STUDY ON COVERGENCE OF CERTAIN LINEAR POSITIVE OPERATORS

Ph.D. THESIS

by

MANJARI SIDHARTH

DEPARTMENT OF MATHEMATICS
INDIAN INSTITUTE OF TECHNOLOGY ROORKEE
ROORKEE – 247667 (INDIA)
SEPTEMBER, 2017

STUDY ON CONVERGENCE OF CERTAIN LINEAR POSITIVE OPERATORS

A THESIS

Submitted in partial fulfilment of the requirements for the award of the degree

of

DOCTOR OF PHILOSOPHY

in

MATHEMATICS

by

MANJARI SIDHARTH

DEPARTMENT OF MATHEMATICS
INDIAN INSTITUTE OF TECHNOLOGY ROORKEE
ROORKEE–247667 (INDIA)
SEPTEMBER, 2017

INDIAN INSTITUTE OF TECHNOLOGY ROORKEE
ROORKEE

CANDIDATE'S DECLARATION

I hereby certify that the work which is being presented in this thesis entitled, **"STUDY ON CONVERGENCE OF CERTAIN LINEAR POSITIVE OPERATORS"** in partial fulfilment of the requirements for the award of the Degree of Doctor of Philosophy and submitted in the Department of Mathematics of the Indian Institute of Technology Roorkee, Roorkee is an authentic record of my own work carried out during a period from January, 2014 to July, 2017 under the supervision of Dr. P. N. Agrawal, Professor, Department of Mathematics, Indian Institute of Technology Roorkee, Roorkee.

The matter presented in this thesis has not been submitted by me for the award of any other degree of this or any other Institution.

(MANJARI SIDHARTH)

This is to certify that the above statement made by the candidate is correct to the best of my knowledge.

(P. N. Agrawal)
Supervisor

Date**: 20/09/2017**

Abstract

In the thesis, we study approximation properties of some well known operators and their q-analogues. We divide the thesis into eight chapters. The chapter 0 includes the literature survey, basic definitions and some basic notations of approximation methods which will be used throughout the thesis.
In the first chapter, we discussed the Schurer type q-Bernstein Kantorovich operator which was introduced by Lin. We obtain a local approximation theorem and the statistical convergence of these operators. We also study the rate of convergence by means of the first order modulus of continuity, Lipschitz class function, the modulus of continuity of the first order derivative and the Voronovskaja type theorem.

The second chapter is concerned with the Stancu-Kantorovich operators based on Pólya-Eggenberger distribution. We obtain some direct results for these operators by means of the Lipschitz class function, the modulus of continuity and the weighted space. Also, we study an approximation theorem with the aid of the unified Ditzian-Totik modulus of smoothness $\omega_{\phi^\tau}(f;t),\ \ 0 \leq \tau \leq 1$ and the rate of convergence of the operators for the functions having a derivative which is locally of bounded variation on $[0, \infty)$.

In the third chapter, we introduce the Szász-Durrmeyer type operators based on Boas-Buck type polynomials which include Brenke-type polynomials, Sheffer polynomials and Appell polynomials. We establish the moments of the operator and a Voronvskaja type asymptotic theorem and then proceed to study the convergence of the operators with the help of Lipschitz type space and weighted modulus of continuity. Next, we obtain a direct approximation theorem with the aid of unified Ditzian-Totik modulus of smoothness. Furthermore, we study the approximation of functions whose derivatives are locally of bounded variation.

In the fourth chapter, we obtain the rate of approximation of the bivariate Bernstein-Schurer-Stancu type operators based on q-integers by means of the moduli of continuity and Lipschitz class. We also estimate the degree of approximation by means of Lipschitz class function and the rate of convergence with the help of mixed modulus of smoothness for the GBS operator of q-Bernstein-Schurer-Stancu type. Furthermore, we show the comparisons by some illustrative graphics in Maple for the convergence of the operators to some functions.

In the fifth chapter we study the approximation properties of the bivariate extension of q-Bernstein-Schurer-Durrmeyer operators and obtained the rate of convergence of the operators with the aid of the Lipschitz class function and the modulus of continuity. Here, we estimate the rate of convergence of these operators by means of Peetre's K-functional. Then, the associated GBS (Generalized Boolean Sum) operator of the q-Bernstein-Schurer-Durrmeyer type is defined and discussed. Furthermore, we illustrate the convergence rate of the bivariate Durrmeyer type operators and the associate GBS operators to certain functions by numerical examples and graphs using Maple algorithm.

In the sixth chapter, We discuss the mixed summation integral type two dimensional q-Lupaş-Phillips-Bernstein operators which was first introduced by Honey Sharma in 2015. We establish a Voronovskaja type theorem and introduce the associated GBS case (Generalized Boolean Sum) of these operators and we study the rate of convergence by utilizing the Lipschitz class and the mixed modulus of smoothness. Furthermore, we show the rate of convergence of the bivariate operators and the corresponding GBS operators by illustrative graphics and numerical examples using Maple algorithms.

In the seventh chapter, we obtain the degree of approximation for the Kantorovich-type q-Bernstein-Schurer operators in terms of the partial moduli of continuity and the Peetre's K-functional. Finally, we construct the GBS (Generalized Boolean Sum) operators of bivariate q-Bernstein-Schurer-Kantorovich type and estimate the rate of convergence for these operators with the help of mixed modulus of smoothness.

In the last chapter, we establish the approximation properties of the bivariate operators which are the combination of Bernstein-Chlodowsky operators and the Szász operators

involving Appell polynomials. We investigate the degree of approximation of the operators with the help of complete modulus of continuity and the partial moduli of continuity. In the last section of the paper, we introduce the Generalized Boolean Sum (GBS) of these bivariate Chlodowsky-Szasz-Appell type operators and examine the order of approximation in the Bögel space of continuous functions by means of mixed modulus of smoothness.

Acknowledgements

First of all my everlasting bow in reverence and gratitude is to "Almighty God" whose gracious blessing enabled me to complete my work in time. Words cannot express the depth of my gratitude for all those who directly or indirectly helped me in my endeavor.

It is with the immense respect that I feel indebted to my supervisor **Prof. P. N. Agrawal**, Department of Mathematics, I.I.T. Roorkee, for considering me worthy of their able guidance. I thank him for lending me constant encouragement, advice support, experience and above all punctuality which have indeed gone a long way in bringing this research to fruition. Without his active guidance, it would not have been possible for me to complete this work.

I sincerely thank my research committee members e.g. **Prof. N. Sukavanam**, SRC Chairperson, **Prof. V. K. Katiyar** and **Prof. M. J. Nigam** for their guidance, cooperation and many valuable comments that have helped me in improving the quality of my research work. I wish to sincerely acknowledgement the continuous financial support of Ministry of Human Resource and Development, India without which it would have become difficult to take up this work.

I would like to thank the Heads of the Department of Mathematics, I.I.T. Roorkee, during the tenure of my research, namely: **Prof. R. C. Mittal** and **Prof. V. K. Katiyar** for their academic help and encouragement.

I extend my sincere thanks to **Prof. Nurhayat Ispir**, Department of Mathematics, Gazi University, Turkey, **Prof. Ana Maria Acu**, Department of Mathematics and Informatics, Lucian Blaga University of Sibiu and **Prof. Serkan Araci**, Department of Economics, Hasan Kalyoncu University, Turkey for providing me a helping hand whenever I asked for.

Heartfelt gratitude goes to my seniors **Dr. A. Sathish Kumar, Dr. Meenu Rani, Dr. Arun Kajla, Ms. Priyanka, Ms. Renu, Mr. Kiran, Mr. Rakesh** for their support. For being the most direct source of inspiration and support, I would like to thank my labmates **Trapti, Sheetal, Pooja, Tarul** and **Ruchi**.

I consider myself especially lucky to be blessed with some truly special friendships. I wish to dedicate this space to **Alka, Juhi, Indira, Nishi, Ankit, Vivek** and **Kavita** and for their unsolicited, unspoken and happily lent friendly support, concern and love.

Above all, the credit of compulsion of this work goes to my **Parents**. I express my humble gratitude to my parents, for their decades of support, whose patience, encouragement, unfailing love and blessings are reflected in each and every word of this work. I would also like to thank my family members especially my sisters **Akanksha, Kanika** and brother **Anirudh** . I am thankful to them for numerous naughty moments, laughs and definitely fights, which keep me going.

Last but not the least I thank all those who have helped me directly or indirectly at various stages of this work.

Roorkee (Manjari Sidharth)

September , 2017

Table of Contents

Introduction

0.1 General Introduction

The field of approximation theory has become so vast that it intersects with every other branch of mathematics territory. Approximation theory has both pragmatic side which is concerned largely with computational practicalities, precise estimations of errors, and so on and also a theoretical side, which is more often concerned with existence and uniqueness questions and application to other theoretical issues. The primary aim of a general approximation is to represent non-arithmetic quantities by arithmetic quantities so that accuracy can be ascertained to a desired degree. For instance, when we try to expand a function in a power series, we are trying to represent the function in terms of polynomials namely, the partial sum of the power series.

The first significant result in Approximation theory is Weierstrass approximation theorem which assumes a key part in the advancement of general estimation hypothesis. In 1885, Weierstrass proved the density of algebraic polynomials in the class of continuous real valued functions on a compact interval, and the density of trigonometric polynomials in the class of 2π periodic continuous real valued functions. It states that, if f is a continuous real valued function defined on $[a, b]$ then there exists an algebraic polynomial $p(x)$ such that

$$|f(x) - p(x)| \leq \epsilon, \quad \forall x \in [a, b] \text{ where } \epsilon > 0.$$

Over the next so many years, the alternative proofs of this result were given by the best analyst of the period. The impact of the theorem in mathematics was immediate. Later, there were proofs by famous mathematicians such as Runge (1885), Picards (1891), Lerch (1892 and 1903), Volterra (1897), Lebesgue (1898), Mittag-Leffler (1900), Fejér (1900), Landau (1908), de la Valleé Poussin (1908), Jackson (1911)

In 1912, a Russian mathematician S. N. Bernstein formulated a sequence of polynomials to prove Weierstarss theorem, as follows:

If f is a real valued bounded function on $[0,1]$, then

$$B_n(f;x)=\sum_{k=0}^{n} b_{n,k}(x) f\left(\frac{k}{n}\right)=\sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k} x^k(1-x)^{n-k},$$

is called the Bernstein polynomial of order n of the function $f(x)$. Here the kernel $b_{n,k}(x)$ are the binomial or Newton probabilities which is very well known in the theory of probability. Bernstein [44] proved that for $f\in C[0,1]$, the sequence $\{B_n(f;x)\}$ converges uniformly to $f(x)$ on $[0,1]$. Because of shape preserving property, Bernstein polynomials have a practical applications. In 1962, Schurer [121] generalized the Bernstein polynomials by extending its domain from $[0,1]$ to $[0,1+p]$, where p is a non-negative integer.

In [54], Davis proved that for any convex function, the classical Bernstein polynomial is also convex and the sequence of Bernstein polynomials is monotonically decreasing. The same author also proved that if the k-th ordinary differences of a function are non-negative on $[0,1]$ then the k-th derivative of the classical Bernstein polynomial is non-negative. Stancu [126] proposed the positive linear operators $B_n^{\alpha,\beta}: C[0,1]\to C[0,1]$ as:

$$B_n^{\alpha,\beta}(f;x)=\sum_{k=0}^{n} b_{n,k}(x) f\left(\frac{k+\alpha}{n+\beta}\right),$$

where α,β satisfy the condition $0\leq\alpha\leq\beta$. If $\alpha=\beta=0$, the above sequence of operators include the Bernstein polynomials. Kantorovich [94] introduced an integral modification of Bernstein polynomials to approximate Lebesgue integrable functions in $[0,1]$. Durrmeyer [65] proposed another modification of Bernstein polynomials to approximate functions in $L_p[0,1], p\geq 1$.

Abel et al. [1] established an estimate of the rate of convergence for functions of bounded variations by the beta operators using the decomposition technique. The complete asymptotic expansion of the sequence of operators for smooth functions, as n tends to infinity, was also obtained. Karsli [95] discussed the rate of pointwise convergence of a new type of Gamma operators for functions with derivatives of bounded variations. Gupta and Beniwal [75] established the rate of convergence in simultaneous approximation by Durrmeyer-Schurer type operators. Wafi and Khatoon [138] obtained the rate of convergence and Voronovskaja type theorems for the first order derivatives of the generalized Baskakov operators for functions of one and two variables in polynomial weighted spaces. Govil et al. [73] obtained an estimate of the rate of convergence for function of bounded variation by a Durrmeyer type modification of the operators introduced by Jain and Pethe. Deo et al.[55] studied the simultaneous approximation for the generalized Bernstein-Durrmeyer operators. Deo and Singh [57] proposed Baskakov-Durrmeyer operators and studied their rate of convergence in simultaneous approximation. Gupta et al. [82] introduced a certain family of mixed summation integral type operators having different weight functions and obtained some local direct theorems in ordinary and simultaneous approximation. Gupta et al. [76] discussed the rate of convergence of the Szász-Mirakyan-Durrmeyer for functions with derivatives of bounded variations. Wafi and Khatoon [137] obtained the rate of convergence, asymptotic formula, direct and inverse theorems for the generalized Baskakov operators introduced by Mihesan [101]. Bivariate extension of these operators was also discussed by the authors in [137].

For other related literature we refer to (cf. [2], [72], [77], [78], [83], [106], [135], [136], [115], [116], [117]).

Quantum calculus is the generalized name for the investigation of calculus without limits. q-calculus appeared as a connection between physics and mathematics, it has a lot of applications in different mathematical areas such as hypergeometric functions, combinatorics, orthogonal polynomials and in other sciences such as quantum theory, mechanics, theory of relativity etc. In the last decade, the application of q-calculus in the area of approximation theory has attracted a lot of interest.

Lupaş was the first person who pioneered work on q-analogue of the Bernstein polynomials. In 1987, he introduced a q-analogue of the Bernstein operator and investigated its approximation properties.

Let $q > 0$ and $f \in C[0,1]$. The linear operator

$$L_{n,q}(x) = \sum_{k=0}^{n} f\left(\frac{[k]_q}{[n]_q}\right) b_{n,k}(q;x), \quad n \in \mathbb{N}$$

where

$$b_{n,k}(q;x) = \begin{bmatrix} n \\ k \end{bmatrix} \frac{q^{k(k-1)/2}x^k(1-x)^{n-k}}{(1-x+qx)...(1-x+q^{n-1})}$$

is called the Lupas q-analogue of the Bernstein operator.

Clearly, if $q = 1$, then the operator $L_{n,q}$ reduce to the classical Bernstein polynomials. In the case $q \neq 1$, operator $L_{n,q}$ are rational functions rather than polynomial. The Limit q-Lupas operator comes out naturally as a limit for a sequence of the Lupas q-analogues of the Bernstein operator.

Later in 1996, Phillips [110] generalized the Bernstein polynomials using q-binomial coefficients which is defined as

$$B_{n,q}(f;x) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q x^k(1-x)_q^{n-k} f\left(\frac{[k]_q}{[n]_q}\right), \text{ for each positive integer } n. \tag{0.1.1}$$

We see that the q-Bernstein polynomials defined in (0.1.1) interpolates f at both end points of $[0,1]$ and it is a linear operator that maps functions defined on [0,1] to the set of polynomials of degree at most n, and for $0 < q \leq 1$, it is monotone operator.

In 1997, Phillips also proved that the q-Bernstein polynomials may be expressed in terms of the q-difference as

$$B_{n,q}(f;x) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \left(\Delta_q^k f_0\right) x^k, \tag{0.1.2}$$

where $\Delta_q^k f_j = \Delta_q^{k-1} f_{j+1} - q^{k-1}\Delta_q^{k-1} f_j, \quad k \geq 1$ with $\Delta_q^0 f_j = f_j = f\left(\frac{[j]_q}{[n]_q}\right)$.

The expression (0.1.2) of $B_{n,q}(f;x)$ in terms of q-difference shows its worthiness as a true q-analogue of the classical Bernstein operators. He evaluated the approximation properties of a function at the interval which are in the form of geometric progression. In [108], Oruç et al. extended the results given in [54] to the generalized Bernstein polynomials.

Muraru [102] introduced the q-Bernstein-Schurer operators and obtained the Korovkin-type approximation theorem and the rate of convergence of the operators in terms of the first order modulus of continuity. Durrmeyer [65] introduced the integral modification of the Bernstein polynomials and in 2005, Derriennic [59] generalized the modified Bernstein polynomials for Jacobi weights using the q-Bernstein basis proposed by Phillips. She extended the various properties of modified Bernstein polynomials to their q-analogues. Adell and Cal [14] considered the Durrmeyer type modification of Bernstein, Szász and Baskakov operators and solved the two different kinds of problems. In this paper, authors used a probabilistic approach to obtain the result concerning the preservation of shape properties, Lipschitz constant and global smoothness as well as convexity.

Dalmanglu [53] defined the q-analogue of the Bernstein-Kantorovich operators and examined the order of approximation of the operators by means of modulus of continuity. Subsequently, Radu [114] investigated the statistical convergence results of these operators. Agratini [15] studied the limit of iterates of q-analogue of the Bernstein polynomials introduced by Lupas and also proposed a new class of q-Bernstein operators depending on a parameter which fix certain polynomials of second degree. Govil and Gupta [71] introduced a new type of q-Meyer-König-Zeller-Durrmeyer operators and established some approximation properties. Gupta and Heping [80] introduced certain q-analogue of Bernstein-Durrmeyer operators for $0 < q < 1$ and investigated the rate of convergence of the operators by using modulus of continuity. After this, in the continuation of Durrmeyer modification of Bernstein operators, Aral and Gupta [29] introduced Szász-Durrmeyer operators based on q-integer on the space of continuous functions on positive semi-axis and studied their approximation properties and established an asymptotic behavior of these operators with respect to weighted norm. In [27], Aral and Gupta considered certain q-Baskakov operators and studied some of their approximation properties. Aral and Gupta [28] represented the q-Baskakov operators in terms of divided differences to discuss the q-derivatives and shape preserving properties. Gupta and Karsli [81] studied some approximation properties of the Szász-Mirakyan-Baskakov-Stancu operators based on q-integers. Mursaleen and Khan [105] studied the statistical approximation properties of Bernstein-Schurer operators based on q-integers. Muraru and Acu [103] proposed a Durrmeyer variant of q-Schurer operators and established a Korovkin-type approximation theorem and the rate of approximation. Ruchi et al. [50] introduced the bivariate case of Stancu type Kantorovich modification of the operators proposed by Ren and Zeng [118].

Karsli et al. [97] introduced the q-analogue of the general Gamma type operators and studied the rate of convergence, weighted approximation and A-statistical convergence of these operators. Recently, Acu [11] introduced a q-analogue of Stancu-Schurer-Kantorovich operators and studied its rate of convergence and the statistical approximation properties. Subsequently, Agrawal et al.[17] constructed a bivariate generalization of a new kind of Kantorovich type q- Bernstein Schurer operators and studied the rate of convergence, the degree of approximation by means of the Lipschitz type class and a Voronovskaja type theorem. Ruchi et al. [120] introduced the Kantorovich variant of Bernstein-Szász operators based on q-integers for functions of one and two variables and also studied the associated GBS operators. For more details about the work on linear positive operators based on q-integers we refer the reader to [30].

0.2 Bivariate extensions of the linear positive operators

In 1951, Kingsley [98] first introduced the bivariate extension of Bernstein polynomials for the functions belonging to the class $C^{(k)}$, where $C^{(k)}$, is the class of all the functions whose derivatives of order $1, 2, ..., k$ exist and are continuous. The Bernstein polynomial associated with the function $f(x, y)$ is defined as

$$B_{m,n}(f; x, y) = \sum_{k_1=0}^{m} \sum_{k_2=0}^{n} b_{m,k_1}(x)\, b_{n,k_2}(y)\, f\left(\frac{k_1}{m}, \frac{k_2}{n}\right)$$

where $b_{m,k_1}(x)$ and $b_{n,k_2}(y)$ are the Bernstein basis.

The main purpose of his research was to show the uniform convergence of these polynomials in $S(S : 0 \leq x, y \leq 1)$. Butzer in [47] gave a more direct proof of Kingsley theorem and prove that if all the partial derivatives of a function $f(x, y)$ of order at most k exist and are continuous in S, then

$$\lim_{m,n\to\infty} \frac{\partial^k}{\partial y^{k-i}} (B_{m,n}f) \longrightarrow \left(\frac{\partial^k}{\partial y^{k-i}}\right) f$$

uniformly in S. Wigert [140] and S. N. Bernstein proved the same result for one variable. In [100], Martinez replaced each coefficient of these polynomials by their integral parts and proved their convergence in both the uniform and L_p norms. Stancu [125] introduced another bivariate extension of the Bernstein polynomials on the isosceles right triangle $\Delta := \{(x, y) : x+y \leq 1, x \geq 0,\ y \geq 0\}$ and indicated a simple method for extending the Bernstein polynomials. After this, Stancu [127] defined another linear positive operators in two or several variables.

Bărbosu [38] extended the q-Bernstein polynomial in two variables by using the parametric extensions. Let $I^2 = [0,1] \times [0,1]$ be the unit square then for any function $f : I^2 \to \mathbb{R}$, the parametric extension of the operator (0.1.1) can be defined as

$$B^{x}_{n_1,q_1} = \sum_{k_1=0}^{n_1} \begin{bmatrix} n_1 \\ k_1 \end{bmatrix}_{q_1} x^{k_1} \, (1-x)^{n_1-k_1}_{q_1} \, f\left(\frac{[k_1]_{q_1}}{[n_1]_{q_1}}\right) \tag{0.2.1}$$

and

$$B^{y}_{n_2,q_2} = \sum_{k_2=0}^{n_2} \begin{bmatrix} n_2 \\ k_2 \end{bmatrix}_{q_2} y^{k_2} \, (1-y)^{n_2-k_2}_{q_2} \, f\left(\frac{[k_2]_{q_2}}{[n_2]_{q_2}}\right) \tag{0.2.2}$$

where $q_1, q_2 > 0$. Clearly, for $q_1 = 1, q_2 = 1$ these parametric extensions reduce to the parametric extension of classical Bernstein polynomials.

After these modifications many researchers also studied the bivariate and multivariate case of the linear positive operators (cf. [12], [17], [51], [58]).

0.3 Linear operators based on orthogonal polynomials

Appell [25], introduced a sequence of polynomials $P_n(x)$ of degree n which satisfies the differential equation

$$D\,P_n(x) = nP_{n-1}(x), \quad D \equiv \frac{d}{dx},$$

known as Appell polynomials. These polynomials have been studied widely because of their remarkable applications not only in mathematics [33] but also in physics and in chemistry. In [124], Sheffer extended the class of Appell polynomials and called these polynomials as zero type polynomials. Using Appell polynomials, Jakimovski and Leviatan [89] introduced a generalization of the Favard-Szász operators as

$$P_n(f;x) = \frac{e^{-nx}}{g(1)} \sum_{k=0}^{\infty} p_k(nx) f\left(\frac{k}{n}\right), \tag{0.3.1}$$

where, $g(u)e^{ux} = \sum_{k=0}^{\infty} p_k(x)u^k$ is the generating function for the Appell polynomials $p_k(x) \geq 0$, with $g(z) = \sum_{n=0}^{\infty} a_n z^n, \ |z| < R, \ R > 1$ and $g(1) \neq 0$.

Subsequently, the Stancu type generalization of the operators (0.3.1) was introduced by Atakut and Büyükyazici [32], wherein the authors established some approximation properties. These generalization of the operators given by (0.3.1) is defined as

$$P_n^*(f;x) = \frac{e^{-b_n x}}{g(1)} \sum_{k=0}^{\infty} p_k(b_n x) f\left(\frac{k}{c_n}\right), \tag{0.3.2}$$

where $(b_n), (c_n)$ denote the unbounded and increasing sequences of positive real numbers such that $b_n \geq 1,\ c_n \geq 1$, and $\lim_{n\to\infty} \frac{1}{c_n} = 0,\ \frac{b_n}{c_n} = 1 + O\left(\frac{1}{c_n}\right)$, as $n \to \infty$. In the special case $g(z) = 1$, these operators reduce to the modified Szász operators studied by Walczak [139]. Also, for $b_n = n = c_n$, these operators coincide with the operators (0.3.1).

Ismail [85] generalized Szász operators by means of the Sheffer polynomials. Varma and Tasdelen [133] defined Szász type operators involving Charlier polynomials. In this same paper, the authors also introduced the Kantorovich type generalization of these operators. Kajla and Agrawal [92] proposed the Szász-Durrmeyer type operators based on Charlier polynomials. Varma et al. [132] introduced Szász type operators involving Brenke-type polynomials. Mursaleen and Ansari [104] studied Chlodowsky type generalization of Szász type operators by involving Brenke type polynomials. Tasdelen et al. [131] proposed a Kantorovich variant of the Szász operators involving Brenke type polynomials. Garg et al. [70] investigated the order of convergence of these operators and the degree of approximation for continuous functions in a weighted space.

Agrawal and Ispir [19] introduced bivariate operators by a combination of Bernstein-Chlodowsky polynomials and the Szász type operators based on Charlier polynomials. For some other related papers one can refer to (cf. [23], [16], etc.,).

0.3.1 Definitions for single variable case

Definition 1. **Modulus of continuity:** For $f(x) \in C[a,b]$, the modulus of continuity is defined as

$$\omega(f;\delta) = \sup\left\{|f(x_2) - f(x_1)| : \quad \text{whenever} \quad |x_2 - x_1| \leq \delta,\ \ \delta > 0\right\}.$$

Then $\omega(\delta)$ is continuous, increases as δ increases and it tends to 0, as $\delta \to 0$. The same definition holds for $f(x) \in C_{2\pi}$, and the greatest value of $\omega(\delta)$ is $\omega(\pi)$.

Properties of modulus of continuity:

- If n is positive integer then $\omega(f; n\,\delta) \leq n\,\omega(f; \delta)$.
- If $k > 0,\ \omega(f; k\,\delta) \leq (k+1)\,\omega(f; \delta)$.
- If $\omega(\delta) = 0$ for some $\delta > 0$, then $f(x)$ is a constant.

Definition 2. **First order modulus of continuity:** The first modulus of continuity of $f \in C(I)$ for $\delta > 0$ is given by

$$\omega(f, \delta) = \max_{0<|h|<\delta, x, x+h \in I} |f(x+h) - f(x)|.$$

We observe that for all $f \in C(I)$, we have

$$\lim_{\delta \to 0^+} \omega(f, \delta) = 0,$$

and for any $\delta > 0$,
On the closed interval $[0, b]$, the usual modulus of continuity of f is defined by

$$\omega_b(f, \delta) = \sup_{0<|u-x|\leq\delta} \ \sup_{x,u \in [0,b]} |f(u) - f(x)|, \quad \delta > 0.$$

Definition 3. **Lipschitz continuity:**
The definition of Lipschitz continuity is due to German mathematician Rudolph Lipschitz who used his concept of continuity to prove existence of solutions to some important differential equations. Quantifying continuous behavior in terms of Lipschitz continuity simplifies many aspects of mathematical analysis and the use of Lipschitz continuity has become ubiquitous in engineering and applied mathematics.
A function f is said to be Lipschitz continuous on an interval I if there is a constant $M > 0$ such that

$$|f(x_1) - f(x_2)| \leq M\,|x_1 - x_2| \qquad \forall \ \ x_1,\ x_2 \in I.$$

Definition 4. **Lipschitz type space:**
The Lipschitz type space [130] was considered by Otto Sz*á*sz to establish the uniform convergence of the Sz*á*sz operators for functions in this space.
For $0 < \xi \leq 1,\ x \in (0, \infty),\ u \in [0, \infty)$ we define

$$Lip_M^*(\xi) := \left\{ f \in C[0,\infty) : |f(u) - f(x)| \leq M_f \frac{|u-x|^\xi}{(u+x)^{\frac{\xi}{2}}};\ \text{where } M_f \text{ is a constant which depends on } f \right\}.$$

Definition 5. **Second order moduli of smoothness:**

To measure the degree of approximation of positive linear operators, the second order modulus of smoothness is used. For $f \in C(I)$ and $\delta > 0$, we have

$$\omega_2(f;\delta) = \sup\left\{|f(x+h) - 2f(x) + f(x-h)| : x, x \pm h \in I,\ 0 < h \leq \delta\right\},$$

or we can write it as

$$\omega_2(f;\delta) = \sup_{0<h\leq\delta}\ \sup_{x\in I} |f(x+h) - 2f(x) + f(x-h)|.$$

Let $C^2(I) := \left\{f \in C(I) : f_{xx}, f_{xy}, f_{yx}, f_{yy} \in C(I)\right\}$.

The norm on the space $C^2(I)$ is defined as

$$||f||_{C^{(2)}(I)} = ||f|| + \sum_{i=1}^{2}\left(\left\|\frac{\partial^i f}{\partial x^i}\right\| + \left\|\frac{\partial^i f}{\partial y^i}\right\|\right).$$

Definition 6. **Peetre's K-functional**: For $f \in C(I)$, let us consider the following K-functional:

$$K_2(f,\delta) = inf\{||f-g|| + \delta||g||_{C^2(I)} : g \in C^2(I)\},$$

where $\delta > 0$.

By [61], there exists an absolute constant $C > 0$ such that

$$K_2(f,\delta) \leq C\omega_2(f,\sqrt{\delta}).$$

Definition 7. **Ditzian-Totik modulus of smoothness**: Guo et al. [74] studied the direct, inverse and equivalence approximation theorems by means of the unified modulus. The definitions of the Ditzian-Totik modulus of smoothness and the Peetre's K-functional are given as: Let $\phi^2(x) = x(1+x)$ and $f \in C_B[0,\infty)$, the space of all bounded and continuous functions on $[0,\infty)$. The moduli $\omega_{\phi^\tau}(f,u),\ 0 \leq \tau \leq 1$, is defined as

$$\omega_{\phi^\tau}(f,u) = \sup_{0\leq h\leq u}\ \sup_{x\pm\frac{h\phi^\tau(x)}{2}\in[0,\infty)}\left|f\left(x + \frac{h\phi^\tau(x)}{2}\right) - f\left(x - \frac{h\phi^\tau(x)}{2}\right)\right|,$$

and the appropriate K-functional is given by

$$K_{\phi^\tau}(f,u) = \inf_{g\in W_\tau}\left\{||f-g|| + u||\phi^\tau g'||\right\},$$

where $W_\tau = \{g : g \in AC_{loc}[0,\infty), \ ||\phi^\tau g'|| < \infty\}$, AC_{loc} denotes the space of locally absolutely continuous functions on $[0,\infty)$.

From [63] there exists a constant $M > 0$ such that

$$M^{-1}\omega_{\phi^\tau}(f,u) \le K_{\phi^\tau}(f,u) \le M\omega_{\phi^\tau}(f,u). \tag{0.3.3}$$

0.4 Weighted Approximation

For $\gamma > 0$, let

$$C_\gamma[0,\infty) := \{f \in C[0,\infty) : |f(u)| \le M(1+u^\gamma), \ \text{for some} \ M > 0\}$$

endowed with the norm $||f||_\gamma = \sup_{u\in[0,\infty)} \dfrac{|f(u)|}{(1+u^\gamma)}$, then

$$C_2^0[0,\infty) := \left\{ f \in C_2[0,\infty); \lim_{x\to\infty} \frac{|f(x)|}{1+x^2} \ \text{exists and is finite} \right\}.$$

We study the approximation of functions in the subspace $C_2^0[0,\infty)$ of $C_2[0,\infty)$. Such type of function spaces have been considered by several researchers (cf. [26], [79]).
It is well known that the classical modulus of continuity of first order $\omega(f;\delta), \delta > 0$ does not tend to zero, as $\delta \to 0$, on an infinite interval. A weighted modulus of continuity $\Omega(f;\delta)$ was defined in [141] which tends to zero as $\delta \to 0$ on $[0,\infty)$. For $f \in C_2^0[0,\infty)$, the weighted modulus of continuity defined by Y$\ddot{u}$ksel and Ispir [141] is given as follows:

$$\Omega(f;\delta) = \sup_{x\in[0,\infty), 0<h\le\delta} \frac{|f(x+h)-f(x)|}{1+(x+h)^2}. \tag{0.4.1}$$

Some properties of $\Omega(f;\delta)$ are collected in the following lemma:

Lemma 0.4.1. *[141] Let $f \in C_2^0[0,\infty)$. Then the following results hold:*

1. *$\Omega(f;\delta)$ is monotonically increasing function of δ.*

2. $\lim_{\delta\to 0^+} \Omega(f;\delta) = 0$.

3. *For each $m \in \mathbb{N}$, $\Omega(f;m\delta) \le m\Omega(f;\delta)$.*

4. *For each $\lambda \in (0,\infty), \Omega(f;\lambda\delta) \le (1+\lambda)\Omega(f;\delta)$.*

0.4.1 Definitions for bivariate case

Definition 8. **Modulus of continuity for functions of two variables**: Let I and J be the two compact intervals on the real line and a function $f : I \times J \to \mathbb{R}$, then the modulus of continuity $\omega(f;\delta_1,\delta_2) : [0,\infty) \times [0,\infty) \to \mathbb{R}$ is defined by

$$\omega(f;\delta_1,\delta_2) = \sup\left\{|f(x_1,y_1)-f(x_2,y_2)| : \quad \text{whenever} \quad |x_1-y_1| \leq \delta_1, |x_2-y_2| \leq \delta_2\right\},$$

where $(x_1,y_1),(x_2,y_2) \in I \times J.$

Definition 9. **Complete modulus of continuity for bivariate case**: For $f \in C(I)$, the complete modulus of continuity for the bivariate case is defined as follows:

$$\bar{\omega}(f;\delta_1,\delta_2) = \sup\left\{|f(u,v) - f(x,y)| : (u,v),(x,y) \in I \text{ and } |u-x| \leq \delta_1, \;\; |v-y| \leq \delta_2\right\},$$

where $\bar{\omega}(f,\delta_1,\delta_2)$ satisfies the following properties:

1. $\bar{\omega}(f,\delta_1,\delta_2) \to 0$, if $\delta_1 \to 0$ and $\delta_2 \to 0$;
2. $|f(u,v) - f(x,y)| \leq \bar{\omega}(f,\delta_1,\delta_2)\left(1+\dfrac{|u-x|}{\delta_1}\right)\left(1+\dfrac{|v-y|}{\delta_2}\right).$

For more details we refer the reader to [24].

Definition 10. **Partial modulus of continuity for bivariate case**: Further, the partial moduli of continuity with respect to x and y is given by

$$\omega_1(f;\delta) = \sup\left\{|f(x_1,y) - f(x_2,y)| : y \in I \;\; and \;\; |x_1-x_2| \leq \delta\right\},$$

and

$$\omega_2(f;\delta) = \sup\left\{|f(x,y_1) - f(x,y_2)| : x \in I \;\; and \;\; |y_1-y_2| \leq \delta\right\}.$$

It is clear that they satisfy the properties of the usual modulus of continuity.

Definition 11. **Lipschitz class**: For $0 < \xi \leq 1$ and $0 < \gamma \leq 1$, for $f \in C(I)$ we define the Lipschitz class $Lip_M(\xi,\gamma)$ for the bivariate case as follows:

$$|f(u,v) - f(x,y)| \leq M|u-x|^{\xi}|v-y|^{\gamma}.$$

Definition 12. **Peetre's K-functional and second order complete modulus of continuity**

The Peetre's K-functional of the function $f \in C(I \times J)$ is defined by

$$\mathcal{K}(f;\delta) = \inf_{g \in C^2(I \times J)}\{||f-g||_{C(I\times J)} + \delta||g||_{C(I\times J)}\}, \delta > 0.$$

where $C^2(I \times J) = \{f \in C(I \times J) : f''_{xx}, f''_{x,y}, f''_{yx}, f''_{yy} \in C(I \times J)\}$ endowed with the norm $||f||_{C(I\times J)} = \sup_{(x,y)\in I\times J} |f(x,y)|$.
Also, from ([48], pp.192) it is known that

$$\mathcal{K}(f;\delta) \leq M\left\{\bar{\omega}_2(f;\sqrt{\delta}) + min(1,\delta)||f||_{C(I\times J)}\right\}, \tag{0.4.2}$$

holds for all $\delta > 0$. The constant M in the above inequality is independent of δ and f and $\bar{\omega}_2(f;\sqrt{\delta})$ is the second order modulus of continuity which is defined as

$$\bar{\omega}_2(f;\sqrt{\delta}) = \sup_{|h|\leq\delta,|k|\leq\delta} \left\{\left|\sum_{\nu=0}^{2}(-1)^{2-\nu} f(x+\nu h, y+\nu k)\right| : (x,y),(x+2h,y+2k) \in I \times J,\right\}.$$

0.4.2 Definitions of q-calculus

1. **The q-integer and q-factorial**: For any fixed real number $q > 0$ satisfying the condition $0 < q < 1$, the q-integer $[n]_q$, for $n \in \mathbb{N}$ and q-factorial $[n]_q!$ are defined as

$$[n]_q = \begin{cases} \dfrac{(1-q^n)}{(1-q)}, & \text{if } q \neq 1 \\ n, & \text{if } q = 1, \end{cases}$$

and

$$[n]_q! = \begin{cases} [n]_q[n-1]_q \ldots 1, & \text{if } n \geq 1 \\ 1, & \text{if } n = 0, \end{cases}$$

respectively.

2. **q-binomial coefficient**: For any integers n, k satisfying $0 \leq k \leq n$, the q-binomial coefficient is given by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[n-k]_q![k]_q!}.$$

3. **The Gauss binomial or q-binomial formula**: The Gauss binomial or q-binomial formula is defined as:

$$(x+a)_q^n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{k(k-1)/2} a^k x^{n-k}.$$

4. q**-derivative of a function**: The q-derivative of a function f is given by

$$(D_q f)(x) = \frac{f(x) - f(qx)}{(1-q)x}, \quad x \neq 0,$$

and $(D_q f)(0) = f'(0)$ provided $f'(0)$ exists. For $n \in \mathbb{N}$, we have

$$D_q(1+x)_q^n = [n]_q(1+qx)_q^{n-1} \quad \text{and} \quad D_q(1+x)_q^{-n} = -[n]_q\,(1+x)_q^{-(n+1)}.$$

5. q**-Riemann integral**: The Riemann type q-integral is defined as

$$\int_c^d \int_a^b f(u,v) d_{q_1}^R u \, d_{q_2}^R v = (1-q_1)(1-q_2)(b-a)(c-d) \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} f\left(a+(b-a){q_1}^i, c+(c-d){q_2}^j\right) {q_1}^i {q_2}^j.$$

provided the series on the right hand side converge.

6. q**-Jackson Integral** Let $0 < a < b$, $0 < q < 1$ and f be real-valued function. The q-Jackson integral of f over the intervals $[0,b]$ and $[a,b]$ are defined by

$$\int_0^b f(u)\, d_q u = (1-q)b \sum_{j=0}^{\infty} f(q^j b) q^j$$

and

$$\int_a^b f(u)\, d_q u = \int_0^b f(u)\, d_q u - \int_0^a f(u)\, d_q u.$$

For further details one can refer to [90].

0.5 Generalized boolean sum operators

In [45] and [46], Bögel introduced the concepts of B-continuity and B-differentiability. Dobrescu and Matei [64] established an approximation theorem involving these kind of functions in which they showed that the bivariate generalization of Bernstein polynomials can be uniformly approximated by the associated GBS (generalized boolean sum) operators using the definitions of B-continuity and B-differentiability. Badea et al. [35] established a Korovkin type theorem (known as Test Function Theorem) on the approximation of B-continuous functions and obtained some sequences of uniformly approximating pseudo polynomials. Badea and Cottin [37] obtained Korovkin type theorems for generalized boolean sum operators. Bărbosu et al. [39] introduced the GBS operators of Durrmeyer-Stancu type based on q integers and obtained the rate of approximation with the aid of the Lipschitz class of B-continuous functions and the mixed modulus of smoothness. Agrawal et al. [20] defined GBS operators of Lupas-Durrmeyer type based on Pólya distribution and discussed the degree of approximation by means of the mixed modulus of smoothnes. Ispir [87] constructed the GBS operators associated with a combination of Chlodowsky and modified Szász operators and studied the rate of convergence for the Bögel continuous and Bögel differentiable functions. For some important contributions in this direction we refer to [cf. [52], [41], [42], [112], [67], [68], [66], [111], [113], [40], [60] and etc.]. Now we give some basic definitions and notions regarding B$\ddot{o}$gel space:

0.5.1 Basic definitions related to B$\ddot{o}$gel space

Definition 13. **B$\ddot{o}$gel continuity and Mixed difference**
Let I, J be compact sub intervals of real axis then the function f on $I \times J$ is called a B-continuous (Bögel continuous) function if for every $(x_0, y_0) \in I \times J$ we have

$$\lim_{(x,y)\to(x_0,y_0)} \Delta f[(x_0, y_0); (x, y)] = 0,$$

where $\Delta f[(x_0, y_0); (x, y)]$ denotes the mixed difference defined by

$$\Delta f[(x_0, y_0); (x, y)] = f(x, y) - f(x, y_0) - f(x_0, y) + f(x_0, y_0).$$

Definition 14. **B$\ddot{o}$gel boundedness**
The function $f : I \times J \longrightarrow \mathbb{R}$ is said to be B-bounded on $I \times J$ iff there exists $M > 0$ such that

$$\|\Delta f[(x_0, y_0); (x, y)]\| \leq M$$

for any $(x_0, y_0); (x, y) \in I \times J$. Let $B_b(I \times J)$, denote the space of all B-bounded functions on $I \times J \to \mathbb{R}$, equipped with the norm

$$\|f\|_B = \sup_{(x,y),(x_0,y_0)\in I\times J} |\Delta f[(x_0, y_0); (x, y)]| .$$

We denote by $C_b(I \times J)$, the space of all B-continuous functions on $I \times J$. $B(I \times J), C(I \times J)$ denote the space of all bounded functions and the space of all continuous (in the usual sense) functions on $I \times J$ endowed with the sup-norm $\|.\|_\infty$ respectively. It is known that $C(I \times J) \subset C_b(I \times J)$ ([35], page 52).

Definition 15. **Bögel differentiability**
A function $f : I \times J \longrightarrow \mathbb{R}$ is called a B-differentiable (Bögel differentiable) function at $(x_0, y_0) \in I \times J$ if the limit

$$\lim_{(x,y)\to(x_0,y_0)} \frac{\Delta f[(x_0, y_0); (x, y)]}{(x - x_0)(y - y_0)}$$

exists and is finite.
The limit is said to be the B-differential of f at the point (x_0, y_0) and is denoted by $D_B(f; x_0, y_0)$ and the space of all B-differentiable functions is denoted by $D_b(I \times J)$.

Definition 16. **Lipschitz class for Bögel space**: For $f \in C_b(I \times J)$, the Lipschitz class $Lip_M(\xi, \gamma)$ with $\xi, \gamma \in (0, 1]$ is defined as

$$Lip_M(\xi, \gamma) = \Big\{ f \in C_b(I \times J) : |\Delta f[(x_0, y_0); (x, y)]| \leq M |x_0 - x|^\xi |y_0 - y|^\gamma, \text{ for } (x_0, y_0), (x, y) \in I \times J \Big\}.$$

Definition 17. **Mixed modulus of smoothness**
The mixed modulus of smoothness of $f \in C_b(I \times J)$ is defined as

$$\omega_{mixed}(f; \delta_1, \delta_2) := \sup \Big\{ |\Delta f[(x_0, y_0); (x, y)]| : |x - x_0| < \delta_1, |y - y_0| < \delta_2 \Big\},$$

for all $(x, y), (x_0, y_0) \in I \times J$ and for any $(\delta_1, \delta_2) \in (0, \infty) \times (0, \infty)$ with $\omega_{mixed} : [0, \infty) \times [0, \infty) \to \mathbb{R}$.

The basic properties of ω_{mixed} were obtained by Badea et al. in [36] and [34] which are similar to the properties of the usual modulus of continuity.

Let $\mathbb{R}^{I\times J} = \{f : I \times J \longrightarrow \mathbb{R}\}$ and $L : \mathbb{R}^{I\times J} \longrightarrow \mathbb{R}^{I\times J}$ be a bivariate linear positive operators. If $f \in \mathbb{R}^{I\times J}$ and $(x, y) \in I \times J$ then the GBS (generalized boolean sum)operator associated to L is defined as

$$U(f; x, y) = L\left((f(*, y) + f(x, \circ) - f(*, \circ)); x, y\right).$$

Badea and Badea [34] gave a Korovkin-type theorem for B-continuous functions which is given as

Theorem 0.1. Let $(L_{m,n})$ be the sequence of positive linear operators such that $L_{m,n} : R^{I\times J} \longrightarrow R^{I\times J}$, for $e_{i,j}(u,v) = u^i\, v^j$ (i, j non-negative integers such that $0 \leq i+j \leq 2$), we have

(i) $L_{m,n}(e_{0,0}; x, y) = L(1; x, y) = 1,$

(ii) $L_{m,n}(e_{1,0}; x, y) = L(u; x, y) = x + u_{m,n}(x, y),$

(iii) $L_{m,n}(e_{0,1}; x, y) = L(v; x, y) = y + v_{m,n}(x, y),$

(iv) $L_{m,n}(e_{0,2} + e_{2,0}; x, y) = L(u^2 + v^2; x, y) = y^2 + x^2 + w_{m,n}(x, y),$

(v) $\lim_{m,n\to\infty} u_{m,n}(x, y) = \lim_{m,n\to\infty} v_{m,n}(x, y) = \lim_{m,n\to\infty} w_{m,n}(x, y) = 0$ uniformly on $I \times J$

then the sequence $\{U_{m,n}f\}$ converges to f uniformly on $I \times J$ for any $f \in C_b(I \times J)$.

Later, Badea et al. proved Shisha Mond theorem for B-continuous functions using GBS operators which is given as follows:

Theorem 0.2. Let $L(f; x, y) : C_b(I \times J) \longrightarrow C_b(I \times J)$ be the bivariate positive linear operators and $U(f; x, y)$ be the associated GBS operator then for $f \in C_b(I \times J)$ and $(x, y) \in I \times J$, we have

$$\begin{aligned} |f(x,y) - U(f;x,y)| \leq\ & |f(x,y)|\,|1 - L(1;x,y)| + \Big\{L(1;x,y) \\ & + \frac{1}{\delta_1}\sqrt{L\left((u-x)^2;x,y\right)} + \frac{1}{\delta_2}\sqrt{L\left((v-y)^2;x,y\right)} \\ & + \frac{1}{\delta_1\delta_2}\sqrt{L\left((u-x)^2;x,y\right)L\left((v-y)^2;x,y\right)}\Big\}\,\omega_{mixed}(f;\delta_1,\delta_2), \end{aligned}$$

where $\delta_1, \delta_2 \geq 0$.

0.6 List of publications

- **M. Sidharth**, N. Ispir and P. N. Agrawal. GBS operators of Bernstein-Schurer-Kantorovich type based on q-integers, ***Appl. Math. Comput.*** 269 (2015) 558-568.

- **M. Sidharth**, N. Ispir and P. N. Agrawal. Approximation of B-continuous and B-Differentiable functions by GBS operators of q-Bernstein-Schurer-Stancu type, ***Turk. J. Math.***, 40 (2016) 1298-1315.

- **M. Sidharth**, N. Ispir and P. N. Agrawal, Blending type Approximation by q-Generalized Boolean Sum of Durrmeyer type, ***Math. Meth. Appl. Sci.*** 40 (11) (2017) 3901-3911.

- **M. Sidharth**, A. M. Acu and P. N. Agrawal, Chlodowsky-Szász- Appell type operators for functions of two variables, ***Ann. Funct. Anal.,*** DOI: 10.1215/20088752-2017-0009, (2017).

- P. N. Agrawal, A. M. Acu and **M. Sidharth**, Approximation degree of a Kantorovich variant of Stancu operators based on Pólya-Eggenberger Distribution, communicated in ***RACSAM*** *(Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matemáticas)*, (2017).

- P. N. Agrawal, N. Ispir and **M. Sidharth**, Quantitative Estimates of Generalized Boolean Sum operators of Blending type, Accepted in ***Numer. Funct. Anal. Optim.*** , (2017).

- **M. Sidharth**, P. N. Agrawal and Sercan Araci, Szász-Durrmeyer operators involving Boas-Buck polynomials of blending type, ***J. Inequal. Appl.,*** 122: (2017).

- **M. Sidharth** and P. N. Agrawal, Rate of Convergence of Modified Schurer- type q-Bernstein Kantorovich operators, **Proceeding of International Mathematical Analysis and Its Applications**, (2015) 243-253 .

0.7 Contents of the Thesis

Chapter 1. In the first chapter, we discuss the Schurer type q-Bernstein Kantorovich operators introduced by Lin [99] and obtain a local approximation theorem and the statistical convergence of these operators. In this chapter we also study the rate of convergence by means of the first order modulus of continuity, Lipschitz class function, the modulus of continuity of the first order derivative and the Voronovskaja type theorem.

The results in this chapter are published in **Proceedings of Mathematical Analysis and its Applications (Springer publications)**.

Chapter 2. The second chapter is concerned with the Stancu-Kantorovich operators based on Pólya-Eggenberger distribution. We obtain some direct results for these operators by means of the Lipschitz class function, the modulus of continuity and the weighted space. Also, we study an approximation theorem with the aid of the unified Ditzian-Totik modulus of smoothness $\omega_{\phi^\tau}(f;t),\ \ 0 \leq \tau \leq 1$ and the rate of convergence of the operators for functions having a derivative of bounded variation on every finite subinterval of $[0, \infty)$.

The results in this chapter are communicated in **RACSAM (Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matemáticas) (Springer Publications)**.

