QUEUEING MODELING OF REPAIRABLE MACHINING SYSTEMS WITH SERVICE INTERRUPTION

Ph.D. THESIS

by

RAKESH KUMAR MEENA

DEPARTMENT OF MATHEMATICS
INDIAN INSTITUTE OF TECHNOLOGY ROORKEE
ROORKEE – 247 667 (INDIA)
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A THESIS

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in

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ROORKEE

CANDIDATE'S DECLARATION

I hereby certify that the work which is being presented in the thesis entitled, **"QUEUEING MODELING OF REPAIRABLE MACHINING SYSTEMS WITH SERVICE INTERRUPTION"** in partial fulfilment of the requirements for the award of the Degree of Doctor of Philosophy and submitted in the Department of Mathematics of the Indian Institute of Technology Roorkee, Roorkee is an authentic record of my own work carried out during a period from January, 2013 to December, 2017 under the supervision of Dr. Madhu Jain, Associate Professor, Department of Mathematics, Indian Institute of Technology Roorkee, Roorkee.

The matter presented in this thesis has not been submitted by me for the award of any other degree of this or any other Institution.

(RAKESH KUMAR MEENA)

This is to certify that the above statement made by the candidate is correct to the best of my knowledge.

(Madhu Jain)
Supervisor

Date**: 27/12/2017**

ABSTRACT

The queueing theory and performance models have become the essential tools for the system designers/organizations to deal with unavoidable interruption of machining systems and also have potential industrial applications in computer and communication networks, traffic control, nuclear and power plants, distribution and power supply systems, production and assembly lines etc. In the industrial scenario, the primary objective of system designers is to design the machining system which may be fault tolerable. Many engineering systems which are prone to failures and operate in a machining environment can be improved by appropriate choice of redundancy as well as maintainability. Keeping in mind the vital role of queueing modeling of machining system with service interruptions due to unavailability of the server, in the present thesis some of queueing models for machining system with service interruption have been explored in different frameworks. In the present work our prime objective is to develop both transient as well as steady state queueing models for the repairable machining system with service interruption to investigate queueing and reliability indices of the concerned system. A variety of prominent features namely control policies, threshold policies, vacation, working vacation, server breakdown, provision of standby support, reboot and recovery process have been incorporated to dealt with the interruptions occurred in machining environment. Furthermore optimal system parameters have been obtained to determine the optimal cost of the system.

The thesis is organized into 10 chapters including the first chapter devoted to general introduction on the relevant topics of work done in the thesis. The literature review, basic concepts and methodology used are also discussed in the first chapter. The chapters 2-8 and chapter 9-10 explore Markovian and non-Markovian models of machining system, respectively.

The study done is concluded by highlighting the noble features and future scope in the end of the thesis. The relevant references are listed in the alphabetical order. The investigation presented in the chapters 1-10 of the thesis are as follows:

Chapter 1 entitled ‘General Introduction’ presents the motivation, the overview of the relevant research works, methodological aspects, solution techniques, survey of the literature and contents of the thesis. *Chapter 2* deals with the time-dependent analysis of an M/M/1 queueing model with state dependent rates and optional multiple working vacations. *Chapter 3* is concerned with admission control of maintenance for unreliable server machining system

with working vacation. The *chapter 4* presents the admission control policy for the fault tolerant system comprising of multi-components operating machines and multi types of warm standbys under the maintenance of single unreliable server. The concepts of F-policy which deals with the controlling of admission of failed machines and imperfect coverage are incorporated to make Markov model more realistic. *Chapter 5* deals with the performance modeling of finite Markov M/M/1/L working vacation model for the fault tolerant machining system (FTMS). The concepts of redundancy along with the provision of dissimilar warm standbys are considered to maintain the pre-required high reliability of the system. In *chapter 6*, Markov model of multi-component machining system comprising of two unreliable heterogeneous servers and mixed type of standby support has been studied. In *chapter 7*, the performance prediction of fault tolerant machining system with imperfect coverage, reboot and server vacation is described. This study is concerned with the performance modeling of a fault tolerant system consisting of operating units supported by a combination of warm and cold spares. In *chapter 8*, Markovian model for FTS multi-component machining system with imperfect coverage, standby support and working vacation is investigated. In *chapter 9*, the availability analysis of M/G/1 FTS system for R-out-of-M: G configuration is described. The study of imperfect fault coverage and availability analysis of redundant machining system having the facility of recovery and replacement has been done. In *chapter 10,* the M/G/1 model for the multi-component fault tolerant machining system by incorporating the features of common cause shock failure and standby support has been investigated. Finally noble features and future scope of investigations done are presented in the conclusion section given after chapter 10.

The queueing models developed in the present thesis provide valuable insights for the system design and may be successfully used in abundant congestion situations encountered in machining environment. Keeping in mind a variety of problems have been explored using different methodologies. It is hoped that the queueing models developed for machining systems in this thesis may be helpful to system analysts, developers, and practitioners to frame more optimal and efficient design of the concerned system.

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"*Strength does not come from winning. Your struggles develop your strengths. When you go through hardships and decide not to surrender, that is Strength*"- Arnold Schwarzenegger

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LIST OF PUBLICATIONS

International Journals/Conference Proceedings

1. Madhu Jain, Chandra Shekhar and **Rakesh Kumar Meena** (2017): Admission control policy of maintenance for unreliable server machining system with working vacation, Arabian Journal for Science and Engineering 42:2993-3005 (Springer).
2. Madhu Jain and **Rakesh Kumar Meena** (2017): Fault tolerant system with imperfect coverage, reboot and server vacation, Journal of Industrial Engineering International 13:171-180 (Springer).
3. Madhu Jain and **Rakesh Kumar Meena** (2017): Markovian analysis of unreliable multi-components redundant fault tolerant system with working vacation and F-policy, Cogent Mathematics 4:1-17 (Taylor & Francis).
4. Madhu Jain and **Rakesh Kumar Meena** (2017): Vacation model for Markov machine repair problem with two heterogeneous unreliable servers and threshold recovery, Journal of Industrial Engineering International, doi.org/10.1007/s40092-017-0214-x (Springer).
5. Madhu Jain, Sudeep Singh Sanga and **Rakesh Kumar Meena** (2016): Control F-policy for Markovian retrial queue with server breakdowns, Proceeding of 1st IEEE International Conference on Power Electronics. Intelligent Control and Energy Systems (ICPEICES-2016), Delhi, 2016, pp. 1-5.
6. Charan Jeet Singh, Madhu Jain, Sandeep Kaur and **Rakesh Kumar Meena** (2016): 'Retrial bulk queue with state dependent arrival and negative customers', In: Deep K. *et al*. (Eds) Proceedings of Sixth International Conference on Soft Computing for Problem Solving. Advances in Intelligent Systems and Computing, 547:290-301. Springer, Singapore.
7. Madhu Jain and **Rakesh Kumar Meena** (2014): Optimal threshold policy for machining system with imperfect coverage reboot and vacation, Proceeding of International Conference on Emerging Trends in Global Management Practices – An Interdisciplinary Approach (INCONSYM 2014), Symbiosis Noida, 2014, pp. 728-740.

Research Paper communicated for publication

1. Madhu Jain and **Rakesh Kumar Meena (2017):** Maintainability of *R*-out-of-*N*:*G* fault tolerant machining system with imperfect fault coverage,
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3. Madhu Jain, Richa Sharma and **Rakesh Kumar Meena (2017):** Performance modeling of fault tolerant machining system with working vacation and working breakdown,
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4. Azhagappan Arumugam, Madhu Jain and **Rakesh Kumar Meena (2017):** Transient analysis of a single server queue with state dependent rates and optional multiple working vacation,
   Communicated to Proceedings of National Academy of Science, Springer.
   Madhu Jain, Chandra Shekhar and **Rakesh Kumar Meena (2017):** Performance analysis and control F-policy for fault folerant system with working vacation, Communicated to OPSEARCH, Springer.
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PARTICIPATION IN CONFERENCES/WORKSHOPS

Participation in Conference

1. Participated and presented a paper entitled "Availability Prediction of M/G/1 Fault Tolerant Machining System with Common Cause Failure and Reboot" at **OR 59 Annual Conference** of the OR society at **Loughborough University**, Loughborough United Kingdom, 12-14 September 2017.
2. Attended an International Conference on "**Recent Trends in Mathematical Analysis and its Applications**" organized by the **Department of Mathematics, IIT Roorkee** during 21-23 Dec., 2014.
3. Attended an International Conference on "**Soft Computing for Problem Solving**" organized by the **Department of Mathematics, IIT Roorkee** at Noida Campus, during 26-28 Dec., 2013.
4. Participated and presented a paper entitled "Optimal threshold policy for machining system with imperfect coverage, reboot and vacation" at International Conference on Emerging Trends in Global Management Practices- An Interdisciplinary Approach organized by **Symbiosis Centre for Management Studies Noida,** during 7-8 March 2014.
5. Participated and presented a paper entitled "Transient Analysis of Machining System with vacation and Imperfect coverage" at National Conference on **Recent Trends and Developments in Operations Research** organized by the **Department of Mathematics, BITS Pilani**, Rajasthan during 22-2 Feb., 2014.
6. Participated and presented a paper entitled "Queueing Analysis of Multi-Component Machining System with Spare Provisioning, vacation and Imperfect coverage" at National Conference on **"Advances in Mathematical Sciences & Their Applications"** organized by the **Department of Mathematics, Hindu Girls College**, **Sonipat (Haryana)** during 28-30 March., 2014.

Participation in workshops

1. Attended a workshop on ‘**Applied Stochastic Models and Optimization**’ organized by the **Department of Mathematics IIT Roorkee**, during 26-27th May, 2017.
2. Attended a workshop on ‘**Modeling, Optimization and Simulation of Stochastic Systems**’ organized by the **Department of Mathematics IIT Roorkee**, during 26th Nov., 2016.
3. Attended a workshop on ‘**Optimization Technique for Solving Industrial Problems**’ organized by the **Department of Mathematics IIT Roorkee**, during 15th Oct., 2016.
4. Attended a workshop on ‘**Author Workshop on Book Publishing**’ organized by **Mahatma Gandhi Central Library, IIT Roorkee** in association with **Elsevier**, 26th Sep. 2016.
5. Attended a workshop on ‘**Optimization**’ organized by the **Industrial Engineering and Operations Research Department, IIT Bomabay**, during 9-21 May, 2016.
6. Attended a workshop on ‘**Training Workshop on Reference Management Software-Mendeley**’ organized by **Mahatma Gandhi Central Library, IIT Roorkee**, 20th Jan. 2016.
7. Attended a workshop on ‘**Introduction to Matlab and Mathematica**’ organized by the Department of Mathematics IIT Roorkee, during 21-22 April, 2012.

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Chapter 1

General Introduction

1.1 Motivation

With the advancement in the technology, the machines have a significant impact on human life. Due to increase in just in time (JIT) demand, the size and complexity of machining systems have grown up extremely which also increase the occurrence of faults/failures in the machines causing interference in the operation of the systems. The occurrence of faults/failures in machining systems is quite common phenomenon but has adverse impact on the revenue as well as output of the concerned organization/industry. To avoid the failures and faults of the machining systems, the reliability, availability, maintainability (RAM) issues need to be predicted during the design, development and operation phases.

In the real life scenarios, we all experience the queueing situations in day-to-day life including machining systems used at toll booth or traffic signal, automatic ticketing system at railway counters or bank ATM, supermarket automatic checkout system, automatic petrol-refueling station, and many other places. The queueing theory and reliability models have become the essential tools for the system designers/organizations to deal with unavoidable interruption of machining systems and also have potential industrial applications in computer and communication networks, traffic control, nuclear and power plants, distribution and power supply systems, production and assembly lines, etc. The primary objective of the system designers is to design the machining system which may be fault tolerable. Many engineering systems operating in a machining environment can be improved by appropriate choice of redundancy as well as maintainability. Keeping in mind the vital role of queueing modeling of machining system with service interruptions due to unavailability of the server, a variety of queueing models for machining system with service interruption have been explored in different frameworks by several researchers working in the area of queueing and reliability theory. The spare provisioning and maintainability can help in reducing the risk of sudden breakdown of the system and increasing the life time of the concerned system. The server vacation or working vacation is also a kind of service interruption caused due to unavailability of the server and can be allowed to improve and enhance the quality of operation mode of any machining system. The performance prediction of repairable redundant machining system with service interruption in different frameworks is the main objective of the research work presented in the thesis.

In modern era of automated life style, the performance modeling of fault tolerance machining system has become more difficult job. In real-time machining systems, the sudden breakdown of machining components or server failure may cause interruptions in the service rendered by the system. There are several reasons of occurrence of interruption in the real time machining systems such as sudden breakdown of machines, server failure, unavailability of the server due to vacation and many others. Sometimes, these interruptions cause severe disaster on the production as well as goodwill of the concerned organizations. In queueing literature some research works can be found on the performance prediction of machining system with service interruption. However there is scarcity of performance models which deal with the fault tolerance and admission control issues to maintain the smooth functioning of machining systems for a long period of time. By literature survey, it is realized that much more research works has to be done in this direction so as to avoid the service interruption during operation phase of the machining system. To deal with these situations, we are interested to develop some queueing models for machining systems with service interruptions which may be of further used to analyze the performance of complex real time machining systems.

From the queueing literature, it is seen that in the past few years, some articles on queueing modeling of repairable machining systems operating under optimal admission policy have appeared. However, there is urgent need of research works which can explore the performance metrics of the unreliable fault tolerant machining system in generic set up. The numerous versatile applications of optimal control policy for fault tolerant machining systems which are prone to failure have motivated us to study queueing modeling of machining system with service interruptions by incorporating the concepts of optimal control threshold policies. In the present study we shall investigate some queueing models for the performance prediction of repairable machining system by incorporating various realistic features such as unreliable server, vacation policies, threshold policies, standby support, reboot and recovery, etc. In the thesis work, our study is devoted to develop some queueing models for repairable multi-component systems in the general frameworks by including the noble concepts of service interruption under the assumptions of vacation, working vacation and unreliable server, etc. To deal with the performance metrics of real-time systems, we have developed the stochastic models under the assumption of F-policy, common cause shock failure, mixed type standby support, etc.

The prime objective of our investigation is to develop both transient as well as steady state queueing models for the multi-component machining systems with service interruption due to either unreliable server or vacation/working vacation. The current introductory chapter provides the basic concepts and overview of the methodology used for the modeling and performance analysis of various models of machining systems operating in Markovian or non-Markovian set up. The remaining contents of the ongoing introductory chapter are organized in the following manner. Section 1.2 describes some important aspects of queueing characterization of repairable machining system. The service interruption factors involved in queueing scenarios of machining systems are discussed in the Section 1.3. Some basic methodological aspects used to carry out the investigation in the thesis work are described in the Section 1.4. Some Markovian queueing models for performance prediction of MRP are presented in Section 1.5. Section 1.6 is devoted for the survey of literature of relevant topics concerned with the performance prediction of repairable machining systems in different frameworks. The content of the thesis work is described in the Section 1.7. Finally, in Section 1.8, the concluding remarks for highlighting the noble feature of work done are given.

1.2 Repairable Machining Systems

We are living in a high-tech and automated machining world, where our dependence on machines in various sphere of life cannot be denied. Since the machines may be prone to failures as such redundancy and maintenance issues are always involved in machining system. The queues formations in the context of repairable machining system are referred as finite source queues. In these systems, the calling population is the machines; an arrival corresponds to a machine breakdown and server corresponds to the repair facility. Due to fact that finite population models can be used for the modeling of wide range of real-time systems, noticeable works have been reported on finite population queueing models. To resolve the problems of blocking and delay at machining system, the queueing theory approach as well as reliability, availability and maintainability (RAM) approaches can be used. The pictorial view of repairable machining system with different stages of repair facility can be seen in Figure 1.1.

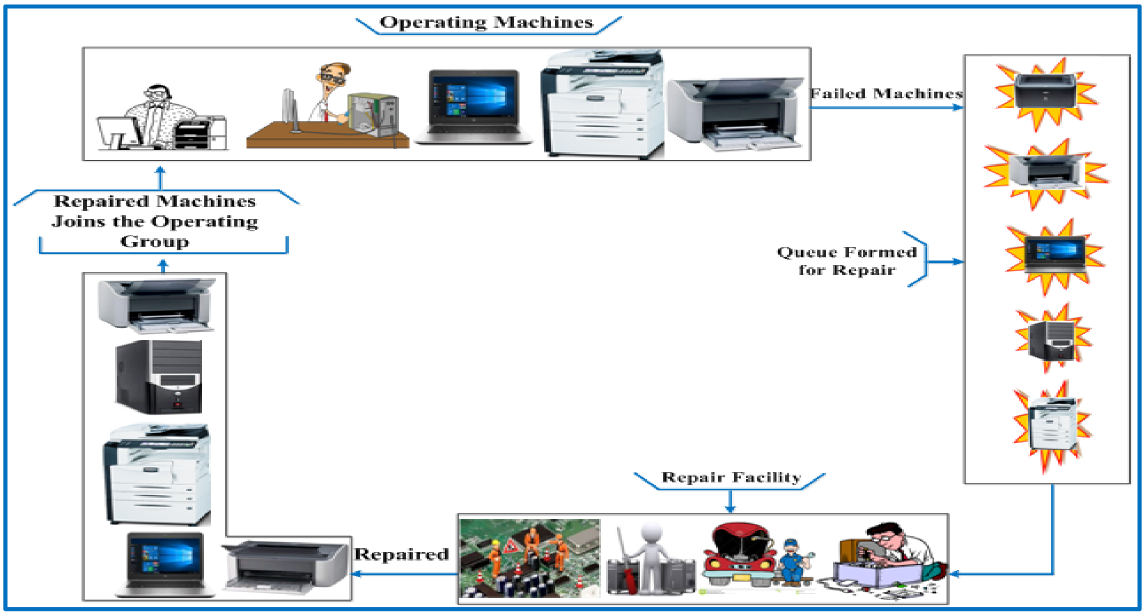


Fig. 1.1: Formation of queue of failed machines at repair shop

1.2.1 Machine repair system with redundancy

The occurrence of fault and failures in real time machining environment is a quite common phenomenon which not only affects the desired output and efficiency of the system but cause increase in down time and economic loss. The system analysts can reduce the risk of fault occurrence and consequently enhancement in the reliability of the system with the standby support. In recent past, some queue theorists have studied finite population queues to facilitate the performance modeling of redundant machining system with standby support. There are several applications of queueing theory in the area of performance prediction of redundant machining system. Queueing theory can be used to analyze the effects of random failures on the functioning of machining system having facility of both redundancy and maintainability. The occurrence of faults in machining system not only affects the system production but also have the adverse effect on the revenue of the system. To avoid these inconvenient situations, the decision makers provide the facility of standby support and repair facility. There are several methods, techniques and terminologies for implementing the redundancy in repairable machining systems. The commonly used redundancy approaches in industries are (i) Standby redundancy, (ii) N Modular redundancy and (iii) K-out-of-N: G redundancy. Here in the thesis work, we have used standby redundancy and K-out-of-N: G redundancy. The brief descriptions of these redundancies are as follows:

- ***Standby redundancy***

Standby redundancy is also known as backup redundancy and can be used when the system has some identical standby machines to back up the active failed machines. The standby machines typically does not monitor the system, but can be used as a spare.

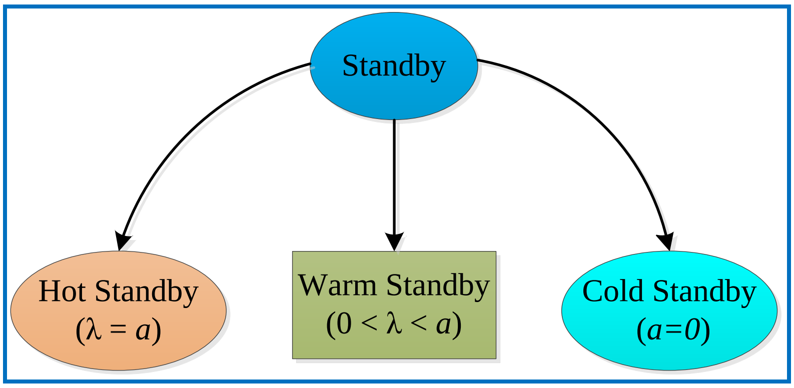


Fig. 1.2: Classification of standby machines based on failure characteristic

The standby machines are broadly classified into three categories on the basis of their failure characteristics which are depicted in Figure 1.2. Here λ and a denote the failure rates of operating and standby machining components, respectively.

(a) ***Hot standby machine*****:** The failure rate of hot standby machine is equivalent to the failure rate of active machine i.e. $\lambda \cong a$.

(b) ***Warm standby machine*****:** The failure rate of warm standby machine is non-zero and less than failure rate of active machine i.e. $0 < a < \lambda$.

(c) ***Cold standby machine*****:** The failure rate of cold standby machine is zero during inactive state i.e. $a = 0$.

- ***K-out-of-N: G Redundancy***

A generalization of N parallel components occurs when a requirements exists for K-out-of-N identical and independent components to function for the system to be in operative state. In special cases, if K=1, complete redundancy occurs via parallel arrangement of the components and if K=N, the N components are arranged in series. In this case, the reliability is obtained by the binomial probability distribution. Thus reliability of K-out-of-N: G system is

$$R_{K-out\,of-N}(t) = \sum_{i=0}^{N-K} \binom{N}{i} (1-R)^i R^{N-i}, \text{where R is a reliability of a component} \qquad (1.1)$$

- ***N Modular redundancy***

N Modular redundancy is also known as parallel redundancy. In this redundancy, the systems have N units running in parallel. For software embedded system, all units are highly synchronized and receive the same input information at the same time.

1.2.2 Machining system under control policies

The control policies can be used to overcome the waste of valuable resources, time and money of an industry or company operating in machining environment. The past research works dealing with the controlling of queueing situations can be divided in two broad categories, first one to control the service and other one to control the arrival. For any machining system, N-policy can be implemented for the economic utilization of the server. Threshold N-policy states that the server is turned on to render repair only when the workload of repair job of failed machines reaches to pre-defined threshold level N. To control the arrivals, F-policy can be used. In case of F-policy, the customer's entry is stopped in the system if it reaches its full capacity level. To allow again customers to join the system, the queue length should be ceases to a predefined threshold value 'F'.

1.2.3 Fault tolerant machining systems

The operation and capacity of machining systems involved in computer or communication networks, manufacturing or production systems and many other systems are highly affected by the failure of machining components. The occurrence of sudden breakdown of machining systems may cause not only a loss of desired output and efficiency but also increase in the down time and cost. To avoid these adverse situations, the system designers may be interested in reboot, recovery and provision of standbys which make the machining system fault tolerable. In the fault tolerant systems (FTSs), some units may fail, but still the system remains operative and continues to perform its assigned job due to the provision of maintainability, optimal control and standbys. In the present scenario of modern technology, computer controlled fault tolerant machining system has become the necessity and it brought a tremendous change in the system design to control the risk of machine failure. Now-a-days, the machines are equipped with an inbuilt fault-handling mechanism which automatically detects the failure of a component and recovers the system by replacing the failed operating unit by a standby unit, if available. In many software embedded systems, in the case when the fault handling mechanism fails to detect and recover the faults, the machines can also reconfigure temporarily by reboot process. But in some practical situations, the fault-handling device may prove inadequate to recover a fault perfectly; this situation is known as imperfect coverage.

1.3 Queueing Model with Service Interruptions

The real time systems may undergo through the several types of service interruptions. There may be sudden occurrence of faults in machines or failure of the server or unavailability of server due to vacation. Here, in this thesis work we have considered the service interruption cause due to the sudden breakdown of server i.e. concept of unreliable server or due to the failure of machines, or due to vacationing server. For the review work on queue with interruption we refer the notable survey work done by Krishnamoorthy *et al*. (2014)

1.3.1 Unreliable server models

In the real life situations, occurrence of faults or failure in machining environment is quite common phenomenon, which results most often in system breakdown. In the present scenario, we have noticed that most of the studies devoted to the queueing analysis of machining systems are restricted to reliable machining systems, but in real life, no systems or machines can be reliable as such the incorporation of an unreliable machining parts or unreliable server concept will be helpful to portray the more versatile queueing scenarios of real time system. In many queueing systems, the server may fail at any instant while rendering the service to the customers; thus by considering the server break down, the queueing model can be developed to depict more real-world situations.

1.3.2 Vacation model

Queueing systems with server vacation have many applications in machining systems working in industrial environments including manufacturing and production systems, computer and telecommunication networks, transportation and service sectors, manufacturing and inventory control systems and many others. In case when there is no work-load of failed machines, there may be issue of server being idle for a long period. The repairmen can utilize this time in rendering some other type of service which may be maintenance job, book keeping etc. in case of a machine interference problem. Over the last few decades, a substantial amount of works have been done for the performance prediction of queueing systems with vacations. The queueing modeling with vacationing server for the fault tolerant systems can also be done to deal with many realistic situations where the server may leave the system to go for vacation in case when the system becomes empty.

There are various schemes to classify the server vacation queue; the most prominent way is service discipline. These service schemes are defined as

(i) Exhaustive service scheme vacation

In the context of exhaustive service, on completion of vacation period, the server renders repair to each failed machines which are waiting to get repair and commences another vacation only when no failed machines for repair job are available in the system.

(ii) Gated service scheme vacation

As the server returns back from the vacation, he renders repair only those failed machines that were queue up when the server arrived.

(iii) Limited service scheme vacation

After completion of vacation period, the server renders repair to a fixed number of failed machines, say *m*, which is pre-defined. In this vacation scheme, the server renders repair to at most *m* failed machines on returning from vacation and then only server commences another vacation.

If the server returns from vacation and finds the repair facility empty, the server follows one of the following three schemes:

(i) ***Multiple vacations***: Under the multiple vacations scheme, the server immediately takes another vacation if finds no job after returning back from vacation.

(ii) ***Single vacation****:* Under the single vacation scheme, the server does not take another vacation in case he finds the repair facility empty on returning back from first vacation until the server renders repair at least one failed machine.

(iii) ***Hybrid vacation***: Under the hybrid single/multiple vacation scheme, after returning back from the first vacation, the server waits for a random duration and if still no failed machine joins the system, takes another vacation.

In another way, the server vacation models can also categorized in the following two broad ways:

(i) If the server will not render any service while on vacation period then this type of vacation period is termed as ***complete vacation***.

(ii) If the server renders service during vacation period with slower rate than that of normal busy period then the server is said to be on ***partially vacation or working vacation***. In many service systems, the server while on vacation may not like to remain idle due to many reasons including the loss of profit in case when some jobs are expected to accumulate during the vacation period. The same is the case with the machine repair systems; in such case, when the failed machines join the system during vacation, the server rather than completely stopping the service provides repair to the failed machines at a different pace and is called on working vacation.

1.4 Methodological Aspects

The stochastic modeling of waiting lines or queues has become an efficient tool to resolve the congestion situations encountered in many machining systems. The blocking and delay problems due to service interruption in machining systems can be tackled using queueing modeling approaches. The system characteristics used to define the queueing structure is visualized in Figure 1.3.

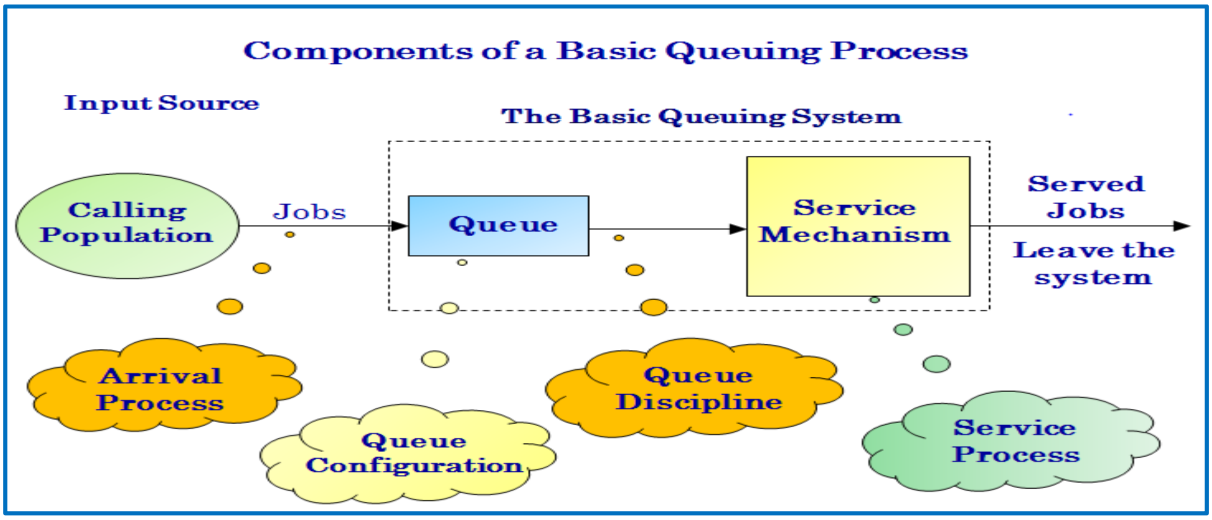


Fig. 1.3: The basic queueing structure

The basic queueing model can be developed using (i) Arrival process (ii) Service process (iii) Queue discipline (iv) System capacity (v) Number of service channels (vi) Number of service stages.

A variety of analytical and numerical methodologies were developed by the queue theorists for the performance prediction of queueing systems. To analyze the machining system with service interruption, we have developed some stochastic models and derived several queueing and reliability metrics. The important indices established in various chapters include mean queue size, throughput, waiting time in the system, failure frequency, machine availability, etc. and others, by using the analytical and numerical techniques. For the transient analysis of queueing models for the machining systems, the analytical methods used includes generating function, modified Bessel function, continued fraction approach, spectral method, etc. The numerical technique namely Runge-Kutta-IV order method has also been used to solve the system of the differential equations governing the concerned model. The analytical techniques namely supplementary variable technique (SVT) and recursive techniques and numerical techniques viz. successive-over-relaxation (SOR) and matrix method have been used for the steady state analysis of machining systems. For the optimal design of repairable machining systems, optimization techniques namely quasi-Newton technique and heuristic search approach are used. The hybrid soft computing technique adaptive-neuro fuzzy inference system (ANFIS) is also employed to compare the numerical results obtained by analytical/numerical method for the concerned model. It is worthwhile to describe stochastic processes and some solution techniques in brief which we have been used to evaluate the performance metrics of some machining systems with service interruption.

1.4.1 Stochastic and Markov Processes

The ***stochastic process*** is a family of random variables $\{\chi(\tau), \tau \in T\}$ such that the state of the system is characterized at every instant over a finite or infinite interval. The set of all possible values of random variable $\{\chi(\tau)\}$ is known as its state space. The state space is discrete if it contains a finite or denumerable infinity of points; otherwise, it is continuous. A stochastic process in continuous time $\{\tau \in T\}$ may have either a discrete or a continuous state space. Some important contributions on stochastic processes to analyze the survival and density functions are due to Chaubey and Sen (1996), Chaubey *et al*. (2011), Chaubey and Zhang (2015) and many other.

Markov process represents the stochastic phenomenon by permitting the output at any instant to be depended only on the outcome that precedes it and none before that. Thus in a Markov process $\chi(\tau)$, the past has no influence on the future if the present is specified.

Given $\tau_1 < \tau_2 ... < \tau_m$, then the random process $\chi(\tau_m)$ represents Markov process if

$$P[\chi(\tau_m) \le x_m / \chi(\tau_{m-1}), \chi(\tau_{m-1}), ..., \chi(\tau_1)] = P[\chi(\tau_m) \le x_m / \chi(\tau_{m-1})]. \qquad (1.2)$$

If the state space is discrete i.e. finite or countable infinite, then *Markov process* is termed as Markov chain. The process of machining systems includes the failure and repair of machines as well as unavailability of the server which may be random in nature. Such processes can be represented by stochastic processes and in specific cases by Markovian processes also. The exponential distribution which is quite often used for the life time and repair time follows the Markovian property.

1.4.2 Analytical techniques

There are several powerful analytical techniques based on stochastic theory, which can be employed to evaluate the queue size distributions and other performance indices of machining systems. In the thesis work, we have used the stochastic process for analyzing both Markov and non-Markovian models by framing Chapman-Kolmogorov equations for the system states (Gross *et al*., 2008). For the performance analysis of machining system, several analytical techniques including Birth death process, generating function approach, recursive method, continued fraction method, modified Bessel's function method, supplementary variable technique are used. The brief descriptions of analytical methodologies used are as follows:

- ***Birth-Death process***

A special case of continuous time *Markov process* where the states represent the current size of population and where the transitions are limited to birth and deaths is governed by *birth-*

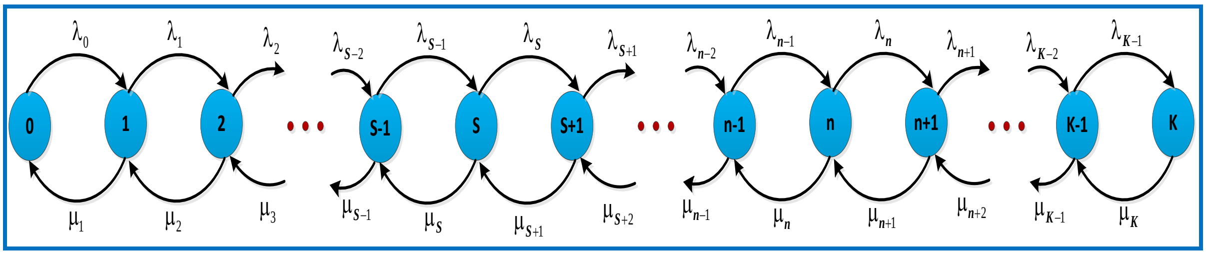


Fig. 1.4: Finite birth-death process

death process. The birth-death process has been widely used in queueing theory for the performance analysis of repairable machining systems. The finite population model called the machine interference problems can be analyzed using birth death process. The machine repair problem (MRP) can be formulated as birth death process for finite calling population K and is depicted in Figure 1.4.

When a birth occurs i.e. a machine fails, the process goes from state n to n+1. When a death occurs i.e. a failed machine is repaired, the process goes from state n to state n-1. Let λ_n and μ_n be the birth and death rates of state 'n' ($0 \leq n \leq K$) and $P_n(t)$ be the probability associated with state 'n' at time t. For finite capacity model, the birth-death process is given by

$$\frac{dP_0(t)}{dt} = -\lambda_0 P_0(t) + \mu_1 P_1(t) \tag{1.3}$$

$$\frac{dP_n(t)}{dt} = -(\lambda_n + \mu_n) P_n(t) + \lambda_{n-1} P_{n-1}(t) + \mu_{n+1} P_{n+1}(t), \quad 1 \leq n \leq K-1 \tag{1.4}$$

$$\frac{dP_K(t)}{dt} = -\mu_K P_K(t) + \lambda_{K-1} P_{K-1}(t) \tag{1.5}$$

- ***Probability Generating Function (PGF)***

The probability generating function is a powerful tool to deal with stochastic processes $\chi(t)$ and is based upon the convergent power series to solve the transient as well as steady state system of equations governing the concerned model to generate the corresponding probabilities. The probability generating function $G(z,t)$ associated with probability $P_n(t)$ $(n \geq 0)$ can be defined by

$$G(z,t) = \sum_{n=1}^{\infty} P_n(t)\, z^n \tag{1.6a}$$

provided, the series (1.6) converges in some interval. Here $P_n(t)$ denotes the sequence of the state dependent probabilities of system being in state 'n'at time t.

The mean $E\{X(t)\}$ and variance $Var\{X(t)\}$ of stochastic process $\chi(t)$ are obtained using PGF as follows

$$E\{X(t)\} = \sum_{n=0}^{\infty} n P_n(t) = \lim_{z \to 1} G'(z,t) \tag{1.6b}$$

$$Var\{X(t)\} == E\{X^2(t)\} - (E\{X(t)\})^2, \tag{1.6 c}$$

- ***Continued fraction method***

Continued fraction is applicable when differential equation has the three-term recurrence structure. The approximations using continued fraction provide a good representation for transcendental functions. Continued fraction is much useful than the classical representation by power series. A systematic study of theory of continued fraction is presented in the book of Jones and Thron (1980).

Continued fraction represents simple and mathematically an elegant method of obtaining transient solution. A continued fraction is given by

$$\cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{b_3 + \dots}}} \tag{1.7}$$

where a_i, b_i, for $i = 1, 2, 3, \dots$ are real or complex numbers. This fraction can be terminated by $a_1, b_1, a_2, b_2, \dots, a_i, b_i$ and dropping all the remaining $a_{i+1}, b_{i+1}, \cdots$. Equivalently, it can be written as

$$\frac{a_1}{b_1 +} \frac{a_2}{b_2 +} \frac{a_3}{b_3 +} \dots, \tag{1.8}$$

The number obtained by this operation is termed as i^{th} convergent and is denoted by A_i / B_i. Recurrence relation plays an important role in the transient analysis of birth-death process. There is hardly a computational task which does not rely on recursive technique techniques at one time or another. Here both A_i and B_i satisfy the recurrence relation

$$U_i = a_i U_{i-2} + b_i U_{i-1} \tag{1.9}$$

with initial value $A_0 = 0$, $A_1 = a_1$, $B_0 = 1$ and $B_1 = b_1$. There is close connection between birth-death processes and continued fraction. We have successfully applied continued fraction approach to find the time-dependent solution of the model developed in chapter 2.

- ***Modified Bessel's function***

The modified Bessel function of the first kind of order *n*, denoted by $I_n(x)$, is defined as

$$I_n(x)=\sum_{k=0}^{\infty}\frac{(x/2)^{2k+n}}{k!\,\Gamma(k+n+1)},\ n>0, \tag{1.10}$$

which is solution of the following Bessel modified equation

$$x^2y''+xy'-(x^2+n^2)y=0, n\geq 0 \tag{1.11}$$

In particular, $I_n(x)=I_{-n}(x)$, for $n\geq 0$.

- ***Supplementary Variable Technique (SVT)***

The non-Markovian process associated with life time of machining parts and repair time make the performance analysis quite cumbersome from analysis point of view. To deal with non-Markovian queueing model, *supplementary variable technique* (SVT) can be used. In SVT, non-Markovian process is converted in to Markov process by introducing one or more *supplementary variables*. The state description of an M/G/1 machining system can be represented by bi-variate stochastic process $\{\xi(t),\eta(t)\}$ here $\xi(t)$ represents the status of the server and $\eta(t)$ specifying the number of failed machines in the system at time *t*. Consider $U(t)$ as supplementary variable denoting the either elapsed service time or remaining service time. The technique was first introduced by Cox (1955) who used supplementary variable technique to study M/G/1 queue. For queueing and reliability modeling of machining system with general repair time in chapters 9 and 10, we have used the non-Markovian process which is tackled using supplementary variable corresponding to remaining repair time.

1.4.3 Numerical techniques

The performance prediction of complex machining systems by using classical analytical queueing techniques is quite tedious task; in such cases numerical techniques are quite useful. To obtain probabilities associated with the state space of governing model various numerical techniques such as ***Runge-Kutta (R-K)***, ***Matrix method***, ***Successive over relaxation***, etc can be employed.

- ***Runge-Kutta 4^th^ Order Method (R-K)***

This method is good choice for solving the set of linear ordinary differential equations governing the model. It is noticed that *Runge-Kutta 4th order method* is quite accurate, stable and easy to implement for obtaining the transient solution of queueing problem. Using MATLAB, Runge-Kutta method of 4th order is implemented to get the unknown probabilities associated with the system state space by using the "Ode 45" function. The iterative steps used to compute the transient probability vector Π in chapters 6, 7 and 8 are as follows:

$$\Pi_i=\Pi_{i-1}+\frac{1}{6}(K_1+2K_2+2K_3+K_4) \tag{1.12}$$

where $K_1 = h\,f(t_{i-1}, \Pi_{i-1})$, $K_2 = h\,f(t_{i-1} + \frac{1}{2}h, \Pi_{i-1} + \frac{1}{2}K_2)$,

$$K_3 = h\,f(t_{i-1}, \frac{1}{2}h, \Pi_{i-1} + \frac{1}{2}K_1),\quad K_4 = h\,f(t_{i-1} + h, \Pi_{i-1} + K_3).$$

(1.13)

- ***Successive over relaxation (SOR) method***

By applying extrapolation to the Gauss-Seidel method, the successive over relaxation (SOR) method, is figured out. This extrapolation takes the form of a weighted average between the previous iterate and compute Gauss-Seidel iterate successively for each component by using $X_i^{(k)} = (1-\omega)\,X_i^{(k-1)} + \omega\,\overline{X}_i^{(k)}$, where $\overline{X}_i$ denotes a Gauss-Seidal iterate, and ω is the extrapolation (or scaling) factor. The idea is to choose a value for ω that will accelerate the rate of convergence of iterates to the solution. The SOR method is successfully employed to solve the set of Chapman-Kolmogorov equations governing the models developed in chapters 3 and 4.

- ***Matrix method***

In this method to solve the difference differential equations, we first take Laplace transform of each of the governing equations system states and obtain the transient state queue size distributions by solving them taking inverse Laplace transform. Jain and Bhargawa (2009) have presented a transient solution for the degraded machine repair problem with unreliable server and mixed standby support. For studying the steady state behavior of any repairable machining system, the matrix method can be easily employed. The utility of matrix recursive technique for the machining system is that it can be employed for such a systems also which are represented with infinite states. The probability vector of system states can be obtained easily by solving the steady state governing equations.

1.4.4 Optimization technique

The cost function constructed for real time system is complex and highly non-linear in nature; therefore, it is not easy to obtain the optimum system cost analytically. For the optimal deign of fault tolerant machining systems, several researchers have used the analytical and fuzzy optimization methods so as to optimize the system cost by evaluating the optimal system parameters (Jha *et al*. 2009; Jha *et al*. 2011; Jha *et al*., 2014). To deal with the highly non-linear functions, numerous numerical methods are available in the literature such as Newton's method, quasi-Newton approach, search approaches, which can be used for the cost optimization of machining system.

In the present investigation, we have used quasi-Newton and direct search methods for the optimization purpose. Quasi-Newton method can be easily used to find the global values of continuous decision variables $(x_1, x_2, ..., x_n)$ by minimizing the cost $TC(x_1, x_2, ..., x_n)$ which is a non-linear convex function and twice continuously differentiable. It is an iterative method with some stopping criterion depending on the tolerance limit. The main advantage for implementing this method is its fast convergence and affine invariance. The quasi-Newton method can be implemented for the optimal design of system in queueing environment (Wang *et al*., 2009). The theoretical basic iterative step of quasi-Newton method is defined as

$$x^{i+1} = x^i - t\nabla^2 TC(x)^{-1} \nabla TC(x) \tag{1.14}$$

The following steps to implement quasi-Newton method are performed to reach the minimum value of $TC(x_1^*, x_2^*, ..., x_n^*)$ and the corresponding optimal parameters $(x_1^*, x_2^*, ..., x_n^*)$.

(i) Let the initial value of decision variables $\overrightarrow{\Omega_0} = [x_1, x_2, ..., x_n]^T$, $i = 0$ and set tolerance ε.

(ii) Set the initial trial solution for $\overrightarrow{\Omega_0}$ and compute $\mathrm{TC}(\overrightarrow{\Omega_0})$.

(iii) Compute the cost gradient $\vec{\nabla} TC(\overrightarrow{\Omega_i}) = \left[\frac{\partial TC}{\partial x_1}, \frac{\partial TC}{\partial x_2}, ..., \frac{\partial TC}{\partial x_n}\right]_{\vec{\Omega} = \vec{\Omega}_i}$ and the cost Hessian matrix .

$$H(\overrightarrow{\Omega_i}) = \begin{bmatrix} \frac{\partial^2 TC}{\partial^2 x_1} & \frac{\partial^2 TC}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 TC}{\partial x_1 \partial x_n} \\ \frac{\partial^2 TC}{\partial x_2 \partial x_1} & \frac{\partial^2 TC}{\partial^2 x_2} & \cdots & \frac{\partial^2 TC}{\partial x_2 \partial x_n} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial^2 TC}{\partial x_n \partial x_1} & \frac{\partial^2 TC}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 TC}{\partial^2 x_n} \end{bmatrix}_{\vec{\Omega} = \vec{\Omega}_i}$$

(iv) Find the new trial solution $\overrightarrow{\Omega_{i+1}} = \Omega_i - \left[\mathrm{H}\left(\overrightarrow{\Omega_i}\right)\right]^{-1} \vec{\nabla}\mathrm{TC}\left(\overrightarrow{\Omega_i}\right)$.

(v) Set $i = i + 1$ and repeat steps (iii) and (iv) until $\max\left(\left|\frac{\partial TC}{\partial x_1}\right|, \left|\frac{\partial TC}{\partial x_2}\right|, ..., \left|\frac{\partial TC}{\partial x_n}\right| \right) < \varepsilon$.

(vi) Find the global minimum value $TC^*(x_1^*, x_2^*, ..., x_n^*) = TC^*(\overrightarrow{\Omega_i^*})$.

1.4.5 Adaptive-neuro fuzzy inference system (ANFIS) model

The combination of neural network and fuzzy logic presents an emerging soft computing technique ANFIS. It is widely used for the performance of complex systems for which analytical formulae cannot be framed. The use of soft computing technique for the performance modeling plays a vital role due to its critical utility in decision analysis, automatic control, fault detection of complex machining systems detection/removable and many more. The noticeable works on ANFIS has been done by Jang (1993). Lin and Liu (2003) have proposed an adaptive Neuro-fuzzy inference system for the optimal analysis of chemical–mechanical polishing process parameters. They examined the machine parameters during the wafer flattering process by chemical polishing and adopted an ANFIS to predict the surface roughness in the absence of CMP experiment. Jain and Upadhyaya (2009) dealt with a multi-component machining system consisting of M operating units along with *k* types of spare machines. They also evaluated the performance indices by using ANFIS which can identify parameters by applying the supervised learning methods. Sharifian *et al*. (2011) have given a predictive and probabilistic load balancing algorithm for the cluster-based web server; this algorithm significantly improves both the throughput and mean response time in contrast of two existing load balancing algorithms. By using the feature of a neural network and fuzzy inference system, some researchers have developed the adaptive neuro-fuzzy inference system (ANFIS) controller for the performance analysis of various embedded systems in different frameworks (cf. Mucsi *et al.*, 2011; Yang and Zhao, 2012). For the performance prediction of degraded multi-component system with standby switching failure and operating under N-policy and multiple vacations, Kumar and Jain (2013) used ANFIS to match the soft computing based results with the results obtained numerically using successive over relaxation (SOR). The performance analysis of machining system with F-policy and retrial attempts has been done using the recursive method by Jain and Sanga (2017). They have compared the results obtained by recursive method with the results of ANFIS hybrid soft computing technique.

The hybrid soft computing approach ANFIS is a neural network based representation of fuzzy systems equipped with learning capabilities. In fuzzy rule-based ANFIS, the rules can be formulated as

$$\text{IF } u_1 \text{ is } A_1 \text{ AND } u_2 \text{ is } A_2 \ldots \text{AND } u_n \text{ is } A_n \text{ THEN } v = f(u_1, u_1, \ldots, u_n)$$

Here f is a linear combination of the input variables $(u_1, u_1, \ldots, u_n)$, and A_i 's are the associated fuzzy sets.

Thus

$$f(u_1, u_1, \ldots, u_n) = w_0 + w_1 u_1 + w_2 u_2 + \ldots + w_n u_n \tag{1.15}$$

where $w_0, w_1, \ldots, w_n$ are real constants. This is a particular case of the weighted average method of defuzzification.

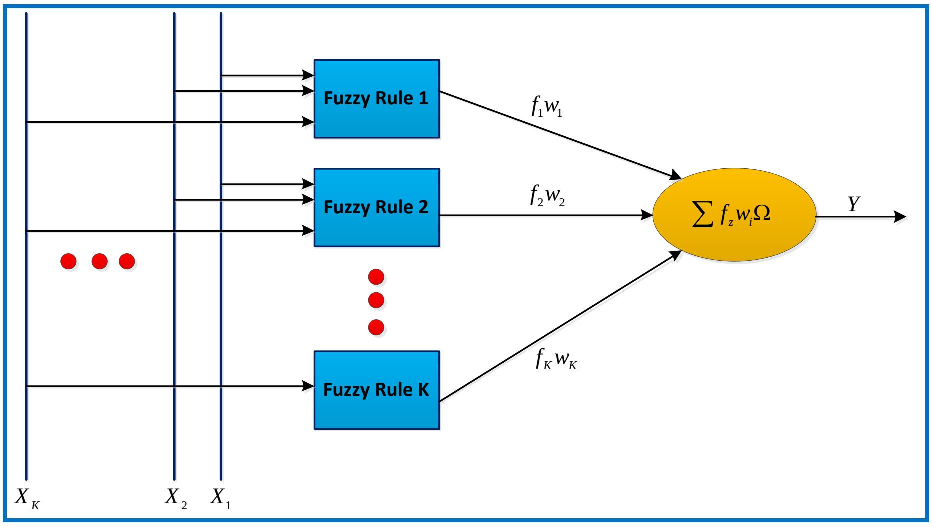


Fig. 1.5: Neuro fuzzy network

The ANFIS has a number of layers where each layer has a number of nodes. For FTS model, a fuzzy inference system with one input parameter (say λ) and one output E(N) can be described by the following n rules (Takagi and Sugeno, 1985):

$$\text{IF } \lambda \text{ is } A_1 \text{ THEN } f_1 = p_1\lambda + r_1$$

$$\text{IF } \lambda \text{ is } A_2 \text{ THEN } f_2 = p_2\lambda + r_2$$

$$\dots$$

$$\dots$$

$$\text{IF } \lambda \text{ is } A_n \text{ THEN } f_n = p_n\lambda + r_n$$

Let $Q_{l,i}$ be the output of node i in layer l. Thus, the functionalities of the layered architecture (see Figure 1.5) of ANFIS can be explained briefly as follows:

Layer 1: Each node in the 1st layer is an adaptive unit with output

$$O_{1,i} = \mu_{A_i(\lambda)} = w_i\ , \quad i = 1, 2, \dots, n$$

Here w_i is the firing strength of each node. The shape of membership function for each A_i can be taken as Gaussian.

Layer 2: Hidden layer 'j': For each node in hidden layer output is obtained using

$$O_{j,i} = \overline{w_i} = \frac{w_i}{\sum_{i=1}^{n} w_i}, \quad i = 1, 2, \dots, n; \quad j = 1, 2, \dots, K-1 \tag{1.16}$$

Layer 3: For each node in the layer 3, the output is obtained as

$$O_{j+1,i} = \overline{w_i}\, f_i = \overline{w_i}(p_i\lambda + r_i), \quad i = 1, 2, \dots, n; \quad j = 1, 2, \dots, K-1 \tag{1.17}$$

Layer 4: Considering the single node in output layer, the overall output is determined by

$$O_{K+1,1} = \sum_{i=1}^{n} \overline{w_i} f_i = \frac{\sum_{i=1}^{n} w_i f_i}{\sum_{i=1}^{n} w_i} \tag{1.18}$$

1.5 Some Markovian Queueing Models of MRP

Here, in this section we provide a brief account of some Markovian models related to our investigation.

1.5.1 M/M/R machine repair model with standby

When we talk about the performance of machining system and its efficiency/availability, we can't ignore the fact that the machines are always prone to failures. These failures can interrupt the services being provided through these machines. To reduce the inconvenience due to failures, the system should be supported with the standbys. Whenever any machine breaks down, which may be due to its component failure or some common cause, it can be replaced by the available standby part.

Consider the Markovian MRP with finite population M. Let the failure rate of operating (standby) machines be denoted by λ (*a*). Then the effective failure rates of (i) MRP without standbys, (ii) MRP with Y standbys and (iii) MRP with mixed standby support of *i* (1≤ *i*≤ *k*) types S_i machines k^{th} type of standby machines are respectively, are given by

(i) For finite population model:

$$\lambda_m = (M-m)\lambda;\ m = 0,1,2,...,M \tag{1.19}$$

(ii) For MRP with cold standbys:

$$\lambda_m = \begin{cases} M\lambda, & 0 \le m \le Y \\ (M-m+Y)\lambda, & Y \le m < Y+M \\ 0, & m \ge Y+M \end{cases} \tag{1.20}$$

(iii) For MRP with mixed standbys:

$$\lambda_m = \begin{cases} M\lambda + \left(\sum_{i=2}^{k} S_i a_i + (S_1 - m) a_1 \right); & 0 \le m \le S_1 \\ M\lambda + \left(\sum_{i=1}^{j} S_i - m \right) a_j + \sum_{i=j+1}^{k} S_i a_i; & S^{(j-1)} \le m \le S^{(j)};\ j = 2,3,4,...,k \\ \left(M + \sum_{i=1}^{k} S_i - m \right) \lambda; & S^{(k)} \le m \le K = M + S^{(k)} - 1 \end{cases} \tag{1.21}$$

where a_i denotes the failure rates of i^{th} type standbys.

1.5.2 Standby switching failure model

The replacement from standby state to operating state is said to be the standby switching. The switching may be either manual or automatic (i.e through an automatic switching device). There is always a probability of imperfect switching i.e. after switching the machine may work or may not work; in case of automatic standby switching, the switching device may also fail due to any reason with some probability (see Figure1.6).

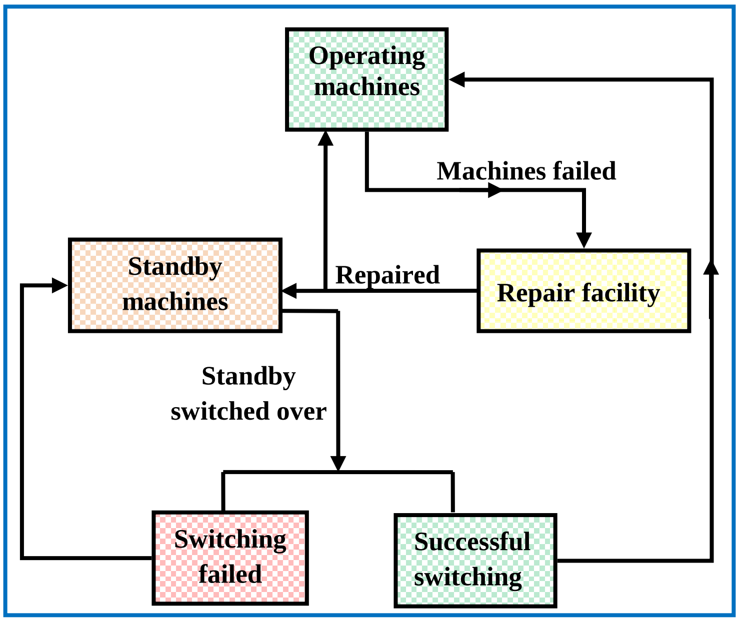


Fig. 1.6: Standby switching failure

Let us consider an example of any machining system, where the main concern is the uninterrupted power supply. When main power supply fails, an alternate power supply source whether it is generator or inverter, is switched on. There is always a possibility that the switching device may also fail with some probability and the interruption problem occurred due to power supply failure may not sorted out due to standby switching failure.

1.5.3 M/M/1 machining system with unreliable server

In finite population machining system, the server may fail at any instant while rendering the service to the failed machines. The state transition diagram of queueing model of machining system with service interruption due to server break down is depicted in Figure 1.7. The status of system with an unreliable server can be expressed by node (m, i), where m $(m=0,1,2,\ldots,K)$ denotes the number of failed machines in the system and i $(i=0,1)$ denotes the status of the server.

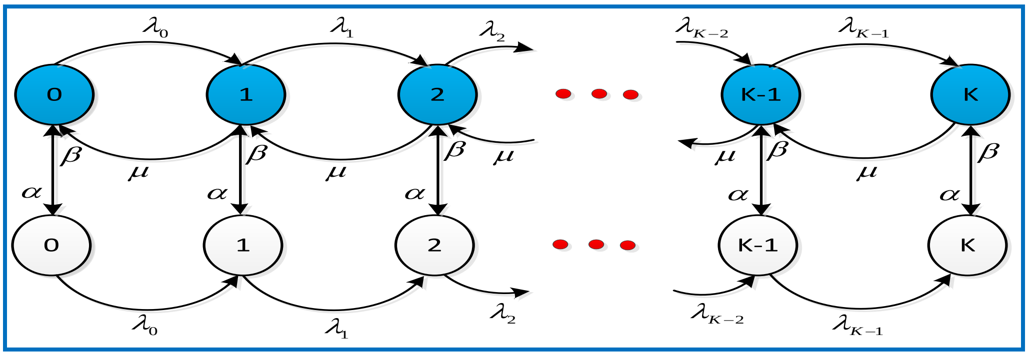


Fig. 1.7: Machining system with server breakdown

The state $i=0(1)$ denotes the repair state (busy state) of the server. Here notations λ_n and μ denotes the state dependent failure rate and service rate, respectively. The life time and repair time of the server are assumed to be exponentially distributed with rate α and β, respectively.

1.5.4 M/M/1 machining system with complete vacation

The queueing modeling of machining system with vacationing server can also be done to deal with many realistic situations where the server may leave the system to go for vacation in case when the system becomes empty. The state transition diagram for multi-component machining system have M similar machines and having single reliable server and provision of complete vacation is shown in Figure 1.8. The states of system with complete vacation can be expressed by node (m, i), where m $(m=0,1,2,\ldots,K)$ denotes the number of failed machines in the system and i $(i=0,1)$ denotes the status of the server. The state $i=0(1)$ denote the complete vacation (busy state) of the server. Here θ denotes the vacation completion rate of the server.

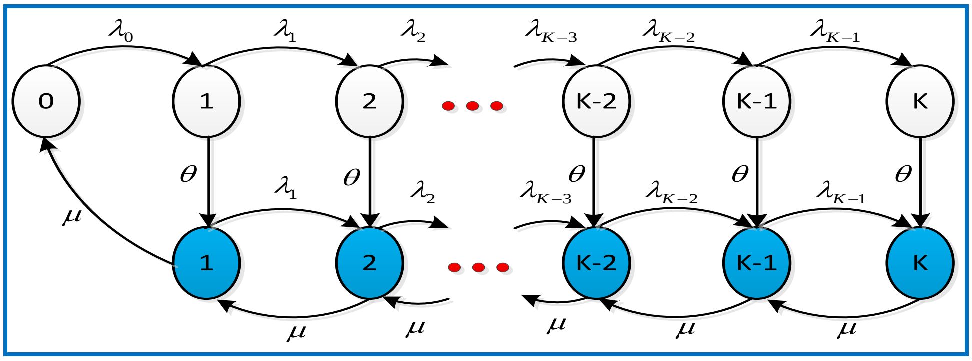


Fig. 1.8: Machining system with complete vacation

1.5.5 M/M/1 queue with working vacation

In many service systems, the server while on vacation may not like to remain idle due to many reasons including the loss of profit in case when some jobs accumulate during the vacation period. The same is the case with the machine repair systems; in such case, when the failed machines join the system during vacation, the server rather than completely

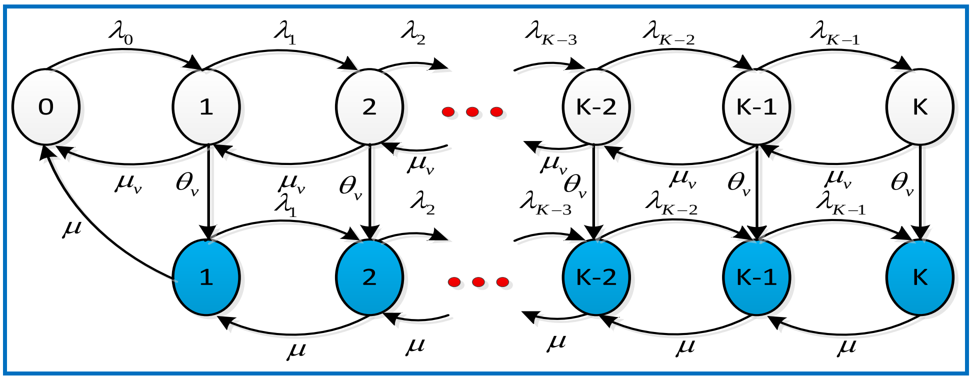


Fig. 1.9: Machining system with working vacation

stopping the repair job, provides repair to the failed machines at a different pace and is called on working vacation. The transition flows among neighboring states for multi-component machining system having M similar machines, single reliable server and operating under working vacation is depicted in Figure 1.9. The states of system with working vacation can be expressed by node (m, i), where m ($m=0,1,2,\ldots,K$) denotes the number of failed machines in the system and i ($i=0,1$) denotes the status of the server. The state $i=0(1)$ denotes the working vacation state (busy state) of the server. Here θ_v and μ_v denote the vacation completion rate and service rate during vacation state, respectively.

1.5.6 M/M/1 machining system operating under F-policy

In real time machining systems, the provision of redundancy as well as maintainability of the failed components can be made so as the system can operate in spite of unpredictable failures of machining components. However, after a certain level, the flow of failed machines for the repair jobs may not be permitted due to capacity constraint of the repair job shop. In the context of real life applications, arrival control is one of the cost-effective as well as managerial efficient approaches which can be used for the optimal utilization of the capacity of the machining system. The system state can be denoted by node (m,i) where m ($m=0,1,2,\ldots,K$) represents the number of failed machines in the system and i ($i=0,1$) denotes the status of the server. Here $i=1(0)$ represents the status of system when failed machines are allowed (not allowed) in the system. The rate transition diagram for machining system under F-policy and single reliable server is depicted in Figure 1.10. The notation γ denotes the set up rate for allowing the failed machines to enter into the system.

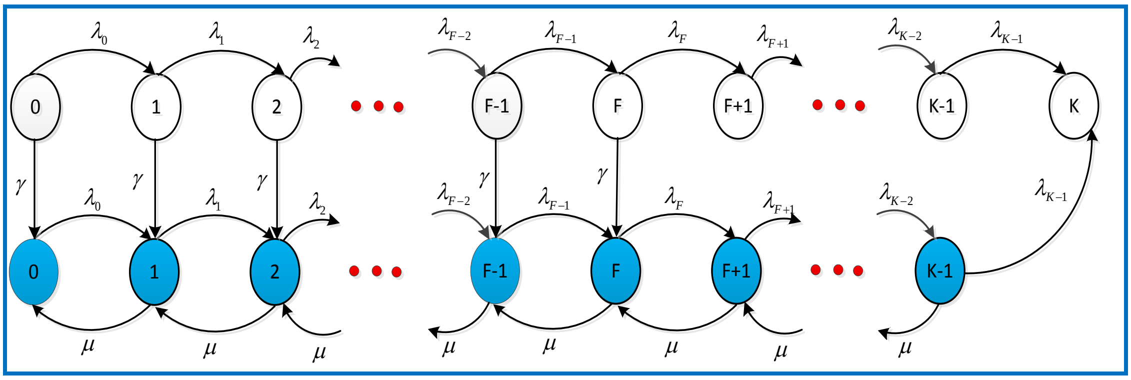


Fig. 1.10: Machining system under F-policy

1.5.7 K-out-of-N:G machining system

The K-out-of-N: G configuration in any machining system states that at least K out of total M machines should be in good condition for the normal functioning of the system. The reliability of K-out-of-N: G configuration is defined as

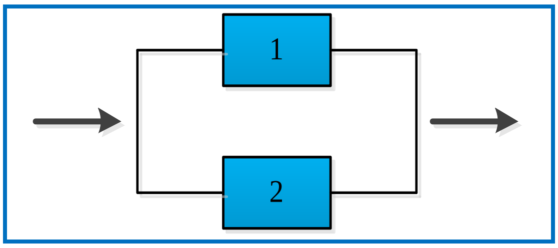


Fig. 1.11: 1-out-of -2 Configuration

For the special case of 1-out-of -2: G configuration, the system consists of total two machines; one of which is in operating mode while other one acts as standby (see Figure 1.11). When a failure occurs, it is repaired while the other one continues the operation. The system is said to be completely fails when both the machines fail.

1.5.8 Non-Markovian model for machining system

Consider a non-Markovian system consisting of '*n*' similar components. Let lifetime and repair time of components are exponentially and general distributed, respectively. To deal with the non-Markovian repair time, we introduce a supplementary variable measuring the remaining repair time for each component at each of the system states. Let us define the system state by $i\left(i=0,1\right)$ which represents either operational $(i=1)$ or failed $(i=0)$ state.

For the illustration purpose, we consider 2-*i.i.d.* components system in a standby configuration. When one of the components is operational, the other component is in standby mode i.e. the component first fails, then the other takes over the responsibility while the first is being repaired. It is assumed that the component failure rate λ is constant and time-to-repair of the component is general distributed having a cumulative distribution function $B(u),(u\geq 0)$ and a probability density function $b(u),(u\geq 0)$ with mean repair rate 'b'.

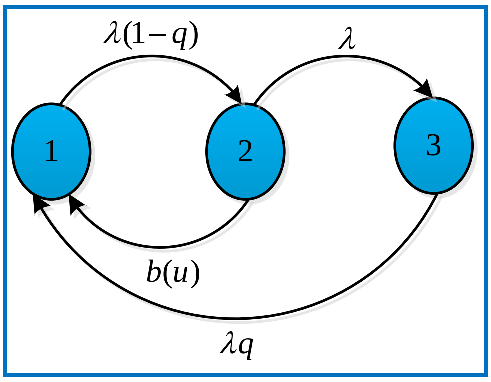


Fig. 1.12: State transition diagram

There may be the possibility of failure during switching from standby to operational state. We assume that 'q' is the probability of a switching failure. The state transition diagram for this system is shown in fig. 1.12.

