CONTROLLABILITY AND STABILITY OF FRACTIONAL ORDER DYNAMICAL SYSTEMS

Ph. D. THESIS

by

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A THESIS

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CANDIDATE'S DECLARATION

I hereby certify that the work which is being presented in the thesis entitled "CONTROLLABILITY AND STABILITY OF FRACTIONAL ORDER DYNAMICAL SYSTEMS" in partial fulfilment of the requirements for the award of the Degree of Doctor of Philosophy and submitted in the Department of Mathematics of the Indian Institute of Technology Roorkee, Roorkee is an authentic record of my own work carried out during a period from July, 2012 to July, 2017 under the supervision of Dr. N. Sukavanam, Professor, Department of Mathematics, Indian Institute of Technology Roorkee, Roorkee.

The matter presented in this thesis has not been submitted by me for the award of any other degree of this or any other Institutions.

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This is to certify that the above statement made by the candidate is correct to the best of my knowledge.

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The Ph.D. Viva-Voce Examination of **Mr. Srinivasan V,** Research Scholar, has been held on …………………………………

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This is to certify that the student has made all the corrections in the thesis.

Signature of Supervisor **Head of the Department**

Dated:________________

Dedicated

to

My Parents

Abstract

This thesis is concerned with the study of qualitative properties of dynamical systems such as controllability, stability, stabilization and synchronization/anti-synchronization with or without delay.

There are nine chapters in the thesis. Chapter 1 contains introductory matter and literature survey related to stability of first order systems, controllability, stability and chaotic synchronization of fractional-order systems. Preliminaries and some basic definitions are given in Chapter 2, which are required in subsequent chapters.

Chapter 3 concerns the development of asymptotic stability and stabilizability of a class of nonlinear dynamical systems with fixed delay in state variable. New sufficient conditions are established in terms of the system parameters such as the eigenvalues of the linear operator, delay parameter, and bounds on the nonlinear parts. Finally, examples are given to testify the effectiveness of the proposed theory.

In Chapter 4 the stability analysis of a class of fractional order bimodal piecewise nonlinear system is presented. The existence and uniqueness of solution of the system is established by assuming continuity condition involving the state variable and Lipschitz continuity of the nonlinear function with respect to the state variable. Then suitable sufficient conditions for the asymptotic stability of the system has been proposed. Finally, two examples with numerical simulations are given to empirically testify the proposed stability conditions.

In Chapter 5, we consider a class of nonlinear fractional-order control system with delay in state variable. Existence and uniqueness of solution is shown by using method of steps. Then the sensitivity of the state is shown with respect to the initial state and perturbed nonlinear function of the system. Finally, numerical examples are given to validate the

analytical results.

Chapter 6, deals with the development of synchronization and anti-synchronization of a fractional-order delay financial system with market confidence by using an active control approach. Firstly, a Gauss-Seidel like predictor-corrector scheme is proposed to solve fractional-order delay systems. Then numerical comparisons of this scheme with the existing two schemes are shown via an example. Furthermore, numerical simulations are given to show that the financial system has chaotic behaviours for different values of time-delay and fractional-order. Then a suitable active control for synchronization/anti-synchronization of the system has been proposed. Finally, the effectiveness and validity of the proposed control are shown with the help of two numerical simulations for different fractional orders and time-delays.

In Chapter 7, a class of fractional-order $\alpha \in (1, 2]$ semilinear control systems with delay in Banach space is considered. Sufficient condition for exact controllability has been established by using Sadovskii's fixed point theorem and the theory of strongly continuous α-order cosine family. An example is given to illustrate the result.

Chapter 8, is concerned with trajectory controllability of a class of fractional-order $\alpha \in (1, 2]$ semilinear control systems with delay in state variable. The nonlinearity is considered with respect to both state and control variables. Firstly, the existence and uniqueness of the system is proved under suitable conditions on the nonlinear term involving state variable. Then the trajectory controllability of this class of systems is studied using Mittag-Leffler functions and Gronwall-Bellman inequality. Finally, examples are given to illustrate the proposed theory.

The conclusion of the thesis and possible directions of future work is given in Chapter 9.

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Srinivasan V

List of Figures

3.1 Solution and attractor graphs of the system (3.3.1) at $\tau = 8.1$. 45

3.2 Solution and attractor graphs of the system (3.3.1) at $\tau = 20$. 46

3.3 Solution and attractor graphs of the system (3.3.2) at $\tau = 1.6$. 48

3.4 Solution and attractor graphs of the system (3.3.2) at $\tau = 3$. 49

3.5 Solution and attractor graphs of the system (3.3.2) at $\tau = 1.3$. 50

3.6 Solution and attractor graphs of the system (3.3.3) at $\tau = 0.1$. 52

3.7 Solution and attractor graphs of the system (3.3.3) at $\tau = 2$. 53

4.1 Nonlinear mechanical system with fractional damping 58

4.2 Bimodal system 62

4.3 Solution and attractor graphs for (i). 69

4.4 Solution and attractor graphs for (ii). 70

4.5 Solution and attractor graphs for (iii). 71

4.6 Solution and attractor graphs for (iv). 72

5.1 Control and error graphs for different values of ϵ in (5.4.5)-(5.4.6). 94

5.2 Solution and error graphs for different values of ϵ in (5.4.5)-(5.4.6). 96

5.3 Control and error graphs for different values of ζ in (5.4.9)-(5.4.10). 99

5.4 Solution and error graphs for different values of ζ in (5.4.9)-(5.4.10). 101

6.1 Solution of the system (6.2.7). 110

6.2 Error of the system (6.2.7). 111

6.3 Chaotic attractors of the system (6.2.1) with $\alpha = 0.95$ and $\tau = 0.05$. 112

6.4 Chaotic attractors of the system (6.2.1) with $\alpha = 0.8$ and $\tau = 0.09$. 113

6.5 Synchronization of drive system (6.3.1) (black color dotted lines) and response system (6.3.2) (red color lines). 119
6.6 Anti-synchronization of drive system (6.3.1) (black color dotted lines) and response system (6.3.6) (red color lines). 120
6.7 Error systems (6.3.3) and (6.3.7) with $\alpha = 0.95$ and $\tau = 0.05$. 121
6.8 Error systems (6.3.3) and (6.3.7) for different fractional-orders. 122
6.9 Synchronization of drive system (6.3.1) (black color dotted lines) and response system (6.3.2) (red color lines). 123
6.10 Anti-synchronization of drive system (6.3.1) (black color dotted lines) and response system (6.3.6) (red color lines). 124
6.11 Error systems (6.3.3) and (6.3.7) with $\alpha = 0.9$ and $\tau = 0.07$. 125
6.12 Error systems (6.3.3) and (6.3.7) for different fractional-orders. 126

8.1 Nonlinear spring-mass system. 149

List of Tables

3.1 Comparison results. 54

4.1 Efficiency of stability conditions. 67

5.1 Numerical values of control $\widetilde{u}(t)$ in the interval $[0, 2]$ for different values of ϵ in (5.4.8). 93

5.2 Numerical values of $z(t)$ in the interval $[0, 2]$ for different values of ϵ in (5.4.7). 95

5.3 Maximum absolute error between solutions of (5.4.1)-(5.4.2) and (5.4.5)-(5.4.6). 95

5.4 Numerical values of control $\approx{u}$ (t) in the interval $[0, 2]$ for different values of ζ in (5.4.12). 98

5.5 Numerical values of $z(t)$ in the interval $[0, 2]$ for different values of ϵ in (5.4.11). 100

5.6 Maximum absolute error between solutions of (5.4.1)-(5.4.2) and (5.4.9)-(5.4.10). 100

6.1 Infinity norm of the absolute errors between analytical solution of (6.2.7) and numerical solutions of (6.2.7) (using M_1, M_2, M_3). 109

Contents

Abstract **i**

Acknowledgement **iii**

List of Figures **vi**

List of Tables **vii**

1 Introduction **1**

1.1 General introduction 1

1.2 Literature survey 5

1.2.1 Fractional-order systems 5

1.2.2 Stability of dynamical systems 5

1.2.3 Controllability of dynamical systems 8

1.2.4 Synchronization of chaotic systems 12

1.3 Organization of the thesis 13

2 Basic concepts **17**

2.1 Basic concepts of control theory 17

2.1.1 Stability of time-delay system 21

2.2 Fractional system theory 23

2.2.1 Basics of fractional calculus 23

2.2.2 Controllability of finite dimensional fractional-order systems . 25

2.2.3 Stability of finite dimensional fractional-order systems 26

2.2.4 Stabilization of fractional-order semilinear system 27
2.3 Infinite dimensional fractional-order systems 27
2.4 Basics of functional analysis 32

3 Asymptotic stability and stabilizability of nonlinear systems with delay 35
3.1 Introduction 35
3.2 Main results 36
3.2.1 Stability of nonlinear delay systems 36
3.2.2 Stabilization of nonlinear delay systems 42
3.3 Examples 43
3.4 Conclusion 55

4 Stability of a class of fractional order bimodal piecewise nonlinear system 57
4.1 Introduction 57
4.2 Existence and uniqueness of solution 61
4.3 Stability of FOBPNLS 64
4.4 Numerical results 65
4.5 Conclusion 74

5 Sensitivity analysis of nonlinear fractional order control systems with delay 75
5.1 Introduction 75
5.2 Existence and uniqueness of solution 76
5.3 Sensitivity of the solution 79
5.3.1 Dependence of solution on small perturbation of ϕ 79
5.3.2 Dependence of solution on small perturbation of f 86
5.4 Numerical results 90
5.5 Conclusion 97

6 Synchronization and anti-synchronization of a chaotic fractional order financial delay system 103

6.1 Introduction 103
6.2 Dynamics and chaos in a financial system 104
6.2.1 Numerical method 105
6.3 Synchronization and anti-synchronization with delay 114
6.3.1 Synchronization between two identical fractional-order financial systems 114
6.3.2 Anti-synchronization between two identical fractional-order financial systems 116
6.4 Simulation results 118
6.5 Conclusion 127

7 Controllability of fractional order $\alpha \in (1, 2]$ systems with delay 129

7.1 Introduction 129
7.2 Main result 130
7.3 Example 134
7.4 Conclusion 136

8 Trajectory controllability of fractional order $\alpha \in (1, 2]$ systems with delay 137

8.1 Introduction 137
8.2 Existence and uniqueness of solution 138
8.3 Trajectory controllability 141
8.3.1 One dimensional system with linear control term 141
8.3.2 n-dimensional system with nonlinear control term 143
8.4 Application 149
8.5 Conclusion 150

9 Conclusion and future scope 151

9.1 Conclusions 151
9.2 Future scope 152

A Algorithm 155

List of publications 158

Bibliography 159

Chapter 1

Introduction

1.1 General introduction

A **system** is a set of entities which interact with each other and produce various outputs in response to different inputs. **Dynamical system** is a natural or man made system whose instantaneous description or state changes with respect to time or some variable. Aircraft, automobile, population dynamics, financial and economic forecasting, biological system etc. are some well known dynamical systems. In general, the study of dynamical systems fall into three categories, namely, to predict, to explain and to understand the phenomena. In first category, the aim is to predict a future state of the system from observations of its past/present states. In second, the aim is to diagnose what possible past states of the system might have led to the present state of the system. In the third category, the aim is neither to predict the future nor explain the past but rather to provide a theory for the physical phenomena.

Mathematical control theory is a branch of mathematics that deals with the behavior of dynamical systems. In real world problems the underlying dynamics may not predict desired future state or explain the past state. To achieve our desired future or explain past states, we need to change behavior of the dynamics by adding suitable inputs or observations of the dynamical systems. Such dynamical systems are called **control systems**. In our everyday life we encounter several control systems. For examples, our body controls its temperature continuously. It may increase or decrease its temperature when it finds that it is too cold or too hot. Predator populations are known to increase or decrease in response to prey availability. This relationship between prey availability and carnivore populations is one of the delicate balance maintained by nature. There are several man made control systems such as refrigerator, automatic water heater, washing machine, motor vehicles, missiles, etc, which are being used in our everyday life. However, whether a control system is natural or man made, their main aim is to control or regulate a particular variable within certain operating limits.

The concepts of controllability, stability and stabilization plays crucial roles in control theory. A system is said to be **controllable** if by means of an input one can transfer the system from any initial state to any other state in a finite time. A critical point of a dynamical system is said to be **stable**, if every solution which is initially close to it remains close to it for all times. For a control system, one can characterize two types of controls, namely open-loop control and closed-loop control or feedback control. In practice, feedback control is widely used. A linear time invariant control system is said to be **stabilizable** if all the unstable states can be made to have stable dynamics by choosing suitable feedback control.

Delay naturally occurs in most of the real world problems. So dynamical systems represented by ordinary or partial differential equations with delay seems appropriate to model the real world problems. The occurrence of delay in a dynamical system may influence the stability property of the system. As time-delay plays an important role in the stability of dynamical system, the stability and stabilizability analysis of delay systems are also important current topics of research in control theory.

On the other hand fractional-order systems are generalization of the classical integer-order systems. In recent years, many researchers believe that fractional-order systems are more appropriate to model the real world problems. In the following, an example is shown to depict the importance of fractional-order systems. Let us consider the following simple population dynamics

$$ {}^{C}D_t^{\alpha}x(t) \quad = \quad cx(t),\ x(t_0) = x_0, \quad 0 < \alpha \le 1, \tag{1.1.1} $$

where $x(t)$ represents the population at time t, x_0 is the initial population at $t = t_0$, $c =$ (birth rate−death rate), and ${}^C D_t^{\alpha}$ is the Caputo fractional derivative of order $0 < \alpha \le 1$ [see Definition 2.2.3]. The solution of the system (1.1.1) can be written as follows:

$$x(t) = E_{\alpha,1}[c(t-t_0)^{\alpha}]x_0, \quad 0 < \alpha \le 1 \tag{1.1.2}$$

where $E_{\alpha,1}[c(t-t_0)^{\alpha}]$ is the Mittag-Leffler function [see Definition 2.2.4], which will reduce to exponential function $e^{c(t-t_0)}$ when the fractional-order $\alpha = 1$. Suppose that z_1 is the actual population for the year t_1. Then the estimated population for the same year t_1 using (1.1.2) with $\alpha = 1$ (integer-order) may not be close to represents the actual population z_1. However, by taking the fractional-order $0 < \alpha < 1$ in the equation (1.1.2), one can get a suitable α such that $x(t_1)$ is equal to z_1, using (1.1.2). Using this α one may be able to predict to some extent, future population. This illustrates that models based on fractional-order systems will yield better results than the integer-order systems. Therefore, fractional-order systems with delay considered to be natural to model some real world problems. Because of this, controllability and stability of fractional-order delay systems are also important current topics of research in control theory.

The general mathematical formulation of finite dimensional nonlinear delay control system is of the form

$${}^C D_t^{\alpha} x(t) = Ax(t) + Bu(t) + F(t, x(t), x(t-\tau)), \quad p-1 < \alpha \le p, \quad t > 0, \tag{1.1.3}$$

where p is a positive integer, $x(t) = \phi(t)$, $t \in [-\tau, 0]$ and $x^{(i)}(0) = x_i$, $(i = 1, 2, \ldots p-1)$ denotes the i^{th} derivative of x. Here $x(t) \in \mathbb{R}^n$ for each $t > 0$ is the state vector, $u(t) \in \mathbb{R}^m$ is the control, A is a constant $n \times n$ matrix, B is a constant $n \times m$ matrix, $F : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is a nonlinear vector function such that $F(t, 0, 0) = 0$, $\tau > 0$ is a real constant and ϕ is a continuous vector valued history function defined on the interval $[-\tau, 0]$. Here the notation ${}^C D_t^{\alpha}$ represent the Caputo fractional derivative of order $\alpha \in (p-1, p]$.

The solution of (1.1.3) is given by

$$
\begin{aligned}
x(t) &= E_{\alpha,1}[At^{\alpha}]\phi(0) + \sum_{i=1}^{p-1} t^{i} E_{\alpha,i+1}[At^{\alpha}]x_i \\
&\quad + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}[A(t-s)^{\alpha}][Bu(s) + F(s, x(s), x(s-\tau))]\mathrm{d}s,
\end{aligned}
$$

where $E_{\alpha,i}[\cdot]$, $i = 1, 2, \ldots, p$ are the Mittag-Leffler functions with two parameters.

Motivation of the thesis

Many real life problems can be modeled by fractional-order systems with or without delay in finite or infinite dimensional spaces. Most of the systems that arise in practice are nonlinear to some extent, at least over a specific range. Since linear systems are much easier to handle mathematically, the first step in dealing with a nonlinear system is usually, if possible, to linearize it around some nominal operating point. A better approximation to nonlinear system is the semilinear system, that is, a system consisting of a linear part as well as a nonlinear part and can be derived from a general nonlinear system by making a local approximation about some nominal trajectory. There are various properties of the system such as existence, uniqueness and regularity of the solutions, stability of equilibrium points etc. Controllability and stability are important areas of study in Control Theory. In many applications the objective of the control action is to drive the system from one state to another in an optimal fashion. However, before we formulate the question of optimality it is necessary to pose the more fundamental question of whether or not it is possible to reach the final state from any arbitrary initial state. Similarly, for the stability, the question arise if every solution which is initially close to zero solution remains close to zero solution for ever or not. Synchronization of two dynamical systems is an important area of research related to stability theory. In most of the real world problems it is difficult to synchronize trajectories of two identical chaotic systems when they start at nearby initial points encompassed by a small region in phase space. The controllability, stability and synchronization are somehow related concepts in control theory. Exhaustive literature is available on controllability and stability of linear systems. However in case of systems involving nonlinearity, delay, fractional derivatives or impulsive conditions there are lot of scope for improvement of existing results as well as for exploring new directions. So, the above facts gives the motivation to study the controllability, stability and synchronization of different kinds of dynamical systems.

1.2 Literature survey

In this section, firstly we give short literature review on existence, uniqueness of solution and numerical schemes for fractional-order systems. Subsequently, detailed literature review on stability, controllability and chaotic synchronization of several kinds of dynamical systems is given.

1.2.1 Fractional-order systems

The applications of fractional-order systems are in many fields like signal processing, economics, population dynamics, viscoelastic materials, astrophysics and control theory (see, Bagley and Torvik [5], Kilbas et al [61] and Rivero et al [106]). The existence and uniqueness of solutions and numerical schemes for solution of fractional-order dynamical systems can be found in Pitcher and Sewell [103], Diethelm [34], Miller and Ross [90] and Podlubny [104]. In nature a nonzero time delay occurs always between the instants at which a cause and its effects take place. There are many papers available in literature on delay systems. Fractional-order systems with delay have been studied by many authors. In 2008, Benchohra et al [9], in 2008, Lakshmikantham [74], and in 2008, Maraaba et al [87] presented an existence and uniqueness theorem for fractional-order differential equations with delay. In 2011, Bhalekar and Gejji [11] provided a numerical scheme for fractional-order differential equations with delay.

1.2.2 Stability of dynamical systems

The origin of stability theory starts from Russian Mathematician Lyapunov's doctoral thesis work "The general problem of the stability of motion", Moscow University, 1892. In

1966, the translated version of his thesis from Russian to French then French to English is published in [82]. Lyapunov proposed two methods for demonstrating stability. The first method developed the solution in a series which was then proved convergent within limits. Second method (direct method) demonstrates stability of a system using a scalar function with some analogies. Later this type of scalar functions are termed as Lyapunov functions.

In 1958, Razumikhin [105] extended Lyapunov's second method to the delay systems. He obtained sufficient conditions for stability of the n-dimensional delay system $\dot{x} = f(t, x(t), x(t-\tau))$ with $f(t, 0, 0) = 0$ on the basis of first approximation of the system, where $f(t, x(t), x(t-\tau))$ is a holomorphic function of the variables $x(t)$ and $x(t-\tau)$. In 1963, Krasovskii [66] generalized Lyapunov's second method and also extended Lyapunov's second method to the delay systems.

In 1974, Winston [127], studied asymptotic stability of an one-dimensional time varying semilinear perturbed system using a direct method of Razumikhin.

In 1976, LaSalle [75] provided an invariant principle which improves the stability conditions obtained through Lyapunov's direct method for the case of nonlinear nonautonomous systems.

In 1983, Mori et al [93] studied stabilizability of linear systems with state delay using linear feedback and Lyapunov functional. Based on this work, in 1994, Lehman and Shujaee [76] presented delay independent sufficient conditions for stability of nonlinear time-varying functional differential equations.

In 1989, Cheres et al [23], in 1999, Hou et al [50] and in 2006, Aleksandrov and Zhabko [4] discuss the stability and stabilization of delay systems with nonlinear perturbations by using the method of Lyapunov functions in the form of Razumikhin.

In the following papers the authors analyze stability of the delay systems based on the structural properties of the systems. In 1985, Mori [94] obtained several sufficient conditions (delay dependent) for the asymptotic stability of linear time-delay systems. In 2009, Choi [24] obtained sufficient condition for delay independent global asymptotic stability of a class of delay systems. In 2014, Liu et al [78] established a new generalized Halanay inequality and using this they studied asymptotic stability of a class of delay differential systems with time-varying structures and delays.

In many papers Lyapunov-Krasovskii functional has been used to analyse the stability of delay systems. For example, in 1994, Trinh and Aldeen [122], in 1998, Győri et al. [45] and in 2002, Ni and Er [99], presented sufficient conditions for asymptotic stability of linear systems with delayed perturbations using Lyapunov functional. In 2002, Fridman [39] obtained necessary and sufficient conditions for singularly perturbed linear systems with delay. In 2004, Haddad and Chellaboina [46] established asymptotic stability conditions for linear and nonlinear non-negative dynamical systems with delay. In 2010, Syed Ali and Balasubramaniam [115] investigated global exponential stability and the exponential convergence rate for time-delay systems with nonlinear uncertainties. In 2014, Thuan et al [119] studied exponential stabilization of time-varying delay systems with nonlinear perturbations. In 2015, Liu [79] established sufficient condition for asymptotic stability of interval time-varying delay systems with nonlinear perturbations. In 2016 and 2017 Lakshmanan and his coworkers [71–73] addressed stability of several time–delay systems.

A switching system consists of a collection of continuous-time subsystems with a switching law that switches the system dynamics in such a way that only one of the subsystem is active at each time instant. It is desirable to get continuous, piecewise smooth trajectories. Bimodal piecewise linear systems are a class of switching systems with two linear subsystems and a state dependent switching law. Last two decades onwards many authors studied various aspects of linear bimodal piecewise systems. For instance, in 2015, Sahan and Edlem [27] studied necessary and sufficient conditions for well-posedness (the existence and uniqueness of solutions) of a class of bimodal piecewise linear systems. In 2004 and 2008, Camblibel et al [17,18] and in 2014, Eren et al [36] established necessary and sufficient conditions for quadratic stability and stabilization of bimodal piecewise linear system in terms of linear matrix inequalities.

For the stability of fractional-order sytems in 2008, Wen et al [126], in 2012, Chen et al [21] and in 2015, Zhang et al [131] studied sufficient conditions for the local asymptotic stability and stabilization of fractional-order $0 < \alpha < 2$ systems by using Gronwall-Bellman lemma. In 2010, Li et al [77] introduced Mittag–Leffler stability and generalized Mittag–Leffler stability and investigated stability of fractional-order nonlinear systems using Lyapunov direct method. In 2010, Sabatier et al [107] presented three linear matrix inequality

(LMI) conditions based on stability domain deformation, characterization of the instability convex domain and generalized LMI framework, respectively. In 2013, HosseinNia et al [49], obtained two sufficient conditions for stability of fractional-order switching systems in terms of LMIs. First one based on common Lyapunov functions which are generalized to fractioal-order systems and the second one based on frequency domain approach.

1.2.3 Controllability of dynamical systems

In 1960–1963 Kalman [54–56] introduced the concept of controllability for the linear finite dimensional system using a rank condition which depends on controllability grammian matrix.

In 1967, Tarnove [118] introduced a method to obtain controllability of nonlinear systems by investigating the existence of a fixed point of a certain set-valued mapping. He used a fixed-point theorem due to Bohnenblust-Karlin to obtain sufficient conditions for A-controllability of the nonlinear system $\dot{x} = f(t, x, u)$, where A is a nonempty, bounded, closed convex subset of continuous functions. Subsequently this idea was used by Dauer [29] in 1972 to obtain a set of sufficient conditions for controllability of the nonlinear control system of the form $\dot{x} = g(t, x) + k(t, u)$ over a bounded interval $[t_0, t_1]$ in the finite dimensional space using Ky-Fan fixed point theorem.

In 1966, Fattorini [38] generalize the controllability of finite dimensional linear system to a class of control systems in Banach space with the state operator (operating on state variable) as an elliptic partial differential operator in a bounded domain of Euclidean space.

In 1972, Lukes [81] has shown that if a linear system $\dot{x}(t) = A(t)x(t) + B(t)u(t)$ is controllable then the perturbed nonlinear system $\dot{x}(t) = A(t)x(t)+B(t)u(t)+f(t, x(t), u(t))$ is also controllable provided the nonlinear function f is continuous and bounded. He used Schauder's fixed-point theorem to show controllability results. For the case where the nonlinear function f is independent of control parameter u, Vidyasagar [123] arrived the same conclusion with the condition $\lim_{\|x\|\to\infty} \|f(t, x)\|/\|x\| = 0$. Modifying Vidyasagar's approach in 1976, Dauer [30] showed that Vidyasagar's condition can be further relaxed. In 1990, Do [35] modified the conditions of Dauer and proved a theorem under which Dauer results easily followed.

A system $\dot{x} = f(x, u)$ is symmetric means that $f(x, -u) = -f(x, u)$. In 1974, Brunovsky [16] proved local controllability for a kind of symmetric nonlinear systems.

For the controllability of a class of nonlinear systems, in 1975 Hirschom [47], obtained an explicit expression for the reachable set.

In 1975, Triggiani [120] extended the classical controllability and observability results in finite-dimensional spaces to the linear abstract systems defined on infinite-dimensional Banach spaces, under the assumption that the operator acting on the state was bounded. Then in 1977, [121] he has proved that exact controllability in finite time for linear control systems given on an infinite dimensional Banach space in integral form (mild solution) can never arise using locally L_1-controls, if the associated C_0 semigroup is compact for all $t > 0$. Based on Triggiani's work in 1983, Louis and Wexler [80] revealed some further restrictive features of the exact controllability concept in the setting of evolution equations in Hilbert spaces.

In 1987, Naito [97] studied approximate controllability of a class of nonlinear systems using Schauder's degree theorem with assumption that the nonlinear function F is uniformly bounded. In 1989, Naito [96] studied approximate controllability of a class of nonlinear systems by replacing uniformly bounded condition on the nonlinear operator F to a inequality condition $1 - kTM\|P\|\exp(kTM) > 0$ along with the conditions $F(0) = 0$ and kT is sufficiently small, where $k > 0$ is a Lipschitz constant of the function F, $T > 0$ is a control time, $M > 1$ is a bound of the semigroup $S(t)$, P is a projective type operator introduced by estimating the control efficiency of the operator B. In 1989, Naito and Park [98] extended this results to a delay Volterra control system.

In 1995, George [41] obtained set of sufficient conditions to the approximate controllability for the time-varying system using the theory of monotone operators and operators of type (M). In 2007, Sharma and George [109] provided a necessary and sufficient condition to the controllability of matrix second-order control systems and presented a numerical scheme for the computation of control.

In 1996, Klamka [62] formulated sufficient conditions for constrained exact local controllability of nonlinear infinite dimensional control system. He proved controllability results using generalized open mapping theorem. In 2008, [63] he presented necessary and sufficient conditions for different kinds of controllability to the control systems considered in both finite and infinite dimensional settings. In 2009, he extended in [64] his earlier results on constrained controllability to the delay systems.

In 2004, Dauer and Mahmudov [31] obtained approximate controllability conditions for the semilinear evolution systems using Rothe's fixed point theorem with assumption that the operator acting on the state is compact. Then in the same paper they obtained exact controllability conditions for semilinear systems with nonlinearity having small Lipschitz constants using Banach fixed point theorem. For this case they assumed the operator acting on the state is not compact. In 2008, Mahmudov [84] studied approximate controllability of semilinear control systems with nonlocal conditions in Hilbert spaces using Schauder and Nussbaum fixed point theorems with the assumption that the linearized system is approximate controllable.

In 2005, Joshi and Kumar [52] examined computation of optimal control for the exact controllability problem governed by the linear parabolic differential equations. Based on this work, in 2007 Kumar et al. [67] provided computation of optimal control for the nonlinear parabolic differential equations. In 2014, Sonawane et al [111] obtained a set of sufficient conditions for the exact controllability of the wave equation with multiplicative controls.

The controllability of fractional-order systems with or without delay have been studied by many authors. In particular, in 2008, Adams and Hartley [1] studied the finite time controllability of fractional-order systems under the condition that lead to a system output remaining at zero with zero input. In 2013, Surendra Kumar and Sukavanam [69] studied controllability of fractional-order system using integral contractor which is a weaker condition than the Lipschitz condition. In 2012, Balachandran and Kokila [7] obtained sufficient conditions for controllability of linear and nonlinear fractional dynamical systems in finite dimensional spaces using Schauder's fixed point theorem. In 2014, Mahmudov and Zorlu [85] extended their earlier results on approximate controllable of semilinear evolution systems to the semilinear fractional-order evolution systems using Schauder fixed point theorem. In 2014, Zhang and Liu [132] studied the controllability of fractional functional differential equations with nondense domain using integrated semigroup theory and

Schauder's fixed point theorem. In 2012, Wei [125] obtained necessary and sufficient conditions for the controllability of fractional linear control systems with control delay. In 2012, Surendra Kumar and Sukavanam [68] studied the existence and uniqueness of mild solution of a class of semilinear fractional-order control systems with delay in state variable using contraction principle and the Schauder fixed point theorem. In 2013, Debbouche and Torres [32] presented existence of solution and approximate controllability of a class of delay fractional systems with nonlocal initial condition in a Hilbert space using Schauder's fixed point theorem. In 2013, Kamaljeet and Bahuguna [57] studied the controllability of the impulsive finite time-delay fractional-order systems with nonlocal condition in Banach spaces. They obtained controllability results using condensing operator and Sadovskii's fixed point theorem via semigroup theory. In 2016, Kamaljeet and Bahuguna [58] and Kamaljeet et al [59] studied approximate controllability of a class of delayed fractional-order control systems with nonlocal condition in Hilbert spaces using Krasnoselskii's fixed point theorem. For the controllability of fractional systems of order $\alpha \in (1, 2]$ with nonlocal conditions: in 2013, Kexue et al [60] used Sadovskii's fixed point theorem to establish sufficient conditions. In 2015, Shukla et al [110] studied sufficient conditions for approximate controllability of the delay fractional systems of order $\alpha \in (1, 2]$ using sequential approach.

A system is said to be trajectory controllable if and only if it is possible, by means of an input, to transfer the system from any initial state to any other desired state along a prescribed trajectory. Therefore trajectory controllability is stronger notion of controllability. In recent years many authors studied trajectory controllability of various kind of dynamical systems. In particular, in 2010, Chalishajar et al. [19] studied trajectory controllability of abstract nonlinear integro–differential system in finite and infinite dimensional space settings. In 2013, Bin and Liu [12] studied trajectory controllability of semilinear evolutions equations with impulses and delay. In 2015, Klamka et al. [65] investigated the trajectory controllability of finite–dimensional semilinear systems with point delay in control and in nonlinear term. In 2016, Govindaraj et al [43] discussed the trajectory controllability of fractional–order $\alpha \in (0; 1]$ systems.

1.2.4 Synchronization of chaotic systems

Synchronization (anti-synchronization) of two systems means that the trajectories of one of the systems will converge to the same values (same values with opposite sign) as the other and they will remain in step with each other in due course [102]. Applications of chaos and synchronization are in many fields like secure communication, modeling brain activity, chemical reactions, ecological systems and financial systems. In literature several types of synchronization have been proposed. Some of them are adaptive feedback control (Zhu and Cao [134] and Odibat [101]), active control (Ho et al [83] and Yassen [129]), back-stepping design method (Wu and Lu [128]) and sliding mode control (Yau [130] and Faieghi and Delavari [37]). In the last few decades many authors focused their research on chaos and synchronization of fractional–order systems with or without time-delay. In particular, in 2010, Bhalekar and Gejji [10], in 2011, Taghvafard and Erjaee [116], in 2013, Agrawal et al [3] and in 2014, Srivastava et al [114] studied synchronization of different fractional–order chaotic systems using active control method. In 2012, Agrawal et al [2] studied synchronization of two different pairs of fractional-order systems namely (i) Lotka-Volterra chaotic system (master system) and Newton-Leipnik chaotic system (slave system) and (ii) Lotka–Volterra chaotic system (master system) and Lorenz chaotic system (slave system), respectively using an active control approach. In 2016, Soukkou et al [113] proposed a fractional–order prediction based feedback control scheme to stabilize unstable equilibrium points and to synchronize the fractional–order chaotic systems. In 2005, Tang [117] studied synchronization of different fractional–order time delayed systems using active control approach.

A history of eccentric behaviours may affect present and future states of the economic dynamical system. Therefore, considering fractional-order financial system with state delay seems more realistic to model a economic problem. Recently, in 2008, Chen [22] investigated dynamic behaviours of a fractional-order financial system. In 2017, Huang and Cao [51] discussed synchronization and anti–synchronization of fractional–order financial system with market confidence using active control strategy. In 2011, Zhen et al [133] investigated complex dynamical behaviours of the fractional–order financial delay system for different fractional–orders and time–delay by numerical simulations, in which they consider the dynamics involving the variables interest rate, investment demand and price index.

In this thesis some results on controllability, stability, stabilization and synchronization/anti-synchronization of dynamical systems with or without delay are presented.

1.3 Organization of the thesis

This thesis consists of nine chapters. **Chapter 1** contains introductory matter and literature survey related to controllability, stability, stabilization and chaotic synchronization of various dynamical systems. Preliminaries and some basic definitions are given in **Chapter 2**, which are required in subsequent chapters.

In **Chapter 3**, we focus on asymptotic stability and stabilization of a class of nonlinear delay systems of the form

$$\left.\begin{array}{rcl} \dot{x}(t) & = & Ax(t) + f(x(t-\tau)) + g(x(t)), \quad t > 0 \\ x(t) & = & \phi(t), \quad t \in [-\tau, 0], \end{array}\right\}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, A is a constant $n \times n$ matrix, $f, g : \mathbb{R}^n \to \mathbb{R}^n$ are the nonlinear vector functions, $\tau > 0$ is a real constant, ϕ is the continuous vector valued history function defined on $[-\tau, 0]$. In general, Lyapunov–Krasovskii stability theory and Razumikhin stability theory are the common approaches to study stability of nonlinear delay systems. Suppose the Lyapunov candidates are simple quadratic functionals, one can easily check negativity of the derivative of Lyapunov candidates using linear matrix inequalities. Constructions of Lyapunov functionals which give stability conditions become very difficult for systems having complicated nonlinear functions. To overcome these difficulties, in this study, we use Gronwall-Bellman lemma and some simple inequalities to obtain sufficient conditions for the delay dependent asymptotic stability and stabilization of some systems. The stability conditions given in this chapter are new and improves some results available in literature for certain class of nonlinearity. These sufficient conditions are established in terms of the system parameters such as the eigenvalues of the linear operator A, delay parameter τ, and bounds on the nonlinear parts f and g. Then some examples are given to testify the effectiveness of the proposed theory.

Chapter 4 is concerned with the stability analysis of a class of fractional-order bimodal piecewise nonlinear system of the form

$$ {}^{C}D_t^{\alpha} x = \begin{cases} A_1 x(t) + f(t, x(t)) & \text{if } c^T x(t) \geq 0 \\ A_2 x(t) + f(t, x(t)) & \text{if } c^T x(t) \leq 0 \end{cases} \tag{1.3.1} $$

with the initial state $x(0) = x_0$, where $x \in \mathbb{R}^n$ is the state, $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is a nonlinear vector function and all other vectors/matrices are of appropriate dimensions.

In literature only stability of linear bimodal systems have been considered so far. In this chapter we consider the stability properties of fractional semilinear bimodal system. The existence and uniqueness of solution for the system (1.3.1) is established by assuming continuity condition involving a state variable x and Lipschitz continuity of the nonlinear function f with respect to x. Then suitable sufficient conditions for the asymptotic stability of (1.3.1) has been proposed. Finally, three examples with some numerical simulations are given to testify the proposed stability conditions.

In **Chapter 5**, we consider a class of nonlinear fractional-order $0 < \alpha < 1$ one-dimensional delayed control system of the form (1.1.3) with the nonlinear function $F \equiv f(t, x(t-\tau))$. By assuming that the nonlinear function f is Lipschitz continuous with respect to the state variable, the existence and uniqueness of solution and the explicit form of the control function are shown in terms of Mittag-Leffer functions using method of steps. Then by using Gronwall's inequality the sensitivity of the state and control with respect to small perturbation of the history function ϕ and small perturbation of the nonlinear function f are shown. The analytical results are substantiated by numerical examples.

Chapter 6, deals with the development of synchronization and anti-synchronization of fractional-order delay financial system with market confidence by using an active control approach. The assumed system is a four-dimensional system of the form (1.1.3) with $A \equiv 0$ and $B \equiv 0$. Firstly, a Gauss-Seidel like predictor-corrector scheme is proposed to solve fractional-order delay systems. Then numerical comparisons of this scheme with the existing two schemes are shown via an example. Furthermore, numerical simulations are given to show that the financial system has chaotic behaviours for different values of time-delay and fractional-order. Finally a suitable active control for synchronization/anti-synchronization of the system has been proposed. The effectiveness and validity of the proposed theory are shown with the help of two numerical simulations for different fractional-orders and time-delays.

In **Chapter 7**, a class of fractional-order $\alpha \in (1, 2]$ semilinear delay control system of the form (1.1.3) with $F \equiv f(t, x(t-\tau))$ in Banach space setting is considered. Sufficient condition for exact controllability have been established by assuming suitable conditions on operators A, B and f. The controllability results are proved using Sadovskii's fixed point theorem and the theory of strongly continuous α-order cosine family. An example is given to illustrate the result.

In **Chapter 8**, a class of fractional-order $\alpha \in (1, 2]$ delay systems of the form (1.1.3) with a nonlinear term $B(t, u(t))$ in place of $Bu(t)$ is considered. Firstly, the existence and uniqueness of solution of the system is proved under suitable conditions on the nonlinear term involving state variable. Then the trajectory controllability of this class of systems is studied using Mittag-Leffler functions and Gronwall-Bellman inequality. The trajectory controllability results are proved in two cases. In case (i), we assume the system to be of dimension one in which the nonlinear control term $B(t, u(t)) = b(t)u(t)$ and the operator A is a simple real scalar a. In this case, the trajectory controllability is proved by assuming the function $b(t)$ is continuous and non-vanishing on the given interval $[0, T]$ and the nonlinear term f is Lipschitz continuous. In case (ii), we considered n-dimensional system with a nonlinear control term $B(t, u(t))$. In this case, the trajectory controllability is proved by assuming that $B(t, u(t))$ satisfies monotonicity and coercivity conditions and f is Lipschitz continuous. Then two examples are given to illustrate the proposed theory.

Finally, the conclusion of the thesis and possible directions of future work are given in **Chapter 9**.

Chapter 2

Basic concepts

In this chapter, basic definitions and preliminaries are given in three sections. In Section 2.1, basic concepts of control theory in finite dimensional setting are given. Some basic definitions of fractional calculus with the preliminary concepts related to controllability, stability and stabilization of fractional-order systems are given in Section 2.2. In Section 2.3, preliminary concepts of infinite dimensional system theory are given. Finally, some basics tools of functional analysis are given in Section 2.4.

2.1 Basic concepts of control theory

The general mathematical formulation of finite dimensional first-order linear control system is of the form

$$\left.\begin{aligned} \dot{x}(t) &= A(t)x(t) + B(t)u(t), \quad t_0 \le t \le T \\ x(t_0) &= x_0, \end{aligned}\right\} \tag{2.1.1}$$

where the state $x(t) \in \mathbb{R}^n$ and the control $u(t) \in \mathbb{R}^m$ for each $t \in [t_0, T]$. Here $A(t)$ and $B(t)$ are $n \times n$ and $n \times m$ matrices, respectively with piecewise continuous elements.

Let $x(\cdot)$ and $u(\cdot)$ belongs to the function spaces $L_2([t_0, T]; \mathbb{R}^n)$ and $L_2([t_0, T]; \mathbb{R}^m)$, respectively. The solution of the control system (2.1.1) is given by (using variation of parameters method)

$$x(t) = \Phi(t, t_0)x_0 + \int_{t_0}^{t} \Phi(t, s)B(s)u(s)\mathrm{d}s, \tag{2.1.2}$$

where $\Phi(t, t_0)$ is an $n \times n$ matrix, called the state transition matrix of the homogeneous system $\dot{x} = A(t)x(t)$. The state transition matrix satisfies the Volterra integral equation

$$\Phi(t, t_0) = I + \int_{t_0}^{t} A(s)\Phi(s, t_0)\mathrm{d}s,$$

where I is the identity matrix of order $n \times n$. By means of the Picard iteration method this leads to Peano-Baker series,

$$\Phi(t, t_0) = I + \int_{t_0}^{t} A(s)ds + \int_{t_0}^{t} A(s_1) \int_{t_0}^{s_1} A(s_2)\mathrm{d}s_2\mathrm{d}s_1 + \cdots . \tag{2.1.3}$$

If A is independent of time then the series (2.1.3) reduces to $e^{A(t-t_0)}$. In general, the state transition matrix satisfies the following properties [15]:

(i) $\Phi(t, \tau)\Phi(\tau, t_0) = \Phi(t, t_0)$,

(ii) $\Phi(t, t) = I$,

(iii) $\Phi^{-1}(t, t_0) = \Phi(t_0, t)$, and

(iv) $\frac{\mathrm{d}}{\mathrm{d}t}\Phi(\tau, t_0) = A\Phi(\tau, t_0)$.

Definition 2.1.1 (Controllability)**.** *The system* (2.1.1) *is said to be controllable on* $[t_0, T]$*, if for every pair of vectors* $x_0, x_T \in \mathbb{R}^n$*, there exists a control* $u(\cdot) \in L_2([t_0, T]; \mathbb{R}^m)$ *such that the solution* $x(\cdot)$ *of (2.1.1) satisfies* $x(T) = x_T$*.*

Define the matrix $G : L_2([t_0, T]; \mathbb{R}^m) \to \mathbb{R}^n$ by

$$Gu = \int_{t_0}^{T} \Phi(t_0, s)B(s)u(s)\mathrm{d}s. \tag{2.1.4}$$

The adjoint (conjugate transpose) matrix $G^* : \mathbb{R}^n \to L_2([t_0, T]; \mathbb{R}^m)$ is defined as

$$(G^*v)(t) = B^*(t)\Phi^*(t_0, t)v,$$

where B^* and Φ^* are the conjugate transpose of the matrices B and Φ, respectively. The matrix G is called controllability matrix. Using the matrices G and G^*, the controllability Grammian $(n \times n)$ matrix for the system (2.1.1) is defined as

$$W(t_0, T) = GG^* = \int_{t_0}^{T} \Phi(t_0, s)B(s)B^*(s)\Phi^*(t_0, s)\mathrm{d}s.$$

Theorem 2.1.1. *[8, 112] The system* (2.1.1) *is controllable if and only if the symmetric controllability Grammian matrix $W(t_0, T)$ is nonsingular. In this case the control*

$$u(t) = -B^*(t)\Phi^*(t_0, t)W^{-1}(t_0, T)[x_0 - \Phi(t_0, T)x_T],$$

defined on $t_0 \leq t \leq T$, drives the system (2.1.1) *from $x(t_0) = x_0$ to $x(T) = x_T$.*

Theorem 2.1.2 (Kalman's Rank condition [8, 112])**.** *If the matrices A and B are time independent then the linear control system* (2.1.1) *is controllable if and only if the rank of the matrix $\begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix} = n$.*

Consider the nonlinear system

$$\dot{x}(t) = f(t, x(t)), \quad x(t_0) = x_0, \tag{2.1.5}$$

where $x \in \mathbb{R}^n$ is the state vector and $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is a nonlinear vector function such that $f(t, 0) = 0$.

Definition 2.1.2 (Stability [8])**.** *For the system (2.1.5), the trivial solution $x(t) = 0$ is said to be:*

- ***stable*** *if for any $t_0 > 0$ and for any given $\epsilon > 0$ there exists a $\delta > 0$ such that $\|x_0\| < \delta$ implies $\|x(t)\| < \epsilon$, $t \geq t_0$.*
- ***asymptotically stable*** *if it is stable and $\lim_{t\to\infty} x(t) = 0$.*
- ***unstable*** *if it is not stable.*

Note that, throughout this section $\| \cdot \|$ denotes the Euclidean norm.

Consider the following linear time invariant system

$$\dot{x} = Ax, \quad x(t_0) = x_0, \tag{2.1.6}$$

where $x \in \mathbb{R}^n$ is the state vector and A is a nonsingular constant $n \times n$ matrix. Then origin $x = 0$ is the equilibrium point of (2.1.6).

Theorem 2.1.3. *[8, 20] The system (2.1.6) is asymptotically stable if and only if all the eigenvalues of A have negative real parts; (2.1.6) is unstable for at least one of the eigenvalues of A have positive real part(s); and completely unstable if all the eigenvalues of A have positive real parts.*

Instead of constant matrix A in (2.1.6), if we consider time varying matrix $A(t)$, then the Theorem 2.1.3 may not hold good. However, if $A(t)$ satisfies the condition

$$\lim_{t\to\infty} A(t) = A_1,$$

where A_1 is a constant $n \times n$ matrix, then the following theorem holds.

Theorem 2.1.4. *[8] If the origin as an asymptotically stable equilibrium point for the system $\dot{x}(t) = A_1 x(t)$, then it also is for $\dot{x}(t) = (A_1 + B(t))x(t)$ provided $\lim_{t\to\infty} \|B(t)\| = 0$ and $B(t)$ is continuous in $[0, \infty)$.*

Suppose in the system (2.1.5) the nonlinear function is $f(x(t))$ then we study stability of the system using Lyapunov function. The Lyapunov function $V(x)$ is defined as follows:

(i) $V(x)$ and all of its partial derivatives are continuous,

(ii) $V(0) = 0$ and $V(x) > 0$ for $x \neq 0$ (positive definite) in some neighbourhood $\|x\| \leq k$ of the origin,

(iii) $\dot{V}(0) = 0$, and $\dot{V}(x) \leq 0$ (negative semi-definite) for $x \neq 0$ in some neighbourhood $\|x\| \leq k$, where $\dot{V}(x) = \dfrac{\mathrm{d}V}{\mathrm{d}t}(x(t))$.

Theorem 2.1.5. *The zero solution of the system is*

(i) ***stable*** *if there exist a Lyapunov function defined as above,*

(ii) ***asymptotically stable*** *if there exits a Lyapunov function whose derivative* $\dot{V} < 0$ *(negative definite).*

Consider the following linear time invariant control system

$$\dot{x} = Ax + Bu, \quad x(t_0) = x_0 \tag{2.1.7}$$

where $x \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}^m$ is the control, A is a constant $n \times n$ matrix and B is a constant $n \times m$ matrix.

Theorem 2.1.6. *[63] Let S be an arbitrary set of n complex numbers which is symmetric with respect to real axis. Let A and B are the matrices of order $n \times n$ and $n \times m$ respectively. Then there exists a constant matrix K such that the spectrum (set of all eigenvalues) of the matrix $(A + BK)$ is the set S if and only if the rank of the matrix $[\,B \quad AB \quad A^2B \quad \cdots \quad A^{n-1}B\,] = n$.*

Definition 2.1.3 (Stabilizability [8,63])**.** *The control system (2.1.7) is said to be stabilizable if there exists a constant feedback matrix K such that the spectrum of $(A + BK)$ entirely lies in the left-hand side of the complex plane.*

Note: From Theorem 2.1.6, it is clear that the system (2.1.7) is **stabilizable** if the rank of $[\,B \quad AB \quad A^2B \quad \cdots \quad A^{n-1}B\,] = n$, i.e., if the system (2.1.7) is **controllable**.

2.1.1 Stability of time-delay system

Consider the time-delay system

$$\left.\begin{array}{rcl} \dot{x}(t) & = & Ax(t) + h(t, x(t), x(t-\tau)), \quad t > 0 \\ x(t) & = & \phi(t), \quad t \in [-\tau, 0], \end{array}\right\} \tag{2.1.8}$$

where $x \in \mathbb{R}^n$ is the state vector, A is a constant $n \times n$ matrix, $h : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is a nonlinear vector function such that $h(t, 0, 0) = 0$, $\tau > 0$ is a real constant, ϕ is the continuous vector valued history function and $\|\phi\| = \sup_{t\in[-\tau,0]} \|\phi(t)\|$.

Definition 2.1.4 (Stability [44]). *For the system (2.1.8), the trivial solution $x(t) = 0$ is said to be:*

- ***stable*** *if for any $t_0 > 0$ and for any given $\epsilon > 0$ there exists a $\delta > 0$ such that $\|\phi\| < \delta$ implies $\|x(t)\| < \epsilon$, $t \geq t_0$.*
- ***asymptotically stable*** *if it is stable and $\lim_{t\to\infty} x(t) = 0$.*
- ***unstable*** *if it is not stable.*

Note that, the stability definition for delay system is same as stability definition for systems without delay, except for the assumption about the initial condition.

Let $\phi \in \mathfrak{C} = C([-r, 0] \to \mathbb{R}^n)$, (the notation $\mathfrak{C}$ denotes the set of all continuous functions from $[-r, 0] \to \mathbb{R}^n$, $r > 0$) with the norm $\|\phi\|_{\mathfrak{C}} = \sup_{-r\leq\theta\leq0} \|\phi(\theta)\|$, where $\|\cdot\|$ is the Euclidean norm on $\mathbb{R}^n$.

If x is a function defined on $[t-r, t] \to \mathbb{R}^n$ then we define a new function $x_t : [-r, 0] \to \mathbb{R}^n$ by $x_t(\theta) = x(t + \theta), \quad -r \leq \theta \leq 0$.

Consider the following differential equation

$$\dot{x} = f(t, x_t), \quad x(t_0 + \theta) = \phi(\theta), \quad -r \leq \theta \leq 0 \tag{2.1.9}$$

where $x(t) \in \mathbb{R}^n$, the nonlinear function $f : \mathbb{R} \times \mathfrak{C} \to \mathbb{R}^n$ and $t_0 \geq 0$. Note that (2.1.9) includes the cases: (i) when $r = 0$ the function $f(t, x_t)$ is equal to $f(t, x(t))$ and (ii) the function with discrete delay terms $f(t, x_t) = f(t, x(t - \tau_1), x(t - \tau_2), \ldots, x(t - \tau_m))$, where $\tau_i > 0$ are constants and $r = \max_{1\leq i\leq m} \tau_i$.

Theorem 2.1.7 (Razumikhin Theorem [44]). *Suppose f takes the values from $\mathbb{R} \times$ (bounded sets of $\mathfrak{C}$) $\to$ (bounded sets of $\mathbb{R}^n$), $v_j(j = 1, 2, 3) : \mathbb{R} \to \mathbb{R}$ are continuous nondecreasing functions, $v_j(t)$ are positive for $t > 0$ and $v_1(0) = v_2(0) = 0$, v_2 are strictly* *increasing. If there exists a continuously differentiable function* $V : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ *and continuous nondecreasing function* $p(s) > s$ *for* $s > 0$ *such that*

$$v_1(\|x\|) \leq V(t, x(t)) \leq v_2(\|x\|), \quad t \in \mathbb{R} \text{ and } x \in \mathbb{R}^n,$$

and

$$\dot{V}(t, x(t)) \leq -v_3(\|x\|), \quad \text{whenever } V(t+\theta, x(t+\theta)) \leq p(V(t, x(t))),$$

for $\theta \in [-r, 0]$*, then the system* (2.1.9) *is uniformly asymptotically stable.*

2.2 Fractional system theory

2.2.1 Basics of fractional calculus

Definition 2.2.1. *[61][Riemann-Liouville fractional integral] Let* $x \in C[t_0, \infty)$ *be the set of all continuous function defined on* $[t_0, \infty)$*. For* $x \in C[t_0, \infty)$ *and* $t \in [t_0, \infty)$*, the Riemann-Liouville fractional integral of order* α*,* $I_t^\alpha x(t)$ *is defined by*

$$I_t^\alpha x(t) = \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} x(s) \mathrm{d}s,$$

where $\alpha > 0$*,* $\Gamma(.)$ *is the gamma function, defined as* $\Gamma(\alpha) = \int_0^\infty e^{-t} t^{\alpha-1} \mathrm{d}t$*.*

Definition 2.2.2. *[61][Riemann-Liouville fractional derivative] Let* $x \in C[t_0, \infty)$*. For* $t \in [t_0, \infty)$*, the Riemann-Liouville fractional derivative* $D_t^\alpha x(t)$ *of order* α *is defined by*

$$\begin{aligned} D_t^\alpha x(t) &= D^n I_t^{n-\alpha} x(t) \\ &= \frac{1}{\Gamma(n-\alpha)} \frac{\mathrm{d}^n}{\mathrm{d}t^n} \int_{t_0}^{t} (t-s)^{n-\alpha-1} x(s) \mathrm{d}s, \end{aligned}$$

where n *is the positive integer such that* $n - 1 \leq \alpha < n$*.*

Definition 2.2.3. *[61][Caputo fractional derivative] Let $x \in C[t_0,\infty)$. For $t \in [t_0,\infty)$, the Caputo fractional derivative ${}^C D_t^\alpha x(t)$ of order α is defined by*

$$
{}^C D_t^\alpha x(t) = \begin{cases} I_t^{n-\alpha} D^n x(t) & \\ = \dfrac{1}{\Gamma(n-\alpha)} \displaystyle\int_{t_0}^{t} (t-s)^{n-\alpha-1} \left[\frac{\mathrm{d}^n}{\mathrm{d}s^n} x(s)\right] \mathrm{d}s, & \text{if } n-1<\alpha<n, \\ \dfrac{\mathrm{d}^n}{\mathrm{d}t^n} x(t), & \text{if } \alpha = n, \end{cases}
$$

if the integral on the right hand side exists. Here n is a positive integer.