Chapter 3. In this chapter, we introduce the Sz$\acute{a}$sz-Durrmeyer type operators based on Boas-Buck type polynomials which include Brenke-type polynomials, Sheffer polynomials and Appell polynomials. We establish the moments of the operator and a Voronvskaja type asymptotic theorem and then proceed to study the convergence of the operators with the help of Lipschitz type space and weighted modulus of continuity. Next, we obtain a direct approximation theorem with the aid of unified Ditzian-Totik modulus of smoothness. Furthermore, we study the approximation of functions whose derivatives are locally of bounded variation.

The results in this chapter are published in **Journal of Inequalities and Applications (Springer Publications)**.

Chapter 4. In the fourth chapter, we obtain the rate of approximation of the bivariate Bernstein-Schurer-Stancu type operators based on q-integers by means of the moduli of continuity and Lipschitz class. We also estimate the degree of approximation by means of Lipschitz class function and the rate of convergence with the help of mixed modulus of smoothness for the GBS operator of q-Bernstein-Schurer-Stancu type. Furthermore, we show the comparisons by some illustrative graphics in Matlab for the convergence of the operators to some functions.

The results in this chapter are published in **Turkish Journal of Mathematics (Scientific and Technological Research Council of Turkey (TÜBITAK) publications)**.

Chapter 5. In this chapter, we study the approximation properties of the bivariate extension of q-Bernstein-Schurer-Durrmeyer operators and obtain the rate of convergence of the operators with the aid of the Lipschitz class function and the modulus of continuity. Here, we estimate the rate of convergence of these operators by means of Peetre's K-functional. Then, the associated GBS operator of the q-Bernstein-Schurer-Durrmeyer type is defined and discussed. The smoothness properties of these operators are improved with the help of mixed K-functional. Furthermore, we show the convergence of the bivariate Durrmeyer type operators and the associated GBS operators to certain functions by illustrative graphics using Matlab algorithm.

The results in this chapter are published in **Mathematical Methods in the Applied Sciences (Wiley Online Library Publications)**.

Chapter 6. In sixth chapter, we study the mixed summation integral type two dimensional q-Lupaş-Phillips-Bernstein operators introduced by Sharma in 2015. We establish a Voronovskaja type theorem and introduce the associated GBS case of these operators and study its properties. Furthermore, we illustrate the rate of convergence of the operators introduced by Sharma and the corresponding GBS operators by numerical examples.

The results in this chapter are accepted in **Numerical Functional Analysis and Optimization (Taylor and Francis Publications)**.

Chapter 7. In the seventh chapter, we deal with the approximation properties of the Kantorovich-type q-Bernstein-Schurer operators by means of the partial moduli of continuity and the Peetre's K-functional. Finally, we construct the GBS operators of bivariate q-Bernstein-Schurer-Kantorovich type and estimate the rate of convergence for these operators with the help of mixed modulus of smoothness.

The results in this chapter are published **Applied Mathematics and Computation (Elsevier Publications)**.

Chapter 8. In this chapter, we establish the approximation properties of the bivariate operators which are the combination of Bernstein-Chlodowsky operators and the Szász operators involving Appell polynomials. We investigate the degree of approximation of the operators with the help of complete modulus of continuity and the partial moduli of continuity. In the last section of the chapter, we introduce the GBS case of the bivariate operators and study the order of approximation in the Bögel space of continuous functions.

The results in this chapter are published **Annals of Functional Analysis (Duke University Press Publications)**.

Chapter 1

Rate of convergence of modified Schurer type q-Bernstein Kantorovich operators

1.1 Introduction

Muraru [102] introduced Bernstein-Schurer polynomials based on q-integers and established the rate of convergence in terms of modulus of the continuity. Agrawal et. al [18] considered the Stancu variant of these operators and discussed some local and global direct results. Later, Agrawal et. al [21] proposed Durrmeyer type modification of these operators and discussed some local direct results and studied the rate of convergence of modified limit q-Bernstein-Schurer type operators.

Recently, Lin [99] introduced a new kind of modified Schurer type q-Bernstein-Kantorovich operators as follows:

Let $p \in \mathbb{N}^0$(the set of non negative integers) be arbitrary but fixed and α, β be integers satisfying $0 \leq \alpha \leq \beta$. For $f \in C[0, 1+p]$

$$K_{n,q}^{(\alpha,\beta)}(f;x) = \sum_{k=0}^{n+p} \bar{p}_{n,k}(q;x) \int_0^1 f\left(\frac{u}{[n+1+\beta]_q} + \frac{q[k+\alpha]_q}{[n+1+\beta]_q}\right) d_q u, x \in [0,1], \tag{1.1.1}$$

where $\bar{p}_{n,k}(q;x) = \begin{bmatrix} n+p \\ k \end{bmatrix}_q x^k \prod_{s=0}^{n+p+k-1}(1-q^s x)$. It is clear that $K_{n,q}^{(\alpha,\beta)}(f;x)$ is a linear positive operator. It is remarked that when $\alpha = \beta = 0$, it reduces to the operator discussed in [134].

In the present chapter, we continue the work done by Lin by discussing the rate of convergence in terms of the modulus of continuity, elements of Lipschitz-type space and Voronovskaja type theorem.

1.2 Preliminaries

In this section, we give some basic results which will be used in the sequel.

Lemma 1.2.1. *[99]. For $K_{n,q}^{(\alpha,\beta)}(u^m;x),\ m=0,1,2,$ we have*

(i) $K_{n,q}^{(\alpha,\beta)}(1;x) = 1,$

(ii) $K_{n,q}^{(\alpha,\beta)}(u;x) = \dfrac{[n+p]_q}{[n+1+\beta]_q} q^{\alpha+1}x + \dfrac{1}{[n+1+\beta]_q}\left(\dfrac{1}{[2]_q} + q[\alpha]_q\right),$

(iii) $K_{n,q}^{(\alpha,\beta)}(u^2;x) = \dfrac{[n+p]_q[n+p-1]_q}{[n+1+\beta]_q^2} q^{2\alpha+3}x^2 + \dfrac{[n+p]_q}{[n+1+\beta]_q^2}\Bigg(\dfrac{2}{[2]_q}q^{\alpha+1} + q^{2+\alpha}(2[\alpha]_q + q^{\alpha})\Bigg)x + \dfrac{1}{[n+1+\beta]_q^2}\left(\dfrac{1}{[3]_q} + \dfrac{2q[\alpha]_q}{[2]_q} + q^2[\alpha]_q^2\right).$

Remark 1.2.1. *For the modified Schurer type q- Bernstein-Kantorovich operators, we have*

(i) $\lim\limits_{n\to\infty}[n]_{q_n}(K_{n,q_n}^{(\alpha,\beta)}((u-x);x) = \left[\dfrac{1+2\alpha}{2} - (\alpha+1)x\right],$

(ii) $\lim\limits_{n\to\infty}[n]_{q_n}(K_{n,q_n}^{(\alpha,\beta)}((u-x)^2;x) = x(1-x).$

1.3 Main results

1.3.1 Rate of convergence

In our next theorem, we will find the rate of convergence of the operator (1.1.1) using first order modulus of continuity.

From the definition of the first order modulus of continuity, we have

$$|f(x_1)-f(x_2)|\leq \omega(f,\delta)\bigg(\frac{|x_1-x_2|}{\delta}+1\bigg). \tag{1.3.1}$$

we will use this inequality in our result.

Theorem 1.3.1. *For $f\in C[0,1+p]$, we have*

$$|K_{n,q}^{(\alpha,\beta)}(f;x)-f(x)|\leq 2\omega\bigg(f;\sqrt{\delta_{n,q}^{(\alpha,\beta)}}\bigg),$$

where $\omega(f,.)$ is the modulus of continuity of f and $\delta_{n,q}^{(\alpha,\beta)}:=K_{n,q}^{(\alpha,\beta)}((u-x)^2;x)$.

Proof. Using the linearity and positivity of the operator, in view of (1.3.1) we get

$$\begin{aligned}
&|K_{n,q}^{(\alpha,\beta)}(f;x)-f(x)|\\
&= \bigg|\sum_{k=0}^{n+p}\bar{p}_{n,k}(q;x)\int_0^1\bigg(f\bigg(\frac{u}{[n+1+\beta]_q}+\frac{q[k+\alpha]}{[n+1+\beta]_q}\bigg)-f(x)\bigg)d_qu\bigg|\\
&\leq \sum_{k=0}^{n+p}\bar{p}_{n,k}(q;x)\int_0^1\bigg|f\bigg(\frac{u}{[n+1+\beta]_q}+\frac{q[k+\alpha]_q}{[n+1+\beta]_q}\bigg)-f(x)\bigg|d_qu\\
&\leq \sum_{k=0}^{n+p}\bar{p}_{n,k}(q;x)\int_0^1\left(\frac{\left|\frac{u}{[n+1+\beta]_q}+\frac{q[k+\alpha]_q}{[n+1+\beta]_q}-x\right|}{\delta}+1\right)\omega(f,\delta)\,d_qu\\
&\leq \omega(f,\delta)\sum_{k=0}^{n+p}\bar{p}_{n,k}(q;x)+\frac{\omega(f,\delta)}{\delta}\bigg(\sum_{k=0}^{n+p}\bar{p}_{n,k}(q;x)\int_0^1\bigg|\frac{u}{[n+1+\beta]_q}\\
&\quad+\frac{q[k+\alpha]_q}{[n+1+\beta]_q}-x\bigg|d_qu\bigg).
\end{aligned}$$

On applying the Cauchy-Schwarz inequality, we have

$$\begin{aligned}
&\int_0^1\bigg|\frac{u}{[n+1+\beta]_q}+\frac{q[k+\alpha]_q}{[n+1+\beta]_q}-x\bigg|d_qu\\
&\qquad\leq \bigg\{\int_0^1\bigg(\frac{u}{[n+1+\beta]_q}+\frac{q[k+\alpha]_q}{[n+1+\beta]_q}-x\bigg)^2d_qu\bigg\}^{1/2}\\
&\qquad= \sqrt{a_{n,k}^{(\alpha,\beta)}(x)}.
\end{aligned}$$

Hence,

$$|K_{n,q}^{(\alpha,\beta)}(f;x)-f(x)| \leq \omega(f,\delta)+\frac{\omega(f,\delta)}{\delta}\sum_{k=0}^{n+p}\{a_{n,k}^{(\alpha,\beta)}\}^{1/2}\bar{p}_{n,k}(q;x).$$

Again applying the Cauchy-Schwarz inequality, we get
$|K_{n,q}^{(\alpha,\beta)}(f;x)-f(x)|$

$$\begin{aligned}
&\leq \omega(f,\delta)+\frac{\omega(f,\delta)}{\delta}\left\{\sum_{k=0}^{n+p}a_{n,k}^{(\alpha,\beta)}\bar{p}_{n,k}(q;x)\right\}^{1/2}\left\{\sum_{k=0}^{n+p}\bar{p}_{n,k}(q;x)\right\}^{1/2}\\
&= \omega(f,\delta)+\frac{\omega(f,\delta)}{\delta}\left\{\sum_{k=0}^{n+p}\bar{p}_{n,k}(q;x)\int_0^1\left(\frac{u}{[n+1+\beta]_q}+\frac{q[k+\alpha]_q}{[n+1+\beta]_q}-x\right)^2 d_qu\right\}^{1/2}\\
&= \omega(f,\delta)+\frac{\omega(f,\delta)}{\delta}\left\{K_{n,q}^{(\alpha,\beta)}((u-x)^2;x)\right\}^{1/2}.
\end{aligned}$$

Choosing $\delta := \delta_{n,q}^{(\alpha,\beta)}(x) = K_{n,q}^{(\alpha,\beta)}((u-x)^2;x)$, we have

$$|K_{n,q}^{(\alpha,\beta)}(f;x)-f(x)| \ \leq 2\omega\left(f,\sqrt{\delta_{n,q}^{(\alpha,\beta)}(x)}\right).$$

Hence, we get the desired result. □

Corollary 1.3.2. *Let $f\in Lip_M^*(\xi)$ for $0<\xi\leq 1$,then*

$$|K_{n,q}^{(\alpha,\beta)}(f;x)-f(x)|\leq 2M\left(\delta_{n,q}^{(\alpha,\beta)}(x)\right)^{\xi/2},$$

where $\delta_{n,q}^{(\alpha,\beta)}(x)=K_{n,q}^{(\alpha,\beta)}((u-x)^2;x)$.

Proof. Since $f\in Lip_M^*(\xi)$, we have $\omega(f,\delta)\leq M\delta^{\xi}$, for any $\delta>0$.
Hence the result follows from Theorem 1. □

Theorem 1.3.3. *If $f(x)$ has a continuous derivative $f'(x)$ and $\omega(f',\delta)$ is the modulus of continuity of $f'(x)$ on $[0,1+p]$, then*

$$|K_{n,q}^{(\alpha,\beta)}(f;x)-f(x)|\leq M\,|\mu_{n,q,p}^{(\alpha,\beta)}|+\omega(f',\delta)\left(1+\sqrt{\delta_{n,q}^{(\alpha,\beta)}(x)}\right),$$

where M is a positive constant such that $|f'(x)|\leq\ M$ and

$$\mu_{n,q,p}^{(\alpha,\beta)}(x)=\left(\frac{q^{\alpha+1}[n+p]_q}{[n+1+\beta]_q}-x\right)x+\frac{1}{[n+1+\beta]_q}\left(\frac{1}{[2]_q}+q[\alpha]_q\right). \tag{1.3.2}$$

Proof. On applying the mean value theorem, we get

$$\begin{aligned} f\left(\frac{u}{[n+1+\beta]_q}+\frac{q[k+\alpha]_q}{[n+1+\beta]_q}\right)-f(x) &= \left(\frac{u}{[n+1+\beta]_q}+\frac{q[k+\alpha]_q}{[n+1+\beta]_q}-x\right)f'(\xi)\\ &= \left(\frac{u}{[n+1+\beta]_q}+\frac{q[k+\alpha]_q}{[n+1+\beta]_q}-x\right)f'(x)\\ &\quad +\left(\frac{u}{[n+1+\beta]_q}+\frac{q[k+\alpha]_q}{[n+1+\beta]_q}-x\right)\\ &\quad (f'(\xi)-f'(x)) \end{aligned}$$

where, ξ lies between $\left(\frac{u}{[n+1+\beta]_q}+\frac{q[k+\alpha]_q}{[n+1+\beta]_q}\right)$ and x.

Hence, we get $|K_{n,q}^{(\alpha,\beta)}(f;x)-f(x)|$

$$\begin{aligned} &= \Bigg|f'(x)\sum_{k=0}^{n+p}\int_0^1\left(\frac{u}{[n+1+\beta]_q}+\frac{q[k+\alpha]_q}{[n+1+\beta]_q}-x\right)\begin{bmatrix}n\\k\end{bmatrix}_q x^k\prod_{s=0}^{n+p+k-1}(1-q^sx)d_qu\\ &\quad +\sum_{k=0}^{n+p}\int_0^1\left(\frac{u}{[n+1+\beta]_q}+\frac{q[k+\alpha]_q}{[n+1+\beta]_q}-x\right)(f'(\xi)-f'(x))\begin{bmatrix}n\\k\end{bmatrix}_q x^k\\ &\quad \prod_{s=0}^{n+p+k-1}(1-q^sx)d_qu\Bigg|\\ &\le |f'(x)||K_{n,q}^{(\alpha,\beta)}((u-x);x)|+\sum_{k=0}^{n+p}\int_0^1\left|\frac{u}{[n+1+\beta]_q}+\frac{q[k+\alpha]_q}{[n+1+\beta]_q}-x\right|\\ &\quad \times|f'(\xi)-f'(x)|\begin{bmatrix}n\\k\end{bmatrix}_q x^k\prod_{s=0}^{n+p+k-1}(1-q^sx)d_qu\\ &\le M|\mu_{n,q,p}^{(\alpha,\beta)}|+\sum_{k=o}^{n+p}\int_0^1\omega(f',\delta)\left(\frac{\left|\frac{u}{[n+1+\beta]_q}+\frac{q[k+\alpha]_q}{[n+1+\beta]_q}-x\right|}{\delta}+1\right)\\ &\quad \times\left|\frac{u}{[n+1+\beta]_q}+\frac{q[k+\alpha]_q}{[n+1+\beta]_q}-x\right|\begin{bmatrix}n\\k\end{bmatrix}_q x^k\prod_{s=0}^{n+p+k-1}(1-q^sx)d_qu\\ &\le M\,|\mu_{n,q,p}^{(\alpha,\beta)}|+\omega(f',\delta)\sum_{k=o}^{n+p}\int_0^1\left|\frac{u}{[n+1+\beta]_q}+\frac{q[k+\alpha]_q}{[n+1+\beta]_q}-x\right|\begin{bmatrix}n\\k\end{bmatrix}_q x^k\\ &\quad \prod_{s=0}^{n+p+k-1}(1-q^sx)d_qu+\frac{\omega(f',\delta)}{\delta}\sum_{k=0}^{n+p}\int_0^1\left(\frac{u}{[n+1+\beta]_q}+\frac{q[k+\alpha]_q}{[n+1+\beta]_q}-x\right)^2\\ &\quad \times\begin{bmatrix}n\\k\end{bmatrix}_q x^k\prod_{s=0}^{n+p+k-1}(1-q^sx)d_qu, \end{aligned}$$

where $\mu_{n,q,p}^{(\alpha,\beta)}$ is given in (1.3.2).

Now, applying Cauchy-Schwarz inequality in second term of the right side of the inequality and using Lemma 1.2.1, we have

$$\begin{aligned}
&|K_{n,q}^{(\alpha,\beta)}(f;x)-f(x)|\\
\leq\ & M\,|\mu_{n,q,p}^{(\alpha,\beta)}|+\omega(f',\delta)\Bigg(\sum_{k=0}^{n+p}\int_0^1\left(\frac{u}{[n+1+\beta]_q}+\frac{q[k+\alpha]_q}{[n+1+\beta]_q}-x\right)^2\\
&\times\begin{bmatrix} n\\ k\end{bmatrix}_q x^k\prod_{s=0}^{n+p+k-1}(1-q^sx)d_qu\Bigg)^{1/2}\\
+\ & \frac{\omega(f',\delta)}{\delta}\sum_{k=0}^{n+p}\int_0^1\left(\frac{u}{[n+1+\beta]_q}+\frac{q[k+\alpha]_q}{[n+1+\beta]_q}-x\right)^2\begin{bmatrix} n\\ k\end{bmatrix}_q x^k\\
&\times\prod_{s=0}^{n+p+k-1}(1-q^sx)d_qu,\\
\leq\ & M\,|\mu_{n,q,p}^{(\alpha,\beta)}|+\omega(f',\delta)\sqrt{K_{n,q}^{(\alpha,\beta)}((u-x)^2;x)}+\frac{\omega(f',\delta)}{\delta}K_{n,q}^{(\alpha,\beta)}((u-x)^2;x).
\end{aligned}$$

Choosing $\delta:=\delta_{n,q}^{(\alpha,\beta)}(x)=K_{n,q}^{(\alpha,\beta)}((u-x)^2;x)$,we have

$$|K_{n,q}^{(\alpha,\beta)}(f;x)-f(x)|\ \leq\ M|\mu_{n,q,p}^{(\alpha,\beta)}|+\omega(f',\delta_{n,q}^{(\alpha,\beta)}(x))(1+\sqrt{\delta_{n,q}^{(\alpha,\beta)}(x)}).$$

Hence, we get the desired result. □

1.3.2 Asymptotic Result

Theorem 1.3.4. *Let $f\in C[0,1+p]$, $0<q_n<1$ be a sequence such that $q_n\to 1$ and $\frac{1}{[n]_{q_n}}\to 0$, as $n\to\infty$. Suppose that $f''(x)$ exist at a point $x\in[0,1]$, then we have*

$$\lim_{n\to\infty}[n]_{q_n}(K_{n,q_n}^{(\alpha,\beta)}(f;x)-f(x))=\left(\frac{1+2\alpha}{2}-(\alpha+1)x\right)f'(x)+\frac{1}{2}(x-x^2)f''(x).$$

Proof. By Taylor's formula we have

$$f(u)=f(x)+(u-x)f'(x)+\frac{1}{2}f''(x)(u-x)^2+r(u-x)^2, \tag{1.3.3}$$

where $r(u,x)$ is the Peano form of the remainder and $\lim_{u\to x}r(u,x)=0$.

Applying $K_{n,q_n}^{(\alpha,\beta)}(.;x)$ on both sides of equation (1.3.3), we get

$$\begin{aligned} K_{n,q_n}^{(\alpha,\beta)}(f;x) - f(x) &= f'(x)K_{n,q_n}^{(\alpha,\beta)}((u-x);x) + \frac{1}{2}f''(x)K_{n,q_n}^{(\alpha,\beta)}((u-x)^2;x) \\ &+ K_{n,q_n}^{(\alpha,\beta)}((u-x)^2 r(u,x);x). \end{aligned}$$

Now taking the limit as $n \to \infty$, on both sides of the above equation, we get

$$\begin{aligned} &\lim_{n\to\infty}[n]_{q_n}(K_{n,q_n}^{(\alpha,\beta)}(f;x) - f(x)) \\ &= \lim_{n\to\infty}[n]_{q_n} f'(x)(K_{n,q_n}^{(\alpha,\beta)}((u-x);x) + \lim_{n\to\infty}[n]_{q_n}\frac{f''(x)}{2}(K_{n,q_n}^{(\alpha,\beta)}((u-x)^2;x) \\ &+ \lim_{n\to\infty}[n]_{q_n} K_{n,q_n}^{(\alpha,\beta)}((u-x)^2 r(u,x);x). \end{aligned} \tag{1.3.4}$$

From the Remark 1.2.1, we have

$$\lim_{n\to\infty}[n]_{q_n}(K_{n,q_n}^{(\alpha,\beta)}((u-x);x) = \left[\frac{1+2\alpha}{2} - (\alpha+1)x\right], \tag{1.3.5}$$

uniformly in [0,1], and

$$\lim_{n\to\infty}[n]_{q_n}(K_{n,q_n}^{(\alpha,\beta)}((u-x)^2;x) = x(1-x) \text{ , uniformly in } [0,1]. \tag{1.3.6}$$

Hence in order to prove the result, it is sufficient to show that

$$[n]_{q_n} K_{n,q_n}^{(\alpha,\beta)}((u-x)^2 r(u,x);x) \to 0, \text{ as } n \to \infty \text{ , uniformly in } [0,1].$$

By using the Cauchy-Schwarz inequality, we have

$$K_{n,q_n}^{(\alpha,\beta)}((u-x)^2 r(u,x);x) \leq \sqrt{K_{n,q_n}^{(\alpha,\beta)}(r^2(u,x);x)}\sqrt{K_{n,q_n}^{(\alpha,\beta)}((u-x)^4;x)}. \tag{1.3.7}$$

We observe that $r^2(x,x) = 0$, and from Theorem 1.3.1, we have

$$\lim_{n\to\infty} K_{n,q_n}^{(\alpha,\beta)}(r^2(u,x);x) = r^2(x,x) = 0. \tag{1.3.8}$$

Hence, from (1.3.7) and (1.3.8), we get

$$\lim_{n\to\infty}[n]_{q_n} K_{n,q_n}^{(\alpha,\beta)}((u-x)^2 r(u,x);x) = 0 \text{ , uniformly in } [0,1], \tag{1.3.9}$$

in view of the fact that

$$K_{n,q_n}^{(\alpha,\beta)}((u-x)^4;x) = O\left(\frac{1}{n^2}\right) \text{ , uniformly in } [0,1].$$

Now, combining (1.3.4)-(1.3.6) and (1.3.9), we get the required result.

This completes the proof of the theorem. □

We observe that due to the presence of x in the denominator on the right side we get only pointwise approximation. In the case of Szász operators [130], this x gets cancelled leading to the uniform convergence.

Theorem 1.3.5. *Let $f \in Lip_M^*(r), r \in (0,1]$. Then $\forall x \in (0,1)$, we have*

$$|K_{n,q}^{(\alpha,\beta)}(f;x) - f(x)| \leq M\left(\frac{\delta_{n,q}^{(\alpha,\beta)}(x)}{x}\right)^{r/2},$$

where $\delta_{n,q}^{(\alpha,\beta)}(x) = K_{n,q}^{(\alpha,\beta)}((u-x)^2;x)$.

Proof. First, we prove the result for $r = 1$.

$$\begin{aligned} |K_{n,q}^{(\alpha,\beta)}(f;x) - f(x)| &\leq \sum_{k=0}^{n+p} \bar{p}_{n,k}(q;x) \int_0^1 \left| f\left(\frac{u}{[n+1+\beta]_q} + \frac{q[k+\alpha]_q}{[n+1+\beta]_q}\right) - f(x)\right| d_q u \\ &\leq M \sum_{k=0}^{n+p} \bar{p}_{n,k}(q;x) \int_0^1 \frac{\left|\frac{u}{[n+1+\beta]_q} + \frac{q[k+\alpha]_q}{[n+1+\beta]_q} - x\right|}{\sqrt{\frac{u}{[n+1+\beta]_q} + \frac{q[k+\alpha]_q}{[n+1+\beta]_q} + x}} d_q u. \end{aligned}$$

Since

$$\frac{1}{\sqrt{\frac{u}{[n+1+\beta]_q} + \frac{q[k+\alpha]_q}{[n+1+\beta]_q} + x}} < \frac{1}{\sqrt{x}},$$

applying the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned} |K_{n,q}^{(\alpha,\beta)}(f;x) - f(x)| &\leq \frac{M}{\sqrt{x}} \sum_{k=0}^{n+p} \bar{p}_{n,k}(q;x) \int_0^1 \left|\frac{u}{[n+1+\beta]_q} + \frac{q[k+\alpha]_q}{[n+1+\beta]_q} - x\right| d_q u \\ &= \frac{M}{\sqrt{x}} K_{n,q}^{(\alpha,\beta)}(|u-x|;x) \\ &= M\sqrt{\frac{\delta_{n,q}^{(\alpha,\beta)}(x)}{x}}. \end{aligned}$$

Hence, the result is true for $r = 1$.

Now, we prove the result for $r \in (0,1)$. Applying the Hölder's inequality for summation with $p = \frac{1}{r}$ and $q = \frac{1}{1-r}$, we get

$$
\begin{aligned}
&|K_{n,q}^{(\alpha,\beta)}(f;x)-f(x)| \\
&\leq \sum_{k=0}^{n+p}\bar{p}_{n,k}(q;x)\int_0^1\left|f\left(\frac{u}{[n+1+\beta]_q}+\frac{q[k+\alpha]_q}{[n+1+\beta]_q}\right)-f(x)\right|d_qu \\
&\leq \left\{\sum_{k=0}^{n+p}\bar{p}_{n,k}(q;x)\left(\int_0^1\left|f\left(\frac{u}{[n+1+\beta]_q}+\frac{q[k+\alpha]_q}{[n+1+\beta]_q}\right)-f(x)\right|d_qu\right)^{1/r}\right\}^r.
\end{aligned}
$$

Again applying Hölder's inequality for integration with $p=\frac{1}{r}$ and $q=\frac{1}{1-r}$, we get

$$
\begin{aligned}
&|K_{n,q}^{(\alpha,\beta)}(f;x)-f(x)| \\
&\leq \left\{\sum_{k=0}^{n+p}\bar{p}_{n,k}(q;x)\int_0^1\left|f\left(\frac{u}{[n+1+\beta]_q}+\frac{q[k+\alpha]_q}{[n+1+\beta]_q}\right)-f(x)\right|^{1/r}d_qu\right\}^r.
\end{aligned}
$$

Since $f\in Lip_M^*(r)$, we have

$$
\begin{aligned}
|K_{n,q}^{(\alpha,\beta)}(f;x)-f(x)| &\leq M\left\{\sum_{k=0}^{n+p}\bar{p}_{n,k}(q;x)\int_0^1\frac{\left|\frac{u}{[n+1+\beta]_q}+\frac{q[k+\alpha]_q}{[n+1+\beta]_q}-x\right|}{\sqrt{\frac{u}{[n+1+\beta]_q}+\frac{q[k+\alpha]_q}{[n+1+\beta]_q}+x}}d_qu\right\}^r \\
&\leq \frac{M}{\sqrt[r]{x}}\left\{\sum_{k=0}^{n+p}\bar{p}_{n,k}(q;x)\int_0^1\left|\frac{u}{[n+1+\beta]_q}+\frac{q[k+\alpha]_q}{[n+1+\beta]_q}-x\right|d_qu\right\}^r \\
&\leq \frac{M}{\sqrt[r]{x}}\left(K_{n,q}^{(\alpha,\beta)}(|u-x|;x)\right)^r.
\end{aligned}
$$

Thus, applying Cauchy-Schwarz inequality

$$
|K_{n,q}^{(\alpha,\beta)}(f;x)-f(x)| \leq M\left(\frac{\delta_{n,q}^{(\alpha,\beta)}(x)}{x}\right)^{r/2}.
$$

Hence, we get the desired result. □

Chapter 2

Stancu-Kantorovich type operators based on Polya-Eggenberger Distribution

2.1 Introduction

Based on Pölya-Eggenberger distribution, Stancu [128] introduced a generalization of the Baskakov operators for a real valued function which is bounded on $[0,\infty)$ as

$$\begin{aligned} V_n^{[\alpha]}(f;x) &= \sum_{k=0}^{\infty} v_{n,k}(x,\alpha) f\left(\frac{k}{n}\right) = \sum_{k=0}^{\infty} \binom{n+k-1}{k} \frac{1^{[n,-\alpha]} x^{[k,-\alpha]}}{(1+x)^{[n+k,-\alpha]}} f\left(\frac{k}{n}\right), \\ &= \frac{1^{[n,-\alpha]}}{(1+x)^{[n,-\alpha]}} \sum_{k=0}^{\infty} \frac{1}{k!} \frac{n^{[k,-1]} \left(\dfrac{x}{\alpha}\right)^{[k,-1]}}{\left(\dfrac{1+x}{\alpha}+n\right)^{[k,-1]}} f\left(\frac{k}{n}\right), \end{aligned} \tag{2.1.1}$$

where $\alpha = \alpha(n) \geq 0$ and $u^{[n,h]} = u(u-h)(u-2h)...(u-\overline{n-1}h)$, $u^{[0,h]} = 1$. If $\alpha = 0$, then the operator (2.1.1) reduces to the Baskakov operators [43]:

$$V_n(f;x) = \sum_{k=0}^{\infty} \binom{n+k-1}{k} \frac{x^k}{(1+x)^{n+k}} f\left(\frac{k}{n}\right). \tag{2.1.2}$$

For the class of bounded and Lebesgue integrable functions on $[0,\infty)$, Deo et al. [56]

defined the Kantorovich variant of the operators (2.1.1) as follows:

$$K_n^{[\alpha]}(f;x) = (n-1)\frac{1^{[n,-\alpha]}}{(1+x)^{[n,-\alpha]}}\sum_{k=0}^{\infty}\frac{1}{k!}\frac{n^{[k,-1]}\left(\dfrac{x}{\alpha}\right)^{[k,-1]}}{\left(\dfrac{1+x}{\alpha}+n\right)^{[k,-1]}}\int_{\frac{k}{n-1}}^{\frac{k+1}{n-1}} f(u)du. \tag{2.1.3}$$

Alternatively, we may write the operator (2.1.3) as

$$\begin{aligned} K_n^{[\alpha]}(f;x) &= (n-1)\sum_{k=0}^{\infty}\binom{n+k-1}{k}\frac{B\left(\dfrac{x}{\alpha}+k,\dfrac{1}{\alpha}+n\right)}{B\left(\dfrac{x}{\alpha},\dfrac{1}{\alpha}\right)}\int_0^{\infty} f(u)\chi_{\left[\frac{k}{n-1},\frac{k+1}{n-1}\right]}(u)du \\ &= \int_0^{\infty} L_n^{[\alpha]}(x,u)f(u)du, \end{aligned} \tag{2.1.4}$$

where, $L_n^{[\alpha]}(x,u) = (n-1)\sum_{k=0}^{\infty}\binom{n+k-1}{k}\frac{B\left(\frac{x}{\alpha}+k,\frac{1}{\alpha}+n\right)}{B\left(\frac{x}{\alpha},\frac{1}{\alpha}\right)}\chi_{\left[\frac{k}{n-1},\frac{k+1}{n-1}\right]}(u),\ \chi_{\left[\frac{k}{n-1},\frac{k+1}{n-1}\right]}(u)$ is the characteristic function of $\left[\dfrac{k}{n-1},\dfrac{k+1}{n-1}\right]$ and $B(m,n)$ is the Beta function.

The authors [56] examined the uniform convergence, Voronovskaja type asymptotic result and the weighted approximation properties for functions in $C_2[0,\infty)$. In this chapter, we examine the approximation behavior of the operators given by (2.1.4) for functions in a weighted space and a direct approximation theorem by means of $\omega_{\phi^\tau}(f,u),\ 0\le\tau\le 1$, which unifies the classical modulus of smoothness $\omega(f,u)$ for $\tau=0$, and the Ditzian-Totik modulus of smoothness $\omega_\phi(f,u)$, for $\tau=1$. Furthermore, the rate of approximation of the functions with derivatives of bounded variation by these operators is also established.

Throughout this chapter, C denotes a constant not necessarily the same at each occurrence.

2.2 Basic results

Lemma 2.2.1. *[56] For the Stancu-Baskakov operators (2.1.1), there hold the equalities:*

(i) $V_n^{[\alpha]}(e_0;x) = 1;$

(ii) $V_n^{[\alpha]}(e_1;x) = \dfrac{x}{(1-\alpha)}$;

(iii) $V_n^{[\alpha]}(e_2;x) = \dfrac{1}{(1-\alpha)(1-2\alpha)}\left[x^2 + \dfrac{x(x+1)}{n} + \alpha\left(1-\dfrac{1}{n}\right)x\right]$.

Lemma 2.2.2. *[56] Let* $e_i(u) = u^i$ $i = 0,1,2,3,4,$ *then for the Stancu-Kantorovich operators (2.1.3), the following equalities hold:*

(i) $K_n^{[\alpha]}(e_0;x) = 1$;

(ii) $K_n^{[\alpha]}(e_1;x) = \dfrac{nx}{(n-1)(1-\alpha)} + \dfrac{1}{2(n-1)}$;

(iii) $K_n^{[\alpha]}(e_2;x) = \dfrac{n(n+1)}{(n-1)^2}\dfrac{x(x+\alpha)}{(1-\alpha)(1-2\alpha)} + \dfrac{2nx}{(n-1)^2(1-\alpha)} + \dfrac{1}{3(n-1)^2}$;

(iv) $K_n^{[\alpha]}(e_3;x) = -\dfrac{1}{4(n-1)^3(1-\alpha)(1-2\alpha)(1-3\alpha)}\{-4x(\alpha+x)(2\alpha+x)n^3$
$+6x(\alpha+x)(5\alpha-2x-3)n^2 - 2x(7-26\alpha+4x^2+23\alpha^2-15\alpha x+9x)n$
$+(\alpha-1)(2\alpha-1)(3\alpha-1)\}$;

(v) $K_n^{[\alpha]}(e_4;x) = -\dfrac{1}{5(n-1)^4(1-\alpha)(1-2\alpha)(1-3\alpha)(1-4\alpha)}\Big\{5x(3\alpha+x)(2\alpha+x)$
$(\alpha+x)n^4 - 10x(-3x-4+7\alpha)(2\alpha+x)(\alpha+x)n^3 + 5x(\alpha+x)(54\alpha^2-57\alpha-$
$41\alpha x+11x^2+15+24x)n^2 - 5x(-6x^3+39\alpha+56\alpha^3-83\alpha^2-15x+57\alpha x-6-$
$16x^2+28x^2\alpha-54\alpha^2 x)n\Big\} + \dfrac{1}{5(n-1)^4}$.

Proof. The identities (i)-(iii) are proved in ([56], Lemma 2.2), so we give the proof of the identity (iv). The identity (v) can be proved in a similar manner. By the definition of the operator (2.1.3), we have

$$K_n^{[\alpha]}(e_3;x) = \frac{1}{(n-1)^3}\left[n^3V_n^{[\alpha]}(e_3;x) + 3\frac{n^2}{2}V_n^{[\alpha]}(e_2;x) + nV_n^{[\alpha]}(e_1;x) + \frac{1}{4}V_n^{[\alpha]}(1;x)\right]. \tag{2.2.1}$$

Since the values of $V_n^{[\alpha]}(e_i;x), i = 0,1,2$ are known from Lemma 2.2.1, we need to compute only

$$V_n^{[\alpha]}(e_3;x) = \frac{1^{[n,-\alpha]}}{(1+x)^{[n,-\alpha]}}\sum_{k=0}^{\infty}\frac{n^{[k,-1]}\left(\frac{x}{\alpha}\right)^{[k,-1]}}{\left(\frac{1+x}{\alpha}+n\right)^{[k,-1]}}\frac{k^3}{n^3}$$
$$= \frac{1}{n^3}\sum_{k=0}^{\infty} v_{n,k}(x,\alpha)k(k-1)(k-2)+\frac{3}{n}V_n^{[\alpha]}(e_2;x)-\frac{2}{n^2}V_n^{[\alpha]}(e_1;x). \tag{2.2.2}$$

Using hypergeometric series we may write

$$\begin{aligned} I_1 &= \sum_{k=0}^{\infty} v_{n,k}(x,\alpha)k(k-1)(k-2) \\ &= \frac{1^{[n,-\alpha]}}{(1+x)^{[n,-\alpha]}}\sum_{k=0}^{\infty}\frac{1}{k!}\frac{n^{[k+3,-1]}\left(\frac{x}{\alpha}\right)^{[k+3,-1]}}{\left(\frac{1+x}{\alpha}+n\right)^{[k+3,-1]}} \\ &= \frac{1^{[n,-\alpha]}}{(1+x)^{[n,-\alpha]}}\frac{n(n+1)(n+2)\left(\frac{x}{\alpha}\right)\left(\frac{x}{\alpha}+1\right)\left(\frac{x}{\alpha}+2\right)}{\left(\frac{1+x}{\alpha}+n\right)\left(\frac{1+x}{\alpha}+n+1\right)\left(\frac{1+x}{\alpha}+n+2\right)} \\ &\quad \sum_{k=0}^{\infty}\frac{(n+3)^{[k,-1]}\left(\frac{x}{\alpha}+3\right)^{[k,-1]}}{\left(\frac{1+x}{\alpha}+n+3\right)^{[k,-1]}} \\ &= \frac{1^{[n,-\alpha]}}{(1+x)^{[n,-\alpha]}}\frac{n(n+1)(n+2)\left(\frac{x}{\alpha}\right)\left(\frac{x}{\alpha}+1\right)\left(\frac{x}{\alpha}+2\right)}{\left(\frac{1+x}{\alpha}+n\right)\left(\frac{1+x}{\alpha}+n+1\right)\left(\frac{1+x}{\alpha}+n+2\right)} \\ &\quad {}_2F_1\Bigg((n+3),\left(\frac{x}{\alpha}+3\right),\left(\frac{1+x}{\alpha}+n+3\right);1\Bigg). \end{aligned}$$

By the representation of hypergeometric series in terms of Gamma function

$${}_2F_1(a,b,c;1)=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)},$$

we have

$$I_1=\frac{n(n+1)(n+2)x(x+\alpha)(x+2\alpha)}{(1-\alpha)(1-2\alpha)(1-3\alpha)}.$$

Hence,

$$V_n^{[\alpha]}(e_3;x) = \frac{1}{n^3}\frac{n(n+1)(n+2)x(x+\alpha)(x+2\alpha)}{(1-\alpha)(1-2\alpha)(1-3\alpha)} + \frac{3}{n}V_n^{[\alpha]}(e_2;x) - \frac{2}{n^2}V_n^{[\alpha]}(e_1;x).$$

Thus

$$\begin{aligned} K_n^{[\alpha]}(e_3;x) &= -\frac{1}{4(n-1)^3(1-\alpha)(1-2\alpha)(1-3\alpha)}\cdot\{-4x(\alpha+x)(2\alpha+x)n^3 \\ &\quad +6x(\alpha+x)(5\alpha-2x-3)n^2 - 2x(7-26\alpha+4x^2+23\alpha^2-15\alpha x+9x)n \\ &\quad +(\alpha-1)(2\alpha-1)(3\alpha-1)\}. \end{aligned}$$

Similarly we can obtain $K_n^{[\alpha]}(e_4;x)$. □

Let us denote the m-th order central moment of the operators (2.1.3) by $\mu_{n,m}^{[\alpha]}(x) = K_n^{[\alpha]}((u-x)^m;x)$. From Lemma 2.2.2, by simple calculations we have the following result:

Lemma 2.2.3. *[56] For the functions $\mu_{n,m}^{[\alpha]}(x)$, there hold the equalities:*

(i) $\mu_{n,1}^{[\alpha]}(x) = \left(\frac{n}{(n-1)(1-\alpha)} - 1\right)x + \frac{1}{2(n-1)};$

(ii) $\mu_{n,2}^{[\alpha]}(x) = x^2\left(\frac{n(n+1)}{(n-1)^2(1-\alpha)(1-2\alpha)} - \frac{2n}{(n-1)(1-\alpha)} + 1\right) + x\left(\frac{\alpha n(n+1)}{(n-1)^2(1-\alpha)(1-2\alpha)} + \frac{2n}{(n-1)^2} - \frac{1}{n-1}\right) + \frac{1}{3(n-1)^2}.$

Lemma 2.2.4. *Assuming that $\alpha(n)\to 0$ and $n\alpha(n)\to l$, as $n\to\infty$, $l\in\mathbb{R}$, we have*

(i) $\lim_{n\to\infty} n\mu_{n,1}^{[\alpha]}(x) = (l+1)x + \frac{1}{2};$

(ii) $\lim_{n\to\infty} n\mu_{n,2}^{[\alpha]}(x) = x(x+1)(l+1);$

(iii) $\lim_{n\to\infty} n^2\mu_{n,4}^{[\alpha]}(x) = 3(l+1)^2x^2(x+1)^2.$

Proof. By simple calculations we obtain the proof of (i) and (ii) hence we discuss the proof of (iii). Using Lemma 2.2.2 we have

$$\begin{aligned}\lim_{n\to\infty} n^2\mu_{n,4}^{[\alpha]}(x) &= \lim_{n\to\infty} n^2\left(K_n^{[\alpha]}(e_4;x) - 4xK_n^{[\alpha]}(e_3;x) + 6x^2K_n^{[\alpha]}(e_2;x) - 4x^3K_n^{[\alpha]}(e_1;x) + x^4\right)\\ &= 3x^2(x+1)^2(l+1)^2.\end{aligned}$$

□

2.3 Main Results

In the following result we obtain a direct approximation theorem for functions in the Lipschitz type space:

Theorem 2.3.1. *Let* $f \in Lip_M^*(r)$ *and* $r \in (0,1]$. *Then, for all* $x \in (0,\infty)$, *we have*

$$|K_n^{[\alpha]}(f;x) - f(x)| \le M\left(\frac{\mu_{n,2}^{[\alpha]}(x)}{x}\right)^{r/2}.$$

Proof. Applying the Hölder's inequality for integration with $p = \dfrac{2}{r}$ and $q = \dfrac{2}{2-r}$, we have

$$\begin{aligned}|K_n^{[\alpha]}(f;x) - f(x)| &\le \sum_{k=0}^{\infty} b_{n,k}^{[\alpha]}(x)\int_{\frac{k}{n-1}}^{\frac{k+1}{n-1}} |f(u)-f(x)|du\\ &\le \sum_{k=0}^{\infty} b_{n,k}^{[\alpha]}(x)\left(\int_{\frac{k}{n-1}}^{\frac{k+1}{n-1}} |f(u)-f(x)|^{\frac{2}{r}}du\right)^{\frac{r}{2}}\left(\int_{\frac{k}{n-1}}^{\frac{k+1}{n-1}} 1du\right)^{\frac{2-r}{2}}\\ &\le \sum_{k=0}^{\infty}\left(b_{n,k}^{[\alpha]}(x)\int_{\frac{k}{n-1}}^{\frac{k+1}{n-1}} |f(u)-f(x)|^{\frac{2}{r}}du\right)^{\frac{r}{2}}\left(b_{n,k}^{[\alpha]}(x)\int_{\frac{k}{n-1}}^{\frac{k+1}{n-1}} 1du\right)^{\frac{2-r}{2}}.\end{aligned}$$

Again applying Hölder's inequality for summation with $p = \dfrac{2}{r}$ and $q = \dfrac{2}{2-r}$, and Lemma 2.2.2 we have

$$\begin{aligned}|K_n^{[\alpha]}(f;x) - f(x)| &\le \left(\sum_{k=0}^{\infty} b_{n,k}^{[\alpha]}(x)\int_{\frac{k}{n-1}}^{\frac{k+1}{n-1}} |f(u)-f(x)|^{\frac{2}{r}}du\right)^{\frac{r}{2}}\left(\sum_{k=0}^{\infty} b_{n,k}^{[\alpha]}(x)\int_{\frac{k}{n-1}}^{\frac{k+1}{n-1}} 1du\right)^{\frac{2-r}{2}}\\ &\le M\left(\sum_{k=0}^{\infty} b_{n,k}^{[\alpha]}(x)\int_{\frac{k}{n-1}}^{\frac{k+1}{n-1}} \frac{(u-x)^2}{(u+x)}du\right)^{\frac{r}{2}}\\ &\le \frac{M}{x^{r/2}}\left(\sum_{k=0}^{\infty} b_{n,k}^{[\alpha]}(x)\int_{\frac{k}{n-1}}^{\frac{k+1}{n-1}} (u-x)^2du\right)^{\frac{r}{2}} = \frac{M}{x^{r/2}}\left(\mu_{n,2}^{[\alpha]}(x)\right)^{\frac{r}{2}} = M\left(\frac{\mu_{n,2}^{[\alpha]}(x)}{x}\right)^{\frac{r}{2}}.\end{aligned}$$

Hence, we reach the desired result. □

2.3.1 Weighted approximation

Our following result is a Voronovskaja type theorem for the operators given by (2.1.3):

Theorem 2.3.2. *Let $f \in C_2[0,\infty)$, $\alpha = \alpha(n) \to 0$, as $n \to \infty$ and $\lim_{n\to\infty} n\alpha(n) = l \in \mathbb{R}$. If f admits a derivative of second order at $x \in [0,\infty)$, then we have*

$$\lim_{n\to\infty} n[K_n^{[\alpha]}(f;x) - f(x)] = \frac{2(l+1)x+1}{2} f'(x) + \frac{x(x+1)(l+1)}{2} f''(x).$$

Proof. By the Taylor's theorem, we may write

$$f(u) = f(x) + f'(x)(u-x) + \frac{1}{2} f''(x)(u-x)^2 + \varepsilon(u,x)(u-x)^2, \tag{2.3.1}$$

where $\varepsilon(u,x) \in C_2[0,\infty)$ and $\lim_{u\to x} \varepsilon(u,x) = 0$.
Applying the operator $K_n^{[\alpha]}(\cdot, x)$ on both sides of (2.3.1), we have

$$\begin{aligned} \lim_{n\to\infty} n\left[K_n^{[\alpha]}(f;x) - f(x)\right] &= \lim_{n\to\infty} n\,\mu_{n,1}^{[\alpha]}(x) f'(x) + \lim_{n\to\infty} n\,\mu_{n,2}^{[\alpha]}(x) \frac{f''(x)}{2} \\ &\quad + \lim_{n\to\infty} n\, K_n^{[\alpha]}(\varepsilon(u,x)(u-x)^2; x). \end{aligned} \tag{2.3.2}$$

Using the Cauchy-Schwarz inequality in the last term of the right side of (2.3.2), we get

$$nK_n^{[\alpha]}(\varepsilon(u,x)(u-x)^2;x) \le \sqrt{K_n^{[\alpha]}(\varepsilon^2(u,x);x)}\sqrt{n^2\mu_{n,4}^{[\alpha]}(x)}.$$

In view of ([56], Theorem 3.2), $\lim_{n\to\infty} K_n^{[\alpha]}(\varepsilon^2(u,x);x) = \varepsilon^2(x,x) = 0$, since $\varepsilon(u,x) \to 0$ as $u \to x$, and by using Lemma 2.2.4, we get

$$\lim_{n\to\infty} nK_n^{[\alpha]}(\varepsilon(u,x)(u-x)^2;x) = 0.$$

Now, from (2.3.2) and Lemma 2.2.4, we have the required result.

□

Theorem 2.3.3. *For $f \in C_2[0,\infty)$, we have the following inequality*

$$|K_n^{[\alpha]}(f;x) - f(x)| \le 4M_f(1+x^2)\mu_{n,2}^{[\alpha]}(x) + 2\omega_{b+1}\left(f; \sqrt{\mu_{n,2}^{[\alpha]}(x)}\right), \tag{2.3.3}$$

where $\omega_b(f;\delta)$ is the modulus of continuity of f on [0,b].

Proof. From [84], for $x \in [0,b]$ and all $u \geq 0$, we have

$$|f(u)-f(x)| \leq 4M_f(u-x)^2(1+x^2)+\left(1+\frac{|u-x|}{\delta}\right)\omega_{b+1}(f,\delta),$$

for any $\delta > 0$. Hence by Cauchy-Schwarz inequality

$$|K_n^{[\alpha]}(f;x)-f(x)| \leq 4M_f(1+x^2)K_n^{[\alpha]}((u-x)^2;x)+\omega_{b+1}(f,\delta)\left(1+\frac{1}{\delta}(\mu_{n,2}^{[\alpha]}(x))^{1/2}\right).$$

Choosing $\delta = \sqrt{\mu_{n,2}^{[\alpha]}(x)}$, we get the desired result.