1.5.9 Fault tolerant machining systems

With the advancement of modern technology, computer controlled fault tolerant machining system has become the necessity and it brought a tremendous change in the system design to control the risk of machine failure. The operation and capacity of fault tolerant systems involved in software embedded machining systems are highly affected by the failure of machining components. The occurrence of faults may cause not only a loss of desired output and efficiency but also an increase in the down time and cost. To avoid these adverse situations, the organizations or industries make provision of standbys and maintenance. In the fault tolerant systems (FTSs), some units may fail, but still the system remains operative and continues to perform its assigned job due to the provision of maintainability, optimal control and standbys. In many FTSs, machines are equipped with an inbuilt fault-handling mechanism which automatically detects the failure of a component and recovers the system by replacing the failed operating unit with a standby unit, if available. In the case when the fault handling mechanism fails to detect and recover the faults, the machines can also reconfigure temporarily by reboot process. But in some practical situations, the fault-handling device may prove inadequate to recover a fault perfectly; this situation is known as imperfect coverage.

1.6 Survey of Literature

Due to wide spread applications of queueing theory in real life scenarios to deal with congestion situations, it has been a deep interest of subject in recent past year. The introductory work on queueing theory was due to a Danish mathematician A.K. Erlang, 1909, who published "The Theory of Probabilities and Telephone Conversations". The waiting in queue phenomenon is not just an experience confined to human beings; the failed machines may also wait to be repaired. In this section, our prime objective is to review the significant contributions of queue theorists in the direction of performance modeling of machining systems in different frameworks which are relevant to the topics investigated in the present thesis work.

1.6.1 Queueing Modeling of Machining System- A brief historical view point

Noticeable works have been reported on finite population queueing models, due to fact that these can be used for the modeling of wide range of real-time systems. From the historical view point, it’s worth-noting to have a look on the important past research works done for finite population queueing models. Early past contributions on machine repair problem (MRP) can be attributed to Palm (1943) who applied the probabilistic model for analyzing the textile problem. A lot of research works have been done on machine interference systems by eminent queue theorists; the notable contributions in this regards can be found in survey article of Haque and Armstrong (2007) and Jain *et al*. (2010). Now we shall provide a brief review of the literature related to our works over the past few decades up to recent contributions. Machines repair problems have been studied by many queue theorists for a variety of congestion situations. They have used different assumptions and approaches for this purpose. The significant contributions of researchers from historical development of the MRP can be attributed to Ashcroft (1950), Taylor and Jackson (1954). The recent past works done by Gross *et al*. (1977), Elsayed (1981), Chelst *et al*. (1981), Wang (1990), Wang and Hsu (1995), Shawky (1997) are worth noting in the area of MRP. Some Markovian models by including the concepts of re-attempts of jobs present in the retrial orbit have been studied by (Phung-Duc *et al*., 2010; Artalejo and Phung-Duc, 2013; Phung-Duc and Kawanishi, 2014).

Wang *et al*. (2011) investigated a machine repair problem incorporating variable server under balking concept. They have used recursive approach to evaluate the steady state probabilities of system and carried out the sensitivity and cost analysis of the investigated model. Nobel (2013) analyzed the queueing model with retrial attempts using discrete time analysis for a fault tolerant server system. Recently, Shekhar *et al*. (2017) presented a survey article on queues in machining systems to compile the more recent works done from the 2010 to 2017.

1.6.2 Machining system with standby support

The random failures of machining parts during the operation have adverse effects on the revenue and production of an industry/organization operating in machining environment. The failure prone machines as well as unreliable server can interrupt not only the smooth functioning of the system but also result in the reputation loss and degradation in the quality of product. To ensure the pre-specified system reliability and to achieve the production goal, the system designers have paid their attentions towards the fault tolerant systems. Therefore, the standby support has become the essential attribute to develop fault tolerant system (FTS) embedded in many industries such as computer/communication, manufacturing/production, transportation/electronic systems and many more industries. To ensure the high reliability of the machining system, the standby support is essential requirement, but the system designers/organizers can use a limited number of standby parts due to overloading, cost constraint, size limitations, weight and space problem, power consumption, and many other factors. In the recent past, many researchers have attracted towards the reliability measures of the machining systems with different configurations having the maintenance and spare provisioning. In the recent past, the significant contributions on repairable machining systems with standby support have been done by Nakagawa and Osaki (1975), Murari and Goyal (1984), Goel *et al*. (1985), Singh and Srinivasu (1987a), Singh and Srinivasu (1987b) and many other. In recent years, finite population M/M/R Markovian models with redundancy and some more features viz. reneging and balking were examined by Wang *et al*. (2007) and Jain *et al*. (2008). Ke and Lin (2008) proposed Markov model to study the queueing characteristics of machining system with standby support for the manufacturing system. They carried out the sensitivity and profit analysis by developing two models with different service interruptions. The cost analysis of Markovian MRP can be seen in queueing models analyzed using recursive approach by Sivazlian and Wang (1989), Wang and Sivazlian (1992) and Wang (1993). Wang (1995) used recursive method and direct search approach to study the performance of multi-component machining system supported by two types of spare. Hsieh and Wang (1995), Wang and Kuo (2000), Wang and Ke (2003) and Jain *et al*. (2004) proposed matrix method to obtained various metrics including MTTF and reliability/availability of repairable redundant system. Chakravarthy and Gómez-Corral (2009) analyzed a repairable *k*-out-of-*N* system with spare part support. The performance and optimization analysis of machining system with standby switching failure have been carried out by Ke *et al*. (2016). Jain (2016) presented the transient study of machining system by incorporating some features, namely service interruption, priority and mixed standbys support. Recently, the reliability analysis of a repairable redundant machining system with standby support, switching failure and geometric reneging has been carried out by Shekhar *et al*. (2017).

1.6.3 Queueing system with unreliable server

The queueing model with server break down can be developed to depict more real-world machining systems. The queueing and reliability characteristics of machine repair problems where the server is subject to random failure have been modeled by several researchers for a various queueing situations.

The concept of server breakdown in queueing theory was first introduced by White and Christie (1958). Since then several queueing models in Markovian and non-Markovian set up have appeared in literature. Several contributors have dealt with the problem of sever breakdown in queueing system (Shogan, 1979; Alam and Mani 1989; Wang, 1995, Wartenhorst, 1995; Ke and Wang 1999). The significant contributions on unreliable server for non-Markovian queueing systems using supplementary variable techniques are due to Choudhury *et al*. (2009), Choudhury and Tadj (2009), Choudhury and Ke (2012) and Singh *et al*. (2013). Chakravarthy and Agarwal (2003) developed a MRP with an unreliable server and phase type repairs and services. Ke (2004) studied a like-queue production system under bi-level control policy by considering an unreliable server (machine) which operates under N policy with an early startup. Ke *et al*. (2009) developed a Markovian model having a finite buffer of the multi-server queueing system in which the servers are unreliable and allowed to go for vacation on the basis of (d, c) vacation policy. Hassan and Hoda Ibrahim (2013) proposed a multi-level queueing model with server breakdown. Yen *et al*. (2016) studied a repairable machining system by considering standby support, working breakdown and unreliable service station. Further, Kuo and Ke (2016) presented the performance results for MRM with standby, switching failure and unreliable server.

1.6.4 Machining system with vacation

In the recent past, the server vacation models have been studied to analyze the system performance in specific situations wherein the server becomes unavailable for some times. From the cost-economic point of view, it is beneficial to send the server on vacation as soon as he becomes idle or no repair job available in the system. Due to its critical applications, queueing model with server vacation can be applied in many systems operating in machining environment in different set up. In most of the machine repair queueing models, it is assumed that if any failed machine joins the queue, the server will be immediately activated for rendering the service. The important contributions on vacation queueing models in different contexts can be found in the survey work done by Doshi (1986), Takagi (1991), Selvam and Sivasankaran (1994), Tian and Zhang (2006) etc. Choudhury and Borthakur (2000) presented stochastic decomposition results using analytical approach for batch arrival queue, single and multiple vacation. In recent past, some noticeable research works on queueing model with vacation in the context of performance modeling of machining system have appeared (cf. Gupta, 1997; Jain *et al*., 2004; Ke and Lin, 2005).

Under the assumption of vacation policy, Ke and Wang (2007) developed the machine repair model having two types of spares. In this study, they have used the matrix geometric method for the prediction of performance measures related to queueing characteristics. Jain and Upadhyaya (2009) investigated the performance of multi-component machining system with multiple vacations of the servers, multiple types of redundant components and operating under N-policy. They obtained the probabilities for the system states and various key metrics by implementing the matrix recursive method. Ke *et al*. (2011) obtained the various system performance measures and presented the cost analysis for the machine repair problem (MRP) with standby support under the assumption of server vacation. Ke and Wu (2012) developed Markovian multi-server machine repair models for the machining system with synchronous multiple vacation and standbys support by implementing the matrix analytical technique to determine the formulae for the performance prediction of the concerned system. A multi-repairmen Markovian machine repair model with standby support and synchronous single vacation policy was proposed by Wu and Ke (2014). Wang *et al*. (2014) investigated M/M/1 Markovian machining system with the provision of standby support, multiple-vacation and unreliable server. Recently, Ke *et al*. (2017) investigated a M/M/c retrial queue with balking and vacation. The matrix-geometric method is used to evaluate the steady state probabilities of the system. A cost analysis is also done using three methods, namely quasi-Newton, Nelder-Mead Simplex, and simulated annealing method.

1.6.5 Machining system with working vacation

In many service systems, the server while on vacation may not like to remain idle due to many reasons including the loss of profit in case when some jobs accumulate during the vacation period. The same is the case with the machine repair systems; in such case, when the failed machines join the system during vacation, the server rather than completely stopping the service provides repair to the failed machines at a different pace and is called on working vacation. The introductory work on the working vacation model in the queueing literature was due to Servi and Finn (2002). They studied the M/M/1 queueing system by incorporating the feature of working vacation. Due to the enormous instances of MRPs with working vacation, the attention of queue theorists have diverted to this issue. A few research papers on working vacations in different contexts have appeared in which researchers have applied matrix geometric approach for the solution purpose (Tian *et al*., 2008; Lin and Ke, 2009; Wang *et al*., 2009; Ayyappan *et al*., 2010; Jain and Jain, 2010). The notable works have also been done on non-Markovian models with working vacation by Baba (2005) Jain and Agrawal (2007), Banik *et al*. (2007) and many more.

Lin and Ke (2009) proposed a MMR queueing model with working vacation and obtained the steady state queue size distribution using matrix-geometric method. Jain and Upadhyaya (2011) developed a unreliable finite buffer Markovian multi-server queueing model with discouragement and synchronous working vacation. Jain (2012) proposed a queueing model with second optional service by including the concepts of unreliable server and working vacation. As far as modeling of queueing system with MRP is concerned, some works related to working vacation can be seen in the queueing literature. To analyze the Markovian queue with working vacation, Selvaraju and Goswami (2013) considered the phenomena of impatient customers. Jain and Preeti (2014) investigated a M/M/1 Markovian machine repair model with standby support, working vacation and server breakdown. They have used recursive matrix method to evaluate the steady state probabilities of the system. Yang and Wu (2015) investigated the N-policy for M/M/1 working vacation queueing model by considering the server breakdowns. They employed the particle swarm optimization algorithm to optimize the cost function and determined the optimal parameters. To study more versatile and real world situations, Liu *et al*. (2015) proposed Markov machine repair model with the provision of cold type spares, working vacations and interruption due to vacation. For developing the queueing model, they considered the phase-type (PH) distribution for the life time of the components and obtained several system indices by employing matrix- analytical approach. Further, Goswami and Selvaraju (2016) studied the phase type of arrival in multi-server queueing model by including the concept of working vacation. The queueing modeling of machining system under the control F-policy and working vacation has been investigated by Jain *et al*. (2016).

1.6.6 Queueing system under F-policy

To control the arrivals of the jobs in case of full capacity of a system, admission control policy based on threshold level ‘F’ was first introduced by Gupta (1995). This policy states that the customer’s entry is stopped in the system if it reaches to its full capacity level. To allow again customers to join the system, it is assumed that the queue length should drop to a predefined threshold value ‘F’. The machine repair model (MRM) with F-policy have been studied by a very few researchers working in the area of queueing models. It is worthwhile to cite some important past works related to queues operating under F-policy. In this area, the significant contributions of queue theorists are as follows. For non-Markovian queue, the works done by Wang *et al*. (2007), Wang *et al*. (2008) are worth-mentioning. Ke *et al*. (2010) and Yang *et al*. (2010) studied single server finite Markovian model with F-policy by considering the optional service and working vacation, respectively. Chang *et al*. (2011) presented $M/H_2/1/K$ queue with F-policy, server startup and server breakdown. Huang *et al*. (2011) and Jain *et al*. (2012) investigated M/M/2/K queueing system under F-policy using genetic algorithm and matrix method, respectively. Jain *et al*. (2012) studied M/M/2/K queueing system under F-policy and N-policy.

Kumar and Jain (2013) obtained the queue size distribution using recursive method for the Markovian repairable machining system operating under both F-policy and N-policy. Chang *et al*. (2014) presented a randomized arrival control policy for a finite capacity queueing model with an unreliable server. They constructed the cost function to determine the optimal control policy at minimum cost. Yang and Chang (2015) studied queueing system under F-policy using fuzzy parameter approach. Recently, Shekhar *et al*. (2017) investigated a time-sharing machining system for the optimal (N, F) policy.

1.6.7 Machining systems with imperfect coverage

In literature, a very few researchers have contributed towards the queueing and reliability analysis of repairable machining system with imperfect coverage. It is worth-noting to cite some prominent works in the recent past on repairable machining system with imperfect fault coverage (cf. Pham, 1992; Shakil, 1994; Moustafa, 1997; Huang *et al*., 2006). The reliability and availability analysis have been carried out for repairable systems with imperfect fault coverage by Wang and Chiu (2006), Ke *et al*. (2008) and Ke *et al*. (2010). Optimal control policies for maximum profit in case of imperfect items under a variant assumptions has been proposed by many researchers (Jaggi *et al*., 2006; Jaggi *et al*., 2008; Jaggi *et al*., 2015).

Wang *et al*. (2012) investigated a repairable machining system using supplementary variable technique with the provision of standby support and imperfect coverage. Wang *et al*. (2013) proposed a Markovian model for MRP by incorporating the realistic assumptions of multiple types of imperfect coverage and state dependent service rate using the pressure conditions. To determine the optimal control parameters, they have used quasi- Newton method and particle swarm optimization (PSO) algorithm by constructing a profit function. The provision of multiple vacation and imperfect coverage for the performance modeling of a repairable machining system was proposed by Jain and Gupta (2013). Ke *et al*. (2013) presented the queueing analysis of unreliable multi-repairmen machining system comprising of operating machines and warm spares by incorporating the concept of imperfect coverage and reboot action. They determined the queue size distributions by employing the matrix recursive approach. Hsu *et al*. (2014) incorporated the noble features of switching failure, reboot delay and standbys support to make their machine repair model closer to realistic machining scenarios. They employed the probabilistic global search Lausanne (PGSL) method for the profit analysis of the system. Jain *et al*. (2014) used matrix method to explore the optimal N-policy for MRP by including some noble features such as unreliable server, imperfect coverage and reboot. Ke and Liu (2014) studied a repairable system operating in failure prone environment with reboot delay, repair facility and imperfect coverage. Wang *et al.* (2014) proposed an M/G/1 machine repair model with imperfect fault coverage. Kumar *et al*. (2015) investigated a computer system with imperfect fault detection of hardware. They have used the semi-Markov process and regenerative point approach for the evaluation of reliability measures of the concerned system. Jain (2016) presented a transient analysis of a repairable redundant system with the provision of mixed standby, imperfect repair, switching failure and reboot. Recently, a recursive method based on supplementary variable technique was used by Kuo and Ke (2016) to develop an M/G/1 model for the repairable system with an unreliable server and switching failure.

1.7 Contents of the Thesis

Queueing models of machining system with service interruption accommodate the real-world queueing scenarios more closely. In the present thesis, we have developed some queueing models with interruptions for the performance prediction of machining systems in different contexts, either by developing unreliable or/and vacationing server. To study the transient as well as steady state behavior of the system, our prime objective is to develop some queueing model for machining system with service interruption. The queue size distributions have been derived for evaluating the queueing and reliability indices. The analytical as well as numerical results are provided to explore how these models can be useful for the better design and operation of real-time machining systems.

The thesis is organized into 10 chapters including the first chapter devoted to general introduction and overview of the topics covered in the thesis. Chapters 2-8 and chapters 9-10 explore Markovian and non-Markovian models of machining system, respectively. The investigations done are arranged chapter wise as follows:

Chapter 1: General introduction.

Chapter 2: State dependent M/M/1 queue with optional working vacation.

Chapter 3: F-policy for unreliable server machining system with working vacation.

Chapter 4: Unreliable server FTS with working vacation.

Chapter 5: Unreliable server FTS with working vacation and working breakdown.

Chapter 6: MRP with unreliable server and threshold recovery.

Chapter 7: FTS with imperfect coverage, reboot and server vacation.

Chapter 8: F-policy for FTS with working vacation.

Chapter 9: Availability of R-out-of-N:G systems with imperfect fault coverage.

Chapter 10: Availability of M/G/1 systems with common cause shock failure.

The chapter-wise brief outlines of the thesis are as follows:

The ongoing ***Chapter* 1** presents the some basic concepts related to queueing modeling of machining systems with service interruption, methodology used and overview of the works done. The survey of relevant literature and summary of the work done in the thesis have also been presented. Moreover, some preliminary concepts such as F-policy, unreliable server, vacation, working vacation, reboot and recovery process, switching failure are described briefly. Several techniques used for the performance analysis of various models presented in the thesis are discussed.

Chapter 2 contains the *transient analysis* of Markovian single server queueing system by incorporating the *state dependent rates* and provision of *complete* and *partial working vacations of the server*. When the system becomes empty, the server has choice to go for either *multiple complete vacations* or multiple working vacations. The time-dependent system size probabilities are derived explicitly in closed form using generating function and continued fraction. Further, the explicit expressions for the stationary distributions are deduced from transient counterpart. Some important queueing measures and cost function are examined by taking numerical example.

Chapter 3 is devoted to the performance analysis of machining system operating under the *admission control* policy. The server is prone to failure and under go for the *working vacation* in case when there is no repair job in the system. The failed machines are allowed to enter the system till the system capacity (K) is full; then after failed machines are not allowed to join the system until the system size again decreases to the pre-specified threshold level ‘F’. At that instant, the server takes startup time in order to start allowing the failed machines to enter into the system for the repair job. A matrix method based on *successive over relaxation* (SOR) is applied to obtain the steady state probabilities and various performance indices including the cost function. Quasi-Newton method and direct search method are used to determine the optimal service rate and threshold parameter.

Chapter 4 is concerned with the performance evaluation of *fault tolerant* system (FTS) having multi-components operating machines and *multi types of warm standby* machines. The repair of failed machines is rendered by the server which is subject to breakdown and repair. The successive over relaxation (SOR) method is used to obtain the system state probabilities at the steady state which are further used to evaluate other system indices including the mean queue length of failed machines, mean number of standby machines, throughput, etc. By constructing the cost function in terms of various performance indices and associated cost elements, the optimal repair rate is determined so that the maintenance of the concerned failure prone FTS can be done in an economic manner. The hybrid soft computing approach based on a neuro-fuzzy inference (ANFIS) system is implemented to compare the numerical results obtained by SOR method and neuro-fuzzy approach.

Chapter 5 presents the performance analysis of *fault tolerant* system (FTS) by developing M/M/1/L finite population queueing model with *working vacation*. There is facility of dissimilar type *warm standby machines* so as to replace the failed machines in order to continue the operation of the system in spite of failures of machines. The repairman is allowed to take a working vacation in case of no workload of broken down machines. The *matrix method* is implemented for evaluating the transient queue size distribution and closed form expressions of the performance metrics of multi-component FTS. Moreover, cost function is constructed which can be further used to control the system cost factors.

In ***chapter 6***, Markov model is developed to facilitate the performance analysis of multi-component machining system operating under the care of *two heterogeneous servers* and with the facility of *mixed type of spares*. The repair job of broken down machines are done on the basis of bi-level threshold policy for the activation of the servers. The server returns back to render repair job when the pre-specified workload of failed machines is build up. The first (second) repairman turns on only when the work load of N_1 (N_2) failed machines are accumulated in the system. The both servers may go for vacation in case when all the machines are in good condition and there are no pending repair jobs for the repairmen. Runge-Kutta method is implemented to solve the set of governing equations used to formulate Markov model. Various system metrics including the mean queue length, machine availability, throughput etc. are derived to determine the performance of the machining system. A cost function is also constructed to determine the optimal repair rate of the server by minimizing the expected cost incurred. The hybrid soft computing method is considered to develop the adaptive neuro-fuzzy inference system (*ANFIS*).

In ***chapter 7*** studies the performance of *fault tolerant* machining system by incorporating the features of *imperfect coverage* and reboot. Markov model for machine repair problem with server vacation is developed under the assumption that the operating units can be replaced by the available cold/warm type standby unit. The on lines as well as warm standby units are subject to failures and are send for the repair to a repair station having single repairman which is prone to failure. If the failed unit is not detected, the system enters into an *unsafe state* from which it is cleared by the *reboot* and *recovery* action. The server is allowed to go for vacation if there is no failed unit present in the system. *Runge-Kutta* method is used to evaluate the system state probabilities and queueing measures.

C*hapter 8* contains the performance analysis of multi-component machining system by developing Markovian model by including the concepts of *imperfect coverage*, *standby support* and *working vacation*. The online and warm standby machines may fail and can be repaired by a single skilled repairman. When the system reaches to its full capacity, no more jobs of repairing of failed machines are accepted until the work load of repair jobs reduces to a threshold level 'F'. Some realistic features such as working vacation, *reboot* and *recovery* processes are included for the formulation of governing equations. Before initiating the repair of the failed machines in case of coming back from the vacation state, the server requires the set up time. The various performance measures including the reliability indices are derived by using the transient probabilities evaluated using *Runge-Kutta* method.

In ***chapter 9,*** *R-out-of-M: G* structure is investigated to analyze the *M/G/1* model of FTS. The system state probabilities are used to determine the availability of the concerned system. The noble concepts of imperfect fault coverage, recovery and replacement have been incorporated. A recursive algorithm has been suggested for *M/G/1* model to evaluate the steady state probabilities and various other indices by treating remaining repair time of the failed machines as *supplementary variable*. The system availability for *R*-out-of-*M*: *G* structure has been derived analytically for specific repair time distributions such as Exponential, Erlang and Deterministic. Furthermore, total cost is framed to obtain the optimal repair rate by using *Newton-quasi method*.

C***hapter 10*** is dealt with the performance prediction of multi-component fault tolerant machining system with common cause shock by developing the M/G/1 model with standby provisioning. By taking the remaining service time as supplementary variable, the queue size distribution of the concerned system is obtained. By using recursive and supplementary variable approaches, the state probabilities and availability indices have been evaluated analytically for the R-out-of-N: G structure. For specific distributions such as Exponential, 3-

stage Erlang and deterministic distribution, the explicit formulae for the availability are also established. The system performance for specific configurations has been examined by computing the numerical results for suitable illustration. For the optimal system design, the cost optimization has been done by using quasi-Newton approach to determine the optimal system descriptors.

1.8 Concluding Remarks

Queueing modeling of failure prone machining system done in the present study provides an effective and powerful tool that can help the system designers and organizers to achieve the pre-defined specific goals. This approach can be easily implemented and has several advantages including the good and rapid estimations of the system performance. Since machining systems are widely used in every sphere of human lives, the system performance can be enhanced by appropriate choice of standby provisioning and maintainability so that the system can operate smoothly without any hindrance. Increasing system utilization and decreasing waiting time can enhance the system productivity also. To improve the availability and efficiency of the machining systems, the provision of maintainability and standby support have been taken into consideration in several Markovian models.

Markov model offers a systematic approach to machine repair problem and allows for the system characterization of an existing system using birth death process. In chapters 2-8, several models are successfully analyzed based on Markovian assumptions of life time and repair time distributions by using suitable birth and death rates. The governing equations of concerned queueing models have framed using birth death process to evaluate the performance metrics. The investigation done in the thesis may be extensively practiced or utilized in industrial settings to resolve the system failure problems and may be helpful for lower downing the cost due to unavailability of server either due to server break down or server being on vacation. The rising cost of the system failure can be attributed not only to the system failure but also production of items which are not up to the marks due to some unnoticed faults in the system operation.

Queueing theoretic modeling proposed is an attempt to minimize the cost via improving the reliability and availability of the system. Various performance measures and cost function are established which are further validated by conducting the numerical simulations. Numerical simulation carried out provides the valuable insights for the sensitivity of several descriptors on the different indices such as the queue length of failed machines, throughput, machine availability, etc. The model developed in generic set up for the machining system depicts the realistic scenarios of many real time systems due to incorporation of spare part support and common cause shock failure. To make investigation done applicable to many embedded systems, the non-Markovian models have also investigated in chapters 9 and 10 by including the features of general repair time distribution.

The integrating analytic formulism and numerical simulation done provides a strong base for the queue modeling of concerned machining system. The combined strengths of analytic and hybrid soft computing approach ANFIS to tackle queueing problems in research and in practice make our study more significant. The numerical simulation results carried out may be helpful to draw out the managerial insights for the understanding real-life machining systems.

The fault tolerant systems studied have many practical applications including in defense where the equipments are prone to failure and high reliability is one of the key concerns of the system engineer. For such failure prone systems, queue theoretic quantitative assessment of performance indices is of vital importance to achieve the pre-specified reliability. We have developed the performance models of fault tolerant system supported by mixed standbys and repair facility. Some realistic concepts such as reboot, recovery, unreliable server and vacation are incorporated in order to develop generalized queueing models.

The performance analysis of FTS presented have potential applications in industrial and manufacturing systems, computer and communications systems and many more real time systems. The investigation done may provide the valuable insights to the system analysts/designers to enhance the performability, reliability and availability of the concerned failure prone systems operating in machining environment. The queueing models developed for the fault tolerant repairable redundant machining systems with service interruptions have enormous applications in the real-time systems such as power transmission lines, distributed data networks, nuclear power plants, data exchange systems, and aerospace applications, etc.

The suggested optimal control policy for maintainability will provide the gainful insight to the system analysts and industrial engineers to design an economical system under some techno-economic constraints. In industrial scenario, the optimal control model for the machining systems can be used to provide the performance indices for the fault tolerant embedded systems wherein the server as well as machining components are failure prone. The optimal repair rate corresponding to minimum total expected cost determined using Newton-quasi method facilitates the valuable performance metrics which can be used in many real time systems to achieve the goal of mission availability and fault tolerance at minimum cost.

In many real time machining systems, the failure of machining components may not be detected or located as such immediate replacement of failed component with standby component cannot be done and the case of imperfect recovery arises. For fault tolerant systems, this situation can be tackled by reboot operation according to which the system restarts depending upon the complexity of system.

Various performance measures viz. queue size distribution, long run probabilities, average queue length, etc. may be helpful to the system designers and decision makers for the improvement and enhancement of the existing systems operating in different machining frameworks. It is hoped that the investigations reported would likely to highlight the significance of queueing models with service interruption for the improvement of the design, development and configuration of machining systems.

Chapter 2

State Dependent M/M/1 Queue with Optional Working Vacation

2.1 Introduction

In many machining systems involved in day-to-day activities as well as in industrial scenarios such as computer and communication networks, flexible manufacturing and production systems, service and distribution systems, etc., there may be situations where the server may not available for the service due to some faults or operator being on vacation. The period of server's unavailability in queueing scenarios can be treated as queue with service interruption. In some queueing systems, the server can do the secondary job during vacation by rendering service with different rate and is called on working vacation. In more general set up there may be vacation in which the servers after becoming idle have the choice either go for complete vacation i.e. remain idle during vacation or serve the jobs with slower rate who arrive during vacation and treated as on working vacation. If there is no customer waiting in the queue for the service while the server returns to the system from vacation, he may start another vacation. This process continues until he finds a customer to be arrived after returning back from the vacation/working vacation. This situation in the queueing system is treated as the server follows the multiple vacations. In multiple vacation queueing system, after returning back from the vacation or working vacation, if somebody is waiting in the queue, the busy period of the server starts. In the recent past a sufficient amount of research work has been done on vacation queueing models (Choudhury and Madan 2005; Ke and Lin 2005; Ke 2007). Due to applicability in many real-world queueing situations, recently many queue theorists have paid their attention and contributed significantly on the performance modeling of queueing system with vacation by Jeyakumar and Senthilnathan (2012).

Working vacation policy is beneficial for both system organizers as well as customers. When the system becomes empty, the service station resumes working vacation in which it can perform the secondary jobs and also available for rendering the service to the new jobs with a slower rate. In many queueing situations, it is noticed that the service station can renders service to the jobs with a service rate which is lower than the normal rate, instead of completely unavailable for the service as in complete vacation policy. Servi and Finn (2002) initiated the working vacation concept for M/M/1 queueing model, where a customer receives service with a rate which is less than the usual one.

The analysis of transient behavior in closed form which is computationally tractable is quite tedious as can be noticed in case of single server Markovian model with vacation. To predict the transient behavior of Markovian single server queue with multiple vacations, a limited effort has been done by using the combined generating function and continued fraction approaches. In this chapter, we derive the time-dependent queue size distributions in explicit form for the state-dependent Markovian queue wherein on finding no customers in the system, the server has the option to go either for vacation or working vacation. A very few research work have done on transient analysis of queueing models (cf. Kumar and Arivudainambi, 2000; Parthasarathy and Selvaraju, 2001; Kumar and Madheswari, 2005; Kumar *et al*., 2007). Parthasarathy and Sudhesh (2007) derived the transient analytical results of system size probabilities for an M/M/1 retrial queueing model with state-dependent rates using a continued fraction. The modified Bessel function is used for the transient analysis of Markovian queueing model (Al-Seedy *et al*., 2009; Kumar *et al*., 2009; Kalidass and Kasturi, 2011; Kalidass and Ramanath, 2012). The transient and stationary probability distributions and system indices are obtained in closed form for M/M/1 queue with working vacations by Sudhesh and Francis Raj (2012). Kalidass *et al*. (2014) analyzed an M/M/1 queueing model where the server can resume multiple vacations. They also computed the steady state as well as time-dependent probabilities, mean and variance of the system size and other indices. Recently, Sudhesh *et al*. (2017) investigated an M/M/1 queueing model with working vacation and customers' impatience in the transient counterpart where they have derived the system size probabilities for both single and multiple working vacation models.

In this chapter, we analyze the transient behavior of multiple vacation M/M/1 queue with state-dependent rates and choice of server to avail either full vacation or go for working vacation in the case when the system becomes empty. The contents of the remaining chapter are arranged as follows. The single server queue having provision of vacation and optional working vacation and state-dependent rates is described in Section 2.2. The transient probabilities are obtained in Section 2.3 explicitly by employing the generating function and continued fraction approaches. In Section 2.4, the system performance measures and cost function are computed. Numerical simulations are carried out in the next Section 2.5 to understand the influence of system parameters.

2.2 Model Description

A single server queueing model with optional vacation/working vacation and state-dependent rates is considered. The in-flow and out-flow rates of the system states are shown in state transition diagram presented in figure 2.1. The assumptions to derive the queue size distribution at transient state are described as follows:

- In M/M/1 queueing system, the jobs join according to Poisson arrival pattern with the rate λ.
- The service of the jobs is rendered by single service station following the exponential distribution with mean μ^{-1}.
- Whenever the service station becomes free, it may go either for working vacation with probability σ or complete vacation with probability $\overline{\sigma} = 1 - \sigma$.
- During the complete vacation period, the arriving jobs are permitted to enter into the system according to Poisson fashion with the rate λ_1. The vacation duration follows an exponential distribution with the rate θ.
- During the working vacation period, the newly arriving jobs are also allowed to join into the system in Poisson fashion with rate λ_0 and are served following exponential distribution with mean μ_v^{-1}, where $\mu_v < \mu$. The working vacation is governed by the exponential distribution with parameter θ_v.
- We assume that the arrival and service times, vacation and working vacation times are all independent to each other. The service discipline is first come first served (FCFS).
- We consider the bivariate stochastic process $\{\xi(\tau), \eta(\tau)\}$ where $\{\eta(\tau), \tau \geq 0\}$ denotes the system size and $\xi(\tau)$ represents the service station state at time τ.

Now we define

$$\xi(\tau) = \begin{cases} 0; \text{ the server is on working vacation at time } \tau, \\ 1; \text{ the server is busy in providing service to the customers at time } \tau, \\ 2; \text{ the server is on complete vacation and is idle at time } \tau. \end{cases}$$

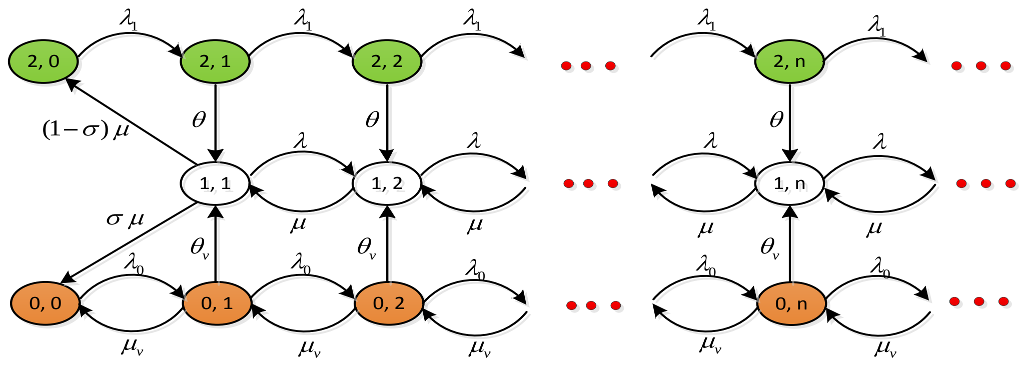


Fig. 2.1: State transition diagram of M/M/1/WV

2.3 Model Governing Equations

For the continuous time Markov chain $\{\xi(\tau),\eta(\tau),\tau\geq 0\}$ over the state space $\Omega=(0,0)\cup(2,0)\cup\{(k,n)/k=0\,\text{or}\,1\,\text{or}\,2;n=1,2,...\}$, we define the state probabilities as follows:

$$P_{0,n}(\tau)=P\{\xi(\tau)=0,\eta(\tau)=n\},n=0,1,2,...,$$

$$P_{1,n}(\tau)=P\{\xi(\tau)=1,\eta(\tau)=n\},n=1,2,3,...,$$

$$P_{2,n}(\tau)=P\{\xi(\tau)=2,\eta(\tau)=n\},n=0,1,2,....$$

The governing Equations for states $P_{k,n}(\tau),k=0,1,2;\ n=0,1,2,...,$ are constructed as follows:

$$P'_{0,0}(\tau)=-\lambda_0P_{0,0}(\tau)+\mu_vP_{0,1}(\tau)+\mu\sigma P_{1,1}(\tau), \tag{2.1}$$

$$P'_{0,n}(\tau)=-(\lambda_0+\mu_v+\theta_v)P_{0,n}(\tau)+\lambda_0P_{0,n-1}(\tau)+\mu_vP_{0,n+1}(\tau),\ n\geq 1, \tag{2.2}$$

$$P'_{1,1}(\tau)=-(\lambda+\mu)P_{1,1}(\tau)+\mu P_{1,2}(\tau)+\theta_vP_{0,1}(\tau)+\theta P_{2,1}(\tau), \tag{2.3}$$

$$P'_{1,n}(\tau)=-(\lambda+\mu)P_{1,n}(\tau)+\lambda P_{1,n-1}(\tau)+\mu P_{1,n+1}(\tau)+\theta_vP_{0,n}(\tau)+\theta P_{2,n}(\tau),\ n\geq 2, \tag{2.4}$$

$$P'_{2,0}(\tau)=-\lambda_1P_{2,0}(\tau)+\mu\bar{\sigma}P_{1,1}(\tau), \tag{2.5}$$

$$P'_{2,n}(\tau)=-(\lambda_1+\theta)P_{2,n}(\tau)+\lambda_1\bar{\sigma}P_{2,n-1}(\tau),\ n\geq 1. \tag{2.6}$$

with $P_{0,0}(0)=1$.

2.3.1 Transient system size distribution

We denote the Laplace transform of $P_{k,n}(\tau)$ by $\hat{P}_{k,n}(s)$. The transient probabilities for various states are derived by solving the governing Equations (2.1)-(2.6).

(i) Evaluation of $P_{2,n}(\tau)$

Taking Laplace transforms of both sides of Equation (2.5) and after simplification, we get

$$\hat{P}_{2,0}(s)=\frac{\mu\bar{\sigma}}{s+\lambda_1}\hat{P}_{1,1}(s) \tag{2.7}$$

Laplace inversion of (2.7) yields

$$P_{2,0}(\tau) = \mu\bar{\sigma}\, e^{-\lambda_1\tau} * P_{1,1}(\tau) \qquad (2.8)$$

Taking Laplace transform of (2.6) and after some algebra, we get

$$\hat{P}_{2,n}(s) = \left(\frac{\lambda_1}{s+\lambda_1+\theta}\right)^n \hat{P}_{2,0}(s) \qquad (2.9)$$

On taking Laplace inversion of (2.9), we get

$$P_{2,n}(\tau) = \lambda_1{}^n e^{-(\lambda_1+\theta)\tau}\, \frac{\tau^{n-1}}{(n-1)!} * P_{2,0}(\tau) \qquad (2.10)$$

Thus, $P_{2,n}(\tau)$ is expressed in terms of $P_{2,0}(\tau)$, and $P_{2,0}(\tau)$ is obtained in terms of $P_{1,1}(\tau)$.

(ii) Evaluation of $P_{1,n}(\tau)$

Let $G(z,\tau) = \sum_{n=1}^{\infty} P_{1,n}(\tau)\, z^n$ be the probability generating function (PGF) corresponding to $P_{1,n}(\tau)$. Applying PGF on (2.3) and (2.4), we get

$$\frac{\partial G(z,\tau)}{\partial \tau} = \left[-(\lambda+\mu)+\lambda z+\frac{\mu}{z}\right]G(z,\tau) - \mu P_{1,1}(\tau) + \theta_v \sum_{n=1}^{\infty} P_{0,n}(\tau)\, z^n + \theta \sum_{n=1}^{\infty} P_{2,n}(\tau)\, z^n \qquad (2.11)$$

Solving Equation (2.11), we obtain

$$G(z,\tau) = \int_0^{\tau} \sum_{n=1}^{\infty} \left[\theta_v P_{0,n}(u) + \gamma_1 P_{2,n}(u)\right] z^n\, e^{-\phi(\tau)} e^{\psi(\tau)} du - \mu \int_0^{\tau} P_{1,1}(u)\, e^{-\phi(\tau)} e^{\psi(\tau)} du \qquad (2.12)$$

where $\phi(\tau) = (\lambda+\mu)(\tau-\mu)$ and $\psi(\tau) = (\lambda z + \mu / z)(\tau-\mu)$.

Denoting $\alpha = \sqrt{\lambda\,\mu}$ and $\beta = \sqrt{\frac{\lambda}{\mu}}$, we have

$$e^{[(\lambda z + \lambda / z)\tau]} = \sum_{n=-\infty}^{\infty} (\beta z)^n I_n(\alpha\tau), \qquad (2.13)$$

Here $I_n(\tau)$ denotes the n^{th} order modified Bessel function of the first kind.

Using (2.13) and (2.12) for $n > 0$, we get

$$\begin{aligned} P_{1,n}(\tau) &= \int_0^{\tau} \sum_{m=1}^{\infty} \left[\theta_v P_{0,m}(u) + \theta P_{2,m}(u)\right] \beta^{n-m} I_{n-m}(\alpha(\tau-u))\, e^{-\phi(\tau)} du \\ &\quad - \mu \int_0^{\tau} P_{1,1}(u) \beta^n I_{n-m}(\alpha(\tau-u))\, e^{-\phi(\tau)} du \end{aligned} \qquad (2.14)$$

For negative integer value of n, Equation (2.14) holds but L.H.S. is zero. Also, we know that $I_{-n}(x) = I_n(x),\ n = 1,2,3,...,.$ Thus we get for $n = -1,-2,-3,...,.$

$$0=\int_0^\tau \sum_{m=1}^{\infty}\left[\theta_v P_{0,m}(u)+\theta P_{2,m}(u)\right]\beta^{-n-m} I_{n+m}(\alpha(\tau-u))\, e^{-(\lambda+\mu)(\tau-\mu)} du$$
$$-\mu\int_0^\tau P_{1,1}(u)\beta^{-n} I_n(\alpha(\tau-u))\, e^{-(\lambda+\mu)(\tau-\mu)} du \qquad (2.15)$$

For positive value of n, from (2.14) and (2.15), we obtain

$$P_{1,n}(\tau)=\int_0^\tau \sum_{m=1}^{\infty}\left[\theta_v P_{0,m}(u)+\theta P_{2,m}(u)\right]\beta^{n-m}[I_{n-m}(\alpha(\tau-t))-I_{n+m}(\alpha(\tau-t))]e^{-\phi(\tau)}dt. \qquad (2.16)$$

Thus, $P_{1,n}(\tau)$ is computed as a function of $P_{0,n}(\tau)$ and $P_{2,n}(\tau)$, for $n=1,2,3,....$

(iii) *Evaluation of* $P_{0,n}(\tau)$.

Laplace transforms on (2.1) and (2.2) yield

$$\hat{P}_{0,0}(s)=\frac{1}{(s+\lambda_0)-\mu_v\dfrac{\hat{P}_{0,1}(s)}{\hat{P}_{0,0}(s)}-\mu\sigma\dfrac{\hat{P}_{1,1}(s)}{\hat{P}_{0,0}(s)}} \qquad (2.17)$$

Also for $n\geq 0$,

$$\frac{\hat{P}_{0,n}(s)}{\hat{P}_{0,n-1}(s)}=\frac{\lambda_0}{(s+\lambda_0+\mu_v+\theta_v)-\mu_v\dfrac{\hat{P}_{0,n+1}(s)}{\hat{P}_{0,n}(s)}} \qquad (2.18)$$

On simplification of (2.18), we get

$$\hat{P}_{0,n}(s)=\beta_0^n\left(\frac{w-\sqrt{w^2-\alpha_0^2}}{\alpha_0}\right)^n \hat{P}_{0,0}(s) \qquad (2.19)$$

where $w=s+\lambda_0+\mu_v+\theta_v$, $\alpha_0=2\sqrt{\lambda_0\mu_v}$ and $\beta_0=\sqrt{\lambda_0/\mu_v}$.

Laplace inversion on (2.19) yields

$$P_{0,n}(\tau)=\lambda_0\beta_0^{n-1}[I_{n-1}(\alpha_0\tau)-I_{n+1}(\alpha_0\tau)]e^{-(\lambda_0+\mu_v+\theta_v)\tau} * P_{0,0}(\tau). \qquad (2.20)$$

(iv) *Evaluation of* $P_{1,1}(\tau)$.

For $n=1$, taking Laplace transform on (2.16), we get

$$\hat{P}_{1,1}(s)=\frac{1}{\mu}\sum_{m=1}^{\infty}\left[\theta_v\hat{P}_{0,m}(s)+\theta\hat{P}_{2,m}(s)\right]\left(\frac{p-\sqrt{p^2-\alpha^2}}{\alpha\beta}\right)^m, \qquad (2.21)$$

where $p=s+\lambda+\mu$.

Using (2.19), (2.9) and (2.7) in (2.21) and after simplification, we obtain

$$\hat{P}_{1,1}(s)=\frac{\theta_v}{\mu}\sum_{m=1}^{\infty}\beta_0^m\left(\frac{w-\sqrt{w^2-\alpha_0^2}}{\alpha_0}\right)^m\left(\frac{p-\sqrt{p^2-\alpha^2}}{\alpha\beta}\right)^m$$
$$\times\sum_{k=0}^{\infty}\theta^k(\bar{\sigma})^k\frac{1}{(s+\lambda_1)^k}\left[\sum_{i=1}^{\infty}\frac{\lambda_1^i}{(s+\lambda_1+\theta)^i}\left(\frac{p-\sqrt{p^2-\alpha^2}}{\alpha\beta}\right)^i\right]\hat{P}_{0,0}(s). \tag{2.22}$$

Laplace inversion of (2.22) gives

$$P_{1,1}(\tau)=\frac{\theta_v}{\mu}\sum_{m=1}^{\infty}\frac{\lambda_0}{\beta_0^{1-m}}\left[I_{m-1}(\alpha_0\tau)-I_{m+1}(\alpha_0\tau)\right]e^{-(\lambda_0+\mu_v+\theta_v)\tau}$$
$$*\frac{\lambda}{\beta^{m+1}}\left[I_{m-1}(\alpha\tau)-I_{m+1}(\alpha\tau)\right]e^{-(\lambda+\mu)\tau}*\sum_{k=0}^{\infty}\theta^k(\bar{\sigma})^k e^{-\lambda_1\tau}\frac{\tau^{k-1}}{(k-1)!} \tag{2.23}$$
$$*\left[\sum_{i=1}^{\infty}\lambda_1^i e^{-(\lambda_1+\theta)\tau}\frac{\tau^{i-1}}{(i-1)!}*\frac{\lambda}{\beta^{i+1}}\left[I_{i-1}(\alpha\tau)-I_{i+1}(\alpha\tau)\right]e^{-(\lambda+\mu)\tau}\right]^{*k}*P_{0,0}(\tau).$$

***(v)* Evaluation of $P_{0,0}(t)$.**

For $n=1$, using (2.19), and (2.22) in (2.17) and after some mathematical manipulations, we get

$$\hat{P}_{0,0}(s)=\sum_{r=0}^{\infty}\frac{1}{(s+\lambda_0)^{r+1}}\sum_{j=0}^{r}\binom{r}{j}(\mu_0\beta_0)^j\left(\frac{w-\sqrt{w^2-\alpha_0{}^2}}{\alpha_0}\right)^j(\mu\sigma)^{r-j}$$
$$\times\left[\frac{\theta_v}{\mu}\sum_{m=1}^{\infty}\beta_0^m\left(\frac{w-\sqrt{w^2-\alpha_0^2}}{\alpha_0}\right)^m\left(\frac{p-\sqrt{p^2-\alpha^2}}{\alpha\beta}\right)^m\hat{P}_{0,0}(s)\right] \tag{2.24}$$
$$\times\left[\sum_{k=0}^{\infty}\theta^k(\bar{\sigma})^k\frac{1}{(s+\theta)^k}\left(\sum_{i=1}^{\infty}\frac{\lambda_1^i}{(s+\lambda_1+\theta)^i}\left(\frac{p-\sqrt{p^2-\alpha^2}}{\alpha\beta}\right)^i\right)^k\right]^{(r-j)}$$

Taking inverse Laplace transform of Equation (2.24), we obtain

$$P_{0,0}(\tau)=\sum_{r=0}^{\infty}e^{-\lambda_0\tau}\frac{\tau^r}{r!}*\sum_{j=0}^{r}\binom{r}{j}(\mu_v\beta_0)^j\frac{\lambda_0}{\beta_0}\left[I_{j-1}(\alpha_0\tau)-I_{j+1}(\alpha_0\tau)\right]e^{-(\lambda_0+\mu_v+\theta_v)\tau}$$
$$*\left[\frac{\theta_v}{\mu}\sum_{m=1}^{\infty}\frac{\lambda_0}{\beta_0^{1-m}}\left[I_{m-1}(\alpha_0\tau)-I_{m+1}(\alpha_0\tau)\right]e^{-(\lambda_0+\mu_v+\theta_v)\tau}\right.$$
$$*\frac{\lambda}{\beta^{m+1}}\left[I_{m-1}(\alpha\tau)-I_{m+1}(\alpha\tau)\right]e^{-(\lambda+\mu)\tau}*\sum_{k=0}^{\infty}\theta^k(\bar{\sigma})^k e^{-\lambda_1\tau}\frac{\tau^{k-1}}{(k-1)!} \tag{2.25}$$
$$\left.*\left(\sum_{i=1}^{\infty}\lambda_1^i e^{-(\lambda_1+\theta)\tau}\frac{\tau^{i-1}}{(i-1)!}*\frac{\lambda}{\beta^{i+1}}[I_{i-1}(\alpha\tau)-I_{i+1}(\alpha\tau)]e^{-(\lambda+\mu)\tau}\right)^{*k}\right]^{*(r-j)}.$$

2.3.2 Stationary system size distribution

By replacing the L.H.S. of Equations from (2.1) to (2.6) by zero, we get the steady-state distribution for this model. In this section, the stationary system size probabilities

$\pi_{0,n}, \pi_{1,n}, \pi_{2,n}$ and $\pi_{0,0}$ are computed from their time-dependent counterparts. It is assumed that $\lambda_0 < \mu_v$ and $\lambda_1 < \mu_1$ for steady state. Let

$$A_0 = \frac{(\lambda_0 + \mu_v + \theta_v) - \sqrt{(\lambda_0 + \mu_v + \theta_v)^2 - 4\lambda_0\mu_v}}{2\mu_v} \tag{2.26}$$

From (2.7), (2.9) and (2.22), we get

$$\pi_{2,n} = \lim_{s\to 0} s\hat{P}_{2,n}(\mathrm{s}) = \left(\frac{\theta_v\bar{\sigma}}{\lambda_1\sigma}\right)\left(\frac{\lambda_1}{\lambda_1+\theta}\right)\sum_{m=1}^{\infty} A_0^m \pi_{0,0} \tag{2.27}$$

From (2.19), we obtain

$$\pi_{0,n} = \lim_{s\to 0} s\hat{P}_{0,n}(\mathrm{s}) = A_0^{\,n}\pi_{0,0} \tag{2.28}$$

Taking Laplace transform on (2.16) and using the expressions for $\pi_{0,n}$ and $\pi_{2,n}$, we get

$$\pi_{1,n} = \lim_{s\to 0} s\hat{P}_{1,n}(\mathrm{s}) = \frac{1}{\mu-\lambda}\sum_{j=1}^{\infty}\left[\left(\frac{\lambda}{\mu}\right)^{n-j} - \left(\frac{\lambda}{\mu}\right)^{n}\right]\left[\theta_v A_0^{\,j} + \left(\frac{\theta_v\theta\bar{\sigma}}{\lambda_1\sigma}\right)\left(\frac{\lambda_1}{\lambda_1+\theta}\right)^{j}\sum_{m=1}^{\infty} A_0^m\right]\pi_{0,0} \tag{2.29}$$

Using the normalizing condition

$$\pi_{0,0} + \sum_{n=1}^{\infty}\pi_{0,\mathrm{n}} + \sum_{n=1}^{\infty}\pi_{1,\mathrm{n}} + \sum_{n=0}^{\infty}\pi_{2,\mathrm{n}} = 1, \tag{2.30}$$

we obtain the expression for $\pi_{0,0} = \lim_{s\to 0} s\hat{P}_{0,0}(\mathrm{s})$ as

$$\begin{aligned}\pi_{0,0} = \Bigg[1 &+ \sum_{n=1}^{\infty} A_0^{\,n} + \left(\frac{\theta_v\bar{\sigma}}{\lambda_1\sigma}\right)\left(\frac{\lambda_1}{\lambda_1+\theta}\right)^{n}\sum_{m=1}^{\infty} A_0^{\,m} \\ &+ \sum_{n=1}^{\infty}\frac{1}{\mu-\lambda}\sum_{j=1}^{\infty}\left[\left(\frac{\lambda}{\mu}\right)^{n-j} - \left(\frac{\lambda}{\mu}\right)^{n}\right]\left[\theta_v A_0^{\,j} + \left(\frac{\theta_v\theta\bar{\sigma}}{\lambda_1\sigma}\right)\left(\frac{\lambda_1}{\lambda_1+\theta}\right)^{j}\sum_{m=1}^{\infty} A_0^{\,m}\right]\Bigg]^{-1}\end{aligned} \tag{2.31}$$

2.3.3 Special cases

Case (*i*): When $\lambda_1 = 0, \theta = 0, \sigma = 1, \lambda = \lambda_0, \gamma = \theta_v$, then

$$P_{1,n}(\tau) = \gamma\int_0^{\tau}\sum_{m=1}^{\infty} P_{0,m}(u)\beta^{n-m}\left[I_{n-m}(\alpha(\tau-u)) - I_{n+m}(\alpha(\tau-u))\right]e^{-(\lambda+\mu)(\tau-u)}du, \tag{2.32}$$

which coincides with the result (5.6) in Sudhesh *et al*. (2017) when $\alpha_1 = \alpha, \beta_1 = \beta$.

Case (*ii*): When $\lambda_1 = 0, \theta = 0, \sigma = 1, \lambda = \lambda_0, \gamma = \theta_v$ then

$$P_{1,n}(\tau) = \gamma\int_0^{\tau}\sum_{m=1}^{\infty} P_{0,m}(u)\,\beta^{n-m}[I_{n-m}(\alpha(\tau-u)) - I_{n+m}(\alpha(\tau-u))]e^{-\phi(\tau)}du, \tag{2.33}$$

$$P_{0,n}(\tau)=\lambda_0\beta_0^{n-1}[I_{n-1}(\alpha_0\tau)-I_{n+1}(\alpha_0\tau)]e^{-(\lambda_0+\mu_v+\gamma)\tau}*P_{0,0}(\tau), \tag{2.34}$$

In this case Equations (2.33) and (2.34) coincide respectively with the results (2.11) and (2.16) in Sudhesh and Raj (2012).

Case (*iii*): When $\lambda_1=0,\theta=0,\mu_v=0,\sigma=1,\lambda=\lambda_0,\gamma=\theta_v$, then

$$P_{1,n}(\tau)=\gamma\int_0^\tau\sum_{m=1}^{\infty}P_{0,m}(u)\,\beta^{n-m}[I_{n-m}(\alpha(\tau-u))-I_{n+m}(\alpha(\tau-u))]e^{-\phi(\tau)}du, \tag{2.35}$$

$$P_{0,n}(\tau)=\lambda^n e^{-(\lambda+\gamma)\tau}\,\frac{\tau^{n-1}}{(n-1)!}*P_{0,0}(\tau), \tag{2.36}$$

which on simplification coincides respectively with Equations (2.15) and (2.13) given in Kalidass and Ramanath (2014).

2.4 System Performance Measures

To predict the transient behavior of the developed model, we formulate some system indices such as mean system size, the variance of the system size, the throughput of the system and different state probabilities as follows:

2.4.1 Mean system size

Let $E\{X(\tau)\}$ be the expected system size at the time τ . Now,

$$E\{X(\tau)\}=\sum_{n=1}^{\infty}n\{P_{0,n}(\tau)+P_{1,n}(\tau)+P_{2,n}(\tau)\},\quad E\{X(0)\}=0. \tag{2.37}$$

From (2.2), (2.3), (2.4) and (2.6), we obtain

$$\begin{aligned}\frac{d}{d\tau}E\{X(\tau)\}&=\lambda_0\sum_{n=0}^{\infty}P_{0,n}(\tau)-\sum_{n=1}^{\infty}(\theta_v n+\mu_v)P_{0,n}(\tau)\\&\quad-\lambda\sum_{n=1}^{\infty}(n-1)P_{1,n}(\tau)+\mu\sum_{n=1}^{\infty}(n+1)P_{1,n}(\tau)+\lambda_1\sum_{n=0}^{\infty}P_{2,n}(\tau).\end{aligned} \tag{2.38}$$

Integrating (2.27), we get

$$\begin{aligned}E\{X(\tau)\}&=\lambda_0\sum_{n=0}^{\infty}\int_0^\tau P_{0,n}(u)du-\sum_{n=1}^{\infty}(\theta_v n+\mu_0)\int_0^\tau P_{0,n}(u)du\\&\quad-\lambda\sum_{n=1}^{\infty}(n-1)\int_0^\tau P_{1,n}(u)du+\mu\sum_{n=1}^{\infty}(n+1)\int_0^\tau P_{1,n}(u)du+\lambda_1\sum_{n=0}^{\infty}\int_0^\tau P_{2,n}(u)du.\end{aligned} \tag{2.39}$$

2.4.2 Variance of system size

To obtain the variance $Var\{X(\tau)\}$ of the system size at time τ , we use

$$Var\{X(\tau)\} = E\{X^2(\tau)\} - (E\{X(\tau)\})^2, \tag{2.40}$$

where $E\{X^2(\tau)\} = \sum_{n=1}^{\infty} n^2\{P_{0,n}(\tau) + P_{1,n}(\tau) + P_{2,n}(\tau)\}$.

From Equations (2.2), (2.3), (2.4) and (2.6), we obtain

$$\begin{aligned} \frac{d}{d\tau}E\{X^2(\tau)\} &= \lambda_0 \sum_{n=0}^{\infty} P_{0,n}(\tau) + \sum_{n=1}^{\infty}\{2(\lambda_0 - \mu_v)n + \mu_v\}P_{0,n}(\tau) \\ &\quad + \sum_{n=1}^{\infty}\{2(\lambda - \mu)n + (\lambda + \mu)\}P_{1,n}(\tau) + 2\lambda_1 \sum_{n=1}^{\infty} P_{2,n}(\tau) + \lambda_1 \sum_{n=0}^{\infty} P_{2,n}(\tau). \end{aligned} \tag{2.41}$$

On integrating Equation (2.39), we get

$$\begin{aligned} E\{X^2(\tau)\} &= \lambda_0 \sum_{n=0}^{\infty} \int_0^{\tau} P_{0,n}(u)du + \sum_{n=1}^{\infty}\{2(\lambda_0 - \mu_v)n + \mu_v\}\int_0^{\tau} P_{0,n}(u)du \\ &\quad + \sum_{n=1}^{\infty}\{2(\lambda - \mu)n + (\lambda + \mu)\}\int_0^{\tau} P_{1,n}(u)du + 2\lambda_1 \sum_{n=1}^{\infty}\int_0^{\tau} P_{2,n}(u)du + \lambda_1 \sum_{n=0}^{\infty}\int_0^{\tau} P_{2,n}(u)du, \end{aligned} \tag{2.42}$$

where $P_{0,n}(\tau)$, $P_{1,n}(\tau)$ and $P_{2,n}(\tau)$, for $n = 1, 2, 3, \ldots$, are given by (2.20), (2.16) and (2.10), respectively.

2.4.3 The throughput

At time τ, the throughput is obtained as

$$TP(\tau) = \mu\{\sum_{n=1}^{\infty} P_{1,n}(\tau)\} + \mu_v\{\sum_{n=0}^{\infty}(P_{2,n}(\tau)\} \tag{2.43}$$

2.4.4 Service station state probabilities

The probabilities of different system status such as in normal busy, vacation and working vacation mode respectively, at time τ are

$$P_B(\tau) = \sum_{n=1}^{\infty} P_{1,n}(\tau),\ P_V(\tau) = \sum_{n=0}^{\infty} P_{0,n}(\tau) \text{ and } P_{WV}(\tau) = \sum_{n=0}^{\infty} P_{2,n}(\tau). \tag{2.44}$$

2.4.5 The cost function

The cost function to determine the total expected cost $TC(\tau)$ incurred by the system, the following cost elements per unit time associated with different system states are considered:

C_H : Per customer holding cost incurred on waiting in the queue for the service.

C_B : Per customer cost incurred on a busy service station.

C_V : Per customer cost incurred when the service station is availing the complete vacation.

C_{WV} : Per customer cost incurred while the service station is on working vacation.

C_1 : Cost involved in rendering the service by the service station with service rate μ.

The cost function is formulated as

$$TC(\tau) = C_H E\{X(\tau)\} + C_B P_B(\tau) + C_V P_V(\tau) + C_{WV} P_{WV}(\tau) + C_1 \mu \qquad (2.45)$$

2.5 Numerical Simulation

The numerical results of various performance indices for the concerned queueing model developed in previous sections are computed. The sensitivity of the system indices for various system descriptors is also explored by performing the numerical experiments using the Matlab software. For the numerical results, we choose the default system parameters as follows:

$$\lambda = 0.6, \lambda_1 = 0.4, \lambda_0 = 0.2, \mu = 1.5, \mu_v = 1.0, \theta = 0.05, \theta_v = 0.03, \sigma = 0.7$$

The numerical results are displayed in Tables 2.1-2.5 and Figures 2.2-2.10. In Tables 2.1-2.5, the effect of system descriptors on the transient system state probabilities { $P_B(\tau)$, $P_V(\tau)$, $P_{WV}(\tau)$ } and system cost $TC(\tau)$ at different time epochs are summarized for varying values of parameters $\lambda, \mu, \mu_v, \gamma,$ and θ_v, respectively.

The trends of various system indices for varying different parameters are as follows:

(i) ***Effect of arrival rate*** (λ) ***and*** (λ_1)***:*** The normal busy state probability $P_B(\tau)$ and total cost $TC(\tau)$ show the increasing trend for the increasing value of arrival rate (λ) but vacation state $P_V(\tau)$ and working vacation state $P_{WV}(\tau)$ probabilities exhibit the reverse trend i.e. decrease as arrival rate (λ) increases. Figures 2.2-2.3 clearly show the increasing trend in mean queue length $E\{X(\tau)\}$ as arrival rates λ and λ_1 grow up.

(ii) ***Effect of service rate*** (μ) ***and*** (μ_v)***:*** In Tables 2.2 and 2.3, the gradually decreasing trend of busy state probability $P_B(\tau)$ for the increasing values of service rate μ and μ_v are seen. The vacation state probability $P_V(\tau)$ and working vacation state probability $P_{WV}(\tau)$ increase as the service rate (μ) increases. From Figures 2.5 and 2.6, it is noted that the throughput $TP(\tau)$ of the system increases as μ and μ_v increase.

(iii) ***Effect of set up rate*** (θ) ***and*** (θ_v)***:*** The setup rates θ and θ_v also have significant effects on various system indices. The busy state probability $P_B(\tau)$ and working vacation state probability $P_{WV}(\tau)$ grow up with an increase in the value of setup rate (θ)**.** The vacation state probability $P_V(\tau)$ and system total cost $TC(\tau)$ show significant decrement as the value of set up rate (θ) grow up. The impact of set up rate (θ_v) on various system indices seems to be negligible. Figure 2.4 clearly depicts the decreasing trends of mean queue length $E\{X(\tau)\}$. Figure 2.7 shows the increasing trend of throughput $TP(\tau)$ of the system for the growing value of setup rate (θ).

(iv) ***System cost*** $TC(\tau)$***:*** The computation of cost results of the developed model makes investigation quite productive, interesting and useful. We have evaluated total expected cost $TC(\tau)$ of the system by framing the cost function. The effect of various system parameters $\lambda, \mu, \mu_v, \theta$ and θ_v on total expected cost $TC(\tau)$ are displayed in Tables 2.1-2.5.