For any functions $x, y \in C[t_0,\infty)$, constants $a, b \in \mathbb{R}$ and $n-1<\alpha\le n$, $(n\in\mathbb{N})$ the following properties hold:

(i) ${}^C D_t^\alpha [ax(t)+by(t)] = a\,{}^C D_t^\alpha x(t) + b\,{}^C D_t^\alpha y(t)$.

(ii) ${}^C D_t^\alpha x(t) = D_t^\alpha x(t) - \sum_{k=0}^{n-1} \frac{x^{(k)}(t_0)}{\Gamma(k-\alpha+1)}(x-t_0)^{k-\alpha}$.

(iii) $I_t^\alpha\, {}^C D_t^\alpha x(t) = x(t) - x(0), \quad 0<\alpha\le 1$.

(iv) ${}^C D_t^\alpha\, {}^C D_t^\beta x(t) \neq {}^C D_t^\beta\, {}^C D_t^\alpha x(t), \quad 0<\alpha,\beta\le 1$ and $\alpha\neq\beta$.

(v) If $\alpha=\beta$, then the property (iv) becomes ${}^C D_t^\alpha\, {}^C D_t^\beta x(t) = {}^C D_t^{\alpha+\beta} x(t) = {}^C D_t^\beta\, {}^C D_t^\alpha x(t)$. In general, ${}^C D_t^{l\alpha} x(t) = {}^C D_t^\alpha\, {}^C D_t^{(l-1)\alpha}, \quad l = 2,3,\ldots$.

Definition 2.2.4 (Mittag-Leffler function)**.** *The Mittag-Leffler function with two parameters is defined by*

$$
E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+\beta)}
$$

where $z, \beta \in \mathbb{C}, \Re(\alpha) > 0$. When $\beta = 1$, one has $E_{\alpha,1}(z) = E_\alpha(z)$, furthermore, $E_{1,1}(z) = e^z$.

Consider the following semilinear fractional-order system

$$
{}^C D_t^\alpha x(t) = Ax(t) + f(t,x(t)), \quad m-1<\alpha\le m, \tag{2.2.1}
$$

with the initial condition $x^{(i)}(t_0) = x_i$, $(i = 0, 1, 2, \dots m-1)$, where A is $n \times n$ constant matrix and $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is a nonlinear function which satisfies Lipschitz condition,

$$\|f(t, y) - f(t, z)\| \leq L\|y - z\|$$

for some constant $L > 0$.

The solution of (2.2.1) is given by [61]

$$x(t) = \sum_{i=0}^{m-1} (t - t_0)^i E_{\alpha,i+1}[A(t - t_0)^\alpha] x_i + \int_{t_0}^{t} (t - s)^{\alpha-1} E_{\alpha,\alpha}[A(t - s)^\alpha] f(s, x(s)) \mathrm{d}s,$$

for $i = 0, 1, 2, \dots, m-1$.

2.2.2 Controllability of finite dimensional fractional-order systems

Consider the fractional-order linear time invariant control system

$$^C D_t^\alpha x(t) = Ax(t) + Bu(t), t_0 \leq t \leq T < \infty \tag{2.2.2}$$

with $x(t_0) = x_0$ and $x'(t_0) = x_1$, where $\alpha \in (0, 2]$, $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, A and B are constant matrices of order $n \times n$ and $n \times m$, respectively.

Definition 2.2.5 (Controllability)**.** *The system* (2.2.1) *with* $0 < \alpha \leq 2$ *is controllable if for any* t_0*, any initial state* $x(t_0) = x_0 : 0 < \alpha \leq 1$*,* $(x(t_0) = x_0$ *and* $x'(0) = x_1 : 1 < \alpha \leq 2)$ *and any final state* x_T *there exist a finite time* $T > t_0$ *and a control* $u(t)$*,* $t_0 \leq t \leq T$*, such that* $x(T) = x_T$*.*

Theorem 2.2.1. *[88, 95] The system* (2.2.2) *is controllable if and only if the* $n \times n$ *symmetric controllability Gramian matrix*

$$U(t_0, T) = \int_{t_0}^{T} S(T - t) BB^T S^T(T - t)(T - t)^{2(1-\alpha)} \mathrm{d}t,$$

where $S(t) = t^{\alpha-1}E_{\alpha,\alpha}[At^{\alpha}]$, *is nonsingular.*

In this case the control

$$u(t) = \begin{cases} -(T-t)^{2(1-\alpha)}B^{T}S^{T}(T-t)U^{-1}(t_0,T)\left[-x_T + E_{\alpha,1}[A(T-t_0)^{\alpha}]x_0\right], & 0<\alpha\leq 1, \\ \begin{cases} -(T-t)^{2(1-\alpha)}B^{T}S^{T}(T-t)U^{-1}(t_0,T)\big[-x_T + E_{\alpha,1}[A(T-t_0)^{\alpha}]x_0 \\ +(T-t_0)E_{\alpha,2}[A(T-t_0)^{\alpha}]x_1\big], \end{cases} & 1<\alpha\leq 2, \end{cases}$$

defined on $t_0 \leq t \leq T$, *transfers the system* (2.2.2) *from initial state at* $t = t_0$ *to final state* $x(T) = x_T$.

Theorem 2.2.2. *[91] The system* (2.2.2) *is controllable if and only if the* $n \times nm$ *matrix*

$$U = [B, AB, A^2B, \ldots, A^{n-1}B]$$

has rank n.

2.2.3 Stability of finite dimensional fractional-order systems

Consider the following nonlinear system

$$^{C}D_t^{\alpha}x(t) = f(t, x(t)), \quad 0<\alpha\leq 1, \tag{2.2.3}$$

with the initial condition $x(0) = x_0$ and $f(t, 0) = 0$ for all t.

Definition 2.2.6. *The trivial solution* $x(t) = 0$ *of* (2.2.3) *is said to be:*

- ***stable*** *if for any given* $\epsilon > 0$ *there exists a* $\delta > 0$ *such that* $\|x(0)\| < \delta$ *implies* $\|x(t)\| < \epsilon$, $t \geq 0$.
- ***asymptotically stable*** *if it is stable and* $\lim_{t\to\infty} x(t) = 0$.
- ***unstable*** *if it is not stable.*

Consider the semilinear system given in (2.2.1) with fractional-order $0 < \alpha \leq 1$.

Theorem 2.2.3. *[126] If* $|\arg(spec(A))| > \frac{\alpha\pi}{2}$*,* $\alpha\|A\| > 1$ *and* $f(t, x(t))$ *satisfies* $\lim_{x(t)\to 0} \frac{\|f(t,x(t))\|}{\|x(t)\|} = 0$*, then the system* (2.2.1) *is asymptotically stable. Here* $spec(A)$ *denotes set of all eigenvalues of the matrix* A*.*

2.2.4 Stabilization of fractional-order semilinear system

Consider the following controlled fractional-order nonlinear system:

$$^{C}D_t^{\alpha}x(t) = Ax(t) + Bu(t) + f(t, x(t)) \tag{2.2.4}$$

where $u(t)$ is the control input, $B \in \mathbb{R}^{n\times m}$ and the remaining terms are same as in (2.2.1). Suppose that we apply linear feedback control input $u(t) = Kx(t)$ in (2.2.4), then the resulting closed loop system is

$$\begin{aligned} ^{C}D_t^{\alpha}x(t) &= (A + BK)x(t) + f(t, x(t)) \\ &= \tilde{A}x(t) + f(t, x(t)) \end{aligned} \tag{2.2.5}$$

where $\tilde{A} = A + BK$ and feedback gain $K \in \mathbb{R}^{m\times n}$ needs to be determined.

Theorem 2.2.4. *[126] If feedback gain* K *is chosen such that* $|\arg(spec(\tilde{A}))| > \frac{\alpha\pi}{2}$*,* $\alpha\|\tilde{A}\| > 1$ *and* $f(t, x(t))$ *satisfies* $\lim_{x(t)\to 0} \frac{\|f(t,x(t))\|}{\|x(t)\|} = 0$*, then controlled system* (2.2.5) *is asymptotically stable.*

2.3 Infinite dimensional fractional-order systems

Let X be a Banach space with the norm $\|\cdot\|$. We denote the space

$$L_p([0,T];X) := \left\{ g : [0,T] \to X \middle| \, g \text{ is Bochner } p\text{-integrable on } [0,T] \right\},$$

with the norm

$$\|g\|_{L_p([0,T];X)} = \left(\int_0^T \|g(t)\|^p \mathrm{d}t \right)^{1/p},$$

where $1 \leq p < \infty$.

The **Hölder space** $\mathcal{C}^\gamma, 0 < \gamma < 1$, is defined by

$$\mathcal{C}^\gamma([0,T];X) := \{g \in C([0,T];X) | \sup_{s,t\in[0,T]} \frac{\|g(t)-g(s)\|_X}{|t-s|^\gamma} < \infty\}$$

with

$$\|g\|_{\mathcal{C}^\gamma} := \sup_{t\in[0,T]} \|g(t)\|_X + \sup_{s,t\in[0,T]} \frac{\|g(t)-g(s)\|_X}{|t-s|^\gamma}.$$

If $\lambda > 1$ we define $\mathcal{C}^\lambda([0,T];X)$ as the space of all functions satisfying $g \in \mathcal{C}^m([0,T];X)$ and $g^{(m)} \in \mathcal{C}^\gamma([0,T];X)$, where $\gamma = \lambda - m, m = \lfloor \lambda \rfloor$ and $\lfloor \lambda \rfloor$ denotes greatest integer less than or equal to λ. It is endowed with the norm

$$\|g\|_{\mathcal{C}^\lambda} = \|g\|_{\mathcal{C}^m} + \|g^{(m)}\|_{\mathcal{C}^\gamma}.$$

Fractional-order $\alpha \in (0,1]$**:**

Consider the linear fractional-order system

$$^C D_t^\alpha x(t) = Ax(t), \tag{2.3.1}$$

with $x(0) = \eta$, $\alpha \in (0,1]$, where $A : D(A) \subset X \to X$ is densely defined and closed linear operator on the Banach space X.

The system (2.3.1) can be rewritten as

$$x(t) = \eta + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} Ax(s)\mathrm{d}s. \tag{2.3.2}$$

Definition 2.3.1. *[6] A strongly continuous α-order semigroup $\{T_\alpha^1(t)\} \subset \mathcal{B}$ is said to generate an operator A if it satisfies the following conditions:*

(i) $T_\alpha^1(t)$ is strongly continuous and $T_\alpha^1(0) = I$;

(ii) if $\eta \in D(A)$ then $T_\alpha^1(t)\eta \in D(A)$ and $AT_\alpha^1(t)\eta = T_\alpha^1(t)A\eta,\ t \geq 0$;

(iii) for every $\eta \in D(A)$, $T^1_\alpha(t)\eta$ *is a solution of* (2.3.2).

Definition 2.3.2. *The mild solution of* (2.3.1) *is given by*

$$x(t) \quad = \quad T^1_\alpha(t)\eta, \tag{2.3.3}$$

where

$$\begin{aligned} T^1_\alpha(t) &= \int_0^\infty \xi_\alpha(\theta) T_1(t^\alpha \theta) \mathrm{d}\theta, \\ \xi_\alpha(\theta) &= \frac{1}{\alpha}\theta^{-1-\frac{1}{\alpha}}\omega_\alpha\left(\theta^{-\frac{1}{\alpha}}\right) \geq 0, \\ \omega_\alpha(\theta) &= \frac{1}{\pi}\sum_{n=0}^{\infty}(-1)^{n-1}\theta^{-\alpha n-1}\frac{\Gamma(n\alpha+1)}{n!}\sin(n\alpha\pi). \end{aligned}$$

Here, ξ_α *is a probability density function defined on* $(0,\infty)$*. According to [86],*

$$\int_0^\infty \theta\xi_\alpha(\theta)\mathrm{d}\theta = \int_0^\infty \frac{1}{\theta^\alpha}\omega_\alpha(\theta)\mathrm{d}\theta = \frac{1}{\Gamma(1+\alpha)}.$$

Now, consider the semilinear fractional-order system

$${}^{C}D^\alpha_t x(t) = Ax(t) + f(t, x(t)), \quad 0 < t \leq T, \quad 0 < \alpha \leq 1 \tag{2.3.4}$$

with $x(0) = x_0$. The state $x(t)$ belongs to the Banach spaces X for each t.

The operator A and f are defined as follows: A generates a strongly continuous α-order semigroup $\{T^1_\alpha(t)\}_{t\geq 0}$ on X, and the map $f : [0, T] \times X \to X$ is a nonlinear operator and the positive constant $T < \infty$.

Definition 2.3.3. *[6] A function* $x(t) \in C([0, T]; X)$ *is said to be the mild solution of* (2.2.2) *if it satisfies*

$$x(t) \quad = \quad T^1_\alpha(t)x_0 + \int_0^t (t-s)^{\alpha-1} T^2_\alpha(t-s) f(s, x(s))\mathrm{d}s, \tag{2.3.5}$$

where

$$T_\alpha^2(t) = \alpha \int_0^\infty \theta \xi_\alpha(\theta) T(t^\alpha \theta) \mathrm{d}\theta, \quad \theta \in (0, \infty).$$

Fractional-order $\alpha \in (1, 2]$**:**

Consider the linear fractional system

$$^C D_t^\alpha x(t) = Ax(t), \tag{2.3.6}$$

with $x(0) = \eta,\ x'(0) = 0, \quad \alpha \in (1, 2]$, where $A : D(A) \subset X \to X$ is densely defined and closed linear operator on the Banach space X.

Definition 2.3.4. *[6, 110] A strongly continuous cosine family of order-α, $\{C_\alpha(t)\}_{t \geq 0} \subset \mathcal{B}(X)$ (the set of all bounded linear operator from X to X) is said to generate an operator A if it satisfies the following conditions:*

(i) for $t \geq 0$, $C_\alpha(t)$ is strongly continuous and $C_\alpha(0) = I$;

(ii) if $\eta \in D(A)$ then $C_\alpha(t)\eta \in D(A)$ and $AC_\alpha(t)\eta = C_\alpha(t)A\eta,\ t \geq 0$;

(iii) for every $\eta \in D(A)$, $C_\alpha(t)\eta$ is a solution of (2.3.6).

Definition 2.3.5. *[60] The sine family of order α, $S_\alpha : [0, \infty) \to \mathcal{B}(X)$ related with C_α is defined by*

$$S_\alpha(t) = \int_0^t C_\alpha(\xi) \mathrm{d}\xi, \ \ t \geq 0.$$

If (2.3.6) has an α-order cosine family $C_\alpha(t)$, then the problem

$$^C D^\alpha x(t) = Ax(t), \quad x(0) = \eta,\ x'(0) = y_0, \quad \alpha \in (1, 2],$$

is uniquely solvable with solution

$$x(t) = C_\alpha(t)\eta + S_\alpha(t)x_0,$$

provided $\eta, x_0 \in D(A)$.

Definition 2.3.6. *[60] The Riemann-Liouville family of order α, $P_\alpha : [0,\infty) \to \mathcal{B}(X)$ related with C_α is defined by*

$$P_\alpha(t) = I_t^{\alpha-1} C_\alpha(t),$$

where $I_t^{\alpha-1}$ denotes the Riemann-Liouville fractional integral of order $\alpha - 1$.

Definition 2.3.7. *[60] $C_\alpha(t)$ is called exponentially bounded if*

$$\|C_\alpha(t)\| \leq Me^{\rho t}, \;\; t \geq 0. \tag{2.3.7}$$

for some constants $M \geq 1$ and $\rho \geq 0$.

If the problem (2.3.6) has a strongly continuous α-order cosine family $C_\alpha(t)$ satisfying (2.3.7), then A is said to belong to $\mathcal{C}^\alpha(X; M, \rho)$. Denote $\mathcal{C}^\alpha(\rho) := \bigcup\{\mathcal{C}^\alpha(X; M, \rho); M \geq 1\}$, $\;\mathcal{C}^\alpha := \bigcup\{\mathcal{C}^\alpha(\rho); \rho \geq 0\}$. In these notations $\mathcal{C}^1$ and $\mathcal{C}^2$ are the sets of all infinitesimal generators of C_0-semigroups and cosine operator families, respectively.

Lemma 2.3.1. *[6] Let $0 < \alpha < \beta \leq 2$, $\gamma = \alpha/\beta$, $\rho \geq 0$. If $A \in \mathcal{C}^\beta(\rho)$ then $A \in \mathcal{C}^\alpha(\rho^{1/\gamma})$ and*

$$C_\alpha(t) := \int_0^\infty \varphi_{t,\gamma}(s) C_\beta(s) \mathrm{d}s, \;\; t > 0, \tag{2.3.8}$$

where $\varphi_{t,\gamma}(s) := t^{-\gamma} \sum_{n=0}^{\infty} \frac{(-st^{-\gamma})^n}{n!\Gamma(-\gamma n + 1 - \gamma)}$, $\;0 < \gamma < 1$; (2.3.8) holds in the strong sense.

Consider the fractional semilinear control system (2.3.4) with $x(0) = x_0,\; x'(0) = x_1$ and $1 < \alpha \leq 2$. The operator A generates a strongly continuous α-order cosine family $\{C_\alpha(t)\}_{t\geq 0}$ on X and the operator f is same as in the system (2.3.4).

Definition 2.3.8. *[60][$1 < \alpha \leq 2$] A function $x \in C([0,T]; X)$ is called a mild solution for (2.3.4) if it satisfies the integral equation*

$$x(t) = C_\alpha(t)\phi(0) + S_\alpha(t)x_0 + \int_0^t P_\alpha(t-s) f(s, x(s)) \mathrm{d}s, \quad t \in [0,T].$$

2.4 Basics of functional analysis

Definition 2.4.1 (Completely continuous operator [25]). *If X_1 and X_2 are Banach spaces and $\Phi : X_1 \to X_2$ is a bounded linear operator, then Φ is said to be completely continuous if for every weakly convergent sequence (x_n) in X_1, the sequence (Φx_n) is norm-convergent in X_2.*

Remark 2.4.1. *If Φ is a compact operator, then Φ is completely continuous. If X_1 is reflexive and Φ is completely continuous, then Φ is compact.*

Remark 2.4.2. *If $\Phi : X_1 \to X_2$ is a constant operator then Φ is completely continuous.*

Definition 2.4.2 (Contraction operator). *Let X be a Banach space. The operator $\Phi : X \to X$ is called contraction if $\|\Phi(y) - \Phi(z)\| \leq k\|y - z\|$, $0 < k < 1$, for all $y, z \in X$.*

Note: [108] Sum of a completely continuous operator and a contraction operator is called a condensing operator.

Theorem 2.4.1 (Sadovskii fixed point theorem [108]). *If Φ is a condensing operator from a convex closed and bounded set M of a Banach space X into itself, then it has at least one fixed point in M.*

Definition 2.4.3. *[53] Let X be a reflexive real Banach space. A mapping $F : X \to X^*$ is said to be of type (M) if the following conditions hold:*

(a) If a sequence $\{x_n\} \in X$ converges weakly to $x \in X$ and $\{Fx_n\}$ converges weakly to $y \in X^*$ and $\lim_n \sup(Fx_n, x_n) \leq (y, x)$, then $Fx = y$.

(b) F is continuous from finite dimensional subspaces of X into X^* endowed with weak topology.*

Lemma 2.4.1. *[53] Let X be a real Banach space and $F : X \to X^*$ be a mapping of type (M). If F is coercive then the range of F is all of X^*.*

Lemma 2.4.2. *[26][Gronwall-Bellman lemma] If*

$$x(t) \leq h(t) + \int_{t_0}^{t} k(s)x(s)\mathrm{d}s, \quad t \in [t_0, T)$$

where $k(t)$ *is nonnegative and all the functions involved are continuous on* $[t_0, T), T \leq +\infty$, *then* $x(t)$ *satisfies*

$$x(t) \leq h(t) + \int_{t_0}^{t} h(s)k(s) \exp\left[\int_{s}^{t} k(s_1)\mathrm{d}s_1\right] \mathrm{d}s, \quad t \in [t_0, T).$$

Chapter 3

Asymptotic stability and stabilizability of nonlinear systems with delay

This Chapter deals with asymptotic stability and stabilizability of a class of nonlinear dynamical systems with fixed delay in state variable. New sufficient conditions are established in terms of the system parameters such as the eigenvalues of the linear operator, delay parameter and bounds on nonlinear parts. Finally, three examples are given to validate the efficiency of the proposed theory.

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3.1 Introduction

In general, Lyapunov–Krasovskii stability theory and Razumikhin stability theory are the common approaches to study stability of nonlinear delay systems. Suppose the Lyapunov candidates are simple quadratic functionals, one can easily check negativity of the derivative of Lyapunov candidates using linear matrix inequalities. Constructions of Lyapunov functionals which give stability conditions is very difficult for systems having complicated nonlinear functions [100]. To overcome these difficulties, in this study, we use Gronwall-Bellman lemma 2.4.2 and some inequalities to obtain sufficient conditions for the delay dependent asymptotic stability and stabilization of some systems. The stability conditions given in this chapter are new and improves some results available in literature for certain class of nonlinearity.

3.2 Main results

3.2.1 Stability of nonlinear delay systems

Consider the following nonlinear delay system

$$\left.\begin{aligned} \dot{x}(t) &= Ax(t) + f(x(t-\tau)) + g(x(t)), \quad t > 0 \\ x(t) &= \phi(t), \quad t \in [-\tau, 0], \end{aligned}\right\} \tag{3.2.1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, A is a constant $n \times n$ matrix, $f, g : \mathbb{R}^n \to \mathbb{R}^n$ are the nonlinear vector functions, $\tau > 0$ is a real constant, $\phi \in \mathfrak{C}$ is the continuous vector valued history function and $\|\phi\|_{\mathfrak{C}} = \sup_{t\in[-\tau,0]} \|\phi(t)\|$. Here $\mathfrak{C} = \{\phi : [-\tau, 0] \to \mathbb{R}^n \mid \phi \text{ is continuous}\}$ and $\|\cdot\|$ denotes the Euclidean norm.

The solution of (3.2.1) is written as

$$x(t) = e^{At}\phi(0) + \int_0^t e^{A(t-s)}[f(x(s-\tau)) + g(x(s))]\mathrm{d}s. \tag{3.2.2}$$

We assume the following conditions to show the asymptotic stability of system (3.2.1) with $\|\phi\|_{\mathfrak{C}}$.

(1) The eigenvalues of A have negative real parts,

(2) the nonlinear functions $f(x(t-\tau))$ and $g(x(t))$ satisfy the conditions $\lim_{x(t)\to 0}(\|g(x(t))\|/\|x(t)\|) = 0$ and $\|f(x(t-\tau))\| \leq e^{-\lambda\tau}\|x(t-\tau)\|$,

(3) $-\lambda + e^{-\lambda\tau} < 0$ and $-\lambda + C_1 < 0$ ($C_1 > 0$ is a constant given in (3.2.3)),

where $\lambda = \min\{\lambda_i, i = 1, 2, \ldots, n\}$, $-\lambda_i$'s ($\lambda_i > 0$) are the real parts of the eigenvalues of A.

By the condition (2), there exist a constat $C_1 > 0$ such that

$$\|g(x(t))\| \leq C_1\|x(t)\| \quad \text{when } \|x(t)\| < \delta, \tag{3.2.3}$$

for sufficiently small $\delta > 0$. Using the conditions (1) and (2) and (3.2.3), in the solution (3.2.2) we get

$$\|x(t)\| \leq e^{-\lambda t}\left[|\phi(0)| + e^{-\lambda\tau}\int_0^t e^{\lambda s}\|x(s-\tau)\|\mathrm{d}s + C_1\int_0^t e^{\lambda s}\|x(s)\|\mathrm{d}s\right]. \tag{3.2.4}$$

When $t \in [0, \tau]$, $x(t-\tau) = \phi(t-\tau)$, hence (3.2.4) becomes

$$\|x(t)\| \leq e^{-\lambda t}\left[|\phi(0)| + e^{-\lambda\tau}\int_0^t e^{\lambda s}\|\phi(s-\tau)\|\mathrm{d}s + C_1\int_0^t e^{\lambda s}\|x(s)\|\mathrm{d}s\right].$$

This implies

$$e^{\lambda t}\|x(t)\| \leq \|\phi\|_{\mathfrak{C}}\left(1 + e^{-\lambda\tau}\left(\frac{e^{\lambda t}-1}{\lambda}\right)\right) + C_1\int_0^t e^{\lambda s}\|x(s)\|\mathrm{d}s. \tag{3.2.5}$$

By using Gronwall's inequality given in Lemma 2.4.2, (3.2.5) becomes

$$\begin{aligned}
e^{\lambda t}\|x(t)\| &\leq \|\phi\|_{\mathfrak{C}}\left(\frac{\lambda - e^{-\lambda\tau}}{\lambda} + \left(\frac{e^{-\lambda\tau}}{\lambda}\right)e^{\lambda t}\right) \\
&\quad + C_1\|\phi\|_{\mathfrak{C}}\int_0^t \left(\frac{\lambda - e^{-\lambda\tau}}{\lambda} + \left(\frac{e^{-\lambda\tau}}{\lambda}\right)e^{\lambda s}\right) e^{C_1(t-s)}\mathrm{d}s \\
&\leq \|\phi\|_{\mathfrak{C}}\left(\frac{\lambda - e^{-\lambda\tau}}{\lambda} + \left(\frac{e^{-\lambda\tau}}{\lambda}\right)e^{\lambda t}\right) + \|\phi\|_{\mathfrak{C}}\left(\frac{-\lambda + e^{-\lambda\tau}}{\lambda}\right)\left(1 - e^{C_1 t}\right) \\
&\quad + \frac{\|\phi\|_{\mathfrak{C}}C_1 e^{-\lambda\tau}}{\lambda(\lambda - C_1)}\left(e^{\lambda t} - e^{C_1 t}\right).
\end{aligned} \tag{3.2.6}$$

By condition (3), inequality (3.2.6) becomes

$$\begin{aligned}
\|x(t)\| &\leq \|\phi\|_{\mathfrak{C}}\left(\left(\frac{\lambda - e^{-\lambda\tau}}{\lambda}\right)e^{-\lambda t} + \left(\frac{e^{-\lambda\tau}}{\lambda}\right)\right) - \|\phi\|_{\mathfrak{C}}\left(\frac{-\lambda + e^{-\lambda\tau}}{\lambda}\right)e^{(-\lambda + C_1)t} \\
&\quad + \frac{\|\phi\|_{\mathfrak{C}}C_1 e^{-\lambda\tau}}{\lambda(\lambda - C_1)}
\end{aligned}$$

$$\begin{aligned}
&= \|\phi\|_{\mathfrak{C}}\left[\frac{e^{-\lambda\tau}}{\lambda}\left(1+\frac{C_1}{\lambda-C_1}\right)+\left(\frac{\lambda-e^{-\lambda\tau}}{\lambda}\right)e^{-\lambda t}+\left(\frac{\lambda-e^{-\lambda\tau}}{\lambda}\right)e^{(-\lambda+C_1)t}\right]\\
&\leq \|\phi\|_{\mathfrak{C}}\left[\frac{e^{-\lambda\tau}}{\lambda-C_1}+\left(\frac{\lambda-e^{-\lambda\tau}}{\lambda}\right)e^{-\lambda t}+\left(\frac{\lambda-e^{-\lambda\tau}}{\lambda}\right)e^{(-\lambda+C_1)t}\right]. \qquad (3.2.7)
\end{aligned}$$

Differentiating the right side of (3.2.7) with respect to t, we get

$$\|\phi\|_{\mathfrak{C}}[(-\lambda+e^{-\lambda\tau})e^{-\lambda t}+\frac{1}{\lambda}(-\lambda+C_1)(\lambda-e^{-\lambda\tau})e^{(-\lambda+C_1)t}].$$

Under the condition (3) it is negative. Therefore, in the interval $[0,\tau]$ the solution of (3.2.1) is bounded by a decreasing function.

When $t\in[\tau,2\tau]$, (3.2.4) becomes

$$\begin{aligned}
\|x(t)\| \leq\ & e^{-\lambda t}\left[|\phi(0)|+e^{-\lambda\tau}\int_0^{\tau}e^{\lambda s}\|\phi(s-\tau)\|\mathrm{d}s+e^{-\lambda\tau}\int_{\tau}^{t}e^{\lambda s}\|x(s-\tau)\|\mathrm{d}s\right.\\
&\left.+C_1\int_0^{\tau}e^{\lambda s}\|x(s)\|\mathrm{d}s+C_1\int_{\tau}^{t}e^{\lambda s}\|x(s)\|\mathrm{d}s\right]. \qquad (3.2.8)
\end{aligned}$$

Substituting (3.2.7) in the third and fourth term of right side of the inequality (3.2.8), we get

$$\begin{aligned}
\|x(t)\| \leq\ & \|\phi\|_{\mathfrak{C}}e^{-\lambda t}\left[1+e^{-\lambda\tau}\frac{(e^{\lambda\tau}-1)}{\lambda}+e^{-\lambda\tau}\frac{e^{-\lambda\tau}}{\lambda(\lambda-C_1)}\left(e^{\lambda t}-e^{\lambda\tau}\right)\right.\\
&+\left(\frac{\lambda-e^{-\lambda\tau}}{\lambda}\right)(t-\tau)+\left(\frac{\lambda-e^{-\lambda\tau}}{\lambda C_1}\right)e^{-C_1\tau}\left(e^{C_1 t}-e^{C_1\tau}\right)\\
&\left.+\frac{C_1e^{-\lambda\tau}}{\lambda(\lambda-C_1)}(e^{\lambda\tau}-1)+\frac{C_1\tau}{\lambda}(\lambda-e^{-\lambda\tau})+\frac{(\lambda-e^{-\lambda\tau})}{\lambda}(e^{C_1\tau}-1)\right]\\
&+C_1e^{-\lambda t}\int_{\tau}^{t}e^{\lambda s}\|x(s)\|\mathrm{d}s.
\end{aligned}$$

Using condition (3), we get

$$\begin{aligned}
e^{\lambda t}\|x(t)\| \leq\ & \|\phi\|_{\mathfrak{C}}\left[\frac{(\lambda-e^{-\lambda\tau})}{\lambda}+\frac{1}{\lambda}+\frac{\tau}{\lambda}(\lambda-e^{-\lambda\tau})\right.\\
&+\frac{C_1e^{-\lambda\tau}}{\lambda(\lambda-C_1)}(e^{\lambda\tau}-1)+\frac{C_1\tau}{\lambda}(\lambda-e^{-\lambda\tau})+\frac{(\lambda-e^{-\lambda\tau})}{\lambda}(e^{C_1\tau}-1)\\
&\left.+\frac{e^{-2\lambda\tau}}{\lambda(\lambda-C_1)}e^{\lambda t}+\frac{(\lambda-e^{-\lambda\tau})}{\lambda C_1}e^{-C_1\tau}e^{C_1 t}\right]+C_1\int_{\tau}^{t}e^{\lambda s}\|x(s)\|\mathrm{d}s. \qquad (3.2.9)
\end{aligned}$$

By Lemma 2.4.2, (3.2.9) becomes

$$\begin{aligned} e^{\lambda t}\|x(t)\| &\leq \|\phi\|_{\mathfrak{C}}\Big[M + \frac{e^{-2\lambda\tau}}{\lambda(\lambda - C_1)}e^{\lambda t} + \frac{(\lambda - e^{-\lambda\tau})}{\lambda C_1}e^{-C_1\tau}e^{C_1 t}\Big] \\ &\quad + C_1\|\phi\|\int_\tau^t \Big[M + \frac{e^{-2\lambda\tau}}{\lambda(\lambda - C_1)}e^{\lambda s} + \frac{(\lambda - e^{-\lambda\tau})}{\lambda C_1}e^{-C_1\tau}e^{C_1 s}\Big]e^{C_1(t-s)}\mathrm{d}s \\ &\leq \|\phi\|_{\mathfrak{C}}\Big[M + \frac{e^{-2\lambda\tau}}{\lambda(\lambda - C_1)}e^{\lambda t} + \frac{(\lambda - e^{-\lambda\tau})}{\lambda C_1}e^{-C_1\tau}e^{C_1 t} - Me^{C_1 t}\left(e^{-C_1 t} - e^{-C_1\tau}\right) \\ &\quad + \frac{C_1 e^{C_1 t}e^{-2\lambda\tau}}{\lambda(\lambda - C_1)^2}\left(e^{(\lambda - C_1)t} - e^{(\lambda - C_1)\tau}\right) + \frac{2\tau}{\lambda}(\lambda - e^{-\lambda\tau})e^{-C_1\tau}e^{C_1 t}\Big]. \end{aligned}$$

This implies

$$\begin{aligned} \|x(t)\| &\leq \|\phi\|_{\mathfrak{C}}\Big[Me^{-\lambda t} + \frac{e^{-2\lambda\tau}}{\lambda(\lambda - C_1)} + \frac{(\lambda - e^{-\lambda\tau})}{\lambda C_1}e^{-C_1\tau}e^{(-\lambda + C_1)t} + Me^{-C_1\tau}e^{(-\lambda + C_1)t} \\ &\quad + \frac{C_1 e^{-2\lambda\tau}}{\lambda(\lambda - C_1)^2} + \frac{2\tau}{\lambda}(\lambda - e^{-\lambda\tau})e^{-C_1\tau}e^{(-\lambda + C_1)t}\Big], \end{aligned} \tag{3.2.10}$$

where $M = \dfrac{(\lambda - e^{-\lambda\tau})}{\lambda} + \dfrac{1}{\lambda} + \dfrac{\tau}{\lambda}(\lambda - e^{-\lambda\tau}) + \dfrac{C_1}{\lambda(\lambda - C_1)} + \dfrac{C_1\tau}{\lambda}(\lambda - e^{-\lambda\tau}) + \dfrac{(\lambda - e^{-\lambda\tau})}{\lambda}e^{C_1\tau} > 0$. The differentiation of right side of the inequality (3.2.10) with respect to t is given by

$$\begin{aligned} &\|\phi\|_{\mathfrak{C}}\Big[-\lambda Me^{-\lambda t} + \frac{(-\lambda + C_1)}{\lambda C_1}(\lambda - e^{-\lambda\tau})e^{-C_1\tau}e^{(-\lambda + C_1)t} + (-\lambda + C_1)Me^{-C_1\tau}e^{(-\lambda + C_1)t} \\ &+ \frac{2\tau}{\lambda}(-\lambda + C_1)(\lambda - e^{-\lambda\tau})e^{-C_1\tau}e^{(-\lambda + C_1)t}\Big]. \end{aligned}$$

Under condition (3) it is negative. This implies that the solution of (3.2.1) in the interval $[\tau, 2\tau]$ is bounded by a decreasing function.

Proceeding in this way the solution of (3.2.1) in the successive intervals $[i\tau, (i-1)\tau], i = 2, 3, 4, \ldots$ are bounded by the decreasing functions.

To show asymptotic stability, we use the induction method. When $t \in [0, \tau]$, (3.2.7) becomes

$$\|x(t)\| \leq \|\phi\|_{\mathfrak{C}}\left[L_1 e^{-\lambda t} + M_1 e^{(-\lambda + C_1)t}\right], \tag{3.2.11}$$

where $L_1 = (\lambda - e^{-\lambda\tau})/\lambda$ and $M_1 = e^{-C_1\tau}/(\lambda - C_1) + L_1$ are positive constants.

When $t \in [\tau, 2\tau]$, by using (3.2.11), the inequality (3.2.8) becomes

$$\begin{aligned} e^{\lambda t}\|x(t)\| &\leq \|\phi\|_{\mathfrak{C}}\Bigg[L_1 + \frac{1}{\lambda} + \tau L_1 + \frac{M_1}{C_1}e^{-C_1\tau}(e^{C_1 t} - e^{C_1\tau}) \\ &\quad + C_1\left(\tau L_1 + \frac{M_1}{C_1}(e^{C_1\tau} - 1)\right)\Bigg] + C_1\int_\tau^t e^{\lambda s}\|x(s)\|\mathrm{d}s \\ &\leq \|\phi\|_{\mathfrak{C}}\left[L_2 + \widetilde{M}_2 e^{C_1 t}\right] + C_1\int_\tau^t e^{\lambda s}\|x(s)\|\mathrm{d}s, \qquad (3.2.12) \end{aligned}$$

where $L_2 = L_1(1 + \tau(1 + C_1)) + (1/\lambda) + M_1 e^{C_1\tau}$ and $\widetilde{M}_2 = (M_1/C_1)e^{-C_1\tau}$ are positive constants.

By Lemma 2.4.2, (3.2.12) becomes

$$e^{\lambda t}\|x(t)\| \leq \|\phi\|_{\mathfrak{C}}\left[L_2 + \widetilde{M}_2 e^{C_1 t}\right] + C_1\|\phi\|_{\mathfrak{C}}\int_\tau^t \left[L_2 + \widetilde{M}_2 e^{C_1 s}\right]e^{C_1(t-s)}\mathrm{d}s. \quad (3.2.13)$$

From (3.2.13),

$$\|x(t)\| \leq \|\phi\|_{\mathfrak{C}}\left[L_2 e^{-\lambda t} + M_2 e^{(-\lambda + C_1)t}\right],$$

where $M_2 = L_2 e^{-C_1\tau} + (1 + C_1\tau)\widetilde{M}_2$ is a positive constant.

Assume when $t \in [(k-1)\tau, k\tau]$,

$$\|x(t)\| \leq \|\phi\|_{\mathfrak{C}}\left[L_k e^{-\lambda t} + M_k e^{(-\lambda + C_1)t}\right],$$

where L_k and M_k are suitable positive constants which depends on τ, λ and C_1.

Now, in the interval $t \in [k\tau, (k+1)\tau]$,

$$\begin{aligned} \|x(t)\| &\leq e^{-\lambda t}\Bigg[|\phi(0)| + e^{-\lambda\tau}\int_0^\tau e^{\lambda s}\|\phi(s-\tau)\|\mathrm{d}s + e^{-\lambda\tau}\sum_{i=2}^{k}\int_{(i-1)\tau}^{i\tau} e^{\lambda s}\|x(s-\tau)\|\mathrm{d}s \\ &\quad + e^{-\lambda\tau}\int_{k\tau}^t e^{\lambda s}\|x(s-\tau)\|\mathrm{d}s + C_1\sum_{i=1}^{k}\int_{(i-1)\tau}^{i\tau} e^{\lambda s}\|x(s)\|\mathrm{d}s + C_1\int_{k\tau}^t e^{\lambda s}\|x(s)\|\mathrm{d}s\Bigg] \end{aligned}$$

$$\leq \; e^{-\lambda t}\|\phi\|_{\mathfrak{C}}\Bigg[1+e^{-\lambda\tau}\left(\frac{e^{\lambda\tau}-1}{\lambda}\right)+\sum_{i=2}^{k}\left(\tau L_{i-1}+\frac{M_{i-1}}{C_1}e^{-C_1\tau}\left(e^{i\tau C_1}-e^{(i-1)\tau C_1}\right)\right)$$
$$+\left(\tau L_k+\frac{M_k}{C_1}e^{-C_1\tau}\left(e^{C_1 t}-e^{k\tau C_1}\right)\right)+\sum_{i=1}^{k}\left(\tau C_1 L_i+M_i\left(e^{i\tau C_1}-e^{(i-1)\tau C_1}\right)\right)\Bigg]$$
$$+C_1 e^{-\lambda t}\int_{k\tau}^{t} e^{\lambda s}\|x(s)\|\mathrm{d}s.$$

This implies

$$e^{\lambda t}\|x(t)\| \;\leq\; \|\phi\|_{\mathfrak{C}}\left[L_{k+1}+\widetilde{M}_{k+1}e^{C_1 t}\right]+C_1\int_{k\tau}^{t} e^{\lambda s}\|x(s)\|\mathrm{d}s, \tag{3.2.14}$$

where

$L_{k+1} = L_1+(1/\lambda)+\sum_{i=2}^{k}\left(\tau L_{i-1}+(M_{i-1}/C_1)e^{(i-1)\tau C_1}\right)+\tau L_k+\sum_{i=1}^{k}\left(\tau C_1 L_i+M_i e^{i\tau C_1}\right)$, $\widetilde{M}_{k+1} = (M_k/C_1)e^{-C_1\tau}$ and the remaining constants L_i, M_i $(i = 3, 4, \ldots, k-1)$ depend on τ, λ and C_1.

By Lemma 2.4.2, (3.2.14) becomes

$$e^{\lambda t}\|x(t)\| \;\leq\; \|\phi\|_{\mathfrak{C}}\left[L_{k+1}+\widetilde{M}_{k+1}e^{C_1 t}\right]$$
$$+C_1\|\phi\|_{\mathfrak{C}}\int_{k\tau}^{t}\left[L_{k+1}+\widetilde{M}_{k+1}e^{C_1 s}\right]e^{C_1(t-s)}\mathrm{d}s,$$

which implies,

$$\|x(t)\| \;\leq\; \|\phi\|_{\mathfrak{C}}\left[L_{k+1}e^{-\lambda t}+M_{k+1}e^{(-\lambda+C_1)t}\right],$$
$$< \; \delta\left[L_{k+1}e^{-\lambda t}+M_{k+1}e^{(-\lambda+C_1)t}\right], \tag{3.2.15}$$

where $M_{k+1} = L_{k+1}e^{-C_1 k\tau}+(1+C_1\tau)\widetilde{M}_{k+1}$ is a positive constant.

Now, from (3.2.15) $x(t)$ tends to zero as $t \to \infty$. Hence the zero solution of the system (3.2.1) is asymptotically stable.

3.2.2 Stabilization of nonlinear delay systems

In the previous subsection we considered stability of the nonlinear delay dynamical system. Now we consider the nonlinear control system with delay given by

$$\left.\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) + f(x(t-\tau)) + g(x(t)), \quad t > 0 \\ x(t) &= \phi(t), \quad t \in [-\tau, 0], \end{aligned}\right\} \tag{3.2.16}$$

where $u(t) \in \mathbb{R}^m$ is the control input, B is a constant $n \times m$ matrix (in practice $m \leq n$) and the remaining terms are same as defined in (3.2.1).

Definition 3.2.1. *The system (3.2.16) is stabilizable if we can find a control function $u(t)$ such that the solution of (3.2.16) corresponding to the control is asymptotically stable. In other words $\|x(t)\| \to 0$ as $t \to \infty$.*

Let us consider the linear feedback control $u(t) = Kx(t)$, then (3.2.16) becomes

$$\left.\begin{aligned} \dot{x}(t) &= \widetilde{A}x(t) + f(x(t-\tau)) + g(x(t)), \quad t > 0 \\ x(t) &= \phi(t), \quad t \in [-\tau, 0], \end{aligned}\right\} \tag{3.2.17}$$

where $\widetilde{A} = A + BK$ and K is the feedback gain matrix of order $m \times n$ which needs to be determined.

If the feedback gain matrix K is chosen such that

1. the eigenvalues of $\widetilde{A}$ entirely lies in the left-hand side of the complex plane,

2. the nonlinear functions f and g satisfy the conditions $\|f(x(t-\tau))\| \leq e^{-\lambda\tau}\|x(t-\tau)\|$ and
   $\lim_{x(t)\to 0} \|g(x(t))\|/\|x(t)\| = 0$,

3. $-\lambda + e^{-\lambda\tau} < 0$ and $-\lambda + C_1 < 0$ (where $\lambda = \min\{\lambda_i, i = 1, 2, \dots, n\}$, $-\lambda_i$'s ($\lambda_i > 0$) are the real parts of the eigenvalues of the matrix $\widetilde{A}$),

then the zero solution of the controlled system (3.2.17) is asymptotically stable. The proof is same as that of the proof of asymptotic stability of the system (3.2.1) in Section 3.2.1, by replacing A with $\widetilde{A}$.

3.3 Examples

In this section we plot all the graphs using Matlab with the algorithm given in [11].

Example 3.3.1. *Consider the following system of nonlinear delay differential equations*

$$\left.\begin{aligned}\frac{\mathrm{d}x_1(t)}{\mathrm{d}t} &= -6x_1(t)+x_2(t)+e^{-0.2\tau}\sin\left(x_2(t-\tau)\right)+x_1^2(t)+x_2^2(t)\\ \frac{\mathrm{d}x_2(t)}{\mathrm{d}t} &= -17x_1(t)+2x_2(t)+x_3(t)+e^{-0.2\tau}\tan^{-1}\left(x_3(t-\tau)\right)+x_2^2(t)+x_3^2(t)\\ \frac{\mathrm{d}x_3(t)}{\mathrm{d}t} &= -0.2x_3(t)+e^{-0.2\tau}x_1(t-\tau)+\sqrt{2x_1^2(t)x_3^2(t)-x_2^4(t)}\end{aligned}\right\}\quad(3.3.1)$$

with the initial conditions

$$\left.\begin{aligned}x_1(t) &= 0.03\\ x_2(t) &= 0.04\\ x_3(t) &= -0.02\end{aligned}\right\}\quad \text{for } t\in[-\tau,0].$$

The system (3.3.1) can be rewritten in the form of (3.2.1) with

$$A = \begin{bmatrix} -6 & 1 & 0\\ -17 & 2 & 1\\ 0 & 0 & -0.2\end{bmatrix},\quad f(x(t-\tau)) = \begin{bmatrix} e^{-0.2\tau}\sin\left(x_2(t-\tau)\right)\\ e^{-0.2\tau}\tan^{-1}\left(x_3(t-\tau)\right)\\ e^{-0.2\tau}x_1(t-\tau)\end{bmatrix},$$

$$g(x(t)) = \begin{bmatrix} x_1^2(t)+x_2^2(t)\\ x_2^2(t)+x_3^2(t)\\ \sqrt{2x_1^2(t)x_3^2(t)-x_2^4(t)}\end{bmatrix},\quad x(t)=\begin{bmatrix}x_1(t)\\ x_2(t)\\ x_3(t)\end{bmatrix},\ \text{and } \phi(t)=\begin{bmatrix}0.03\\ 0.04\\ -0.02\end{bmatrix}.$$

The eigenvalues of A *are* $-2\pm i$ *and* -0.2. *This gives* $\lambda = \min\{2, 0.2\} = 0.2$. *Now,* $\|f(x(t-\tau))\| \le e^{-0.2\tau}\|x(t-\tau)\|$ *and*

$$\begin{aligned}\lim_{x(t)\to 0}\frac{\|g(x(t)\|}{\|x(t)\|} &= \lim_{x(t)\to 0}\frac{\sqrt{(x_1^2(t)+x_2^2(t))^2+(x_2^2(t)+x_3^2(t))^2+2x_1^2(t)x_3^2(t)-x_2^4(t)}}{\sqrt{x_1^2(t)+x_2^2(t)+x_3^2(t)}}\\ &= \lim_{x(t)\to 0}\frac{x_1^2(t)+x_2^2(t)+x_3^2(t)}{\sqrt{x_1^2(t)+x_2^2(t)+x_3^2(t)}} = \lim_{x(t)\to 0}\|x(t)\| = 0.\end{aligned}$$

If we choose $\delta < 0.1$, *then by* (3.2.3), $C_1 = 0.0539$ *because* $\|g(x(t))\| = \|x(t)\|^2 =$

$0.0539\|x(t)\|$. *This implies* $-\lambda + C_1 = -0.2 + 0.0539 = -0.1461 < 0$. *Hence all the conditions given in Section 3.2.1 are satisfied for this example except the condition* $-\lambda + e^{-\lambda\tau} < 0$ *(or)* $\tau > -(\ln\lambda)/\lambda$. *If we choose* $\tau > -(\ln\lambda)/\lambda = -(\ln 0.2)/0.2 = 8.0472$, *then the system (3.3.1) becomes asymptotically stable. For instance, Figures 3.1(a) and 3.2(a) shows the solution of the system (3.3.1) corresponding to* $\tau = 8.1$ *and* $\tau = 20$ *respectively. Figures 3.1(b) and 3.2(b) shows the attractor graphs of the system (3.3.1) corresponding to* $\tau = 8.1$ *and* $\tau = 20$ *respectively. By observing Figures 3.1 and 3.2, it is clear that the zero solution of the system (3.3.1) is asymptotically stable.*

Example 3.3.2. *Consider the following system of nonlinear delay differential equations*

$$\left.\begin{aligned} \frac{\mathrm{d}x_1(t)}{\mathrm{d}t} &= -0.5x_1(t) + e^{-0.5\tau}x_3(t-\tau) + x_3^2(t) \\ \frac{\mathrm{d}x_2(t)}{\mathrm{d}t} &= -0.5x_2(t) + e^{-0.5\tau}x_1(t-\tau) + x_2(t)x_3(t) \\ \frac{\mathrm{d}x_3(t)}{\mathrm{d}t} &= -0.5x_3(t) + e^{-0.5\tau}x_2(t-\tau) + x_1(t)x_3(t) \end{aligned}\right\} \tag{3.3.2}$$

with the initial conditions

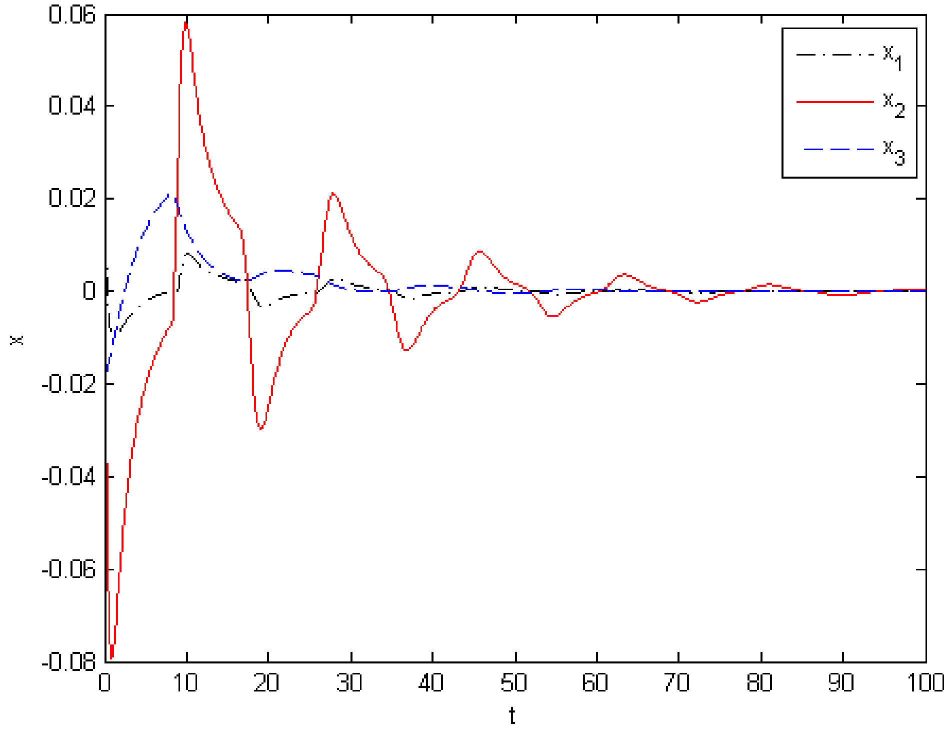
$$\left.\begin{aligned} x_1(t) &= 0.05 \\ x_2(t) &= -0.07 \\ x_3(t) &= 0.04 \end{aligned}\right\} \quad \text{for } t \in [-\tau, 0].$$

The system (3.3.2) can be rewritten in the form of (3.2.1) with

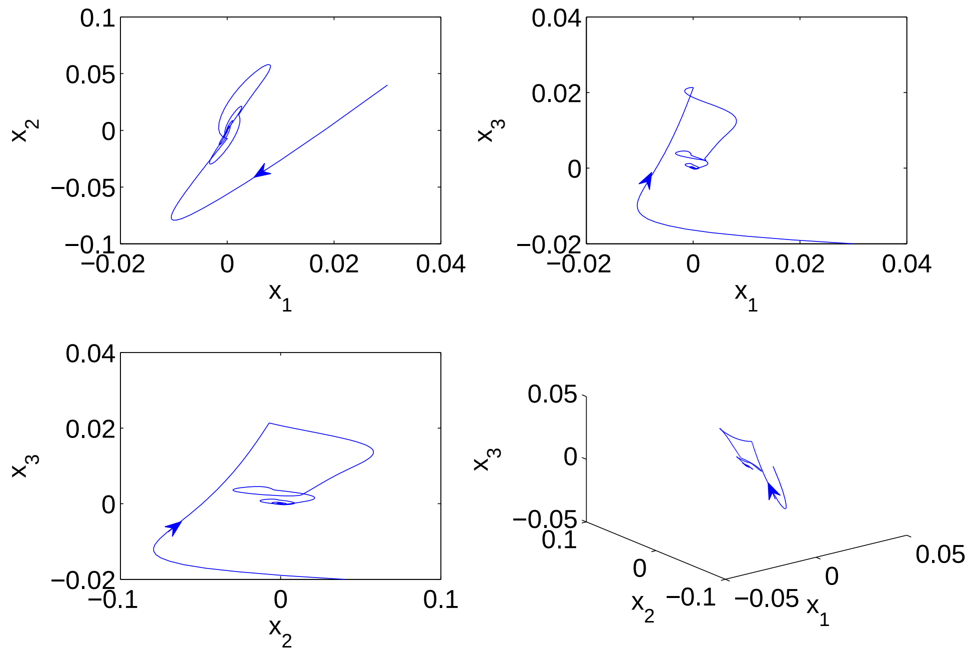
$$A = \begin{bmatrix} -0.5 & 0 & 0 \\ 0 & -0.5 & 0 \\ 0 & 0 & -0.5 \end{bmatrix}, \quad f(x(t-\tau)) = \begin{bmatrix} e^{-0.5\tau}x_3(t-\tau) \\ e^{-0.5\tau}x_1(t-\tau) \\ e^{-0.5\tau}x_2(t-\tau) \end{bmatrix},$$

$$g(x(t)) = \begin{bmatrix} x_3^2(t) \\ x_2(t)x_3(t) \\ x_1(t)x_3(t) \end{bmatrix}, \quad x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}, \quad \text{and } \phi(t) = \begin{bmatrix} 0.05 \\ -0.07 \\ 0.04 \end{bmatrix}.$$

The eigenvalues of A are $-0.5, -0.5$ *and* -0.5. *This gives* $\lambda = 0.5$. *Now,* $\|f(x(t-\tau))\| \leq$

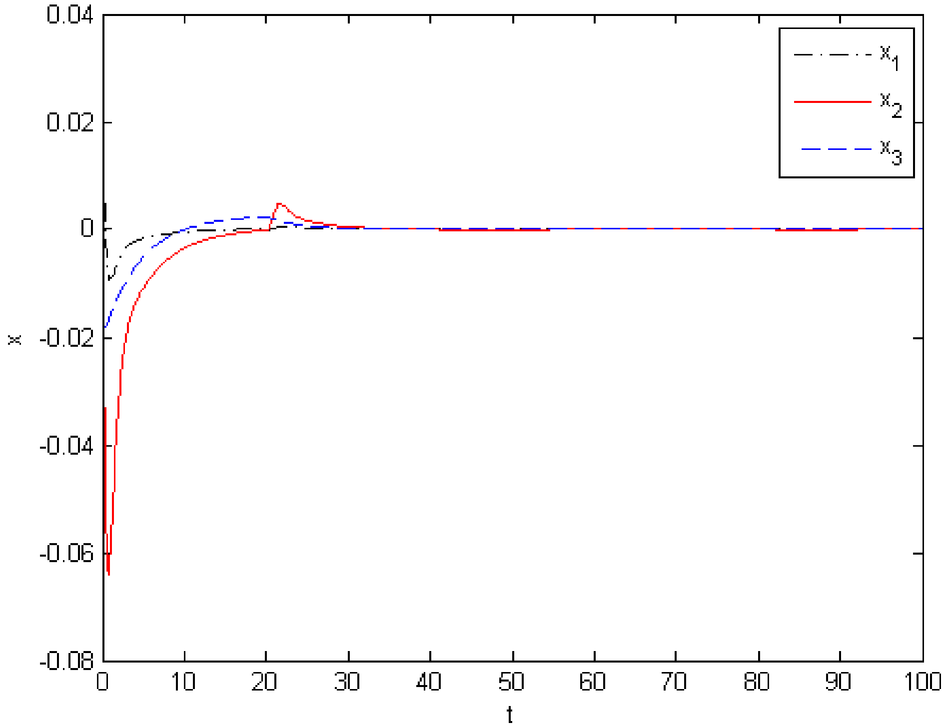


(a) solution graph

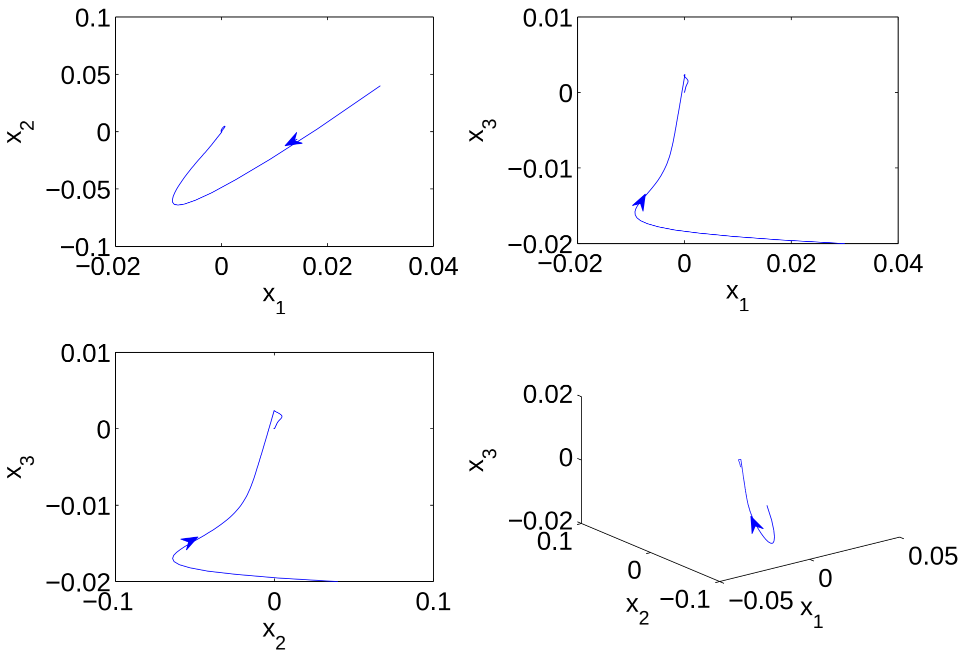


(b) Attractor graph

Figure 3.1: Solution and attractor graphs of the system (3.3.1) at $\tau = 8.1$.