□

Theorem 2.3.4. *Let $f \in C_2^0[0,\infty)$ and $\alpha = \alpha(n) \to 0$, as $n \to \infty$. Then, we have the following result:*

$$\lim_{n\to\infty} ||K_n^{[\alpha]}(f)-f||_2 = 0.$$

Proof. From [69], in order to prove this result it is sufficient to show that

$$\lim_{n\to\infty} ||K_n^{[\alpha]}(u^m;x)-x^m||_2 = 0, \quad m = 0,1,2. \tag{2.3.4}$$

From Lemma 2.2.2, $K_n^{[\alpha]}(1;x) = 1$, therefore the condition (2.3.4) holds for $m = 0$.

Using Lemma 2.2.2, we have

$$\begin{aligned}
||K_n^{[\alpha]}(u;x)-x||_2 &= \sup_{x\geq 0}\frac{1}{(1+x^2)}\left|\frac{nx}{(n-1)(1-\alpha)}+\frac{1}{2(n-1)}-x\right| \\
&= \sup_{x\geq 0}\frac{1}{(1+x^2)}\left(\frac{x(1+\alpha(n-1))}{(n-1)(1-\alpha)}+\frac{1}{2(n-1)}\right) \\
&= \frac{1+\alpha(n-1)}{2(n-1)(1-\alpha)}+\frac{1}{2(n-1)}.
\end{aligned}$$

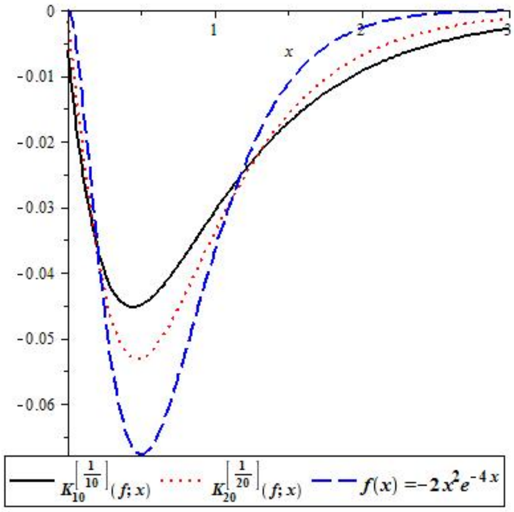
Hence, $\lim_{n\to\infty} ||K_n^{[\alpha]}(u;x)-x||_2 = 0$.

Now

$$\begin{aligned}
||K_n^{[\alpha]}(u^2;x)-x^2||_2 &= \sup_{x\geq 0}\frac{1}{(1+x^2)}\left|\frac{n(n+1)}{(n-1)^2}\frac{x(x+\alpha)}{(1-\alpha)(1-2\alpha)}+\frac{2nx}{(n-1)^2(1-\alpha)}\right. \\
&\quad \left. +\frac{1}{3(n-1)^2}-x^2\right| \\
&= \left(\frac{n(n+1)}{(n-1)^2(1-\alpha)(1-2\alpha)}-1\right)+\frac{1}{2}\left(\frac{n(n+1)\alpha}{(n-1)^2(1-\alpha)(1-2\alpha)}\right. \\
&\quad \left. +\frac{2n}{(n-1)^2(1-\alpha)}\right)+\frac{1}{3(n-1)^2}.
\end{aligned}$$

Therefore, $\lim_{n\to\infty} ||K_n^{[\alpha]}(u^2;x)-x^2||_2 = 0$. This completes the proof of the theorem. □

Example 1. The convergence of the operators $K_n^{[\alpha]}(f;x)$ is illustrated in Figure 1, where $f(x) = -2x^2e^{-4x}$, $\alpha = 1/n$, $n = 10$ and $n = 20$, respectively. We can see that when the values of n are increasing, the graphs of operators $K_n^{[\alpha]}(f;x)$ are going to the graph of the function f.



The convergence of $K_n^{[\alpha]}(f;x)$ to $f(x)$

In Table 1, we have computed the error of approximation for $K_n^{[\alpha]}$ at certain points, for $n = 10$ and $n = 20$, respectively.

Table 1. Error of approximation for $K_n^{[\alpha]}$

x	$\lvert K_{10}^{[1/10]}(f;x)-f(x)\rvert$	$\lvert K_{20}^{[1/20]}(f;x)-f(x)\rvert$
0.4	0.019565407970	0.012368435090
0.6	0.022150921080	0.014168200500
0.8	0.014892362830	0.009011449298
1.0	0.006013519940	0.002842511327
1.4	0.004823881541	0.004095282537
1.6	0.006630626569	0.004928260795
1.8	0.006993133290	0.004797156210
2.0	0.006563403422	0.004200434530
2.2	0.005780754788	0.003454436427
2.4	0.004903691508	0.002729426270
2.6	0.004066205408	0.002100415754
2.8	0.003326488994	0.001588409028
3.0	0.002700935903	0.001187735798

Theorem 2.3.5. *Let $f \in C_2^0[0,\infty)$, $\alpha=\alpha(n)\to 0$, as $n\to\infty$ and $a>0$. Then*

$$\lim_{n\to\infty}\sup_{x\in[0,\infty)}\frac{\left|K_n^{[\alpha]}(f;x)-f(x)\right|}{(1+x^2)^{1+a}}=0.$$

Proof. Let $x_0\in[0,\infty)$ be an arbitrary but fixed point. Then

$$\begin{aligned}\sup_{x\in[0,\infty)}\frac{\left|K_n^{[\alpha]}(f;x)-f(x)\right|}{(1+x^2)^{1+a}} &\le \sup_{x\le x_0}\frac{\left|K_n^{[\alpha]}(f;x)-f(x)\right|}{(1+x^2)^{1+a}}+\sup_{x>x_0}\frac{\left|K_n^{[\alpha]}(f;x)-f(x)\right|}{(1+x^2)^{1+a}}\\ &\le ||K_n^{[\alpha]}(f;x)-f(x)||_{C[0,x_0]}+||f||_2\sup_{x>x_0}\frac{K_n^{[\alpha]}(1+u^2;x)}{(1+x^2)^{1+a}}\\ &\quad+\sup_{x>x_0}\frac{|f(x)|}{(1+x^2)^{1+a}}.\end{aligned} \tag{2.3.5}$$

Since $|f(x)|\le ||f||_2(1+x^2)$, we have

$$\sup_{x>x_0}\frac{|f(x)|}{(1+x^2)^{1+a}}\le\frac{||f||_2}{(1+x_0^2)^a}. \tag{2.3.6}$$

Let $\epsilon > 0$, be arbitrary. We choose x_0 to be so large that

$$\frac{||f||_2}{(1+x_0^2)^a} < \frac{\epsilon}{4}. \tag{2.3.7}$$

Since $\lim_{n\to\infty} \sup_{x>x_0} \frac{K_n^{[\alpha]}(1+u^2;x)}{1+x^2} = 1$, it follows that

$$\sup_{x>x_0} \frac{K_n^{[\alpha]}(1+u^2;x)}{1+x^2} \leq \frac{(1+x_0^2)^a}{\|f\|_2}\frac{\epsilon}{4} + 1,$$

for sufficiently large n. Therefore,

$$\|f\|_2 \sup_{x>x_0} \frac{K_n^{[\alpha]}(1+u^2;x)}{(1+x^2)^{a+1}} \leq \frac{\|f\|_2}{(1+x_0^2)^a} \sup_{x>x_0} \frac{K_n^{[\alpha]}(1+u^2;x)}{(1+x^2)} \leq \frac{\epsilon}{4} + \frac{||f||_2}{(1+x_0^2)^a}. \tag{2.3.8}$$

Applying Theorem 2.3.4 , we can find for sufficiently large n

$$||K_n^{[\alpha]}(f;x) - f(x)||_{C[0,x_0]} < \frac{\epsilon}{4}. \tag{2.3.9}$$

Combining (2.3.5)-(2.3.9), we obtain

$$\sup_{x\in[0,\infty)} \frac{\left|K_n^{[\alpha]}(f;x) - f(x)\right|}{(1+x^2)^{1+a}} < \epsilon.$$

This proves the required result. □

Theorem 2.3.6. *Let $f \in C_2^0[0,\infty)$. If $\alpha = \alpha(n) \to 0$ as $n \to \infty$ and $\lim_{n\to\infty} n\alpha(n) = l \in \mathbb{R}$. Then for sufficiently large n, we have*

$$\sup_{x\in[0,\infty)} \frac{\left|K_n^{[\alpha]}(f;x) - f(x)\right|}{(1+x^2)^{5/2}} \leq \tilde{C}(l)\Omega\left(f;\frac{1}{\sqrt{n}}\right), \tag{2.3.10}$$

where $\tilde{C}(l)$ is a positive constant depending on l.

Proof. For $x \in (0,\infty)$ and $\delta > 0$, using (0.4.1) and Lemma 0.4.1, we have

$$\begin{aligned}|f(u)-f(x)| &\leq \left(1+(x+|x-u|)^2\right)\Omega(f;|u-x|)\\ &\leq 2(1+x^2)(1+(u-x)^2)\left(1+\frac{|u-x|}{\delta}\right)\Omega(f;\delta).\end{aligned}$$

Applying $K_n^{[\alpha]}(\cdot;x)$ both sides, we can write

$$\begin{aligned}|K_n^{[\alpha]}(f;x)-f(x)| \leq\ & 2(1+x^2)\Omega(f;\delta)\Big(1+K_n^{[\alpha]}((u-x)^2;x)\\ &+K_n^{[\alpha]}\left((1+(u-x)^2)\frac{|u-x|}{\delta};x\right)\Big).\end{aligned} \tag{2.3.11}$$

From Lemma 2.2.4, for sufficiently large n, it follows

$$n\mu_{n,2}^{[\alpha]}(x) \leq C(l)(1+x^2) \text{ and } n^2\mu_{n,4}^{[\alpha]}(x) \leq C(l)(1+x^2)^2, \tag{2.3.12}$$

where $C(l)$ is a positive constant depending on l.

Now, applying the Cauchy-Schwarz inequality in the last term of (2.3.11), we obtain

$$K_n^{[\alpha]}\left((1+(u-x)^2)\frac{|u-x|}{\delta};x\right) \leq \frac{1}{\delta}\left(\mu_{n,2}^{[\alpha]}(x)\right)^{1/2}+\frac{1}{\delta}\left(\mu_{n,4}^{[\alpha]}(x)\right)^{1/2}\left(\mu_{n,2}^{[\alpha]}(x)\right)^{1/2}. \tag{2.3.13}$$

Combining the estimates (2.3.11)-(2.3.13) and taking

$$\tilde{C}(l) = 2\left(1+\sqrt{C(l)}+2C(l)\right) \text{ and } \delta = \frac{1}{\sqrt{n}},$$

we reach the required result.

□

2.3.2 Unified modulus theorem

In this section, we shall investigate a direct theorem with the aid of the unified Ditzian-Totik modulus of smoothness $\omega_{\phi^\tau}(f,t),\ 0 \leq \tau \leq 1$.

Theorem 2.3.7. *Let $f \in C_B[0,\infty)$, then for sufficiently large n*

$$|K_n^{[\alpha]}(f;x)-f(x)| \leq C\omega_{\phi^\tau}\left(f;\frac{\phi^{1-\tau}(x)}{\sqrt{n}}\right), \tag{2.3.14}$$

where C is independent of f and n.

Proof. By the definition of $K_{\phi^\tau}(f,t)$, for a fixed n,x,τ we can choose $g = g_{n,x,\tau} \in W_\tau$ such that

$$||f-g|| + \frac{\phi^{1-\tau}(x)}{\sqrt{n}}||\phi^\tau g'|| \le 2K_{\phi^\tau}\left(f;\frac{\phi^{1-\tau}(x)}{\sqrt{n}}\right).$$

We can write

$$\begin{aligned}|K_n^{[\alpha]}(f;x)-f(x)| &\le |K_n^{[\alpha]}(f-g;x)| + |K_n^{[\alpha]}(g;x)-g(x)| + |g(x)-f(x)| \\ &\le 2||f-g|| + |K_n^{[\alpha]}(g;x)-g(x)|. \end{aligned} \tag{2.3.15}$$

Since $g \in W_\tau$, we have

$$g(u) = g(x) + \int_x^u g'(v)dv$$

and so

$$|K_n^{[\alpha]}(g;x)-g(x)| \le K_n^{[\alpha]}\left(\left|\int_x^u g'(v)dv\right|;x\right). \tag{2.3.16}$$

By applying Hölder's inequality, we get

$$\begin{aligned}\left|\int_x^u g'(v)dv\right| &\le ||\phi^\tau g'|| \left|\int_x^u \frac{dv}{\phi^\tau(v)}\right| \\ &\le ||\phi^\tau g'||\,|u-x|^{1-\tau}\left|\int_x^u \frac{dv}{\phi(v)}\right|^\tau,\end{aligned}$$

we can write

$$\begin{aligned}\left|\int_x^u \frac{dv}{\phi(v)}\right| &\le \left|\int_x^u \frac{dv}{\sqrt{v}}\right|\left(\frac{1}{\sqrt{1+x}}+\frac{1}{\sqrt{1+u}}\right) \\ &\le \frac{2|u-x|}{\sqrt{x}}\left(\frac{1}{\sqrt{1+x}}+\frac{1}{\sqrt{1+u}}\right).\end{aligned}$$

Hence

$$\begin{aligned}\left|\int_x^u g'(v)dv\right| &\le \frac{2^\tau||\phi^\tau g'||\,|u-x|}{x^{\tau/2}}\left(\frac{1}{\sqrt{1+x}}+\frac{1}{\sqrt{1+u}}\right)^\tau \\ &\le \frac{2^\tau||\phi^\tau g'||\,|u-x|}{x^{\tau/2}}\left(\frac{1}{(1+x)^{\tau/2}}+\frac{1}{(1+u)^{\tau/2}}\right),\end{aligned} \tag{2.3.17}$$

on applying the inequality

$$|a+b|^r \le |a|^r + |b|^r, \;\; 0 \le r \le 1.$$

Thus, from (2.3.16) and (2.3.17) and using Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}|K_n^{[\alpha]}(g;x)-g(x)| &\leq \frac{2^\tau||\phi^\tau g'||}{x^{\tau/2}}K_n^{[\alpha]}\left(|u-x|\left(\frac{1}{(1+x)^{\tau/2}}+\frac{1}{(1+u)^{\tau/2}}\right);x\right)\\ &\leq \frac{2^\tau||\phi^\tau g'||}{x^{\tau/2}}\left(\frac{1}{(1+x)^{\tau/2}}\sqrt{\mu_{n,2}^{[\alpha]}(x)}+\sqrt{\mu_{n,2}^{[\alpha]}(x)}\sqrt{K_n^{[\alpha]}\left((1+u)^{-\tau};x\right)}\right)\\ &\leq 2^\tau||\phi^\tau g'||\sqrt{\mu_{n,2}^{[\alpha]}(x)}\left\{\phi^{-\tau}(x)+x^{-\tau/2}\sqrt{K_n^{[\alpha]}\left((1+u)^{-\tau};x\right)}\right\}.\end{aligned} \tag{2.3.18}$$

Therefore, using ([56], Theorem 3.2) for sufficiently large n, we have

$$|K_n^{[\alpha]}(g;x)-g(x)|\leq 2^\tau||\phi^\tau g'||C\sqrt{\frac{\phi^2(x)}{n}}\left\{\phi^{-\tau}(x)+x^{-\tau/2}(1+x)^{-\tau/2}\right\}, \tag{2.3.19}$$

for sufficiently large n.

Thus, combining (2.3.15) and (2.3.19), we find

$$\begin{aligned}|K_n^{[\alpha]}(f;x)-f(x)| &\leq 2||f-g||+2^{\tau+1}||\phi^\tau g'||\frac{C\phi^{1-\tau}(x)}{\sqrt{n}}\\ &\leq C\left\{||f-g||+\frac{\phi^{1-\tau}(x)}{\sqrt{n}}||\phi^\tau g'||\right\}\\ &\leq CK_{\phi^\tau}\left(f;\frac{\phi^{1-\tau}(x)}{\sqrt{n}}\right)\leq C\omega_{\phi^\tau}\left(f;\frac{\phi^{1-\tau}(x)}{\sqrt{n}}\right).\end{aligned}$$

This completes the proof of the theorem. □

2.3.3 Rate of Convergence of Stancu-Kantorovich type operators based on Polya-Eggenberger Distribution

This section is devoted to the discussion of approximation of functions having derivatives of bounded variation. For papers related to the study in this direction we refer the reader to (cf. [9], [62], [75], [86], [91], [96], [107] and [109] etc.). Let

$$DBV[0,\infty):=\left\{f\in C_2[0,\infty):f'\quad \text{is of bounded variation in } [a,b]\subset[0,\infty)\right\}.$$

A function $f\in DBV[0,\infty)$ can be represented as

$$f(x)=\int_0^x g(u)du+f(0),$$

where $g\in BV[a,b]$, the space of bounded variation function on $[a,b]$.

Lemma 2.3.1. *Let* $\alpha = \alpha(n) \to 0$, *as* $n \to \infty$ *and* $\lim_{n\to\infty} n\alpha(n) = l \in \mathbb{R}$. *For every* $x > 0$ *and sufficiently large* n,

$$\text{(i)}\ \ \xi_n^{[\alpha]}(x,u) = \int_0^u L_n^{[\alpha]}(x,v)dv \le \frac{C(l)}{(x-u)^2}\frac{(1+x^2)}{n},\ \ 0 \le u < x,$$

$$\text{(ii)}\ \ 1-\xi_n^{[\alpha]}(x,u) = \int_u^\infty L_n^{[\alpha]}(x,v)dv \le \frac{C(l)}{(u-x)^2}\frac{(1+x^2)}{n},\ \ x \le u < \infty,$$

where $C(l) > 0$ *is constant and depends on* l.

Proof. Using Lemma 2.2.2 and (2.3.12), we have

$$\begin{aligned}\xi_n^{[\alpha]}(x,u) &= \int_0^u L_n^{[\alpha]}(x,v)dv \le \int_0^u \left(\frac{x-v}{x-u}\right)^2 L_n^{[\alpha]}(x,v)dv \\ &\le \frac{1}{(x-u)^2}K_n^{[\alpha]}((u-x)^2;x) \le \frac{C(l)}{(x-u)^2}\frac{1+x^2}{n},\end{aligned}$$

when n is large enough. Similarly, we can prove (ii). □

Theorem 2.3.8. *Let* $f \in DBV[0,\infty)$, $\alpha = \alpha(n) \to 0$, *as* $n \to \infty$ *and* $\lim_{n\to\infty} n\alpha(n) = l \in \mathbb{R}$. *Then, for every* $x \in (0,\infty)$ *and sufficiently large* n, *we have*

$$\begin{aligned}|K_n^{[\alpha]}(f;x) - f(x)| \le\ & \left[\left(\frac{n}{(n-1)(1-\alpha)} - 1\right)x + \frac{1}{2(n-1)}\right]\left|\frac{f'(x+)+f'(x-)}{2}\right| \\ & + \sqrt{C(l)\frac{1+x^2}{n}}\left|\frac{f'(x+)-f'(x-)}{2}\right| + C(l)\frac{1+x^2}{nx}\sum_{k=1}^{[\sqrt{n}]}\left(\bigvee_{x-\frac{x}{k}}^{x} f'_x\right) \\ & + \frac{x}{\sqrt{n}}\left(\bigvee_{x-\frac{x}{\sqrt{n}}}^{x} f'_x\right) + \left(4M_f + \frac{M_f + |f(x)|}{x^2}\right)C(l)\frac{1+x^2}{n} \\ & + |f'(x+)|\sqrt{C(l)\frac{1+x^2}{n}} + C(l)\frac{1+x^2}{nx^2}|f(2x) - f(x) - xf'(x+)| \\ & + \frac{x}{\sqrt{n}}\left(\bigvee_{x}^{x+\frac{x}{\sqrt{n}}} f'_x\right) + C(l)\frac{1+x^2}{nx}\sum_{k=1}^{[\sqrt{n}]}\left(\bigvee_{x}^{x+\frac{x}{\sqrt{n}}} f'_x\right),\end{aligned}$$

Where $\bigvee_a^b f$ *is the total variation of* f *on* $[a,b]$ *and* f'_x *is given by*

$$f'_x(u) = \begin{cases} f'(u) - f'(x-), & 0 \le u < x \\ 0, & u = x, \\ f'(u) - f'(x+) & x < u < \infty. \end{cases} \tag{2.3.20}$$

Proof. For any $f \in DBV[0,\infty)$, from (2.3.20), we may write

$$\begin{aligned} f'(v) &= \frac{1}{2}\left(f'(x+)+f'(x-)\right)+f'_x(v)+\frac{1}{2}\left(f'(x+)-f'(x-)\right)\operatorname{sgn}(v-x) \\ &\quad +\delta_x(v)\left(f'(v)-\frac{1}{2}(f'(x+)+f'(x-))\right), \end{aligned} \tag{2.3.21}$$

where

$$\delta_x(v)=\begin{cases} 1, & v=x \\ 0, & v\neq x. \end{cases}$$

Since $K_n^{[\alpha]}(e_0;x)=1$, using (2.3.21) for every $x\in(0,\infty)$, we get

$$\begin{aligned} K_n^{[\alpha]}(f;x)-f(x) &= \int_0^\infty L_n^{[\alpha]}(x,u)(f(u)-f(x))du=\int_0^\infty L_n^{[\alpha]}(x,u)\left(\int_x^u f'(v)dv\right)du \\ &= -\int_0^x\left(\int_u^x f'(v)dv\right)L_n^{[\alpha]}(x,u)du+\int_x^\infty\left(\int_x^u f'(v)dv\right)L_n^{[\alpha]}(x,u)du. \end{aligned} \tag{2.3.22}$$

Let $I_1:=\int_0^x\left(\int_u^x f'(v)dv\right)L_n^{[\alpha]}(x,u)du,\quad I_2:=\int_x^\infty\left(\int_x^u f'(v)dv\right)L_n^{[\alpha]}(x,u)du.$

Since $\int_x^u \delta_x(v)dv=0$, using (2.3.22) we have

$$\begin{aligned} I_1 &= \int_0^x\left\{\int_u^x\left(\frac{1}{2}\left(f'(x+)+f'(x-)\right)+f'_x(v)\right.\right. \\ &\quad \left.\left.+\frac{1}{2}\left(f'(x+)-f'(x-)\right)\operatorname{sgn}(v-x)\right)dv\right\}L_n^{[\alpha]}(x,u)du \\ &= \frac{1}{2}\left(f'(x+)+f'(x-)\right)\int_0^x(x-u)L_n^{[\alpha]}(x,u)du+\int_0^x\left(\int_u^x f'_x(v)dv\right)L_n^{[\alpha]}(x,u)du \\ &- \frac{1}{2}\left(f'(x+)-f'(x-)\right)\int_0^x(x-u)L_n^{[\alpha]}(x,u)du. \end{aligned} \tag{2.3.23}$$

Similarly, we have

$$\begin{aligned} I_2 &= \int_x^\infty\left\{\int_x^u\left(\frac{1}{2}\left(f'(x+)+f'(x-)\right)+f'_x(u)\right.\right. \\ &\quad \left.\left.+\frac{1}{2}\left(f'(x+)-f'(x-)\right)\operatorname{sgn}(v-x)\right)dv\right\}L_n^{[\alpha]}(x,u)du \\ &= \frac{1}{2}\left(f'(x+)+f'(x-)\right)\int_x^\infty(u-x)L_n^{[\alpha]}(x,u)du+\int_x^\infty\left(\int_x^u f'_x(v)dv\right)L_n^{[\alpha]}(x,u)du \\ &+ \frac{1}{2}\left(f'(x+)-f'(x-)\right)\int_x^\infty(u-x)L_n^{[\alpha]}(x,u)du. \end{aligned} \tag{2.3.24}$$

Combining the relation (2.3.22)-(2.3.24), we get

$$
\begin{aligned}
& K_n^{[\alpha]}(f;x) - f(x) \\
&= \frac{1}{2}\left(f'(x+)+f'(x-)\right)\int_0^\infty (u-x)L_n^{[\alpha]}(x,u)du + \frac{1}{2}\Big(f'(x+)-f'(x-)\Big) \\
&\quad \int_0^\infty |u-x| L_n^{[\alpha]}(x,u)du - \int_0^x \left(\int_u^x f_x'(v)dv\right) L_n^{[\alpha]}(x,u)du \\
&\quad + \int_x^\infty \left(\int_x^u f_x'(v)dv\right) L_n^{[\alpha]}(x,u)du.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
& |K_n^{[\alpha]}(f;x) - f(x)| \\
&\leq \left|\frac{f'(x+)+f'(x-)}{2}\right| |K_n^{[\alpha]}(u-x;x)| + \left|\frac{f'(x+)-f'(x-)}{2}\right| K_n^{[\alpha]}(|u-x|;x) \\
&\quad + \left|\int_0^x \left(\int_u^x f_x'(v)dv\right) L_n^{[\alpha]}(x,u)du\right| + \left|\int_x^\infty \left(\int_x^u f_x'(v)dv\right) L_n^{[\alpha]}(x,u)du\right|.
\end{aligned} \tag{2.3.25}
$$

Now, assume that

$$
C_n^{[\alpha]}(f_x', x) = \int_0^x \left(\int_u^x f_x'(v)dv\right) L_n^{[\alpha]}(x,u)du,
$$

and

$$
D_n^{[\alpha]}(f_x', x) = \int_x^\infty \left(\int_x^u f_x'(v)dv\right) L_n^{[\alpha]}(x,u)du.
$$

Now the problem is reduced to estimate $C_n^{[\alpha]}(f_x', x)$ and $D_n^{[\alpha]}(f_x', x)$.
Using the definition of $\xi_n^{[\alpha]}(x,u)$ given in Lemma 2.3.1 and applying the integration by parts, we can write

$$
C_n^{[\alpha]}(f_x', x) = \int_0^x \left(\int_u^x f_x'(v)dv\right) \frac{\partial \xi_n^{[\alpha]}(x,u)}{\partial u}du = \int_0^x f_x'(u)\xi_n^{[\alpha]}(x,u)du.
$$

Thus,

$$
\begin{aligned}
|C_n^{[\alpha]}(f_x', x)| &= \int_0^x |f_x'(u)| \xi_n^{[\alpha]}(x,u)du \\
&\leq \int_0^{x-\frac{x}{\sqrt{n}}} |f_x'(u)| \xi_n^{[\alpha]}(x,u)du + \int_{x-\frac{x}{\sqrt{n}}}^x |f_x'(u)| \xi_n^{[\alpha]}(x,u)du.
\end{aligned}
$$

Since $f'_x(x) = 0$ and $\xi_n^{[\alpha]}(x,u) \le 1$,

$$\int_{x-\frac{x}{\sqrt{n}}}^{x} |f'_x(u)|\xi_n^{[\alpha]}(x,u)du = \int_{x-\frac{x}{\sqrt{n}}}^{x} |f'_x(u) - f'_x(x)|\xi_n^{[\alpha]}(x,u)du \le \int_{x-\frac{x}{\sqrt{n}}}^{x} \left(\bigvee_u^x f'_x\right) du$$
$$\le \frac{x}{\sqrt{n}} \left(\bigvee_{x-\frac{x}{\sqrt{n}}}^{x} f'_x\right).$$

Using Lemma 2.3.1, and assuming $u = x - \frac{x}{v}$, we have

$$\int_0^{x-\frac{x}{\sqrt{n}}} |f'_x(u)|\xi_n^{[\alpha]}(x,u)du \le C(l)\frac{1+x^2}{n}\int_0^{x-\frac{x}{\sqrt{n}}} \frac{|f'_x(u)|}{(x-u)^2}du$$
$$\le C(l)\frac{1+x^2}{n}\int_0^{x-\frac{x}{\sqrt{n}}} \left(\bigvee_u^x f'_x\right)\frac{du}{(x-u)^2}$$
$$= C(l)\frac{1+x^2}{nx}\int_1^{\sqrt{n}} \left(\bigvee_{x-\frac{x}{v}}^{x} f'_x\right) \le C(l)\frac{(1+x^2)}{nx}\sum_{k=1}^{[\sqrt{n}]} \left(\bigvee_{x-\frac{x}{k}}^{x} f'_x\right).$$

Therefore,

$$|C_n^{[\alpha]}(f'_x,x)| = C(l)\frac{1+x^2}{nx}\sum_{k=1}^{[\sqrt{n}]} \left(\bigvee_{x-\frac{x}{k}}^{x} f'_x\right) + \frac{x}{\sqrt{n}}\left(\bigvee_{x-\frac{x}{\sqrt{n}}}^{x} f'_x\right).$$

Using integration by parts in $D_n^{[\alpha]}(f'_x,x)$ and applying Lemma 2.3.1, we have

$$\begin{aligned}
|D_n^{[\alpha]}(f'_x,x)| &\le \left|\int_x^{2x}\left(\int_x^u f'_x(v)dv\right)\frac{\partial}{\partial u}(1-\xi_n^{[\alpha]}(x,u))du\right| + \left|\int_{2x}^{\infty}\left(\int_x^u f'_x(v)dv\right)L_n^{[\alpha]}(x,u)du\right| \\
&\le \left|\int_x^{2x} f'_x(v)dv\right||1-\xi_n^{[\alpha]}(x,2x)| + \int_x^{2x}|f'_x(u)|(1-\xi_n^{[\alpha]}(x,u))du \\
&\quad + \left|\int_{2x}^{\infty}(f(u)-f(x))L_n^{[\alpha]}(x,u)du\right| + |f'(x+)|\left|\int_{2x}^{\infty}(u-x)L_n^{[\alpha]}(x,u)du\right|.
\end{aligned}$$

We have

$$\begin{aligned}
&\int_x^{2x}|f'_x(u)|(1-\xi_n^{[\alpha]}(x,u))du \\
&= \int_x^{x+\frac{x}{\sqrt{n}}}|f'_x(u)|(1-\xi_n^{[\alpha]}(x,u))du + \int_{x+\frac{x}{\sqrt{n}}}^{2x}|f'_x(u)|(1-\xi_n^{[\alpha]}(x,u))du \\
&= J_1 + J_2.
\end{aligned} \tag{2.3.26}$$

Since $f'_x(x) = 0$ and $1 - \xi_n^{[\alpha]}(x,u) \leq 1$,

$$J_1 = \int_x^{x+\frac{x}{\sqrt{n}}} |f'_x(u) - f'_x(x)|(1 - \xi_n^{[\alpha]}(x,u))du \leq \frac{x}{\sqrt{n}} \left(\bigvee_x^{x+\frac{x}{\sqrt{n}}} f'_x \right).$$

Using Lemma 2.3.1 and assuming $u = x + \frac{x}{v}$, we obtain

$$J_2 \leq C(l)\frac{1+x^2}{n} \int_{x+\frac{x}{\sqrt{n}}}^{2x} \frac{1}{(u-x)^2}|f'_x(u) - f'_x(x)|du \leq C(l)\frac{1+x^2}{n} \int_{x+\frac{x}{\sqrt{n}}}^{2x} \frac{1}{(u-x)^2} \left(\bigvee_x^{t} f'_x \right) du$$
$$= C(l)\frac{1+x^2}{nx} \int_1^{\sqrt{n}} \left(\bigvee_x^{x+\frac{x}{v}} f'_x \right) dv \leq C(l)\frac{1+x^2}{nx} \sum_{k=1}^{[\sqrt{n}]} \int_k^{k+1} \left(\bigvee_x^{x+\frac{x}{v}} f'_x \right) dv$$
$$\leq C(l)\frac{1+x^2}{nx} \sum_{k=1}^{[\sqrt{n}]} \left(\bigvee_x^{x+\frac{x}{k}} f'_x \right).$$

Putting the values of J_1 and J_2 in (2.3.26), we have

$$\int_x^{2x} |f'_x(u)|(1 - \xi_n^{[\alpha]}(x,u))du \leq \frac{x}{\sqrt{n}} \left(\bigvee_x^{x+\frac{x}{\sqrt{n}}} f'_x \right) + C(l)\frac{1+x^2}{nx} \sum_{k=1}^{[\sqrt{n}]} \left(\bigvee_x^{x+\frac{x}{k}} f'_x \right).$$

Therefore, applying Cauchy-Schwarz inequality and Lemma 2.3.1, we get

$$\begin{aligned} |D_n^{[\alpha]}(f'_x, x)| \;\leq\; & M_f \int_{2x}^{\infty} (u^2+1)L_n^{[\alpha]}(x,u)du + |f(x)| \int_{2x}^{\infty} L_n^{[\alpha]}(x,u)du \\ & + |f'(x+)| \sqrt{C(l)\frac{1+x^2}{n}} + C(l)\frac{1+x^2}{nx^2}|f(2x) - f(x) - xf'(x+)| \\ & + \frac{x}{\sqrt{n}} \left(\bigvee_x^{x+\frac{x}{\sqrt{n}}} f'_x \right) + C(l)\frac{1+x^2}{nx} \sum_{k=1}^{[\sqrt{n}]} \left(\bigvee_x^{x+\frac{x}{k}} f'_x \right). \end{aligned} \tag{2.3.27}$$

Since $u \geq 2x$, we have

$$M_f \int_{2x}^{\infty} (u^2+1)L_n^{[\alpha]}(x,u)du + |f(x)| \int_{2x}^{\infty} L_n^{[\alpha]}(x,u)du$$
$$\begin{aligned} &\leq (M_f + |f(x)|) \int_{2x}^{\infty} L_n^{[\alpha]}(x,u)du + 4M_f \int_{2x}^{\infty} (u-x)^2 L_n^{[\alpha]}(x,u)du \\ &\leq \frac{M_f + |f(x)|}{x^2} \int_0^{\infty} (u-x)^2 L_n^{[\alpha]}(x,u)du + 4M_f \int_0^{\infty} (u-x)^2 L_n^{[\alpha]}(x,u)du \\ &\leq \left(4M_f + \frac{M_f + |f(x)|}{x^2}\right) C(l)\frac{1+x^2}{n}. \end{aligned}$$

Using the above inequality, we have

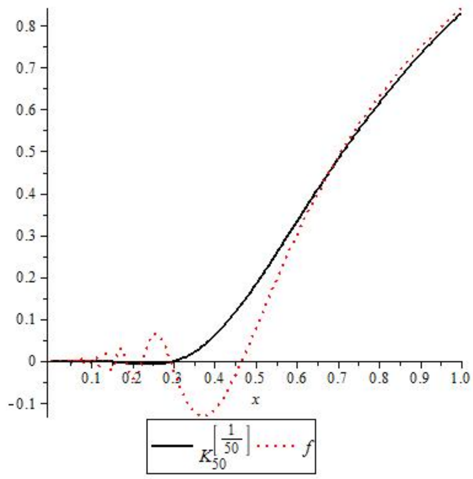
$$|D_n^{[\alpha]}(f'_x,x)| \leq \left(4M_f+\frac{M_f+|f(x)|}{x^2}\right)C(l)\frac{1+x^2}{n}+|f'(x+)|\sqrt{C(l)\frac{1+x^2}{n}}$$
$$+C(l)\frac{1+x^2}{nx^2}|f(2x)-f(x)-xf'(x+)|+\frac{x}{\sqrt{n}}\left(\bigvee_x^{x+\frac{x}{\sqrt{n}}} f'_x\right)$$
$$+C(l)\frac{1+x^2}{nx}\sum_{k=1}^{[\sqrt{n}]}\left(\bigvee_x^{x+\frac{x}{k}} f'_x\right). \tag{2.3.28}$$

Now from (2.3.25), (2.3.27) and (2.3.28), we reach the required result. □

Example 2. Let us consider the following function

$$f:[0,1]\to\mathbb{R}, f(x)=\begin{cases} x^2\sin\frac{1}{x}, x\neq 0,\\ 0,\ x=0.\end{cases}$$

The function f is differentiable and of bounded variation on $[0,1]$. For $n=50$ and $\alpha=1/n$, the convergence of $K_{50}^{[1/50]}(f;x)$ to $f(x)$ is illustrated in Figure 2.



The convergence of $K_{50}^{[1/50]}(f;x)$ to $f(x)$

2.4 Better Approximation

In 2011, Cárdenas-Morales et al. [49] considered the sequence of linear Bernstein-type operators defined for $f\in C[0,1]$ by $B_n(f\circ\tau^{-1})\circ\tau$, τ being any function that is continuously differentiable ∞ times on $[0,1]$, such that $\tau(0)=0$, $\tau(1)=1$ and $\tau'(x)>0$

for $x \in [0,1]$. A Durrmeyer type generalization of $B_n(f \circ \tau^{-1}) \circ \tau$ was also studied in [8]. Recently, Aral et al. [31] introduced similar modifications of the Szász-Mirakyan operators and the Durrmeyer modifications of these operators were introduced in [10].

Assume that ρ is any function satisfying the conditions:

$\rho_1)$ ρ is a continously differentiable function;

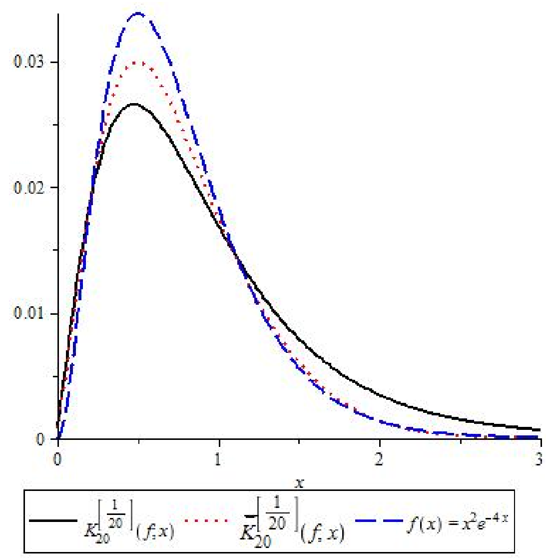
$\rho_2)$ $\rho(0) = 0, \inf\limits_{x \in [0,\infty)} \rho'(x) \geq 1.$

So, using the technique proposed in [49], we modify the operators defined in (2.1.4) as follows:

$$\overline{K}_n^{[\alpha]}(f;x) = \sum_{k=0}^{\infty} \overline{b}_{n,k}^{[\alpha]}(x) \int_{\frac{k}{n-1}}^{\frac{k+1}{n-1}} \left(f \circ \rho^{-1}\right)(t)dt,$$

where $\overline{b}_{n,k}^{[\alpha]}(x) = (n-1)\binom{n+k-1}{k} \dfrac{B\left(\frac{\rho(x)}{\alpha}+k, \frac{1}{\alpha}+n\right)}{B\left(\frac{\rho(x)}{\alpha}, \frac{1}{\alpha}\right)}.$

Example 3. We compare the convergence of Kantorovich variant of Stancu operators based on Polya-Eggenberger distribution $K_n^{[\alpha]}$ defined in (2.1.4) with the modified operators $\overline{K}_n^{[\alpha]}$. We have considered the function $f(x) = x^2e^{-4x}$ and $\rho(x) = x^2 + x$. For $x \in [0,3]$, $n = 20$, $\alpha = \frac{1}{20}$, the convergence of the operators $K_n^{[\alpha]}$ and $\overline{K}_n^{[\alpha]}$ to the function f is illustrated in Figure 3. Note that the approximation by $\overline{K}_n^{[\alpha]}$ is better then using the operators $K_n^{[\alpha]}$.



Approximation process by $K_n^{[\alpha]}$ and $\overline{K}_n^{[\alpha]}$

Chapter 3

Szász-Durrmeyer operators involving Boas-Buck polynomials of blending type

3.1 Introduction

In [129], Sucu et al. introduced the Sz$\acute{a}$sz operators involving Boas-Buck type polynomials as follows:

$$B_n(f;x) := \frac{1}{A(1)G(nxH(1))}\sum_{k=0}^{\infty} p_k(nx) f\left(\frac{k}{n}\right), \quad x \geq 0,\ n \in \mathbb{N}, \tag{3.1.1}$$

where generating function of the Boas-Buck type polynomials is given by

$$A(u)G(xH(u)) = \sum_{k=0}^{\infty} p_k(x)u^k, \tag{3.1.2}$$

and $A(u), G(u)$ and $H(u)$ are analytic functions described as
$A(u) = \sum_{k=0}^{\infty} a_k u^k,\ (a_0 \neq 0),\quad G(u) = \sum_{k=0}^{\infty} g_k u^k,\ (g_k \neq 0,\ \forall\ k),$
$H(u) = \sum_{k=1}^{\infty} h_k u^k,\ (h_1 \neq 0).$

Motivated by the above work, in the present chapter we define Sz$\acute{a}$sz-Durrmeyer type operators based on Boas-Buck type polynomials as follows:

For a function $f \in C_\gamma[0,\infty)$, we define

$$M_n(f;x) = \frac{1}{A(1)G(nxH(1))}\sum_{k=1}^{\infty}\frac{p_k(nx)}{B(k,n+1)}\int_0^\infty \frac{u^{k-1}}{(1+u)^{n+k+1}}f(u)du + \frac{a_0 b_0}{A(1)G(nxH(1))}f(0), \tag{3.1.3}$$

where $B(k,n+1)$ is the beta function and $x \geq 0,\ n \in \mathbb{N}$.

Alternatively, we may write the operator (3.1.3) as

$$M_n(f;x) := \int_0^\infty W(n,x,u)f(u)du, \tag{3.1.4}$$

where

$$W(n,x,u) := \frac{1}{A(1)G(nxH(1))}\sum_{k=1}^{\infty}\frac{p_k(nx)}{B(k,n+1)}\frac{u^{k-1}}{(1+u)^{n+k+1}} + \frac{a_0 b_0}{A(1)G(nxH(1))}\delta(u),$$

and $\delta(u)$ being the Dirac-delta function.

First we show the uniform convergence of the operators (3.1.3) by means of the Bohman Korovkin theorem on compact subsets of $[0,\infty)$ for functions in $C_\gamma[0,\infty)$. Then a Voronovskaja type asymptotic theorem and the rate of convergence by means of the weighted modulus of continuity are established. Further, by means of the unified Ditzian-Totik modulus of smoothness, we obtain a direct approximation theorem. The approximation of functions with derivatives of bounded variation is also studied.

3.2 Preliminaries

In this section, we study the approximation properties of the operators M_n for functions belonging to different function spaces.

Lemma 3.2.1. *[129] For the operators B_n, one has*

(i) $B_n(1;x) = 1$,

(ii) $B_n(s;x) = \dfrac{G'(nxH(1))}{G(nxH(1))}x + \dfrac{A'(1)}{nA(1)}$,

(iii) $B_n(s^2;x) = \frac{G''(nxH(1))}{G(nxH(1))}x^2 + \frac{(2A'(1)+(1+H''(1))A(1))}{nA(1)}\frac{G'(nxH(1))}{G(nxH(1))}x + \frac{A''(1)+A(1)}{n^2A(1)}$,

(iv) $$B_n(s^3;x) = \frac{G'''(nxH(1))}{G(nxH(1))}x^3 + \frac{(3A(1)+3A'(1)+3A(1)H''(1))}{nA(1)}\frac{G''(nxH(1))}{G(nxH(1))}x^2 + \frac{(6A'(1)+A(1)+3A(1)H''(1)+3A''(1)+3A'(1)H''(1)+A(1)H'''(1))}{n^2A(1)}\frac{G'(nxH(1))}{G(nxH(1))}x + \frac{A'''(1)+3A''(1)+A'(1)}{n^3A(1)},$$

(v) $$B_n(s^4;x) = \frac{G^{iv}(nxH(1))}{G(nxH(1))}x^4 + \left(\frac{4A'(1)+6A(1)H''(1)+6A(1)}{nA(1)}\right)\frac{G'''(nxH(1))}{G(nxH(1))}x^3 + \left(\frac{6A''(1)+12A'(1)H''(1)+4A(1)H'''(1)+3A(1)(H''(1))^2+7A(1)+18A'(1)}{n^2A(1)} + \frac{18A(1)H''(1)}{n^2A(1)}\right)\frac{G''(nxH(1))}{G(nxH(1))}x^2 + \left(\frac{4A'''(1)+6A''(1)H''(1)+4A'(1)H'''(1)}{n^3A(1)} + \frac{+A(1)H^{iv}(1)36A'(1)+A(1)+7A(1)H''(1)+18A''(1)+18A'(1)H''(1)}{n^3A(1)} + \frac{6A(1)H'''(1)-22A'(1)}{n^3A(1)}\right)\frac{G'(nxH(1))}{G(nxH(1))}x + \left(\frac{13A''(1)+A'(1)+A^{iv}}{A(1)}\right).$$

Proof. Since the identities (i)-(iii) are proved in [129], we give below the proof of only (iv): The identity (v) follows similarly.
It is easily seen that

$$\begin{aligned}\sum_{k=0}^{\infty} k^3 p_k(nx) &= (4A''(1)+A'(1))G(nxH(1)) + \Big(6A'(1)+A(1)+3A(1)H''(1)+3A''(1) \\ &\quad +3A'(1)H''(1)+A(1)H'''(1)\Big)G'(nxH(1))nx + \Big(3A(1)+3A'(1) \\ &\quad +3A(1)H''(1)\Big)G''(nxH(1))n^2x^2 + A(1)G'''(nxH(1))n^3x^3,\end{aligned}$$

and

$$\begin{aligned}
\sum_{k=0}^{\infty} k^4 p_k(nx) &= A(1)G^{iv}(nxH(1))n^4x^4 + \Big(4A'(1) + 6A(1)H''(1) + 6A(1)\Big) \\
&\times G'''(nxH(1))n^3x^3 + \Big(6A''(1) + 12A'(1)H''(1) + 4A(1)H'''(1) \\
&+3A(1)(H''(1))^2 + 7A(1) + 18A'(1) + 18A(1)H''(1)\Big)G''(nxH(1))n^2x^2 \\
&+\Big(4A'''(1) + 6A''(1)H''(1) + 4A'(1)H'''(1) + A(1)H^{iv}(1) + 36A'(1) \\
&+A(1) + 7A(1)H''(1) + 18A''(1) + 18A'(1)H''(1) + 6A(1)H'''(1) \\
&-22A'(1)\Big)G'(nxH(1))nx + (13A''(1) + A'(1) + A^{iv}(1))G(nxH(1)).
\end{aligned}$$

□

Now, by simple calculations we obtain the identity (iii) and (iv). Hence the details are omitted.