To examine the cost function, the surface graphs are displayed in Figures 2.8-2.10 for three cost sets as given below:

Set I: $C_H = 90,\ C_B = 80,\ C_V = 60, C_{WV} = 40, C_1 = 45.$

Set II: $C_H = 80,\ C_B = 70,\ C_V = 60, C_{WV} = 40, C_1 = 40.$

Set II: $C_H = 90,\ C_B = 70,\ C_V = 65, C_{WV} = 45, C_1 = 40.$

Table 2.1: Effect of arrival rate (λ) on system indices and total cost

τ	λ	$P_B(\tau)$	$P_V(\tau)$	$P_{WV}(\tau)$	$TC(\tau)$
2	0.6	0.009076	0.986966	0.003958	162.91
	0.8	0.009385	0.986812	0.003803	163.04
	1.0	0.009681	0.986664	0.003655	163.17
6	0.6	0.036845	0.925081	0.038074	225.51
	0.8	0.040386	0.923318	0.036296	226.82
	1.0	0.04405	0.921494	0.034456	228.37
10	0.6	0.061441	0.849679	0.088881	275.25
	0.8	0.069683	0.845622	0.084695	278.36
	1.0	0.078822	0.841119	0.080059	282.28

Table 2.2: Effect of service rate (μ) on system indices and total cost

τ	μ	$P_B(\tau)$	$P_V(\tau)$	$P_{WV}(\tau)$	$TC(\tau)$
2	2	0.007638	0.987685	0.004677	185.18
	4	0.004458	0.989275	0.006267	274.71
	6	0.003074	0.989967	0.00696	364.52
6	2	0.027939	0.929507	0.042554	246.25
	4	0.013605	0.936621	0.049773	333.79
	6	0.008911	0.938948	0.052141	423.08
10	2	0.044648	0.857904	0.097449	294.07
	4	0.020723	0.869576	0.109701	379.59
	6	0.013431	0.873122	0.113448	468.40

Table 2.3: Effect of service rate (μ_v) on system indices and total cost

τ	μ_v	$P_B(\tau)$	$P_V(\tau)$	$P_{WV}(\tau)$	$TC(\tau)$

2	1	0.009076	0.986966	0.003958	162.91
	3	0.009074	0.986966	0.00396	162.89
	5	0.009073	0.986966	0.003961	162.88
6	1	0.036845	0.925081	0.038074	225.51
	3	0.03672	0.925012	0.038268	224.77
	5	0.036683	0.924986	0.038331	224.59
10	1	0.061441	0.849679	0.088881	275.25
	3	0.060904	0.849202	0.089893	272.81
	5	0.060779	0.849049	0.090172	272.31

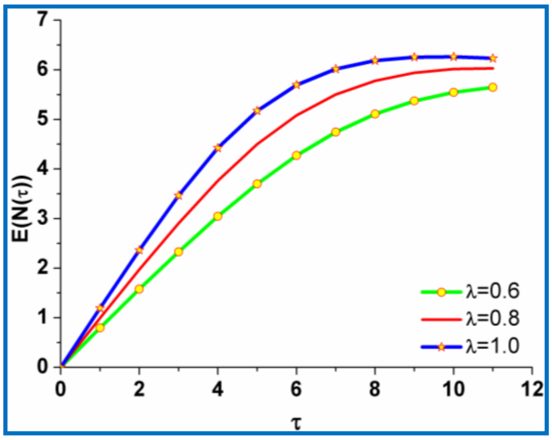
Table 2.4: Effect of set up rate (θ) on system indices and total cost

τ	θ	$P_B(\tau)$	$P_V(\tau)$	$P_{WV}(\tau)$	$TC(\tau)$
2	0.05	0.009076	0.986966	0.003958	162.91
	0.1	0.017463	0.974829	0.007708	162.35
	0.15	0.025215	0.963521	0.011264	161.82
6	0.05	0.036845	0.925081	0.038074	225.51
	0.1	0.064722	0.864779	0.070499	217.09
	0.15	0.085769	0.815988	0.098243	209.98
10	0.05	0.061441	0.849679	0.088881	275.25
	0.1	0.097896	0.744559	0.157545	251.49
	0.15	0.118917	0.66981	0.211275	233.72

Table 2.5: Effect of set up rate (θ_v) on system indices and total cost

τ	θ_v	$P_B(\tau)$	$P_V(\tau)$	$P_{WV}(\tau)$	$TC(\tau)$
2	0.03	0.009076	0.986966	0.003958	162.9116
	0.09	0.009087	0.986969	0.003944	162.912
	0.15	0.009097	0.986971	0.003931	162.9124
6	0.03	0.036845	0.925081	0.038074	225.5066
	0.09	0.037255	0.92536	0.037385	225.5202
	0.15	0.037624	0.925619	0.036758	225.5326
10	0.03	0.061441	0.849679	0.088881	275.2495
	0.09	0.062928	0.851333	0.085741	275.3111
	0.15	0.064182	0.852805	0.083015	275.3656

(i) (i)



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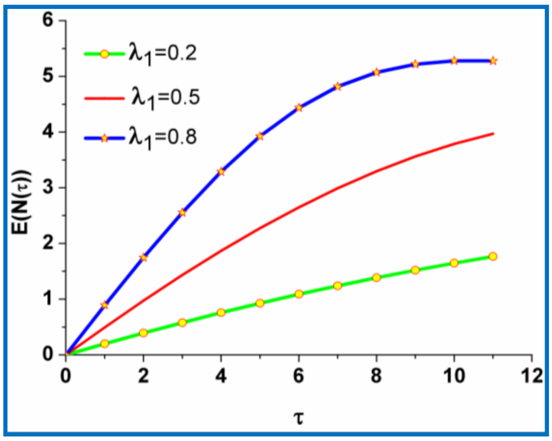

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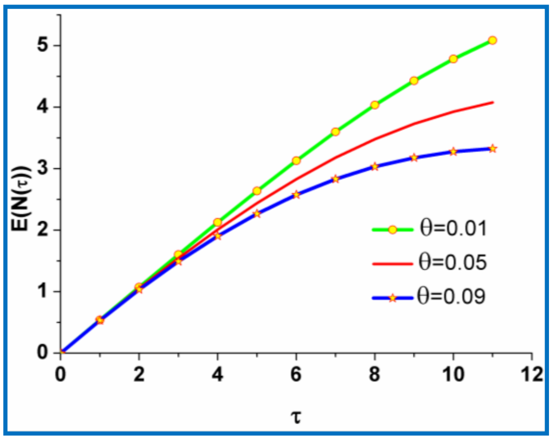
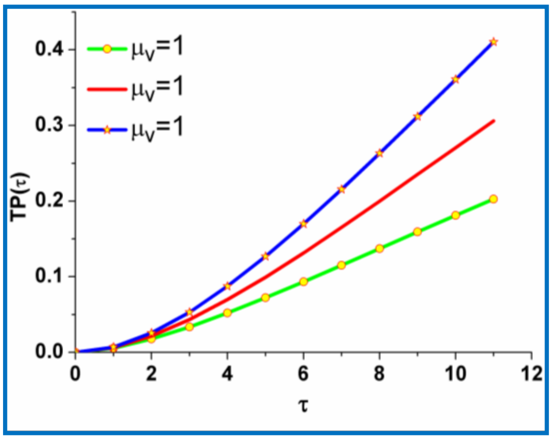


Fig. 2.2: $E(N(\tau))$ **for different value of (i) λ (ii) λ_1 (iii)** θ



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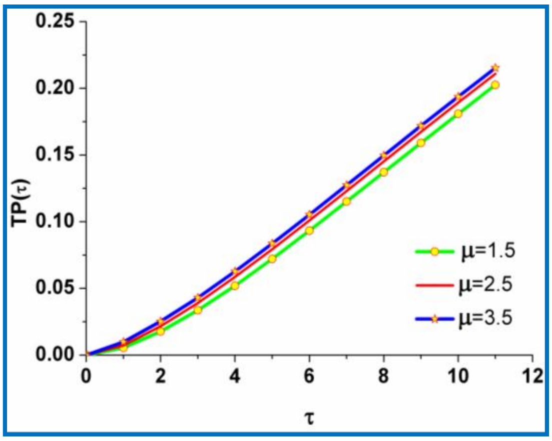


(iii)

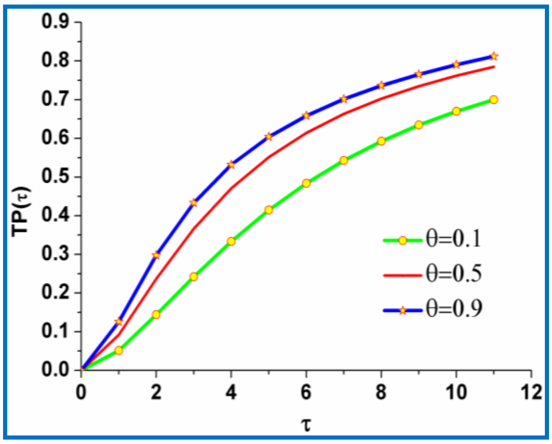
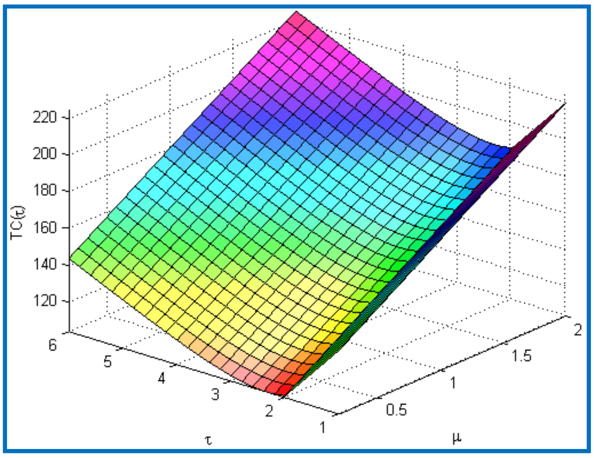
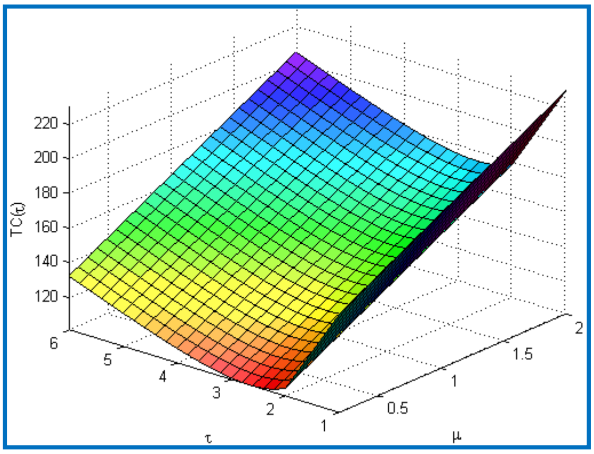


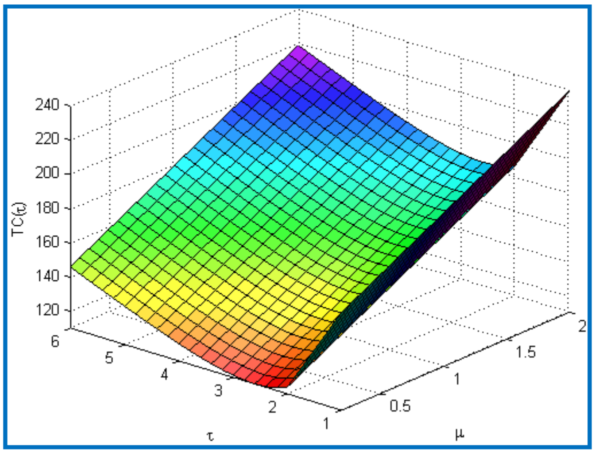
Fig. 2.3: $TP(\tau)$ **for different value of (i) μ_v (ii) μ (iii)** θ



(i)



(ii)



(iii)

Fig. 2.4: The system cost $TC(\tau)$ and μ for (i) cost sets I (ii) cost sets II (iii) cost sets III.

Chapter 3

F-policy for Unreliable Server Machining System with Working Vacation

3.1 Introduction

In the context of real queueing applications, controlling arrival is one of the cost-effective as well as managerial efficient approaches which can be used for the optimal utilization of the machining system. To control the arrivals of the jobs in case of full capacity of a system, F-policy concept which was, first introduced by Gupta (1995) states that the arriving customers are not allowed to enter the system if it reaches its capacity level, but when the queue length again decreases to a predefined threshold value ‘F’, the customers are further allowed to enter the system to form a queue. The machine repair problems with F-policy, which is quite common phenomenon of real systems, have been rarely investigated in queueing literature. Yang *et al*. (2010) analyzed F-policy queueing system with exponential startup time and evaluated the optimal parameters by minimizing the cost function using quasi-Newton and direct search methods.

In the queueing literature, it is noticed that most of the studies devoted to the performance analysis of queues with server vacation are restricted to reliable server model. However, this is not case in real life; no server can be perfect and as such the incorporation of the unreliable server concept for the performance modeling will be helpful to portrait the more versatile and realistic queueing scenarios. Excellent surveys on the queueing models with server breakdowns and other features were presented by Ke (2003, 2004, 2005) . Jain and Jain (2010) used matrix geometric approach to explore various system characteristics of an unreliable server queueing system with working vacation.

From the literature survey, it is evident that in the past few years, a few research articles on the machine repair problem with vacation policy have appeared. It is noticed that no research work has been done to explore the performance metrics of the unreliable server machine repair problem by incorporating the realistic features of working vacation and F-policy together along with the startup time. Motivated by this fact, we have framed machine repair problem in the general framework by including the noble concepts of F-policy, working vacation, set up, and an unreliable server. The numerical technique based on successive over relaxation (SOR) method is used to determine the probabilities associated with different system states. To explore the impact of system parameters on the performance indices, numerical simulation has been conducted by taking an illustration. The direct search method and quasi-Newton method have also been used to obtain the optimal values of the decision parameters. The rest of the chapter is organized in the following manner. In Section 3.2, we present the model description by mentioning the assumptions and mathematical notations. In Section 3.3, mathematical formulation and governing equations are constructed on the basis of birth-death process. In Section 3.4, various performance indices of the concerned queueing system are formulated explicitly in terms of steady state probabilities. By implementing the direct search and quasi-Newton methods, the cost analysis is carried out in Section 3.5. In Section 3.6, we present the numerical simulation results and sensitivity analysis.

3.2 Model Description

In order to investigate an unreliable fault tolerant machining system with working vacation under admission control F-policy, we develop a finite Markov M/M/1/K/WV model by using birth-death process. The governing steady state equations are framed for the system states on the basis of appropriate transition rates. To avoid the overload and stress on the machining system, the concept of admission control F-policy is incorporated so as to maintain the smooth functioning of the machining system. The concept of working vacation is added to enhance the maintainability and reliability of the machining system at optimum cost. For the model development, we assume that all the underlying processes involved in the mathematical formulation, i.e. arrival, service and vacation, are statistically independent.

- The machining system consists of a finite number (say M) of machines. The machine may fail according to a Poisson process with parameter λ so that the effective failure rate when there are 'n' failed machines in the system is $\lambda_n = (M-n)\lambda$ for $0 \le n < K$, where $K = M - m + 1$.
- For the smooth functioning of the machining system, the system is maintained by providing the repair to the failed machines. The repair time of failed machines is assumed to be exponentially distributed with rate μ. The failed machines arriving at the repair facility for their repair job, forms a single queue and repair rendering to the failed machine to restore its functioning is done according to first in first out (FIFO) discipline.
- When there is no job for the repair in the queue after completion of the repair of last failed machine, the server goes for the working vacation. For being more productive and to increase the system capacity, during the vacation period rather than completely terminating the repair job, the server continues to render the repair job or do some

additional job such as maintenance, book keeping, etc. with slower rate μ_v in comparison to that of normal busy rate μ. The repair times of the failed machines follow the exponential distribution with rate μ_v when the server is providing service during the working vacation period. In case when any failed machine arrives in the system for the repair, the vacationing server returns back to the system with rate θ_v.

- In order to avoid the overload or stress of failed machines arriving for the repair in the system, the admission control following F-policy is quite useful. The overload of the system can be shared with outsource repair facility at extra cost incurred to get failed machines repaired timely. The capacity of the repair facility is assumed to be finite (say K). If the number of failed machines joining the repair facility reaches its full capacity (K), the entry of any new failed machines in the system is stopped until and unless the number of failed machines in the system decreases to the threshold value F, $(0 \le F \le K-1)$. At that instant, the server requires the start-up time which is assumed to be exponentially distributed with the parameter ψ; then after the failed machines are allowed to join the system for the repair job.
- The server is subject to breakdown; the server's lifetime is exponentially distributed with parameter α. The broken down server is immediately sent for the repair job which is done according to an exponential distribution with rate β_1 when failed in a normal busy period, and with rate β_2 when failed during the working vacation period.

3.3 Mathematical Formulation and Analysis

Consider the working vacationing server queueing model operating under F - policy for the admission of failed machines by incorporating the realistic feature of unreliable server. In order to obtain the performance indices of the machining system, we frame the governing equations for the mathematical formulation of the finite population Markov queueing model. Using the notations and assumptions described in the previous Section 3.2, the model is formulated as follows:

Consider Markov chain model of a bivariate stochastic process $\{\xi(\tau), \eta(\tau)\}$ where $\xi(\tau)$ represents the status of the server at time τ and $\eta(\tau)$ denotes the number of failed machines in the system at time τ. The steady state probabilities of the system for the state space

$$S = \{[(j,n) \mid 0 \le j \le 7; 1 \le n \le K-1] \cap [(j,0) \mid j = 1,3,5,7] \cap [(j,K) \mid j = 0,1,6,7]\}$$

are represented $P_{j,n}$. The state-transition diagram of continuous time Markov chain is depicted in figure 3.1.

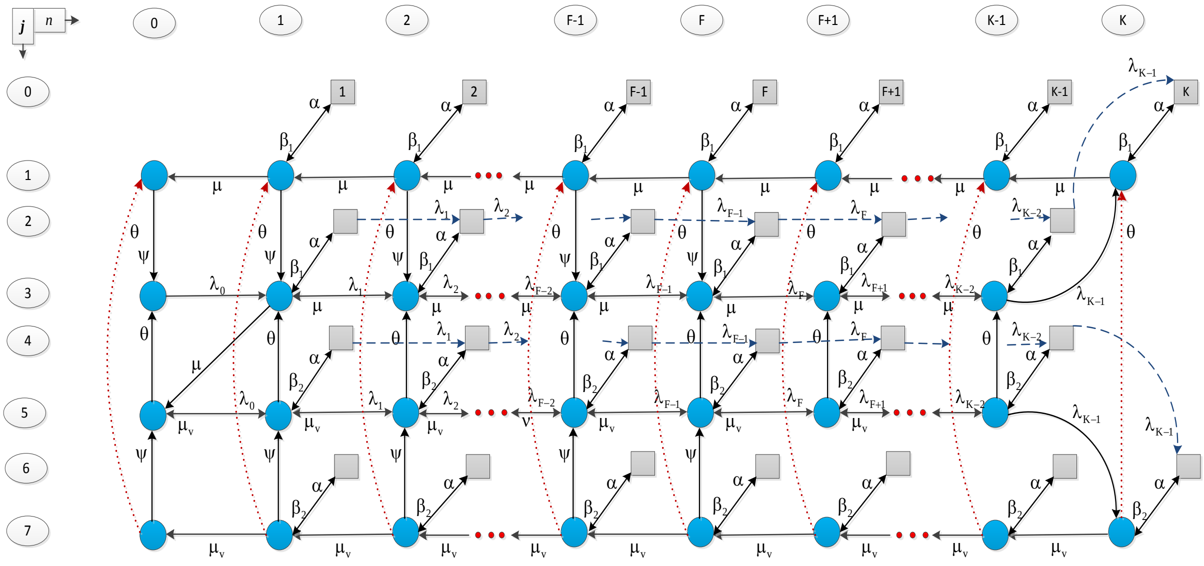


Fig. 3.1: State transition diagram for M/M/1/WV model

Now we define $\xi(\tau)$ as

$$\xi(\tau)=\begin{cases}0, & \text{Server is under repair when broken down during the normal busy state and the joining}\\ & \text{of failed machines are not allowed in the system;}\\ 1, & \text{Server is in normal busy state and the joining of failed machines are not allowed in the system;}\\ 2, & \text{Server is under repair when broken down during the normal busy state and the joining}\\ & \text{of failed machines are allowed in the system;}\\ 3, & \text{Server is in normal busy state and the the joining of failed machines are allowed in the system;}\\ 4, & \text{Server is under repair when broken down during the working vacation state while the}\\ & \text{joining of failed machines are allowed in the system;}\\ 5, & \text{Server is in working vacation state and the joining of failed machines are allowed in the system;}\\ 6, & \text{Server is under repair when broken down during the working vacation state and the joining}\\ & \text{of failed machines are not allowed in the system;}\\ 7, & \text{Server is in working vacation state and the the joining of the failed machines are not allowed}\\ & \text{in the system.}\end{cases}$$

3.3.1 Governing equations

For evaluating the probabilities associated with different system states, Chapman-Kolmogorov equations are constructed by using the appropriate rates of underlying birth-death process. Based on the law of conservation of transition flow, the following difference equations for server's different states $\xi(\tau)$ are framed:

(i) When $\xi(\tau)=0$.

By equating the out-flows from the state $(0,\mathrm{n})$ having probability $P_{0,n}$ $(1\le \mathrm{n}\le \mathrm{K})$ with the in-flows from the state $(1,\mathrm{n})$ and $(2,\mathrm{n})$, the governing equations for the states $(0,\mathrm{n})$ for $\mathrm{n}=1,2,3,\ldots,\mathrm{K}$ are obtained as:

$$\beta_1 P_{0,n} = \alpha P_{1,n};\ 1 \le n \le K-1 \tag{3.1}$$

$$\beta_1 P_{0,K} = \alpha P_{1,K} + \lambda_{K-1} P_{2,K-1} \tag{3.2}$$

(ii) When $\xi(\tau) = 1$.

The following equations are obtained by equating the in-flows and out-flows for the states $(1,n),\ 0 \le n \le K$:

$$\psi P_{1,0} = \mu P_{1,1} + \theta_v P_{7,0} \tag{3.3}$$

$$(\alpha + \mu + \psi) P_{1,n} = \beta_1 P_{o,n} + \mu P_{1,n+1} + \theta_v P_{7,n};\ 1 \le n \le F \tag{3.4}$$

$$(\alpha + \mu) P_{1,n} = \beta_1 P_{o,n} + \mu P_{1,n+1} + \theta_v P_{7,n};\ F+1 \le n \le K-1 \tag{3.5}$$

$$(\alpha + \mu) P_{1,K} = \beta_1 P_{o,K} + \lambda_{K-1} P_{3,K-1} + \theta_v P_{7,K} \tag{3.6}$$

(iii)When $\xi(\tau) = 2$.

For states $(2,n),\ n = 1,2,...,K-1,$ the following equations are constructed using the law of conservation of total flows:

$$(\beta_1 + \lambda_1) P_{2,1} = \alpha P_{3,1} \tag{3.7}$$

$$(\beta_1 + \lambda_n) P_{2,n} = \lambda_{n-1} P_{2,n-1} + \alpha P_{3,n};\ 2 \le n \le K-1 \tag{3.8}$$

(iv) When $\xi(\tau) = 3$.

On equating the in-flows and out-flows for the system states $(3,n),\ n = 0,1,2,...,K-1,$ we obtain the following governing equations:

$$\lambda_0 P_{3,0} = \psi P_{1,0} + \theta_v P_{5,0} \tag{3.9}$$

$$(\alpha + \lambda_n + \mu) P_{3,n} = \psi P_{1,n} + \beta_1 P_{2,n} + \lambda_{n-1} P_{3,n-1} + \mu P_{3,n+1} + \theta_v P_{5,n};\ 1 \le n \le F \tag{3.10}$$

$$(\alpha + \lambda_n + \mu) P_{3,n} = \beta_1 P_{2,n} + \lambda_{n-1} P_{3,n-1} + \mu P_{3,n+1} + \theta_v P_{5,n};\ F+1 \le n \le K-2 \tag{3.11}$$

$$(\alpha + \lambda_{K-1} + \mu) P_{3,K-1} = \beta_1 P_{2,K-1} + \lambda_{K-2} P_{3,K-2} + \theta_v P_{5,K-1} \tag{3.12}$$

(v) When $\xi(\tau) = 4$.

For states $(4,n),\ n = 1,2,...,K-1,$ following the law of conservation of flows, we get

$$(\beta_2 + \lambda_1) P_{4,1} = \alpha P_{5,1} \tag{3.13}$$

$$(\beta_2 + \lambda_n) P_{4,n} = \lambda_{n-1} P_{4,n-1} + \alpha P_{5,n};\ 2 \le n \le K-1 \tag{3.14}$$

(vi) When $\xi(\tau) = 5$.

On the basis of the law of conservation of flows, for states $(5,n),\ 0 \le n \le K-1,$ we obtain

$$(\theta + \lambda_0) P_{5,0} = \mu P_{3,1} + \mu_v P_{5,1} + \psi P_{7,0} \tag{3.15}$$

$$(\alpha + \theta + \lambda_n + \mu_v) P_{5,n} = \beta_2 P_{4,n} + \lambda_{n-1} P_{5,n-1} + \mu_v P_{5,n+1} + \psi P_{7,n};\ 1 \le n \le F \tag{3.16}$$

$$(\alpha + \theta_v + \lambda_n + \mu_v) P_{5,n} = \beta_2 P_{4,n} + \lambda_{n-1} P_{5,n-1} + \mu_v P_{5,n+1};\ F+1 \le n \le K-2 \tag{3.17}$$

$$(\alpha+\theta_v+\lambda_{K-1}+\mu_v)P_{5,K-1}=\beta_2 P_{4,K-1}+\lambda_{K-2}P_{5,K-2} \tag{3.18}$$

(vii) When $\xi(\tau)=6$.

On equating the in-flows and out-flows for the states $(6,n), 1\le n\le K$, we get

$$\beta_2 P_{6,n}=\alpha P_{7,n};\ 1\le n\le K-1 \tag{3.19}$$

$$\beta_2 P_{6,K}=\alpha P_{7,K}+\lambda_{K-1}P_{4,K-1} \tag{3.20}$$

(viii) When $\xi(\tau)=7$.

The following governing equations are constructed from states $(7,n), 0\le n\le K$:

$$(\theta_v+\psi)P_{7,0}=\mu_v P_{7,1}(t) \tag{3.21}$$

$$(\alpha+\theta_v+\psi+\mu_v)P_{7,n}=\beta_2 P_{6,n}+\mu_v P_{7,n+1};\ 1\le n\le F \tag{3.22}$$

$$(\alpha+\theta_v+\mu_v)P_{7,n}=\beta_2 P_{6,n}+\mu_v P_{7,n+1};\ F+1\le n\le K-1 \tag{3.23}$$

$$(\alpha+\theta_v+\mu_v)P_{7,K}=\lambda_{K-1}P_{5,K-1}+\beta_2 P_{6,K} \tag{3.24}$$

By applying the classical analytical methods for solving the set of difference equations such as recursive method or probability generating function approach, the explicit results for the probability distribution are difficult to be derived, in particular when the numbers of states are large and flow rates are state dependent. In such situations, the numerical method can be easily implemented to solve the set of simultaneous linear difference equations. Here we shall employ the numerical technique based on successive over relaxation (SOR) method to solve the system of Equations (3.1) to (3.24). It is to be worth noting that SOR method is an extrapolation to Gauss-Seidal method in which the convergence rate is accelerated by taking the relaxation parameter $1<\omega<1.25$ (*cf.* Jain *et al*. 2011,Hadjidimos 2000). This numerical approach is suitable for determining the solution of system of equations. After evaluating the probabilities, we are in a position to analyze the system behavior by determining the different performance measures. The set of difference Equations (3.1)-(3.24) is solved for the probabilities for different states by converting in matrix form. We denote the coefficient matrix '$\mathbf{P}$' which consists of coefficients of all the probabilities as described in Chapman-Kolmogorov Equations (3.1)-(3.24). Let $\boldsymbol{\pi}$ be a probability vector which consists of all the unknown probabilities $P_{j,n}$ for feasible index set $\{j,n\}$. Now, we have

$$\mathbf{P}\boldsymbol{\pi}=\mathbf{0} \tag{3.25}$$

The normalizing condition is given by

$$\boldsymbol{\pi}\mathbf{e}=1 \tag{3.26}$$

where '$\mathbf{e}$' denotes the unit row vector of size (8K). Replacing the last row of the matrix Equation (3.25) by (3.26), we get new matrix equation as

$$\mathbf{Q}\boldsymbol{\pi} = \mathbf{B} \tag{3.27}$$

Here first (8K-1) rows of the coefficient matrix **Q** have same elements as that of **P** but the last $(8\text{K})^{\text{th}}$ row has all elements equal to 1. Also $\mathbf{B} = [0, 0, 0, ..., 0, 1]^{\text{T}}$ is column vector of size 8K. The above Chapman-Kolmogorov differential-difference Equations (3.1)-(3.24) represent the continuous time Markov chain involved in the concerned model. Figure 3.1 exhibits the feasible transitions with appropriate rates among neighboring states of the system for different events.

3.4 System Performance Measures

In order to explore the system characteristics and to examine the performance, we formulate the various performance indices in terms of steady state probabilities of the system states. The expressions for the expected number of failed machines in the system, expected waiting time of failed machines in the system, effective joining rate of the failed machines in the system, probabilities of the server being in different states and other performance indices are established as follows:

(i) The expected number of failed machines in the system is

$$E(N) = \sum_{j=0}^{1}\sum_{n=1}^{K} n\, P_{j,n} + \sum_{j=2}^{5}\sum_{n=1}^{K-1} n\, P_{j,n} + \sum_{n=1}^{K-1} n\, P_{6,n} + \sum_{n=1}^{K} n\, P_{7,n} \tag{3.28}$$

(ii) The probability that the server being operative in normal and working vacation mode is

$$P_B = \sum_{j=0}^{1}\sum_{n=0}^{K} P_{6j+1,n} + \sum_{j=1}^{2}\sum_{n=0}^{K-1} P_{2j+1,n} \tag{3.29}$$

(iii) The probability that the server is in idle state is

$$P_I = \sum_{j=0}^{3} P_{2j+1,0} \tag{3.30}$$

(iv) The probability that the server being in the broken down state is

$$P_D = \sum_{j=0}^{1}\sum_{n=1}^{K} P_{6j,n} + \sum_{j=1}^{2}\sum_{n=1}^{K-1} P_{2j,n} \tag{3.31}$$

(v) The probability that the server in operating (inclusive working vacation) state is blocked for the joining of failed machines is

$$P_L = \sum_{j=0}^{1}\sum_{n=0}^{K} P_{6j+1,n} \tag{3.32}$$

(vi) Once the system capacity is full, due to implementation the F-policy, the failed units are not allowed to enter into the system until the queue size of failed machines ceases to threshold level ‘F’. However, some set up time is needed to further allow the failed machines into the system as such there is chance that the repair job of some more machines is completed after threshold level is reached. Thus the probability that the server starts to allow the failed machines entering into the system, is

$$P_S = \sum_{j=0}^{1} \sum_{n=0}^{F} P_{6j+1,n} \tag{3.33}$$

(vii) The effective joining rate of failed machines in the system, is given by

$$\lambda_{eff} = \sum_{j=2}^{5} \sum_{n=0}^{K-1} (M-n)\lambda P_{j,n} \tag{3.34}$$

(viii) The expected waiting time of the failed machines in the system, is

$$E(W) = \frac{E(N)}{\lambda_{eff}} \tag{3.35}$$

3.5 Cost Function and Optimal Parameters

The system designer may be interested in determining the optimal parameters in order to reduce the total cost incurred in the system. Now, we construct the cost function by considering the three decision variables (F, μ, ν) and various cost elements involved in different activities. Here variable 'F' is discrete whereas other two variables (μ, ν) are continuous. The cost elements associated with different activities are considered to be linear and defined as follows:

C_H : Holding cost per unit time for each failed machines present in the system.

C_B : Cost per unit time to maintain the system in the operating state.

C_D : Cost per unit time incurred on a failed server.

C_I : Cost per unit time for an idle server.

C_L : Fixed cost for every lost failed machines when the system is blocked.

C_S : Start up cost per unit time incurred on the system to prepare it for allowing the failed machines to enter the system.

C_W : Waiting cost per unit time of each failed machine while waiting for the service.

C_K : The fixed cost incurred on the system capacity.

C_1 : Cost per unit time of the server for providing service during normal busy period.

C_2 : Cost per unit time of the server for providing service during a working vacation period.

C_3 : Cost per unit time associated with the startup time which is required for allowing the failed machine to enter the system as per F-policy.

Based on the definitions of the each cost element listed above and various system indices established in the previous section, the total expected cost per unit time is constructed as-

$$\begin{aligned}\mathrm{TC}(F,\mu,\mu_v) &= C_H \cdot E(N) + C_B \cdot P_B + C_I \cdot P_I + C_D \cdot P_D + C_S \cdot P_S + C_L \cdot P_L \\ &\quad + C_W \cdot E(W) + C_K \cdot K + C_1 \cdot \mu + C_2 \cdot \mu_v + C_3 \cdot \psi\end{aligned} \tag{3.36}$$

In order to determine the optimal control parameters (F^*, μ^*, μ_v^*), we consider optimization problem:

$$\mathrm{TC}(F^*, \mu^*, \mu_v^*) = \min \mathrm{TC}(F, \mu, \mu_v) \tag{3.37}$$

In long run, the various events get statistically stabilize. Thus, the expected total cost is considered in terms of different cost elements for per unit time. Due to the unstructured multivariate function $\mathrm{TC}(F,\mu,\mu_v)$, it is not feasible to develop analytical results for the optimum parameter values using a classical optimization approach; however search method can be easily implemented.

3.5.1 Direct search method

The cost function is highly non-linear and complex. Therefore, in order to obtain minimum of the cost function, we perform some numerical computations; thus we can obtain a global minimum. To obtain the optimum value of discrete variable 'F', we first use direct substitution of successive values of 'F' into the cost function. Based on the trend of the total cost with the changes in 'F', the optimal decision variable 'F^*' is determined for different service rates μ and μ_v.

3.5.2 Quasi-Newton method

Quasi-Newton method can be easily used to find the global values of continuous decision variables (μ, μ_v) by minimizing the cost $\mathrm{TC}(F^*, \mu, \mu_v)$ which is a non-linear convex function and twice continuously differentiable. It is an iterative method with some stopping criterion depending on the tolerance limit. The main advantage for implementing this method is its fast convergence and affine invariance. The theoretical basic iterative step is defined as

$$x^+ = x - t\nabla^2 f(x)^{-1} \nabla f(x) \tag{3.38}$$

The following steps to implement quasi-Newton method are performed to reach the minimum value of $\mathrm{TC}^*(F^*, \mu^*, \mu_v^*)$ and the corresponding optimal service rates (μ^*, μ_v^*).

(i) Let the initial value of decision variables $\overrightarrow{\Omega_0} = [\mu, \mu_v]^T$, $i = 0$ and the tolerance $\varepsilon = 10^{-6}$.

(ii) Set the initial trial solution for $\overrightarrow{\Omega_0}$ and compute $\mathrm{TC}(\overrightarrow{\Omega_0})$.

(iii) Compute the cost gradient $\vec{\nabla}\mathrm{TC}(\overrightarrow{\Omega_i}) = \left[\dfrac{\partial \mathrm{TC}}{\partial \mu}, \dfrac{\partial \mathrm{TC}}{\partial \mu_v}\right]_{\vec{\Omega}=\vec{\Omega}_i}$ and the cost Hessian matrix

$$H(\overrightarrow{\Omega_i}) = \begin{bmatrix} \dfrac{\partial^2 TC}{\partial^2 \mu} & \dfrac{\partial^2 TC}{\partial \mu \partial \mu_v} \\ \dfrac{\partial^2 TC}{\partial v \partial \mu} & \dfrac{\partial^2 TC}{\partial^2 \mu_v} \end{bmatrix}_{\overrightarrow{\Omega}=\overrightarrow{\Omega_i}}$$

(iv) Find the new trial solution $\overrightarrow{\Omega_{i+1}} = \Omega_i - \left[H\left(\overrightarrow{\Omega_i}\right)\right]^{-1} \vec{\nabla} TC\left(\overrightarrow{\Omega_i}\right)$.

(v) Set $i = i+1$ and repeat steps (iii) and (iv) until $\max\left(\left|\dfrac{\partial TC}{\partial \mu}\right|, \left|\dfrac{\partial TC}{\partial \mu_v}\right|\right) < \varepsilon$.

(vi) Find the global minimum value $TC^*(F^*, \mu^*, \mu_v^*) = TC^*(\overrightarrow{\Omega_i^*})$.

3.6 Illustration and Numerical Simulation

The analytical results of the system characteristics are not sufficient to establish the worthiness of the model developed. To explore the practical applicability of the proposed M/M/1/WV model with F-policy and server breakdown in a real-time machining system, we consider the machine repair problem encountered in automated manufacturing systems, having total machine M=20 and a finite capacity (say K=19) repair shop. The failed machines arrive at the repair shop with rate $\lambda = 2$ machines per day. Once the capacity is full, the failed machines are not allowed to join the system until the number of failed machines in the repair shop decreases to predefined threshold value say F=5 machines. The operator (i.e. server) takes startup time, before again allowing the failed machines to enter into the system for the repair job. During the normal busy state, the operator provides service with rate $\mu = 25$ machines per day. During the working vacation period, the operator also provides the service with lower rate $\mu_v = 20$ machines per day. The Markov model deals with more realistic and practical scenario by assuming that the repairman is prone to be unpredictable breakdowns during any stage of service with rate $\alpha = 0.05$ by noting that the operator is in broken down state on average 2 hours in a week of 5 days, while working 8 hours in a day. For the smooth functioning of the system, there is the provision of recovery of failed operator by providing repair immediately; the average recovery time of the operator is 1 minutes 15 seconds and 1 minute while broken down during normal busy and working vacation period, respectively. For the cited illustration, the probabilities corresponding to 8K=152 states are obtained using SOR method. Then, by using Equations (3.28)-(3.31) and (3.35), we compute performance indices E(N), P_B, P_I, P_D and E(W) as follows:

$$E(N) = 7.76,\ P_B = 0.9179,\ P_I = 0.0126,\ P_D = 0.000669 \text{ and } E(W) = 0.319.$$

3.6.1 Sensitivity of system descriptors with respect to performance indices

In order to compute the numerical results for the probabilities for different states of the system, the coding of SOR program is done in MATLAB software. The numerical simulation has been done to explore the effect of varying parameters on different performance indices.

In Tables 3.1-3.4 and Figures 3.2-3.5, different system performance indices are displayed for the default parameters set as-

$$C_H = 335; C_B = 100; C_I = 200; C_S = 400; C_D = 300; C_L = 26; C_W = 40; C_1 = 5; C_2 = 2; C_3 = 1;$$
$$M = 20; F = 5; m = 2; \lambda = 2; \psi = 2; \mu = 25; \mu_v = 20; \theta = 3; \alpha = 0.05; \beta_1 = 75; \beta_2 = 60.$$

The trends of variations in different system indices viz. long run system state probabilities P_B, P_D, P_I and mean queue length $E(N)$ and expected waiting time $E(W)$ by varying the different parameters are displayed in Tables 3.1 to 3.4 and Figures 3.2 to 3.5. Based on numerical results, we summarize the observations based on numerical results as follows:

(i) Effect of number of machines and threshold parameter (M, F)*:* It is noticed from the Tables 3.1-3.4 that on increasing the number of machines (M), performance indices namely expected number of failed machines E(N), expected waiting time in the system E(W) and state probability P_B show an increasing trend, whereas probabilities P_I and P_D decrease. From Figures 3.2(i)-3.2(iii) and 3.4(i)-3.4(iii) we also notice that expected system length E(N) and expected waiting time E(W) show increasing trend on increasing M. Figures 3.3(i)-3.3(iii) and 3.5(i)-3.5(iii) depict that on increasing the value of F, both E(N) and E(W) increase. The increasing trends in E(N) and E(W) by the increment in M and F are expected due to the fact that there are more failed machines in case of a large number of operative machines in the system.

(ii) Effect of the failure rates of machines and the server $(\boldsymbol{\lambda}, \boldsymbol{\alpha})$*:* It is clear from Table 3.1 and Figures 3.2(i) and 3.4(i) that the system indices namely expected number of failed machines in the system E(N), expected waiting time in the system E(W) and long run probabilities P_B, P_D increase whereas the state probability of server being remained idle (P_I) decreases on increasing the value of λ. The effect of variation of α on E(N) as well as E(W) is almost negligible; this pattern can be attributed to the choice of a parameter (α) which is set very low by considering the realistic scenario that server rarely fails. From Table 3.2 and Figures 3.2(iii) and 3.4(iii), it is also clearly noticed that the expected number of the failed machines E(N) and expected waiting time of failed machines in the system E(W) increase slowly as the failure rate (α) of server increases. For increasing value of failure rate (λ, a), the increasing pattern in E(N) and E(W) are more prominent for the higher rates which match with the observations in many real-time systems.

(iii) Effect of repair rates (μ, μ_v)**:**

Tables 3.3 and 3.4 show that on increasing the repair rate μ and μ_v, there is a decrement in the expected number of failed machines E(N) as well as in the expected waiting time of the failed machines. Figures 3.2(ii), 3.4(ii) and 3.3(iii), 3.5(iii) also depict the similar trends which match with our expectations. It is quite interesting finding that the expected number of failed machines E(N) and expected waiting time E(W) decrease up to a certain limit by improving the service rate μ and μ_v. Figures 3.3(iii) and 3.5(iii) show that the expected number of failed machines E(N) and expected waiting time E(W) gradually decrease, on increasing the value of μ_v These types of variations tally with the realistic experience of machine repair problems of machining systems.

(iv) Effect of startup rate and vacation completion rate (ψ, θ_v)**:**

Figure 3.3(i) displays that E(N) increases sharply in the beginning on increasing the value of ψ but for higher values of ψ, E(N) increases gradually. On increasing the value of θ_v in the Figure 3.3(ii), the value of E(N) decreases sharply in the beginning but for further higher values of θ_v, it attains almost constant value. In Figure 3.5(i) on increasing the value of ψ, the expected waiting time E(W) increases quickly for the lower value of F; however as F becomes large, it shows almost constant value. In the Figure 3.5(ii), for rising the value of θ_v expected waiting time E(W) initially decreases sharply, then after lowers down slowly.

3.6.2 Expected cost and optimal cost parameters

The cost function is evaluated by setting various cost elements and then optimal parameter value ‘F’ is determined by heuristic search approach. We first use direct substitution of successive values of F into the cost function given in the Equation (3.36) and examine the variations in the expected total cost with the increasing value of ‘F’. The value of ‘F’ corresponding to minimum expected cost is taken as ‘F^*’. Then by using quasi-Newton method, the repair rates of failed machines during normal busy and working vacation states (μ, μ_v) are determined by minimizing the cost function $TC(F^*, \mu, \mu_v)$. Quasi-Newton approach is implemented by fixing the $F = F^*$ to minimize the cost $TC(F^*, \mu, \mu_v)$ until $TC(F^*, \mu^*, \mu_v^*)$ is obtained. The implementation of quasi-Newton approach is done by fixing the maximum tolerance limit as 10^{-6}. For finding the global minimum of expected cost

$TC(F^*, \mu, \mu_v^*)$ and the corresponding optimal parameters μ^* and μ_v^*, we assign following two sets of cost elements:

Case I: C_H=\$50, C_B=\$100, C_I=\$200, C_S=\$350, C_D=\$400, C_L=\$26, C_W=\$40, C_K=\$5, C_1=\$3;

Case II: C_H=\$25, C_B=\$100, C_I=\$100, C_S=\$400, C_D=\$500, C_L=\$26, C_W=\$35, C_K=\$5, C_1=\$3;

For computing the cost function, other default parameters are chosen as-

$$M=16,\ m=2,\ \lambda=2,\ \psi=2,\ \mu=40,\ \mu_v=20,\ \theta_v=3,\ \alpha=0.05,\ \beta_1=75,\ \beta_2=60.$$

For cost sets I and II, heuristic search based on the direct allocation of the threshold parameter F, yields the optimal values of F as $F^*=9$ and $F^*=13$, respectively. It is clear from Table 3.5 that for fixing $F^*=9$, the optimal expected cost TC=\$376.57 is achieved at optimal parameters $\mu^*=29.47$ and $\mu_v^*=24.95$ using the quasi-Newton method.

For the cost set II in Table 3.6, it is observed that the minimum expected total cost converges to TC=\$365.37 at the optimal parameters $(F^*, \mu^*, \mu_v^*)=(13, 29.469, 24.95)$. For different values of F, the variations of the total cost of the system with other parameters (μ, μ_v^*), (λ, ψ) and (θ_v, α) are depicted in Figures 3.6(i-ii)-3.8(i-ii), respectively. It is observed from the Figures 3.6(i), 3.6(ii) and 3.7(ii) that TC is a convex function with respect to parameters μ and μ_v^* respectively. From Figures 3.6(i-ii) and 3.7(i-ii), it is worth noting that the cost function is convex in nature for the chosen parameters. Thus the value of optimal parameters for minimum expected total cost can be evaluated with some classical, heuristic or meta-heuristic techniques. It is observed that TC increases remarkably for the increasing value of the failure rate (see Fig. 3.7(i)) which indicates that the more often failures of machines are the costly affair which is quite common observation in real time system too.

Table 3.1: Variations in performance indices for different values of M and λ

M	λ	P_B	P_I	P_D	E(N)	E(W)
15	1	0.2501	0.3786	0.000476	1.33	0.097
	2	0.6485	0.0776	0.000660	4.08	0.188
	3	0.8557	0.0526	0.000646	6.25	0.269
20	1	0.4034	0.2200	0.000582	2.39	0.136
	2	0.9179	0.0126	0.000669	7.76	0.319
	3	0.9456	0.0379	0.000644	10.59	0.442
25	1	0.6085	0.1057	0.000643	4.08	0.195
	2	0.9909	0.0025	0.000666	12.45	0.500
	3	0.9546	0.0356	0.000644	15.02	0.625

Table 3.2: Variations in performance indices for different value of M and α

M	α	P_B	P_I	P_D	E(N)	E(W)
15	0.05	0.6485	0.0776	0.00066	4.08	0.188
	0.10	0.6488	0.0761	0.00656	4.12	0.191
	0.15	0.6498	0.0629	0.06209	4.50	0.217
20	0.05	0.9179	0.0126	0.00067	7.76	0.319
	0.10	0.9146	0.0123	0.00665	7.82	0.323
	0.15	0.8804	0.0103	0.06259	8.39	0.366
25	0.05	0.9909	0.0025	0.00067	12.45	0.500
	0.10	0.9851	0.0026	0.00661	12.52	0.506
	0.15	0.9301	0.0036	0.06234	13.15	0.564

Table 3.3: Variations in performance indices for different values of M and μ_v

M	μ_v	P_B	P_I	P_D	E(N)	E(W)
15	4	0.7029	0.0527	0.0006717	5.16	0.278
	12	0.6870	0.0533	0.0006738	4.67	0.230
	20	0.6485	0.0776	0.0006600	4.08	0.188
20	4	0.9242	0.0109	0.0006697	8.12	0.347
	12	0.9234	0.0101	0.0006704	7.96	0.333
	20	0.9179	0.0126	0.0006693	7.76	0.319
25	4	0.9910	0.0025	0.0006656	12.49	0.504
	12	0.9911	0.0024	0.0006656	12.47	0.502
	20	0.9909	0.0025	0.0006656	12.45	0.500

Table 3.4: Variations in performance indices for different values of M and μ

M	μ	P_B	P_I	P_D	E(N)	E(W)
15	22	0.749	0.0584	0.000659	4.68	0.231
	25	0.648	0.0776	0.000660	4.08	0.188
	28	0.559	0.0959	0.000660	3.68	0.163
20	22	0.960	0.0097	0.000665	8.93	0.411
	25	0.918	0.0126	0.000669	7.76	0.319
	28	0.852	0.0204	0.000674	6.77	0.257
25	22	0.991	0.0053	0.000663	13.77	0.630
	25	0.991	0.0025	0.000666	12.45	0.500
	28	0.979	0.0025	0.000667	11.08	0.399

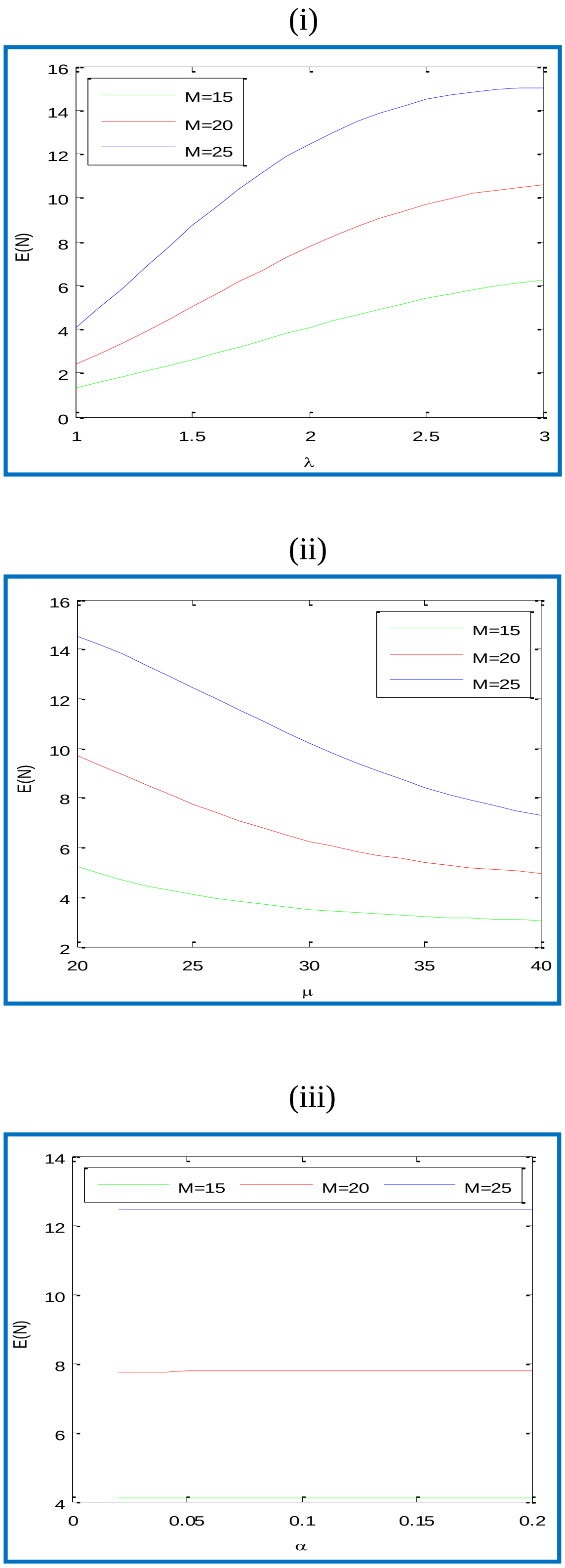


Fig. 3.2: Expected number of failed machine E(N) by varying (i) λ (ii) μ (iii) α for different values of M

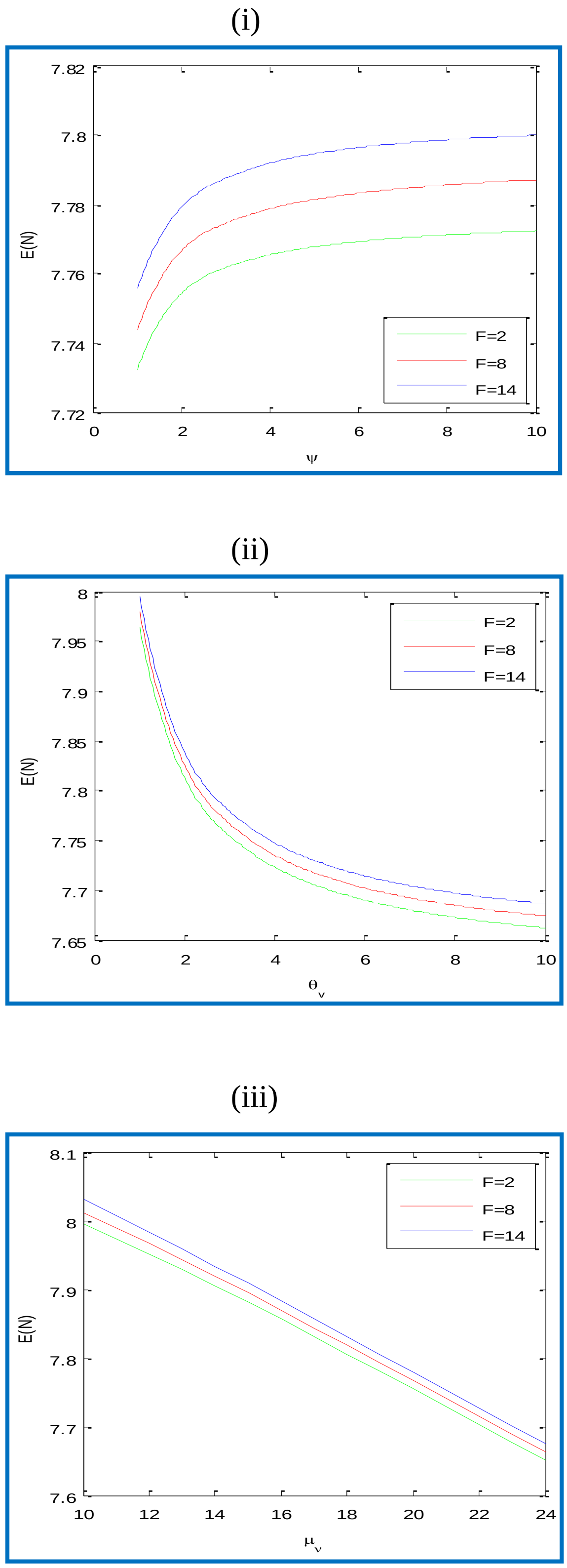
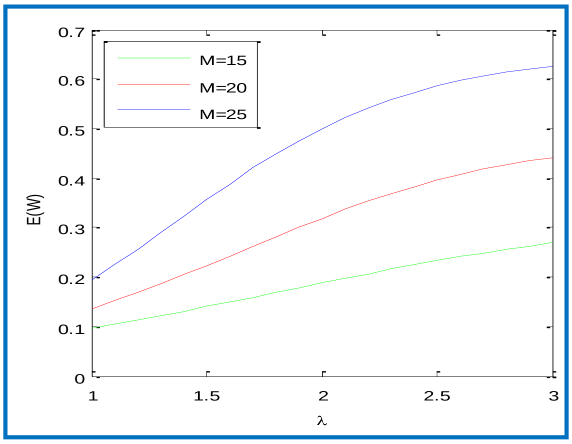
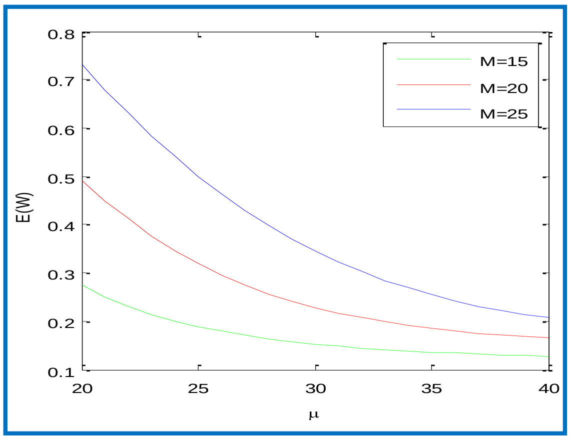


Fig. 3.3: Expected number of failed machine E(N) by varying (i) ψ (ii) θ_v (iii) μ_v for different values of F

(i)



(ii)



(iii)

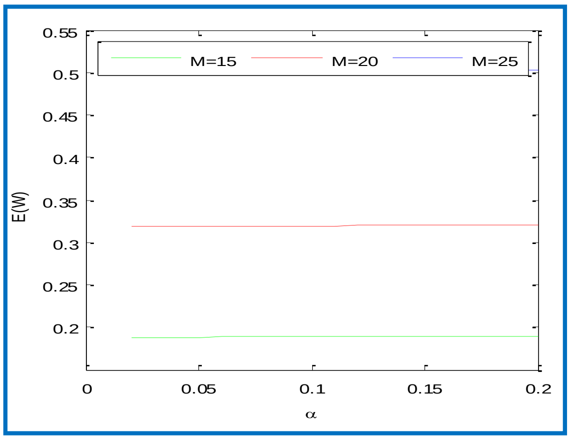
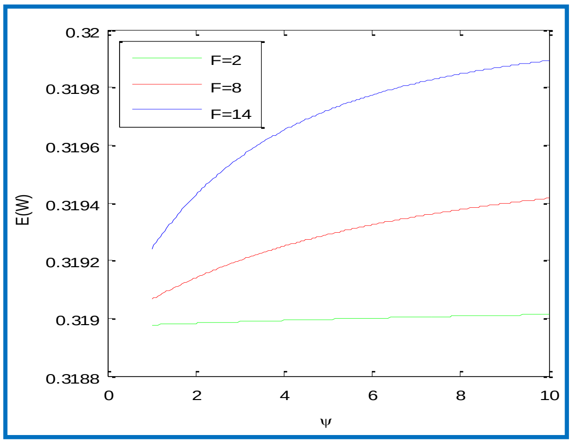
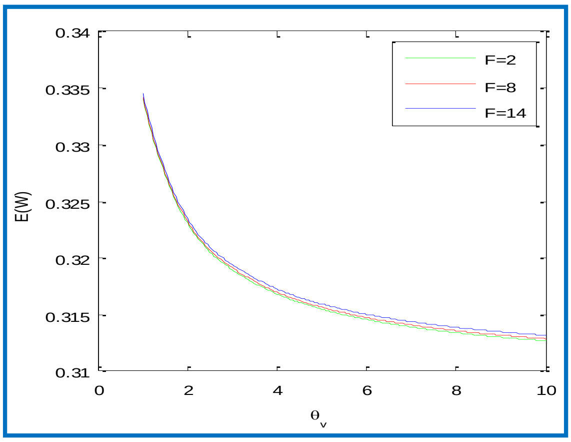


Fig. 3.4: Expected waiting time of failed machines E(W) **by varying (i) λ (ii) μ (iii) α for different values of M**

(i)



(ii)



(iii)

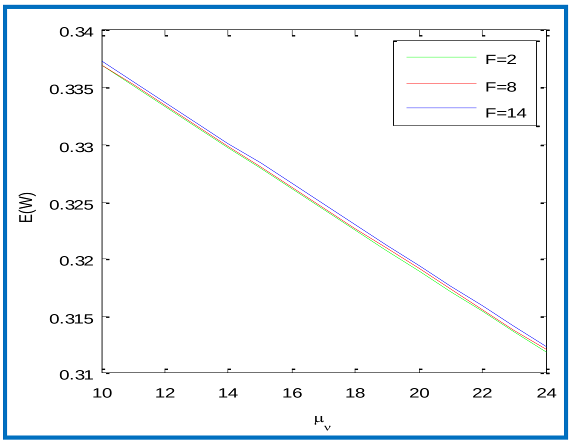
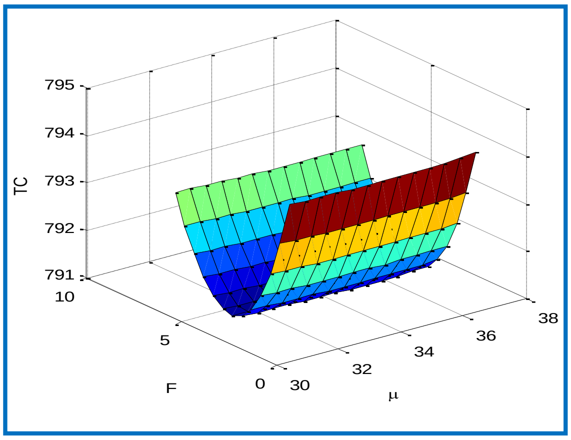


Fig. 3.5: Expected waiting time of failed machines E(W) **by varying (i) ψ (ii) θ_v (iii) μ_v for different values of F**

(i)



(i)

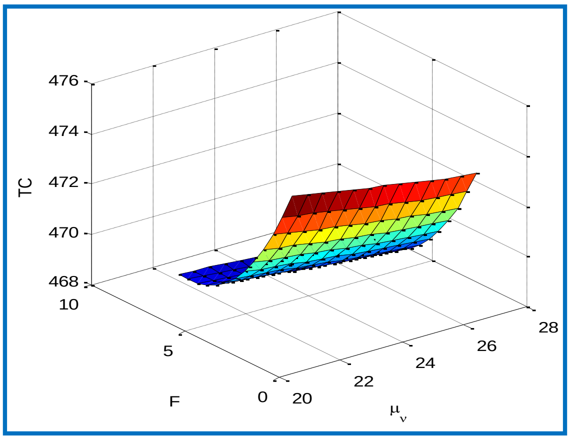
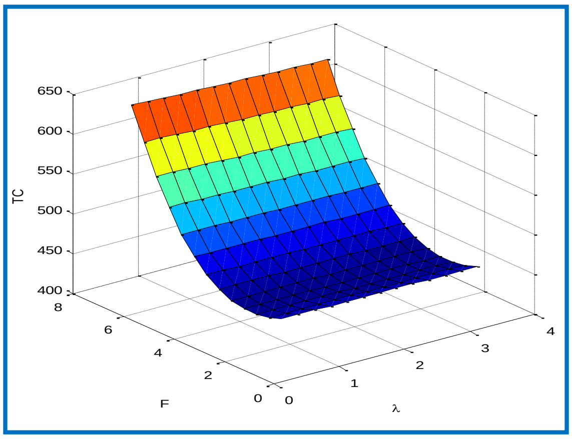


Fig. 3.6: Total cost of the system by varying F for different value of (i) μ (ii) μ_v

(ii)



(ii)

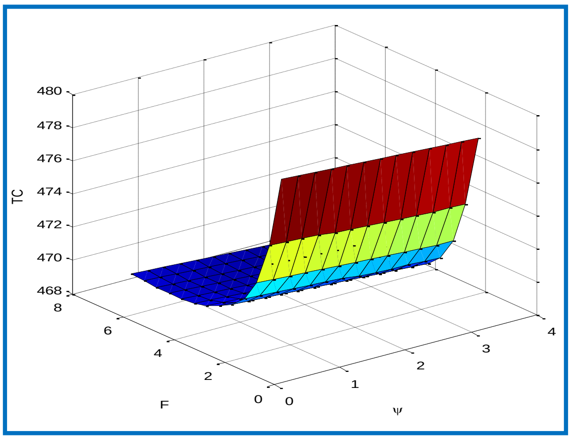
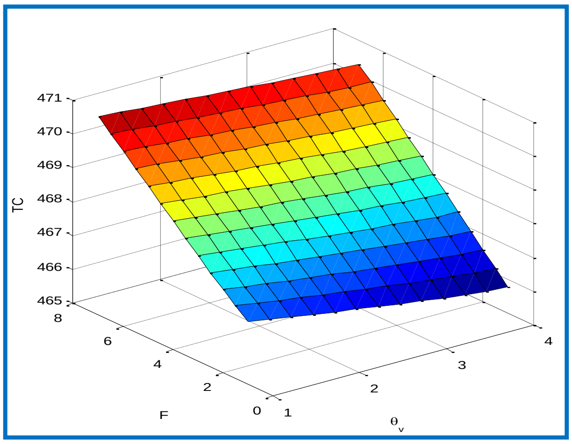


Fig. 3.7: Total cost of the system by varying F for different value of (i) λ (ii) ψ

(iii)



(iii)

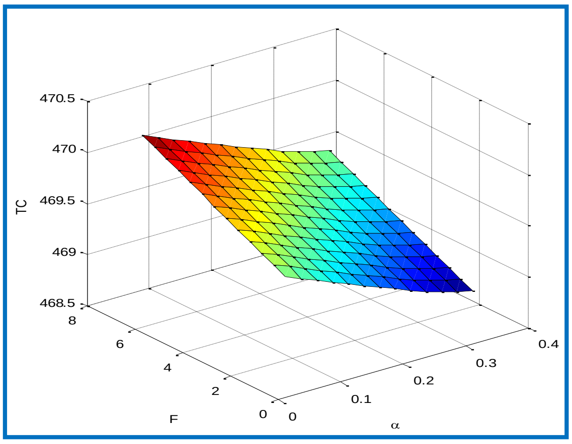


Fig. 3.8: Total cost of the system by varying F for different value of (i) θ_v (ii) α

Chapter 4

Unreliable Server FTS with Working Vacation

4.1 Introduction

The finite population queueing models for the machining systems with standby support have been developed by many queue theorists due to its applicability in real time systems having failure prone components. For example, standby power equipment is required during the operation of a patient in any hospital due to random power breakdown. Many more instances of fault occurrence can be noticed in real time systems, such as power stations, manufacturing and production units, nuclear and power plant systems, call centers etc. Liu *et al*. (2015) studied a Markovian repairable system with cold standbys and having single repairman which is allowed to take working vacation and vacation interruptions after each repair according to Bernoulli rule.

In any machining system while providing service, the server may break down; the service interruption due to server failure for a long time directly affects the profit/goodwill as well as the hindrance in achieving desired output. Excellent works on the machine repair problem with server breakdowns in different contexts were presented by (Shree *et al*., 2015; Kuo and Ke 2016), and many others.

For the modeling of FTS, the neuro-fuzzy inference model having the provision of neural network trained by using the available input/output data sets can be developed. In the context of automated machine repair system, the adaptive neuro-fuzzy controller can be easily designed for the prediction of optimal control parameters (Lin and Liu, 2003). Jain and Upadhyaya (2009) used ANFIS to match the soft computing based results with the analytical results obtained by matrix recursive method for the performance prediction of degraded multi-component machining system with switch over failure. They have developed Markov model under more realistic assumptions such as N-policy and multiple vacations. K-heterogeneous servers and multiple vacations Markov model were proposed by Kumar and Jain (2013) to analyze the machining system having operating as well as inventory of standby machines. Further, they have matched their results obtained by SOR with ANFIS generated results.

From the literature survey, it is evident that a very few research articles have appeared on the performance analysis of machining system with spare provisioning and operating under vacation policy in particular when features of reboot and recovery are included to make system fault tolerant. It is noticed that there is a research gap in the area of MRP with the option of working vacation or complete vacation. In many real time systems, whenever server becomes idle, it may have the option to go for either complete vacation or working vacation. This situation of choice of vacation and working vacation can also be realized in machine repair systems. From the literature review, it is noticed that there is no work on queueing models developed so far by taking combination of vacation and working vacation. In this chapter, we are concerned with Markov analysis for the performance prediction of FTS by developing machine repair model with unreliable server and provision of standby machines. We have also incorporated the feature of server's choice of either go for the complete vacation or opt for working vacation in case when the system becomes empty i.e. there is no repair job of failed machines. In case of no line up repair job, the server can either take complete vacation and remain idle or go for working vacation after taking set up time. The modeling of MRP with reboot and recovery processes can be implemented for the performance improvement of FTSs. To explore the performance metrics of the unreliable server machining system with standby support by incorporating assumptions of imperfect coverage, reboot and recovery along with the option of complete vacation or working vacation, a Markov model in general set up can be framed. Motivated by this fact, in the present chapter, we develop Markov model for the unreliable multi-component fault tolerant system by including the features of (i) multiple types of warm standbys, (ii) F-policy, (iii) optional working vacation, (iv) startup time (v) imperfect coverage. The noble feature of the present investigation is to allow the server, either to take full vacation or to continue the repairs to failed machines with lower rate (i.e. working vacation) during the vacation also. For the maintainability of FTS at optimum cost, the optimal value of control repair parameter is suggested.

The successive over relaxation (SOR) method has been used to solve the set of equations governing the model in order to determine the steady state probabilities associated with different system states. After solving the set of equations governing the concerned FT model, the impact of system descriptors on the performance metrics is examined by taking an illustration and conducting numerical simulation. The hybrid soft computing technique known as adaptive neuro-fuzzy inference system (ANFIS) is implemented to compare the results obtained by SOR method. The remaining contents of the chapter are structured in different sections. In Section 4.2, we describe the model whereas in Section 4.3, difference equations are constructed on the basis of birth-death process. In Section 4.4, various system performance metrics are formulated in terms of the steady state probabilities. In Section 4.5, we present the numerical simulation results and sensitivity analysis.