(a) solution graph



(b) Attractor graph

Figure 3.2: Solution and attractor graphs of the system (3.3.1) at $\tau = 20$.

$e^{-0.5\tau}\|x(t-\tau)\|$ *and*

$$\lim_{x(t)\to 0}\frac{\|g(x(t)\|}{\|x(t)\|}=\lim_{x(t)\to 0}\frac{|x_3(t)|\sqrt{x_3^2(t)+x_2^2(t)+x_1^2(t)}}{\sqrt{x_1^2(t)+x_2^2(t)+x_3^2(t)}}=0.$$

If we choose $\delta<0.1$, *then by* (3.2.3), $C_1=0.04$ *because* $\|g(x(t))\|=|x_3(t)|\|x(t)\|=0.04\|x(t)\|$. *This implies* $-\lambda+C_1=-0.5+0.04=-0.46<0$. *Hence all the conditions given in Section 3.2.1 are satisfied for this example except the condition* $-\lambda+e^{-\lambda\tau}<0$ *(or)* $\tau>-(\ln\lambda)/\lambda$. *If we choose* $\tau>-(\ln\lambda)/\lambda=-(\ln 0.5)/0.5=1.3863$ *then the system (3.3.2) becomes asymptotically stable. Here, Figures 3.3(a), 3.4(a) and 3.5(a) shows the solution of the system (3.3.2) corresponding to* $\tau=1.6, 3$ *and* 1.3 *respectively. Figures 3.3(b),3.4(b) and 3.5(b)shows the attractor graphs of the system (3.3.2) corresponding to* $\tau=1.6,3$ *and* 1.3 *respectively. By observing Figures 3.3 and 3.4, it is clear that the zero solution of the system (3.3.2) is asymptotically stable.*

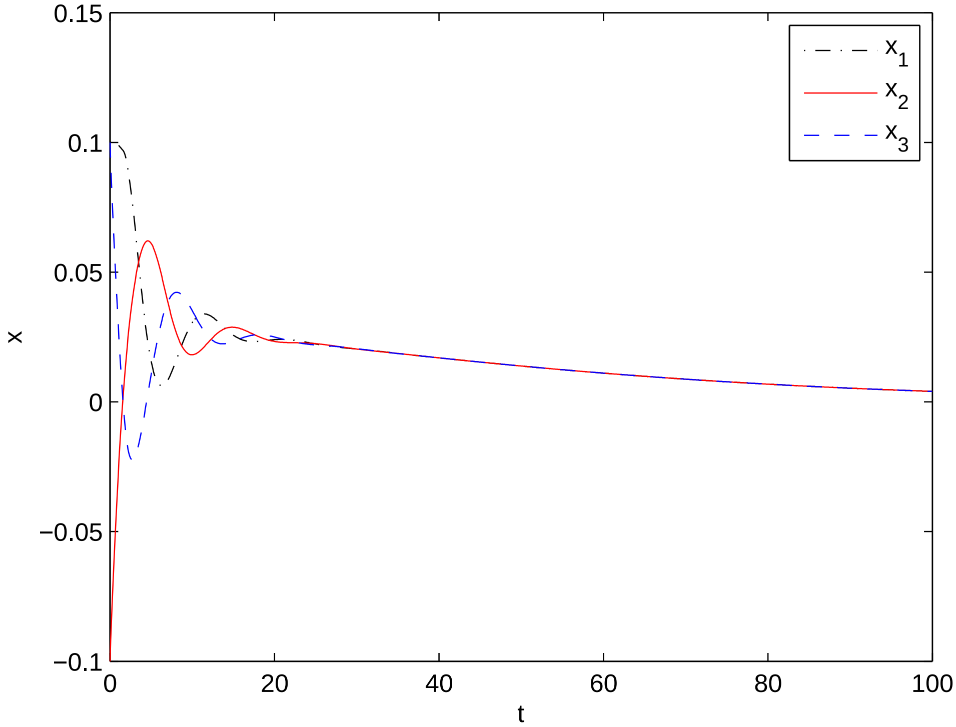
Figure 3.5 shows the system (3.3.2) is unstable at $\tau=1.3$ *(i.e. when* $\tau<1.3863$).

Example 3.3.3. *Consider the following control system*

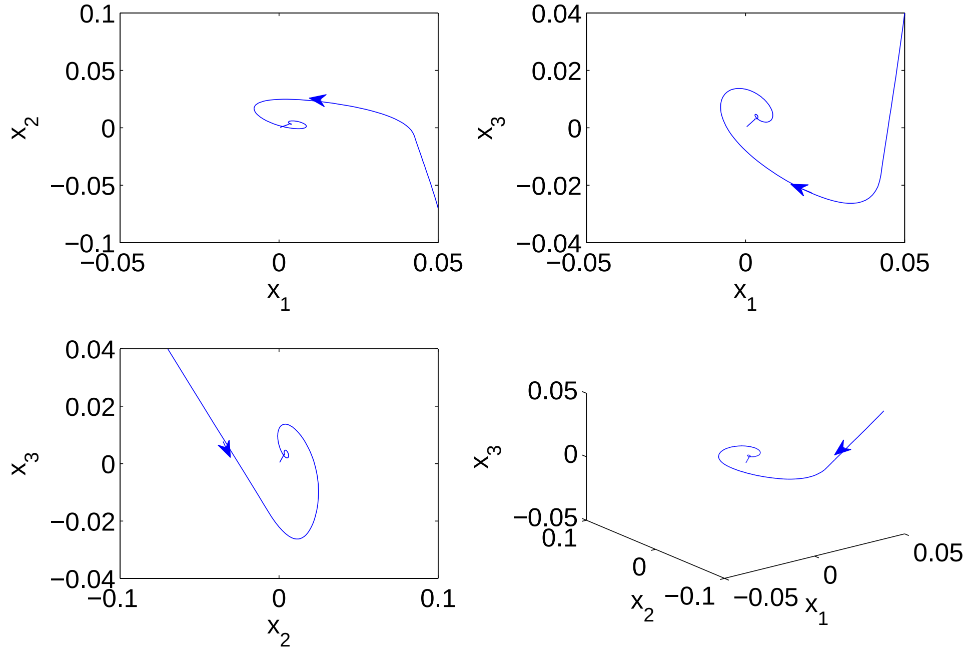
$$\left.\begin{aligned}\dot{x}_1(t) &= x_1(t)+x_2(t)+x_3(t)+u_1(t)+e^{-2\tau}x_1(t-\tau)+x_1^2(t)+x_2^2(t)\\ \dot{x}_2(t) &= 2x_1(t)+x_2(t)-x_3(t)+e^{-2\tau}\sin(x_3(t-\tau))+(x_1(t)-x_2(t))x_3(t)\\ \dot{x}_3(t) &= -2x_1(t)+x_2(t)-2x_3(t)+u_3(t)+e^{-2\tau}x_2(t-\tau)+\sqrt{2x_1(t)x_2(t)}\,x_3(t)\end{aligned}\right\}\quad(3.3.3)$$

with the initial conditions

$$\left.\begin{aligned}x_1(t) &= 0.1\\ x_2(t) &= -0.1\\ x_3(t) &= 0.1\end{aligned}\right\}\quad \text{for } t\in[-\tau,0].$$

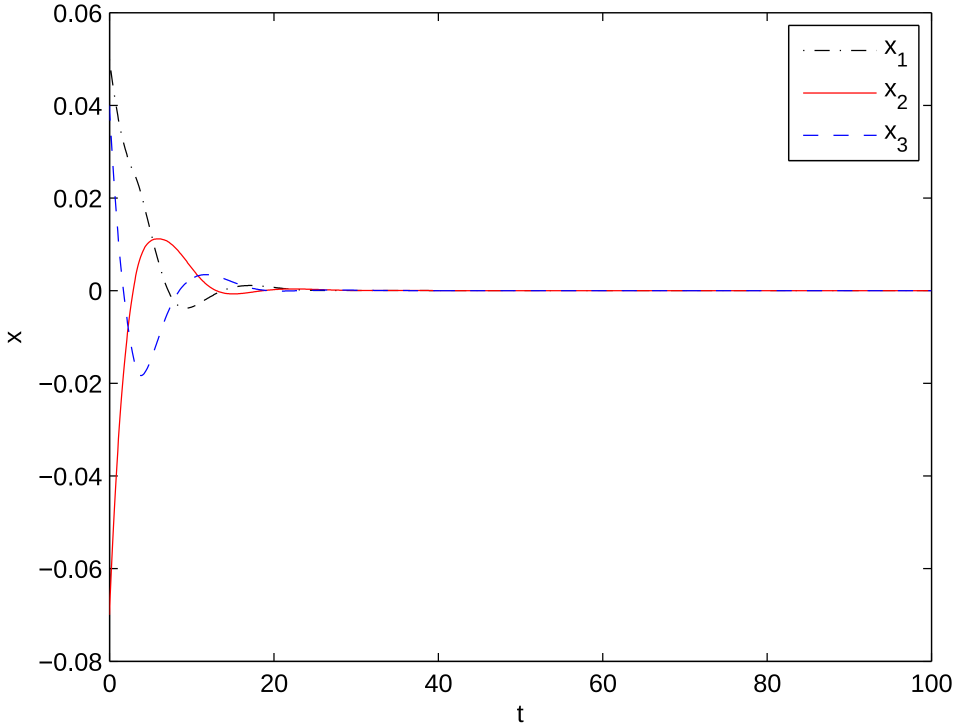


(a) solution graph

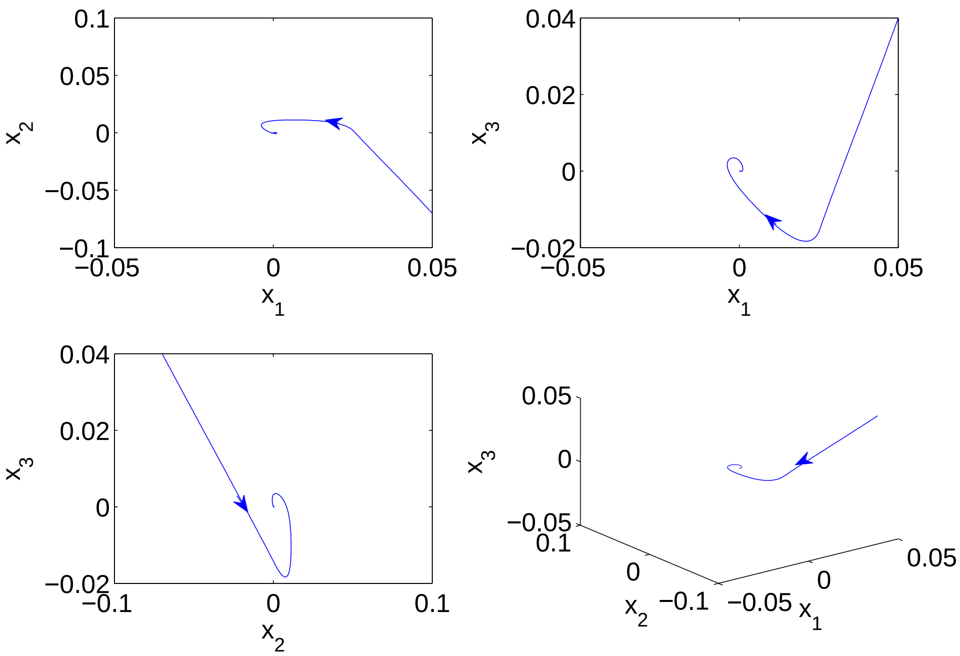


(b) Attractor graph

Figure 3.3: Solution and attractor graphs of the system (3.3.2) at $\tau = 1.6$.

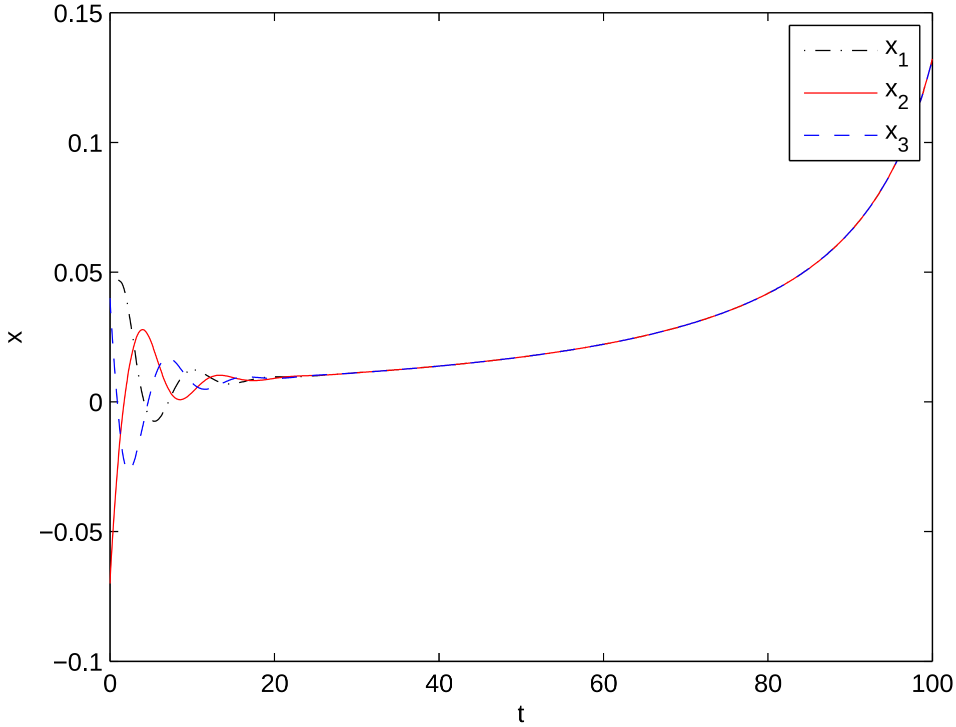


(a) solution graph

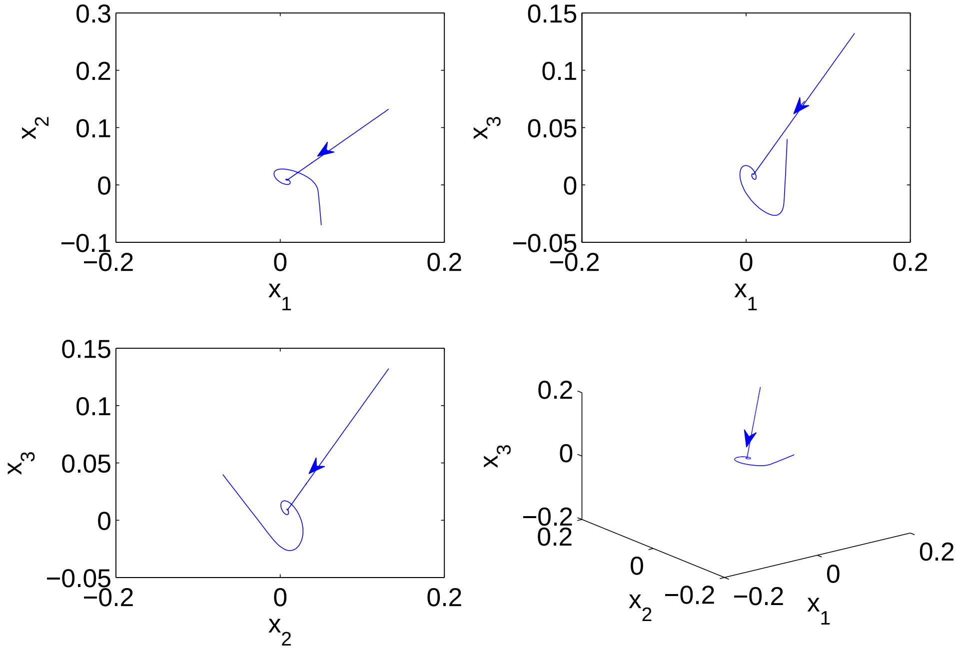


(b) Attractor graph

Figure 3.4: Solution and attractor graphs of the system (3.3.2) at $\tau = 3$.



(a) solution graph



(b) Attractor graph

Figure 3.5: Solution and attractor graphs of the system (3.3.2) at $\tau = 1.3$.

The system (3.3.3) can be rewritten in the form of (3.2.16) with

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ -2 & 1 & -2 \end{bmatrix}, \quad f(x(t-\tau)) = \begin{bmatrix} e^{-2\tau}x_1(t-\tau) \\ e^{-2\tau}\sin(x_3(t-\tau)) \\ e^{-2\tau}x_2(t-\tau) \end{bmatrix},$$

$$g(x(t)) = \begin{bmatrix} x_1^2(t)+x_2^2(t) \\ (x_1(t)-x_2(t))x_3(t) \\ \sqrt{2x_1(t)x_2(t)}\;x_3(t) \end{bmatrix}, \quad x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}, \quad \phi(t) = \begin{bmatrix} 0.1 \\ -0.1 \\ 0.1 \end{bmatrix},$$

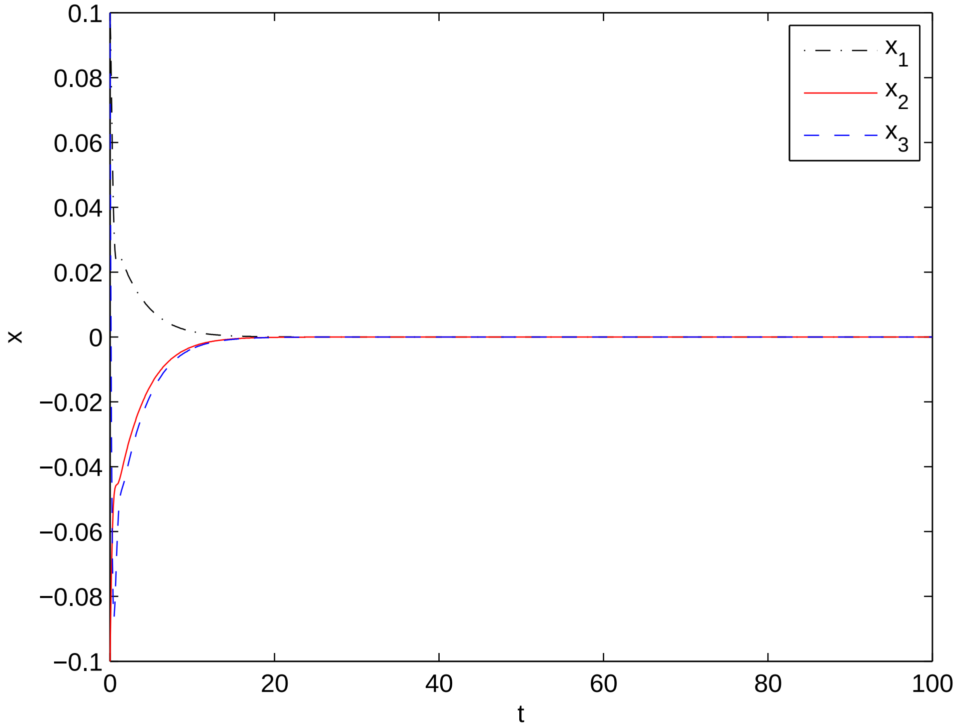
$$B = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \text{ and } u(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \end{bmatrix}.$$

Note that the matrices A and B satisfy the rank condition in Theorem 2.1.6.

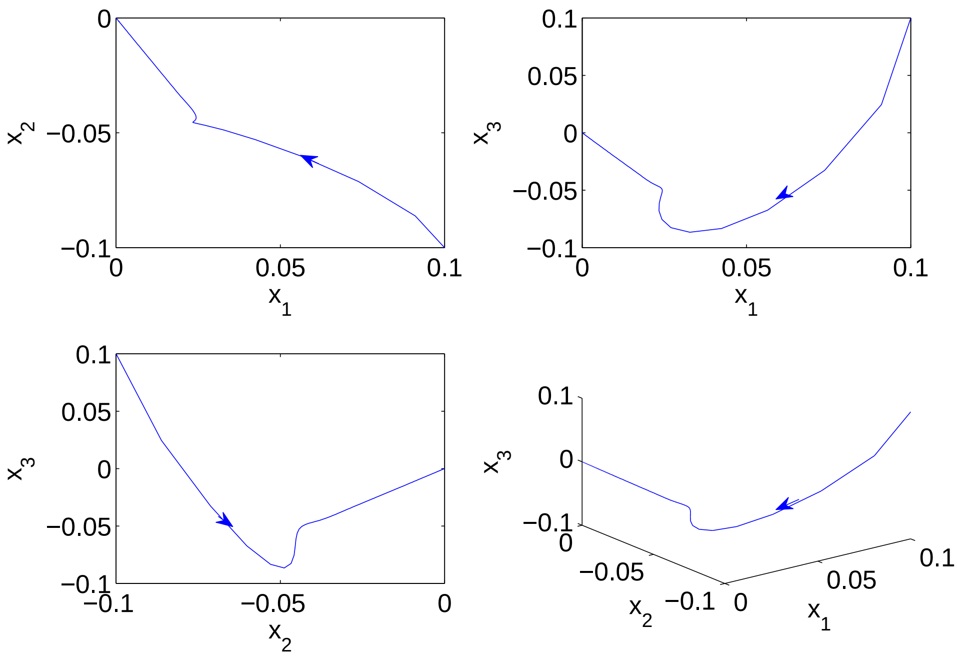
Let us consider the linear feedback control $u(t) = Kx(t)$. Suppose we choose the feedback gain matrix $K = 0.5\begin{bmatrix} -19 & -14 & 1 \end{bmatrix}$ such that the matrix $\widetilde{A} = A + BK$ has eigenvalues $-2, -3$ and -4. Then the system (3.3.3) becomes the controlled system of the form (3.2.17) with $\widetilde{A} = \begin{bmatrix} -8.5 & -6 & 1.5 \\ 2 & 1 & -1 \\ -11.5 & -6 & -1.5 \end{bmatrix}$. From this, $\lambda = \min\{2,3,4\} = 2$. Now, $\|f(x(t-\tau))\| \leq e^{-2\tau}\|x(t-\tau)\|$ and

$$\lim_{x(t)\to 0} \frac{\|g(x(t)\|}{\|x(t)\|} = \lim_{x(t)\to 0} \sqrt{x_1^2(t)+x_2^2(t)} = 0.$$

If we choose $\delta < 0.2$, then by (3.2.3), $C_1 = 0.1414$ because $\|g(x(t))\| = \sqrt{(x_1^2(t)+x_2^2(t))}\|x(t)\| = 0.1414\|x(t)\|$. This implies $-\lambda + C_1 = -1.8586 < 0$. Hence all the conditions given in Section 3.2.2 are satisfied for this example except the condition $-\lambda + e^{-\lambda\tau} < 0$ (or) $\tau > -(\ln\lambda)/\lambda$. By the condition $-\lambda + e^{-\lambda\tau} < 0$, τ is greater than $-(\ln 2)/2 = -0.3466$. Therefore, for any value of $\tau > 0$ the zero solution of the controlled system of (3.3.3) becomes asymptotically stable. For instance, the Figures 3.6 and 3.7 shows the asymptotic stability of the controlled system of (3.3.3). Here, Figures 3.6(a) and 3.7(a) shows the solutions of the controlled system of (3.3.3) corresponding to $\tau = 0.1$ and $\tau = 2$ respectively.

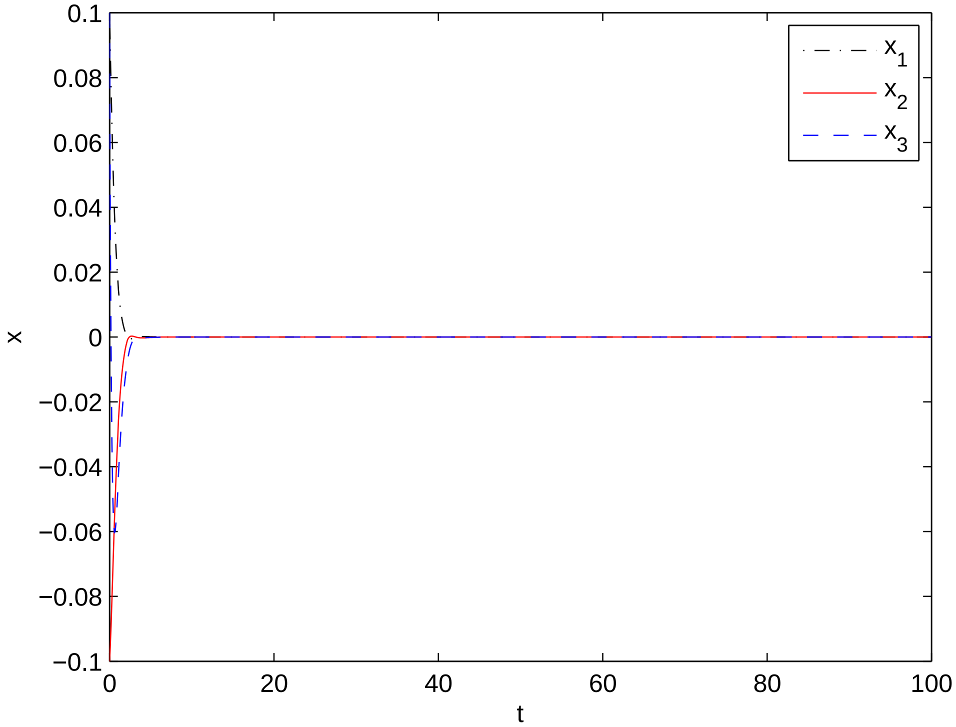


(a) solution graph

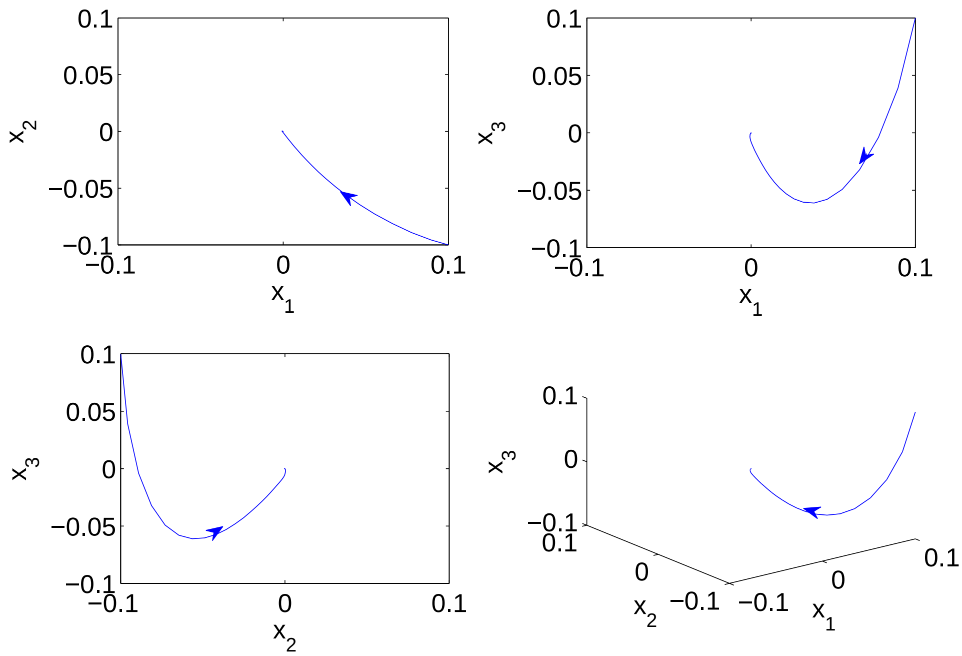


(b) Attractor graph

Figure 3.6: Solution and attractor graphs of the system (3.3.3) at $\tau = 0.1$.



(a) solution graph



(b) Attractor graph

Figure 3.7: Solution and attractor graphs of the system (3.3.3) at $\tau = 2$.

Figures 3.6(b) and 3.7(b) shows the attractor graphs of the controlled system of (3.3.3) corresponding to $\tau = 0.1$ and $\tau = 2$ respectively.

To compare our proposed theory with the Razumikhin stability theory (see, Theorem 2.1.7), we consider the Lyapunov-Razumikhin function $V = x^T x$. This implies $\dot{V}(t, x(t)) = \dot{x}^T(t)x(t) + x^T(t)\dot{x}(t)$. From (3.2.1), we obtain

$$\begin{aligned} \dot{V} &= x^T(t)A^T x(t) + f^T(x(t-\tau))x(t) + g^T(x(t))x(t) + x^T(t)Ax(t) + x^T(t)f(x(t-\tau)) \\ &\quad + x^T(t)g(x(t)). \end{aligned}$$

Taking norm on left hand side of the above equation and using the assumptions in Section 3.2, we get

$$\dot{V}(t, x(t)) \leq 2\Big(-\lambda\|x(t)\|^2 + e^{-\lambda\tau}\|x(t-\tau)\|\|x(t)\| + C_1\|x(t)\|^2 \Big). \tag{3.3.4}$$

Since $\|x(t-\tau)\| \leq \sup_{t-\tau \leq t^* \leq t} \|x(t^*)\|$, (3.3.4) becomes

$$\dot{V}(t, x(t)) \leq 2\big(-\lambda + e^{-\lambda\tau} + C_1 \big)\|x(t)\|^2, \quad t > \tau. \tag{3.3.5}$$

By Theorem 2.1.7, the expression $-\lambda + e^{-\lambda\tau} + C_1$ in (3.3.5) should be less than zero. This implies $\tau > (1/\lambda)(\ln(\lambda - C_1))$. For the asymptotic stability of the systems given in examples 3.3.1-3.3.3, the minimum value of τ are given in Table 3.1 using Razumikhin technique and our method.

Table 3.1: Comparison results.

	τ minimum using Razumikhin technique	**τ minimum through our result**
Example 3.3.1	9.4856	8.0472
Example 3.3.2	1.5531	1.3863
Example 3.3.3	0	0

3.4 Conclusion

In this chapter, the asymptotic stability and stabilizability of a class of nonlinear systems with fixed delay in the state variable has been studied. A set of sufficient conditions was developed by assuming conditions on the system parameters such as eigenvalues of the linear operator, delay parameter and bound on the nonlinear part. Then, three examples were given to testify the effectiveness of the proposed theory. The Table 3.1 shows the minimum value of τ for these three examples using Razumikhin technique with function $V = x^T x$ and our proposed theory, respectively. The system given in Example 3.3.1 is asymptotically stable when $\tau = 8.1$ which can be predicted by our method. But Razumikhin method predicts stability only for $\tau > 9.4856$. In Example 3.3.2, the system is unstable for $\tau = 1.3$ which is close to the minimum value of delay i.e., 1.3863, whereas by Lyapunov method minimum value of τ is 1.5531. Hence, Lyapunov method predicts stability at $\tau = 1.3$ which is not true.

Chapter 4

Stability of a class of fractional order bimodal piecewise nonlinear system

This chapter is concerned with the stability analysis of a class of fractional-order bimodal piecewise nonlinear systems. Firstly, the existence and uniqueness of solutions of this class of systems is established by assuming continuity of the state variable and Lipschitz continuity of the nonlinear function with respect to the state variable. Then suitable sufficient conditions for the asymptotic stability of fractional-order bimodal piecewise nonlinear systems have been proposed. Finally, two examples with numerical simulations are given to empirically testify the proposed stability conditions.

4.1 Introduction

In the existing literature, there are many works related to bimodal piecewise linear systems of the form

$$\dot{x} = \begin{cases} A_1 x(t) & \text{if } c^T x(t) \geq 0 \\ A_2 x(t) & \text{if } c^T x(t) \leq 0 \end{cases} \tag{4.1.1}$$

with the initial state $x(0) = x_0$, where $x(t) \in \mathbb{R}^n$ is the state at time t, A_1, A_2 are $n \times n$ matrices with real entries and the vector $c \in \mathbb{R}^n$.

In this chapter, we consider the following fractional-order bimodal piecewise nonlinear systems (FOBPNLS),

$$^{C}D_t^{\alpha}x = \begin{cases} A_1x(t) + f(t, x(t)) & \text{if } c^Tx(t) \geq 0 \\ A_2x(t) + f(t, x(t)) & \text{if } c^Tx(t) \leq 0 \end{cases} \tag{4.1.2}$$

with the initial state $x(0) = x_0$, where $x(t) \in \mathbb{R}^n$ is the state at time t, $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is a nonlinear vector function, A_1, A_2 are $n \times n$ matrices with real entries and the vector $c \in \mathbb{R}^n$.

Motivated by the work related to the linear model on oscillatory processes with or without fractional damping in [5, 14, 42], we consider the following nonlinear mechanical system with fractional damping to illustrate the application of FOBPNLS (4.1.2).

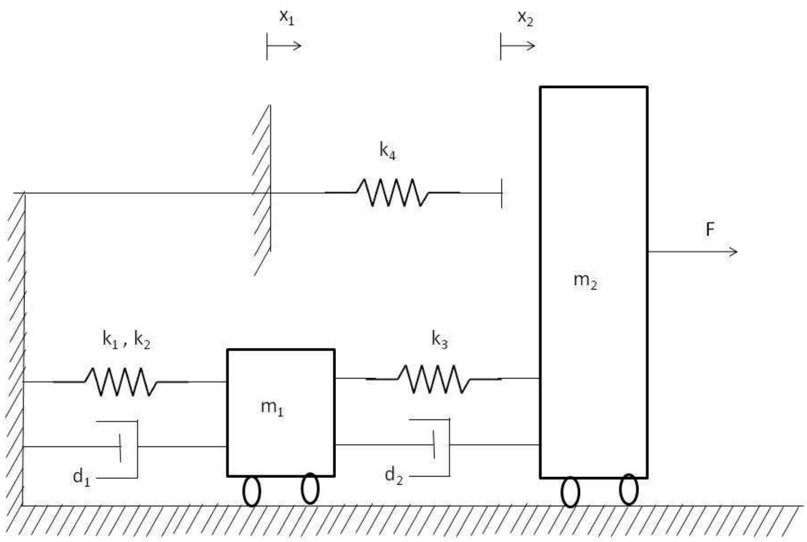


Figure 4.1: Nonlinear mechanical system with fractional damping

In the mechanical system shown in Fig. 4.1, m_1, m_2 denote the masses of the left and right carts respectively. The left end of the cart m_1 is connected to a non-linear spring with the forcing displacement relation $k_1x_1(t) + k_2x_1(t)^3$ and a damper with coefficient d_1, respectively. A linear spring with stiffness k_3 and a damper with coefficient d_2 are connected between the carts m_1 and m_2. A one-sided linear spring with stiffness k_4 is also connected with the left end of the cart m_2. Let $x_1(t)$ and $x_2(t)$ be the displacements of the carts m_1

and m_2, respectively, from the tip of the leftmost spring. If the force F is applied to the card m_2, then the equations of motion for this system is given by

$$\begin{aligned} m_1\ddot{x}_1 + k_1x_1 + k_2x_1^3 + d_1\ {}^CD_t^{\frac{3}{2}}x_1 - k_3(x_2 - x_1) - d_2\left({}^CD_t^{\frac{3}{2}}x_2 -^C D_t^{\frac{3}{2}}x_1\right) &= 0, \\ m_2\ddot{x}_2 + k_3(x_2 - x_1) + d_2\left({}^CD_t^{\frac{3}{2}}x_2 -^C D_t^{\frac{3}{2}}x_1\right) - k_4\max(-x_2, 0) &= F \end{aligned}$$

or

$$\left.\begin{aligned} m_1\ddot{x}_1 + (d_1 + d_2)\ {}^CD_t^{\frac{3}{2}}x_1 - d_2\ {}^CD_t^{\frac{3}{2}}x_2 + (k_1 + k_3)x_1 - k_3x_2 &= -k_2x_1^3 \\ m_2\ddot{x}_2 - d_2\ {}^CD_t^{\frac{3}{2}}x_1 + d_2\ {}^CD_t^{\frac{3}{2}}x_2 - k_3x_1 + k_3x_2 - k_4\max(-x_2, 0) &= F \end{aligned}\right\}, \quad (4.1.3)$$

with the initial conditions $x_1(0) = x_2(0) = \dot{x}_1(0) = \dot{x}_2(0) = 0$.

Note that x_1 and x_2 are differentiable and by considering $\ddot{x}_1 =^C \mathfrak{D}_t^{4\left(\frac{1}{2}\right)}x_1$, $\ddot{x}_2 =^C \mathfrak{D}_t^{4\left(\frac{1}{2}\right)}x_2$ then the pair of fractional differential equations (4.1.3) reduces to the following pair of sequential fractional differential equations

$$\left.\begin{aligned} m_1\ {}^C\mathfrak{D}_t^{4\alpha}x_1 + (d_1 + d_2)\ {}^C\mathfrak{D}_t^{3\alpha}x_1 - d_2\ {}^C\mathfrak{D}_t^{3\alpha}x_2 & \\ +(k_1 + k_3)x_1 - k_3x_2 &= -k_2x_1^3 \\ m_2\ {}^C\mathfrak{D}_t^{4\alpha}x_2 - d_2\ {}^C\mathfrak{D}_t^{3\alpha}x_1 + d_2\ {}^C\mathfrak{D}_t^{3\alpha}x_2 & \\ -k_3x_1 + k_3x_2 - k_4\max(-x_2, 0) &= F \end{aligned}\right\}, \quad (4.1.4)$$

where $\alpha = \frac{1}{2}$, the sequential fractional derivatives ${}^C\mathfrak{D}_t^{\alpha} =^C D_t^{\alpha}$ and ${}^C\mathfrak{D}_t^{l\alpha} =^C \mathfrak{D}_t^{\alpha}\ {}^C\mathfrak{D}_t^{(l-1)\alpha}$ $(l = 2, 3, \ldots)$ (see, fractional derivative properties in Chapter 2).

Taking ${}^C\mathfrak{D}_t^{i\alpha}x_1 = z_{i+1}$ and ${}^C\mathfrak{D}_t^{i\alpha}x_2 = \overline{z}_{i+1}$ $(i = 0, 1, 2, 3)$, equation (4.1.4) can be written in the form (4.1.2), where $A_1 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ -\frac{(k_1+k_3)}{m_1} & 0 & 0 & -\frac{(d_1+d_2)}{m_1} & \frac{k_3}{m_1} & 0 & 0 & \frac{d_2}{m_1} \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ \frac{k_3}{m_2} & 0 & 0 & \frac{d_2}{m_2} & -\frac{k_3}{m_2} & 0 & 0 & -\frac{d_2}{m_2} \end{bmatrix}$,

$$A_2 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ -\frac{(k_1+k_3)}{m_1} & 0 & 0 & -\frac{(d_1+d_2)}{m_1} & \frac{k_3}{m_1} & 0 & 0 & \frac{d_2}{m_1} \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ \frac{k_3}{m_2} & 0 & 0 & \frac{d_2}{m_2} & -\frac{(k_3+k_4)}{m_2} & 0 & 0 & -\frac{d_2}{m_2} \end{bmatrix}, \; f(t,x(t)) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -k_2 z_1^3 \\ 0 \\ 0 \\ 0 \\ F \end{bmatrix},$$

$c = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ and $x = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ \overline{z}_1 \\ \overline{z}_2 \\ \overline{z}_3 \\ \overline{z}_4 \end{bmatrix}$. The initial conditions $x_1(0) = x_2(0) = \dot{x}_1(0) = \dot{x}_2(0) = 0$ implies that $x_1(t)$ and $x_2(t)$ must be differentiable in $[0, t]$. Therefore ${}^C\mathfrak{D}_t^{i\alpha} x_1(0) = z_{i+1}(0) = {}^C\mathfrak{D}_t^{i\alpha} x_2(0) = \overline{z}_{i+1}(0) = 0, \;\; (i = 0, 1, 2, 3)$. This implies $x(0) = 0$.

We refer [13,90] to know more about sequential fractional differential equations and how to reduce higher order sequential fractional differential equations with constant coefficients to system of fractional differential equations with constant coefficients.

We state the following stability definition for the system (4.1.2) by assuming the system is well-posed i.e., the system (4.1.2) has a unique solution for the given initial state.

Definition 4.1.1. *For the system (4.1.2) with $f(t, 0) = 0$, the trivial solution $x(t) = 0$ is said to be:*

- ***stable*** *if for any given $\epsilon > 0$ there exists a $\delta > 0$ such that $\|x(0)\| < \delta$ implies $\|x(t)\| < \epsilon$, $t \geq 0$.*
- ***asymptotically stable*** *if it is stable and $\lim_{t\to\infty} x(t) = 0$.*
- ***unstable*** *if it is not stable.*

Here $\| \cdot \|$ denotes the Euclidean-norm.

4.2 Existence and uniqueness of solution

In this section, we assume without loss of generality the initial state lies in the region $c^T x \geq 0$ and then investigate the existence and uniqueness of solution.

We assume the following conditions on the matrices A_1 and A_2 and the nonlinear function f.

(A) The right hand side of (4.1.2) is continuous along the hyperplane $\{x \mid c^T x = 0\}$, i.e.

$$c^T x = 0 \quad \Rightarrow A_1 x = A_2 x,$$

(B) $f : [0, \infty) \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous in the first variable and Lipschitz continuous in the second variable. That is,

$$|f(t, y_1) - f(t, y_2)| \leq L|y_1 - y_2|,$$

where $L > 0$ is the Lipschitz constant.

Further, throughout this chapter we assumed $t_0 = 0$ as the initial time, $t_1, t_3, t_5, \ldots$ are the time instances when the solution of the system switches from the region $c^T x \geq 0$ to the region $c^T x \leq 0$ and $t_2, t_4, t_6, \ldots$ are the time instances when the solution of the system switches from the region $c^T x \leq 0$ to the region $c^T x \geq 0$ (For instance, see Figure 4.2 for two-dimensional system. In that the notation $x(t_j) \equiv (x_1(t_j), x_2(t_j))$, $j = 0, 1, 2, 3, \ldots$).

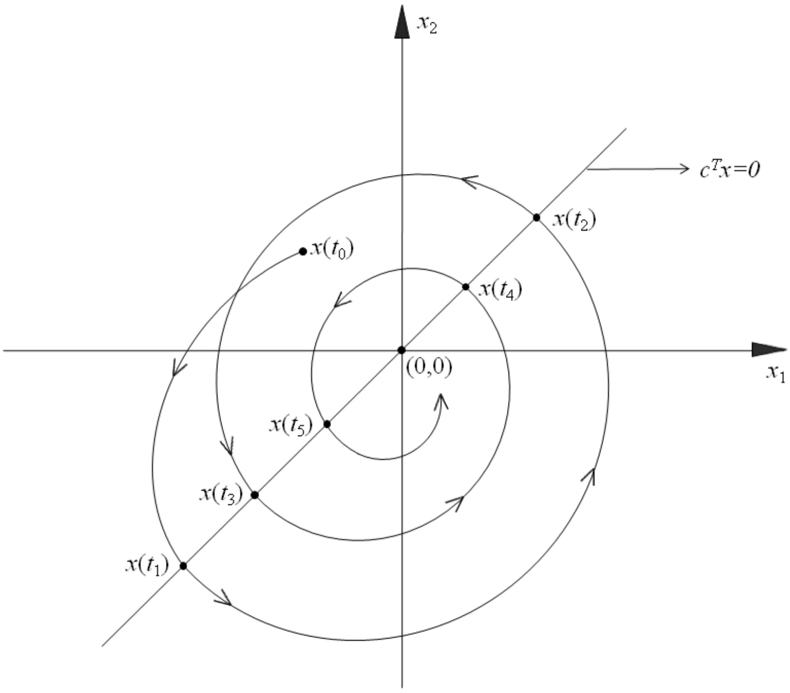


Figure 4.2: Bimodal system

When $t \in [0, t_1]$ the solution of (4.1.2) is given by (see, [61])

$$x(t) = E_{\alpha,1}[A_1 t^\alpha]x(0) + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}[A_1(t-s)^\alpha] f(s, x(s)) ds \tag{4.2.1}$$

When $t \in [t_1, t_2]$ the solution of (4.1.2) is given by

$$x(t) = E_{\alpha,1}[A_2(t-t_1)^\alpha]x(t_1) + \int_{t_1}^t (t-s)^{\alpha-1} E_{\alpha,\alpha}[A_2(t-s)^\alpha] f(s, x(s)) ds.$$

Substituting $x(t_1)$ from (4.2.1), we obtain

$$\begin{aligned} x(t) &= E_{\alpha,1}[A_2(t-t_1)^\alpha]E_{\alpha,1}[A_1t_1^\alpha]x(0) \\ &+E_{\alpha,1}[A_2(t-t_1)^\alpha]\int_0^{t_1}(t_1-s)^{\alpha-1}E_{\alpha,\alpha}[A_1(t_1-s)^\alpha]f(s,x(s))ds \\ &+\int_{t_1}^t(t-s)^{\alpha-1}E_{\alpha,\alpha}[A_2(t-s)^\alpha]f(s,x(s))ds. \end{aligned} \quad (4.2.2)$$

Proceeding in a similar way for the succeeding intervals $[t_i, t_{i+1}]$, $i = 2, 3, \ldots$, we can easily prove the following theorem:

Theorem 4.2.1. *For each $x_0 \in \mathbb{R}^n$ with the conditions (A) and (B) there exists a unique absolutely continuous solution on the interval $[0, t_{2p}], (p = 1, 2, 3, \ldots)$ for the system (4.1.2) given by*

$$x(t) = \begin{cases} z_1(t), & t \in [0, t_1], & c^T x \geq 0, \\ z_2(t), & t \in [t_1, t_2], & c^T x \leq 0, \\ z_{2i-1}(t), & t \in [t_{2i-2}, t_{2i-1}], & c^T x \geq 0, \\ z_{2i}(t), & t \in [t_{2i-1}, t_{2i}], & c^T x \leq 0, \end{cases} \quad (4.2.3)$$

where

$$\begin{aligned} z_1(t) &= E_{\alpha,1}[A_1t^\alpha]x(0) + \int_0^t(t-s)^{\alpha-1}E_{\alpha,\alpha}[A_1(t-s)^\alpha]f(s,x(s))ds, \\ z_2(t) &= E_{\alpha,1}[A_2(t-t_1)^\alpha]E_{\alpha,1}[A_1t_1^\alpha]x(0) \\ &+E_{\alpha,1}[A_2(t-t_1)^\alpha]\int_0^{t_1}(t_1-s)^{\alpha-1}E_{\alpha,\alpha}[A_1(t_1-s)^\alpha]f(s,x(s))ds \\ &+\int_{t_1}^t(t-s)^{\alpha-1}E_{\alpha,\alpha}[A_2(t-s)^\alpha]f(s,x(s))ds, \end{aligned}$$

$$
\begin{aligned}
z_{2i-1}(t) &= E_{\alpha,1}[A_1(t-t_{2i-2})^{\alpha}]E_{\alpha,1}[A_2(t_{2i-2}-t_{2i-3})^{\alpha}]\dots E_{\alpha,1}[A_2(t_2-t_1)^{\alpha}]E_{\alpha,1}[A_1t_1^{\alpha}]x(0) \\
&\quad +E_{\alpha,1}[A_1(t-t_{2i-2})^{\alpha}]E_{\alpha,1}[A_2(t_{2i-2}-t_{2i-3})^{\alpha}]\dots E_{\alpha,1}[A_2(t_2-t_1)^{\alpha}] \\
&\quad \times \int_0^{t_1}(t_1-s)^{\alpha-1}E_{\alpha,\alpha}[A_1(t_1-s)^{\alpha}]f(s,x(s))ds \\
&\quad +E_{\alpha,1}[A_1(t-t_{2i-2})^{\alpha}]E_{\alpha,1}[A_2(t_{2i-2}-t_{2i-3})^{\alpha}]\dots E_{\alpha,1}[A_1(t_3-t_2)^{\alpha}] \\
&\quad \times \int_{t_1}^{t_2}(t_2-s)^{\alpha-1}E_{\alpha,\alpha}[A_2(t_2-s)^{\alpha}]f(s,x(s))ds+\cdots \\
&\quad +E_{\alpha,1}[A_1(t-t_{2i-2})^{\alpha}]\int_{t_{2i-3}}^{t_{2i-2}}(t_{2i-2}-s)^{\alpha-1}E_{\alpha,\alpha}[A_2(t_{2i-2}-s)^{\alpha}]f(s,x(s))ds \\
&\quad +\int_{t_{2i-2}}^{t}(t-s)^{\alpha-1}E_{\alpha,\alpha}[A_1(t-s)^{\alpha}]f(s,x(s))ds,
\end{aligned}
$$

and

$$
\begin{aligned}
z_{2i}(t) &= E_{\alpha,1}[A_2(t-t_{2i-1})^{\alpha}]E_{\alpha,1}[A_1(t_{2i-1}-t_{2i-2})^{\alpha}]\dots E_{\alpha,1}[A_2(t_2-t_1)^{\alpha}]E_{\alpha,1}[A_1t_1^{\alpha}]x(0) \\
&\quad +E_{\alpha,1}[A_2(t-t_{2i-1})^{\alpha}]E_{\alpha,1}[A_1(t_{2i-1}-t_{2i-2})^{\alpha}]\dots E_{\alpha,1}[A_2(t_2-t_1)^{\alpha}] \\
&\quad \times \int_0^{t_1}(t_1-s)^{\alpha-1}E_{\alpha,\alpha}[A_1(t_1-s)^{\alpha}]f(s,x(s))ds \\
&\quad +E_{\alpha,1}[A_2(t-t_{2i-1})^{\alpha}]E_{\alpha,1}[A_1(t_{2i-1}-t_{2i-2})^{\alpha}]\dots E_{\alpha,1}[A_1(t_3-t_2)^{\alpha}] \\
&\quad \times \int_{t_1}^{t_2}(t_2-s)^{\alpha-1}E_{\alpha,\alpha}[A_2(t_2-s)^{\alpha}]f(s,x(s))ds \\
&\quad +\cdots+E_{\alpha,1}[A_2(t-t_{2i-1})^{\alpha}]\int_{t_{2i-2}}^{t_{2i-1}}(t_{2i-1}-s)^{\alpha-1}E_{\alpha,\alpha}[A_1(t_{2i-1}-s)^{\alpha}]f(s,x(s))ds \\
&\quad +\int_{t_{2i-1}}^{t}(t-s)^{\alpha-1}E_{\alpha,\alpha}[A_2(t-s)^{\alpha}]f(s,x(s))ds,
\end{aligned}
$$

for all $i=2,3,\dots$.

