BIORTHOGONALITY OF R_I, R_{II} POLYNOMIALS AND COMPLEMENTARY CHAIN SEQUENCES

Ph. D. THESIS

by

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A THESIS

Submitted in partial fulfilment of the requirements for the award of the degree

of

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CANDIDATE'S DECLARATION

I hereby certify that the work which is being presented in the thesis entitled "**BIORTHOGONALITY OF R_I, R_{II} POLYNOMIALS AND COMPLEMENTARY CHAIN SEQUENCES**" in partial fulfilment of the requirements for the award of the Degree of Doctor of Philosophy and submitted in the Department of Mathematics of the Indian Institute of Technology Roorkee, Roorkee is an authentic record of my own work carried out during a period from January, 2013 to April, 2018 under the supervision of Dr. A. Swaminathan, Associate Professor, Department of Mathematics, Indian Institute of Technology Roorkee, Roorkee.

The matter presented in this thesis has not been submitted by me for the award of any other degree of this or any other Institution.

(KIRAN KUMAR BEHERA)

This is to certify that the above statement made by the candidate is correct to the best of my knowledge.

(A. Swaminathan)
Supervisor

Date:**April , 2018**

Dedicated

to

My Parents

Dr. Shreedhar Prasad Behera and Mrs. Bichitra Behera

(*who allowed me to follow my heart,*

get lost in the middle and find my way back again)

Abstract

The classical theory of orthogonal polynomials has found several applications in recent years, particularly, in the areas of spectral theory and mathematical physics. Many advancements are obtained through the matrix representations and associated eigenvalue problems of the orthogonal polynomials as well as the continued fraction expansions of the special functions that arise in such studies. The underlying theme of the thesis on one hand is to explore both structural and qualitative aspects of perturbations of the continued fraction parameters in case of special functions and recursion coefficients in case of orthogonal polynomials and on the other hand to obtain biorthogonality relations of the related functions.

The structural and qualitative aspects of two perturbations in the parameters of a g-fraction is studied. The first perturbation is when a finite number of parameters g_j are missing in which case we call the corresponding g-fraction a *gap-g-fraction*. Using one of the gap-g-fractions, a class of Pick functions is identified. The second case is replacing $\{g_n\}_{n=0}^{\infty}$ by a new sequence $\{g_n^{(\beta_k)}\}_{n=0}^{\infty}$ in which the j^{th} term g_j is replaced by $g_j^{(\beta_k)}$ and the results are illustrated using the Schur and Carathéodory functions.

The consequences of the map $m_n \mapsto 1 - m_n$, where m_n, $n \geq 0$, is the minimal parameter sequence of a chain sequence, are explored in case of polynomials orthogonal both on the real line and on the unit circle. In this context, the concept of complementary chain sequence is introduced. It is shown, in particular, how the map can be useful in characterizing chain sequences with a single parameter sequences.

The map $\mathcal{F}(\lambda) \mapsto \mathcal{F}(\lambda^2)$, where $\mathcal{F}(\lambda)$ is a general T-fraction, is used to define generalized Jacobi pencil matrices. The denominators of the approximants of a T-fraction satisfy a recurrence relation of R_I type, with which is associated a sequence of Laurent polynomials. The biorthogonality relations of these Laurent polynomials are

discussed. This serves as the bridge between the two parts of the thesis and provides a gradual transition from perturbation theory to the concept of biorthogonality.

The sequence $\{\mathcal{Q}_n(\lambda)\}_{n=0}^{\infty}$, where $\mathcal{Q}_n(\lambda) := \mathcal{P}_n(\lambda)+\alpha_n\mathcal{P}_{n-1}(\lambda)$, $\alpha_n \in \mathbb{R}\backslash\{0\}$, $n \geq 0$, is considered with $\{\mathcal{P}_n(\lambda)\}_{n=0}^{\infty}$ satisfying

$$\mathcal{P}_{n+1}(\lambda) = \rho_n(\lambda - \beta_n)\mathcal{P}_n(\lambda) + \tau_n(\lambda - \gamma_n)\mathcal{P}_{n-1}(\lambda), \quad n \geq 1.$$

A unique sequence $\{\alpha_n\}_{n=0}^{\infty}$ is constructed such that $\{\mathcal{Q}_n(\lambda)\}_{n=0}^{\infty}$ not only satisfies mixed recurrence relations of R_I and R_{II} type but also $\mathcal{Q}_n(1) = 0$, $\forall n \geq 1$. The polynomials $\mathcal{Q}_n(\lambda)$, $n \geq 1$, are shown to satisfy biorthogonality relations with respect to a discrete measure that follows from their eigenvalue representations. With certain additional conditions, a para-orthogonal polynomial is also obtained from $\mathcal{Q}_n(\lambda)$.

The recurrence relation of R_{II} type

$$\mathcal{O}_{n+1}(\lambda) = \rho_n(\lambda - \nu_n)\mathcal{O}_n(z) + \tau_n(\lambda - a_n)(\lambda - b_n)\mathcal{O}_{n-1}(\lambda), \qquad n \geq 1,$$

is used to construct a sequence of orthogonal rational functions $\{\varphi_n(\lambda)\}$ satisfying two properties. First, the related matrix pencil has the numerator polynomials $\mathcal{O}_n(\lambda)$ as the characteristic polynomials and $\varphi_n(\lambda)$ as components of the eigenvectors. Second, the orthogonal sequence $\{\varphi_n(\lambda)\}$ is also biorthogonal to another sequence of rational functions. A Christoffel type transformation of the orthogonal rational functions so constructed is also obtained, illustrating the differences with the results available in the literature.

There is a conscious effort to give the results obtained in the thesis a proper context in the vast theory of orthogonal polynomials and biorthogonality. At the same time, future direction of research is also provided wherever possible.

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I was introduced to the area of orthogonal polynomials and special functions by Dr. A Swaminathan who is my supervisor for both Master's dissertation and Ph.D. thesis. It has been a journey of ten years and I learnt a lot from him. I thank him for his patience and insights into the ways to do a fruitful research.

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I am indebted to my parents Dr. Shreedhar Prasad Behera and Mrs. Bichitra Behera and sister Dr. Pragyan Padmaja Behera for their constant encouragement and support. They always gave me the freedom to choose my own path to destiny and I hope I have found the right way forward.

Contents

List of Symbols

x	Real variable
z	Complex variable
λ	Real or complex variable
$\mathbb{R}$	Real line
$\mathbb{C}$	Complex plane
$\mathbb{D}$	Unit disc $\{z : \lvert z\rvert < 1\}$
$\partial\mathbb{D}$	Unit circle $\{z : \lvert z\rvert = 1\}$
$f(z)$	Schur function
$\mathcal{C}(z)$	Carathéodory function
$\mathcal{F}(\lambda)$	Analytic function represented by a continued fraction
d_n, a_n	Chain sequences
m_n, k_n	Minimal parameters of chain sequence
M_n	Maximal parameters of chain sequence
$\mathrm{P}_n(x)$	Orthogonal polynomial on real line of degree n
$\mathrm{K}_n(x)$	Kernel polynomial for $\mathrm{P}_n(\lambda)$
$\mathcal{S}_n(x)$	Symmetric polynomial on real line
$R_n(z)$	Para-orthogonal polynomial on the unit line
$\Phi_n(z)$	Szegő polynomial of degree n
$\phi_n(z)$	Orthonormal Szegő polynomials
$\varphi(z)$	Orthogonal rational function
$\hat{\varphi}(z)$	Christoffel type transform of orthogonal rational function
$\mathcal{G}, \mathcal{H}$	Tridiagonal matrices
$\mathcal{G}_n, \mathcal{H}_n$	n^{th} principal minor of $\mathcal{G}, \mathcal{H}$ respectively

J	Jacobi matrix
$\mathcal{J}^{L(R)}(\lambda)$	Generalized Jacobi pencil matrix
$\boldsymbol{\varrho}$	Eigen vector with rational functions as components
σ_i^R, σ_i^L	Components of right and left eigenvectors
Π_n	Space of polynomials of degree at most n
$\bigwedge$	Space of Laurent polynomials
$\mathcal{L}$	Space of rational functions
$\mathcal{N}$	Moment functional on space of Laurent polynomials
$\mathfrak{N}$	Moment functional on space of rational functions
$\mathfrak{N}_o$, $\mathfrak{N}_e$	Moment functionals for Christoffel type transforms
$\Gamma^{e(o)}$, $\Lambda^{e(o)}$	Matrix operators on space of rational functions
α_n, β_n,	Poles of the rational functions
γ_n, ρ_n, ν_n	Parameters used in recurrence relations
κ_n	Leading coefficient of a polynomial

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Chapter 1

Introduction

1.1 Orthogonal Polynomials

The theory of orthogonal polynomials is a classical one and has found several applications, among others, in numerical interpolation and quadrature (Gautschi [79]), dynamical systems and control theory (Datta and Mohan [49]), rational approximation (Nikishin and Sorokin [136], Bultheel and Barel [31]), special functions (Askey [6]), non-linear differential equations (Van Assche [178]), random matrix theory and integrable systems (Deift [57]), statistical quantum mechanics, spectral theory and other areas of mathematical physics (Simon [158]). There have been special applications of orthogonal polynomials in analytic function theory, particularly, de Branges proof [50] of the Bieberbach conjecture and finding zero free regions of polynomials (Lewis [121]).

While the theory of orthogonal polynomials on the real line can be traced to the initial developments in the areas of planetary motion and continued fractions (Khrushchev [106]), the study of orthogonal polynomials on the unit circle was initiated by Szegő [167,168]. We refer to the monographs of Freud [74], Geronimus [81]and Szegő [169] for early developments in the area of orthogonal polynomials on the unit circle, while a compendium of modern research in the area as well as historical notes, can be found in the two volumes of Simon [156,157].

In this section, we give a brief overview of basic properties of polynomials orthogonal on both the real line and the unit circle. These properties, among several other related concepts will be mainly used in the thesis.