4.2 Model Description

Consider a finite population Markov M/M/1/K/V+WV model under admission control *F*-policy for the performance analysis of the multi component fault tolerant system. The fault tolerant machining system consists of M identical operating machines and is supported with k types of warm standbys and an unreliable server. There are S_i $(1 \le i \le k)$ standby machines of type i such that the total standby machines are $S = S^{(k)} = S_1 + S_2 + ... + S_k$. It is assumed that i^{th} $(1 \le i \le k-1)$ type standbys are used before $(i+1)^{th}$ type standbys to replace the failed machines. The operating as well as standby machines are prone to failure. The life time of operating (standby) machines are assumed to be exponentially distributed with parameter λ (a). Whenever an operating machine breaks down, it is immediately replaced by the i^{th} type of standby machine, if available. If all the standby machines are used in replacing the failed machines and some more machines fail, then the system operates in short mode till there are $m(< M)$ operating machines in the system. The system fails with the failure of $(M + S - m)^{th}$ machine, *i.e.* as soon as number of operating machines drops below m. The switchover of failed machines is not perfect *i.e.* the switch over of the failed machine takes place by standby machine with the coverage probability c. Whenever the switchover of failed machine by standby machine is unsuccessful with probability $(1-c)$, the system goes to unsafe mode. The recovery as well as reboot processes are governed by the exponential distribution. We assume that in the unsafe mode, the system is automatically cleared by a reboot process with rate r.

Once the system becomes empty, i.e. there is no job of repair, the server can take either complete vacation with probability $\bar{p}(=1-p)$ or working vacation with probability p, But before going for the vacation (working vacation), the system also needs some set up time which is exponentially distributed with rate ε_0 (ε). The repair time of failed machines during normal busy period (working vacation) is assumed to be governed by exponential distribution with mean $1/\mu(1/\mu_v)$. The duration of the working vacation period (vacation period) follows the exponential distribution with mean θ_v^{-1}. The server from vacation returns to working vacation (normal busy) mode with rate $\theta(\theta_v)$ after completing a random duration which is exponential distributed. The life time and repair time of the server are assumed to be exponentially distributed with mean rate a and b, respectively.

The control of the arrivals of failed machines in the system is done according to the F-policy which states that when the system capacity becomes full, from the working vacation (normal busy period) the server moves to F-policy mode by taking set up time ξ_w^{-1} (ξ_b^{-1}). In F-policy mode, the system forbids any broken down machines from entering in the system until workload of repair job of failed machines ceases to a pre-specified threshold level $F(0 \le F \le K-1)$.When the system again reaches to the threshold level 'F' of the queue length, the server takes a startup time governed by exponential distribution with parameter ψ; after completion of set up, the failed machines start to enter in the system. It is assumed that all the stochastic processes, associated with the set up and vacation (working vacation), reboot and recovery, and life time and repair times of machines, which are involved in the system, are independent and follow the Markovian property.

The bivariate Markov process $\chi(\tau) = \{(\xi(\tau), \eta(\tau)); \tau \ge 0\}$ is used to develop the Markov model. Here $\eta(\tau)$ and $\xi(\tau)$ denote the number of failed machines in the system and the state of the server at time τ, respectively.

$$\xi(\tau)=\begin{cases} 0, \text{ Server is on vaction mode (VAC);} \\ 1, \text{ Server is operating in working vaction mode (BWV);} \\ 2, \text{ Server is brokendown while failed from normal busy} \\ \quad \text{mode and working vacation mode (DBW);} \\ 3, \text{ Server is operating in normal busy (NOB);} \\ 4, \text{ Server is operating in busy mode and the failed machines are} \\ \quad \text{not allowed due to F-policy (BNF);} \\ 5, \text{ Server is under repair when broken down during the busy mode} \\ \quad \text{of the server and the failed machines are not allowed due to} \\ \quad \text{F-policy (DBF);} \\ 6, \text{ Server is in recovery state during the vacation period (RCV);} \\ 7, \text{ Server is in reboot state during vacation (RBV);} \\ 8, \text{ Server is in recovery state during working vacation period (RCW);} \\ 9, \text{ Server is in reboot state during working vacation period (RBW);} \\ 10, \text{ Server is in recovery state during brokendown period (RCD);} \\ 11, \text{ Server is in reboot state during brokendown period (RBD);} \\ 12, \text{ Server is in recovery state while operating in normal busy mode (RCB);} \\ 13, \text{ Server is under reboot state while rendering the service in normal} \\ \quad \text{busy mode (RBB);} \end{cases}$$

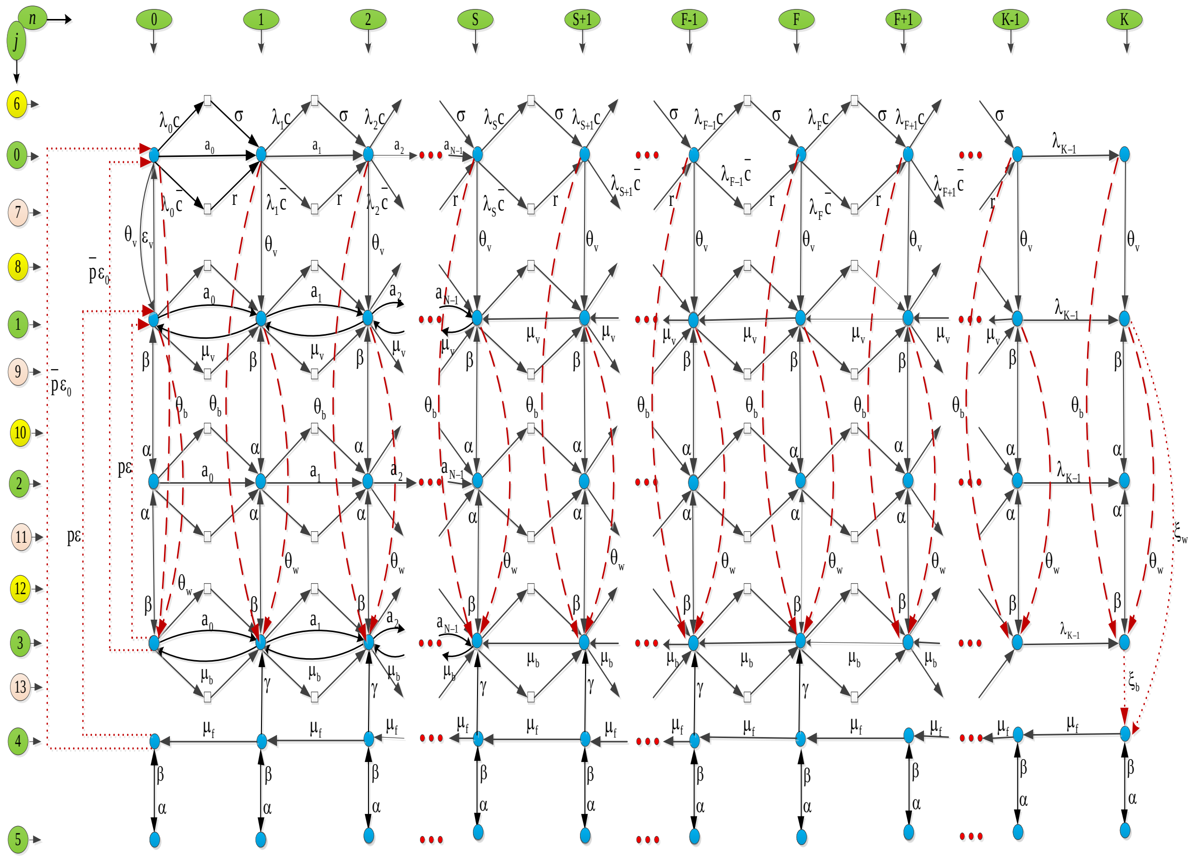


Fig. 4.1: State transition diagram

The steady state probabilities associated with state space (see Fig. 4.1.)

$$\chi=\{(i,n)\,|\,i=0,1,2,...,5;n=0,1,...,K\}\cup\{(i,n)\,|\,i=6,7,...,13;n=1,1,...,K-1\}$$

are denoted by $P_{i,n}=\lim\limits_{t\to\infty}\ \text{Prob.}\left\{\xi(\tau)=i,\eta(\tau)=n\right\}$.

4.3 Governing Equations

To evaluate the queue size distribution of the number of failed machines in the system, the governing equations are framed by setting the appropriate rates of underlying birth-death process. The law of flow balance is used to construct these equations. Now, we define the failure rate depending upon the number of failed machines as follows:

$$\lambda_n=\begin{cases} M\lambda+\left(\sum\limits_{i=2}^{k}S_i a_i+(S_1-n)a_1\right); & 0\le n<S_1 \\ M\lambda+\left(S^{(j-1)}-n\right)a_{j-1}+\sum\limits_{i=j}^{k-1}S_i a_i; & S^{(j-1)}\le n<S^{(j)};\ j=2,3,4,...,k-1 \\ M\lambda+\left(S^{(k)}-n\right)\delta_k; & S^{(k-1)}\le n<S^{(k)} \\ \left(M+S^{(k)}-n\right)\lambda; & S^{(k)}\le n<K=M+S^{(k)}-(m-1) \end{cases}$$

For the brevity, we have used the notation for failure rate of standby machines by

$$a_n = \begin{cases} \left(\sum_{i=2}^{k} S_i\delta_i + (S_1 - n)\delta_1\right); & 0 \le n < S_1 \\ \left(S^{(j-1)} - n\right)\delta_{j-1} + \sum_{i=j}^{k-1} S_i\delta_i; & S^{(j-1)} \le n < S^{(j)};\ j = 2,3,4....,k-1 \\ \left(S^{(k)} - n\right)\delta_k; & S^{(k-1)} \le n < S^{(k)} \end{cases}$$

(i) Server vacation (VAC) state when $\xi(\tau) = 0$ ***:***

As soon as the server becomes free i.e. when there is no job of repairing of the failed machines in the system, it reaches to vacation state. Now, we frame Chapman-Kolmogorov equations for the states $\{(0,n);\ n = 0,1,...,K\}$ as follows:

$$(\lambda_0 + a_0 + \theta_b + \theta)P_{0,0} = \varepsilon_v P_{1,0} + \bar{p}\,\varepsilon_0 P_{3,0} + \bar{p}\,\varepsilon_0 P_{4,0} \tag{4.1}$$

$$(\lambda_n + a_n + \theta_b + \theta)P_{0,n} = a_{n-1}P_{0,n-1} + \sigma P_{0,n-1} + r\,P_{0,n-1}; \qquad 1 \le n \le S-1 \tag{4.2}$$

$$(\lambda_S + \theta + \theta_b)P_{0,S} = a_{S-1}P_{0,S-1} + \sigma P_{0,S-1} + r\,P_{0,S-1} \tag{4.3}$$

$$(\lambda_n + \theta + \theta_b)P_{0,n} = \sigma P_{0,n-1} + r\,P_{0,n-1}; \qquad S+1 \le n \le K-1 \tag{4.4}$$

$$(\theta + \theta_b)P_{0,K} = \lambda_{K-1}P_{0,K-1} \tag{4.5}$$

(ii) Server working vacation (BWV) state when $\xi(\tau) = 1$***:***

In case of working vacation state of the server, the following steady state equations hold:

$$(\lambda_0 + \varepsilon_v + \alpha + a_0 + \theta_v)P_{1,0} = \mu_v P_{1,1} + \beta P_{2,0} + p\,\varepsilon P_{3,0} + p\varepsilon P_{4,0} + \theta P_{0,0} \tag{4.6}$$

$$\begin{aligned}(\lambda_n + a_n + \alpha + \theta_v + \mu_v)P_{1,n} &= \theta P_{0,n} + \sigma P_{1,n-1} + r\,P_{1,n-1} + a_{n-1}P_{1,n-1} \\ &\quad + \mu_v P_{1,n+1} + \beta P_{2,n}; 1 \le n \le S-1\end{aligned} \tag{4.7}$$

$$(\lambda_S + \alpha + \theta_v + \mu_v)P_{1,S} = \theta P_{0,S} + \sigma P_{1,S-1} + r\,P_{1,S-1} + a_{N-1}P_{1,S-1} + \mu_v P_{1,S+1} + \beta P_{2,S} \tag{4.8}$$

$$(\lambda_n + \alpha + \theta_v + \mu_v)P_{1,n} = \theta P_{0,n} + \sigma P_{1,n-1} + r\,P_{1,n-1} + \mu_v P_{1,n+1} + \beta P_{2,n}; \quad S+1 \le n \le K-1 \tag{4.9}$$

$$(\alpha + \xi_w + \theta_v)P_{1,K} = \theta P_{0,K} + \lambda_{K-1}P_{1,K-1} + \beta P_{2,K} \tag{4.10}$$

(iii) Server broken down state (DBW) when $\xi(\tau) = 2$ ***:***

When the server is under repair while broken down during working vacation $J(\tau) = 1$and normal busy state $\xi(\tau) = 3$, for the states $\{(2,n); n = 0,1,2,...,K\}$, we frame the steady state equations as follows:

$$(\lambda_0 + a_0 + 2\beta)P_{2,0} = \alpha P_{1,0} + \alpha P_{3,0} \tag{4.11}$$

$$(\lambda_n + a_n + 2\beta)P_{2,n} = \alpha P_{1,n} + \alpha P_{3,n} + \sigma P_{2,n-1} + r\,P_{2,n-1} + a_{n-1}P_{2,n-1} \qquad 1 \le n \le S-1 \tag{4.12}$$

$$(\lambda_S + 2\beta)P_{2,S} = \alpha P_{1,S} + \alpha P_{3,S} + \sigma P_{2,S-1} + r\,P_{2,S-1} + a_{S-1}P_{2,S-1} \tag{4.13}$$

$$(\lambda_n + 2\beta)P_{2,n} = \alpha P_{1,n} + \alpha P_{3,n} + \sigma P_{2,n-1} + r\,P_{2,n-1} \qquad S+1 \le n \le K-1 \tag{4.14}$$

$$2\beta P_{2,K} = \alpha P_{1,K} + \alpha P_{3,K} + \lambda_{K-1} P_{2,K-1} \tag{4.15}$$

(iv) Server normal busy (NOB) state when $\xi(\tau) = 3$:

When the server is operating in busy mode to provide the repair of the failed machines, the steady state equations are framed for the states $\{(3,n);\ n = 0,1,...,K\}$ as follows:

$$(\lambda_0 + a_0 + \alpha + p\varepsilon + \overline{p}\,\varepsilon_0)P_{3,0} = \beta P_{2,0} + \mu P_{3,1} + \theta_b P_{0,0} + \theta_v P_{1,0} \tag{4.16}$$

$$\begin{aligned}(\lambda_n + a_n + \alpha + \mu)P_{3,n} &= \theta_b P_{0,n} + \theta_v P_{1,n} + \beta P_{2,n} + \sigma P_{3,n-1} + r\,P_{3,n-1} + a_{n-1} P_{3,n-1} \\ &\quad + \mu\, P_{3,n+1} + \psi P_{4,n}; \qquad 1 \le n \le S-1\end{aligned} \tag{4.17}$$

$$\begin{aligned}(\lambda_S + \alpha + \mu)P_{3,S} &= \theta_b P_{0,S} + \theta_v P_{1,S} + \beta P_{2,S} + \sigma P_{3,S-1} + r\,P_{3,S-1} \\ &\quad + a_{S-1} P_{3,S-1} + \mu P_{3,S+1} + \psi\, P_{4,S}\end{aligned} \tag{4.18}$$

$$\begin{aligned}(\lambda_n + a + \mu)P_{3,n} &= \theta_b P_{0,n} + \theta_v P_{1,n} + \beta P_{2,n} + \sigma R_{3,n-1} + r\,Q_{3,n-1} \\ &\quad + \mu\, P_{3,n+1} + \psi\, P_{4,n}; \qquad S+1 \le n \le F\end{aligned} \tag{4.19}$$

$$\begin{aligned}(\lambda_n + a_n + \alpha + \mu)P_{3,n} &= \theta P_{0,n} + \theta_v P_{1,n} + \beta P_{2,n} + \sigma P_{3,n-1} \\ &\quad + r\,P_{3,n-1} + \mu P_{3,n+1}; \quad F+1 \le n \le K-1\end{aligned} \tag{4.20}$$

$$(\alpha + \xi_b)P_{3,K} = \theta P_{3,K} + \theta_v P_{1,K} + \beta P_{2,K} + \lambda_{K-1}\, P_{3,K-1} \tag{4.21}$$

(v) *Server is in normal busy state and the admission of failed machines is not allowed (BNF) when $\xi(\tau) = 4$*:

Here steady state equations are framed for the states $((4,n);\ n = 0,1,...,K)$ as

$$(\alpha + p\varepsilon + \overline{p}\,\varepsilon_0)P_{4,0} = \mu_f P_{4,1} + \beta P_{5,0} \tag{4.22}$$

$$(\alpha + \psi + \mu_f)P_{4,n} = \mu_f P_{4,n+1} + \beta P_{5,n} \qquad 1 \le n \le F \tag{4.23}$$

$$(\alpha + \mu_f)P_{4,n} = \mu_f P_{4,n+1} + \beta P_{5,n} \qquad F+1 \le n \le K-1 \tag{4.24}$$

$$(\alpha + \mu_f)P_{4,K} = \xi_w P_{1,K} + \xi_b P_{3,K} + \beta P_{5,n} \tag{4.25}$$

(vi) Server is in broken down state from busy state (DBF) when $\xi(\tau) = 5$:

In this case for states $\xi(\tau) = 5$, the steady state equation is:

$$\beta P_{5,n} = \alpha P_{4,n}; \qquad 0 \le n \le K \tag{4.26}$$

(vii) Server is in recovery state when $\xi(\tau) = 6,8,10,12$:

From $\xi(\tau) = 0,1,2,3$ states, due to failure detection with probability c, the system can go to recovery state $\xi(\tau) = 6,8,10,12$. For the recovery states (i,n), the steady state equations are framed as:

$$\sigma P_{6+2i,n+1} = \lambda_n c P_{i,n}; \quad i = 0,1,2,3; \qquad 1 \le n \le K-1 \tag{4.27}$$

(viii) Server is in reboot state when $\xi(\tau) = 7,9,11,13$:

From $\xi(\tau) = 0,1,2,3$ state, due to unsuccessful failure detection with probability $(1-c)$, the system can go to reboot state i.e. $\xi(\tau) = 7,9,11,13$ states. Here, the steady state equations are framed as:

$$r P_{7+2i,n+1} = \lambda_n \bar{c} P_{i,n}; \quad i = 0,1,2,3; \qquad 1 \le n \le K-1 \tag{4.28}$$

Chapman-Kolmogorov equations framed for the continuous Markov chain of concerned model are difficult to solve analytically due to cumbersome algebraic manipulation involved in recursive approach. However, numerical method can be easily implemented to obtain the probabilities associated with a large state space for which steady sate equations are already constructed. In the present investigation, the steady state equations (4.1)-(4.28) are solved numerically by using well known successive over relaxation (SOR) method, which is a powerful tool for the computation purpose of set of equations. In this method, the convergence rate is accelerated by selecting the appropriate relaxation parameter lying in interval [1, 1.25].

4.4 Performance Measures

The prime aim of determining probabilities in previous section is to formulate various metrics to examine the performances of the concerned fault tolerant system. The expressions for the mean queue length of the failed machines in the system, effective joining rate of failed machines, throughput of the system etc. are established as follows:

4.4.1 Queueing indices

(i) Mean queue size of failed machines E(N) is

$$E(N) = \sum_{i=0}^{5}\sum_{n=0}^{K} n P_{i,n} + \sum_{i=6}^{13}\sum_{n=1}^{K-1} n P_{i,n} \tag{4.29}$$

(ii) The throughput τ is

$$TP = \sum_{n=1}^{K-1} \mu_v P_{1,n} + \sum_{n=1}^{K-1} \mu P_{3,n} + \sum_{n=1}^{K} \mu_f P_{4,n} \tag{4.30}$$

(iii) Average number of available standby machines is

$$E(S) = \sum_{i=0}^{3}\sum_{n=0}^{S} (S-n) P_{i,n} + \sum_{i=6}^{13}\sum_{n=0}^{S} (S-n) P_{i,n} \tag{4.31}$$

(iv) The effective rate by which failed machines join the waiting queue, is given by

$$\lambda_{eff} = \sum_{i=0}^{3}\sum_{n=0}^{S-1} (\lambda_n + a_n) P_{i,n} + \sum_{i=0}^{3}\sum_{n=S}^{K} \lambda_{n-S} P_{i,n} \tag{4.32}$$

(v) Machine availability is obtained using

$$MA = 1 - \frac{E(N)}{M+S} \tag{4.33}$$

4.4.2 Long run probabilities

Now we establish long run probabilities associated with different states of the server which may be busy (P_B), broken down and under repair (P_{BD}), on vacation (P_V) and on working vacation (P_{WV}) respectively. Thus

$$P_B = \sum_{n=1}^{K-1} P_{1,n} + \sum_{n=1}^{K-1} P_{3,n} + \sum_{n=1}^{K} P_{4,n} \tag{4.34}$$

$$P_{BD} = \sum_{n=0}^{K} P_{2,n} + \sum_{n=1}^{K-1} P_{10,n} + \sum_{n=1}^{K-1} P_{11,n} + \sum_{n=0}^{K} P_{5,n} \tag{4.35}$$

$$P_V = \sum_{n=0}^{K} P_{0,n} + \sum_{n=1}^{K-1} P_{6,n} + \sum_{n=1}^{K-1} P_{7,n} \tag{4.36}$$

$$P_{WV} = \sum_{n=0}^{K} P_{1,n} + \sum_{n=1}^{K-1} P_{8,n} + \sum_{n=1}^{K-1} P_{9,n} \tag{4.37}$$

4.4.3 Cost function

To quantify per unit time total cost $TC\ (\mu)$ spent for the system, the various cost factors related to several system indices of Markovian model of fault tolerant system are taken into consideration. Now, we define per unit cost related to different activities as follows:

C_H : Holding cost per unit time associated with each down machine.

C_B : Cost per unit time incurred when the server is in normal busy state.

C_{BD} : Cost per unit time incurred when the server is broken down and is under repair.

C_v : Cost per unit time incurred when the server is on vacation.

C_{wv} : Cost incurred when the server is in working vacation state.

C_F : Cost incurred for providing service to the failed machines when the admission of failed machines are not allowed.

C_A : Cost incurred for providing the repair to the failed machines when the admission of failed machines are allowed.

The total cost per unit time incurred on the system is framed by summing different cost factors multiplied by respective system indices as follows:

$$TC\ (\mu) = C_H E(N) + C_B P_B + C_{BD} P_{BD} + C_v P_V + C_{wv} P_{wv} + C_F \mu_f + C_A \mu \tag{4.38}$$

4.5 Numerical Results

To reveal the practical applicability of multi-component fault tolerant system operating in real time machining environment, numerical illustration is taken. To compute numerical results, we fix the various parameters as

$K=10, M=8, S=4,\ m=2,\ c=0.5,\ a=0.2, \lambda=0.5,\ \mu=\mu_f=4, \alpha=0.05,\ \ \beta=2,$

$\sigma=1,\ r=1.5,\ \psi=1.5,\ \theta=\theta_v=1.5,\ \xi_b=0.5, \xi_w=0.5, \varepsilon=1.5, \varepsilon_0=1.0,\ p=0.6.$

The sensitivity of parameters has been examined to reveal the impact of varying system descriptors on different system metrics. The numerical results displayed in the form of graphs can be easily interpreted to understand the behavior of FTS system.

Table 4.1: Cost elements associated with various system indices

Cost Set	C_H	C_B	C_{BD}	C_V	C_{WV}	C_F	C_A
I	\$80	\$20	\$30	\$5	\$20	\$1	\$6
II	\$70	\$10	\$10	\$5	\$20	\$1	\$6

(ψ,θ)	(0.5, 1)	(1, 1)	(1.5, 1)	(1.5, 0.5)	(1.5, 1.5)	(1.5, 2.5)
μ^*	7.967	8.479	8.802	8.084	8.366	8.518
Optimal cost	300.7518	295.1175	330.6238	330.6238	337.9267	343.4445

Table 4.2: Optimal repair rate μ^* and optimal cost (\$)

The optimal repair rate and associated minimum total cost are obtained for the two sets of cost factors and are displayed in the Table 4.1.

The optimal repair rate 'μ' is obtained by computing the cost TC which is also depicted in Figures 4.2 (i-ii). Table 4.2 depicts the optimal repair rate and corresponding optimal total cost $TC(\mu^*)$ for different sets of cost elements.

The impact of system descriptors on different indices are examined by displaying the numerical results in Tables 4.4-4.5 and Figures 4.4(i-ii)-4.5(i-ii). The expected number of failed machines summarized in Tables 4.4-4.5 indicates that E(N) increases as λ grows up but decreases as μ increases. The long run probabilities P_{BD}, P_B and total cost incurred on the system also increases as λ increases but lowers down as μ increases. It is also found that the mean number of standby machines decreases (increases) as $\lambda\,(\mu)$ increases. From Tables 4.4-4.5, we notice that the impact of c on various system indices is also significant.

Neuro-fuzzy technique is used to demonstrate the feasibility of soft computing approach for the quantitative assessment of various performance indices of the fault tolerant MRP in particular when input parameters are not crisp. The results by ANFIS approach have been computed by using neuro-fuzzy tool in Matlab software. The failure rate (λ) is treated as linguistic variable in the context of the fuzzy system. The membership function for the failure rate of operating machine (λ) is considered as Gaussian function.

Table 4.3: Linguistic values of the membership functions for input parameter λ

Input variable	No. of membership functions	Linguistic values
Failure rate of operating unit λ	5	• Very low • Low• Average • High • Very high

Table 4.3 provides the linguistic values of membership functions corresponding to the input parameter λ. The shape of the corresponding membership function treated as Gaussian function is depicted in Figure 4.3. The numerical results corresponding to ANFIS are plotted by tick marks in Figures 4.4(i-ii)-4.5(i-ii) whereas the continuous curves are drawn for the results computed by using SOR method.

From Figure 4.4(i), we see that as the rate of failed machine (λ) increases, E(N) initially increases rapidly and then after becomes almost constant. The trend of E(N) is plotted in Figure 4.4(i); a sharp increment is noticed up to $\lambda=1$, and then after it becomes asymptotically stable as λ grows. From Figures 4.4(ii) and 4.5(i), it is clear that the expected number of standby machines *E(S)* and machine availability (*MA*) decrease rapidly initially but as λ grows, these indices become almost constant i.e. the higher value of λ has negligible impact on *E(S)* and machine availability (MA). From Figure 4.5(ii), it is clearly seen that as failure rate of operating unit (λ) grows up, the throughput of the system (TP) increases rapidly initially and then after gradually becomes almost constant.

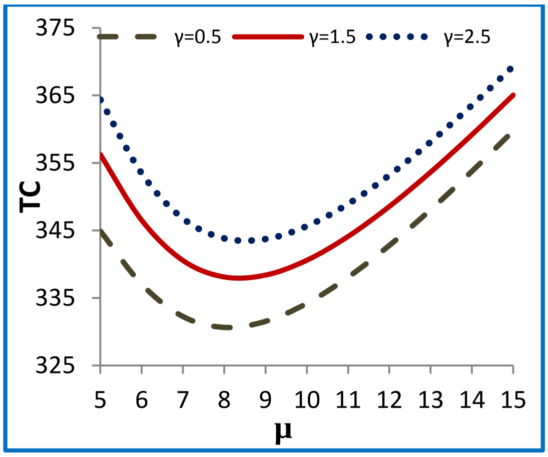
The Figures 4.4(i-ii)-4.5(i-ii) exhibit almost coincident values for both analytical and ANFIS results. Based on critical and comparative analysis of graphs, we conclude that the SOR results are very close to the results shown by neuro-fuzzy results as such neuro-fuzzy controller can be developed for the FTS to track the performance of many real time embedded systems.

Table 4.4: Effect of λ and c on various system indices

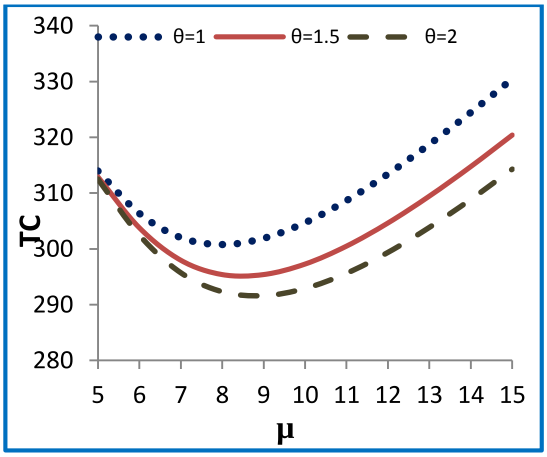
c	λ	E(N)	E(S)	P_{BD}	P_B	TC
0.3	0.5	3.21	2.712	0.00938	0.1732	277.4
	1	4.11	2.042	0.01398	0.2241	322.5
	1.5	4.51	1.773	0.01668	0.2475	342.3
0.6	0.5	3.21	2.710	0.00935	0.1722	277.4
	1	4.10	2.044	0.01393	0.2226	322.2
	1.5	4.49	1.779	0.01661	0.2457	341.6
0.8	0.5	3.21	2.707	0.00933	0.1713	277.4
	1	4.10	2.046	0.01388	0.2211	321.8
	1.5	4.48	1.784	0.01654	0.2440	341.0

Table 4.5: Effect of μ and c on various system indices

c	μ	E(N)	E(S)	P_{BD}	P_B	TC
0.3	2	5.082	0.868	0.0335	0.496	280.3
	3	4.732	1.293	0.0284	0.412	269.3
	4	4.631	1.496	0.0252	0.359	271.2
0.6	2	5.076	0.869	0.0334	0.491	279.9
	3	4.722	1.296	0.0283	0.407	268.8
	4	4.618	1.501	0.0251	0.355	270.5
0.8	2	5.070	0.871	0.0333	0.486	279.5
	3	4.713	1.299	0.0282	0.403	268.2
	4	4.606	1.506	0.0251	0.351	269.8



(i)



(ii)

Fig. 4.2: Variation of TC with respect to μ for different value of (i) γ and (ii) θ

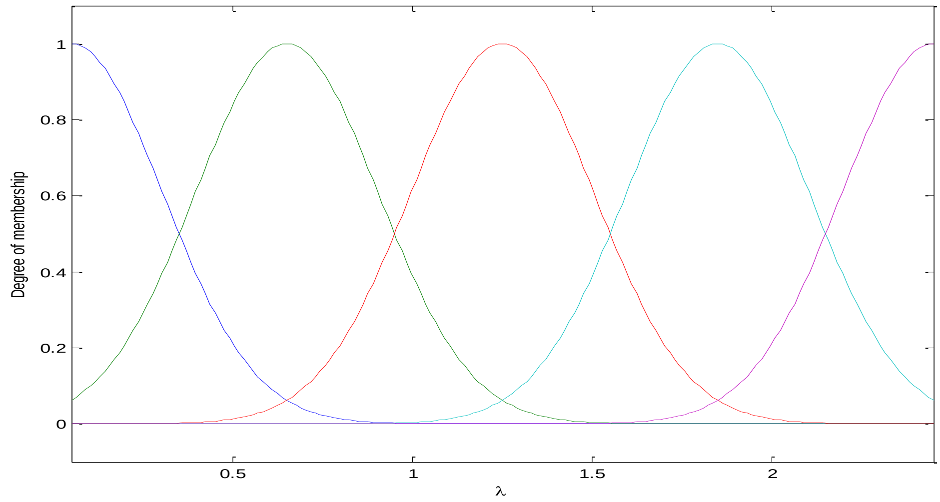
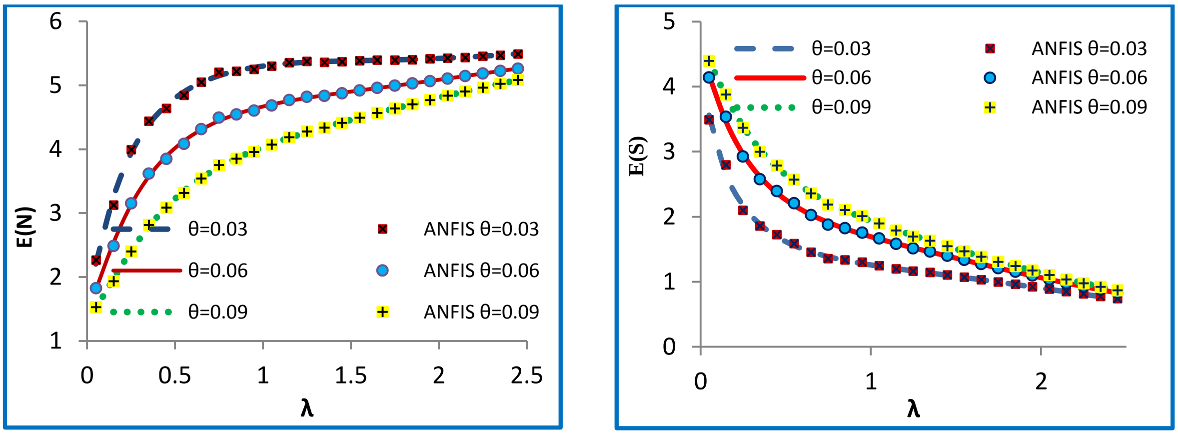
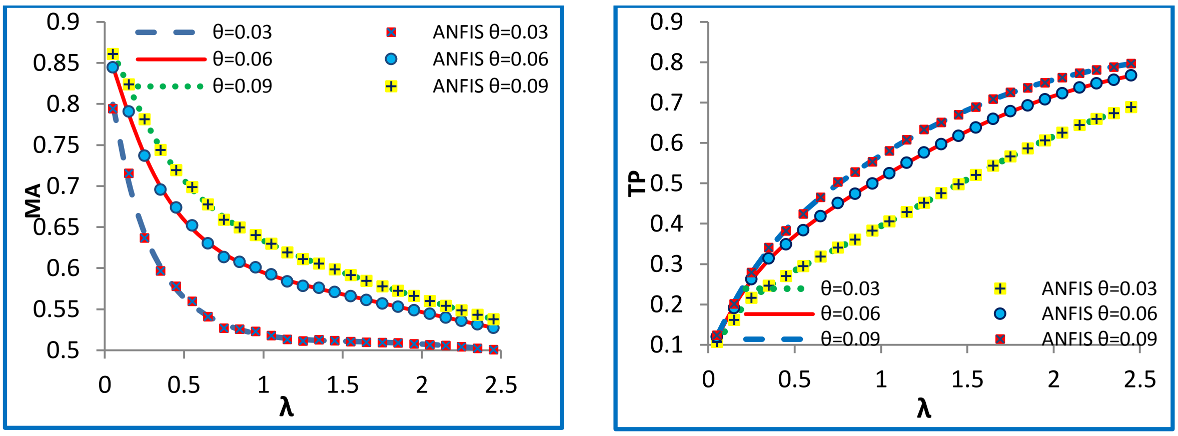


Fig. 4.3: Membership functions for input variable λ



(i) (ii)

Fig. 4.4: Variation of (i) *E(N)* and (ii) *E(S)* with respect to λ for different value of θ used in SOR and ANFIS



(i) (ii)

Fig. 4.5: Variation of (i) *MA* and (ii) *TP* with respect to λ for different value of θ used in SOR and ANFIS

Chapter 5

Unreliable Server FTS with Working vacation and Working Breakdown

5.1 Introduction

In queueing literature, some works have been done for the finite population unreliable queueing models with standby support by many prominent researchers to analyze the reliability indices. Wang and Kuo (2000) studied a series system with the provision of the mixed standby support. They have evaluated MTTF and availability for four configuration of series system. Further, Jain *et al*. (2004) developed the finite population model by considering the N-policy for machine repair problems. They have used the concept of reneging and warm standby in order to evaluate MTTF and reliability indices of machining system. To make the system more closer to realistic scenarios, Jain (2016) studied a repairable redundant system by incorporating mixed standbys, imperfect repair, reboot and switching failure.

The failure process of any machining system can be governed by ‘the statistical properties of the time between consecutive failures’. In many machining systems, it is assumed that upon a failure, the machining system stops service completely and goes under a repair process. Also, a failed component becomes operational again after it has been repaired. The concepts of failure prone machines in machining system maintained by repairman and supported by standbys, are worthwhile for the modeling of many real time systems. In queueing literature, there has been a growing interest in the Markov analysis of unreliable server machining system which has numerous applications in real time fault tolerant system. The related contributions of the queue theorists on the machining system with unreliable server can be seen in the works of (Ke *et al.*, 2014; Yang and Chiang, 2014). Markov model for the machine repair problem for the manufacturing system with N-policy, imperfect coverage, reboot delay and server breakdown was proposed by Jain *et al*. (2014). They have used the matrix method to evaluate the queue size distribution for the repairable machining system.

This chapter presents a finite population Markov machine repair model for the machining system with the provision of fault tolerance. The proposed model incorporates many realistic assumptions viz. (i) working breakdown of the server, (ii) warm standbys and (iii) working vacation. The performance model of the fault tolerant machining system can facilitate the performance metrics which may be helpful in upgrading the maintainability and redundancy policies of the concerned system. The remaining contents of the chapter are structured in different sections. In Section 5.2, we describe the model whereas in Section 5.3, differential difference equations for the system states are formulated on the basis of birth-death process. The spectral theory based matrix method is employed in Section 5.4 to evaluate the queue size distribution. In Section 5.5, performance metrics are formulated in terms of transient probabilities. In Section 5.6, we provide numerical simulation by taking an illustration.

5.2 System Description

Consider a finite Markov model for multi-component machining system comprising of M operating and total $S\,(=\sum_{i=1}^{k} S_i)$ mixed standbys where k^{th} type of standby machines available are S_k. The operating (standby) machines are assumed to be prone to breakdown with rate $\lambda(a)$, where $(0 < a < \lambda)$. The life time of operating (standby) machines are assumed to be exponentially distributed with rate $\lambda(a)$. In case when the operating machine fails, it is immediately replaced with i^{th} $(1 \le i \le k)$ type of standby machine (S_i) available. The replaced standby machines have the similar failure characteristic as that of operating machine. The system works in degraded mode, when the number of operating machine lowers down to $M\,(\ge l)$.

The repair time of failed machines during operating state, working vacation and working breakdown are assumed to be governed by exponential distribution with rate $\mu, \mu_{\mathrm{v}}, \mu_{\mathrm{d}}$, respectively. As soon as there is no repair job left in the system, the server goes for working vacation state with rate ε and provides service with rate $\mu_{\mathrm{v}} \le \mu$.The server is prone to failure during operation and working vacation state. When the server fails during the busy state, it can also repair the failed units with degraded repair rate μ_{d}. The life time and repair time of the server are assumed to be exponentially distributed with mean rate α and β, respectively. The duration of the working vacation period follows the exponential distribution with mean $1/\theta$.

The bivariate stochastic process $\chi(\tau) = \{\xi(\tau), \eta(\tau); \tau \ge 0\}$ is used to develop Markov model. Here $\xi(\tau)$ denote the number of failed units in the system and $\eta(\tau)\,(\eta(\tau) = 0,1,2,...,L)$ and the state of the server respectively at time τ. Now we define $\xi(\tau)$ as

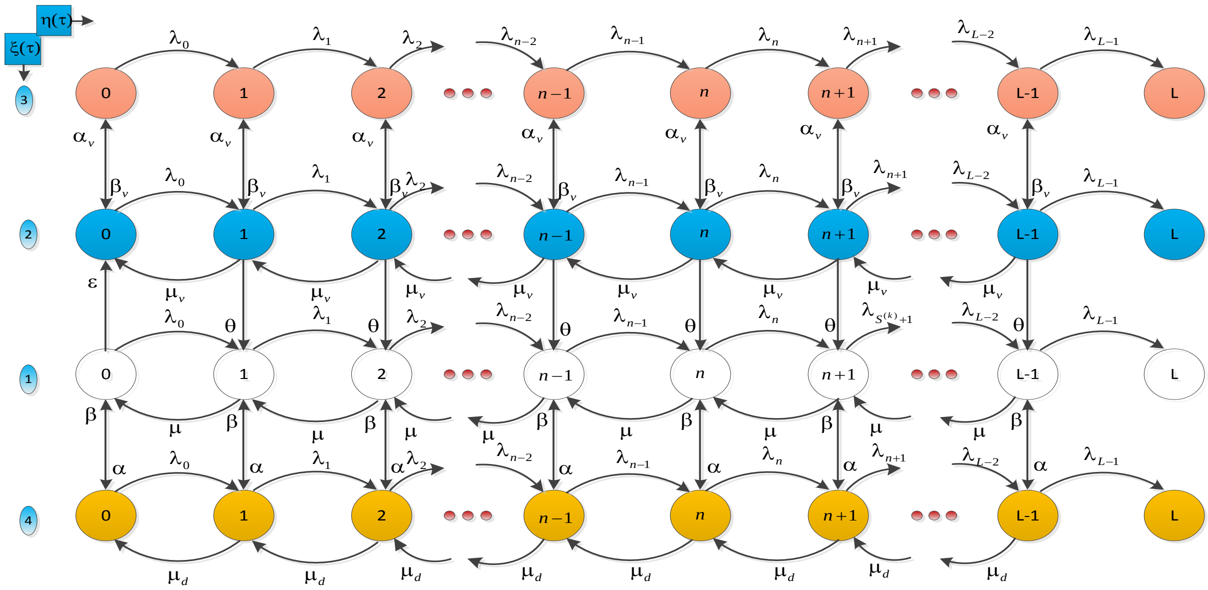


Fig. 5.1: State transition diagram of M/M/1/WV FTS

$$\xi(\tau)=\begin{cases}1, \text{ Server is in operating state;}\\ 2, \text{ Server is in working vacation state;}\\ 3, \text{ Server is under repair when broken down during working vacation state;}\\ 4, \text{ Server is under repair when broken down during operating state and}\\ \quad \text{can perform repair job with degraded rate.}\end{cases}$$

The failure rate of units depends upon the number of operating as well as available standby machines, and is given by:

$$\lambda_m=\begin{cases}M\lambda+\left(\sum_{i=2}^{k}S_i a_i+(S_1-m)a_1\right); & 0\le m\le S_1\\ M\lambda+\left(S^{(j)}-m\right)a_j+\sum_{i=j+1}^{k}S_i a_i; & S^{(j-1)}\le m\le S^{(j)};\ j=2,3,4,\dots,k\\ \left(M+S^{(k)}-m\right)\lambda; & S^{(k)}\le m\le L=M+S^{(k)}-(l-1)\end{cases}$$

5.3 Governing Equations

To evaluate the system size distribution associated with the number of failed machines, and different value of $\xi(\tau)=1,2,3,4$, the differential difference equations are framed by setting the appropriate rates of underlying birth-death process. We apply the law of flow balance to construct the following governing equations:

(i) $\xi(\tau)=1$: ***The server is in operating state.***

$$\frac{dP_{1,0}(\tau)}{d\tau}=-(\lambda_0+\alpha+\varepsilon)P_{1,0}(\tau)+\mu P_{1,1}(\tau)+\beta P_{4,0}(\tau) \tag{5.1}$$

$$\frac{dP_{1,m}(\tau)}{d\tau}=-(\lambda_m+\alpha+\mu)P_{1,m}(\tau)+\lambda_{m-1}P_{1,m-1}(\tau)+\theta P_{2,m}(\tau)$$
$$+\mu P_{1,m+1}(\tau)+\beta P_{4,m}(\tau);\ 1\le l\le L-2 \tag{5.2}$$

$$\frac{dP_{1,L-1}(\tau)}{d\tau}=-(\lambda_{L-1}+\alpha+\mu)P_{1,L-1}(\tau)+\lambda_{L-2}P_{1,L-2}(\tau)+\theta P_{2,L-1}(\tau)+\beta P_{4,L-1}(\tau) \tag{5.3}$$

$$\frac{dP_{1,L}(\tau)}{d\tau}=\lambda_{L-1}P_{1,L-1}(\tau) \tag{5.4}$$

(ii) $\xi(\tau)=2$ **: *The server is in working vacation state.***

$$\frac{dP_{2,0}(\tau)}{d\tau}=-(\lambda_0+\alpha_v)P_{2,0}(\tau)+\mu_v P_{2,1}(\tau)+\beta_v P_{3,0}(\tau)+\varepsilon P_{1,0}(\tau) \tag{5.5}$$

$$\frac{dP_{2,m}(\tau)}{d\tau}=-(\lambda_m+\alpha_v+\mu_v+\theta)P_{2,m}(\tau)+\lambda_{m-1}P_{2,m-1}(\tau)$$
$$+\mu_v P_{2,m+1}(\tau)+\beta_v P_{3,m}(\tau);\ 1\le l\le L-2 \tag{5.6}$$

$$\frac{dP_{2,L-1}(\tau)}{d\tau}=-(\lambda_{L-1}+\alpha_v+\mu_v+\theta)P_{2,L-1}(\tau)+\lambda_{L-2}P_{2,L-2}(\tau)+\beta_v P_{3,L-1}(\tau) \tag{5.7}$$

$$\frac{dP_{2,L}(\tau)}{d\tau}=\lambda_{L-1}P_{2,L-1}(\tau) \tag{5.8}$$

(iii) $\xi(\tau)=3$ **: *The server is broken down while failed during working vacation state.***

$$\frac{dP_{3,0}(\tau)}{d\tau}=-(\lambda_0+\beta_v)P_{3,0}(\tau)+\alpha_v P_{2,0}(\tau) \tag{5.9}$$

$$\frac{dP_{3,m}(\tau)}{d\tau}=-(\lambda_m+\beta_v)P_{3,m}(\tau)+\lambda_{m-1}P_{3,m-1}(\tau)+\alpha_v P_{3,m}(\tau);\ \ 1\le m\le L-2 \tag{5.10}$$

$$\frac{dP_{3,L}(\tau)}{d\tau}=\lambda_{L-1}P_{3,\,L-1}(\tau) \tag{5.11}$$

(iv) $\xi(\tau)=4$ **: *The server partially broken down while failed during operating state and performing repair job in degraded mode.***

$$\frac{dP_{4,0}(\tau)}{d\tau}=-(\lambda_0+\beta)P_{4,0}(\tau)+\mu_d P_{4,1}(\tau)+\alpha P_{1,0}(\tau) \tag{5.12}$$

$$\frac{dP_{4,m}(\tau)}{d\tau}=-(\lambda_n+\beta+\mu_d)P_{4,m}(\tau)+\lambda_{m-1}P_{4,m-1}(\tau)+\mu_d P_{4,m+1}(\tau)$$
$$+\alpha P_{4,m}(\tau);\ \ 1\le m\le L-2 \tag{5.13}$$

$$\frac{dP_{4,L-1}(\tau)}{d\tau}=-(\lambda_{L-1}+\beta+\mu_d)P_{4,L-1}(\tau)+\lambda_{L-2}P_{4,L-2}(\tau)+\alpha P_{4,L-1}(\tau) \tag{5.14}$$

$$\frac{dP_{4,L}(\tau)}{d\tau}=\lambda_{L-1}P_{4,L-1}(\tau) \tag{5.15}$$

5.4 The Mathematical Analysis

In this section, we use spectral theory by employing the '*matrix method*' for solving differential-difference Equations (5.1)-(5.15). First, we take Laplace transforms of Equations (5.1)-(5.15) and then put them in the form of block matrix equations.

Laplace transforms of Equations (5.1)-(5.15) yield the following set of equations:

(i) $\xi(\tau)=1$: ***The server is in operating state.***

$$(s+\lambda_0+\alpha+\varepsilon)P_{1,0}^*(s)-\mu P_{1,1}^*(s)-\beta P_{4,0}^*(s)=P_{1,0}(0) \tag{5.16}$$

$$\begin{aligned}&(s+\lambda_m+\alpha+\mu)P_{1,m}^*(s)-\lambda_{m-1}P_{2,m-1}^*(s)-\theta P_{2,m}^*(s)-\mu P_{1,m+1}^*(s)-\beta P_{4,m}^*(s)\\&=P_{1,m}(0);\ \ 1\le m\le L-2\end{aligned} \tag{5.17}$$

$$(s+\lambda_n+\alpha+\mu)P_{1,L-1}^*(s)-\lambda_{L-2}P_{2,L-2}^*(s)-\theta P_{2,L-1}^*(s)-\beta P_{4,L-1}^*(s)=P_{1,L-1}(0) \tag{5.18}$$

$$sP_{1,L}^*(s)-\lambda_{L-1}P_{1,L-1}^*(s)=P_{1,L}(0) \tag{5.19}$$

(ii) $\xi(\tau)=2$: ***The server is in working vacation state.***

$$(s+\lambda_0+\alpha_v)P_{2,0}^*(s)-\mu_v P_{2,1}^*(s)-\beta_v P_{3,0}^*(s)-\varepsilon P_{1,0}^*(s)=P_{2,0}(0) \tag{5.20}$$

$$\begin{aligned}&(s+\lambda_m+\alpha_v+\mu_v+\theta)P_{2,m}^*(s)-\lambda_{m-1}P_{2,m-1}^*(s)-\mu_v P_{2,m+1}^*(s)-\beta_v P_{3,m}^*(s)\\&=P_{2,m}(0);\ 1\le m\le L-2\end{aligned} \tag{5.21}$$

$$(s+\lambda_{L-1}+\alpha_v+\mu_v+\theta)P_{2,L-1}^*(s)-\lambda_{L-2}P_{2,L-2}^*(s)-\beta_v P_{3,L-1}^*(s)=P_{2,L-1}(0) \tag{5.22}$$

$$sP_{2,L}^*(s)-\lambda_{L-1}P_{2,L-1}^*(s)=P_{2,L}(0) \tag{5.23}$$

(iii) $\xi(\tau)=3$: ***The server is broken down while failed during working vacation state.***

$$(s+\lambda_0+\beta_v)P_{3,0}^*(s)-\alpha_v P_{2,0}(t)=P_{3,0}(0) \tag{5.24}$$

$$(s+\lambda_m+\beta_v)P_{3,m}^*(s)-\lambda_{m-1}P_{3,m-1}^*(s)-\alpha_v P_{3,m}^*(s)=P_{3,m}(0);\ \ 1\le m\le L-1 \tag{5.25}$$

$$sP_{3,L}^*(s)-\lambda_{L-1}P_{3,L-1}^*(s)=P_{3,L}(0) \tag{5.26}$$

(iv) $\xi(\tau)=4$: ***The server partially broken down while failed during operating state and performing repair job in degraded mode.***

$$(s+\lambda_0+\beta)P_{4,0}^*(s)-\mu_d P_{4,1}^*(s)-\alpha P_{1,0}^*(s)=P_{4,0}(0) \tag{5.27}$$

$$\begin{aligned}&(s+\lambda_m+\beta+\mu_d)P_{4,m}^*(s)-\lambda_{m-1}P_{4,m-1}^*(s)-\mu_d P_{4,m+1}^*(s)-\alpha P_{4,m}^*(s)\\&=P_{4,m}(0);\ 1\le m\le L-2\end{aligned} \tag{5.28}$$

$$(s+\lambda_{L-1}+\beta+\mu_d)P_{4,L-1}^*(s)-\lambda_{L-2}P_{4,L-2}^*(s)-\alpha P_{4,L-1}^*(s)=P_{4,L-1}(0) \tag{5.29}$$

$$sP_{4,L}^*(s)-\lambda_{L-1}P_{4,L-1}^*(s)=P_{4,L}(0) \tag{5.30}$$

The above set of Equations (5.16)-(5.30) can be written in matrix form as:

$$\mathbf{Q}(s)\,\mathbf{P}^{*}(s)=\mathbf{P}(0) \tag{5.31}$$

where $$\mathbf{Q}(s)=\begin{bmatrix} \mathbf{\Lambda}_{11} & \mathbf{\Lambda}_{12} & \mathbf{0} & \mathbf{\Lambda}_{14} \\ \mathbf{0} & \mathbf{\Lambda}_{22} & \mathbf{\Lambda}_{23} & \mathbf{0} \\ \mathbf{0} & \mathbf{\Lambda}_{32} & \mathbf{\Lambda}_{33} & \mathbf{0} \\ \mathbf{\Lambda}_{41} & \mathbf{0} & \mathbf{0} & \mathbf{\Lambda}_{44} \end{bmatrix}$$

$\mathbf{Q}(s)$ is square matrix of order 4(L+1) and is constructed by taking coefficients of unknown probabilities. Here all sub matrices $[\Lambda_{ij}]$ and null matrixes '**0**' are of order $(\mathrm{L}+1)$ and are given by

$$\mathbf{\Lambda}_{11}=\begin{bmatrix} -\begin{pmatrix} s+\lambda_0 \\ +\alpha+\varepsilon \end{pmatrix} & \mu & 0 & \cdots & 0 & 0 \\ \lambda_0 & -\begin{pmatrix} s+\lambda_1 \\ +\alpha+\mu \end{pmatrix} & \mu & \cdots & 0 & 0 \\ 0 & \lambda_1 & -\begin{pmatrix} s+\lambda_2 \\ +\alpha+\mu \end{pmatrix} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -\begin{pmatrix} s+\lambda_{L-1} \\ +\alpha+\mu \end{pmatrix} & \mu \\ 0 & 0 & 0 & \cdots & \lambda_{L-1} & -\begin{pmatrix} s+\alpha \\ +\mu \end{pmatrix} \end{bmatrix};$$

$$\mathbf{\Lambda}_{12}=\begin{bmatrix} \varepsilon & 0 & 0 & \cdots & 0 & 0 \\ 0 & \theta & 0 & \cdots & 0 & 0 \\ 0 & 0 & \theta & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \theta & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix};$$

$\mathbf{\Lambda}_{13}=\beta\,\mathbf{I}_{(\mathrm{L+1})}$, $\mathbf{\Lambda}_{23}=\beta_{\mathrm{v}}\,\mathbf{I}_{(\mathrm{L+1})}$, $\mathbf{\Lambda}_{32}=\alpha_{\mathrm{v}}\,\mathbf{I}_{(\mathrm{L+1})}$, $\mathbf{\Lambda}_{41}=\alpha\,\mathbf{I}_{(\mathrm{L+1})}$;

where $\mathbf{I}_{(\mathrm{L+1})}$ is unit matrix of order (L+1).

$$\mathbf{\Lambda}_{21}=\begin{bmatrix} -\begin{pmatrix} s+\lambda_0 \\ +\alpha_v \end{pmatrix} & \mu_v & 0 & \cdots & 0 & 0 \\ \lambda_0 & -\begin{pmatrix} s+\lambda_1+\alpha_v \\ +\mu_v+\theta \end{pmatrix} & \mu_v & \cdots & 0 & 0 \\ 0 & \lambda_1 & -\begin{pmatrix} s+\lambda_2+\alpha_v \\ +\mu_v+\theta \end{pmatrix} & \cdots & \mu_v & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -\begin{pmatrix} s+\lambda_{L-1}+\alpha_v \\ +\mu_v+\theta \end{pmatrix} & \mu_v \\ 0 & 0 & 0 & \cdots & \lambda_{L-1} & -\begin{pmatrix} s+\alpha_v \\ +\mu_v+\theta \end{pmatrix} \end{bmatrix};$$

$$\mathbf{\Lambda}_{33}=\begin{bmatrix} -(s+\lambda_0+\beta_v) & 0 & 0 & \cdots & 0 & 0 \\ \lambda_0 & -(s+\lambda_1+\beta_v) & 0 & \cdots & 0 & 0 \\ 0 & \lambda_1 & -(s+\lambda_2+\beta_v) & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -(s+\lambda_{L-1}+\beta_v) & 0 \\ 0 & 0 & 0 & \cdots & \lambda_{L-1} & -(s+\beta_v) \end{bmatrix};$$

$$\mathbf{\Lambda}_{44}=\begin{bmatrix} -(s+\lambda_0+\beta) & \mu_d & 0 & \cdots & 0 & 0 \\ \lambda_0 & -\begin{pmatrix} s+\lambda_1 \\ +\beta+\mu_d \end{pmatrix} & \mu_d & \cdots & 0 & 0 \\ 0 & \lambda_1 & -\begin{pmatrix} s+\lambda_2 \\ +\beta+\mu_d \end{pmatrix} & \cdots & \mu_d & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -\begin{pmatrix} s+\lambda_{L-1} \\ +\beta+\mu_d \end{pmatrix} & \mu_d \\ 0 & 0 & 0 & \cdots & \lambda_{L-1} & -\begin{pmatrix} s+\beta \\ +\mu_d \end{pmatrix} \end{bmatrix};$$

Also, denotes the unknown vector $\mathbf{P}^*(\mathrm{s})$ in partitioned form as

$$\mathrm{P}^*(s)=[\mathrm{P}^*_{1,m}(s),\mathrm{P}^*_{2,m}(s),\mathrm{P}^*_{3,m}(s),\mathrm{P}^*_{4,m}(s)]^T$$

where

$$\mathbf{P}^*_{1,\mathrm{m}}(\mathrm{s})=[\mathbf{P}^*_{1,0}(\mathrm{s}),\mathbf{P}^*_{1,1}(\mathrm{s}),...,\mathbf{P}^*_{1,\mathrm{L}\text{-}1}(\mathrm{s}),\mathbf{P}^*_{1,\mathrm{L}}(\mathrm{s})]^{\mathrm{T}};\ \ \mathbf{P}^*_{2,\mathrm{m}}(\mathrm{s})=[\mathbf{P}^*_{2,0}(\mathrm{s}),\mathbf{P}^*_{2,1}(\mathrm{s}),...,\mathbf{P}^*_{2,\mathrm{L}\text{-}1}(\mathrm{s}),\mathbf{P}^*_{2,\mathrm{L}}(\mathrm{s})]^{\mathrm{T}}$$

$$\mathbf{P}^*_{3,\mathrm{m}}(\mathrm{s})=[\mathbf{P}^*_{3,0}(\mathrm{s}),\mathbf{P}^*_{3,1}(\mathrm{s}),...,\mathbf{P}^*_{3,\mathrm{L}\text{-}1}(\mathrm{s}),\mathbf{P}^*_{3,\mathrm{L}}(\mathrm{s})]^{\mathrm{T}};\ \ \mathbf{P}^*_{4,\mathrm{m}}(\mathrm{s})=[\mathbf{P}^*_{4,0}(\mathrm{s}),\mathbf{P}^*_{4,1}(\mathrm{s}),...,\mathbf{P}^*_{4,\mathrm{L}\text{-}1}(\mathrm{s}),\mathbf{P}^*_{4,\mathrm{L}}(\mathrm{s})]^{\mathrm{T}}.$$

Here $\mathbf{P}(0)=[1,0,0,...,0,0,0,0,...,0,0,0,0,...,0]_{(4\mathrm{L}+4)\times 1}$ is an initial vector.

Now, we apply Crammer's rule on matrix $\mathbf{Q}(s)$ to compute the probabilities $\mathrm{P}^*_{i,m}(s)$,

$(i=1,2,3,4; m=0,1,...,L)$ as:

$$\mathbf{P}^{*}_{\mathrm{i,m}}(\mathrm{s}) = \frac{\det[\mathbf{Q}_{\mathrm{j+1}}(\mathrm{s})]}{\det[\mathbf{Q}(\mathrm{s})]},\ (j=(i-1)(L+1)+m;\ i=1,2,3,4\ \ m=0,1,2,...,L) \tag{5.32}$$

where $\mathbf{Q}_{\mathrm{j+1}}(\mathrm{s})$ $(j=(i-1)(L+1)+m+1;\ i=,1,2,3,4\ \ m=0,1,2,...,L)$ is obtained by replacing j^{th} column of det[$\mathbf{Q}(\mathrm{s})$] with the elements of initial vector $\mathbf{P}(0)$.

To solve the Equation (5.31), we proceed to calculate the characteristic roots of matrix $\mathbf{Q}(\mathrm{s})$. It is noted that $s=0$ is one of the roots. Let $s=(-d)$, so that we have

$$\mathbf{Q}\left(-d\right) = (\mathbf{Q} - d\ \boldsymbol{I}) \tag{5.33}$$

Now Equation (5.31) converts into

$$Q(-d)\,P^{*}(s) = (Q - d\,\mathrm{I})\,P^{*}(s) = P(0) \tag{5.34}$$

Suppose that other roots in which r are real roots and n are complex roots in pairs are denoted by:

$d_1, d_2, ..., d_r$ and $(d_{r+1}, \bar{d}_{r+1}), (d_{r+2}, \bar{d}_{r+2}), ..., (d_{r+n}, \bar{d}_{r+n})$, respectively.

Now, we get

$$|\mathbf{Q}(\mathrm{s})| = s\left[\prod_{j=1}^{r}(s+d_j)\right]\left[\prod_{j=1}^{n}(s+d_{r+j})(s+\bar{d}_{r+j})\right] \tag{5.35}$$

From Equations (5.32) and (5.35), we get

$$P^{*}_{i,m}(\mathrm{s}) = \frac{|\mathrm{Q(s)}|}{\mathrm{s}\left[\prod_{\mathrm{j=1}}^{r}(\mathrm{s+d_j})\right]\left[\prod_{\mathrm{j=1}}^{n}(\mathrm{s+d_{r+j}})(\mathrm{s}+\bar{\mathrm{d}}_{\mathrm{r+j}})\right]},\ i=1,\ 2,\ 3,\ 4,\ m=0,1,2,...,\mathrm{L} \tag{5.36}$$

Using partial fractions, we expand Equation (5.36), as follows

$$\boldsymbol{P}^{*}_{\boldsymbol{i,m}}(\boldsymbol{s}) = \frac{\mathrm{a}_0}{s} + \frac{\mathrm{a}_1}{s+\mathrm{d}_1} + ... + \frac{\mathrm{a}_{\boldsymbol{r}}}{s+\mathrm{d}_{\boldsymbol{r}}} + \frac{\mathrm{b}_{\boldsymbol{r}}s+\mathrm{c}_{\boldsymbol{r}}}{(s+\mathrm{d}_{\boldsymbol{r}+1})(s+\bar{\mathrm{d}}_{\boldsymbol{r}+1})} + ... + \frac{\mathrm{b}_{\boldsymbol{n}}s+\mathrm{c}_{\boldsymbol{n}}}{(s+\mathrm{d}_{\boldsymbol{r+n}})(s+\bar{\mathrm{d}}_{\boldsymbol{r+n}})} \tag{5.37}$$

Here a_0 and a_q $(q=1,2,...,r)$ are real numbers calculated as:

$$\mathrm{a}_0 = \frac{\left|\mathbf{Q}_{\boldsymbol{j}+1}(\mathrm{s})\right|}{\left(\prod_{\mathrm{j=1}}^{1}\mathrm{d_j}\right)\left(\prod_{\mathrm{j=1}}^{\boldsymbol{n}}\mathrm{d_{l+j}}\bar{\mathrm{d}}_{\mathrm{l+j}}\right)} \tag{5.38}$$

$$\mathrm{a}_q = \frac{\left|\mathbf{Q}_{j+1}(\mathrm{-d_p})\right|}{(\mathrm{-d_q})\left[\prod_{\substack{\mathrm{j=1}\\ \mathrm{j^1q}}}^{1}(\mathrm{d_j\text{-}d_q})\right]\left[\prod_{\substack{\mathrm{j=1}\\ \mathrm{j^1q}}}^{n}(\mathrm{d_{l+j}\text{-}d_q})(\bar{\mathrm{d}}_{\mathrm{l+j}}\text{-}\mathrm{d_q})\right]},\mathrm{q}=1,2,...,r. \tag{5.39}$$

Let complex characteristic root d_{r+p} is a combination of real part u_p and imaginary part v_p.

Then

$$\mathrm{b_p}(-\mathbf{d_{l+p}})+\mathrm{c_p} = \frac{\left|Q_j(-d_{r+p})\right|}{(-d_{r+p})\left[\prod_{\substack{j=1\\ j\neq p}}^{r}(d_j-d_{r+p})\right]\left[\prod_{\substack{j=1\\ j\neq p}}^{m}(d_{r+j}-d_{r+p})(\bar{d}_{r+j}-d_{r+p})\right]}; \qquad (5.40)$$

$p=1,2,\ldots,n$

On taking inverse Laplace transform of Equation (5.37), we get

$$P_{i,m}(\mathrm{t}) = \mathrm{a}_0+\sum_{\mathrm{q=1}}^{r}\mathrm{a}_q\mathrm{e}^{-\mathrm{d}_q\mathrm{t}}+\sum_{\mathrm{p=1}}^{n}\left[\mathrm{b_p e^{-u_p t}cos(v_p t)}+\frac{\mathrm{c_p\text{-}b_p u_p}}{\mathrm{v_p}}\mathrm{e^{-u_p t}sin(v_p t)}\right]; \qquad (5.41)$$

$i=1,2,\ 3,\ 4\ \ n=0,1,2,....,L$

where $a_0, a_p, d_q, b_p, c_p, u_p$ and v_p all are real numbers.