In the following lemma we obtain the moments for the operators defined by (3.1.3), utilizing Lemma 3.2.1:

Lemma 3.2.2. *For the operators M_n there hold the equalities:*

(i) $M_n(1;x) = 1$;

(ii) $M_n(u;x) = \frac{1}{n}\left(\frac{G'(nxH(1))}{G(nxH(1))}nx + \frac{A'(1)}{A(1)}\right)$;

(iii) $M_n(u^2;x) = \frac{1}{n(n-1)}\left[\frac{G''(nxH(1))}{G(nxH(1))}n^2x^2 + \left(2\frac{A'(1)}{A(1)} + H''(1) + 2\right)\frac{G'(nxH(1))}{G(nxH(1))}nx + 2\frac{A'(1)}{A(1)} + \frac{A''(1)}{A(1)}\right]$;

(iv) $M_n(u^3;x) = \frac{1}{n(n-1)(n-2)}\left[\frac{G'''(nxH(1))}{G(nxH(1))}n^3x^3 + \left(3\frac{A'(1)}{A(1)} + 6 + 3H''(1)\right) \times \frac{G''(nxH(1))}{G(nxH(1))}n^2x^2 + \left(12\frac{A'(1)}{A(1)} + H''(1) + 3\frac{A'(1)}{A(1)}H''(1) + H'''(1) + 4\right)\frac{G'(nxH(1))}{G(nxH(1))}nx + 7\frac{A''(1)}{A(1)} + 6\frac{A'(1)}{A(1)}\right]$,

(v) $M_n(u^4;x) = \frac{1}{n(n-1)(n-2)(n-3)}\left[\frac{G^{iv}(nxH(1))}{G(nxH(1))}n^4x^4 + \left(4\frac{A'(1)}{A(1)} + 6H''(1) + 12\right)\frac{G'''(nxH(1))}{G(nxH(1))}n^3x^3 + \left(6\frac{A''(1)}{A(1)} + 12\frac{A'(1)}{A(1)}H''(1) + 21\frac{A'(1)}{A(1)} + 3\frac{H''(1)}{A(1)} + 4H'''(1) + \right.\right.$

$$18H''(1) + 3(H''(1))^2 + 21\Big)\frac{G''(nxH(1))}{G(nxH(1))}n^2x^2 + \Big(4\frac{A''(1)}{A(1)} + 6\frac{A''(1)}{A(1)}H''(1) + 36\frac{A''(1)}{A(1)}H''(1) + 42\frac{A'(1)}{A(1)} + 4\frac{A'(1)}{A(1)}H'''(1) + 36\frac{A''(1)}{A(1)} + H^{iv}(1) + 12H'''(1) + 36H''(1) + 24\Big)\frac{G'(nxH(1))}{G(nxH(1))}nx + \frac{A^{iv}(1)}{A(1)} + 48\frac{A''(1)}{A(1)} + 13\frac{A'(1)}{A(1)} + 11\Big].$$

Hence as a consequence of Lemma 3.2.2, we find:

Lemma 3.2.3. *For the operator (3.1.3), we have the following results:*

(i) $M_n((u-x);x) = \left(\frac{G'(nxH(1))}{G(nxH(1))} - 1\right)x + \frac{A'(1)}{nA(1)}$,

(ii) $$M_n((u-x)^2;x) = \left(\frac{n}{n-1}\frac{G''(nxH(1))}{G(nxH(1))} - 2\frac{G'(nxH(1))}{G(nxH(1))} + 1\right)x^2 + \left(\frac{1}{n-1}\left(2\frac{A'(1)}{A(1)} + H''(1) + 2\right)\frac{G'(nxH(1))}{G(nxH(1))} - \frac{2}{n}\frac{A'(1)}{A(1)}\right)x + \frac{1}{n(n-1)}\left(2\frac{A'(1)}{A(1)} + \frac{A''(1)}{A(1)}\right),$$

(iii) $$M_n((u-x)^4;x) = \left\{\frac{n^3}{(n-1)(n-2)(n-3)}\frac{G^{iv}(nxH(1))}{G(nxH(1))} - \frac{4n^2}{(n-1)(n-2)}\frac{G'''(nxH(1))}{G(nxH(1))} + \frac{6n}{(n-1)}\frac{G''(nxH(1))}{G(nxH(1))} - 4\frac{G'(nxH(1))}{G(nxH(1))} + 1\right\}x^4 + \left\{\frac{n^2}{(n-1)(n-2)(n-3)}\frac{G'''(nxH(1))}{G(nxH(1))}\left(4\frac{A'(1)}{A(1)} + 6H''(1) + 12\right) - \frac{4n}{(n-1)(n-2)}\frac{G''(nxH(1))}{G(nxH(1))}\left(3\frac{A'(1)}{A(1)} + 3H''(1) + 6\right) + \frac{6}{(n-1)}\frac{G'(nxH(1))}{G(nxH(1))}\left(2\frac{A'(1)}{A(1)} + H''(1) + 2\right) - \frac{4}{n}\frac{A'(1)}{A(1)}\right\}x^3$$
$$+ \left\{\frac{n}{(n-1)(n-2)(n-3)}\frac{G''(nxH(1))}{G(nxH(1))}\left(6\frac{A''(1)}{A(1)} + 12\frac{A'(1)}{A(1)}H''(1) + 21\frac{A'(1)}{A(1)} + 3\frac{H''(1)}{A(1)} + 4H'''(1) + 18H''(1) + 3(H''(1))^2 + 21\right) - \frac{4}{(n-1)(n-2)}\frac{G'(nxH(1))}{G(nxH(1))}\left(12\frac{A'(1)}{A(1)} + H''(1) + 3\frac{A'(1)}{A(1)}H''(1) + H''(1) + 4\right)\right\}x^2$$
$$+ \left\{\frac{1}{(n-1)(n-2)(n-3)}\frac{G'(nxH(1))}{G(nxH(1))}\left(4\frac{A''(1)}{A(1)} + 6\frac{A''(1)}{A(1)}H''(1) + 36\frac{A''(1)}{A(1)}H''(1) + 42\frac{A'(1)}{A(1)} + 4\frac{A'(1)}{A(1)}H'''(1) + 36\frac{A''(1)}{A(1)} + H^{iv}(1) + 12H'''(1) + 36H''(1) + 24\right) - \frac{4}{n(n-1)(n-2)}\left(7\frac{A''(1)}{A(1)} + 6\frac{A'(1)}{A(1)}\right)\right\}x$$

$$+\frac{1}{(n-1)(n-2)(n-3)}\left(\frac{A^{iv}(1)}{A(1)}+48\frac{A''(1)}{A(1)}+13\frac{A'(1)}{A(1)}+11\right).$$

Now, in order to study the approximation properties of the considered operators (3.1.3), we make the following assumptions on the analytic functions $A(u), H(u)$ and $G(u)$. It is to be noted that the following assumptions are valid pointwise. These assumptions will be needed to prove the Theorems 3.3.3, 3.3.6 and 3.3.8 of this chapter which are pointwise results.

$$\lim_{n\to\infty} n\left\{\frac{G'(nxH(1))}{G(nxH(1))}-1\right\}=l_1(x),$$

$$\lim_{n\to\infty} n\left\{\frac{n}{n-1}\frac{G''(nxH(1))}{G(nxH(1))}-2\frac{G'(nxH(1))}{G(nxH(1))}+1\right\}=l_2(x),$$

$$\lim_{n\to\infty} n^2\left\{\frac{n^2}{(n-1)(n-2)(n-3)}\frac{G'''(nxH(1))}{G(nxH(1))}\left(4\frac{A'(1)}{A(1)}+6H''(1)+12\right)\right.$$
$$-\frac{4n}{(n-1)(n-2)}\frac{G''(nxH(1))}{G(nxH(1))}\left(3\frac{A'(1)}{A(1)}+3H''(1)+6\right)$$
$$\left.+\frac{6}{(n-1)}\frac{G'(nxH(1))}{G(nxH(1))}\left(2\frac{A'(1)}{A(1)}+H''(1)+2\right)-\frac{4}{n}\frac{A'(1)}{A(1)}\right\}=l_3(x),$$

$$\lim_{n\to\infty} n^2\left\{\frac{n^3}{(n-1)(n-2)(n-3)}\frac{G^{iv}(nxH(1))}{G(nxH(1))}-\frac{4n^2}{(n-1)(n-2)}\frac{G'''(nxH(1))}{G(nxH(1))}+\frac{6n}{(n-1)}\frac{G''(nxH(1))}{G(nxH(1))}-\right.$$
$$\left.4\frac{G'(nxH(1))}{G(nxH(1))}+1\right\}=l_4(x).$$

As a result of the above assumptions and applying Lemma 3.2.3, we reach the following important result:

Lemma 3.2.4. *For the operator (3.1.3) we have*

(i) $\lim_{n\to\infty} nM_n((u-x);x)=l_1(x)x+\frac{A'(1)}{A(1)}$,

(ii) $\lim_{n\to\infty} nM_n((u-x)^2;x)=l_2(x)x^2+x(H''(1)+2)=\eta(x)$, *(say)*

(iii) $\lim_{n\to\infty} n^2M_n((u-x)^4;x)=l_4(x)x^4+l_3(x)x^3+\left(6\frac{A''(1)}{A(1)}-27\frac{A'(1)}{A(1)}+\frac{H''(1)}{A(1)}+\right.$ $\left.14H''(1)+3(H''(1))^2+5\right)=\nu(x)$, *(say).*

3.3 Main Results

Throughout the chapter, we assume $\delta_n(x) = M_n((u-x)^2;x)$. In the following theorem we show that the operators defined by (3.1.3) are an approximation process for $f \in C_\gamma[0,\infty)$, using the Bohman-Korovkin theorem.

Theorem 3.3.1. *Let $f \in C_\gamma[0,\infty)$. Then,*

$$\lim_{n\to\infty} M_n(f;x) = f(x),$$

holds uniformly in $x \in [0,a]$, $a > 0$.

Proof. From Lemma 3.2.2, it follows that

$$\lim_{n\to\infty} M_n(u^i;x) = x^i, \ \ i = 0,1,2$$

uniformly in $x \in [0,a]$. Hence by Bohman-Korovkin theorem, the required result is immediate. □

In the following theorem we find the rate of convergence of the operators M_n for functions in $Lip_M^*(\xi)$.

Theorem 3.3.2. *Let $f \in Lip_M^*(\xi)$ and $\xi \in (0,1]$. Then, for all $x \in (0,\infty)$, we have*

$$|M_n(f;x) - f(x)| \ \leq \ M\left(\frac{\delta_n(x)}{x}\right)^{\frac{\xi}{2}}.$$

Proof. By the linearity and positivity of the operators M_n, from (3.1.4) we obtain

$$|M_n(f;x) - f(x)| \ \leq \ \int_0^\infty W(n,x,u)|f(u) - f(x)|du.$$

Applying the Hölder's inequality with $p = \dfrac{2}{\xi}$ and $q = \dfrac{2}{2-\xi}$ and Lemma 3.2.2, we have

$$\begin{aligned}
|M_n(f;x) - f(x)| \ &\leq \ \left(\int_0^\infty W(n,x,u)|f(u) - f(x)|^{\frac{2}{\xi}}dt\right)^{\frac{\xi}{2}}\left(\int_0^\infty W(n,x,u)du\right)^{\frac{2-\xi}{2}} \\
&\leq \ \left(\int_0^\infty W(n,x,u)|f(u) - f(x)|^{\frac{2}{\xi}}du\right)^{\frac{\xi}{2}} \\
&\leq \ M\left(\int_0^\infty W(n,x,u)\frac{(u-x)^2}{(u+x)}du\right)^{\frac{\xi}{2}} \\
&\leq \ M\left(\frac{\delta_n(x)}{x}\right)^{\frac{\xi}{2}}.
\end{aligned}$$

Thus, we reach the desired result. □

In our next result, we establish a Voronovskaja type approximation theorem.

Theorem 3.3.3. *Let $f \in C_\gamma[0,\infty)$, admitting a derivative of second order at a point $x \in [0,\infty)$, then there holds*

$$\lim_{n\to\infty} n(M_n(f;x) - f(x)) = \left\{l_1(x)x + \frac{A'(1)}{A(1)}\right\} f'(x) + \left\{l_2(x)x^2 + x(H''(1)+2)\right\} \frac{f''(x)}{2}.$$

If f'' is continuous on $[0,\infty)$ then the limit in (3.3.3) holds uniformly in $x \in [0,a] \subset [0,\infty)$, $a > 0$.

Proof. By the Taylor's theorem

$$f(u) = f(x) + f'(x)(u-x) + \frac{1}{2}f''(x)(u-x)^2 + \varepsilon(u,x)(u-x)^2, \tag{3.3.1}$$

where, $\varepsilon(u,x) \in C_\gamma[0,\infty)$ and $\lim_{u\to x}\varepsilon(u,x) = 0$.
Applying the operator $M_n(.,x)$ on both sides of (3.3.1), we have

$$\begin{aligned}\lim_{n\to\infty} n\left(M_n(f;x) - f(x)\right) &= \lim_{n\to\infty} n\, M_n(u-x;x)f'(x) + \lim_{n\to\infty} n\, M_n((u-x)^2;x)\frac{f''(x)}{2}\\ &\quad + \lim_{n\to\infty} n\, M_n(\varepsilon(u,x)(u-x)^2;x).\end{aligned} \tag{3.3.2}$$

Using Cauchy-Schwarz inequality in the last term of the right side of (3.3.2), we get

$$nM_n(\varepsilon(u,x)(u-x)^2;x) \leq \sqrt{M_n(\varepsilon^2(u,x);x)}\sqrt{n^2M_n((u-x)^4;x)}.$$

Since $\varepsilon(u,x) \to 0$, as $u \to x$, applying Theorem 3.3.1, for every $x \in [0,\infty)$ we obtain $\lim_{n\to\infty} M_n(\varepsilon^2(u,x);x) = \varepsilon^2(x,x) = 0$.
Next applying Lemma 3.2.4, for sufficiently large n and every $x \in [0,\infty)$

$$n^2M_n((u-x)^4;x) = O(1).$$

Hence,

$$\lim_{n\to\infty} nM_n(\varepsilon(u,x)(u-x)^2;x) = 0. \tag{3.3.3}$$

Now, from (3.3.2), (3.3.3) and Lemma 3.2.4, the required result follows. □

The uniformity assertion follows from the uniform continuity of f'' on $[0,a]$ and the fact that all the other estimates hold uniformly in $x \in [0,a]$.

In our next theorem, we obtain the degree of approximation of the M_n operators for functions in the space $C_2[0,\infty)$ in terms of the classical modulus of continuity.

Theorem 3.3.4. *For $f \in C_2[0,\infty)$, we have the following inequality*

$$|M_n(f;x) - f(x)| \leq 4M_f(1+x^2)\delta_n(x) + 2\omega_{b+1}\left(f; \sqrt{\delta_n(x)}\right). \tag{3.3.4}$$

where, $\omega(f;\delta_n(x))$ is the modulus of continuity of f on [0,b+1].

Proof. From [84], for $u \in [0,\infty)$ and $x \in [0,b]$, we obtain

$$|f(u) - f(x)| \leq 4M_f(u-x)^2(1+x^2) + \left(1 + \frac{|u-x|}{\delta}\right)\omega_{b+1}(f,\delta), \quad \delta > 0.$$

Hence, by applying Cauchy-Schwarz inequality

$$\begin{aligned}
|M_n(f;x) - f(x)| &\leq 4M_f(1+x^2)M_n((u-x)^2;x) + \omega_{b+1}(f,\delta) \\
&\quad \left(1 + \frac{1}{\delta}(M_n((u-x)^2;x))^{1/2}\right) \\
&= M_f(1+x^2)\delta_n(x) + \omega_{b+1}(f,\delta)\left(1 + \frac{1}{\delta}\sqrt{\delta_n(x)}\right).
\end{aligned}$$

Choosing $\delta = \sqrt{\delta_n(x)}$, we get the desired result.

□

The next section is devoted to the weighted approximation properties of the operators M_n.

3.3.1 Weighted approximation

Firstly, we establish the following basic approximation theorem for functions in the weighted space of continuous functions $C_2^0[0,\infty)$ by the operators M_n:

Theorem 3.3.5. *For $f \in C_2^0[0,\infty)$, and $a > 0$, we have*

$$\lim_{n\to\infty} \sup_{x\in[0,\infty)} \frac{|M_n(f;x) - f(x)|}{(1+x^2)^{1+a}} = 0.$$

Proof. Let $x_0 \in [0,\infty)$ be an arbitrary but fixed point. Then

$$\begin{aligned}\sup_{x\in[0,\infty)} \frac{|M_n(f;x)-f(x)|}{(1+x^2)^{1+a}} &\le \sup_{x\le x_0} \frac{|M_n(f;x)-f(x)|}{(1+x^2)^{1+a}} + \sup_{x>x_0} \frac{|M_n(f;x)-f(x)|}{(1+x^2)^{1+a}} \\ &\le ||M_n(f;.)-f||_{C[0,x_0]} + ||f||_2 \sup_{x>x_0} \frac{M_n(1+u^2;x)}{(1+x^2)^{1+a}} \\ &\quad + \sup_{x>x_0} \frac{|f(x)|}{(1+x^2)^{1+a}}.\end{aligned} \tag{3.3.5}$$

Since $|f(x)| \le ||f||_2(1+x^2)$, we have

$$\sup_{x>x_0} \frac{|f(x)|}{(1+x^2)^{1+a}} \le \frac{||f||_2}{(1+x_0^2)^a}.$$

Let $\epsilon > 0$, be arbitrary. We choose x_0 to be so large that

$$\frac{||f||_2}{(1+x_0^2)^a} < \frac{\epsilon}{6} \text{ so that } \sup_{x>x_0} \frac{|f(x)|}{(1+x^2)^{1+a}} < \frac{\epsilon}{6}. \tag{3.3.6}$$

From Theorem 3.3.1, there exists $n_1 \in \mathbb{N}$ such that

$$\begin{aligned}||f||_2 \frac{M_n(1+u^2;x)}{(1+x^2)^{1+a}} &\le \frac{||f||_2}{(1+x^2)^a}\left(1+x^2+\frac{\epsilon}{3||f||_2}\right), \ \forall n > n_1 \\ &\le \frac{||f||_2}{(1+x_0^2)^a} + \frac{\epsilon}{3}, \ \forall n > n_1 \text{ and } x > x_0.\end{aligned}$$

Hence,

$$||f||_2 \sup_{x>x_0} \frac{M_n(1+u^2;x)}{(1+x^2)^{1+a}} \le \frac{\epsilon}{2}, \ \forall\, n > n_1. \tag{3.3.7}$$

Applying Theorem 3.3.4, we can find $n_2 \in \mathbb{N}$ such that

$$||M_n(f;.)-f||_{C[0,x_0]} < \frac{\epsilon}{3}, \forall\, n > n_2. \tag{3.3.8}$$

Let $n_0 = \max(n_1, n_2)$. Combining (3.3.5)-(3.3.8), we obtain

$$\sup_{x\in[0,\infty)} \frac{|M_n(f;x)-f(x)|}{(1+x^2)^{1+a}} < \epsilon, \ \forall n > n_0.$$

Hence the required result is obtained. □

In our next theorem, we determine the order of approximation for functions in a weighted space of continuous functions on $[0,\infty)$ by M_n operators:

Theorem 3.3.6. *Let $f \in C_2^0[0,\infty)$. Then for sufficiently large n, we have*

$$|M_n(f;x) - f(x)| \leq C(x)\Omega\left(f; \frac{1}{\sqrt{n}}\right), \tag{3.3.9}$$

where $C(x) = 2(1+x^2)\left(1 + C_1\,|\eta(x)| + \sqrt{C_1}\,|\eta(x)|^{1/2}\left(1+\sqrt{C_2}\,|\nu(x)|^{1/2}\right)\right)$, C_1, C_2 *are constants independent of x and n and $\eta(x), \nu(x)$ are as given in Lemma 3.2.4.*

Proof. For $x \in (0,\infty)$ and $\delta > 0$, using (0.4.1) and Lemma (0.4.1)

$$|f(u) - f(x)| \leq 2(1+x^2)(1+(u-x)^2)\left(1 + \frac{|u-x|}{\delta}\right)\Omega(f;\delta).$$

Applying $M_n(\cdot;x)$ both sides, we can write

$$\begin{aligned} |M_n(f;x) - f(x)| &\leq 2(1+x^2)\Omega(f;\delta)\left(1 + M_n((u-x)^2;x)\right. \\ &\quad \left. + M_n\left((1+(u-x)^2)\frac{|u-x|}{\delta};x\right)\right). \end{aligned} \tag{3.3.10}$$

From Lemma 3.2.4, for sufficiently large n, it follows

$$nM_n((u-x)^2;x) \leq C_1\,|\eta(x)| \text{ and } n^2M_n((u-x)^4;x) \leq C_2\,|\nu(x)|, \tag{3.3.11}$$

Now, applying the Cauchy-Schwarz inequality in the last term of (3.3.10), we obtain

$$\begin{aligned} M_n\left((1+(u-x)^2)\frac{|u-x|}{\delta};x\right) &\leq \frac{1}{\delta}\left(M_n((u-x)^2;x)\right)^{1/2} + \frac{1}{\delta}\left(M_n((u-x)^4;x)\right)^{1/2} \\ &\quad \left(M_n((u-x)^2;x)\right)^{1/2}. \end{aligned} \tag{3.3.12}$$

Combining the estimates (3.3.10)-(3.3.12) and taking

$$\delta = \frac{1}{\sqrt{n}},$$

we reach the required result. □

3.3.2 Unified modulus theorem

Theorem 3.3.7. *Let $f \in C_B[0,\infty)$, then for sufficiently large n*

$$|M_n(f;x) - f(x)| \;\; \leq \;\; C\omega_{\phi^\tau}\left(f; \frac{\phi^{1-\tau}(x)}{\sqrt{n}}\right),$$

where C is independent of f and n.

Proof. By the definition of $K_{\phi^\tau}(f,u)$, for a fixed n, x, τ we can choose $g = g_{n,x,\tau} \in W_\tau$ such that

$$||f-g|| + \frac{\phi^{1-\tau}(x)}{\sqrt{n}}||\phi^\tau g'|| \leq 2K_{\phi^\tau}\left(f; \frac{\phi^{1-\tau}(x)}{\sqrt{n}}\right). \tag{3.3.13}$$

We may write

$$\begin{aligned} |M_n(f;x) - f(x)| \;\; &\leq \;\; |M_n(f-g;x)| + |M_n(g;x) - g(x)| + |g(x) - f(x)| \\ &\leq \;\; 2||f-g|| + |M_n(g;x) - g(x)|. \end{aligned} \tag{3.3.14}$$

Since $g \in W_\tau$, we have

$$g(u) = g(x) + \int_x^u g'(v)dv$$

and so

$$|M_n(g;x) - g(x)| \;\; \leq \;\; M_n\left(\left|\int_x^u g'(v)dv\right|; x\right). \tag{3.3.15}$$

By applying Hölder's inequality, we get

$$\begin{aligned} \left|\int_x^u g'(v)dv\right| \;\; &\leq \;\; ||\phi^\tau g'|| \left|\int_x^u \frac{dv}{\phi^\tau(v)}\right| \\ &\leq \;\; ||\phi^\tau g'|| \, |u-x|^{1-\tau} \left|\int_x^u \frac{dv}{\phi(v)}\right|^\tau, \end{aligned}$$

we may write

$$\left|\int_x^u \frac{dv}{\phi(v)}\right| \;\; \leq \;\; \left|\int_x^u \frac{dv}{\sqrt{v}}\right| \left(\frac{1}{\sqrt{1+x}} + \frac{1}{\sqrt{1+u}}\right).$$

Hence, on using the inequality $|a+b|^r \leq |a|^r + |b|^r,\ \ 0 \leq r \leq 1$.

$$\begin{aligned} \left|\int_x^u g'(v)dv\right| &\leq \frac{2^\tau ||\phi^\tau g'||\,|u-x|}{x^{\tau/2}}\left(\frac{1}{\sqrt{1+x}}+\frac{1}{\sqrt{1+u}}\right)^\tau \\ &\leq \frac{2^\tau ||\phi^\tau g'||\,|u-x|}{x^{\tau/2}}\left(\frac{1}{(1+x)^{\tau/2}}+\frac{1}{(1+u)^{\tau/2}}\right). \end{aligned} \tag{3.3.16}$$

Thus, from (3.3.15), (3.3.16),Cauchy-Schwarz inequality and using Theorem 3.3.1, we obtain

$$\begin{aligned} |M_n(g;x)-g(x)| &\leq \frac{2^\tau ||\phi^\tau g'||}{x^{\tau/2}} M_n\left(|u-x|\left(\frac{1}{(1+x)^{\tau/2}}+\frac{1}{(1+u)^{\tau/2}}\right);x\right) \\ &\leq \frac{2^\tau ||\phi^\tau g'||}{x^{\tau/2}}\left(\frac{1}{(1+x)^{\tau/2}}\sqrt{M_n((u-x)^2;x)}\right. \\ &\quad \left. +\sqrt{M_n((u-x)^2;x)}\sqrt{M_n\left((1+u)^{-\tau};x\right)}\right) \\ &\leq 2^\tau ||\phi^\tau g'||\sqrt{M_n((u-x)^2;x)}\left\{\phi^{-\tau}(x)+x^{-\tau/2}\sqrt{M_n\left((1+u)^{-\tau};x\right)}\right\} \\ &\leq 2^\tau C||\phi^\tau g'||\frac{\phi(x)}{\sqrt{n}}\left\{\phi^{-\tau}(x)+x^{-\tau/2}(1+x)^{-\tau/2}\right\}, \\ &= 2^{\tau+1}\frac{||\phi^\tau g'||\phi^{1-\tau}(x)}{\sqrt{n}}, \end{aligned} \tag{3.3.17}$$

for sufficiently large n.

Hence, combining (3.3.13)-(3.3.15) and (3.3.17), we find

$$\begin{aligned} |M_n(f;x)-f(x)| &\leq 2||f-g|| + 2^{\tau+1}C||\phi^\tau g'||\frac{\phi^{1-\tau}}{\sqrt{n}} \\ &\leq C\left\{||f-g|| + \frac{\phi^{1-\tau}(x)}{\sqrt{n}}||\phi^\tau g'||\right\} \\ &\leq 2CK_{\phi^\tau}\left(f;\frac{\phi^{1-\tau}(x)}{\sqrt{n}}\right) \\ &\leq C\omega_{\phi^\tau}\left(f;\frac{\phi^{1-\tau}(x)}{\sqrt{n}}\right). \end{aligned}$$

This completes the proof of the theorem. □

3.3.3 Rate of Convergence of Szász-Durrmeyer operators based on Boas Buck polynomials

In this section, we discuss the approximation of functions with a derivative of bounded variation. We show that the points x where $f'(x+)$ and $f'(x-)$ exist, the operators $M_n(f;x)$ converge to the function $f(x)$, as $n \to \infty$.

Lemma 3.3.1. *Let $\alpha = \alpha(n) \to 0$, as $n \to \infty$ and $\lim_{n\to\infty} n\alpha(n) = l \in \mathbb{R}$. For every $x > 0$ and sufficiently large n, we have*

$$\text{(i)}\ \ \xi_n(x,u) = \int_0^u W(n,x,u)dv \leq \frac{C_1\,|\eta(x)|}{(x-u)^2},$$

$$\text{(ii)}\ \ 1-\xi_n(x,u) = \int_u^\infty W(n,x,u)dv \leq \frac{C_1\,|\eta(x)|}{(u-x)^2},$$

where $\eta(x)$ is as given in Lemma 3.2.4.

Proof. Using Lemma 3.2.2 and (3.3.11), we have

$$\begin{aligned}\xi_n(x,u) = \int_0^u W(n,x,u)dv &\leq \int_0^u \left(\frac{x-v}{x-u}\right)^2 W(n,x,u)(x,v)dv\\ &\leq \frac{1}{(x-u)^2} M_n((u-x)^2;x) \leq \frac{C_1\,|\eta(x)|}{(x-u)^2},\end{aligned}$$

when n is large enough.

Similarly, we can prove (ii). □

Theorem 3.3.8. *Let $f \in DBV[0,\infty)$. Then, for every $x > 0$ and sufficiently large n*

$$\begin{aligned}|M_n(f;x) - f(x)| \ \leq\ & \left[\left(\frac{G'(nxH(1))}{G(nxH(1))} - 1\right)x + \frac{A'(1)}{nA(1)}\right]\left|\frac{f'(x+)+f'(x-)}{2}\right|\\ & + \sqrt{C_1\,|\eta(x)|}\left|\frac{f'(x+)-f'(x-)}{2}\right| + \frac{C_1\,|\eta(x)|}{x}\sum_{k=1}^{[\sqrt{n}]}\left(\bigvee_{x-\frac{x}{k}}^{x} f'_x\right)\\ & + \frac{x}{\sqrt{n}}\left(\bigvee_{x-\frac{x}{\sqrt{n}}}^{x} f'_x\right) + \left(4M_f + \frac{M_f + |f(x)|}{x^2}\right)C_1\,|\eta(x)|\\ & + |f'(x+)|\sqrt{C_1\,|\eta(x)|} + \frac{C_1\,|\eta(x)|}{x^2}|f(2x) - f(x) - xf'(x+)|\\ & + \frac{x}{\sqrt{n}}\left(\bigvee_{x}^{x+\frac{x}{\sqrt{n}}} f'_x\right) + \frac{C_1\,|\eta(x)|}{x}\sum_{k=1}^{[\sqrt{n}]}\left(\bigvee_{x}^{x+\frac{x}{\sqrt{n}}} f'_x\right),\end{aligned}$$

where C_1 is a positive constant and $\bigvee_a^b f$ denotes the total variation of f on $[a,b]$ and f'_x is defined by

$$f'_x(u) = \begin{cases} f'(u) - f'(x-), & 0 \le u < x \\ 0, & u = x, \\ f'(u) - f'(x+) & x < u < \infty. \end{cases} \tag{3.3.18}$$

Proof. For any $f \in DBV\,[0,\infty)$, from (3.3.18), we may write

$$\begin{aligned} f'(v) &= \frac{1}{2}\left(f'(x+) + f'(x-)\right) + f'_x(v) + \frac{1}{2}\left(f'(x+) - f'(x-)\right)\text{sgn}(v-x) \\ &\quad + \delta_x(v)\left(f'(v) - \frac{1}{2}(f'(x+) + f'(x-))\right), \end{aligned} \tag{3.3.19}$$

$$\text{where} \quad \delta_x(v) = \begin{cases} 1, & v = x \\ 0, & v \neq x. \end{cases}$$

Since $M_n(e_0; x) = 1$, using (3.3.19) for every $x \in (0,\infty)$, we get

$$\begin{aligned} M_n(f;x) - f(x) &= \int_0^\infty W(n,x,t)(f(u) - f(x))du = \int_0^\infty W(n,x,u)\left(\int_x^u f'(v)dv\right)du \\ &= -\int_0^x \left(\int_u^x f'(v)dv\right) W(n,x,u)du \\ &\quad + \int_x^\infty \left(\int_x^u f'(v)dv\right) W(n,x,u)du. \end{aligned} \tag{3.3.20}$$

Let

$$I_1 := \int_0^x \left(\int_u^x f'(v)dv\right) W(n,x,u)du, \quad I_2 := \int_x^\infty \left(\int_x^u f'(v)dv\right) W(n,x,u)du.$$

Since $\int_x^u \delta_x(v)dv = 0$, and using (3.3.19), we have

$$\begin{aligned} I_1 &= \int_0^x \left\{\int_u^x \left(\frac{1}{2}\left(f'(x+) + f'(x-)\right) + f'_x(v)\right.\right. \\ &\quad \left.\left. + \frac{1}{2}\left(f'(x+) - f'(x-)\right)\text{sgn}(v-x)\right)dv\right\} W(n,x,u)du \\ &= \frac{1}{2}\left(f'(x+) + f'(x-)\right)\int_0^x (x-u)W(n,x,u)du + \int_0^x \left(\int_u^x f'_x(v)dv\right) W(n,x,u)du \\ &\quad - \frac{1}{2}\left(f'(x+) - f'(x-)\right)\int_0^x (x-u)W(n,x,u)du. \end{aligned} \tag{3.3.21}$$

Similarly, we have

$$\begin{aligned} I_2 &= \int_x^\infty \left\{ \int_x^u \left(\frac{1}{2}\left(f'(x+)+f'(x-)\right) + f'_x(v) \right.\right. \\ &\quad \left.\left. + \frac{1}{2}\left(f'(x+)-f'(x-)\right)\operatorname{sgn}(v-x) \right) dv \right\} W(n,x,u)du \\ &= \frac{1}{2}\left(f'(x+)+f'(x-)\right)\int_x^\infty (u-x)W(n,x,u)du + \int_x^\infty \left(\int_x^u f'_x(v)dv \right) W(n,x,u)du \\ &+ \frac{1}{2}\left(f'(x+)-f'(x-)\right)\int_x^\infty (u-x)W(n,x,u)du. \end{aligned} \tag{3.3.22}$$

Combining the relation (3.3.20)-(3.3.22), we get

$$\begin{aligned} M_n(f;x)-f(x) &= \frac{1}{2}\left(f'(x+)+f'(x-)\right)\int_0^\infty (u-x)W(n,x,u)du + \frac{1}{2}\left(f'(x+)-f'(x-)\right) \\ &\times \int_0^\infty |u-x|W(n,x,u)du - \int_0^x \left(\int_u^x f'_x(v)dv \right) W(n,x,u)du \\ &+ \int_x^\infty \left(\int_x^u f'_x(v)dv \right) W(n,x,u)du. \end{aligned}$$

Hence,

$$\begin{aligned} &|M_n(f;x)-f(x)| \\ &\leq \left| \frac{f'(x+)+f'(x-)}{2} \right| |M_n(u-x;x)| + \left| \frac{f'(x+)-f'(x-)}{2} \right| M_n(|u-x|;x) \\ &+ \left| \int_0^x \left(\int_u^x f'_x(v)dv \right) W(n,x,u)du \right| + \left| \int_x^\infty \left(\int_x^u f'_x(v)dv \right) W(n,x,u)du \right|. \end{aligned} \tag{3.3.23}$$

Now, assume that

$$C_n(f'_x,x) = \int_0^x \left(\int_u^x f'_x(v)dv \right) W(n,x,u)du,$$

and

$$D_n(f'_x,x) = \int_x^\infty \left(\int_x^u f'_x(v)dv \right) W(n,x,u)du.$$

Now the problem is reduced to estimate $C_n(f'_x,x)$ and $D_n(f'_x,x)$.

Using the definition of $\xi_n(x,u)$ given in Lemma 3.3.1 and applying the integration by parts, we can write

$$C_n(f'_x,x) = \int_0^x \left(\int_u^x f'_x(v)dv \right) \frac{\partial \xi_n(x,u)}{\partial u} du = \int_0^x f'_x(u)\xi_n(x,u)du.$$

Thus,

$$\begin{aligned}|C_n(f'_x,x)| &= \int_0^x |f'_x(u)|\xi_n(x,u)du \\ &\le \int_0^{x-\frac{x}{\sqrt{n}}} |f'_x(u)|\xi_n(x,u)du + \int_{x-\frac{x}{\sqrt{n}}}^x |f'_x(u)|\xi_n(x,u)du.\end{aligned}$$

Since $f'_x(x) = 0$ and $\xi_n(x,u) \le 1$, we get

$$\begin{aligned}\int_{x-\frac{x}{\sqrt{n}}}^x |f'_x(u)|\xi_n(x,u)du &= \int_{x-\frac{x}{\sqrt{n}}}^x |f'_x(u) - f'_x(x)|\xi_n(x,u)du \le \int_{x-\frac{x}{\sqrt{n}}}^x \left(\bigvee_u^x f'_x\right) du \\ &\le \left(\bigvee_{x-\frac{x}{\sqrt{n}}}^x f'_x\right) \int_{x-\frac{x}{\sqrt{n}}}^x du = \frac{x}{\sqrt{n}} \left(\bigvee_{x-\frac{x}{\sqrt{n}}}^x f'_x\right).\end{aligned}$$

Using Lemma 3.3.1, and assuming $u = x - \dfrac{x}{v}$, we have

$$\begin{aligned}\int_0^{x-\frac{x}{\sqrt{n}}} |f'_x(u)|\xi_n(x,u)du &\le C_1\,|\eta(x)| \int_0^{x-\frac{x}{\sqrt{n}}} \frac{|f'_x(u)|}{(x-u)^2}du \\ &\le C_1\,|\eta(x)| \int_0^{x-\frac{x}{\sqrt{n}}} \left(\bigvee_u^x f'_x\right) \frac{du}{(x-u)^2} \\ &= \frac{C_1\,|\eta(x)|}{x} \int_1^{\sqrt{n}} \left(\bigvee_{x-\frac{x}{v}}^x f'_x\right) \le \frac{C_1\,|\eta(x)|}{x} \sum_{k=1}^{[\sqrt{n}]} \left(\bigvee_{x-\frac{x}{k}}^x f'_x\right).\end{aligned}$$

Therefore,

$$|C_n(f'_x,x)| = \frac{C_1\,|\eta(x)|}{x} \sum_{k=1}^{[\sqrt{n}]} \left(\bigvee_{x-\frac{x}{k}}^x f'_x\right) + \frac{x}{\sqrt{n}} \left(\bigvee_{x-\frac{x}{\sqrt{n}}}^x f'_x\right).$$

Using integration by parts in $D_n(f'_x,x)$ and applying Lemma 3.3.1, we have

$$\begin{aligned}|D_n(f'_x,x)| \;\; &\le \;\; \left|\int_x^{2x} \left(\int_x^u f'_x(v)dv\right) \frac{\partial}{\partial u}(1-\xi_n(x,u))du\right| \\ &\quad + \left|\int_{2x}^{\infty} \left(\int_x^u f'_x(v)dv\right) W(n,x,u)du\right| \\ &\le \;\; \left|\int_x^{2x} f'_x(v)dv\right| |1-\xi_n(x,2x)| + \int_x^{2x} |f'_x(u)|(1-\xi_n(x,u))du \\ &\quad + \left|\int_{2x}^{\infty} (f(u)-f(x))W(n,x,u)du\right| \\ &\quad + |f'(x+)| \left|\int_{2x}^{\infty} (u-x)W(n,x,u)(x,u)du\right|.\end{aligned}$$

We have

$$\int_x^{2x} |f_x'(u)|(1-\xi_n(x,u))du = \int_x^{x+\frac{x}{\sqrt{n}}} |f_x'(u)|(1-\xi_n(x,u))d \quad (3.3.24)$$

$$u + \int_{x+\frac{x}{\sqrt{n}}}^{2x} |f_x'(u)|(1-\xi_n](x,u))du$$

$$= J_1 + J_2. \quad (3.3.25)$$

Since $f_x'(x) = 0$ and $1-\xi_n(x,u) \leq 1$, we have

$$J_1 = \int_x^{x+\frac{x}{\sqrt{n}}} |f_x'(u) - f_x'(x)|(1-\xi_n(x,u))du \leq \int_x^{x+\frac{x}{\sqrt{n}}} \left(\bigvee_x^{x+\frac{x}{\sqrt{n}}} f_x'\right) du = \frac{x}{\sqrt{n}} \left(\bigvee_x^{x+\frac{x}{\sqrt{n}}} f_x'\right).$$

Using Lemma 3.3.1 and assuming $u = x + \frac{x}{v}$, we obtain

$$J_2 \leq C_1\,|\eta(x)| \int_{x+\frac{x}{\sqrt{n}}}^{2x} \frac{1}{(u-x)^2} |f_x'(u) - f_x'(x)|du \leq C_1\,|\eta(x)| \int_{x+\frac{x}{\sqrt{n}}}^{2x} \frac{1}{(u-x)^2} \left(\bigvee_x^{u} f_x'\right) du$$

$$= \frac{C_1\,|\eta(x)|}{x} \int_1^{\sqrt{n}} \left(\bigvee_x^{x+\frac{x}{v}} f_x'\right) dv \leq \frac{C_1\,|\eta(x)|}{x} \sum_{k=1}^{[\sqrt{n}]} \int_k^{k+1} \left(\bigvee_x^{x+\frac{x}{v}} f_x'\right) dv$$

$$\leq \frac{C_1\,|\eta(x)|}{x} \sum_{k=1}^{[\sqrt{n}]} \left(\bigvee_x^{x+\frac{x}{k}} f_x'\right).$$

Putting the values of J_1 and J_2 in (3.3.24), we have

$$\int_x^{2x} |f_x'(u)|(1-\xi_n(x,u))du \leq \frac{x}{\sqrt{n}} \left(\bigvee_x^{x+\frac{x}{\sqrt{n}}} f_x'\right) + \frac{C_1\,|\eta(x)|}{x} \sum_{k=1}^{[\sqrt{n}]} \left(\bigvee_x^{x+\frac{x}{k}} f_x'\right).$$

Therefore, applying Cauchy-Schwarz inequality and Lemma 3.3.1, we get

$$|D_n(f_x', x)| \leq M_f \int_{2x}^{\infty} (u^2+1) W(n,x,u)du + |f(x)| \int_{2x}^{\infty} W(n,x,u)du$$

$$+|f'(x+)|\sqrt{C_1\,|\eta(x)|} + \frac{C_1\,|\eta(x)|}{x^2} |f(2x) - f(x) - xf'(x+)|$$

$$+\frac{x}{\sqrt{n}} \left(\bigvee_x^{x+\frac{x}{\sqrt{n}}} f_x'\right) + \frac{C_1\,|\eta(x)|}{x} \sum_{k=1}^{[\sqrt{n}]} \left(\bigvee_x^{x+\frac{x}{k}} f_x'\right). \quad (3.3.26)$$

Since $u \geq 2x$, we have

$$\begin{aligned}
& M_f \int_{2x}^{\infty} (u^2+1)W(n,x,u)du + |f(x)| \int_{2x}^{\infty} W(n,x,u)du \\
& \leq (M_f + |f(x)|) \int_{2x}^{\infty} W(n,x,u)du + 4M_f \int_{2x}^{\infty} (u-x)^2 W(n,x,u)du \\
& \leq \frac{M_f + |f(x)|}{x^2} \int_{0}^{\infty} (u-x)^2 W(n,x,u)du + 4M_f \int_{0}^{\infty} (u-x)^2 W(n,x,u)du \\
& \leq \left(4M_f + \frac{M_f + |f(x)|}{x^2}\right) C_1 \, |\eta(x)|.
\end{aligned}$$

Using the above inequality, we have

$$\begin{aligned}
|D_n(f'_x, x)| \quad \leq \quad & \left(4M_f + \frac{M_f + |f(x)|}{x^2}\right) C_1 \, |\eta(x)| + |f'(x+)| \sqrt{C_1 \, |\eta(x)|} \\
& + C_1 \frac{1+x^2}{nx^2} |f(2x) - f(x) - xf'(x+)| + \frac{x}{\sqrt{n}} \left(\bigvee_{x}^{x+\frac{x}{\sqrt{n}}} f'_x \right) \\
& + \frac{C_1 \, |\eta(x)|}{x} \sum_{k=1}^{[\sqrt{n}]} \left(\bigvee_{x}^{x+\frac{x}{k}} f'_x \right).
\end{aligned} \tag{3.3.27}$$

Now from (3.3.23), (3.3.26) and (3.3.27), we reach the required result. □

Chapter 4

Approximation by q-GBS Bernstein-Schurer-Stancu type operators in a Bögel space

4.1 Introduction

Very Recently Bărbosu and Muraru [40] defined the q-Bernstein-Schurer-Stancu operators for the bivariate case as follows:

Let p_1, p_2 be non-negative integers, $I = [0, 1 + p_1] \times [0, 1 + p_2]$ and $J = [0, 1] \times [0, 1]$. Let $\{q_m\}$ and $\{q_n\}$ be sequences in $(0, 1)$ such that $q_m \to 1$, $q_m^m \to a\,(0 \leq a < 1)$, as $m \to \infty$ and $q_n \to 1$, $q_n^n \to b\,(0 \leq b < 1)$, as $n \to \infty$. Further, let $0 \leq \alpha_1 \leq \beta_1$, $0 \leq \alpha_2 \leq \beta_2$ and $S_{m,n,p_1,p_2}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)} : C(I) \to C(J)$ then for any $f \in C(I)$ we have,

$$\begin{aligned} S_{m,n,p_1,p_2}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}(f; q_m, q_n, x, y) &= \sum_{k_1=0}^{m+p_1} \sum_{k_2=0}^{n+p_2} \begin{bmatrix} m+p_1 \\ k_1 \end{bmatrix}_{q_m} \begin{bmatrix} n+p_2 \\ k_2 \end{bmatrix}_{q_n} \prod_{s=0}^{m+p_1-k_1-1} (1 - q_m^s x) \\ &\quad \times \prod_{r=0}^{n+p_2-k_2-1} (1 - q_n^r y)\, x^{k_1}\, y^{k_2} f_{k_1,k_2}, \end{aligned} \tag{4.1.1}$$

where $f_{k_1,k_2} = f\left(\dfrac{[k_1]_{q_m} + \alpha_1}{[m]_{q_m} + \beta_1}, \dfrac{[k_2]_{q_n} + \alpha_2}{[n]_{q_n} + \beta_2}\right)$.

4.2 Moments

Lemma 4.2.1. *[40] Let* $e_{i.j} : I \to I, e_{i,j}(x,y) = x^i y^j (0 \leq i+j \leq 2, i, j(integers))$ *be the test functions. Then the following equalities hold for the operators given by (4.1.1):*

(i) $S^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}_{m,n,p_1,p_2}(e_{0,0}; q_m, q_n, x, y) = e_{0,0}(x,y)$,

(ii) $S^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}_{m,n,p_1,p_2}(e_{1,0}; q_m, q_n, x, y) = \dfrac{[m+p_1]_{q_m} x + \alpha_1}{[m]_{q_m} + \beta_1}$,

(iii) $S^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}_{m,n,p_1,p_2}(e_{0,1}; q_m, q_n, x, y) = \dfrac{[n+p_2]_{q_n} y + \alpha_2}{[n]_{q_n} + \beta_2}$,

(iv) $S^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}_{m,n,p_1,p_2}(e_{2,0}; q_m, q_n, x, y) = \dfrac{1}{([m]_{q_m} + \beta_1)^2}\Big([m+p_1]^2_{q_m} x^2 + [m+p_1]_{q_m} x(1-x)$
$+ 2\alpha_1 [m+p_1]_{q_m} x + \alpha_1^2\Big)$,

(v) $S^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}_{m,n,p_1,p_2}(e_{0,2}; q_m, q_n, x, y) = \dfrac{1}{([n]_{q_n} + \beta_2)^2}\Big([n+p_2]^2_{q_n} y^2 + [n+p_2]_{q_n} y(1-y)$
$+ 2\alpha_2 [n+p_2]_{q_n} y + \alpha_2^2\Big)$.

Lemma 4.2.2. *For* $(x,y) \in J$, *we have*

(i) $S^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}_{m,n,p_1,p_2}((u-x)^2; q_m, q_n, x, y) = \dfrac{1}{([m+\beta_1]_{q_m})^2}\{((q_m^m [p_1]_{q_m} - \beta_1)x + \alpha_1)^2 +$
$[m+p_1]_{q_m} x(1-x)\}$,

(ii) $S^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}_{m,n,p_1,p_2}((v-y)^2; q_m, q_n, x, y) = \dfrac{1}{([n+\beta_2]_{q_n})^2}\{((q_n^n [p_2]_{q_n} - \beta_2)y + \alpha_2)^2 + [n+$
$p_2]_{q_n} y(1-y)\}$.

Lemma 4.2.3. *For* $(x,y) \in J$, *we have*

(i) $\lim\limits_{m\to\infty} [m]_{q_m} S^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}_{m,n,p_1,p_2}((u-x); q_m, q_n, x, y) = \alpha_1 - \beta_1 x$,

(ii) $\lim\limits_{n\to\infty} [n]_{q_n} S^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}_{m,n,p_1,p_2}((v-y); q_m, q_n, x, y) = \alpha_2 - \beta_2 y$,

(iii) $\lim\limits_{m\to\infty} [m]_{q_m} S^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}_{m,n,p_1,p_2}((u-x)^2; q_m, q_n, x, y) = x(1-x)$,

(iv) $\lim\limits_{n\to\infty} [n]_{q_n} S^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}_{m,n,p_1,p_2}((v-y)^2; q_m, q_n, x, y) = y(1-y)$.

Similarly, it can be shown that

$$S^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}_{m,n,p_1,p_2}((u-x)^4;q_m,q_n,x,y) = O\left(\frac{1}{[m]^2_{q_m}}\right), \text{ as } m\to\infty, \text{ uniformly in } x\in[0,1], \tag{4.2.1}$$

and

$$S^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}_{m,n,p_1,p_2}((v-y)^4;q_m,q_n,x,y) = O\left(\frac{1}{[n]^2_{q_n}}\right), \text{ as } n\to\infty, \text{ uniformly in } y\in[0,1]. \tag{4.2.2}$$

4.3 Main results

Let δ_m and δ_n be defined as

$$\begin{aligned}
\delta_m &= max_{x\in[0,1]}\{S^{(\alpha_1,\beta_1)}_{m,p_1}((u-x)^2;q_m,x)\}^{1/2}\\
&= \frac{1}{[m]_{q_m}+\beta_1}\sqrt{4\text{max}_{x\in[0,1]}\left(((q_m^m[p_1]_{q_m}-\beta)x+\alpha_1)^2+[m+p_1]_{q_m}\right)},\\
\text{and}\quad \delta_n &= max_{y\in[0,1]}\{S^{(\alpha_2,\beta_2)}_{n,p_2}((v-y)^2;q_n,y\}^{1/2}\\
&= \frac{1}{[n]_{q_n}+\beta_2}\sqrt{4\text{max}_{y\in[0,1]}\left(((q_n^n[p_2]_{q_n}-\beta_2)y+\alpha_2)^2+[n+p_2]_{q_n}\right)}.
\end{aligned}$$

Theorem 4.3.1. *Let $f\in C(I)$. Then we have the inequality*

$$||S^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}_{m,n,p_1,p_2}(f;q_m,q_n,.,.)-f|| \leq 2(\omega_1(f;\delta_m)+\omega_2(f;\delta_n)).$$

Proof. By the definition of partial moduli of continuity, Lemma 4.2.1 and using Cauchy-Schwarz inequality we may write

$$
\begin{aligned}
&|S_{m,n,p_1,p_2}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}(f;q_m,q_n,x,y)-f(x,y)|\\
&\leq S_{m,n,p_1,p_2}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}(|f(u,v)-f(x,y)|;q_m,q_n,x,y)\\
&\leq S_{m,n,p_1,p_2}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}(|f(u,v)-f(u,y)|;q_m,q_n,x,y)\\
&\quad +S_{m,n,p_1,p_2}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}(|f(u,y)-f(x,y)|;q_m,q_n,x,y)\\
&\leq S_{m,n,p_1,p_2}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}(\omega_2(f;|v-y|);q_m,q_n,x,y)\\
&\quad +S_{m,n,p_1,p_2}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}(\omega_1(f;|u-x|);q_m,q_n,x,y)\\
&\leq \omega_2(f;\delta_n)\left[1+\frac{1}{\delta_n}S_{m,n,p_1,p_2}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}(|v-y|;q_m,q_n,x,y)\right]\\
&\quad +\omega_1(f;\delta_m)\left[1+\frac{1}{\delta_m}S_{m,n,p_1,p_2}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}(|u-x|;q_m,q_n,x,y)\right]\\
&\leq \omega_2(f;\delta_n)\left[1+\frac{1}{\delta_n}\left(S_{m,n,p_1,p_2}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}((v-y)^2;q_m,q_n,x,y)\right)^{1/2}\right]\\
&\quad +\omega_1(f;\delta_m)\left[1+\frac{1}{\delta_m}\left(S_{m,n,p_1,p_2}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}((u-x)^2;q_m,q_n,x,y)\right)^{1/2}\right]\\
&\leq \omega_2(f;\delta_n)\left(1+\frac{1}{\delta_n}\frac{1}{[n]_{q_n}+\beta_2}\sqrt{4\max_{y\in[0,1]}\left(((q_n^n[p_2]_{q_n}-\beta_2)y+\alpha_2)^2+[n+p_2]_{q_n}\right)}\right)\\
&\quad +\omega_1(f;\delta_m)\left(1+\frac{1}{\delta_m}\frac{1}{[m]_{q_m}+\beta_1}\sqrt{4\max_{x\in[0,1]}\left(((q_m^m[p_1]_{q_m}-\beta_1)x+\alpha_1)^2+[m+p_1]_{q_m}\right)}\right).
\end{aligned}
$$

Hence, we reach the desired result. □

Theorem 4.3.2. *Let $f\in C(I)$ and $0<q_m,q_n<1$. Then for all $(x,y)\in J$, we have*

$$||S_{m,n,p_1,p_2}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}(f;q_m,q_n,.,.)-f||\leq 4\,\bar{\omega}(f,\delta_m,\delta_n).$$

Proof. We have

$$
\begin{aligned}
&|S_{m,n,p_1,p_2}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}(f;q_m,q_n,x,y)-f(x,y)|\\
&\leq S_{m,n,p_1,p_2}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}(|f(u,v)-f(x,y)|;q_m,q_n,x,y)\\
&\leq \omega(f;\delta_m,\delta_n)\left(S_{m,p_1}^{(\alpha_1,\beta_1)}(f_0;q_{m_1},x)+\frac{1}{\delta_m}S_{m,p_1}^{(\alpha_1,\beta_1)}(|u-x|;q_m,x)\right)\\
&\quad\times\left(S_{n,p_2}^{(\alpha_2,\beta_2)}(f_0;q_n,y)+\frac{1}{\delta_n}S_{n,p_2}^{(\alpha_2,\beta_2)}(|v-y|;q_n,y)\right).
\end{aligned}
$$

Applying Cauchy-Schwarz inequality, we have

$$
\begin{aligned}
&|S_{m,n,p_1,p_2}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}(f;q_m,q_n,x,y)-f(x,y)| \\
&\qquad \leq \bar{\omega}(f;\delta_m,\delta_n)\Bigg\{\left(1+\frac{1}{\delta_m}\sqrt{S_{m,p_1}^{(\alpha_1,\beta_1)}((u-x)^2;q_m,x)}\right) \\
&\qquad\quad \times\left(1+\frac{1}{\delta_n}\sqrt{S_{n,p_2}^{(\alpha_2,\beta_2)}((v-y)^2;q_n,y)}\right)\Bigg\} \\
&\qquad \leq 4\,\bar{\omega}(f;\delta_m,\delta_n).
\end{aligned}
$$

This completes the proof. □

Example 1. For $n,\ m=10,\ p_1,\ p_2=2, \alpha_1=3,\ \beta_1=4,\ \alpha_2=5,\ \beta_2=7,\ q_1,\ q_2=0.5, q_1,\ q_2=0.7$ and $q_1,\ q_2=0.9$ the convergence of the operators $S_{10,10,2,2}^{(3,4,5,7)}(f;.5,.5,x,y)$ (yellow), $S_{10,10,2,2}^{(3,4,5,7)}(f;.7,.7,x,y)$ (pink), $S_{10,10,2,2}^{(3,4,5,7)}(f;.9,.9,x,y)$ (blue) to $f\,(x,y)=x\left(x-\frac{1}{4}\right)\left(y-\frac{3}{7}\right)$ (red) is illustrated by Figure 1.

