system* (5.3.1) *has uniformly bounded solution.*

Proof. Consider a function $\psi(t)$ such that

$$\begin{aligned} \psi(t) &= n(t) + \frac{1}{(p-T)}E(t), \\ \frac{d\psi(t)}{dt} &= n'(t) + \frac{1}{(p-T)}E'(t), \\ &= rn\left(1 - \frac{n}{K}\right) - \frac{cE}{(p-T)} \end{aligned}$$

Introduce a positive constant M and rewrite the above equation as follows:

$$\frac{d\psi(t)}{dt} + M\psi(t) \leq -\frac{r}{K}\left[n^2 - \left(K + \frac{KM}{r}\right)n\right] - \frac{(c-M)E}{(p-T)}$$

Choosing $M = \frac{c}{2}$, further simplification yields

$$\begin{aligned} \frac{d\psi(t)}{dt} + M\psi(t) &\leq -\frac{r}{K}\left(n - \left(\frac{K}{2} + \frac{Kc}{4r}\right)\right)^2 + \left(\frac{K}{2} + \frac{Kc}{4r}\right)^2 \\ \frac{d\psi(t)}{dt} + M\psi(t) &\leq N; \quad N = \left(\frac{K}{2} + \frac{Kc}{4r}\right)^2 \end{aligned}$$

Solution of above differential inequality gives,

$$\begin{aligned} \psi(t) &\leq \frac{2N}{c}\left(1 - e^{-ct/2}\right) + \psi(0)e^{-ct/2}, \\ 0 < \lim_{t\to\infty} \psi(t) &\leq \frac{2N}{c}. \end{aligned}$$

Accordingly, all the solutions of (5.3.1) initiating from $\mathbb{R}^2_+$ are confined in the region

$$R = \left\{(n, E) \in \mathbf{R}; 0 < n(t) + \frac{1}{(p-T)}E(t) \leq \frac{2N}{c} + \phi \text{ for any } \phi > 0\right\}.$$

□

5.5 Existence of Equilibrium States of the Aggregated Model

(i) $P_0(0,0)$ is the trivial equilibrium point and always exists.

(ii) $P_1(K,0)$, is fishing-free equilibrium point and it always exists.

(iv) $P_2(n^*,E^*)$ is the interior equilibrium point of the system (5.3.1) and is obtained as:

$$E^* = \frac{r(b+an^*)}{Q(n^*)}\left(1-\frac{n^*}{K}\right) \tag{5.5.1}$$

$$n^*Q(n^*)(p-T) - c(b+an^*) = 0 \tag{5.5.2}$$

The value of n^* is obtained from the following :

$$n^*Q(n^*)(p-T) - c(b+an^*) = 0. \tag{5.5.3}$$

After solving the equation (5.5.3), a quadratic equation can be obtained as follow:

$$\Big((p-T)Q_1 - acQ_{11}\Big)n^{*2} + \Big((p-T)Q_0 - bcQ_{11} - acQ_{00}\Big)n^* - bcQ_{00} = 0.$$

$$n^* = \frac{-\Big((p-T)Q_0 - bcQ_{11} - acQ_{00}\Big) + \sqrt{\Delta}}{2\Big((p-T)Q_1 - acQ_{11}\Big)} \tag{5.5.4}$$

where

$$\Delta = \Big((p-T)Q_0 - bcQ_{11} - acQ_{00}\Big)^2 + 4bcQ_{00}\Big((p-T)Q_1 - acQ_{11}\Big)$$

The value n^* is a unique positive solution of (5.5.4) provided

$$(p-T)Q_1 - acQ_{11} > 0.$$

This gives

$$T < T_1; \quad T_1 = p - \frac{acQ_{11}}{Q_1} = p - p_1 \tag{5.5.5}$$

The value of E^* is positive for

$$n^* < K \tag{5.5.6}$$

The expression (5.5.6) gives the following condition

$$T < T_2; \tag{5.5.7}$$

where

$$\begin{aligned} T_2 &= p - \frac{c(b+aK)(Q_{11}K+Q_{00})}{K(Q_1K+Q_0)} \\ &= p - \left(\frac{acQ_{11}}{Q_1} + \left(\frac{c(bQ_1Q_{11}+KaQ_1Q_{00}+abQ_1-KQ_0Q_{11})}{KQ_1(Q_1K+Q_0)}\right)\right) \end{aligned}$$

Therefore, the interior equilibrium point $P_2(n^*, E^*)$ of the system (5.3.1) is feasible for the following conditions

$$T < min(T_1, T_2) = T_2 \tag{5.5.8}$$

It may be noted that if $p < p_1$, then the interior equilibrium point will not exist at all. However, if $p > p_1$, then there should be $T < min(T_1, T_2) = T_2$ for existence of this point.

5.6 Stability analysis and Bifurcations of Equilibrium States

The Jacobian matrix of the system (5.3.1) at any point (n, E) is given by

$$J(n,E) = \begin{bmatrix} n\left(-\frac{r}{K} - E\left(\frac{(b+an)Q'(n)-aQ(n)}{(b+an)^2}\right)\right) + F & -\frac{Q(n)n}{b+an} \\ E(p-T)\left(\frac{bQ(n)+bnQ'(n)+an^2Q'(n)}{(b+an)^2}\right) & G \end{bmatrix}$$

Following some theorems are stated for the stability of various equilibrium states.

Theorem 5.6.1. *The equilibrium point $P_0(0,0)$ is always saddle with unstable manifold in n-direction and stable manifold in E-direction.*

Proof. The Jacobian matrix of the system (5.3.1) at the point $(0,0)$ is given by

$$J(0,0) = \begin{bmatrix} r & 0 \\ 0 & -c \end{bmatrix}$$

The two eigenvalues of Jacobian matrix $J(0,0)$ are r and $-c$. Thus, the equilibrium point $(0,0)$ is always saddle with unstable manifold in n-direction and stable manifold in E-direction. □

Theorem 5.6.2. *The axial equilibrium point $P_1(K,0)$ is locally asymptotically stable provided*

$$T > T_2; \quad T_2 = p - \frac{c(b+aK)(Q_{11}K + Q_{00})}{K(Q_1K + Q_0)} = p - p_2. \tag{5.6.1}$$

Proof. The Jacobian matrix of the system (5.3.1) at the point $P_1(K,0)$ is given by

$$J(K,0) = \begin{bmatrix} -r & -\dfrac{Q(K)K}{b+aK} \\ 0 & \dfrac{Q(K)(p-T)K}{b+aK} - c \end{bmatrix}$$

The eigenvalues of Jacobian matrix about the fishery free state $P_1(K,0)$ are

$$\lambda_1 = -r < 0 \quad \textit{and} \quad \lambda_2 = \frac{Q(K)(p-T)K}{b+aK} - c$$

Accordingly, $P_1(K,0)$ is locally asymptotically stable for the condition (5.6.1).

□

Remark 5.6.3. *The equilibrium point $P_1(K,0)$ becomes saddle when the condition* (5.6.1) *is violated.*

If $p < p_2$, then condition (5.6.1) *is trivially satisfied and the fishery free point $P_1(K,0)$ is locally asymptotically stable. This means fishery is not profitable and it will be closed. However, when $p > p_2$ and tax T is sufficiently high to satisfy* (5.6.1), *then also, the fisheries will not be profitable and ultimately they will be closed. The state P_1 has a transcritical bifurcation at*

$$T = T_2. \tag{5.6.2}$$

Theorem 5.6.4. *The equilibrium point (n^*, E^*) if exists, is locally asymptotically stable provided*

$$n^*\left(-\frac{r}{K}-E^*\left(\frac{(b+an^*)Q'(n^*)-aQ(n^*)}{(b+an^*)^2}\right)\right)<0 \tag{5.6.3}$$

Proof. For the local stability, the Jacobian matrix of the system (5.3.1) at an interior point (n^*, E^*) is given by

$$J(n^*, E^*) = \begin{bmatrix} n^*\left(-\frac{r}{K}-E^*\left(\frac{(b+an^*)Q'(n^*)-aQ(n^*)}{(b+an^*)^2}\right)\right) & -\frac{Q(n^*)n^*}{b+an^*} \\ E^*(p-T)\left(\frac{bQ(n^*)+bn^*Q'(n^*)+an^{*2}Q'(n^*)}{(b+an^*)^2}\right) & 0 \end{bmatrix}$$

The stability conditions of interior point $P_2(n^*, E^*)$ are

$$\begin{aligned} tr(J(n^*, E^*)) &= n^*\left(-\frac{r}{K}-E^*\left(\frac{(b+an^*)Q'(n^*)-aQ(n^*)}{(b+an^*)^2}\right)\right)<0, \\ det(J(n^*, E^*)) &= E^*\frac{Q(n^*)n^*}{b+an^*}\left[\frac{(p-T)\big(bQ(n^*)+bn^*Q'(n^*)+an^{*2}Q'(n^*)\big)}{(b+an^*)^2}\right]>0. \end{aligned}$$

Accordingly, the equilibrium state (n^*, E^*) is locally asymptotically stable provided the condition (5.6.3) holds.

□

Remark 5.6.5. *When the condition* (5.6.3) *is violated, the interior state P_2 becomes unstable via Hopf bifurcation (periodic solutions). Therefore, $tr(J(n^*, E^*)) = 0$ gives the Hopf point at $T = T_3$. Since it is quite difficult to obtain the Hopf point analytically. Therefore, it is calculated numerically in the Numerical Section.*

Note that if $(b+an^*)Q'(n^*)-aQ(n^*)>0$, then trace will always be negative and interior point is always stable. However, when $(b+an^*)Q'(n^*)-aQ(n^*)<0$, then the interior state is stable provided

$$\frac{r}{K} > E^*\left(\frac{aQ(n^*)-(b+an^*)Q'(n^*)}{(b+an^*)^2}\right) \tag{5.6.4}$$

5.7 Global Stability

Theorem 5.7.1. *The fishing free equilibrium point $P_1(K, 0)$ is globally asymptotically stable for $T > T_2$.*

Proof. To demonstrate the global stability of the system (5.3.1) at $(K, 0)$, construct a suitable Lyapunov function as follows:

$$V(n, E) = \int_K^n \frac{x - K}{x}\, dx + d_1 \int_0^E dy, \tag{5.7.1}$$

where d_1 is positive a constant. The function $V(n, E)$ is positive definite and $V(K, 0) = 0$. Differentiate the function $V(n, E)$ (5.11.3) w.r.t. time t and the following is obtained

$$\begin{aligned} \frac{dV}{dt} &= \frac{n-K}{n}\frac{dn}{dt} + d_1\frac{dE}{dt} \\ &= (n-K)\left[r\left(1 - \frac{n}{K}\right) - \frac{qE}{b+an}\right] + d_1 E\left[\frac{q(p-T)n}{b+an} - c\right] \end{aligned}$$

Take $d_1 = \dfrac{1}{p-T}$, solving the above equation and the following is obtained

$$\begin{aligned} \frac{dV}{dt} &= -\frac{r}{K}(n-K)^2 - \frac{qKE}{b+an} - \frac{cE}{p-T} \\ \frac{dV}{dt} &< 0 \end{aligned}$$

This shows that $\dfrac{dV}{dt}$ is negative definite.

Accordingly, the fishing free equilibrium point $(K, 0)$ is globally asymptotically stable for $T > T_2$. □

5.8 Maximum Sustainable Yield (MSY)

The maximum rate of harvesting of any biological resource biomass is called maximum sustainable yield (MSY) at equilibrium such that harvesting higher than MSY may lead to extinction of resources. The total yield function at equilibrium is given as follows:

$$Y^* = H(n^*, E^*) = \frac{Q(n^*)n^*E^*}{b + an^*} = rn^*\left(1 - \frac{n^*}{K}\right) \tag{5.8.1}$$

such that

$$\frac{\partial Y^*}{\partial n^*} = 0 \quad and \quad \frac{\partial^2 Y^*}{\partial n^{*2}} < 0$$

This gives

$$n^* = \frac{K}{2} \quad and \quad \frac{\partial^2 Y^*}{\partial n^{*2}} = \frac{-2r}{K}.$$

Accordingly, the amount $n^* = \frac{K}{2}$, gives the maximum yield and it is given as follows:

$$Y^*_{MSY} = \frac{rK}{4} \quad at \quad n^* = \frac{K}{2}. \tag{5.8.2}$$

The effects of taxation on total yield is observed and it is shown in the figure - 5.5. It is observed that the yield increases with increase in taxation $\left(\frac{dY^*}{dT} > 0\right)$ initially. Further, $\frac{dY^*}{dT} < 0$, for sufficiently large value of T. For $T = 2.4315319$, the yield becomes maximum and the corresponding yield is verified as maximum sustainable yield $Y^*_{MSY} = 0.25$. Accordingly, there exists a positive optimal tax that will maximize the total yield and the optimal level of taxation is obtained as follows:

$$T_{opt} = 2.4315319 \tag{5.8.3}$$

Substituting $n^* = \frac{K}{2}$ in (5.5.4) gives T_{opt} as

$$T = T_{opt} = p - \frac{c(Ka + 2b)(KQ_{11} + 2Q_{00})}{K(KQ_1 + 2Q_0)}. \tag{5.8.4}$$

Substituting parametric values in (5.8.4) verify (5.8.3). Moreover, the total yield will becomes zero for $T = 3.090186$. The higher taxation will lead to closure of fishery.

5.9 Bionomic Equilibrium

The net economic revenue to the society is represented as follows:

$$P(t, n, E, T) = \left(\frac{(p - T)Q(n)n}{b + an} - c\right)E + \frac{Q(n)TnE}{b + an} = \left(\frac{pQ(n)n}{b + an} - c\right)E \tag{5.9.1}$$

The bionomic equilibrium $P_{BE}(n_{BE}, E_{BE})$ can be calculated by solving the following:

$$\frac{dn}{dt} = \frac{dE}{dt} = P = 0$$

It gives

$$\begin{aligned} n_{BE} &= \frac{-\Big(pQ_0 - bcQ_{11} - acQ_{00}\Big) + \sqrt{\triangle_1}}{2\Big(pQ_1 - acQ_{11}\Big)} \\ E_{BE} &= \frac{r(b + an_{BE})}{Q(n_{BE})}\Big(1 - \frac{n_{BE}}{K}\Big) \end{aligned}$$

where

$$\triangle_1 = \Big(pQ_0 - bcQ_{11} - acQ_{00}\Big)^2 + 4bcQ_{00}\Big(pQ_1 - acQ_{11}\Big)$$

The bionomic equilibrium $P_{BE}(n_{BE}, E_{BE})$ is feasible provided

$$p > \frac{acQ_{11}}{Q_1}. \tag{5.9.2}$$

5.10 Optimal Taxation Policy

An optimal taxation policy for the system (5.3.1) is investigated to maximize the total discounted net revenue using taxation as a control instrument. The optimal control problem over an infinite time horizon is given by

$$\max_{T_{min} < T(t) < T_{max}} I = \int_0^\infty e^{-\delta t}\Big(\frac{Q(n)pn}{b + an} - c\Big)E\,dt \tag{5.10.1}$$

The constant δ is the instantaneous annual rate of discount decided by harvesting agencies. The main objective is to determine an optimal taxation policy $T = T(t)$ to maximize (5.10.1) subject to the state equations in the system (5.3.1) and the control constraints $T_{min} < T(t) < T_{max}$.