5.5 Performance Measures

In this section, we determine various metrics for the FTS in order to analyze the system performance. For this purpose, we develop some system metrics in terms of transient probabilities as follows:

(i) The expected number of failed machines in the system at time τ

$$EN(\tau)=\sum_{i=0}^{4}\sum_{m=0}^{L} mP_{1,m}(\tau) \qquad (5.42)$$

(ii) Machine availability at time τ *is given by*

$$MA(\tau)=1-\frac{EN(\tau)}{M+S} \qquad (5.43)$$

(iii) Throughput of the system at time τ *is as*

$$TP(\tau)=\sum_{m=1}^{L}\mu P_{1,m}(\tau)+\sum_{m=1}^{L}\mu_v P_{2,m}(\tau)+\sum_{m=1}^{L}\mu_d P_{4,m}(\tau) \qquad (5.44)$$

(iv) Reliability of FTS at time τ *is given by*

$$R_Y(\tau)=1-\sum_{i=1}^{4}P_{i,L}(\tau) \qquad (5.45)$$

(v) Mean time to system failure (MTTF) is obtained as

$$MTTF=\int_0^{\infty}R_Y(\tau)\,dt=\lim_{s\to 0}[1/s-\sum_{i=1}^{4}P_{i,L}(\tau)] \qquad (5.46)$$

(vi) The probability of the server being busy in normal and working vacation mode are given by

$$P_B(\tau)=\sum_{m=0}^{L}P_{1,m}(\tau) \text{ and } P_{WV}(\tau)=\sum_{m=1}^{L}P_{2,m}(\tau)\text{, respectively.} \qquad (5.47)$$

(vii) The probability that the server is under repair while broken down during working vacation at time τ

$$P_{BD}(\tau)=\sum_{m=0}^{L}P_{4,m}(\tau) \tag{5.48}$$

(viii) The probability that the server is in working broken down state and under repair at time τ

$$P_{d}(\tau)=\sum_{m=0}^{L}P_{4,m}(\tau) \tag{5.49}$$

(ix) System cost

It is always beneficial to quantify the total cost incurred on the system so that the industrial engineer or system designers may get insight for the up gradation of future design of the system. The total expected system cost is composition of various cost elements associated to specific states or activities of the concerned system. The total cost incurred per unit time of the system is determined by framing the expected cost function $TC(\tau)$. The cost elements are defined as follows.

C_H : Holding cost associated with each failed machine.

C_B : Cost incurred when the server is in normal busy state.

C_P : Penalty cost incurred while the server is under broken down state.

C_d : Cost incurred when the server is brokendown and is under repair.

C_{wv} : Cost incurred when the server is in working vacation state.

C_m : Cost involved in the repair of each failed machine with service rate μ.

The cost function is framed as follows:

$$TC(\tau)=C_H EN(\tau)+C_B P_B(\tau)+C_P P_{BD}(\tau)+C_d P_d(\tau)+C_{wv}P_{wv}(\tau)+C_m\mu \tag{5.50}$$

5.6 Numerical Simulation

In this section, to predict the various system metrics with respect to system parameters, numerical simulation is provided by taking a suitable illustration. The system behavior is examined by computing the numerical results which are displayed in Tables 5.2-5.3 and Figures 5.2-5.6. To illustrate the analytical results of FTMS derived in the earlier sections, we have developed the code for generating the numerical results in software 'MATLAB' using 'Pentium IV'. For computation purpose, we fix the default parameter as follows:

$$L=8, M=6, S=2, k=2, l=1, \lambda=0.5, a=0.02, b=4, \mu=5,$$
$$\mu_v=3, \mu_d=1, \alpha_1=0.03, \alpha_2=0.02,\ \varepsilon=0.3, \theta=1.0.$$

Table 5.1: Cost elements incurred with various system metrics

Cost Set	C_H	C_B	C_p	C_d	C_{wv}	C_m
I	$70	$30	$140	$180	$250	$20
II	$80	$30	$140	$150	$250	$15
III	$80	$30	$120	$160	$260	$20
IV	$70	$30	$140	$180	$250	$20

Table 5.2: Effect of service rate (μ_v) on various performance indices

μ_v	τ	$EN(\tau)$	$MA(\tau)$	$R_Y(\tau)$	$P_{WV}(\tau)$	$P_d(\tau)$	TC(τ)
2	2	2.37	0.66	0.995	0.828	0.0022	388.90
	4	2.48	0.65	0.951	0.300	0.0043	346.28
	6	2.61	0.63	0.915	0.290	0.0046	356.15
3	2	2.12	0.70	0.996	0.906	0.0020	375.92
	4	2.32	0.67	0.961	0.344	0.0042	337.41
	6	2.46	0.65	0.929	0.317	0.0045	346.42
4	2	1.91	0.73	0.997	0.978	0.0019	365.59
	4	2.18	0.69	0.968	0.394	0.0041	330.95
	6	2.34	0.67	0.939	0.348	0.0044	339.38

Table 5.3: Effect of service rate (θ) on various performance indices

θ	τ	$EN(\tau)$	$MA(\tau)$	$R_Y(\tau)$	$P_{WV}(\tau)$	$P_d(\tau)$	TC(τ)
2.5	2	2.12	0.70	0.996	0.906	0.0020	375.92
	4	2.32	0.67	0.961	0.344	0.0042	337.41
	6	2.46	0.65	0.929	0.317	0.0045	346.42
3.5	2	2.03	0.71	0.997	0.744	0.0024	354.42
	4	2.23	0.68	0.965	0.285	0.0044	324.94
	6	2.38	0.66	0.935	0.270	0.0046	336.10
4.5	2	1.97	0.72	0.997	0.638	0.0026	340.23
	4	2.18	0.69	0.967	0.253	0.0045	318.40
	6	2.34	0.67	0.938	0.243	0.0047	330.46

(i) (ii) (iii)

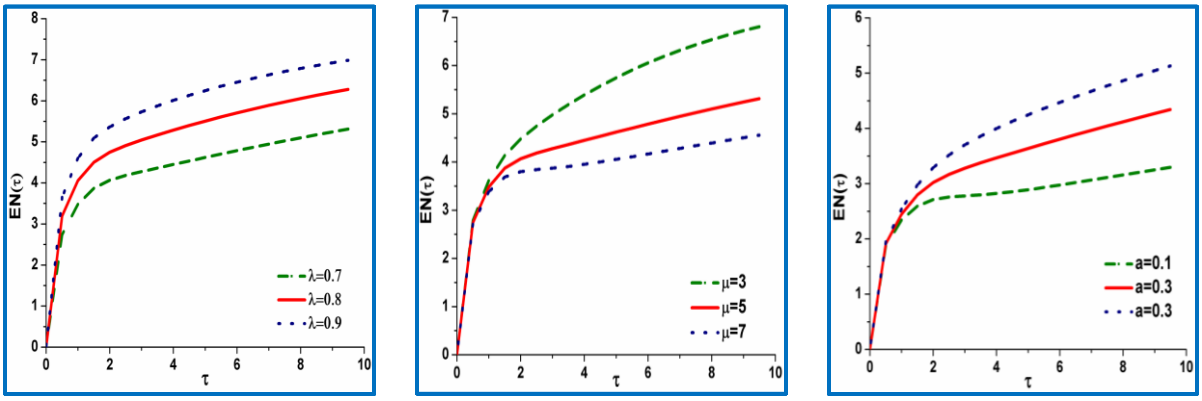


Fig. 5.2: $EN(\tau)$ **vs** τ **with variation in (i) λ (ii) μ and (iii) a**

(i) (ii) (iii)

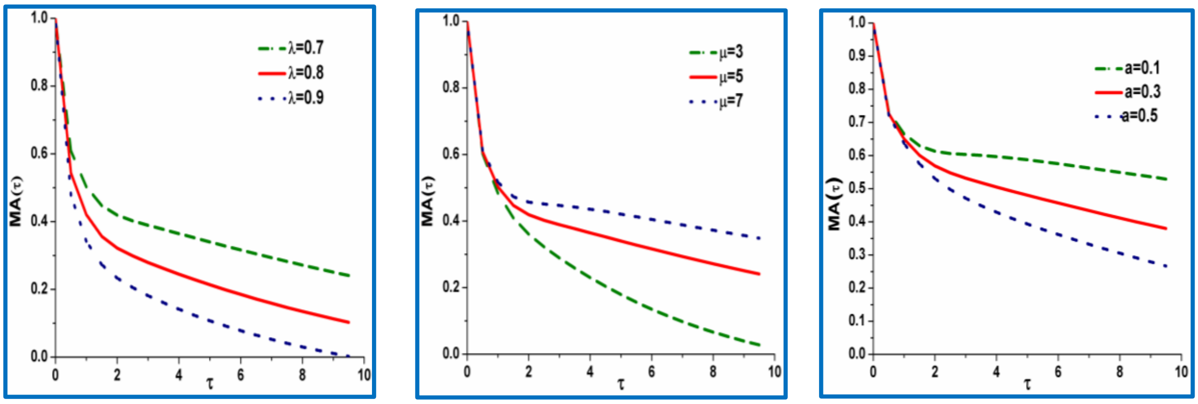


Fig. 5.3: $MA(\tau)$ **vs** τ **with variation in (i) λ (ii) μ and (iii) a**

(i) (ii) (iii)

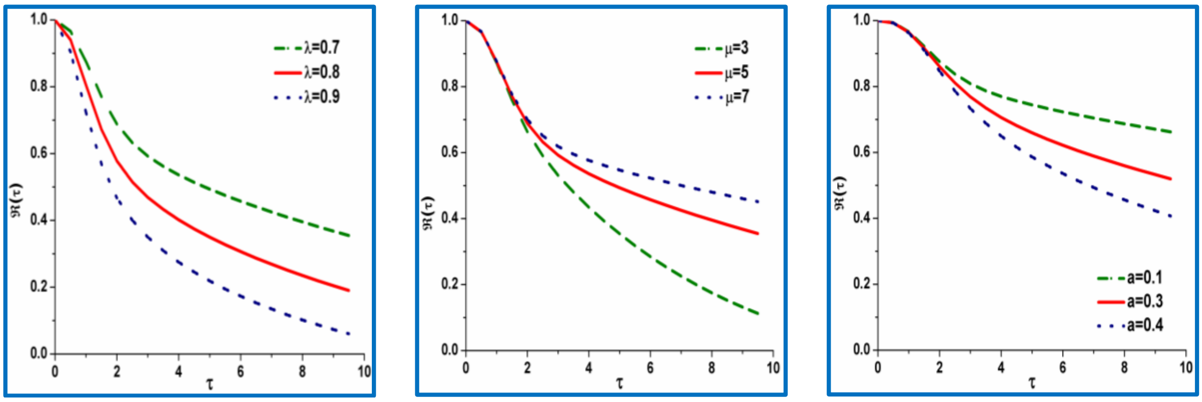


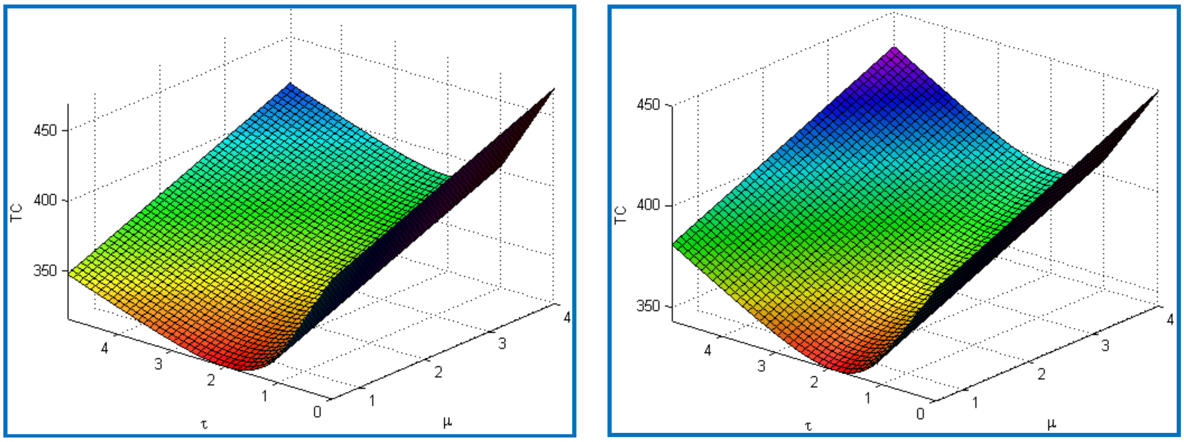
Fig. 5.4: $R_Y(\tau)$ **vs** τ **with variation in (i) λ (ii) μ and (iii) a**

Table 5.1 shows the different sets of cost elements which are used to determine the total cost of FTMS consisting of six operating machines and two different types of standby machines. Tables 5.2-5.3 demonstrate the effect of service and vacation rates on the expected number of failed machines, machine availability, reliability, system cost and long run probabilities of FTMS. From Tables 5.2-5.3, it is clearly observed that the expected number of failed machines $EN(\tau)$ and probability of working broken down state $P_d(\tau)$ show the increasing trend as time $'\tau'$ grows. The decreasing trend is observed in the values of $EN(\tau)$ and $P_d(\tau)$ for the increasing values of μ_v and θ. Moreover, the system reliability $R_Y(\tau)$, machine availability $MA(\tau)$ and the probability of server being on working vacation $P_{WV}(\tau)$ decrease (increase) as we increase the values of $'\tau'$ (μ_v and θ).

From Figures 5.2 (i-iii), we can clearly notice that the expected number of failed machines $EN(\tau)$ shows the increasing trend with respect to time $'\tau'$, and failure rate $\lambda(a)$ of operating machine (server). The accumulation of failed machines in case of high failure rate of both operating and standby machines is quite obvious. For increasing values of service rate (μ), $EN(\tau)$ decreases as expected and tally with many real life scenarios. The decreasing trend of machine availability $MA(\tau)$ has been observed from Figures 5.3 (i-iii) as $'\tau'$ attains the higher values. It can be realized in many machining systems, the machine availability $MA(\tau)$ decreases as the value of failure rate of operating machine (λ) as well as failure rate of server (a) grow up. On the other hand, $MA(\tau)$ shows increasing trend for the increasing value of service rate (μ) which can be treated as control parameter to enhance the system availability.

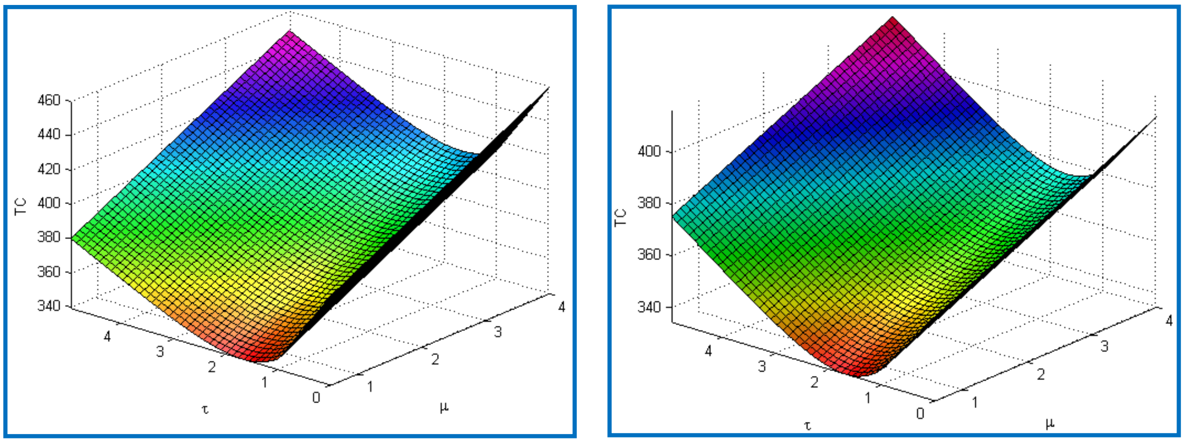
The trend of reliability $R_Y(\tau)$ for different system parameters are depicted in Figures 4(i-iii). From these Figures, we found that $R_Y(\tau)$ decreases as time $'\tau'$ passes. Also, as λ and a grow up, the system reliability $R_Y(\tau)$ seems to decrease. The system reliability $R_Y(\tau)$ is higher for the higher value of service rate (μ) which is quite obvious. Thus, the service rate can play a critical role to achieve the pre-specified mission reliability in many complex systems.

The combine effect of μ and τ for the different sets of cost element provided in Table 1 are given in Figures 5.5 (i-ii)-6.6(i-ii). The total expected cost first decreases and then increases sharply for increasing values of time along with the service rate. The optimal value of service rate can be easily determined by using numerical optimization technique viz. quasi-Newton approach. Finally, based on numerical experiment performed, we conclude our findings as follows.



(i) (ii)

Fig. 5.5: Total system cost for various values of service rate μ with respect to time τ for (i) Cost set I (ii) Cost set II



(i) (ii)

Fig. 5.6: Total system cost for various values of service rate μ with respect to time τ for (i) Cost set III (ii) Cost set IV

By choosing the higher values of service rate, the system reliability of FTMS can be increased significantly, but simultaneously cost also increases. To resolve this situation, the optimal service rate can be evaluated at minimum cost. Also, the cost of considered FTMS can be further reduced by taking care of some other sensitive parameters such as repair rate of server and service rate during working vacation.

Chapter 6

MRP with Unreliable Server and Threshold Recovery

6.1 Introduction

The threshold policy for starting of service can be used to overcome the waste of valuable resources, time and money of an industry or company operating in machining environment. For any single machine repair system, N-policy can be implemented for the economic utilization of the server. Threshold N-policy states that the server is turned on to render repair only when the workload of repair job of failed machines reaches to pre-defined threshold level *N*. To explore the performance of a Markovian machining system operating under N-policy, reneging and the provision of warm standbys, Jain *et al*. (2004) proposed a finite source queueing model. The matrix method is employed by Jain and Upadhyaya (2009) to evaluate the steady state probabilities and other system indices of a degraded system by including the realistic concepts of threshold N-policy, multiple vacations and multiple type spare support. Further, Yang and Chang (2014) examined Markov machine repair model with threshold recovery policy to facilitate the performance analysis by taking some realistic factors into account. They also developed the queueing model for the cost analysis of multi-component machining system by particle swarm optimization. Recently, a time shared Markov study of machine repair problem having some realistic features such as threshold policy, additional repairman and mixed spares have been carried out by Jain, Shekhar and Shukla (2016).

The provision of more servers is always helpful in reducing the work load and to facilitate the faster service. However, to keep the server active in case of less work load, is costly affair. In this chapter, we study the transient analysis of machine repair problem having mixed warm standby support, and two heterogeneous unreliable servers (Nobel and Tijms 2000). The first (second) server is activated only when workload of N_1 (N_2) failed machines is accumulated in the system. As soon as the server becomes idle, he goes for vacation. The organization of the chapter is structured as follows. The system description to formulate the mathematical model is presented in Section 6.2. In Section 6.3, Chapman-Kolmogorov equations at transient state are constructed. To predict the performance of the developed model, some metrics have been constructed. Furthermore, total expected cost function is also constructed to evaluate the optimal service rate in Section 6.4. The architecture of ANFIS model is also briefly described. In Section 6.5, numerical illustration and cost analysis have been presented.

6.2 Model Description

Notations

M: The number of operating machines.

S: The number of warm standbys machines i.e. $S=\sum_{j=1}^{k} S_j \ (j=1,2,3,...,k)$.

K: The sum of both operating and warm standby machines i.e. $K=M+S$.

λ: The failure rate of operating machines.

a_j: The failure rate of $j^{th}(j=1,2,3,...,k)$ types of warm standby machines.

υ_i: Mean vacation rate by which the $i^{th}(i=1,2)$ server returns from the vacation.

λ_d: Degraded failure rate of machines.

μ_i: Mean service rate of $i^{th}(i=1,2)$ server

α_i: The life time of $i^{th}(i=1,2)$ server.

β_i: Repair rate of $i^{th}(i=1,\ 2)$ server.

To study the machine repair problem (MRP), we develop Markov machining system with vacation. For the maintenance purpose, there is provision of two unreliable servers and mixed type warm standbys. Markovian model is formulated considering the following assumptions:

- The system consists of *M* operating and *k*-types of warm standbys machines having the failure characteristic. At least *M* operating machines are required for the normal operation of the system, however the system can operate in short mode with at least *l* (< *M*) operating machines. The operating machine may fail in Poisson pattern with failure rate λ. The j^{th} type of standbys machines fail according to Poisson process with rates a_j (j=1, 2,3,…,k).
- The repair facility consists of two heterogeneous repairmen. The first (second) server becomes activate after taking exponentially distributed set up time $\nu_i^{-1}(i=1,2)$ to render repair of failed machines when the workload of $N_1(N_2)$ failed machines have accumulated in the system.
- The repair job of the failed machines is done by the i^{th} (i=1, 2) repairman according to exponential distribution with repair rate μ_i. The repairman follows first in first out (FIFO) discipline to render the repair to the failed machines and can repair only one failed machine at a time.

- The switchover time from standby state to operating or from repair to standby state is negligible and assumed to be perfect.
- While rendering the service to the failed machine, the i^{th} (i=1, 2) server may fail following Poisson distribution with rate α_i.
- When the server fails during the busy period, the i^{th} (i=1, 2) repair of the broken down server is done immediately by the repairman according to exponential distribution with rate β_i (i=1, 2).

The following indicator function $\xi(\tau)$ is used to define the server status at time epoch $'\tau'$:

$$\xi(\tau)=\begin{cases} 0, \text{ when both the servers are on vacation .} \\ 1, \text{ when the server 1 is in busy state and server 2 is on vacation.} \\ 2, \text{when the server 1 is brokendown and server 2 is on vacation.} \\ 3, \text{when both the servers are busy.} \\ 4, \text{when the server 1 is brokendown and server 2 is busy.} \\ 5, \text{when the server 2 is brokendown and server 1 is busy.} \\ 6, \text{when both the servers are brokendown.} \end{cases}$$

The transient state probabilities of the system states are defined as follows:

$P_{0,m}(\tau)$: The probability that at time τ there are m ($0\leq m \leq K$) failed machines in the system and both the servers are unavailable due to vacation.

$P_{i,m}(\tau)$: The probability that at time τ there are m ($1\leq m \leq K$) failed machines in the system and the server is in state $\xi(\tau)=i;\ 1\leq i\leq 6.$

6.3 Governing Equations of the Model

To develop Markov model for the transient behavior of machining system described in previous section, the state dependent transition rates for all the system states are to be specified. By using these rates, the governing Chapman-Kolmogorov differential difference equations can be easily constructed to formulate the model using birth-death process. For notational convenience, we shall use $\mu^{(2)}=\mu_1+\mu_2,\ \ \alpha=\alpha_1+\alpha_2$ and $S^{(k)}=\sum_{j=1}^{k} S_j$ .The failure rate of operating machines λ_m is given as

$$\lambda_m=\begin{cases} M\lambda+a_n, & 0\leq m<S_1 \\ M\lambda+a_n, & S^{(j-1)}\leq m<S^{(j)},\ 2\leq j\leq k \\ (K-n)\lambda_d, & S^{(k)}\leq m<M+S^{(k)}=K \\ 0, & \text{otherwise} \end{cases}$$

where

$$a_m = \begin{cases} (S_1 - m)\delta_1 + \sum_{i=2}^{k} S_i\delta_i, & 0 \le m < S_1 \\ (S^{(j)} - m)\delta_j + \sum_{i=j+1}^{k} S_i\delta_i, & S^{(j-1)} \le m < S^{(j)},\ 2 \le j \le k \end{cases}$$

The repair rate $\mu_{i,m}$ depends upon the server status $'\xi(\tau) = i'$, defined in previous section. Now we define

$$\mu_{i,m} = \begin{cases} \mu^{(2)}; \ 2 \le m \le K,\ \xi(\tau) = 3 \\ \mu_1; \ 1 \le m \le K,\ \xi(\tau) = 1,5 \\ \mu_2; \ 1 \le m \le K,\ \xi(\tau) = 4 \end{cases}$$

The transient equations to frame the Markov model are constructed by following the flow conversation law. Now we frame the equations using the appropriate transition rates for different level i $(0 \le i \le 6)$ as follows:

(i) For $\xi(\tau) = 0$***: When both servers are on vacation.***

$$\frac{dP_{0,0}(\tau)}{d\tau} = -\lambda_0 P_{0,0}(\tau) + \mu_1 P_{1,1}(\tau) \tag{6.1}$$

$$\frac{dP_{0,m}(\tau)}{d\tau} = -\lambda_n P_{0,m}(\tau) + \lambda_{m-1} P_{0,m-1}(\tau);\ 1 \le m \le N_1 - 1 \tag{6.2}$$

$$\frac{dP_{0,m}(\tau)}{d\tau} = -(\lambda_m + \upsilon_1) P_{0,m}(\tau) + \lambda_{m-1} P_{0,m-1}(\tau); N_1 \le m \le K - 1 \tag{6.3}$$

$$\frac{dP_{0,K}(\tau)}{dt} = -\upsilon_1 P_{0,K}(\tau) + \lambda_{K-1} P_{0,K-1}(\tau) \tag{6.4}$$

(ii) For $\xi(\tau) = 1$***: Busy state for server 1 while the server 2 is on vacation.***

$$\frac{dP_{1,1}(\tau)}{d\tau} = -(\lambda_1 + \mu_1 + \alpha_1) P_{1,1}(\tau) + \mu_1 P_{1,2}(\tau) + \beta_1 P_{2,1}(\tau) + \mu_2 P_{3,2}(\tau) \tag{6.5}$$

$$\begin{aligned} \frac{dP_{1,m}(\tau)}{d\tau} &= -(\lambda_m + \mu_1 + \alpha_1) P_{1,m}(\tau) + \lambda_{m-1} P_{1,m-1}(\tau) \\ &\quad + \mu_1 P_{1,m+1}(\tau) + \beta_1 P_{2,m}(\tau); 2 \le m \le N_1 - 1 \end{aligned} \tag{6.6}$$

$$\begin{aligned} \frac{dP_{1,m}(\tau)}{d\tau} &= -(\lambda_m + \mu_1 + \alpha_1) P_{1,m}(\tau) + \lambda_{m-1} P_{1,m-1}(\tau) + \mu_1 P_{1,m+1}(\tau) \\ &\quad + \beta_1 P_{2,m}(\tau) + \upsilon_1 P_{0,m}(\tau); N_1 \le m \le N_2 - 1 \end{aligned} \tag{6.7}$$

$$\begin{aligned} \frac{dP_{1,m}(\tau)}{d\tau} &= -(\lambda_m + \mu_1 + \alpha_1 + \upsilon_2) P_{1,m}(\tau) + \lambda_{m-1} P_{1,m-1}(\tau) + \mu_1 P_{1,m-3}(\tau) \\ &\quad + \beta_1 P_{2,m}(\tau) + \upsilon_1 P_{0,m}(\tau);\ N_2 \le m \le K - 1 \end{aligned} \tag{6.8}$$

$$\frac{dP_{1,K}(\tau)}{d\tau}=-(\mu_1+\alpha_1+\upsilon_2)P_{1,K}(\tau)+\lambda_{K-1}P_{1,K-1}(\tau)+\beta_1P_{2,K}(\tau)+\upsilon_1P_{0,K}(\tau) \tag{6.9}$$

(iii) For* $\xi(\tau)=2$*: Broken down state for server 1while the server 2 is on vacation.

$$\frac{dP_{2,1}(\tau)}{d\tau}=-(\lambda_1+\beta_1)P_{2,1}(\tau)+\alpha_1P_{2,1}(\tau) \tag{6.10}$$

$$\frac{dP_{2,m}(\tau)}{d\tau}=-(\lambda_m+\beta_1)P_{2,m}(\tau)+\lambda_{m-1}P_{2,m-1}(\tau)+\alpha_1P_{1,m}(\tau);2\le m\le K-1 \tag{6.11}$$

$$\frac{dP_{2,K}(\tau)}{d\tau}=-\beta_1P_{2,K}(\tau)+\lambda_{K-1}P_{2,K-1}(\tau)+\alpha_1P_{1,K}(\tau) \tag{6.12}$$

(iv) For* $\xi(\tau)=3$*: When both servers are busy.

$$\frac{dP_{3,2}(\tau)}{d\tau}=-(\lambda_2+\alpha+\mu_2)P_{3,2}(\tau)+\mu^{(2)}P_{3,3}(\tau)+\beta_1P_{4,2}(\tau)+\beta_2P_{5,2}(\tau) \tag{6.13}$$

$$\begin{aligned}\frac{dP_{3,m}(\tau)}{d\tau}&=-(\lambda_m+\alpha+\mu^{(2)})P_{i,m}(\tau)+\lambda_{m-1}P_{i,m-1}(\tau)+\mu^{(2)}P_{i,m+1}(\tau)+\beta_1P_{i+1,m}(\tau)\\&\quad+\beta_2P_{i+2,m}(\tau);\ 3\le m\le N_2-1\end{aligned} \tag{6.14}$$

$$\begin{aligned}\frac{dP_{3,m}(\tau)}{d\tau}&=-(\lambda_m+\alpha+\mu^{(2)})P_{i,m}(\tau)+\lambda_{m-1}P_{i,m-1}(\tau)+\mu^{(2)}P_{i,m+1}(\tau)+\beta_1P_{i+1,m}(\tau)\\&\quad+\beta_2P_{i+2,m}(\tau)+\upsilon_2P_{i-2,m}(\tau);\ N_2\le m\le K-1\end{aligned} \tag{6.15}$$

$$\begin{aligned}\frac{dP_{3,K}(\tau)}{d\tau}&=-(\alpha+\mu^{(2)})P_{i,K}(\tau)+\lambda_{K-1}P_{i,K-1}(\tau)+\mu^{(2)}P_{i,K+1}(\tau)+\beta_1P_{i+1,K}(\tau)\\&\quad+\beta_2P_{i+2,K}(\tau)+\upsilon_2P_{i-2,K}(\tau)\end{aligned} \tag{6.16}$$

(v) For* $\xi(\tau)=4$*: When server 1 is broken down and server 2 is busy.

$$\frac{dP_{4,2}(\tau)}{d\tau}=-(\lambda_2+\beta_1+\alpha_2)P_{4,2}(\tau)+\mu_2P_{4,3}(\tau)+\alpha_1P_{3,2}(\tau)+\beta_2P_{6,2}(\tau) \tag{6.17}$$

$$\begin{aligned}\frac{dP_{4,m}(\tau)}{d\tau}&=-(\lambda_m+\beta_1+\mu_2+\alpha_2)P_{i,m}(\tau)+\lambda_{m-1}P_{i,m-1}(\tau)+\mu_2P_{i,m+1}(\tau)+\alpha_1P_{i-1,m}(\tau)\\&\quad+\beta_2P_{i+2,m}(\tau);\ N_1\le m\le K-1\end{aligned} \tag{6.18}$$

$$\frac{dP_{4,K}(\tau)}{d\tau}=-(\beta_1+\mu_2+\alpha_2)P_{4,K}(\tau)+\lambda_{K-1}P_{4,K-1}(\tau)+\alpha_1P_{3,K}(\tau)+\beta_2P_{6,K}(\tau) \tag{6.19}$$

(vi) For* $\xi(\tau)=5$*: When server 2 is broken down and server 1 is busy.

$$\frac{dP_{5,2}(\tau)}{d\tau}=-(\lambda_2+\beta_2+\alpha_1)P_{5,2}(\tau)+\mu_1P_{5,3}(\tau)+\alpha_2P_{3,2}(\tau)+\beta_1P_{6,2}(\tau) \tag{6.20}$$

$$\begin{aligned}\frac{dP_{5,m}(\tau)}{d\tau}&=-(\lambda_m+\beta_2+\mu_1+\alpha_1)P_{5,m}(\tau)+\lambda_{m-1}P_{5,m-1}(\tau)+\mu_1P_{5,m+1}(\tau)+\alpha_2P_{3,m}(\tau)\\&\quad+\beta_1P_{6,m}(\tau);N_1\le m\le K-1\end{aligned} \tag{6.21}$$

$$\frac{dP_{5,K}(\tau)}{d\tau} = -(\beta_2 + \mu_1 + \alpha_1)P_{5,K}(\tau) + \lambda_{K-1}P_{5,K-1}(\tau) + \alpha_2 P_{3,K}(\tau) + \beta_1 P_{6,K}(\tau) \quad (6.22)$$

(vii) For $\xi(\tau) = 6$***: When both servers are broken down.***

$$\frac{dP_{6,2}(\tau)}{d\tau} = -(\lambda_2 + \beta_1 + \beta_2)P_{6,2}(\tau) + \alpha_2 P_{4,2}(\tau) + \alpha_1 P_{5,2}(\tau) \quad (6.23)$$

$$\begin{aligned}\frac{dP_{6,m}(\tau)}{d\tau} &= -(\lambda_m + \beta_1 + \beta_2)P_{6,m}(\tau) + \lambda_{m-1}P_{6,m-1}(\tau) + \alpha_2 P_{4,m}(\tau) \\ &\quad + \alpha_1 P_{5,m}(\tau);\ N_1 \le m \le K-1\end{aligned} \quad (6.24)$$

$$\frac{dP_{6,K}(\tau)}{d\tau} = -(\beta_1 + \beta_2)P_{6,K}(\tau) + \lambda_{K-1}P_{6,K-1}(\tau) + \alpha_2 P_{4,K}(\tau) + \alpha_1 P_{5,K}(\tau) \quad (6.25)$$

6.4 Performance Measures

The performance of any real time system can be assessed in terms of metrics which reveal the system's operating behavior in different scenarios.

6.4.1 Queueing Indices

To predict and explore the behavior of the system, we formulate transient performance indices viz. (i) Expected number of broken down machines $EN(\tau)$, (ii) Machine availability $MA(\tau)$, (iii) Carried load $\lambda_{eff}(\tau)$, and (iv) Throughput $TP(\tau)$ at time epoch $'\tau'$ as follows:

(i) $$EN(\tau) = \sum_{m=0}^{K} m(P_{0,n}(\tau)) + \sum_{i=1}^{2}\sum_{m=1}^{K} m(P_{i,m}(\tau)) + \sum_{i=3}^{6}\sum_{m=2}^{K} m(P_{i,m}(\tau)) \quad (6.26)$$

(ii) $$MA(\tau) = 1 - \frac{EN(\tau)}{M+S} \quad (6.27)$$

(iii) $$\lambda_{eff}(\tau) = \sum_{m=0}^{K} \lambda_m P_{0,m}(\tau) + \sum_{i=1}^{2}\sum_{m=1}^{K} \lambda_m P_{i,m}(\tau) + \sum_{i=3}^{6}\sum_{m=2}^{K} \lambda_m P_{i,m}(\tau) \quad (6.28)$$

(iv) $$TP(\tau) = \sum_{n=1}^{K} \mu_1 P_{1,m}(\tau) + \sum_{m=2}^{K} \mu^{(2)} P_{3,m}(\tau) + \sum_{m=2}^{K} \mu_2 P_{4,m}(\tau) + \sum_{m=2}^{K} \mu_1 P_{5,m}(\tau) \quad (6.29)$$

6.4.2 Long-run system sates probabilities

The long run probabilities of the server being in different states i.e. (i) both servers being on vacation $P_v(\tau)$, (ii) only server 1 being busy $P_{B1}(\tau)$, (iii) only server 2 being busy $P_{B2}(\tau)$, (iv) both servers being busy $P_B(\tau)$, (v) only server 1 is under repair $P_{D1}(\tau)$, (vi) only server 2 is under repair $P_{D2}(\tau)$, and (vii) both servers are broken down $P_D(\tau)$ respectively, at time epoch $'\tau'$ are constructed as follows:

$$P_v(\tau) = \sum_{m=0}^{K} P_{0,m}(\tau) \quad (6.30)$$

(i) $$P_{B1}(\tau) = \sum_{m=1}^{K} P_{1,m}(\tau) + \sum_{m=2}^{K} P_{5,m}(\tau) \tag{6.31}$$

(ii) $$P_{B2}(\tau) = \sum_{m=2}^{K} P_{4,m}(\tau) \tag{6.32}$$

(iii) $$P_{B}(\tau) = \sum_{m=2}^{K} P_{3,m}(\tau) \tag{6.33}$$

(iv) $$P_{D1}(\tau) = \sum_{m=1}^{K} P_{2,m}(\tau) + \sum_{m=2}^{K} P_{4,m}(\tau) \tag{6.34}$$

(v) $$P_{D2}(\tau) = \sum_{m=2}^{K} P_{5,m}(\tau) \tag{6.35}$$

(vi) $$P_{D}(\tau) = \sum_{m=2}^{K} P_{6,m}(\tau) \tag{6.36}$$

6.4.3 System cost

To determine the cost incurred for the system operation, we formulate a cost function involving some cost elements associated to different states of the machining system. The various cost elements per unit time associated with different states of the system, are defined as follows:

C_H : Holding cost of one failed unit in the system.

C_V : Cost spent on the system when both servers are on vacation.

C_{B1} : Cost spent on the system when server 1 is busy.

C_{B2} : Cost spent on the system when server 2 is busy.

C_B : Cost spent on the system when both servers are busy.

C_{D1} : Cost spent on the system while only server 1 is under repair.

C_{D2} : Cost spent on the system while only server 2 is under repair.

C_D : Cost spent on the system when both the servers are under repair.

Now we frame the cost function TC(t) which involves the total cost per unit time by considering the above cost elements and respective performance measures as follows:

$$\begin{aligned} TC(\tau) = {} & C_H\, EN(\tau) + C_V\, P_V(\tau) + C_{B1}\, P_{B1}(\tau) + C_{B2}\, P_{B2}(\tau) \\ & + C_B\, P_B(\tau) + C_{D1}\, P_{D1}(\tau) + C_{D2}\, P_{D2}(\tau) + C_D\, P_D(\tau) \end{aligned} \tag{6.37}$$

6.4.4 Neuro-fuzzy based ANFIS model

Now we outline a brief concept of ANFIS approach which is based on a neural network underlying the fuzzified parameters.

The fuzzy rules employed in ANFIS can be formulated as

$$\text{IF } (e_1 \text{ is } E_1) \text{ AND } (e_2 \text{ is } E_2) \ldots \text{AND } (e_n \text{ is } E_n) \text{THEN } G = F(e_1, e_1, ..., e_n). \quad (6.38)$$

Here F is a linear combination of the input variables $(e_1, e_2, ..., e_n)$, and $\mathrm{E_i}$'s are the respective fuzzy sets. Now output is obtained by using weighted average for the defuzzification method, given by

$$F(e_1, e_2, ..., e_n) = h_0 + h_1 e_1 + h_2 e_2 + ... + h_n e_n \quad (6.39)$$

where h_i represents the weights corresponding to input parameter e_i $(i = 0, 1, 2, ..., n)$.

For our FTS model, an adaptive neuro-fuzzy inference system is constructed by considering the input parameters λ and ν to produce output $EN(\tau)$.

6.5 Numerical Simulation

The numerical results are presented to explore the sensitivity of the system descriptors for the various performance measures and to facilitate the cost analysis. The numerical computation has been done by using Runge-Kutta method to provide transient solution. For numerical simulation purpose, Runge- Kutta 4th order algorithm is implemented by using the ode 45 function of MATLAB software. To characterize the system behavior for different system descriptors, the numerical results are presented in Tables 6.1-6.2 and Figures 6.2-6.4.

In Tables 6.1 and 6.2, we display the trends of the mean number of failed machines $EN(\tau)$, availability of machines $MA(\tau)$, throughput $TP(\tau)$ and system state long run probabilities by varying different input parameters. The default parameters are chosen as:

$$M = 5, l = 2, R = 2, k = 2, S = 2, \lambda = 0.5, a_1 = 0.02, a_2 = 0.04,$$

$$\lambda_d = 0.5,\ \mu_1 = 2, \mu_2 = 3, \alpha_1 = 0.1, \alpha_2 = 0.3, \beta_1 = 10, \beta_2 = 8.$$

As we expect, it is noted in Tables 6.1-6.2 that as time increases, both $EN(\tau)$ and $TP(\tau)$ increase whereas $MA(\tau)$ decreases.

To compute the ANFIS results, neuro-fuzzy tool in Matlab is used by considering Gaussian membership function for the input parameters (λ and ν). For fuzzification of λ and ν, we opt the five members which are shown in Figure 6.1. The members taken for each λ and ν are (i) very low (ii) low (iii) average (iv) high (v) very high.

To plot the results evaluated by R-K method, the continuous lines are used, whereas the numerical results obtained by using ANFIS are depicted by tick marks in Figures 6.2-6.6. From Figures 6.2 (i-ii), it is noticed that as time grows up, the number of failed machines $EN(\tau)$ increases; this trend matches with the realistic situation also. From Figures 6.2 (i), we see that as failure rate (λ) of machine increases, the number of failed machines $EN(\tau)$ also becomes higher. It is clear from Figure 6.2 (ii) that the average number of failed machine lowers down as the value of vacation rate ν increases; the trend for increasing the values of time τ is also observed in the figures. The Figures 6.3 (i-ii) show the decreasing trend for the system availability $MA(\tau)$ with respect to time τ. The system availability $MA(\tau)$ significantly decreases (increases) as value of λ (ν) increases. The throughput $TP(\tau)$ plotted in Figures 6.4 (i-ii), is significantly increases as λ and ν increase. In these figures, it is quite clear that the effects of parameters λ and ν on throughput $TP(\tau)$ are much prevalent as time τ grows; however after a certain time, the impact seems to be stabilized.

In Figures 6.2 (i-ii), Figures 6.3 (i-ii) and Figures 6.4 (i-ii) numerical results for $EN(\tau)$, $MA(\tau)$ and $TP(\tau)$ respectively are plotted by using both Runge-Kutta method (curve) and ANFIS (ticked marks) approach. From these figures, we can easily see that the ANFIS results are at par with the results obtained by Runge-Kutta method. Also, we conclude that the neuro-fuzzy controller can be developed for the quantitative assessment of metrics of unreliable machining system to track the system performance.

The total expected cost incurred on the system TC(μ) can be minimized with respect to the decision parameter repair rate (μ) of the failed machines using heuristic search approach. To search the optimal value of repair rate 'μ*', we choose three sets of cost elements (in $) as given in Table 6.3. To make the study more useful from the cost-benefit view point, the total cost function is plotted in Figures 6.5 (i-iii) for three cost sets I, II and III respectively and varying values of μ and τ. It is noticed that the TC(μ*) is a convex function with respect to μ and τ both which can be seen in Figures 6.5 (i-iii). The results obtained are quite interesting and can be applied to any real-time machining systems for upgrading the system by suitable choice of service/repair rate.

The minimum expected cost of the system is obtained as TC(μ*) = $190.59 at time τ =1 and the corresponding optimal repair rate is μ*=1.54485 for cost set I. For cost set II, the minimum expected cost of the system obtained is TC(μ*) = $150.37 and the associated optimal repair rate is μ*=1.485456 at time τ *=1*. The minimum expected cost of the system is TC(μ*) =$128.07 and the corresponding optimal repair rate is achieved μ*=1.24788 at time τ *=1* for the cost set III.

Table 6.1: Variations in system indices by varying time for different values of α

α	τ	$EN(\tau)$	$TP(\tau)$	$MA(\tau)$	$P_B(\tau)$	$P_V(\tau)$	$TC(\tau)$
0.02	2	0.549268	0.001812	0.921533	7.50E-07	0.998181	206.6814
	6	1.502964	0.057002	0.785291	3.36E-04	0.94283	275.096
	10	2.029223	0.163365	0.710111	2.85E-03	0.836967	302.3159
0.04	2	0.705696	0.003618	0.899186	2.91E-06	0.996354	218.9347
	6	1.878615	0.096852	0.731626	1.10E-03	0.902545	299.466
	10	2.400913	0.243451	0.657012	8.03E-03	0.757294	321.1928
0.06	2	0.861606	0.006133	0.876913	8.06E-06	0.993796	231.0421
	6	2.229894	0.141126	0.681444	2.59E-03	0.857623	321.2991
	10	2.72298	0.318345	0.611003	1.65E-02	0.683456	337.2844

Table 6.2: Variations in system indices by varying time for different values of μ

μ	τ	$EN(\tau)$	$TP(\tau)$	$MA(\tau)$	$P_B(\tau)$	$P_V(\tau)$	$TC(\tau)$
1	1	1.647587	0.030931	0.76463	5.01E-05	0.984513	312.5974
	3	3.971559	0.43216	0.432634	0.009167	0.787161	471.2221
	5	4.605148	0.827214	0.342122	0.042498	0.604173	499.1506
5	1	0.868953	0.012663	0.875864	2.00E-06	0.996826	292.0626
	3	2.30164	0.247366	0.671194	4.90E-04	0.938067	398.3246
	5	3.175741	0.609982	0.546323	5.43E-03	0.849098	456.0356
9	1	0.867817	0.017336	0.876026	1.47E-06	0.997103	332.0112
	3	2.26336	0.275525	0.676663	2.70E-04	0.953961	437.5128
	5	2.916208	0.51288	0.583399	1.30E-03	0.914607	484.2185

Table 6.3 Cost elements (in $) associated with various system indices

Cost Set	C_H	C_V	C_{B1}	C_{B2}	C_B	C_{D1}	C_{D2}	C_D	C_m
I	170	70	50	60	70	80	90	130	30
II	120	70	50	60	70	80	90	130	25
III	80	70	50	60	70	80	90	130	20

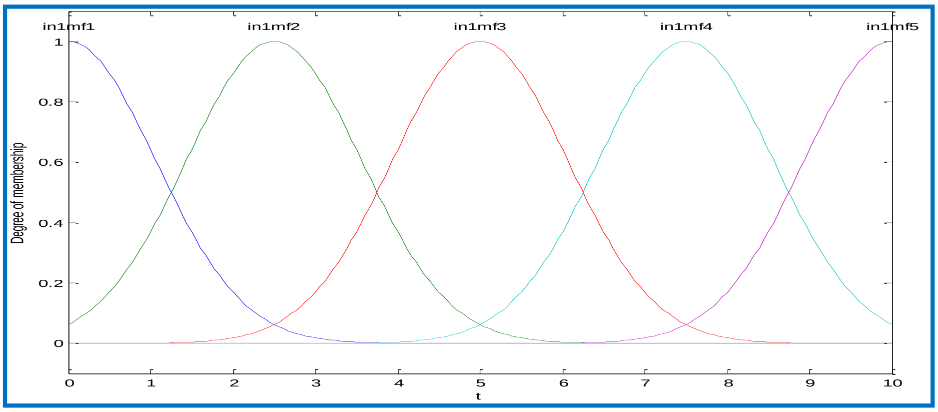
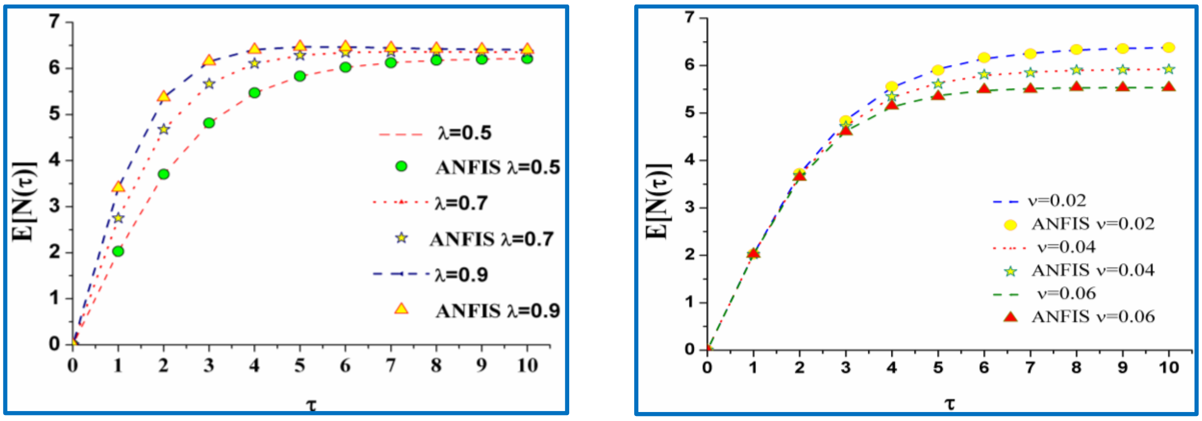
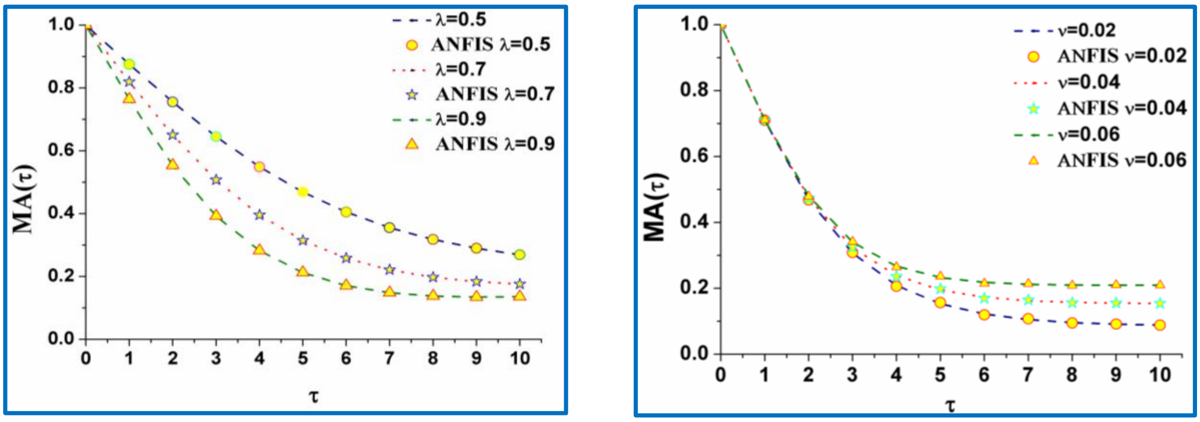


Fig. 6.1: Membership function for input variable λ and ν



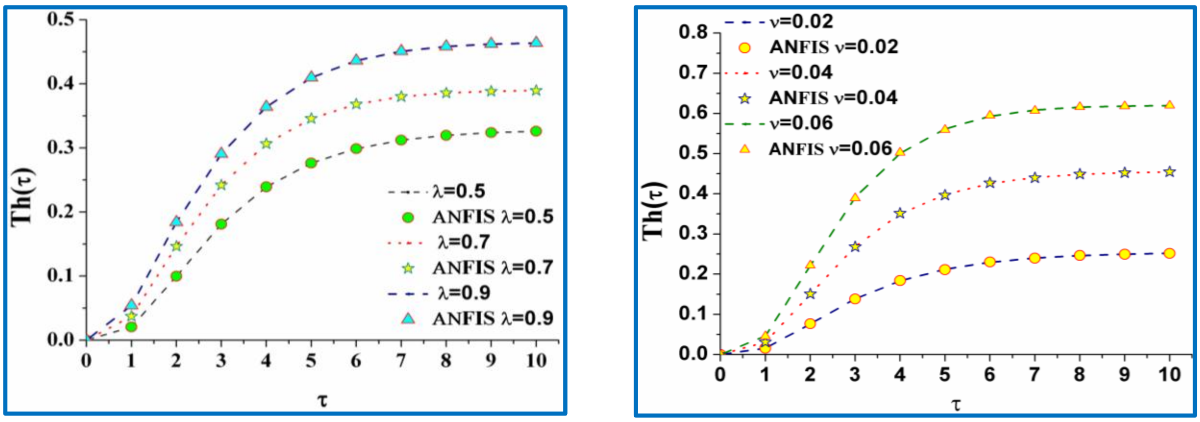
(i) (ii)

Fig. 6.2: $EN(\tau)$ Vs t for different value of (i) λ and (ii) ν



(i) (ii)

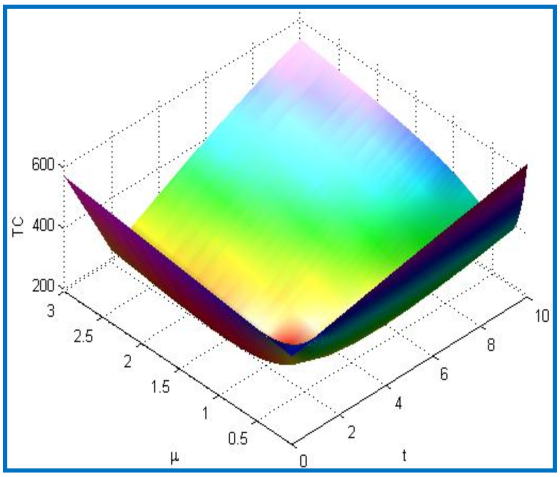
Fig. 6.3: $MA(\tau)$ Vs t for different value of (i) λ and (ii) ν



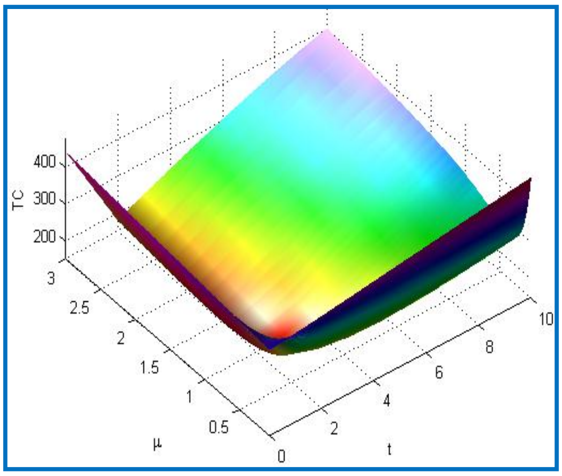
(i) (ii)

Fig. 6.4: TP(τ) Vs t for different value of (i) λ and (ii) ν

(i)



(ii)



(iii)

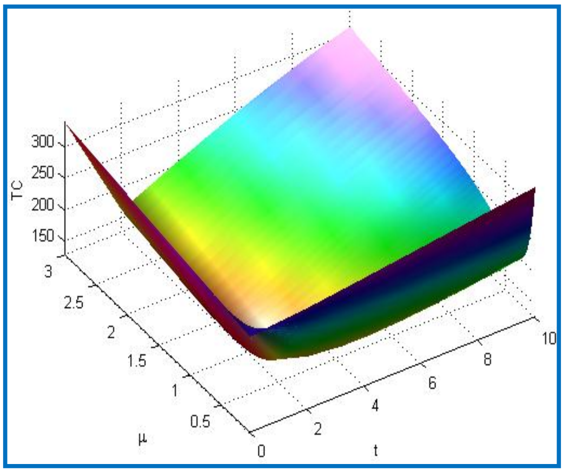


Fig. 6.5: $TC(\tau)$ for varying values of μ (i) cost set I (i) cost set II (iii) cost set III.

Chapter 7

FTS with Imperfect Coverage, Reboot and ServerVacation

7.1 Introduction

For the smooth functioning and to achieve desired availability of the concerned machining system, the concepts of redundancy and maintainability have drawn the attention of practitioners as well as researchers. To reduce the maintainability and operating cost, the provision of server vacation is a key feature which has been included in many queueing models to analyze the congestion problems in different contexts.

The timings of reboot operation may vary from a few seconds to long hours, depending upon the complexity of the machining system. In many industries, an extensive loss of production as well as cost occurs due to the failure of some malicious components if not tackled properly with the help of suitable mechanism. But in some practical situations, the fault handling device may prove inadequate to recover a fault perfectly. These types of situations are called as system with imperfect coverage. In literature, some research works can be found on the reliability analysis of the machine repair problems with imperfect fault coverage. In this context, we cite some recent works which are relevant to present investigation. The concept of the reboot was discussed by Trivedi (2002) for the analysis of some reliability models in his book on 'Probability and Statistics with Reliability, Queueing and Computer Science'. Ke *et al*. (2008) has done the performance analysis of a repairable system by including the features of detection, imperfect coverage and reboot. A statistical model for a standby system involving reboot, switch failure and unreliable repair was presented by Hsu *et al*. (2011). The queueing and reliability indices of the machine repair systems with imperfect coverage and reboot have been studied by Jain *et al.*(2012), Jain (2013), Jain and Gupta (2013), Wang *et al.*(2013), and many others. Further, Ke and Liu (2014) investigated the machine repair system with imperfect coverage incorporating the reboot delay concept by taking the illustration of gamma and exponential time distributions.

In the present investigation, we provide the performance indices of the machining system supported by a repair facility and mixed standby units operating under vacation policy. A few research papers on the machine repair problem with vacation policy in different frameworks have appeared in the past few years as mentioned earlier. But to the best of authors' knowledge, no research article explores the transient study of the machine repair problem combined with vacation, mixed standbys, imperfect coverage and server breakdown. Further, the implementation of ANFIS technique to match soft computing results with the analytic results make our study to deal with complex dynamic behavior of the machining system in efficient computational manner by incorporating many realistic features.The remaining contents of the chapter are structured in different sections. In Section 7.2, we provide notations and assumptions to formulate the Markov model. In Section 7.3, Chapman-Kolmogorov equations are constructed for the transient state to develop Markov model which are further solved numerically with the help of Runge-Kutta method. In Section 7.4, some performance indices have been established explicitly by using the transient probabilities of the system states. In Section 7.5, the cost function is constructed. In Section 7.6, numerical illustration and sensitivity analysis are provided.

7.2 Model Description

In this section, we develop a Markov queueing model by defining the appropriate transition rates of the concerned birth-death process for the performance analysis of fault tolerant system. The assumptions and notations used for developing the model are as follows:

- The system consists of M operating and S_n $(1 \le n \le l)$ of n^{th} type standby units.
- The operating units are prone to failure and have the life time exponentially distributed with mean $1/\lambda$. When an operating unit fails, it is immediately backed up by an available standby unit. When a standby unit moves into an operating state, it has the same failure characteristic as that of an operating on line unit.
- The n^{th} type standby unit may also fail; the life time of n^{th} type unit follows the exponential distribution with mean $1/\alpha_n$, $(0 \le \alpha_n \le \lambda)$.
- When all the spares are used, the system operates in short (i.e. degraded) mode with at least $m(< N)$ operating units and its failure rate becomes λ_d $(> \lambda)$.
- The failed units are repired by the repairman in the same order in which failure occurs, i.e. repaire is performed by following the FIFO service discipline. The repair of failed units is rendered by the server according to exponentially distribution with rate μ.
- The operating unit can be successfully recovered with probability c; the recovery time of operating unit is exponentially distributed with parameter σ.
- The server is allowed to go for vacation if there is no work load of repair job of failed units in the system and returns back from the vacation as soon as any unit fails.

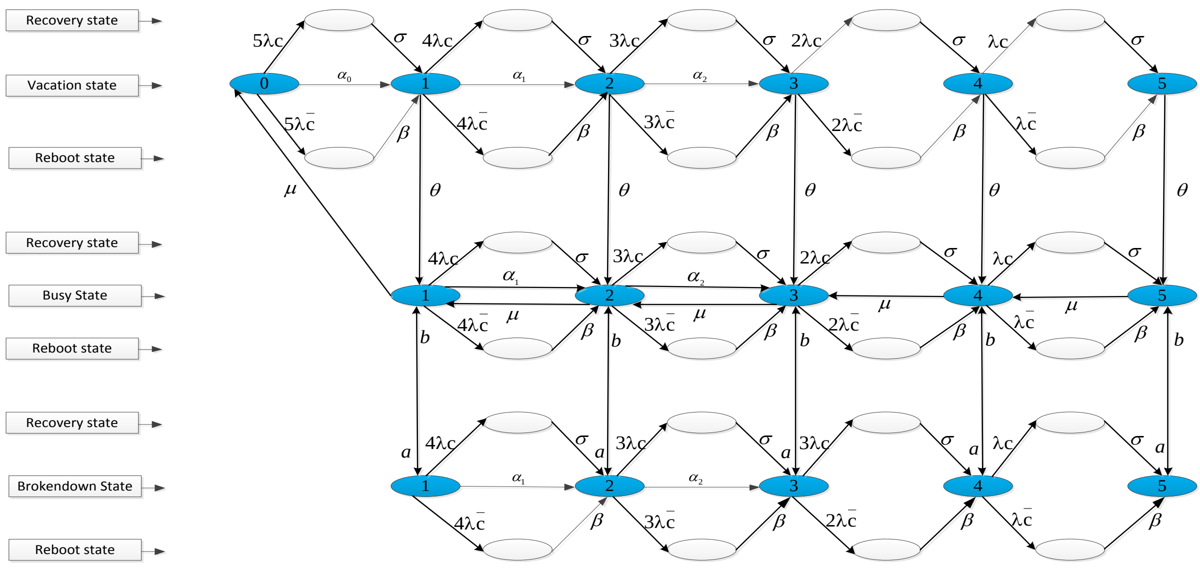


Fig. 7.1: Transition state diagram of M/M/1 FTS

To develop the model, Markov process for the three mutually exclusive system states i.e. (i) operating state, (ii) recovery state and (iii) reboot state is taken into account. The transient probabilities of the failed units at time t for different states are defined for the scenarios when the server is in (i) vacation state, (ii) busy state, and (iii) broken down state. Let $P_{i,j,k}(t)$ denote the probability that there are $i, (1 \le i \le L)$ failed units when the server is in $j^{\text{th}}\,(j = 0,1,2)$ state and the system is operating in $k^{\text{th}}(k = 0,1,2)$ mode. Here indices $j = 0,1,2$ when the server is on vacation, busy and broken down states, respectively. Also indices $k = 0,1,2$ represent the system status in operating state, recovery state and reboot state, respectively.

The failure rate of the operating units which depends upon the number of already failed units, is given by

$$\lambda_i = \begin{cases} M\lambda & i < S^{(l)} \\ (M + S^{(l)} - i)\lambda_d & S^{(l)} \le i < L \end{cases} \tag{7.1}$$

where $S^{(l)} = \sum_{n=1}^{l} S^{(n)}$ and λ_d is the degraded failure rate. The failure rate of the standby units is given by

$$\alpha_i = \begin{cases} \sum_{x=2}^{l} S_x \nu_x + (S_1 - i)\nu_1, & i < S^{(x)} \\ \sum_{y=x+1}^{l} S_y \nu_y + (S^{(x)} - i)\nu_x, & S^{(x-1)} \le i < S^{(x)} \quad x = 2,3,\ldots,l-1 \\ (S^{(l)} - i)\nu_l, & S^{(l-1)} \le i < S^{(l)} \\ 0, & S^{(l)} \le i < L \end{cases} \tag{7.2}$$

7.3 Model Governing Equations

For evaluating the probabilities associated with different server states, Kolmogorov Chapman equations have been constructed using the transition flow rates of birth death process specifying the Markov model.The state transition for in-flow and out-flow rates of specific model when $M=5, S_1=2, S_2=1, l=2, m=1$ is shown in Figure 7.1.

(i) Server vacation state when $j=0, k=0$.

As soon as the server becomes free when there is no job of repairing the failed machines in the system, it reaches to vacation state $(0,0,0)$. In this case, the server is in vacation state and reaches to other state using appropriate transition rates. Now, we frame the Chapman-Kolmogorov equation for state $(0,0,0)$ as follows:

$$\frac{dP_{0,0,0}(t)}{dt}=-(\lambda_0+\alpha_0)P_{0,0,0}(t)+\mu P_{1,1,0}(t) \tag{7.3}$$

The server returns back to busy state with rate θ when a failed machine joins the system; in between, some more mahines may fail so that during vacation period, the system states may be $(i,0,0), i=1,2,...,L$. Using appropriate in-flow and out-flow rates, we formulate the governing equations for $(i,0,0), i=1,2,...,L$ as follows:

$$\begin{aligned}\frac{dP_{i,0,0}(t)}{dt}&=-(\lambda_i+\theta+\alpha_i)P_{i,0,0}(t)+\alpha_{i-1}P_{i,0,0}(t)+\sigma P_{i-1,0,1}(t)\\&\quad+\beta P_{i-1,0,2}(t);\quad 1\le i\le S^{(l)}-1\end{aligned} \tag{7.4}$$

$$\frac{dP_{S^{(l)},0,0}(t)}{dt}=-(\lambda_{S^{(l)}}+\theta)P_{S^{(l)},0,0}(t)+\alpha_{S^{(l)}-1}P_{S^{(l)},0,0}(t)+\sigma P_{S^{(l)}-1,0,1}(t)+\beta P_{S^{(l)}-1,0,2}(t) \tag{7.5}$$

$$\frac{dP_{i,0,0}(t)}{dt}=-(\lambda_i+\theta)P_{i,0,0}(t)+\sigma P_{i-1,0,1}(t)+\beta P_{i-1,0,2}(t);\quad S^{(l)}+1\le i\le L-1 \tag{7.6}$$

$$\frac{dP_{L,0,0}(t)}{dt}=-\theta P_{L,0,0}(t)+\sigma P_{L-1,0,1}(t)+\beta P_{L-1,0,2}(t) \tag{7.7}$$

(ii) Server being in busy state when $j=1, k=0$.