4.3 Stability of FOBPNLS

The Theorem (2.2.3) gives sufficient condition for the asymptotic stability of usual fractional-order system (2.2.1) with $0<\alpha\leq 1$. In this section, we propose the following conditions for the asymptotic stability for system (4.1.2):

Suppose that the assumption (A) and (B) hold and $\|x(0)\|<\delta$ ($\delta>0$ is sufficiently small). Furthermore,

1. $|\arg(\lambda_i(A_1))| > \alpha\pi/2$ and $|\arg(\tilde{\lambda}_i(A_2))| > \alpha\pi/2$, where $\lambda_i, \tilde{\lambda}_i (i = 1, 2, \ldots, n)$ denotes the eigenvalues of matrices A_1 and A_2 respectively,
2. $\alpha m > 1$, where $m = \min\{\|A_1\|, \|A_2\|\}$,
3. the nonlinear function f satisfies the condition $\lim_{x(t)\to 0}(\|f(t, x(t))\|/\|x(t)\|) = 0$.

Then, the zero solution of the nonlinear system (4.1.2) is asymptotically stable.

In the following section, we empirically validate the efficiency of the above conditions. A rigourous proof is open for research.

4.4 Numerical results

In this section, we plot all the figures using Matlab with the predictor-corrector method [33].

Example 4.4.1. ***2-dimensional FOBPNLS:***

Consider the FOBPNLS (4.1.2) *with* $x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$, $c = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ *and* $f(t, x(t)) = \begin{bmatrix} x_1(t)x_2(t) \\ x_2^2(t) \end{bmatrix}$. *Here, the nonlinear function* $f(t, x(t))$ *satisfies the condition*

$$\lim_{x(t)\to 0}\left(\frac{\|f(t, x(t))\|}{\|x(t)\|}\right) = \lim_{x_1(t), x_2(t)\to 0}\left(\frac{|x_2(t)|\sqrt{x_1(t)^2 + x_2(t)^2}}{\sqrt{x_1(t)^2 + x_2(t)^2}}\right) = 0.$$

In the following we give four pair of matrices A_1 *and* A_2 *and different fractional-orders* α *to test the validity of the stability conditions given in Section 4.3:*

(i) $A_1 = \begin{bmatrix} -3 & 2 \\ -1 & -2 \end{bmatrix}$, $A_2 = \begin{bmatrix} -1 & 0 \\ 1 & -4 \end{bmatrix}$ *and* $\alpha = 0.7$,

(ii) $A_1 = \begin{bmatrix} -3 & 2 \\ -1 & -2 \end{bmatrix}$, $A_2 = \begin{bmatrix} -1 & 0 \\ 1 & -4 \end{bmatrix}$ *and* $\alpha = 0.15$,

(iii) $A_1 = \begin{bmatrix} 10 & -6 \\ 2 & -1 \end{bmatrix}$, $A_2 = \begin{bmatrix} -1 & 5 \\ 5 & -4 \end{bmatrix}$ *and* $\alpha = 0.95$,

(iv) $A_1 = \begin{bmatrix} -4 & 5 \\ -1 & -2 \end{bmatrix}$, $A_2 = \begin{bmatrix} -3 & 4 \\ 1 & -4 \end{bmatrix}$ *and* $\alpha = 0.85$.

Table 4.1: Efficiency of stability conditions.

	(i)	(ii)	(iii)	(iv)
eigenvalues of A_1	$-2.5 \pm 1.3229i$	$-2.5 \pm 1.3229i$	$8.772, 0.228$	$-3 \pm 2i$
eigenvalues of A_2	$-4, -1$	$-4, -1$	$-7.7202, 2.7202$	$-1.4384, -5.5616$
$\lvert\arg(\lambda_1(A_1))\rvert$, $\lvert\arg(\lambda_2(A_1))\rvert$	$2.6549 > 1.1$	$2.6549 > 0.3926$	$0 \not> 1.4922$	$2.5536 > 1.3352$
$\lvert\arg(\lambda_1(A_2))\rvert$, $\lvert\arg(\lambda_2(A_2))\rvert$	$\pi > 1.1$	$\pi > 0.3926$	$0 \not> 1.4922, \pi > 1.4922$	$\pi > 1.3352$
$m = \min\{\lVert A_1\rVert, \lVert A_2\rVert\}$	3.6226	3.6226	7.7202	6.3574
αm	$2.5358 > 1$	$0.5433 \not> 1$	$7.3341 > 1$	$5.4038 > 1$
nature of stability	stable	not stable	not stable	stable

It can be seen that the conditions (A) and (B) given in Section 4.2 and conditions 1-3 given in Section 4.3 are satisfied.

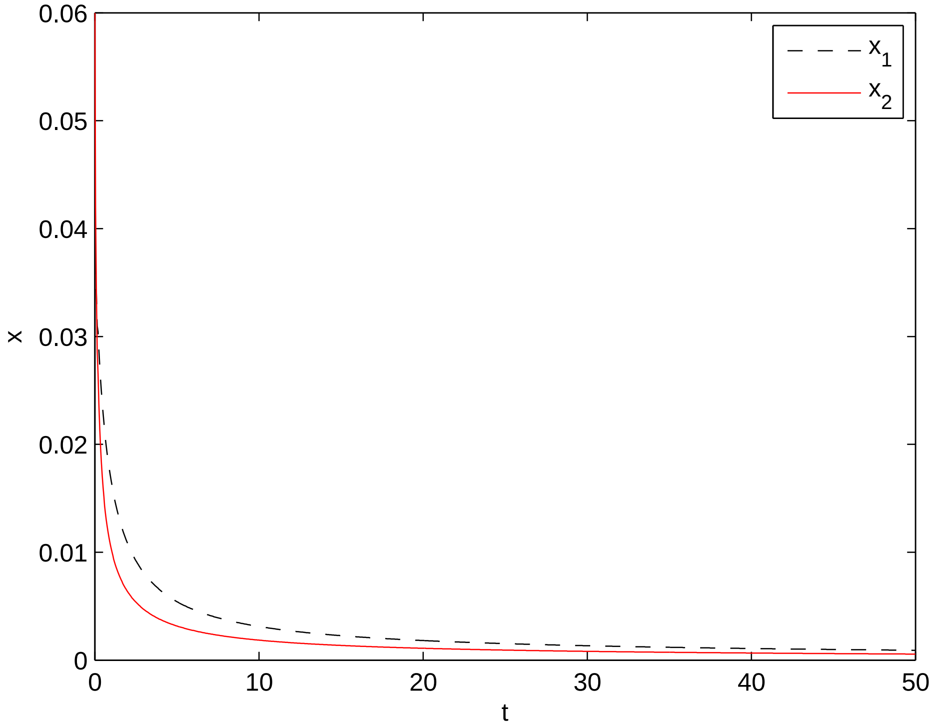
Figures 4.3–4.6 show the solution and attractor graphs for the systems (i)–(iv) respectively with initial condition $\begin{bmatrix} 0.04 \\ 0.06 \end{bmatrix}$ *and establish that the zero solution of the systems (i) and (iv) are asymptotically stable and the zero solution of the systems (ii) and (iii) are unstable.*

Example 4.4.2. *Consider the system (4.1.3) with masses* $m_1 = m_2 = 1$ *Kg, the spring constants* $k_1 = k_4 = 2$ *N/m,* $k_2 = 0.1$ *N/m,* $k_3 = 3$ *N/m, the damping co-efficients* $d_1 = 1$ *N-s/m,* $d_2 = 2$ *N-s/m and the force* $F = 0$ *N then (4.1.4) becomes*

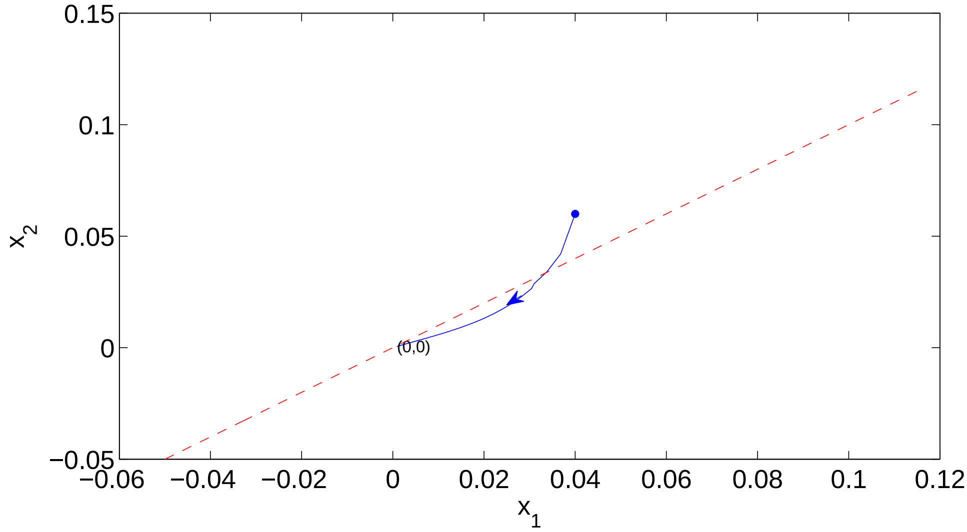
$$\left.\begin{array}{rcl} {}^C\mathfrak{D}_t^{4\alpha}x_1 + 3\,{}^C\mathfrak{D}_t^{3\alpha}x_1 - 2\,{}^C\mathfrak{D}_t^{3\alpha}x_2 + 5x_1 - 3x_2 &=& -0.1x_1^3 \\ {}^C\mathfrak{D}_t^{4\alpha}x_2 - 2\,{}^C\mathfrak{D}_t^{3\alpha}x_1 + 2\,{}^C\mathfrak{D}_t^{3\alpha}x_2 - 3x_1 + 3x_2 - 2\max(-x_2, 0) &=& 0 \end{array}\right\}, \quad (4.4.1)$$

where $\alpha = \frac{1}{2}$. *Now, (4.4.1) can be written in the form of (4.1.2) with the initial condition* $x(0) = 0$, *where*

$$A_1 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ -5 & 0 & 0 & -3 & 3 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 3 & 0 & 0 & 2 & -3 & 0 & 0 & -2 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ -5 & 0 & 0 & -3 & 3 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 3 & 0 & 0 & 2 & -5 & 0 & 0 & -2 \end{bmatrix},$$

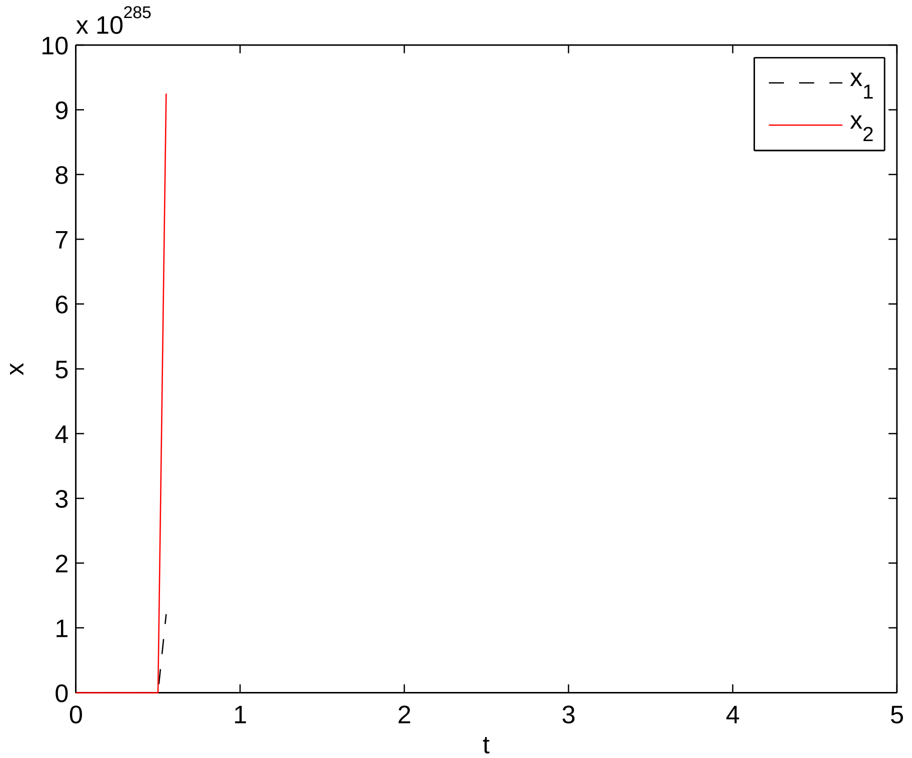


(a) Solution

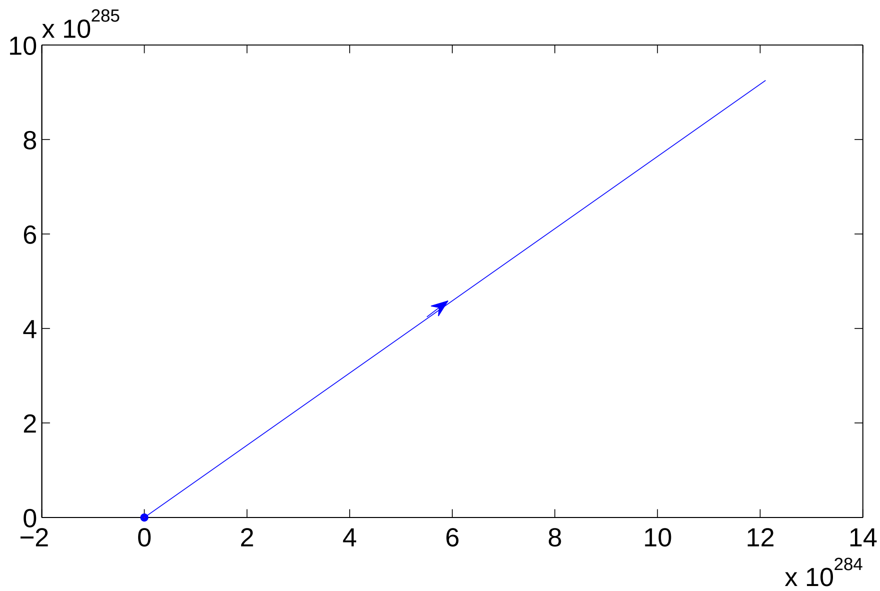


(b) Attractor

Figure 4.3: Solution and attractor graphs for (i).

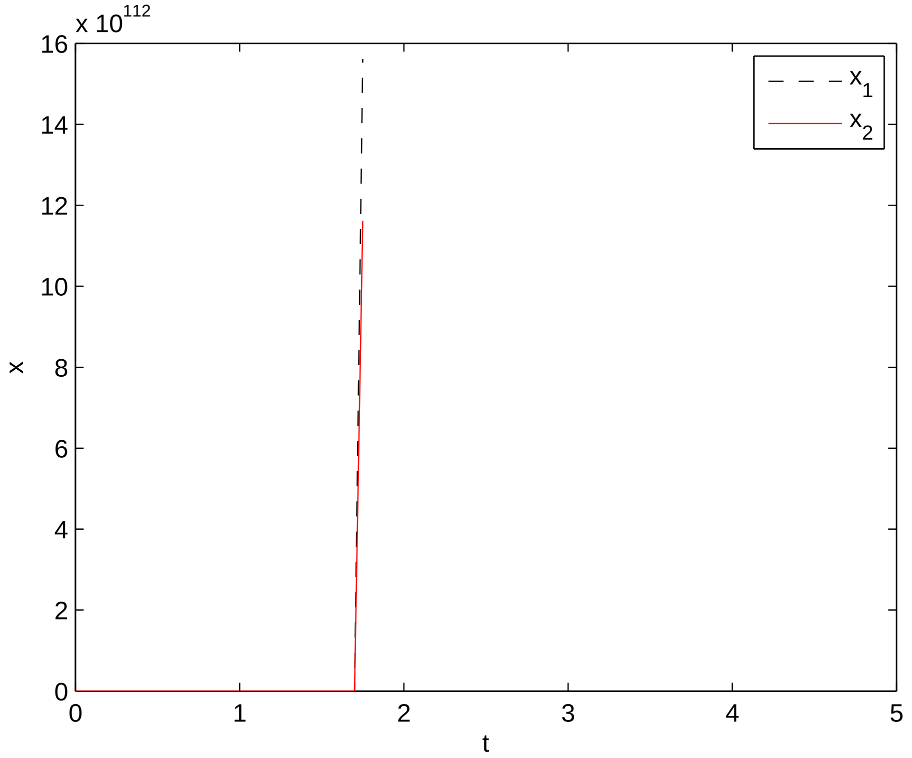


(a) Solution

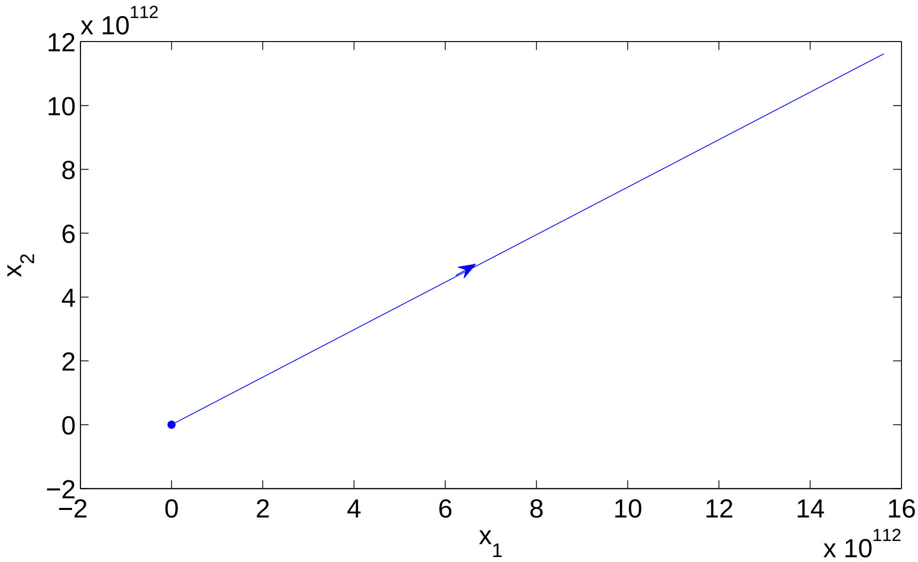


(b) Attractor

Figure 4.4: Solution and attractor graphs for (ii).

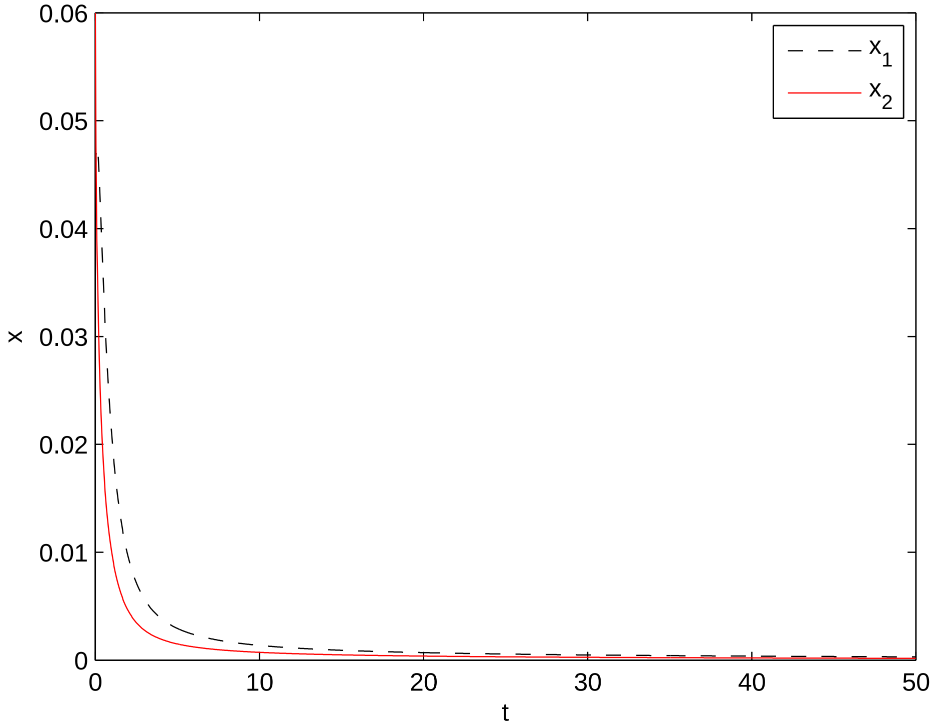


(a) Solution

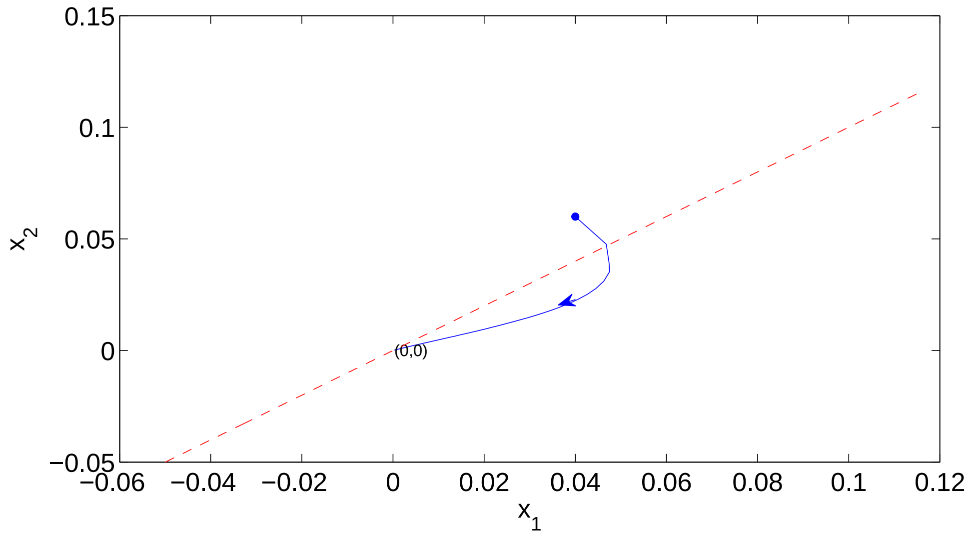


(b) Attractor

Figure 4.5: Solution and attractor graphs for (iii).



(a) Solution



(b) Attractor

Figure 4.6: Solution and attractor graphs for (iv).

$c = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, $x = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ \overline{z}_1 \\ \overline{z}_2 \\ \overline{z}_3 \\ \overline{z}_4 \end{bmatrix}$ *and* $f(t, x(t)) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -0.1z_1^3 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$. *The eigenvalues of* A_1 *are* $\{-4.4821, -1.2976, -0.8005 \pm 0.6603i, 0.6121 \pm 0.9256i, 0.5782 \pm 0.6661i\}$ *and the eigenvalues of* A_2 *are* $\{-4.4732, -1.3424, -0.9693 \pm 0.8316i, 0.6484 \pm 0.9363i, 0.7287 \pm 0.8535i\}$. *Now,*

$$\begin{aligned}
|\arg(-4.4821)| &= 3.1416 > \alpha\frac{\pi}{2} = 0.7854 \\
|\arg(-1.2976)| &= 3.1416 > \alpha\frac{\pi}{2} = 0.7854 \\
|\arg(-0.8005 \pm 0.6603i)| &= 2.4519 > 0.7854 \\
|\arg(0.6121 \pm 0.9256i)| &= 0.9865 > 0.7854 \\
|\arg(0.5782 \pm 0.6661i)| &= 0.8559 > 0.7854 \\
|\arg(-4.4732)| &= 3.1416 > 0.7854 \\
|\arg(-1.3424)| &= 3.1416 > 0.7854 \\
|\arg(-0.9693 \pm 0.8316i)| &= 2.4325 > 0.7854 \\
|\arg(0.6484 \pm 0.9363i)| &= 0.9651 > 0.7854 \\
|\arg(0.7287 \pm 0.8535i)| &= 0.8641 > 0.7854
\end{aligned}$$

and $m = \min\{\|A_1\|, \|A_2\|\} = \min\{8.5074, 9.2100\} = 8.5074 > 1$. *Here, the function* $f(t, x(t))$ *satisfies the condition*

$$\lim_{x(t)\to 0} \frac{0.1z_1^3}{\sqrt{\sum_{i=1}^{4}(z_i^2 + \overline{z}_1^2)}} = 0.$$

Therefore, using stability conditions given in Section 4.3 the non-linear system (4.4.1) is asymptotically stable.

4.5 Conclusion

In this chapter, a class of fractional-order bimodal piecewise nonlinear systems is considered. Firstly, the existence and uniqueness of solution for this class of systems has been established by assuming continuity of the state variable and Lipschitz continuity of the nonlinear function with respect to the state variable. Furthermore, a set of sufficient conditions for the stability of the systems has been proposed. Then numerically validate the efficiency of these conditions through some examples.

Chapter 5

Sensitivity analysis of nonlinear fractional order control systems with delay

In this chapter, we consider a class of nonlinear fractional order control system with delay in state variable. Existence and uniqueness of solution is shown by using method of steps. The sensitivity of the state and control with respect to the parameters of the system is shown. Finally, analytical results are substantiated by numerical examples.

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5.1 Introduction

In [92], Morgado et al. studied the existence and uniqueness of solution of a linear fractional order differential equation with delay by using method of steps and also studied dependence of the solution on the parameters of the equation. To the best of our knowledge sensitivity analysis of solution of nonlinear fractional order control system with delay in state variable is not available in the literature so far.

In this chapter, we extend the analysis and numerical methods of linear fractional order differential equations with delay presented in [92] to a class of nonlinear fractional order control system. For that, we consider the following nonlinear fractional order control system with finite delay $\tau > 0$:

$$ {}^{C}D^{\alpha}y(t) = ay(t) + bu(t) + f(t, y(t-\tau))\ ; \quad 0 < t \leq T \tag{5.1.1} $$

$$ y(t) = \phi(t), \quad t \in [-\tau, 0] \quad \text{and} \quad y(T) = y_T \tag{5.1.2} $$

where a and b are scalar constants, $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ is a nonlinear function which is Lipschitz continuous in second variable, the initial function ϕ is continuous on $[-\tau, 0]$, $u(t), 0 \leq t \leq T$ is the control variable which drives given system from $\phi(0)$ to y_T and ${}^{C}D^{\alpha}$ is the Caputo derivative of order α, $\quad 0 < \alpha < 1$.

5.2 Existence and uniqueness of solution

In what follows we assume the following condition on the nonlinear function f.

(A) $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ is continuous in the first variable and Lipschitz continuous in the second variable. That is,

$$ |f(t, x_1) - f(t, x_2)| \leq L|x_1 - x_2|, $$

where L is the Lipschitz constant.

Our aim is to prove the existence and uniqueness of solution of (5.1.1)-(5.1.2) and find a control $u(t)$ suitably. Let $y_1, y_2, \ldots, y_k$ be arbitrary real numbers, $T \in [k\tau, (k+1)\tau]$; $y(i\tau) = y_i, i = 1, 2, \ldots, k$.

First consider the interval $0 \leq t \leq \tau$. Let us assume $y(\tau) = y_1$. Here, since $y(t-\tau) = \phi(t-\tau)$, (5.1.1) becomes

$$ {}^{C}D^{\alpha}y(t) = ay(t) + bu_1(t) + g_\tau(t, y(t-\tau)), \quad 0 < t \leq \tau, $$

where $g_\tau(t, y(t-\tau)) = f(t, \phi(t-\tau))$. Since f is Lipschitz continuous, g_τ is continuous function of t, for a given control function $u_1(t)$ there exists a unique solution for (5.1.1) in the interval $[0, \tau]$. Its solution is of the form [21, 61],

$$y_\tau(t) = E_\alpha(at^\alpha)\phi(0) + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(a(t-s)^\alpha)[bu_1(s) + f(s, \phi(s-\tau))]\mathrm{d}s. \quad (5.2.1)$$

Now, if we take the control

$$\begin{aligned} u_1(t) &= -(\tau-t)^{1-\alpha} b E_{\alpha,\alpha}[a(\tau-t)^\alpha] Q_\tau^{-1} \Bigg[-y_1 + E_\alpha[a\tau^\alpha]\phi(0) + \int_0^\tau (\tau-s)^{\alpha-1} \\ & \quad \times E_{\alpha,\alpha}[a(\tau-s)^\alpha] f(s, \phi(s-\tau))\mathrm{d}s \Bigg], \end{aligned} \quad (5.2.2)$$

where $Q_\tau = b^2 \int_0^\tau \left[E_{\alpha,\alpha}[a(\tau-s)^\alpha]\right]^2 \mathrm{d}s$, then the system is drives from $\phi(0)$ to y_1.

Hence in the interval $[0, \tau]$, the solution of (5.1.1)-(5.1.2) exists and is unique.

Now, in the interval $[0, 2\tau]$, let us assume $y(2\tau) = y_2$. The equation (5.1.1) may be written as

$${}^C D^\alpha y(t) = ay(t) + bu_{2\tau}(t) + g_{2\tau}(t, y(t-\tau)), \quad 0 < t \le 2\tau,$$

where

$$g_{2\tau}(t, y(t-\tau)) = \begin{cases} f(t, \phi(t-\tau)), & 0 < t \le \tau \\ f(t, y_\tau(t-\tau)), & \tau < t \le 2\tau \end{cases}$$

and the control

$$u_{2\tau}(t) = \begin{cases} u_1(t), & 0 \le t \le \tau \\ u_2(t), & \tau \le t \le 2\tau \end{cases}.$$

Then, the solution of (5.1.1) in the interval $[0, 2\tau]$ is given by

$$y(t) = \begin{cases} y_\tau(t), & 0 \le t \le \tau \\ y_{2\tau}(t), & \tau \le t \le 2\tau \end{cases} \quad (5.2.3)$$

where $y_{2\tau}(t) = E_\alpha[a(t-\tau)^\alpha]y_1 + \int_\tau^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(a(t-s)^\alpha)[bu_2(s) + f(s, y_\tau(s-\tau))]\mathrm{d}s, \ \tau \le t \le 2\tau$ and $y_\tau(t)$ is given by (5.2.1).

If we take

$$u_2(t) = -(2\tau - t)^{1-\alpha} b E_{\alpha,\alpha}[a(2\tau - t)^{\alpha}] Q_{2\tau}^{-1}\Big[- y_2 + E_{\alpha}[a\tau^{\alpha}]y_1 + \int_{\tau}^{2\tau} (2\tau - s)^{\alpha-1} \times E_{\alpha,\alpha}[a(2\tau - s)^{\alpha}] f(s, y_{\tau}(s-\tau))\mathrm{d}s\Big], \quad (5.2.4)$$

where $Q_{2\tau} = b^2 \int_{\tau}^{2\tau} \big[E_{\alpha,\alpha}[a(2\tau - s)^{\alpha}]\big]^2 \mathrm{d}s$ and $u_1(t)$ is as given in (5.2.2), then the system reaches the point y_2 at $t = 2\tau$. Thus the control $u_{2\tau}(t)$ drives the given system from $\phi(0)$ to y_2.

Proceeding in a similar way we can easily prove the following theorem:

Theorem 5.2.1. *Let $y_{0\tau}(t) = \phi(t)$ and let k be the greatest positive integer such that $k\tau \leq T$ and let*

$$g_{k\tau}(t, y(t-\tau)) = \begin{cases} f(t, y_{0\tau}(t-\tau)), & 0 < t \leq \tau \\ f(t, y_{\tau}(t-\tau)), & \tau < t \leq 2\tau \\ \quad \vdots & \\ f(t, y_{(k-1)\tau}(t-\tau)), & (k-1)\tau < t \leq k\tau \end{cases} \quad (5.2.5)$$

be continuous.

Then, there exists a unique solution on the interval $[0, T]$ *for the control system* (5.1.1)-(5.1.2) *with* $y(i\tau) = y_i$ *given by*

$$y(t) = \begin{cases} y_{i\tau}(t), & t \in [(i-1)\tau, i\tau], \ i = 1, 2, \ldots k, \\ y_T(t), & t \in [k\tau, T] \end{cases}$$

where

$$y_{i\tau}(t) = E_{\alpha}[a(t-(i-1)\tau)^{\alpha}]y_{i-1} + \int_{(i-1)\tau}^{t} (t-s)^{\alpha-1} E_{\alpha,\alpha}(a(t-s)^{\alpha})[bu_i(s) + g_{i\tau}(s, y(s-\tau))]\mathrm{d}s, \quad t \in [(i-1)\tau, i\tau],$$

$$y_T(t) = E_{\alpha}[a(t-k\tau)^{\alpha}]y_k + \int_{k\tau}^{t} (t-s)^{\alpha-1} E_{\alpha,\alpha}(a(t-s)^{\alpha})[bu_T(s) + f(s, y_k(s-\tau))]\mathrm{d}s, \ t \in [k\tau, T]$$

and the control function is given by

$$\begin{aligned} u_i(t) &= -(i\tau - t)^{1-\alpha} b E_{\alpha,\alpha}[a(i\tau - t)^\alpha] Q_{i\tau}^{-1} \Bigg[- y_i + E_\alpha[a\tau^\alpha] y_{i-1} + \int\limits_{(i-1)\tau}^{i\tau} (i\tau - s)^{\alpha-1} \\ &\quad \times E_{\alpha,\alpha}[a(i\tau - s)^\alpha] f(s, y_{(i-1)\tau}(s-\tau)) \mathrm{d}s \Bigg], i = 1, 2, \ldots, k, \\ u_T(t) &= -(T - t)^{1-\alpha} b E_{\alpha,\alpha}[a(T - t)^\alpha] Q_T^{-1} \Bigg[- y_T + E_\alpha[a(T - k\tau)^\alpha] y_k + \int\limits_{k\tau}^{T} (T - s)^{\alpha-1} \\ &\quad \times E_{\alpha,\alpha}[a(T - s)^\alpha] f(s, y_{k\tau}(s-\tau)) \mathrm{d}s \Bigg], \end{aligned}$$

where the modified controllability Grammian are defined by $Q_{i\tau} = b^2 \int\limits_{(i-1)\tau}^{i\tau} \big[E_{\alpha,\alpha}[a(i\tau - s)^\alpha]\big]^2 \mathrm{d}s$ *and* $Q_T = b^2 \int\limits_{k\tau}^{T} \big[E_{\alpha,\alpha}[a(T - s)^\alpha]\big]^2 \mathrm{d}s$*,*
for all $i = 1, 2, \ldots, k$*.*

5.3 Sensitivity of the solution

5.3.1 Dependence of solution on small perturbation of ϕ

Let $y(t)$ be the unique solution of the control system (5.1.1)-(5.1.2) and $z(t)$ be the solution of

$$\begin{aligned} {}^C D^\alpha z(t) &= a z(t) + b\widetilde{u}(t) + f(t, z(t-\tau)), \quad 0 < t \leq T \qquad (5.3.1) \\ z(t) &= \widetilde{\phi}(t), \quad t \in [-\tau, 0], z(T) = y_T \qquad (5.3.2) \end{aligned}$$

where $\widetilde{\phi}$ is also continuous on $[-\tau, 0]$. Assume that

$$\|\phi - \widetilde{\phi}\| = \max_{-\tau \leq t \leq 0} |\phi(t) - \widetilde{\phi}(t)| \leq \epsilon, \qquad (5.3.3)$$

for some $\epsilon > 0$.

Now the dependence of $|y(t) - z(t)|$ on $\|\phi - \widetilde{\phi}\|$ is stated in the following theorem.

Theorem 5.3.1. *Let us assume that there exists a unique solution $y(t)$ and $z(t)$ for the systems* (5.1.1)-(5.1.2) *and* (5.3.1)-(5.3.2), *respectively on the interval* $[0,T]$. *Assume that* (5.3.3) *holds. Then, we have*

$$|y(t)-z(t)| \leq \epsilon A_k E_\alpha[|a|(k\tau)^\alpha], \quad t \in [(k-1)\tau, k\tau], \tag{5.3.4}$$

and

$$|y(t)-z(t)| \leq \epsilon H E_\alpha[|a|T^\alpha], \quad t \in [k\tau, T], \tag{5.3.5}$$

where $A_0 = 1$,

$$A_k = 1 + \frac{1}{\Gamma(\alpha+1)} \sum_{j=0}^{k-1} \left\{ [(k-j)\tau]^\alpha \left[|b|^2 P_{j+1} + L A_j E_\alpha[|a|(j\tau)^\alpha] \right] \right\}, \tag{5.3.6}$$

$$\begin{aligned} P_m &= (m\tau)^{1-\alpha} E_{\alpha,\alpha}[|a|(m\tau)^\alpha] Q_{m\tau}^{-1} A_{m-1} E_\alpha[|a|((m-1)\tau)^\alpha] L \\ &\quad \times \int_{(m-1)\tau}^{m\tau} (m\tau - s_1)^{\alpha-1} E_{\alpha,\alpha}[|a|(m\tau - s_1)^\alpha] \mathrm{d}s_1 \end{aligned} \tag{5.3.7}$$

and

$$\begin{aligned} H &= 1 + \frac{1}{\Gamma(\alpha+1)} \sum_{j=1}^{k} T^\alpha \left\{ |b|^2 P_j + L A_{j-1} E_\alpha[|a|((j-1)\tau)^\alpha] \right\} + \frac{T^\alpha L A_k E_\alpha[|a|(k\tau)^\alpha]}{\Gamma(\alpha+1)} \\ &\quad \times \left[|b|^2 T^{1-\alpha} E_{\alpha,\alpha}[|a|T^\alpha] Q_T^{-1} \int_{k\tau}^{T} (T-s_1)^{\alpha-1} E_{\alpha,\alpha}[|a|(T-s_1)^\alpha] \mathrm{d}s_1 + 1 \right] \end{aligned} \tag{5.3.8}$$

for all $k = 1, 2, 3, \ldots,$ *and* $m = 1, 2, 3, \ldots, k$.

Proof. Taking fractional integral I_t^α on both sides of (5.1.1) and (5.3.1), we get

$$\begin{aligned} y(t) - z(t) &= \phi(0) - \widetilde{\phi}(0) + \frac{a}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} (y(s) - z(s)) \mathrm{d}s + \frac{b}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \\ &\quad \times [u(s) - \widetilde{u}(s)] \mathrm{d}s + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} [f(s, y(s-\tau)) - f(s, z(s-\tau))] \mathrm{d}s, \end{aligned}$$

$$\begin{aligned}|y(t)-z(t)| &\leq |\phi(0)-\widetilde{\phi}(0)| + \frac{|a|}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|y(s)-z(s)|\mathrm{d}s + \frac{|b|}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} \\ &\quad \times |u(s)-\widetilde{u}(s)|\mathrm{d}s + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left[L|y(s-\tau)-z(s-\tau)|\right]\mathrm{d}s, \qquad (5.3.9)\end{aligned}$$

where L is a Lipschitz constant as in assumption (A).

We use induction on k to prove this theorem.

Let $k=1$ and $t\in[0,\tau]$. From Theorem 5.2.1, the control for the system (5.3.1)-(5.3.2) is given by

$$\begin{aligned}\widetilde{u}_1(t) &= -(\tau-t)^{1-\alpha} b E_{\alpha,\alpha}[a(\tau-t)^\alpha] Q_\tau^{-1}\Bigg[-y_1 + E_\alpha[a\tau^\alpha]\widetilde{\phi}(0) \\ &\quad + \int_0^\tau (\tau-s)^{\alpha-1}E_{\alpha,\alpha}[a(\tau-s)^\alpha] f(s,\widetilde{\phi}(s-\tau))\mathrm{d}s\Bigg],\end{aligned}$$

where $Q_\tau = b^2\int_0^\tau \left[E_{\alpha,\alpha}[a(\tau-s)^\alpha]\right]^2\mathrm{d}s$. Now,

$$\begin{aligned}|u_1(t)-\widetilde{u}_1(t)| &\leq \tau^{1-\alpha}|b|E_{\alpha,\alpha}[|a|\tau^\alpha]Q_\tau^{-1}\Bigg[E_\alpha[|a|\tau^\alpha]\epsilon \\ &\quad + L\epsilon\int_0^\tau (\tau-s)^{\alpha-1}E_{\alpha,\alpha}[|a|(\tau-s)^\alpha]\mathrm{d}s\Bigg] \\ &= \epsilon|b|P_1,\end{aligned}$$

where

$$P_1 = \tau^{1-\alpha}E_{\alpha,\alpha}[|a|\tau^\alpha]Q_\tau^{-1}\left[E_\alpha[|a|\tau^\alpha] + L\int_0^\tau (\tau-s)^{\alpha-1}E_{\alpha,\alpha}[|a|(\tau-s)^\alpha]\mathrm{d}s\right].$$

Then from (5.3.9), we have

$$
\begin{aligned}
|y(t)-z(t)| \;\leq\; & |\phi(0)-\widetilde{\phi}(0)| + \frac{|a|}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|y(s)-z(s)|\mathrm{d}s + \frac{|b|}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} \\
& \times |u_1(s)-\widetilde{u}_1(s)|\mathrm{d}s + \frac{1}{\Gamma(\alpha)}\int_0^t (s-\tau)^{\alpha-1}\left[L|\phi(s-\tau)-\widetilde{\phi}(s-\tau)|\right]\mathrm{d}s
\end{aligned}
$$

$$
\begin{aligned}
\leq\; & |\phi(0)-\widetilde{\phi}(0)| + \frac{|a|}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|y(s)-z(s)|\mathrm{d}s + \frac{|b|^2}{\Gamma(\alpha)}\epsilon P_1\int_0^t (t-s)^{\alpha-1}\mathrm{d}s \\
& + \frac{L}{\Gamma(\alpha)}\max_{s\in[0,t]}|\phi(s-\tau)-\widetilde{\phi}(s-\tau)|\int_0^t (t-s)^{\alpha-1}\mathrm{d}s \\
\leq\; & \|\phi-\widetilde{\phi}\| + \frac{|b|^2}{\Gamma(\alpha)}\epsilon P_1\frac{t^\alpha}{\alpha} + \frac{L}{\Gamma(\alpha)}\|\phi-\widetilde{\phi}\|\frac{t^\alpha}{\alpha} + \frac{|a|}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|y(s)-z(s)|\mathrm{d}s \\
\leq\; & \epsilon A_1 + \frac{|a|}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|y(s)-z(s)|\mathrm{d}s.
\end{aligned}
$$

By Gronwall inequality for fractional integral [34], we get

$$
|y(t)-z(t)| \;\leq\; A_1 E_\alpha[|a|\tau^\alpha]\epsilon, \quad t\in[0,\tau],
$$

where $A_1 = 1 + \dfrac{|b|^2 P_1 \tau^\alpha}{\Gamma(\alpha+1)} + \dfrac{L\tau^\alpha}{\Gamma(\alpha+1)}$. Therefore, (5.3.4) holds for $k=1$.

Now we assume that (5.3.4) holds for $(k-1)$. So if $t\in[(k-2)\tau,(k-1)\tau]$, then the following inequality is satisfied:

$$
|y(t)-z(t)| \;\leq\; \epsilon A_{k-1} E_\alpha[|a|((k-1)\tau)^\alpha], \tag{5.3.10}
$$

with $A_{k-1} = 1 + \frac{1}{\Gamma(\alpha+1)} \sum_{j=0}^{k-2} \Big\{ ((k-1-j)\tau)^{\alpha} \big[|b|^2 P_{j+1} + L A_j E_{\alpha}[|a|(j\tau)^{\alpha}]\big] \Big\}$, where

$$P_{j+1} = ((j+1)\tau)^{1-\alpha} E_{\alpha,\alpha}[|a|((j+1)\tau)^{\alpha}] Q^{-1}_{(j+1)\tau} A_j E_{\alpha}[|a|(j\tau)^{\alpha}] L \int_{j\tau}^{(j+1)\tau} ((j+1)\tau - s_1)^{\alpha-1} \times E_{\alpha,\alpha}[|a|((j+1)\tau - s_1)^{\alpha}] \mathrm{d}s_1.$$

Now, we prove that it will also be valid for k.

When $t \in [(k-1)\tau, k\tau],\ k \geq 2$, the control for the system (5.3.1)-(5.3.2) is

$$\widetilde{u}_k(t) = -(k\tau - t)^{1-\alpha} b E_{\alpha,\alpha}[a(k\tau - t)^{\alpha}] Q^{-1}_{k\tau} \Bigg[- y_k + E_{\alpha}[a\tau^{\alpha}] y_{k-1} + \int_{(k-1)\tau}^{k\tau} (k\tau - s)^{\alpha-1} \times E_{\alpha,\alpha}[a(k\tau - s)^{\alpha}] f(s, y_{k-1}(s-\tau)) \mathrm{d}s \Bigg],$$

where $Q_{k\tau} = b^2 \int_{(k-1)\tau}^{k\tau} \big[E_{\alpha,\alpha}[a(k\tau - s)^{\alpha}]\big]^2 \mathrm{d}s$. Now,

$$\begin{aligned} |u_k(t) - \widetilde{u}_k(t)| &\leq (k\tau)^{1-\alpha} |b| E_{\alpha,\alpha}[|a|(k\tau)^{\alpha}] Q^{-1}_{k\tau} L \epsilon A_{k-1} E_{\alpha}[|a|((k-1)\tau)^{\alpha}] \\ &\quad \times \int_{(k-1)\tau}^{k\tau} (k\tau - s)^{\alpha-1} E_{\alpha,\alpha}[|a|(k\tau - s)^{\alpha}] \mathrm{d}s \\ &= \epsilon |b| P_k, \end{aligned}$$

where P_k is given in (5.3.7) when $m = k$.

From (5.3.9), we have

$$
\begin{aligned}
|y(t)-z(t)| \;\le\; & |\phi(0)-\widetilde{\phi}(0)| + \frac{|a|}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|y(s)-z(s)|\mathrm{d}s \\
& + \frac{|b|}{\Gamma(\alpha)}\sum_{j=1}^{k-1}\int_{(j-1)\tau}^{j\tau}(t-s)^{\alpha-1}|u_j(s)-\widetilde{u}_j(s)|\mathrm{d}s \\
& + \frac{|b|}{\Gamma(\alpha)}\int_{(k-1)\tau}^{t}(t-s)^{\alpha-1}|u_k(s)-\widetilde{u}_k(s)|\mathrm{d}s \\
& + \frac{L}{\Gamma(\alpha)}\sum_{j=1}^{k-1}\int_{(j-1)\tau}^{j\tau}(t-s)^{\alpha-1}|y(s-\tau)-z(s-\tau)|\mathrm{d}s \\
& + \frac{L}{\Gamma(\alpha)}\int_{(k-1)\tau}^{t}(t-s)^{\alpha-1}|y(s-\tau)-z(s-\tau)|\mathrm{d}s.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
|y(t)-z(t)| \;\le\; & \epsilon + \frac{|a|}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|y(s)-z(s)|\mathrm{d}s \\
& + \frac{|b|^2\epsilon}{\Gamma(\alpha)}\sum_{j=1}^{k-1}\Bigg\{(j\tau)^{1-\alpha}E_{\alpha,\alpha}[a(j\tau)^{\alpha}]Q_{j\tau}^{-1}LA_{j-1}E_{\alpha}[|a|((j-1)\tau)^{\alpha}] \\
& \times \int_{(j-1)\tau}^{j\tau}(j\tau-s_1)^{\alpha-1}E_{\alpha,\alpha}[|a|(j\tau-s_1)^{\alpha}]\mathrm{d}s_1 \int_{(j-1)\tau}^{j\tau}(t-s)^{\alpha-1}\mathrm{d}s\Bigg\} \\
& + \frac{|b|^2\epsilon}{\Gamma(\alpha)}(k\tau)^{1-\alpha}E_{\alpha,\alpha}[|a|(k\tau)^{\alpha}]Q_{k\tau}^{-1}LA_{k-1}E_{\alpha}[|a|((k-1)\tau)^{\alpha}] \\
& \times \int_{(k-1)\tau}^{k\tau}(k\tau-s_1)^{\alpha-1}E_{\alpha,\alpha}[|a|(k\tau-s_1)^{\alpha}]\mathrm{d}s_1 \int_{(k-1)\tau}^{t}(t-s)^{\alpha-1}\mathrm{d}s \\
& + \frac{L}{\Gamma(\alpha)}\sum_{j=1}^{k-1}\max_{s\in[(j-1)\tau,j\tau]}|y(s-\tau)-z(s-\tau)|\int_{(j-1)\tau}^{j\tau}(t-s)^{\alpha-1}\mathrm{d}s \\
& + \frac{L}{\Gamma(\alpha)}\max_{s\in[(k-1)\tau,k\tau]}|y(s-\tau)-z(s-\tau)|\int_{(k-1)\tau}^{t}(t-s)^{\alpha-1}\mathrm{d}s.
\end{aligned}
$$

Now, using (5.3.10),

$$
\begin{aligned}
|y(t)-z(t)| \;\leq\; & \epsilon\Bigg\{1+\frac{|b|^2}{\Gamma(\alpha+1)}\sum_{i=0}^{k-2}\Bigg[((i+1)\tau)^{1-\alpha}E_{\alpha,\alpha}[a((i+1)\tau)^{\alpha}]Q^{-1}_{(i+1)\tau}LA_iE_{\alpha}[|a|(i\tau)^{\alpha}] \\
& \times\int\limits_{i\tau}^{(i+1)\tau}((i+1)\tau-s_1)^{\alpha-1}E_{\alpha,\alpha}[|a|((i+1)\tau-s_1)^{\alpha}]\mathrm{d}s_1[(k-i)\tau]^{\alpha}\Bigg]+\frac{|b|^2}{\Gamma(\alpha+1)} \\
& \times(k\tau)^{1-\alpha}E_{\alpha,\alpha}[|a|(k\tau)^{\alpha}]Q^{-1}_{k\tau}LA_{k-1}E_{\alpha}[|a|((k-1)\tau)^{\alpha}]\int\limits_{(k-1)\tau}^{k\tau}(k\tau-s_1)^{\alpha-1} \\
& \times E_{\alpha,\alpha}[|a|(k\tau-s_1)^{\alpha}]\mathrm{d}s_1\tau^{\alpha}+\frac{L}{\Gamma(\alpha+1)}\sum_{i=0}^{k-2}A_iE_{\alpha}[|a|(i\tau)^{\alpha}]((k-i)\tau)^{\alpha} \\
& +\frac{L}{\Gamma(\alpha+1)}A_{k-1}E_{\alpha}[|a|((k-1)\tau)^{\alpha}]\tau^{\alpha}\Bigg\}+\frac{|a|}{\Gamma(\alpha)}\int\limits_0^t(t-s)^{\alpha-1}|y(s)-z(s)|\mathrm{d}s.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
|y(t)-z(t)| \;\leq\; & \epsilon\Bigg\{1+\frac{1}{\Gamma(\alpha+1)}\sum_{j=0}^{k-1}\Big[|b|^2P_{j+1}+LA_jE_{\alpha}[|a|(j\tau)^{\alpha}][(k-j)\tau]^{\alpha}\Big]\Bigg\} \\
& +\frac{|a|}{\Gamma(\alpha)}\int\limits_0^t(t-s)^{\alpha-1}|y(s)-z(s)|\mathrm{d}s \\
= \;& \epsilon A_k+\frac{|a|}{\Gamma(\alpha)}\int\limits_0^t(t-s)^{\alpha-1}|y(s)-z(s)|\mathrm{d}s.
\end{aligned}
$$

From this, (5.3.4) follows by a Gronwall inequality [34].