The taxation policy [70] is assumed as follows:

$$T(t) = \begin{cases} \overline{T}(t) & for \quad t \in [0, t_1] \\ T^* & for \quad t > t_1 \end{cases}$$

Pontryagin's Maximum Principle is used to obtain the optimal level of the solution of the problem (5.10.1). Let $\lambda_1(t)$ and $\lambda_2(t)$ are adjoint variables w.r.t. the time t corresponding to the variables n and E, respectively. The associated Hamiltonian function is given by

$$\begin{aligned} \mathcal{H}(t, n, E, T) &= e^{-\delta t}\Big(\frac{Q(n)pn}{b + an} - c\Big)E + \lambda_1\Big[rn\Big(1 - \frac{n}{k}\Big) - \frac{Q(n)En}{b + an}\Big] + \\ &\quad \lambda_2\Big[E\Big(\frac{Q(n)(p - T)n}{b + an} - c\Big)\Big] \end{aligned} \tag{5.10.2}$$

It can be observed that Hamiltonian is linear equation in control variable T. The optimal control problem involves singular and bang-bang controls. Also, the optimal control must satisfy the following conditions that maximizes $\mathcal{H}$ such that:

$$\overline{T} = \begin{cases} T_{max} & \forall t \in [0, t_1] \quad with \quad \dfrac{d\mathcal{H}}{dT} > 0 \\ T_{min} & \forall t \in [0, t_1] \quad with \quad \dfrac{d\mathcal{H}}{dT} < 0 \end{cases}$$

The Hamiltonian in (5.10.2) must be maximized for $T \in [T_{min}, T_{max}]$. Assume that the optimal solution does not occur at T_{min} or T_{max}. Therefore, the considered control problem gives a singular solution on the control set (T_{min}, T_{max}) if

$$\begin{aligned} \frac{\partial \mathcal{H}}{\partial T} &= 0, \\ i.e., \frac{-Q(n)n}{b+an}\lambda_2(t) &= 0 \quad \Rightarrow \quad \lambda_2(t) = 0. \end{aligned} \tag{5.10.3}$$

Pontryagin's Maximum Principle [104] is utilized to find the singular control and the adjoint variables must satisfy the adjoint equations such that:

$$\frac{d\lambda_1}{dt} = -\frac{\partial \mathcal{H}}{\partial n}, \quad \frac{d\lambda_2}{dt} = -\frac{\partial \mathcal{H}}{\partial E} \tag{5.10.4}$$

The adjoint equations are:

$$\begin{aligned} \frac{d\lambda_1}{dt} &= -\frac{\partial \mathcal{H}}{\partial n} = -\Bigg[e^{-\delta t} pE\Bigg(\frac{bQ(n) + bnQ'(n) + an^2Q'(n)}{(b+an)^2}\Bigg) + \\ &\quad \lambda_1\Bigg(-\frac{rn}{k} - nE\Bigg(\frac{(b+an)Q'(n) - aQ(n)}{(b+an)^2}\Bigg)\Bigg)\Bigg] \end{aligned} \tag{5.10.5}$$

$$\begin{aligned} \frac{d\lambda_2}{dt} &= -\frac{\partial \mathcal{H}}{\partial E} = -\Bigg[e^{-\delta t}\Bigg(\frac{pQ(n)n}{(b+an)} - c\Bigg) + \lambda_1\Bigg(\frac{Q(n)n}{b+an}\Bigg)\Bigg] \\ &= -\Bigg[e^{-\delta t}\Bigg(p - \frac{c(b+an)}{Q(n)n}\Bigg) - \lambda_1\Bigg] \end{aligned} \tag{5.10.6}$$

The control problem admits a singular solution on the control set $[0, E_{max}]$ if $\dfrac{\partial \mathcal{H}}{\partial E} = 0$,

$$\Rightarrow \lambda_1(t) = e^{-\delta t}\Bigg(p - \frac{c(b+an)}{Q(n)n}\Bigg) \tag{5.10.7}$$

Let $\lambda_i(t) = \mu_i(t)e^{-\delta t}$, where $\mu_i(t) = \lambda_i(t))e^{\delta t}$ for $i = 1, 2$ are called the shadow prices and these are constant over time.

Solving (5.10.5), a linear differential equation is obtained in λ_1 and in the interior equilibrium (n^*, E^*) such that

$$\frac{d\lambda_1}{dt} - A_1\lambda_2 = -e^{-\delta t}A_2 \tag{5.10.8}$$

with

$$\begin{aligned} A_1 &= \frac{rn^*}{K} + n^*E^*\left(\frac{(b+an^*)Q'(n^*) - aQ(n^*)}{(b+an^*)^2}\right) \quad \textit{and} \\ A_2 &= pE^*\left(\frac{bQ(n^*) + bn^*Q'(n^*) + an^{*2}Q'(n^*)}{(b+an^*)^2}\right) \end{aligned}$$

Solving equation (5.10.8),

$$\lambda_1(t) = \frac{A_1}{A_2+\delta}e^{-\delta t} \tag{5.10.9}$$

Using (5.10.7) and (7.5.11),

$$p - \frac{c(b+an^*)}{Q(n^*)n^*} = \frac{A_1}{A_2+\delta} \tag{5.10.10}$$

Accordingly, (5.10.10) gives the desired singular path. Substituting the values of A_1, A_2 and $Q(n^*)$, n^*, an expression for taxation T can be obtained. $T = T_\delta$ be the solution (if exists) of the equation (5.10.10). Then, using the value of $T = T_\delta$ in the interior equilibrium point, the optimal equilibrium point (n_δ, E_δ) can be obtained.

It is assumed that no subsidy is provided to the fishery then possible range of tax is determined by using the condition $0 < T < T_2$. Accordingly, the optimal taxation $T = T_\delta$ is supposed to be the optimal solution of the equation (5.10.10) and it must lies in the range $0 < T < T_2$.

5.11 A Special Case

From the model (5.3.1), a special case can be obtained for the particular choice of parameters. Considering the case when $\alpha = \beta = 0$ but $\alpha_0 \neq 0$ and $\beta_0 \neq 0$, This gives us the density independent migration rate of fishing vessels. In this case, the fast equilibria will take the form

$$E_1^* = \frac{\alpha_0}{\alpha_0+\beta_0}E \quad \textit{and} \quad E_2^* = \frac{\beta_0}{\alpha_0+\beta_0}E \tag{5.11.1}$$

These also gives the fast fast equilibria for effort is proportional to total density of effort with the proportional terms (constants) $\frac{\alpha_0}{\alpha_0+\beta_0}$ and $\frac{\beta_0}{\alpha_0+\beta_0}$. In this case, the aggregated model (5.3.1) will take the following form:

$$\begin{aligned} \frac{dn}{dt} &= rn\left(1-\frac{n}{K}\right) - \frac{qnE}{b+an} \\ \frac{dE}{dt} &= E\left(\frac{q(p-T)n}{b+an} - c\right) \end{aligned} \tag{5.11.2}$$

with

$$q = \frac{Q_0}{Q_{00}} = \frac{q_1 v_1 \alpha_0 + q_2 v_2 \beta_0}{\alpha_0+\beta_0}$$

The interior equilibrium point (n^*, E^*) of the system (5.11.2) can be obtained by substituting $Q(n^*) = \frac{Q_0}{Q_{00}} = q$ in (5.5.3) and it is obtained as follows:

$$(n^*, E^*) = \left(\frac{bc}{q(p-T)-ac}, r(b+an^*)\left(1-\frac{n^*}{K}\right)\right) \tag{5.11.3}$$

The interior equilibrium point (n^*, E^*) is positive for

$$T < min(T_1{}', T_2{}') = T_2{}'; \quad T_1{}' = p - \frac{ac}{q} \quad and \quad T_2{}' = p - \left(\frac{ac}{q} + \frac{bc}{qK}\right)$$

The equation (5.11.3) shows that with increasing tax, the fish biomass increases whereas the fishing effort decreases.

The effects of taxation on yield is shown in the figure- 5.6. The figure gives the maximum sustainable yield $Y^*{}_{MSY} = 0.25$ for $T'_{opt} = 2.377$. Comparing it with the figure 5.5, it can be concluded that the MSY occurs at higher level of taxation ($T_{opt} = 2.4315319$) in case of density dependent migration rate.

The stability of $(0, 0)$ will remain unchanged for the model (5.11.2). The stability condition (5.6.1) for the boundary equilibrium point $(K, 0)$ will become

$$T > T_2{}'; \quad T_2{}' = p - \left(\frac{ac}{q} + \frac{bc}{qK}\right). \tag{5.11.4}$$

If the condition (5.11.4) is violated then the point $(K, 0)$ will become unstable. Therefore, there is transcritical bifurcation at

$$T = T_2{}' \tag{5.11.5}$$

Substituting $Q(n^*) = \dfrac{Q_0}{Q_{00}} = q$ and $Q'(n^*) = 0$, the trace (5.6.4) and determinant (5.6.4) of the interior equilibrium point (n^*, E^*) will take the form as follow:

$$tr(J(n^*, E^*)) = n^*\bigg(-\frac{r}{K} + \frac{E^* aq}{(b + an^*)^2}\bigg), \tag{5.11.6}$$

$$det(J(n^*, E^*)) = \frac{(p - T)bq^2 E^* n^*}{(b + an^*)^3} > 0. \tag{5.11.7}$$

The $tr(J(n^*, E^*)) < 0$ shows that system (5.11.2) is locally asymptotically stable provided

$$T > T_3{}'; \quad T_3{}' = p - \bigg(\frac{ac}{q} + \frac{bc}{qK} + \frac{bc}{qK}\bigg(\frac{aK + b)}{qaK - b)}\bigg)\bigg) \tag{5.11.8}$$

The $tr(J(n^*, E^*)) = 0$ shows that system (5.11.2) exhibits the periodic solutions around (n^*, E^*) at the point $T = T^H = T_3{}'$.

5.12 Numerical Simulations

The dynamics of system is carried out using the software package MATCONT [29], [43]. Using the software package MATCONT, in the continuation of the interior equilibrium point, some bifurcation points of codimension-1 are detected w.r.t. the bifurcation parameter T and p for the following choice of data:

$$\begin{aligned} &K = 1, r = 1, a = 2, b = 1, q_1 = 1, q_2 = 1.2, c = 0.4, \\ &v_1{}^* = 0.5, v_2{}^* = 0.6, \alpha = 2, \beta = 3, \alpha_0 = 1, \beta_0 = 1. \end{aligned} \tag{5.12.1}$$

The table- 5.1 describes the behavior of the system (5.3.1) w.r.t. the parameters T and p. For this data set (5.12.1) with $T = 2$ and $p = 5$, the boundary equilibrium point $P_1 = (1, 0)$ is found to be unstable [see condition (5.6.1)]. The interior equilibrium point exists and it is computed as $P_2 = (0.37645557, 1.7605287)$. It is locally asymptotically stable according to the condition-(5.6.4). The phase portrait shown in Fig - 5.1. In the continuation of the interior equilibrium point P_2, some bifurcation diagrams of codimension-1 w.r.t. the parameters T and p are obtained in the figures 5.2(A) and 5.2(B), respectively. Moreover, the figure- 5.2(A) shows that level of fish stock increases with the increase of tax level and the figure- 5.2(B) shows that level of fish stock decreases with the increase of price level. The Hopf point is

Parameter-T (fix $p=5$)	Parameter- p (fix $T=2$)	Behavior
$T_2 = 3.090186$	$p = p^{tc} = 3.909814$	Transcritical bifurcation [see (5.6.2)]
$T_3 = 0.993663$	$p = p^H = 6.006337$	Hopf Point
$T > T_2$	$p < p^{tc}$	P_1 is L.A.S [see (5.6.1)]
$T_3 < T < T_2$	$p^{tc} < p < p^H$	P_2 is L.A.S [see (5.6.4)]
$T < T_3$	$p > p^H$	Periodic Solution around P_2

Table 5.1: Dynamical behavior of the system (5.3.1) w.r.t. the parameters T and p for the data set (5.12.1).

Parameter-T (fix $p=5$)	Parameter- p (fix $T=2$)	Behavior
$T_2' = 3.032787$	$p = {p_1}^{tc} = 3.967213$	Transcritical bifurcation [see (5.11.5)]
$T_3' = 1.065572$	$p = {p_1}^H = 5.934427$	Hopf Point
$T > T_2'$	$p < {p_1}^{tc}$	P_1 is L.A.S [see (5.11.4)]
$T_3' < T < T_2'$	${p_1}^{tc} < p < {p_1}^H$	P_2 is L.A.S [see (5.11.8)]
$T < T_3'$	$p > {p_1}^H$	Periodic Solution around P_2

Table 5.2: Dynamical behavior of the system (5.11.2) w.r.t. the parameters T and p for the data set (5.12.1) with $\alpha = 0$ and $\beta = 0$.

computed as $T = T_3$. The periodic solutions w.r.t. the Hopf point $T_3 = 0.993663$ are drawn in the figure-5.3 (B). Moreover, the solutions for $T < T_3$ are also shown in the figure-5.4. The analysis combined with numerical simulation complete the Table-5.1.

Keeping all parameters fixed of the data set (5.12.1), the value of $q = \dfrac{Q_0}{Q_{00}} = 0.61$ is calculated for the aggregated system (5.11.2) for the density independent migration rate of fishing vessels. This gives the interior equilibrium point $P_2 = (0.388808045, 1.779606416)$ which is locally asymptotically stable. The table- 5.2 describes the changes in the behavior of the system (5.11.2) (or the changes in the behavior of the system (5.3.1) when the migration rate of the fishing vessels becomes constant) w.r.t. the different values of parameters T and p.

The dynamics of the system (5.11.2) are shown in the figures 5.9-5.10. From the above these two Tables-5.1 and 5.2, the following conclusions can be obtained.

$$T_3 < T_3' < T_2' < T_2$$

Therefore, it can be concluded that the range of taxation (T_3', T_2') for the system (5.11.2) is subset of the (T_3, T_2). Accordingly, the range of taxation is enhanced due to the density dependent migration of fishing vessels.

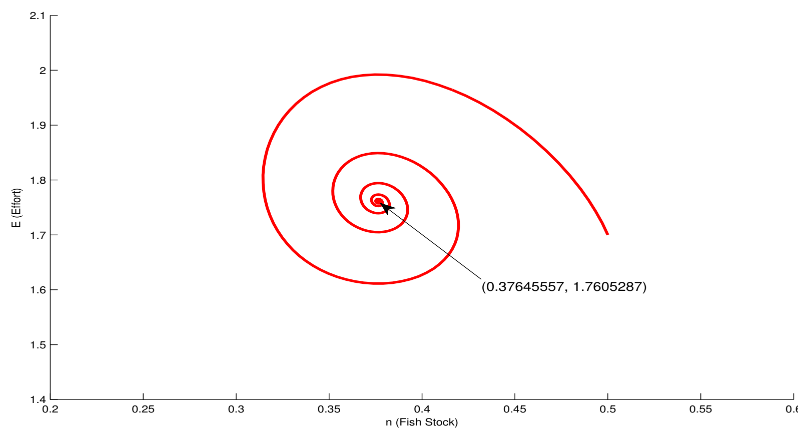

Figure 5.1: Phase portrait for the system (5.3.1) using data set (5.12.1).

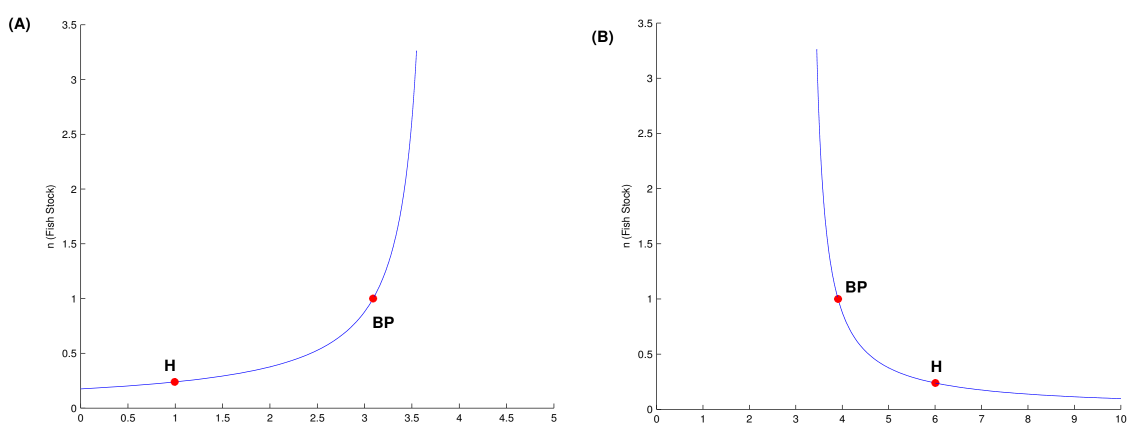

Figure 5.2: Bifurcation of co-dim-1 in Fig-(A). w.r.t. parameter T and Fig-(B). w.r.t. price (p) in the continuation of equilibrium point $(n^*, E^*) = (0.37645557, 1.7605287)$ for the system (5.3.1) using data set (5.12.1).

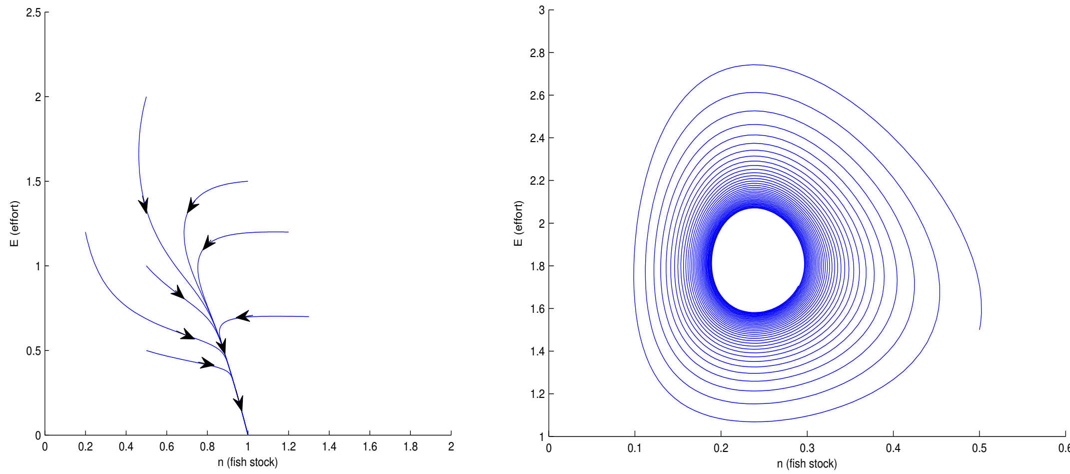

Figure 5.3: Phase portrait for (A). Fishery Free Equilibrium Point $(K, 0)$ for the different initial values shows that it is globally asymptotically stable for $T > T_2$ and (B). Stable limit cycle for Hopf point $T = 0.993663$ (or $p = 6.006337$), using data set (5.12.1) for the system (5.3.1).