Example 2. For $m,\ n=10,\ \alpha_1,\alpha_2=1,\ \beta_1,\beta_2=2,\ p_1,p_2=1$, the comparison of the convergence of q -Bernstein-Schurer-Stancu (blue) given by $S_{m,n,p_1,p_2}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}(f;q_m,q_n,x,y)$ and the operators bivariate q-Bernstein-Schurer (green), q-Bernstein-Stancu (red), to $f\,(x,y)=2x\cos(\pi x)\,y^3$(yellow) with $q_m=m/(m+1), q_n=1-1/\sqrt{n}$ are illustrated in the Figure 2.

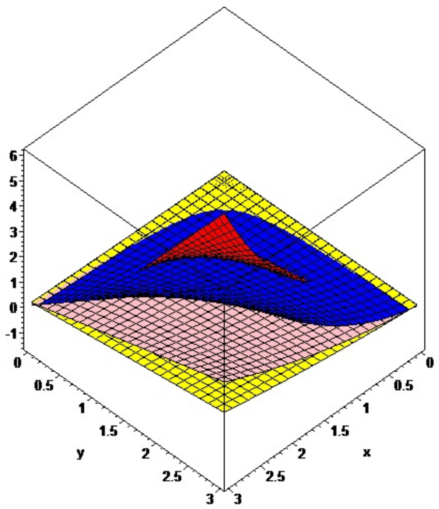


Figure 1

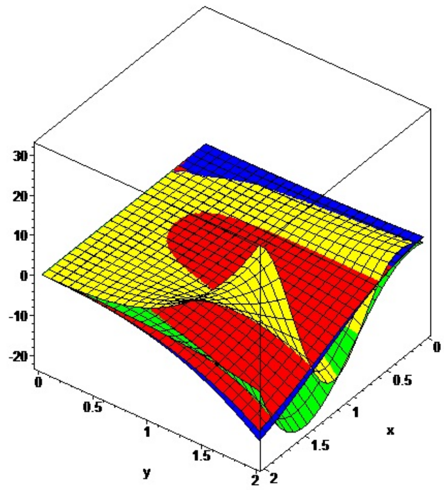


Figure 2

4.3.1 Degree of approximation

Now, we estimate the degree of approximation for the bivariate operators (4.1.1) by means of the Lipschitz class.

Theorem 4.3.3. *Let $f \in Lip_M(\xi,\gamma)$. Then, we have*

$$||S^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}_{m,n,p_1,p_2}(f;q_m,q_n,.,.)-f|| \leq M\delta_m^{\xi}\delta_n^{\gamma}.$$

Proof. By our hypothesis, we may write

$$\begin{aligned} |S^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}_{m,n,p_1,p_2}(f;q_m,q_n,x,y)-f(x,y)| &\leq S^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}_{m,n,p_1,p_2}(|f(u,v)-f(x,y)|;q_m,q_n,x,y) \\ &\leq MS^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}_{m,n,p_1,p_2}(|u-x|^{\xi}|v-y|^{\gamma};q_m,q_n,x,y) \\ &= MS^{(\alpha_1,\beta_1)}_{m,p_1}(|u-x|^{\xi};q_m,x)S^{(\alpha_2,\beta_2)}_{n,p_2}(|v-y|^{\gamma};q_n,y). \end{aligned}$$

Now, using the Hölder's inequality with $u_1=\frac{2}{\xi}$, $v_1=\frac{2}{2-\xi}$ and $u_2=\frac{2}{\gamma}$ and $v_2=\frac{2}{2-\gamma}$, we have

$$\begin{aligned} S^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}_{m,n,p_1,p_2}(f;q_m,q_n,x,y)-f(x,y)| &\leq MS^{(\alpha_1,\beta_1)}_{m,p_1}((u-x)^2;q_m,x)^{\frac{\xi}{2}}S^{(\alpha_1,\beta_1)}_{m,p_1}(f_0;q_m,x)^{\frac{2-\xi}{2}} \\ &\quad\times S^{(\alpha_2,\beta_2)}_{n,p_2}((v-y)^2;q_n,y)^{\frac{\gamma}{2}}S^{(\alpha_2,\beta_2)}_{n,p_2}(f_0;q_n,y)^{\frac{2-\gamma}{2}} \\ &\leq M\delta_m^{\xi}\delta_n^{\gamma}. \end{aligned}$$

Hence, the proof is completed. □

Theorem 4.3.4. *Let $f \in C(I)$ and $(x,y) \in J$. Then, we have*

$$|S^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}_{m,n,p_1,p_2}(f;q_m,q_n,x,y)-f(x,y)| \leq \|f_x\|\delta_m+\|f_y\|\delta_n.$$

Proof. Let $(x,y) \in J$ be a fixed point. Then, we can write

$$f(u,v)-f(x,y)=\int_x^u f_t(t,v)d_qt+\int_y^v f_s(x,s)d_qs.$$

Now applying $S^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}_{m,n,p_1,p_2}(.;q_m,q_n,x,y)$ on both sides, we have

$$\begin{aligned} |S^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}_{m,n,p_1,p_2}(f;q_m,q_n,x,y)-f(x,y)| &\leq S^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}_{m,n,p_1,p_2}\left(\int_x^u f_t(t,v)d_qt;q_m,q_n,x,y\right) \\ &\quad+S^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}_{m,n,p_1,p_2}\left(\int_y^v f_s(x,s)d_qs;q_m,q_n,x,y\right). \end{aligned}$$

Since

$\left|\int_x^u f_t(t,v)d_qt\right| \leq \|f_x\||u-x|$ and $\left|\int_y^v f_s(x,s)d_qs\right| \leq \|f_y\||v-y|$, we have

$$\begin{aligned}|S_{m,n,p_1,p_2}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}(f;q_m,q_n,x,y)-f(x,y)| &\leq \|f_x\|S_{m,p_1}^{(\alpha_1,\beta_1)}(|u-x|;q_m,x)\\ &\quad +\|f_y\|S_{n,p_2}^{(\alpha_2,\beta_2)}(|v-y|;q_n,y).\end{aligned}$$

Now, applying the Cauchy-Schwarz inequality, we get

$$\begin{aligned}&|S_{m,n,p_1,p_2}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}(f;q_m,q_n,x,y)-f(x,y)|\\ &\leq \|f_x\|S_{m,p_1}^{(\alpha_1,\beta_1)}((u-x)^2;q_m,x)^{1/2}S_{m,p_1}^{(\alpha_1,\beta_1)}(f_0;q_m,x)^{1/2}\\ &\quad +\|f_y\|S_{n,p_2}^{(\alpha_2,\beta_2)}((v-y)^2;q_n,y)^{1/2}S_{n,p_2}^{(\alpha_2,\beta_2)}(f_0;q_n,y)^{1/2}\\ &\leq \|f_x\|\delta_m+\|f_y\|\delta_n.\end{aligned}$$

Hence the theorem is proved. □

Theorem 4.3.5. *If $f \in C(I)$, we obtain*

$$\begin{aligned}|S_{m,n,p_1,p_2}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}(f;q_m,q_n,x,y)-f(x,y)| &\leq M\left\{\bar{\omega_2}(f;\sqrt{C_{m,n}})+min\{1,C_{m,n}\}||f||_{C(I^2)}\right\}\\ &\quad +\omega(f;\psi_{m,n}),\end{aligned}$$

where

$$\psi_{m,n}=\sqrt{max_{(x,y)\in J}\left\{\left(\frac{[m+p_1]_{q_m}x+\alpha_1}{[m]_{q_m}+\beta_1}-x\right)^2+\left(\frac{[n+p_2]_{q_n}y+\alpha_2}{[n]_{q_n}+\beta_2}-y\right)^2\right\}},$$

$$C_{m,n}=\delta_m^2+\delta_n^2+\psi_{m,n}^2$$

and the constant $M>0$, is independent of f and $C_{m,n}$.

Proof. We introduce the auxiliary operators as follows:

$$\begin{aligned}S_{m,n,p_1,p_2}^{*(\alpha_1,\beta_1,\alpha_2,\beta_2)}(f;q_m,q_n,x,y) &= S_{m,n,p_1,p_2}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}(f;q_m,q_n,x,y)\\ &\quad -f\left(\frac{[m+p_1]_{q_m}x+\alpha_1}{[m]_{q_m}+\beta_1},\frac{[n+p_2]_{q_n}y+\alpha_2}{[n]_{q_n}+\beta_2}\right)+f(x,y),\end{aligned}$$

then using Lemma 4.2.1, we have
$S_{m,n,p_1,p_2}^{*(\alpha_1,\beta_1,\alpha_2,\beta_2)}((u-x);q_m,q_n,x,y)=0$ and $S_{m,n,p_1,p_2}^{*(\alpha_1,\beta_1,\alpha_2,\beta_2)}((v-y);q_m,q_n,x,y)=0$.

Let $g \in C^2(I)$ and $u, v \in I$. Using the Taylor's theorem, we may write

$$\begin{aligned} g(u,v)-g(x,y) &= g(u,y)-g(x,y)+g(u,v)-g(u,y) \\ &= \frac{\partial g(x,y)}{\partial x}(u-x)+\int_x^u (u-t)\frac{\partial^2 g(t,y)}{\partial t^2}dt \\ &\quad +\frac{\partial g(x,y)}{\partial y}(v-y)+\int_y^v (v-s)\frac{\partial^2 g(x,s)}{\partial s^2}ds. \end{aligned}$$

Applying the operator $S^{*(\alpha_1,\beta_1,\alpha_2,\beta_2)}_{m,n,p_1,p_2}(.,q_m,q_n,x,y)$ on both sides, we get

$$\begin{aligned} S^{*(\alpha_1,\beta_1,\alpha_2,\beta_2)}_{m,n,p_1,p_2}(f;q_m,q_n,x,y)-f(x,y) &= S^{*(\alpha_1,\beta_1,\alpha_2,\beta_2)}_{m,n,p_1,p_2}\left(\int_x^u (u-t)\frac{\partial^2 g(t,y)}{\partial t^2}dt;q_m,q_n,x,y\right) \\ &\quad +S^{*(\alpha_1,\beta_1,\alpha_2,\beta_2)}_{m,n,p_1,p_2}\left(\int_y^v (v-s)\frac{\partial^2 g(x,s)}{\partial s^2}ds;q_m,q_n,x,y\right) \\ &= S^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}_{m,n,p_1,p_2}\left(\int_x^u (u-t)\frac{\partial^2 g(t,y)}{\partial t^2}dt;q_m,q_n,x,y\right) \\ &\quad -\int_x^{\frac{[m+p_1]_{q_m}x+\alpha_1}{[m]_{q_m}+\beta_1}}\left(\frac{[m+p_1]_{q_m}x+\alpha_1}{[m]_{q_m}+\beta_1}-t\right)\frac{\partial^2 g(t,y)}{\partial t^2}dt \\ &\quad +S^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}_{m,n,p_1,p_2}\left(\int_y^v (v-s)\frac{\partial^2 g(x,s)}{\partial s^2}ds;q_m,q_n,x,y\right) \\ &\quad -\int_x^{\frac{[n+p_2]_{q_n}y+\alpha_2}{[n]_{q_n}+\beta_2}}\left(\frac{[n+p_2]_{q_n}y+\alpha_2}{[n]_{q_n}+\beta_2}-s\right)\frac{\partial^2 g(x,s)}{\partial s^2}ds. \end{aligned}$$

Hence
$|S^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}_{m,n,p_1,p_2}(f;q_m,q_n,x,y)-f(x,y)|$

$$\begin{aligned} &\leq S^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}_{m,n,p_1,p_2}\left(\left|\int_x^u |u-t|\left|\frac{\partial^2 g(t,y)}{\partial t^2}\right|dt\right|;x,y\right) \\ &\quad +\left|\int_x^{\frac{[m+p_1]_{q_m}x+\alpha_1}{[m]_{q_m}+\beta_1}}\left|\frac{[m+p_1]_{q_m}x+\alpha_1}{[m]_{q_m}+\beta_1}-t\right|\left|\frac{\partial^2 g(t,y)}{\partial t^2}\right|dt\right| \\ &\quad +S^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}_{m,n,p_1,p_2}\left(\left|\int_y^v |v-s|\left|\frac{\partial^2 g(x,s)}{\partial s^2}\right|ds\right|;x,y\right) \\ &\quad +\left|\int_x^{\frac{[n+p_2]_{q_n}y+\alpha_2}{[n]_{q_n}+\beta_2}}\left|\frac{[n+p_2]_{q_n}y+\alpha_2}{[n]_{q_n}+\beta_2}-s\right|\left|\frac{\partial^2 g(x,s)}{\partial s^2}\right|ds\right| \end{aligned}$$

$$
\begin{aligned}
&\leq \left\{S^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}_{m,n,p_1,p_2}((u-x)^2;q_m,q_n,x,y)+\left(\frac{[m+p_1]_{q_m}x+\alpha_1}{[m]_{q_m}+\beta_1}-x\right)^2\right\}||g||_{C^2(I)}\\
&\quad+\left\{S^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}_{m,n,p_1,p_2}((v-y)^2;q_m,q_n,x,y)+\left(\frac{[n+p_2]_{q_n}y+\alpha_2}{[n]_{q_n}+\beta_2}-y\right)^2\right\}||g||_{C^2(I)}\\
&\leq (\delta_m^2+\delta_n^2+\psi_{m,n}^2)||g||_{C^2(I)}\\
&= C_{m,n}||g||_{C^2(I)}
\end{aligned}
\tag{4.3.1}
$$

Also, using Lemma 4.2.1

$$
\begin{aligned}
|S^{*(\alpha_1,\beta_1,\alpha_2,\beta_2)}_{m,n,p_1,p_2}(f;q_m,q_n,x,y)| &\leq |S^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}_{m,n,p_1,p_2}(f;q_m,q_n,x,y)|\\
&\quad+\left|f\left(\frac{[m+p_1]_{q_m}x+\alpha_1}{[m]_{q_m}+\beta_1},\frac{[n+p_2]_{q_n}y+\alpha_2}{[n]_{q_n}+\beta_2}\right)\right|+|f(x,y)|\\
&\leq 3||f||_{C(I)}.
\end{aligned}
\tag{4.3.2}
$$

Hence in view of (4.3.1) and (4.3.2), we get

$$
\begin{aligned}
&|S^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}_{m,n,p_1,p_2}(f;q_m,q_n,x,y)-f(x,y)|\\
&= \Bigg|S^{*(\alpha_1,\beta_1,\alpha_2,\beta_2)}_{m,n,p_1,p_2}(f;q_m,q_n,x,y)-f(x,y)\\
&\quad+f\left(\frac{[m+p_1]_{q_m}x+\alpha_1}{[m]_{q_m}+\beta_1},\frac{[n+p_2]_{q_n}y+\alpha_2}{[n]_{q_n}+\beta_2}\right)-f(x,y)\Bigg|\\
&\leq |S^{*(\alpha_1,\beta_1,\alpha_2,\beta_2)}_{m,n,p_1,p_2}(f-g;q_m,q_n,x,y)|+|S^{*(\alpha_1,\beta_1,\alpha_2,\beta_2)}_{m,n,p_1,p_2}(g;q_m,q_n,x,y)-g(x,y)|\\
&\quad+|g(x,y)-f(x,y)|+\left|f\left(\frac{[m+p_1]_{q_m}x+\alpha_1}{[m]_{q_m}+\beta_1},\frac{[n+p_2]_{q_n}y+\alpha_2}{[n]_{q_n}+\beta_2}\right)-f(x,y)\right|
\end{aligned}
$$

$$
\begin{aligned}
&\leq 4||f-g||_{C(I)}+|S^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}_{m,n,p_1,p_2}(g;q_m,q_n,x,y)-g(x,y)|\\
&\quad+\left|f\left(\frac{[m+p_1]_{q_m}x+\alpha_1}{[m]_{q_m}+\beta_1},\frac{[n+p_2]_{q_n}y+\alpha_2}{[n]_{q_n}+\beta_2}\right)-f(x,y)\right|\\
&\leq \left(4||f-g||_{C(I)}+C_{m,n}||g||_{C^2(I)}\right)+\omega(f;\psi_{m,n})\\
&\leq 4K_2(f;C_{m,n})+\omega(f;\psi_{m,n})\\
&\leq M\left\{\bar{\omega}_2(f;\sqrt{C_{m,n}})+min\{1,C_{m,n}\}||f||_{C(I)}\right\}+\omega(f;\psi_{m,n}).
\end{aligned}
$$

Hence, we get the desired result.

□

4.3.2 Voronovskaja type theorem

In this section, we obtain a Voronovskaja type asymptotic theorem for the bivariate operators $S_{m,n,p_1,p_2}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}$.

Theorem 4.3.6. *Let $f \in C^2(I)$. Then, we have*

$$\lim_{n\to\infty}[n]_{q_n}(S_{n,n,p_1,p_2}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}(f;q_n,x,y)-f(x,y))$$
$$= (\alpha_1-\beta_1 x)f_x(x,y)+(\alpha_2-\beta_2 y)f_y(x,y)+\frac{f_{xx}(x,y)}{2}x(1-x)+\frac{f_{yy}(x,y)}{2}y(1-y),$$

uniformly on J.

Proof. Let $(x,y)\in J$. By the Taylor's theorem, we have

$$\begin{aligned}S_{n,n,p_1,p_2}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}(f(u,v);q_n,x,y) &= f(x,y)+f_x(x,y)S_{n,p_1}^{(\alpha_1,\beta_1,)}((u-x);q_n,x)\\ &+f_y(x,y)S_{n,p_2}^{(\alpha_2,\beta_2)}((v-y);q_n,y)\\ &+\frac{1}{2}\{f_{xx}(x,y)S_{n,p_1}^{(\alpha_1,\beta_1)}((u-x)^2;q_n,x)\\ &+2f_{xy}(x,y)S_{n,n,p_1,p_2}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}((u-x)(v-y);q_n,x,y)\\ &+f_{yy}(x,y)S_{n,p_2}^{(\alpha_2,\beta_2)}((v-y)^2;q_n,y)\}\\ &+S_{n,n,p_1,p_2}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}(\varepsilon(u,v;x,y)\sqrt{(u-x)^4+(v-y)^4};q_n,x,y).\end{aligned}$$

Applying Lemma 4.2.3, where $\varepsilon(u,v;x,y)\in C(I)$ and $\varepsilon(u,v;x,y)\to 0$ as $(u,v)\to(x,y)$.

$$\lim_{n\to\infty}[n]_{q_n}(S_{n,n,p_1,p_2}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}(f;q_n,x,y)-f(x,y))$$
$$\begin{aligned}&= (\alpha_1-\beta_1 x)f_x(x,y)+(\alpha_2-\beta_2 y)f_y(x,y)+\frac{f_{xx}(x,y)}{2}x(1-x)+\frac{f_{yy}(x,y)}{2}y(1-y)\\ &\quad+\lim_{n\to\infty}[n]_{q_n}S_{n,n,p_1,p_2}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}(\varepsilon(u,v;x,y)\sqrt{(u-x)^4+(v-y)^4};q_n,x,y).\end{aligned}$$

Now, applying Hölder inequality, we have

$$\left|S_{n,n,p_1,p_2}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}(\varepsilon(u,v;x,y)\sqrt{(u-x)^4+(v-y)^4};q_n,x,y)\right|$$
$$\begin{aligned}&\leq \{S_{n,n,p_1,p_2}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}(\varepsilon^2(u,v;x,y)q_n,x,y)\}^{1/2}\left[\left\{S_{n,n,p_1,p_2}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}((u-x)^4;q_n,x,y)\right\}^{1/2}\right.\\ &\quad\left.+\left\{S_{n,n,p_1,p_2}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}((v-y)^4;q_n,x,y)\right\}^{1/2}\right]\end{aligned}$$

Since, $\varepsilon^2(u,v;x,y) \to 0$ as $(u,v) \to (x,y)$, applying Theorem 4.3.1, we get

$$\lim_{n\to\infty} S_{n,n,p_1,p_2}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}(\varepsilon^2(u,v;x,y),x,y) = 0.$$

Further, in view of (4.2.1) and (4.2.2),

$$[n]_{q_n}\left\{S_{n,p_1}^{(\alpha_1,\beta_1)}((u-x)^4;q_n,x) + S_{n,p_1}^{(\alpha_2,\beta_2)}((v-y)^4;q_n,y)\right\}^{1/2}$$
$$= O(1), \text{ as } n\to\infty, \text{ uniformly in } (x,y)\in J.$$

Hence

$$\lim_{n\to\infty}[n]_{q_n} S_{m,n,p_1,p_2}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}(\varepsilon(u,v;x,y)\sqrt{(u-x)^4+(v-y)^4};q_n,x,y) = 0,$$

uniformly in $(x,y)\in J$. This completes the proof. □

4.4 Construction of q-GBS-Bernstein-Schurer-Stancu operator

For any $(x,y)\in J$, the q-GBS operator of Bernstein-Schurer-Stancu type $U_{m,n,p_1,p_2}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)} : C_b(I)\to C_b(J)$, associated to $S_{m,n,p_1,p_2}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}$ is defined as:

$$\begin{aligned} U_{m,n,p_1,p_2}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}(f(u,v);q_m,q_n,x,y) &= S_{m,n,p_1,p_2}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}(f(u,y)+f(x,v)-f(u,v);q_m,q_n,x,y) \\ &= \sum_{k_1=0}^{m+p_1}\sum_{k_2=0}^{n+p_2}\begin{bmatrix} m+p_1 \\ k_1\end{bmatrix}_{q_m}\begin{bmatrix} n+p_2 \\ k_2\end{bmatrix}_{q_n}\prod_{s=0}^{m+p_1-k_1-1}(1-q_m^s x) \\ &\times \prod_{r=0}^{n+p_2-k_2-1}(1-q_n^r y)x^{k_1}y^{k_2}\{f_{k_1}+f_{k_2}-f_{k_1,k_2}\}, \end{aligned}$$

$$\begin{aligned} \text{where } f_{k_1}(y) &= f\left(\frac{[k_1]_{q_m}+\alpha_1}{[m]_{q_m}+\beta_1},y\right),\ f_{k_2}(x) = f\left(x,\frac{[k_2]_{q_n}+\alpha_2}{[n]_{q_n}+\beta_2}\right), \\ f_{k_1,k_2} &= f\left(\frac{[k_1]_{q_m}+\alpha_1}{[m]_{q_m}+\beta_1},\frac{[k_2]_{q_n}+\alpha_2}{[n]_{q_n}+\beta_2}\right). \end{aligned}$$

Next theorem gives the degree of approximation for the operators $U_{m,n,p_1,p_2}^{*(\alpha_1,\beta_1,\alpha_2,\beta_2)}$ by means of the Lipschitz class of Bögel continuous functions.

Theorem 4.4.1. *Let $f \in Lip_M(\xi,\gamma)$ then we have*

$$\left|U_{m,n,p_1,p_2}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}(f;q_m,q_n,x,y)-f(x,y)\right| \leq M\delta_m^{\xi/2}\delta_n^{\gamma/2}.$$

for $M>0,\xi,\gamma\in(0,1]$.

Proof. By the definition of the operator $U_{m,n,p_1,p_2}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}(f;q_m,q_n,x,y)$ and by linearity of the operator $S_{m,n,p_1,p_2}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}$ given by (4.1.1), we can write

$$\begin{aligned} U_{m,n,p_1,p_2}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}(f;q_m,q_n,x,y) &= f(x,y)\,S_{m,n,p_1,p_2}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}(e_{00};q_m,q_n,x,y)\\ &\quad -S_{m,n,p_1,p_2}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}\left(\Delta_{(x,y)}f[u,v;x,y];q_m,q_n,x,y\right). \end{aligned}$$

By the hypothesis, we get

$$\begin{aligned} &\left|U_{m,n,p_1,p_2}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}(f;q_m,q_n,x,y)-f(x,y)\right|\\ &\quad\leq S_{m,n,p_1,p_2}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}\left(\left|\Delta_{(x,y)}f[u,v;x,y]\right|;q_m,q_n,x,y\right)\\ &\quad\leq MS_{m,n,p_1,p_2}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}\left(|u-x|^{\xi}|v-y|^{\gamma};q_m,q_n,x,y\right)\\ &\quad= MS_{m,n,p_1,p_2}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}\left(|u-x|^{\xi};q_m,x\right)S_{m,n,p_1,p_2}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}\left(|v-y|^{\gamma};q_n,y\right). \end{aligned}$$

Now, using the Hölder's inequality with $t_1=2/\xi, s_1=2/(2-\xi)$ and $t_2=2/\gamma, s_2=2/(2-\gamma)$, we have

$$\begin{aligned} &\left|U_{m,n,p_1,p_2}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}(f;q_m,q_n,x,y)-f(x,y)\right|\\ &\quad\leq M\left(S_{m,n,p_1,p_2}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}(u-x)^2;q_m,x\right)^{\xi/2}S_{m,n,p_1,p_2}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}(e_0;q_m,x)^{(2-\xi)/2}\\ &\quad\quad\times S_{m,n,p_1,p_2}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}\left((v-y)^2;y\right)^{\gamma/2}S_{m,n,p_1,p_2}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}(e_0;q_n,y)^{(2-\gamma)/2}. \end{aligned}$$

Considering Lemma 4.2.1, we obtain the degree of local approximation for B-continuous functions belonging to $Lip_M(\xi,\gamma)$. □

Theorem 4.4.2. *Let the function $f\in D_b(I)$ with $D_Bf\in B(I)$. Then, for each $(x,y)\in J$, we have*

$$\begin{aligned} &|U_{m,n,p_1,p_2}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}(f;q_m,q_n,x,y)-f(x,y)|\\ &\quad\leq \frac{C}{[m]_{q_m}^{1/2}[n]_{q_n}^{1/2}}\left\{||D_Bf||_\infty+\omega_{mixed}(D_Bf;[m]_{q_m}^{-1/2}[n]_{q_n}^{-1/2})\right\}. \end{aligned}$$

Proof. By our hypothesis

$$\Delta_{x,y} f[u,v;x,y] = (u-x)(v-y)D_B f(\xi,\eta),\ \ with\ x<\xi<t\,;\ y<\eta<s.$$

It is clear that

$$D_B f(\xi,\eta) = \Delta_{(x,y)} D_B f(\xi,\eta) + D_B f(\xi,y) + D_B f(x,\eta) - D_B f(x,y).$$

Since $D_B f \in B(I)$, by above relations, we can write

$$\begin{aligned}
&|S_{m,n,p_1,p_2}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}(\Delta_{(x,y)} f[u,v;x,y];q_m,q_n,x,y)| \\
&= |S_{m,n,p_1,p_2}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}((u-x)(v-y)D_B f(\xi,\eta);q_m,q_n,x,y)| \\
&\leq S_{m,n,p_1,p_2}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}(|u-x||v-y||\Delta_{(x,y)} D_B f(\xi,\eta)|;q_m,q_n,x,y) \\
&\quad + S_{m,n,p_1,p_2}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}(|u-x||v-y|(|D_B f(\xi,y)| \\
&\quad + |D_B f(x,\eta)| + |D_B f(x,y)|);q_m,q_n,x,y) \\
&\leq S_{m,n,p_1,p_2}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}(|u-x||v-y|\omega_{mixed}(D_B f;|\xi-x|,|\eta-y|);q_m,q_n,x,y) \\
&\quad + 3\ ||D_B f||_\infty\ S_{m,n,p_1,p_2}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}(|u-x||v-y|;q_m,q_n,x,y).
\end{aligned}$$

We have

$$\begin{aligned}
\omega_{mixed}(D_B f;|\xi-x|,|\eta-y|) &\leq \omega_{mixed}(D_B f;|u-x|,|v-y|) \\
&\leq (1+\delta_m^{-1}|u-x|)(1+\delta_n^{-1}|v-y|)\ \omega_{mixed}(D_B f;\delta_m,\delta_n).
\end{aligned}$$

Substituting in the above inequality, using the linearity of $S_{m,n,p_1,p_2}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}$ and applying the Cauchy-Schwarz inequality we obtain

$$\begin{aligned}
&|U_{m,n,p_1,p_2}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}(f;q_m,q_n,x,y) - f(x,y)| \\
&= |S_{m,n,p_1,p_2}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}\Delta_{(x,y)} f[u,v;x,y];q_m,q_n,x,y| \\
&\leq 3||D_B f||_\infty \sqrt{S_{m,n,p_1,p_2}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}((u-x)^2(v-y)^2;q_m,q_n,x,y)} \\
&\quad + \Bigg(S_{m,n,p_1,p_2}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}(|u-x||v-y|;q_m,q_n,x,y) \\
&\quad + \delta_m^{-1} S_{m,n,p_1,p_2}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}((u-x)^2|v-y|;q_m,q_n,x,y) \\
&\quad + \delta_n^{-1} S_{m,n,p_1,p_2}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}(|u-x|(v-y)^2;q_m,q_n,x,y) \\
&\quad + \delta_m^{-1}\delta_n^{-1} S_{m,n,p_1,p_2}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}((u-x)^2(v-y)^2;q_m,q_n,x,y) \Bigg) \omega_{mixed}(D_B f;\delta_m,\delta_n)
\end{aligned}$$

$$
\begin{aligned}
\leq\ & 3||D_B f||_\infty \sqrt{S^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}_{m,n,p_1,p_2}((u-x)^2(v-y)^2;q_m,q_n,x,y)} \\
& +\Bigg(\sqrt{S^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}_{m,n,p_1,p_2}((u-x)^2(v-y)^2;q_m,q_n,x,y)} \\
& +\delta_m^{-1}\sqrt{S^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}_{m,n,p_1,p_2}((u-x)^4(v-y)^2;q_m,q_n,x,y)} \\
& +\delta_n^{-1}\sqrt{S^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}_{m,n,p_1,p_2}((u-x)^2(v-y)^4;q_m,q_n,x,y)} \\
& +\delta_m^{-1}\delta_n^{-1} S^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}_{m,n,p_1,p_2}((u-x)^2(v-y)^2;q_m,q_n,x,y)\Bigg)\omega_{mixed}(D_B f;\delta_m,\delta_n).
\end{aligned}
$$

It is observed that for $(x,y),(u,v)\in J$ and $j,k\in\{1,2\}$

$$
\begin{aligned}
& S^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}_{m,n,p_1,p_2}((u-x)^{2j}(v-y)^{2k};q_m,q_n,x,y) \\
& \qquad = S^{(\alpha_1,\beta_1)}_{m,p_1}((u-x)^{2j};q_m,x,y)\ S^{(\alpha_2,\beta_2)}_{n,p_2}((v-y)^{2k};q_n,x,y).
\end{aligned}
$$

Hence choosing $\delta_m = \dfrac{1}{[m]_{q_m}^{1/2}}$, $\delta_n = \dfrac{1}{[n]_{q_n}^{1/2}}$. and using Lemma 4.2.3, we get the required result. □

Example 3. In Figures 3 and 4, respectively, for $m,n = 10,\ \alpha_1,\alpha_2 = 1, \beta_2,\beta_2 = 2,$ $p_1,p_2 = 1$ and for $m,n = 5, \alpha_1 = 0.4,\ \beta_1 = 0.7,\ \alpha_2 = 0.5,\ \beta_2 = 0.9,\ p_1,p_2 =$ 2, the comparison of convergence of the operators $S^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}_{m,n,p_1,p_2}(f;q_m,q_n,x,y)$ (green) and its GBS type operators $U^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}_{m,n,p_1,p_2}(f;q_m,q_n,x,y)$ (pink) to $f(x,y) = x\sin(\pi x)y$; (yellow) with $q_m = m/(m+1), q_n = 1-1/\sqrt{n}$ is illustrated. It is clearly seen that the operator $U^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}_{m,n,p_1,p_2}$ gives a better approximation than the operator $S^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}_{m,n,p_1,p_2}$.

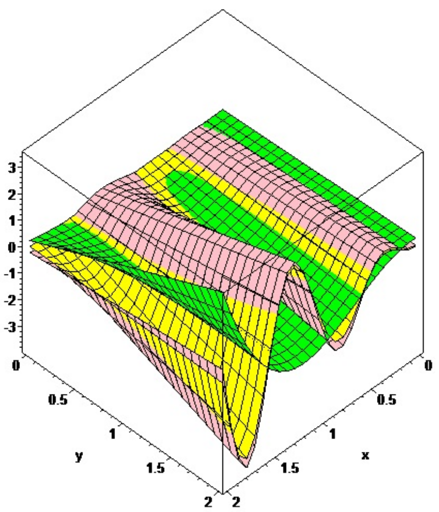


Figure 3

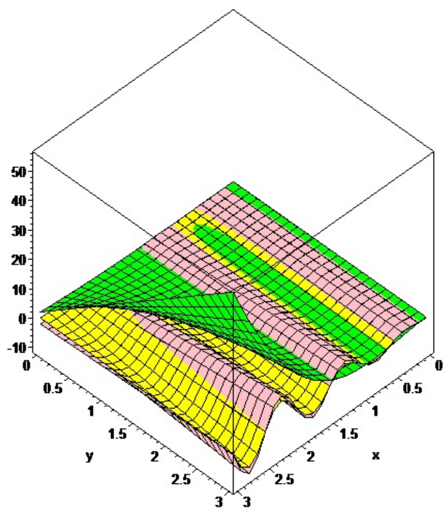


Figure 4

Chapter 5

Blending type approximation by q-Generalized Boolean Sum of Durrmeyer type

5.1 Introduction

Ren and Zeng [119] introduced a new kind of q-Bernstein-Schurer operators. Acu et al. [13] defined the q-Durrmeyer modification of these operators and investigated the statistical convergence in terms of modulus of continuity and a Lipschitz class function and the asymptotic result. Recently, Kajla et al. [93] investigated some local and global results for these operators and also studied the rate of convergence for the bivariate generalization of these operators. Recently, significant contributions have been made to study the approximation properties of some other operators defined for Lebesgue integrable functions in the literature (cf. [3], [4], [5], [6] and [7]).

In this chapter, we investigate the approximation properties of the bivariate q-Bernstein-Schurer-Durrmeyer type operators and also define the Generalized boolean sum operators.

5.2 Basic results

In [93], Kajla et al. constructed a bivariate extension of the q-Bernstein-Schurer-Durrmeyer operators introduced by Acu et al.[13] as follows:

For $p_1, p_2 \in \mathbb{N} \cup \{0\}, q_1, q_2 \in (0,1)$ and $f \in L_1(I_1 \times I_2)$, $I_1 = [0, 1+p_1]$ and $I_2 = [0, 1+p_2]$, the space of bounded and Lebesgue integrable functions on $I_1 \times I_2$, the bivariate extension of the q-Bernstein-Schurer-Durrmeyer operators is defined as:

$$\begin{aligned} D_{n_1,n_2,p_1,p_2}(f;q_1,q_2,x,y) &= \frac{[n_1+p_1+1]_{q_1}[n_1]_{q_1}}{[n_1+p_1]_{q_1}} \frac{[n_2+p_2+1]_{q_2}[n_2]_{q_2}}{[n_2+p_2]_{q_2}} \\ &\times \sum_{k_1=0}^{n_1+p_1} \sum_{k_2=0}^{n_2+p_2} \tilde{b}_{n_1,k_1}(q_1,x)\tilde{b}_{n_2,k_2}(q_2,y) q_1^{-k_1} q_2^{-k_2} \\ &\times \int_0^{\frac{[n_1+p_1]_{q_1}}{[n_1]_{q_1}}} \int_0^{\frac{[n_2+p_2]_{q_2}}{[n_2]_{q_2}}} f(u,v)\tilde{b}_{n_1,k_1}(q_1,q_1u)\,\tilde{b}_{n_2,k_2}(q_2,q_2v)\,d_{q_1}u\,d_{q_2}v, \end{aligned} \tag{5.2.1}$$

where, $\tilde{b}_{n,k}(q,x) = \frac{[n]_q^{n+p}}{[n+p]_q^{n+p}} \begin{bmatrix} n+p \\ k \end{bmatrix}_q x^k \left(\frac{[n+p]_q}{[n]_q} - x \right)_q^{n+p-k}$, $(x,y) \in J^2$, J being $[0,1]$.

Lemma 5.2.1. *[93] Let $e_{ij}(x,y) = x^i y^j$, $i,j \in \mathbb{N}$, $x,y \in \mathbb{R}$ be the two dimensional test functions. The bivariate q-Bernstein-Schurer-Durrmeyer type operators defined by (5.2.1) satisfy the equalities:*

1. $D_{n_1,n_2,p_1,p_2}(e_{0,0};q_1,q_2,x,y) = 1;$

2. $D_{n_1,n_2,p_1,p_2}(e_{1,0};q_1,q_2,x,y) = \frac{[n_1+p_1]_{q_1}}{[n_1+p_1+2]_{q_1}[n_1]_{q_1}}(1+q_1x[n_1]_{q_1});$

3. $D_{n_1,n_2,p_1,p_2}(e_{0,1};q_1,q_2,x,y) = \frac{[n_2+p_2]_{q_2}}{[n_2+p_2+2]_{q_2}[n_2]_{q_2}}(1+q_2y[n_2]_{q_2});$

4. $D_{n_1,n_2,p_1,p_2}(e_{2,0};q_1,q_2,x,y) = \frac{q_1^4[n_1+p_1]_{q_1}[n_1+p_1-1]_{q_1}x^2 + q_1\frac{[n_1+p_1]_{q_1}^2}{[n_1]_{q_1}}(q_1+1)^2x}{[n_1+p_1+3]_{q_1}[n_1+p_1+2]_{q_1}} +$
$\frac{[n_1+p_1]_{q_1}^2(1+q_1)}{[n_1]_{q_1}^2\,[n_1+p_1+3]_{q_1}[n_1+p_1+2]_{q_1}};$

5. $D_{n_1,n_2,p_1,p_2}(e_{0,2};q_1,q_2,x,y) = \dfrac{q_2^4[n_2+p_2]_{q_2}[n_2+p_2-1]_{q_2}y^2 + q_2\dfrac{[n_2+p_2]_{q_2}^2}{[n_2]_{q_2}}(q_2+1)^2y}{[n_2+p_2+3]_{q_2}[n_2+p_2+2]_{q_2}} +$
$\dfrac{[n_2+p_2]_{q_2}^2(1+q_2)}{[n_2]_{q_2}^2\,[n_2+p_2+3]_{q_2}[n_2+p_2+2]_{q_2}}.$

Lemma 5.2.2. *[93] The bivariate q-Bernstein-Schurer-Durrmeyer type operators satisfy the equalities:*

1. $D_{n_1,n_2,p_1,p_2}(u-x;q_1,q_2,x,y) = \left(q_1\dfrac{[n_1+p_1]_{q_1}}{[n_1+p_1+2]_{q_1}} - 1\right)x + \dfrac{[n_1+p_1]_{q_1}}{[n_1]_{q_1}[n_1+p_1+2]_{q_1}};$

2. $D_{n_1,n_2,p_1,p_2}(v-y;q_1,q_2,x,y) = \left(q_2\dfrac{[n_2+p_2]_{q_2}}{[n_2+p_2+2]_{q_2}} - 1\right)y + \dfrac{[n_2+p_2]_{q_2}}{[n_2]_{q_2}[n_2+p_2+2]_{q_2}};$

3. $D_{n_1,n_2,p_1,p_2}((u-x)^2;q_1,q_2,x,y) = \left(q_1^4\dfrac{[n_1+p_1]_{q_1}[n_1+p_1-1]_{q_1}}{[n_1+p_1+2]_{q_1}[n_1+p_1+3]_{q_1}} - 2q_1\dfrac{[n_1+p_1]_{q_1}}{[n_1+p_1+2]} + 1\right)x^2$
$+ \dfrac{[n_1+p_1]_{q_1}}{[n_1]_{q_1}}\left(q_1(q_1+1)^2\dfrac{[n_1+p_1]_{q_1}}{[n_1+p_1+2]_{q_1}[n_1+p_1+3]_{q_1}} - \dfrac{2}{[n_1+p_1+1]_{q_1}}\right)x +$
$\dfrac{[n_1+p_1]_{q_1}^2}{[n_1]_{q_1}^2}\dfrac{(q_1+1)}{[n_1+p_1+2]_{q_1}[n_1+p_1+3]_{q_1}};$

4. $D_{n_1,n_2,p_1,p_2}((v-y)^2;q_1,q_2,x,y) = \left(q_2^4\dfrac{[n_2+p_2]_{q_2}[n_2+p_2-1]_{q_2}}{[n_2+p_2+2]_{q_2}[n_2+p_2+3]_{q_2}} - 2q_2\dfrac{[n_2+p_2]_{q_2}}{[n_2+p_2+2]} + 1\right)y^2$
$+ \dfrac{[n_2+p_2]_{q_2}}{[n_2]_{q_2}}\left(q_2(q_2+1)^2\dfrac{[n_2+p_2]_{q_2}}{[n_2+p_2+2]_{q_2}[n_2+p_2+3]_{q_2}} - \dfrac{2}{[n_2+p_2+1]_{q_2}}\right)x +$
$\dfrac{[n_2+p_2]_{q_2}^2}{[n_2]_{q_2}^2}\dfrac{(q_2+1)}{[n_2+p_2+2]_{q_2}[n_2+p_2+3]_{q_2}}.$

Consequently, for every $(x,y) \in J^2$

$$D_{n_1,n_2,p_1,p_2}((u-x)^2;q_1,q_2,x,y) = O\left(\frac{1}{[n_1]_{q_1}}\right),$$

$$D_{n_1,n_2,p_1,p_2}((v-y)^2;q_1,q_2,x,y) = O\left(\frac{1}{[n_2]_{q_2}}\right),$$

and

$$D_{n_1,n_2,p_1,p_2}((u-x)^4;q_1,q_2,x,y) = O\left(\frac{1}{[n_1]_{q_1}^2}\right),$$

$$D_{n_1,n_2,p_1,p_2}((v-y)^4;q_1,q_2,x,y) = O\left(\frac{1}{[n_2]_{q_2}^2}\right),$$

as $n_1, n_2 \to \infty$.

5.3 Main Results

5.3.1 Local Approximation Theorem

In what follows, let

$$\delta_{n_1} = \delta_{n_1}(x) = D_{n_1,n_2,p_1,p_2}((u-x)^2;q_1,q_2,x,y)$$

and

$$\delta_{n_2} = \delta_{n_2}(y) = D_{n_1,n_2,p_1,p_2}((v-y)^2;q_1,q_2,x,y).$$

Theorem 5.3.1. *For the function* $f \in C(I_1 \times I_2)$, *we have the following inequality*

$$\begin{aligned} &|D_{n_1,n_2,p_1,p_2}(f;q_1,q_2,x,y) - f(x,y)| \\ &\leq M\Big\{\bar{\omega_2}(f;\sqrt{A_{p_1,p_2}^{n_1,n_2}(q_1,q_2,x,y)}) + min\{1, A_{p_1,p_2}^{n_1,n_2}(q_1,q_2,x,y)\}||f||_{C(I)}\Big\} \\ &+\omega\Bigg(f;\Bigg(\bigg(\frac{[n_1+p_1]_{q_1}(1+q_1x[n_1]_{q_1})}{[n_1+p_1+2]_{q_1}} - x\bigg)^2 \\ &+\bigg(\frac{[n_2+p_2]_{q_2}(1+q_2y[n_2]_{q_2})}{[n_2+p_2+2]_{q_2}} - y\bigg)^2\Bigg)^{1/2}\Bigg), \end{aligned} \tag{5.3.1}$$

where

$$\begin{aligned} A_{p_1,p_2}^{n_1,n_2}(q_1,q_2,x,y) &= \Big\{\delta_{n_1}^2 + \delta_{n_2}^2 + \bigg(\frac{[n_1+p_1]_{q_1}(1+q_1x[n_1]_{q_1})}{[n_1+p_1+2]_{q_1}} - x\bigg)^2 \\ &+\bigg(\frac{[n_2+p_2]_{q_2}(1+q_2y[n_2]_{q_2})}{[n_2+p_2+2]_{q_2}} - y\bigg)^2\Big\} \\ &= \Big\{\delta_{n_1}^2 + \delta_{n_2}^2 + (D_{n_1,n_2,p_1,p_2}(u-x;q_1,q_2,x,y))^2 \\ &+(D_{n_1,n_2,p_1,p_2}(v-y;q_1,q_2,x,y))^2\Big\}, \end{aligned}$$

and the constant $M > 0$, *is independent of* f *and* $A_{p_1,p_2}^{n_1,n_2}(q_1,q_2,x,y)$.

Remark 1. Taking $q_1 = q_{n_1}$, $q_2 = q_{n_2}$ in Theorem 5.3.1, where $\{q_{n_1}\}$ and $\{q_{n_2}\}$ are sequences such that $q_{n_i} \to 1,\ q_{n_i}^{n_i} \to 0$, as $n_i \to \infty,\ i = 1, 2$, it follows that the right hand side of the inequality (5.3.1) tends to zero as $n_i \to \infty,\ i = 1, 2$.

Proof. To prove this, let us define:

$$\begin{aligned} D^*_{n_1,n_2,p_1,p_2}(f; q_1, q_2, x, y) &= D_{n_1,n_2,p_1,p_2}(f; q_1, q_2, x, y) - f\Bigg(\frac{[n_1+p_1]_{q_1}(1+q_1x[n_1]_{q_1})}{[n_1+p_1+2]_{q_1}}, \\ &\quad \frac{[n_2+p_2]_{q_2}(1+q_2y[n_2]_{q_2})}{[n_2+p_2+2]_{q_2}}\Bigg) + f(x,y), \end{aligned}$$

then using Lemma 5.2.1, we have

$D^*_{n_1,n_2,p_1,p_2}(1; q_1, q_2, x, y) = 1$

$D^*_{n_1,n_2,p_1,p_2}((u-x); q_1, q_2, x, y) = 0$ and $D^*_{n_1,n_2,p_1,p_2}((v-y); q_1, q_2, x, y) = 0$.