When the server is busy in providing repair of the failed machines, the transient equations are framed by using appropriate transition rates for states $(i,1,0), i=1,2,...,L$ as follows:

$$\frac{dP_{1,1,0}(t)}{dt}=-(\lambda_i+a+\mu+\alpha_i)P_{1,1,0}(t)+\theta P_{1,0,0}(t)+\mu P_{2,1,0}(t)+bP_{1,2,0}(t) \tag{7.8}$$

$$\begin{aligned}\frac{dP_{i,1,0}(t)}{dt}&=-(\lambda_i+a+\mu+\alpha_i)P_{i,1,0}(t)+\theta P_{i,0,0}(t)+\mu P_{i+1,1,0}(t)+bP_{i,2,0}(t)\\&\quad+\alpha_{i-1}P_{i,1,0}(t)+\sigma P_{i-1,1,1}(t)+\beta P_{i-1,1,2}(t);\quad 1<i\le S^{(l)}-1\end{aligned} \tag{7.9}$$

$$\frac{dP_{S^{(l)},1,0}(t)}{dt}=-(\lambda_{S^{(l)}}+a+\mu)P_{S^{(l)},1,0}(t)+\theta P_{S^{(l)},0,0}(t)+\mu P_{S^{(l)}+1,1,0}(t)+bP_{S^{(l)},2,0}(t)$$
$$+\alpha_{S^{(l)}-1}P_{i,1,0}(t)+\sigma P_{S^{(l)}-1,1,1}(t)+\beta P_{S^{(l)}-1,1,2}(t) \tag{7.10}$$

$$\frac{dP_{i,1,0}(t)}{dt}=-(\lambda_i+a+\mu)P_{i,1,0}(t)+\theta P_{i,0,0}(t)+\mu P_{i+1,1,0}(t)+bP_{i,2,0}(t)$$
$$+\sigma P_{i-1,1,1}(t)+\beta P_{i-1,1,2}(t); \quad S^{(l)}+1\le i\le L-1 \tag{7.11}$$

$$\frac{dP_{L,1,0}(t)}{dt}=-(\mu+a)P_{L,1,0}(t)+\theta P_{L,0,0}(t)+bP_{L,2,0}(t)+\sigma P_{L-1,1,1}(t)+\beta P_{L-1,1,2}(t) \tag{7.12}$$

***(iii)* The failed server is under repair state when $j=2, k=0$.**

In this case the server is broken down and the repairman is performing the repair job to restore it. Now for states $(i,2,0), i=1,2,...,L$ the transient equations are framed by law of conservation of flows as follows:

$$\frac{dP_{1,2,0}(t)}{dt}=-(\lambda_1+b+\alpha_1)P_{1,2,0}(t)+aP_{1,1,0}(t) \tag{7.13}$$

$$\frac{dP_{i,2,0}(t)}{dt}=-(\lambda_i+b+\alpha_i)P_{i,2,0}(t)+aP_{i,1,0}(t)+\sigma P_{i-1,2,1}(t)+\alpha_{i-1}P_{i,2,0}(t)$$
$$+\beta P_{i-1,2,2}(t); \quad 1<i\le S^{(l)}-1 \tag{7.14}$$

$$\frac{dP_{S^{(l)},2,0}(t)}{dt}=-(\lambda_{S^{(l)}}+b)P_{S^{(l)},2,0}(t)+aP_{S^{(l)},1,0}(t)+\sigma P_{S^{(l)}-1,2,1}(t)+\beta P_{S^{(l)}-1,2,2}(t)$$
$$+\alpha_{S^{(l)}-1}P_{S^{(l)},2,0}(t) \tag{7.15}$$

$$\frac{dP_{i,2,0}(t)}{dt}=-(\lambda_i+b)P_{i,2,0}(t)+aP_{i,1,0}(t)+\sigma P_{i-1,2,1}(t)+\beta P_{i-1,2,2}(t); \quad S^{(l)}+1\le i\le L-1 \tag{7.16}$$

$$\frac{dP_{L,2,0}(t)}{dt}=-bP_{L,2,0}(t)+aP_{L,1,0}(t)+\sigma P_{L-1,2,1}(t)+\beta P_{L-1,2,2}(t) \tag{7.17}$$

***(iv)* For $k=1, j=0,1,3$ when the system is in recovery state.**

From $(i,j,0)$ state, due to perfect failure detection, the system can go to recovery state $(i,j,1)$, for $i=1,2,...,L-1$; $j=0,1,2$. For the recovery states, the transient equations are framed as:

$$\frac{dP_{i,j,1}(t)}{dt}=-\sigma P_{i,j,1}(t)+\lambda_i cP_{i,j,0}(t); \; j=0,1,2; \; i=1,2,...,L-1. \tag{7.18}$$

***(v)* For $k=2, j=0,1,3$, when the system is in reboot state.**

From $(i,j,0)$ state, due to imperfect failure detection, the system can go to reboot state $(i,j,2)$, for $i=1,2,...,L-1$; $j=0,1,2$. For these states, the transient equations are constructed as:

$$\frac{dP_{i,j,2}(t)}{dt} = -\beta P_{i,j,2}(t) + \lambda_i \bar{c} P_{i,j,0}(t); \qquad j = 0,1,2;\ i = 1,2,...,L-1 \tag{7.19}$$

The Equations (7.3)-(7.19) have been solved numerically by using Runge-Kutta 4^{th} order method, which is a powerful tool to solve the ordinary differential equations of first order. It is a good choice to employ this technique to solve the set of differential equations governing the system state probabilities. It is worth noting that the Runge-Kutta method is quite accurate, stable and easy to implement in comparison to other methods available to solve the differential equations. For the coding purpose, we have chosen this particular method here and MATLAB's 'ode45' function is exploited for the programming purpose.

7.4 Performance Indices

To analyze the transient system behavior, we derive various performance indices using the probabilities which can be evaluated as described in previous section. The expressions for the expected number of failed units in the system, failure frequency of the system, availability of the server and the system state probabilities for the server being in different states and other performance metrics are established as follows:

(i) The average number of failed units in the system at time t is

$$EN(t) = \sum_{i=1}^{L}\sum_{j=0}^{2} iP_{i,j,0}(t) + \sum_{i=1}^{L}\sum_{j=0}^{2} i\left\{P_{i,j,1}(t) + P_{i,j,2}(t)\right\} \tag{7.20}$$

(ii) Failure frequency of the server at time t is

$$f(t) = a\sum_{i=1}^{L} P_{i,2,0}(t) \tag{7.21}$$

(iii) System availability of the at time t is

$$A(t) = 1 - \left(\sum_{i=0}^{L} P_{i,2,0}(t) + \sum_{i=1}^{L-1}\left\{P_{i,2,1}(t) + P_{i,2,2}(t)\right\}\right) \tag{7.22}$$

(iv) The transient probability that the system is in recovery state

$$P_{RC}(t) = \sum_{i=0}^{L-1} P_{i,j,1}(t) + \sum_{i=1}^{L-1} P_{i,j,1}(t) + \sum_{i=1}^{L-1} P_{i,j,1}(t) \tag{7.23}$$

(v) The transient probability that the system is in reboot state

$$P_{R}(t) = \sum_{i=0}^{L-1} P_{i,j,2}(t) + \sum_{i=1}^{L-1} P_{i,j,2}(t) + \sum_{i=1}^{L-1} P_{i,j,2}(t) \tag{7.24}$$

(vi) The transient probability that the server is in broken down state

$$P_{BD}(t) = \sum_{i=1}^{L} P_{i,2,0}(t) + \sum_{i=1}^{L-1} P_{i,2,1}(t) + \sum_{i=1}^{L-1} P_{i,2,2}(t) \tag{7.25}$$

(vii) The transient probability that the server is in busy state

$$P_B(t)=\sum_{i=1}^{L}P_{i,1,0}(t)+\sum_{i=1}^{L-1}P_{i,1,1}(t)+\sum_{i=1}^{L-1}P_{i,1,2}(t) \tag{7.26}$$

(viii) The transient probability that the serverbeing on vacation state

$$P_V(t)=\sum_{i=0}^{L}P_{i,1,0}(t)+\sum_{i=1}^{L-1}P_{i,1,1}(t)+\sum_{i=1}^{L-1}P_{i,1,2}(t) \tag{7.27}$$

7.5 Cost Function

The system designer may be interested to determine the proper combination of the spares and repairmen so as the total cost incurred on the system should be minimum. While making such decisions, it should be kept in mind that the organization should not bear the burden of excessive costs of keeping the spares and repairmen. To determine the optimal number of repairmen and spare machines, we construct the cost function by considering various cost elements involved in different activities. We denote the various cost elements incurred on different activities as follows:

C_V: Cost per unit time of the server when he is on vacation;

C_B: Cost per unit time of the server when he is in busy state;

C_H: Holding cost of one failed unit per unit time in the system;

C_{BD}: Cost of repairing of a broken down server per unit time.

Now we evaluate the total cost function by considering the above cost elements and respective performance measures as follows:

$$\mathrm{C}(\mathrm{t})=\mathrm{C}_V P_V(t)+\mathrm{C}_\mathrm{B} P_B(t)+\mathrm{C}_\mathrm{H} E[N(t)]+\mathrm{C}_{BD} P_{BD}(t) \tag{7.28}$$

7.6 Numerical Illustration

To reveal the practical applicability of the underlying model in real time machining system, we consider an illustration from flexible manufacturing systems where robots are used for the packing purpose. In normal functioning mode, the system requires 5 robots, however, the system can work in degraded mode i.e. in short mode if there are less than 5 but at least one operating robot is in active state. The system has two types of standby robots which acts as back up unit in case of failure of any operating unit and immediately put in place of broken down robot. The failure rate of the operating robot is $\lambda=0.1$ robots per day. To maintain the desired level of availability, three standby robots having failure rate $\alpha_0=0.7, \alpha_1=0.4,$ and $\alpha_2=0.2$ robots per day and 1 robot which cannot fail in active mode, are taken as warm and cold standby units, respectively. Whenever a robot fails, its failure is detected, diagnosed and recovered with probability $c=0.5$ and recovery rate is assumed to be $\sigma=0.8$. If fault is not detected due to imperfect coverage, it is cleared by the reboot or reset operation with a rate $\beta=10$ per day. To evaluate the performance results, coding of the program is done in MATLAB software. The subroutine ode45 is employed for solving the set of equations associated with the probabilities of different system states. For the computation of results, we fixed the various parameters as

$$\text{Cost Set: } C_V=\$10, C_B=\$100, C_H=\$20, C_{BD}=\$200$$

$$M=5,\quad S_1=2,\quad S_2=1,\quad l=3,\quad m=1,\ c=0.5, a=0.02,\ b=1, \mu=2,$$
$$\lambda=0.1,\quad \alpha_0=0.07,\quad \alpha_1=0.04,\quad \alpha_2=0.02,\ \sigma=0.8,\ \beta=10,\ \nu=0.02$$

The sensitivity analysis has been done to explore the effect of varying parameters on different performance indices. The numerical results displayed in form of tables and graphs are quite easy to understand the behavior of system. The numerical results obtained have been displayed in graphs and tables.

The numerical results for various performance indices for varying values of different parameters are presented in Tables 7.1-7.3. The effects of different parameters are examined by displaying the numerical results for the availability and average number of failed units in the system in Figures 7.2 (i-iv) and 7.3 (i-ii). From Figures 7.2 (i-iv), we observe that as time increases, the system availability initially decreases rapidly and then after becomes almost constant. In the case of curves of $EN(t)$ plotted in Figures 7.3 (i-ii), sharp increment is noticed up to $t=20$, then after it becomes asymptotically stable as time passes.

The trends of various performance indices for varying different parameters are as follows:

(i) Effect of failure rate of operating unit and repairman (λ, a).

From Figures 7.2 (i) and 7.2 (ii), it is noted that the availability of the system decreases as λ and a increase. From Table 7.1, it is clear that the average number of failed units increases as λ increases.

(ii) Effect of reboot rate and service rate (β, μ).

Table 7.2 displays that the average number of failed units $EN(t)$ exhibits the increasing trend as reboot rate β increases; on the contrary, the availability $A(t)$ of the system decreases as reboot rate β increases. It is seen in Figure 7.2 (iv) that as μ increases, the availability $A(t)$ of the system increases while $EN(t)$ decreases.

(iii) Effect of repair rate of repairman and coverage factor (b, c).

From Figures 7.2 (iii) and 7.3 (ii), it is noticed that the availability $A(t)$ of the system increases as repair rate (b) increases. The average number of failed machines $EN(t)$ reveals the increasing trend as the coverage factor c increases.

Neuro-fuzzy technique is applied to compute the performance indices of the fault tolerant MRP. The membership function of input parameters λ, a and b are taken as trapezoidal function by taking the very low, low, average, high and very high values as depicted in Figure 7.4. The numerical results by ANFIS approach have been computed by using the neuro-fuzzy tool in Matlab software. To facilitate the comparison of results obtained by Runge-Kutta method with neuro-fuzzy results, we plot the availability by both approaches in Figures 7.5-7.7.

The sensitivity of availability obtained by R-K method is depicted by continuous line for varying parameters λ, a and b. The numerical results obtained by using neuro-fuzzy technique are shown by tick marks. From these figures, we notice that both R-K and ANFIS results are quite close to each other.

Table 7.1: Effect of failure rate of operating unit (λ) on various performance indices

λ	t	$P_{RC}(t)$	$P_R(t)$	$P_{BD}(t)$	EN(t)	f(t)	A(t)	$P_B(t)$	TC(t)
0.1	10	0.13	0.0094	0.00051	2.75	0.00049	0.99949	0.0289	67.6
	30	0.06	0.0045	0.00095	3.96	0.00085	0.99905	0.0486	93.9
	50	0.05	0.0041	0.00100	4.07	0.00088	0.99900	0.0506	96.2
0.3	10	0.14	0.0093	0.00090	4.08	0.00079	0.99910	0.0545	96.7
	30	0.07	0.0057	0.00156	4.41	0.00119	0.99844	0.0836	106.0
	50	0.07	0.0058	0.00156	4.41	0.00119	0.99844	0.0837	106.2
0.5	10	0.12	0.0073	0.00112	4.47	0.00096	0.99888	0.0731	106.1
	30	0.09	0.0074	0.00215	4.43	0.00152	0.99785	0.1238	110.2
	50	0.09	0.0075	0.00218	4.43	0.00153	0.99782	0.1250	110.3

Table 7.2: Effect of reboot rate (β) on various performance indices

β	t	$P_{RC}(t)$	$P_R(t)$	$P_{BD}(t)$	EN(t)	f(t)	A(t)	$P_B(t)$	TC(t)
2	10	0.1255	0.0478	0.00049	2.6	0.00045	0.99951	0.0278	65.5
	30	0.0561	0.0222	0.00097	3.9	0.00083	0.99903	0.0494	93.3
	50	0.0501	0.0200	0.00102	4.0	0.00087	0.99898	0.0517	95.6
6	10	0.1266	0.0157	0.00051	2.7	0.00049	0.99949	0.0287	67.3
	30	0.0567	0.0074	0.00096	4.0	0.00084	0.99904	0.0488	93.8
	50	0.0509	0.0068	0.00100	4.1	0.00088	0.99900	0.0508	96.1
10	10	0.1268	0.0094	0.00051	2.7	0.00049	0.99949	0.0289	67.6
	30	0.0569	0.0045	0.00095	4.0	0.00085	0.99905	0.0486	93.9
	50	0.0510	0.0041	0.00100	4.1	0.00088	0.99900	0.0506	96.2

Table 7.3: Effect of failure rate of standby unit (α) on various performance indices

α	T	$P_{RC}(t)$	$P_R(t)$	$P_{BD}(t)$	EN(t)	f(t)	A(t)	$P_B(t)$	TC(t)
0.03	10	0.127	0.0094	0.00051	2.75	0.00049	0.99949	0.0289	67.64
	30	0.057	0.0045	0.00095	3.96	0.00085	0.99905	0.0486	93.90
	50	0.051	0.0041	0.00100	4.07	0.00088	0.99900	0.0506	96.23
0.06	10	0.120	0.0088	0.00055	2.89	0.00053	0.99945	0.0310	70.79
	30	0.054	0.0043	0.00098	4.02	0.00087	0.99902	0.0498	95.03
	50	0.049	0.0039	0.00102	4.11	0.00090	0.99898	0.0517	97.18
0.09	10	0.114	0.0084	0.00059	3.02	0.00056	0.99941	0.0329	73.41
	30	0.052	0.0041	0.00100	4.06	0.00089	0.99900	0.0510	95.97
	50	0.047	0.0038	0.00104	4.15	0.00092	0.99896	0.0527	97.98

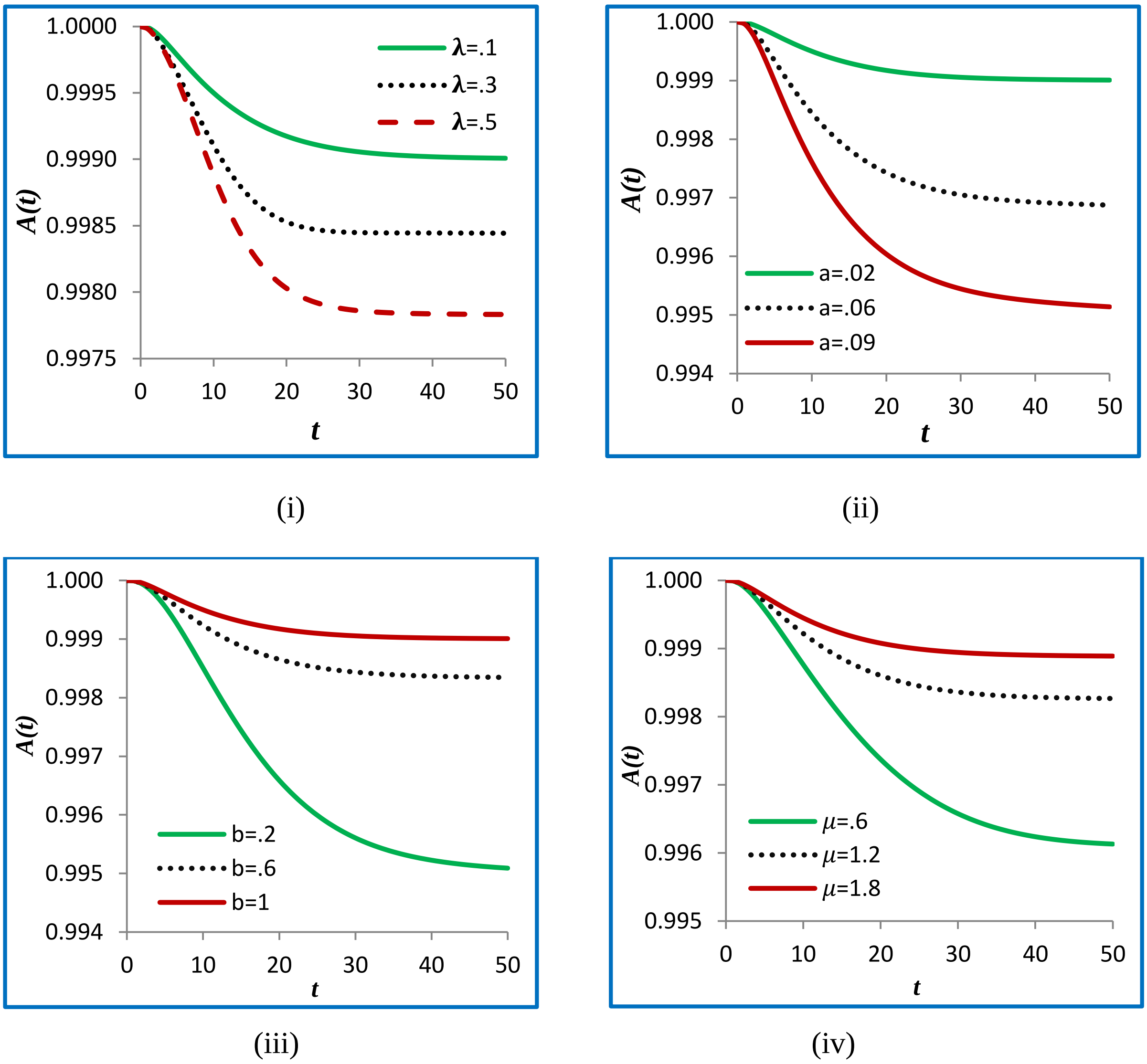
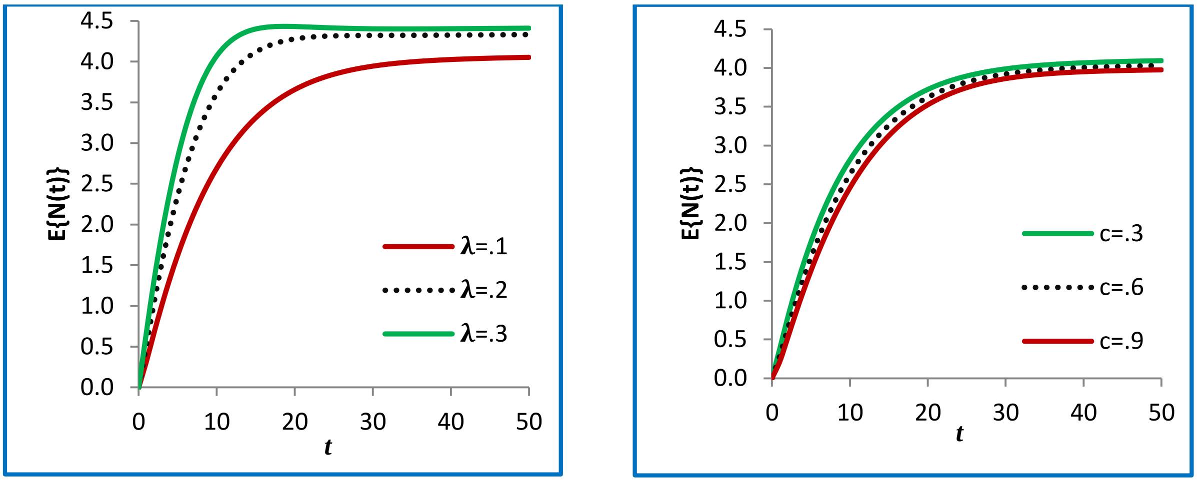


Fig. 7.2: Variation in $A(t)$ for different value of (i) λ (ii) a (iii) b (iv) μ



(i) (ii)

Fig. 7.3: Variation in $EN(t)$ for different value of (i) λ and (ii) c

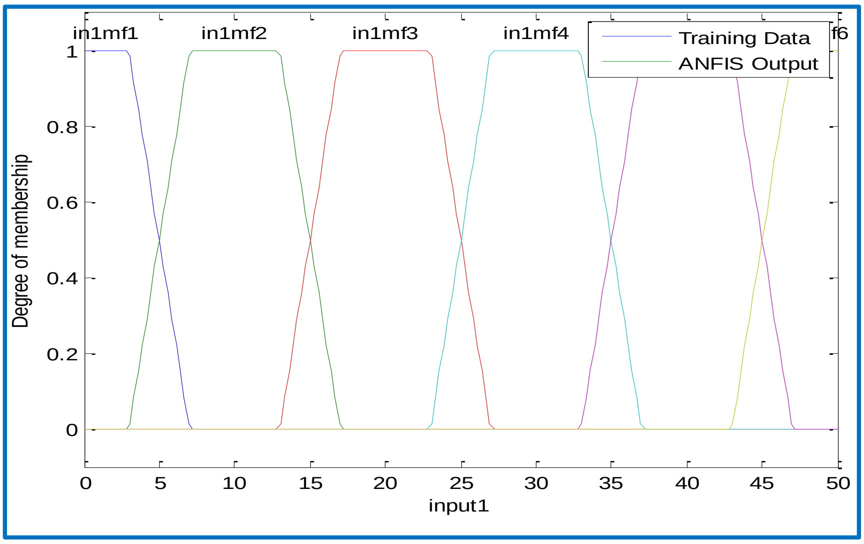


Fig. 7.4: Membership function for λ, a and b

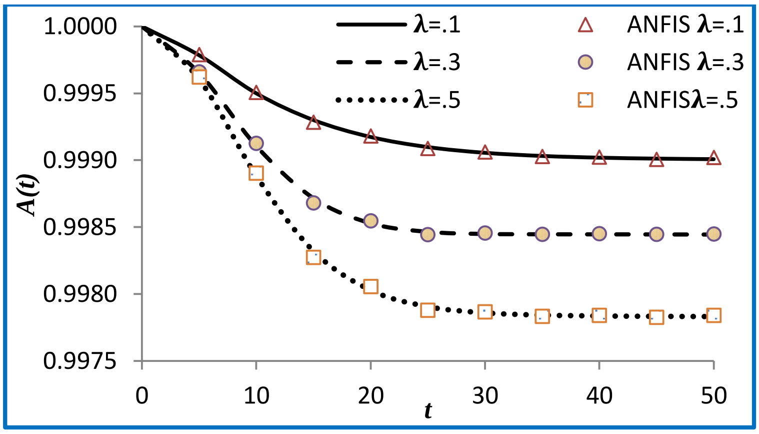


Fig. 7.5: $A(t)$ vs t for different values of λ

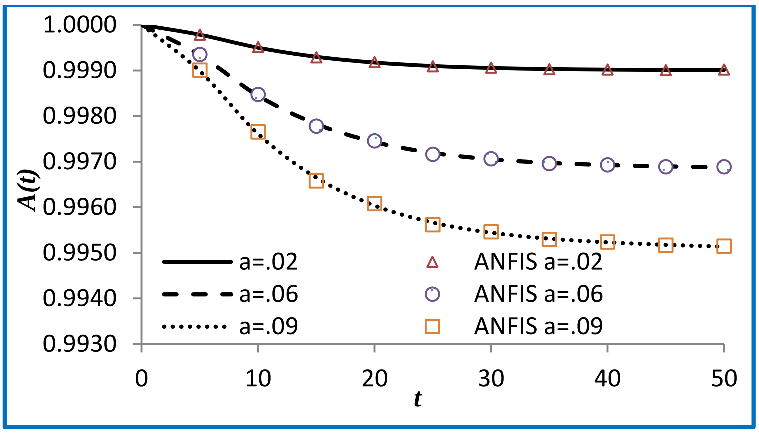


Fig. 7.6: $A(t)$ **vs** t **for different values of** a

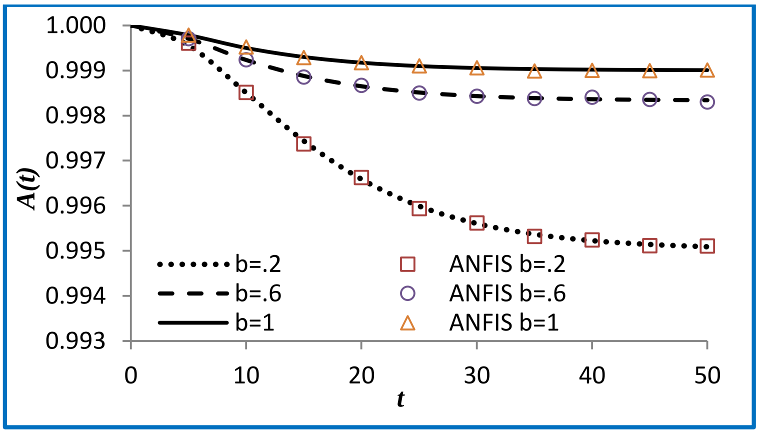


Fig. 7.7: $A(t)$ **vs** t **for different values of** b

Chapter 8

F-policy for Fault Tolerant System with Working Vacation

8.1 Introduction

Machine failure is the common phenomenon in the realistic scenarios of automated machining systems. The fault tolerant system (FTS) designed for variant purposes, can automatically detect the fault of the component and may be capable to recover the system by switch over of the failed machine with standby machine, with coverage probability c. Sometimes the system is not able to switch over the failed machines; in that case the system reaches to unsafe mode with complementary probability $\bar{c} = 1 - c$, and that can be cleared by the reboot process automatically. The queueing and reliability models by including the feature of imperfect coverage have attracted some researchers (Trivedi, 2002;Wang and Chiu, 2006; Wang *et al.* 2013). Jain and Gupta (2013) suggested an optimal policy for the maintainability of a repairable system by including the noble features of imperfect fault coverage and multiple vacations. Jain *et al*. (2014) proposed N-policy for Markovian machining system under the assumptions of unreliable server, imperfect coverage and reboot. Finite population Markovian modeling is commonly used to predict the reliability, MTTF and other performance indices and facilitates various realistic and more consistent queueing metrics in practice. In the present model, we consider Markovian models for the queueing and reliability prediction of fault tolerant machining system by incorporating many noble features which were not taken together by other researchers working on the same lines. The key concepts incorporated for the modeling of the concerned FTS are (i) dis-similar warm standby machines (ii) working vacation (iii) imperfect coverage (iv) reboot delay (v) time varying system, etc. Without loss of generality, we study the fault tolerant system operating in machining environment under the assumption of Markovian processes for the life and repair times of the failed machines. The recovery and reboot are taken into account to make the model to be applicable in real time systems. The remaining content of this chapter is structured in the following Sections. In Section 8.2, assumptions and notations are presented to model the multi-component machining problem of FTS. To analyze the FTS, Chapman-Kolmogorov differential difference equations are constructed in Section 8.3. In Section 8.4, performance of the system is assessed by deriving some indices. The sensitivity and relative sensitivity are presented in Section 8.5. The next Section 8.6 contains the numerical simulation results to check the sensitivity of the performance indices.

8.2 Model Description

In order to study the concept of threshold 'F' policy for the admission of failed machine for the repair job of the fault tolerant machining system, we develop Markovian model for the machining system with standby support and a single repairman. The finite population queueing model is formulated under the following assumptions:

Consider a finite capacity (K) fault tolerant machining system comprising of M online machines and total $S \equiv S^{(k)} = S_1 + S_2 + ... + S_k$ warm standby machines, where S_k represents the k^{th} type of warm standby machine. To repair the failed machines, the system has the facility of single skilled failure prone repairman. When the repair system reaches to its full capacity, no more failed machines will be accepted for the repair job until the system capacity further reduces to level 'F'. In case of failure of an operating machine, a standby unit of i^{th} type $\{i = \min(1, 2, ..., k)\}$ if available is used to replace the failed machine with probability c. The provision of reboot and recovery processes are taken into account while switch over of failed machine takes place. The server takes the startup time before initiating the repair job after returning from the vacation.

The operating machines and standby machines are subject to breakdowns and have the life time governed by exponential distributions with mean λ^{-1} and a^{-1}, respectively. When an operating machine breaks down, it is immediately switched-over by an available standby machine with probability c. If switchover of failed machines is not successfully completed, then the system goes to unsafe mode with probability $\bar{c}$ and the systems reconfigure itself by reboot process. The recovery time and reboot process are exponentially distributed with rates σ and r, respectively.

The server is also likely to go for working vacation on finding no failed machines in the system. The vacation duration of the server is exponentially distributed with mean parameter θ^{-1}. The time to serve the failed machines during working vacation and normal busy period are exponentially distributed with repair rates μ_v and μ_b, respectively. The repair of the failed machines is done on the basis of FCFS discipline.

To control the admission of failed machines, once the capacity (K) of the system is full, the joining of failed machines in the system is stopped and will be further initiated by taking some start up time ν only when the queue size level of failed machines in the system drops to level $F(<K)$. The life time and repair time of the repairman are exponentially distributed with mean parameter α^{-1} and β^{-1}.

It is noted that $\{(\xi(t),\eta(t)):t\geq 0\}$ is a bi-variate Markov process which is discrete in state space and continuous in time. The system at time t, is defined in terms of transient state probabilities associated with the random variables $\xi(t)$ and $\eta(t)$.

$$\chi(t)=[\{(i,n)\,|\,i=0,2,3,4,7,8,10,12;n=1,2,3,...,K-1\}\cup\{(i,n)\,|\,i=5,9;n=0,1,2,...,K-1\}$$
$$\cup\{(i,n)\,|\,i=1,11;\ n=0,1,2,...,K-1,K\}\cup\{(i,n)\,|\,i=6;\ n=0,1,2,...,K-1,K\}]$$

We define the probabilities of the system states at epoch t as follows:

$P_{n,i}(t)$: Probability that there are n failed machines in the system at time t at i^{th} level $(i=0,1,2,..,12)$.

$$\xi(t)=\begin{cases} 0;\ \text{Server is in under repair when broken down during normal busy mode} \\ \quad \text{and failed machines are not allowed to enter the system (DBN).} \\ 1;\ \text{Server is in normal busy state when failed machines are not permitted} \\ \quad \text{to enter the system (NBN).} \\ 2;\ \text{Server is in recovery state during normal busy mode (NBC).} \\ 3;\ \text{Server is in normal busy mode (NBO).} \\ 4;\ \text{Server is in reboot state during normal busy mode (NBR).} \\ 5;\ \text{Server is in recovery state during broken down mode (BDC).} \\ 6;\ \text{Server is in under repair when broken down during normal busy and} \\ \quad \text{working vacation state (BDD).} \\ 7;\ \text{Server is in reboot state during broken down mode (BDR).} \\ 8;\ \text{Server is in recovery state during working vacation mode (WVC).} \\ 9;\ \text{Server is on a working vacation period and the failed machines are} \\ \quad \text{allowed to enter the system (WVO).} \\ 10;\ \text{Server is in reboot state during working vacation mode (WVC)} \\ 11;\ \text{Server is on a working vacation period and the failed machines are not} \\ \quad \text{allowed to enter the system (WVN).} \\ 12;\ \text{Server is in under repair when broken down during} \\ \quad \text{working vacation state (DWV).} \end{cases}.$$

The mean failure rate of online failed machine (λ_n) and standby machine (a_n) are given by:

$$\lambda_n=\begin{cases} M\lambda+\left(\sum_{i=2}^{k}S_i a_i+(S_1-n)a_1\right); & 0\leq n<S_1 \\ M\lambda+\left(S^{(j-1)}-n\right)a_{j-1}+\sum_{i=j}^{k-1}S_i a_i; & S^{(j-1)}\leq n<S^{(j)};\ j=2,3,4,...,k-1 \\ M\lambda+\left(S^{(k)}-n\right)\delta_k; & S^{(k-1)}\leq n<S^{(k)} \\ \left(M+S^{(k)}-n\right)\lambda; & S^{(k)}\leq n<K=M+S^{(k)}-(m-1) \end{cases}$$

For the brevity, we have used the notation for failure rate of standby machines by

$$a_n = \begin{cases} \left(\sum_{i=2}^{k} S_i \delta_i + (S_1 - n)\delta_1 \right); & 0 \le n < S_1 \\ \left(S^{(j-1)} - n \right) \delta_{j-1} + \sum_{i=j}^{k-1} S_i \delta_i; & S^{(j-1)} \le n < S^{(j)};\ j = 2,3,4....,k-1 \\ \left(S^{(k)} - n \right) \delta_k; & S^{(k-1)} \le n < S^{(k)} \end{cases}$$

8.3 Model Governing Equations

Chapman-Kolmogorov differential-difference equations for the transient probabilities associated with system states are constructed by taking in-flow and out-flow transition rates. For Markov modelling of the fault tolerant machining system, the governing equations for the states $\xi(t)$ of the server at time t are framed as follows:

(i) $\xi(t) = 0$ **: *When the server is under repair while broken down during busy state (DBN).***

$$\frac{dP_{n,0}(t)}{dt} = -\beta P_{0,n}(t) + \alpha P_{1,n}(t); \qquad 1 \le n \le K-1 \tag{8.1}$$

(ii) $\xi(t) = 1$ ***: When the server is in busy state and the entry of failed machines are not permitted in the system (NBN).***

$$\frac{dP_{1,n}(t)}{dt} = -(\alpha + \mu_b + \nu)P_{1,n}(t) + \beta P_{0,n}(t) + \mu_b P_{1,n+1}(t); \quad 1 \le n \le F \tag{8.2}$$

$$\frac{dP_{1,n}(t)}{dt} = -(\alpha + \mu_b)P_{1,n}(t) + \beta P_{0,n}(t) + \mu_b P_{1,n+1}(t); \quad F+1 \le n \le K-1 \tag{8.3}$$

$$\frac{dP_{1,K}(t)}{dt} = -(\alpha + \mu_b)P_{1,K}(t) + \lambda_{K-S-1} P_{K-1,3}(t) \tag{8.4}$$

(iii) $\xi(t) = 2$ **: *When the server is in recovery state for normal busy state (NBC).***

$$\frac{dP_{2,n}(t)}{dt} = -\sigma P_{2,n}(t) + \lambda_0 c P_{3,n}(t); \quad 1 \le n \le S \tag{8.5}$$

$$\frac{dP_{2,n}(t)}{dt} = -\sigma P_{2,n}(t) + \lambda_{n-S} c P_{3,n}(t); \quad S+1 \le n \le K-1 \tag{8.6}$$

(iv) $\xi(t) = 3$ **: *When the server is in normal busy and the failed machines are permitted to enter the system (NBO).***

$$\frac{dP_{3,0}(t)}{dt} = -(\alpha + \lambda_0 + \varepsilon + a_0)P_{3,0}(t) + \mu_b (P_{3,1}(t) + P_{1,1}(t)) + \beta P_{6,0}(t) \tag{8.7}$$

$$\begin{aligned} \frac{dP_{3,n}(t)}{dt} &= -(\alpha + \lambda_0 + \mu_b + a_n)P_{3,n}(t) + \nu P_{1,n}(t) + \sigma P_{2,n-1}(t) + r P_{4,n-1}(t) \\ &+ a_{n-1} P_{3,n-1}(t) + \gamma P_{9,n}(t) + \mu_b P_{3,n+1}(t) + \beta P_{6,n}(t); \ 1 \le n \le S-1 \end{aligned} \tag{8.8}$$

$$\frac{dP_{3,S}(t)}{dt} = -(\alpha + \lambda_0 + \mu_b)P_{3,S}(t) + \nu P_{1,S}(t) + \sigma P_{2,S-1}(t) + r P_{4,S-1}(t) + a_{S-1}P_{3,S-1}(t) + \mu_b P_{3,S+1}(t) + \beta P_{6,S}(t) + \gamma P_{9,S}(t) \tag{8.9}$$

$$\frac{dP_{3,n}(t)}{dt} = -(\alpha + \lambda_{n-S} + \mu_b)P_{3,n}(t) + \nu P_{1,n}(t) + \sigma P_{2,n-1}(t) + r P_{4,n-1}(t) + \mu_b P_{3,n+1}(t) + \beta P_{6,n}(t) + \gamma P_{9,n}(t); \quad S+1 \le n \le F \tag{8.10}$$

$$\frac{dP_{3,n}(t)}{dt} = -(\alpha + \lambda_{n-S} + \mu_b)P_{3,n}(t) + \sigma P_{2,n-1}(t) + r P_{4,n-1}(t) + \mu_b P_{3,n+1}(t) + \beta P_{6,n}(t) + \gamma P_{9,n}(t); \quad F+1 \le n \le K-2 \tag{8.11}$$

$$\frac{dP_{3,K-1}(t)}{dt} = -(\alpha + \lambda_{K-S-1} + \mu_b)P_{3,K-1}(t) + \sigma P_{2,K-2}(t) + r P_{4,K-2}(t) + \beta P_{6,K-1}(t) + \gamma P_{9,K-1}(t) \tag{8.12}$$

(v) $\xi(t) = 4$ ***: When the server is in reboot state for normal busy mode (NBR).***

$$\frac{dP_{4,n}(t)}{dt} = -r P_{4,n}(t) + \lambda_0 \bar{c} P_{3,n}(t); \ \ 1 \le n \le S \tag{8.13}$$

$$\frac{dP_{4,n}(t)}{dt} = -r P_{4,n}(t) + \lambda_{n-S} \bar{c} P_{3,n}(t); \ \ S+1 \le n \le K-1 \tag{8.14}$$

(vi) $\xi(t) = 5$ ***: When the server is in recovery state during break down (BDC) period.***

$$\frac{dP_{5,n}(t)}{dt} = -\sigma P_{5,n}(t) + \lambda_0 c P_{6,n}(t); \ \ 1 \le n \le S \tag{8.15}$$

$$\frac{dP_{5,n}(t)}{dt} = -\sigma P_{5,n}(t) + \lambda_{n-S} c P_{6,n}(t); \ \ S+1 \le n \le K-1 \tag{8.16}$$

(vii) $\xi(t) = 6$ ***: When the server is in breakdown state (BDD).***

$$\frac{dP_{6,0}(t)}{dt} = -(\lambda_0 + 2\beta + a_0)P_{6,0}(t) + \alpha P_{3,0}(t) + \alpha P_{9,0}(t) \tag{8.17}$$

$$\frac{dP_{6,n}(t)}{dt} = -(\lambda_0 + 2\beta + a_n)P_{6,n}(t) + a_{n-1}P_{6,n-1}(t) + \sigma P_{5,n-1}(t) + r P_{7,n-1}(t) + \alpha P_{3,n}(t) + \alpha P_{9,n}(t); \ \ 1 \le n \le S-1 \tag{8.18}$$

$$\frac{dP_{6,S}(t)}{dt} = -(\lambda_0 + 2\beta)P_{6,S}(t) + \sigma P_{5,S-1}(t) + r P_{7,S-1}(t) + \alpha P_{3,S}(t) + \alpha P_{9,S}(t) + a_{S-1}P_{6,S-1}(t) \tag{8.19}$$

$$\frac{dP_{6,n}(t)}{dt} = -(\lambda_{n-S} + 2\beta)P_{6,n}(t) + \sigma P_{5,n-1}(t) + r P_{7,n-1}(t) + \alpha P_{3,n}(t) + \alpha P_{9,n}(t); \ \ S+1 \le n \le K-2 \tag{8.20}$$

$$\frac{dP_{6,K-1}(t)}{dt} = -(\lambda_{K-S-1}c + 2\beta)P_{6,K-1}(t) + \sigma P_{5,K-2}(t) + r\,P_{7,K-2}(t) + \alpha P_{3,K-1}(t) + \alpha P_{9,K-1}(t) \tag{8.21}$$

$$\frac{dP_{6,K}(t)}{dt} = \lambda_{K-S-1}P_{6,K}(t) + \alpha P_{1,K}(t) + \alpha P_{11,K}(t) \tag{8.22}$$

(viii) $\xi(t) = 7$ ***: When the server is in reboot state during break down period (BDR).***

$$\frac{dP_{n,7}(t)}{dt} = -r\,P_{n,7}(t) + \lambda_0 \bar{c}\,P_{n,6}(t);\;\; 1 \le n \le S \tag{8.23}$$

$$\frac{dP_{7,n}(t)}{dt} = -r\,P_{7,n}(t) + \lambda_{n-S}\bar{c}\,P_{6,n}(t);\;\; S+1 \le n \le K-1 \tag{8.24}$$

(ix) $\xi(t) = 8$ ***: When the server is in recovery state during working vacation (WVC).***

$$\frac{dP_{8,n}(t)}{dt} = -\sigma P_{8,n}(t) + \lambda_0 c\,P_{9,n}(t);\;\; 1 \le n \le S \tag{8.25}$$

$$\frac{dP_{8,n}(t)}{dt} = -\sigma P_{8,n}(t) + \lambda_{n-S} c\,P_{9,n}(t);\;\; S+1 \le n \le K-1 \tag{8.26}$$

(x) $\xi(t) = 9$ ***: When the server is in the working vacation and the failed machines are permitted to join the system (WVO).***

$$\frac{dP_{9,0}(t)}{dt} = -(\alpha + \lambda_0 + a_0)P_{3,0}(t) + \varepsilon\,P_{3,0}(t) + \mu_v(P_{9,1}(t) + P_{11,1}(t)) + \beta P_{6,0}(t) \tag{8.27}$$

$$\frac{dP_{9,n}(t)}{dt} = -(\alpha + \lambda_0 + \mu_v + \gamma + a_n)P_{9,n}(t) + \nu P_{11,n}(t) + \sigma P_{8,n-1}(t) + r\,P_{10,n-1}(t) + a_{n-1}P_{9,n-1}(t) + \mu_v P_{9,n+1}(t) + \beta P_{6,n}(t);\;\; 1 \le n \le S-1 \tag{8.28}$$

$$\frac{dP_{9,S}(t)}{dt} = -(\alpha + \lambda_0 + \mu_v + \gamma)P_{9,S}(t) + \nu\,P_{11,S}(t) + \sigma P_{8,S-1}(t) + r\,P_{10,S-1}(t) + a_{S-1}P_{9,S-1}(t) + \mu_v P_{9,S+1}(t) + \beta P_{6,S}(t) \tag{8.29}$$

$$\frac{dP_{9,n}(t)}{dt} = -(\alpha + \lambda_{n-S} + \gamma + \mu_v)P_{9,n}(t) + \nu\,P_{11,n}(t) + \sigma P_{8,n-1}(t) + r\,P_{10,n-1}(t) + \mu_v P_{9,n+1}(t) + \beta P_{6,n}(t);\qquad S+1 \le n \le F \tag{8.30}$$

$$\frac{dP_{9,n}(t)}{dt} = -(\alpha + \lambda_{n-S} + \mu_v + \gamma)P_{9,n}(t) + \sigma P_{8,n-1}(t) + r\,P_{10,n-1}(t) + \mu_v P_{9,n+1}(t) + \beta P_{6,n}(t);\qquad F+1 \le n \le K-2 \tag{8.31}$$

$$\frac{dP_{9,K-1}(t)}{dt} = -(\alpha + \lambda_{K-S-1}c + \mu_v + \gamma)P_{9,K-1}(t) + \sigma P_{8,K-2}(t) + r\,P_{10,K-2}(t) + \beta P_{6,K-1}(t) \tag{8.32}$$

(xi) $\xi(t) = 10$ ***: When the server is in reboot state during working vacation (WVR).***

$$\frac{dP_{10,n}(t)}{dt} = -r\,P_{10,n}(t) + \lambda_0 \bar{c}\,P_{9,n}(t);\;\; 1 \le n \le S \tag{8.33}$$

$$\frac{dP_{10,n}(t)}{dt} = -r\,P_{10,n}(t) + \lambda_{n-S}\bar{c}\,P_{9,n}(t); \quad S+1 \le n \le K-1 \tag{8.34}$$

(xii) $\xi(t)=11$**: *When the server is on working vacation when no more failed machines can enter the system (WVN).***

$$\frac{dP_{11,n}(t)}{dt} = -(\alpha+\mu_v+\nu)P_{11,n}(t) + \beta P_{12,n}(t) + \mu_v P_{11,n+1}(t); \quad 1 \le n \le F \tag{8.35}$$

$$\frac{dP_{11,n}(t)}{dt} = -(\alpha+\mu_v)P_{11,n}(t) + \beta P_{12,n}(t) + \mu_v P_{11,n+1}(t); \quad F+1 \le n \le K-1 \tag{8.36}$$

$$\frac{dP_{11,K}(t)}{dt} = -(\alpha+\mu_v)P_{11,K}(t) + \lambda_{K-S-1}P_{9,K-1}(t) \tag{8.37}$$

(xiii) $\xi(t)=12$**: *When the server is under repair while breaks down during working vacation (DWV).***

$$\frac{dP_{12,n}(t)}{dt} = -\beta P_{12,n}(t) + \alpha P_{11,n}(t); \qquad 1 \le n \le K-1 \tag{8.38}$$

8.4 Performance Measures

The system characteristics can be examined by deriving the performance indices by considering the transient state probabilities. These measures have played significant role in achieving the high reliability and can be used as useful tools by the industrial engineers and system managers for the improvement of the grade of service (GoS) by predicting the reliability and queueing indices of the concerned fault tolerant system.

8.4.1 The transient probabilities at time 't' of *server being in broken down state* $P_{BD}(t)$, *server being busy* $P_B(t)$, *server rendering service during vacation period* $P_{BW}(t)$, *server being idle* $P_I(t)$, *server is in reboot state* $P_{RB}(t)$, *server is in recovery state* $P_{RC}(t)$ are obtained as follows:

$$P_{BD}(t) = \sum_{n=1}^{K-1} P_{0,n}(t) + \sum_{n=0}^{K-2} P_{5,n}(t) + \sum_{n=0}^{K} P_{6,n}(t) + \sum_{n=0}^{K-2} P_{7,n}(t) + \sum_{n=1}^{K-1} P_{12,n}(t) \tag{8.39}$$

$$P_B(t) = \sum_{n=1}^{K} P_{1,n}(t) + \sum_{n=0}^{K-1} P_{3,n}(t) \tag{8.40}$$

$$P_{BW}(t) = \sum_{n=0}^{K-1} P_{9,n}(t) + \sum_{n=1}^{K} P_{11,n}(t) \tag{8.41}$$

$$P_I(t) = P_{3,0}(t) + P_{9,0}(t) \tag{8.42}$$

$$P_{RB}(t) = \sum_{n=1}^{K-1}\left(P_{4,n}(t) + P_{7,n}(t) + P_{10,n}(t)\right) \tag{8.43}$$

$$P_{RC}(t)=\sum_{n=1}^{K-1}\left(P_{2,n}(t)+P_{5,n}(t)+P_{8,n}(t)\right) \tag{8.44}$$

8.4.2 Queueing indices

The queueing characteristics can be used for the performance evaluation of the fault tolerant system (FTS). Now, using transient probabilities, we establish the various queueing indices such as mean queue length of failed machines present in the system, throughput, mean number of available standbys, expected delay time, expected waiting time, etc.

(i) The mean queue length of failed machines in the system

$$\begin{aligned} EN(t)=&\sum_{j=0}^{12}\sum_{n=1}^{K-2} n\,P_{j,n}(t)+(K-1)\left(P_{0,K}(t)+P_{1,K}(t)+P_{3,K}(t)+P_{6,K}(t)\right.\\ &\left.+P_{9,K}(t)+P_{11,K}(t)+P_{12,K}(t)\right)+K\left(P_{1,K}(t)+P_{6,K}(t)+P_{11,K}(t)\right) \end{aligned} \tag{8.45}$$

(ii) The throughput can be obtained using

$$TP(t)=\sum_{n=1}^{K}\mu_b P_{1,n}(t)+\sum_{n=1}^{K-1}\mu_b P_{3,n}(t)+\sum_{n=1}^{K-1}\mu_v P_{9,n}(t)+\sum_{n=1}^{K}\mu_v P_{11,n}(t) \tag{8.46}$$

(iii) Mean number of available standby machines

$$ES(t)=\sum_{j=2}^{10} SP_{j,0}(t)+\sum_{j=2}^{12}\sum_{n=1}^{S-1}(S-n)P_{j,n}(t) \tag{8.47}$$

(iv) Mean number of operating machines in the system is

$$\begin{aligned} EO(t)=&M\sum_{j=2}^{10}P_{j,0}(t)+M\sum_{j=0}^{12}\sum_{n=1}^{S}P_{j,n}(t)+\sum_{j=0}^{12}\sum_{n=S+1}^{K-2}(M+S-n)P_{j,n}(t)\\ &+(M+S-K+1)\left(P_{0,K-1}(t)+P_{1,K-1}(t)+P_{3,K-1}(t)+P_{6,K-1}(t)+P_{9,K-1}(t)\right.\\ &\left.+P_{11,K-1}(t)+P_{12,K-1}(t)\right)+(M+S-K)\left(P_{1,K}(t)+P_{6,K}(t)+P_{11,K}(t)\right) \end{aligned} \tag{8.48}$$

(v) The effective rate which is the overall rate of joining of failed machines in the system

$$\begin{aligned} \lambda_{eff}=&\sum_{n=0}^{S-1}(\lambda_0+a_n)P_{3,n}(t)+\sum_{n=S}^{K-1}\lambda_{n-S}\,P_{3,n}(t)+\sum_{n=0}^{S-1}(\lambda_0+a_n)P_{6,n}(t)\\ &+\sum_{n=S}^{K-1}\lambda_{n-S}\,P_{6,n}(t)+\sum_{n=0}^{S-1}(\lambda_0+a_n)P_{9,n}(t)+\sum_{n=S}^{K-1}\lambda_{n-S}\,P_{9,n}(t) \end{aligned} \tag{8.49}$$

(vi) The expected waiting time for the failed machines in the system, is

$$EW(t)=\frac{EN(t)}{\lambda_{eff}} \tag{8.50}$$

(vii) The expected delay time is

$$ED(t)=\frac{E\{N(t)\}}{TP(t)} \tag{8.51}$$

8.4.3 Reliability indices

These measures can play significant role to increase the capacity and reliability of the fault tolerant system. The performance measures based on reliability issues are also helpful in improving the system availability during operational phase at time t.

(i) *Machine availability at time t*

$$MA(t)=1-\frac{E\{N(t)\}}{M+S} \tag{8.52}$$

(ii) *Reliability of the system at time t is*

$$R_Y(t)=1-P_{6,K}(t) \tag{8.53}$$

(iii) *Mean time-to-failure (MTTF) is obtained using*

$$MTTF=\int_0^\infty R_Y(t)\,dt \tag{8.54}$$

8.5 Sensitivity Analysis

The system reliability, throughput of the system and MTTF for varying values of different system descriptor θ which are taken specifically $\lambda,\alpha,\mu_b,\beta,\mu_b$ can be examined by using the derivatives of performance indices. Now, we obtain

$$\Phi_\theta(t)\equiv\frac{\partial R_Y(t)}{\partial\theta}=-\frac{\partial P_{6,K}(t)}{\partial\theta} \tag{8.56}$$

$$\Delta_\theta(t)\equiv\frac{\partial TP(t)}{\partial\theta} \tag{8.57}$$

$$\Psi_\theta\equiv\frac{\partial MTTF}{\partial\theta}=\frac{\partial\left(\int_{t=0}^{\infty}R_Y(t)\,dt\right)}{\partial\theta} \tag{8.58}$$

where $\frac{\partial R_y(t)}{\partial\theta}$, $\frac{\partial TP(t)}{\partial\theta}$ and $\frac{\partial MTTF}{\partial\theta}$ are computed numerically.

The relative sensitivity related to the reliability function, throughput of the system and MTTF are also presented by evaluating the following metrics:

$$\Omega_\theta(t)\equiv\frac{\partial R_Y(t)/R_Y(t)}{\partial\theta/\theta}=\frac{\partial R_Y(t)}{\partial\theta}.\frac{\theta}{R_Y(t)}=\Phi_\theta(t).\frac{\theta}{R_Y(t)} \tag{8.59}$$

$$\Xi_\theta(t)\equiv\frac{\partial\tau(t)/\tau(t)}{\partial\theta/\theta}=\frac{\partial\tau(t)}{\partial\theta}.\frac{\theta}{\tau(t)}=\Delta_\theta(t).\frac{\theta}{\tau(t)} \tag{8.60}$$

$$\Gamma_\theta\equiv\frac{\partial MTTF/MTTF}{\partial\theta/\theta}=\frac{\partial MTTF}{\partial\theta}.\frac{\theta}{MTTF}=\Psi_\theta\frac{\theta}{MTTF} \tag{8.61}$$

8.6 Numerical Simulation

The tractability and implementation of the suggested method are demonstrated by computing the various performance indices namely mean queue length of failed machines, reliability, mean number of available standby machines, cost function with respect to varying values of different parameters. The trends of performance indices based on numerical experiment have been display graphically. By conducting numerical experiment, we have also explored the sensitivity and relative sensitivity of some indices viz. the throughput, reliability and MTTF by varying values of different parameters.

8.6.1 Effect of parameters on performance indices

To visualize the impact of different parameters, the graphs of the queue length of failed machines, reliability and MTTF have been computed for the transient state. The sensitivity of the system indices with respect to different system descriptors $\lambda, \alpha, \mu_b, \beta, \mu_v$ is presented by varying time. We set the default system parameters as $M = 25, F = 8, S = 4, m = 2,$ $\lambda = 0.75, \mu_b = 5, \mu_v = 2, \; \nu = 0.5, \; \beta = 10, \; \gamma = 10, \sigma = 12, r = 1$ and the results computed are depicted in Figs. 8.1-8.2. It is observed that the mean queue length of failed machines $EN(t)$ increases with the increasing values of λ but decreases with the increasing repair rate μ_b. There is no remarkable effect of β and μ_v on $EN(t)$. The mean queue length of failed machines $EN(t)$ increases for the higher value of the number of operating machines (M) as well as the standby machines (S).

As we expect, the system reliability $R_Y(t)$, lowers down with the increase in the failure rates λ of operating as well as standbys machines. The numerical results for the reliability match with our expectations, as clearly seen in Fig. 8.2(i) and 8.2(iii); the system reliability $R_Y(t)$ decreases with the increase in the time for different values of λ and α, respectively. Initially there is no effect of changing value of λ and α but the decreasing effect with respect to λ and α becomes more prominent as time grows up. From Fig. 8.2 (ii), the system reliability $R_Y(t)$ grows up as time passes for different values of repair rate of failed machines (μ_b). Initially the system reliability $R_Y(t)$ shows no significant change for the different values of β but later on shows increasing trend as time increases. The reliability of the system $R_Y(t)$ increases as the number of operating machines as well as standby machines increase.

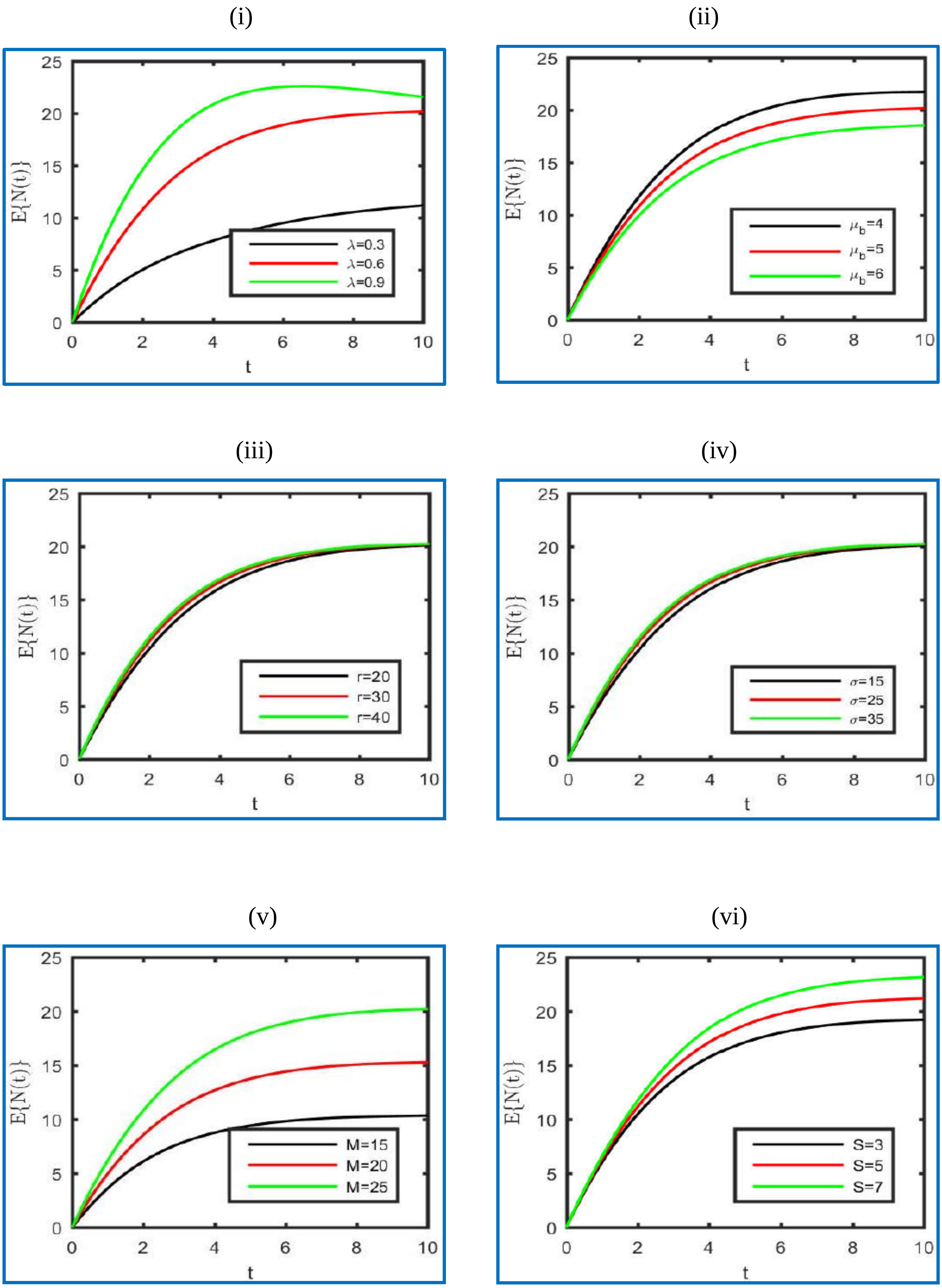


Fig. 8.1: Expected number of failed machines $E\{N(t)\}$ **vs.** t **by varying parameters (i)** λ **(ii)** μ_b **(iii)** r **(iv)** σ **(v)** M **(vi)** S

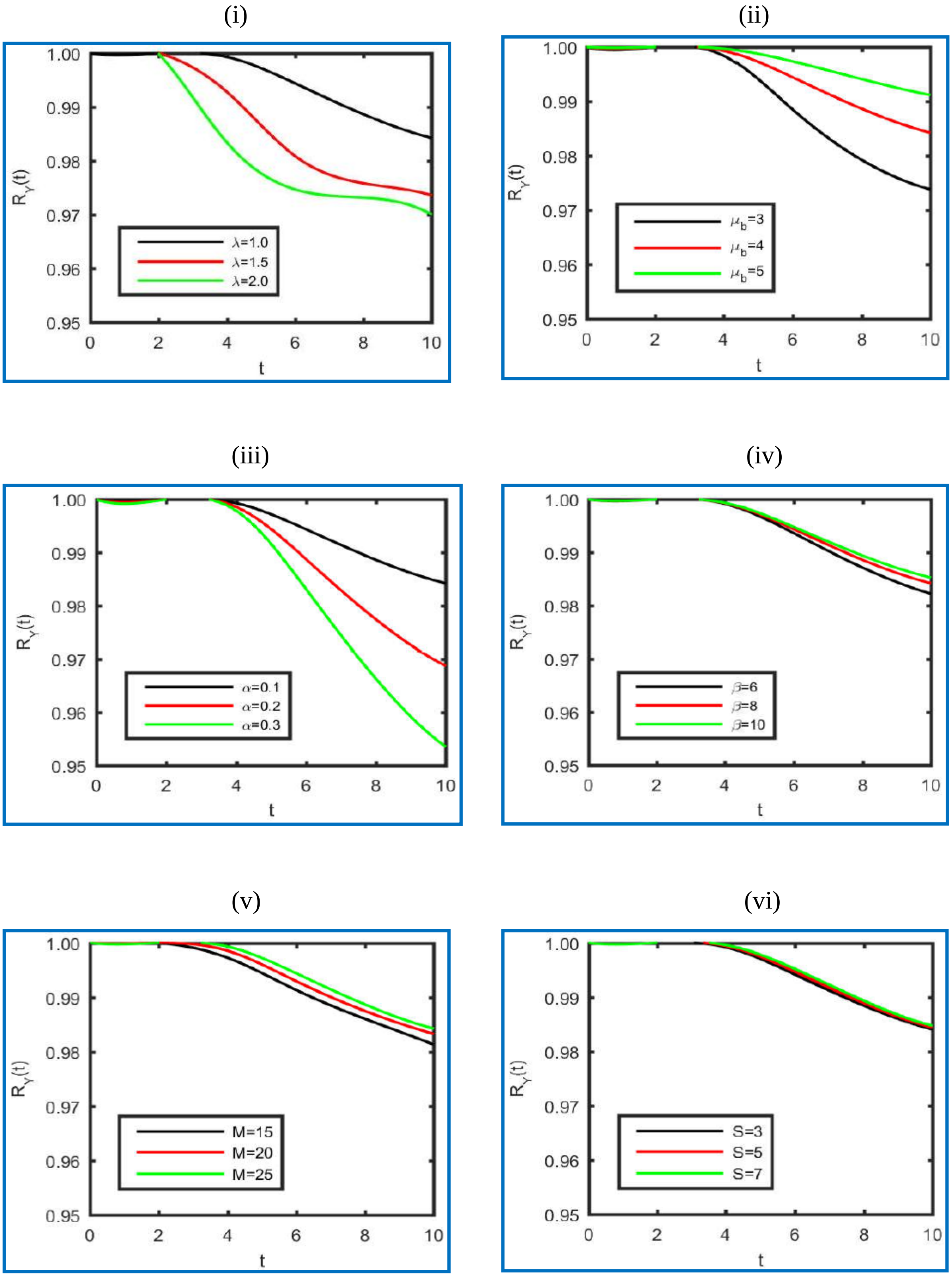


Fig. 8.2: System reliability $R_Y(t)$ Vs. t by varying parameters (i) λ (ii) μ_b (iii) α (iv) β (v) M (vi) S

8.6.2 Sensitivity of system reliability

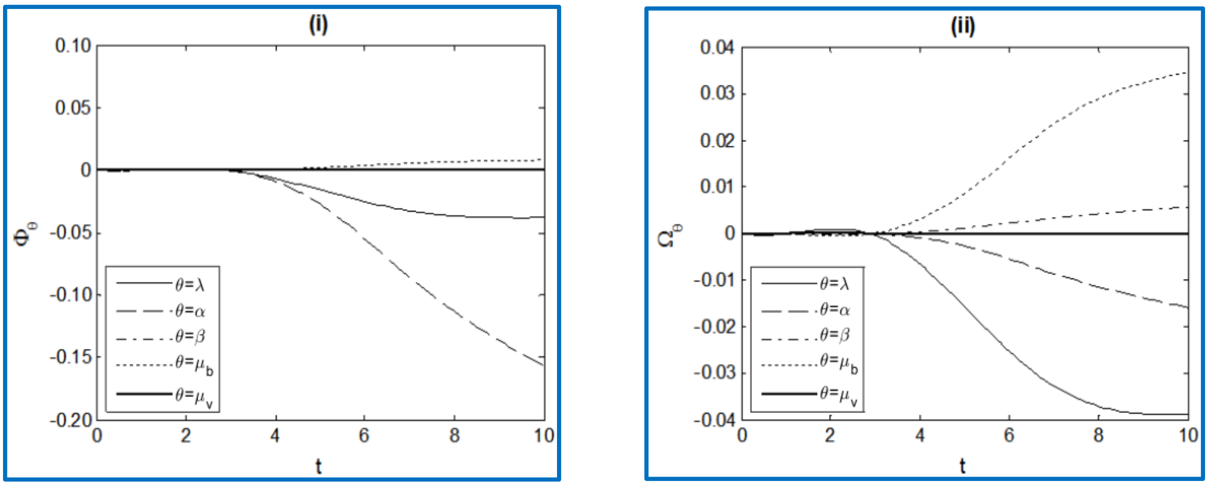


Fig. 8.3 (i) Reliability sensitivity $\Phi_\theta(t)$ and (ii) Relative sensitivity $\Phi_\theta(t)$ vs. t for different system parameters

The pattern of reliability function is depicted in Figure 8.2 for various system descriptors. The trend of the reliability for the varying values of the system parameters is depicted in Figure 8.2 by setting default parameters as *M=25; F=8; S=4; m=2;* $\lambda=0.75$; $\mu_b=5$; $\mu_v=2$; $v=0.5$; $c=0.4$; $\theta=14$; $\alpha=0.2$; $\beta=10$; $\gamma=10$; $\sigma=12$; $r=1$. It can be easily seen that the reliability increases with the decrease in failure rates (λ and α) and increase in repair rate (μ and β). It is clear from the figure that the desired reliability can be easily achieved by improving the system repair facility which matches with the experience on real time systems.