1.1.1 Orthogonal Polynomials on the real line (OPRL)

Let N be a linear functional defined on the linear space $\mathbb{P}$ of polynomials with complex coefficients. Defining the moments associated with N as $\mu_n = \mathrm{N}(x^n)$, the Gram matrix $[\mathrm{N}(x^{i+j})]_{i,j=0}^{\infty}$ associated with N is given by

$$\boldsymbol{H} = \begin{pmatrix} \mu_0 & \mu_1 & \cdots & \mu_n & \cdots \\ \mu_1 & \mu_2 & \cdots & \mu_{n+1} & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mu_n & \mu_{n+1} & \cdots & \mu_{2n} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}.$$

If the determinants Δ_n of the principal leading $(n+1)\times(n+1)$ submatrices of $\boldsymbol{H}$ are such that $\Delta_n \neq 0$, $n \geq 0$, then N is called quasi-definite. In such a case, a sequence $\{\mathrm{P}_n(x)\}_{n=0}^{\infty}$ of unique (up to a non-zero factor) polynomials can be defined satisfying

$$\mathrm{N}(\mathrm{P}_n(x)\,\mathrm{P}_m(x)) = \boldsymbol{h}_n\delta_{n,m}, \qquad \boldsymbol{h}_n \neq 0.$$

The polynomial $\mathrm{P}_n(x)$, $n \geq 1$, is of degree n and $\{\mathrm{P}_n(x)\}_{n=0}^{\infty}$ is called a sequence of orthogonal polynomials on the real line (OPRL). In case the leading coefficient of $\mathrm{P}_n(x)$ is unity, $\{\mathrm{P}_n(x)\}_{n=0}^{\infty}$ is called a sequence of monic OPRL.

We would like to note that, in general, for a $m \times n$ matrix H, the matrix H^*H is called the Gram matrix of H and many of their properties, particularly their inverse, are studied, for example, in Kurmayya and Ramesh [116] and Reddy and Kurmayya [147]. We would also like to refer to Kulkarni and Sukumar [112, 113] for spectral analysis in associative algebras.

Three term recurrence relation

A fundamental property of a sequence of monic OPRL is that any three polynomials of consecutive degrees are connected by a linear relation of the form

$$\mathrm{P}_{n+1}(x) = (x - b_{n+1})\mathrm{P}_n - a_n^2\mathrm{P}_{n-1}, \quad n \geq 0 \tag{1.1.1}$$

with the initial conditions $\mathrm{P}_{-1}(x) = 0$ and $\mathrm{P}_0(x) = 1$. This is because the polynomial $x\mathrm{P}_n(x)$ belongs to the space $\mathbb{P}_{n+1}$ of polynomials of degree at most $n+1$, which is spanned by the orthogonal basis $\{\mathrm{P}_0(x), \mathrm{P}_1(x), \cdots, \mathrm{P}_{n+1}(x)\}$. The relation (1.1.1) follows from the orthogonality of the polynomials $\mathrm{P}_n(x)$, $n \geq 0$.

Conversely, if a sequence of polynomials $\{\mathrm{P}_n(x)\}_{n=0}^{\infty}$ satisfies (1.1.1) with the same initial conditions as above and where $\{b_n\}_{n=1}^{\infty}$ and $\{a_n^2\}_{n=1}^{\infty}$ are arbitrary sequences of complex numbers with $a_n^2 \neq 0$, $n \geq 0$, then, there exists a quasi-definite functional N with $\{\mathrm{P}_n(x)\}_{n=0}^{\infty}$ as its corresponding sequence of OPRL. This result is referred in the literature as Favard's Theorem.

Existence of a positive measure

The functional N is said to be positive definite if $\det \Delta_n > 0$. In this case, there exists a non-trivial positive measure $d\mu(x)$ supported on some subset $\mathbb{E}'$ of the real line such that N has the integral representation

$$\mathrm{N}(x^n) = \int_{\mathbb{E}'} x^n d\mu(x), \quad n \geq 1.$$

Favard's Theorem holds for a positive definite functional if, and only if $b_n \in \mathbb{R}$ and $a_n^2 > 0$, $n \geq 0$, where b_n and a_n^2 occur in (1.1.1). Further, $\{\mathrm{P}_n(x)\}_{n=0}^{\infty}$ is the monic sequence of OPRL with respect to N.

Jacobi matrix

The recurrence relation (1.1.1) written as

$$x\mathrm{P}_n(x) = \mathrm{P}_{n+1}(x) + b_{n+1}\mathrm{P}_n(x) + a_n^2\mathrm{P}_{n-1}(x), \quad n \geq 0,$$

yields the matrix representation $x\mathbf{P}(x) = \boldsymbol{J}\mathbf{P}(x)$, where

$$\boldsymbol{J} := \begin{pmatrix} b_1 & 1 & 0 & \cdots \\ a_0^2 & b_2 & 1 & \cdots \\ 0 & a_1^2 & b_3 & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix} \quad \text{and} \quad \boldsymbol{P} := \begin{pmatrix} \mathrm{P}_0(x) & \mathrm{P}_1(x) & \mathrm{P}_2(x) & \cdots \end{pmatrix}.$$

The matrix $\boldsymbol{J}$ is called the monic Jacobi matrix associated with N. It is also the matrix representation of the multiplication operator $x \mapsto x\mathrm{p}(x)$ with respect to the basis $\{1, x, x^2, \cdots\}$. A fundamental property of the matrix $\boldsymbol{J}$ is that $\mathrm{P}_n(x)$ is the characteristic polynomial of the $n \times n$ leading principal submatrix $\boldsymbol{J}_n$ of $\boldsymbol{J}$, so that the eigen-values of $\boldsymbol{J}_n$ are precisely the zeros of $\mathrm{P}_n(x)$, $n \geq 1$.

Interlacing of zeros

Let the functional N be such that $\mathrm{N}(\pi(x)) > 0$ for every real polynomial $\pi(x)$ which is non-negative on $\mathbb{E} \subseteq (-\infty, \infty)$ and which does not vanish identically on $\mathbb{E}$. Then N is said to be positive definite on the interval $\mathbb{E}$. In this case, the zeros of the associated OPRL $\mathrm{P}_n(x)$, $n \geq 1$, are real, simple and lie in the interior of $\mathbb{E}$. Further, if $\{x_{n,j}\}_{j=1}^{n}$ denote the zeros of $\mathrm{P}_n(x)$, then, the following relation

$$x_{n+1,1} < x_{n,1} < x_{n+1,2} < \cdots < x_{n+1,n} < x_{n,n} < x_{n+1,n+1}, \quad n \geq 1, \tag{1.1.2}$$

called as the interlacing of zeros of $\mathrm{P}_n(x)$ and $\mathrm{P}_{n+1}(x)$ holds. Moreover, the interval $[\xi_1, \eta_1]$, where $\xi_i = \lim_{n\to\infty} x_{n,i}$ and $\eta_j = \lim_{n\to\infty} x_{n,n-j+1}$, $i, j = 1, 2, \cdots$, is called the true interval of orthogonality of the sequence of OPRL and is the smallest closed interval containing all the zeros of all the polynomials $\mathrm{P}_n(x)$.

Conversely, if (1.1.2) holds, Wendroff [185] proved that these zeros can be embedded in an orthogonal sequence, that is, there exist a sequence $\{\mathrm{P}_k(x)\}_{k=0}^{\infty}$, such that $\mathrm{P}_n(x) = (x - x_{n,1}) \cdots (x - x_{n,n})$ and $\mathrm{P}_{n+1}(x) = (x - x_{n+1,1}) \cdots (x - x_{n+1,n+1})$. We also refer to Beardon et al. [15] who investigated the relation between the zeros and the recursion coefficients for such embedding to occur.

Kernel polynomials

Let $\mathrm{p}_n(x)$ be the orthonormal polynomial obtained from $\mathrm{P}_n(x)$, $n \geq 1$. The polynomials

$$\mathcal{K}_n(y, x) = \sum_{i=0}^{n} \mathrm{p}_i(y)\mathrm{p}_i(x), \qquad n \geq 1,$$

where $\mathrm{p}_n(x)$ is a real polynomial and $y \in \mathbb{R}$, are called kernel polynomials. Further, they have the reproducing property that, for any polynomial $\pi(x)$, $\pi(t) = \mathrm{N}(\pi(x)\mathcal{K}_n(t, x))$, where N is positive definite and operates on x. The monic form $\mathrm{K}_n(y,x)$ of the kernel polynomials $\mathcal{K}_n(t,x))$ is given by

$$\mathrm{K}_n(y,x) = \frac{\mathrm{P}_{n+1}(x) - \frac{\mathrm{P}_{n+1}(y)}{\mathrm{P}_n(y)}\mathrm{P}_n(x)}{x-y}, \quad n \geq 1.$$

This follows from the fact that $\mathrm{K}_n(y,x)$, $n \geq 1$, are orthogonal with respect to the functional N^*_y, where

$$\mathrm{N}^*_y(x^n) = \mu_{n+1} - \mu_n; \quad \mathrm{N}^*_y(\pi(x)) = \mathrm{N}[(x-y)\pi(x)].$$

Symmetric OPRL

Let N be a functional defined as $\mathrm{N}(x^n) = \mathrm{S}(x^{2n})$, where S is a symmetric quasi-definite functional (all odd ordered moments are zero). If $\{\mathcal{S}_n(x)\}_{n=0}^{\infty}$ is the sequence of monic OPRL with respect to S, then

$$\mathcal{S}_{2m}(x) = \mathrm{P}_m(x^2) \quad \text{and} \quad \mathcal{S}_{2m+1}(x) = x\mathrm{K}_m(x^2), \quad m \geq 0, \tag{1.1.3}$$

where $\{\mathrm{P}_n(x)\}_{n=0}^{\infty}$ and $\{\mathrm{K}_n(x) := \mathrm{K}_n(0,x)\}_{n=0}^{\infty}$ are respectively, the sequences of monic OPRL and the corresponding kernel polynomials with respect to N.

Conversely, with N a quasi-definite functional, let a symmetric moment functional S be defined by $\mathrm{S}(x^{2i}) = \mathrm{N}(x^i)$, $\mathrm{S}(x^{2i+1}) = 0$, $i = 0, 1, \cdots$. Let $\{\mathrm{P}_n(x)\}_{n=0}^{\infty}$ be the sequence of monic OPRL with respect to N and $\{\mathrm{K}_n(x)\}_{n=0}^{\infty}$ be the corresponding kernel polynomials. If $\{\mathcal{S}_n(x)\}_{n=0}^{\infty}$ is a new sequence of monic polynomials defined as in (1.1.3) then, $\mathrm{S}(\mathcal{S}_{2m}(x)\mathcal{S}_{2n+1}(x)) = 0$, $m, n = 0, 1, \cdots$, which shows that $\{\mathcal{S}_n(x)\}$ is the sequence of monic OPRL with respect to S.

Classical orthogonal polynomials

The classification of classical orthogonal polynomials initially arose in Bochner's [20] investigation of polynomial solutions of degree n of the Sturm-Liouville equation

$$a_2(x)y''(x) + a_1(x)y'(x) + a_0(x)y(x) + \lambda y(x) = 0,$$

for each eigenvalue $\lambda = \lambda_n$, $n \geq 0$.