Let $g \in C^2(I_1 \times I_2)$ and $u, v \in I_1 \times I_2$. By the Taylor's theorem, we may write

$$\begin{aligned} g(u,v) - g(x,y) &= g(u,y) - g(x,y) + g(u,v) - g(u,y) \\ &= \frac{\partial g(x,y)}{\partial x}(u-x) + \int_x^u (u-t)\frac{\partial^2 g(t,y)}{\partial t^2}dt \\ &\quad + \frac{\partial g(x,y)}{\partial y}(v-y) + \int_y^v (v-s)\frac{\partial^2 g(x,s)}{\partial s^2}ds. \end{aligned}$$

Applying the operator $D^*_{n_1,n_2,p_1,p_2}(., q_1, q_2, x, y)$ to both sides of the above equality, we get

$$\begin{aligned} D^*_{n_1,n_2,p_1,p_2}(g; q_1, q_2, x, y) - g(x,y) &= D^*_{n_1,n_2,p_1,p_2}\left(\int_x^u (u-t)\frac{\partial^2 g(t,y)}{\partial t^2}dt; q_1, q_2, x, y\right) \\ &\quad + D^*_{n_1,n_2,p_1,p_2}\left(\int_y^v (v-s)\frac{\partial^2 g(x,s)}{\partial s^2}ds; q_1, q_2, x, y\right) \\ &= D_{n_1,n_2,p_1,p_2}\left(\int_x^u (u-t)\frac{\partial^2 g(t,y)}{\partial t^2}dt; q_1, q_2, x, y\right) \\ &\quad - \int_x^{\frac{[n_1+p_1]_{q_1}(1+q_1x[n_1]_{q_1})}{[n_1+p_1+2]_{q_1}}} \left(\frac{[n_1+p_1]_{q_1}(1+q_1x[n_1]_{q_1})}{[n_1+p_1+2]_{q_1}} - t\right)\frac{\partial^2 g(t,y)}{\partial t^2}dt \\ &\quad + D_{n_1,n_2,p_1,p_2}\left(\int_y^v (v-s)\frac{\partial^2 g(x,s)}{\partial s^2}ds; q_1, q_2, x, y\right) \\ &\quad - \int_y^{\frac{[n_2+p_2]_{q_2}(1+q_2y[n_2]_{q_2})}{[n_2+p_2+2]_{q_2}}} \left(\frac{[n_2+p_2]_{q_2}(1+q_2y[n_2]_{q_2})}{[n_2+p_2+2]_{q_2}} - s\right)\frac{\partial^2 g(x,s)}{\partial s^2}ds. \end{aligned}$$

Hence

$$
\begin{aligned}
&|D^{*}_{n_1,n_2,p_1,p_2}(g;q_1,q_2,x,y)-g(x,y)|\\
&\le D_{n_1,n_2,p_1,p_2}\left(\left|\int_x^u |u-t|\left|\frac{\partial^2 g(t,y)}{\partial t^2}\right|dt\right|;x,y\right)\\
&+\left|\int_x^{\frac{[n_1+p_1]_{q_1}(1+q_1x[n_1]_{q_1})}{[n_1+p_1+2]_{q_1}}}\left|\frac{[n_1+p_1]_{q_1}(1+q_1x[n_1]_{q_1})}{[n_1+p_1+2]_{q_1}}-t\right|\left|\frac{\partial^2 g(t,y)}{\partial t^2}\right|dt\right|\\
&+K_{n_1,n_2,p}\left(\left|\int_y^v |v-s|\left|\frac{\partial^2 g(x,s)}{\partial s^2}\right|ds\right|;x,y\right)\\
&+\left|\int_y^{\frac{[n_2+p_2]_{q_2}(1+q_2y[n_2]_{q_2})}{[n_2+p_2+2]_{q_2}}}\left|\frac{[n_2+p_2]_{q_2}(1+q_2y[n_2]_{q_2})}{[n_2+p_2+2]_{q_2}}-s\right|\left|\frac{\partial^2 g(x,s)}{\partial s^2}\right|ds\right|\\
&\le \left\{D_{n_1,n_2,p_1,p_2}((u-x)^2;q_1,q_2,x,y)+\left(\frac{[n_1+p_1]_{q_1}(1+q_1x[n_1]_{q_1})}{[n_1+p_1+2]_{q_1}}-x\right)^2\right\}||g||_{C^2(I_1\times I_2)}\\
&+\left\{D_{n_1,n_2,p_1,p_2}((v-y)^2;q_1,q_2,x,y)+\left(\frac{[n_2+p_2]_{q_2}(1+q_2y[n_2]_{q_2})}{[n_2+p_2+2]_{q_2}}-y\right)^2\right\}||g||_{C^2(I_1\times I_2)}
\end{aligned}
\tag{5.3.2}
$$

$$
\begin{aligned}
&\le \left\{\delta_{n_1}^2+\left(\frac{[n_1+p_1]_{q_1}(1+q_1x[n_1]_{q_1})}{[n_1+p_1+2]_{q_1}}-x\right)^2\right\}||g||_{C^2(I_1\times I_2)}\\
&\quad+\left\{\delta_{n_2}^2+\left(\frac{[n_2+p_2]_{q_2}(1+q_2y[n_2]_{q_2})}{[n_2+p_2+2]_{q_2}}-y\right)^2\right\}||g||_{C^2(I_1\times I_2)}\\
&= \left\{\delta_{n_1}^2+\delta_{n_2}^2+\left(\frac{[n_1+p_1]_{q_1}(1+q_1x[n_1]_{q_1})}{[n_1+p_1+2]_{q_1}}-x\right)^2\right.\\
&\quad\left.+\left(\frac{[n_2+p_2]_{q_2}(1+q_2y[n_2]_{q_2})}{[n_2+p_2+2]_{q_1}}-y\right)^2\right\}||g||_{C^2(I_1\times I_2)}\\
&= A^{n_1,n_2}_{p_1,p_2}(q_1,q_2,x,y)||g||_{C^2(I_1\times I_2)}.
\end{aligned}
\tag{5.3.3}
$$

Also, using Lemma 5.2.1

$$
\begin{aligned}
|D^{*}_{n_1,n_2,p_1,p_2}(f;q_1,q_2,x,y)| &\le |D_{n_1,n_2,p_1,p_2}(f;q_1,q_2,x,y)|\\
&\quad+\left|f\left(\frac{[n_1+p_1]_{q_1}(1+q_1x[n_1]_{q_1})}{[n_1+p_1+2]_{q_1}},\frac{[n_2+p_2]_{q_2}(1+q_2y[n_2]_{q_2})}{[n_2+p_2+2]_{q_2}}\right)\right|\\
&\quad+|f(x,y)|\\
&\le 3||f||_{C(I_1\times I_2)}.
\end{aligned}
\tag{5.3.4}
$$

Hence, in view of (5.3.3)-(5.3.4) for every $g\in C^2(I_1\times I_2)$, we get

$$\begin{aligned}
&|D_{n_1,n_2,p_1,p_2}(f;q_1,q_2,x,y)-f(x,y)|\\
=\ & \Bigg|D^*_{n_1,n_2,p_1,p_2}(f;q_1,q_2,x,y)-f(x,y)\\
&+f\left(\frac{[n_1+p_1]_{q_1}(1+q_1x[n_1]_{q_1})}{[n_1+p_1+2]_{q_1}},\frac{[n_2+p_2]_{q_2}(1+q_2y[n_2]_{q_2})}{[n_2+p_2+2]_{q_2}}\right)-f(x,y)\Bigg|\\
\leq\ & |D^*_{n_1,n_2,p_1,p_2}(f-g;q_1,q_2,x,y)|+|D^*_{n_1,n_2,p_1,p_2}(g;q_1,q_2,x,y)-g(x,y)|\\
&+|g(x,y)-f(x,y)|+\Bigg|f\left(\frac{[n_1+p_1]_{q_1}(1+q_1x[n_1]_{q_1})}{[n_1+p_1+2]_{q_1}},\frac{[n_2+p_2]_{q_2}(1+q_2y[n_2]_{q_2})}{[n_2+p_2+2]_{q_2}}\right)\\
&-f(x,y)\Bigg|\\
\leq\ & 4||f-g||_{C(I_1\times I_2)}+|D^*_{n_1,n_2,p_1,p_2}(g;q_1,q_2,x,y)-g(x,y)|\\
&+\Bigg|f\left(\frac{[n_1+p_1]_{q_1}(1+q_1x[n_1]_{q_1})}{[n_1+p_1+2]_{q_1}},\frac{[n_2+p_2]_{q_2}(1+q_2y[n_2]_{q_2})}{[n_2+p_2+2]_{q_2}}\right)-f(x,y)\Bigg|\\
\leq\ & \left(4||f-g||_{C(I)}+A^{n_1,n_2}_{p_1,p_2}(q_{n_1},q_{n_2},x,y)||g||_{C^2(I_1\times I_2)}\right)\\
&+\omega\left(f;\sqrt{\left(\frac{[n_1+p_1]_{q_1}(1+q_1x[n_1]_{q_1})}{[n_1+p_1+2]_{q_1}}-x\right)^2+\left(\frac{[n_2+p_2]_{q_2}(1+q_2y[n_2]_{q_2})}{[n_2+p_2+2]_{q_2}}-y\right)^2}\right).
\end{aligned}$$

Using (0.4.2), we have

$$\begin{aligned}
&|D_{n_1,n_2,p_1,p_2}(f;q_1,q_2,x,y)-f(x,y)|\\
&\quad\leq\ 4\mathcal{K}(f;A^{n_1,n_2}_{p_1,p_2}(q_1,q_2,x,y))+\omega\left(f;\left\{\left(\frac{[n_1+p_1]_{q_1}(1+q_1x[n_1]_{q_1})}{[n_1+p_1+2]_{q_1}}-x\right)^2\right.\right.\\
&\qquad\left.\left.+\left(\frac{[n_2+p_2]_{q_2}(1+q_2y[n_2]_{q_2})}{[n_2+p_2+2]_{q_2}}-y\right)^2\right\}^{1/2}\right)
\end{aligned}$$

Hence, we reach the desired result.

□

5.4 Construction of GBS operators of q-Bernstein-Schurer-Durmeyer type

For $f \in C(I_1 \times I_2)$, the parametric extensions of the univariate operator introduced by Acu et al. [13] are the operators $D^x_{n_1,p_1}, D^y_{n_2,p_2} : C(I_1 \times I_2) \to C(I_1 \times I_2)$, defined for each positive integers n_1, n_2 as follows

$$D^x_{n,p_1}(f;q_1,x,y) = \sum_{k_1=0}^{n+p_1} \tilde{b}_{n,k_1}(q_1,x)q_1^{-k_1} \int_0^{\frac{[n+p_1]_{q_1}}{[n]_{q_1}}} f(u,y)\tilde{b}_{n,k_1}(q_1,q_1u)d_{q_1}u,$$

$$D^y_{n,p_2}(f;q_2,x,y) = \sum_{k_2=0}^{n+p_2} \tilde{b}_{n,k_2}(q_2,y)q_2^{-k_2} \int_0^{\frac{[n+p_2]_{q_2}}{[n]_{q_2}}} f(x,v)\tilde{b}_{n,k_2}(q_2,q_2v)d_{q_2}v.$$

For any $(x,y) \in J^2$, the GBS operator of q-Bernstein-Schurer-Durrmeyer type $U_{n_1,n_2,p_1,p_2} : C_b(I_1 \times I_2) \to C(J^2)$, associated to D_{n_1,n_2,p_1,p_2} is defined as:

$$\begin{aligned} &U_{n_1,n_2,p_1,p_2}(f;q_1,q_2,x,y) \\ &= \frac{[n_1+p_1+1]_{q_1}[n_1]_{q_1}}{[n_1+p_1]_{q_1}} \frac{[n_2+p_2+1]_{q_2}[n_2]_{q_2}}{[n_2+p_2]_{q_2}} \sum_{k_1=0}^{n_1+p_1}\sum_{k_2=0}^{n_2+p_2} \tilde{b}_{n_1,k_1}(q_1,x)\tilde{b}_{n_2,k_2}(q_2,y)q_1^{-k_1}q_2^{-k_2} \\ &\quad\times \int_0^{\frac{[n_1+p_1]_{q_1}}{[n_1]_{q_1}}} \int_0^{\frac{[n_2+p_2]_{q_2}}{[n_2]_{q_2}}} [f(x,v)+f(u,y)-f(u,v)]\tilde{b}_{n_1,k_1}(q_1,q_1u)\tilde{b}_{n_2,k_2}(q_2,q_2v)d_{q_1}u d_{q_2}v. \end{aligned} \tag{5.4.1}$$

It can be easily observed that the GBS q-Durrmeyer operators is the boolean sum of parametric extensions D^x_{n,p_1}, D^y_{n,p_2}, i.e

$$U_{n_1,n_2,p_1,p_2} = D^x_{n,p_1} \oplus D^y_{n,p_2} = D^x_{n,p_1} + D^y_{n,p_2} - D_{n_1,n_2,p_1,p_2}$$

Our next theorem gives the degree of approximation for the operators U_{n_1,n_2,p_1,p_2} by means of the Lipschitz class of Bögel continuous functions.

Theorem 5.4.1. *Let $f \in Lip_M(\xi,\gamma)$ then we have*

$$\left|U_{n_1,n_2,p_1,p_2}(f;q_1,q_2,x,y) - f(x,y)\right| \le M\delta_{n_1}^{\xi/2}\delta_{n_2}^{\gamma/2},$$

for $M>0, \xi,\gamma \in (0,1]$.

Proof. We may write

$$\begin{aligned} U_{n_1,n_2,p_1,p_2}(f;q_1,q_2,x,y) &= D_{n_1,n_2,p_1,p_2}(f(x,s)+f(u,y)-f(u,v);q_1,q_2,x,y) \\ &= D_{n_1,n_2,p_1,p_2}(f(x,y)-\Delta f[(u,v);(x,y)];q_1,q_2,x,y) \\ &= f(x,y)D_{n_1,n_2,p_1,p_2}(e_{00};q_1,q_2,x,y) \\ &\quad -D_{n_1,n_2,p_1,p_2}(\Delta f[(u,v);(x,y)];q_1,q_2,x,y). \end{aligned}$$

Since $f \in Lip_M(\xi, \gamma)$, we have

$$\begin{aligned}
|U_{n_1,n_2,p_1,p_2}(f; q_1, q_2, x, y) - f(x, y)| &\leq D_{n_1,n_2,p_1,p_2}\left(|\Delta f[(u,v);(x,y)]|; q_1, q_1, x, y\right) \\
&\leq M D_{n_1,n_2,p_1,p_2}\left(|u-x|^{\xi}|v-y|^{\gamma}; q_1, q_1, x, y\right) \\
&= M D_{n_1,p_1}\left(|u-x|^{\xi}; q_1, x\right) D_{n_2,p_2}\left(|v-y|^{\gamma}; q_2, y\right).
\end{aligned}$$

Now, using the Hölder's inequality with $t_1 = 2/\xi, s_1 = 2/(2-\xi)$ and $t_2 = 2/\gamma, s_2 = 2/(2-\gamma)$, we have

$|U_{n_1,n_2,p_1,p_2}(f; q_1, q_2, x, y) - f(x, y)|$

$$\begin{aligned}
&\leq M\left(D_{n_1,p_1}((u-x)^2; q_1, x)\right)^{\xi/2}\left(D_{n_1,p_1}(e_0; q_1, x)\right)^{(2-\xi)/2} \\
&\quad \times \left(D_{n_2,p_2}((v-y)^2; q_2, y)\right)^{\gamma/2}\left(D_{n_2,p_2}(e_0; q_2, y)\right)^{(2-\gamma)/2}.
\end{aligned}$$

Now, considering Lemma 5.2.1, we obtain the degree of local approximation for B-continuous functions belonging to $Lip_M(\xi, \gamma)$. □

In the following theorem we estimate the rate of convergence by the operators U_{n_1,n_2,p_1,p_2} for Bögel differentiable functions.

Theorem 5.4.2. *Let the function $f \in D_b(I_1 \times I_2)$ with $D_B f \in B(I_1 \times I_2)$. Then, for each $(x, y) \in J$, we have*

$$\begin{aligned}
|U_{n_1,n_2,p_1,p_2}(f; q_1, q_2, x, y) - f(x, y)| &\leq \frac{C}{[n_1]_{q_1}^{1/2}[n_2]_{q_2}^{1/2}}\Big\{||D_B f||_{\infty} \\
&\quad + \omega_{mixed}(D_B f; [n_1]_{q_1}^{-1/2}[n_2]_{q_2}^{-1/2})\Big\}.
\end{aligned}$$

Proof. Since $f \in D_b(I_1 \times I_2)$, we have the identity

$$\Delta f[(u,v);(x,y)] = (u-x)(v-y)D_B f(\xi, \eta), \ \ with\ x < \xi < u;\ y < \eta < v.$$

It is clear that

$$D_B f(\xi, \eta) = \Delta D_B f(\xi, \eta) + D_B f(\xi, y) + D_B f(x, \eta) - D_B f(x, y).$$

Since $D_B f \in B(I_1 \times I_2)$, by above relations, we can write

$$
\begin{aligned}
&|D_{n_1,n_2,p_1,p_2}(\Delta f[(t,s);(x,y)];q_1,q_2,x,y)| \\
&\quad = |D_{n_1,n_2,p_1,p_2}((u-x)(v-y)D_B f(\xi,\eta);q_1,q_2,x,y)| \\
&\quad \leq D_{n_1,n_2,p_1,p_2}(|u-x||v-y||\Delta D_B f(\xi,\eta)|;q_1,q_2,x,y) \\
&\quad + D_{n_1,n_2,p_1,p_2}(|u-x||v-y|(|D_B f(\xi,y)|+|D_B f(x,\eta)| \\
&\quad + |D_B f(x,y)|);q_1,q_2,x,y) \\
&\quad \leq D_{n_1,n_2,p_1,p_2}(|u-x||v-y|\omega_{mixed}(D_B f;|\xi-x|,|\eta-y|);q_1,q_2,x,y) \\
&\quad + 3\ ||D_B f||_\infty\ D_{n_1,n_2,p_1,p_2}(|u-x||v-y|;q_1,q_2,x,y).
\end{aligned}
\tag{5.4.2}
$$

Since the mixed modulus of smoothness ω_{mixed} is non-decreasing, we have

$$
\begin{aligned}
\omega_{mixed}(D_B f;|\xi-x|,|\eta-y|) &\leq \omega_{mixed}(D_B f;|u-x|,|v-y|) \\
&\leq (1+\delta_1^{-1}|u-x|)(1+\delta_2^{-1}|v-y|)\ \omega_{mixed}(D_B f;\delta_1,\delta_2).
\end{aligned}
\tag{5.4.3}
$$

Combining (5.4.2)-(5.4.3), and proceeding along the lines of the proof of Theorem 4.4.2, on taking $\delta_1 = \frac{1}{[n_1]_{q_1}^{1/2}}$, $\delta_2 = \frac{1}{[n_2]_{q_2}^{1/2}}$ and using Lemma 5.2.2, we obtain the required result.

□

Some graphs on GBS operators of bivariate q-Bernstein-Schurer-Durrmeyer type

Example 1. Let $p_1 = p_2 = 2$. For $n_1, n_2 = 15, q_1 = .20, q_2 = .25$ (yellow) and $n_1, n_2 = 25, q_1 = .35, q_2 = .40$ (pink), the convergence of the operators $U_{n_1,n_2,p_1,p_2}(f;q_1,q_2,x,y)$ to $f(x,y) = \cos(\pi x^2) + y^2$ (blue) is illustrated in Figure 1. In Figure 2, for $n_1, n_2 = 35, q_1 = .45, q_2 = .55$ (grey) and $n_1, n_2 = 35, q_1 = q_2 = 1$ (pink), the convergence of the operators $U_{n_1,n_2,p_1,p_2}(f;q_1,q_2,x,y)$ to the same function is shown. It is observed that, as the values of n_1, n_2 and q_1, q_2 increase, the convergence of

U_{n_1,n_2,p_1,p_2} to $f(x,y)$ becomes better.

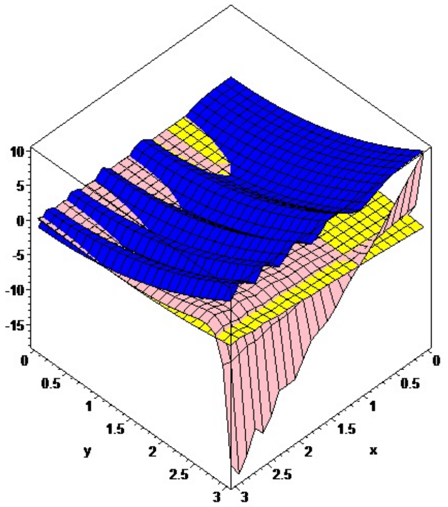


Figure 1

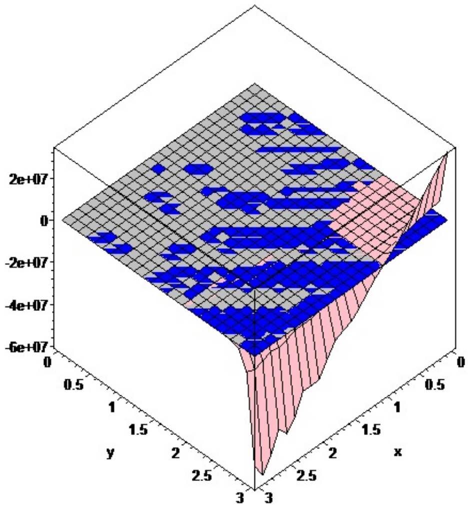


Figure 2

Example 2. Let $p_1 = 2, p_2 = 3$ and $f(x,y) = x + \sin(3y)$. For $n_1 = 25, n_2 = 10$ and $q_1 = .50, q_2 = .60$, the convergence of the operators $D_{n_1,n_2,p_1,p_2}(f; q_1, q_2, x, y)$ (grey) given by (5.2.1) and its GBS operators $U_{n_1,n_2,p_1,p_2}(f; q_1, q_2, x, y)$ (yellow) to $f(x,y)$ (blue) is given in Figure 3. It is observed that the convergence of $U_{n_1,n_2,p_1,p_2}(f; q_1, q_2, x, y)$ to $f(x,y)$ is at least as good as that of $D_{n_1,n_2,p_1,p_2}(f; q_1, q_2, x, y)$.

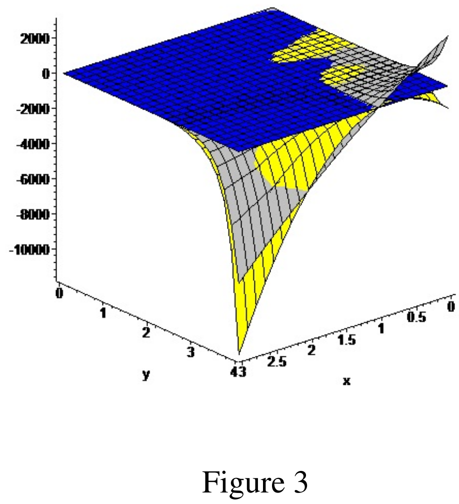


Figure 3

Chapter 6

Quantitative Estimates of Generalized Boolean Sum operators of Blending type

6.1 Introduction

In [123], Sharma and Aujla introduced a mixed summation-integral-type Lupaş-Phillips-Bernstein operator wherein they proved the statistical convergence of these operators and estimated the rate of convergence by using modulus of continuity. Later on, Sharma [122] introduced the bivariate case of these operators and determined the rate of convergence by means of the complete and the partial modulii of continuity and the Peetre's K-functional. In this chapter, we extend the work done by these researchers by introducing the GBS operator of q-Lupaş-Phillips-Bernstein type and obtain the degree of approximation by means of the mixed modulus of smoothness.

In [122], the two dimensional mixed summation-integral type q-Lupaş-Phillips-Bernstein operators is defined as follows: Let $(q_n)_n, (q_m)_m$ be sequences of real numbers such that $0 < q_n, q_m < 1$, for all n, m and $\lim_{n\to\infty} q_n = 1, \lim_{m\to\infty} q_m = 1$, and $q_n^n \to a, q_m^m \to b \ (0 \leq a, b < 1)$, as $n, m \to \infty$. For $(x, y) \in \square$, where $\square = [0, 1] \times [0, 1]$, and for any $f \in L_B(\square)$, the space of bounded and Lebesgue integrable functions in $\square$, we have

$$\begin{aligned}\widetilde{D}_{n,m}^{q_n,q_m}(f; x, y) &= [n+1]_{q_n}[m+1]_{q_m} \sum_{k=0}^{n}\sum_{j=0}^{m} q_n^{-k}\, q_m^{-j}\, \Phi_{n,m}^{q_n,q_m}(x, y) \\ &\quad \int_{u=0}^{1}\int_{v=0}^{1} \Psi_{n,m}^{q_n,q_m}(q_n u, q_m v) f(u, v) d_{q_n}u\, d_{q_m}v,\end{aligned} \tag{6.1.1}$$

where, $\Phi^{q_n,q_m}_{n,m}(t,s) = b^{q_n}_{n,k}(t)b^{q_m}_{m,j}(s)$ and $\Psi^{q_n,q_m}_{n,m}(t,s) = p^{q_n}_{n,k}(t)p^{q_m}_{m,j}(s)$. Further, $b^q_{n,k}(x), p^q_{n,k}(x)$ are the Lupas and Phillips-Bernstein bases respectively, given by

$$b^q_{n,k}(x) = \frac{\left[{n \atop k}\right]_q q^{k(k-1)/2} x^k (1-x)^{n-k}_q}{\prod_{i=0}^{n-1}(1-x+q^i x)}$$

and

$$p^q_{n,k}(x) = \begin{bmatrix} n \\ k \end{bmatrix}_q x^k (1-x)^{n-k}_q.$$

For any $f \in C_b(\square)$, the space of Bögel continuous function on the $\square$, the GBS operator of the operators given by (6.1.1) is defined as follows:

$$\begin{aligned} \widetilde{T}^{q_n,q_m}_{n,m}(f;x,y) &= [n+1]_{q_n}[m+1]_{q_m} \sum_{k=0}^{n}\sum_{j=0}^{m} q_n^{-k}\, q_m^{-j}\, \Phi^{q_n,q_m}_{n,m}(x,y) \int_{u=0}^{1}\int_{v=0}^{1} \Psi^{q_n,q_m}_{n,m}(q_n u, q_m v) \\ &\quad \times (f(x,u) + f(v,y) - f(u,v))\, d_{q_n}u\, d_{q_m}v, \quad (x,y) \in \square. \end{aligned} \tag{6.1.2}$$

Now, we present a lemma which will be used in the sequel.

Lemma 6.1.1. *[122] Let $(x,y) \in \square, m,n \in \mathbb{N}$. For the operator $\widetilde{D}^{q_n,q_m}_{n,m}$ there holds the equalities:*

(i) $\widetilde{D}^{q_n,q_m}_{n,m}(1;x,y) = 1;$

(ii) $\widetilde{D}^{q_n,q_m}_{n,m}(u;x,y) = \dfrac{[n]_{q_n}}{[n+2]_{q_n}} q_n x + \dfrac{1}{[n+2]_{q_n}};$

(iii) $\widetilde{D}^{q_n,q_m}_{n,m}(v;x,y) = \dfrac{[m]_{q_m}}{[m+2]_{q_m}} q_m y + \dfrac{1}{[m+2]_{q_m}};$

(iv) $\widetilde{D}^{q_n,q_m}_{n,m}(u^2;x,y) = \dfrac{[n]^2_{q_n}}{[n+2]_{q_n}[n+3]_{q_n}} q_n^3 \left\{ x^2 + \dfrac{x(1-x)}{[n]_{q_n}} - \dfrac{x^2(1-x)(1-q_n)}{(1-x+xq_n)} \left(1 - \dfrac{1}{[n]_{q_n}}\right) \right\} + \dfrac{[n]_{q_n}}{[n+2]_{q_n}[n+3]_{q_n}} q_n(2q_n+1)x + \dfrac{1}{[n+2]_{q_n}[n+3]_{q_n}}(q_n+1);$

(v) $\widetilde{D}^{q_n,q_m}_{n,m}(v^2;x,y) = \dfrac{[m]^2_{q_m}}{[m+2]_{q_m}[m+3]_{q_m}} q_m^3 \left\{ y^2 + \dfrac{y(1-y)}{[m]_{q_m}} - \dfrac{y^2(1-y)(1-q_m)}{(1-y+yq_m)} \left(1 - \dfrac{1}{[m]_{q_m}}\right) \right\} + \dfrac{[m]_{q_m}}{[m+2]_{q_m}[m+3]_{q_m}} q_m(2q_m+1)y + \dfrac{1}{[m+2]_{q_m}[m+3]_{q_m}}(q_m+1).$

Remark 6.1.1. *For the operator* $\widetilde{D}_{n,m}^{q_n,q_m}$, *we have*

(i) $\lim\limits_{n\to\infty}[n]_{q_n}\widetilde{D}_{n,m}^{q_n,q_m}((u-x);x)=(a-1)x,$

(ii) $\lim\limits_{n\to\infty}[n]_{q_n}\widetilde{D}_{n,m}^{q_n,q_m}((u-x)^2;x)=x^3-3x^2+2x+ax^2(1-x).$

(iii) $\lim\limits_{n\to\infty}[n]_{q_n}\widetilde{D}_{n,m}^{q_n,q_m}((u-x)^4;x)=-12x^4+56x^2-24ax^3(1-x).$

6.2 Main Results

In what follows, let $\delta_n=\widetilde{D}_{n,m}^{q_n,q_m}\left((u-x)^2;x,y\right)$ and $\delta_m=\widetilde{D}_{n,m}^{q_n,q_m}\left((v-y)^2;x,y\right)$.

6.2.1 Asymptotic Result

For the operators defined by (6.1.1), to prove the Voronovskaja type asymptotic result, let us take $n=m$.

Theorem 6.2.1. *For* $f\in C^2(\Box)$, *we have*

$$\begin{aligned}&\lim_{n\to\infty}[n]_{q_n}\left\{\widetilde{D}_{n,n}^{q_n}(f;x,y)-f(x,y)\right\}\\ =\ & f_x(x,y)(a-1)x+f_y(x,y)(a-1)y+\frac{1}{2}\Big\{f_{xx}(x,y)\left(x^3-3x^2+2x+ax^2(1-x)\right)\\ &+f_{yy}(x,y)\left(y^3-3y^2+2y+ay^2(1-y)\right)\Big\},\end{aligned}$$

uniformly in $(x,y)\in\Box$.

Proof. By our hypothesis

$$\begin{aligned}\widetilde{D}_{n,n}^{q_n}(f(u,v);x,y)\ =\ & f(x,y)+f_x(x,y)\widetilde{D}_{n,n}^{q_n}((u-x);x)+f_y(x,y)\widetilde{D}_{n,n}^{q_n}((v-y);y)\\ &+\frac{1}{2}\Big\{f_{xx}(x,y)\widetilde{D}_{n,n}^{q_n}((u-x)^2;x)+2f_{xy}(x,y)\widetilde{D}_{n,n}^{q_n}((u-x);x,y)\\ &\widetilde{D}_{n,n}^{q_n}(v-y);x,y)+f_{yy}(x,y)\widetilde{D}_{n,n}^{q_n}(x,y)((v-y)^2;y)\Big\}\\ &+\widetilde{D}_{n,n}^{q_n}\left(\phi(u,v;x,y)\{(u-x)^2+(v-y)^2\};x,y\right).\end{aligned} \tag{6.2.1}$$

Where $\lim_{(u,v)\to(x,y)} \phi(u,v;x,y) = 0$.

Now, using Remark 6.1.1, we may write

$$\lim_{n\to\infty}[n]_{q_n}\{\widetilde{D}^{q_n}_{n,n}(f(u,v);x,y) - f(x,y)\}$$

$$\begin{aligned} &= f_x(x,y)(a-1)x + f_y(x,y)(a-1)y \\ &+\frac{1}{2}\Big\{ f_{xx}(x,y)\left(x^3 - 3x^2 + 2x + ax^2(1-x)\right) \\ &+f_{yy}(x,y)\left(y^3 - 3y^2 + 2y + ay^2(1-y)\right) \Big\} \\ &+ \lim_{n\to\infty}[n]_{q_n}\widetilde{D}^{q_n}_{n,n}\left(\phi(u,v;x,y)\{(u-x)^2 + (v-y)^2\};x,y\right), \end{aligned} \tag{6.2.2}$$

uniformly in $(x,y) \in \square$.

Applying the Hölder's inequality, we have

$$|\widetilde{D}^{q_n}_{n,n}\left(\phi(u,v;x,y)\left\{(u-x)^2 + (v-y)^2\right\};x,y\right)|$$

$$\leq \left\{\widetilde{D}^{q_n}_{n,n}(\phi^2(u,v;x,y);x,y)\right\}^{1/2}\left[\left\{\widetilde{D}^{q_n}_{n,n}((u-x)^4;x)\right\}^{1/2} + \left\{\widetilde{D}^{q_n}_{n,n}((v-y)^4;y)\right\}^{1/2}\right].$$

By the Korovkin-type theorem ([122], Theorem 1. page.754), we have

$$\lim_{n\to\infty}\widetilde{D}^{q_n}_{n,n}(\phi^2(u,v;x,y);x,y) = \phi^2(u,v;x,y) = 0,$$

uniformly in $(x,y) \in \square$.

Again using Remark 6.1.2,

$\widetilde{D}^{q_n}_{n,n}((u-x)^4;x) = O\left(\frac{1}{[n]^2_{q_n}}\right)$ and $\widetilde{D}^{q_n}_{n,n}((v-y)^4;x) = O\left(\frac{1}{[n]^2_{q_n}}\right)$,

hence

$$\lim_{n\to\infty}[n]_{q_n}\widetilde{D}^{q_n}_{n,n}\left(\phi(u,v;x,y)\left\{(u-x)^2 + (v-y)^2\right\};x,y\right) = 0, \tag{6.2.3}$$

uniformly for all $(x,y) \in \square$.

By combining (6.2.2) and (6.2.3), we reach the desired result. □

Theorem 6.2.2. *For $f \in C_b(\square)$ and each $(x,y) \in \square$, there holds the following inequality*

$$|\widetilde{T}^{q_n,q_m}_{n,m}(f;x,y) - f(x,y)| \leq 4\,\omega_{mixed}(f;\delta_n,\delta_m).$$

Proof. Since

$$\omega_{mixed}(f;\lambda_1\delta_n,\lambda_2\delta_m)\leq(1+\lambda_1)(1+\lambda_2)\;\omega_{mixed}(f;\delta_n,\delta_m);\;\;\lambda_1,\lambda_2>0,$$

we have,

$$\begin{aligned}|\Delta_{(x,y)}f(u,v)| &\leq \omega_{mixed}(f;|u-x|,|v-y|)\\ &\leq \left(1+\frac{|u-x|}{\delta_n}\right)\left(1+\frac{|u-y|}{\delta_m}\right)\omega_{mixed}(f;\delta_n,\delta_m), \qquad (6.2.4)\end{aligned}$$

for every $(x,y),(u,v)\in\square$ and for any $\delta_n,\delta_m>0$. Further, by the definition of $\Delta_{(x,y)}f(u,v)$, we get

$$f(x,v)+f(u,y)-f(u,v)=f(x,y)-\Delta_{(x,y)}f(u,v).$$

Hence

$$\widetilde{T}_{n,m}^{q_n,q_m}(f;x,y)=f(x,y)\,\widetilde{D}_{n,m}^{q_n,q_m}(1;x,y)-\;\widetilde{D}_{n,m}^{q_n,q_m}\left(\Delta_{(x,y)}f(u,v);x,y\right).$$

Note that, $\widetilde{D}_{n,m}^{q_n,q_m}(1;x,y)=1$, hence using (6.2.4) and the Cauchy-Schwarz inequality we obtain,

$$\begin{aligned}|\widetilde{T}_{n,m}^{q_n,q_m}(f;x,y)-f(x,y)| &\leq \widetilde{D}_{n,m}^{q_n,q_m}\left(|\Delta_{(x,y)}f(u,v)|;x,y\right)\\ &\leq \Bigg(\widetilde{D}_{n,m}^{q_n,q_m}(1;x,y)+\delta_n^{-1}\sqrt{\widetilde{D}_{n,m}^{q_n,q_m}((u-x)^2;x,y)}\\ &\quad+\delta_m^{-1}\sqrt{\widetilde{D}_{n,m}^{q_n,q_m}((v-y)^2;x,y)}+\delta_n^{-1}\delta_m^{-1}\sqrt{\widetilde{D}_{n,m}^{q_n,q_m}((u-x)^2;x,y)}\\ &\quad\times\sqrt{\widetilde{D}_{n,m}^{q_n,q_m}((v-y)^2;x,y)}\Bigg)\,\omega_{mixed}(f;\delta_n,\delta_m)\\ &\leq 4\;\omega_{mixed}(f;\delta_n,\delta_m),\end{aligned}$$

which gives us the desired result.

□

In our next result, the degree of approximation for the GBS operators $\widetilde{T}_{n,m}^{q_n,q_m}$ is obtained by using the Lipschitz class $Lip_M(\alpha,\beta), 0<\alpha,\beta\leq 1$ for Bögel continuous function.

Theorem 6.2.3. *For $f\in Lip_M\,(\alpha,\beta)$, we have*

$$\left|\widetilde{T}_{n,m}^{q_n,q_m}\,(f;x,y)-f\,(x,y)\right|\leq M\delta_n^{\alpha/2}\delta_m^{\beta/2},$$

where M is a certain positive constant.

Proof. Since

$$\begin{aligned}\widetilde{T}_{n,m}^{q_n,q_m}(f;x,y) &= \widetilde{D}_{n,m}^{q_n,q_m}(f(x,v)+f(u,y)-f(u,v);x,y)\\ &= \widetilde{D}_{n,m}^{q_n,q_m}\left(f(x,y)-\Delta_{(x,y)}f(u,v);x,y\right)\\ &= f(x,y)\,\widetilde{D}_{n,m}^{q_n,q_m}(1;x,y)-\widetilde{D}_{n,m}^{q_n,q_m}\left(\Delta_{(x,y)}f(u,v);x,y\right),\end{aligned}$$

by our hypothesis, we find

$$\begin{aligned}\left|\widetilde{T}_{n,m}^{q_n,q_m}(f;x,y)-f(x,y)\right| &\leq \widetilde{D}_{n,m}^{q_n,q_m}\left(\left|\Delta_{(x,y)}f(u,v)\right|;x,y\right)\\ &\leq M\widetilde{D}_{n,m}^{q_n,q_m}\left(|u-x|^{\alpha}|v-y|^{\beta};x,y\right)\\ &= M\widetilde{D}_{n,m}^{q_n,q_m}\left(|u-x|^{\alpha};x,y\right)\widetilde{D}_{n,m}^{q_n,q_m}\left(|v-y|^{\beta};x,y\right).\end{aligned}$$

Now, applying the Hölder's inequality with $p_1 = 2/\alpha, q_1 = 2/(2-\alpha)$ and $p_2 = 2/\beta, q_2 = 2/(2-\beta)$, we obtain

$$\begin{aligned}\left|\widetilde{T}_{n,m}^{q_n,q_m}(f;x,y)-f(x,y)\right| \leq\ & M\left(\widetilde{D}_{n,m}^{q_n,q_m}(u-x)^2;x,y\right)^{\alpha/2}\widetilde{D}_{n,m}^{q_n,q_m}(1;x,y)^{(2-\alpha)/2}\\ &\times\widetilde{D}_{n,m}^{q_n,q_m}\left((v-y)^2;x,y\right)^{\beta/2}\widetilde{D}_{n,m}^{q_n,q_m}(1;x,y)^{(2-\beta)/2}.\end{aligned}$$

Using Lemma 6.1.1, we reach the desired result. □

Theorem 6.2.4. *If $f \in D_b(\square)$ and $D_B f \in B(\square)$, then for each $(x,y) \in \square$, we get*

$$|\widetilde{T}_{n,m}^{q_n,q_m}(f;x,y)-f(x,y)| \leq \frac{M}{[n]_{q_n}^{1/2}[m]_{q_m}^{1/2}}\left(||D_B f||_{\infty}+\omega_{mixed}(D_B f;[n]_{q_n}^{-1/2},[m]_{q_m}^{-1/2})\right).$$

Proof. By our hypothesis

$$\Delta_{x,y}f(u,v) = (u-x)(v-y)D_B f(\xi,\eta),\ \ with\ x<\xi<u;\ y<\eta<s.$$

Clearly,

$$D_B f(\xi,\eta) = \Delta_{(x,y)}D_B f(\xi,\eta)+D_B f(\xi,y)+D_B f(x,\eta)-D_B f(x,y).$$

Since $D_B f \in B(\Box)$, from the above equalities, we have

$$\begin{aligned} |\widetilde{D}^{q_n,q_m}_{n,m}(\Delta_{(x,y)}f(u,v);x,y)| &= |\widetilde{D}^{q_n,q_m}_{n,m}((u-x)(v-y)D_Bf(\xi,\eta);x,y)| \\ &\le \widetilde{D}^{q_n,q_m}_{n,m}(|u-x||v-y||\Delta_{(x,y)}D_Bf(\xi,\eta)|;x,y) \\ &\quad +\widetilde{D}^{q_n,q_m}_{n,m}\Big(|u-x||v-y|(|D_Bf(\xi,y)| \\ &\quad +|D_Bf(x,\eta)|+|D_Bf(x,y)|);x,y\Big) \\ &\le \widetilde{D}^{q_n,q_m}_{n,m}(|u-x||v-y|\omega_{mixed}(D_Bf;|\xi-x|,|\eta-y|);x,y) \\ &\quad +3\ ||D_Bf||_\infty\ \widetilde{D}^{q_n,q_m}_{n,m}(|u-x||v-y|;x,y). \end{aligned} \tag{6.2.5}$$

By the properties of mixed modulus of smoothness ω_{mixed}, we can write

$$\begin{aligned} \omega_{mixed}(D_Bf;|\xi-x|,|\eta-y|) &\le \omega_{mixed}(D_Bf;|u-x|,|v-y|) \\ &\le (1+\delta_n^{-1}|u-x|)(1+\delta_m^{-1}|v-y|)\ \omega_{mixed}(D_Bf;\delta_n,\delta_m). \end{aligned} \tag{6.2.6}$$

Combining (6.2.5), (6.2.6)and using the Cauchy-Schwarz inequality we find

$$\begin{aligned} &|\widetilde{T}^{q_n,q_m}_{n,m}(f;x,y)-f(x,y)| \\ &= |\widetilde{D}^{q_n,q_m}_{n,m}\Delta_{(x,y)}f(u,v);x,y| \\ &\le 3||D_Bf||_\infty\sqrt{\widetilde{D}^{q_n,q_m}_{n,m}((u-x)^2(v-y)^2;x,y)}+\Big(\widetilde{D}^{q_n,q_m}_{n,m}(|u-x||v-y|;x,y) \\ &\quad +\delta_n^{-1}\widetilde{D}^{q_n,q_m}_{n,m}((u-x)^2|s-y|;x,y)+\delta_m^{-1}\widetilde{D}^{q_n,q_m}_{n,m}(|u-x|(v-y)^2;x,y) \\ &\quad +\delta_n^{-1}\delta_m^{-1}\widetilde{D}^{q_n,q_m}_{n,m}((u-x)^2(v-y)^2;x,y)\Big)\omega_{mixed}(D_Bf;\delta_n,\delta_m) \\ &\le 3||D_Bf||_\infty\sqrt{\widetilde{D}^{q_n,q_m}_{n,m}((u-x)^2(v-y)^2;x,y)}+\Big(\sqrt{\widetilde{D}^{q_n,q_m}_{n,m}((u-x)^2(v-y)^2;x,y)} \\ &\quad +\delta_n^{-1}\sqrt{\widetilde{D}^{q_n,q_m}_{n,m}((u-x)^4(v-y)^2;x,y)}+\delta_m^{-1}\sqrt{\widetilde{D}^{q_n,q_m}_{n,m}((u-x)^2(v-y)^4;x,y)} \\ &\quad +\delta_n^{-1}\delta_m^{-1}\widetilde{D}^{q_n,q_m}_{n,m}((u-x)^2(v-y)^2;x,y)\Big)\omega_{mixed}(D_Bf;\delta_n,\delta_m). \end{aligned} \tag{6.2.7}$$

We note that for $(x,y),(t,s)\in\Box$ and $i,j\in\{1,2\}$

$$\widetilde{D}^{q_n,q_m}_{n,m}((u-x)^{2i}(v-y)^{2j};x,y)=\widetilde{D}^{q_n}_{n}((u-x)^{2i};x)\widetilde{D}^{q_m}_{m}((v-y)^{2j};y). \tag{6.2.8}$$

From Remark 6.1.1, for sufficiently large n

$$\widetilde{D}_n^{q_n}((u-x)^2;x) \leq \frac{M_1}{[n]_{q_n}}, \quad \widetilde{D}_m^{q_m}((v-y)^2;y) \leq \frac{M_2}{[m]_{q_m}} \tag{6.2.9}$$

$$\widetilde{D}_n^{q_n}((u-x)^4;x) \leq \frac{M_3}{[n]_{q_n}^2}, \quad \widetilde{D}_m^{q_m}((v-y)^4;y) \leq \frac{M_4}{[m]_{q_m}^2}, \tag{6.2.10}$$

where $M_i > 0$, for $i = 1,2,3,4$

Let $\delta_n = \frac{1}{[n]_{q_n}^{1/2}}$, and $\delta_m = \frac{1}{[m]_{q_m}^{1/2}}$.

Then, combining (6.2.7)-(6.2.10) lead us to

$$\begin{aligned} |\widetilde{T}_{n,m}^{q_n,q_m}(f;x,y) - f(x,y)| &= 3||D_B||_\infty O\left(\frac{1}{[n]_{q_n}^{1/2}}\right) O\left(\frac{1}{[m]_{q_m}^{1/2}}\right) \\ &+ O\left(\frac{1}{[n]_{q_n}^{1/2}}\right) O\left(\frac{1}{[m]_{q_m}^{1/2}}\right) \omega_{mixed}(D_B f; [n]_{q_n}^{-1/2}, [m]_{q_m}^{-1/2}), \end{aligned}$$

from which the required result is immediate.

□

Now, we examine the rate of convergence of the operators $\widetilde{D}_{n,m}^{q_n,q_m}$ and $\widetilde{T}_{n,m}^{q_n,q_m}$ by some illustrations and numerical examples.

Example 1. For $n = m = 5$ (green) and $n = m = 10$ (orange) with $q_n = (n-1)/n$ and $q_m = m/(m+1)$, the convergence of the operator $\widetilde{D}_{n,m}^{q_n,q_m}$ given by (6.1.1) to $f(x,y) = \cos(\pi x^2)\, y^3$ (yellow) is shown in Figure 1. In Figure 2, for $n = m = 5$ and the same q_n, q_m, we compare the rate of convergence of the operator $\widetilde{D}_{n,m}^{q_n,q_m}$(green) and its GBS operator $\widetilde{T}_{n,m}^{q_n,q_m}$ (grey) to the same function f (yellow). It is observed that the rate of convergence of $\widetilde{T}_{n,m}^{q_n,q_m}$ appears to be as good as $\widetilde{D}_{n,m}^{q_n,q_m}$.

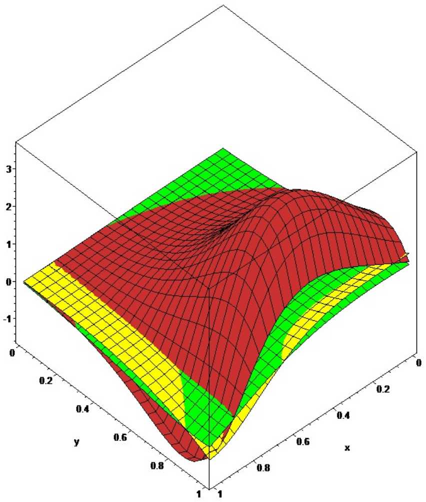


Figure 1 : For $n = m = 5$ (green), $n = m = 10$ (orange) the convergence of $\widetilde{D}_{n,m}^{q_n,q_m}$ to function f

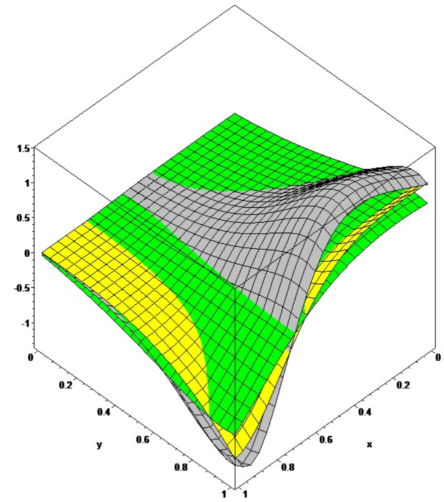


Figure 2 : For $n = m = 5,$ the convergence of $\widetilde{D}_{n,m}^{q_n,q_m}$ (green) and its GBS operator $\widetilde{T}_{n,m}^{q_n,q_m}$ (grey)

Example 2. Let $q_n = (n-1)/n$ and $q_m = m/(m+1)$. For $n = m = 5$ (grey) and $n = m = 15$ (pink), the convergence of $\widetilde{T}_{n,m}^{q_n,q_m}(f;x,y)$ to $f(x,y) = \cos(\pi x^2)/(1+y)$ (yellow) is illustrated in Figures 3 and 4, respectively. It is clear that the degree of approximation becomes better on increasing the values of n, m.

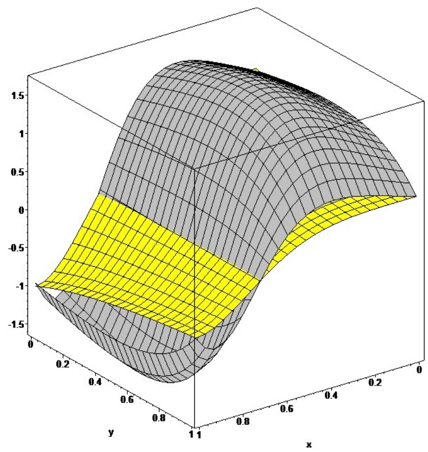


Figure 3 : For $n = m = 5,$ the convergence of $\widetilde{T}_{n,m}^{q_n,q_m}(f;x,y)$ (grey) to $f(x,y)$ (yellow)

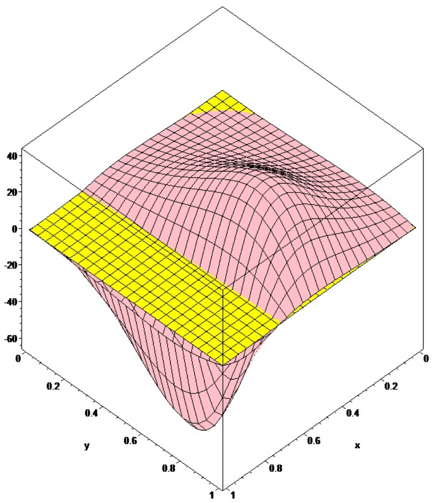


Figure 4 : For $n = m = 15,$ the convergence of $\widetilde{T}_{n,m}^{q_n,q_m}(f;x,y)$ (pink) to $f(x,y)$ (yellow)

Example 3. In Figure 5, for $n = m = 5$ and $q_n = .10, q_m = .20$ and $n = m = 15$ and $q_n = .90, q_m = .95$, the convergence of $\widetilde{T}_{n,m}^{q_n,q_m}(f;x,y)$ (respectively, green and grey) to $f(x,y) = xy^2 + x^2y$ (yellow) is shown.
It is observed that, on increasing the values of n, m and the corresponding q_n, q_m, the convergence of $\widetilde{T}_{n,m}^{q_n,q_m}(f;x,y)$ to $f(x,y)$ becomes better.