The reliability is examined by presenting sensitivity and the relative sensitivity as shown in Fig. 8.3(i) and 8.3(ii). It is noticed that the reliability is quit sensitive for λ while ν has least impact on the reliability. From the magnitude $|\Phi_\theta(t)|$, we can conclude that for different parameters, the reliability is sensitive in order $\lambda > \alpha > \mu_b > \beta > \mu_v$.

8.6.3 Sensitivity of throughput

The sensitivity of throughput (τ) is depicted in Figure 8.4(i) with respect to the system parameters for default parameters fixed as *M=25; F=8; S=4; m=2;* $\lambda=0.75$; $\mu_b=5$; $\mu_v=2$; $v=0.5$; $c=0.4$; $\theta=14$; $\alpha=0.2$; $\beta=10$; $\gamma=10$; $\sigma=12$; $r=1$.

It can be seen that the throughout is highest sensitive to μ while least sensitive to ν. From the magnitude $|\Delta_\theta(t)|$, we can conclude that throughput function with respect to different parameters is sensitive in order $\mu_b > \alpha > \lambda > \beta > \mu_v$. The relative sensitivity of throughput function $\Xi_\theta(t)$ is shown in Fig 8.4(ii).

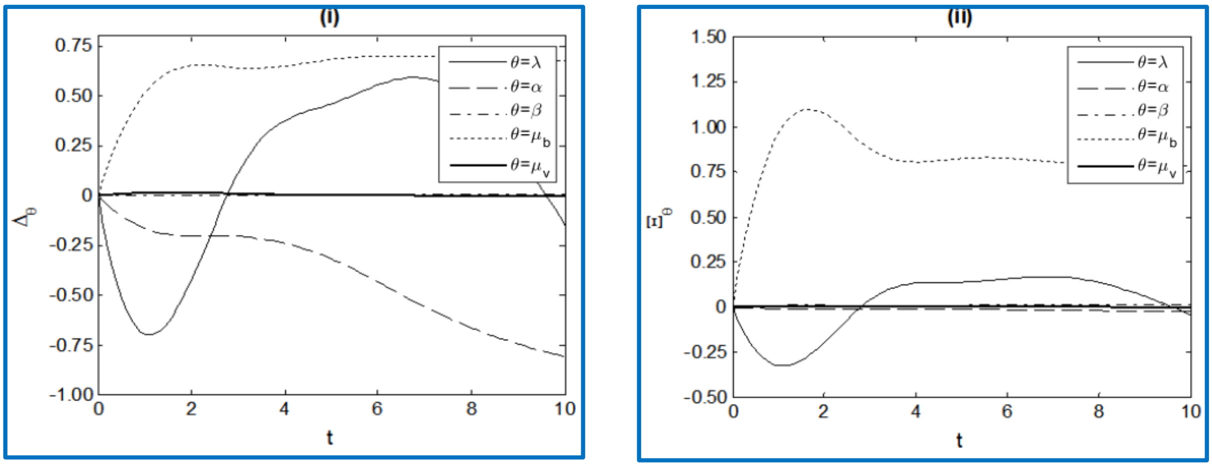


Fig. 8.4: (i) Sensitivity of throughput $\Delta_\theta(t)$ and (ii) Relative sensitivity of the throughput $\Xi_\theta(t)$

8.6.4 Sensitivity and relative sensitivity of MTTF

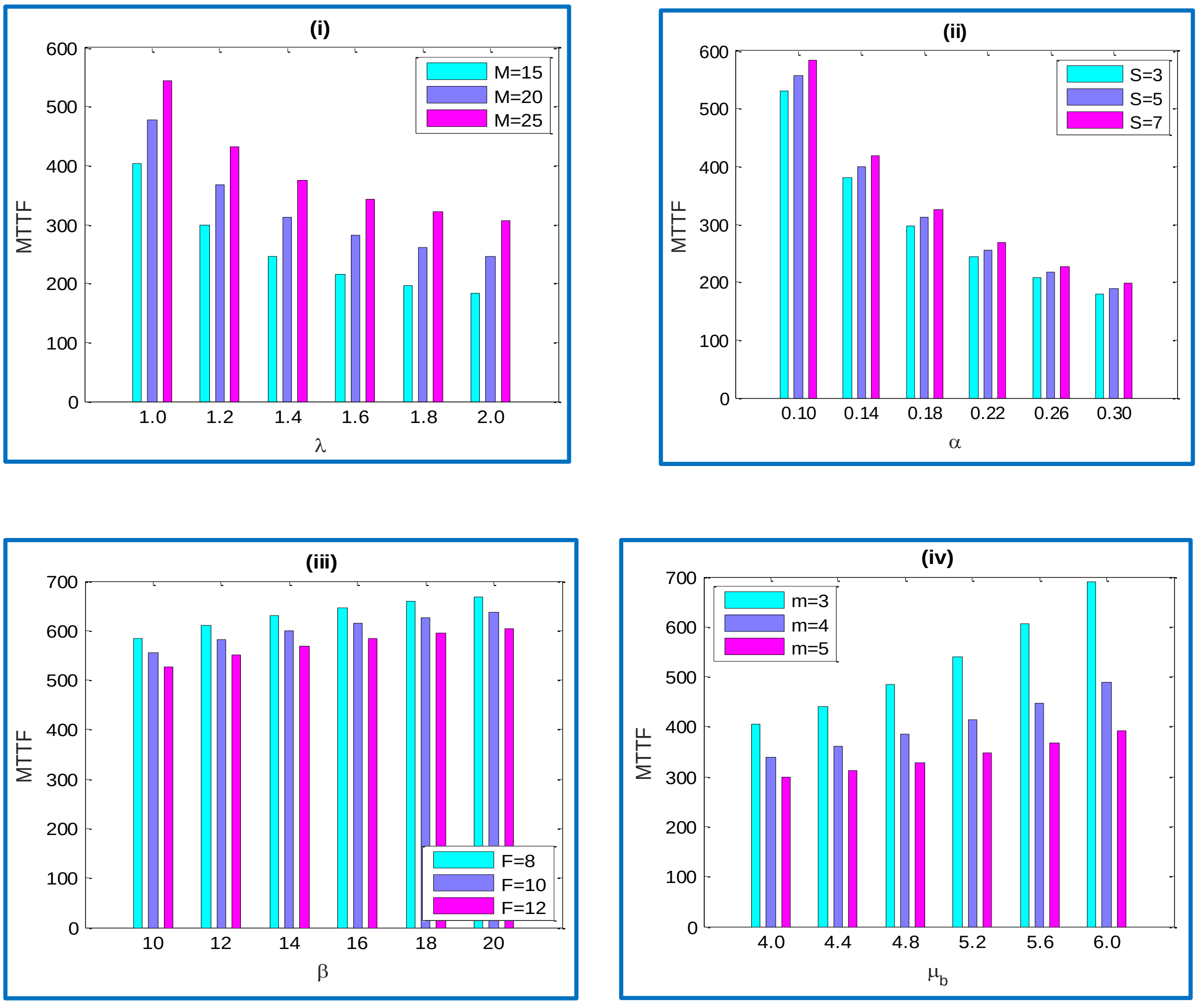


Fig. 8.5: Mean time to failure (MTTF) for (i) λ (ii) α (iii) β (iv) μ_b

Table 8.1: Sensitivity analysis $\Gamma_\theta(t)$ of MTTF

M	α	β	λ	μ_b	μ_v
15	-4005.64	17.43	-789.51	177.29	-0.10
20	-4733.27	20.54	-815.64	180.15	-0.06
25	-5396.58	23.37	-831.84	180.93	-0.04

Table 8.2: Relative sensitivity analysis $\Psi_\theta(t)$ of MTTF

M	α	β	λ	μ_b	μ_v
15	-9.90739E-01	3.44864E-01	-1.95274E+00	1.75404E+00	-1.28475E-04
20	-9.90948E-01	3.43960E-01	-1.70760E+00	1.50861E+00	-6.13667E-05
25	-9.90993E-01	3.43258E-01	-1.52754E+00	1.32901E+00	-3.65608E-05

To explore the sensitivity of MTTF, we set the default parameters as M=20; F=8; S=4; m=2; λ=01.3; μ_b=5; μ_v=3; ν=0.5; c=0.4; θ=14; α=0.2; β=10; γ=10; σ=12; r=1. Numerical results for the MTTF are depicted in Figure 8.5 for varying different parameters. It is noted from Figs 8.5(i)-8.5(ii) that the MTTF grows up as the number of operating as well as standby machines increases. On the contrarily MTTF decreases as F increases. Moreover, there is increment in MTTF as value of M increases. In Tables 8.1-8.2, we display the sensitivity $\Gamma_\theta(t)$ and relative sensitivity $\Psi_\theta(t)$ of MTTF for varying values of different system parameters. It is evident that magnitude of MTTF $|\Gamma_\theta(t)|$ is sensitive in the following order of parameters $\lambda > \alpha > \mu_b > \beta > \mu_v$ when $M = 15, 20, 25$. Moreover, the order of magnitude of relative sensitivity of MTTF $|\Psi_\theta(t)|$ is $\alpha > \beta > \lambda > \mu_b > \mu_v$ when $M = 15, 20, 25$.

Chapter 9

Availability of R-out-of-M:G FTS with Imperfect Fault Coverage

9.1 Introduction

The automated machining systems can be upgraded by using the fault tolerant mechanism which reduces the risk of immediate breakdowns and improves the system availability. The prime objective of system developers is to design redundant and repairable systems by incorporating all those features which helps to upgrade the efficiency, reliability as well as availability related metrics of concerned machining system. In queueing and reliability literature, several studies are devoted to the performance analysis of the redundant machining systems in Markovian frameworks (cf. Sivazlian and Wang, 1989; Wang and Kuo, 1997; Wang and Ke, 2003; Karmeshu and Sharma, 2006; Haque and Armstrong, 2007; Ke and Wu, 2012; Hsu *et al*., 2014) and many others. To enhance the performability of the machining systems, the concept of non-exponential distribution for life time/repair time distributions should be used. From the queue management point of view, Singh *et al*. (2015) and Karmeshu *et al*. (2017) contributed significantly to analyze the finite buffer queue and adaptive mean queue size, respectively.

The performance prediction via non-Markovian queueing analysis of fault tolerable machining system has attracted a few authors due to its critical applications in several real time systems. In the present study, we implement supplementary variable and recursive approaches to determine the availability and the performance metrics of the machining system operating in fault tolerant environment by developing M/G/1 model and treating the remaining repair time as a supplementary variable. The supplementary variable technique to solve the non-Markovian systems was first introduced by Cox (1951) which later on, implemented by many researchers to analyze the service systems in different frameworks (cf. Hokstad, 1975; Choi and Park, 1990; Gupta and Srinivasa Rao, 1994; Gupta and Srinivasa Rao, 1996; Nobel and Tijms, 2000; Madan, 2000; Wang and Ke, 2000; Ke and Wang, 2002; Ke, 2003) . The notable contributions on M/G/1 model for the repairable machining system via supplementary variable technique can be found in work of (Yamashiro, 1981; Goel and Gupta, 1983; Goel *et al*., 1983; Gupta and Agarwal, 1984; Mahmoud *et al*., 1987; Mahmoud *et al*., 1988; Srinivasa Rao and Gupta, 2000; Jain *et al*., 2004; Wang *et al*., 2005; Wang and

Chiu, 2006). By incorporating many important features viz. N-policy and vacation, Ke (2003) developed a G/M/1 queueing model to study the probability distribution at steady state to compute average queue size by using supplementary variable technique. Wang *et al*. (2005) carried out the availability analysis of three different redundant machining system configurations for the recovery of failed components by employing supplementary variable technique. The M(n)/G/1/K queueing model with state dependent arrival rate and removal server has been studied by Chao and Rahman (2006). They have obtained the steady state queue size distributions of system state using recursive and supplementary variable technique. The finite capacity non-Markovian queueing systems combined with F-policy have been studied in Wang *et al*. (2007) by employing the supplementary variable technique. Wang and Chen (2009) studied and compared the availability of three different systems incorporating reboot delay and switching failure. They have used supplementary variable approach to evaluate the stationary explicit expressions for the availability of three configurations. The availability prediction of a repairable system with imperfect fault coverage and common cause failure have been investigated using supplementary variable technique by Jain (2013). The availability prediction of K-out-of-(M+W) configuration was studied by Ke *et al*. (2013) of a repairable retrial system with the provision of standby support. Wang *et al*. (2014) presented a steady state analysis of a M/G/1 machining system using supplementary variable technique. Ke *et al*. (2016) proposed the queueing model for the analysis of a machining system with standbys support and imperfect switch over facility. In this study, they have used the recursive method based on supplementary variable technique and carried out cost analysis using Probabilistic Global Search Lausanne method (PGSLM). Recently, Jain and Sanga (2017) have used supplementary variable technique for the steady state analysis of fault tolerant machining system under the control F-policy.

In the available works on M/G/1 queue with redundancy and imperfect fault coverage, the provision of replacement along with recovery of failed machines has not made. However, in many systems, the failed components can be replaced due to fact that repair process may not adequate for the recovery of failed components. Sometime replacement seems to good option in comparison to repair due to techno-economic reason. The M/G/1 queueing model for the fault tolerant machining system is considered by including the features of repair, reboot, recovery and replacement processes to deal with more versatile and realistic scenarios of real-time systems. The queueing and availability analysis of fault tolerant machining system is presented by using recursive method to solve the difference differential equations obtained, after introducing the supplementary variable corresponding to remaining repair time. The cost optimization is also done by Newton-quasi method. Furthermore, numerical experiments are conducted to explore the sensitivity of parameters on the system availability and various other performance metrics. The contents of rest of the chapter are managed in the following manner. In Section 9.2, the notations and assumptions for formulating the model are presented. Section 9.3 provides the queue size distribution of the system. The solution algorithm to compute the steady state probabilities associated to different states of the system is described in Section 9.4. The system availability analysis is carried out for *R*-out of-*M*: *G* configuration in Section 9.5. In Section 9.6, various system indices are established to characterize the system reliability aspects and system cost function is formulated. The numerical simulation is performed to study the sensitivity of system descriptors on performance indices in Section 9.7.

9.2 Description of the Model

The M/G/1 queueing model is considered for the performance modeling and availability analysis of fault tolerant repairable system consisting of M identical operating machines. The operating machine may fail in Poisson fashion with rate λ. The repair time of failed machines is assumed to be independent and identically distributed (i.i.d.) random variable with cumulative distribution function $G(u)(u \geq 0)$, probability density function $g(u)(u \geq 0)$, and mean repair rate g. The failed machines are repaired by the single server in order of their breakdowns i.e. following the first in first out (FIFO) discipline. The failure of operating machine is detected with probability c and after detection recovered successfully with probability p by rate σ. In case if with probability $(1-\mathrm{c})$, the failure of the operating machine is not recovered successfully then the system will immediately takes reconfiguration operation to restore its functioning by reboot processes with rate r. When the fault is not recovered successfully with probability $(1-p)$, the failed machine is replaced with rate η by a new machine to maintain the functioning of the fault tolerant machining system.

Let $\chi = \{\eta(\tau), \xi(\tau); \tau \geq 0\}$ be a continuous time stochastic process where $\eta(\tau)$ denotes the number of failed machines in the system at time τ and $\xi(\tau)$ denotes the operation mode of the system at time τ which takes values 0, 1, 2, 3 when the system is in operating, reboot, recovery and replacement modes, respectively. For the analysis of non-Markovian process, the supplementary variable is introduced. Let $U(\tau)$ represents the supplementary variable corresponding to remaining time of repair of the failed machines at time τ.

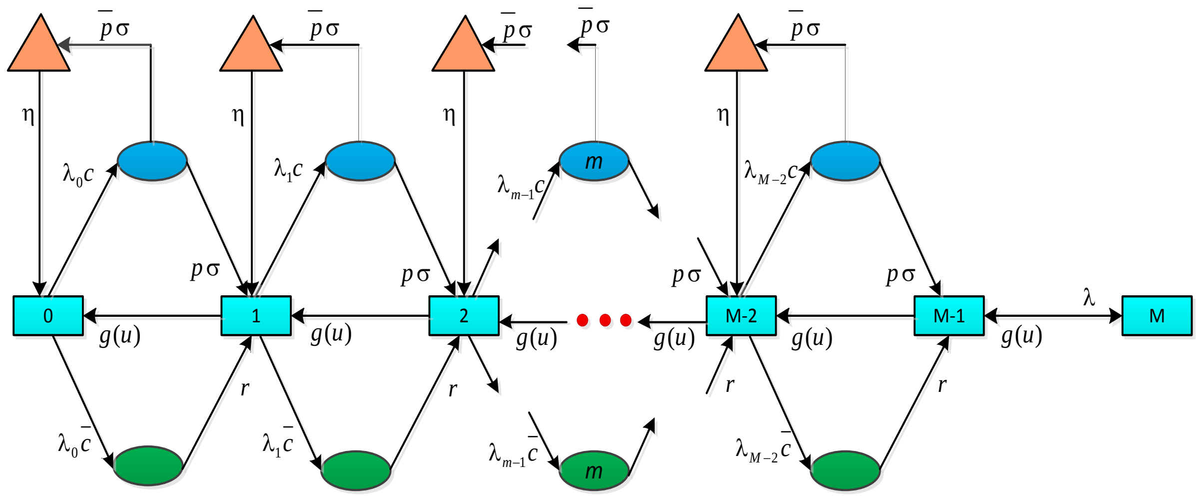


Fig. 9.1: Transition state diagram for M/G/1 FTS

The system state probabilities at time τ for different system modes from operation view point are defined as follows (See fig. 9.1):

(i) Operating state: $P_m(u,\tau)du = \Pr ob.\{\eta(\tau)=m, \xi(\tau)=0,\ u<U(\tau)<u+du\};\ 0\le m\le M$

(ii) Reboot state: $Q_m(\tau) = \Pr ob.\{\eta(\tau)=m, \xi(\tau)=1\}; 1\le m\le M-1$

(iii) Recovery state: $R_m(\tau) = \Pr ob.\{\eta(\tau)=m, \xi(\tau)=2\};\ 1\le m\le M-1$

(iv) Replacement state: $W_m(\tau) = \text{Prob.}\{\eta(\tau)=m, \xi(\tau)=3\}; 1\le m\le M-1$

It is assumed that initially all the machines are in good state. The state dependent failure rate λ_n for component of the fault tolerant repairable system is given as

$$\lambda_m = (M-m)\lambda, m = 0,1,2,...,M-1.$$

9.3 Governing Equations and Queue Size Distribution

The transient state Chapman-Kolmogorov equations for state space $\Omega = [(0,0)\bigcap\{(i,m), i=0,1,2,3;\ m=1,2,3,...,M-1\}\bigcap(0,M)]$ are constructed using the appropriate birth and death rates. After introducing the supplementary variable, the governing equations are framed using the state-transition rate relating to the individual states of the system at time τ and $\tau+d\tau$ as follows:

$$\frac{d}{dt}P_0(\tau) = -(\lambda_0+\lambda_p)P_0(u,\tau)+P_1(0,\tau)+\eta W_1(\tau) \tag{9.1}$$

$$\left(\frac{\partial}{\partial\tau}-\frac{\partial}{\partial u}\right)P_m(u,\tau) = -(M-n)\lambda P_n(u,\tau)+p\theta g(u)Q_n(\tau)+\sigma g(u)R_n(\tau)$$
$$+\eta g(u)W_{n+1}(\tau)+g(u)P_{n+1}(0,\tau), \quad 1\le m\le M-2, \tag{9.2}$$

$$\left(\frac{\partial}{\partial\tau}-\frac{\partial}{\partial u}\right)P_{M-1}(u,\tau) = -\lambda P_{M-1}(u,\tau)+p\theta g(u)Q_n(\tau)+\sigma b(u)R_{M-1}(\tau)+g(u)P_M(0,\tau) \tag{9.3}$$

$$\left(\frac{\partial}{\partial\tau}-\frac{\partial}{\partial u}\right)P_M(u,\tau)=\lambda P_{M-1}(u,\tau) \tag{9.4}$$

$$\frac{d}{dt}R_m(\tau)=-\sigma R_m(\tau)+(M-(m-1))\lambda c P_{m-1}(\tau),\ 1\le m\le M-1 \tag{9.5}$$

$$\frac{d}{dt}Q_m(\tau)=-\theta Q_m(\tau)+(M-(m-1))\lambda c P_{m-1}(\tau),\quad 1\le m\le M-1 \tag{9.6}$$

$$\frac{d}{dt}W_m(\tau)=-\eta W_m(\tau)+(1-p)\sigma R_m(\tau),\qquad 1\le m\le M-1 \tag{9.7}$$

For the steady state, the system state probabilities are denoted by

$$P_m(u)=\lim_{\tau\to\infty}P_m(u,\tau),\ 0\le m\le M;\qquad Q_m=\lim_{\tau\to\infty}Q_m(\tau),\ 1\le m\le M-1;$$

$$R_m=\lim_{\tau\to\infty}R_m(\tau),\ 1\le m\le M-1;\qquad W_m=\lim_{\tau\to\infty}W_m(\tau),\ 1\le m\le M-1.$$

$$P_m=\lim_{\tau\to\infty}P_m(\tau),\ 0\le m\le M;$$

The transient Equations (9.1)–(9.7) can be written in steady state form as follows:

$$0=-M\lambda P_0+\eta W_1+P_1(0) \tag{9.8}$$

$$\begin{aligned}0=\frac{d}{du}P_m(u)-(M-m)\lambda P_m(u)+r\,g(u)Q_m+p\sigma\,g(u)R_m\\ +\eta g(u)W_{m+1}+g(u)P_{m+1}(0),\quad 1\le m\le M-2,\end{aligned} \tag{9.9}$$

$$0=\frac{d}{du}P_{M-1}(u)-\lambda P_{M-1}(u)+r\,g(u)Q_{M-1}+p\,\sigma b(u)R_{M-1}+g(u)P_M(0), \tag{9.10}$$

$$0=\frac{d}{du}P_M(u)+\lambda P_{M-1}(u) \tag{9.11}$$

$$0=(M-(m-1))\lambda(1-c)P_{m-1}-\theta Q_m,\ 1\le m\le M-1 \tag{9.12}$$

$$0=(M-(m-1))\lambda c P_{m-1}-\sigma R_m,\ 1\le m\le M-1 \tag{9.13}$$

$$0=(1-p)\sigma R_m-\eta W_m,\ 1\le m\le M-1 \tag{9.14}$$

Solving (9.12), (9.13) and (9.14), we obtain

$$Q_m=\frac{(M-m+1)\lambda(1-c)}{r}P_{m-1},\ 1\le m\le M-1 \tag{9.15}$$

$$R_m=\frac{(M-m+1)\lambda c}{\sigma}P_{m-1},\ 1\le m\le M-1 \tag{9.16}$$

$$W_m=\frac{(M-m+1)(1-p)\lambda c}{\eta}P_{m-1},\ 1\le m\le M-1 \tag{9.17}$$

In particular when m=1, we have

$$W_1 = \frac{M(1-p)\lambda c}{\eta} P_0 \tag{9.18}$$

Now, using (9.1) and (9.10), we get

$$P_1(0) = M\lambda(1-c(1-p))P_0 \tag{9.19}$$

Denote the following Laplace-Stieltjes transforms (LST) of CDF *G(τ)* and probabilities as

$$L\{G(t)\} = G^*(s) = \int_0^\infty e^{-su} dG(u) = \int_0^\infty e^{-su} g(u)\, du \tag{9.20}$$

$$\text{and } P^*(s) = \int_0^\infty e^{-su} P_m(u)\, du;\ P_m = P_m^*(0) = \int_0^\infty P_m(u)\, du \tag{9.21}$$

Taking Laplace-Stieltjes transforms of (9.9)-(9.11), we obtain

$$\begin{aligned} 0 &= sP_m^*(s) - P_m(0) - (M-m)\lambda P_m^*(s) + p(M-m+1)\lambda c G^*(s)P_{m-1} \\ &\quad + (M-m+1)\lambda(1-c)G^*(s)P_{m-1} + (1-p)(M-m)\lambda c G^*(s)P_m \\ &\quad + G^*(s)P_{m+1}(0),\ 1 \le m \le M-2, \end{aligned} \tag{9.22}$$

$$\begin{aligned} 0 &= sP_{M-1}^*(s) - P_{M-1}(0) - \lambda P_{M-1}^*(s) + p(M-m+1)\lambda c G^*(s)P_{m-1} \\ &\quad + (M-m+1)\lambda(1-c)G^*(s)P_{m-1} + G^*(s)P_M(0), \end{aligned} \tag{9.23}$$

$$0 = sP_M^*(s) - P_M(0) + \lambda P_{M-1}^*(s) \tag{9.24}$$

Setting $s = 0$ in (9.22), we obtain

$$\begin{aligned} 0 = P_{m+1}(0) - P_m(0) + (M-m)\lambda(c(1-p)-1)P_n^*(0) + (M-m+1)\lambda(cp+(1-c))P_{m-1}, \\ 1 \le m \le M-2, \end{aligned} \tag{9.25}$$

Using (9.8) and (9.21) in (9.25) for $m = 1, 2, ..., M-1$, we find

$$P_{m+1}(0) = (M-m)\lambda(1-c(1-p))P_m,\quad 1 \le m \le M-1 \tag{9.26}$$

Setting $s = (M-m)\lambda$ in (9.22), we obtain

$$\begin{aligned} 0 &= -P_m(0) + p(M-m+1))\lambda c G^*\{(M-m)\lambda\}P_{m-1} \\ &\quad + (M-m+1))(1-c)\lambda G^*\{(M-m)\lambda\}P_{m-1} + (1-p)(M-m+1))\lambda c G^*\{(M-m)\lambda\}P_m \\ &\quad + G^*\{(M-m)\lambda\}P_{m+1}(0),\quad 1 \le m \le M-2, \end{aligned} \tag{9.27}$$

Using (9.26) in (9.27), we get

$$P_m = f_m P_0,\quad 1 \le m \le M-2 \tag{9.28}$$

$$\text{where}\quad f_m = \frac{M(1-c(1-p))^m}{(M-m)} \prod_{i=1}^{m} \frac{(1-G^*\{(M-i)\lambda\})}{G^*\{(M-i)\lambda\}};\ 1 \le m \le M-2. \tag{9.29}$$

Using (9.20), (9.22) and (9.24) in (9.12), we obtain

$$Q_m = g_m P_0,\quad 1 \le m \le M-1 \tag{9.30}$$

where $g_m = \dfrac{M(M-m+1)(1-c(1-p))^m(1-c)\lambda}{r(M-m)} \prod_{i=1}^{m} \dfrac{(1-G^*\{(M-i)\lambda\})}{G^*\{(M-i)\lambda\}}; 1 \le m \le M-2.$ (9.31)

Using (9.20), (9.22) and (9.24) in (9.13), we obtain

$$R_m = h_m P_0, \quad 1 \le m \le M-1 \tag{9.32}$$

where $h_m = \dfrac{M(M-m+1))(1-c(1-p))^m \lambda c}{\sigma(M-m)} \prod_{i=1}^{m} \dfrac{(1-G^*\{(M-i)\lambda\})}{G^*\{(M-i)\lambda\}}.$ (9.33)

Using (9.20), (9.22) and (9.24) in (9.14), we obtain

$$W_m = w_m P_0, \quad 1 \le m \le M-1 \tag{9.34}$$

where $w_m = \dfrac{M(M-m+1)(1-c(1-p))^m(1-p)\lambda c}{\eta(M-m)} \prod_{i=1}^{m} \dfrac{(1-G^*\{(M-i)\lambda\})}{G^*\{(M-i)\lambda\}}.$ (9.35)

Setting $s = \lambda$ in (9.23), we get

$$0 = -\lambda(1-c(1-p))P_{M-2} + \lambda((1-c)+cp)G^*(\lambda)P_{M-2} + \lambda(1-c(1-p))G^*(\lambda)P_{M-1} \tag{9.36}$$

Solving (9.36) using (9.26), we obtain

$$P_{M-1} = f_{M-1} P_0 \tag{9.37}$$

where $f_{M-1} = \dfrac{(1-G^*(\lambda))}{G^*(\lambda)} \dfrac{M(1-c(1-p))^{M-2}}{2} \prod_{i=1}^{M-2} \dfrac{(1-G^*\{(M-i)\lambda\})}{G^*\{(M-i)\lambda\}}.$ (9.38)

Now, differentiating (9.23) and (9.24) w. r. t. $'s'$ and then setting $s = 0$, we get

$$\lambda P_{M-1}^{*1} = [1-\beta\lambda(1-c(1-p))]P_{M-1} - 2\lambda\beta((1-c)+cp)]P_{M-2}, \tag{9.39}$$

$$P_M = -\lambda P_{M-1}^{*1}, \tag{9.40}$$

Using (9.28), (9.37) and (9.39) in (9.40), we obtain

$$P_M = f_M P_0 \tag{9.41}$$

where $f_M = \left(2\lambda\beta d - [1-\beta\lambda d]\dfrac{(1-G^*(\lambda))}{G^*(\lambda)}\right) \dfrac{Md^{M-2}}{2} \prod_{i=1}^{M-2} \dfrac{[1-G^*\{(M-i)\lambda\}]}{G^*\{(M-i)\lambda\}}.$ (9.42)

The normalizing condition is used to determine P_0. Thus, we obtain

$$P_0 = \left[1 + \sum_{m=1}^{M-1}(f_{m-1} + g_m + h_m + w_m) + f_{M-1} + f_M\right]^{-1}. \tag{9.43}$$

9.4 Algorithm to Compute Steady State Probabilities

In this section, we outline the computational algorithm to evaluate the state probabilities using recursive approach as:

Input parameters: $\mathrm{M},\ \lambda,\ \mu, \eta, \sigma,\ \mathrm{r}, \mathrm{c}, \text{and}\, \mathrm{p}.$

Step I: Evaluate f_m, g_m, h_m and $w_m (m=1,2,...,M-2)$ using (9.29), (9.31), (9.33), and (9.35), respectively.

Step II: Evaluate f_{M-1} using (9.38) and f_M using (9.42).

Step III: Evaluate P_0 using (9.43).

Step IV: For $m=1$ compute $P_1(0)$ using (9.19). Also evaluate $P_m(0)\,(m=2,3,...,M)$ using (9.26).

Step V: Evaluate for P_m, Q_m, R_m, W_m for $(1 \le n \le M-2)$ and P_{M-1}, P_M using (9.28), (9.30), (9.32), (9.34), (9.37) and (9.41), respectively.

9.5 Availability Analysis of *R*-out-of-*M*:*G* Configuration System

In this section, availability evaluation of *R-out-of-M: G* structure for multi-component machining system is determined for specific distribution of repair times. Now, for illustration, consider a machining system comprising of total *M*=3 operative machines. For parallel configuration, the system works if and only if at least 1 out of total 3 machines are working, and is called a 1-out-of-3: *G* system. We consider a machining system consisting of M machines; the system works if and only if at least R out of total M machines are working. For the special case when $M = 3$, we evaluate the explicit results for the availability indices by considering series and parallel configuration also. To demonstrate the working procedure of evaluating the availability of the system, three different repair time distributions viz. (i) Exponential (ii) deterministic and (iii) 3-stage Erlang are taken into account. For series configuration, all units will be in working mode as such we have 3-out-of-3: *G* system. In case of 2-out-of-3: *G* structure, the system will work if at least 2 out of 3 units are working.

For brevity, we use the following notations to evaluate the analytical results for the availability:

$$\rho = \frac{\lambda}{\mu},\ d = (1 - c(1-p)),\ \ a = 3d\rho$$

9.5.1 Exponential repair time distribution

Consider the exponential distribution for the repair with pdf, CDF and LST of repair time given by $g(u) = \mu e^{-\mu x}$, $G(u) = 1 - e^{\mu x}$ and $G^*(\lambda) = \dfrac{\mu}{\mu + \lambda}$, respectively.

For R-out-of-M: G structure, the availability is obtained using

$$A_{(R\text{-}M)}(\infty) = P_0 + \sum_{m=1}^{M-R} \left(P_m + R_m\right) \tag{9.44}$$

The system availability and steady state probabilities associated to different states of the 3 unit system are evaluated by setting M=3 in previous section as follows:

(a) Operating state probabilities.

$$P_1 = 3aP_0;\; P_2 = 3a\rho P_0;\; P_3 = \left(6a^2 - 3a\rho - 3a^2\rho\right)P_0 \tag{9.45}$$

(b) Reboot state probabilities.

$$Q_1 = \frac{9\lambda(1-c)a}{r}P_0;\; Q_2 = \frac{12\lambda(1-c)a^2}{r}P_0 \tag{9.46}$$

(c) Recovery state probabilities.

$$R_1 = \frac{9\lambda c a}{\sigma}P_0;\; R_2 = \frac{12\lambda c a^2}{\sigma}P_0 \tag{9.47}$$

(d) Replacement state probabilities.

$$W_1 = \frac{9\lambda c(1-p)a}{\eta}P_0;\; W_2 = \frac{12\lambda c(1-p)a^2}{\eta}P_0 \tag{9.48}$$

Here, for M=3, P_0 can be determined using normalizing condition given in (9.43)

$$P_0 = \left(1 + 3a + 6a^2 - 3a^2\rho + 3a\lambda\left(\frac{c}{\sigma} + \frac{(1-c)}{r} + \frac{c(1-p)}{\eta}\right)(3+4a)\right)^{-1} \tag{9.49}$$

The system availability for *R-out-of-3: G (R=1, 2, 3)* configuration is determined using

$$A_{(1-3)}(\infty) = \frac{1 + 3a + 3a\rho + B_1\left(\dfrac{c}{\sigma}\right)}{1 + 3a + 6a^2 - 3a^2\rho + B_1\left(\dfrac{c}{\sigma} + \dfrac{(1-c)}{r} + \dfrac{c(1-p)}{\eta}\right)} \tag{9.50}$$

$$A_{(2-3)}(\infty) = \frac{1 + 3a\left(1 + \dfrac{3\lambda c}{\sigma}\right)}{1 + 3a + 6a^2 - 3a^2\rho + B_1\left(\dfrac{c}{\sigma} + \dfrac{(1-c)}{r} + \dfrac{c(1-p)}{\eta}\right)} \tag{9.51}$$

$$A_{(3-3)}(\infty) = \left(1 + 3a + 6a^2 - 3a^2\rho + 3a\lambda\left(\frac{c}{\sigma} + \frac{(1-c)}{r} + \frac{c(1-p)}{\eta} \right)(3+4a) \right)^{-1} \quad (9.52)$$

where $B_1 = 3a\lambda(3+4a)$.

9.5.2 Deterministic repair time distribution

The system state probabilities and availability of *R-out-of-3:G* configurations system for deterministic repair time distribution are obtained by taking the LST of CDF $G(\mu)$ of repair time defined as

$$G^*(\lambda) = e^{-\frac{\lambda}{\mu}}.$$

(a) Operating state probabilities.

$$\left.\begin{aligned} &P_1 = \frac{3}{2}d\left(e^{2\rho} - 1\right)P_0;\ P_2 = \frac{3}{2}d\left(1 - e^{\rho} - e^{2\rho} + e^{3\rho}\right)P_0; \\ &P_3 = \frac{3}{2}d\left((a-1)(1 - e^{\rho} + e^{3\rho}) + (1+a)e^{2\rho}\right)P_0 \end{aligned}\right\} \quad (9.53)$$

(b) Reboot state probabilities.

$$Q_1 = \frac{9d\lambda(1-c)}{2r}\left(e^{2\rho} - 1\right)P_0;\ Q_2 = \frac{6d^2\lambda(1-c)}{r}\left(1 - e^{\rho} - e^{2\rho} + e^{3\rho}\right)P_0 \quad (9.54)$$

(c) Recovery state probabilities.

$$R_1 = \frac{9d\lambda c}{2\sigma}\left(e^{2\rho} - 1\right)P_0;\quad R_2 = \frac{6d^2\lambda c}{\sigma}\left(1 - e^{\rho} - e^{2\rho} + e^{3\rho}\right)P_0 \quad (9.55)$$

(d) Replacement state probabilities.

$$W_1 = \frac{9d\lambda c(1-p)}{2\eta}\left(e^{2\rho} - 1\right)P_0;\ W_2 = \frac{6d^2\lambda c(1-p)}{\eta}\left(1 - e^{\rho} - e^{2\rho} + e^{3\rho}\right)P_0 \quad (9.56)$$

Also P_0 is given by

$$P_0 = [\,1 + \frac{3}{2}d(a(1 - e^{\rho} - e^{2\rho} + e^{3\rho}) - (1 - e^{2\rho})) + 3d\lambda(\frac{c}{\sigma} + \frac{(1-c)}{r} + \frac{c(1-p)}{\eta})(e^{2\rho} - 1)(\frac{3}{2} + 2d(e^{\rho} - 1))]^{-1} \quad (9.57)$$

The system availability of *R-out-of-3:G* (*R*=1, 2, 3) structure is evaluated as

$$A_{(1-3)}(\infty) = \frac{1 - \frac{3}{2}d\left(e^{\rho} - e^{3\rho}\right) + B_2\left(\frac{c}{\sigma}\right)}{1 + \frac{3}{2}d\left(a\left(1 - e^{\rho} - e^{2\rho} + e^{3\rho}\right) - \left(1 - e^{2\rho}\right)\right) + B_2\left(\frac{c}{\sigma} + \frac{(1-c)}{r} + \frac{c(1-p)}{\eta}\right)} \quad (9.58)$$

$$A_{(2-3)}(\infty)=\frac{1+\frac{3}{2}d\left(e^{2\rho}-1\right)\left(1+\frac{3\lambda c}{\sigma}\right)}{1+\frac{3}{2}d\left(a\left(1-e^{\rho}-e^{2\rho}+e^{3\rho}\right)-\left(1-e^{2\rho}\right)\right)+B_2\left(\frac{c}{\sigma}+\frac{(1-c)}{r}+\frac{c(1-p)}{\eta}\right)} \tag{9.59}$$

$$\begin{aligned}A_{(3-3)}(\infty)=[\,1+\frac{3}{2}d(a(1-e^{\rho}-e^{2\rho}+e^{3\rho})-(1-e^{2\rho}))+3d\lambda(\frac{c}{\sigma}+\frac{(1-c)}{r}\\ +\frac{c(1-p)}{\eta})(e^{2\rho}-1)(\frac{3}{2}+2d(e^{\rho}-1))]^{-1}\end{aligned} \tag{9.60}$$

where $B_2=3d\lambda\left(e^{2\rho}-1\right)\left[\frac{3}{2}+2d(e^{\rho}-1)\right]$.

9.5.3 3-stage Erlang repair time distribution

The system state probabilities and availability of *R-out-of-3:G (R=1, 2, 3)* configuration for 3-stage Erlang distribution for repair time are obtained by taking the LST of CDF $\mathrm{G}(\mu)$ for repair time as

$$G^*(\lambda)=\left(\frac{3\mu}{3\mu+\lambda}\right)^3$$

(a) Operating state probabilities.

$$\left.\begin{aligned}&P_1=a\left(\frac{4}{9}\rho^2+2\rho+3\right)P_0;\;\; P_2=3a\rho\left(\frac{4}{729}\rho^4+\frac{6}{81}\rho^3+\frac{11}{27}\rho^2+\rho+1\right)P_0\\ &P_3=3a\left(2a\left(\frac{4}{27}\rho^2+\frac{2}{3}\rho+1\right)-(1-a)\rho\left(\frac{4}{27}\rho^2+\frac{2}{3}\rho+1\right)\left(\frac{1}{27}\rho^2+\frac{1}{3}\rho+1\right)\right)P_0\end{aligned}\right\} \tag{9.61}$$

(b) Reboot state probabilities.

$$\left.\begin{aligned}&Q_1=\frac{9\lambda(1-c)a}{r}\left(\frac{4}{27}\rho^2+\frac{2}{3}\rho+1\right)P_0\\ &Q_2=\frac{12\lambda(1-c)a^2}{r}\left(\frac{4}{729}\rho^4+\frac{6}{81}\rho^3+\frac{11}{27}\rho^2+\rho+1\right)P_0\end{aligned}\right\} \tag{9.62}$$

(c) Recovery state probabilities

$$R_1=\frac{9\lambda ca}{\sigma}\left(\frac{4}{27}\rho^2+\frac{2}{3}\rho+1\right)P_0,\; R_2=\frac{12\lambda ca^2}{\sigma}\left(\frac{4}{729}\rho^4+\frac{6}{81}\rho^3+\frac{11}{27}\rho^2+\rho+1\right)P_0 \tag{9.63}$$

(d) Replacement state probabilities.

$$\left.\begin{aligned}&W_1=\frac{9\lambda c(1-p)a}{\eta}\left(\frac{4}{27}\rho^2+\frac{2}{3}\rho+1\right)P_0,\\ &W_2=\frac{12\lambda c(1-p)a^2}{\eta}\left(\frac{4}{729}\rho^4+\frac{6}{81}\rho^3+\frac{11}{27}\rho^2+\rho+1\right)P_0\end{aligned}\right\} \tag{9.64}$$

Using normalizing condition, P_0 is obtained as

$$P_0 = [1+3a+6a^2+2a\rho+7a^2\rho+\frac{4a\rho^2}{9}+\frac{35a^2\rho^2}{9}$$
$$+\zeta_2(\frac{a^2\rho^3}{9}+\zeta(\frac{4a^2\rho^2\lambda}{9}))+\zeta_1\zeta(3a\lambda)+\zeta(\frac{4a\rho^2\lambda}{3})]^{-1} \tag{9.65}$$

where $\zeta = \frac{c}{\sigma}+\frac{\bar{c}p}{\eta}+\frac{\bar{c}}{r}$, $\zeta_1 = 3+4a+2\rho+4a\rho$, $\zeta_2 = 11+2\rho+\frac{4\rho^2}{27}$.

The system availability are obtained as

$$\left.\begin{aligned} A_{(1-3)}(\infty) &= [1+3a+5a\rho+\frac{31}{9}a\rho^2+\frac{11}{9}a\rho^3+\frac{2}{9}a\rho^4+\frac{4}{243}a\rho^5+\frac{9ac\lambda}{\sigma}+\frac{12a^2c\lambda}{\sigma} \\ &\quad +\frac{6a\rho c\lambda}{\sigma}+\frac{12a^2\rho c\lambda}{\sigma}+\frac{4a\rho^2c\lambda}{3\sigma}+\frac{44a^2\rho^2c\lambda}{9\sigma}+\frac{8a^2\rho^3c\lambda}{9\sigma}+\frac{16a^2\rho^4c\lambda}{243\sigma}]P_0 \end{aligned}\right\} \tag{9.66}$$

$$A_{(2-3)}(\infty) = \left(1+a(3+2\rho+\frac{4\rho^2}{9})+\frac{a(27+2\rho(9+2\rho))c\lambda}{3\sigma}\right)P_0 \tag{9.67}$$

$$A_{(1-3)}(\infty) = [1+3a+6a^2+2a\rho+7a^2\rho+\frac{4a\rho^2}{9}+\frac{35a^2\rho^2}{9}$$
$$+\zeta_2(\frac{a^2\rho^3}{9}+\zeta(\frac{4a^2\rho^2\lambda}{9}))+\zeta_1(\zeta(3a\lambda))+\zeta(\frac{4a\rho^2\lambda}{3})]^{-1} \tag{9.68}$$

9.6 System Performance Measures

To explore the queueing and reliability characteristics of an M/G/1 fault tolerant machining system at steady state, we formulate various performance indices which can be further used to characterize the system behavior. Now we establish the formulae in terms of steady state probabilities which are already determined in Section 3. Furthermore, the cost function is also established.

(i) Expected number of failed machines in the system is

$$E[N] = \sum_{m=1}^{M-1} m.(P_m+Q_m+R_m+W_m)+(M-1)P_{M-1}+MP_M \tag{9.69}$$

(ii) The long run probabilities that the system is in reboot state, recovery state and replacement state

$$P_{RC} = \sum_{m=1}^{M-1} R_m = \frac{(M-m+1)\lambda c}{\sigma}P_{m-1} \tag{9.70}$$

$$P_{RB} = \sum_{m=1}^{M-1} Q_m = \frac{(M-m+1)(1-c)\lambda}{r}P_{m-1} \tag{9.71}$$

$$P_{RP} = \sum_{m=1}^{M-1} W_m = \frac{(M-m+1)(1-p)\lambda c}{\eta}P_m \tag{9.72}$$

(iii) System Cost Optimization

The total cost incurred per unit time of the system is determined by framing the expected cost function by treating repair rate (μ) as decision variable (Wang and Yang 2009). The cost function $TC(\mu)$ is minimized in order to find the optimal repair rate (μ^*). The cost elements associated with different system metrics are used to frame the cost function $TC(\mu)$. The following cost elements per unit time constitute the cost function:

C_H : Holding cost incurred on each failed machine waiting for the repair.

C_{RB} : Reboot cost incurred to the one unit of the system.

C_{RC} : Cost associated with the recovery of single failed machines.

C_{RP} : Cost associated with the replacement of one failed machine.

C_m : Cost involved in the repair of each failed machine with repair rate μ.

C_c : Cost incurred in detecting each failed machine.

We formulate the cost function as follows:

The optimization problem is

$$TC(\mu) = C_H E(N) + C_{RC} P_{RC} + C_{RB} P_{RB} + C_{RP} P_{RP} + \mu C_m + c\, C_c$$

$$TC(\mu^*) = \min.\ TC(\mu) \tag{9.73}$$

$$\text{subject to:} \quad 0 \le \lambda \le \mu \quad \text{and } 0 < c < 1.$$

The analytical result for the optimal value of μ is not easy to derive due to fact that the total cost $\mathrm{TC}(\mu)$ is non-linear in nature. Quasi-Newton technique is applied to obtain the optimal value of continuous decision variable (μ) until the minimum value of cost function $\mathrm{TC}(\mu)$, say $\mathrm{TC}(\mu^*)$ is attained. Quasi-Newton method is an iterative method with some stopping criterion depending on the tolerance limit and can be used to find the μ^*. The main advantage for implementing this method is its fast convergence and affine invariance. The iterative steps to implement Newton-quasi method are as follows:

(i) Set the initial value of decision variable as μ_0 for $\mathrm{i} = 0$ and the tolerance $\varepsilon = 10^{-7}$.

(ii) Compute total cost $\mathrm{TC}(\mu_0)$ for initial value of μ_0.

(iii) Evaluate the cost gradient $\vec{\nabla}\mathrm{TC}(\mu_{\mathrm{i}}) = \left[\dfrac{\partial \mathrm{TC}}{\partial \mu}\right]_{\mu=\mu_{\mathrm{i}}}$ and

Hessian matrix $\mathrm{H}(\mu_{\mathrm{i}}) = \left[\dfrac{\partial^2 \mathrm{TC}}{\partial^2 \mu}\right]_{\mu=\mu_{\mathrm{i}}}$.

(iv) Determine new trial solution $\mu_{i+1} = \mu_i - \left[H(\mu_i)\right]^{-1} \vec{\nabla} TC(\mu_i)$.

(v) Set $i = i+1$ and repeat steps (iii) and (iv) until $\max\left(\left| \frac{\partial TC}{\partial \mu} \right| \right) < \varepsilon$.

(vi) Find the global minimum value $TC(\mu^*) = TC(\mu_i)$.

9.7 Numerical Simulation

In this section, we present numerical simulation and cost optimization results by taking numerical illustration of the fault tolerant M/G/1machining system with imperfect coverage. For a specific problem of manufacturing system, we consider the application of packing with the help of robots.

9.7.1 Application of M/G/1 fault tolerant system in manufacturing system

The practical applicability of imperfect coverage and replacement processes in fault tolerant M/G/1 machining system is demonstrated by taking an illustration of manufacturing system where robots are used for packing purpose. The normal functioning of the system is done with the help of M operating robots whose failure rate is assumed to be λ robots per day. Whenever any fault occurs in the robots, its fault is detected and diagnosed. Then after, the recovery of failed robots are done with probability c and recovery rate σ; if failed robots are not recovered successfully with probability $(1-p)$ then replacement of the failed robot is done with rate η. If fault is not detected due to imperfect coverage, it is cleared by reboot or reset operation with rate r. To maintain the pre-requisite level of availability of robots in the system, we provide the repair to the failed robots with mean repair rate μ per day.

9.7.2 System availability

This section explores the numerical results of availability for specific configuration *R*-out-of-*M*: *G* (*R*=1,2,3). For facilitating the computational results, we consider the repair time governed the Exponential, Deterministic and 3-stage Erlang distributions. For the numerical computations, software 'MATLAB' is used by fixing default system parameters as:

$$M = 3,\ \lambda = 0.6,\ \mu = 10, \eta = 2, \sigma = 0.6,\ r = 0.7,\ c = 0.5,\ p = 0.6.$$

From the Tables 9.1-9.2, it can be clearly seen that the availability for each of the three configurations *R-out-of-3: G (R=1, 2, 3)* for exponential, deterministic and 3-stage Erlang distributions grows up as the value of reboot rate (r) and repair rate (μ) increase. From figures 9.2(i-iii)-9.3(i-iii), it is seen that the availability $A_{R-3}(\infty)\,(R = 1, 2, 3)$ shows decreasing trend as failure rate λ of operating machines increases.

Table 9.1: Effect of r on the system availability of R-out-of 3: G configuration

Systm config.	1-out-of-3 (Parallel)			2-out-of-3 (Series)			3-out-of-3 (Series)		
r	Exp	E_3	D	Exp	E_3	D	Exp	E_3	D
2.0	0.6182	0.6171	0.6165	0.6087	0.6072	0.6064	0.4661	0.4631	0.4615
2.5	0.6415	0.6403	0.6396	0.6317	0.6300	0.6292	0.4837	0.4805	0.4789
3.0	0.6581	0.6568	0.6561	0.6480	0.6462	0.6453	0.4962	0.4929	0.4912
3.5	0.6705	0.6691	0.6683	0.6601	0.6583	0.6574	0.5055	0.5021	0.5003
4.0	0.6800	0.6786	0.6778	0.6696	0.6677	0.6667	0.5127	0.5093	0.5074

Table 9.2: Effect of μ on the system availability of R-out-of 3: G configuration

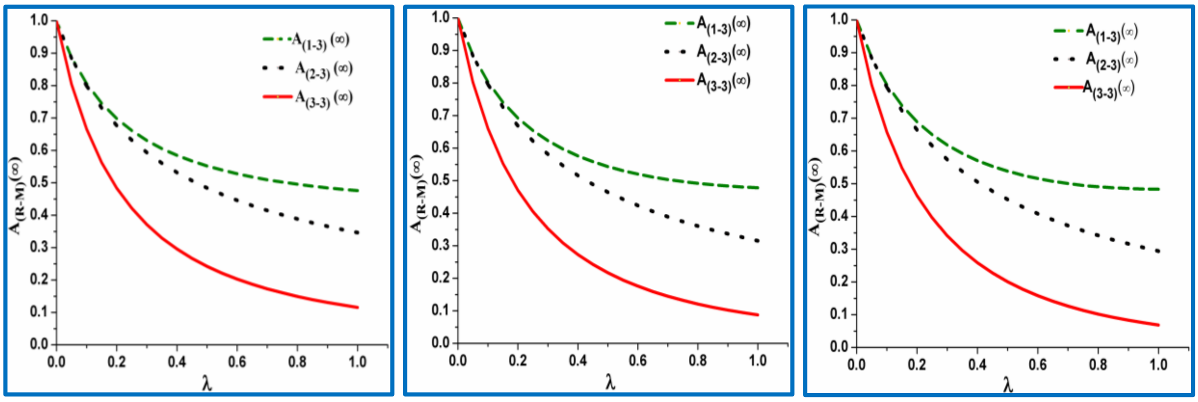
System config.	1-out-of-3 (Parallel)			2-out-of-3 (Series)			3-out-of-3 (Series)		
μ	Exp	E_3	D	Exp	E_3	D	Exp	E_3	D
3.0	0.6340	0.6249	0.6195	0.5935	0.5798	0.5717	0.3709	0.3496	0.3370
3.5	0.6475	0.6398	0.6353	0.6131	0.6019	0.5954	0.3981	0.3803	0.3700
4.0	0.6586	0.6520	0.6482	0.6288	0.6195	0.6142	0.4206	0.4055	0.3969
4.5	0.6678	0.6621	0.6589	0.6416	0.6338	0.6293	0.4394	0.4266	0.4193
5.0	0.6756	0.6707	0.6679	0.6522	0.6455	0.6418	0.4554	0.4444	0.4381

Table 9.3: System indices for three different repair time distributions with failure rate λ

λ	E[N]			P_{RB}			P_{RP}		
	Exp	E_3	D	Exp	E_3	D	Exp	E_3	D
0.5	0.7473	1.1255	1.2119	0.0447	0.0611	0.0617	0.2233	0.3054	0.3087
1	1.2052	1.5598	1.7105	0.0630	0.0697	0.0695	0.3148	0.3486	0.3473
1.5	1.4580	1.7878	1.9413	0.0699	0.0731	0.0730	0.3496	0.3656	0.3651
2	1.6152	1.9173	2.0545	0.0735	0.0754	0.0755	0.3675	0.3771	0.3774

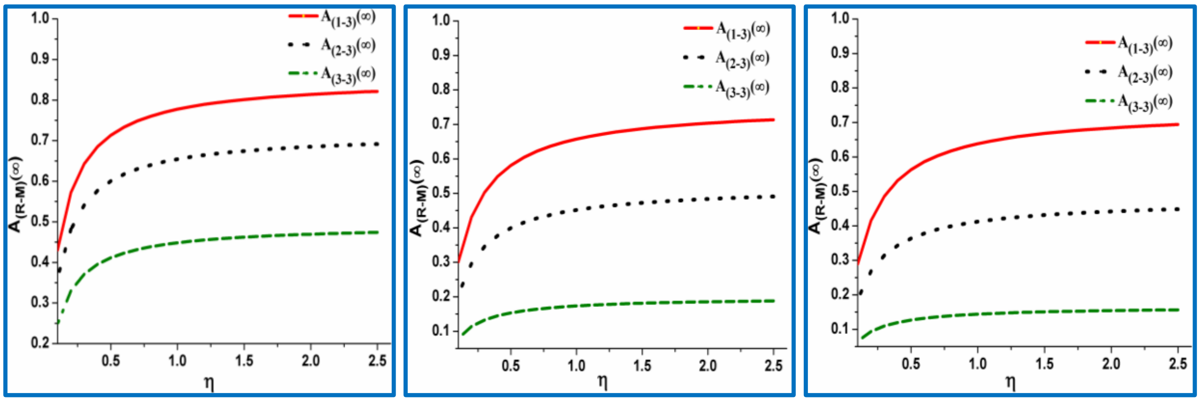
Table 9.4: System indices for three different repair time distributions with failure rate μ

μ	E[N]			P_{RB}			P_{RP}		
	Exp	E_3	D	Exp	E_3	D	Exp	E_3	D
1	1.3254	1.6747	1.8411	0.0583	0.0624	0.0620	0.2914	0.3121	0.3101
2	0.7473	1.1255	1.2119	0.0447	0.0611	0.0617	0.2233	0.3054	0.3087
3	0.4573	0.8697	0.9162	0.0322	0.0580	0.0590	0.1609	0.2899	0.2952
4	0.3015	0.7291	0.7588	0.0233	0.0545	0.0556	0.1164	0.2724	0.2778
5	0.2109	0.6380	0.6594	0.0173	0.0511	0.0521	0.0864	0.2556	0.2607



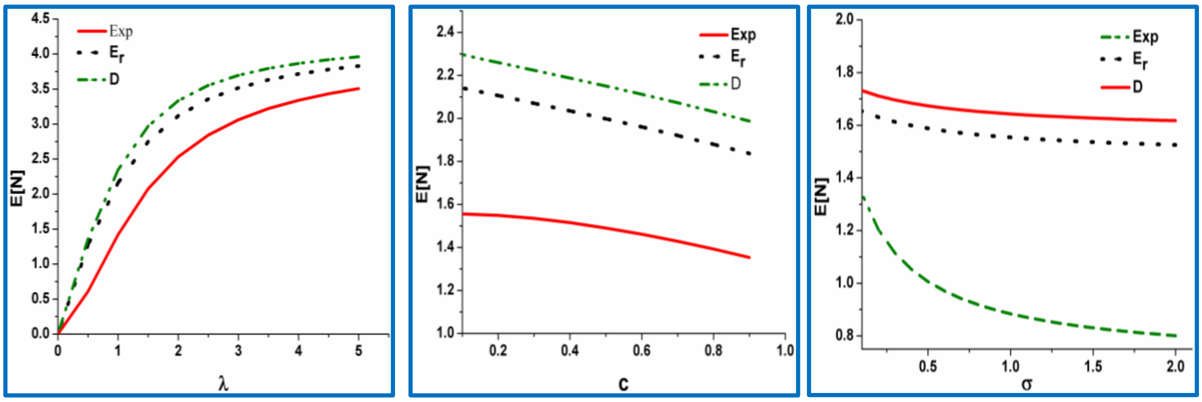
(i) (ii) (iii)

Fig. 9.2: System availability vs λ for different distributions (i) Exponential (ii) 3-stage Erlang (iii) Deterministic



(i) (ii) (iii)

Fig. 9.3: System availability vs η for different distributions (i) Exponential (ii) 3-stage Erlang (iii) Deterministic



(i) (ii) (iii)

Fig. 9.4: Effect of *λ*, *σ* and *c* on *E[N]* for different distributions.

Based on distributions and configuration wise the availability trends obtained are $A^{E}_{R-3}(\infty) > A^{Er}_{R-3}(\infty) > A^{D}_{R-3}(\infty)$ $(R=1,2,3)$, which can be clearly seen in Tables 9.1-9.2 and Figures 9.2(i-iii)-9.3(i-iii).

9.7.3 Performance indices of the system

In this section, the system indices by computing different state probabilities are obtained to characterize the system behavior by fixing default system parameters as $\mathrm{M}=5,\ \lambda=0.5,\ \mu=6, \eta=2, \sigma=3,\ \mathrm{r}=2,\ \mathrm{c}=0.7,\ \mathrm{p}=0.6$. Table 9.1 indicates the increasing trend of mean queue length *E[N]* of the failed machines in the system with the increase in failure rate λ. Figure 9.4(i) depicts that the mean queue length *E[N]* increases rapidly initially for the increasing value of failure rate of machine from $\lambda=0$ to $\lambda=4$. But beyond that, the mean queue length gradually increases for the further increment in the failure rate of machine from $\lambda=4$ to $\lambda=5$. From Table 9.1, it can be noticed that the probabilities of reboot and replacement (P_{RB} and P_{RP}) increase with the increase in the failure rate λ. As we expect, on increasing the repair rate of any real time system, the number of failed machines lowers down. The results obtained in Table 9.2 for *E[N]* with reference to repair rate μ matches with our expectations. The Table 9.2 clearly depicts the decreasing trend of probabilities of reboot and replacement (P_{RB} and P_{RP}) with the increase in repair rate (μ). The impact of fault coverage probability (*c*) can be seen in Figure 9.4(ii) for all the three distributions; The mean queue length of failed machines *E[N]* lowers down with an increment in the fault coverage probability (*c*). The Figure 9.4(iii) clearly shows that the mean queue length *E[N]* slightly decreases on increasing the value of recovery rate (σ).

9.6.4 System cost analysis

The cost function $TC(\mu)$ seems to be non-linear and complex in nature; therefore it is quite tedious task. to optimize such a function analytically. Here, we determine the optimal repair rate (μ^{*}) to minimize the cost function $TC(\mu)$ of the concerned system. The Matlab software is used to minimize the cost function by employing quasi-Newton approach. To minimize the total cost of the concerned system, three sets of cost elements are taken in to consideration as given in Table 9.5. For the computation purpose, the set of default parameters chosen are as

$$M=5, \lambda=0.1, \mu=2.5, \eta=0.9, r=1.8, \sigma=1.5, c=0.8, p=0.6.$$

Table 9.5: Cost elements associated to different states of the system

Cost set	C_H	C_{RB}	C_{RC}	C_{RP}	C_m	C_c
I	\$20	\$10	\$15	\$12	\$8	\$5
II	\$25	\$15	\$15	\$12	\$9	\$5
III	\$30	\$15	\$15	\$12	\$10	\$5

Table 9.6: The iterative results of quasi-Newton method for Exponential distribution for cost set I.

Iteration (i)	μ_i	$TC(\mu_i)$	First order optimality
0	2.5	33.9958	7.1908
1	1.5	27.8062	4.2477
2	1.3557	27.2793	2.9757
3	1.1488	26.9395	0.0254
4	1.147	26.9394	0.0083
5	1.1474	26.9394	1.6×10^{-5}
6	1.1474	26.9394	2.08×10^{-7}

Table 9.7: The iterative results of quasi-Newton method for 3-stage Erlang distribution for cost set I.

Iteration (i)	μ_i	$TC(\mu_i)$	First order optimality
0	2.5000	35.7608	6.6962
1	1.5000	30.1888	3.5314
2	1.0773	29.6591	2.0640
3	1.2332	29.5744	0.7645
4	1.1911	29.5555	0.1204
5	1.1832	29.5550	0.0087
6	1.1837	29.5550	9.18×10^{-5}
7	1.1837	29.5550	2.01×10^{-7}

Table 9.8: The iterative results of quasi-Newton method for Deterministic distribution for cost set I

Iteration (i)	μ_i	$TC(\mu_i)$	First order optimality
0	2.5000	35.8385	6.6252
1	1.5000	30.4456	3.1240
2	1.1574	30.0364	1.4219
3	1.2645	29.9908	0.4727
4	1.2378	29.9837	0.0527
5	1.2345	29.9836	0.0023
6	1.2346	29.9836	1.04×10^{-5}
7	1.2346	29.9836	3.86×10^{-7}

Table 9.9: The minimum cost TC (μ^*) and corresponding optimal repair rate (μ^*)

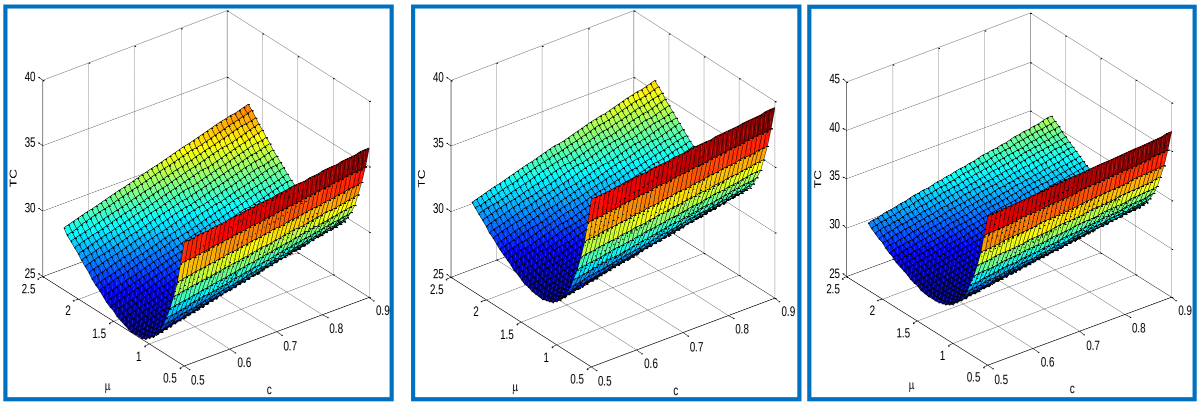
Cost set	Exp	E_3	D
	μ^*, TC(μ^*)	μ^*, TC(μ^*)	μ^*, TC(μ^*)
I	1.1474, \$26.93	1.1837, \$29.55	1.2346, \$29.98
II	1.1944, \$30.56	1.2380, \$33.80	1.2892, \$34.28
III	1.2318, \$34.55	1.2819, \$38.43	1.3333, \$38.97

The quasi-Newton approach is implemented using software Matlab to optimize the cost function. The minimum total costs obtained are presented in Tables 9.6-9.9 and Figs 9.5(i-iii)-9.10(i-iii). From Table 9.6, it is noticed that the optimal repair rate (μ^*) for Exponential distribution (E) is 1.1474 and corresponding minimum cost is TC (μ^*) = \$26.93 for cost set I. For 3-stage Erlang (E_r) and Deterministic (D) distributions, the optimal repair rate and associated minimum cost obtained are (1.1837, 1.2346) and (\$29.56, \$29.98) respectively for cost set I. For the cost set II, the optimal repair rate (μ^*) to the system are 1.1944, 1.2380 and 1.2892 and the corresponding minimum cost incurred are \$30.56, \$33.80 and \$34.29, respectively for Exponential, 3-stage Erlang and Deterministic distributions, respectively.

In Table 9.9, the optimal repair rate (μ^*) and associated minimum cost for cost set III are also displayed for each of the three distributions (i) Exponential, (ii) 3-stage Erlang (E_r) and (iii) Deterministic (D) distributions. The trends of the cost results obtained for cost set III is depicted in Figures 9.7(i), 9.7(ii) and 9.7(iii) for Exponential, 3-stage Erlang (E_r) and Deterministic (D) distributions, respectively.