After several classification theorems provided for example, in Kwon and Littlejohn [118], Marcellán et al. [128] and references therein, the Hermite, Laguerre and Jacobi polynomials orthogonal on the real line with respect to the normal, gamma and beta distributions respectively are characterized as the classical orthogonal polynomials. The classical orthogonal polynomials satisfy several common properties. They include the following:

- They satisfy a second order Sturm-Liouville differential equation of the form $\mathfrak{a}_2(x)y''(x) + \mathfrak{a}_1(x)y'(x) + \lambda_n y(x) = 0$, where $\mathfrak{a}_2(x)$ is a polynomial of degree at most two, $\mathfrak{a}_1(x)$ is linear polynomial and λ_n depends only on n.

- They are orthogonal with respect to a positive weight function $\omega(x)$ which satisfies Pearson's differential equation $\mathfrak{p}(x)\omega'(x) = \mathfrak{q}(x)\omega(x)$, where $\mathfrak{p}(x)$ and $\mathfrak{q}(x)$ are polynomials of degree at most two and of exact degree one respectively.

- They satisfy the Rodrigue's formula $\frac{d^n}{dx^n}[\mathfrak{p}^n(x)\omega(x)] = e_n\omega(x)P_n(x)$, $n = 0, 1, \cdots$, where e_n is a normalization constant.

- Their derivatives also form an orthogonal sequence but with different parameters.

For various results in this direction, we refer to Atia and Alaya [9], Littlejohn [123], Vinet and Zhedanov [182] and references therein. We would also like to add that the Hermite and Laguerre polynomials are examples of Sheffer sequences arising in combinatorics. The Sheffer sequences are studied, for example, in Jana et al. [95] and Rapeli et al. [145].

1.1.2 Orthogonal Polynomials on the unit circle (OPUC)

Let $\mathfrak{N}$ be a Hermitian linear moment functional defined on the linear space of Laurent polynomials $\bigwedge = \text{span } \{z^n\}_{n\in\mathbb{Z}}$. Then, an inner product associated with $\mathfrak{N}$ can be defined on $\bigwedge$ as

$$\langle \mathfrak{p}(z), \mathfrak{q}(z)\rangle_{\mathfrak{N}} = \langle \mathfrak{N}, \mathfrak{p}(z)\overline{\mathfrak{q}(1/\bar{z})}\rangle, \quad \mathfrak{p}(z), \mathfrak{q}(z) \in \bigwedge.$$

The sequence of complex numbers $\{c_n\}_{n=-\infty}^{\infty}$ defined by

$$c_n = \langle z^n, 1\rangle_{\mathfrak{N}} = \langle \mathfrak{N}, z^n\rangle = \overline{\langle \mathfrak{N}, z^{-n}\rangle} = \bar{c}_{-n}, \quad n \in \mathbb{Z},$$

is said to be the moment sequence associated with $\mathfrak{N}$. The Gram matrix $[\langle z^i, z^j\rangle_{\mathfrak{N}}]_{i,j=0}^{\infty}$ associated with the functional $\mathfrak{N}$ with respect to the basis $\{z^n\}_{n=0}^{\infty}$ of the linear space $\mathbb{P}$ is given by

$$\boldsymbol{T} = \begin{vmatrix} c_0 & c_1 & \cdots & c_n & \cdots \\ c_{-1} & c_0 & \cdots & c_{n-1} & \cdots \\ \vdots & \vdots & \vdots & \ddots & \cdots \\ c_{-n} & c_{-n+1} & \cdots & c_0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{vmatrix},$$

and is known in the literature as Toeplitz matrix. We note that $\mathfrak{N}$ being a Hermitian functional implies that $\mathbf{T}$ is Hermitian. If $\det \boldsymbol{T}_n \neq 0$, $n \geq 0$, where $\boldsymbol{T}_n$ is the $(n+1)\times(n+1)$ principal leading submatrix of $\boldsymbol{T}$, then, there exists a sequence $\{\Phi_n(z)\}_{n=0}^{\infty}$ of monic polynomials satisfying

$$\langle \Phi_n(z), \Phi_m(z)\rangle_{\mathfrak{N}} = \boldsymbol{t}_n\delta_{n,m}, \quad \boldsymbol{t}_n \neq 0, \quad n \geq 0,$$

and hence orthogonal with respect to $\mathfrak{N}$.

Existence of a positive measure

If $c_0 = 1$ and $\det \boldsymbol{T}_n > 0$, $n \geq 0$, then $\mathfrak{N}$ is said to be positive definite. There exists a positive measure $d\mu(z)$ supported on the unit circle $\partial\mathbb{D}$ such that the sequence $\{\Phi_n(z)\}_{n=0}^{\infty}$ with positive leading coefficient satisfy the orthogonality property

$$\int_{\partial\mathbb{D}} (\bar{z})^j \Phi_n(z) d\mu(z) = \int_{\partial\mathbb{D}} (z)^{-j} \Phi_n(z) d\mu(z) = 0 \quad j = 0, 1, \ldots, n-1, \quad n \geq 1,$$

on the unit circle and are also called Szegő polynomials. Further, the moments are defined as $\mu_n = \int_{\partial\mathbb{D}} e^{-in\theta} d\mu(\theta)$, $n = 0, \pm 1, \ldots$, where $\mu_{-n} = \bar{\mu}_n$.

Szegő recurrence

The monic Szegő polynomials satisfy the first order recurrence relations

$$\Phi_n(z) = z\Phi_{n-1}(z) - \bar{\alpha}_{n-1}\Phi^*_{n-1}(z), \quad \Phi^*_n(z) = -\alpha_{n-1}z\Phi_{n-1}(z) + \Phi^*_{n-1}(z), \quad n \geq 1,$$

where $\Phi^*_n(z) = z^n\overline{\Phi_n(1/\bar{z})}$. The complex numbers $\alpha_{n-1} = -\overline{\Phi_n(0)}$ are called Verblunsky coefficients in Simon [156]. The Verblunsky coefficients completely characterize the Szegő polynomials in the sense that any sequence $\{\alpha_{n-1}\}_{n=1}^{\infty}$ lying within the unit circle gives rise to a unique probability measure $\mu(z)$ which leads to a unique sequence of Szegő polynomials. The above result, called the Verblunsky theorem in Simon [156], is the analogue of Favard's theorem on the real line.

The Szegő polynomials also satisfy the three term recurrence relation

$$\Phi_{n+1}(z) = \left(\frac{\Phi_{n+1}(0)}{\Phi_n(0)} + z\right)\Phi_n(z) - \frac{(1-|\Phi_n(0)|^2)\Phi_{n+1}(0)}{\Phi_n(0)} z\Phi_{n-1}(z), \quad n \geq 1, \tag{1.1.4}$$

with $\Phi_0(z) = 1$ and $\Phi_1(z) = z + \Phi_1(0)$. We note that if $\Phi_n(0) = 0$, $n \geq 1$, then the three term recurrence relation ceases to exist. In such a case, we have $\Phi_n(z) = z^n$, which is given as the free case in Simon [156, p. 85]. Further, $\langle \Phi_n(z), \Phi_n(z)\rangle = \left\|\Phi_n(z)\right\|^2 = \boldsymbol{t}_n$, which using (1.1.4) is given by

$$\boldsymbol{t}_n^2 = \left(1 - |\Phi_1(0)|^2\right)\left(1 - |\Phi_2(0)|^2\right)\cdots\left(1 - |\Phi_n(0)|^2\right), \quad n \geq 1. \tag{1.1.5}$$

Para-orthogonal Polynomials

In order to develop a quadrature formula on the unit circle, Jones et al. [101] introduced the para-orthogonal polynomials which vanish only on the unit circle. With $|\tau_n| = 1$ and $|\omega_n| = 1$, the para-orthogonal polynomials have the representation

$$\mathcal{X}_n(z, \omega_n) = \Phi_n(z) + \tau_n\Phi^*_n(z), \quad n \geq 1.$$

and satisfy the orthogonality properties

$$\langle \mathcal{X}_n, z^m\rangle = 0, \quad m = 1, 2, \ldots, n-1, \quad \langle \mathcal{X}_n, 1\rangle \neq 0, \quad \langle \mathcal{X}_n, z^n\rangle \neq 0.$$

The para-orthogonality in Jones et al. [101] was proved using the concept of τ_n-invariant polynomials. We note that a sequence of polynomials $\{\mathcal{Y}_n\}$ is called τ_n-invariant if $\mathcal{Y}_n^*(z) = \tau_n \mathcal{Y}_n(z)$, $n \geq 1$.

Kernel polynomials

The kernel polynomials $K_n(z, \omega)$ satisfy the Christoffel–Darboux formula

$$K_n(z,\omega) = \sum_{k=0}^{n} \phi_k(z)\overline{\phi_k(\omega)} = \frac{\phi_{n+1}^*(z)\overline{\phi_{n+1}^*(\omega)} - \phi_{n+1}(z)\overline{\phi_{n+1}(\omega)}}{1 - z\bar{\omega}}. \tag{1.1.6}$$

Let $\tau_n(\omega) = \Phi_n(\omega)/\Phi_n^*(\omega)$, $n \geq 1$, where $|\omega| = 1$. The monic kernel polynomials related to the Szegő polynomials is given by

$$P_n(\omega; z) = \frac{z\Phi_n(z) - \omega\tau_n(\omega)\Phi_n^*(z)}{z - \omega}, \quad n \geq 1, \tag{1.1.7}$$

which satisfy a three term recurrence relation (Costa et al. [40]) of the form

$$P_{n+1}(\omega; z) = [z + b_{n+1}(\omega)]P_n(\omega; z) - a_{n+1}(\omega) z P_{n-1}(\omega; z), \quad n \geq 1, \tag{1.1.8}$$

where $b_n(\omega) = \frac{\tau_n(\omega)}{\tau_{n-1}(\omega)}$ and $a_{n+1} = [1 + \tau_n(\omega)\alpha_{n-1}]\big[1 - \overline{\omega\tau_n(\omega)\alpha_n}\big]\omega$, $n \geq 1$. The polynomials $P_n(\omega; z)$ are $\overline{\tau_n(w)}$-invariant sequences of polynomials which can be easily verified from (1.1.7).