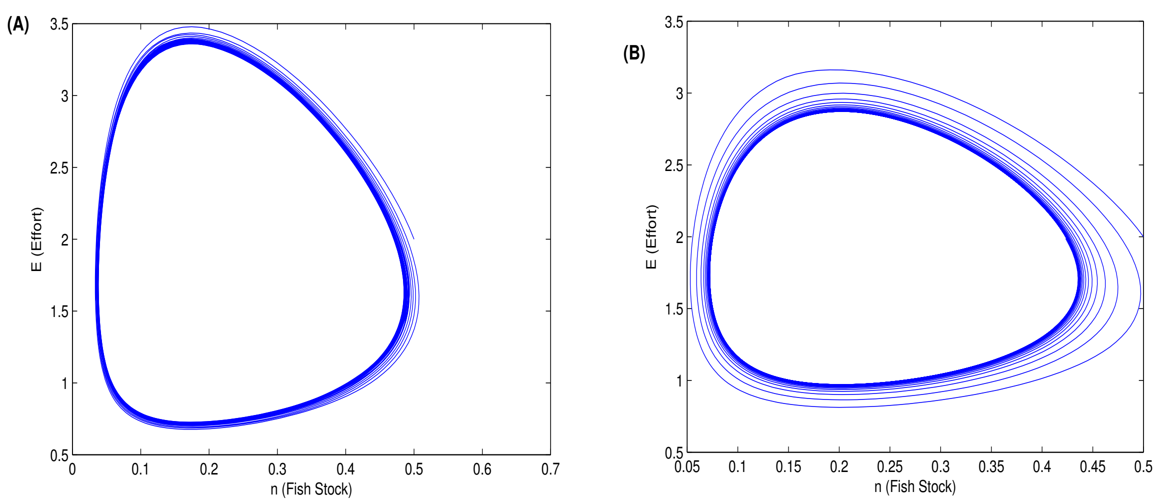

Figure 5.4: Periodic Solutions for the case $T < T_3$ i.e., in the figure-(A) T=0 and figure-(B) T=0.5 , for the system (5.3.1).

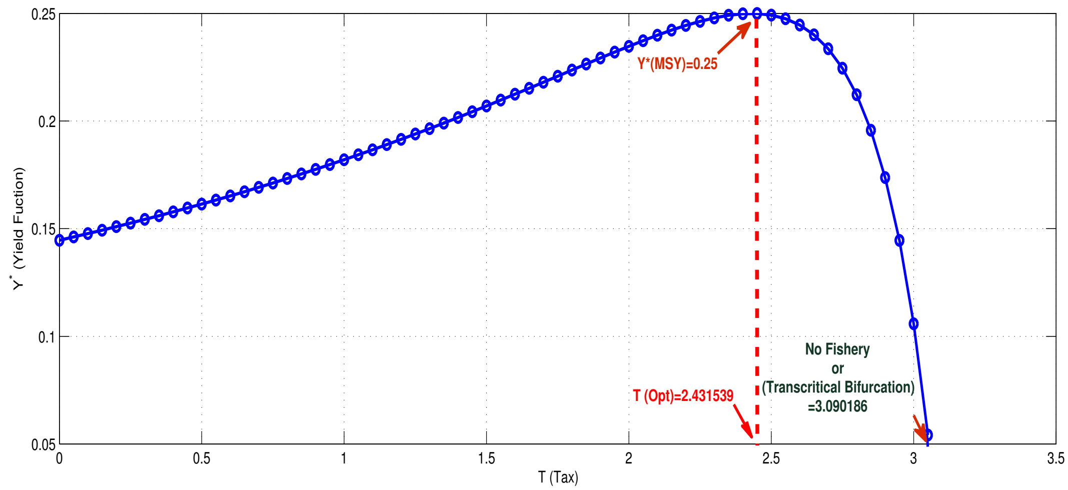

Figure 5.5: Effects of the taxation on Maximum sustainable yield for density independent case for the system (5.3.1) using data set (5.12.1).

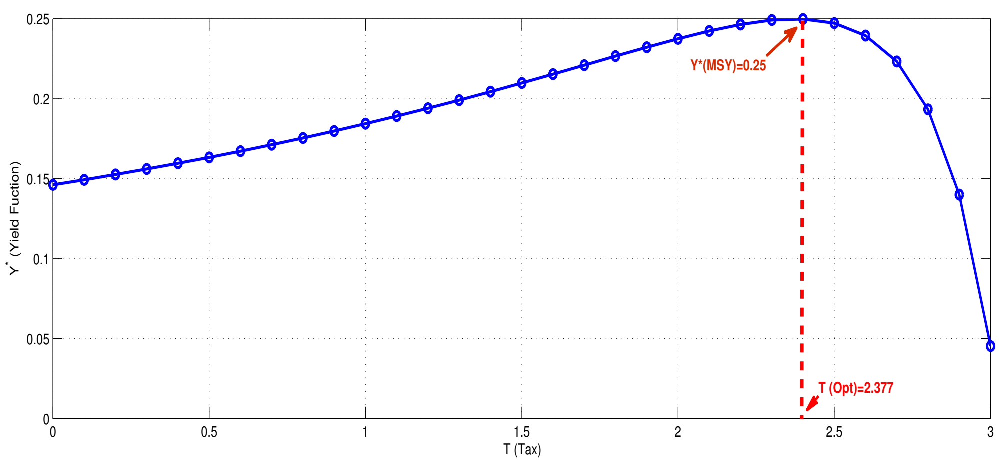

Figure 5.6: Effects of the taxation on Maximum sustainable yield for density independent case for the system- (5.11.2) using data set (5.12.1) with $\alpha = 0$ and $\beta = 0$.

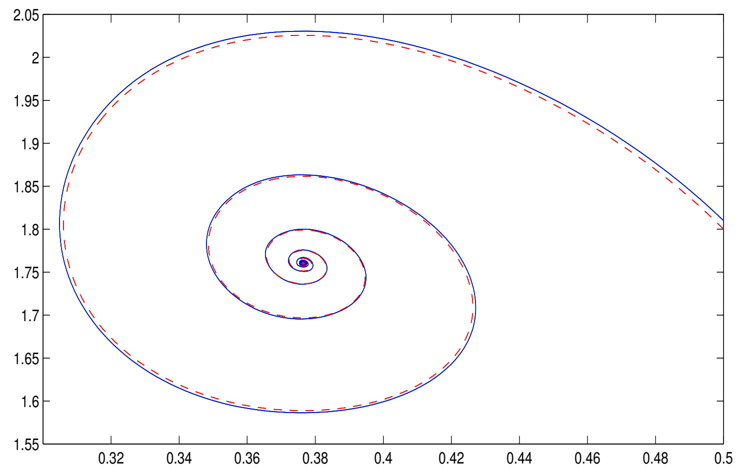

Figure 5.7: Solution curve for the aggregated system (5.3.1) with blue line and the complete system (5.2.1) with red dots are drawn for $\varepsilon = 0.05$.

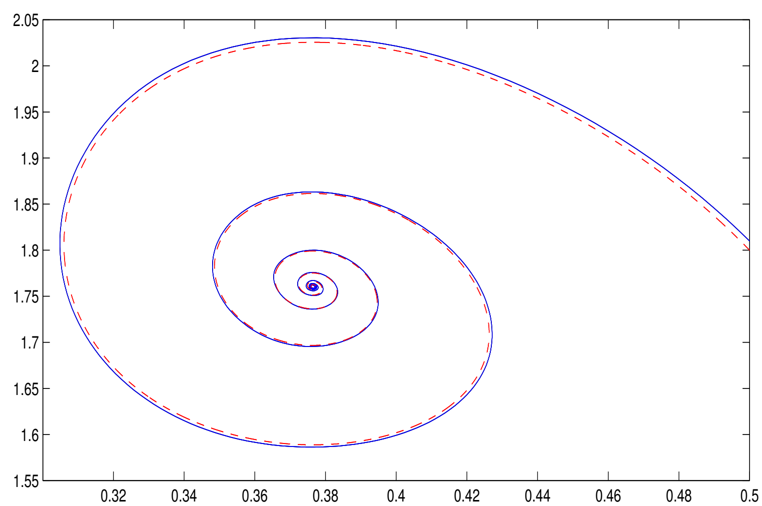

Figure 5.8: Solution curve for the aggregated system (5.3.1) with blue line and the complete system (5.2.1) with red dots are drawn for $\varepsilon = 0.001$.

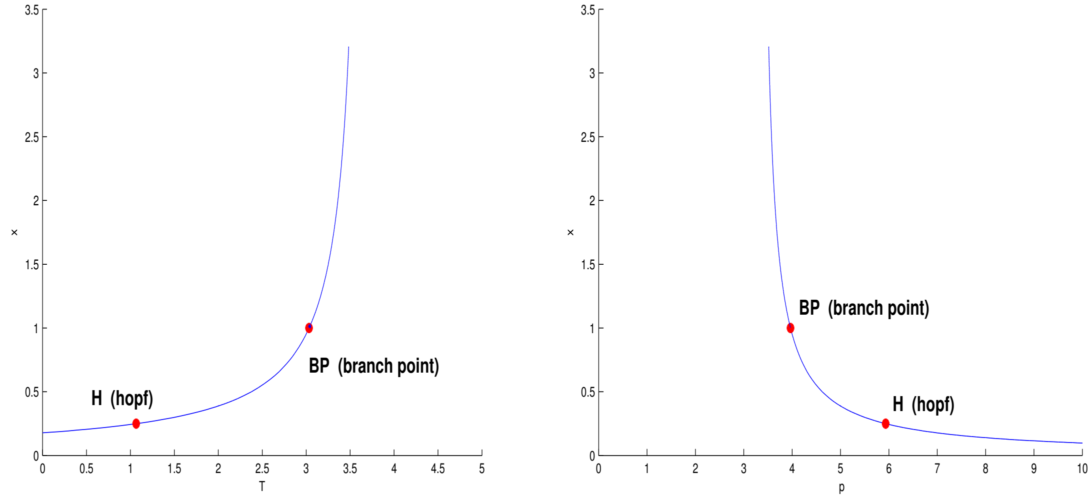

Figure 5.9: Bifurcation of co-dim-1 (A). w.r.t. parameter T (B). w.r.t. price (p) in the continuation of equilibrium point $(n^*, E^*) = (0.388808045, 1.779606416)$ for the system (5.11.2) using data set (5.12.1) with $\alpha = 0$ and $\beta = 0$.

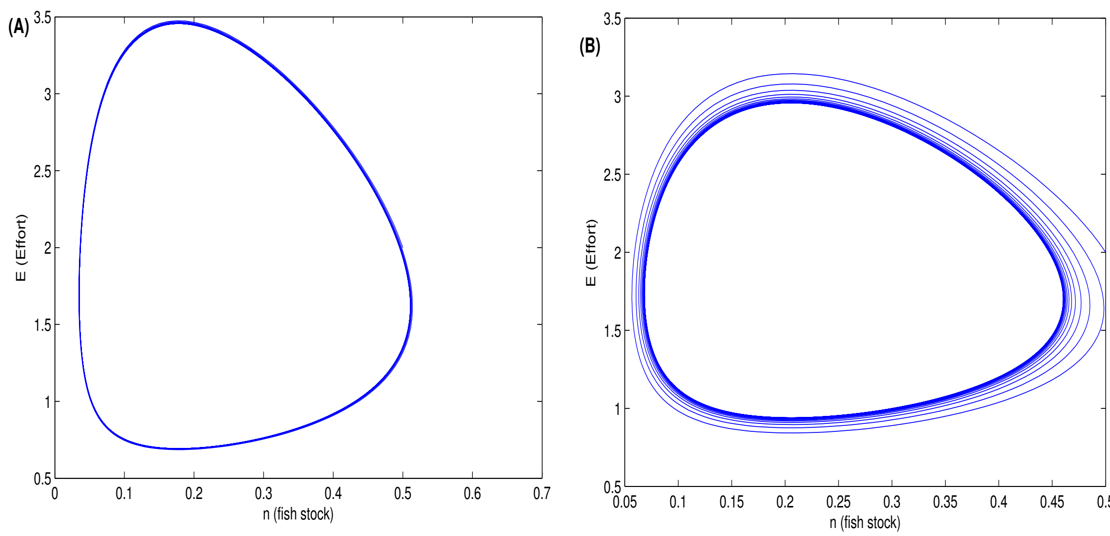

Figure 5.10: Periodic Solutions for the case $T < T_3{}'$ i.e., in the figure- (A) T=0 and figure-(B) T=0.5, for the system (5.11.2) using data set (5.12.1) with $\alpha = 0$ and $\beta = 0$.

5.13 Conclusion

In this chapter, a Stock–Effort dynamical system has been proposed and analyzed. This system is considered in two different fishing zones where fishing vessels move between two zones to increase their revenue. The migration rate of fishing vessels are assumed to be stock dependence. In this system, two different time scales are assumed, a fast one for movements of fish and boats between two zones and a slow one corresponding to fish population growth and fishery dynamics. The aggregation method is used to simplify the mathematical analysis of the complete model. The aggregated model is studied analytically and threshold conditions for existence and stability of various steady states are derived. Taxation policy can be used as an effective control instrument. System (5.3.1) exhibits several bifurcations. Existence of transcritical bifurcation indicates that there will closure of fishery for high taxes. However, the Maximum Sustainable Yield (MSY) and Optimal Taxation Policy is discussed for the aggregated model. Further, some special cases are also discussed including constant migration rate of fishing vessels between two zones. Some numerical results are also illustrated to verify analytical results.

Chapter 6

Predator–Prey model in a Heterogeneous Habitat with Prey Refuge in the presence of Toxicity

6.1 Introduction

Several species are extinct due to over predation, unregulated harvesting and pollution. To protect the species from extinction, several measures like imposing restriction on harvesting of species, creating natural reserves for species, establishment of protected/reserve areas, etc. have been suggested and implemented in literatures. Many mathematical studies of different ecological systems with these strategies have been incorporated. Most of the researchers [116], [64], [65], [127], [62] and[55] have shown that refugia can stabilize the predator-prey model. Kar [65] proposed a predator–prey model where refuge is considered for prey along with independent harvesting of either species. He showed that using the harvesting efforts as control instrument, it is possible to break the cyclic behavior to drive the system to a required state. Wang et al. [127] considered a prey-predator system where individuals from prey fish population could hide in holes and predators are unable to enter there. Ji and Wu [62] considered a predator-prey system using a prey refuge in a constant amount and a constant-rate prey harvesting. It is shown that the system can be controlled by using constant harvesting or constant prey refuge.

In the present chapter, two spatial patches are considered for prey population. The second patch provides refuge for prey reducing risk from predation as well as risk of being infected from the toxicant. The complete model includes the two time scales. One is the fast one and other is the slow one. The fast one corresponds to the migration of prey species between the patches and the slow one corresponds to the growth and interactions between them. The aggregation method is used to reduce the system. The reduced system gives us the approximation solution of the complete system. The aggregated model is discussed analytically as well as numerically.

6.2 Mathematical Model

Consider a prey–predator model in a water body with two different patches where patch-2 works as refuge for prey. Let $n_1(t)$ and $n_2(t)$ are density of prey in patch-1 and patch-2 at time t. Patch-1 is infected directly by some toxicant substances whereas predator also infected indirectly with toxicity (as predator use infected prey for their food). Two different kinds of processes are proposed at two different time scale. At the fast time scale, the displacement of prey between the two patches is considered in the model. Slow time scale includes the growth and the interaction between prey and predator. Therefore, taking t is the slow time scale and τ is the fast one, introduce the fast time scale $\tau = t/\varepsilon$ (ε is the small dimensional less parameter). Accordingly, the complete model is described by the following set of three equations:

$$\begin{aligned} \frac{dn_1}{d\tau} &= (kn_2 - \hat{k}n_1) + \varepsilon\left(r_1 n_1\left(1 - \frac{n_1}{K_1}\right) - a_1 n_1 p - \alpha {n_1}^3\right) \\ \frac{dn_2}{d\tau} &= (\hat{k}n_1 - kn_2) + \varepsilon\left(1 - \frac{n_2}{K_2}\right) \\ \frac{dp}{d\tau} &= \varepsilon(-dp + a_2 n_1 p - \beta p^2) \end{aligned} \tag{6.2.1}$$

$$n_1(0) > 0, \quad n_2(0) > 0, \quad p(0) > 0$$

The constant r_i represents the intrinsic growth rate and $K_i (i = 1, 2)$ as the carrying capacity of the prey in patch-i, respectively. The constants a_1 and a_2 are the interaction parameters between prey–predator. The constant d is the mortality of predator. These parameters are assumed positive. The parameter values k and $\hat{k}$ are the migration rates of prey from the patch-2 to patch-1 and patch-1 to patch-2.