Now, whenever $t\in[k\tau,T]$ the control for the system (5.3.1)-(5.3.2) is

$$
\begin{aligned}
\widetilde{u}_T(t) \;=\; & -(T-t)^{1-\alpha}bE_{\alpha,\alpha}[a(T-t)^{\alpha}]Q_T^{-1}\Bigg[-y_T+E_{\alpha}[a(T-k\tau)^{\alpha}]y_k+\int\limits_{(k)\tau}^{T}(T-s)^{\alpha-1} \\
& \times E_{\alpha,\alpha}[a(T-s)^{\alpha}]f(s,y_k(s-\tau))\mathrm{d}s\Bigg],
\end{aligned}
$$

where $Q_T = b^2 \int\limits_{k\tau}^{T} \left[E_{\alpha,\alpha}[a(T-s)^\alpha]\right]^2 \mathrm{d}s$. Now,

$$|u_T(t) - \widetilde{u}_T(t)| \leq T^{1-\alpha}|b|E_{\alpha,\alpha}[|a|T^\alpha]Q_T^{-1}L\epsilon A_k E_\alpha[|a|(k\tau)^\alpha] \times \int\limits_{k\tau}^{T}(T-s)^{\alpha-1}E_{\alpha,\alpha}[|a|(T-s)^\alpha]\mathrm{d}s.$$

From (5.3.9) and the steps similar to the above we obtain

$$|y(t) - z(t)| \leq \epsilon H + \frac{|a|}{\Gamma(\alpha)}\int\limits_{0}^{t}(t-s)^{\alpha-1}|y(s)-z(s)|\mathrm{d}s,$$

where H is given by (5.3.8) and the result follows by the Gronwall inequality. ■

5.3.2 Dependence of solution on small perturbation of f

Let us consider the function f on the right-hand side of (5.1.1) has small perturbation. Assume that $y(t)$ is the solution of the problem (5.1.1)-(5.1.2) and $z(t)$ is the solution of problem

$$^{C}D^\alpha z(t) = az(t) + b\,\widetilde{\widetilde{u}}(t) + \widetilde{f}(t, z(t-\tau)), \quad 0 < t \leq T \tag{5.3.11}$$

$$z(t) = \phi(t), \quad t \in [-\tau, 0]. \tag{5.3.12}$$

Let the functions f and $\widetilde{f}$ satisfy a Lipschitz condition (assumption (A)) and the assumption:

(B) $\|f(t, z(t-\tau)) - \widetilde{f}(t, z(t-\tau)\| = \max\limits_{t\in[0,T]} |f(t, z(t-\tau)) - \widetilde{f}(t, z(t-\tau)| \leq \epsilon_1$,

where $\epsilon_1 > 0$ is very small.

From (5.1.1)-(5.1.2) and (5.3.11)-(5.3.12)

$$^{C}D^\alpha[y(t) - z(t)] = a[y(t) - z(t)] + b[u(t) - \widetilde{\widetilde{u}}(t)] + f(t, y(t-\tau)) - \widetilde{f}(t, z(t-\tau)) \tag{5.3.13}$$

Taking Riemann-Liouville's fractional integral of order α on both sides of (5.3.13), we get [see for example, proof of Theorem 3.4 in [33]],

$$\begin{aligned}
|y(t)-z(t)| &= \Big|\frac{a}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}[y(s)-z(s)]\mathrm{d}s + \frac{b}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}[u(s)-\stackrel{\approx}{u}(s)]\mathrm{d}s \\
&\quad + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\big[f(s,y(s-\tau))-f(s,z(s-\tau)) \\
&\quad + f(s,z(s-\tau))-\widetilde{f}(s,z(s-\tau))\big]\mathrm{d}s\Big| \\
&\leq \frac{|a|}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|y(s)-z(s)|\mathrm{d}s + \frac{|b|}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|u(s)-\stackrel{\approx}{u}(s)|\mathrm{d}s \\
&\quad + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|f(s,y(s-\tau))-f(s,z(s-\tau))|\mathrm{d}s \\
&\quad + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|f(t,z(t-\tau))-\widetilde{f}(t,z(t-\tau))|\mathrm{d}s. \qquad (5.3.14)
\end{aligned}$$

When $t\in[0,\tau]$, the control for the system (5.3.11)-(5.3.12) is

$$\begin{aligned}
\stackrel{\approx}{u}_1(t) &= -(\tau-t)^{1-\alpha}bE_{\alpha,\alpha}[a(\tau-t)^{\alpha}]Q_\tau^{-1}\Big[-y_1+E_\alpha[a\tau^\alpha]\phi(0)+\int_0^\tau (\tau-s)^{\alpha-1} \\
&\quad \times E_{\alpha,\alpha}[a(\tau-s)^{\alpha}]\widetilde{f}(s,\phi(s-\tau))\mathrm{d}s\Big],
\end{aligned}$$

where $Q_\tau = b^2\int_0^\tau \big[E_{\alpha,\alpha}[a(\tau-s)^\alpha]\big]^2\mathrm{d}s$. Now,

$$|u_1(t)-\stackrel{\approx}{u}_1(t)| \leq \epsilon_1\tau^{1-\alpha}|b|E_{\alpha,\alpha}[|a|\tau^\alpha]Q_\tau^{-1}\int_0^\tau (\tau-s)^{\alpha-1}E_{\alpha,\alpha}[|a|(\tau-s)^\alpha]\mathrm{d}s. \qquad (5.3.15)$$

From (5.3.14), we have

$$\begin{aligned}|y(t)-z(t)| &\leq \frac{|a|}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|y(s)-z(s)|\mathrm{d}s + \frac{|b|}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|u_1(s)- \stackrel{\approx}{u}_1(s)|\mathrm{d}s \\ &+\frac{L}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|\phi(s-\tau)-\phi(s-\tau)|\mathrm{d}s + \frac{\epsilon_1\tau^\alpha}{\Gamma(\alpha+1)} \\ &\leq \frac{\epsilon_1\tau^\alpha}{\Gamma(\alpha+1)}\left[1+|b|^2\tau^{1-\alpha}E_{\alpha,\alpha}[|a|\tau^\alpha]Q_\tau^{-1}\int_0^\tau (\tau-s)^{\alpha-1}E_{\alpha,\alpha}[|a|(\tau-s)^\alpha]\mathrm{d}s\right] \\ &+\frac{|a|}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|y(s)-z(s)|\mathrm{d}s.\end{aligned}$$

Using Gronwall inequality for fractional integral [34], we have

$$|y(t)-z(t)| \leq \epsilon_1 M_1 E_\alpha\left[|a|\tau^\alpha\right], \tag{5.3.16}$$

where $M_1 = \frac{\tau^\alpha}{\Gamma(\alpha+1)}\left[1+|b|^2\tau^{1-\alpha}E_{\alpha,\alpha}[|a|\tau^\alpha]Q_\tau^{-1}\int_0^\tau(\tau-s)^{\alpha-1}E_{\alpha,\alpha}[|a|(\tau-s)^\alpha]\mathrm{d}s\right]$.

When $t\in[\tau,2\tau]$, from Theorem 5.2.1 the control for the system (5.3.11)-(5.3.12) is

$$\begin{aligned}\stackrel{\approx}{u}_2(t) &= -(2\tau-t)^{1-\alpha}bE_{\alpha,\alpha}[a(2\tau-t)^\alpha]Q_{2\tau}^{-1}\Bigg[-y_2+E_\alpha[a\tau^\alpha]y_1+\int_\tau^{2\tau}(2\tau-s)^{\alpha-1} \\ &\times E_{\alpha,\alpha}[a(2\tau-s)^\alpha]\widetilde{f}(s,y(s-\tau))\mathrm{d}s\Bigg],\end{aligned}$$

where $Q_{2\tau} = b^2\int_\tau^{2\tau}\left[E_{\alpha,\alpha}[a(2\tau-s)^\alpha]\right]^2\mathrm{d}s$. Now,

$$\begin{aligned}|u_2(t)- \stackrel{\approx}{u}_2(t)| &\leq \epsilon_1(2\tau)^{1-\alpha}|b|E_{\alpha,\alpha}[|a|(2\tau)^\alpha]Q_{2\tau}^{-1}\left[LM_1E_\alpha[|a|\tau^\alpha]+1\right] \\ &\times\int_\tau^{2\tau}(2\tau-s)^{\alpha-1}E_{\alpha,\alpha}[|a|(2\tau-s)^\alpha]\mathrm{d}s.\end{aligned} \tag{5.3.17}$$

From (5.3.14), we have

$$
\begin{aligned}
|y(t)-z(t)| \;\leq\; & \frac{|a|}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|y(s)-z(s)|\mathrm{d}s + \frac{|b|}{\Gamma(\alpha)}\left\{\int_0^\tau (t-s)^{\alpha-1}|u_1(s)-\stackrel{\approx}{u}_1(s)|\mathrm{d}s\right.\\
& \left.+\int_\tau^t (t-s)^{\alpha-1}|u_2(s)-\stackrel{\approx}{u}_2(s)|\mathrm{d}s\right\}\\
& +\frac{L}{\Gamma(\alpha)}\left\{\int_0^\tau (t-s)^{\alpha-1}|\phi(s-\tau)-\phi(s-\tau)|\mathrm{d}s\right.\\
& \left.+\int_\tau^t (t-s)^{\alpha-1}|y(s-\tau)-z(s-\tau)|\mathrm{d}s\right\} + \frac{\epsilon_1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\mathrm{d}s. \qquad (5.3.18)
\end{aligned}
$$

Substituting (5.3.16), (5.3.15) and (5.3.17) in (5.3.18), we get

$$
\begin{aligned}
|y(t)-z(t)| \;\leq\; & \frac{|a|}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|y(s)-z(s)|\mathrm{d}s + \frac{|b|^2\epsilon_1}{\Gamma(\alpha+1)}\left\{\tau^{1-\alpha}E_{\alpha,\alpha}[a\tau^\alpha]Q_\tau^{-1}\right.\\
& \times\int_0^\tau (\tau-s)^{\alpha-1}E_{\alpha,\alpha}[|a|(\tau-s)^\alpha]\mathrm{d}s\times(2\tau)^\alpha + (2\tau)^{1-\alpha}E_{\alpha,\alpha}[|a|(2\tau)^\alpha]Q_{2\tau}^{-1}\\
& \left.\times\big[1+LM_1E_\alpha[|a|\tau^\alpha]\big]\int_\tau^{2\tau}(2\tau-s)^{\alpha-1}E_{\alpha,\alpha}[|a|(2\tau-s)^\alpha]\mathrm{d}s\times\tau^\alpha\right\}\\
& +\frac{L}{\Gamma(\alpha+1)}\epsilon_1\tau^\alpha M_1E_\alpha[|a|\tau^\alpha] + \frac{\epsilon_1(2\tau)^\alpha}{\Gamma(\alpha+1)}.
\end{aligned}
$$

Using Gronwall inequality for fractional integral, we get

$$
|y(t)-z(t)| \;\leq\; \epsilon_1 M_2 E_\alpha[|a|(2\tau)^\alpha],
$$

where

$$
\begin{aligned}
M_2 &= \frac{|b|^2}{\Gamma(\alpha+1)}\Bigg\{2^{\alpha}\tau E_{\alpha,\alpha}[a\tau^{\alpha}]Q_{\tau}^{-1}\int_0^{\tau}(\tau-s)^{\alpha-1}E_{\alpha,\alpha}[|a|(\tau-s)^{\alpha}]\mathrm{d}s + 2^{1-\alpha}\tau E_{\alpha,\alpha}[|a|(2\tau)^{\alpha}] \\
&\times Q_{2\tau}^{-1}\big[1+LM_1E_{\alpha}[|a|\tau^{\alpha}]\big]\int_{\tau}^{2\tau}(2\tau-s)^{\alpha-1}E_{\alpha,\alpha}[|a|(2\tau-s)^{\alpha}]\mathrm{d}s\Bigg\} \\
&+\frac{L}{\Gamma(\alpha+1)}\tau^{\alpha}M_1E_{\alpha}[|a|\tau^{\alpha}] + \frac{(2\tau)^{\alpha}}{\Gamma(\alpha+1)}.
\end{aligned}
$$

Similarly using (5.3.14), one can conclude the following theorem in the successive intervals $[i\tau,(i+1)\tau],\ i=2,3,\ldots$.

Theorem 5.3.2. *Assume that $y(t)$ is the solution of problem (5.1.1)-(5.1.2) and $z(t)$ is the solution of the problem (5.3.11)-(5.3.12) on the interval $[0,T]$. Let f and $\widetilde{f}$ satisfy Lipschitz condition in the second variable. Then, on each intervals $[(k-1)\tau,k\tau]$ and $[k\tau,T]$, $k=1,2,3,\ldots$, both solutions exist and*

$$
\begin{aligned}
\|y-z\|_{[(k-1)\tau,k\tau]} &\le \epsilon_1K_1E_{\alpha}[|a|((k-1)\tau)^{\alpha}], \\
\|y-z\|_{[k\tau,T]} &\le \epsilon_1K_2E_{\alpha}[|a|(k\tau)^{\alpha}],
\end{aligned}
$$

where K_1 and K_2 are suitable constants which dependents on L,k and α.

5.4 Numerical results

In this section, we plot all the figures using Microcal Origin 6.0 and for computing Mittag-Leffler function Podlubny's MATLAB algorithm has been used.

Example. Consider the fractional order delay differential control system

$$
\begin{aligned}
{}^{C}D^{0.85}y(t) &= \frac{2y(t-2)}{1+y(t-2)^{9.65}} - y(t) + u(t), && (5.4.1)\\
y(t) &= 0.5,\ \ t\in[-2,0] \text{ and } y(2)=1. && (5.4.2)
\end{aligned}
$$

Using (5.2.1), the analytical solution of (5.4.1)-(5.4.2) in the interval $t \in [0,2]$ is

$$y(t) = \frac{1}{2}E_{0.85}[-t^{0.85}] + \int_0^t (t-s)^{-0.15} E_{0.85,0.85}[-(t-s)^{0.85}]\left[u(s) + \frac{1}{1+(0.5)^{9.65}}\right] \mathrm{d}s, \quad (5.4.3)$$

where the control

$$\begin{aligned} u(t) &= -(2-t)^{0.15} E_{0.85,0.85}[-(2-t)^{0.85}] Q_2^{-1}\Bigg[-1 + 0.5E_{0.85}[-2^{0.85}] \\ &\quad + \frac{1}{1+(0.5)^{9.65}} \int_0^2 (2-s)^{-0.15} E_{0.85,0.85}[-(2-s)^{0.85}] \mathrm{d}s\Bigg] \qquad (5.4.4) \end{aligned}$$

and $Q_2 = \int_0^2 \left[E_{0.85,0.85}[-(2-s)^{0.85}]\right]^2 \mathrm{d}s.$

Let $y(t)$ be the solution of the control system (5.4.1)-(5.4.2) in the interval $[0,2]$ and $z(t)$ be the solution of the control system

$$\begin{aligned} {}^C D^{0.85} z(t) &= \frac{2z(t-2)}{1+z(t-2)^{9.65}} - z(t) + \widetilde{u}(t), \qquad (5.4.5) \\ z(t) &= 0.5(1+\epsilon), \quad t \in [-2,0] \text{ and } z(2) = 1. \qquad (5.4.6) \end{aligned}$$

Using (5.2.1), the analytical solution of (5.4.5)-(5.4.6) in the interval $[0,2]$ is

$$\begin{aligned} z(t) &= \frac{(1+\epsilon)}{2} E_{0.85}[-t^{0.85}] + \int_0^t (t-s)^{-0.15} E_{0.85,0.85}[-(t-s)^{0.85}] \\ &\quad \times \left[\widetilde{u}(s) + \frac{(1+\epsilon)}{1+(0.5(1+\epsilon))^{9.65}}\right] \mathrm{d}s, \qquad (5.4.7) \end{aligned}$$

where the control

$$\begin{aligned}
\widetilde{u}(t) &= -(2-t)^{0.15}E_{0.85,0.85}[-(2-t)^{0.85}]Q_2^{-1}\Bigg[-1+\frac{(1+\epsilon)}{2}E_{0.85}[-2^{0.85}] \\
&\quad +\frac{(1+\epsilon)}{1+(0.5(1+\epsilon))^{9.65}}\int_0^2 (2-s)^{-0.15}E_{0.85,0.85}[-(2-s)^{0.85}]\mathrm{d}s\Bigg] \\
&= -3.17581\Bigg[-1+0.202714\left(\frac{1+\epsilon}{2}\right)+\frac{0.797286(1+\epsilon)}{1+(0.5(1+\epsilon))^{9.65}}\Bigg] \\
&\quad \times(2-t)^{0.15}E_{0.85,0.85}[-(2-t)^{0.85}] \qquad (5.4.8)
\end{aligned}$$

and $Q_2 = \int_0^2 \Big[E_{0.85,0.85}[-(2-s)^{0.85}]\Big]^2 \mathrm{d}s = 0.31488$. Here the numerical solution of $\widetilde{u}(t)$ is computed using MATLAB. In Table 5.1 the numerical values of the control $\widetilde{u}(t)$ in the interval $[0,2]$ for different values of ϵ in (5.4.8) is given with step size $h = 0.1$. In Figure 5.1(a), the control $\widetilde{u}(t)$ in the interval $[0,2]$ for different values of ϵ in (5.4.5)-(5.4.6) are given. In Figure 5.1(b), the error between control of (5.4.1)-(5.4.2) (given in (5.4.4)) and the control of (5.4.5)-(5.4.6) (given in (5.4.8)) for different values of ϵ are presented.

Using MATLAB with the algorithm in Appendix A, we compute the numerical solution $z(t)$ in (5.4.7) in the interval $[0,2]$ for different values of ϵ with step size $h = 0.1$ (see Table 5.2). Figure 5.2(a) shows the solution $z(t)$ of (5.4.5)-(5.4.6) in $[0,2]$ for different values of ϵ. Figure 5.2(b) shows error between solution of (5.4.1)-(5.4.2) and the solution of (5.4.5)-(5.4.6) for different values of ϵ. To verify results of Theorem 5.3.1, in Table 5.3 we give the maximum absolute error between solutions of problems (5.4.1)-(5.4.2) and (5.4.5)-(5.4.6) in the interval $[0,2]$ for different values of ϵ.

Now, let $z(t)$ be the solution of the control system

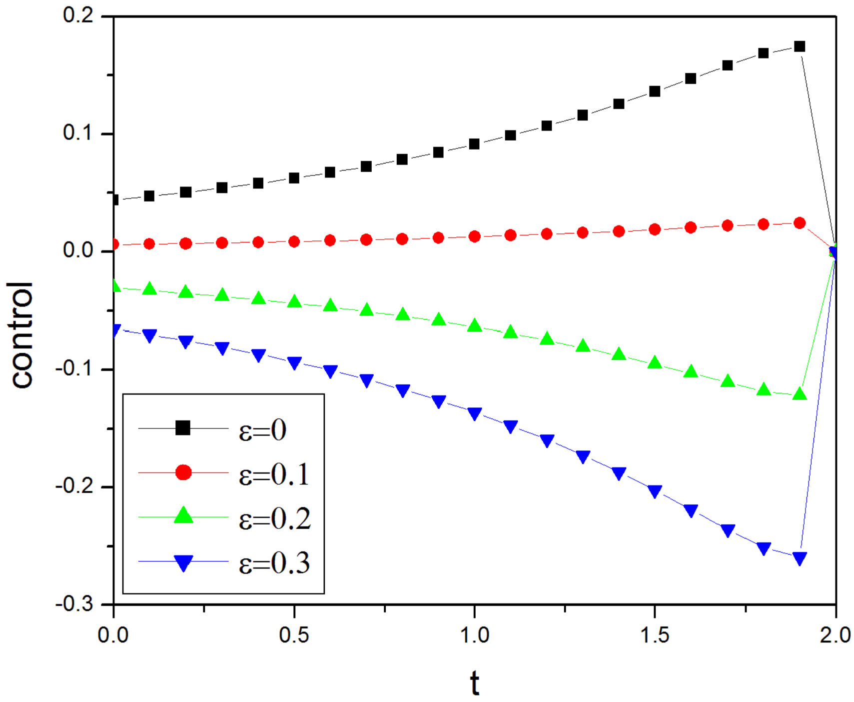
$$\begin{aligned}
{}^{C}D^{0.85}z(t) &= \frac{2(1+\zeta)z(t-2)}{1+z(t-2)^{9.65}} - z(t) + \widetilde{\widetilde{u}}(t), \qquad (5.4.9)\\
z(t) &= 0.5, \quad t\in[-2,0] \text{ and } z(2)=1. \qquad (5.4.10)
\end{aligned}$$

Using (5.2.1), the analytical solution of (5.4.9)-(5.4.10) in the interval $[0,2]$ is

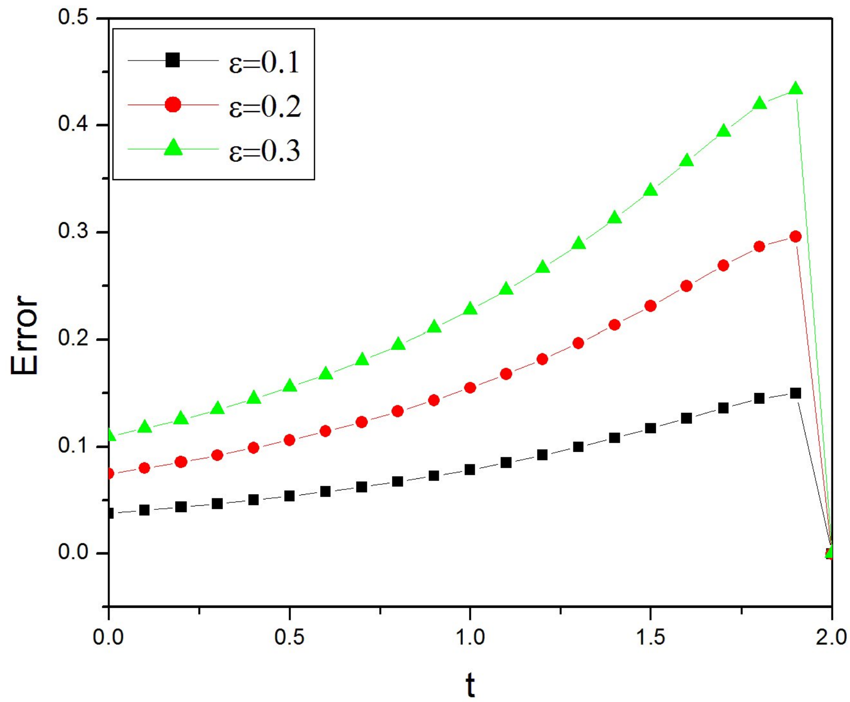
$$z(t) = \frac{1}{2}E_{0.85}[-t^{0.85}] + \int_0^t (t-s)^{-0.15}E_{0.85,0.85}[-(t-s)^{0.85}]\Bigg[\widetilde{\widetilde{u}}(s)+\frac{(1+\zeta)}{1+(0.5)^{9.65}}\Bigg]\mathrm{d}s, \qquad (5.4.11)$$

Table 5.1: Numerical values of control $\widetilde{u}(t)$ in the interval $[0, 2]$ for different values of ϵ in (5.4.8).

$h = 0.1$	$\epsilon = 0$	$\epsilon = 0.1$	$\epsilon = 0.2$	$\epsilon = 0.3$
0	0.043973	0.006111	−0.030721	−0.065418
0.1	0.047079	0.006542	−0.032891	−0.070039
0.2	0.050459	0.007012	−0.035252	−0.075067
0.3	0.05414	0.007524	−0.037824	−0.080544
0.4	0.058153	0.008081	−0.040628	−0.086514
0.5	0.062532	0.00869	−0.043687	−0.093028
0.6	0.067314	0.009354	−0.047028	−0.100142
0.7	0.07254	0.01008	−0.050678	−0.107916
0.8	0.078254	0.010874	−0.054671	−0.116417
0.9	0.084505	0.011743	−0.059038	−0.125717
1	0.091344	0.012693	−0.063816	−0.135891
1.1	0.098824	0.013733	−0.069042	−0.147019
1.2	0.106998	0.014869	−0.074752	−0.159179
1.3	0.115908	0.016107	−0.080977	−0.172434
1.4	0.12558	0.017451	−0.087734	−0.186823
1.5	0.135992	0.018898	−0.095009	−0.202313
1.6	0.147021	0.020431	−0.102714	−0.21872
1.7	0.158286	0.021996	−0.110584	−0.23548
1.8	0.168683	0.023441	−0.117847	−0.250947
1.9	0.174197	0.024207	−0.121699	−0.259149
2	0	0	0	0



(a) Control $\widetilde{u}(t)$ of the system (5.4.5)-(5.4.6).



(b) Error between control of (5.4.1)-(5.4.2) and the control of (5.4.5)-(5.4.6) for different values of ϵ.

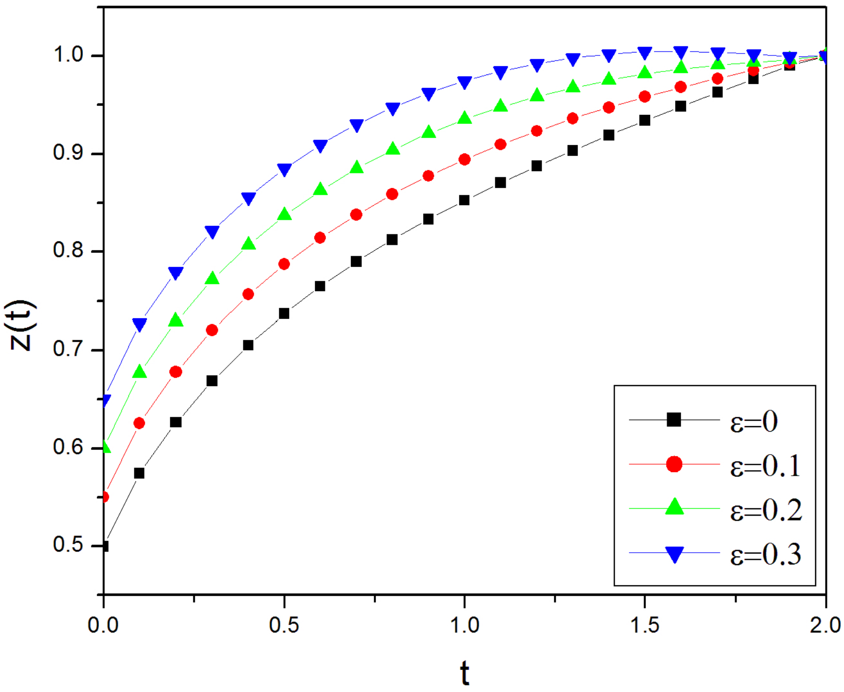
Figure 5.1: Control and error graphs for different values of ϵ in (5.4.5)-(5.4.6).

Table 5.2: Numerical values of $z(t)$ in the interval $[0, 2]$ for different values of ϵ in (5.4.7).

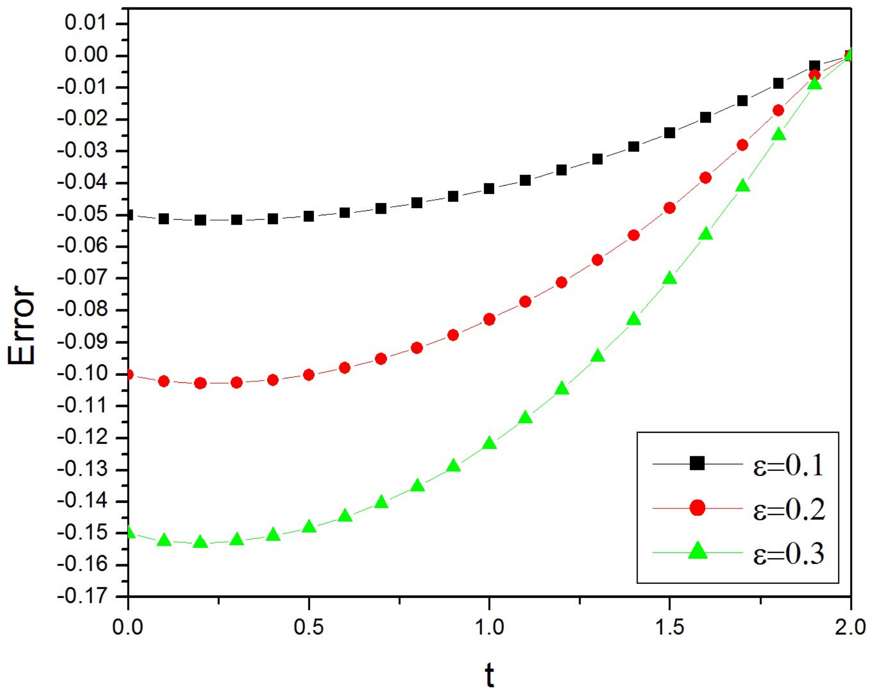
$h = 0.1$	$\epsilon = 0$	$\epsilon = 0.1$	$\epsilon = 0.2$	$\epsilon = 0.3$
0	0.5	0.55	0.6	0.65
0.1	0.574713	0.625876	0.676772	0.727118
0.2	0.626532	0.678118	0.729265	0.779504
0.3	0.669	0.72056	0.771551	0.821364
0.4	0.705208	0.756363	0.80685	0.855954
0.5	0.73677	0.787176	0.83684	0.884968
0.6	0.764717	0.814046	0.862581	0.909469
0.7	0.789775	0.837707	0.88481	0.930196
0.8	0.812492	0.858704	0.90407	0.947684
0.9	0.833294	0.877458	0.920776	0.962339
1	0.85253	0.894307	0.935251	0.97447
1.1	0.870487	0.909521	0.947751	0.984318
1.2	0.887411	0.923328	0.958486	0.992071
1.3	0.903509	0.935916	0.967622	0.997876
1.4	0.918962	0.947444	0.975297	1.001851
1.5	0.933924	0.958047	0.981631	1.004098
1.6	0.948514	0.967841	0.986731	1.004718
1.7	0.962802	0.976918	0.990714	1.003848
1.8	0.976749	0.985347	0.993751	1.001755
1.9	0.989973	0.993136	0.996233	0.999192
2	1	1	1	1

Table 5.3: Maximum absolute error between solutions of (5.4.1)-(5.4.2) and (5.4.5)-(5.4.6).

ϵ	$\|y - z\|$
0.1	5.16×10^{-2}
0.2	1.03×10^{-1}
0.3	1.53×10^{-1}



(a) Solution $z(t)$ of the system (5.4.5)-(5.4.6).



(b) Error between solution of (5.4.1)-(5.4.2) and the solution of (5.4.5)-(5.4.6) for different values of ϵ.

Figure 5.2: Solution and error graphs for different values of ϵ in (5.4.5)-(5.4.6).

where the control

$$\begin{aligned}\widetilde{\widetilde{u}}(t) &= -(2-t)^{0.15}E_{0.85,0.85}[-(2-t)^{0.85}]Q_2^{-1}\Bigg[-1+\frac{1}{2}E_{0.85}[-2^{0.85}] \\ &\quad +\frac{(1+\zeta)}{1+(0.5)^{9.65}}\int_0^2(2-s)^{-0.15}E_{0.85,0.85}[-(2-s)^{0.85}]\mathrm{d}s\Bigg] \\ &= -3.17581[-0.898643+0.796295(1+\zeta)](2-t)^{0.15}E_{0.85,0.85}[-(2-t)^{0.85}] \quad (5.4.12)\end{aligned}$$

and $Q_2 = \int_0^2 \Big[E_{0.85,0.85}[-(2-s)^{0.85}]\Big]^2 \mathrm{d}s = 0.31488$. Here the numerical solution of $\widetilde{\widetilde{u}}\ (t)$ is computed using MATLAB. In Table 5.4 the numerical values of the control $\widetilde{\widetilde{u}}\ (t)$ in the interval $[0, 2]$ for different values of ζ in (5.4.12) is given with step size $h = 0.1$. In Figure 5.3(a), the control of $\widetilde{\widetilde{u}}\ (t)$ in $[0, 2]$ for different values of ζ in (5.4.9)-(5.4.10) are given. Figure 5.3(b), shows the error between control of (5.4.1)-(5.4.2) (given in (5.4.4)) and the control of (5.4.9)-(5.4.10) (given in (5.4.12)) for different values of ζ.

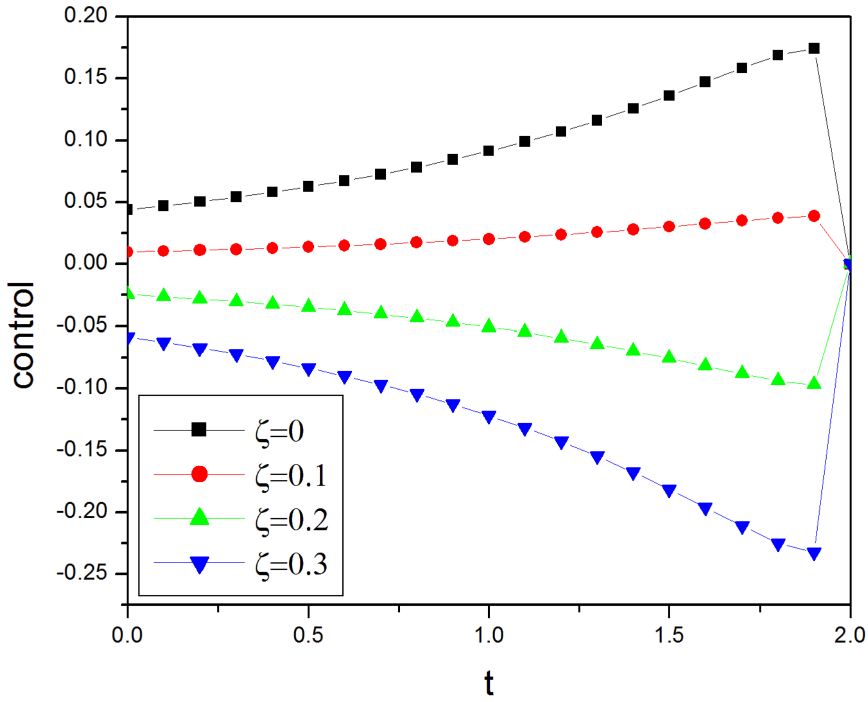
Using MATLAB with the algorithm in Appendix A, we compute the numerical solution $z(t)$ in (5.4.11) in the interval $[0, 2]$ for different values of ζ with step size $h = 0.1$ (see Table 5.5). Figure 5.4(a) shows the solution $z(t)$ in $[0, 2]$ for different values of ζ in (5.4.9)-(5.4.10). In Figure 5.4(b), error between solution of (5.4.1)-(5.4.2) and the solution of (5.4.9)-(5.4.10) for different values of ζ are given. To verify results of Theorem 5.3.2, in Table 5.6 we give the maximum absolute error between solutions of problems (5.4.1)-(5.4.2) and (5.4.9)-(5.4.10) in the interval $[0, 2]$ for different values of the parameter ζ.

5.5 Conclusion

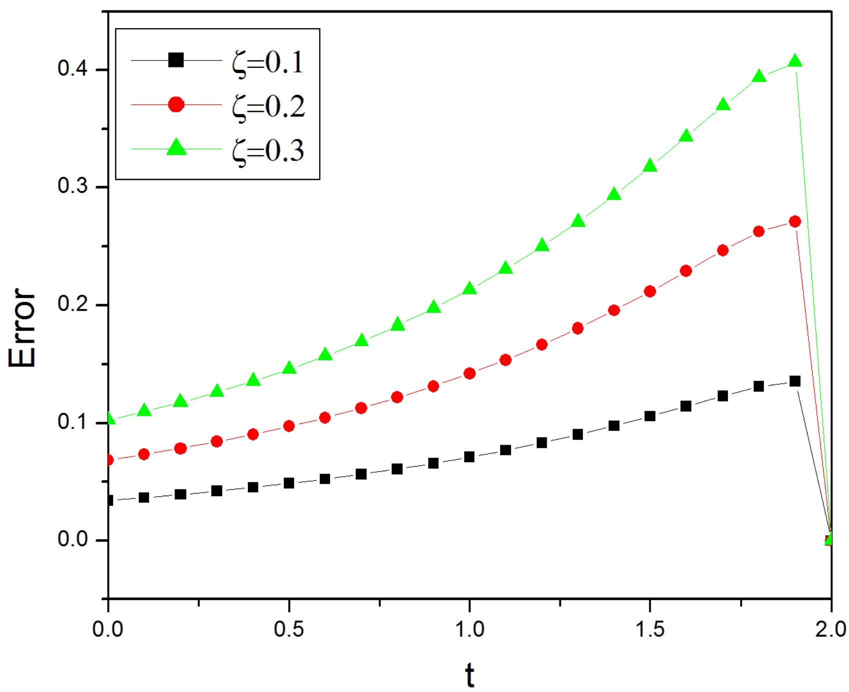
In this chapter, a class of finite dimensional fractional order semilinear control systems with delay has been considered. Firstly, existence and uniqueness of the solution has been derived using method of steps. Then sensitivity of the state has been shown with respect to the initial condition and the perturbed nonlinear function of the system. Finally, some examples were given to show to illustrate the analytical results.

Table 5.4: Numerical values of control $\overset{\approx}{u}(t)$ in the interval $[0,2]$ for different values of ζ in (5.4.12).

$h = 0.1$	$\zeta = 0$	$\zeta = 0.1$	$\zeta = 0.2$	$\zeta = 0.3$
0	0.043973	0.009761	-0.024452	-0.058664
0.1	0.047079	0.01045	-0.026179	-0.062808
0.2	0.050459	0.011201	-0.028058	-0.067317
0.3	0.05414	0.012018	-0.030105	-0.072228
0.4	0.058153	0.012908	-0.032336	-0.077581
0.5	0.062532	0.01388	-0.034771	-0.083423
0.6	0.067314	0.014942	-0.03743	-0.089802
0.7	0.07254	0.016102	-0.040336	-0.096773
0.8	0.078254	0.01737	-0.043513	-0.104397
0.9	0.084505	0.018758	-0.046989	-0.112736
1	0.091344	0.020276	-0.050792	-0.12186
1.1	0.098824	0.021936	-0.054952	-0.13184
1.2	0.106998	0.023751	-0.059496	-0.142743
1.3	0.115908	0.025728	-0.064451	-0.15463
1.4	0.12558	0.027875	-0.069829	-0.167533
1.5	0.135992	0.030187	-0.075619	-0.181425
1.6	0.147021	0.032635	-0.081751	-0.196138
1.7	0.158286	0.035135	-0.088016	-0.211167
1.8	0.168683	0.037443	-0.093797	-0.225036
1.9	0.174197	0.038667	-0.096863	-0.232392
2	0	0	0	0



(a) Control $\tilde{\tilde{u}}(t)$ of the system (5.4.9)-(5.4.10).



(b) Error between control of (5.4.1)-(5.4.2) and the control of (5.4.9)-(5.4.10) for different values of ζ.

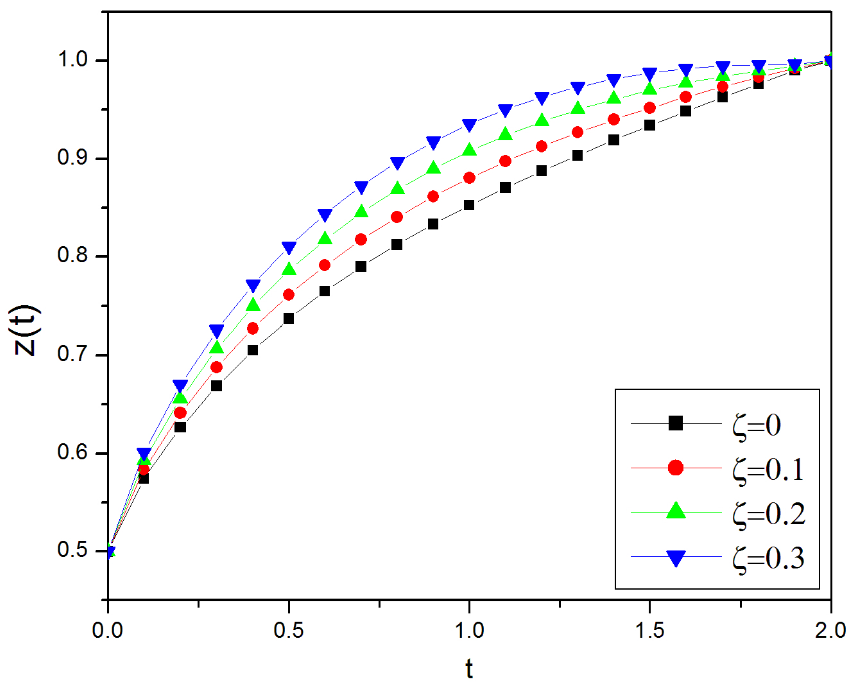
Figure 5.3: Control and error graphs for different values of ζ in (5.4.9)-(5.4.10).

Table 5.5: Numerical values of $z(t)$ in the interval $[0, 2]$ for different values of ϵ in (5.4.11).

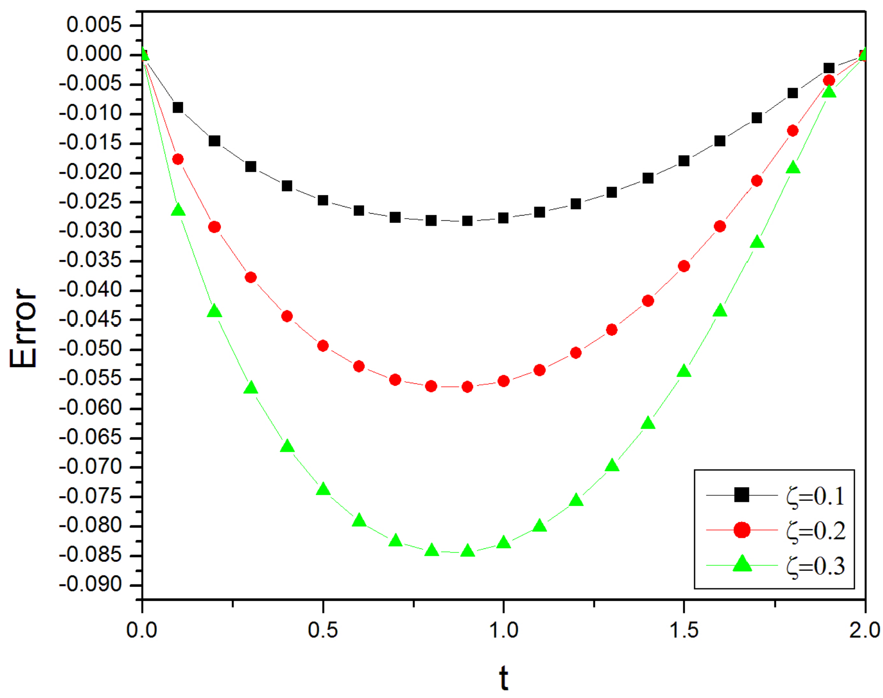
$h = 0.1$	$\zeta = 0$	$\zeta = 0.1$	$\zeta = 0.2$	$\zeta = 0.3$
0	0.5	0.5	0.5	0.5
0.1	0.574713	0.58354	0.592366	0.601193
0.2	0.626532	0.641089	0.655646	0.670203
0.3	0.669	0.687872	0.706743	0.725615
0.4	0.705208	0.727367	0.749527	0.771686
0.5	0.73677	0.76139	0.786011	0.810631
0.6	0.764717	0.791094	0.817471	0.843847
0.7	0.789775	0.817285	0.844795	0.872305
0.8	0.812492	0.840566	0.86864	0.896714
0.9	0.833294	0.861402	0.889509	0.917616
1	0.85253	0.880163	0.907796	0.935429
1.1	0.870487	0.897152	0.923817	0.950482
1.2	0.887411	0.91262	0.937829	0.963038
1.3	0.903509	0.926775	0.950041	0.973307
1.4	0.918962	0.939795	0.960628	0.98146
1.5	0.933924	0.951831	0.969738	0.987645
1.6	0.948513	0.963009	0.977504	0.991999
1.7	0.962802	0.97343	0.984059	0.994688
1.8	0.976749	0.983162	0.989574	0.995986
1.9	0.989973	0.992182	0.994391	0.996599
2	1	1	1	1

Table 5.6: Maximum absolute error between solutions of (5.4.1)-(5.4.2) and (5.4.9)-(5.4.10).

ζ	$\|y - z\|$
0.1	2.81×10^{-2}
0.2	5.62×10^{-2}
0.3	8.43×10^{-2}



(a) Solution $z(t)$ of the system (5.4.9)-(5.4.10).



(b) Error between solution of (5.4.1)-(5.4.2) and the solution of (5.4.9)-(5.4.10) for different values of ζ.

Figure 5.4: Solution and error graphs for different values of ζ in (5.4.9)-(5.4.10).

Chapter 6

Synchronization and anti-synchronization of a chaotic fractional order financial delay system

This chapter deals with the development of synchronization and anti-synchronization of fractional-order delay financial system. Firstly, a Gauss-Seidel like predictor-corrector scheme is introduced to solve fractional-order delay systems. Then, an example is given to show that the proposed numerical scheme is better than the existing numerical schemes. Furthermore, numerical simulations are given to show that the financial system has chaotic behaviours for different values of time-delay and fractional-order. Then a suitable active control for synchronization/anti-synchronization of the system has been proposed. Finally, two examples with numerical simulations are given to validate the effectiveness of the proposed theory for different fractional-orders and time-delays.

[The content of this chapter is communicated to *Nonlinear Dynamics*]

6.1 Introduction

Synchronization (anti-synchronization) of two systems means that the trajectories of one of the systems will converge to the same values (same values but opposite signs) as the other and they will remain in step with each other in due course [102]. In this chapter, we assume that the fractional-order time-delay financial system dynamics contains the following variables (state) namely, the interest rate, investment demand, price index and market confidence. Our aim is to study the chaotic behaviors, synchronization and anti-synchronization of the assumed system. For that, we introduce Gauss-Seidel like predictor-corrector method to solve fractional-order systems with delay.

6.2 Dynamics and chaos in a financial system

Consider the fractional-order financial delay system

$$\left.\begin{aligned} {}^{C}D_t^{\alpha}x(t) &= z(t) + (y(t-\tau) - a)x(t) + m_1 w(t), \\ {}^{C}D_t^{\alpha}y(t) &= 1 - by(t) - x^2(t-\tau) + m_2 w(t), \\ {}^{C}D_t^{\alpha}z(t) &= -x(t-\tau) - cz(t) + m_3 w(t), \\ {}^{C}D_t^{\alpha}w(t) &= -x(t)y(t-\tau)z(t), \end{aligned}\right\} \tag{6.2.1}$$

where the fractional-order $\alpha \in (0,1)$, x denotes the interest rate, y denotes the investment demand, z denotes the price index, w denotes the market confidence, $a \geq 0$ is the saving amount, $b \geq 0$ is the cost per investment, $c \geq 0$ is the demand elasticity of commercial markets, the three constants m_1, m_2, m_3 are the impact factors and $\tau > 0$ is the time delay.
Note: If we choose the control $w(t)$ as the state dependent feedback (linear or nonlinear) control then the system (6.2.1) becomes

$$\left.\begin{aligned} {}^{C}D_t^{\alpha}x(t) &= A_0 x(t) + A_1 x(t-\tau) + f(t, x(t), x(t-\tau)), \quad t > T, \quad T < \infty \\ x(t) &= \phi(t), \quad t \in [-\tau, 0], \end{aligned}\right\}$$

with $0 < \alpha < 1$, the state $x(t) \in \mathbb{R}^4$, the history function $\phi(t) \in \mathbb{R}^4$ for each $t \in [-\tau, 0]$, the nonlinear function $f : [0,T] \times \mathbb{R}^4 \times \mathbb{R}^4 \to \mathbb{R}^4$, A_0 and A_1 are the real matrices of order 4×4 and τ is a positive constant. The existence and uniqueness of solution to this class of systems is given in [124] under the assumption that the nonlinear function is Lipschitz continuous with respect to the state variable.

6.2.1 Numerical method

In this subsection, we introduce Gauss-Seidel like method to solve fractional-order delay systems. For that, we consider the following fractional-order delayed system

$$\left.\begin{array}{rcl} {}^{C}D_t^{\alpha}x(t) &=& f_1(t,x(t),y(t),z(t),w(t),x(t-\tau),y(t-\tau),z(t-\tau),w(t-\tau)),\\ {}^{C}D_t^{\alpha}y(t) &=& f_2(t,x(t),y(t),z(t),w(t),x(t-\tau),y(t-\tau),z(t-\tau),w(t-\tau)),\\ {}^{C}D_t^{\alpha}z(t) &=& f_3(t,x(t),y(t),z(t),w(t),x(t-\tau),y(t-\tau),z(t-\tau),w(t-\tau)),\\ {}^{C}D_t^{\alpha}w(t) &=& f_4(t,x(t),y(t),z(t),w(t),x(t-\tau),y(t-\tau),z(t-\tau),w(t-\tau)), \end{array}\right\}, \quad (6.2.2)$$

for $t > 0$, with the history function $x(t) = \phi_1(t)$, $y(t) = \phi_2(t)$, $z(t) = \phi_3(t)$, $w(t) = \phi_4(t)$ for all $t \in [-\tau, 0]$, where $0 < \alpha \leq 1$, state variables $x, y, z, w \in \mathbb{R}$, the nonlinear functions $f_i : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $(i = 1, 2, 3, 4)$, the history function $\phi_i(t) \in \mathbb{R}$, $(i = 1, 2, 3, 4)$, $\phi \in C[-\tau, 0]$ and the time delay $\tau > 0$.

The system (6.2.2) can written as

$$\left.\begin{array}{rcl} {}^{C}D_t^{\alpha}X(t) &=& F(t,X(t),X(t-\tau)), \quad t>0\\ X(t) &=& \Phi(t), \quad t\in[-\tau,0], \end{array}\right\} \quad (6.2.3)$$

where $X(t) = \begin{bmatrix} x(t)\\ y(t)\\ z(t)\\ w(t) \end{bmatrix}$, $\Phi(t) = \begin{bmatrix} \phi_1(t)\\ \phi_2(t)\\ \phi_3(t)\\ \phi_4(t) \end{bmatrix}$ and

$$F(t,X(t),X(t-\tau)) = \begin{bmatrix} f_1(t,x(t),y(t),z(t),w(t),x(t-\tau),y(t-\tau),z(t-\tau),w(t-\tau))\\ f_2(t,x(t),y(t),z(t),w(t),x(t-\tau),y(t-\tau),z(t-\tau),w(t-\tau))\\ f_3(t,x(t),y(t),z(t),w(t),x(t-\tau),y(t-\tau),z(t-\tau),w(t-\tau))\\ f_4(t,x(t),y(t),z(t),w(t),x(t-\tau),y(t-\tau),z(t-\tau),w(t-\tau)) \end{bmatrix}.$$

Taking Riemann-Liouville fractional integral on both sides of (6.2.3), we get

$$X(t) = \Phi(0) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}F(s,X(s),X(s-\tau))\mathrm{d}s, \quad (6.2.4)$$

Now to solve (6.2.4) we propose the following method.