Carathéodory functions

Let $\mathcal{C}(z)$ be a complex valued function defined as

$$\mathcal{C}(z) = \left\langle \mathfrak{N}, \frac{\omega + z}{\omega - z} \right\rangle = \int_{\partial\mathbb{D}} \frac{\omega + z}{\omega - z} d\mu(z),$$

where the integral representation holds if $\mathfrak{N}$ is a positive definite functional and is called the Riesz-Herglotz transformation of the measure $d\mu(z)$. The function $\mathcal{C}(z)$ for $|z| < 1$ is called a Carathéodory function with $\mathrm{Re}(\mathcal{C}(z)) > 0$ for $|z| < 1$. Further, $\mathcal{C}(z)$

has the power series representation

$$\mathcal{C}(z) = 1 + 2\sum_{k=1}^{\infty} c_{-k} z^k, \quad |z| < 1,$$

where $\{c_{-k}\}$ are the moments associated with the functional $\mathfrak{N}$.

1.2 Special functions and continued fraction representations

Let $\mathrm{A}_n(\omega)$ and $\mathrm{B}_n(\omega)$ be, respectively, the numerator and denominator of the n^{th} approximant of the continued fraction

$$f_n(\omega) = b_0(\omega) + \cfrac{a_1(\omega)}{b_1(\omega) + \cfrac{a_2(\omega)}{b_2(\omega) + \ddots}} = b_0(\omega) + \frac{a_1(\omega)}{b_1(\omega)} \underset{+}{} \frac{a_2(\omega)}{b_2(\omega)} \underset{+}{} \cdots,$$

that is

$$\frac{\mathrm{A}_n(\omega)}{\mathrm{B}_n(\omega)} = b_0(\omega) + \frac{a_1(\omega)}{b_1(\omega)} \underset{+}{} \frac{a_2(\omega)}{b_2(\omega)} \underset{+}{} \cdots \underset{+}{} \frac{a_n(\omega)}{b_n(\omega)}.$$

With the initial values $A_{-1}(\omega) = 1$, $A_0(\omega) = b_0(\omega)$, $B_{-1}(\omega) = 0$ and $B_0(\omega) = 1$, they satisfy the linear difference equations

$$\begin{aligned}
\mathrm{A}_n(\omega) &= b_n(\omega)A_{n-1}(\omega) + a_n(\omega)A_{n-2}(\omega), \\
\mathrm{B}_n(\omega) &= b_n(\omega)B_{n-1}(\omega) + a_n(\omega)B_{n-2}(\omega), \quad n = 1, 2\cdots,
\end{aligned}$$

as well as the determinant formula

$$\mathrm{A}_n(\omega)\mathrm{B}_{n-1}(\omega) - \mathrm{B}_n(\omega)\mathrm{A}_{n-1}(\omega) = (-1)^{n-1}\prod_{k=1}^{n} a_k(\omega) \neq 0, \quad n = 1, 2, \cdots. \tag{1.2.1}$$

The determinant formula (1.2.1) plays a fundamental role in the *correspondence* of a continued fraction to a power series. The series

$$L = c_m\omega^m + c_{m+1}\omega^{m+1} + c_{m+2}\omega^{m+2} + \cdots, \quad c_m \in \mathbb{C}\setminus\{0\}, \quad m \in \mathbb{Z} \tag{1.2.2}$$

is called a formal Laurent series (fLs) with $L(f)$ denoting the fLs of the function f merormophic at the origin. A sequence $\{\mathrm{R}_n := \mathrm{R}_n(\omega)\}$ of functions meromorphic at the origin is said to correspond to a formal Laurent series L at $\omega = 0$ if

$$\lim_{n\to\infty} \boldsymbol{\lambda}(L - L(\mathrm{R})) = \infty, \quad \text{where} \quad \boldsymbol{\lambda}(L) = \begin{cases} \infty, & L = 0; \\ m, & L \neq 0. \end{cases}$$

Here, m is as defined in (1.2.2). The order of correspondence of $\mathrm{R}_n(\omega)$ is defined to be

$$\boldsymbol{\nu}_n = \boldsymbol{\lambda}(L - L(\mathrm{R}_n)),$$

which signifies that, $L(\mathrm{R}_n)$ and L agrees term by term up to and including the term containing $\omega^{\boldsymbol{\nu}_n - 1}$. Many properties of correspondence of a continued fraction are studied in Jones and Thron [102, Chapter 5]. We state one such fundamental result.

Theorem 1.2.1. [102, Theorem 5.1] *Given a sequence $\{R_m(\omega)\}$ of functions meromorphic at the origin, there exists a formal Laurent series L such that $\{R_m(\omega)\}$ corresponds to L if, and only if*

$$\lim_{n\to\infty} \boldsymbol{\lambda}(L(R_{n+1}) - L(R_n)) = \infty. \tag{1.2.3}$$

If (1.2.3) holds, then the L to which $\{R_n(\omega)\}$ corresponds is determined uniquely. Further, if the sequence $\{\boldsymbol{\lambda}(L(R_{n+1}) - L(R_n))\}$ tends monotonically to infinity, then the order of correspondence of $R_n(\omega)$ is given by $\boldsymbol{\nu}_n = \boldsymbol{\lambda}(L(R_{n+1}) - L(R_n))$.

For a comprehensive study of continued fractions, we refer to the monographs of Jones and Thron [102], Lorentzen and Waadeland [124] and Wall [184].

Now, we give a brief review of the continued fractions that are used in this thesis and the special functions they represent.

Schur function and Schur fractions

The class of Schur functions was studied extensively by J. Schur [153]. A function f which is analytic in the unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ such that $|f(\mathbb{D})| \leq 1$ is called a Schur function. The Schur function is further said to be normalized if $f(0) \in (-1, 1)$.

Using Schwarz's lemma and the fact that disk automorphisms are given by the bilinear transformations $\dfrac{z-\alpha}{1-\bar{\alpha}z}$, $|\alpha| < 1$, Schur gave the "continued fraction like" algorithm

$$f_0(z) = f(z), \quad f_{n+1}(z) = \frac{f_n(z) - \alpha_n}{z(1 - \bar{\alpha}_n f_n(z))}, \quad \alpha_n := f_n(o), \quad n \geq 0, \tag{1.2.4}$$

to obtain a sequence of Schur functions $\{f_n(z)\}$. It is clear that $|\alpha_n| = |f_n(0)| \leq 1$. However, the algorithm terminates if a Schur function f_N is obtained in (1.2.4) such that $|f_N| = 1$.

Wall [184] converted the algorithm (1.2.4) into the continued fraction

$$\alpha_0 + \frac{(1-|\alpha_0|^2)z}{\bar{\alpha}_0 z} \, {}_{+} \, \frac{1}{\alpha_1} \, {}_{+} \, \frac{(1-|\alpha_1|^2)z}{\bar{\alpha}_1 z} \, {}_{+} \, \frac{1}{\alpha_2} \, {}_{+} \, \frac{(1-|\alpha_2|^2)z}{\bar{\alpha}_2 z} \, {}_{+} \, \cdots. \tag{1.2.5}$$

The continued fraction (1.2.5) is called a Schur fraction in Jones et al. [99] and is completely determined by $\{\alpha_n\}_{n=0}^{\infty}$. In case $|\alpha_n| < 1$, $n \geq 0$, the Schur fraction is called positive.

Carathéodory function and g-fractions

Related to the Schur function $f(z)$ are the Carathéodory function $\mathcal{C}(z)$ given by

$$\mathcal{C}(z) = \frac{1 + zf(z)}{1 - zf(z)}, \quad z \in \mathbb{D}. \tag{1.2.6}$$

From (1.2.6), it is clear that $\mathcal{C}(0) = 1$ and $\text{Re}\ \mathcal{C}(z) = (1 - |zf(z)|^2)/(|1 - zf(z)|^2)$. Hence for $|z| < 1$, $|f(z)| \leq 1 \iff \text{Re}\ \mathcal{C}(z) \geq 0$. In fact, (1.2.6) describes an one-one correspondence between the class of Carathéodory functions and the class of Schur functions.

To derive a continued fraction for Carathéodory functions, Wall [183] defined the sequence $\{\gamma_n\}_{n=0}^{\infty}$ satisfying the recurrence relation

$$\gamma_0 := 0, \quad \gamma_{n+1} := \frac{\gamma_n - \bar{\alpha}_n}{1 - \alpha_n \beta_n}, \quad n = 0, 1, \cdots .$$

and introduced the functions $h_n(z)$ as

$$h_n(z) := \frac{1 - \gamma_n f_n(z)}{1 + z\beta_n f_n(z)}, \quad n = 0, 1, \cdots , \tag{1.2.7}$$

usually referred to as the Wall Ansatz (Derevyagin [62]). Here, $f_n(z)$ are the Schur functions obtained from the Schur algorithm (1.2.4) and $\{\alpha_n\}_{n=0}^{\infty}$ are the parameters appearing in the Schur fraction (1.2.5). Using (1.2.4), and the Wall Ansatz (1.2.7) the following continued fraction

$$\mathcal{C}(z) = \frac{1+z}{1-z} + \frac{2(\gamma_0 - \bar{\alpha}_0)z}{\gamma_1 - \gamma_0 z}\,_{+}\, \frac{(\gamma_1 + \bar{\alpha}_0)(\gamma_1 - \bar{\alpha}_1)z}{\gamma_2 - \gamma_1 z}\,_{+}\, \frac{(\gamma_2 + \bar{\alpha}_1)(\gamma_2 - \bar{\alpha}_2)z}{\gamma_3 - \gamma_2 z}\,_{+}\, \cdots ,$$

was obtained for a Carathéodory function $\mathcal{C}(z)$. Further, if $\mathcal{C}(z)$ is such that $\mathcal{C}(\mathbb{R}) \subseteq \mathbb{R}$ and normalized by $\mathcal{C}(0) = 1$, then the following continued fraction expansion

$$\frac{1-z}{1+z}\mathcal{C}(z) = \frac{1}{1}\,_{-}\, \frac{(1-g_0)g_1\omega}{1}\,_{-}\, \frac{(1-g_1)g_2\omega}{1}\,_{-}\, \frac{(1-g_2)g_3\omega}{1}\,_{-}\, \cdots , \quad z \in \mathbb{D}, \tag{1.2.8}$$

can be derived (Wall [183]), where $g_0 = 0$, $g_p = (1 - \alpha_{p-1})/2$, $p = 1, 2, \cdots$ and $w = -4z/(1-z)^2$. Note that $|\alpha_n| \leq 1$ implies $0 \leq g_n \leq 1$ with the continued fraction terminating in case equality holds. In general,the continued fraction appearing in the right hand side of (1.2.8) with $0 \leq g_n \leq 1$, $n = 0, 1, \cdots$, is called the g-fraction and are used, in particular, to represent analytic functions on bounded domains.