The term αn_1^3 represents the infection of prey species by some external toxic substances, for example, industrial waste [27]. Since $\frac{d^2(\alpha n_1^3)}{dn_1^2} = 6\alpha n_1 > 0$. This shows that there is an accelerated growth in the toxic substances to the density of prey $n_1(t)$, as more and more of prey consume the infected food. Predator is also infected by toxicant indirectly with the term βp^2 $(0 < \beta < \alpha < 1)$. To study the fast dispersal model, neglect the slow part of the complete system (6.2.1). Therefore, the fast system for $\varepsilon = 0$ gives the following fast equilibrium points:

$$n_1^* = \frac{k}{k+\hat{k}} n = v_1^* n, \quad n_2^* = \frac{\hat{k}}{k+\hat{k}} n = v_2^* n$$

The constants v_1^* and v_2^* represent the proportion of the prey in each patch at the fast equilibria. Over the fast time scale the total prey n and predator p are constant. However, at slow time scale these populations are not constant.

6.3 Aggregated Model

Let $n = n_1(t) + n_2(t)$ be the aggregated variable. Adding first two equations of the system (6.2.1) and substituting the fast variables in the complete model (6.2.1) is reduced to the following set of differential equations:

$$\begin{aligned} \frac{dn}{dt} &= rn\left(1 - \frac{n}{K}\right) - \alpha_1 n^3 = n.F(n,p) \\ \frac{dp}{dt} &= -dp + bnp - \beta p^2 = p.G(n,p) \end{aligned} \tag{6.3.1}$$

$$n(0) > 0, p(t) > 0,$$

with

$$r = r_1 v_1^* + r_2 v_2^*, \quad \frac{r}{K} = \frac{r_1 v_1^{*2}}{K_1} + r_2 v_2^{*2} K_2, \quad a = a_1 v_1^*, \quad b = b_1 v_1^*, \quad \alpha_1 = \alpha v_1^{*3}.$$

The dynamics of the system (6.3.1) is an approximation of the dynamics of the global variables in the complete model (6.2.1).

6.4 Steady States and Stability Analysis

The possible steady states of the system (6.3.1) are given as below:

1. $P_0(0,0)$ is a trivial equilibrium point.

2. $P_1(\tilde{n},0)$ is the boundary equilibrium point in the absence of predator and it is obtained as follows:

$$(\tilde{n},0)=\left(\frac{-r+\sqrt{r^2+4\alpha_1K^2r}}{2\alpha_1K},0\right)$$

3. $P^*(n^*,p^*)$ is the unique interior equilibrium point and it is obtained as follows:

$$n^*=\frac{(-A+\sqrt{(A^2+4\alpha_1\beta K^2A)}}{2\alpha_1K};\quad A=(r+abK).$$

and

$$p^*=\frac{bn^*-d}{\beta}$$

The interior equilibrium point (n^*,p^*) is positive provided

$$n^*>\frac{d}{b} \tag{6.4.1}$$

6.5 Local Stability Analysis

The local stability conditions for the feasible equilibrium points of the system (6.3.1) are investigated using the nature of eigenvalues of the Jacobian matrix evaluated at the corresponding equilibrium points.

$$J(n,p)=\begin{bmatrix} n\left(-\dfrac{r}{K}-\alpha_1n\right)+F & -an \\ bp & (-\beta p)+G \end{bmatrix}$$

Theorem 6.5.1. *The equilibrium point $P_0(0,0)$ is always saddle with unstable manifold in n-direction and stable manifold in p-direction.*

Proof. For the equilibrium point $(0,0)$, the Jacobian matrix is

$$J_0(0,0) = \begin{bmatrix} r & 0 \\ 0 & -d \end{bmatrix}$$

The Eigen values of J_0 are: $r > 0$ and $-d < 0$. Hence, the trivial equilibrium point $(0,0)$ is always saddle point with unstable manifold in n-direction and stable manifold in p-direction. □

Theorem 6.5.2. *The axial equilibrium point $P_1(\tilde{n}, 0)$ is locally asymptotically stable provided*

$$\tilde{n} < \frac{d}{b} \tag{6.5.1}$$

Proof. For the equilibrium point $(\tilde{n}, 0)$ the Jacobian matrix is

$$J(\tilde{n},0) = \begin{bmatrix} \tilde{n}\Big(-\dfrac{r}{K} - \alpha_1\tilde{n}\Big) & -a\tilde{n} \\ 0 & -d + b\tilde{n} \end{bmatrix}$$

The equilibrium point $(\tilde{n}, 0)$ is locally asymptotically stable for the condition (6.5.1) If the condition (6.5.1) is violated then the point $(\tilde{n}, 0)$ becomes saddle. Therefore, there is a transcritical bifurcation around $(\tilde{n}, 0)$ for $\tilde{n} = \dfrac{d}{b}$ □

Theorem 6.5.3. *The equilibrium point (n^*, E^*) is always locally asymptotically stable.*

Proof. The Jacobian matrix about the interior equilibrium point (n^*, p^*) is given by

$$J(n^*,p^*) = \begin{bmatrix} n^*\Big(-\dfrac{r}{K} - \alpha_1 n^*\Big) & -an^* \\ bp^* & (-\beta p^*) \end{bmatrix}$$

The trace and determinant of the above matrix is given below:

$$trJ^*(n^*,p^*) = -\frac{rn}{K} - 2\alpha_1 n^{2^*} - \beta p^* < 0,$$

$$detJ^*(n^*,p^*) = \left(\frac{rn}{K} + 2\alpha_1 n^*\right)(\beta p^*) + abn^*p^* > 0.$$

Using Routh Hurwitz Criteria, this shows that the unique interior equilibrium point (n^*, p^*) is locally asymptotically stable. □

6.6 Global Stability

In this section, the global stability of the aggregated model (6.3.1) is discussed for a suitable Lyponouv function:

$$V(n,p) = \left(n - n^* - n^* \log\frac{n}{n^*}\right) + d_1\left(p - p^* - p^* \log\frac{p}{p^*}\right)$$

Where d_1 is a suitable constant to be determined, It can be seen that $V(n,p)$ is positive definite for all (n,p) and zero at equilibrium point(n^*, p^*). Differentiate the function $V(n,p)$ w.r.t. time t.

$$\frac{dV}{dt} = \frac{(n-n^*)}{n}\frac{dn}{dt} + d_1\frac{(p-p^*)}{p}\frac{dp}{dt}$$

The above equation will become as:

$$\frac{dV}{dt} = (n-n^*)\left(r(1-\frac{n}{K}) - ap - \alpha_1 n^2\right) + d_1(p-p^*)(-d + bn - \beta p)$$

Since the equilibrium point (n^*, p^*) satisfy the equations of the system (6.3.1). Therefore, choosing $d_1 = \frac{a}{b}$, the above equation will take the form:

$$\frac{dV}{dt} = -\left(\frac{n}{K} + \alpha_1(n+n^*)\right)(n-n^*)^2 - \frac{a}{b}(p-p^*)^2$$

This implies that

$$\frac{dV}{dt} < 0$$

This shows that $\frac{dV}{dt}$ is negative definite in some neighborhood of point (n^*, p^*).

Accordingly, the interior equilibrium point (n^*, p^*) is globally asymptotically stable.

6.7 Existence of Limit Cycles

Bendixon–Dulac Criteria is used to look for existence of limit cycle. Since the functions $F(n,p)$ and $G(n,p)$ in the system (6.3.1) are smooth in a simply connected region D of the first quadrant of the (n,p) plane. Consider a function as follows:

$$H(n,p) = \frac{1}{np}$$

This is also smooth in the region D.

$$B(n,p) = \frac{\partial(FH)}{\partial n} + \frac{(\partial(GH))}{\partial p}$$

This gives us

$$B(n,p) = -\left(\frac{np}{K} + \frac{(2\alpha_1 n)}{p} + \frac{\beta}{n}\right) < 0$$

It can be seen that the above expression will remain negative for all choice of positive parameters. This shows that there is no change in the sign of the above expression. Accordingly, there are no closed orbits (periodic solutions) lying entirely in the region-D.

6.8 Numerical Simulations

Consider the following data set for the suitable choice of parameters in appropriate units:

$$K = 100, r = 8, a = 2, b = 3, \alpha_1 = 1.5, \beta = 0.7, d = 0.0007 \tag{6.8.1}$$

For this data set, the boundary equilibrium point $(\tilde{n}, 0) = (2.282888, 0)$ is obtained. The unique interior equilibrium point for the initial condition $(1.5, 3)$ is computed as $(n^*, p^*) = (0.09333.9897)$ and it is locally asymptotically stable. The time series and phase portrait of the solution of the system (6.3.1) is shown in the figure-6.1. The global stability of the point (n^*, p^*) for the different initial values is shown in the figure-6.2. The figure-6.3 describes the solution curve for complete model (6.2.1) and the aggregated model (6.3.1) for $\varepsilon = 0.01$. This shows that the solution of aggregated model is the good approximation solution of the complete model for the very small value of $\varepsilon = 0.01$. There is transcritical bifurcation w.r.t.

the parameter d around the equilibrium point $(\tilde{n}, 0) = (2.282888, 0)$ at $d = 6.848665$ which is detected using the software MATCONT in the figure-6.4. The corresponding eigenvalues are -15.8174 and 0.

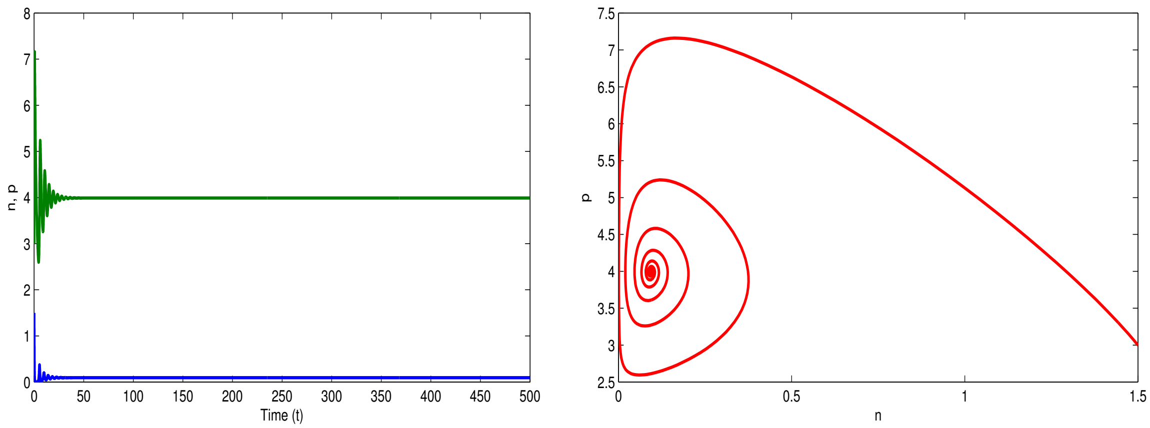

Figure 6.1: Time series and phase portrait of the system (6.3.1) for the given data set.

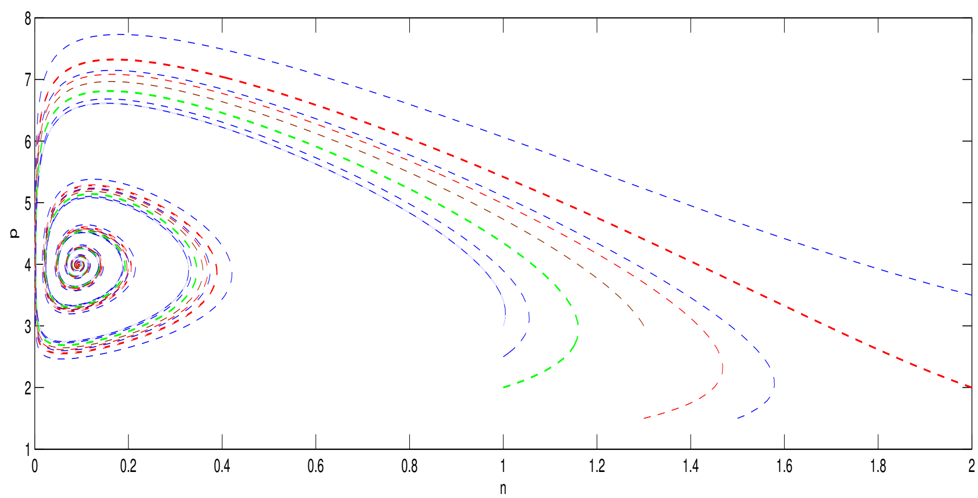

Figure 6.2: The solution curve for different initial values shows the global stability of (n^*, p^*) of aggregated system (6.3.1) using data set (6.8.1).

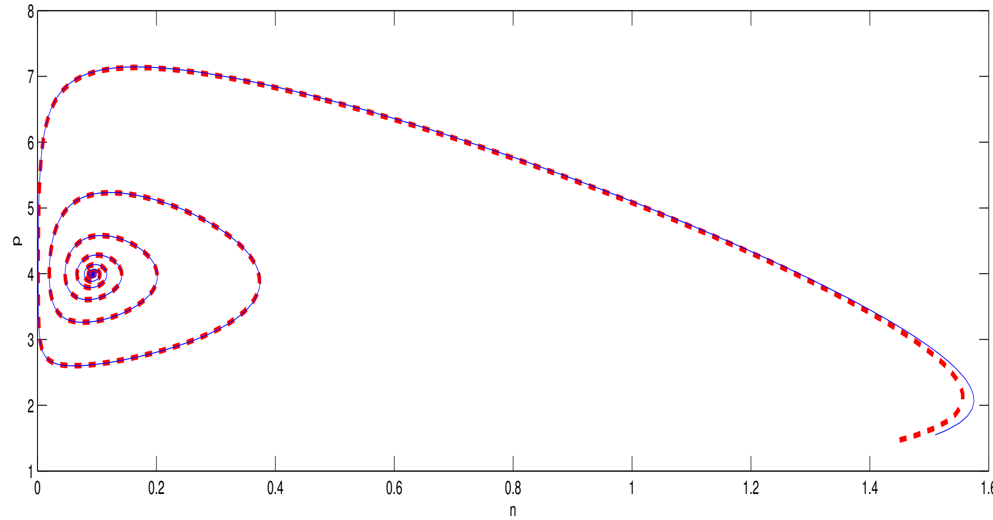

Figure 6.3: The solution curve for complete model (6.2.1) (with blue lines) and aggregated system (6.3.1) (with red dotted lines) for $\varepsilon = 0.001$.

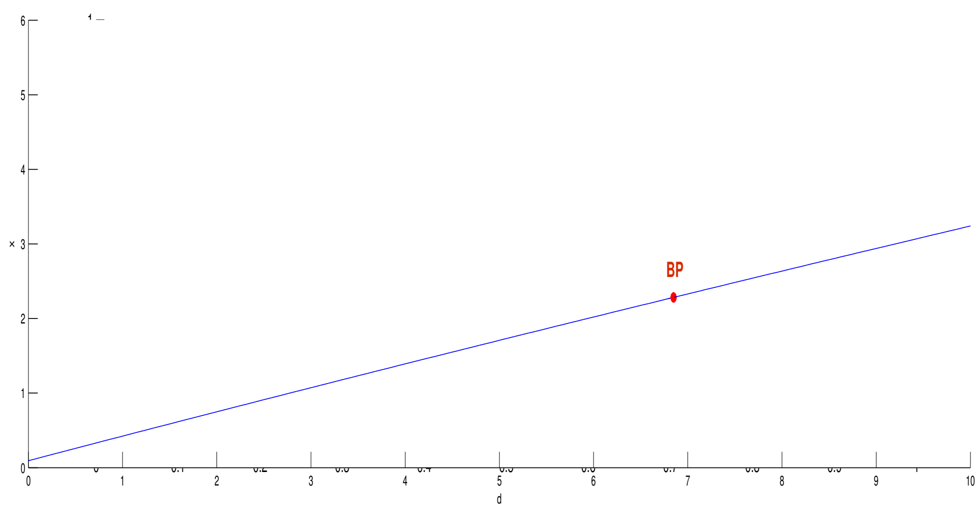

Figure 6.4: Bifurcation diagram of co-dimension-1 w.r.t. the bifurcation parameter d.

6.9 Conclusion

In this chapter, a spatial predator–prey mathematical model has been proposed and analyzed. The effect of toxicity is considered in the system. Using the Routh-Hurwitz criteria, it has been shown that the unique interior equilibrium point exists under certain condition and it is globally asymptotically stable. There does not exist any periodic solutions in the interior and it is also confirmed through Bendixon–Dulac Criteria. This system is based upon the two time scales: fast one for the movement of prey species between the patches and slow one corresponds to the growth of prey-predator and their interactions. Aggregation method is used for mathematical analysis. Numerically it is shown that for $\varepsilon = 0.01$ or $\varepsilon = 0.001$, the approximation made for the aggregating the complete model into reduced one is relevant. This means that the trajectories of aggregated model remain close to the trajectories of the complete model.

Chapter 7

A Dynamic Reaction Model in a Heterogeneous Habitat considering Prey Refuge and Alternate Food to Predator

7.1 Introduction

The present chapter deals with a dynamic reaction model in a heterogeneous patchy habitat. This is a predator–prey fishery system where only prey population is subjected to harvesting. The prey is migrating between patches in search of food or to take refuge. The predator is assumed to depend on some alternate food resource [113] to avoid extinction because of reduced availability of prey due to refuge and harvesting. The alternate food is not the preferred food and is available in abundance so the separate dynamics of alternate food is not considered in the model. A two-time scale model is developed. The reduced model based on aggregation method is analyzed using perturbation techniques and Center Manifold Theorem.