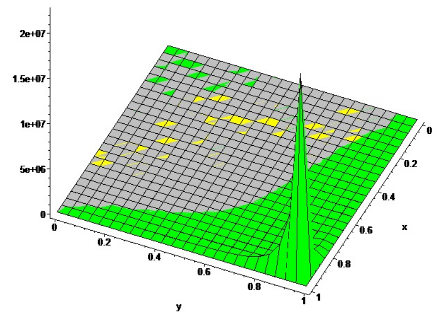


Figure 5 : For $n = m = 5, q_n = .10, q_m = .20$ (green)
and $n = m = 15, q_n = .90, q_m = .95$ (grey),
convergence of $\widetilde{T}_{n,m}^{q_n,q_m}$ to $f(x,y)$

Example 4. For $n = 5000$ and the different values of q_n, q_m, the error of the approximation of $\widetilde{T}_{n,m}^{q_n,q_m}(f;x,y)$ to $f(x,y) = xy^2 + x^2y$, by using the mixed modulus of continuity of f is listed in the following table:

q_n	q_m	error bound
.75	.85	3.785685010
.80	.85	3.349448117
.885	.889	2.111541166
.890	.975	0.941783434
.905	.989	0.570639803
.975	.998	0.118876832
.987	.9989	0.062860082
.995	.9987	0.041912622
.995	.9989	0.038577083
.9975	.9989	0.027140508
.9995	.9997	0.007416630
1	1	0.003226274

Table: the degree of
approximation of $\widetilde{T}_{n,m}^{q_n,q_m}(f;x,y)$ to $f(x,y)$ in terms of ω_{mixed}

Chapter 7

Generalized boolean sum operators of q-Bernstein-Schurer-Kantorovich type

7.1 Introduction

In [102], Muraru constructed the q-Bernstein-Schurer operators defined by

$$\overline{B}_{n,p}(f;x)=\sum_{k=0}^{n+p}\overline{b}_{n+p,k}(q;x)f\left(\frac{[k]_q}{[n]_q}\right),\quad x\in J, \tag{7.1.1}$$

where $\overline{b}_{n+p,k}(q;x)=\begin{pmatrix}n+p\\k\end{pmatrix}_q x^k(1-x)_q^{n+p-k}$ and $J=[0,1]$ and established a Korovkin type approximation theorem and the rate of convergence in terms of the first order modulus of continuity.

Recently, Agrawal et al.[17] constructed a bivariate case of a new kind of Kantorovich type generalization of the operators $\bar{B}_{n,p}$ as follows:

Let $I=[0,1+p], I^2=I\times I$, and $C(I^2)$ be the space of all real valued continuous functions on I^2 endowed with the norm $\|f\|=\sup\limits_{(x,y)\in I^2}|f(x,y)|$.

For $f\in C(I^2)$ and $0<q_{n_1},q_{n_2}<1$, the bivariate generalization of Kantorovich type q-Bernstein-Schurer operators is defined by

$$K_{n_1,n_2,p}(f;q_{n_1},q_{n_2},x,y) = [n_1+1]_{q_{n_1}}[n_2+1]_{q_{n_2}}\sum_{k_1=0}^{n_1+p}\sum_{k_2=0}^{n_2+p} b^{q_{n_1},q_{n_2}}_{n_1+p,n_2+p,k_1,k_2}(x,y)q_{n_1}^{-k_1}q_{n_2}^{-k_2}$$
$$\times\int_{[k_2]_{q_{n_2}}/[n_2+1]_{q_{n_2}}}^{[k_2+1]_{q_{n_2}}/[n_2+1]_{q_{n_2}}}\int_{[k_1]_{q_{n_1}}/[n_1+1]_{q_{n_1}}}^{[k_1+1]_{q_{n_1}}/[n_1+1]_{q_{n_1}}} f(u,v)d^R_{q_{n_1}}(u)d^R_{q_{n_2}}(v), \tag{7.1.2}$$

where $b^{q_{n_1},q_{n_2}}_{n_1+p,n_2+p,k_1,k_2}(x,y)=\binom{n_1+p}{k_1}_{q_{n_1}}\binom{n_2+p}{k_2}_{q_{n_2}} x^{k_1}y^{k_2}(1-x)^{n_1+p-k_1}_{q_{n_1}}(1-y)^{n_2+p-k_2}_{q_{n_2}}$

and $x,y\in J^2,\ J$ being $[0,1]$.
and studied some of their approximation properties. In the present chapter we continue the work done by Agrawal et al.[17] by discussing the rate of convergence in terms of the partial moduli of continuity and the Peetre's K-functional. In the last section of the paper, we construct the GBS (Generalized Boolean Sum) operators of the q-Bernstein-Schurer-Kantorovich type defined by (7.1.2) and obtain the order of approximation in terms of the mixed modulus of smoothness.

7.2 Preliminaries

Now, we present a lemma which will be used in the sequel. In what follows, for $i=1,2$, let (q_{n_i}) be a sequence in $(0,1)$ satisfying $q_{n_i}\to 1$ and $q_{n_i}^{n_i}\to a_i,(0\le a_i<1)$ as $n_i\to\infty$.

Lemma 7.2.1. *[17] Let $e_{ij}=x^iy^j$, $(i,j)\in\mathbb{N}^0\times\mathbb{N}^0$ with $i+j\le 2$ be the two dimensional test functions. Then the following equalities hold for the operators given by (7.1.2):*

(i) $K_{n_1,n_2,p}(e_{00};q_{n_1},q_{n_2},x,y)=1;$

(ii) $K_{n_1,n_2,p}(e_{10};q_{n_1},q_{n_2},x,y)=\frac{[n_1+p]_{q_{n_1}}}{[n_1+1]_{q_{n_1}}}\frac{2q_{n_1}}{[2]_{q_{n_1}}}x+\frac{1}{[2]_{q_{n_1}}[n_1+1]_{q_{n_1}}};$

(iii) $K_{n_1,n_2,p}(e_{01};q_{n_1},q_{n_2},x,y)=\frac{[n_2+p]_{q_{n_2}}}{[n_2+1]_{q_{n_2}}}\frac{2q_{n_2}}{[2]_{q_{n_2}}}y+\frac{1}{[2]_{q_{n_2}}[n_2+1]_{q_{n_2}}};$

(iv) $K_{n_1,n_2,p}(e_{20};q_{n_1},q_{n_2},x,y)=\frac{1}{[n_1+1]^2_{q_{n_1}}[3]_{q_{n_1}}}+\frac{q_{n_1}(3+5q_{n_1}+4q_{n_1}^2)}{[2]_{q_{n_1}}[3]_{q_{n_1}}}\frac{[n_1+p]_{q_{n_1}}}{[n_1+1]_{q^2_{n_1}}}x$
$+\frac{q_{n_1}^2(1+q_{n_1}+4q_{n_1}^2)}{[2]_{q_{n_1}}[3]_{q_{n_1}}}\frac{[n_1+p]_{q_{n_1}}[n_1+p-1]_{q_{n_1}}}{[n_1+1]_{q^2_{n_1}}}x^2;$

(v) $K_{n_1,n_2,p}(e_{02};q_{n_1},q_{n_2},x,y) = \frac{1}{[n_2+1]_{q_{n_2}}^2[3]_{q_{n_2}}} + \frac{q_{n_2}(3+5q_{n_2}+4q_{n_2}^2)}{[2]_{q_{n_2}}[3]_{q_{n_2}}}\frac{[n_2+p]_{q_{n_2}}}{[n_2+1]_{q_{n_2}}^2}y$
$+ \frac{q_{n_2}^2(1+q_{n_2}+4q_{n_2}^2)}{[2]_{q_{n_2}}[3]_{q_{n_2}}}\frac{[n_2+p]_{q_{n_2}}[n_2+p-1]_{q_{n_2}}}{[n_2+1]_{q_{n_2}^2}}y^2.$

7.3 Main results

In what follows, let

$$\delta_{n_1} = \delta_{n_1}(x) = \left(K_{n_1,n_2,p}((u-x)^2;q_{n_1},q_{n_2},x,y)\right)^{1/2},$$

and

$$\delta_{n_2} = \delta_{n_2}(y) = \left(K_{n_1,n_2,p}((v-y)^2;q_{n_1},q_{n_2},x,y)\right)^{1/2}.$$

First, we obtain an estimate of the rate of convergence of the bivariate operators in terms of the partial moduli of continuity.

Theorem 7.3.1. *For $f \in C(I^2)$, there holds*

$$|K_{n_1,n_2,p}(f;q_{n_1},q_{n_2},x,y) - f(x,y)| \leq 2(\omega_1(f;\delta_{n_1}) + \omega_2(f;\delta_{n_2})),$$

Proof. By the definition of partial moduli of continuity, Lemma 7.2.1 and using Cauchy-Schwarz inequality we may write

$$\begin{aligned}
&|K_{n_1,n_2,p}(f;q_{n_1},q_{n_2},x,y) - f(x,y)| \\
&\leq K_{n_1,n_2,p}(|f(u,v)-f(x,y)|;q_{n_1}q_{,n_2},x,y) \\
&\leq K_{n_1,n_2,p}(|f(u,v)-f(v,y)|;q_{n_1},q_{n_2},x,y) + K_{n_1,n_2,p}(|f(u,y)-f(x,y)|;q_{n_1},q_{n_2},x,y) \\
&\leq K_{n_1,n_2,p}(\omega_2(f;|v-y|);q_{n_1},q_{n_2},x,y) + K_{n_1,n_2,p}(\omega_1(f;|u-x|);q_{n_1},q_{n_2},x,y) \\
&\leq \omega_2(f;\delta_{n_2})\left[1+\frac{1}{\delta_{n_2}}K_{n_1,n_2,p}(|v-y|;q_{n_1},q_{n_2},x,y)\right] \\
&\quad +\omega_1(f;\delta_{n_1})\left[1+\frac{1}{\delta_{n_1}}K_{n_1,n_2,p}(|u-x|;q_{n_1},q_{n_2},x,y)\right] \\
&\leq \omega_2(f;\delta_{n_2})\left[1+\frac{1}{\delta_{n_2}}\left(K_{n_1,n_2,p}((v-y)^2;q_{n_1},q_{n_2},x,y)\right)^{1/2}\right] \\
&\quad +\omega_1(f;\delta_{n_1})\left[1+\frac{1}{\delta_{n_1}}\left(K_{n_1,n_2,p}((u-x)^2;q_{n_1},q_{n_2},x,y)\right)^{1/2}\right].
\end{aligned}$$

In [17], it is shown that $\delta_{n_1}, \delta_{n_2} \to 0$ as $n_1, n_2 \to \infty$.
Hence, we reach the desired result. □

Theorem 7.3.2. *If $f \in C(I^2)$, Then we obtain*

$$|K_{n_1,n_2,p}(f;q_{n_1},q_{n_2},x,y)-f(x,y)|$$

$$\leq M\bigg\{\bar{\omega}_2(f;\sqrt{A_{n_1,n_2,p}(q_{n_1},q_{n_2},x,y)})+min\{1,A_{n_1,n_2,p}(q_{n_1},q_{n_2},x,y)\}||f||_{C(I^2)}\bigg\}$$
$$+\omega\bigg(f;\sqrt{\bigg(\frac{[n_1+p]_{q_{n_1}}2q_{n_1}x+1}{[n_1+1]_{q_{n_1}}[2]_{q_{n_1}}}-x\bigg)^2+\bigg(\frac{[n_2+p]_{q_{n_2}}2q_{n_2}y+1}{[n_2+1]_{q_{n_2}}[2]_{q_{n_2}}}-y\bigg)^2}\bigg),$$

where

$$\begin{aligned}A_{n_1,n_2,p}(q_{n_1},q_{n_2},x,y) &= \bigg\{\delta_{n_1}^2+\delta_{n_2}^2+\bigg(\frac{[n_1+p]_{q_{n_1}}2q_{n_1}x+1}{[n_1+1]_{q_{n_1}}[2]_{q_{n_1}}}-x\bigg)^2\\ &\quad+\bigg(\frac{[n_2+p]_{q_{n_2}}2q_{n_2}y+1}{[n_2+1]_{q_{n_2}}[2]_{q_{n_2}}}-y\bigg)^2\bigg\}\end{aligned}$$

and the constant $M>0$, is independent of f and $A_{n_1,n_2,p}(q_{n_1},q_{n_2},x,y)$.

Proof. Let us define the auxiliary operators

$$K^*_{n_1,n_2,p}(f;q_{n_1},q_{n_2},x,y)$$

$$=K_{n_1,n_2,p}(f;q_{n_1},q_{n_2},x,y)-f\bigg(\frac{[n_1+p]_{q_{n_1}}2q_{n_1}x+1}{[n_1+1]_{q_{n_1}}[2]_{q_{n_1}}},\frac{[n_2+p]_{q_{n_2}}2q_{n_2}y+1}{[n_2+1]_{q_{n_2}}[2]_{q_{n_2}}}\bigg)$$

$$+f(x,y),$$

then using Lemma 7.2.1, we have

$$K^*_{n_1,n_2,p}((u-x);q_{n_1},q_{n_2},x,y)=0 \text{ and } K^*_{n_1,n_2,p}((v-y);q_{n_1},q_{n_2},x,y)=0.$$

Let $g\in C^2(I^2)$ and $u,v\in I$. Using the Taylor's theorem, we may write

$$\begin{aligned}g(u,v)-g(x,y) &= g(u,y)-g(x,y)+g(u,v)-g(u,y)\\ &= \frac{\partial g(x,y)}{\partial x}(u-x)+\int_x^u(u-t)\frac{\partial^2 g(t,y)}{\partial t^2}dt\\ &\quad+\frac{\partial g(x,y)}{\partial y}(v-y)+\int_y^v(v-s)\frac{\partial^2 g(x,s)}{\partial s^2}ds.\end{aligned}$$

Applying the operator $K^*_{n_1,n_2,p}(.,q_{n_1},q_{n_2},x,y)$ on both sides, we get

$$
\begin{aligned}
K^*_{n_1,n_2,p}(g;q_{n_1},q_{n_2},x,y)-g(x,y) &= K^*_{n_1,n_2,p}\bigg(\int_x^u (u-t)\frac{\partial^2 g(t,y)}{\partial t^2}dt;q_{n_1},q_{n_2},x,y\bigg)\\
&\quad +K^*_{n_1,n_2,p}\bigg(\int_y^v (v-s)\frac{\partial^2 g(x,s)}{\partial s^2}ds;q_{n_1},q_{n_2},x,y\bigg)\\
&= K_{n_1,n_2,p}\bigg(\int_x^u (u-t)\frac{\partial^2 g(t,y)}{\partial t^2}dt;q_{n_1},q_{n_2},x,y\bigg)\\
&\quad -\int_x^{\frac{[n_1+p]_{q_{n_1}}2q_{n_1}x+1}{[n_1+1]_{q_{n_1}}[2]_{q_{n_1}}}}\bigg(\frac{[n_1+p]_{q_{n_1}}2q_{n_1}x+1}{[n_1+1]_{q_{n_1}}[2]_{q_{n_1}}}-t\bigg)\frac{\partial^2 g(t,y)}{\partial t^2}dt\\
&\quad +K_{n_1,n_2,p}\bigg(\int_y^v (v-s)\frac{\partial^2 g(x,s)}{\partial s^2}ds;q_{n_1},q_{n_2},x,y\bigg)\\
&\quad -\int_y^{\frac{[n_2+p]_{q_{n_2}}2q_{n_2}y+1}{[n_2+1]_{q_{n_2}}[2]_{q_{n_2}}}}\bigg(\frac{[n_2+p]_{q_{n_2}}2q_{n_2}x+1}{[n_2+1]_{q_{n_2}}[2]_{q_{n_2}}}-s\bigg)\frac{\partial^2 g(x,s)}{\partial s^2}ds.
\end{aligned}
$$

Hence
$|K^*_{n_1,n_2,p}(g;q_{n_1},q_{n_2},x,y)-g(x,y)|$

$$
\begin{aligned}
&\leq K_{n_1,n_2,p}\bigg(\bigg|\int_x^u |u-t|\bigg|\frac{\partial^2 g(t,y)}{\partial t^2}\bigg|dt\bigg|;x,y\bigg)\\
&\quad +\bigg|\int_x^{\frac{[n_1+p]_{q_{n_1}}2q_{n_1}x+1}{[n_1+1]_{q_{n_1}}[2]_{q_{n_1}}}}\bigg|\frac{[n_1+p]_{q_{n_1}}2q_{n_1}x+1}{[n_1+1]_{q_{n_1}}[2]_{q_{n_1}}}-t\bigg|\bigg|\frac{\partial^2 g(t,y)}{\partial t^2}\bigg|dt\bigg|\\
&\quad +K_{n_1,n_2,p}\bigg(\bigg|\int_y^v |v-s|\bigg|\frac{\partial^2 g(x,s)}{\partial s^2}\bigg|ds\bigg|;x,y\bigg)\\
&\quad +\bigg|\int_y^{\frac{[n_2+p]_{q_{n_2}}2q_{n_2}y+1}{[n_2+1]_{q_{n_2}}[2]_{q_{n_2}}}}\bigg|\frac{[n_2+p]_{q_{n_2}}2q_{n_2}y+1}{[n_2+1]_{q_{n_2}}[2]_{q_{n_2}}}-s\bigg|\bigg|\frac{\partial^2 g(x,s)}{\partial s^2}\bigg|ds\bigg|\\
&\leq \bigg\{K_{n_1,n_2,p}((u-x)^2;q_{n_1},q_{n_2},x,y)+\bigg(\frac{[n_1+p]_{q_{n_1}}2q_{n_1}x+1}{[n_1+1]_{q_{n_1}}[2]_{q_{n_1}}}-x\bigg)^2\bigg\}||g||_{C^2(I^2)}\\
&\quad +\bigg\{K_{n_1,n_2,p}((v-y)^2;q_{n_1},q_{n_2},x,y)+\bigg(\frac{[n_2+p]_{q_{n_2}}2q_{n_2}y+1}{[n_2+1]_{q_{n_2}}[2]_{q_{n_2}}}-y\bigg)^2\bigg\}||g||_{C^2(I^2)}
\end{aligned}
$$

$$\begin{aligned}
&\leq \left\{\delta_{n_1}^2+\left(\frac{[n_1+p]_{q_{n_1}}2q_{n_1}x+1}{[n_1+1]_{q_{n_1}}[2]_{q_{n_1}}}-x\right)^2\right\}||g||_{C^2(I^2)}+\left\{\delta_{n_2}^2\right.\\
&\left.+\left(\frac{[n_2+p]_{q_{n_2}}2q_{n_2}y+1}{[n_2+1]_{q_{n_2}}[2]_{q_{n_2}}}-y\right)^2\right\}||g||_{C^2(I^2)}\\
&= \left\{\delta_{n_1}^2+\delta_{n_2}^2+\left(\frac{[n_1+p]_{q_{n_1}}2q_{n_1}x+1}{[n_1+1]_{q_{n_1}}[2]_{q_{n_1}}}-x\right)^2\right.\\
&\left.+\left(\frac{[n_2+p]_{q_{n_2}}2q_{n_2}y+1}{[n_2+1]_{q_{n_2}}[2]_{q_{n_1}}}-y\right)^2\right\}||g||_{C^2(I^2)}\\
&= A_{n_1,n_2,p}(q_{n_1},q_{n_2},x,y)||g||_{C^2(I^2)}.
\end{aligned} \tag{7.3.1}$$

Also, using Lemma 7.2.1

$$\begin{aligned}
&|K^*_{n_1,n_2,p}(f;q_{n_1},q_{n_2},x,y)|\\
&\leq |K_{n_1,n_2,p}(f;q_{n_1},q_{n_2},x,y)|+\left|f\left(\frac{[n_1+p]_{q_{n_1}}2q_{n_1}x+1}{[n_1+1]_{q_{n_1}}[2]_{q_{n_1}}},\frac{[n_2+p]_{q_{n_2}}2q_{n_2}y+1}{[n_2+1]_{q_{n_2}}[2]_{q_{n_1}}}\right)\right|\\
&+|f(x,y)|\leq 3||f||_{C(I^2)}.
\end{aligned} \tag{7.3.2}$$

Hence in view of (7.3.1), (7.3.2) and using the relation (0.4.2), we get

$$\begin{aligned}
&|K_{n_1,n_2,p}(f;q_{n_1},q_{n_2},x,y)-f(x,y)|\\
&= \left|K^*_{n_1,n_2,p}(f;q_{n_1},q_{n_2},x,y)-f(x,y)+f\left(\frac{[n_1+p]_{q_{n_1}}2q_{n_1}x+1}{[n_1+1]_{q_{n_1}}[2]_{q_{n_1}}},\frac{[n_2+p]_{q_{n_2}}2q_{n_2}y+1}{[n_2+1]_{q_{n_2}}[2]_{q_{n_1}}}\right)\right.\\
&\left.-f(x,y)\right|\\
&\leq |K^*_{n_1,n_2,p}(f-g;q_{n_1},q_{n_2},x,y)|+|K^*_{n_1,n_2,p}(g;q_{n_1},q_{n_2},x,y)-g(x,y)|+|g(x,y)-f(x,y)|\\
&+\left|f\left(\frac{[n_1+p]_{q_{n_1}}2q_{n_1}x+1}{[n_1+1]_{q_{n_1}}[2]_{q_{n_1}}},\frac{[n_2+p]_{q_{n_2}}2q_{n_2}y+1}{[n_2+1]_{q_{n_2}}[2]_{q_{n_2}}}\right)-f(x,y)\right|\\
&\leq 4||f-g||_{C(I^2)}+|K^*_{n_1,n_2,p}(g;q_{n_1},q_{n_2},x,y)-g(x,y)|\\
&+\left|f\left(\frac{[n_1+p]_{q_{n_1}}2q_{n_1}x+1}{[n_1+1]_{q_{n_1}}[2]_{q_{n_1}}},\frac{[n_2+p]_{q_{n_2}}2q_{n_2}y+1}{[n_2+1]_{q_{n_2}}[2]_{q_{n_2}}}\right)-f(x,y)\right|\\
&\leq \left(4||f-g||_{C(I^2)}+A_{n_1,n_2,p}(q_{n_1},q_{n_2},x,y)||g||_{C^2(I^2)}\right)\\
&+\omega\left(f;\sqrt{\left(\frac{[n_1+p]_{q_{n_1}}2q_{n_1}x+1}{[n_1+1]_{q_{n_1}}[2]_{q_{n_1}}}-x\right)^2+\left(\frac{[n_2+p]_{q_{n_2}}2q_{n_2}y+1}{[n_2+1]_{q_{n_2}}[2]_{q_{n_2}}}-y\right)^2}\right)
\end{aligned}$$

$$\leq \quad 4\mathcal{K}(f; A_{n_1,n_2,p}(q_{n_1}, q_{n_2}, x, y)) + \omega\Bigg(f; \Bigg\{\Bigg(\frac{[n_1+p]_{q_{n_1}} 2q_{n_1}x + 1}{[n_1+1]_{q_{n_1}}[2]_{q_{n_1}}} - x\Bigg)^2 + \Bigg(\frac{[n_2+p]_{q_{n_2}} 2q_{n_2}y + 1}{[n_2+1]_{q_{n_2}}[2]_{q_{n_2}}} - y\Bigg)^2\Bigg\}^{1/2}\Bigg)$$

$$\leq \quad M\Bigg\{\bar{\omega}_2\Big(f; \sqrt{A_{n_1,n_2,p}(q_{n_1}, q_{n_2}, x, y)}\Big) + min\{1, A_{n_1,n_2,p}(q_{n_1}, q_{n_2}, x, y)\}||f||_{C(I^2)}\Bigg\}$$
$$+\omega\Bigg(f; \sqrt{\Bigg(\frac{[n_1+p]_{q_{n_1}} 2q_{n_1}x + 1}{[n_1+1]_{q_{n_1}}[2]_{q_{n_1}}} - x\Bigg)^2 + \Bigg(\frac{[n_2+p]_{q_{n_2}} 2q_{n_2}y + 1}{[n_2+1]_{q_{n_2}}[2]_{q_{n_2}}} - y\Bigg)^2}\Bigg).$$

Thus, we get the desired result.

□

7.4 GBS operator of q-Bernstein-Schurer-Kantorovich type

We define the GBS operator of the operator $K_{n_1,n_2,p}$ given by (7.1.2), for any $f \in C_b(I^2)$ and $m, n \in \mathbb{N}$, by

$$T_{n_1,n_2,p}(f(u,v); q_{n_1}, q_{n_2}, x, y) := K_{n_1,n_2,p}(f(u,y) + f(x,v) - f(u,v); q_{n_1}, q_{n_2}, x, y),$$

for all $(x, y) \in J^2$.

More precisely for any $f \in C_b(I^2)$, the GBS operator of q-Bernstein-Schurer-Kantorovich type is given by

$$T_{n_1,n_2,p}(f; q_{n_1}, q_{n_2}, x, y)$$
$$= [n_1+1]_{q_{n_1}}[n_2+1]_{q_{n_2}} \sum_{k_1=0}^{n_1+p}\sum_{k_2=0}^{n_2+p} b^{q_{n_1},q_{n_2}}_{n_1+p,n_2+p,k_1,k_2}(x,y) q_{n_1}^{-k_1} q_{n_2}^{-k_2}$$
$$\times \int_{[k_2]_{q_{n_2}}/[n_2+1]_{q_{n_2}}}^{[k_2+1]_{q_{n_2}}/[n_2+1]_{q_{n_2}}} \int_{[k_1]_{q_{n_1}}/[n_1+1]_{q_{n_1}}}^{[k_1+1]_{q_{n_1}}/[n_1+1]_{q_{n_1}}} [f(x,v) + f(u,y) - f(u,v)] d^R_{q_{n_1}}(u) d^R_{q_{n_2}}(v), \tag{7.4.1}$$

where $b^{q_{n_1},q_{n_2}}_{n_1+p,n_2+p,k_1,k_2}(x,y) = \begin{pmatrix} n_1+p \\ k_1 \end{pmatrix}_{q_{n_1}} \begin{pmatrix} n_2+p \\ k_2 \end{pmatrix}_{q_{n_2}} x^{k_1} y^{k_2} (1-x)^{n_1+p-k_1}_{q_{n_1}} (1-y)^{n_2+p-k_2}_{q_{n_2}}$.

Here the operator $T_{n_1,n_2,p}$ is a linear positive operator and is well defined from the space $C_b(I^2)$ into $C(J^2)$.

Theorem 7.4.1. *For every $f \in C_b(I^2)$, at each point $(x,y) \in J^2$, the operator (7.4.1) verifies the following inequality*

$$|T_{n_1,n_2,p}(f;q_{n_1},q_{n_2},x,y) - f(x,y)| \leq 4\,\omega_{mixed}(f;\delta_{n_1},\delta_{n_2}).$$

Proof. We may write

$$\begin{aligned} T_{n_1,n_2,p}(f;q_{n_1},q_{n_2},x,y) &= f(x,y)\,K_{n_1,n_2,p}(e_{00};q_{n_1},q_{n_2},x,y) \\ &\quad - K_{n_1,n_2,p}\left(\Delta_{(x,y)} f[u,v;x,y];q_{n_1},q_{n_2},x,y\right). \end{aligned}$$

Since $K_{n_1,n_2,p}(e_{00};q_{n_1},q_{n_2},x,y) = 1$, considering the inequality (6.2.4) and applying the Cauchy-Schwarz inequality we obtain,

$|T_{n_1,n_2,p}(f;q_{n_1},q_{n_2},x,y) - f(x,y)|$

$$\begin{aligned} &\leq K_{n_1,n_2,p}\left(|\Delta_{(x,y)} f[u,v;x,y]|;q_{n_1},q_{n_2},x,y\right) \\ &\leq \Bigg(K_{n_1,n_2,p}(e_{00};q_{n_1},q_{n_2},x,y) \\ &\quad +\delta_{n_1}^{-1}\sqrt{K_{n_1,n_2,p}((u-x)^2;q_{n_1},q_{n_2},x,y)} \\ &\quad +\delta_{n_2}^{-1}\sqrt{K_{n_1,n_2,p}((v-y)^2;q_{n_1},q_{n_2},x,y)} \\ &\quad +\delta_{n_1}^{-1}\delta_{n_2}^{-1}\sqrt{K_{n_1,n_2,p}((u-x)^2;q_{n_1},q_{n_2},x,y)} \\ &\quad \times\sqrt{K_{n_1,n_2,p}((v-y)^2;q_{n_1},q_{n_2},x,y)}\Bigg)\,\omega_{mixed}(f;\delta_{n_1},\delta_{n_2})) \\ &\leq 4\,\omega_{mixed}(f;\delta_{n_1},\delta_{n_2}), \end{aligned}$$

from which the desired result is immediate.

□

Next theorem gives the degree of approximation for the operators $T_{n_1,n_2,p}$ by means of the Lipschitz class of functions in $C_b(I^2)$.

Theorem 7.4.2. *Let $f \in Lip_M\,(\alpha,\beta)$ then we have*

$$|T_{n_1,n_2,p}\,(f;q_{n_1},q_{n_2},x,y) - f\,(x,y)| \leq M\delta_{n_1}^{\alpha/2}\delta_{n_2}^{\beta/2}$$

for $M > 0$, $\alpha,\beta \in (0,1]$.

Proof. By the same reasoning as in Theorem 6.2.3, we have

$$\begin{aligned}\left|T_{n_1,n_2,p}\left(f;q_{n_1},q_{n_2},x,y\right)-f\left(x,y\right)\right| &\leq K_{n_1,n_2,p}\left(\left|\Delta_{(x,y)}f\left[u,v;x,y\right]\right|;x,y\right)\\ &\leq MK_{n_1,n_2,p}\left(\left|u-x\right|^{\alpha}\left|v-y\right|^{\beta};x,y\right)\\ &= MK_{n_1,n_2,p}\left(\left|u-x\right|^{\alpha};x\right)K_{n_1,n_2,p}\left(\left|v-y\right|^{\beta};y\right).\end{aligned}$$

Now, using the H$\ddot{o}$lder's inequality with $p_1=2/\alpha, q_1=2/\left(2-\alpha\right)$ and $p_2=2/\beta, q_2=2/\left(2-\beta\right)$, we have

$$\begin{aligned}\left|T_{n_1,n_2,p}\left(f;q_{n_1},q_{n_2},x,y\right)-f\left(x,y\right)\right| &\leq M\left(K_{n_1,n_2,p}(u-x)^2;x\right)^{\alpha/2}K_{n_1,n_2,p}\left(e_0;x\right)^{(2-\alpha)/2}\\ &\quad\times K_{n_1,n_2,p}\left((v-y)^2;y\right)^{\beta/2}K_{n_1,n_2,p}\left(e_0;y\right)^{(2-\beta)/2}.\end{aligned}$$

Considering Lemma 7.2.1, we obtain the degree of local approximation for B-continuous functions belonging to $Lip_M\left(\alpha,\beta\right)$. □

Theorem 7.4.3. *Let the function $f\in D_b(I^2)$ with $D_Bf\in B(I^2)$. Then, if $(x,y)\in J^2$, there holds:*

$$\begin{aligned}|T_{n_1,n_2,p}(f;q_{n_1},q_{n_2},x,y)-f(x,y)| &\leq \frac{M}{[n_1]_{q_{n_1}}^{1/2}[n_2]_{q_{n_2}}^{1/2}}\Big(||D_Bf||_\infty\\ &\quad+\omega_{mixed}(D_Bf;[n_1]_{q_{n_1}}^{-1/2},[n_2]_{q_{n_2}}^{-1/2})\Big).\end{aligned}$$

Proof. Since $f\in D_b(I^2)$, we have the identity

$$\Delta_{x,y}f[u,v;x,y]=(u-x)(v-y)D_Bf(\xi,\eta),\ \ with\ x<\xi<u;\ \ y<\eta<v.$$

It is clear that

$$D_Bf(\xi,\eta)=\Delta_{(x,y)}D_Bf(\xi,\eta)+D_Bf(\xi,y)+D_Bf(x,\eta)-D_Bf(x,y).$$

Since $D_Bf\in B(I^2)$, by above relations, we can write

$$\begin{aligned}&|K_{n_1,n_2,p}(\Delta_{(x,y)}f[u,v;x,y];q_{n_1},q_{n_2},x,y)|\\ &\quad= |K_{n_1,n_2,p}((u-x)(v-y)D_Bf(\xi,\eta);q_{n_1},q_{n_2},x,y)|\\ &\quad\leq K_{n_1,n_2,p}(|u-x||v-y||\Delta_{(x,y)}D_Bf(\xi,\eta)|;x,y)\\ &\qquad+K_{n_1,n_2,p}(|u-x||v-y|(|D_Bf(\xi,y)|\\ &\qquad+|D_Bf(x,\eta)|+|D_Bf(x,y)|);q_{n_1},q_{n_2},x,y)\\ &\quad\leq K_{n_1,n_2,p}(|u-x||v-y|\omega_{mixed}(D_Bf;|\xi-x|,|\eta-y|);x,y)\\ &\qquad+3\ ||D_Bf||_\infty\ K_{n_1,n_2,p}(|u-x||v-y|;q_{n_1},q_{n_2},x,y).\end{aligned}\tag{7.4.2}$$

Since the mixed modulus of smoothness ω_{mixed} is non-decreasing, we have

$$\begin{aligned}\omega_{mixed}(D_B f; |\xi - x|, |\eta - y|) &\leq \omega_{mixed}(D_B f; |u - x|, |v - y|) \\ &\leq (1 + \delta_{n_1}^{-1}|u - x|)(1 + \delta_{n_2}^{-1}|v - y|)\ \omega_{mixed}(D_B f; \delta_{n_1}, \delta_{n_2}).\end{aligned}$$

It is known from ([17], Theorem. 4.5) that

$$K_{n_1,p}((u-x)^2; q_{n_1}, x) \leq \frac{M_1}{[n_1]_{q_{n_1}}}, \quad K_{n_2,p}((v-y)^2; q_{n_2}, y) \leq \frac{M_2}{[n_2]_{q_{n_2}}}. \tag{7.4.3}$$

$$K_{n_1,p}((u-x)^4; q_{n_1}, x) \leq \frac{M_2}{[n_1]^2_{q_{n_1}}}, \quad K_{n_2,p}((v-y)^4; q_{n_2}, y) \leq \frac{M_4}{[n_2]^2_{q_{n_2}}} \tag{7.4.4}$$

for some constants $M_i > 0$, for $i = 1, 2, 3, 4$.

Hence applying the Cauchy-Schwarz inequality and using (7.4.3)-(7.4.4) in (7.4.2), on choosing $\delta_{n_1} = \dfrac{1}{[n_1]^{1/2}_{q_{n_1}}}$, and $\delta_{n_2} = \dfrac{1}{[n_2]^{1/2}_{q_{n_2}}}$,
we obtain the required result.

□

7.5 Applications

1. The Stancu variant of the operator $K_{n_1,n_2,p}$ defined by (7.1.2) is given by

$$K_{n_1,n_2,p}(f; q_{n_1}, q_{n_2}, x, y)_{\alpha,\sigma,\beta,\gamma}$$

$$\begin{aligned}&= [n_1+\beta+1]_{q_{n_1}}[n_2+\gamma+1]_{q_{n_2}} \sum_{k_1=0}^{n_1+p}\sum_{k_2=0}^{n_2+p} b^{q_{n_1},q_{n_2}}_{n_1+p,n_2+p,k_1,k_2}(x,y) q_{n_1}^{-k_1} q_{n_2}^{-k_2} \\ &\quad \times \int_{[k_2+\sigma]_{q_{n_2}}/[n_2+\gamma+1]_{q_{n_2}}}^{[k_2+\sigma+1]_{q_{n_2}}/[n_2+\gamma+1]_{q_{n_2}}} \int_{[k_1+\alpha]_{q_{n_1}}/[n_1+\beta+1]_{q_{n_1}}}^{[k_1+\alpha+1]_{q_{n_1}}/[n_1+\beta+1]_{q_{n_1}}} f(u,v) d^R_{q_{n_1}}(u) d^R_{q_{n_2}}(v),\end{aligned}$$

where $b^{q_{n_1},q_{n_2}}_{n_1+p,n_2+p,k_1,k_2}(x,y) = \binom{n_1+p}{k_1}_{q_{n_1}} \binom{n_2+p}{k_2}_{q_{n_2}} x^{k_1} y^{k_2} (1-x)^{n_1+p-k_1}_{q_{n_1}} (1-y)^{n_2+p-k_2}_{q_{n_2}}$, $0 \leq \alpha \leq \beta$ and $0 \leq \sigma \leq \gamma$.

For B-continuous functions, the GBS operator of Stancu type $K_{n_1,n_2,p}(f; q_{n_1}, q_{n_2}, x, y)_{\alpha,\sigma,\beta,\gamma}$ is defined by

$T_{n_1,n_2,p}(f(u,v);q_{n_1},q_{n_2},x,y)_{\alpha,\sigma,\beta,\gamma}$

$$= [n_1+\beta+1]_{q_{n_1}}[n_2+\gamma+1]_{q_{n_2}}\sum_{k_1=0}^{n_1+p}\sum_{k_2=0}^{n_2+p} b^{q_{n_1},q_{n_2}}_{n_1+p,n_2+p,k_1,k_2}(x,y)q_{n_1}^{-k_1}q_{n_2}^{-k_2}$$
$$\times\int_{\frac{[k_2+\sigma]_{q_{n_2}}}{[n_2+\gamma+1]_{q_{n_2}}}}^{\frac{[k_2+\sigma+1]_{q_{n_2}}}{[n_2+\gamma+1]_{q_{n_2}}}}\int_{\frac{[k_1+\alpha]_{q_{n_1}}}{[n_1+\beta+1]_{q_{n_1}}}}^{\frac{[k_1+\alpha+1]_{q_{n_1}}}{[n_1+\beta+1]_{q_{n_1}}}}(f(x,v)+f(u,y)-f(u,v))d^R_{q_{n_1}}(u)d^R_{q_{n_2}}(v),$$

which is the Stancu variant of the operator $T_{n_1,n_2,p}(f;q_{n_1},q_{n_2},x,y)$ given by (7.4.1). Theorems 7.4.1 - 7.4.3 can similarly be obtained for the operator $T_{n_1,n_2,p}(f;q_{n_1},q_{n_2},x,y)_{\alpha,\sigma,\beta,\gamma}$ with $f\in C_b(I^2)$ and all $(x,y)\in J^2$.

2. Agrawal et al. [22], introduced the Stancu type generalization of modified Schurer operators based on q-integers as

$$S^{(\alpha,\beta)}_{n,p}(f,q,x)=\frac{[n+p+1]_q}{(1+p)^{2n+2p+1}}\sum_{k=0}^{n+p} b^q_{n+p,k}(x)q^{-k}\int_0^{1+p} f\left(\frac{[n]_q u+\alpha}{[n]_q+\beta}\right)b^q_{n+p,k}(qu)d_qu, ,$$

$\forall\ x\in[0,1+p]$ where $f\in C(I)$ and $b^q_{n+p,k}=\binom{n+p}{k}_q x^k(1+p-x)_q^{n+p-k}$ and $0\leq\alpha\leq\beta$.

We define bivariate case of this operator
$S_{n_1,n_2,p}(f,q_{n_1},q_{n_2},x,y)_{(\alpha_1,\alpha_2,\beta_1,\beta_2)}$

$$= \frac{[n_1+p+1]_{q_{n_1}}}{(1+p)^{2n_1+2p+1}}\frac{[n_2+p+1]_{q_{n_2}}}{(1+p)^{2n_2+2p+1}}\sum_{k_1=0}^{n_1+p}\sum_{k_1=0}^{n_1+p} b^{q_{n_1},q_{n_2}}_{n_1+p,n_2+p,k_1,k_2}(x,y)q_{n_1}^{-k_1}q_{n_2}^{-k_2}$$
$$\times\int_0^{1+p}\int_0^{1+p} f\left(\frac{[n_1]_{q_{n_1}}u+\alpha_1}{[n_1]_{q_{n_1}}+\beta_1},\frac{[n_2]_{q_{n_2}}v+\alpha_2}{[n_2]_{q_{n_2}}+\beta_2}\right)b^q_{n_1+p,n_2+p,k_1,k_2}(q_{n_1}u,q_{n_2}v)d_{q_1}u\,d_{q_2}v,$$

where $b^q_{n_1+p,n_2+p,k_1,k_2}(x,y)=\binom{n_1+p}{k_1}_{q_{n_1}}\binom{n_2+p}{k_2}_{q_{n_2}}x^{k_1}(1+p-x)^{n_1+p-k_1}_{q_{n_1}}y^{k_2}(1+p-y)^{n_2+p-k_2}_{q_{n_2}}$, and $0\leq\alpha_1\leq\beta_1,\ 0\leq\alpha_2\leq\beta_2;\alpha_i,\beta_i\in\mathbb{R},\ i=1,2.$

The GBS operator of Stancu type $S_{n_1,n_2,p}(f,q_{n_1},q_{n_2},x,y)_{(\alpha_1,\alpha_2,\beta_1,\beta_2)}$ is defined by

$T_{n_1,n_2,p}(f(u,v),q_{n_1},q_{n_2},x,y)_{(\alpha_1,\alpha_2,\beta_1,\beta_2)}$

$$= S_{n_1,n_2,p}(f,q_{n_1},q_{n_2},x,y)_{(\alpha_1,\alpha_2,\beta_1,\beta_2)}(f(u,y)+f(x,v)-f(u,v),q_{n_1},q_{n_2},x,y)$$

$\forall \ (x, y) \in J^2$. The approximation behavior these operators will considered elsewhere.

Chapter 8

Chlodowsky-Szász-Appell type operators for functions of two variables

8.1 Introduction

On the interval $[0, a_n]$ with $a_n \to \infty$, as $n \to \infty$, the Bernstein-Chlodowsky polynomials are defined by

$$B_n(f;x) = \sum_{k=0}^{n} \binom{n}{k} \left(\frac{x}{a_n}\right)^k \left(1 - \frac{x}{a_n}\right)^{n-k} f\left(k\frac{a_n}{n}\right), \tag{8.1.1}$$

where $x \in [0, a_n]$ and $\lim\limits_{n\to\infty} \dfrac{a_n}{n} = 0$.

By combining the Bernstein-Chlodowsky operators (8.1.1) and the operators (0.3.2), we introduce the bivariate operators as follows:

$$T_{n,m}(f;x,y) = \sum_{k=0}^{n}\sum_{j=0}^{\infty} \binom{n}{k} \left(\frac{x}{a_n}\right)^k \left(1 - \frac{x}{a_n}\right)^{n-k} \frac{e^{-b_m y}}{g(1)} p_j(b_m y) f\left(k\frac{a_n}{n}, \frac{j}{c_m}\right), \tag{8.1.2}$$

for all $n, m \in \mathbb{N},\ f \in C(A)_{a_n}$ with $A_{a_n} = \{(x,y) : 0 \le x \le a_n, 0 \le y < \infty\}$, and $C(A_{a_n}) := \{f : A_{a_n} \to R \text{ is continuous}\}$. Note that the operator (8.1.2) is the tensorial product of ${}_xB_n$ and ${}_yP^*_m$, i.e., $T_{n,m} = {}_xB_n \circ {}_yP^*_m$, where

$${}_xB_n(f;x,y) = \sum_{k=0}^{n} \binom{n}{k} \left(\frac{x}{a_n}\right)^k \left(1 - \frac{x}{a_n}\right)^{n-k} f\left(k\frac{a_n}{n}, y\right),$$

and

$$_yP_m^*(f;x,y) = \frac{e^{-b_m y}}{g(1)} \sum_{k=0}^{\infty} p_k(b_m y) f\left(x, \frac{k}{c_m}\right).$$

The purpose of the present chapter is to establish the degree of approximation for the bivariate operators (8.1.2) by means of the moduli of continuity and the Lipschitz class. The rate of convergence of these operators for a weighted space is studied with the aid of modulus of continuity defined in [88]. Subsequently, the GBS case of these operators (8.1.2) is introduced and the approximation degree for the GBS operators is obtained by means of the mixed modulus of smoothness.

8.2 Preliminaries

To examine the approximation properties of the operators (8.1.2), we give some basic results using the test functions $e_{i,j} = u^i v^j$ $(i, j = 0, 1, 2, 3, 4)$ as follows:

Lemma 8.2.1. *For the operators $T_{n,m}$ there holds the identities:*

(i) $T_{n,m}(e_{0,0}; x, y) = 1,$

(ii) $T_{n,m}(e_{1,0}; x, y) = x,$

(iii) $T_{n,m}(e_{0,1}; x, y) = \frac{b_m}{c_m} y + \frac{1}{c_m} \frac{g'(1)}{g(1)},$

(iv) $T_{n,m}(e_{2,0}; x, y) = x^2 + \frac{x}{n}(a_n - x),$

(v) $T_{n,m}(e_{0,2}; x, y) = \frac{b_m^2}{c_m^2} y^2 + \frac{b_m}{c_m^2}\left(2\frac{g'(1)}{g(1)} + 1\right) y + \frac{1}{c_m^2}\left(\frac{g''(1)}{g(1)} + \frac{g'(1)}{g(1)}\right),$

(vi) $T_{n,m}(e_{3,0}; x, y) = x^3 + \frac{x(a_n - x)}{n^2}\left(x(3n-2) + a_n\right),$

(vii) $T_{n,m}(e_{0,3}; x, y) = \frac{b_m^3}{c_m^3} y^3 + \frac{b_m^2}{c_m^3}\left(3\frac{g'(1)}{g(1)} + 4\right) y^2 + \frac{b_m}{c_m^3}\left(3\frac{g''(1)}{g(1)} + 8\frac{g'(1)}{g(1)} + 1\right) y$
$+ \frac{1}{c_m^3}\left(\frac{g'''(1)}{g(1)} + 4\frac{g''(1)}{g(1)} + \frac{g'(1)}{g(1)}\right),$

(viii) $T_{n,m}(e_{4,0}; x, y) = x^4 + \frac{x^3(a_n - x)}{n^3}(6n^2 - 5n + 2) + \frac{x(a_n - x)^2}{n^3}\left(2x(3n-2) + a_n\right) + \frac{x^2 a_n (a_n - x)(n-1)}{n^3},$

(ix) $T_{n,m}(e_{0,4};x,y)=\frac{b_m^4}{c_m^4}y^4+\frac{b_m^3}{c_m^4}y^3\left(4\frac{g'(1)}{g(1)}+10\right)+\frac{b_m^2}{c_m^4}y^2\left(6\frac{g''(1)}{g(1)}+30\frac{g'(1)}{g(1)}+14\right)+\frac{b_m}{c_m^4}y\left(4\frac{g'''(1)}{g(1)}+30\frac{g''(1)}{g(1)}+28\frac{g'(1)}{g(1)}+1\right)+\frac{1}{c_m^4}\left(\frac{g^{(4)}(1)}{g(1)}+10\frac{g'''(1)}{g(1)}+14\frac{g''(1)}{g(1)}+\frac{g'(1)}{g(1)}\right).$

Proof. By simple calculations, we can easily prove the above results. Hence the details are omitted. □

As a consequence of Lemma 8.2.1, we obtain:

Lemma 8.2.2. *For the operator (8.1.2), we have the following results:*

(i) $T_{n,m}((e_{1,0}-x)^2;x,y)=\frac{x}{n}(a_n-x),$

(ii) $T_{n,m}((e_{0,1}-y)^2;x,y)=\left(\frac{b_m}{c_m}-1\right)^2y^2+\left(2\frac{b_m}{c_m^2}\frac{g'(1)}{g(1)}-\frac{2}{c_m}\frac{g'(1)}{g(1)}+\frac{b_m}{c_m^2}\right)y+\frac{1}{c_m^2}\left(\frac{g'(1)}{g(1)}+\frac{g''(1)}{g(1)}\right),$

(iii) $T_{n,m}((e_{1,0}-x)^4;x,y)=\left(\frac{3}{n^2}-\frac{6}{n^3}\right)x^4-\frac{6a_n(n-2)}{n^3}x^3+a_n^2\left(\frac{3}{n^2}-\frac{7}{n^3}\right)x^2+\frac{a_n^3}{n^3}x,$

(iv) $T_{n,m}((e_{0,1}-y)^4;x,y)=\left(\frac{b_m}{c_m}-1\right)^4y^4+\left\{\frac{b_m^3}{c_m^4}\left(4\frac{g'(1)}{g(1)}+10\right)+6\frac{b_m}{c_m^2}\left(2\frac{g'(1)}{g(1)}+1\right)-4\frac{b_m^2}{c_m^3}\left(3\frac{g'(1)}{g(1)}+4\right)-\frac{4}{c_m}\frac{g'(1)}{g(1)}\right\}y^3+\left\{\frac{b_m^2}{c_m^4}\left(6\frac{g''(1)}{g(1)}+30\frac{g'(1)}{g(1)}+14\right)-4\frac{b_m}{c_m^3}\left(3\frac{g''(1)}{g(1)}+8\frac{g'(1)}{g(1)}+1\right)+\frac{6}{c_m^2}\left(\frac{g''(1)}{g(1)}+\frac{g'(1)}{g(1)}\right)\right\}y^2+\left\{\frac{b_m}{c_m^4}\left(4\frac{g'''(1)}{g(1)}+30\frac{g''(1)}{g(1)}+28\frac{g'(1)}{g(1)}+1\right)-\frac{4}{c_m^3}\left(\frac{g'''(1)}{g(1)}+4\frac{g''(1)}{g(1)}+\frac{g'(1)}{g(1)}\right)\right\}y+\frac{1}{c_m^4}\left(\frac{g^4(1)}{g(1)}+10\frac{g'''(1)}{g(1)}+14\frac{g''(1)}{g(1)}+\frac{g'(1)}{g(1)}\right).$

Lemma 8.2.3. *Taking into account the conditions on $(a_n),(b_n),(c_n)$ and using Lemma 8.2.1 and Lemma 8.2.2, we may write*

(i) $T_{n,m}((e_{1,0}-x)^2;x,y)=O\left(\frac{a_n}{n}\right)(x^2+x),$ *as* $n\to\infty,$

(ii) $T_{n,m}((e_{0,1}-y)^2;x,y)\leq\frac{\eta(g)}{c_m}(y^2+y+1),$

(iii) $T_{n,m}((e_{1,0}-x)^4;x,y)=O\left(\frac{a_n}{n}\right)(x^4+x^3+x^2+x),$ *as* $n\to\infty,$

(iv) $T_{n,m}((e_{0,1}-y)^4;x,y) \leq \dfrac{\mu(g)}{c_m}(y^4+y^3+y^2+y+1)$,

where $\eta(g)$ *and* $\mu(g)$ *are certain constants depending on* g.