Consider the uniform grid $\{t_n = nh : n = -l, -l+1, \ldots, -1, 0, 1, \ldots, N\}$ where l and N are positive integers such that $h = T/N = \tau/l$, (T is suitably chosen). Let

$$X_h(t_j) = \Phi(t_j), \quad j = -l, -l+1, \ldots, -1, 0$$

and

$$X_h(t_j - \tau) = X_h(jh - lh) = X_h(t_{j-l}),\ j = 0, 1, 2, \ldots, N.$$

Suppose $X_h(t_j) \approx X(t_j), \quad (j = -l, -l+1, \ldots, -1, 0, 1, \ldots, n)$ are known and we wish to calculate $X_h(t_{n+1})$ using

$$X(t_{n+1}) \;=\; \Phi(0) + \frac{1}{\Gamma(\alpha)} \int_0^{t_{n+1}} (t_{n+1} - s)^{\alpha-1} F(s, X(s), X(s-\tau)) \mathrm{d}s. \qquad (6.2.5)$$

To replace the integral in (6.2.5), we use product trapezoidal quadrature formula, where the nodes t_j $(j = 0, 1, 2, \ldots, n+1)$ are taken with respect to the weight function $(t_{n+1} - s)^{\alpha-1}$. Thus

$$\begin{aligned} X(t_{n+1}) \;=\;& \Phi(0) + \frac{h^\alpha}{\Gamma(\alpha+2)} \sum_{j=0}^{n} d_{j,n+1} F(t_j, X_j, X(t_j - \tau)) \\ &+ \frac{h^\alpha}{\Gamma(\alpha+2)} F(t_{n+1}, X_{n+1}, X(t_{n+1} - \tau)), \end{aligned} \qquad (6.2.6)$$

where

$$d_{j,n+1} = \begin{cases} n^{\alpha+1} - (n-\alpha)(n+1)^\alpha, & \text{if } j = 0, \\ (n-j+2)^{\alpha+1} + (n-j)^{\alpha+1} - 2(n-j+1)^{\alpha+1}, & \text{if } 1 \le j \le n. \end{cases}$$

To solve the nonlinear functional system (6.2.6), a two-step predictor-corrector formula was given by Gejji et al. in [40]. Now we give a Gauss-Seidel like two-step predictor-corrector formula to solve the system (6.2.6).

The predictors are

$$\begin{aligned}
x_{n+1}^p &= \phi_1(0) \\
&\quad + \frac{h^\alpha}{\Gamma(\alpha+2)} \sum_{j=0}^{n} d_{j,n+1} f_1(t_j, x_j^p, y_j^p, z_j^p, w_j^p, x(t_j-\tau), y(t_j-\tau), z(t_j-\tau), w(t_j-\tau)), \\
y_{n+1}^p &= \phi_2(0) \\
&\quad + \frac{h^\alpha}{\Gamma(\alpha+2)} \sum_{j=0}^{n} d_{j,n+1} f_2(t_j, x_j^p, y_j^p, z_j^p, w_j^p, x(t_j-\tau), y(t_j-\tau), z(t_j-\tau), w(t_j-\tau)), \\
z_{n+1}^p &= \phi_3(0) \\
&\quad + \frac{h^\alpha}{\Gamma(\alpha+2)} \sum_{j=0}^{n} d_{j,n+1} f_3(t_j, x_j^p, y_j^p, z_j^p, w_j^p, x(t_j-\tau), y(t_j-\tau), z(t_j-\tau), w(t_j-\tau)), \\
w_{n+1}^p &= \phi_4(0) \\
&\quad + \frac{h^\alpha}{\Gamma(\alpha+2)} \sum_{j=0}^{n} d_{j,n+1} f_4(t_j, x_j^p, y_j^p, z_j^p, w_j^p, x(t_j-\tau), y(t_j-\tau), z(t_j-\tau), w(t_j-\tau)),
\end{aligned}$$

and

$$\begin{aligned}
\bar{x}_{n+1}^p &= \frac{h^\alpha}{\Gamma(\alpha+2)} f_1(t_{n+1}, x_{n+1}^p, y_{n+1}^p, z_{n+1}^p, w_{n+1}^p, \\
&\quad x(t_{n+1}-\tau), y(t_{n+1}-\tau), z(t_{n+1}-\tau), w(t_{n+1}-\tau)), \\
\bar{y}_{n+1}^p &= \frac{h^\alpha}{\Gamma(\alpha+2)} f_2(t_{n+1}, x_{n+1}^p, y_{n+1}^p, z_{n+1}^p, w_{n+1}^p, \\
&\quad x(t_{n+1}-\tau), y(t_{n+1}-\tau), z(t_{n+1}-\tau), w(t_{n+1}-\tau)), \\
\bar{z}_{n+1}^p &= \frac{h^\alpha}{\Gamma(\alpha+2)} f_3(t_{n+1}, x_{n+1}^p, y_{n+1}^p, z_{n+1}^p, w_{n+1}^p, \\
&\quad x(t_{n+1}-\tau), y(t_{n+1}-\tau), z(t_{n+1}-\tau), w(t_{n+1}-\tau)), \\
\bar{w}_{n+1}^p &= \frac{h^\alpha}{\Gamma(\alpha+2)} f_4(t_{n+1}, x_{n+1}^p, y_{n+1}^p, z_{n+1}^p, w_{n+1}^p, \\
&\quad x(t_{n+1}-\tau), y(t_{n+1}-\tau), z(t_{n+1}-\tau), w(t_{n+1}-\tau)),
\end{aligned}$$

where $(x_0^p, y_0^p, z_0^p, w_0^p) = (\phi_1(0), \phi_2(0), \phi_3(0), \phi_4(0))$. The corrector is

$$\begin{aligned}
x_{n+1}^c &= x_{n+1}^p + \frac{h^\alpha}{\Gamma(\alpha+2)} f_1\big(t_{n+1}, x_{n+1}^p + \bar{x}_{n+1}^p, y_{n+1}^p + \bar{y}_{n+1}^p, z_{n+1}^p + \bar{z}_{n+1}^p, w_{n+1}^p + \bar{w}_{n+1}^p,\\
&\quad x(t_{n+1}-\tau), y(t_{n+1}-\tau), z(t_{n+1}-\tau), w(t_{n+1}-\tau)\big),\\
y_{n+1}^c &= y_{n+1}^p + \frac{h^\alpha}{\Gamma(\alpha+2)} f_2\big(t_{n+1}, x_{n+1}^c, y_{n+1}^p + \bar{y}_{n+1}^p, z_{n+1}^p + \bar{z}_{n+1}^p, w_{n+1}^p + \bar{w}_{n+1}^p,\\
&\quad x(t_{n+1}-\tau), y(t_{n+1}-\tau), z(t_{n+1}-\tau), w(t_{n+1}-\tau)\big),\\
z_{n+1}^c &= z_{n+1}^p + \frac{h^\alpha}{\Gamma(\alpha+2)} f_3\big(t_{n+1}, x_{n+1}^c, y_{n+1}^c, z_{n+1}^p + \bar{z}_{n+1}^p, w_{n+1}^p + \bar{w}_{n+1}^p,\\
&\quad x(t_{n+1}-\tau), y(t_{n+1}-\tau), z(t_{n+1}-\tau), w(t_{n+1}-\tau)\big),\\
w_{n+1}^c &= w_{n+1}^p + \frac{h^\alpha}{\Gamma(\alpha+2)} f_4\big(t_{n+1}, x_{n+1}^c, y_{n+1}^c, z_{n+1}^c, w_{n+1}^p + \bar{w}_{n+1}^p,\\
&\quad x(t_{n+1}-\tau), y(t_{n+1}-\tau), z(t_{n+1}-\tau), w(t_{n+1}-\tau)\big).
\end{aligned}$$

Note that for this method we considered four fractional-order delay differential equations, but one can easily extend it for r number of equations, where r is any positive integer.

In the following example, we compare this scheme with existing numerical schemes to solve fractional-order delay differential systems.

Example 6.2.1. *Consider the following system*

$$\left.\begin{aligned}
{}^C D_t^{1/2} x(t) &= x(t)y(t) - 3x(t) + 4y(t) - x(t-1)y(t-1)\\
&\quad -y(t-1) - 2 + \frac{\Gamma(3)}{\Gamma(5/2)} t^{3/2}\\
{}^C D_t^{1/2} y(t) &= -x(t-1) - y(t-1) + x(t) - y(t) + \frac{2}{\Gamma(1/2)} t^{1/2}
\end{aligned}\right\}, t > 0, \qquad (6.2.7)$$

with the history function $x(t) = t^2$ and $y(t) = t$ for $t \in [-1, 0]$.

The analytical solution for the system (6.2.7) *is $x(t) = t^2$ and $y(t) = t$ for all $t \geq 0$. If $(x^M(t), y^M(t))$ is the numerical solution of* (6.2.7) *in the interval $t \in [0, 2]$ using a method M, then the errors $\|Ex^M\|$ and $\|Ey^M\|$ are defined by the infinity norms as*

$$\begin{aligned}
\|Ex^M\| &= \max_k \left|x(t_k) - x^M(t_k)\right| \qquad \textit{and}\\
\|Ey^M\| &= \max_k \left|y(t_k) - y^M(t_k)\right|,
\end{aligned}$$

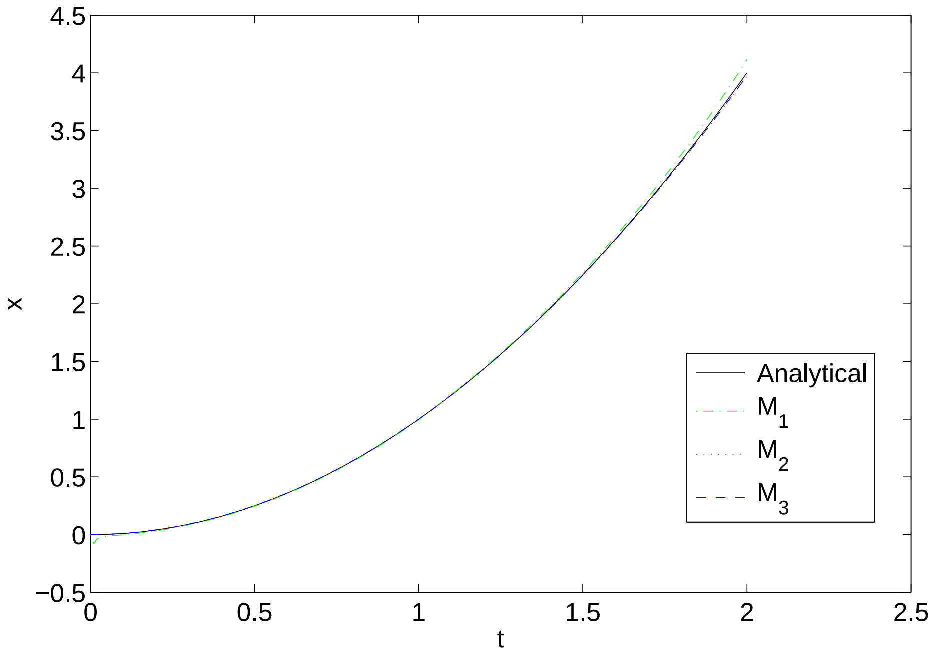
where k denote the partition points of the interval $[0, 2]$.

Table 6.1: Infinity norm of the absolute errors between analytical solution of (6.2.7) and numerical solutions of (6.2.7) (using M_1, M_2, M_3).

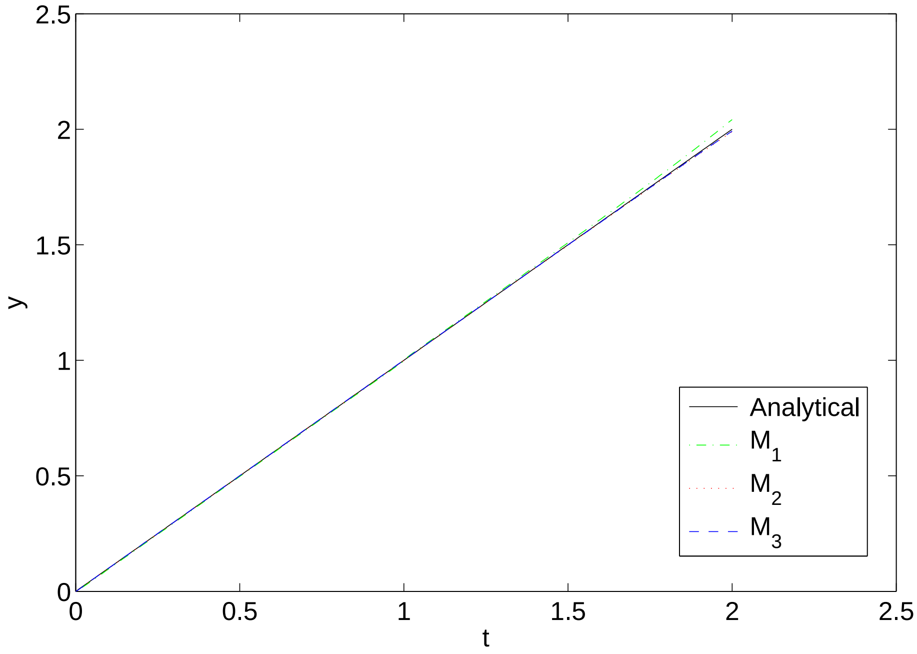
h	M_1		M_2		M_3	
	$\|Ex_1^M\|$	$\|Ey_1^M\|$	$\|Ex_2^M\|$	$\|Ey_2^M\|$	$\|Ex_3^M\|$	$\|Ey_3^M\|$
0.01	1.15×10^{-1}	4.21×10^{-2}	2.95×10^{-2}	1.03×10^{-2}	2.49×10^{-2}	8.19×10^{-3}
0.005	5.76×10^{-2}	2.12×10^{-2}	1.09×10^{-2}	3.81×10^{-3}	8.11×10^{-3}	2.64×10^{-3}
0.0025	3.80×10^{-2}	1.07×10^{-2}	3.93×10^{-3}	1.38×10^{-3}	2.67×10^{-3}	8.62×10^{-4}

Figure 6.1 shows that the numerical solution of (6.2.7) *in the interval* $[0, 2]$ *using methods (i) modified Adams-Bashforth predictor-corrector method (M_1) given in [11], (ii) the predictor-corrector method (M_2) given in [40] and (iii) our proposed Gauss-Seidel like method (M_3), are very close to the analytical solution of* (6.2.7). *Figure 6.2 shows error graphs using these three methods and it indicates that error graphs using the proposed Gauss-Seidel like method (M_3) are close to zero as compared to other two methods. Infinity norm of the absolute errors* $\|Ex^{M_i}\|$ *and* $\|Ey^{M_i}\|$, $i = 1, 2, 3$ *of the system* (6.2.7) *in the interval* $[0, 2]$ *using the three methods* M_1, M_2 *and* M_3*, respectively for different values of step size* h *are given in Table 6.1.* ■

Using Gauss-Seidel like method it has been observed that the fractional-order financial delayed system (6.2.1) is chaotic for different values of time-delays and fractional-orders (see, Figure 6.3 for $\tau = 0.05$, $\alpha = 0.95$ and Figure 6.4 for $\tau = 0.09$, $\alpha = 0.8$) with $a = 2.1$, $b = 0.3$, $c = 2.6$, $m_1 = 8.4$, $m_2 = 6.4$, $m_3 = 2.2$ and history function $[\ x(t) \quad y(t) \quad z(t) \quad w(t)\ ]^T = [\ 1 \quad 4 \quad 5 \quad 3\ ]^T$ for all $t \in [-\tau, 0]$.

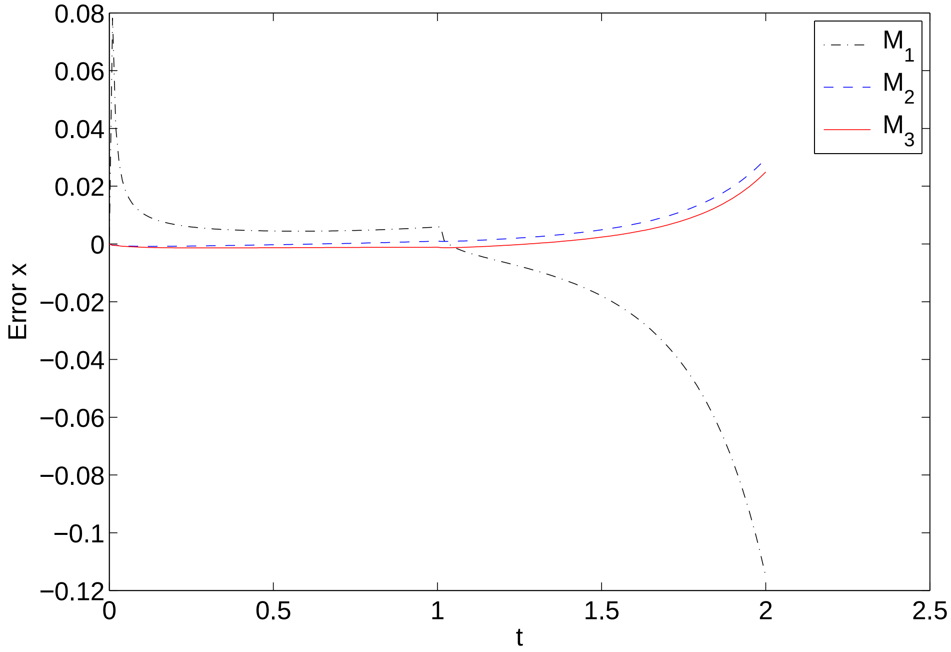


(a) Solution x

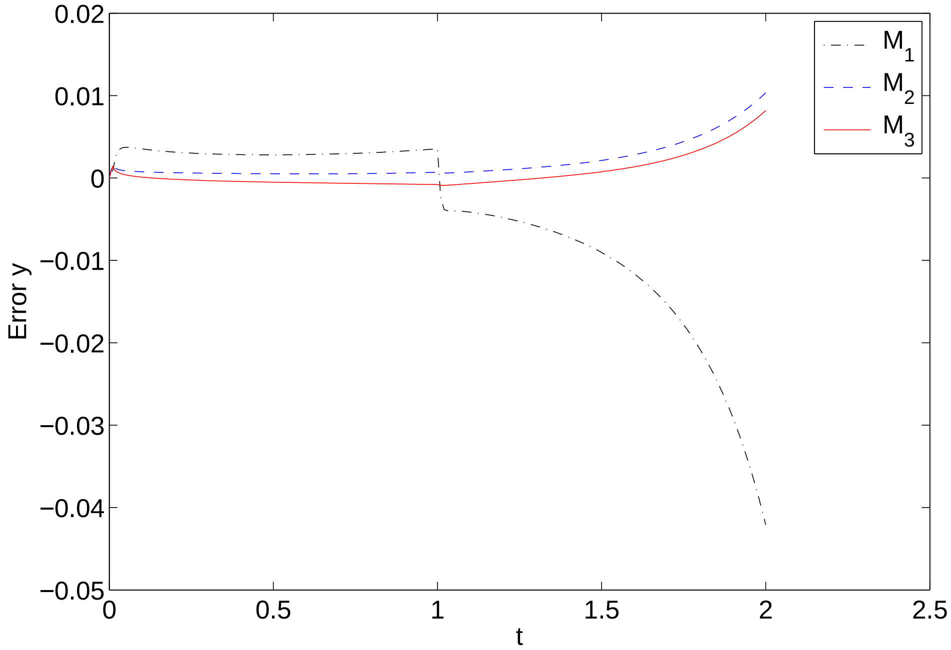


(b) Solution y

Figure 6.1: Solution of the system (6.2.7).

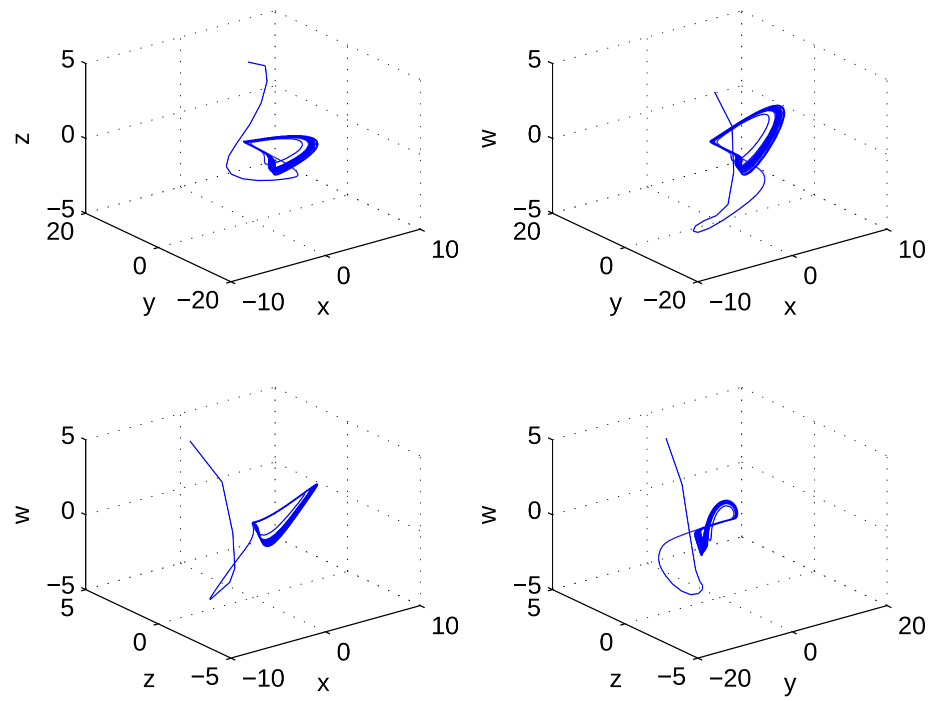


(a) Error x

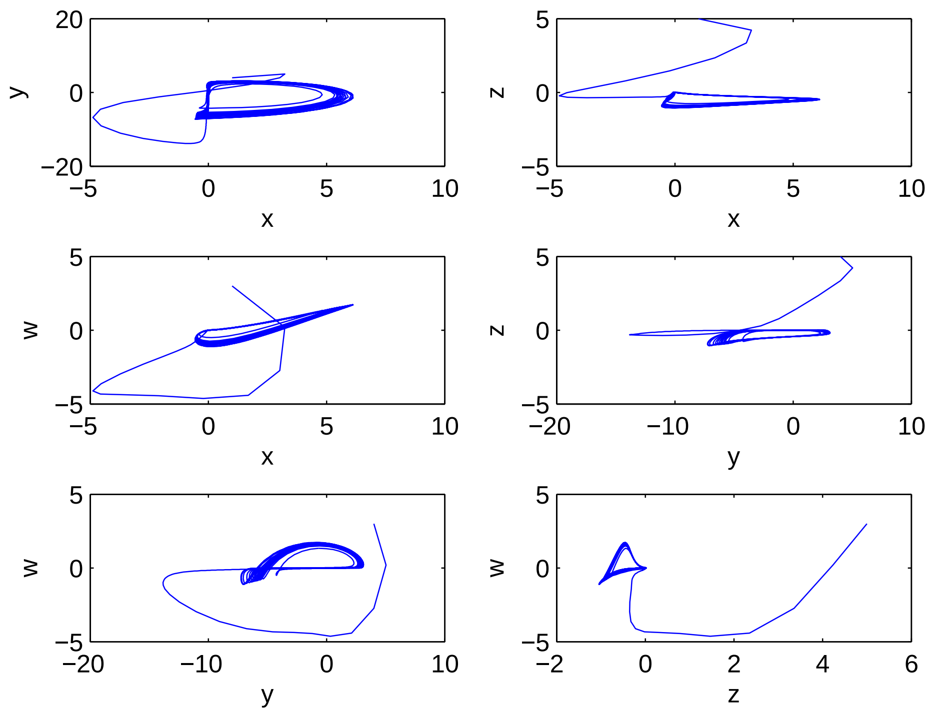


(b) Error y

Figure 6.2: Error of the system (6.2.7).

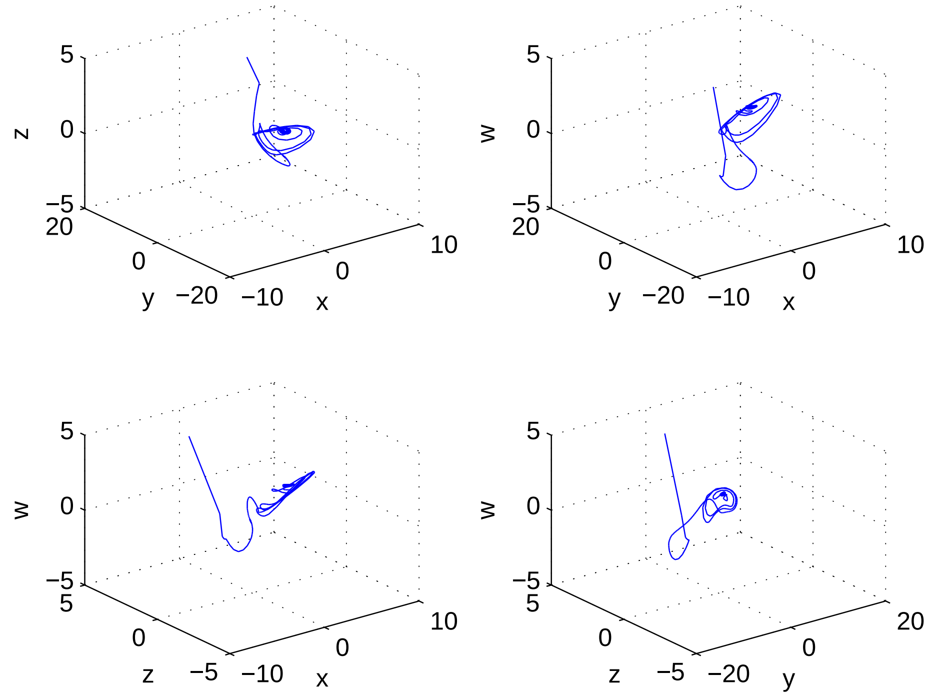


(a) Chaotic attractors

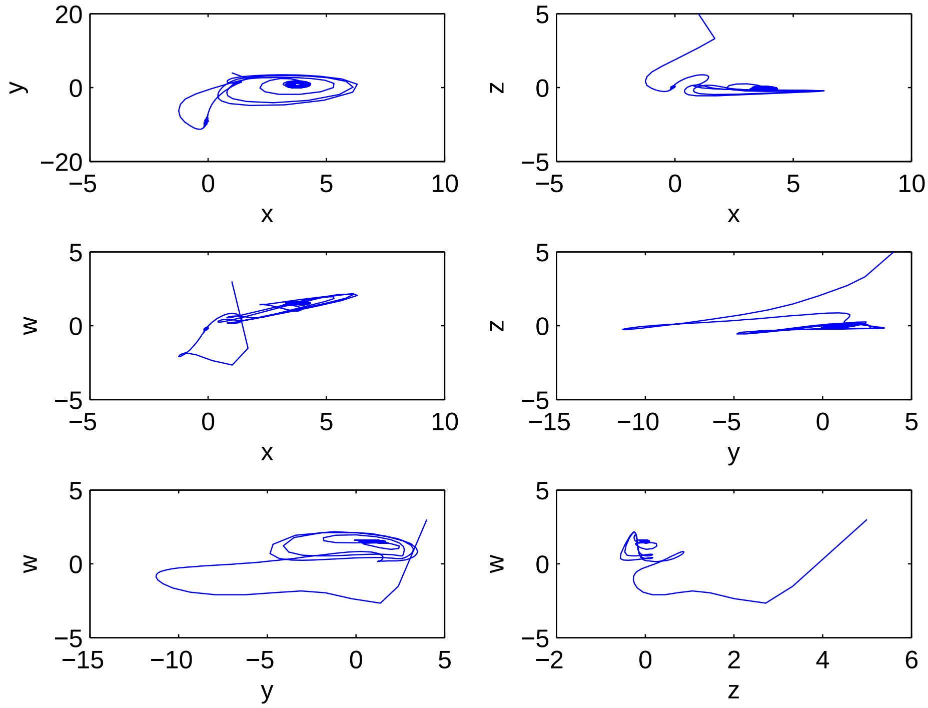


(b) 2D chaotic attractors

Figure 6.3: Chaotic attractors of the system (6.2.1) with $\alpha = 0.95$ and $\tau = 0.05$.



(a) Chaotic attractors



(b) 2D chaotic attractors

Figure 6.4: Chaotic attractors of the system (6.2.1) with $\alpha = 0.8$ and $\tau = 0.09$.

6.3 Synchronization and anti-synchronization with delay

6.3.1 Synchronization between two identical fractional-order financial systems

We define drive and response systems as in the following:

$$\left.\begin{aligned} {}^{C}D^{\alpha}x_1(t) &= z_1(t) + (y_1(t-\tau) - a)x_1(t) + m_1 w_1(t), \\ {}^{C}D_t^{\alpha}y_1(t) &= 1 - by_1(t) - x_1^2(t-\tau) + m_2 w_1(t), \\ {}^{C}D_t^{\alpha}z_1(t) &= -x_1(t-\tau) - cz_1(t) + m_3 w_1(t), \\ {}^{C}D_t^{\alpha}w_1(t) &= -x_1(t)y_1(t-\tau)z_1(t), \end{aligned}\right\} \tag{6.3.1}$$

$$\left.\begin{aligned} {}^{C}D_t^{\alpha}x_2(t) &= z_2(t) + (y_2(t-\tau) - a)x_2(t) + m_1 w_2(t) + u_1(t), \\ {}^{C}D_t^{\alpha}y_2(t) &= 1 - by_2(t) - x_2^2(t-\tau) + m_2 w_2(t) + u_2(t), \\ {}^{C}D_t^{\alpha}z_2(t) &= -x_2(t-\tau) - cz_2(t) + m_3 w_2(t) + u_3(t), \\ {}^{C}D_t^{\alpha}w_2(t) &= -x_2(t)y_2(t-\tau)z_2(t) + u_4(t), \end{aligned}\right\} \tag{6.3.2}$$

where the unknowns $u_i(t), i = 1,2,3,4$ are active control functions for the responding system. Now by taking the error functions as $e_1 = x_2 - x_1$, $e_2 = y_2 - y_1$, $e_3 = z_2 - z_1$, $e_4 = w_2 - w_1$, we obtain the error system from (6.3.2) and (6.3.1) as

$$\left.\begin{aligned} {}^{C}D_t^{\alpha}e_1(t) &= e_3(t) - ae_1(t) + y_2(t-\tau)x_2(t) - y_1(t-\tau)x_1(t) + m_1 e_4(t) + u_1(t), \\ {}^{C}D_t^{\alpha}e_2(t) &= -be_2(t) - x_2^2(t-\tau) + x_1^2(t-\tau) + m_2 e_4(t) + u_2(t), \\ {}^{C}D_t^{\alpha}e_3(t) &= -e_1(t-\tau) - ce_3(t) + m_3 e_4(t) + u_3(t), \\ {}^{C}D_t^{\alpha}e_4(t) &= -x_2(t)y_2(t-\tau)z_2(t) + x_1(t)y_1(t-\tau)z_1(t) + u_4(t). \end{aligned}\right\} \tag{6.3.3}$$

The active control $u_i(t)$ can be chosen as

$$\left.\begin{aligned} u_1(t) &= \mu_1(t) - y_2(t-\tau)x_2(t) + y_1(t-\tau)x_1(t), \\ u_2(t) &= \mu_2(t) + x_2^2(t-\tau) - x_1^2(t-\tau), \\ u_3(t) &= \mu_3(t) + e_1(t-\tau), \\ u_4(t) &= \mu_4(t) + e_1(t) + e_3(t) + x_2(t)y_2(t-\tau)z_2(t) - x_1(t)y_1(t-\tau)z_1(t), \end{aligned}\right\} \tag{6.3.4}$$

where the terms $\mu_i(t), i = 1,2,3,4$ are some linear functions of $e_i(t), i = 1,2,3,4$. Then the error system (6.3.3) becomes

$$\begin{aligned} {}^C D_t^\alpha e_1(t) &= e_3(t) - ae_1(t) + m_1 e_4(t) + \mu_1(t), \\ {}^C D_t^\alpha e_2(t) &= -be_2(t) + m_2 e_4(t) + \mu_2(t), \\ {}^C D_t^\alpha e_3(t) &= -ce_3(t) + m_3 e_4(t) + \mu_3(t), \\ {}^C D_t^\alpha e_4(t) &= e_1(t) + e_3(t) + \mu_4(t). \end{aligned}$$

The above system can be written as

$${}^C D_t^\alpha E(t) = AE(t) + M(t), \tag{6.3.5}$$

where

$$A = \begin{bmatrix} -a & 0 & 1 & m_1 \\ 0 & -b & 0 & m_2 \\ 0 & 0 & -c & m_3 \\ 1 & 0 & 1 & 0 \end{bmatrix}, \quad E(t) = \begin{bmatrix} e_1(t) \\ e_2(t) \\ e_3(t) \\ e_4(t) \end{bmatrix}, \quad \text{and} \quad M(t) = \begin{bmatrix} \mu_1(t) \\ \mu_2(t) \\ \mu_3(t) \\ \mu_4(t) \end{bmatrix}.$$

Here, we have to choose the input function $M(t)$ in such a way that the system (6.3.5) is asymptotically stable. For that we chosen the feedback control $M(t) = KE(t)$, such that $|\arg \lambda_i(A+K)| > \alpha\pi/2$, $(0 < \alpha < 1)$, where $K \in \mathbb{R}^4$ is gain matrix and $\lambda_i, (i = 1,2,3,4)$ are the eigenvalues of the matrix $A + K$ [see, Theorem 2.2.4]. In particular, if we choose the matrix

$$K = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & -1 & -1 \end{bmatrix}$$

then the eigenvalues of the matrix $A + K$ becomes $-a, -b, -c$ and -1 (a, b, c are positive constants), which satisfies $|\arg \lambda_i(A+K)| > \alpha\pi/2$, $(0 < \alpha < 1$ and $i = 1, 2, 3, 4)$. Thus the error system (6.3.3) converges to zero as $t \to \infty$. This implies that the synchronization between the drive system (6.3.2) and response system (6.3.3) is achieved.

6.3.2 Anti-synchronization between two identical fractional-order financial systems

For the anti-synchronization, we consider system (6.3.1) as the drive system and the following systems as the response system

$$\left.\begin{aligned} {}^C D_t^\alpha x_2(t) &= z_2(t) + (y_2(t-\tau) - a)x_2(t) + m_1 w_2(t) + v_1(t), \\ {}^C D_t^\alpha y_2(t) &= 1 - b y_2(t) - x_2^2(t-\tau) + m_2 w_2(t) + v_2(t), \\ {}^C D_t^\alpha z_2(t) &= -x_2(t-\tau) - c z_2(t) + m_3 w_2(t) + v_3(t), \\ {}^C D_t^\alpha w_2(t) &= -x_2(t) y_2(t-\tau) z_2(t) + v_4(t), \end{aligned}\right\} \tag{6.3.6}$$

Now by taking the error functions as

$$e_1 = x_2 + x_1,\ e_2 = y_2 + y_1,\ e_3 = z_2 + z_1,\ e_4 = w_2 + w_1,$$

we obtain the error system from (6.3.6) and (6.3.1) as

$$\left.\begin{aligned}
{}^{C}D_t^{\alpha}e_1(t) &= e_3(t) - ae_1(t) + m_1e_4(t) + y_2(t-\tau)x_2(t) + y_1(t-\tau)x_1(t) + v_1(t),\\
{}^{C}D_t^{\alpha}e_2(t) &= -be_2(t) + m_2e_4(t) + 2 - x_2^2(t-\tau) - x_1^2(t-\tau) + v_2(t),\\
{}^{C}D_t^{\alpha}e_3(t) &= -ce_3(t) + m_3e_4(t) - e_1(t-\tau) + v_3(t),\\
{}^{C}D_t^{\alpha}e_4(t) &= -x_2(t)y_2(t-\tau)z_2(t) - x_1(t)y_1(t-\tau)z_1(t) + v_4(t).
\end{aligned}\right\}\quad (6.3.7)$$

We choose the active control $v_i(t)$ as

$$\left.\begin{aligned}
v_1(t) &= \xi_1(t) - y_2(t-\tau)x_2(t) - y_1(t-\tau)x_1(t),\\
v_2(t) &= \xi_2(t) - 2 + x_2^2(t-\tau) + x_1^2(t-\tau),\\
v_3(t) &= \xi_3(t) + e_1(t-\tau),\\
v_4(t) &= \xi_4(t) + e_2(t) + e_4(t) + x_2(t)y_2(t-\tau)z_2(t) + x_1(t)y_1(t-\tau)z_1(t),
\end{aligned}\right\}\quad (6.3.8)$$

where the terms $\xi_i(t), i = 1, 2, 3, 4$ are the linear functions of $e_i(t), i = 1, 2, 3, 4$. Then the error system (6.3.7) becomes

$$\begin{aligned}
{}^{C}D_t^{\alpha}e_1(t) &= e_3(t) - ae_1(t) + m_1e_4(t) + \xi_1(t),\\
{}^{C}D_t^{\alpha}e_2(t) &= -be_2(t) + m_2e_4(t) + \xi_2(t),\\
{}^{C}D_t^{\alpha}e_3(t) &= -ce_3(t) + m_3e_4(t) + \xi_3(t),\\
{}^{C}D_t^{\alpha}e_4(t) &= e_2(t) + e_4(t) + \xi_4(t),
\end{aligned}$$

Now, the above system can be written in the form of system (6.3.5) with

$$A = \begin{bmatrix} -a & 0 & 1 & m_1\\ 0 & -b & 0 & m_2\\ 0 & 0 & -c & m_3\\ 0 & 1 & 0 & 1 \end{bmatrix} \quad \text{and} \quad M(t) = \begin{bmatrix} \xi_1(t)\\ \xi_2(t)\\ \xi_3(t)\\ \xi_4(t) \end{bmatrix}.$$

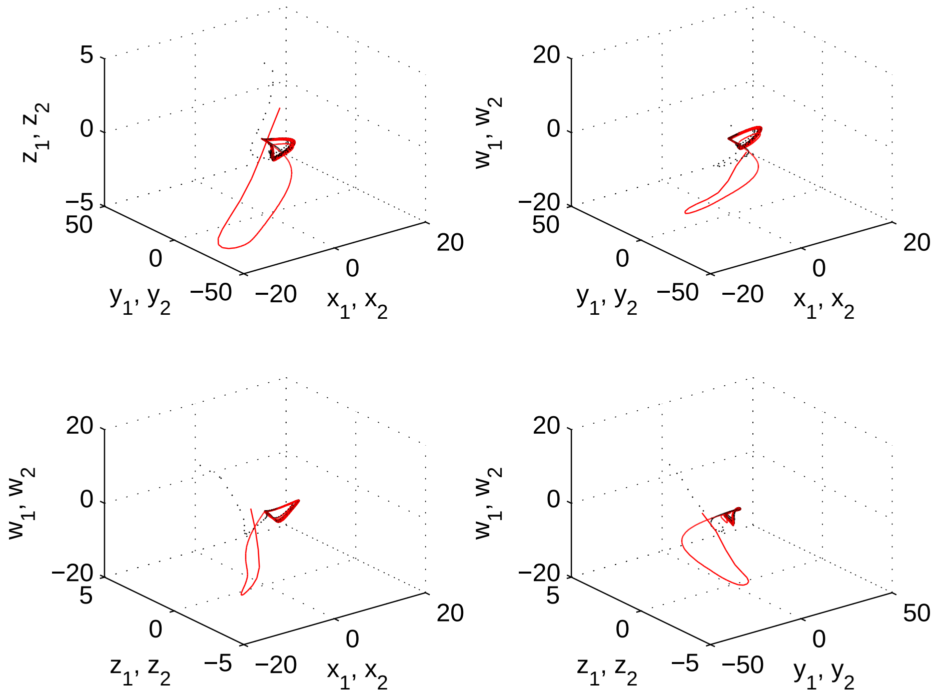
Similar to Section 6.3.1, it is easy to show that the anti-synchronization between the drive system (6.3.2) and response system (6.3.6) is achieved.

6.4 Simulation results

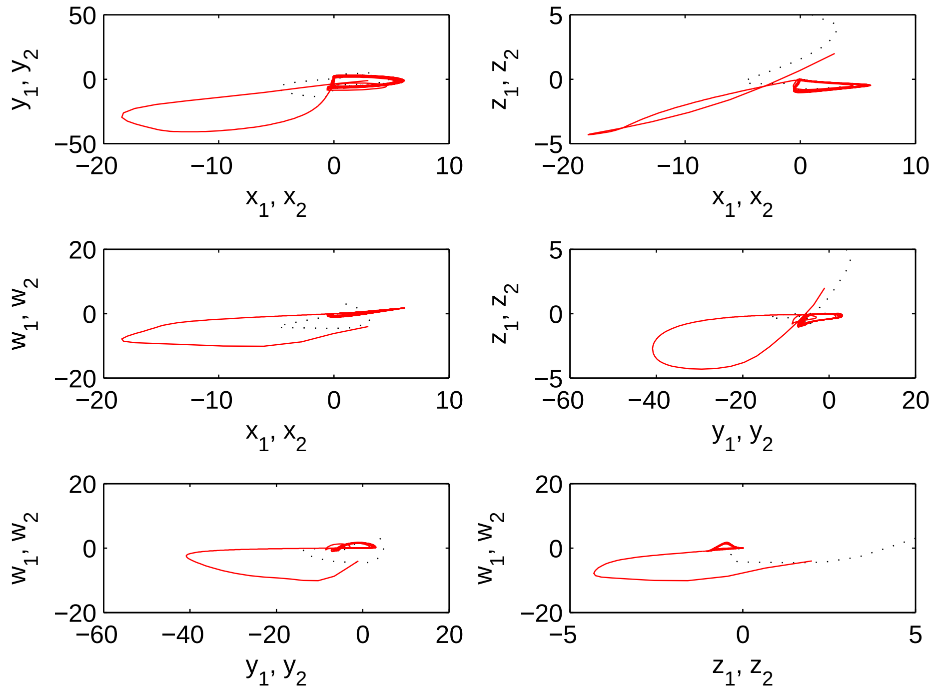
In this section, we give two examples to testify the proposed theory in Section 6.3. Throughout this section, we use Matlab software with the numerical scheme given in Section 6.2.1 to solve fractional-order delay systems.

Example 6.4.1. *Consider the parameter values* $a = 2.1$, $b = 0.3$, $c = 2.6$, $m_1 = 8.4$, $m_2 = 6.4$, $m_3 = 2.2$ *with fractional-order* $\alpha = 0.95$ *and time-delay* $\tau = 0.05$ *of the drive system* (6.3.1) *and response system* (6.3.2). *The history functions for the drive system* (6.3.1) *and response system* (6.3.2) *are chosen as* $[\ x_1(t) \quad y_1(t) \quad z_1(t) \quad w_1(t)\ ] = [\ 1 \quad 4 \quad 5 \quad 3\ ]$ *and* $[\ x_2(t) \quad y_2(t) \quad z_2(t) \quad w_2(t)\ ] = [\ 3 \quad -1 \quad 2 \quad -4\ ]$, *respectively for all* $t \in [-0.05, 0]$. *The synchronization between two systems* (6.3.1) *and* (6.3.2) *are depicted in Figure 6.5 and the anti-synchronization between two systems* (6.3.1) *and* (6.3.6) *are depicted in Figure 6.6. The solution of the error systems* (6.3.3) *and* (6.3.7) *are shown in Figure 6.7(a) and Figure 6.7(b), respectively. Figures 6.8 shows that the error systems* (6.3.3) *and* (6.3.7) *converges fastly when the fractional-order* α *approaches 1.*

Example 6.4.2. *Consider the parameter values* $a = 3.0$, $b = 0.1$, $c = 1.0$, $m_1 = 5.3$, $m_2 = 8.7$, $m_3 = 2.4$ *with fractional-order* $\alpha = 0.92$ *and time-delay* $\tau = 0.07$ *of the drive system* (6.3.1) *and response system* (6.3.2). *The history functions for the drive system* (6.3.1) *and response system* (6.3.2) *are chosen as* $[\ x_1(t) \quad y_1(t) \quad z_1(t) \quad w_1(t)\ ] = [\ 1 \quad -1.5 \quad 2 \quad -0.5\ ]$ *and* $[\ x_2(t) \quad y_2(t) \quad z_2(t) \quad w_2(t)\ ] = [\ -1 \quad 2 \quad 1 \quad 1.5\ ]$, *respectively for all* $t \in [-0.07, 0]$. *The synchronization of the drive system* (6.3.1) *and response system* (6.3.2) *are depicted in Figure 6.9 and the anti-synchronization of the drive system* (6.3.1) *and response system* (6.3.6) *are depicted in Figure 6.10. The solution of the error systems* (6.3.3) *and* (6.3.7) *are shown in Figure 6.11(a) and Figure 6.11(b), respectively. Figures 6.12 shows that the error systems* (6.3.3) *and* (6.3.7) *converges fastly when the fractional-order* α *approaches 1.*

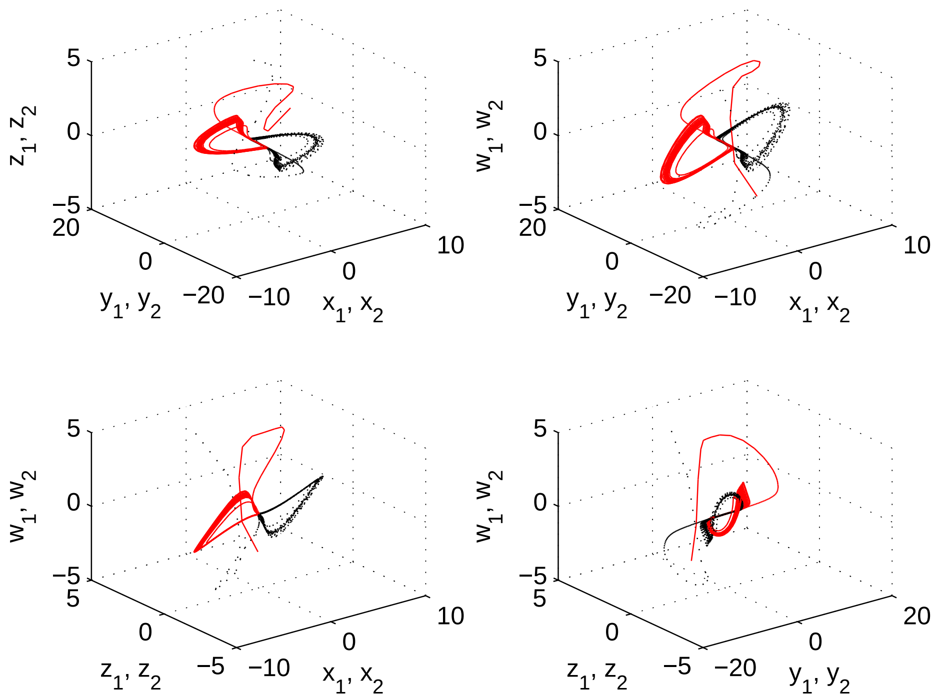


(a)

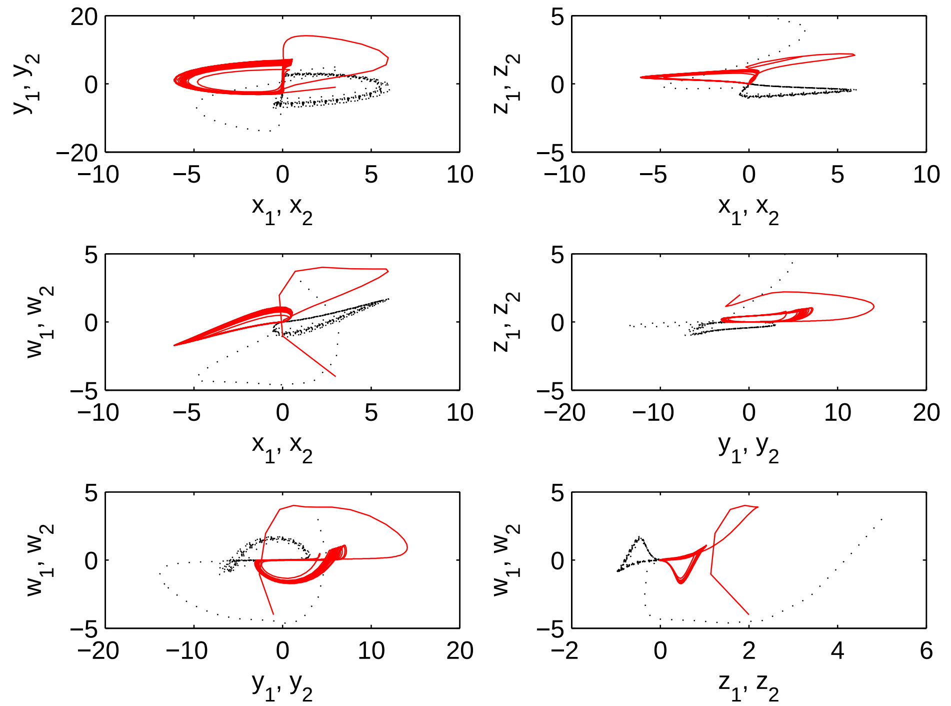


(b)

Figure 6.5: Synchronization of drive system (6.3.1) (black color dotted lines) and response system (6.3.2) (red color lines).

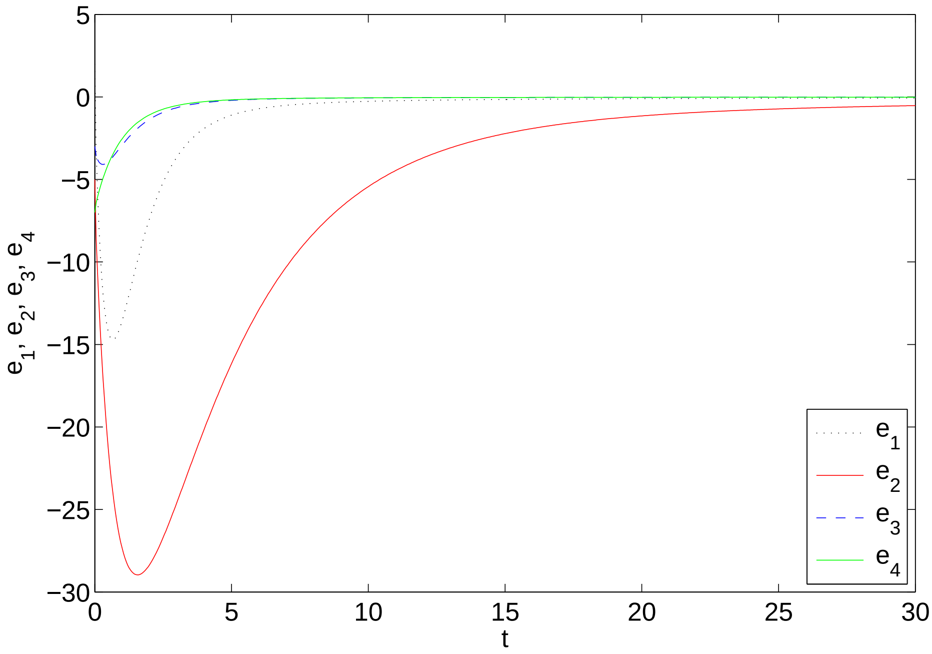


(a)

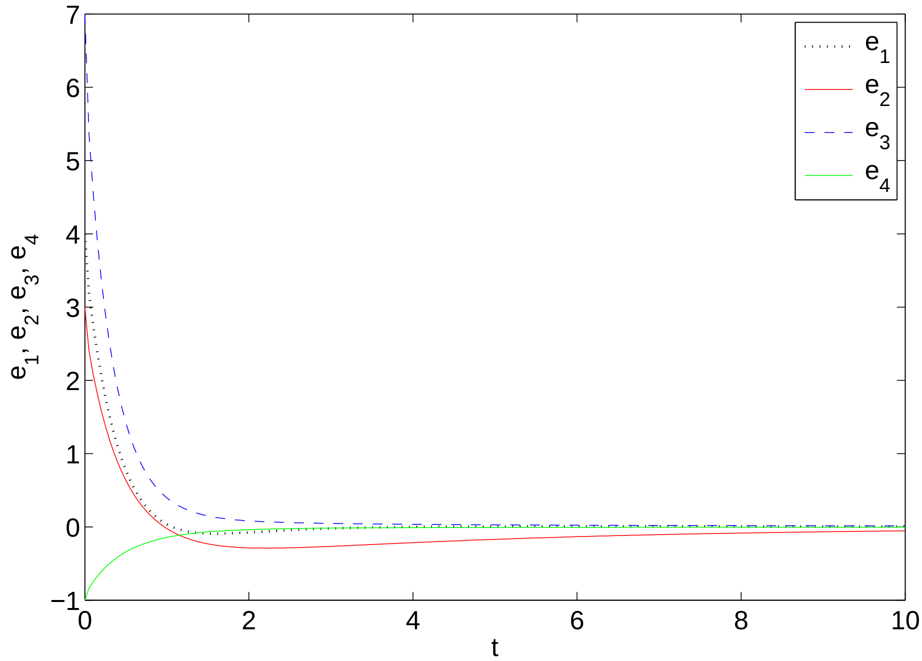


(b)

Figure 6.6: Anti-synchronization of drive system (6.3.1) (black color dotted lines) and response system (6.3.6) (red color lines).