7.2 The Mathematical Model

Let the heterogenous habitat (water body) consists of two spatial patches or layers: the first layer is near the surface and contains food, the second layer corresponds to deeper water. Although, there is the scarcity of food in the deeper layer but provides refuge for prey reducing risk from predation. Let $n_i(t)$ be the density of the prey in layer $i(i = 1, 2)$ and $p(t)$ the density of the predator in layer-1 at time t. The dynamics of corresponding prey- predator system is described as follows:

$$\frac{dn_1}{dt} = r_1 n_1 - A a n_1 p - q_1 E n_1 + (k n_2 - \hat{k}(p) n_1) \tag{7.2.1}$$

$$\frac{dn_2}{dt} = -r_2 n_2 - q_2 E n_2 + (\hat{k}(p) n_1 - k n_2) \tag{7.2.2}$$

$$\frac{dp}{dt} = -\mu p + A b n_1 p + \beta(1 - A) p \tag{7.2.3}$$

The constant $r_1 > 0$ represents the intrinsic growth rate of the prey in layer- 1. The first term in second equation represents the mortality of prey in layer-2 due to non- availability of food in that layer. The constant μ is the natural mortality rate for predator. The predation rate is given by a for the prey in the layer–1 and the constant b is the food conversion rate by predator w.r.t. prey $n_1(t)$. The dependence of predator on alternate food resource is considered in predator dynamics by the term $\beta(1 - A)p$, $(0 < A < 1)$ [113]. If $A = 1$, the predator depends only on the prey species in layer-1. If $A = 0$, then the predator depends on the alternate food resources only.

The parameter k represents the migration rate of prey from layer-2 to layer-1 which is assumed to be constant. The main motivating factors for this movement of the prey are food and light. The migration rate of prey from layer-1 to layer-2 is assumed to be predator–density dependent [6]:

$$\hat{k}(p) = \begin{cases} \alpha p & for \quad p > 0 \\ 0 & for \quad p = 0. \end{cases}$$

The parameters c and p_0 are assumed to be the cost and price of per unit harvest. Therefore, the dynamics of fishing effort with proportional harvesting function is described as follows:

$$\frac{dE}{dt} = (-cE + p_0 q_1 E n_1 + p_0 q_2 E n_2) \tag{7.2.4}$$

It is observed that the migration of species between two layers take place at much faster time scale as compared to the growth of fish population, interaction between the prey predator and harvesting effort. Accordingly, the system includes two time scales; A fast time scale is associated with the movements of fish population between two layers and a slow one is associated with the growth of fish population, interaction between the prey predator and variations of the total numbers of vessels involved in fisheries. Therefore, taking t is the slow time scale and τ is fast one, introduce fast time scale $\tau = t/\varepsilon$ (ε is a small dimensionless parameter).

The complete system, combining the equations (7.2.1)-(7.2.4) with the set of four ordinary differential equations, is described as follows:

$$\begin{aligned}
\frac{dn_1}{d\tau} &= \varepsilon(r_1 n_1 - Aan_1 p - q_1 E n_1) + (kn_2 - \hat{k}(p)n_1) \\
\frac{dn_2}{d\tau} &= \varepsilon(-r_2 n_2 - q_2 E n_2) + (\hat{k}(p)n_1 - kn_2) \\
\frac{dp}{d\tau} &= \varepsilon(-\mu p + Abn_1 p + \beta(1-A)p) \\
\frac{dE}{d\tau} &= \varepsilon(-cE + p_0 q_1 E n_1 + p_0 q_2 E n_2)
\end{aligned} \tag{7.2.5}$$

$$n_1(0) > 0, \quad n_2(0) > 0, \quad p(0) > 0, \quad E(0) > 0.$$

7.2.1 The Fast System

To study the fast dispersal model, neglect the slow part of the system (7.2.5) by taking $\varepsilon << 1$:

$$\begin{aligned}
\frac{dn_1}{d\tau} &= (kn_2 - \hat{k}(p)n_1) \\
\frac{dn_2}{d\tau} &= (\hat{k}(p)n_1 - kn_2) \\
\frac{dp}{d\tau} &= 0 \\
\frac{dE}{d\tau} &= 0
\end{aligned} \tag{7.2.6}$$

The fast equilibria for the fast system (7.2.6) are obtained as given below:

$$\begin{aligned}
n_1^* &= \frac{k}{k+\hat{k}(p)} n = f(p)n = v_1^* n; \qquad f(p) = \frac{k}{k+\hat{k}(p)} \\
n_2^* &= \frac{\hat{k}(p)}{k+\hat{k}(p)} n = (1-f(p))n = v_2^* n
\end{aligned}$$

The equilibrium frequencies v_1^* and v_2^* will take the following form:

$$v_1^* = f(p) = \frac{n_1^*}{n} \quad and \quad v_2^* = (1 - f(p)) = \frac{n_2^*}{n} \tag{7.2.7}$$

From above expression (7.2.7), it is observed that the equilibrium densities (n_1^*, n_2^*) for the fast system are proportional to the total population. The equilibrium frequencies v_1^* and v_2^* represent the proportions of prey in each layer at the fast equilibrium. The sum of the frequencies for layers is always equal to one and sum of their derivatives is always equal to zero. It can be seen that these equilibrium frequencies are functions of the slow variable p which is assumed to be constant at fast time scale. For each set of values of slow variables n, p and E, the fast system approach to an equilibrium. This equilibrium is always different for each set of slow variables. The addition of first two equations of system (7.2.6) gives $n(t) = n_1(t) + n_2(t)$ as constant for the fast part of the complete system. Slow variables i.e., n, p and E are chosen as constant of motion for the fast part of the complete system. The aggregated model is discussed for the slow time scale in the next section.

7.3 The Aggregated Model

Applying aggregation method, the complete system (7.2.5) is reduced to a system of three ordinary differential equations. Let $n(t) = n_1(t) + n_2(t)$ be the aggregated variable. Further, introduce $r(p)$ and $q(p)$ as given below:

$$r(p) = \frac{r_1 k - r_2 \alpha p}{k + \alpha p} \quad and \quad q(p) = \frac{q_1 k + q_2 \alpha p}{k + \alpha p}$$

The following aggregated system is obtained by adding two prey equations and substituting the fast equilibrium in the complete model (7.2.5):

$$\begin{aligned} \frac{dn}{dt} &= n(r(p) - Aaf(p)p - q(p)E) = n \cdot F(p, E) \\ \frac{dp}{dt} &= p(-\mu + Abf(p)n + \beta(1 - A)) = p \cdot G(n, p) \\ \frac{dE}{dt} &= E(-c + p_0 q(p)n) = E \cdot H(n, p) \end{aligned} \tag{7.3.1}$$

$$n(0) > 0, \quad p(0) > 0, \quad E(0) > 0.$$

This aggregated model (7.3.1) includes new and different terms w.r.t. slow part of the complete model. This is because of the density dependence of equilibrium frequencies which leads to new terms in the reduced system (7.3.1). This process is called the functional emergence in the approximated aggregated system. In the next section, we will study the dynamical behavior of this aggregated system.

7.4 Equilibrium States of Aggregated System (7.3.1)

For the system (7.3.1), there exists four non-negative equilibrium points which are given below:

1. The trivial equilibrium point $P_0(0,0,0)$ always exists.

2. The predator free boundary equilibrium point $P_1(\hat{n},0,\hat{E}) = \left(\frac{c}{q_1}, 0, \frac{r_1}{p_0 q_1}\right)$ exists in positive nE-plane.

3. The boundary equilibrium point in positive np-plane is $P_2(\overline{n},\overline{p},0)$ such that $P_2(\overline{n},\overline{p},0) = \left(\frac{k\beta(A-A_0)}{Aak+\alpha r_2}, \frac{kr_1}{Aak+\alpha r_2}, 0\right)$; $\quad A_0 = 1-\frac{\mu}{\beta}$ and it is positive for
$$A > A_0. \tag{7.4.1}$$

4. The unique interior equilibrium point $P^*(n^*,p^*,E^*)$ of the system (7.3.1) exists, where n^*, p^* and E^* are given as below:
$$\begin{aligned} n^* &= \frac{c(k+\alpha p^*)}{p_0 q_1 k + p_0 q_2 \alpha p^*} \\ p^* &= \frac{(A_2 - A)}{p_0 q_2 \alpha (A - A_0)}; \quad A_2 = \frac{\beta-\mu}{\beta - \frac{bc}{p_0 q_1}} \\ E^* &= \frac{r(p^*) - Aaf(p^*)p^*}{q(p^*)} = \frac{r_1 k - (r_2\alpha + Aak)p^*}{q_1 k + q_2 \alpha p^*} \end{aligned}$$
It can be observed that
$$1 - \frac{\mu}{\beta} < \frac{\beta-\mu}{\beta - \frac{bc}{p_0 q_1}}$$
Accordingly, n^* and p^* are positive for
$$A_0 < A < A_2 \tag{7.4.2}$$

The value of E^* is positive for

$$p^* < \frac{r_1 k}{r_2\alpha + Aak} < \frac{r_1 k}{r_2\alpha} \quad \Rightarrow \quad \frac{[Abkc - p_0 q_1 k(\mu - \beta(1-A))]}{p_0 q_2 \alpha[\mu - \beta(1-A)]} < \frac{r_1 k}{r_2 \alpha}$$

which gives

$$A > \frac{\beta - \mu}{\beta - \dfrac{r_2 bc}{p_0(r_2 q_1 + r_1 q_2)}} (= A_1) \tag{7.4.3}$$

Since $A_0 < A_1$, the interior equilibrium point $P_3(n^*, p^*, E^*)$ is feasible for the the condition:

$$\frac{\beta - \mu}{\beta - \dfrac{r_2 bc}{p_0(r_2 q_1 + r_1 q_2)}} < A < \frac{\beta - \mu}{\beta - \dfrac{bc}{p_0 q_1}} \tag{7.4.4}$$

i.e.,

$$A_1 < A < A_2$$

It is noted that the dependence of predator on alternate food (A) is critical for existence of various equilibrium points. Therefore, the following cases can be investigated:

$$0 < A < A_0 \tag{7.4.5}$$

$$A_0 < A < A_1 \tag{7.4.6}$$

$$A_1 < A < A_2 \tag{7.4.7}$$

$$A_2 < A < 1 \tag{7.4.8}$$

It can be observed that the equilibrium points P_0 and P_1 may exist irrespective of A. However, the equilibrium point P_2 exists for the condition (7.4.6). There is coexistence of all the species for condition (7.4.7).The predator may or may not survive under the conditions (7.4.5) and (7.4.8). However, the predator can survive for (7.4.6) and (7.4.7). Accordingly, prey and predator population will not extinct for the cases (7.4.6) and (7.4.7). It can be observed that for the case (7.4.6), no harvesting of prey species is possible because of non-availability of sufficient amount of prey.

For the interior equilibrium point $P^*(n^*, p^*, E^*)$, the following can be easily obtained:

$$\begin{aligned}
\frac{dn^*}{dA} &= \frac{k(q_2 - q_1)(A_2 - A)}{q_2\big(p_0 k q_1 (A - A_0) + (A_2 - A)\big)^2} \\
\frac{dp^*}{dA} &= -\frac{(A_2 - A_0)}{p_0 \alpha q_2 (A - A_0)^2} \\
\frac{dp^*}{d\alpha} &= -\frac{(A_2 - A)}{p_0 \alpha^2 q_2 (A - A_0)} \\
\frac{dn^*}{dp^*} &= -\frac{c(q_2 - q_1)}{{p^2}_0 (q_1 k + q_2 \alpha p^*)^2} \\
\frac{dE^*}{dp^*} &= -Ap_0 \left[\frac{q_1 k r_2 \alpha + q_2 \alpha r_1 k}{(q_1 k + q_2 \alpha p^*)^2} + \alpha(\beta(1 - A) - \mu) \right]
\end{aligned}$$

Accordingly, the value of n^* increases monotonically w.r.t. A provided $q_2 > q_1$, otherwise it decreases. However, the value of p^* decreases monotonically w.r.t. A as well as w.r.t. the migration rate α. The value of E^* decreases with the increase of predator since $A > A_0$.

7.5 Stability Analysis of Feasible Equilibrium Points of Aggregated System

The local stability conditions for feasible equilibrium points of the system (7.3.1) are investigated by governing the nature of eigenvalues of the Jacobian matrix evaluated at the corresponding equilibrium points.

The Jacobian matrix of the system (7.3.1) at (n, p, E) is given by

$$J(n, p, E) = \begin{bmatrix} F & n(r'(p) - Aa(f'(p)p + f(p)) - q'(p)E) & -nq(p) \\ Abf(p)p & Abf'(p)pn + G & 0 \\ Ep_0 q(p) & Ep_0 q'(p)n & H \end{bmatrix}$$

Also,

$$f'(p) = \frac{-\alpha k}{(k + \alpha p)^2} < 0, \quad r'(p) = \frac{-(k + \alpha r_1 k)}{(k + \alpha p)^2} < 0 \quad and$$

$$q'(p) = \frac{k\alpha(q_2 - q_1)}{(k + \alpha p)^2} > 0 \quad provided \quad q_2 > q_1$$

Theorem 7.5.1. *The origin equilibrium state $P_0(0,0,0)$ is always a saddle point with unstable manifold in n-direction and stable manifold in E-direction. It has a stable manifold in p-direction provided*

$$A > 1 - \frac{\mu}{\beta} = A_0 \tag{7.5.1}$$

Proof. The jacobian matrix evaluated at $(0,0,0)$ is given by

$$J_0(0,0,0) = \begin{bmatrix} r_1 & 0 & 0 \\ 0 & -\mu + \beta(1-A) & 0 \\ 0 & 0 & -c \end{bmatrix}$$

The eigenvalues of J_0 are: $\lambda_1 = r_1 > 0$, $\lambda_2 = -\mu + \beta(1-A)$ and $\lambda_3 = -c < 0$. Hence, the origin $(0,0,0)$ is always a saddle point with unstable manifold in n-direction and stable manifold in E-direction. Further, it has a stable manifold in p-direction for the condition (7.5.1).

□

Remark 7.5.2. *If the condition* (7.5.1) *is violated, then P_0 has unstable manifold in p-direction. Accordingly, the trajectories along $p = 0$, starting in neighborhood of P_0 may be attracted to P_1 when the condition* (7.5.1) *is violated.*

Theorem 7.5.3. *There always exists periodic solutions around the boundary equilibrium point $P_1(\hat{n}, 0, \hat{E})$ in nE-plane.*

Proof. The Jacobian matrix evaluated at $(\hat{n}, 0, \hat{E})$ is given by

$$J_1(\hat{n},0,\hat{E}) = \begin{bmatrix} 0 & \hat{n}\left(r_1 - Aa - \dfrac{\hat{E}k\alpha(q_2-q_1)}{k^2}\right) & -\hat{n}q_1 \\ 0 & -\mu + Ab\hat{n} + \beta(1-A) & 0 \\ \hat{E}p_0q_1 & \dfrac{\hat{E}p_0k\hat{n}\alpha(q_2-q_1)}{k^2} & 0 \end{bmatrix}$$

The characteristic equation of the above matrix $P_1(\hat{n}, 0, \hat{E})$ is given by

$$\lambda^3 + B_1\lambda^2 + B_2\lambda + B_3 = 0 \tag{7.5.2}$$

with

$$\begin{aligned} B_1 &= -a_{22} \\ B_2 &= -a_{13}a_{31} \\ B_3 &= a_{13}a_{31}a_{22} \end{aligned}$$

and

$$B_1B_2 - B_3 \;=\; a_{22}a_{13}a_{31} - a_{13}a_{31}a_{22} = 0 \tag{7.5.3}$$

Since $B_1B_2 = B_3$, the equation (7.5.2) may be rewritten as

$$(\lambda^2 + B_2)(\lambda + B_1) = 0$$

This gives

$$\lambda_1 = -B_1 < 0 \quad and \quad \lambda_{2,3} = \pm i\sqrt{B_2}$$

This implies that there exists periodic solutions around the equilibrium point $P_1(\hat{n}, 0, \hat{E})$ in nE-plane.