Szegő polynomials and PC-fractions

In a series of papers, Jones et al. [99–101], investigated the connection between Szegő polynomials and continued fractions. In this context, the following continued fraction

$$\delta_0 - \frac{2\delta_0}{1}\,_{+}\, \frac{1}{\bar{\delta}_1 z}\,_{+}\, \frac{(1 - |\delta_1|^2)z}{\delta_1}\,_{+}\, \frac{1}{\bar{\delta}_2 z}\,_{+}\, \frac{(1 - |\delta_2|^2)z}{\delta_2 z}\,_{+}\, \cdots . \tag{1.2.9}$$

was introduced and was called Hermitian Perron–Carathéodory fractions or HPC-fractions. They are completely determined by $\delta_n \in \mathbb{C}$, where $\delta_0 \neq 0$ and $|\delta_n| \neq 1$ for $n \geq 1$. Under the stronger conditions $\delta_0 > 0$ and $|\delta_n| < 1$, for $n \geq 1$, (1.2.9) is called a positive PC fraction (PPC-fractions).

Let $\mathrm{A}_n(z)$ and $\mathrm{B}_n(z)$ be respectively the numerator and denominator of the n^{th} approximant of a PPC-fraction. Then, by Jones et al. [101, Theorems 3.1 and 3.2], the Szegő polynomials $\Phi_n(z)$ are precisely the odd ordered denominators $\mathrm{B}_{2n+1}(z)$ while $\Phi_n^*(z)$ are the even ordered denominators $\mathrm{B}_{2n}(z)$. The $\delta_n's$ are then given by $\delta_n = \Phi_n(0)$ and are called the Schur parameters or the reflection coefficients.

For $|\zeta| < 1$, the polynomials $\Psi_n(z) = \int_{\partial\mathbb{D}} \frac{z+\zeta}{z-\zeta}(\Phi_n(z) - \Phi_n(\zeta))d\mu(\zeta)$, $n \geq 1$, are known in the literature as the associated Szegő polynomials or polynomials of the second kind (Geronimus [81]). They arise as the odd ordered numerators of (1.2.9). The function $-\Psi_n^*(z)$ is called the polynomial associated with $\Phi_n^*(z)$ respectively and these $-\Psi_n^*(z)'s$ are the even ordered numerators in (1.2.9). It is also known that for $|z| < 1$, $\mathcal{C}(z) - \frac{\Psi_n^*(z)}{\Phi_n^*(z)} = \mathcal{O}\big(z^{n+1}\big)$, where $\mathcal{C}(z)$ is the Carathéodory function.

In another direction of study of PC fractions, Korteweg-de Vries (KdV) equations related to linear evolution of the orthogonality measure on the unit circle and their integrable discretization leading to an algorithm for a modified PC-fractions are presented in Mukaihira and Nakamura [135]. We refer to Joshi and Srinivasan [105] and Kichenassamy and Srinivasan [107] for information of KdV equations and related Painlevé expansions.

Hypergeometric functions and T- fractions

The Gaussian hypergeometric function is defined by the power series

$$ {}_2F_1(a,b;c;\omega) := F(a,b;c;\omega) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n(1)_n}\omega^n, \quad |\omega| < 1, $$

where $a, b \in \mathbb{C}$, $c \in \mathbb{C} \setminus \{0, -1, -2, \cdots\}$ and

$$ (\lambda)_0 = 1, \quad (\lambda)_n = \lambda(\lambda+1)\cdots(\lambda+n-1) = \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)}, $$

is called the Pochhammer symbol. The series converges absolutely for $|\omega| = 1$ if $\text{Re}\,(c-a-b) > 0$ and converges conditionally if $\omega = e^{i\theta} \neq 1$ and $-1 < \text{Re}\,(c-a-b) \leq 0$. The series diverges if $\text{Re}\,(c-a-b) \leq -1$.

The Gaussian hypergeometric function has the following Euler integral representation

$$F(a,b;c;\omega) = \frac{\Gamma(c)}{\Gamma(c)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-\omega t)^{-a}dt, \quad \text{Re}\,(c) > \text{Re}\,(b) > 0,$$

where $\Gamma(z)$ is the Gamma function (Srinivasan [164]). The Euler integral provides an analytic continuation of $F(a,b;c;\omega)$ to the entire complex plane except for a branch cut along the real axis from 1 to ∞ and has been interpreted as a fractional integral in Andrews et al. [4, Section 2.9]. The Gaussian hypergeometric function is also given by Barnes integral (Srinivasan [165]) which is a contour integral involving products of gamma functions.

The Gaussian hypergeometric function satisfies the second order differential equation

$$\omega(1-\omega)y'' + [c-(a+b+1)\omega]y' - aby = 0,$$

which has three regular singular points at $0, 1$ and ∞ with exponents $0, 1-c$; $0, c-a-b$ and a, b respectively. It is in fact the canonical form (Andrews et al. [4]) of any second order differential equation with three regular singular points.

The Gaussian hypergeometric polynomial as well as the differential equation have been studied in various directions. In case $b, c \in \mathbb{R}$, the orthogonality of the polynomial $F(-n,b;c;\omega)$ is stated in Dominici et al. [67, Theorem 1].

Theorem 1.2.2. [67] *Let $n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, $b, c \in \mathbb{R}$ and $c \neq 0, -1, -2, \cdots$. Then $F(-n,b;c;\omega)$ is the n^{th} degree orthogonal polynomial for the n-dependent positive weight function $|\omega^{c-1}(1-\omega)^{b-c-n}|$ on the intervals*

(i) $(-\infty, 0)$ *for* $c > 0$ *and* $b < 1-n$,

(ii) $(0, 1)$ *for* $c > 0$ *and* $b > c+n-1$,

(iii) $(1, \infty)$ *for* $c+n-1 < b < 1-n$.

As a consequence of orthogonality, $F(-n, b; c; \omega)$ has n zeros, real, simple and lying in its interval of orthogonality for the corresponding ranges. Another context is the properties of the zeros of the solutions of the hypergeometric differential equation, obtained in Deaño et al. [52] and Dimitrov and Sri Ranga [66], through classical Sturm theorems and their extensions.

Two Gaussian hypergeometric functions are said to be contiguous if they are functions of the same variable and the difference between the corresponding parameters is at most unity. A linear relation exists between two contiguous Gaussian hypergeometric functions which can be iterated to yield a linear relation between any three Gaussian hypergeometric functions whose parameters differ by integers. Such relations are called contiguous relations and have been used to explore many properties of the function $F(a, b; c; \omega)$; for example, many special functions can be represented by ratios of Gaussian hypergeometric functions. For more details, we refer to Andrews [4].

In fact, the contiguous relations are used to derive continued fractions of the form

$$\mathcal{F}(\lambda) = \frac{f_1\lambda}{1+g_1\lambda}\,\underset{+}{}\,\frac{f_2\lambda}{1+g_2\lambda}\,\underset{+}{}\,\frac{f_3\lambda}{1+g_3\lambda}\,\underset{+}{}\,\cdots, \qquad \lambda \in \mathbb{C}, \tag{1.2.10}$$

for ratios of Gaussian hypergeometric functions, where $f_n > 0$ and $g_n > 0$. Such continued fractions are called T-fractions and are used in simultaneous correspondence to formal Laurent series at $\lambda = 0$ and $\lambda = \infty$. We refer to Jones and Thron [102]and Wall [184] for further information on the convergence and correspondence properties of such T-fractions.

1.3 Spectral Transformations on unit circle

In the theory of orthogonal polynomials, both on the real line and on the unit circle, different kinds of perturbations are studied leading to new spectral properties of the associated matrices. They are perturbations of either the orthogonality measure or the recurrence coefficients or the moment sequence. In case of perturbations of the orthogonality measure, three canonical transformations are usually studied which include multiplying the measure by a polynomial, addition of one or two mass points, and division of the measure by a polynomial along with the addition of mass point(s).

In this section, we briefly explain spectral transformations both of the functional $\mathfrak{N}$ as well the positive measure that exists in case of the functional being positive definite. However, we restrict ourselves to the unit circle case only.

Canonical transformations of linear functional

Given the linear functional $\mathfrak{N}$ and $\alpha \in \mathbb{C}$, the three transformations usually studied in the literature are

- *Canonical Christoffel transformation*:

$$\langle \mathfrak{p}(z), \mathfrak{q}(z) \rangle_{\mathfrak{N}_C} = \langle (z-\alpha)\mathfrak{p}(z), (z-\alpha)\mathfrak{q}(z) \rangle_{\mathfrak{N}}, \quad \alpha \in \mathbb{C}. \tag{1.3.1}$$

- *Canonical Uvarov transformation*:

$$\begin{aligned} \langle \mathfrak{p}(z), \mathfrak{q}(z) \rangle_{\mathfrak{N}_U} &= \langle \mathfrak{p}(z), \mathfrak{q}(z) \rangle_{\mathfrak{N}} + \mathfrak{m}\mathfrak{p}(\alpha)\overline{\mathfrak{q}(1/\bar{\alpha})} + \bar{\mathfrak{m}}\mathfrak{p}(1/\bar{\alpha})\overline{\mathfrak{q}(\alpha)}, \\ |\alpha| &> 1, \quad \mathfrak{m} \in \mathbb{C} \setminus \{0\}. \end{aligned} \tag{1.3.2}$$

- *Canonical Geronimus transformation*:

$$\langle (z-\alpha)\mathfrak{p}(z), (z-\alpha)\mathfrak{q}(z) \rangle_{\mathfrak{N}_G} = \langle \mathfrak{p}(z), \mathfrak{q}(z) \rangle_{\mathfrak{N}}, \quad \alpha \in \mathbb{C}. \tag{1.3.3}$$

The canonical Uvarov transformation (1.3.2) denotes the addition of two mass points which are symmetric about the unit circle. In case of addition of a single mass point, we have $\mathfrak{m} \in \mathbb{R}$ and

$$\langle \mathfrak{p}(z), \mathfrak{q}(z) \rangle_{\mathfrak{N}_U} = \langle \mathfrak{p}(z), \mathfrak{q}(z) \rangle_{\mathfrak{N}} + \mathfrak{m}\mathfrak{p}(\alpha)\overline{\mathfrak{q}(1/\bar{\alpha})}, \quad \alpha \in \mathbb{C}.$$

Canonical transformations of measure

In the case $\mathfrak{N}$ is a positive definite linear functional, there exist a positive measure μ supported on the unit circle such that

$$c_n = \int_{\partial\mathbb{D}} z^n d\mu(t), \quad n \in \mathbb{Z}, \quad \text{and} \quad \langle \mathfrak{p}(z), \mathfrak{q}(z) \rangle_{\mathfrak{N}} = \int_{\partial\mathbb{D}} \mathfrak{p}(z)\overline{\mathfrak{q}(1/\bar{z})}d\mu(t).$$