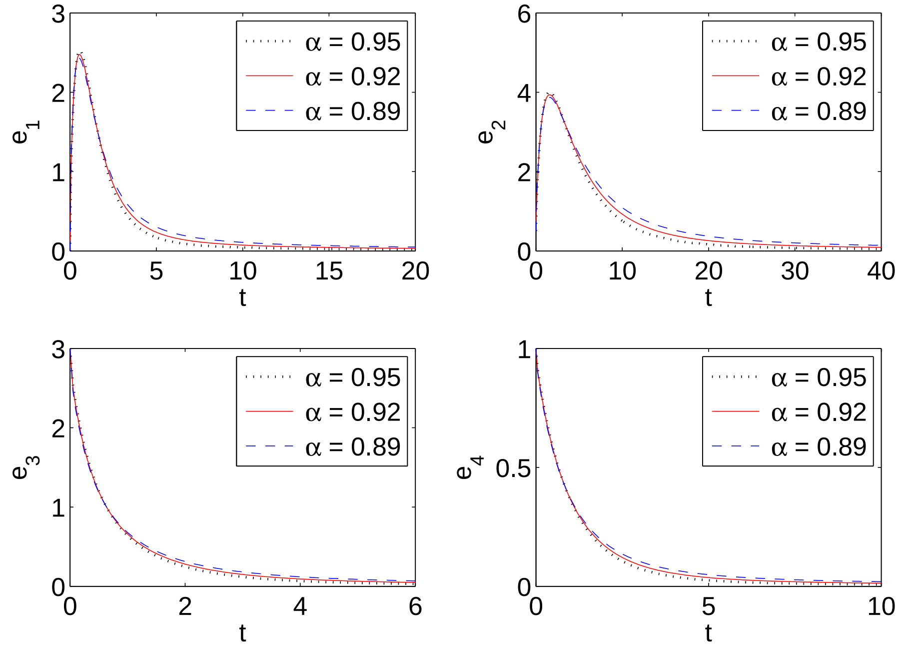


(a) Error system (6.3.3)

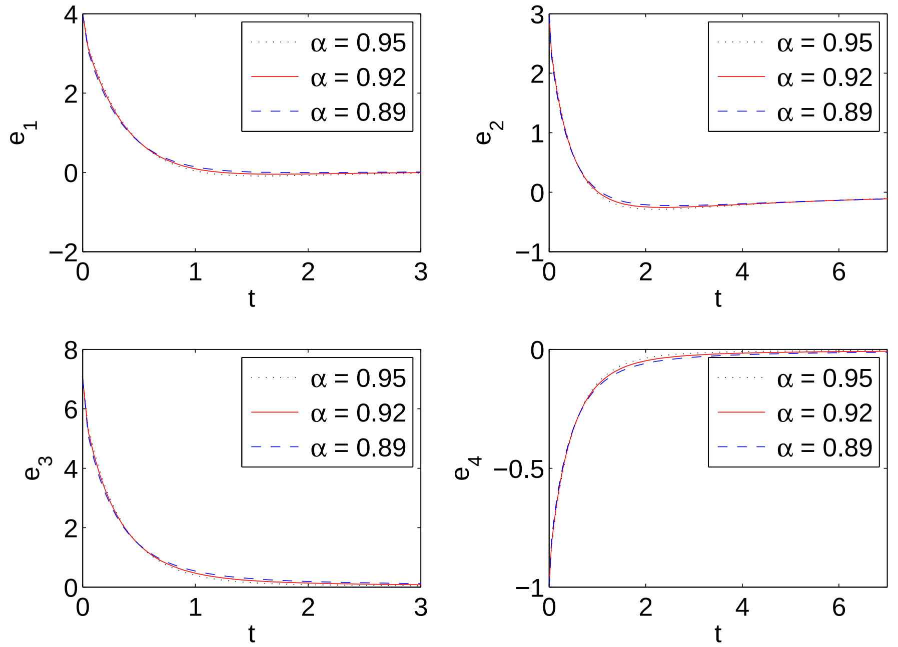


(b) Error system (6.3.7)

Figure 6.7: Error systems (6.3.3) and (6.3.7) with $\alpha = 0.95$ and $\tau = 0.05$.



(a) Error system (6.3.3)



(b) Error system (6.3.7)

Figure 6.8: Error systems (6.3.3) and (6.3.7) for different fractional-orders.

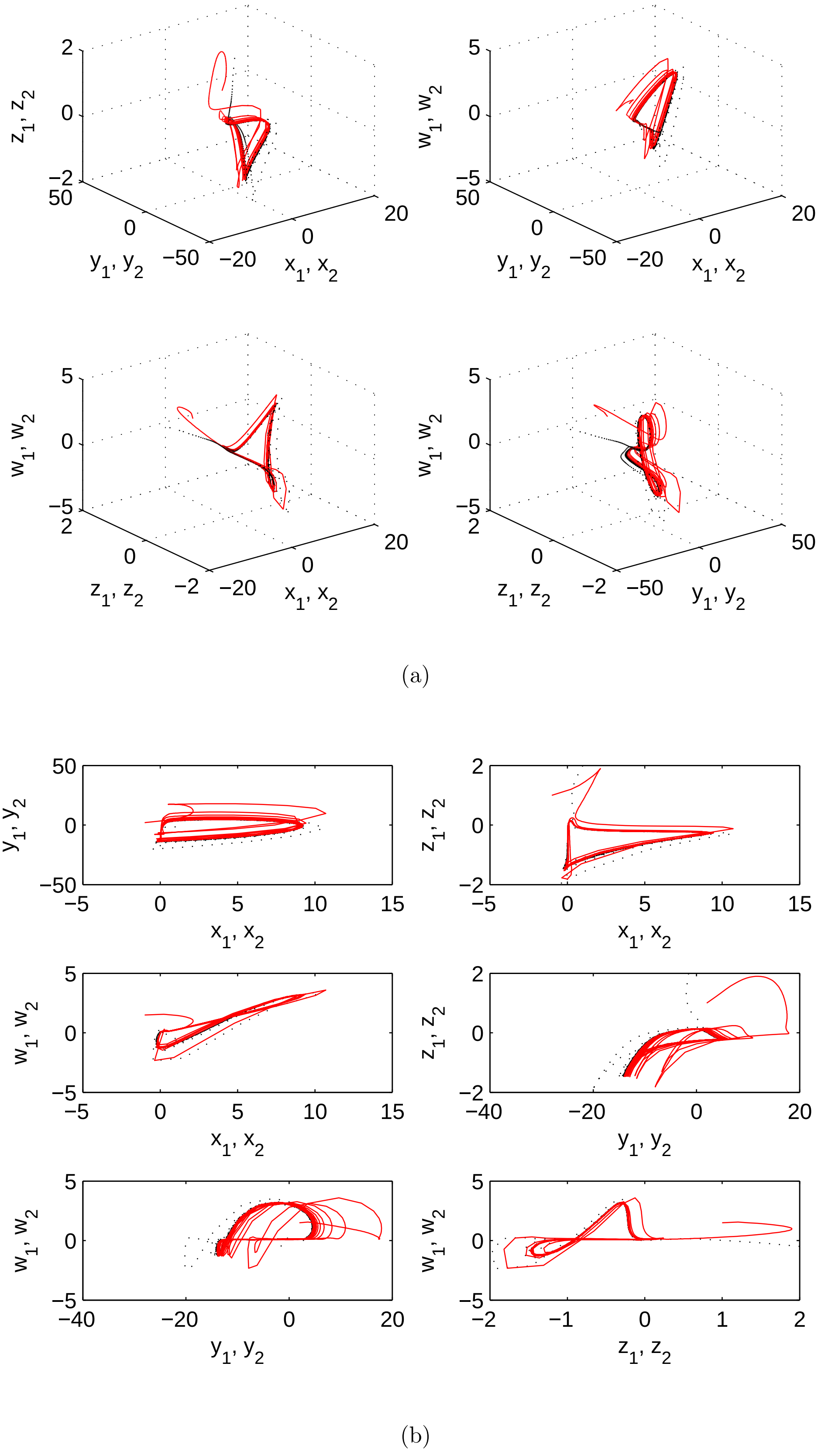
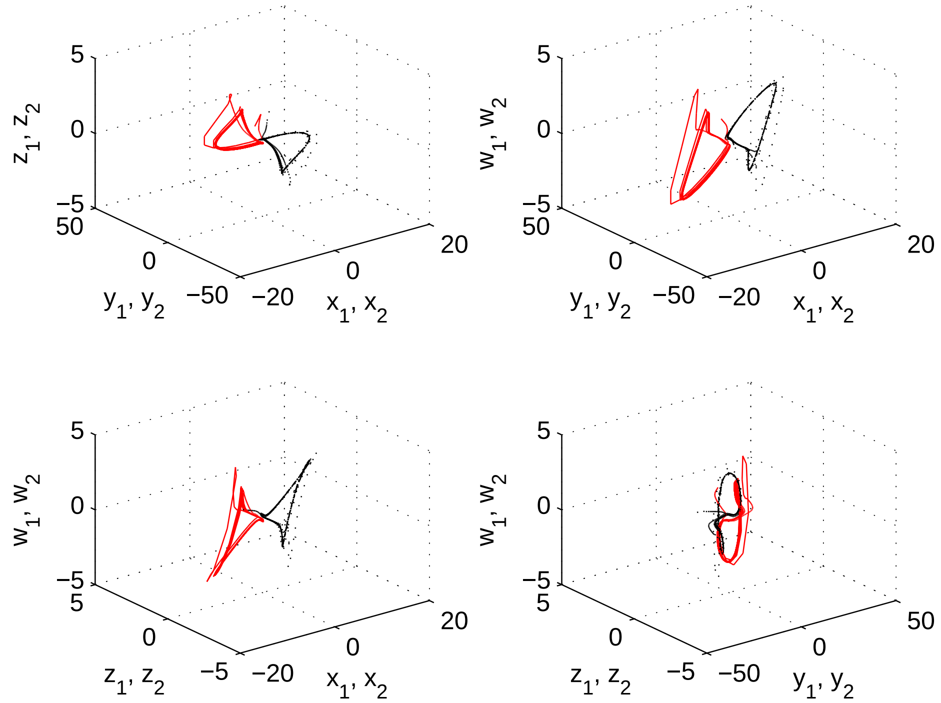
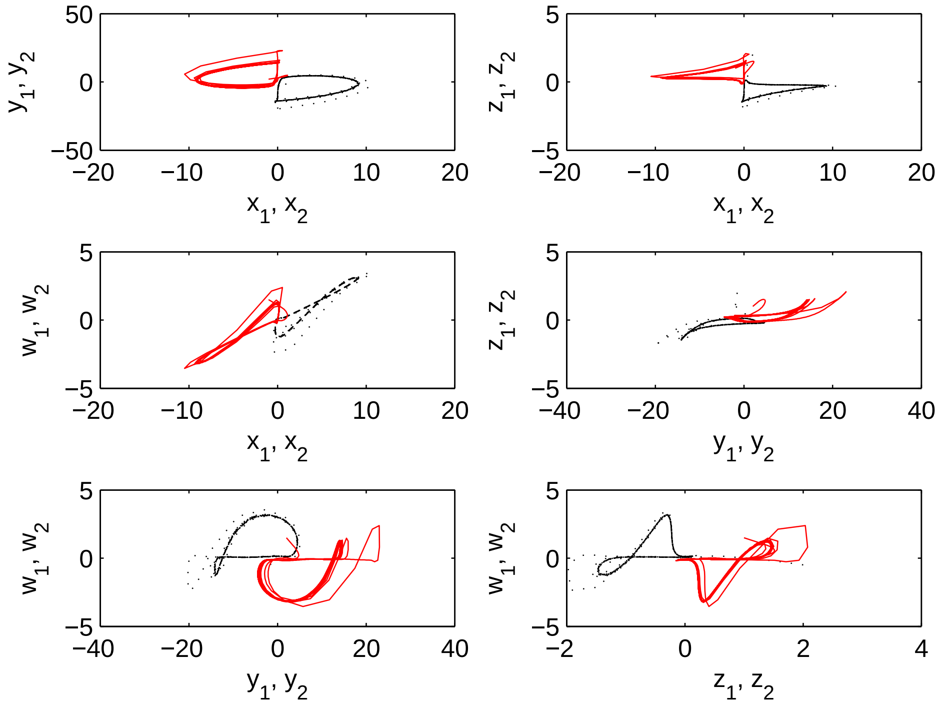


Figure 6.9: Synchronization of drive system (6.3.1) (black color dotted lines) and response system (6.3.2) (red color lines).

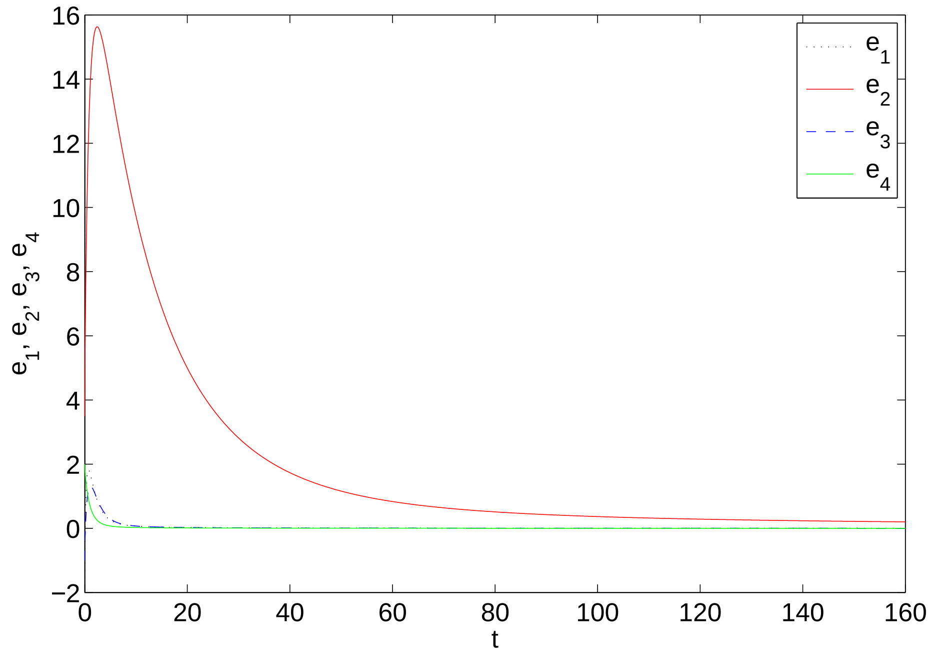


(a)

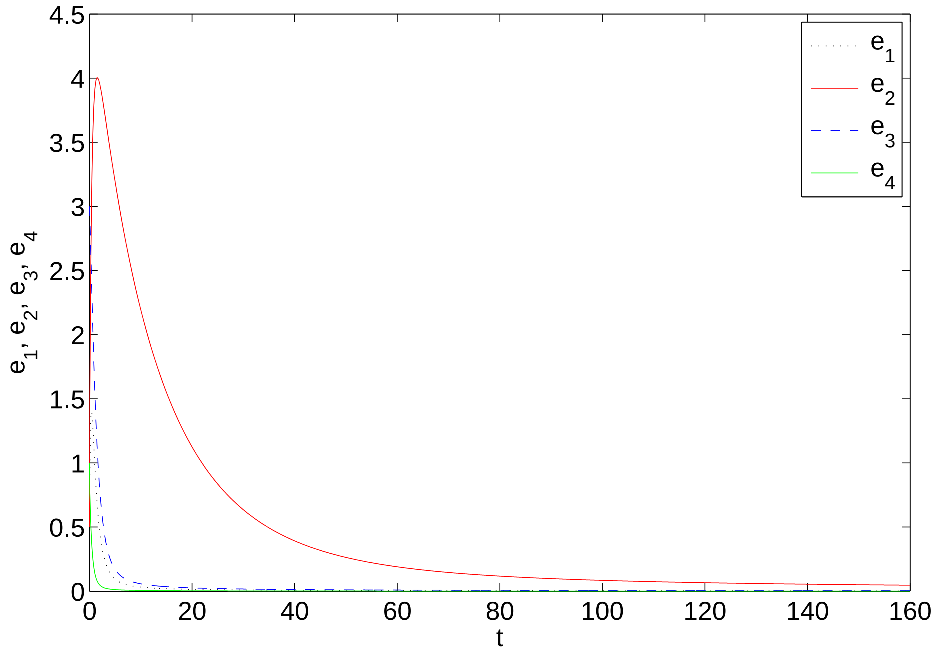


(b)

Figure 6.10: Anti-synchronization of drive system (6.3.1) (black color dotted lines) and response system (6.3.6) (red color lines).

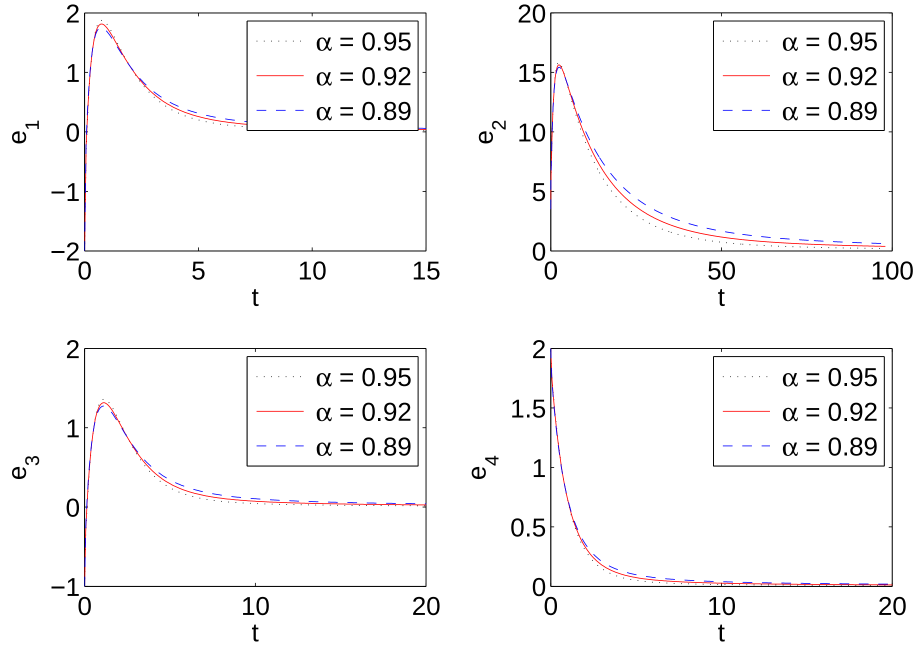


(a) Error system (6.3.3)

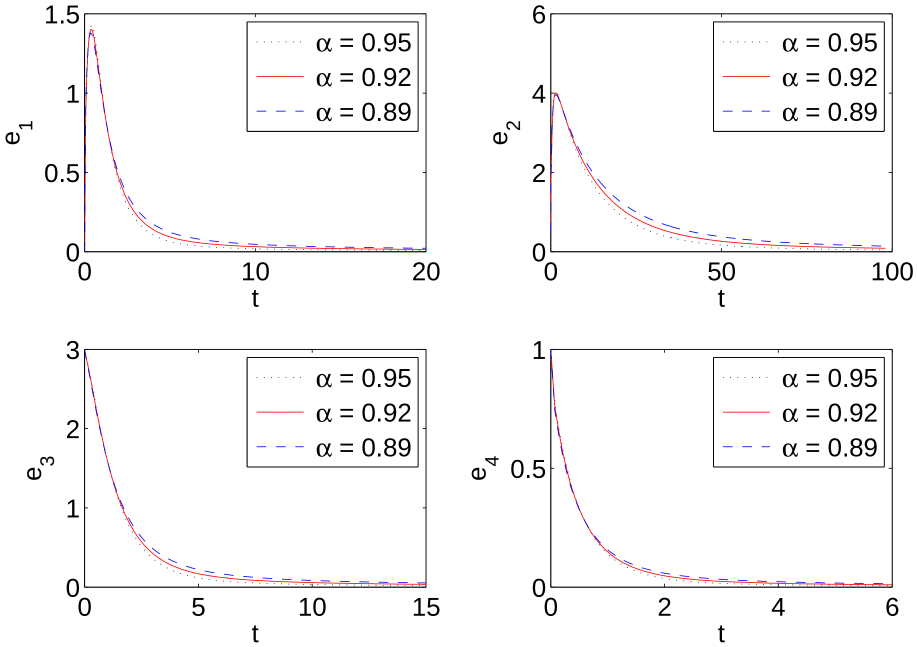


(b) Error system (6.3.7)

Figure 6.11: Error systems (6.3.3) and (6.3.7) with $\alpha = 0.9$ and $\tau = 0.07$.



(a) Error system (6.3.3)



(b) Error system (6.3.7)

Figure 6.12: Error systems (6.3.3) and (6.3.7) for different fractional-orders.

6.5 Conclusion

In this chapter, we consider a time-delayed fractional-order financial chaotic system. Firstly, the chaotic behaviours of the system are shown via numerical simulations using proposed Gauss-Seidel like predictor-corrector method. Then an active control has been proposed to achieve synchronization/anti-synchronization of the system. Finally, two examples are given to validate and to test the efficiency of the proposed theory. Using numerical simulations it has been observed that synchronization/anti-synchronization of the system are more faster when the fractional-order α of the system approaches 1.

Chapter 7

Controllability of fractional order $\alpha \in (1, 2]$ systems with delay

In this Chapter, a class of fractional-order systems of order $\alpha \in (1, 2]$ with delay in Banach spaces is considered. For the exact controllability of this class of systems a set of sufficient condition has been established by using Sadovskii's fixed point theorem and the theory of strongly continuous α-order cosine family. An example is given to illustrate the result.

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7.1 Introduction

Consider the following semilinear delay fractional system:

$$\left.\begin{array}{rcl} {}^{C}D_t^{\alpha}y(t) &=& Ay(t) + Bu(t) + f(t, y(t-\tau)); \;\; 0 < t \leq T, \\ y(t) &=& \phi(t); \;\; -\tau \leq t \leq 0, \\ y'(0) &=& y_0, \end{array}\right\} \qquad (7.1.1)$$

where the state $y(t)$ and the control $u(t)$ belongs to the Banach spaces Y and U respectively, for each t. Here, the positive constant $T < \infty$ and ${}^{C}D_t^{\alpha}$ is the Caputo fractional derivative of order $\alpha \in (1, 2]$.

The operators A, B and f are defined as follows: A generates a strongly continuous cosine family $\{C_\alpha(t)\}_{t\geq 0}$ of order α on Y (see, [6]), the operator B from U to Y is linear and bounded, and the map $f : [0,T] \times Y \to Y$ is a nonlinear operator. We denote by $C = C([-\tau, 0]; Y)$ the Banach space which contains continuous functions from $[-\tau, 0]$ to Y with the norm

$$\|\phi\|_C = \sup\{|\phi(t)|, -\tau \leq t \leq 0\},$$

and $\tau > 0$ is the delay time.

In this chapter, we investigate the exact controllability of (7.1.1). To the best of our knowledge exact controllability of semilinear systems of fractional-order $\alpha \in (1,2]$ with state delay using Sadovskii's fixed point theorem have not been proved by any author so far.

Definition 7.1.1. *The system (7.1.1) is said to be exactly controllable on* $[0,T]$*, if for every* $\phi \in C([-\tau, 0]; Y)$ *with* $\phi(0), y_0 \in D(A)$ *and* $y_T \in Y$*, there exists a control* $u(\cdot) \in L_2([0,T]; U)$ *such that the mild solution* $y(\cdot)$ *of (7.1.1) satisfies* $y(T) = y_T$*.*

7.2 Main result

In what follows, we assume the following hypotheses to prove the main results:

($\mathbf{H_1}$) A generates an α-order cosine family $C_\alpha(t)$ on the Banach space Y and there exists a constant $M_0 \geq 1$ such that

$$\|C_\alpha(t)\| \leq M_0.$$

($\mathbf{H_2}$) The linear operator $W : L_2([0,T], U) \to Y$ defined by

$$Wu = \int_0^T P_\alpha(T-s)Bu(s)\mathrm{d}s$$

has an induced inverse W^{-1} which takes its values in $L_2([0,T]; U)/\ker W$, and there exists constants $M_1 > 0$ and $M_2 > 0$ such that

$$\|B\| \leq M_1, \qquad \|W^{-1}\| \leq M_2.$$

($\mathbf{H_3}$) The function $f(t, y)$ is continuous in the second variable for a.e. $t \in [0, T]$ and measurable in the first variable for all $y \in Y$.

($\mathbf{H_4}$) There exists a function $L_f(t) \in L_1([0, T], \mathbb{R}^+)$ such that the nonlinear function $f(t, y)$ satisfies the condition

$$\|f(t, y) - f(t, z)\| \leq L_f(t)\|y - z\|,$$

for all $y, z \in Y, \;\; 0 \leq t \leq T$.

Theorem 7.2.1. *Suppose* ($\mathbf{H_1}$)–($\mathbf{H_4}$) *are satisfied. Then, the system (7.1.1) is controllable on* $[0, T]$ *provided that*

$$\left[1 + \frac{M_0 M_1 M_2 T^{\alpha}}{\Gamma(\alpha)}\right] \frac{M_0 T^{\alpha-1}}{\Gamma(\alpha)} \|L_f\|_{L^1} < 1. \tag{7.2.1}$$

Proof: Using ($\mathbf{H_2}$), for any $y(\cdot) \in C([-\tau, T]; Y)$, define the control

$$\begin{aligned} u(t) &= W^{-1}\Big\{ y_1 - C_\alpha(T)\phi(0) - S_\alpha(T)y_0 \\ &\quad - \int_0^T P_\alpha(T - s) f(s, y(s - \tau))\mathrm{d}s \Big\}(t), \end{aligned} \tag{7.2.2}$$

for $t \in [0, T]$.

We shall show that, using the control (7.2.2), the operator Φ defined by

$$(\Phi y)(t) = \begin{cases} C_\alpha(t)\phi(0) + S_\alpha(t)y_0 + \int_0^t P_\alpha(t - s)[Bu(s) + f(s, y(s - \tau))]\mathrm{d}s, \; t \in [0, T], \\ \phi(t), \quad t \in [-\tau, 0]. \end{cases}$$

has a fixed point $y(\cdot)$.

For $\delta > 0$, we define a ball

$$B_\delta = \{y \in C([-\tau, T]; Y) : \|y\| \le \delta\}.$$

Then, for each δ, B_δ is a convex, closed and bounded set in $C([-\tau, T]; Y)$. To apply Sadovskii fixed point theorem 2.4.1, first we show that there exist $\delta > 0$ such that $\Phi(B_\delta) \subset B_\delta$. If it is not true then for each $\delta > 0$, there exists a function $y^\delta(\cdot) \in B_\delta$, but $\Phi(y^\delta) \notin B_\delta$, that is, $\|(\Phi y^\delta(t)\| > \delta$ for some $t \in [-\tau, T]$. From $(\mathbf{H_1})$, $(\mathbf{H_2})$, $(\mathbf{H_4})$ and for $t \in [0, T]$, we have

$$\begin{aligned}
\delta &< \|(\Phi y^\delta)(t)\| \\
&\le M_0|\phi(0)| + M_0 T\|y_0\| + \frac{M_0 M_1 M_2 T^\alpha}{\Gamma(\alpha)}\Bigg[\|y_1\| + M_0|\phi(0)| + M_0 T\|y_0\| \\
&\quad + \frac{M_0 T^{\alpha-1}}{\Gamma(\alpha)}\int_0^T \|f(s, y^\delta(s-\tau))\|\mathrm{d}s\Bigg] + \frac{M_0 T^{\alpha-1}}{\Gamma(\alpha)}\int_0^T \|f(s, y^\delta(s-\tau))\|\mathrm{d}s \\
&\le \frac{M_0 M_1 M_2 T^\alpha\|y_1\|}{\Gamma(\alpha)} + \left[1 + \frac{M_0 M_1 M_2 T^\alpha}{\Gamma(\alpha)}\right]\Bigg[M_0|\phi(0)| + M_0 T\|y_0\| \\
&\quad + \frac{M_0 T^{\alpha-1}}{\Gamma(\alpha)}\int_0^T \big[L_f(s)\|y^\delta(s-\tau)\| + \|f(s,0)\|\big]\mathrm{d}s\Bigg].
\end{aligned} \tag{7.2.3}$$

Since $\|y^\delta\| \le \delta$, (7.2.3) becomes

$$\begin{aligned}
\delta &< \|(\Phi y^\delta)(t)\| \\
&\le \frac{M_0 M_1 M_2 T^\alpha\|y_1\|}{\Gamma(\alpha)} + \left[1 + \frac{M_0 M_1 M_2 T^\alpha}{\Gamma(\alpha)}\right]\Bigg[M_0|\phi(0)| + M_0 T\|y_0\| \\
&\quad + \frac{M_0 T^{\alpha-1}}{\Gamma(\alpha)}\int_0^T \|f(s,0)\|\mathrm{d}s\Bigg] + \left[1 + \frac{M_0 M_1 M_2 T^\alpha}{\Gamma(\alpha)}\right]\frac{M_0 T^{\alpha-1}}{\Gamma(\alpha)}\delta\|L_f\|_{L^1}.
\end{aligned}$$

Dividing both sides by δ and taking limit as $\delta \to \infty$, we get

$$\left[1 + \frac{M_0 M_1 M_2 T^\alpha}{\Gamma(\alpha)}\right]\frac{M_0 T^{\alpha-1}}{\Gamma(\alpha)}\|L_f\|_{L^1} \ge 1.$$

This contradicts condition (7.2.1). When $t \in [-\tau, 0]$,

$$\delta < \|(\Phi y^\delta)(t)\| = \|\phi(t)\| \qquad \text{or} \qquad 1 < \|\phi(t)\|/\delta.$$

Taking limit as $\delta \to \infty$, we get $1 < 0$. This is impossible.

Therefore, for some $\delta > 0$, $\Phi(B_\delta) \subset B_\delta$.

Now, we define operators Φ_1 and Φ_2 so that $\Phi_1 + \Phi_2 = \Phi$, as follows

$$\begin{aligned}
(\Phi_1 y)(t) &= \begin{cases} C_\alpha(t)\phi(0) + S_\alpha(t)y_0, & t \in [0,T], \\ \phi(t), & t \in [-\tau, 0] \end{cases} \\
(\Phi_2 y)(t) &= \begin{cases} \int_0^t P_\alpha(t-s)\big[Bu(s) + f(s, y(s-\tau))\big]\mathrm{d}s, & t \in [0,T], \\ 0, & t \in [-\tau, 0]. \end{cases}
\end{aligned}$$

Here, Φ_1 is completely continuous in $C([-\tau,T];Y)$, because for every weakly convergent sequences (y_n) in $B_\delta \subset Y$ their images $(\Phi_1 y_n)$ is a constant function in $C([-\tau,T];Y)$.

Next, we prove that Φ_2 is a contraction operator. Let $y, z \in B_\delta$, then for each $t \in [0,T]$ we have

$$\begin{aligned}
\|(\Phi_2 y)(t) - (\Phi_2 z)(t)\| &\le \int_0^t \|P_\alpha(t-s)\|\|B\|\|W^{-1}\| \\
&\quad \times \left\| - \int_0^T P_\alpha(T-s)[f(s, y(s-\tau)) - f(s, z(s-\tau))]\mathrm{d}s \right\| \mathrm{d}s \\
&\quad + \int_0^t \|P_\alpha(t-s)\|\|f(s, y(s-\tau)) - f(s, z(s-\tau))\|\mathrm{d}s \\
&\le \frac{M_0 M_1 M_2 T^\alpha}{\Gamma(\alpha)} \int_0^T \frac{M_0 T^{\alpha-1}}{\Gamma(\alpha)} L_f(s)\|y-z\|\mathrm{d}s \\
&\quad + \frac{M_0 T^{\alpha-1}}{\Gamma(\alpha)} \int_0^T L_f(s)\|y-z\|\mathrm{d}s \\
&\le \left[1 + \frac{M_0 M_1 M_2 T^\alpha}{\Gamma(\alpha)}\right] \frac{M_0 T^{\alpha-1}}{\Gamma(\alpha)} \|L_f\|_{L_1}\|y-z\|. \qquad (7.2.4)
\end{aligned}$$

From (7.2.1) and (7.2.4), it is clear that Φ_2 is a contraction operator.

Thus by $\Phi = \Phi_1 + \Phi_2$ is a condensing operator on B_δ. Hence, from the Sadovskii's fixed point theorem, Φ has a fixed point $y(\cdot)$ on B_δ which is the mild solution of (7.1.1). Now, it is easy to prove that the mild solution of (7.1.1) satisfies $y(T) = y_T$. This proves the exact controllability of system (7.1.1).

7.3 Example

Let us consider the following system with fractional-order $\alpha \in (1, 2]$,

$$\left.\begin{aligned} {}^{C}D_t^{\alpha} v(t,z) &= \frac{\partial^2 v}{\partial z^2}(t,z) + \sigma(t, v(t-\tau, z)) + B\mu(t,z),\ z \in (0,\pi),\ t \in (0,T], \\ v(t,0) &= v(t,\pi) = 0, \quad t \in (0,T], \\ v(t,z) &= \phi(t,z), \quad t \in [-\tau, 0], \quad z \in (0,\pi), \\ \frac{\partial v}{\partial t}(0,z) &= y_0(z), \quad z \in (0,\pi). \end{aligned}\right\} \tag{7.3.1}$$

Case (i) for $\alpha = 2$:

Let $Y = L_2(0,\pi)$ and $A = \frac{\mathrm{d}^2}{\mathrm{d}z^2}$ with

$$D(A) = \left\{ v \in Y : v, \frac{\mathrm{d}v}{\mathrm{d}z} \text{ are absolutely continuous, } \quad \frac{\mathrm{d}^2 v}{\mathrm{d}z^2} \in Y \text{ and } v(0) = v(\pi) = 0 \right\}.$$

Let $\phi_n(z) = \sqrt{\frac{2}{\pi}} \sin nz, \quad 0 \le z \le \pi, \quad n = 1, 2, 3, \ldots$. Here, ϕ_n is the eigenfunction corresponding to the eigenvalue $-n^2$ of the operator A and $\{\phi_n\}$ ia an orthonormal base for Y. Then

$$\begin{aligned} \zeta &= \sum_{n=1}^{\infty} (\zeta, \phi_n)\phi_n \qquad \text{and} \\ A\zeta &= -\sum_{n=1}^{\infty} n^2 (\zeta, \phi_n)\phi_n, \qquad \zeta \in D(A). \end{aligned}$$

It is easy to show that (see, [70]) A is an infinitesimal generator of a strongly continuous cosine family $C(t)$ and

$$C(t)\zeta = \sum_{n=1}^{\infty} \cos nt (\zeta, \phi_n)\phi_n, \qquad \zeta \in Y, \quad t \in \mathbb{R}. \tag{7.3.2}$$

Let $V : [0,T] \to Y$ be defined as

$$[V(t)](z) = v(t,z); \quad z \in (0,\pi) \tag{7.3.3}$$

and $f : [0,T] \times Y \to Y$ be defined as

$$f(t, V(t-\tau))(z) = \sigma(t, v(t-\tau, z)), \tag{7.3.4}$$

for $z \in (0, \pi)$.

Let $U = L_2(0, \pi)$ and $u : [0,T] \to U$ be defined as

$$(u(t))(z) = \mu(t, z), \quad z \in (0, \pi), \tag{7.3.5}$$

where $\mu : [0,T] \times (0,\pi) \to L_2([0,T]; U)$ is continuous in t.

Now, for $\alpha = 2$, (7.3.1) can be represented in the form (7.1.1) as

$$\left.\begin{aligned} {}^C D_t^\alpha V(t) &= AV(t) + Bu(t) + f(t, V(t-\tau)); \;\; 0 < t \leq T, \\ V(t) &= \phi(t); \;\; -\tau \leq t \leq 0, \\ V'(0) &= y_0. \end{aligned}\right\}$$

Hence for $\alpha = 2$ by Theorem 7.2.1, the system (7.3.1) is controllable, provided f satisfies the conditions ($\mathbf{H_3}$) and ($\mathbf{H_4}$).

Case (ii) for $\alpha \in (1, 2)$:

As A is the generator of the cosine family $C(t)$, from Lemma 2.3.1, we infer that for $\alpha \in (1, 2)$ A generates a continuous α-order cosine family $C_\alpha(t)$ which is exponentially bounded and

$$C_\alpha(t) := \int_0^\infty \varphi_{t,\alpha/2}(\xi) C(\xi) \mathrm{d}\xi, \;\; t > 0,$$

where $C(\xi)$ is given in (7.3.2), $\varphi_{t,\alpha/2}(\xi) = t^{-\alpha/2}\Psi_{\alpha/2}(\xi t^{-\alpha/2})$ and $\Psi_\gamma(s) = \sum_{m=0}^{\infty} \frac{(-s)^m}{m!\Gamma(-\gamma m + 1 - \gamma)}$, $0 < \gamma < 1$ (see [6] for details). Now, the system (7.3.1) with (7.3.3), (7.3.4) and (7.3.5) can be formulated as (7.1.1) in Banach space Y. Therefore by Theorem 7.2.1, the system (7.3.1) is controllable.

7.4 Conclusion

In this chapter, a class of fractional-order $\alpha \in (1, 2]$ semilinear control systems with delay in Banach spaces is considered. Based on suitable assumptions on the system operators A, B and f and using Sadovskii's fixed point theorem with the theory of strongly continuous α-order cosine family, the exact controllability of the system has been studied. An example has been given to illustrate the result.

Chapter 8

Trajectory controllability of fractional order $\alpha \in (1, 2]$ systems with delay

This chapter is concerned with trajectory controllability of a class of fractional-order systems of order $\alpha \in (1, 2]$ with delay in state variable and with a nonlinear control term. Firstly, the existence and uniqueness of solution of the system is proved under suitable conditions on the nonlinear term involving state variable. Then the trajectory controllability is studied using Mittag-Leffler functions and Gronwall-Bellman inequality. Finally, examples are given to illustrate the proposed theory.

[The content of this chapter is to be published in *Journal of Applied Nonlinear Dynamics*]

8.1 Introduction

A system is said to be trajectory controllable if and only if it is possible, by means of an input, to transfer the system from any initial state to any other desired state along a prescribed trajectory. Recently, the authors in [43] studied the trajectory controllability of fractional–order $\alpha \in (0; 1]$ systems. In this chapter we prove the trajectory controllability of the following fractional-order system with delay:

$$\left.\begin{array}{rcl} {}^{C}D_t^{\alpha}x(t) & = & Ax(t) + B(t, u(t)) + f(t, x(t-\tau), x(t)), \quad 0 < t \le T < \infty \\ x(t) & = & \phi(t), \ t \in [-\tau, 0] \quad x'(0) = \widetilde{x}_0 \end{array}\right\}, \tag{8.1.1}$$

where $\alpha \in (1, 2]$, $A \in \mathbb{R}^{n\times n}$ is a constant matrix, the state $x(t)$ and the control $u(t)$ takes their values in $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively for each t. The operators B and f are defined as follows: $B : [0, T] \times \mathbb{R}^m \to \mathbb{R}^n$ and the nonlinear function $f : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$. Here, the initial function ϕ is continuous on $[-\tau, 0]$ (τ is a positive constant).

8.2 Existence and uniqueness of solution

In this section we prove the existence and uniqueness of solution of (8.1.1) by using method of steps.

In what follows we assume the following conditions on the nonlinear functions B and f.

(A1) $B(t, u(t))$ is measurable with respect to t for all $u(t) \in \mathbb{R}^m$ for each t and continuous with respect to u for almost all $t \in [0, T]$ and it satisfies the growth condition

$$\|B(t, u(t))\|_{\mathbb{R}^n} \le b_0(t) + b_1\|u(t)\|_{\mathbb{R}^m}, \quad \forall\, u(t) \in \mathbb{R}^m, \ t \in [0, T].$$

(A2) $f(t, r, s)$ is measurable with respect to t for all $r, s \in \mathbb{R}^n$ and continuous with respect to r and s respectively for almost all $t \in [0, T]$ and it satisfies the growth condition

$$\|f(t, r, s)\|_{\mathbb{R}^n} \le f_0(t) + C_1\|r\|_{\mathbb{R}^n} + C_2\|s\|_{\mathbb{R}^n}, \quad \forall\, r, s \in \mathbb{R}^n, \ t \in [0, T],$$

where $C_1 > 0, C_2 > 0$ and $f_0(t)$ is continuous in the interval $[0, T]$.

(A3) $f : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous in the first variable and Lipschitz continuous in the second and third variables. That is,

$$\|f(t, r_1, s_1) - f(t, r_2, s_2)\| \le L_1\|r_1 - r_2\| + L_2\|s_1 - s_2\|,$$

where $L_1 > 0$ and $L_2 > 0$ are Lipschitz constants.

First consider the interval $0 \leq t \leq \tau$. Here, since $y(t-\tau) = \phi(t-\tau)$, (8.1.1) becomes

$$ {}^{C}D^{\alpha}x(t) = Ax(t) + B(t,u(t)) + g_{\tau}(t,x(t)), \quad 0 < t \leq \tau, $$

where $g_{\tau}(t,x(t)) = f(t,\phi(t-\tau),x(t))$. Assumption (A3) implies that g_{τ} is Lipschitz continuous in x and continuous function of t. Hence for each control function $u(t)$ there exists a unique solution for (8.1.1) in the interval $[0,\tau]$. Its solution in $[0,\tau]$ is of the form (see, [61])

$$ \begin{aligned} x_{\tau}(t) &= E_{\alpha,1}[At^{\alpha}]\phi(0) + tE_{\alpha,2}[At^{\alpha}]x'(0) \\ &\quad + \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}[A(t-s)^{\alpha}][B(s,u(s)) + f(s,\phi(s-\tau),x(s))]\mathrm{d}s. \quad (8.2.1) \end{aligned} $$

Hence in the interval $[0,\tau]$ the solution of (8.1.1) exists and is unique.

Now, in the interval $[0,2\tau]$, the system (8.1.1) may be written as

$$ {}^{C}D^{\alpha}x(t) = Ax(t) + B(t,u(t)) + g_{2\tau}(t,x(t)), \quad 0 < t \leq \tau, $$

where $g_{2\tau}(t,x(t)) = \begin{cases} f(t,\phi(t-\tau),x(t)), & 0 < t \leq \tau \\ f(t,x_{\tau}(t-\tau),x(t)), & \tau < t \leq 2\tau \end{cases}$. Then the solution of (8.1.1) in the interval $[0,2\tau]$ for each u is given by

$$ x(t) = \begin{cases} x_{\tau}(t), & 0 \leq t \leq \tau \\ x_{2\tau}(t), & \tau \leq t \leq 2\tau \end{cases} \quad (8.2.2) $$

where $x_{2\tau}(t) = E_{\alpha,1}[A(t-\tau)^{\alpha}]x(\tau) + (t-\tau)E_{\alpha,2}[A(t-\tau)^{\alpha}]x'(\tau) + \int_{\tau}^{t}(t-s)^{\alpha-1}E_{\alpha,\alpha}[A(t-s)^{\alpha}][B(s,u(s)) + f(s,x_{\tau}(s-\tau),x_{\tau}(s))]\mathrm{d}s, \ \ \tau \leq t \leq 2\tau$ and $x_{\tau}(t)$ is given by (8.2.1).

Proceeding in a similar way we can easily prove the following theorem:

Theorem 8.2.1. *Let $x_{0\tau}(t) = \phi(t)$, $x'_{0\tau}(t) = \widetilde{x}_0$ and let k be the greatest positive integer* *such that* $k\tau \le T$ *and let*

$$g_{k\tau}(t,x(t)) = \begin{cases} f(t, x_{0\tau}(t-\tau), x(t)), & 0 < t \le \tau \\ f(t, x_{\tau}(t-\tau), x(t)), & \tau < t \le 2\tau \\ \quad\vdots & \\ f(t, x_{(k-1)\tau}(t-\tau), x(t)), & (k-1)\tau < t \le k\tau \end{cases} \tag{8.2.3}$$

be continuous and satisfy a Lipschitz condition of the form (A3). Then there exists a unique solution on the interval $[0,T]$ *for the system* (8.1.1) *for each* u *and it is given by*

$$x(t) \;=\; \begin{cases} x_{i\tau}(t), & t \in [(i-1)\tau, i\tau],\ i = 1,2,\dots k, \\ x_T(t), & t \in [k\tau, T] \end{cases}$$

where

$$\begin{aligned} x_{i\tau}(t) &= E_{\alpha,1}[A(t-(i-1)\tau)^{\alpha}]x((i-1)\tau) + (t-(i-1)\tau)E_{\alpha,2}[A(t-(i-1)\tau)^{\alpha}]x'((i-1)\tau) \\ &\quad + \int_{(i-1)\tau}^{t} (t-s)^{\alpha-1}E_{\alpha,\alpha}[A(t-s)^{\alpha}][B(s,u(s)) + g_{i\tau}(s,x(s))]\mathrm{d}s, \quad t \in [(i-1)\tau, i\tau], \\ x_T(t) &= E_{\alpha,1}[A(t-k\tau)^{\alpha}]x(k\tau) + (t-t_k)E_{\alpha,2}[A(t-k\tau)^{\alpha}]x'(k\tau) \\ &\quad + \int_{k\tau}^{t} (t-s)^{\alpha-1}E_{\alpha,\alpha}[A(t-s)^{\alpha}][B(s,u(s)) + f(s,x_k(s-\tau),x(s))]\mathrm{d}s,\ t \in [k\tau, T]. \end{aligned}$$

Definition 8.2.1. *The system* (8.1.1) *is said to be controllable on* $[0,T]$*, if for every* $\phi \in C([-\tau,0];\mathbb{R}^n)$ *with* $\phi(0), \widetilde{x}_0 \in \mathbb{R}^n$ *and* $x_T \in \mathbb{R}^n$*, there exists a control* $u(t) \in \mathbb{R}^m$, $0 \le t \le T$ *such that the solution* $x(\cdot)$ *of* (8.1.1) *satisfies* $x(T) = x_T$*.*

Let Z be the set of all functions $z(\cdot)$ defined on $[0,T]$ such that $z(0) = \phi(0)$, $z'(0) = \widetilde{x}_0$ and $z(T) = x_T$ and let the fractional derivative ${}^{C}D_t^{\alpha}z$, $1 < \alpha \le 2$ exist almost everywhere. We call Z, the set of all trajectories of (8.1.1).

Definition 8.2.2 (Trajectory controllability)**.** *The system* (8.1.1) *is said to be trajectory controllable if for any* $z \in Z$*, there exists a control* $u(t) \in \mathbb{R}^m$, $0 \le t \le T$ *such that the corresponding solution* $x(\cdot)$ *of* (8.1.1) *satisfies* $x(t) = z(t)$ *almost everywhere.*

8.3 Trajectory controllability

8.3.1 One dimensional system with linear control term

Consider the following simple system

$$\left.\begin{array}{rcl} {}^{C}D_t^{\alpha}x(t) & = & ax(t)+b(t)u(t)+f(t,x(t-\tau),x(t)), \quad 0<t\leq T<\infty \\ x(t) & = & \phi(t),\ t\in[-\tau,0] \quad x'(0)=\widetilde{x}_0 \end{array}\right\}, \tag{8.3.1}$$

where $\alpha \in (1,2]$, $a \in \mathbb{R}$ is a constant, the state $x(t)$ and the control $u(t)$ takes their values in $\mathbb{R}$ respectively for each t. The nonlinear function $f : \mathbb{R}\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$. Here, the initial function ϕ is continuous on $[-\tau, 0]$ and $b : [0,T] \to \mathbb{R}$ is continuous.

Here we assume the following conditions to prove the trajectory controllability of (8.3.1):

(i) The function $b(t)$ is continuous on $[0,T]$ and $b(t) \neq 0$ for all $t \in [0,T]$,

(ii) f is Lipschitz continuous with respect to the second and third argument, i.e., there exist positive real numbers l_1 and l_2 such that

$$|f(t,x_1,y_1)-f(t,x_2,y_2)| \leq l_1\|x_1-x_2\|+l_2\|y_1-y_2\|,$$

for all $x_1,x_2,y_1,y_2 \in \mathbb{R},\ t\in[0,T]$.

Theorem 8.3.1. *If the assumptions (i) and (ii) hold, then the system* (8.3.1) *is trajectory controllable on* $[0,T]$.

Proof. Let z be a given trajectory in Z. We define a control $u(t)$ by

$$u(t)=\frac{1}{b(t)}\left[{}^{C}D_t^{\alpha}z(t)-az(t)-f\left(t,z(t-\tau),z(t)\right)\right]. \tag{8.3.2}$$

Substituting (8.3.2) in (8.3.1), we get

$${}^{C}D_t^{\alpha}w(t) \;=\; aw(t)+f\left(t,x(t-\tau),x(t)\right)-f\left(t,z(t-\tau),z(t)\right), \tag{8.3.3}$$

where $w(t)=x(t)-z(t)$ and $w(0)=0$ and $w'(0)=0$.

The solution of (8.3.3) in the interval $t \in [0, \tau]$ is

$$w(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}[a(t-s)^\alpha] \left[f\left(s, x(s-\tau), x(s)\right) - f\left(s, z(s-\tau), z(s)\right) \right] \mathrm{d}s.$$

This implies

$$\begin{aligned} |w(t)| &\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |E_{\alpha,\alpha}[a(t-s)^\alpha]| [l_1 |\phi(s-\tau) - \phi(s-\tau)| + l_2 |w(s)|] \mathrm{d}s \\ &= \frac{l_2}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |E_{\alpha,\alpha}[a(t-s)^\alpha]| |w(s)| \mathrm{d}s. \end{aligned}$$

Hence by Gronwall's inequality for fractional integral (see, [34]) it follows that

$$|x(t) - z(t)| = 0.$$

or, $x(t) = z(t)$ for $t \in [0, \tau]$. Now, assume that $x(t) = z(t)$ for $t \in [(k-2)\tau, (k-1)\tau]$.

Then the solution of (8.3.3) in the interval $t \in [(k-1)\tau, k\tau]$ is

$$w(t) = \frac{1}{\Gamma(\alpha)} \int_{(k-1)\tau}^t (t-s)^{\alpha-1} E_{\alpha,\alpha}[a(t-s)^\alpha] \left[f\left(s, x(s-\tau), x(s)\right) - f\left(s, z(s-\tau), z(s)\right) \right] \mathrm{d}s.$$

This implies

$$|w(t)| \leq \frac{1}{\Gamma(\alpha)} \int_{(k-1)\tau}^t (t-s)^{\alpha-1} |E_{\alpha,\alpha}[a(t-s)^\alpha]| [l_1 |x(s-\tau) - z(s-\tau)| + l_2 |w(s)|] \mathrm{d}s$$

Since $x(t) = z(t)$ in the interval $t \in [(k-2)\tau, (k-1)\tau]$, it follows that

$$|w(t)| \leq \frac{l_2}{\Gamma(\alpha)} \int_{(k-1)\tau}^t (t-s)^{\alpha-1} |E_{\alpha,\alpha}[a(t-s)^\alpha]| |w(s)| \mathrm{d}s.$$

Hence by Gronwall's inequality it follows that

$$|x(t) - z(t)| = 0.$$

or, $x(t) = z(t)$ for $t \in [(k-1)\tau, k\tau]$.

Similarly, we find that $x(t) = z(t)$ in the interval $[k\tau, T]$.

Hence, the system (8.3.1) is trajectory controllable on $[0, T]$.

Example 8.3.1. *Consider the following system*

$$\begin{aligned} {}^C D_t^\alpha x(t) &= x(t) + u(t) + \sin\left(x(t-\tau) + x(t)\right),\ t \in [0, 1], \\ x(t) &= \frac{\pi}{2},\ t \in [-0.5, 0], \qquad x'(0) = 0. \end{aligned}$$

This system satisfy all the assumption in the above theorem. Hence the system is trajectory controllable.