□

Theorem 7.5.4. *The planar equilibrium state $P_2(\overline{n}, \overline{p}, 0)$ is locally asymptotically stable for the condition*

$$\overline{n} < \frac{c}{p_0 q(\overline{p})} \tag{7.5.4}$$

Proof. The Jacobian matrix evaluated at the point $(\overline{n}, \overline{p}, 0)$ is given by

$$J_2(\overline{n}, \overline{p}, 0) = \begin{bmatrix} 0 & \overline{n}(r'(\overline{p}) - Aa(f'(\overline{p})\overline{p} + f(\overline{p}))) & -\overline{n}q(\overline{p}) \\ Abf(\overline{p})\overline{p} & Abf'(\overline{p})\overline{p}\overline{n} & 0 \\ 0 & 0 & -c + p_0\overline{n}q(\overline{p}) \end{bmatrix}$$

The characteristic equation associated to the matrix $J_2(\overline{n}, \overline{p}, 0)$ is given by

$$\Big(\lambda - (-c + p_0\overline{n}q(\overline{p}))\Big)\Big(\lambda^2 - (Ab\overline{p}f'(\overline{p})\overline{n})\lambda + Ab\overline{p}f(\overline{p})\overline{n}\Big(r'(\overline{p}) - Aa(f'(\overline{p})\overline{p} + f(\overline{p}))\Big)\Big) = 0$$

One eigen value is $\lambda_1 = -c + p_0\overline{n}q(\overline{p})$ and other two eigen values can be obtained from the following characteristic equation

$$\Big(\lambda^2 - (Ab\overline{p}f'(\overline{p})\overline{n})\lambda + Ab\overline{p}f(\overline{p})\overline{n}\Big(r'(\overline{p}) - Aa(f'(\overline{p})\overline{p} + f(\overline{p}))\Big) = 0 \tag{7.5.5}$$

The $tr(J_2)$ and $det(J_2)$ of the characteristic equation are computed as below:

$$\begin{aligned} tr(J_2) &= Ab\overline{p}f'(\overline{p})\overline{n} = -\frac{Aa\alpha k^2((\mu - \beta(1-A)))}{(Aak + \alpha r_2)^2(k + \alpha\overline{p})^2} \\ det(J_2) &= Ab\overline{p}f(\overline{p})\overline{n}\Big(r'(\overline{p}) - Aa(f'(\overline{p})\overline{p} + f(\overline{p}))\Big) \\ &= \frac{Aa\alpha k^3((\mu - \beta(1-A)))}{(Aak + \alpha r_2)^2(k + \alpha\overline{p})^2}\left[\frac{k + \alpha r_1 k}{k + \alpha\overline{p}} + \frac{Aak^2 r\alpha}{(k + \alpha\overline{p})(Aak + \alpha r_2)} + Aak\right] \end{aligned}$$

Due to existence condition for the point P_2, $tr(J_2) < 0$ and $det(J_2) > 0$. Accordingly, the point $P_2(\overline{n}, \overline{p}, 0)$ is locally asymptotically stable for $\lambda_1 < 0$ which gives the condition (7.5.4).

□

Remark 7.5.5. *The point $P_2(\overline{n}, \overline{p}, 0)$ is saddle for*

$$\overline{n} > \frac{c}{p_0 q(\overline{p})} \tag{7.5.6}$$

The bifurcation will occur around $P_2(\overline{n}, \overline{p}, 0)$ when

$$\overline{n} = \frac{c}{p_0 q(\overline{p})} \tag{7.5.7}$$

Theorem 7.5.6. *The interior equilibrium state $P^*(x^*, y^*, E^*)$ is locally asymptotically stable for the following sufficient condition:*

$$k\alpha^2 Ab(r_1 + r_2)p^* + A^2 abk^2 \alpha p^* + (q_2 - q_1)\alpha(k^2 Abp^* - p_0 E^*(q_1 k + q_2 \alpha p^*)) > 0 \tag{7.5.8}$$

Proof. The Jacobian matrix evaluated at the point (n^*, p^*, E^*) is given by

$$J^*(n^*, p^*, E^*) = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$J^*(n^*, p^*, E^*) = \begin{bmatrix} 0 & n^*(r'(p^*) - Aa(f'(p^*)p^* + f(p^*)) - q'(p^*)E^*) & -n^*q(p^*) \\ Abf(p^*)p^* & Abf'(p^*)p^*n^* & 0 \\ E^*p_0q(p^*) & E^*p_0q'(p^*)n^* & 0 \end{bmatrix}$$

The characteristics equation of the jacobian matrix J^* about $P^*(n^*, p^*, E^*)$ is given by

$$\lambda^3 + B_1\lambda^2 + B_2\lambda + B_3 = 0 \tag{7.5.9}$$

with

$$\begin{aligned} B_1 &= -a_{22} = -Abf'(p^*)p^*n^* = \frac{Abkp^*n^*}{(k + \alpha p^*)^2} > 0 \\ B_2 &= -a_{13}a_{31} - a_{12}a_{21} = -(-ve)(+ve) - (-ve)(+ve) > 0 \\ B_3 &= a_{13}a_{31}a_{22} - a_{13}a_{31}a_{21} = Abp_0 n^{*2} p^* E^* q(p^*)\left[\frac{\alpha k(q_2 - q_1)}{(k + \alpha p^*)^3} + \frac{\alpha k(q_1 k + q_2 \alpha p^*)}{(k + \alpha p^*)^2}\right] > 0 \end{aligned}$$

Also,

$$\begin{aligned} B_1B_2 - B_3 &= a_{12}a_{21}a_{22} + a_{13}a_{31}a_{21} \\ &= \frac{Abkn^{*2}p^*}{(k+\alpha p^*)^4}\Big[k\alpha^2 Ab(r_1+r_2)p^* + A^2abk^2\alpha p^* + (q_2-q_1)\alpha(k^2Abp^* \\ &\quad -p_0E^*(q_1k+q_2\alpha p^*))\Big] > 0 \end{aligned} \tag{7.5.10}$$

Applying Rowth-Harwitz Criteria, the interior equilibrium point (n^*, p^*, E^*) is locally asymptotically stable iff the condition (7.5.10) is satisfied. □

Remark 7.5.7. *If the condition* (7.5.10) *is violated, the point* (n^*, p^*, E^*) *can be unstable. It may be noted that the condition* (7.5.10) *is always satisfied when* $q_1 = q_2$*. Accordingly, the interior point is always locally asymptotically stable in this case. In general, it is difficult to analyze the condition* (7.5.10) *to determine the stability. The stability of the equilibrium point* (n^*, p^*, E^*) *is discussed for the particular choice of data [see in the Section-7.6]. The numerical section shows that for a given data set, the system exhibits complex and chaotic dynamics.*

Existence of Hopf Bifurcation

The possibility of Hopf bifurcation is investigated w.r.t. to the parameter A around the equilibrium point (n^*, p^*, E^*).

Theorem 7.5.8. *The necessary and sufficient conditions for the occurrence of Hopf bifurcation from the interior equilibrium point* (n^*, p^*, E^*) *are that there exists a* $A = A^H$ *such that:*

$$(i) \quad B_1(A^H) > 0, \quad B_3(A^H) > 0$$

$$(ii) \quad B_1(A^H)B_2(A^H) - B_3(A^H) = 0$$

$$(iii) \quad Re\left[\frac{d\lambda_j}{dA}\right]_{A=A_H} \neq 0 \quad for \quad j = 1, 2.$$

Proof. The Hopf bifurcation occurs at $A = A^H$ and at this point $B_1B_2 = B_3$. Using this, the characteristic equation (7.5.9) will reduce to the following form:

$$(\lambda + B_1)(\lambda^2 + B_2) = 0 \tag{7.5.11}$$

This gives the following three roots:

$$\lambda_1 = -B_1 \quad and \quad \lambda_{2,3} = \pm i\sqrt{B_2} \tag{7.5.12}$$

Transversality Condition: Let the characteristic equation be such that it contains a pair of purely imaginary roots and one real root. Therefore, to show the transversality condition substitute $\lambda = u(A) + iv(A)$ in equation (7.5.9) and separating the real and imaginary part, the following can be obtained:

$$u^3(A) - 3u(A)v^2(A) + B_1(A)(u^2(A) - v^2(A)) + B_2(A)u(A) + B_3(A) = 0 \tag{7.5.13}$$

$$3u^2(A)v(A) - v^3(A) + 2B_1(A)u(A)v(A) + B_2(A)v(A) = 0 \tag{7.5.14}$$

Now differentiating (7.5.13) and (7.5.14) w.r.t. A, the following is obtained:

$$\begin{aligned} F_1(A)u^{'}(A) - F_2(A)v^{'}(A) + F_3(A) = 0, \\ F_2(A)u^{'}(A) + F_1(A)v^{'}(A) + F_4(A) = 0, \end{aligned} \tag{7.5.15}$$

where

$$\begin{aligned} F_1(A) &= 3(u^2(A) - v^2(A)) + 2B_1(A)u(A) + B_2(A) \\ F_2(A) &= 6u(A)v(A) + 2B_1(A)v(A) \\ F_3(A) &= B_3^{'}(A) + B_1^{'}(A)(u^2(A) - v^2(A)) + u(A)B_2^{'}(A) \\ F_4(A) &= 2u(A)v(A)B_1^{'}(A) + v(A)B_2^{'}(A) \end{aligned}$$

$$u^{'} = Re\left[\frac{d\lambda_j}{dA}\right]_{A=A_H} = -\frac{F_1(A)F_3(A) + F_2(A)F_4(A)}{F_1^2(A) + F_2^2(A)} \neq 0 \quad for \quad j = 2, 3.$$

and

$$\lambda_1(A_H) = -B_1(A_H) \neq 0.$$

Therefore, the transversality condition for Hopf bifurcation holds. Accordingly, the Hopf bifurcation occurs at $A = A_H$ around the point (n^*, p^*, E^*). □

It can be concluded that effects of harvesting effort (E) on the dynamics of a predator-prey system with a suitable amount of alternate food resource to predator play very important role. Different Dynamics of the system can be observed using various effects of harvesting effort. The predator- prey system with effort dynamics is considered by the set of differential equations (7.3.1). The analysis of this system concludes that the system can be unstable for $q_2 > q_1$. Thus, consideration of effort dynamics may destabilize the system. This is shown numerically for a choice of data in section 7.6. The dynamics about the interior state is complex and chaotic. This is confirmed by bifurcation diagrams and Lypunouv exponents. It can be observed that for the catch-ability coefficients ($q_1 = q_2$), the interior equilibrium point of this system is always LAS. However, using $E = 0$ and $E = constant$ in the system (7.3.1), it can be easily derived that the predator- prey system is stable. Accordingly, the harvesting effort have a very effective role in the dynamics of a this predator-prey system.

7.6 Numerical Simulations

Consider the following choice of hypothetical data in appropriate units:

$$\begin{aligned} k = 0.5, r_1 = 3, r_2 = 1, a = 2, \alpha = 2.5, q_1 = 1, \\ q_2 = 1.5, \mu = 2, b = 4, \beta = 3, c = 4, p_0 = 10 \end{aligned} \tag{7.6.1}$$

The critical values of A are computed as below:

$$A_0 = 0.333, \quad A_1 = 0.37, \quad A_2 = 0.714$$

For the aggregated system (7.3.1) the boundary equilibrium $P_1 = (4, \quad 0, \quad 0.3)$ is obtained. Choosing $A = 0.35$ $(A > A_0)$, another boundary equilibrium point $P_2 = (0.1297, \quad 0.5263, \quad 0)$ exists. Next, for $A = 0.38$ $(A > A_1)$, the interior equilibrium point $P^* = (0.29967804, \quad 0.43207403, \quad 0.029233241)$ exists.

The dynamics of the system is further explored using software package MATCONT [43], [29]. This package MATCONT is a collection of numerical algorithms implemented as a MATLAB toolbox for the detection, continuation and identification of limit cycles (periodic orbits). In the continuation of the interior equilibrium point P^*, some bifurcation points of codimension-1 are detected in the fig -7.1 w.r.t. the bifurcation parameter A. There is a branch point- $BP1$ (transcritical bifurcation) at $A = A^{tc} = 0.373911$ around the equilibrium point $P_2 \approx$

$(0.293797, \quad 0.521937, \quad 0)$. One Hopf point is obtained at $A = A_H = 0.388133$ in the interior $\mathbb{R}^3_+$ around equilibrium point $P^* \approx (0.301964, \quad 0.370331, \quad 0.227896)$. For this Hopf point, the corresponding first Lyapunov coefficient is $(-2.912374e - 002) < 0$, indicating a supercritical Hopf bifurcation. Thus, there should exist a stable limit cycle, bifurcating from the equilibrium. Another branch point $BP2$ occurs at $A = A^{tc} = 0.0.714286$ around the equilibrium point $P_1 \approx (0.4, \quad 0, \quad 3)$. Since, the fig- 7.1 shows that as the value of A increases close to the branch point- $BP2$ around $A = 0.714$, the solutions in the interior $\mathbb{R}^3_+$ vanishes and it will appear in nE-plane.

The bifurcation diagram w.r.t. parameter A for the system (7.3.1) is drawn in the fig- 7.2 for A in the interval $(0.37, \quad 0.714)$. The complex dynamics is evident from this diagram. To confirm the complexity and chaotic dynamics of system (7.3.1), the dynamics of Lyapunov exponent with their dimension are computed in the figures-7.3 and 7.4. The Lyapunov exponents for $A = 0.39$ and $A = 0.4$ are computed in the fig- 7.3 (A) and 7.3 (B) with their dimensions $D_L = 2.4305$ and $D_L = 2.0952$, respectively. The figure-7.4 shows the Lyapunov exponents for $A = 0.47$ and $A = 0.6$ and the dimensions are computed as $D_L = 2.25545$ and $D_L = 2.4157$, respectively. The presence of positive Lyapunov exponent confirms the complex dynamical behavior in the system for certain values of parameter A.

Different dynamic behaviors observed in fig -7.1 are confirmed by drawing phase portraits w.r.t. different values of A in figures 7.5-7.8. In particular, Fig 7.5 (A) shows local asymptotic stability of boundary equilibrium state $P_2(\overline{n}, \overline{p}, 0)$ for $A = 0.368$. Due to availability of some alternate food, the predator survives and it also takes food from prey. However, harvesting effort diminishes to zero. Increasing A beyond $BP1$, at $A = 0.374$ the interior point P^* is locally asymptotically stable [See fig 7.5 (B)]. Considering $A > A_H$, the system destabilizes and strange attractors are obtained in the figures- 7.6 and 7.7. It can be observed that as the value of A increases close to the branch point- $BP2$ around $A = 0.714$, the chaotic solution in the interior $\mathbb{R}^3_+$ vanishes and the periodic solutions will appear in nE-plane. Analytically result (7.5.3) in the stability of the $(\hat{n}, 0, \hat{E})$ also confirms the the periodic solutions in nE-plane. The change in behavior of solution can be seen in the figure-7.8.

Further, the bifurcations of co-dimension-1 are also explored w.r.t. mortality rate of predator μ in the fig 7.9. The Branch point $BP1$ at $\mu = 1.983924$ around $P_2 \approx (0.293843, \quad 0.520833, \quad 0)$ is computed. The Hopf points $H1$ and $H2$ are detected where Hopf point $H1$ at $\mu = 2.019724$ around $P^* \approx (0.301694, 0.374208, 0.221869)$ and Hopf point $H2$ at $\mu = 2.468001$ around $P_1 \approx (0.4, 0, 3.000001)$ are obtained. For these Hopf points, the corresponding first Lyapunov coefficients are computed as $(-2.964738e - 002) < 0$ and $8.841215e - 004 > 0$, respectively. Accordingly, $H1$ is supercritical and $H2$ is subcritical Hopf bifurcation points.

Next, the numerical simulations of the system (7.3.1) is discussed when the migration rate is not predator density dependent (i.e., $\hat{k}(p)$=constant). In this case, the functions $r(p)$, $q(p)$ and $f(p)$ will assume constant values. Let these values are chosen as:

$$r = 5, q = 2, v_1 = 0.4 \tag{7.6.2}$$

The dynamical behavior of this system using data set (7.6.1) and (7.6.2) is illustrated in the figures- 7.10 to 7.12. These figures shows the complex dynamics even with constant migration rates. This complexity occurs due to effort dynamics. The fig-7.13 considers the case for $A = 0$ i.e., when predators have no interaction with prey and it depends only on alternative food. This shows that predator can survive in the absence of prey because of availability of alternate food resource. Also, for the case $A = 1$ i.e., when predators have no dependence on alternate food and predators have interaction with prey species only. Due to refuge and harvesting of prey, there is scarcity of food (prey) for predators. Therefore, there can be extinction of predators and it can be seen in the fig- 7.14.

The different bifurcation diagrams for this system are drawn in the figure- 7.15 w.r.t. the bifurcation parameter A in the interval $(0.3357, \quad 0.35)$ for the different values of catch-ability coefficient q. In the figures 7.15 (A) to 7.15 (D), different kind of complexity can se observed for different values of q.

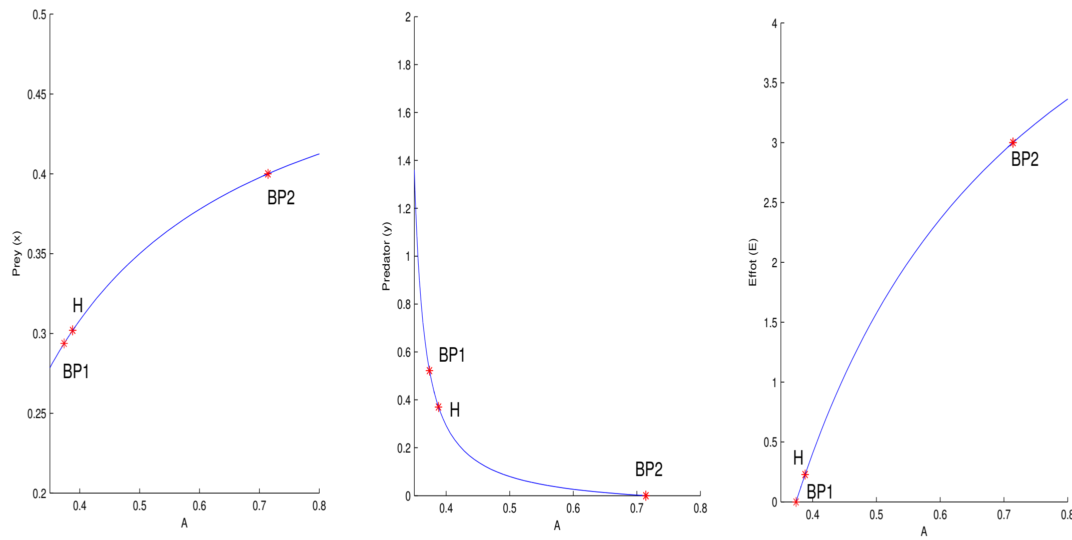

Figure 7.1: One parametric bifurcation diagram for the system (7.3.1) w.r.t. A for the given data set (7.6.1).