The spectral transformation of the orthogonality measure $d\mu$ corresponding to (1.3.1), (1.3.2) and (1.3.3) are

$$d\mu_C = |z-\alpha|^2 d\mu, \quad \alpha \in \mathbb{C}, \tag{1.3.4a}$$

$$d\mu_U = d\mu + \mathfrak{m}\delta(z-\alpha) + \bar{\mathfrak{m}}\delta(z-1/\bar{\alpha}), \quad |\alpha| \in \mathbb{R}_+ \setminus \{0,1\}, \quad \mathfrak{m} \in \mathbb{C}, \tag{1.3.4b}$$

$$d\mu_G = \frac{1}{|z-\alpha|^2} d\mu. \tag{1.3.4c}$$

The canonical Geronimus transformation can be seen to be the inverse of the canonical Christoffel transformation. However, as shown in Marcellán [127], the inverse transform $\mathfrak{N}_G \mapsto \mathfrak{N}_C$ is not unique and they are defined up to the addition of a linear functional $\mathfrak{m}\delta(z-\alpha)+\bar{\mathfrak{m}}\delta(z-1/\bar{\alpha})$. Hence, in the positive definite case, the canonical Geronimus transformation (1.3.4c) can be written as

$$d\mu_{(G,\mathfrak{m})} = \frac{d\mu}{|z-\alpha|^2} + \mathfrak{m}\delta(z-\alpha) + \bar{\mathfrak{m}}\delta(z-1/\bar{\alpha}), \quad |\alpha|>1, \quad \mathfrak{m} \in \mathbb{C}\setminus\{0\}, \tag{1.3.5}$$

with the integral representation

$$\langle \mathfrak{p}(z), \mathfrak{q}(z)\rangle_{\mathfrak{N}_{G,\mathfrak{m}}} = \int_{\partial\mathbb{D}} \mathfrak{p}(z)\overline{\mathfrak{q}(z)}\frac{d\mu}{|z-\alpha|^2} + \mathfrak{m}\mathfrak{p}(1/\bar{\alpha})\overline{\mathfrak{p}(\alpha)} + \bar{\mathfrak{m}}\mathfrak{p}(1/\bar{\alpha})\overline{\mathfrak{q}(\alpha)}.$$

If we denote the canonical Christoffel transformation (1.3.4a), the canonical Uvarov transformation (1.3.4b) and the Geronimus transformation (1.3.5) by $\Upsilon_C(\alpha)$, $\Upsilon_U(\alpha,\mathfrak{m})$ and $\Upsilon_G(\alpha,\mathfrak{m})$ respectively, then

$$\Upsilon_C(\alpha)\circ\Upsilon_G(\alpha,\mathfrak{m}) = \mathcal{I} \quad \text{and} \quad \Upsilon_G(\alpha,\mathfrak{m})\circ\Upsilon_C(\alpha) = \Upsilon_U(\alpha,\mathfrak{m}),$$

where $\mathcal{I}$ is the identity transformation.

The structure of the perturbed orthogonal polynomials as well as the associated Verblunsky coefficients and Carathéodory functions corresponding to the canonical Christoffel, Geronimus and Uvarov transformations are studied, for example, in Garza and Marcellán [77, 78]. We note that in all three cases, the respective perturbed

Carathéodory function $\mathcal{C}_C(z)$, $\mathcal{C}_G(z)$ and $\mathcal{C}_U(z)$ are of the form

$$\frac{\mathfrak{a}(z)\mathcal{C}(z)+\mathfrak{b}(z)}{\mathfrak{c}(z)\mathcal{C}(z)+\mathfrak{d}(z)}$$

where $\mathfrak{a}(z)$, $\mathfrak{b}(z)$, $\mathfrak{c}(z)$ and $\mathfrak{d}(z)$ are polynomials. In case, $\mathfrak{a}(z)\mathfrak{d}(z)-\mathfrak{b}(z)\mathfrak{c}(z)\neq 0$, the spectral transformation of the Carathéodory function is said to be rational. One notable example of such rational spectral transformation is the Aleksandrov transformation of $\mathcal{C}(z)$ defined by

$$\mathcal{C}_{(\lambda)}(z)=\frac{(1-\lambda)+(1+\lambda)\mathcal{C}(z)}{(1+\lambda)+(1-\lambda)\mathcal{C}(z)},\quad |\lambda|=1. \tag{1.3.6}$$

For more information on Aleksandrov transformation, Aleksandrov measures and the related orthogonal polynomials on the unit circle, we refer to Simon [156, p. 35]. The above transformations have also been interpreted in terms of matrix decompositions. We refer to Castillo et al. [38], Daruis et al. [48] Garza and Marcellán [78] and references therein for a matrix perspective of spectral transformations in general.

1.4 Chain Sequences, DG1POP, DG2POP

An important concept that is used in the present work is the theory of chain sequences. We give a brief introduction to chain sequences and then illustrate the role played by them in the theory of orthogonal polynomials on the unit circle. A sequence $\{d_n\}_{n=1}^{\infty}$ which satisfies

$$d_n=(1-g_{n-1})g_n,\qquad n\geq 1,$$

is called a positive chain sequence (Chihara [42]) if $\{g_n\}_{n=0}^{\infty}$, called the parameter sequence, is such that $0\leq g_0<1$, $0<g_n<1$ for $n\geq 1$. This is a stronger condition than the one used in Wall [184], where $0\leq g_n\leq 1$, $n\geq 0$. The parameter sequence $\{g_n\}_{n=0}^{\infty}$ is called a minimal parameter sequence and denoted by $\{m_n\}_{n=0}^{\infty}$ if $m_0=0$. Further, for a fixed chain sequence $\{d_n\}_{n=1}^{\infty}$, let M be the set of all parameter sequences

$\{g_k\}$ of $\{d_n\}_{n=1}^{\infty}$. Let the sequence $\{M_n\}_{n=0}^{\infty}$ be defined by

$$M_n = \sup\{g_n, \text{ for each } n, \{g_k\} \in \mathrm{M}\}, \quad n \geq 0,$$

where sup is supremum of the set. Then, $\{M_n\}_{n=0}^{\infty}$ is called the maximal parameter sequence of $\{d_n\}_{n=1}^{\infty}$. Every chain sequence has a minimal and a maximal parameter sequence which is unique to the chain sequence. For instance, the constant sequence $\{1/4\}$ is a chain sequence with $m_n = n/2(n+1)$, $n \geq 0$, as the minimal and $M_n = 1/2$, $n \geq 0$, as the maximal parameter sequences.

An important property of a chain sequence $\{d_n\}_{n=1}^{\infty}$ is that $\{d_{n+1}\}_{n=1}^{\infty}$ is again a chain sequence with parameter sequence $\{g_{k+1}\}_{k=0}^{\infty}$, where $\{g_k\}_{k=0}^{\infty}$ is any parameter sequence of $\{d_n\}_{n=1}^{\infty}$. Further, $\{M_{k+1}\}_{k=0}^{\infty}$ is the maximal parameter sequence of $\{d_{n+1}\}_{n=1}^{\infty}$. However, if $\{m_{1,k}\}_{k=0}^{\infty}$ denotes the minimal parameter sequence of $\{d_{n+1}\}_{n=1}^{\infty}$, then $m_{1,k} < m_{k+1}$, $k \geq 0$. The proofs of these results can be found in Chihara [42, Chapter III, Theorem 5.4]. There is also a nice section in Ismail [90, Section 7.2] about chain sequences and their properties.

A positive chain sequence $\{d_{n+1}\}_{n=1}^{\infty}$ appears in the three term recurrence relation satisfied by the sequence of polynomials $\{R_n(z)\}_{n=0}^{\infty}$. where $R_n(z)$ is a normalization of the kernel polynomial $P_n(1, z)$ defined in (1.1.7). Hence, using (1.1.8), we obtain the following recurrence relation

$$R_{n+1}(z) = [(1 + ic_{n+1})z + (1 - ic_{n+1})]R_n(z) - 4d_{n+1}zR_{n-1}(z), \quad n \geq 1, \tag{1.4.1}$$

with $R_0(z) = 1$ and $R_1(z) = (1 + ic_1)z + (1 - ic_1)$. Indeed, it is shown in Costa et al. [40] that for $n \geq 1$,

$$R_n(z) = \frac{\prod_{j=0}^{n-1}[1 - \tau_j\alpha_j]}{\prod_{j=0}^{n-1}[1 - \mathrm{Re}(\tau_j\alpha_j)]}P_n(1; z), \quad \text{where} \quad \tau_j = \tau_j(1) = \prod_{k=1}^{j}\frac{1 - ic_k}{1 + ic_k}.$$

Further, $\{c_n\}_{n=1}^{\infty}$ is a real sequence where

$$c_n = \frac{-\mathrm{Im}(\tau_{n-1}\alpha_{n-1})}{1 - \mathrm{Re}(\tau_{n-1}\alpha_{n-1})}, \quad n \geq 1,$$

and $d_{n+1}=(1-g_n)g_{n+1}$, $n\geq 1$, is a chain sequence with parameter sequence

$$g_n=\frac{1}{2}\frac{|1-\tau_{n-1}\alpha_{n-1}|^2}{[1-\mathrm{Re}(\tau_{n-1}\alpha_{n-1})]},\quad n\geq 1.$$

It is also not difficult to verify from (1.4.1) that $R_n(z)$ has $r_{n,n}=\prod_{k=1}^{n}(1+ic_k)$ as the leading coefficient and $r_{n,0}=\bar{r}_{n,n}=\prod_{k=1}^{n}(1-ic_k)$ as the constant term. Further, $R_n^*(z)=z^n\overline{R_n(1/\bar{z})}=R_n(z)$, $n\geq 1$, a property due to which $R_n(z)$ is called a self-inversive polynomial.

The three term recurrence relation (1.4.1) has been studied extensively in Bracciali et al. [25], Castillo et al. [37] and Costa et al. [40] wherein it is shown that $R_n(z)$, $n\geq 1$, can be used to obtain a sequence of Szegő polynomials with respect to the positive non-trivial measure $\mu(z)$ and having $\{\alpha_{n-1}\}_{n=1}^{\infty}$ as the sequence of Verblunsky coefficients. In fact, $R_n(z)$ are special forms of para-orthogonal polynomials and we briefly illustrate the role of chain sequences in this context.