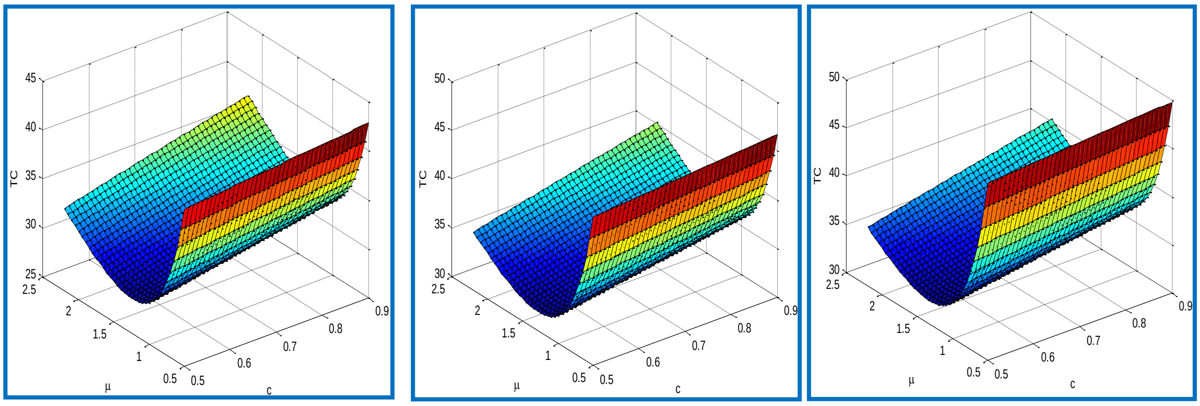
The surface graphs are plotted for the cost results to makes cost study more clear and useful for real time systems. The trends of the total cost *TC* by varying parameters μ, c and η are depicted in Figures 9.5(i-iii)-9.10(i-iii) for all three cost sets and repair time distributions taken as Exponential, 3-stage Erlang (E_r) and Deterministic (D). It is noticed from the Figures 9.5-9.10, that the cost function has the convex nature with respect to the repair rate μ.

The comparative study of optimal repair rate (μ^*) and associated minimum cost TC (μ^*) is done among the three distributions considered and the trend found is Exponential (E) < 3-stage Erlang (E_r) < Deterministic (D).



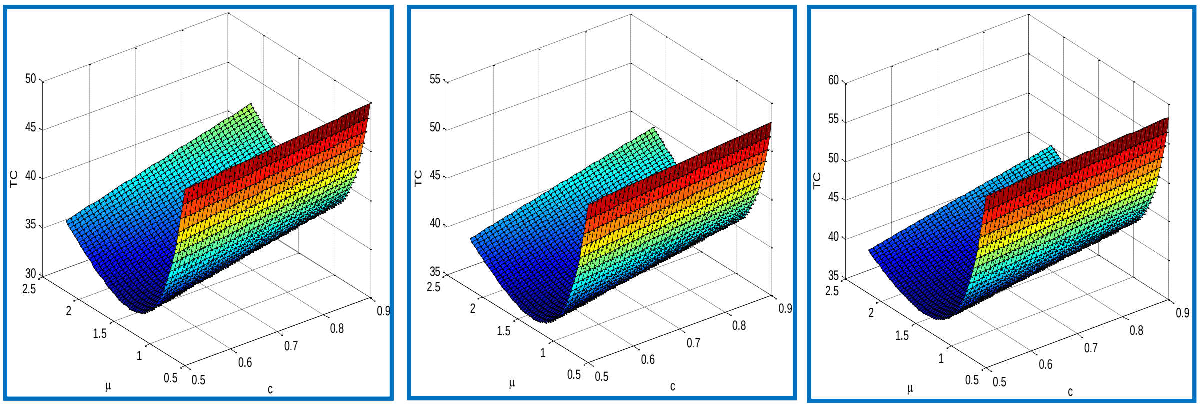
(i) (ii) (iii)

Fig. 9.5: TC vs μ and c (i) Exponential distribution (ii) 3-stage Erlang distribution (iii) Deterministic distribution, for cost set I



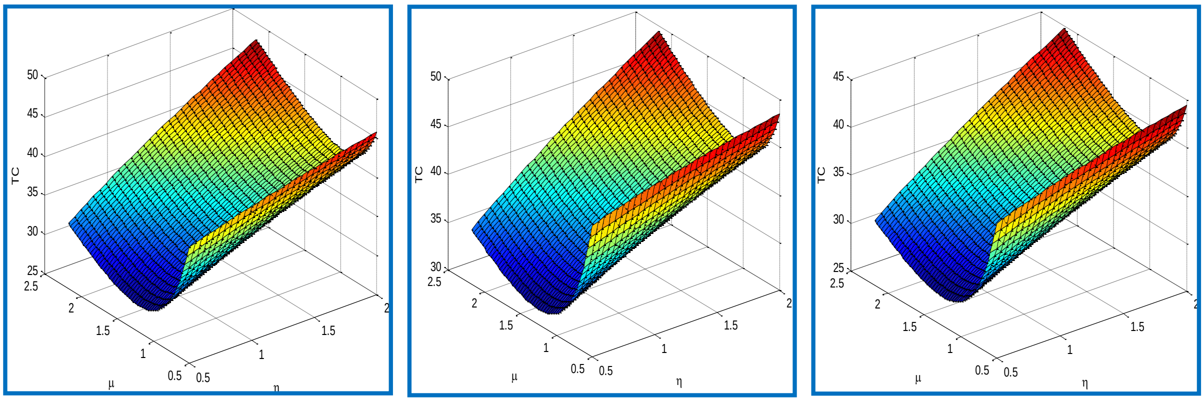
(i) (ii) (iii)

Fig. 9.6: TC vs μ and c (i) Exponential distribution (ii) 3-stage Erlang distribution (iii) Deterministic distribution, for cost set II



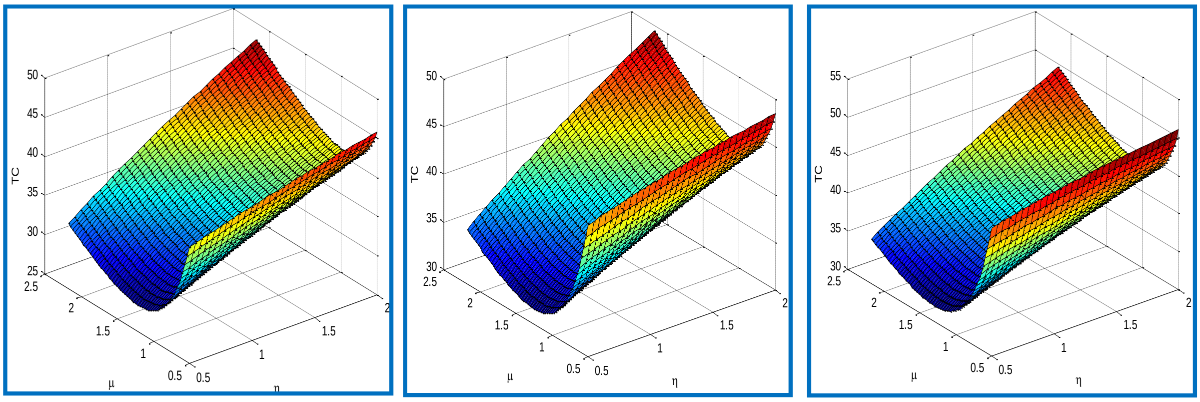
(i) (ii) (iii)

Fig. 9.7: TC vs μ and c (i) Exponential distribution (ii) 3-stage Erlang distribution (iii) Deterministic distribution, for cost set III



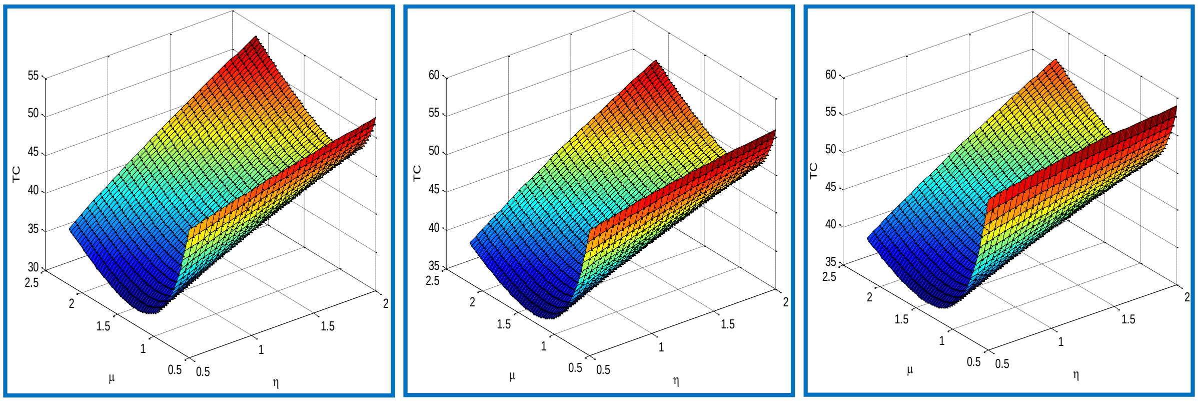
(i) (ii) (iii)

Fig. 9.8: TC vs μ and η (i) Exponential distribution (ii) 3-stage Erlang distribution (iii) Deterministic distribution, for cost set I



(i) (ii) (iii)

Fig. 9.9: TC vs μ and η (i) Exponential distribution (ii) 3-stage Erlang distribution (iii) Deterministic distribution, for cost set II



(i) (ii) (iii)

Fig. 9.10: TC vs μ and η (i) Exponential distribution (ii) 3-stage Erlang distribution (iii) Deterministic distribution, for cost set III

Chapter 10

Availability of M/G/1 FTS with Common Shock Cause Failure

10.1 Introduction

Due to sudden shock, the catastrophic failure may occur in the machining system; the reason of shock may be voltage fluctuation, humidity lightening, human error, thermal issues, etc. The increasingly modern technology of fault tolerance is expected to provide more functionality by maintenance in spite of unavoidable individual and common cause shock failure. The reliability modeling of machining systems exposed to a random shock environment provides more realistic performance metrics for the availability prediction (Lehmann, 2009; Caballé *et al*., 2015). For electronic equipments which are prone to degradation and shock failure, Amy *et al*. (2009) presented a review article on the reliability modeling. Some reliability models under the assumptions of both degradation and random shocks called degradation-threshold-shock models have been developed to study the reliability measures by some researchers (Li and Pham, 2005; Ye *et al*., 2011; Wang and Pham, 2012; Chakravarthy, 2012). Song *et al*. (2016) developed reliability models for the performance analysis of machining system having multi-components in series by considering the shock effects for both hard and soft failures.

In the queueing and reliability literature on machining system, there is no work related to system performability by incorporating shock failure and partially active state in case of minor fault. The non-Markovian models deal with more realistic situation of maintainability as suitable distribution as a specific case of general repair time distribution can be fitted in real time system. The scope of generic non-Markovian model in many real time embedded machining system has motivated us to develop M/G/1 multi-component fault tolerant machining system subject to individual and shock failures by considering the concept of partially active state.

The contents of rest part of the chapter are presented in different sections as follows. The system notations and model description are presented in Section 10.2. In Section 10.3, the probabilities which are further used to design the system metrics, have been obtained using combination of both approaches namely, supplementary variable and recursive method by considering the remaining repair time as supplementary variable. The availability results are presented in Section 10.4 whereas the other system indices and cost function are designed to predict the behaviors of the system in Section 10.5. Next Section 10.6 is devoted to numerical results and evaluation of optimal repair rate using quasi-Newton approach.

10.2 Model Description

The finite capacity M/G/1 fault tolerant machining system comprising of M operating and S warm standby machines is considered by taking the features of partial active and shock failure into account. The assumptions and notations used to develop of the non-Markovian fault tolerant machining model are described as follows.

The operating (warm standby) machines are subjected to failure following Poisson process with parameter $\lambda(a)$. The switching time of the failed operating machine to warm standby after repair is assumed to be negligible. As operating machine fails, it is replaced by the standby machine if available and the replaced standby machines is assumed to have same characteristic as that of failed operating machine. When the fault occurred in the system, it is covered by detaching the failed machine with coverage probability c. In case when the fault is not covered due to imperfect detachment of failed machine, the system goes to partially active state with probability $\bar{c}$. From partial active state, the fault is recovered i.e. the failed machine is removed following the exponential distribution with parameter β. The machining system may also fail due to common cause shock failure with rate λ_p and is recovered from it after repair. To maintain the functioning of the fault tolerant machining system from shock failure, the immediate repair is provided; the shock repair time follows the exponential distribution with parameter μ_p. The repair time of operating machine is assumed to govern by general distribution with cumulative distribution function $B(u)\,(u \geq 0)$, probability density function $b(u)\,(u \geq 0)$, and mean repair rate μ. The repairs of failed operating machines are done according to their failure order i.e. first come first serve (FCFS) basis.

To analyze the non-Markovian M/G/1 fault tolerant multi-component system, we implement supplementary variable technique by introducing $U(t)$ as supplementary variable for the remaining repair time.

Let $\{\eta(t), \xi(t); t \geq 0\}$ be a continuous time bi-variate stochastic process where $\eta(t)$ denotes the number of failed machines in the system at time t and $\xi(t)$ represents the status of the system at time t, which holds values $0, 1$ and 2 for operating, partially active and shock failure mode of the system, respectively. The state transition diagram depicting the in-flow and out-flow rates is shown in Figure 10.1.

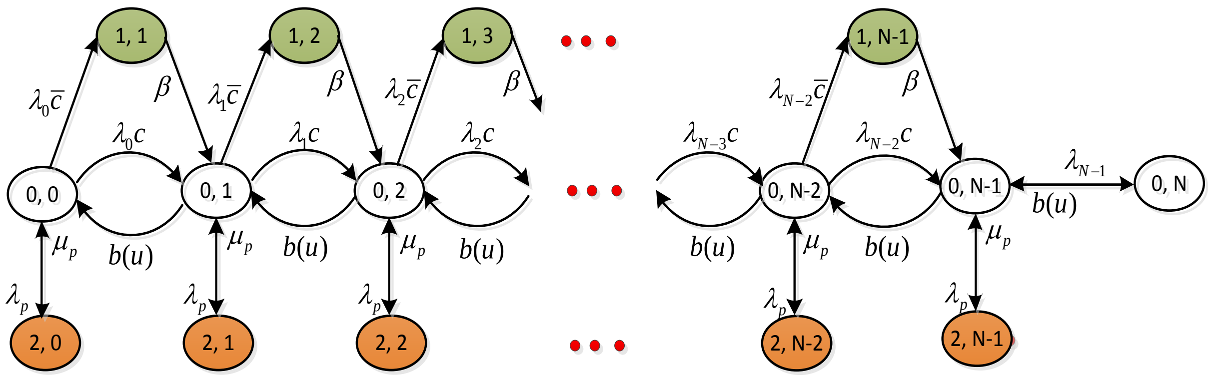


Fig. 10.1: State transition diagram for M/G/1 FTS

Let us define the probabilities of different states as follows:

(i) Active state: $Q_n(u,t)du = \text{Prob.}\{\eta(t)=n, \xi(t)=0,\ u<U(t)<u+du\};\ 0\le n\le N=M+S$

(ii) Partially active state: $R_n(t) = \text{Prob.}\{\eta(t)=n, \xi(t)=1\}; 1\le n\le N-1$

(iii) Common cause shock failure state: $P_n(t) = \text{Prob.}\{\eta(t)=n, \xi(t)=2\};\ 0\le n\le N-1.$

The state dependent failure rate λ_n of the operating machine is defined as:

$$\lambda_n = \begin{cases} M\lambda + (S-n)a, & 0\le n<S \\ M-(n-S), & S\le n<M+S \\ 0, & \text{Othewise} \end{cases}$$

10.3 The Governing Equation and Queue Size Distribution

Chapman-Kolmogorov equations governing the states of the M/G//1 model are framed using the state-transition rate relating to the individual states of the system at time t and $t+dt$ as follows:

$$\frac{d}{dt}Q_0(t) = -(\lambda_0+\lambda_p)Q_0(t) + Q_1(0,t) + \mu_p P_0 \tag{10.1}$$

$$\left(\frac{\partial}{\partial t}-\frac{\partial}{\partial u}\right)Q_n(u,t) = -(\lambda_n+\lambda_p)Q_n(u,t) + c\lambda_{n-1}b(u)Q_{n-1}(u,t) + \mu_p b(u)P_n(t)$$
$$+\beta b(u)R_n(t) + b(u)Q_{n+1}(0,t),\quad 1\le n\le N-1 \tag{10.2}$$

$$\left(\frac{\partial}{\partial t}-\frac{\partial}{\partial u}\right)Q_N(u,t) = \lambda_{N-1}b(u)Q_{N-1}(u,t) \tag{10.3}$$

$$\frac{d}{dt}R_n(t) = (1-c)\lambda_{n-1}Q_{n-1}(t) - \beta R_n(t),\qquad 1\le n\le N-1 \tag{10.4}$$

$$\frac{d}{dt}P_n(t) = -\mu_p P_n(t) + \lambda_p Q_n(t),\quad 0\le n\le N-1 \tag{10.5}$$

Define the steady state probabilities as follows:

$$Q_n(u) = \lim_{t\to\infty} Q_n\left(u,t\right),\ Q_n = \lim_{t\to\infty} Q_n(t) \qquad for \qquad 0 \le n \le N;$$

$$R_n = \lim_{t\to\infty} R_n(t),\ \ 1 \le n \le N-1; P_n = \lim_{t\to\infty} P_n(t),\ \ 0 \le n \le N-1$$

Also,

$$Q_n(u) = b(u)Q_n,\quad 0 \le n \le N$$

Laplace-Stieltjes transforms (LST) of CDF $B(t)$ is denoted by $B^*(s)$, so that

$$L\{B(t)\} = B^*(s) = \int_0^\infty e^{-su} dB(u) = \int_0^\infty e^{-su} b(u)\,du \tag{10.6}$$

Also $$Q^*(s) = \int_0^\infty e^{-su} Q_n(u)\,du;\ Q_n = Q_n^*(0) = \int_0^\infty Q_n(u)\,du \tag{10.7}$$

For different states, the transient Equations (10.1)–(10.6) can be written in steady state form as follows:

$$0 = -(\lambda_0 + \lambda_p)Q_0 + Q_1\left(0\right) + \mu_p P_0 \tag{10.8}$$

$$\begin{aligned} -\frac{d}{du}Q_n\left(u\right) &= -(\lambda_n + \lambda_p)Q_n\left(u\right) + c\lambda_{n-1} b(u)\,Q_{n-1}\left(u\right) + \mu_p b(u)\,P_n \\ &\quad + \beta b(u) R_n + b(u) Q_{n+1}\left(0\right),\quad 1 \le n \le N-1 \end{aligned} \tag{10.9}$$

$$-\frac{d}{du}Q_N\left(u\right) = \lambda_{N-1} b(u)\,Q_{N-1} \tag{10.10}$$

$$0 = (1-c)\lambda_{n-1}Q_{n-1} - \beta R_n,\qquad 1 \le n \le N-1 \tag{10.11}$$

$$0 = -\mu_p P_n + \lambda_p Q_n,\qquad 0 \le n \le N-1 \tag{10.12}$$

Taking Laplace–Stieltjes transform (LST) on both sides of (10.9)–(10.10) and using (10.8), (10.11) and (10.12), we obtain

$$-[sQ^*{}_n(s) - Q_n(0)] = -\lambda_n Q^*{}_n\left(s\right) + \lambda_{n-1} B^*(s)\,Q_{n-1} + B^*(s) Q_{n+1}\left(0\right),\qquad 1 \le n \le N-1 \tag{10.13}$$

$$-[sQ^*{}_N(s) - Q_N(0)] = \lambda_{N-1} B^*(s) Q_{N-1} \tag{10.14}$$

Setting $s = 0$ in (10.13), we obtain

$$Q_{n+1}\left(0\right) = \lambda_n Q_n \tag{10.35}$$

Setting $s = \lambda_n$ in (10.13) and using (10.15),

$$Q_n = \frac{\lambda_{n-1}[1 - B^*(\lambda_n)]}{\lambda_n B^*(\lambda_n)} Q_{n-1},\qquad 1 \le n \le N-1 \tag{10.16}$$

Using (10.16), we have

$$Q_n = \frac{\lambda_0}{\lambda_n}\prod_{i=1}^{n}\frac{[1-B^*(\lambda_i)]}{B^*(\lambda_i)}Q_0, \qquad 1 \le n \le N-1 \tag{10.17}$$

Differentiating (10.13) and (10.14) with respect to s and then setting $s = 0$, we get

$$Q_n = -\lambda_n Q_n^{*^1}(0) - \delta\lambda_{n-1}Q_{n-1} - \delta Q_{n+1}(0) \tag{10.18}$$

$$Q_N = \delta\,\lambda_{N-1}Q_{N-1} \text{where } \delta = -B^{*^1}(0) \tag{10.19}$$

Now, Equation (10.14) and (10.16) yield

$$Q_N = \delta\lambda_0\prod_{i=1}^{n}\frac{[1-B^*(\lambda_i)]}{B^*(\lambda_i)}Q_0 \tag{10.20}$$

Q_0 can be evaluated by using normalizing condition given by

$$Q_0 + \left(\sum_{n=1}^{N-1}(Q_n + R_n) + \sum_{n=0}^{N-1}(P_n)\right) + Q_N = 1 \tag{10.21}$$

10.4 Availability Analysis of R-out-of-N: G Configuration

The structure R-out-of-N: G specifies that if R machines out of total N machines are in good condition, then the system will be in good (i.e. operating) state. The availability of R-out-of-N: G configuration of concerned model is obtained using

$$A_{(R-N)}(\infty) = Q_0 + \sum_{n=1}^{N}Q_n + \sum_{n=1}^{N-1}R_n \tag{10.22}$$

For the illustration purpose, the availability of R-out-of-5: G structures are analytically evaluated for three specific repair time distributions viz. Exponential (Exp), 3-stage Erlang (E_3) and Deterministic (D).

10.4.1 Exponential repair time distribution (Exp):

The CDF and LST for exponential distributed repair time are $B(u) = 1 - e^{\mu x}$ and $B^*(s) = \frac{\mu}{\mu+\lambda}$, respectively. For specific R-out-of-5: G configuration, we find

$$A_{(R-5)}(\infty) = Q_0 + \sum_{n=1}^{4}Q_n + \sum_{n=1}^{3}R_n \tag{10.23}$$

For the availability analysis of 5 unit machining system, the steady state probabilities of different states of the system are determined as follows:

(a) Active state probabilities

$$Q_1 = \frac{\lambda_0}{\mu}Q_0, \quad Q_2 = \frac{\lambda_0\lambda_1}{\mu^2}Q_0, \quad Q_3 = \frac{\lambda_0\lambda_1\lambda_2}{\mu^3}Q_0, \quad Q_4 = \frac{\lambda_0\lambda_1\lambda_2\lambda_3}{\mu^4}Q_0 \tag{10.24a}$$

(b) Partially active state probabilities

$$R_1=\frac{(1-c)\lambda_0}{\beta}Q_0\ ,\ \ R_2=\frac{(1-c)\lambda_0\lambda_1}{\mu\beta}Q_0\ ,\ \ R_3=\frac{(1-c)\lambda_0\lambda_1\lambda_2}{\mu^2\beta}Q_0 \qquad (10.24b)$$

(c) Common cause shock failure state probabilities

$$P_0=\frac{\lambda_p}{\mu_P}Q_0\,,\ P_1=\frac{\lambda_p\lambda_0}{\mu_P\mu}Q_0\,,\ P_2=\frac{\lambda_p\lambda_0\lambda_1}{\mu_P\mu^2}Q_0\ ,\ \ P_3=\frac{\lambda_p\lambda_0\lambda_1\lambda_2}{\mu_P\mu^3}Q_0 \qquad (10.24c)$$

For the brevity, we use notations

$$\rho_i=\frac{\lambda_i}{\mu},\quad 0\le i\le 4 \text{ and } \rho_p=\frac{\lambda_p}{\mu_p}.$$

Now Q_0 can be determined using normalizing condition given in (10.21), as

$$Q_0=\left[1+\rho_0\rho_1\rho_2\rho_3+(1+\rho_p)(\rho_0\rho_1\rho_2+\rho_0\rho_1+\rho_0)+\rho_p+\frac{\lambda_0}{\beta}(1-c)(1+\rho_1\rho_2+\rho_1)\right]^{-1} \quad (10.25)$$

The system availability for *R-out-of-5: G for R=1, 2, 3, 4, 5* are obtained as

(i) $$A_{(1-5)}(\infty)=\left[1+\rho_0\rho_1\rho_2\rho_3+\rho_0\rho_1\rho_2+\rho_0\rho_1+\rho_0+\frac{\lambda_0}{\beta}(1-c)(1+\rho_1\rho_2+\rho_1)\right]Q_0 \quad (10.26a)$$

(ii) $$A_{(2-5)}(\infty)=Q_0=\left[1+\rho_0\rho_1\rho_2+\rho_0\rho_1+\rho_0+\frac{\lambda_0}{\beta}(1-c)(1+\rho_1)\right]Q_0 \qquad (10.26b)$$

(iii) $$A_{(3-5)}(\infty)=\left[1+\rho_0\rho_1+\rho_0+\frac{\lambda_0}{\beta}(1-c)\right]Q_0 \qquad (10.26c)$$

(iv) $$A_{(4-5)}(\infty)=\left[1+\rho_0\right]Q_0 \qquad (10.26d)$$

(v) $$A_{(5-5)}(\infty)=Q_0 \qquad (10.26e)$$

10.4.2 Deterministic repair time distribution

In case of deterministic repair time distribution, the LST of CDF of repair time is defined as

$$B^*(s)=e^{-\frac{\lambda}{\mu}} \qquad (10.27)$$

Now, we obtain the system probabilities as

(a) ***Active state probabilities***

$$\left.\begin{aligned}&Q_1=\frac{\lambda_0}{\lambda_1}\left(e^{\rho_1}-1\right)Q_0,\quad Q_2=\frac{\lambda_0}{\lambda_2}\left(e^{\rho_1}-1\right)\left(e^{\rho_2}-1\right)Q_0\\&Q_3=\frac{\lambda_0}{\lambda_3}\left(e^{\rho_1}-1\right)\left(e^{\rho_2}-1\right)\left(e^{\rho_3}-1\right)Q_0,\ \ Q_4=\rho_0\left(e^{\rho_1}-1\right)\left(e^{\rho_2}-1\right)\left(e^{\rho_3}-1\right)Q_0\end{aligned}\right\} \qquad (10.28a)$$

(b) Partially active state probabilities

$$\left.\begin{aligned} &R_1=(1-c)\frac{\lambda_0}{\beta}Q_0 \;\; ; \;\; R_2=(1-c)\frac{\lambda_0}{\beta}\left(e^{\rho_1}-1\right)Q_0 \\ &R_3=(1-c)\frac{\lambda_0}{\beta}\left(e^{\rho_1}-1\right)\left(e^{\rho_2}-1\right)Q_0 \end{aligned}\right\} \tag{10.28b}$$

(d) Common cause shock failure state probabilities

$$\left.\begin{aligned} &P_0=\rho_p Q_0, \qquad P_1=\rho_p\frac{\lambda_0}{\lambda_1}\left(e^{\rho_1}-1\right)Q_0,\; P_2=\rho_p\frac{\lambda_0}{\lambda_2}\left(e^{\rho_1}-1\right)\left(e^{\rho_2}-1\right)Q_0 \\ &P_3=\rho_p\frac{\lambda_0}{\lambda_3}\left(e^{\rho_1}-1\right)\left(e^{\rho_2}-1\right)\left(e^{\rho_3}-1\right)Q_0 \end{aligned}\right\} \tag{10.28c}$$

Also Q_0 is given by

$$\begin{aligned} Q_0=\Bigg[1&+\left(\frac{\lambda_0}{\lambda_3}+\frac{\lambda_0}{\mu}+\rho_p\frac{\lambda_0}{\lambda_3}+\right)\left(e^{\rho_1}-1\right)\left(e^{\rho_2}-1\right)\left(e^{\rho_3}-1\right)+\left(\frac{\lambda_0}{\lambda_2}+(1-c)\frac{\lambda_0}{\beta}+\rho_p\frac{\lambda_0}{\lambda_2}\right) \\ &\left(e^{\rho_1}-1\right)\left(e^{\rho_2}-1\right)+\left(\frac{\lambda_0}{\lambda_1}+(1-c)\frac{\lambda_0}{\beta}+\rho_p\frac{\lambda_0}{\lambda_1}\right)\left(e^{\rho_1}-1\right)+\rho_p+(1-c)\frac{\lambda_0}{\beta}\Bigg]^{-1} \end{aligned} \tag{10.29}$$

The system availability of *R-out-of-5: G* structure by taking *R*=1, 2, 3, 4, 5 structure is evaluated as

$$\begin{aligned} A_{(1-5)}(\infty)=\Bigg[1&+\left(\frac{\lambda_0}{\lambda_3}+\frac{\lambda_0}{\mu}\right)\left(e^{\rho_1}-1\right)\left(e^{\rho_2}-1\right)\left(e^{\rho_3}-1\right)+\left(\frac{\lambda_0}{\lambda_2}+(1-c)\frac{\lambda_0}{\beta}\right) \\ &\left(e^{\rho_1}-1\right)\left(e^{\rho_2}-1\right)+\left(\frac{\lambda_0}{\lambda_1}+(1-c)\;\frac{\lambda_0}{\beta}\right)\left(e^{\rho_1}-1\right)+(1-c)\;\frac{\lambda_0}{\beta}\Bigg]Q_0 \end{aligned} \tag{10.30a}$$

$$\begin{aligned} A_{(2-5)}(\infty)=\Bigg[1&+\frac{\lambda_0}{\lambda_3}\left(e^{\rho_1}-1\right)\left(e^{\rho_2}-1\right)\left(e^{\rho_3}-1\right)+\frac{\lambda_0}{\lambda_2}\left(e^{\rho_1}-1\right)\left(e^{\rho_2}-1\right) \\ &+\left(\frac{\lambda_0}{\lambda_1}+\bar{c}\frac{\lambda_0}{\beta}\right)\left(e^{\rho_1}-1\right)\left(e^{\rho_2}-1\right)+(1-c)\frac{\lambda_0}{\beta}\Bigg]Q_0 \end{aligned} \tag{10.30b}$$

$$A_{(3-5)}(\infty)=\left[1+\frac{\lambda_0}{\lambda_2}\left(e^{\rho_1}-1\right)\left(e^{\rho_2}-1\right)+\frac{\lambda_0}{\lambda_1}\left(e^{\rho_1}-1\right)+(1-c)\frac{\lambda_0}{\beta}\right]Q_0 \tag{10.30c}$$

$$A_{(4-5)}(\infty)=\left[1+\frac{\lambda_0}{\lambda_1}\left(e^{\rho_1}-1\right)\right]Q_0 \tag{10.30d}$$

$$A_{(5-5)}(\infty)=Q_0 \tag{10.30e}$$

10.4.3 3-stage Erlang repair time distribution

The system state probabilities and availability of *R-out-of-5:G* configuration for 3-stage Erlang distribution for repair time are obtained by using

$$B^*(s)=\left(\frac{3\mu}{3\mu+\lambda}\right)^3. \qquad (10.31)$$

(a) Active state probabilities

$$\left.\begin{aligned}
&Q_1=\frac{\lambda_0}{\lambda_1}\left\{\left(1+\frac{\rho_1}{3}\right)^3-1\right\}Q_0\;,\;\; Q_2=\frac{\lambda_0}{\lambda_2}\left\{\left(1+\frac{\rho_1}{3}\right)^3-1\right\}\left\{\left(1+\frac{\rho_2}{3}\right)^3-1\right\}Q_0\\
&Q_3=\frac{\lambda_0}{\lambda_3}\left\{\left(1+\frac{\rho_1}{3}\right)^3-1\right\}\left\{\left(1+\frac{\rho_2}{3}\right)^3-1\right\}\left\{\left(1+\frac{\rho_3}{3}\right)^3-1\right\}Q_0\\
&Q_4=\rho_0\left\{\left(1+\frac{\rho_1}{3}\right)^3-1\right\}\left\{\left(1+\frac{\rho_2}{3}\right)^3-1\right\}\left\{\left(1+\frac{\rho_3}{3}\right)^3-1\right\}Q_0
\end{aligned}\right\} \qquad (10.32a)$$

(b) Partial active state probabilities

$$\left.\begin{aligned}
&R_1=(1-c)\frac{\lambda_0}{\beta}Q_0,\;\; R_2=(1-c)\frac{\lambda_0}{\beta}\left\{\left(1+\frac{\rho_1}{3}\right)^3-1\right\}Q_0\\
&R_3=(1-c)\frac{\lambda_0}{\beta}\left\{\left(1+\frac{\rho_1}{3}\right)^3-1\right\}\left\{\left(1+\frac{\rho_2}{3}\right)^3-1\right\}Q_0
\end{aligned}\right\} \qquad (10.32b)$$

(e) Common cause shock failure state probabilities

$$\left.\begin{aligned}
&P_0=\rho_p Q_0,\; P_1=\rho_p\frac{\lambda_0}{\lambda_1}\left\{\left(1+\frac{\rho_1}{3}\right)^3-1\right\}Q_0,\\
&P_2=\rho_p\frac{\lambda_0}{\lambda_2}\left\{\left(1+\frac{\rho_1}{3}\right)^3-1\right\}\left\{\left(1+\frac{\rho_2}{3}\right)^3-1\right\}Q_0,\\
&P_3=\rho_p\frac{\lambda_0}{\lambda_3}\left\{\left(1+\frac{\rho_1}{3}\right)^3-1\right\}\left\{\left(1+\frac{\rho_2}{3}\right)^3-1\right\}\left\{\left(1+\frac{\rho_3}{3}\right)^3-1\right\}Q_0
\end{aligned}\right\} \qquad (10.32c)$$

Using normalizing condition, Q_0 is obtained as

$$\begin{aligned}
Q_0=\Bigg[1&+\left(\frac{\lambda_0}{\lambda_3}+\frac{\lambda_0}{\mu}+\rho_p\frac{\lambda_0}{\lambda_3}+\right)\left\{\left(1+\frac{\rho_1}{3}\right)^3-1\right\}\left\{\left(1+\frac{\rho_2}{3}\right)^3-1\right\}\left\{\left(1+\frac{\rho_3}{3}\right)^3-1\right\}\\
&+\left(\frac{\lambda_0}{\lambda_2}+(1-c)\frac{\lambda_0}{\beta}+\rho_p\frac{\lambda_0}{\lambda_2}\right)\left\{\left(1+\frac{\rho_1}{3}\right)^3-1\right\}\left\{\left(1+\frac{\rho_2}{3}\right)^3-1\right\}\\
&+\left(\frac{\lambda_0}{\lambda_1}+(1-c)\frac{\lambda_0}{\beta}+\rho_p\frac{\lambda_0}{\lambda_1}\right)\left\{\left(1+\frac{\rho_1}{3}\right)^3-1\right\}+\rho_p+(1-c)\frac{\lambda_0}{\beta}\Bigg]^{-1}
\end{aligned} \qquad (10.33)$$

The system availability for different system configurations are obtained as

$$A_{(1-5)}(\infty)=\left[1+\left(\frac{\lambda_0}{\lambda_3}+\frac{\lambda_0}{\mu}\right)\left\{\left(1+\frac{\rho_1}{3}\right)^3-1\right\}\left\{\left(1+\frac{\rho_2}{3}\right)^3-1\right\}\left\{\left(1+\frac{\rho_3}{3}\right)^3-1\right\}\right.$$
$$+\left(\frac{\lambda_0}{\lambda_2}+(1-c)\frac{\lambda_0}{\beta}\right)\left\{\left(1+\frac{\rho_1}{3}\right)^3-1\right\}\left\{\left(1+\frac{\rho_2}{3}\right)^3-1\right\} \tag{10.34a}$$
$$\left.+\left(\frac{\lambda_0}{\lambda_1}+(1-c)\ \frac{\lambda_0}{\beta}\right)\left\{\left(1+\frac{\rho_1}{3}\right)^3-1\right\}+(1-c)\ \frac{\lambda_0}{\beta}\right]Q_0$$

$$A_{(2-5)}(\infty)=\left[1+\frac{\lambda_0}{\lambda_3}\left\{\left(1+\frac{\rho_1}{3}\right)^3-1\right\}\left\{\left(1+\frac{\rho_2}{3}\right)^3-1\right\}\left\{\left(1+\frac{\rho_3}{3}\right)^3-1\right\}\right.$$
$$+\frac{\lambda_0}{\lambda_2}\left\{\left(1+\frac{\rho_1}{3}\right)^3-1\right\}\left\{\left(1+\frac{\rho_2}{3}\right)^3-1\right\} \tag{10.34b}$$
$$\left.+\left(\frac{\lambda_0}{\lambda_1}+(1-c)\frac{\lambda_0}{\beta}\right)\left\{\left(1+\frac{\rho_1}{3}\right)^3-1\right\}\left\{\left(1+\frac{\rho_2}{3}\right)^3-1\right\}+(1-c)\frac{\lambda_0}{\beta}\right]Q_0$$

$$A_{(3-5)}(\infty)=\left[1+\frac{\lambda_0}{\lambda_2}\left\{\left(1+\frac{\rho_1}{3}\right)^3-1\right\}\left\{\left(1+\frac{\rho_2}{3}\right)^3-1\right\}+\frac{\lambda_0}{\lambda_1}\left\{\left(1+\frac{\rho_1}{3}\right)^3-1\right\}+(1-c)\frac{\lambda_0}{\beta}\right]Q_0 \tag{10.34c}$$

$$A_{(4-5)}(\infty)=\left[1+\frac{\lambda_0}{\lambda_1}\left\{\left(1+\frac{\rho_1}{3}\right)^3-1\right\}\right]Q_0 \tag{10.34d}$$

$$A_{(5-5)}(\infty)=Q_0 \tag{10.34e}$$

10.5 System Performance Measures and Cost Function

To study and predict the behavior of multi-component fault tolerant machining system, the system performance measures such as mean queue length of failed machines, availability and long run probabilities of different status of the system are formulated.

(i) The mean queue length of failed machines in the system is

$$E(N)=\sum_{n=0}^{N}nQ_n+\sum_{n=1}^{N-2}nR_n+\sum_{n=0}^{N}nP_n \tag{10.35}$$

(ii) The system availability is

$$MA=1-\frac{E(N)}{N} \tag{10.36}$$

(iii) The long run probability of the system being in failed state due to shock failure, normal busy state and reboot state respectively, are

$$P_C=\sum_{n=0}^{N}P_n,\ \ P_B=\sum_{n=0}^{N}Q_n \text{ and } P_{RB}=\sum_{n=1}^{N-2}R_n. \tag{10.37a-c}$$

10.5.1 System cost

Now, we construct a cost function to analyze the cost associated with different activities of the machining system. To make machining system cost-economic, it is necessary to optimize the system parameters of the concerned system. The cost elements incurred per unit time on different activities of the system are considered as:

C_H : Holding cost incurred on each failed machine waiting for the repair.

C_{RB} : Reboot cost incurred per machine of the system.

C_C : Cost incurred on the repair per machine failed due to shock failure.

C_m : Cost involved on the repair per failed machine with repair rate μ.

C_c : Cost incurred in detecting each failed machine.

We construct the cost function TC(μ) by considering repair rate μ as decision variable.

The cost function is given by

$$TC(\mu) = C_H E(N) + C_C P_C + C_{RB} P_{RB} + \mu C_m + c\, C_c \tag{10.38}$$

Now we formulate the cost optimization problem (OP) as follows:

$$\text{(OP)} \quad TC(\mu^*) = \min. TC(\mu)$$

$$\text{subject to:} \;\; 0 \le \lambda \le \mu \;\; \text{and} \;\; 0 < c < 1.$$

It is quite tedious task to optimize $\mathrm{TC}(\mu)$ analytically due to highly complex and non-linear nature of the cost function. Therefore, we employ the numerical approach of optimization viz. Newton-quasi method to determine the minimum expected total cost $\mathrm{TC}(\mu^*)$ and corresponding optimal value of decision variable $'\mu^*'$. The iterative steps described in chapter 9 to implement Newton-quasi method are implemented.

10.6 Numerical Simulation

To reveal the practical applicability of developed M/G/1 model for fault tolerant multi-component machining system, the numerical simulation has been carried out using Matlab software. For the purpose of numerical computations, the default parameters chosen are as follows:

$$\lambda = 0.5, \lambda_p = 0.2, \alpha_1 = 0.01, \alpha_2 = 0.02, \mu = 3, \mu_p = 3, N = 4, M = 2, S = 2, c = 0.6, r = 2.$$

The numerical results are depicted in Figures 10.2-10.3 to explore the $E(N)$ and machine system availability respectively for (i) Exponential (Exp), (ii) 3-stage Erlang (E_3) and (iii) Deterministic (D) distributions for varying values of system parameters λ, c and μ.

It is observed from Figure 10.2 (i) that as failure rate λ of machine grows, the mean queue length $E(N)$ shows significant increment which is quite similar trends as can be noticed in real time machining system. It is also true due to the fact that with the increment in the failure rate of machines, the number of failed machines increases. In Figure 10.2 (ii), the mean queue length $E(N)$ shows slight increment as value of c increases. The decreasing trend of mean queue length $E(N)$ depicted in Figure 10.2 (iii) is quite significant for lower values of repair rate μ i.e. the mean queue length $E(N)$ decreases sharply as repair rate μ increases but deceasing trend diminishes for higher values of μ and finally $E(N)$ becomes almost constant. The system availability (MA) plotted in Figure 10.3 (i-iii) exhibits the reverse trend for varying value of parameter λ, c and μ respectively, i.e. MA decreases (increases) as λ, c (μ) increases.

10.6.1 Availability analysis

In this section, the computational tractability of steady state availability $Av_{R-N}(\infty)$ is validated for three distributions i.e. Exponential (Exp), 3-stage Erlang (E_3) and Deterministic (Det). The default parameters chosen for the numerical results are defined as follows:

$\lambda = 0.2, \lambda_p = 0.04, \alpha_1 = 0.03, \alpha_2 = 0.02, \mu = 1.5, \mu_p = 1, N = 4, M = 2, S = 2, c = 0.9, r = 0.6.$

The numerical results of the availability $Av_{R-N}(\infty)$ are displayed in Figures 10.4 and 10.5 for varying values of λ and μ respectively, for different system configuration.

10.6.2 System cost analysis

The cost benefit analysis plays a significant role to improve the future system design. To determine the economic system cost, we shall obtain the optimal repair rate μ^* using quasi-Newton method for the following cost sets:

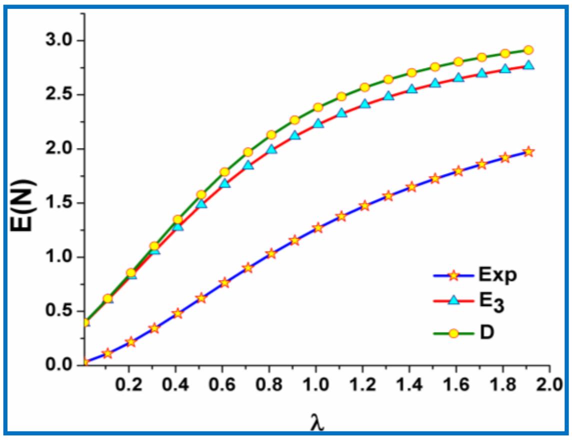
Cost Set I: $C_H = 70, C_B = 20, C_P = 30, C_C = 40,\ C_m = 45,$

Cost Set II: $C_H = 80, C_B = 40, C_P = 30, C_C = 40,\ C_m = 50,$

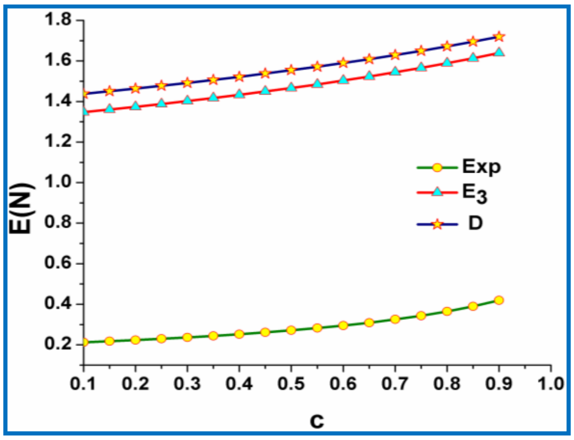
Cost Set III: $C_H = 80, C_B = 45, C_P = 30, C_C = 70,\ C_m = 55.$

The results obtained by implementing quasi-Newton method for three cost sets are summarized in Tables 10.1-10.2 and displayed in Figures 10.6-10.8. The minimum expected and corresponding optimal repair rate (μ^*) for cost sets I, II and III are obtained using quasi-Newton method.

(i)



(ii)



(iii)

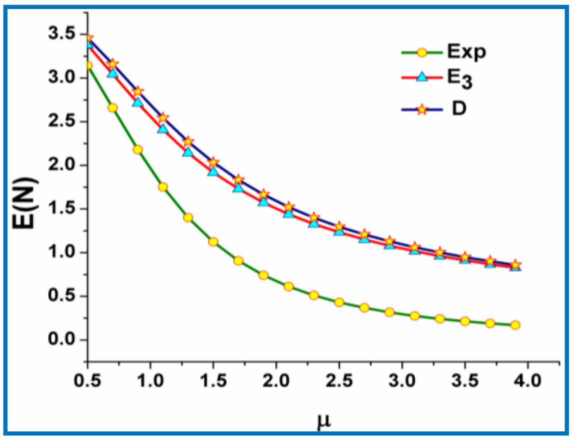
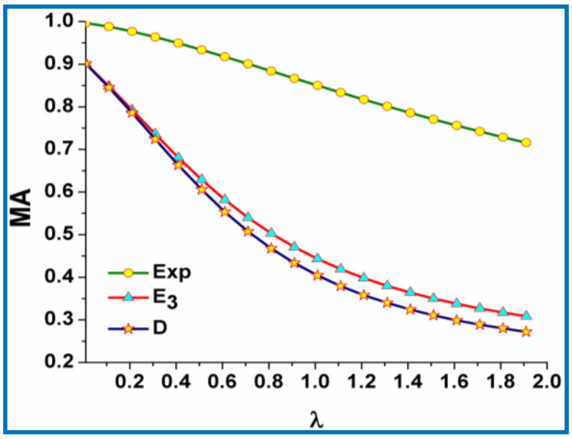
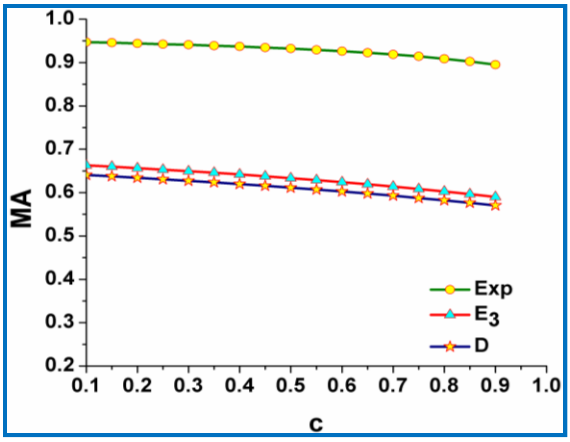


Fig. 10.2: E(N) vs. (i) λ (ii) c and (iii) μ

(i)



(ii)



(iii)

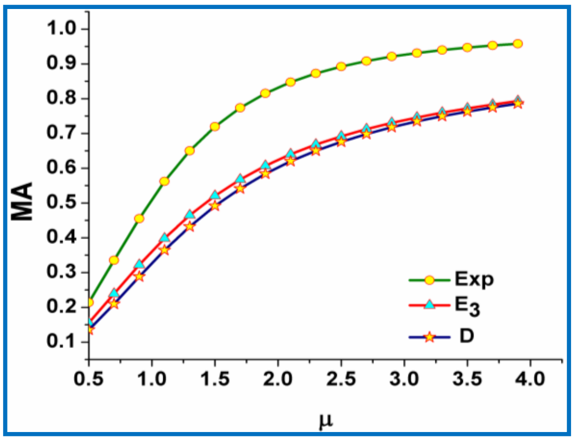
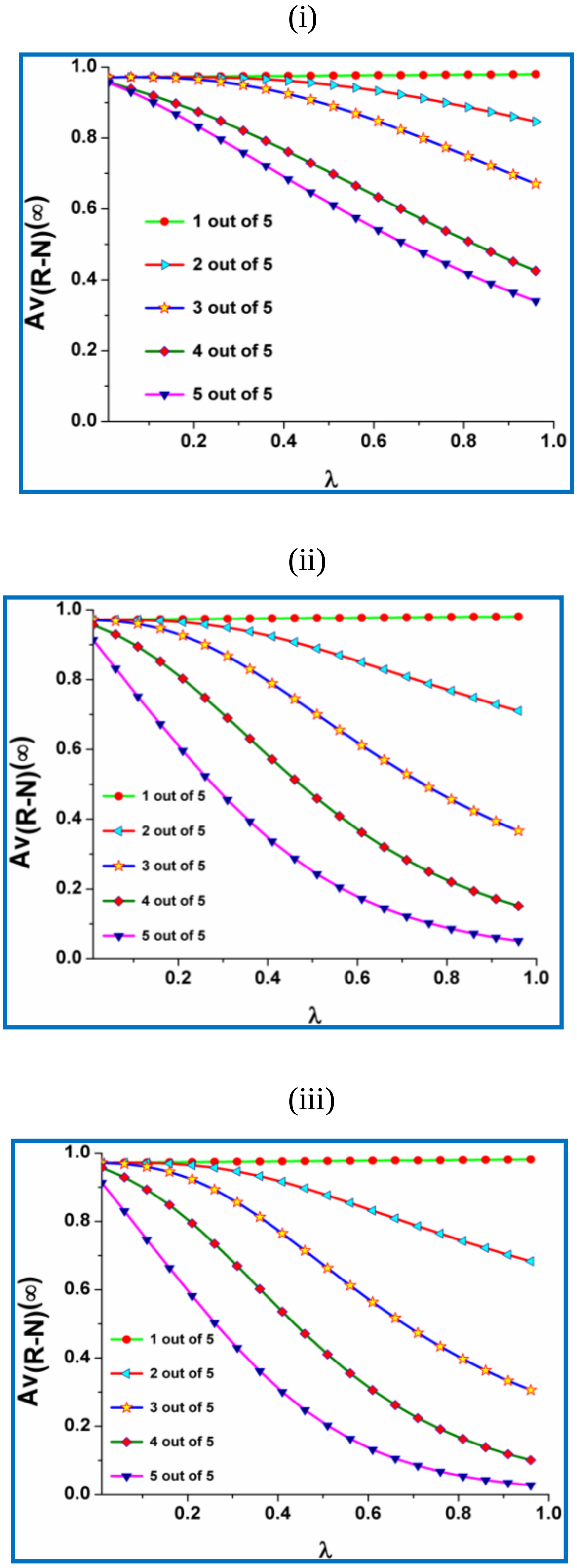


Fig. 10.3: MA vs. (i) λ (ii) c and (iii) μ



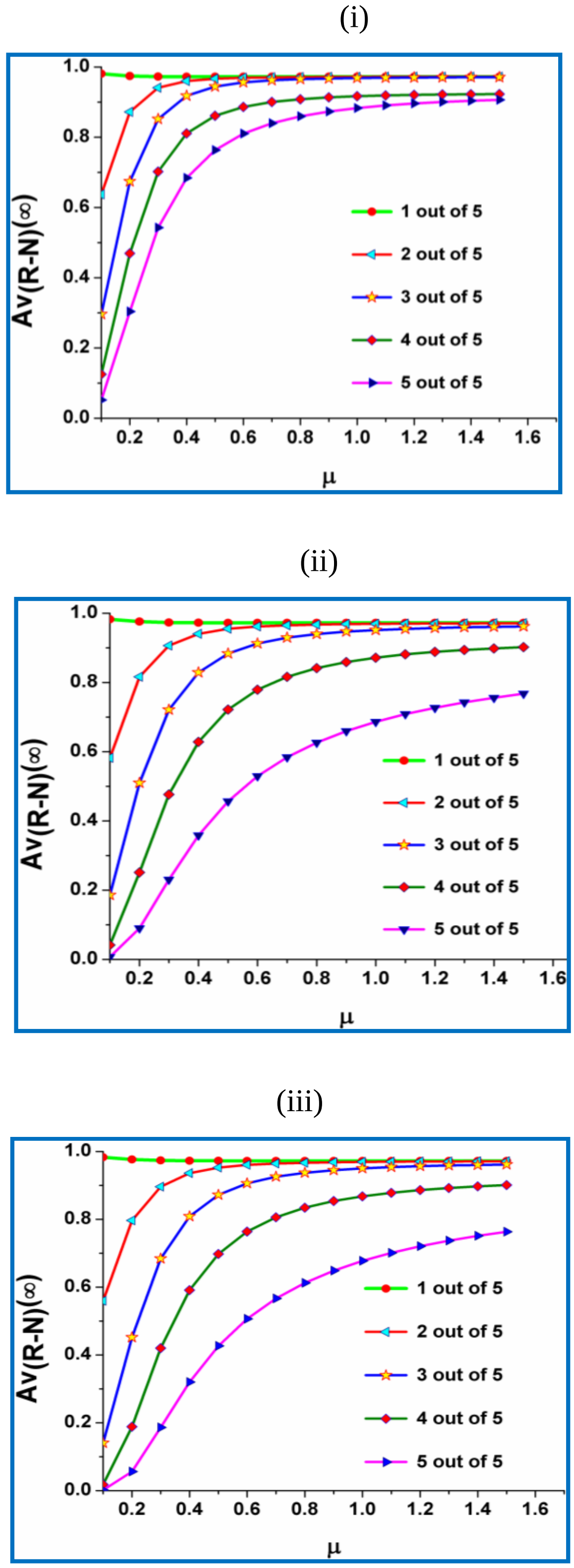


Fig. 10.4: Variations in $Av_{(R-N)}(\infty)$ for different value of λ for (i) Exponential, (ii) 3-stage Erlang and (iii) Deterministic distributions

Fig. 10.5: Variations in $Av_{R-N}(\infty)$ for different value of μ for (i) Exponential, (ii) 3-stage Erlang and (iii) Deterministic distributions

Table 10.1: The iterative results of QNM for different distributions and three cost sets

Cost Sets Dist.		Cost Set I			Cost Set II			Cost Set III		
	Itet. (*i*)	μ_i	TC(μ_i)	Ist Order Optimality	μ_i	TC(μ_i)	Ist Order Optimality	μ_i	TC(μ_i)	Ist Order Optimality
Exp	0	3.00	185.85	33.41507	3.00	205.99	35.0327	3.00	234.55	40.7449
	1	2.00	162.74	6.292147	2.00	183.85	1.272956	2.00	206.27	8.002101
	2	1.89	162.38	0.002514	1.98	183.84	2.89E-06	1.89	205.80	0.003269
	3	1.89	162.38	0.000221	-	-	-	1.89	205.80	0.000301
	4	1.89	162.38	0	-	-	-	1.89	205.80	1.01E-06
E_3	0	3.00	231.49	24.8619	3.00	266.01	24.96106	3.00	291.25	30.71783
	1	2.00	216.64	0.4357	2.00	252.92	3.863119	2.00	272.32	1.999172
	2	1.98	216.64	0.2975	2.13	252.81	2.052461	1.93	272.30	1.450335
	3	1.99	216.64	0.0018	2.09	252.76	0.100149	1.96	272.28	0.033077
	4	1.99	216.64	0	2.09	252.76	0.002781	1.96	272.28	0.000528
	5	-	-	-	2.09	252.76	3.66E-06	1.96	272.28	3.89E-06
D	0	3.00	234.44	23.1001	3.00	269.70	22.84805	-	-	-
	1	2.00	222.40	3.7382	2.00	259.93	8.720655	3.00	294.81	28.63223
	2	2.14	222.28	1.9388	2.28	259.29	3.524157	2.00	279.19	2.876093
	3	2.09	222.23	0.0947	2.20	259.14	0.359621	2.09	279.14	1.585545
	4	2.09	222.23	0.0026	2.19	259.14	0.017306	2.06	279.11	0.0491
	5	2.09	222.23	0	2.19	259.14	8.19E-05	2.06	279.11	0.000879
	6	-	-	-	2.19	259.14	3.49E-06	2.06	279.11	1.85E-06

Table 10.2. Minimum $TC(\mu^*)$ and optimal rate μ^* for cost sets I, II and III

Distributions	Cost set I ($TC(\mu^*)$, μ^*)	Cost set II ($TC(\mu^*)$, μ^*)	Cost set III ($TC(\mu^*)$, μ^*)
Exponential (Exp)	(162.38, 1.89)	(183.84, 1.98)	(205.80, 1.89)
Deterministic (D)	(216.64, 1.99)	(252.76, 2.09)	(272.28, 1.96)
3-stage Erlang (E_3)	(222.23, 2.09)	(259.14, 2.19)	(279.11, 2.06)

The iterative procedure of QNM to examine the iteration counts and minimum cost along with optimal repair rate ($TC(\mu^*), \mu^*$) are summarized in Tables 10.1 and 10.2, respectively. The convex nature of TC with respect to service rate is clearly visible in all Figures 10.6-10.8.

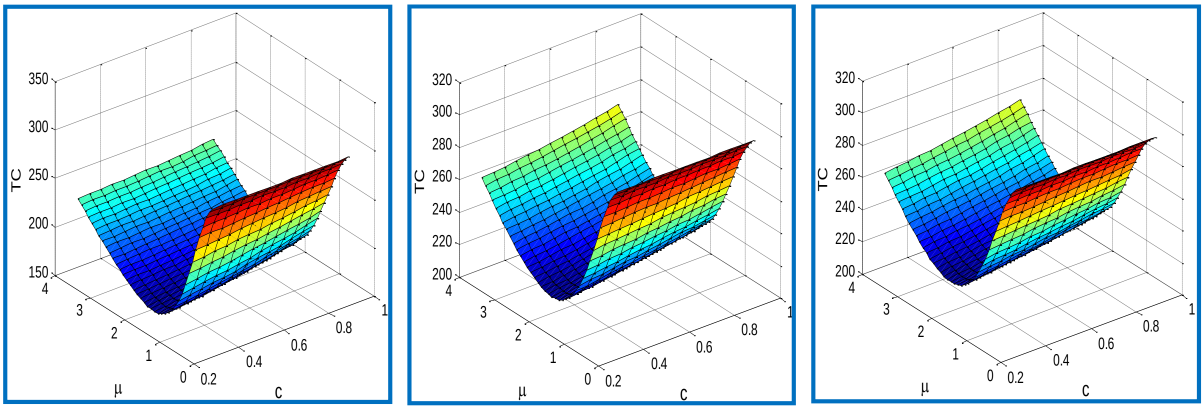


Fig. 10.6: TC vs (μ, c) for cost set I and repair time distributions (i) Exponential (ii) 3-stage Erlang (iii) Deterministic

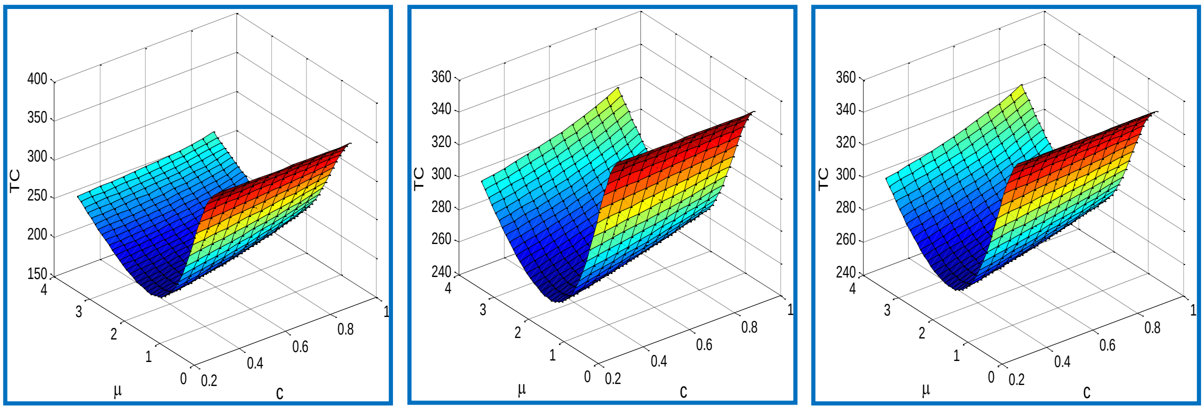


Fig. 10.7: TC vs (μ, c) for cost set II and repair time distributions (i) Exponential (ii) 3-stage Erlang (iii) Deterministic

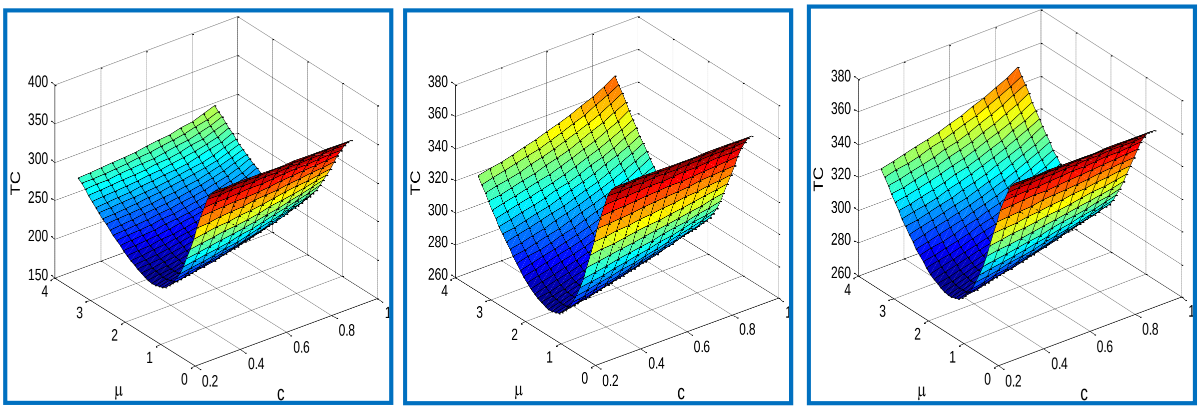


Fig. 10.8: TC vs (μ, c) for cost set III and repair time distributions (i) Exponential (ii) 3-stage Erlang (iii) Deterministic

CONCLUSIONS

The occurrence of service interruption in machining environment is quite common phenomenon. The queueing models developed for repairable machining system with service interruption have enormous applications in the real-time systems such as computer and communications, nuclear power plants, data exchange systems, and automobile repair shops, etc. The research work presented in the thesis will provide a valuable insight to the system analysts/designers to enhance the performability, reliability and availability of the concerned unreliable systems operating in machining environment. To analyze the machining system with service interruption, several queueing and reliability metrics including mean queue size, throughput, waiting time in the system, failure frequency, machine availability, delay time, effective arrival rate, etc. have been established. The highlights of the noble features of work done in the present thesis work are as follows:

- Vacation of the server may be useful to maintain the functioning of real time machining system for a longer time. It is worth noting that the concept of ***server vacation*** is taken into account of all the models developed in chapters 2-8.
- The feature of ***working vacation*** queueing models have been considered in chapters 2-5, and 8 which have numerous applications in real life situations such as computer and communication systems, manufacturing and many others.
- The provision of ***standby support*** in machining systems is taken into account to maintain the smooth function of any real time systems over a long period of time. Based on cost and other physical constraints, the standby support is more effective means and is incorporated in models investigated chapters 4-8 and 10.
- The control policies in the real life scenarios play a crucial role to optimize both time and money. Here in the present thesis work we have incorporated a concept of admission control policy i.e ***F-policy*** to control arrivals which are discussed in chapters 3, 4 and 8.
- The sudden breakdown of server has adverse impact on production and efficiency of any machining system. Therefore, unreliability of server cannot be neglected and thus queueing systems with ***unreliable server*** are more consistent enough with the real time machining systems. The concept of unreliable server presented in chapters 3-8 portrays the modeling of real time machining systems.

- The repair of broken down server is key issue in order to continue the service of the customers waiting in the queue. ***Threshold based recovery policy*** can be used by technicians and maintenance engineers to repair the broken down server. This policy is incorporated in chapter 6.
- To make machining system fault tolerable, the provision of ***reboot*** and ***recovery*** are done by the system designers. The fault tolerable machines are capable of performing its functions even some faults occurred which reduce the risk of sudden breakdown of system production. Here, in the present thesis work we have incorporated ***fault tolerant system*** feature in chapters 4, 5, 7 and 8.
- The feature of ***common cause shock failure*** can be realized in many machining systems due to humidity, temperature change, voltage fluctuation, etc. The concept of common cause failure is some time crucial and should be incorporated while evaluating the system availability or reliability. The common cause shock failure feature is included to formulate model in the chapter 10.
- The ***transient analysis*** is desirable but essential to system designers to understand the nature of system operation at a particular instant of time. Therefore, in the present doctoral work, we have presented transient study of machining system in chapter 2, 5, 6, 7 and 8.
- The hybrid soft computing technique ANFIS employed to compare the numerical results may be used for the design of server controller to provide online performance metrics.
- For the optimal design of repairable machining systems, optimization techniques namely quasi-Newton technique and heuristic search approach are used successfully which reveal the computational tractability to determine the optimal parameters.
- Our investigation of queueing models developed for machining systems with service interruption will provide a valuable insights and also helpful to design and built highly complex and sophisticated real time systems which are used to resolve many real life congestion situations occurred in real time machining systems.

The repairable machining system can be noticed in almost every sphere of life from daily routine activities to various complex real life situations. The queueing modeling of repairable machining systems with service interruption presented in the thesis work have incorporated many realistic features. However, there is scope to enrich the models developed in the present thesis work by incorporating bulk failure/repair. For the optimal design of machining systems developed, we can further extended our study by implementing soft computing techniques for the optimization purpose of system cost and to determine corresponding optimal system parameters. We can also enrich present investigation presented in the thesis by considering the general distribution for life/repair time of the machining parts. The work done in the thesis will be of great importance not only from theoretical point of view but will strongly reflect the practical and managerial implementation.

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