8.3 Main results

In this section, we establish the degree of approximation of the operators given by (8.1.2) in the space of continuous functions on compact set $I_{ab} := [0,a]\times[0,b] \subset A_{a_n}$.

Theorem 8.3.1. *For all* $(x,y)\in I_{ab}$ *and* $f\in C(I_{ab})$, *we have the following inequality:*

$$|T_{n,m}(f;x,y)-f(x,y)| \leq 2\omega(f;\delta_{n,m}),$$

where $\delta_{n,m} = \left(O\left(\frac{a_n}{n}\right)(x^2+x)+\frac{\eta(g)}{c_m}(y+1)^2\right)^{1/2}$.

Proof. Using the Cauchy-Schwarz inequality and Lemma 8.2.3, we have

$$\begin{aligned}
&|T_{n,m}(f;x,y)-f(x,y)|\\
&\leq \omega(f;\delta_{n,m})\left[1+\frac{1}{\delta_{n,m}}\left\{T_{n,m}((e_{1,0}-x)^2+(e_{0,1}-y)^2;x,y)\right\}^{1/2}\right]\\
&\leq \omega(f;\delta_{n,m})\left[1+\frac{1}{\delta_{n,m}}\left\{T_{n,m}((e_{1,0}-x)^2;x,y)+T_{n,m}((e_{0,1}-y)^2;x,y)\right\}^{1/2}\right]\\
&\leq \omega(f;\delta_{n,m})\left[1+\frac{1}{\delta_{n,m}}\left\{O\left(\frac{a_n}{n}\right)(x^2+x)+\frac{\eta(g)}{c_m}(y+1)^2\right\}^{1/2}\right],
\end{aligned}$$

from which the desired result is immediate.

□

In the following theorem, we obtain the rate of convergence of the operators defined by (8.1.2) in terms of the partial moduli of continuity.

Theorem 8.3.2. *For* $f\in C(I_{ab})$ *and all* $(x,y)\in I_{ab}$, *the following result holds:*

$$|T_{n,m}(f;x,y)-f(x,y)| \leq 2\left(\omega_1(f;\delta_n)+\omega_2(f;\delta_m)\right),$$

where $\delta_n^2 = \frac{x}{n}(a_n-x)$ *and* $\delta_m^2 = \left(\frac{b_m}{c_m}-1\right)^2 y^2 + \left(2\frac{b_m}{c_m^2}\frac{g'(1)}{g(1)} - \frac{2}{c_m}\frac{g'(1)}{g(1)} + \frac{b_m}{c_m^2}\right)y + \frac{1}{c_m^2}\left(\frac{g'(1)}{g(1)}+\frac{g''(1)}{g(1)}\right)$.

Proof. Using the definition of partial moduli of continuity, Lemma 8.2.2 and the Cauchy-Schwarz inequality, we have

$$\begin{aligned}
|T_{n,m}(f;x,y)-f(x,y)| &\leq T_{n,m}(|f(u,v)-f(x,v)|;x,y)+T_{n,m}(|f(x,v)-f(x,y)|;x,y)\\
&\leq T_{n,m}\left(\omega_1(f;|u-x|);x,y\right)+T_{n,m}\left(\omega_2(f;|v-y|);x,y\right)\\
&\leq \omega_1(f;\delta_n)\left[1+\frac{1}{\delta_n}T_{n,m}(|u-x|;x,y)\right]\\
&\quad+\omega_2(f;\delta_m)\left[1+\frac{1}{\delta_m}T_{n,m}(|v-y|;x,y)\right]\\
&\leq \omega_1(f;\delta_n)\left[1+\frac{1}{\delta_n}\left(T_{n,m}((e_{1,0}-x)^2;x,y)\right)^{1/2}\right]\\
&\quad+\omega_2(f;\delta_m)\left[1+\frac{1}{\delta_m}\left(T_{n,m}((e_{0,1}-y)^2;x,y)\right)^{1/2}\right],
\end{aligned}$$

choosing $\delta_n^2=\frac{x}{n}(a_n-x)$ and $\delta_m^2=\left(\frac{b_m}{c_m}-1\right)^2y^2+\left(2\frac{b_m}{c_m^2}\frac{g'(1)}{g(1)}-\frac{2}{c_m}\frac{g'(1)}{g(1)}+\frac{b_m}{c_m^2}\right)y+\frac{1}{c_m^2}\left(\frac{g'(1)}{g(1)}+\frac{g''(1)}{g(1)}\right)$, we obtain the required result. □

Now, we establish the degree of approximation for the bivariate operators (8.1.2) with the aid of Lipschitz class.

Theorem 8.3.3. *Let $f\in Lip_M(\gamma_1,\gamma_2)$. Then*

$$|T_{n,m}(f;x,y)-f(x,y)|\leq M\delta_n^{\gamma_1}\delta_m^{\gamma_2},$$

where δ_n and δ_m are the same as in Theorem 8.3.2 and $0<\gamma_1,\gamma_2\leq 1$.

Proof. Since $f\in Lip_M(\gamma_1,\gamma_2)$, we may write

$$\begin{aligned}
|T_{n,m}(f;x,y)-f(x,y)| &\leq T_{n,m}\left(M|u-x|^{\gamma_1}|v-y|^{\gamma_2};x,y\right)\\
&\leq M\ {}_xB_n\left(|u-x|^{\gamma_1};x,y\right)\ {}_yP_m^*\left(|v-y|^{\gamma_2};x,y\right).
\end{aligned}$$

Considering the Hölder's inequality with $(p_1,q_1)=\left(\frac{2}{\gamma_1},\frac{2}{2-\gamma_1}\right)$ and $(p_2,q_2)=\left(\frac{2}{\gamma_2},\frac{2}{2-\gamma_2}\right)$ and Lemma 8.2.1, we have

$$\begin{aligned}
|T_{n,m}(f;x,y)-f(x,y)| &\leq M\ {}_xB_n\left((e_{1,0}-x)^2;x,y\right)^{\gamma_1/2}\ {}_xB_n\left(e_{0,0};x,y\right)^{(2-\gamma_1)/2}\\
&\quad {}_yP_m^*\left((e_{0,1}-y)^2;x,y\right)^{\gamma_2 y/2}\ {}_yP_m^*\left(e_{0,0};x,y\right)^{(2-\gamma_2)/2}\\
&= M\delta_n^{\gamma_1}\,\delta_m^{\gamma_2}.
\end{aligned}$$

This proves the theorem. □

Now, we estimate the degree of approximation of the bivariate operators (8.1.2) in a weighted space. Let B_ρ be the space of all functions f defined on $\mathbb{R}_0^+ \times \mathbb{R}_0^+$, $\mathbb{R}_0^+ = [0, \infty)$ having the property $|f(x,y)| \leq M_f\, \rho(x,y)$, where $M_f > 0$ is a constant which depends on f and $\rho(x,y) = 1 + x^2 + y^2$ is a weight function. Let C_ρ be the subspace of B_ρ of all continuous functions with the norm $||f||_\rho = \sup\limits_{x,y \in \mathbb{R}_0^+} \dfrac{|f(x,y)|}{\rho(x,y)}$ and let C_ρ^0 be the subspace of all functions $f \in C_\rho$ such that $\lim\limits_{x \to \infty} \dfrac{|f(x,y)|}{\rho(x,y)}$ exists finitely. For all $f \in C_\rho^0$, the weighted modulus of continuity is defined by

$$\omega_\rho(f; \delta_1, \delta_2) = \sup_{x,y \in \mathbb{R}_0^+} \sup_{|h_1| \leq \delta_1, |h_2| \leq \delta_2} \frac{|f(x+h_1, y+h_2) - f(x,y)|}{\rho(x,y)\rho(h_1,h_2)}. \tag{8.3.1}$$

Theorem 8.3.4. *If $f \in C_\rho^0$, then for sufficiently large n, m, the following inequality holds:*

$$\sup_{x,y \in \mathbb{R}_0^+} \frac{|T_{n,m}(f; x, y) - f(x,y)|}{\rho(x,y)^3} \leq C\, \omega_\rho(f; \delta_n, \delta_m),$$

where $\delta_n = \left(\dfrac{a_n}{n}\right)^{1/2}$, $\delta_m = \left(\dfrac{\sigma(g)}{c_m}\right)^{1/2}$, $\sigma(g) = \max\{\eta(g), \mu(g)\}$ and C is a constant depending on n, m.

Proof. From ([88], pp.577), we may write

$$|f(u,v) - f(x,y)| \leq 8(1 + x^2 + y^2)\omega_\rho(f; \delta_n, \delta_m)\left(1 + \frac{|u-x|}{\delta_n}\right)\left(1 + \frac{|v-y|}{\delta_m}\right)$$
$$\times (1 + (u-x)^2)(1 + (v-y)^2).$$

Thus,

$$|T_{n,m}(f; x, y) - f(x,y)|$$
$$\leq 8(1 + x^2 + y^2)\omega_\rho(f; \delta_n, \delta_m) \sum_{k=0}^{n} \binom{n}{k} \left(\frac{x}{\alpha_n}\right)^k \left(1 - \frac{x}{\alpha_n}\right)^{n-k} \left(1 + \frac{1}{\delta_n}\left|k\frac{a_n}{n} - x\right|\right)$$
$$\times \left(1 + \left(k\frac{a_n}{n} - x\right)^2\right) \sum_{j=0}^{\infty} \frac{e^{-b_m y}}{g(1)} p_j(b_m y) \left(1 + \frac{1}{\delta_m}\left|\frac{j}{c_m} - y\right|\right)\left(1 + \left(\frac{j}{c_m} - y\right)^2\right).$$

Applying the Cauchy-Schwarz inequality

$$\begin{aligned}
&|T_{n,m}(f;x,y)-f(x,y)|\\
&\leq 8(1+x^2+y^2)\omega_\rho(f;\delta_n,\delta_m)\left[1+T_{n,m}((e_{1,0}-x)^2;x,y)+\frac{1}{\delta_n}\sqrt{T_{n,m}((e_{1,0}-x)^2;x,y)}\right.\\
&\times\frac{1}{\delta_n}\sqrt{T_{n,m}((e_{1,0}-x)^2;x,y)\,T_{n,m}((e_{1,0}-x)^4;x,y)}\\
&\times\left[1+T_{n,m}((e_{0,1}-y)^2;x,y)+\frac{1}{\delta_m}\sqrt{T_{n,m}((e_{0,1}-y)^2;x,y)}\right.\\
&\left.\times\frac{1}{\delta_m}\sqrt{T_{n,m}((e_{0,1}-y)^2;x,y)\,T_{n,m}((e_{0,1}-y)^4;x,y)}\right].
\end{aligned}$$

Using Lemma 8.2.3, we have

$$\begin{aligned}
&|T_{n,m}(f;x,y)-f(x,y)|\\
\leq\;& 8(1+x^2+y^2)\omega_\rho(f;\delta_n,\delta_m)\left[1+O\left(\frac{a_n}{n}\right)(x^2+x)+\frac{1}{\delta_n}\sqrt{O\left(\frac{a_n}{n}\right)(x^2+x)}\right.\\
&\left.+\frac{1}{\delta_n}\sqrt{O\left(\frac{a_n}{n}\right)(x^2+x).O\left(\frac{a_n}{n}\right)(x^4+x^3+x^2+x)}\right]\\
&\times\left[1+\frac{\eta(g)}{c_m}(y+1)^2+\frac{1}{\delta_m}\sqrt{\frac{\eta(g)}{c_m}(y+1)^2}+\frac{1}{\delta_m}\sqrt{\frac{\eta(g)}{c_m}(y+1)^2\,\frac{\mu(g)}{c_m}(y+1)^4}\right].
\end{aligned}$$

Taking $\delta_n=\left(\frac{a_n}{n}\right)^{1/2}$, $\delta_m=\left(\frac{\sigma(g)}{c_m}\right)^{1/2}$ with $\sigma(g)=\max\{\eta(g),\mu(g)\}$, we reach the desired result.

□

8.4 Construction of GBS operators of Chlodowsky-Szàsz-Appell type

In this section, we introduce the GBS case of the operators defined in (8.1.2).

For every $f\in C_b(A_{a_n})$, the GBS operator associated with the operator $T_{n,m}(f;x,y)$ is defined as follows:

$$\begin{aligned}
U_{n,m}(f;x,y) &= \sum_{k=0}^{n}\sum_{j=0}^{\infty}\binom{n}{k}\left(\frac{x}{a_n}\right)^k\left(1-\frac{x}{a_n}\right)^{n-k}\frac{e^{-b_m y}}{g(1)}p_j(b_m y)\\
&\left[f\left(k\frac{a_n}{n},y\right)+f\left(x,\frac{j}{c_m}\right)-f\left(k\frac{a_n}{n},\frac{j}{c_m}\right)\right].
\end{aligned}\tag{8.4.1}$$

Let $I_{cd} := [0,c] \times [0,d] \subset A_{a_n}$.

Theorem 8.4.1. *For every $f \in C_b(I_{cd})$ and $\forall \ (x,y) \in I_{cd}$, we have the following inequality for the operator defined in (8.4.1)*

$$|U_{n,m}(f;x,y) - f(x,y)| \leq 4\omega_{mixed}(f;\delta_n,\delta_m),$$

where $\delta_n = \left(\frac{a_n}{n}(c^2+c)\right)^{1/2}$, $\delta_m := \delta_m(g) = \left(\frac{\rho(g)}{c_m}\right)^{1/2}$ and $\rho(g)$ is a constant depending on g.

Proof. We may write

$$U_{n,m}(f;x,y) = f(x,y)\, T_{n,m}(e_{0,0};x,y) - \ T_{n,m}\left(\Delta_{(x,y)} f(u,v);x,y\right).$$

Since $T_{n,m}(e_{0,0};x,y) = 1$, using 6.2.4 and applying the Cauchy-Schwarz inequality

$$\begin{aligned}
&|U_{n,m}(f;x,y) - f(x,y)| \\
&\quad \leq T_{n,m}\left(|\Delta_{(x,y)} f(u,v)|;x,y\right) \\
&\quad \leq \Bigg(T_{n,m}(e_{0,0};x,y) + \delta_n^{-1}\sqrt{T_{n,m}((e_{1,0}-x)^2;x,y)} + \delta_m^{-1}\sqrt{T_{n,m}((e_{0,1}-y)^2;x,y)} \\
&\quad + \delta_n^{-1}\delta_m^{-1}\sqrt{T_{n,m}((e_{1,0}-x)^2;x,y)}\,\sqrt{T_{n,m}((e_{0,1}-y)^2;x,y)}\Bigg)\ \omega_{mixed}(f;\delta_n,\delta_m).
\end{aligned} \tag{8.4.2}$$

By Lemma 8.2.2 and for all $(x,y) \in I_{cd}$,

$$\begin{aligned}
T_{n,m}((e_{1,0}-x)^2;x,y) &= \frac{x(a_n - x)}{n} \\
&\leq \frac{a_n}{n}(x^2+x) \leq \frac{a_n}{n}(c^2+c).
\end{aligned} \tag{8.4.3}$$

Similarly

$$\begin{aligned}
T_{n,m}((e_{0,1}-y)^2;x,y) &\leq \frac{\eta(g)}{c_m}(y^2+y+1) \\
&\leq \frac{\eta(g)}{c_m}(d^2+d+1) = \frac{\rho(g)}{c_m},
\end{aligned} \tag{8.4.4}$$

where $\rho(g)$ is a constant depending on g.

Combining (8.4.2)-(8.4.4) and choosing $\delta_n = \left(\frac{a_n}{n}(c^2+c)\right)^{1/2}$ and $\delta_m := \delta_m(g) = \left(\frac{\rho(g)}{c_m}\right)^{1/2}$, we get the required result. □

Theorem 8.4.2. *For $f \in Lip_M(\xi_1, \xi_2), 0 < \xi_1, \xi_2 \leq$ and $(x, y) \in I_{cd}$, we have*

$$|U_{n,m}(f; x, y) - f(x, y)| \leq M\delta_n^{\xi_1/2}\delta_m^{\xi_2/2},$$

where $\delta_n = ||_x B_n((u-x)^2; \cdot)||_{C(I_{cd})}$, $\delta_m = ||_y P_m^*((v-y)^2; \cdot)||_{C(I_{cd})}$ and M is a certain positive constant.

Proof. We may write

$$\begin{aligned} U_{n,m}(f; x, y) &= T_{n,m}\left(f(x, y) - \Delta_{(x,y)}f(u, v); x, y\right) \\ &= f(x, y)\, T_{n,m}(e_{0,0}; x, y) - T_{n,m}\left(\Delta_{(x,y)}f(u, v); x, y\right). \end{aligned}$$

By our hypothesis, we get

$$\begin{aligned} |U_{n,m}(f; x, y) - f(x, y)| &\leq T_{n,m}\left(\left|\Delta_{(x,y)}f(u, v)\right|; x, y\right) \\ &\leq MT_{n,m}\left(|u-x|^{\xi_1}|v-y|^{\xi_2}; x, y\right) \\ &= MT_{n,m}\left(|u-x|^{\xi_1}; x, y\right) T_{n,m}\left(|v-y|^{\xi_2}; x, y\right). \end{aligned}$$

Now, applying the Hölder's inequality with $(p_1, q_1) = (2/\xi_1, 2/(2-\xi_1))$ and $(p_2, q_2) = (2/\xi_2, 2/(2-\xi_2))$, we obtain

$$\begin{aligned} |U_{n,m}(f; x, y) - f(x, y)| \leq\ & M\, {}_xB_n\left((u-x)^2; x\right)^{\xi_1/2} {}_xB_n(e_0; x)^{(2-\xi_1)/2} \\ & \times {}_yP_m^*\left((v-y)^2; y\right)^{\xi_2/2} {}_yP_m^*(e_0; y)^{(2-\xi_2)/2}. \end{aligned}$$

Taking $\delta_n = ||_x B_n((u-x)^2; \cdot)||_\infty$ and $\delta_m = ||_y P_m^*((v-y)^2; \cdot)||_\infty$, we get the desired result. □

Theorem 8.4.3. *If $f \in D_b(I_{cd})$ and $D_B f \in B(I_{cd})$, then for each $(x, y) \in I_{cd}$, we get*

$$\begin{aligned} |U_{n,m}(f; x, y) - f(x, y)| \leq C &\left\{3||D_B f||_\infty + 2\omega_{mixed}(f; \delta_n, \delta_m)\sqrt{x^2+x}\sqrt{y^2+y+1}\right\}\delta_n\delta_m \\ &+ \left\{\omega_{mixed}(f; \delta_n, \delta_m)\left(\delta_m\sqrt{x^4+x^3+x^2+x}\sqrt{y^2+y+1}\right.\right. \\ &\left.\left. + \delta_n\sqrt{y^4+y^3+y^2+y+1}\sqrt{x^2+x}\right)\right\}, \end{aligned}$$

where $\delta_n = \sqrt{\dfrac{a_n}{n}}$, $\delta_m = \sqrt{\dfrac{\sigma(g)}{c_m}}$, $\sigma(g) = \max\{\eta(g), \mu(g)\}$ and C is a constant depending on n, m only.

Proof. By our hypothesis

$$\Delta_{(x,y)}f(u,v) = (u-x)(v-y)D_Bf(\alpha,\beta), \; with \; x < \alpha < u\,; \; y < \beta < v.$$

Clearly,

$$D_Bf(\alpha,\beta) = \Delta_{(x,y)}D_Bf(\alpha,\beta) + D_Bf(\alpha,y) + D_Bf(x,\beta) - D_Bf(x,y).$$

Since $D_Bf \in B(I_{cd})$, from the above equalities, we have

$$\begin{aligned} |T_{n,m}(\Delta_{(x,y)}f(u,v);x,y)| &= |T_{n,m}((u-x)(v-y)D_Bf(\alpha,\beta);x,y)| \\ &\leq T_{n,m}(|u-x||v-y||\Delta_{(x,y)}D_Bf(\alpha,\beta)|;x,y) \\ &+ T_{n,m}(|u-x||v-y|(|D_Bf(\alpha,y)| + |D_Bf(x,\beta)| \\ &+ |D_Bf(x,y)|);x,y) \\ &\leq T_{n,m}(|u-x||v-y|\omega_{mixed}(D_Bf;|\alpha-x|,|\beta-y|);x,y) \\ &+ 3\;||D_Bf||_\infty\; T_{n,m}(|u-x||v-y|;x,y). \end{aligned} \tag{8.4.5}$$

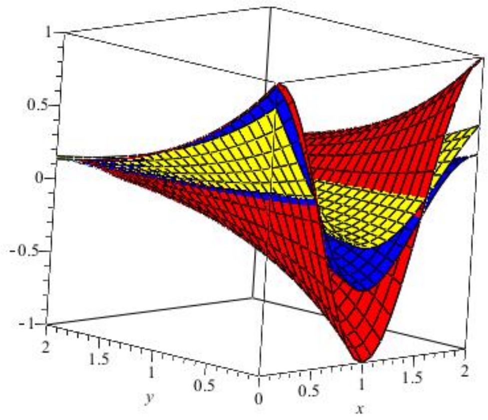
Hence using (6.2.4), applying Cauchy-Schwarz inequality and Lemma 8.2.3, on taking $\delta_n = \sqrt{\dfrac{a_n}{n}}, \delta_m = \sqrt{\dfrac{\sigma(g)}{c_m}}$, we reach the required result.

□

8.4.1 Numerical Examples

In this section we give some numerical results regarding the approximation properties of Chlodowsky-Szász-Appell operators defined in (8.1.2).

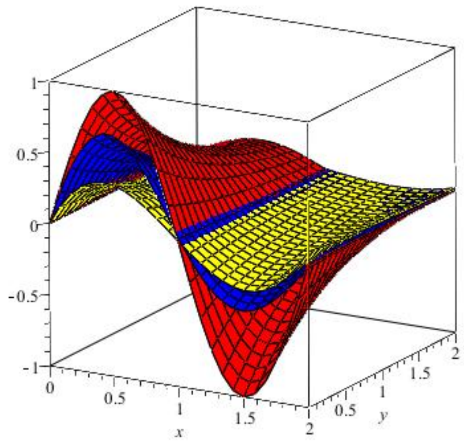
Example 1. Let us consider the function $f(x,y) = e^{-y}\cos(\pi x)$, $g(u) = u$ and $a_n = \sqrt{n}$, $b_n = n$, $c_n = n + \dfrac{1}{\sqrt{n}}$. For $n = m = 5$ and $n = m = 40$, the convergence of $T_{n,m}(f;x,y)$ to $f(x,y)$ is illustrated in Figure 1.



The convergence of $T_{n,m}(f;x,y)$ to $f(x,y)$ (red f, blue $T_{40,40}$, yellow $T_{5,5}$)

Figure 1

Example 2. Let us consider the function $f(x,y) = e^{-y}\sin(\pi x)$, $g(u) = u$ and $a_n = \sqrt{n}$, $b_n = n$, $c_n = n + e^{-n}$. For $n = m = 5$ and $n = m = 40$ the convergence of $T_{n,m}(f;x,y)$ to $f(x,y)$ is illustrated in Figure 2.



The convergence of $T_{n,m}(f;x,y)$ to $f(x,y)$ (red f, blue $T_{40,40}$, yellow $T_{5,5}$)Figure 2

We notice from the above examples that for $n = m = 40$, the approximation of the operator $T_{n,m}$ to the function f is better than $n = m = 5$.

Example 3. If $f \in C(I_{ab})$, then

$$|T_{n,m}(f;x,y) - f(x,y)| \leq 2\left(\omega_1(f;\delta_n) + \omega_2(f;\delta_m)\right)$$
$$\leq 2\left(\|f^{(1,0)}\|_\infty \delta_n + \|f^{(0,1)}\|_\infty \delta_m\right),$$

where δ_n and δ_m are defined in Theorem 8.3.2.

In Table 1 we compute the error of approximation of $f(x,y) = xye^{-y}$ by using the above relation for $I_{ab} = [0,4] \times [0,4]$.

Table 1. Error of approximation for $T_{n,m}$

$n = m$	$a_n = \sqrt{n}, \underline{b}_n = n,\ c_n = n + \frac{1}{\sqrt{n}}$	$a_n = \sqrt{n}, \underline{b}_n = n,\ c_n = n + e^{-n}$
20	3.9062769320	3.9678794400
50	2.6253029800	2.6362709420
100	1.9595674840	1.9624455190
500	0.9975826189	0.9977048873
1000	0.7505644223	0.7505954100
1500	0.6370476284	0.6370614836
2000	0.5677575415	0.5677653628
2500	0.5196173630	0.5196223806
3000	0.4835609538	0.4835644444

Conclusion

The present thesis is an investigation of the approximation properties of the Kantorovich and Durrmeyer variants of the Bernstein-Schurer and Szász type operators. The corresponding bivariate operators and the bivariate operators defined by combining Bernstein -Chlodowsky Szász type operators involving Appell polynomials have been introduced and their approximation behavior for functions of two variables has been investigated. The associated GBS operators have also been considered and their degree of approximation for functions in a Bögel space has been obtained by means of the Lipschitz class modulus of smoothness. The rate of convergence of the considered operators has and mixed been examined by numerical examples and illustrations using Matlab algorithms.

Bibliography

[1] U. Abel, V. Gupta, and R. N. Mohapatra. Local approximation by beta operators. *Nonlinear Anal.*, 62(1):41–52, 2005.

[2] U. Abel, V. Gupta, and R. N. Mohapatra. Local approximation by a variant of Bernstein–Durrmeyer operators. *Nonlinear Anal.*, 68(11):3372–3381, 2008.

[3] T. Acar. (p, q)-Generalization of Szász–Mirakyan operators. *Math. Meth. Appl. Sci.*, 39(10):2685–2695, 2016.

[4] T. Acar. Quantitative q-Voronovskaya and q-Grüss–Voronovskaya-type results for q-Szász operators. *Georgian Math. J.*, 23(4):459–468, 2016.

[5] T. Acar and A. Aral. On pointwise convergence of q-Bernstein operators and their q-derivatives. *Numer. Funct. Anal. Optim.*, 36(3):287–304, 2015.

[6] T. Acar, A. Aral, and V. Gupta. On approximation properties of a new type of Bernstein-Durrmeyer operators. *Math. Slovaca*, 65(5):1107–1122, 2015.

[7] T. Acar, A. Aral, and S. A. Mohiuddine. On Kantorovich modification of (p, q)-Baskakov operators. *J. Inequal. Appl.*, 2016(1):98, 2016.

[8] T. Acar, A. Aral, and I. Rasa. Modified Bernstein-Durrmeyer operators. *Gen. Math*, 22(1):27–41, 2014.

[9] T. Acar, V. Gupta, and A. Aral. Rate of convergence for generalized Szász operators. *Bull . Math. Sci.*, 1(1):99–113, 2011.

[10] T. Acar and G. Ulusoy. Approximation by modified Szász-Durrmeyer operators. *Period. Math. Hungar.*, 72(1):64–75, 2016.

[11] A. M. Acu. Stancu–Schurer–Kantorovich operators based on q-integers. *Appl. Math. Comput.*, 259:896–907, 2015.

[12] A. M. Acu and C. V. Muraru. Approximation properties of bivariate extension of q-Bernstein–Schurer–Kantorovich operators. *Results Math.*, 67(3-4):265–279, 2015.

[13] A. M. Acu, C. V. Muraru, D. F. Sofonea, and V. A. Radu. Some approximation properties of a Durrmeyer variant of q-Bernstein–Schurer operators. *Math. Meth. Appl. Sci.*, 39(18):5636–5650, 2016.

[14] J. A. Adell and J. De la Cal. Bernstein-Durrmeyer operators. *Comput. Math. Appl.*, 30(3-6):1–14, 1995.

[15] O. Agratini. On certain q-analogues of the Bernstein operators. *Carpathian J. Math.*, 24:281–286, 2008.

[16] P. N. Agrawal, B. Baxhaku, and R. Chauhan. The approximation of bivariate Chlodowsky-Szász-Kantorovich-Charlier-type operators. *J. Inequal. Appl.*, 2017(1):195, 2017.

[17] P. N. Agrawal, Z. Finta, and A. S. Kumar. Bivariate q-Bernstein-Schurer-Kantorovich operators. *Results Math.*, 67(3-4):365–380, 2015.

[18] P. N. Agrawal, V. Gupta, and A. S. Kumar. On q-analogue of Bernstein–Schurer–Stancu operators. *Appl. Math. Comput.*, 219(14):7754–7764, 2013.

[19] P. N. Agrawal and N. İspir. Degree of approximation for bivariate Chlodowsky–Szász–Charlier type operators. *Results Math.*, 69(3-4):369–385, 2016.

[20] P. N. Agrawal, N. Ispir, and A. Kajla. GBS operators of Lupaş–Durrmeyer type based on Polya distribution. *Results Math.*, 69(3-4):397–418, 2016.

[21] P. N. Agrawal, A. S. Kumar, and T. A. K. Sinha. Stancu type generalization of modified Schurer operators based on q-integers. *Appl. Math. Comput.*, 226:765–776, 2014.

[22] P. N. Agrawal, A. S. Kumar, and T. A. K. Sinha. Stancu type generalization of modified Schurer operators based on q-integers. *Appl. Math. Comput.*, 226:765–776, 2014.

[23] P. N. Agrawal, D. Kumar, and S. Araci. Linking of Bernstein-Chlodowsky and Szász-Appell-Kantorovich type operators. *J. Nonlinear Sci. Appl.*, 10:15, 2017.

[24] G. A. Anastassiou and S. G. Gal. *Approximation Theory: Moduli of Continuity and Global Smoothness Preservation*. Springer Science & Business Media, 2012.

[25] P. Appell. *Sur une Classe de Polynômes*. Gauthier-Villars, 1880.

[26] A. Aral and T. Acar. Weighted approximation by new Bernstein-Chlodowsky-Gadjiev operators. *Filomat*, 27(2):371–380, 2013.

[27] A. Aral and V. Gupta. On q-Baskakov type operators. *Demonstratio Math.*, 42(1):109–122, 2009.

[28] A. Aral and V. Gupta. Generalized q-Baskakov operators. *Math. Slovaca*, 61(4):619–634, 2011.

[29] A. Aral and V. Gupta. Generalized Szász-Durrmeyer operators. *Lobachevskii J. Math.*, 32(1):23–31, 2011.

[30] A. Aral, V. Gupta, and R. P. Agarwal. *Applications of q-Calculus in Operator Theory*. Springer, 2013.

[31] A. Aral, D. Inoan, and I. Raşa. On the generalized Szász–Mirakyan operators. *Results Math.*, 65(3-4):441–452, 2014.

[32] Ç. Atakut and İ. Büyükyazıcı. Stancu type generalization of the Favard–Szász operators. *Appl. Math. Lett.*, 23(12):1479–1482, 2010.

[33] F. Avram and M. S. Taqqu. Noncentral limit theorems and Appell polynomials. *Ann.Probab.*, pages 767–775, 1987.

[34] C. Badea, I. Badea, and H. Gonska. Notes on degree of approximation of B-continuous and B-differentiable functions. *J. Approx. Theory Appl.*, 4:95–108, 1988.

[35] C. Badea, I. Badea, and H. H. Gonska. A test function theorem and approximation by pseudopolynomials. *Bull. Aust. Math. Soc.*, 34(1):53–64, 1986.

[36] C. Badea and C. Cottin. Korovkin-type theorems for generalized boolean sum operators. In *Colloquia Mathematica Societatis Janos Bolyai*, volume 58, pages 51–67, 1990.

[37] C. Badea and C. Cottin. Korovkin-type theorems for generalized boolean sum operators, Approximation theory (kecskemét, 1990), 51-68. In *Colloq. Math. Soc. János Bolyai, North-Holland, Amsterdam*, volume 58, 1991.

[38] D. Barbosu. Some generalized bivariate Bernstein operators. *Math. Notes*, 1(1):3–10, 2000.

[39] D. Barbosu, A. M. Acu, and C. V. Muraru. On certain GBS-Durrmeyer operators based on q-integers. *Turk. J. Math.*, 41(2):368–380, 2017.

[40] D. Bărbosu and C. V. Muraru. Approximating B-continuous functions using GBS operators of Bernstein–Schurer–Stancu type based on q-integers. *Appl. Math. Comput.*, 259:80–87, 2015.

[41] D. Bărbosu and O. T. Pop. A note on the GBS Bernstein's approximation formula. *Annals Univ. of Craiova, Math. Comp. Sci. Ser.*, 35:1–6, 2008.

[42] I. Barsan, P. Braica, and M. Farcas. About approximation of B-continuous functions of several variables by generalized boolean sum operators of Bernstein type on a simplex. *Creat. Math. Inform*, 20(1):20–23, 2011.

[43] V. A. Baskakov. An instance of a sequence of linear positive operators in the space of continuous functions. *Doklady Akademii Nauk SSSR*, 113(2):249–251, 1957.

[44] S. Bernstein. Démonstration du theéorèm de weierstrass fondée sur le calcul des probabilités. *Comm. Soc. Math. Kharkov,*, 13(1):1–2, 1912.

[45] K. Bögel. Mehrdimensionale differentiation von funktionen mehrerer reeller veränderlichen. *J. Reine Angew. Math.*, 170:197–217, 1934.

[46] K. Bögel. Über mehrdimensionale differentiation, integration und beschränkte variation. *J. Reine Angew. Math.*, 173:5–30, 1935.

[47] P. L. Butzer. On two-dimensional Bernstein polynomials. *Canad. J. Math*, 5:107–113, 1953.

[48] P. L. Butzer and H. Berens. *Semi-groups of Operators and Approximation*, volume 145. Springer Science & Business Media, 2013.

[49] D. Cárdenas-Morales, P. Garrancho, and I. Raşa. Bernstein-type operators which preserve polynomials. *Comput. Math. Appl.*, 62(1):158–163, 2011.

[50] R. Chauhan, N. Ispir, and P. N. Agrawal. A new kind of Bernstein-Schurer-Stancu-Kantorovich-type operators based on q-integers. *J. Inequal. Appl.*, 2017(1):50, 2017.

[51] S. Cooper and S. Waldron. The diagonalisation of the multivariate Bernstein operator. *J. Approx. Theory*, 117(1):103–131, 2002.

[52] C. Cottin. Mixed K-functionals: A measure of smoothness for blending-type approximation. *Math. Z.*, 204(1):69–83, 1990.

[53] Ö. Dalmanoglu. Approximation by Kantorovich type q-Bernstein operators. In *Proceedings of the 12th WSEAS International Conference on Applied Mathematics*, pages 113–117. World Scientific and Engineering Academy and Society (WSEAS), 2007.

[54] P. J. Davis. *Interpolation and Approximation*. Courier Corporation, 1975.

[55] N. Deo, N. Bhardwaj, and S. P. Singh. Simultaneous approximation on generalized Bernstein–Durrmeyer operators. *Afrika Mat.*, pages 1–6, 2013.

[56] N. Deo, M. Dhamija, and D. Miclăuş. Stancu–Kantorovich operators based on inverse Pólya–Eggenberger distribution. *Appl. Math. Comput.*, 273:281–289, 2016.

[57] N. Deo and S. P. Singh. On the degree of approximation by new Durrmeyer type operators. *Gen. Math.*, 18(2):195–209, 2010.

[58] M. M. Derriennic. On multivariate approximation by Bernstein-type polynomials. *J. Approx. Theory*, 45(2):155–166, 1985.

[59] M. M. Derriennic. Modified Bernstein polynomials and Jacobi polynomials in q-calculus. *Rend. Circ. Mat. Palermo (2) Suppl.*, 2004.

[60] S. Deshwal, N. Ispir, and P. N. Agrawal. Blending type approximation by bivariate Bernstein-Kantorovich operators. *Appl. Math*, 11(2):423–432, 2017.

[61] R. A. DeVore and G. G. Lorentz. *Constructive Approximation*, volume 303. Springer Science & Business Media, 1993.

[62] M. Dhamija and N. Deo. Jain–Durrmeyer operators associated with the inverse Pólya–Eggenberger distribution. *Appl. Math. Comput.*, 286:15–22, 2016.

[63] Z. Ditzian and V. Totik. *Moduli of Smoothness*, volume 9. Springer Science & Business Media, 1987.

[64] E. Dobrescu and I. Matei. The approximation by Bernstein type polynomials of bidimensionally continuous functions. *An. Univ. Timişoara Ser. Sti. Mat.-Fiz*, 4:85–90, 1966.

[65] J. L. Durrmeyer. *Une Formule d'inversion de la Transformée de Laplace: Applications à la Théorie des moments*, Ph.D. thesis. 1967.

[66] M. D. Farcas. About approximation of B-continuous and B-differential functions of three variables by GBS operators of Bernstein-Schurer type. *Bul. Stiint. Univ. Politeh. Timis. Ser. Mat. Fiz*, 52(66):13–22, 2007.

[67] M. D. Farcas. About approximation of B-continuous and B-differentiable functions of three variables by GBS operators of Bernstein type. *Creat. Math. Inform.*, 17:20–27, 2008.

[68] M. D. Farcas. About approximation of B-continuous functions of three variables by GBS operators of Bernstein type on a tetrahedron. *Acta Univ. Apulensis Math. Inform*, 16:93–102, 2008.

[69] A. D. Gadzhiev. Theorems of Korovkin type. *Math. Notes*, 20(5):995–998, 1976.

[70] T. Garg, P. N. Agrawal, and S. Araci. Rate of convergence by Kantorovich-Szász type operators based on Brenke type polynomials. *J. Inequal. Appl.*, 2017(1):156, 2017.

[71] N. K. Govil and V. Gupta. Convergence of q-Meyer-König-Zeller-Durrmeyer operators. *Adv. Stud. Contemp. Math.*, 19(1):97–108, 2009.

[72] N. K. Govil and V. Gupta. Rate of approximation for Szász-Mirakyan-Stancu-Durrmeyer operators. *Lobachevskii J. Math.*, 33(3):249–254, 2012.

[73] N. K. Govil, V. Gupta, and D. Soybaş. Certain new classes of Durrmeyer type operators. *Appl. Math. Comput.*, 225:195–203, 2013.

[74] S. Guo, Q. Qi, and G. Liu. The central approximation theorems for Baskakov–Bézier operators. *J. Approx. Theory*, 147(1):112–124, 2007.

[75] M. K. Gupta and M. S. Beniwal. Rate of approximation of derivatives for Durrmeyer-Schurer type operators. *Southeast Asian Bull. Math.*, 32(1), 2008.

[76] M. K. Gupta, M. S. Beniwal, and P. Goel. Rate of convergence for Szász–Mirakyan–Durrmeyer operators with derivatives of bounded variation. *Appl. Math. Comput.*, 199(2):828–832, 2008.

[77] M. K. Gupta and S. Somya. On L_p-approximation for modified Baskakov-Durrmeyer type operators. *J. Int. Acad. Phys. Sci.*, 13(3), 2009.

[78] V. Gupta and R. P. Agarwal. *Convergence Estimates in Approximation Theory*. Springer Science & Business Media, 2014.

[79] V. Gupta and A. Aral. Convergence of the q- analogue of Szász-beta operators. *Appl. Math. Comput.*, 216(2):374–380, 2010.

[80] V. Gupta and W. Heping. The rate of convergence of q-Durrmeyer operators for $0 < q < 1$. *Math. Meth. Appl. Sci.*, 31(16):1946–1955, 2008.

[81] V. Gupta and H. Karsli. Some approximation properties by q-Szász-Mirakyan-Baskakov-Stancu operators. *Lobachevskii J. Math.*, 33(2):175–182, 2012.

[82] V. Gupta, R. N. Mohapatra, and Z. Finta. A certain family of mixed summation-integral type operators. *Math. Comput. Modelling*, 42(1-2):181–191, 2005.

[83] V. Gupta and G. Tachev. *Approximation with Positive Linear Operators and Linear Combinations*. Springer, 2017.

[84] E. Ibikli and E. A. Gadjieva. The order of approximation of some unbounded functions by the sequences of positive linear operators. *Turk. J. Math*, 19(3):331–337, 1995.

[85] M. E. H. Ismail. On a generalization of Szász operators. *Mathematica (Cluj)*, 39:259–267, 1974.

[86] N. Ispir. Rate of convergence of generalized rational type Baskakov operators. *Math. Comput. Modelling*, 46(5):625–631, 2007.

[87] N. Ispir. Quantitative estimates for GBS operators of Chlodowsky-Szász type. *Filomat*, 31(5):1175–1184, 2017.

[88] N. Ispir and C. Atakut. Approximation by modified Szász-Mirakjan operators on weighted spaces. *Proc. Indian Acad. Sci. Math. Sci.*, 112(4):571–578, 2002.

[89] A. Jakimovski and D. Leviatan. Generalized Szász operators for the approximation in the infinite interval. *Mathematica (Cluj)*, 34:97–103, 1969.

[90] V. Kac and P. Cheung. *Quantum Calculus*. Springer Science & Business Media, 2001.

[91] A. Kajla, A. M. Acu, and P. N. Agrawal. Baskakov–Szász-type operators based on inverse Pólya–Eggenberger distribution. *Ann. Funct. Anal.*, 8(1):106–123, 2017.

[92] A. Kajla and P. N. Agrawal. Szász–Durrmeyer type operators based on Charlier polynomials. *Appl. Math. Comput.*, 268:1001–1014, 2015.

[93] A. Kajla, N. Ispir, P. N. Agrawal, and M. Goyal. q- Bernstein–Schurer–Durrmeyer type operators for functions of one and two variables. *Appl. Math. Comput.*, 275:372–385, 2016.

[94] L. V. Kantorovich. Sur certain développements suivant les polynômes de la forme de s. *Bernstein, I, II, CR Acad. URSS*, 563:568, 1930.

[95] H. Karsli. Rate of convergence of new gamma type operators for functions with derivatives of bounded variation. *Math. Comput. Modelling*, 45(5):617–624, 2007.

[96] H. Karsli. Rate of convergence of new gamma type operators for functions with derivatives of bounded variation. *Math. Comput. Modelling*, 45(5):617–624, 2007.

[97] H. Karsli, P. N. Agrawal, and M. Goyal. General gamma type operators based on q-integers. *Appl. Math. Comput.*, 251:564–575, 2015.

[98] E. H. Kingsley. Bernstein polynomials for functions of two variables of class $C^{(k)}$. *Proc. Amer. Math. Soc.*, 2(1):64–71, 1951.

[99] Q. Lin. Statistical approximation of modified Schurer-type q-Bernstein-Kantorovich operators. *J. Inequal. Appl.*, 2014(1):465, 2014.

[100] F. L. Martinez. Some properties of two-dimensional Bernstein polynomials. *J. Approx. Theory*, 59(3):300–306, 1989.

[101] V. Mihesan. Uniform approximation with positive linear operators generated by generalized Baskakov method. *Automat. Comput. Appl. Math*, 7(1):34–37, 1998.

[102] C. V. Muraru. Note on q-Bernstein-Schurer operators. *Stud. Univ. Babes-Bolyai Math*, 56(2):489–495, 2011.

[103] C. V. Muraru and A. M. Acu. Some approximation properties of q-Durrmeyer-Schurer operators. *Scient. Stud. Res. Series Math. Inf.*, 23(1), 2013.

[104] M. Mursaleen and K. J. Ansari. On Chlodowsky variant of Szász operators by Brenke type polynomials. *Appl. Math. Comput.*, 271:991–1003, 2015.

[105] M. Mursaleen and A. Khan. Generalized-Bernstein-Schurer operators and some approximation theorems. *J. Funct. Spaces Appl.*, 2013, 2013.

[106] M. Mursaleen, A. Khan, H. M. Srivastava, and K. S. Nisar. Operators constructed by means of q-Lagrange polynomials and A-statistical approximation. *Appl. Math. Comput.*, 219(12):6911–6918, 2013.

[107] T. Neer, A. M. Acu, and P. N. Agrawal. Bézier variant of genuine-Durrmeyer type operators based on Pólya distribution. *Carpathian J. Math.*, 33(1):73–86, 2017.

[108] H. Oruç, G. M. Phillips, and P. J. Davis. A generalization of the Bernstein polynomials. *Proc. Edinburgh Math. Soc.*, 42(2):403–413, 1999.

[109] M. A. Özarslan and O. Duman. Local approximation behavior of modified SMK operators. *Miskolc Math. Notes*, 11(1):87–99, 2010.

[110] G. M. Phillips. On generalized Bernstein polynomials. *Numerical analysis*, 98:263–269, 1996.

[111] O. T. Pop. Approximation of B-continuous and B-differentiable functions by GBS operators defined by infinite sum. *J. Inequal. Pure Appl. Math*, 10(1), 2009.

[112] O. T. Pop and D. Barbosu. GBS operators of Durrmeyer-Stancu type. *Miskolc Math. Notes*, 9(1):53–60, 2008.

[113] O. T. Pop and M. D. Farcas. About the bivariate operators of Kantorovich type. *Acta Math. Univ. Comenianae*, 78(1):43–52, 2009.

[114] C. Radu. Statistical approximation properties of Kantorovich operators based on q-integers. *Creat. Math. Inform*, 17(2):75–84, 2008.

[115] R. K. S. Rathore and P. N. Agrawal. Inverse and saturation theorems for derivatives of exponential type operators. *Indian J. Pure Appl. Math*, 13:476–490, 1982.

[116] R. K. S. Rathore and O. P. Singh. On convergence of derivatives of post-Widder operators. *Indian J. pure appl. Math*, 11:547–561, 1980.

[117] R. K. S. Rathore and O. P. Singh. Limits of certain iterates of post-Widder operators. *Monatsh. Math.*, 91(2):131–138, 1981.

[118] M. Y. Ren and X. M. Zeng. On statistical approximation properties of modified q-Bernstein-Schurer operators. *Bull. Korean Math. Soc.*, 50(4):1145–1156, 2013.

[119] M. Y. Ren and X. M. Zeng. Approximation of Schurer type q-Bernstein-Kantorovich operators. *J. Inequal. Appl.*, 2015(1):82, 2015.

[120] R. Ruchi, N. Ispir, and P. N. Agrawal. Kantorovich variant of a new kind of q-Bernstein–Schurer operators. *Math. Meth. Appl. Sci.*, 2017.

[121] F. Schurer. Positive Linear Operators in Approximation Theory. *Math. Inst. Tech. Univ. Delft, Report*, 1962.

[122] H. Sharma. On certain family of mixed summation integral type two-dimensional q-Lupaş–Phillips–Bernstein operators. *Appl. Math. Comput.*, 259:741–752, 2015.

[123] H. Sharma and J. S. Aujla. A certain family of mixed summation-integral-type Lupaş-Phillips-Bernstein operators. *Math. Sci.*, 6(1):26, 2012.

[124] I. M. Sheffer. Some properties of polynomial sets of type zero. *Duke Math. J.*, 5(3):590–622, 1939.

[125] D. D. Stancu. A method for obtaining polynomials of Bernstein type of two variables. *Amer. Math. Monthly*, 70(3):260–264, 1963.

[126] D. D. Stancu. Approximation of functions by a new class of linear polynomial operators. *Rev. Roumaine Math. Pures Appl*, 13(8):1173–1194, 1968.

[127] D. D. Stancu. A new class of uniform approximating polynomial operators in two and several variables. In *Proc. of the Conf. on Constr. Theory of Functions, Budapest*, 1969.

[128] D. D. Stancu. Two classes of positive linear operators. *Analele Univ. din Timisoara (Seria St. Mat.)*, pages 213–220, 1970.

[129] S. Sucu, G. İçöz, and S. Varma. On some extensions of Szász operators including Boas-Buck-type polynomials. In *Abstr. Appl. Anal.*, volume 2012. Hindawi Publishing Corporation, 2012.

[130] O. Szász. Generalization of S. Bernstein's polynomials to the infinite interval. *J. Res. Nat. Bur. Standards*, 45:239–245, 1950.

[131] F. Taşdelen, R. Aktaş, and A. Altın. A Kantorovich type of Szász operators including Brenke-type polynomials. In *Abstr. Appl. Anal.*, volume 2012. Hindawi Publishing Corporation, 2012.

[132] S. Varma, S. Sucu, and G. İçöZ. Generalization of Szász operators involving Brenke type polynomials. *Comput. Math. Appl.*, 64(2):121–127, 2012.

[133] S. Varma and F. Taşdelen. Szász type operators involving Charlier polynomials. *Math. Comput. Modelling*, 56(5):118–122, 2012.

[134] T. Vedi. Some Schurer type q-Bernstein operators, Ph. D. Thesis. 2011.

[135] A. Wafi and S. Khatoon. Approximation by generalized Baskakov operators for functions of one and two variables in exponential and polynomial weight spaces. *Thai. J. Math*, 2(2):203–216, 2004.

[136] A. Wafi and S. Khatoon. Direct and inverse theorems for generalized Baskakaov operators in polynomial weight spaces. *An. Ştiinţ. Univ. Al. I. Cuza laşi. Mat*, 50:159–173, 2004.

[137] A. Wafi and S. Khatoon. On the order of approximation of functions by generalized Baskakov operators. *Indian J. Pure Appl. Math*, 35(3):347–358, 2004.

[138] A. Wafi and S. Khatoon. Convergence and Voronovskaja-type theorems for derivatives of generalized Baskakov operators. *Open Math. J.*, 6(2):325–334, 2008.

[139] Z. Walczak. On approximation by modified Szász-Mirakyan operators. *Glas. Mat. Ser III*, 37(2):303–319, 2002.

[140] S. Wigert. *Sur l'approximation par Polynomes des Fonctions Continues*. Almqvist & Wiksell, 1932.

[141] I. Yuksel and N. Ispir. Weighted approximation by a certain family of summation integral-type operators. *Comput. Math. Appl.*, 52(10):1463–1470, 2006.