Chain sequences and para-orthogonality

The two sequences of para-orthogonal polynomials

$$R_n^{(1)}(z)=z\Phi_{n-1}(z)+\Phi_{n-1}^*(z),\quad \text{and}\quad (z-1)R_n^{(2)}(z)=z\Phi_n(z)-\Phi_n^*(z),\quad n\geq 1,$$

are considered by Delsarte and Genin [58], wherein they are referred to as first and second kind singular predictor polynomials respectively. They are further shown to satisfy the three term recurrence relations

$$\begin{aligned}R_n^{(1)}(z)&=(z+1)R_{n-1}^{(1)}(z)-4d_{n+1}^{(1)}zR_{n-1}^{(1)}(z),\\ R_n^{(2)}(z)&=(z+1)R_{n-1}^{(2)}(z)-4d_{n+1}^{(2)}zR_{n-1}^{(2)}(z),\quad n\geq 1,\end{aligned}\tag{1.4.2}$$

where $R_0^{(1)}(z)=R_0^{(1)}(z)=1$, $R_1^{(2)}(z)=R_1^{(2)}(z)=z+1$,

$$d_{n+1}^{(1)}=\frac{1}{4}(1-\alpha_{n-2})(1+\alpha_{n-1})\quad\text{and}\quad d_{n+1}^{(2)}=\frac{1}{4}(1+\alpha_{n-1})(1-\alpha_n),\quad n\geq 1.$$

Following Bracciali et al. [25], we will refer to $R_n^{(1)}(z)$ and $R_n^{(2)}(z)$ as Delsarte and Genin 1 para-orthogonal polynomials (DG1POP) and Delsarte and Genin 2 para-orthogonal polynomials (DG2POP) respectively. It may also be observed that (1.4.1) reduces to either of the recurrence relations (1.4.2) if $c_n = 0$, $n \geq 1$.

Such para-orthogonality arises when the measure satisfy the symmetry property $d\mu(e^{i(2\pi-\theta)}) = -d\mu(e^{i\theta})$. The Verblunsky coefficients are real, given by $\alpha_{n-1} = -\Phi_n(0)$, $n \geq 1$ and in such a case, both $d_{n+1}^{(1)}$ and $d_{n+1}^{(2)}$ become chain sequences. Later, Delsarte and Genin [59] extended the results to include complex Verblunsky coefficients using the recurrence relation

$$\hat{R}_{n+1}(z) = (\bar{\beta}_n z + \beta_n)\hat{R}_n(z) - z\hat{R}_{n-1}(z), \quad n \geq 1,$$

which is a special case of (1.4.1), with $\beta_n \in \mathbb{C}$, $n \geq 1$ and observing that the chain sequence used is the constant chain sequence $\{d_{n+1} = 1/4\}_{n=1}^{\infty}$.

Extension to include complex measures

The fundamental idea in such extension is that while the kernel polynomial $P_n(1; z)$ is unique for a sequence of Szegő polynomials, the converse is not true. That is, there can be more than one sequence of Szegő polynomials that have the same sequence of kernel polynomials. In fact, it was shown in Costa et al. [40], that the family of Szegő polynomials $\Phi_n^{(t)}(z)$, $n \geq 1$, corresponding to the family of non-trivial measures on the unit circle given by

$$\int_0^{2\pi} f(e^{i\theta}) d\mu^{(t,1)}(e^{i\theta}) = \frac{1-t}{1-\delta} \int_0^{2\pi} f(e^{i\theta}) d\mu_{(\delta)}(e^{i\theta}) + \frac{t-\delta}{1-\delta} f(1),$$

lead to the same kernel polynomial sequence $\{P_n(z; 1)\}$. Here, it is assumed that the measure $\mu_{(\delta)}$ (with total mass 1) has a jump δ, $0 \leq \delta < 1$ at $\theta = 0$ so that $\mu^{(t,1)}$ has a jump t, $0 \leq t < 1$ also at $\theta = 0$.

The case $\delta = 0$ was considered in Castillo et al. [37], wherein it was proved that the sequence of polynomials $\{\Phi_n^{(t)}(z)\}$ where

$$\Phi_0^{(t)}(z) = 1 \quad \text{and} \quad \Phi_n^{(t)}(z) = \frac{R_n(z) - 2(1 - m_n^{(t)})R_{n-1}(z)}{\prod_{k=1}^{n}(1 + ic_k)}, \quad n \geq 1, \tag{1.4.3}$$

is a sequence of Szegő polynomials with the Verblunsky coefficients given by

$$\alpha_{n-1}^{(t)} = \frac{1 - 2m_n^{(t)} - ic_n}{1 + ic_n} \prod_{k=1}^{n} \frac{1 + ic_k}{1 - ic_k}, \quad n \geq 1. \tag{1.4.4}$$

Here, $\{m_n^{(t)}\}_{n=0}^{\infty}$ is the minimal parameter sequence of the chain sequence $\{d_n\}_{n=1}^{\infty}$ obtained from the chain sequence $\{d_{n+1}\}_{n=1}^{\infty}$ by defining $d_1 = (1-t)M_1^{(t)}$.

Further, if $c_n = 0$, $n \geq 1$, in (1.4.3), it follows that $(z-1)R_n(z) = z\Phi_n^{(t)}(z) - (\Phi_n^{(t)}(z))^*$ and hence provides an extension of DG2POP to include complex Verblunsky coefficients given by (1.4.4). In the present thesis, we will be concerned only with DG2POP and the associated chain sequences.

Similarly, an extension of DG1POP is studied in Bracciali et al. [25, Theorem 3.1]. Choosing the additional term d_1 in this case such that $d_1 \neq 0$, the sequence of Szegő polynomials are given by

$$\hat{\Phi}_n(z) = \frac{R_{n+1}(z) - 2(1 - \mathfrak{m}_n)R_n(z)}{(z-1)\prod_{k=1}^{n+1}(1 + ic_k)}, \quad n \geq 0,$$

with the Verblunsky coefficients

$$\hat{\alpha}_{n-1} = -\prod_{k=1}^{n} \frac{1 + ic_k}{1 - ic_k} \frac{1 - 2\mathfrak{m}_n - ic_{n+1}}{1 - ic_{n+1}}, \quad n \geq 1.$$

Here $\{\mathfrak{m}_n\}_{n=0}^{\infty}$ is the minimal parameter sequence of the chain sequence $\{d_{n+1}\}_{n=1}^{\infty}$ and is given by $\mathfrak{m}_n = 1 - \frac{R_{n+1}(1)}{2R_n(1)}$, $n \geq 0$.

1.5 Generalized recurrence relations

Recurrence relations of the form

$$\mathcal{P}_{n+1}(\lambda) = \rho_n(\lambda - \beta_n)\mathcal{P}_n(\lambda) + \tau_n(\lambda - \gamma_n)\mathcal{P}_{n-1}(\lambda), \quad n \geq 1, \tag{1.5.1}$$

with $\mathcal{P}_1(\lambda) = \rho_0(\lambda - \beta_0)$ and $\mathcal{P}_0(\lambda) = 1$ are studied in Ismail and Masson [91]. In addition to the restrictions $\rho_n \neq 0$ and $\tau_n \neq 0$, $n \geq 0$, it was shown that if one also assumes $\mathcal{P}_n(\gamma_n) \neq 0$, $n \geq 1$, in (1.5.1), then there exists a linear functional $\mathcal{M}$ such that the orthogonality relations

$$\mathcal{M}[\gamma_0] \neq 0, \quad \mathcal{M}\left[\frac{\lambda^k}{\prod_{k=1}^{n}(\lambda-\gamma_k)}\mathcal{P}_n(\lambda)\right] \neq 0, \quad 0 \leq k < n,$$

hold. Following Ismail and Masson [91], the recurrence relation (1.5.1) will be referred to as recurrence relation of R_I type and $\mathcal{P}_n(\lambda)$, $n \geq 1$, generated by this recurrence relation as R_I polynomials.

A non-trivial positive measure of orthogonality defined either on the unit circle or on the real axis is associated with (1.5.1) whenever $\gamma_n = 0$, $n \geq 1$ and the parameters satisfy certain positivity conditions. For instance, if $\rho_n > 0$, $\beta_n > 0$ and $\tau_n < 0$, then it is shown in Jones et al. [103] that the corresponding polynomials satisfy the Laurent orthogonality property

$$\int_0^\infty t^{-n+s}\mathcal{P}_n(t)d\phi(t) = 0, \quad s = 0, 1, \cdots, n-1.$$

Similarly, when $\rho_n = 1$, $\beta_n \neq 0$ and $\tau_n \neq 0$, there exists (Sri Ranga [162, Theorem 2.1]) a positive measure μ on the unit circle such that $\{\mathcal{P}_n(\lambda)\}_{n=1}^{\infty}$ is a sequence of Szegő polynomials whenever the inequality $0 < \tau_n\beta_n^{-1} < 1 - |\mathcal{P}_n(0)|^2$, $n \geq 1$, holds.

Recurrence relations of the form

$$\mathcal{Q}_{n+1}(\lambda) = \rho_n(\lambda-\nu_n)\mathcal{Q}_n(\lambda) + \tau_n(\lambda-a_n)(\lambda-b_n)\mathcal{Q}_{n-1}(\lambda), \quad n \geq 1, \tag{1.5.2}$$

with initial conditions $\mathcal{Q}_0(\lambda) = 1$ and $\mathcal{Q}_1(\lambda) = \rho_0(\lambda-\nu_0)$ are also studied in Ismail and Masson [91]. It was shown that if $\mathcal{Q}_n(a_n) \neq 0$, $\mathcal{Q}_n(b_n) \neq 0$, $\tau_n \neq 0$ and $\rho_n \neq 0$, $n \geq 0$, then, there exists a rational function

$$\phi_0(\lambda) = 1, \quad \phi_n(\lambda) = \prod_{k=1}^{n}\frac{\mathcal{Q}_n(\lambda)}{(\lambda-a_k)(\lambda-b_k)}, \quad n \geq 1,$$

and a linear functional M defined on the span $\{\lambda^k\phi_n(\lambda) : 0 \leq k \leq n\}$ such that the relation $\mathcal{N}(\lambda^k\phi_n(\lambda)) = 0$, for $0 \leq k < n$ holds. Conversely, we can always obtain (1.5.2) from a sequence of rational functions $\{\phi_n(z)\}_{n=0}^{\infty}$ having poles at $\{a_k\}_{k=0}^{\infty}$ and $\{b_k\}_{k=0}^{\infty}$ and satisfying a three term recurrence relation. Following Ismail and Masson [91], we call (1.5.2) as recurrence relations of R_{II} type and $\mathcal{Q}_n(\lambda)$ the R_{II} polynomials.