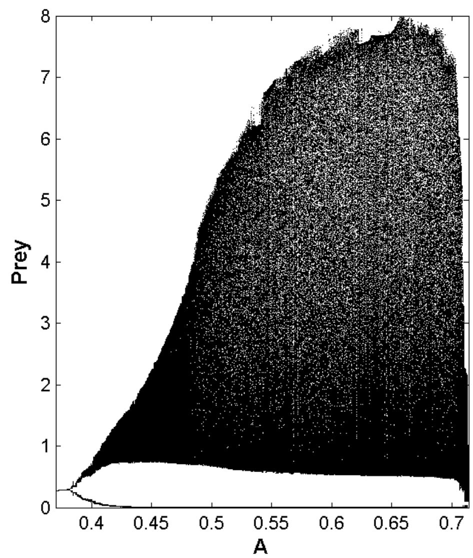

Figure 7.2: Bifurcation diagram w.r.t. A for $A \in (0.37, 0.714)$.

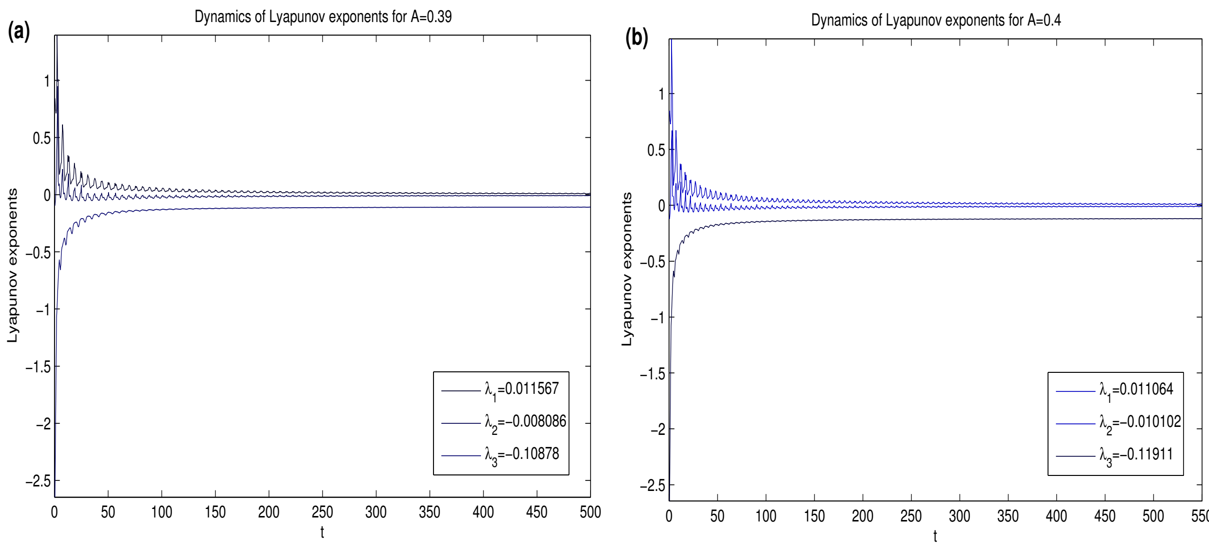

Figure 7.3: Dynamics of Lyponouv exponent for the aggregated model (7.3.1) for (a). $A = 0.39$ and (b). $A = 0.4$.

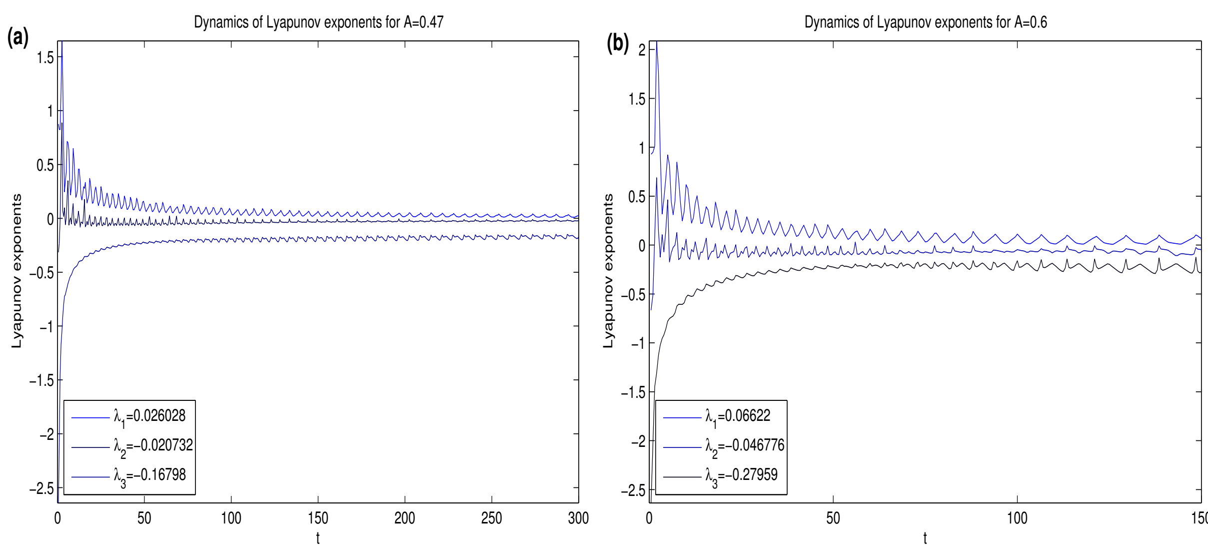

Figure 7.4: Dynamics of Lyponouv exponent for the aggregated model (7.3.1) for (a). $A = 0.47$ and (b). $A = 0.6$.

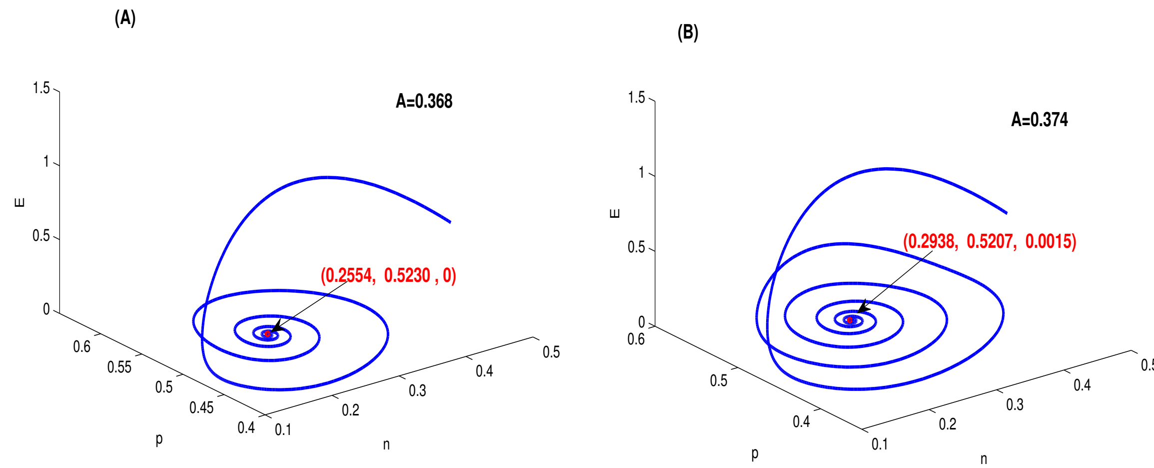

Figure 7.5: Phase portrait for the aggregated model (7.3.1) at (A). $A = 0.368$ and (B). $A = 0.374$.

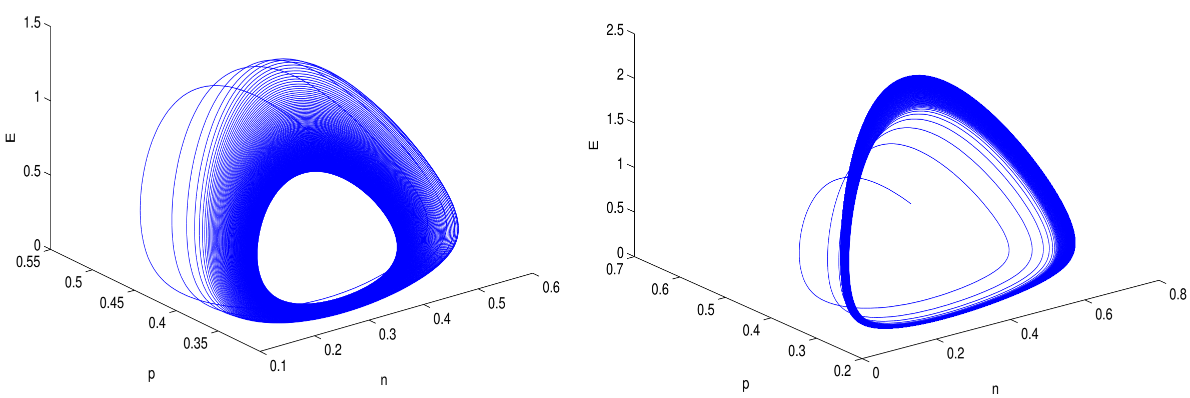

Figure 7.6: Phase portrait for the aggregated model (7.3.1) at (C). $A = 0.39$ and (D). $A = 0.4$.

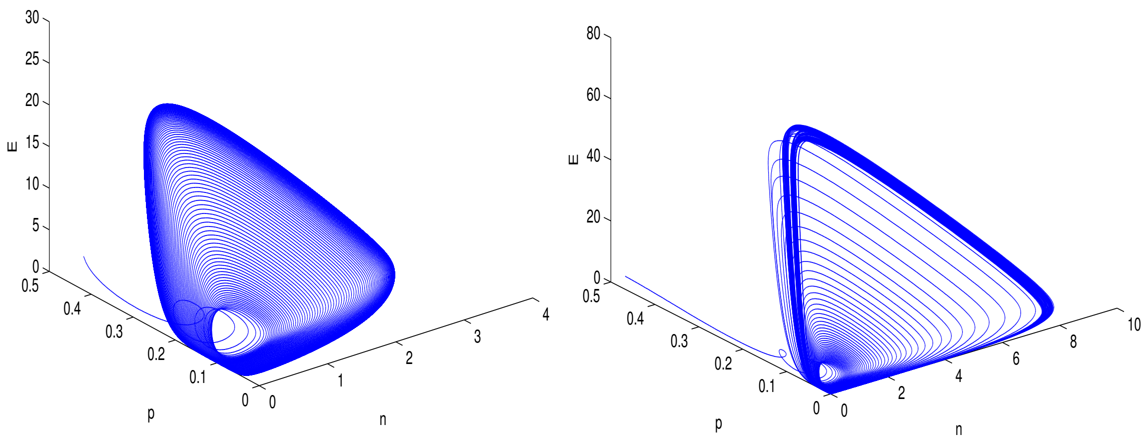

Figure 7.7: Phase portrait for the aggregated model (7.3.1) at (E). $A = 0.47$ and (F). $A = 0.6$

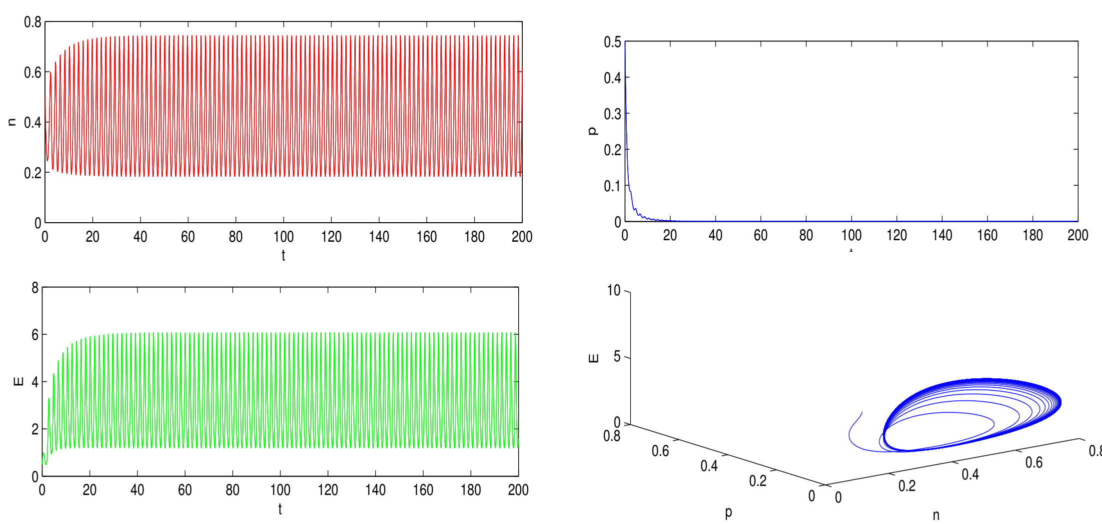

Figure 7.8: Time series plot and phase portrait of prey population $n(t)$, predator population $p(t)$ and Effort $E(t)$ for the aggregated model (7.3.1) at $A = 0.8$.

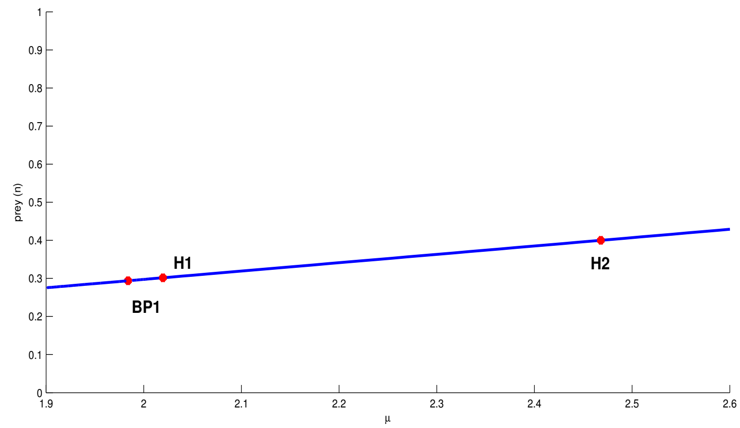

Figure 7.9: One parametric bifurcation diagram for the system (7.3.1) w.r.t. μ for the given data set (7.6.1) .

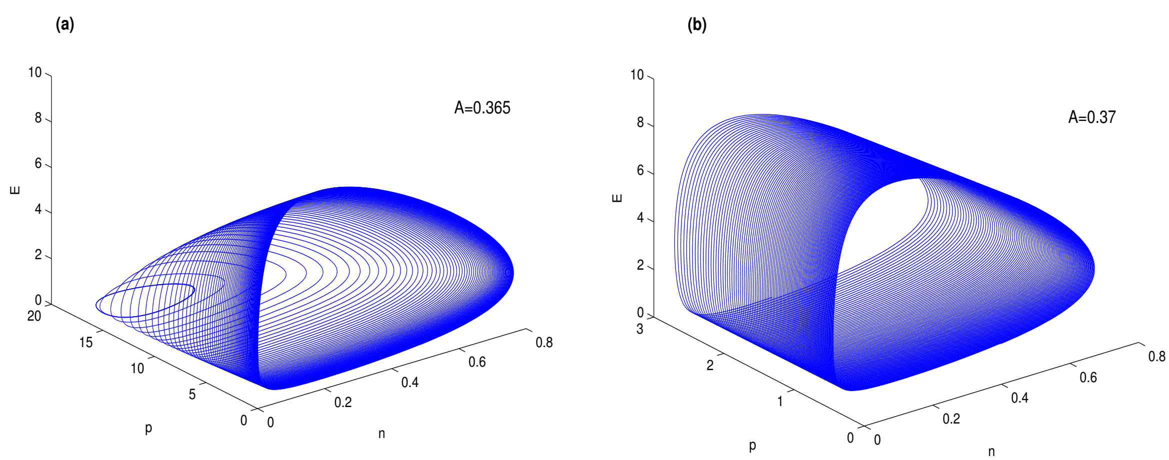

Figure 7.10: Phase portrait for the aggregated model (7.3.1) with constant migration rate at (a). $A = 0.365$ and (b). $A = 0.37$ using data set (7.6.1) and (7.6.2).