8.3.2 n-dimensional system with nonlinear control term

Theorem 8.3.2. *Suppose that the assumptions (A1)–(A3) hold and*

(A4) $B(t, v)$ *satisfies monotonicity and coercivity conditions. That is*

$$< B(t, v_1) - B(t, v_2), v_1 - v_2 > \geq 0, \quad \forall\, v_1, v_2 \in \mathbb{R}^m, t \in [0, T]$$

and

$$\lim_{\|v\| \to \infty} \frac{< B(t, v), v >}{\|v\|} = \infty.$$

Then the system (8.1.1) *is trajectory controllable on* $[0, T]$.

Proof. Let z be a given trajectory in Z. We wish to find a control

$$u(t) = \begin{cases} u_i(t), & t \in [(i-1)\tau, i\tau],\ i = 1, 2, \dots k, \\ u_T(t), & t \in [k\tau, T] \end{cases}$$

satisfying

$$z(t) = \begin{cases} z_{i\tau}(t), & t \in [(i-1)\tau, i\tau],\ i = 1, 2, \dots k, \\ z_T(t), & t \in [k\tau, T] \end{cases}$$

where

$$
\begin{aligned}
z_{i\tau}(t) &= E_{\alpha,1}[A(t-(i-1)\tau)^{\alpha}]z((i-1)\tau) + (t-(i-1)\tau)E_{\alpha,2}[A(t-(i-1)\tau)^{\alpha}]z'((i-1)\tau) \\
&\quad + \int\limits_{(i-1)\tau}^{t} (t-s)^{\alpha-1}E_{\alpha,\alpha}[A(t-s)^{\alpha}][B(s,u_i(s)) \\
&\quad + g_{i\tau}(s, z(s-\tau), z(s))]\mathrm{d}s, \quad t \in [(i-1)\tau, i\tau], \\
z_T(t) &= E_{\alpha,1}[A(t-k\tau)^{\alpha}]z(k\tau) + (t-k\tau)E_{\alpha,2}[A(t-k\tau)^{\alpha}]z'(k\tau) \\
&\quad + \int\limits_{k\tau}^{t}(t-s)^{\alpha-1}E_{\alpha,\alpha}[A(t-s)^{\alpha}][B(s,u_T(s)) + f(s, z_k(s-\tau), z(s))]\mathrm{d}s, \quad t \in [k\tau, T],
\end{aligned}
$$

(k is the greatest positive integer such that $k\tau \le T$).

To find $u(t)$:

The solution of (8.1.1) in the interval $t \in [0,\tau]$ is

$$
\begin{aligned}
z_{\tau}(t) &= E_{\alpha,1}[At^{\alpha}]\phi(0) + tE_{\alpha,2}[At^{\alpha}]\widetilde{x}_0 \\
&\quad + \int\limits_{0}^{t}(t-s)^{\alpha-1}E_{\alpha,\alpha}[A(t-s)^{\alpha}][B(s,u_1(s)) + f(s,\phi(s-\tau), z_{\tau}(s))]\mathrm{d}s. \quad (8.3.4)
\end{aligned}
$$

Taking Caputo's fractional derivative of order $\alpha \in (1,2]$ on both sides of (8.3.4), we obtain

$$
{}^{C}D_t^{\alpha} z_{\tau}(t) = AE_{\alpha,1}[At^{\alpha}]\phi(0) + tAE_{\alpha,2}[At^{\alpha}]\widetilde{x}_0 + {}^{C}D_t^{\alpha}(I_2 + I_1), \qquad (8.3.5)
$$

where

$$
I_1 = \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}[A(t-s)^{\alpha}]f(s,\phi(s-\tau), z_{\tau}(s))\mathrm{d}s \qquad (8.3.6)
$$

and

$$
I_2 = \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}[A(t-s)^{\alpha}]B(s,u_1(s))\mathrm{d}s. \qquad (8.3.7)
$$

The term $tE_{\alpha,\alpha}[At^{\alpha}]$, $1 < \alpha \le 2$ is bounded for all $t \in [0,T]$ and the function $z \in Z$ is also bounded for all $t \in [0,T]$. Then with assumption (A2) the integrand of integral (8.3.6) is bounded. Therefore the integral

$$\begin{aligned} I_1 &= \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}[A(t-s)^\alpha] f(s,\phi(s-\tau),z_\tau(s))\mathrm{d}s \\ &= \int_0^t \sum_{k=0}^{\infty} \frac{A^k (t-s)^{\alpha k+\alpha-1}}{\Gamma(\alpha k+\alpha)} f(s,\phi(s-\tau),z_\tau(s))ds \end{aligned}$$

is follows by applying integration by parts

$$\begin{aligned} I_1 &= \sum_{k=0}^{\infty} \frac{A^k t^{\alpha k+\alpha}}{\Gamma(\alpha k+\alpha+1)} f(0,\phi(-\tau),z_\tau(0)) \\ &+ \int_0^t \sum_{k=0}^{\infty} \frac{A^k (t-s)^{\alpha k+\alpha}}{\Gamma(\alpha k+\alpha+1)} \frac{d}{ds} f(s,\phi(s-\tau),z_\tau(s))ds. \end{aligned} \tag{8.3.8}$$

Taking Caputo fractional derivative of order $\alpha : 1 < \alpha \leq 2$ on both sides (8.3.8), we get

$$ {}^C D_t^\alpha I_1 = I_{1a} + I_{1b}, \tag{8.3.9}$$

where

$$\begin{aligned} I_{1a} &= \frac{1}{\Gamma(2-\alpha)} \int_0^t (t-s)^{1-\alpha} \\ &\times \frac{d^2}{ds^2} \left(\sum_{k=0}^{\infty} \frac{A^k s^{\alpha k+\alpha}}{\Gamma(\alpha k+\alpha+1)} \right) f(0,\phi(-\tau),\phi(0))ds, \end{aligned} \tag{8.3.10}$$

and

$$\begin{aligned} I_{1b} &= \frac{1}{\Gamma(2-\alpha)} \int_0^t (t-s)^{1-\alpha} \frac{d^2}{ds^2} \left(\int_0^s \sum_{k=0}^{\infty} \frac{A^k (s-s_1)^{\alpha k+\alpha}}{\Gamma(\alpha k+\alpha+1)} \right. \\ &\left. \times \frac{d}{ds_1} f(s_1,\phi(s_1-\tau),z_\tau(s_1))ds_1 \right) ds. \end{aligned} \tag{8.3.11}$$

From (8.3.10), we get

$$I_{1a} = E_{\alpha,1}[At^\alpha] f(0,\phi(-\tau),\phi(0)). \tag{8.3.12}$$

Using integration by parts to inner integral of (8.3.11), we get

$$\begin{aligned} I_{1b} &= \frac{1}{\Gamma(2-\alpha)} \int_0^t (t-s)^{1-\alpha} \frac{d^2}{ds^2} \Big(- \sum_{k=0}^{\infty} \frac{A^k s^{\alpha k+\alpha}}{\Gamma(\alpha k+\alpha+1)} f(0, \phi(-\tau), \phi(0)) \\ &+ \int_0^s \sum_{k=0}^{\infty} \frac{A^k (s-s_1)^{\alpha k+\alpha-1}}{\Gamma(\alpha k+\alpha)} f(s_1, \phi(s_1-\tau), z_\tau(s_1)) ds_1 \Big) ds \end{aligned}$$

It follows from the Leibniz integral rule that

$$\begin{aligned} I_{1b} &= -\frac{1}{\Gamma(2-\alpha)} \int_0^t (t-s)^{1-\alpha} \sum_{k=0}^{\infty} \frac{A^k s^{\alpha k+\alpha-2}}{\Gamma(\alpha k+\alpha-1)} f(0, \phi(-\tau), \phi(0)) ds \\ &+ \frac{1}{\Gamma(2-\alpha)} \int_0^t (t-s)^{1-\alpha} \\ &\times \frac{d}{ds} \Big(\int_0^s \sum_{k=0}^{\infty} \frac{A^k (s-s_1)^{\alpha k+\alpha-2}}{\Gamma(\alpha k+\alpha-1)} f(s_1, \phi(s_1-\tau), z_\tau(s_1)) ds_1 \Big) ds. \end{aligned} \tag{8.3.13}$$

Using integration by parts to inner integral of the second term of (8.3.13) then applying Leibniz integral rule to resulting (8.3.13), we find that

$$\begin{aligned} I_{1b} &= \frac{1}{\Gamma(2-\alpha)} \int_0^t (t-s)^{1-\alpha} \int_0^s \sum_{k=0}^{\infty} \frac{A^k (s-s_1)^{\alpha k+\alpha-2}}{\Gamma(\alpha k+\alpha-1)} \frac{d}{ds_1} f(s_1, \phi(s_1-\tau), z_\tau(s_1)) ds_1 ds \\ &= \frac{1}{\Gamma(2-\alpha)} \int_0^t \int_{s_1}^t (t-s)^{1-\alpha} \sum_{k=0}^{\infty} \frac{A^k (s-s_1)^{\alpha k+\alpha-2}}{\Gamma(\alpha k+\alpha-1)} \frac{d}{ds_1} f(s_1, \phi(s_1-\tau), z_\tau(s_1)) ds ds_1 \\ &= \int_0^t \sum_{k=0}^{\infty} \frac{A^k (t-\tau)^{\alpha k}}{\Gamma(\alpha k)} \frac{d}{ds_1} f(s_1, \phi(s_1-\tau), z_\tau(s_1)) ds_1 \\ &= f(t, \phi(t-\tau), z_\tau(t)) - E_{\alpha,1}[At^\alpha] f(0, \phi(-\tau), \phi(0)) \\ &+ A \int_0^t (t-s_1)^{\alpha-1} E_{\alpha,\alpha}[A(t-s_1)^\alpha] f(s_1, \phi(s_1-\tau), z_\tau(s_1)) ds_1. \end{aligned} \tag{8.3.14}$$

Substituting (8.3.14) and (8.3.12) in (8.3.9), we get

$$\begin{aligned} {}^C D_t^\alpha I_1 &= f(t, \phi(t-\tau), z_\tau(t)) \\ &+ A \int_0^t (t-s_1)^{\alpha-1} E_{\alpha,\alpha}[A(t-s_1)^\alpha] f(s_1, \phi(s_1-\tau), z_\tau(s_1)) ds_1. \end{aligned} \tag{8.3.15}$$

Similarly

$$^{C}D_t^{\alpha} I_2 \;=\; B(t,u_1(t)) + A\int_0^t (t-s_1)^{\alpha-1}E_{\alpha,\alpha}[A(t-s_1)^{\alpha}]B(s_1,u_1(s_1))ds_1. \quad (8.3.16)$$

Substituting (8.3.15) and (8.3.16) in (8.3.5), we obtain that

$$\begin{aligned} ^{C}D_t^{\alpha} z_{\tau}(t) &= AE_{\alpha,1}[At^{\alpha}]\phi(0) + tAE_{\alpha,2}[At^{\alpha}]\widetilde{x}_0 + B(t,u_1(t)) \\ &\quad + A\int_0^t (t-s_1)^{\alpha-1}E_{\alpha,\alpha}[A(t-s_1)^{\alpha}]B(s_1,u_1(s_1))ds_1 + f(t,\phi(t-\tau),z_{\tau}(t)) \\ &\quad + A\int_0^t (t-s_1)^{\alpha-1}E_{\alpha,\alpha}[A(t-s_1)^{\alpha}]f(s_1,\phi(s_1-\tau),z_{\tau}(s_1))ds_1 \\ &= Az_{\tau}(t) + B(t,u_1(t)) + f(t,\phi(t-\tau),z_{\tau}(t)) \end{aligned} \quad (8.3.17)$$

This implies

$$B(t,u_1(t)) = y_1(t), \quad (8.3.18)$$

where $y_1(t) = {}^{C}D_t^{\alpha} z_{\tau}(t) - Az_{\tau}(t) - f(t,\phi(t-\tau),z_{\tau}(t))$.

Similarly, when $t \in [(i-1)\tau, i\tau],\ i = 2,3,\ldots,k$ we obtain

$$B(t,u_i(t)) = y_i(t), \quad (8.3.19)$$

where $y_i(t) = {}^{C}D_t^{\alpha} z_{i\tau}(t) - Az_{i\tau}(t) - f(t, z_{(i-1)\tau}(t-\tau), z_{i\tau}(t))$.

Now, when $t \in [k\tau, T]$, we obtain that

$$B(t,u_T(t)) = y_T(t), \quad (8.3.20)$$

where $y_T(t) = {}^{C}D_t^{\alpha} z_T(t) - Az_{i\tau}(t) - f(t, z_{k\tau}(t-\tau), z_T(t))$.

Now, the trajectory controllability follows if we can extract

$$u(t) = \begin{cases} u_i(t), & t \in [(i-1)\tau, i\tau], \quad i = 1,2,\ldots,k \\ u_T(t), & t \in [k\tau, T], \end{cases}$$

from the relation

$$B(t,u(t)) = y(t), \tag{8.3.21}$$

where,

$$y(t) = \begin{cases} y_i(t), & t \in [(i-1)\tau, i\tau], \quad i = 1,2,\dots,k \\ y_T(t), & t \in [k\tau, T], \end{cases}$$

$y_i, (i = 1,2,\dots,k)$ and y_T are given in (8.3.18), (8.3.19) and (8.3.20).

To see this, define an operator $N : \mathbb{R}^n \to \mathbb{R}^n$ by

$$(Nu)(t) = B(t,u(t)).$$

Assumptions (A1) and (A2) imply that N is well defined, continuous and bounded operator. Assumption (A4) shows that N is monotone and coercive. A hemi-continuous monotone mapping is of type (M) (see Definition 2.4.3). Therefore, by Lemma 2.4.1, the nonlinear map N is onto. Hence there exists a control $u(t)$ satisfying (8.3.21). The measurability of $u(t)$ follows as $u(t)$ is in $\mathbb{R}^m$ for each $t \in [0,T]$. This proves the trajectory controllability of the system (8.1.1).

Example 8.3.2. *Consider the following system*

$$\begin{aligned} {}^C D_t^{3/2} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} &= \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + 3 \begin{bmatrix} u_1^3(t) \\ u_2^3(t) \end{bmatrix} \\ &\quad + \begin{bmatrix} -\sin(x_1(t-1) + x_2(t)) \\ \cos(x_2(t-1) + x_1(t)) \end{bmatrix}, \quad t \in [0,1] \\ \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} &= \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad t \in [-1,0]. \end{aligned}$$

It can be easily verified that the above system satisfies the hypotheses (A1)–(A3) and

$3\begin{bmatrix} u_1^3(t) \\ u_2^3(t) \end{bmatrix}$ *is monotone and coercive. Hence by Theorem 8.3.2 it is trajectory controllable on* $[0,1]$.

8.4 Application

Motivated by the work related to the oscillatory processes [28, 42, 48, 89], we give the following nonlinear mechanical system:

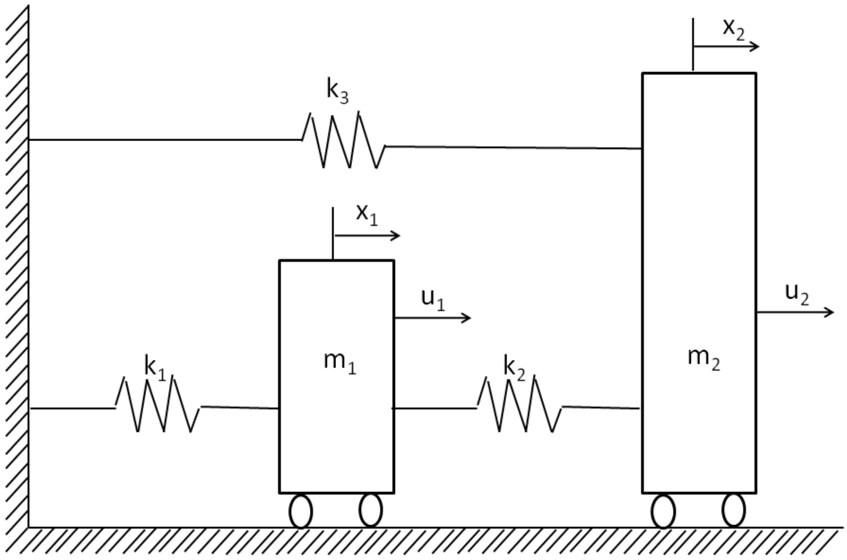

Figure 8.1: Nonlinear spring-mass system.

In the mechanical system shown in Fig. 8.1, m_1, m_2 denote the masses of the left and right carts respectively. The left end of the cart m_1 is connected to a non-linear spring with the forcing displacement relation $k_1(x_1(t)+\sin(x_1(t-\tau)))$. A linear spring with stiffness k_2 is connected between the carts m_1 and m_2. A non-linear spring is connected to the left end of the cart m_2 with forcing displacement relation $k_3 \sin(x_2(t-\tau))$. Let $x_1(t)$ and $x_2(t)$ be the displacements of the carts m_1 and m_2, respectively, from the tip of the leftmost spring. If the forces u_1 and u_2 are applied to the card m_1 and m_2, respectively, then the equations of motion for this system is given by

$$\left.\begin{aligned} m_1\, {}^C D_t^\alpha x_1 + k_1(x_1 + \sin(x_1(t-\tau))) - k_2(x_2 - x_1) &= u_1^3, \\ m_2\, {}^C D_t^\alpha x_2 + k_2(x_2 - x_1) + k_3 \sin(x_2(t-\tau)) &= u_2^3 \end{aligned}\right\}, \quad t \in [0,1], \qquad (8.4.1)$$

with the state history $x_1(t-1) = 0.05$ and $x_2(t) = 0.02$ for $t \in [-1, 0]$, $x_1'(0) = 0$ and $x_2'(0) = 0$ and $1 < \alpha \leq 2$.

This can be rewritten in the form of (8.1.1) with $A = \begin{bmatrix} -\frac{(k_1+k_2)}{m_1} & \frac{k_2}{m_1} \\ \frac{k_2}{m_2} & -\frac{k_2}{m_2} \end{bmatrix}$, $B(t, u(t)) = \begin{bmatrix} u_1^3(t) \\ u_2^3(t) \end{bmatrix}$, $x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$ and $f(t, x(t-1), x(t)) = \begin{bmatrix} -\frac{k_1}{m_1}\sin(x_1(t-\tau)) \\ -\frac{k_3}{m_2}\sin(x_2(t-\tau)) \end{bmatrix}$. If the masses $m_1 = m_2 = 1$ Kg, the spring constants $k_1 = k_3 = 1$ N/m, and $k_2 = 2$ N/m then it can be easily verified that the operators $B(t, u(t))$ and $f(t, x(t-1), x(t))$ satisfy the assumptions (A1) – (A4). Hence, by Section 8.3 the system (8.4.1) is trajectory controllable on $[0, 1]$.

8.5 Conclusion

In this chapter, trajectory controllability of a class of fractional systems of order $\alpha \in (1, 2]$ systems with delay was discussed. The existence and uniqueness of the system has been proved using growth condition on the nonlinear control term and growth and Lipschitz conditions on the nonlinear term involving state variable. Then sufficient conditions for the trajectory controllability of this class of systems has been proved. Two examples were given to validate the proposed theory. For the application an undamped spring-mass system was discussed to illustrate the theory.

Chapter 9

Conclusion and future scope

9.1 Conclusions

In the following the main conclusions of the thesis are presented chapter wise.

In Chapter 3, the asymptotic stability and stabilizability of a class of nonlinear systems with fixed delay in the state variable has been studied. A set of sufficient conditions was developed by assuming conditions on the system parameters such as eigenvalues of the linear operator, delay parameter and bound on the nonlinear part. Then, three examples were given to testify the effectiveness of the proposed theory. It has been observed that the minimum value of τ is smaller than the value obtained by using Razumikhin technique with the Lyapunov function $V = x^T x$. Thus our method can establish asymptotic stability when history function is defined on smaller intervals where Razumikhin method may not.

In Chapter 4, the stability analysis of a class of FOBPNLS was proposed by using Gronwall's lemma and some bounds on the system parameters. Firstly, we addressed the existence and uniqueness of solution of the continuous FOBPNLS. Then a set of sufficient conditions was proposed to guarantee the asymptotic stability of the continuous FOBPNLS. Finally, two example with numerical simulations were given to demonstrate the merits of our proposed stability conditions.

A class of nonlinear fractional–order control system with state delay is considered in Chapter 5. Firstly, using method of steps, the existence and uniqueness of solution has been proposed. Then sensitivity of the state and control with respect to small perturbations of history function and small perturbations of the nonlinear function was studied. Numerical examples were given to test the efficiency of obtained analytical results.

In Chapter 6, the market confidence was introduced to the time-delayed fractional–order financial chaotic system. Firstly, the chaotic behaviours of the system are shown via numerical simulations using proposed Gauss-Seidel like predictor-corrector method. Then an active control has been proposed to achieve synchronization/anti-synchronization of the system. Finally, two examples are given to validate and to test the efficiency of the proposed theory. Using numerical simulations it has been observed that synchronization/anti-synchronization of the system are more faster when the fractional-order α of the system approaches 1.

In Chapter 7, the exact controllability of fractional–order $\alpha \in (1, 2]$ delayed semilinear control system (7.1.1) is proved. The controllability results are obtained using theory of α-order cosine family and Sadovskii's fixed point theorem. The semilinear control system has been taken as a perturbed system of the corresponding linear system which preserves the exact controllability. The use of developed theory has been demonstrated by controlled wave equation.

Trajectory controllability is a strong notion than controllability. In Chapter 8, trajectory controllability of a class of fractional-order $\alpha \in (1, 2]$ systems with delay is considered. The existence and uniqueness of solution of the system has been proved under suitable conditions on the nonlinear term involving state variable. Then sufficient conditions for the trajectory controllability of this class of systems has been proposed.

9.2 Future scope

There is numerous scope for further research on the controllability and stability of fractional-order dynamical systems. Our future work will focus on sensitivity analysis of fractional higher order delayed control systems with respect to small perturbations in the fractional-order, history function and the nonlinear function of the system. Asymptotic stability and stabilization of semilinear fractional-order delayed systems can be studied using system parameters such as eigenvalues of the linear operator, delay parameter and bounds on nonlinear functions. The controllability, stability and stabilization of a class of fractional-order switching systems with or without delay is also the subject of our future research. Throughout the thesis, we consider fixed time delay of various dynamical systems to analyse their qualitative properties. In future one can investigate controllability, stability, stabilizability and synchronization/anti-synchronization of various dynamical systems with different types of delays such as variable time delay and mixed time delay etc with suitable modifications in the procedure adopted in this thesis.

Appendix A

Algorithm

To solve the integral equation of the form

$$y(t) = E_\alpha(at^\alpha)\phi(0) + \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(a(t-s)^\alpha)[bu(s) + f(s, y(s-\tau))]\mathrm{d}s,$$

we give the following algorithm which is modified predictor-corrector algorithm of [11,33].

Consider the uniform grid $\{t_n = nh : n = -l, -l+1, \ldots, -1, 0, 1, \ldots, N\}$ where l and N are integers such that $h = T/N$ and $h = \tau/l$. Let

$$y_h(t_j) = \phi(t_j), \quad j = -l, -l+1, \ldots, -1, 0$$

and

$$y_h(t_j - \tau) = y_h(jh - lh) = y_h(t_{j-l}),\ j = 0, 1, 2, \ldots, N.$$

Suppose $y_h(t_j) \approx y(t_j), \quad (j = -l, -l+1, \ldots, -1, 0, 1, \ldots, n)$ and we wish to calculate $y_h(t_{n+1})$ using

$$\begin{aligned} y(t_{n+1}) &= E_\alpha(at_{n+1}^\alpha)\phi(0) + \int_0^{t_{n+1}} (t_{n+1}-s)^{\alpha-1}E_{\alpha,\alpha}(a(t_{n+1}-s)^\alpha) \\ &\quad \times[bu(s) + f(s, y(s-\tau))]\mathrm{d}s. \end{aligned} \tag{A.0.1}$$

To replace the integral in (A.0.1), we use product trapezoidal quadrature formula, where the nodes t_j $(j = 0, 1, 2, \dots, n+1)$ are taken with respect to the weight function $(t_{n+1} - \cdot)^{\alpha-1} E_{\alpha,\alpha}[a(t_{n+1} - \cdot)^{\alpha}]$. Thus the corrector formula is

$$\begin{aligned} y_h(t_{n+1}) &= E_{\alpha,1}[a(t_j)^{\alpha}]y_0 + h^{\alpha} E_{\alpha,\alpha+2}[ah^{\alpha}][bu(t_{n+1}) + f(t_{n+1}, y_h^P(t_{n+1}))] \\ &\quad + h^{\alpha} \sum_{j=0}^{n} c_{j,n+1}[bu(t_j) + f(t_j, y_h(t_j))], \end{aligned}$$

where

$$c_{j,n+1} = \begin{cases} \left.\begin{array}{l} (n+1)^{\alpha} E_{\alpha,\alpha+1}[a((n+1)h)^{\alpha}] + n^{\alpha+1} E_{\alpha,\alpha+2}[a(nh)^{\alpha}] \\ -(n+1)^{\alpha+1} E_{\alpha,\alpha+2}[a((n+1)h)^{\alpha}] \end{array}\right\}, & \text{if } j = 0, \\ \left.\begin{array}{l} (n-j)^{\alpha+1} E_{\alpha,\alpha+2}[a((n-j)h)^{\alpha}] \\ -2(n+1-j)^{\alpha+1} E_{\alpha,\alpha+2}[a((n+1-j)h)^{\alpha}] \\ +(n+2-j)^{\alpha+1} E_{\alpha,\alpha+2}[a((n+2-j)h)^{\alpha}] \end{array}\right\}, & \text{if } 1 \le j \le n, \\ E_{\alpha,\alpha+2}[ah^{\alpha}], & \text{if } j = n+1 \end{cases}$$

and the predictor formula for (A.0.1) using product rectangle rule is

$$y_h^P(t_{n+1}) = E_{\alpha,1}[a(t_j)^{\alpha}]y_0 + h^{\alpha} \sum_{j=0}^{n} d_{j,n+1}[bu(t_j) + f(t_j, y(t_j))],$$

where

$$d_{j,n+1} = (n+1-j)^{\alpha} E_{\alpha,\alpha+1}[a((n+1-j)h)^{\alpha}] - (n-j)^{\alpha} E_{\alpha,\alpha+1}[a((n-j)h)^{\alpha}].$$

In the following we state the above algorithm in a pseudo-code type notation.

Input variables:

F - real valued function of three real variables that defines the right hand side of the differential equation

α - fractional order $(0 < \alpha \le 1)$ of the differential equation

$y0$ - the history function $\phi(t)$, $t \in [-\tau, 0]$

τ - time delay (a positive real number)

T - the upper bound of the integral (a positive real number)

N - the number of time step that the algorithm used to take (a positive integer)

l - is a positive integer such that $\tau \in [-l, 0]$

Output variable:

y - an array of $N+1$ real numbers that contains the approximate solutions $y(jT/N)$, $j = 0, 1, 2, \ldots, N$.

Internal variables:

h - the step size of the algorithm (a positive real number)

j, k - integer variables used as indices

$y0$ - the history function

yp - the predicted value (a real variable)

c, d - arrays of $N+1$ real number that contains the weights of corrector and predictor, respectively.

m - is a real number such that $m - l \leq 0$

Body of the procedure:

$h := T/N;$

$h := \tau/l;$

FOR $k = 1$ to N

$$\begin{aligned} c(k) &= k^{\alpha} E_{\alpha,\alpha+1}[a(kh)^{\alpha}] - (k-1)^{\alpha} E_{\alpha,\alpha+1}[a((k-1)h)^{\alpha}]; \\ d(k) &= (k+1)^{\alpha+1} E_{\alpha,\alpha+2}[a((k+1)h)^{\alpha}] - 2k^{\alpha+1} E_{\alpha,\alpha+2}[a(kh)^{\alpha}] \\ &\quad + (k-1)^{\alpha+1} E_{\alpha,\alpha+2}[a((k-1)h)^{\alpha}]; \end{aligned}$$

END

IF $m - l <= 0$

$$y0 := \phi((m-l)h)$$

END

FOR $j = 1$ to N

$$
\begin{aligned}
yp &= E_{\alpha,1}[a(jh)^{\alpha}]y0 + h^{\alpha}\sum_{k=0}^{j-1} d[j-k]F(kh, y(k), y(k-l));\\
y(j) &= E_{\alpha,1}[a(jh)^{\alpha}]y0\\
&\quad +h^{\alpha}\Bigg(E_{\alpha,\alpha+2}[ah^{\alpha}]F(jh, yp, y(j-l))\\
&\quad +\big(j^{\alpha}E_{\alpha,\alpha+1}[a(jh)^{\alpha}] + (j-1)^{\alpha+1}E_{\alpha,\alpha+2}[a((j-1)h)^{\alpha}]\\
&\quad -j^{\alpha+1}E_{\alpha,\alpha+2}[a(jh)^{\alpha}]\big)F(0, \phi(0), \phi(0))\\
&\quad +\sum_{k=1}^{j-1} c[j-k]F(kh, y(k), y(k-l))\Bigg);
\end{aligned}
$$

END

List of publications/communications

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- V. Srinivasan, N. Sukavanam, Trajectory controllability of fractional-order $\alpha \in (1, 2]$ systems with delay, *Journal of Applied Nonlinear Dynamics*, To be published.
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Bibliography

[1] Adams, J. and Hartley, T. Finite time controllability of fractional order systems. *J. Comput. Nonlinear Dynam.*, 3(2):021402–1–021402–5, 2008.

[2] Agrawal, S. K., Srivastava, M., and Das, S. Synchronization of fractional order chaotic systems using active control method. *Chaos, Solitons and Fractals*, 45(6):737–752, 2012.

[3] Agrawal, S. K., Srivastava, M., and Das, S. A modified adaptive control method for synchronization of some fractional chaotic systems with unknown parameters. *Nonlinear Dyn.*, 73(1–2):907–919, 2013.

[4] Aleksandrov, A. Y. and Zhabko, A. P. On stability of the solutions of a class of nonlinear delay systems. *Automation and Remote Control*, 67(9):1355–1365, 2006.

[5] Bagley, R. L. and Torvik, P. J. On the fractional calculus model of viscoelastic behavior. *Journal of Rheology*, 30:133–155, 1986.

[6] Bajlekova, E. G. *Fractional evolution equations in Banach spaces.* University Press Facilities, Eindhoven University of Technology, 2001.

[7] Balachandran, K. and Kokila, J. On the controllability of fractional dynamical systems. *Int. J. Appl. Math. Comput. Sci.*, 22(3):523–531, 2012.

[8] Barnett, S. *Introduction to mathematical control theory.* Oxford University Press, London, 1975.

[9] Benchohra, M., Henderson, J., Ntouyas, S., and Ouahab, A. Existence results for fractional order functional differential equations with infinite delay. *J. Math. Anal. Appl.*, 338:1340–1350, 2008.

[10] Bhalekar, S. and Gejji, V. D. Synchronization of different fractional order chaotic systems using active control. *Commun Nonlinear Sci Numer Simulat*, 15:3536–3546, 2010.

[11] Bhalekar, S. and Gejji, V. D. A predictor-corrector scheme for solving nonlinear delay differential equations of fractional order. *J. Fract. Calc. App.*, 1(5):1–9, 2011.

[12] Bin, M. and Liu, Y. Trajectory controllability of semilinear differential evolution equations with impulses and delay. *Open Journal of Applied Sciences*, 3:37–43, 2013.

[13] Bonilla, B., Rivero, M., and Trujillo, J. On systems of linear fractional differential equations with constant coefficients. *Appl. Math. Comput.*, 187:68–78, 2007.

[14] Branner, B. and Hjorth, P. G. *Real and Complex Dynamical Systems*. Kluwer Academic, Netherlands, 1995.

[15] Brockett, R. W. *Finite dimensional linear systems*. John Wiley & Sons, Inc., New York, 1970.

[16] Brunovsky, P. Local controllability of odd systems. *Banach Center Publications*, 1:39–45, 1974.

[17] Camblibel, M. K., Heemels, W. P. M. H., and Schumacher, J. On the controllability of bimodal piecewise linear sytems. In Alur, R. and Pappas, G. J., editors, *Hybrid Systems: Computation and Control, Lecture Notes in Computer Science series*, volume 2993. Springer, Berlin, 2004.

[18] Camblibel, M. K., Heemels, W. P. M. H., and Schumacher, J. A full characterization of stabilizability of bimodal piecewise linear systems with scalar inputs. *Automatica*, 44:1261–1267, 2008.

[19] Chalishajar, D., George, R., Nandakumaran, A., and Acharya, F. Trajectory controllability of nonlinear integro-differential systems. *J. Franklin Institute*, 347:1065–1075, 2010.

[20] Chen, C.-T. *Linear system theory and design*. Oxford University Press, New York, 1999.

[21] Chen, L., Chai, Y., Wu, R., and Yang, J. Stability and stabilization of a class of nonlinear fractional-order systems with caputo derivative. *IEEE Trans. Circuits Syst. II, Exp. Briefs*, 59:602–606, 2012.

[22] Chen, W. C. Nonlinear dynamics and chaos in a fractional-order financial system. *Chaos, Solitons and Fractals*, 36:1305–1314, 2008.

[23] Cheres, E., Palmor, Z. J., and Gutman, S. Quantitative measures of robustness for systems including delayed perturbations. *IEEE Transactions on Automatic Control*, 34(11):1203–1204, 1989.

[24] Choi, J.-Y. Global stability analysis scheme for a class of nonlinear time delay systems. *Automatica*, 45(10):2462–2466, 2009.

[25] Conway, J. B. *A course in functional analysis*. Springer, New York, 1997.

[26] Corduneanu, C. *Principles of Differential and Integral Equations*. Chelsea Publishing Company, New York, 1971.

[27] Şahan, G. and Eldem, V. Well posedness conditions for bimodal piecewise affine systems. *Systems & Control Letters*, 83:9–18, 2015.

[28] Damgaard, P. H., Hjorth, P. G., and Thejll, P. A. A note on positivity of equilibrium populations in radiative transfer theory. *Astronomy and Astrophysics*, 254:422–425, 1992.

[29] Dauer, J. P. A controllability technique for nonlinear systems. *J. Math. Anal. Appl.*, 37:442–451, 1972.

[30] Dauer, J. P. Nonlinear perturbations of quasilinear control systems. *J. Math. Anal. Appl.*, 54:717–725, 1976.

[31] Dauer, J. P. and Mahmudov, N. I. Controllability of some nonlinear systems in hilbert spaces. *Journal of Optimization Theory and Applications*, 123:319–329, 2004.

[32] Debbouche, A. and Torres, D. F. Approximate controllability of fractional nonlocal delay semilinear systems in hilbert spaces. *International Journal of Control*, 86(9):1577–1585, 2013.

[33] Diethelm, K., Ford, N. J., and Freed, A. D. A predictor-corrector approach for the numerical solution of fractional differential equations. *Nonlinear Dynamics*, 29:3–22, 2002.

[34] Diethelm, K. The analysis of fractional differential equations. In Morel, J.-M., Takens, F., and Teissier, B., editors, *Lecture Notes in Mathematics*, volume 2004. Springer, Heidelberg, 2010.

[35] Do, V. N. Controllability of semilinear systems. *Journal of Optimization Theory and Applications*, 65(1):41–52, 1990.

[36] Eren, Y., Shen, J., and Camlibel, K. Quadratic stability and stabilization of bimodal piecewise linear systems. *Automatica*, 50:1444–1450, 2014.

[37] Faieghi, M. R. and Delavari, H. Chaos in fractional-order genesiotesi system and its synchronization. *Communications in Nonlinear Science and Numerical Simulation*, 17(2):731–741, 2012.

[38] Fattorini, H. O. Some remarks on complete controllability. *SIAM Journal on Control*, 4(4):686–694, 1966.

[39] Fridman, E. Effects of small delays on stability of singularly perturbed systems. *Automatica*, 38(5):897–902, 2002.

[40] Gejji, V. D., Sukale, Y., and Bhalekar, S. Solving fractional delay differential equations: A new approach. *Fractional Calculus and Applied Analysis*, 18(2):400–418, 2015.

[41] George, R. Approximate controllability of nonautonomous semilinear systems. *Nonlinear Andysis: Theory, Methods & Applications*, 24(9):1377–1393, 1995.

[42] Gómez-Aguilar, J. F., Yépez-Martínez, H., Calderón-Ramón, C., Cruz-Orduña, I., Escobar-Jiménez, R. F., and Olivares-Peregrino, V. H. Modeling of a mass-spring-damper system by fractional derivatives with and without a singular kernel. *Entropy*, 17:6289–6303, 2015.

[43] Govindaraj, V., Malik, M., and George, R. Trajectory controllability of fractional dynamical systems. *J. Control and Decision*, 2016.

[44] Gu, K., Kharitonov, V. L., and Chen, J. *Stability of time-delay systems*. Birkhäuser, Boston, 2003.

[45] Győri, I., Hartung, F., and Turi, J. Preservation of stability in delay equations under delay perturbations. *J. Math. Anal. Appl.*, 220:290–312, 1998.

[46] Haddad, W. M. and Chellaboina, V. S. Stability theory for nonnegative and compartmental dynamical systems with time delay. *Systems and Control Letters*, 51(5):355–361, 2004.

[47] Hirschom, R. M. Controllability in nonlinear systems. *Journal of Differential Equations*, 19:46–61, 1975.

[48] Hjorth, P. G., Kokkendorff, S. L., and Markvorsen, S. Hyperbolic spaces are of strictly negative type. *Proc. Amer. Math. Soc.*, 130(1):175–181, 2002.

[49] HosseinNia, S. H., Tejado, I., and Vinagre, B. M. Stability of fractional-order swithcing systems. *Comput. Math. Appl.*, 66:585–596, 2013.

[50] Hou, C., Gao, F., and Qian, J. Stability criterion for linear systems with nonlinear delayed perturbations. *Journal of mathematical analysis and applications*, 237(2):573–582, 1999.

[51] Huang, C. and Cao, J. Active control strategy for synchronization and anti-synchronization of a fractional chaotic financial system. *Physica A*, 473:262–275, 2017.

[52] Joshi, M. C. and Kumar, A. Approximation of exact controllability problem involving parabolic differential equations. *IMA Journal of Mathematical Control and Information*, 22:350–363, 2005.

[53] Joshi, M. and Bose, R. *Some topics in nonlinear functional analysis.* Wiley Eastern Ltd., New Delhi, 1985.

[54] Kalman, R. E. Contributions to the theory of optimal control. *Boletn de la Sociedad Matemtica Mexicana*, 5:102–119, 1960.

[55] Kalman, R. E. Mathematical description of linear dynamical systems. *J. SIAM Series A Control*, 1(2):152–192, 1963.

[56] Kalman, R. E., Ho, T., and Narendra, K. Controllability of linear dynamical systems. *Contributions to Differential Equations*, 1:190–213, 1963.

[57] Kamaljeet and Bahuguna, D. Controllability of the impulsive finite delay differential equations of fractional order with nonlocal conditions. *Neural, parallel* & *scientific computations*, 21:517–532, 2013.

[58] Kamaljeet and Bahuguna, D. Approximate controllability of nonlocal neutral fractional integro-differential equations with finite delay. *Journal of Dynamical and Control Systems*, 22(3):485–504, 2016.

[59] Kamaljeet, Bahuguna, D., and Shukla, R. K. Approximate controllability of finite delay fractional functional integro-differential equations with nonlocal condition. *Differential Equations and Dynamical Systems*, pages 1–15, 2016.

[60] Kexue, L., Jigen, P., and Jinghuai, G. Controllability of nonlocal fractional differential systems of order $\alpha \in (1, 2]$ in banach spaces. *Reports on Mathematical Physics*, 71(1):33–43, 2013.

[61] Kilbas, A. A., Srivastava, H. M., and Trujillo, J. J. Theory and applications of fractional differential equations. In Mill, J. V., editor, *North-Holland Mathematics Studies*, volume 204. Elsevier, Amsterdam, 2006.

[62] Klamka, J. Constrained controllability of nonlinear systems. *J. Math. Anal. Appl.*, 201:365–374, 1996.

[63] Klamka, J. Controllability of dynamical systems. *Matematyka Stosowana*, 9:57–75, 2008.

[64] Klamka, J., Ferenstein, E., Babiarz, A., Czornik, A., and Niezabitowsk, M. Constrained controllability of semilinear systems with delays. *Nonlinear Dyn.*, 56(1):169177, 2009.

[65] Klamka, J., Ferenstein, E., Babiarz, A., Czornik, A., and Niezabitowsk, M. Trajectory controllability of semilinear systems with delay in control and state. *Appl. Mech. Mat.*, 789–790:1045–1051, 2015.

[66] Krasovskii, N. N. *Stability of Motion*. Stanford University Press, California, 1963.

[67] Kumar, A., Joshi, M. C., and Pani, A. K. On approximation theorems for controllability of non-linear parabolic problems. *IMA Journal of Mathematical Control and Information*, 24(1):115–136, 2007.

[68] Kumar, S. and Sukavanam, N. Approximate controllability of fractional order semilinear systems with bounded delay. *J. Differential Equations*, 252:6163–6174, 2012.

[69] Kumar, S. and Sukavanam, N. Controllability of fractional order system with nonlinear term having integral contractor. *Fract. Calc. Appl. Anal.*, 16(4):791–801, 2013.

[70] Kumar, S. and Sukavanam, N. Controllability of second-order systems with nonlocal conditions in banach spaces. *Numerical Functional Analysis and Optimization*, 35(4):423431, 2014.

[71] Lakshmanan, S., Lim, C. P., Nahavandi, S., Prakash, M., and Balasubramaniam, P. Dynamical analysis of the hindmarsh-rose neuron with time delays. *IEEE Transactions on Neural Networks and Learning Systems*, 2016.

[72] Lakshmanan, S., Lim, C. P., Prakash, M., Nahavandi, S., and Balasubramaniam, P. Neutral-type of delayed inertial neural networks and their stability analysis using the lmi approach. *Neurocomputing*, 230:243–250, 2017.

[73] Lakshmanan, S., Prakash, M., Lim, C. P., Rakkiyappan, R., Balasubramaniam, P., and Nahavandi, S. Synchronization of an inertial neural network nith time–varying delays and its application to secure communication. *IEEE Transactions on Neural Networks and Learning Systems*, 2016.

[74] Lakshmikantham, V. Theory of fractional functional differential equations. *Nonlinear Analysis*, 69:3337–3343, 2008.

[75] LaSalle, J. P. Stability of nonautonomous systems. *Nonlinear Analysis: Theory, Methods & Applications*, 1(1):83–91, 1976.

[76] Lehman, B. and Shujaee, K. Delay independent stability conditions and decay estimates for time-varying functional differential equations. *IEEE Transactions On Automatic Control*, 39(8):1673–1676, 1994.

[77] Li, Y., Chen, Y.-Q., and Podlubny, I. Stability of fractional-order nonlinear dynamic systems: Lyapunov direct method and generalizaed mittag-leffler stability. *Comput. Math. Appl.*, 59:1810–1821, 2010.

[78] Liu, B., Lu, W., and Chen, T. New criterion of asymptotic stability for delay systems with time-varying structures and delays. *Neural Networks*, 54:103–111, 2014.

[79] Liu, P.-L. New results on delay-range-dependent stability analysis for interval time-varying delay systems with non-linear perturbations. *ISA transactions*, 57:93–100, 2015.

[80] Louis, J. C. and Wexler, D. On exact controllability in hilbert spaces. *Journal of Differential Equations*, 49:258–269, 1983.

[81] Lukes, D. L. Global controllability of nonlinear systems. *SIAM Journal on Control*, 10(1):112–126, 1972.

[82] Lyapunov, A. M. Stability of motion. Academic Press, New York, 1966.

[83] M.-C., H., Y.-C., H., and Chou, C.-H. Phase and anti-phase synchronization of two chaotic systems by using active control. *Physics Letters A*, 296(1):43–48, 2002.

[84] Mahmudov, N. I. Approximate controllability of evolution systems with nonlocal conditions. *Nonlinear Analysis: Theory, Methods & Applications*, 68(3):536–546, 2008.

[85] Mahmudov, N. I. and Zorlu, S. On the approximate controllability of fractional evolution equations with compact analytic semigroup. *Journal of Computational and Applied Mathematics*, 259:194–204, 2014.

[86] Mainardi, F., Paradisi, P., and R., G. Probability distributions generated by fractional diffusion equations. In Kertesz, J. and Kondor, I., editors, *Econophysics: An Emerging Science*. Kluwer, Dordrecht, 2000.

[87] Maraaba, T., Jarad, F., and Baleanu, D. On the existence and the uniqueness theorem for fractional differential equations with bounded delay within caputo derivatives. *Science in China Series A: Mathematics*, 51(10):1775–1786, 2008.

[88] Matignon, D. and Andréa-Novel, B. Some results on controllability and observability of finite-dimensional fractional differential systems. In *IMACS, IEEE-SMC Proceedings Conference*, pages 952–956. France, 1996.

[89] McKibben, M. A. *Discovering evolution equations with applications: Volume 1-Deterministic equations.* Chapman & Hall/CRC, Boca Raton, 2011.

[90] Miller, K. S. and Ross, B. *An Introduction to the Fractional Calculus and Fractional Differential Equations.* John Wiley & sons, Inc., New York, 1993.

[91] Monje, C. A., Chen, Y. Q., Vinagre, B. M., Xue, D., and Feliu, V. Fractional-order systems and controls fundamentals and applications. In Grimble, M. J. and Johnson, M. A., editors, *Advances in Industrial Control*, chapter 14–18, pages 259–389. Springer-Verlag, London, 2010.

[92] Morgado, M. L., Ford, N. J., and Lima, P. Analysis and numerical methods for fractional differential equations with delay. *J. Comput. Appl. Math.*, 252:159–168, 2013.

[93] Mori, T., Noldus, E., and Kuwahara, M. A way to stabilize linear systems with delayed state. *Automatica*, 19(5):571–573, 1983.

[94] Mori, T. Criteria for asymptotic stability of linear time-delay systems. *IEEE Transactions on Automatic Control*, AC–30(2):158–161, 1985.

[95] Mozyrska, D. and Torres, D. F. M. Minimal modified energy control for fractional linear control systems with the caputo derivative. *Carpathian Journal of Mathematics*, 26(2):210–221, 2010.

[96] Naito, K. An inequality condition for approximate controllability of semilinear control systems. *J. Math. Anal. Appl.*, 138(1):129–136, 1989.

[97] Naito, K. Controllability of semilinear control systems dominated by the linear part. *SIAM J. Control Optim.*, 25(3):715–722, 2002.

[98] Naito, K. and Park, J. Y. Approximate controllability for trajectories of a delay volterra control system. *Journal of Optimization Theory and Applications*, 61(2):271–279, 1989.

[99] Ni, M.-L. and Er, M. J. Stability of linear systems with delayed perturbations: an lmi approach. *IEEE Transactions on Circuits and Systems I: Fundamental Theory and Applications*, 49(1):108–112, 2002.

[100] Niculescu, S.-I. *Delay effects on stability: a robust control approach.* Springer, London, 2001.

[101] Odibat, Z. M. Adaptive feedback control and synchronization of non-identical chaotic fractional order systems. *Nonlinear Dynamics*, 60(4):479–487, 2010.

[102] Pecora, L. M. and Carroll, T. L. Synchronization in chaotic systems. *Physical Review Letters*, 64(8):821–825, 1990.

[103] Pitcher, E. and Sewell, W. E. Existence theorems for solutions of differential equations of non-integral order. *Bull. Amer. Math. Soc.*, 44(2):100–107, 02 1938.

[104] Podlubny, I. *Fractional Differential Equations.* Academic Press, San Diego, 1999.

[105] Razumikhin, B. S. Stability in first approximation of systems with lag. *J. Appl. Math. Mech.*, 22(2):215–229, 1958.

[106] Rivero, M., Trujillo, J. J., Vázquez, L., and Velasco, M. P. Fractional dynamics of populations. *Appl. Math. Comput.*, 218:1089–1095, 2011.

[107] Sabatier, J., Moze, M., and Farges, C. Lmi stability conditions for fractional order systems. *Comput. Math. Appl.*, 59:1594–1609, 2010.

[108] Sadovskii, B. N. A fixed-point principle. *Voronezh State University, Translated from Funktsional'nyi Analiz i Ego Prilozheniya*, 1(2):74–76, 1967.

[109] Sharma, J. P. and George, R. K. Controllability of matrix second order systems: A trigonometric matrix approach. *Electronic J. Diff. Equations*, 80:1–14, 2007.

[110] Shukla, A., Sukavanam, N., and Pandey, D. N. Approximate controllability of semi-linear fractional control systems of order $\alpha \in (1, 2]$. In *2015 Proceedings of the Conference on Control and its Applications*, pages 175–180. SIAM, 2015.

[111] Sonawane, R. B., Kumar, A., and Nimse, S. B. Exact controllability of wave equation with multiplicative controls. *Applied Mathematics E-Notes*, 14:45–52, 2014.

[112] Sontag, E. D. *Mathematical Control Theory*. Springer, New York, 1998.

[113] Soukkou, A., Boukabou, A., and Leulmi, S. Prediction-based feedback control and synchronization algorithm of fractional-order chaotic systems. *Nonlinear Dynamics*, 85:2183–2206, 2016.

[114] Srivastava, M., Ansari, S. P., Agrawal, S. K., Das, S., and Leung, A. Y. T. Anti-synchronization between identical and non-identical fractional-order chaotic systems using active control method. *Nonlinear Dyn.*, 76(2):905–914, 2014.

[115] Syed Ali, M. and Balasubramaniam, P. Exponential stability of time-delay systems with nonlinear uncertainties. *International Journal of Computer Mathematics*, 87(6):1363–1373, 2010.

[116] Taghvafard, H. and Erjaee, G. H. Phase and anti-phase synchronization of fractional order chaotic systems via active control. *Commun Nonlinear Sci Numer Simulat*, 16:4079–4088, 2011.

[117] Tang, X. Asymptotic behavior of delay differential equations with instantaneously terms. *Journal of mathematical analysis and applications*, 302(2):342–359, 2005.

[118] Tarnove, I. A controllability problem for nonlinear systems. In Balachandran, L. W., A. V.and Neustadt, editor, *Mathematical Theory of Control*, pages 170–179. Acadamic Press, New York, 1967.

[119] Thuan, M. V., Phat, V. N., Fernando, T., and Trinh, H. Exponential stabilization of time-varying delay systems with non-linear perturbations. *IMA Journal of Mathematical Control and Information*, 31(4):441–464, 2014.

[120] Triggiani, R. Controllability and observability in banach space with bounded operators. *SIAM Journal on Control*, 13(2):462–491, 1975.

[121] Triggiani, R. A note on the lack of exact controllability for mild solutions in banach spaces. *SIAM J. Control Optim.*, 15(3):407411, 1977.

[122] Trinh, H. and Aldeen, M. On the stability of linear systems with delayed perturbations. *IEEE Transactions on Automatic Control*, 39(9):1948–1951, 1994.

[123] Vidyasagar, M. A controllability condition for nonlinear systems. *IEEE Transactions on Automatic Control*, 17(4):569–570, 1972.

[124] Wang, F. F., Chen, D. Y., Zhang, X. G., and Wu, Y. The existence and uniqueness theorem of the solution to a class of nonlinear fractional order system with time delay. *Applied Mathematics Letters*, 53:45–51, 2016.

[125] Wei, J. The controllability of fractional control systems with control delay. *Computers and Mathematics with Applications*, 64:3153–3159, 2012.

[126] Wen, X.-J., Wu, Z.-M., and Lu, J.-G. Stability analysis of a class of nonlinear fractional-order systems. *IEEE Trans. Circuits Syst. II, Exp. Briefs*, 55:1178–1182, 2008.

[127] Winston, E. Asymptotic stability for ordinary differential equations with delayed perturbations. *SIAM Journal on Mathematical Analysis*, 5(2):303–308, 1974.

[128] Wu, X.-Q. and Lu, J.-A. Parameter identification and backstepping control of uncertain lu system. *Chaos, Solitons & Fractals*, 18(4):721–729, 2003.

[129] Yassen, M. T. Chaos synchronization between two different chaotic systems using active control. *Chaos, Solitons & Fractals*, 23(1):131–140, 2005.

[130] Yau, H.-T. Design of adaptive sliding mode controller for chaos synchronization with uncertainties. *Chaos, Solitons & Fractals*, 22(2):341–347, 2004.

[131] Zhang, R., Tian, G., Yang, S., and Cao, H. Stability analysis of a class of fractional-order nonlinear systems with order lying in $(0, 2)$. *ISA Trans.*, 56:102–110, 2015.

[132] Zhang, Z. and Liu, B. Controllability results for fractional functional differential equations with nondense domain. *Numerical Functional Analysis and Optimization*, 35(4):443–460, 2014.

[133] Zhen, W., Xia, H., and Guodong, S. Analysis of nonlinear dynamics and chaos in a fractional order financial system with time delay. *Computers and Mathematics with Applications*, 62:1531–1539, 2011.

[134] Zhu, Q. and Cao, J. Adaptive synchronization of chaotic cohen–crossberg neural networks with mixed time delays. *Nonlinear Dynamics*, 61(3):517–534, 2010.