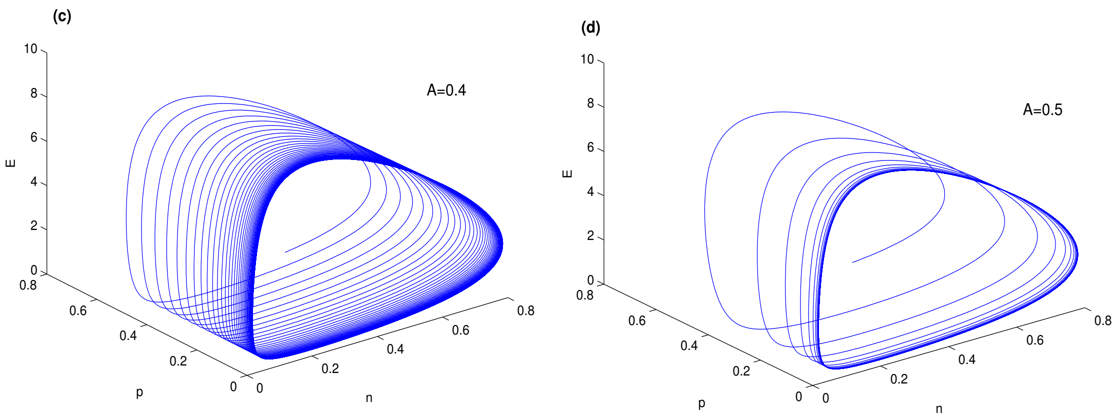

Figure 7.11: Phase portrait for the aggregated model (7.3.1) with constant migration rate at (c). $A = 0.4$ and (d). $A = 0.5$ using data set (7.6.1) and (7.6.2).

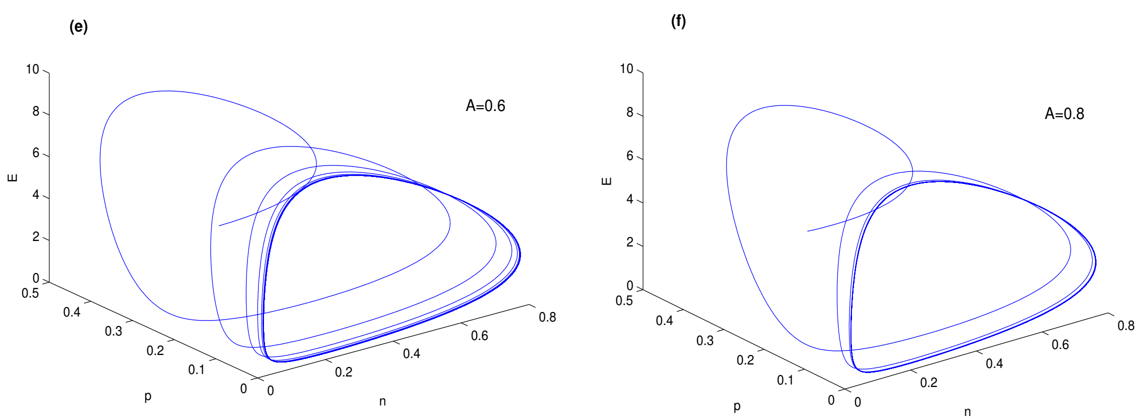

Figure 7.12: Phase portrait for the aggregated model (7.3.1) with constant migration rate at (e). $A = 0.6$ and (f). $A = 0.8$ using data set (7.6.1) and (7.6.2).

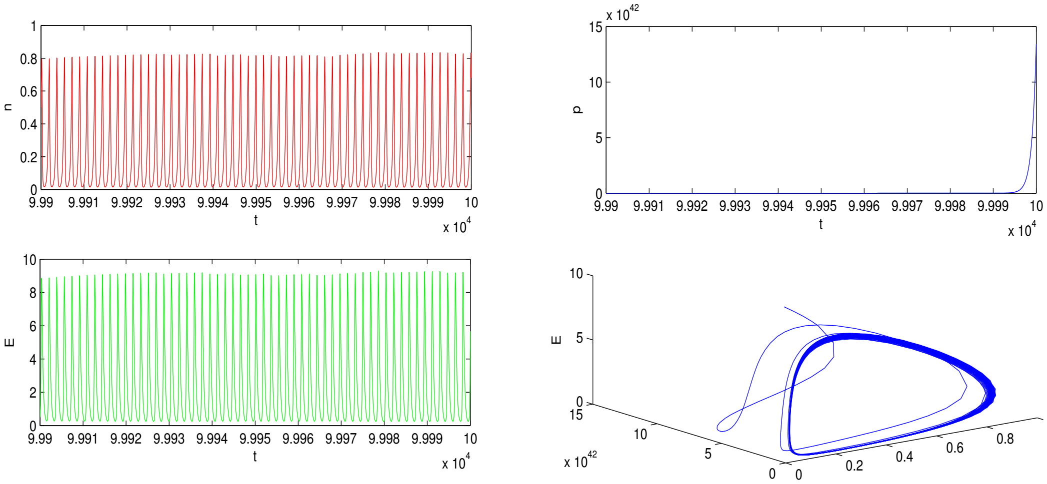

Figure 7.13: Time series plot and phase portrait of prey population $n(t)$, predator population $p(t)$ and Effort $E(t)$ for the aggregated model (7.3.1) with constant migration rate at $A = 0$ using data set (7.6.1) and (7.6.2).

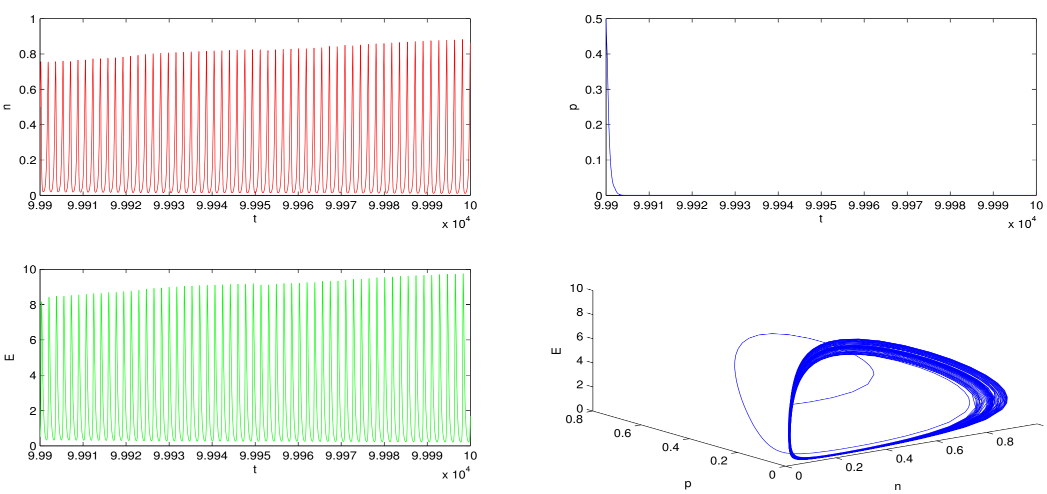

Figure 7.14: Time series plot and phase portrait of prey population $n(t)$, predator population $p(t)$ and Effort $E(t)$ for the aggregated model (7.3.1) with constant migration rate at $A = 1$ using data set (7.6.1) and (7.6.2).

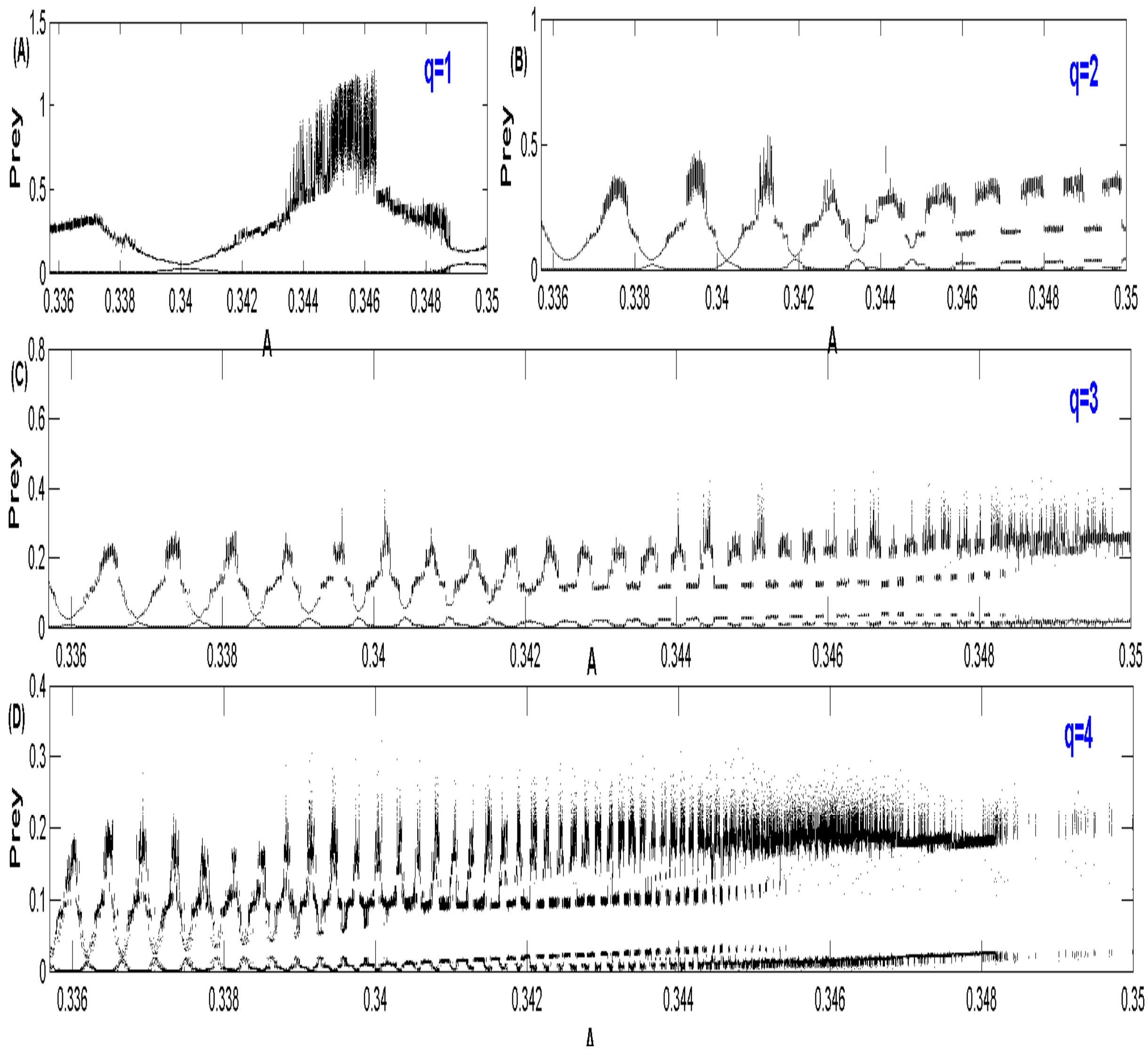

Figure 7.15: Bifurcation diagram for the aggregated model (7.3.1) with constant migration rate w.r.t. the parameter A in the interval $(0.3357, 0.35)$ for $q = 1$ $q = 2$, $q = 3$ and $q = 4$ using data set (7.6.1) and (7.6.2).

7.7 Conclusion

This chapter is concerned with a dynamic reaction predator-prey model with the refuge for the prey. It also incorporates the presence of alternate food for the predator. Meanwhile, prey is subjected to harvesting in both layers with effort dynamics. Two different time scales are considered in the dynamics of the model. Aggregated method is used to reduce the dimension of the model. Considering the dynamics of effort in the predator-prey system given by set of equations (7.3.1) can destabilize the system. However, using $E = 0$ and $E = constant$ in the system (7.3.1), it can be derived that this predator-prey system is stable. The feasibility and dynamical behavior of the aggregated system (7.3.1) is discussed w.r.t. the different values of parameter A. The coexistence of aggregated system is discussed for $A_1 < A < A_2$ (given in (7.4.4)). To validate the analytical results, numerical simulations are carried out for different value of A. From numerical part, it can be seen that the long term behavior of solutions are complex and chaotic. The bifurcation diagrams for density dependent and density independent migration rates are also discussed w.r.t. the parameter A.

Chapter 8

Conclusion and Future plans

8.1 Conclusion

In this thesis, some mathematical models are proposed and analyzed for management of renewable resources. In these models, harvesting play an important role. Various aspects of dynamical behavior of these models are discussed from both ecological and economical point of view.

In chapter 2, a Modified Leslie–Gower predator–prey model with non–linear harvesting of prey is analyzed. In this model, harvesting effort is considered as a dynamical variable. It is observed that unique interior equilibrium point is locally as well as globally stable under certain conditions. It is concluded that the level of harvesting effort decreases with the increasing cost and it will not remain profitable for high value of cost.

The model of chapter 2 is extended incorporating taxation as a control instrument in chapter 3. The system exhibits transcritical bifurcation for some parametric values. Conditions for persistence, bionomic equilibrium and optimal taxation are also derived. The impact of taxation on the system shows that the density of harvesting effort decreases as the tax rates increases. This increases the densities of the prey and predator populations. It can be concluded that the equilibrium level of predator–prey system can be increased by increasing tax level.

In chapter 4, a two-dimensional predator-prey model with combined harvesting is investigated in the presence of toxicity. In this model, predator is provided with some additional food resource. Various types of bifurcations such as saddle-node, transcritical, Hopf, Bogdanov-Takens bifurcation (BT) and Generalized Hopf bifurcation (GH) have been analyzed. Permanence conditions, Bionomic Equilibrium and Optimal Harvesting Policy of the system are also investigated.

In chapter 5, a Stock–Effort dynamical model is investigated in a non-homogenous habitat with non-linear harvesting of stock. The habitat is divided into two fishing zones. The constant fish displacements and movements of fishing vessels between zones is assumed to be at a faster time scale as compared to the local growth / interaction. In this model, it can be observed that the range of taxation can be enhanced due to the density dependent migration of fishing vessels. The Maximum Sustainable yield, Bionomic Equilibrium and Optimal Taxation are obtained for the model. The nonlinear harvesting term plays an important role in determining the dynamics and bifurcations of system.

Chapter 6 analyzed a predator–prey model where two spatial patches are considered for prey population. The patch 1 is assumed to be polluted with toxicants and patch 2 provides refuge for prey reducing risk from predation as well as risk of being infected from the toxicant. Aggregation method is used for mathematical analysis. It can be seen that there does not be any periodic solutions in the interior of first quadrant of aggregated system.

In chapter 7, a predator–prey fishery system is studied where only prey population is subjected to harvesting. The harvesting effort is taken as dynamical variable. The prey is migrating between patches in search of food or to take refuge. The predator is assumed to depend on some alternate food resource to avoid extinction. It is observed that the long term behavior of the solutions of aggregated model are complex and chaotic. Therefore, consideration of the dynamics of harvesting effort in the predator-prey system can destabilize the system.

The numerical simulations have been carried out to validate the analytical results throughout the chapters.

8.2 Future Plans and Possible Extensions

- The chapter 4 can be extended by taking effort as a dynamic variable and taxation as a control instrument similarly as chapters $2, 3, 5$ and 7. The model can be studied for its rich and complex dynamical behavior and optimal taxation policy.

- Moreover, the chapter 5 can be extended by considering market price as a dynamic variable.

- The chapter 5 can be extended for N-number of fishing zones. The optimal number of fishing zones can be obtained that can give Maximum yield at equilibrium.

- The chapter 6 can be extended by considering the harvesting of prey in both patches where fishing vessels also move between the patches with density dependent or constant migration rates.

- A Lotka-Volterra type predator–prey model can be considered in place of a single species model given in chapter 5 on FADs (Fish Aggregating Devices). and the model can be extended for N-number of fishing zones. This model can be studied for the proportional and nonlinear harvesting effort.

List of Publications

Publications in Journals

1. Reenu Rani, Sunita Gakkhar, Non-Linear Effort dynamics for Harvesting in a Predator–Prey System, *Journal of Natural science and research*, **5** (2015), No.3, 2224-3186.

2. Sunita Gakkhar, Reenu Rani, Non-linear harvesting of prey with dynamically varying effort in a modified Leslie-Gower predator-prey system, *Communications in Mathematical Biology and Neuroscience*, **2** (2017), 2052-2541.

3. R. Rani, S. Gakkhar and Ali Moussaoui, The Dynamics of a Fishery System in a Patchy Environment with Non–linear Harvesting, *Journal of Mathematical Methods in Applied Sciences.* (Under Revision)

4. R. Rani, S. Gakkhar, A Dynamic Reaction Model in a Heterogeneous Habitat considering Prey Refuge and Alternate Food to Predator, *Applied Mathematical modeling.* (Under Review)

Publication in Conference

1. R. Rani, S. Gakkhar, Predator–Prey Model in a Heterogenous Habitat with prey refuge in the presence of Toxicity, *Proceedings in an International Conference on Computational Modeling and Simulation*, 17-19 May 2017, University of colombo, Sri Lanka.

Publication to be Communicated

1. R. Rani, S. Gakkhar, The Impact of Provision of Additional Food to Predator in Predator-Prey Model with Combined Harvesting in the Presence of Toxicity.

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