From the R_{II} type recurrence relations, it is clear that the R_{II} polynomials appear as denominators of the approximants of the continued fraction

$$\frac{1}{\zeta_0(\lambda)} - \frac{\sigma_1^L(\lambda)\sigma_1^R(\lambda)}{\zeta_1(\lambda)} - \frac{\sigma_2^L(\lambda)\sigma_2^R(z)}{\zeta_2(\lambda)} - \cdots,$$

where $\sigma_k^L(\lambda)$, $\sigma_k^R(\lambda)$ and $\zeta_k(\lambda)$ are polynomials of degree one. In case of $\{\mathcal{Q}_n(z)\}$ satisfying (1.5.2), $\sigma_k^L(\lambda) = \tau_k^L(\lambda - a_k)$, $\sigma_k^R(\lambda) = \tau_k^R(\lambda - b_k)$ and $\zeta_k(\lambda) = \rho_k(\lambda - \nu_k)$.

Related to the recurrence relations of R_I and R_{II} type, are important concepts of linear pencil matrix, and rational functions satisfying both orthogonality and biorthogonality properties which are briefly explained below.

1.5.1 Orthogonal rational functions

The orthogonal rational functions has been studied extensively and there is a vast literature available on the subject. The theory of rational functions orthogonal on the unit circle is developed parallel to that of polynomials orthogonal on the unit circle and is available in the monograph by Bultheel et al. [33]. A sequence of orthogonal rational functions is obtained from the Gram-Schmidt orthonormalization process in the linear space of rational functions which, in fact, can be characterized by the poles of the basis elements as well. In this direction, for example, in Bultheel et al. [32] and Li [122], starting from a set of pre-defined poles, the rational functions are characterized by Favard type theorems as well as in terms of three-term recurrence relations similar to that of orthogonal polynomials on the real line but with rational coefficients.

In other directions of study, the effect of poles on the asymptotics of the Christoffel functions associated with the orthogonal rational functions and their interval of orthogonality was studied in Deckers and Lubinsky [56]. The spectral methods in case of orthogonal rational functions are illustrated in Velázquez [181]. Linear combinations of orthogonal rational functions and the corresponding rational functions of the second kind with rational coefficients are studied in Deckers et al. [55]. Boyd [28], Langer and Lasarow [119] and Pan [138] found many applications of orthogonal rational functions in the direction of numerical methods.

1.5.2 Linear pencil matrix

Unlike the case for orthogonal polynomials on the real line or on the unit circle, the R_{II} polynomials $\mathcal{Q}_n(z)$, $n \geq 1$, satisfying (1.5.2) is the characteristic polynomial of a matrix pencil $\mathcal{G}_n - \lambda\mathcal{H}_n$, where both $\mathcal{G}_n$ and $\mathcal{H}_n$ are tridiagonal matrices. This was proved in Zhedanov [192], wherein the biorthogonality and Christoffel type transformation of rational functions obtained from R_{II} polynomials were illustrated.

A particular case of linear pencil matrix is considered in Ismail and Ranga [94], where both $\mathcal{G}_n$ and $\mathcal{H}_n$ are Hermitian while H_n is positive definite as well. Such generalized eigenvalue problem is related to the R_{II} type recurrence relation

$$\mathcal{Q}_{n+1}(x) = (x - c_{n+1})\mathcal{Q}_n(x) - d_{n+1}(x^2 + 1)\mathcal{Q}_{n-1}, \quad n \geq 1,$$

where $\{c_n\}_{n=1}^{\infty}$ is a real sequence and $\{d_{n+1}\}_{n=1}^{\infty}$ is a positive chain sequence. Using the associated linear matrix pencil, it is shown that a positive measure on the unit circle can always be associated to such recurrence relations of R_{II} type. In a different context of obtaining Nevanlinna functions (functions mapping upper half plane to upper half plane) from Carathéodory functions, it is shown in Derevyagin [62] that $\mathcal{Q}_n(x)$ appear as denominators of the approximants of continued fraction representations of such Nevanlinna functions. We also refer to Bora et al. [21, 22] for numerical aspects of the eigenvalue problems associated with matrix pencils.

1.5.3 Biorthogonality of polynomials

The term "biorthogonality" has been defined in the literature in different ways. Among these, we will use the following definition formulated in Konhauser [108].

Let $p(x)$ and $q(x)$ be polynomials in x with real coefficients and degree $\mathfrak{p}$ and $\mathfrak{s}$ respectively. If $\mathrm{P}_m(x)$ and $\mathrm{Q}_n(x)$ are polynomials in $p(x)$ and $q(x)$ of degree m and n respectively, then $p(x)$ and $q(x)$ are called basic polynomials.

Definition 1.5.1. [108] *The sequences $\{\mathrm{P}_m(x)\}_{m=0}^{\infty}$ and $\{\mathrm{Q}_n(x)\}_{n=0}^{\infty}$ are biorthogonal over the interval (a, b) with respect to the weight function $\omega(x)$ and the basic polynomials*

$p(x)$ and $q(x)$ provided the orthogonality conditions

$$\int_a^b \mathrm{P}_m(x)\mathrm{Q}_n(x)\omega(x)dx \begin{cases} = 0, & m,n = 0,1,\cdots,\ m \neq n; \\ \neq 0, & m = n. \end{cases} \tag{1.5.3}$$

are satisfied.

Note that, in contrast to the usual orthogonality condition, two different sequences of polynomials are used for the biorthogonality condition (1.5.3).

Further, the real valued function $\omega(x)$ of the real variable x on the interval (a,b) is such that the moments

$$I_{i,j} = \int_a^b [p(x)]^i [q(x)]^j \omega(x)dx, \quad i,j = 0,1,\cdots,$$

exist with $I_{0,0} = \int_a^b \omega(x)dx \neq 0$. In this case, $\omega(x)$ is called an admissible weight function. Using the generalized moments $I_{i,j}$, Konhauser [108] provided necessary and sufficient conditions for the existence of biorthogonal polynomials.

Theorem 1.5.1. [108] *Given the basic polynomials $p(x)$ and $q(x)$ and an admissible weight function $\omega(x)$ on the interval (a,b), the polynomial sequences $\{P_m(x)\}_{m=0}^{\infty}$ and $\{Q_n(x)\}_{n=0}^{\infty}$ satisfying the biorthogonality condition (1.5.3) exist if, and only if, the determinant*

$$\begin{vmatrix} I_{0,0} & I_{0,1} & \cdots & I_{0,n-1} \\ I_{1,0} & I_{1,1} & \cdots & I_{1,n-1} \\ \vdots & \vdots & \ddots & \vdots \\ I_{n-1,0} & I_{n-1,1} & \cdots & I_{n-1,n-1} \end{vmatrix} \neq 0, \quad n = 1,2,\cdots.$$

Moreover, the polynomials are unique, each up to a multiplicative constant.

Biorthogonal polynomials were also obtained as solutions of differential equations. In fact, Preiser [142] proved that there exists only one third order differential equation

$$a(x)y_n'''(x) + b(x)y_n''(x) + c(x)y_n'(x) = \lambda_n y_n(x),$$

having biorthogonal polynomials solutions of degree n in x^m and such that its adjoint differential equation

$$-[p(x)a(x)z_n(x)]''' + [p(x)b(x)z_n(x)]'' - [p(x)c(x)z_n(x)]' = \lambda_n p(x)z_n(x),$$

has biorthogonal polynomial solutions of degree n in x. Here, $a(x)$, $b(x)$, $c(x)$ are functions of x independent of n and λ_n is a parameter independent of x.

Several families of biorthogonal polynomials are known in the literature, some of them having explicit representations. For instance, a great deal of study has been made on the polynomials

$$\mathrm{P}_m(z;\alpha,\beta) = F(-m, \alpha+\beta+1; 2\alpha+1; 1-z); \quad \mathrm{Q}_n(z) = \mathcal{P}_n(z;\alpha,-\beta),$$

which were proved to be biorthogonal in Askey [7] (see also Borrego-Morell and Rafaeli [23]) with respect to the weight function

$$\omega(\theta) = (2-2\cos\theta)^\alpha(-e^{i\theta})^\beta, \quad \theta\in[-\pi,\pi], \quad \operatorname{Re}\alpha > -1/2.$$

Sri Ranga [162] later proved that the sequence $\{\mathrm{P}_m(z;\alpha,\beta)\}_{m=0}^{\infty}$ is also orthogonal with respect to the weight function $\hat{\omega}(\theta) = 2^{2\alpha}e^{(\pi-\theta)\mathrm{Im}\beta}\sin^{2\alpha}\theta/2$ if $\alpha\in\mathbb{R}$, $\alpha > -1/2$ and $i\beta\in\mathbb{R}$. We further refer to Askey [8] and Temme [171] for proofs of the biorthogonality of the polynomials $\mathrm{P}_m(z;\alpha,\beta)$ and $\mathrm{Q}_n(z)$ and related discussions.

In some recent advancements, the zero distribution of polynomials that are biorthogonal to exponentials is analyzed in Lubinsky and Sidi [125] while integral representations of biorthogonal functions using generalized Hankel determinants are found in Ismail and Simeonov [93]. Further, while biorthogonality of rational functions is studied in Rosengren [151], biorthogonality of Laurent polynomials, is studied in Cruz-Barroso et al. [43], Pastro [139] and Zhedanov [190].

As mentioned earlier, different conditions of biorthogonality are also studied in various areas of rational interpolation, Padé approximation, non linear PDEs and random matrix theory. We make no mention of these definitions and refer to Bertola et al. [18], Iserles and Nørsett [88], Iserles and Saff [89] and references therein.

1.6 Motivation and outline of the thesis

The underlying theme of the thesis on one hand is to explore both structural and qualitative aspects of perturbations of the continued fraction parameters in case of special functions and recursion coefficients in case of orthogonal polynomials and on the other hand to obtain biorthogonality relations of Laurent polynomials. One of the perturbations that is studied in the thesis is the map $\mathcal{F}(\lambda) \mapsto \mathcal{F}(\lambda^2)$, where $\mathcal{F}(\lambda)$ is the T-fraction defined in (1.2.10). This leads us to introduce generalized linear Jacobi pencil matrices and study biorthogonality relations of the associated Laurent polynomials. Hence, it is natural to study further biorthogonality relations using the recurrence relations of R_I and R_{II} types. Illustrations are provided for results obtained in the thesis, mostly using the hypergeometric functions. This is due to the fact that many special functions, can be represented as either hypergeometric functions or their ratios. We would also like to add that the thesis involves mostly infinite matrices, whose recent applications can be found in the monograph by Shivakumar et al. [155].

A brief overview of the chapters

From the point of view of their applications, it is obvious that the parameters g_n of the g-fraction occurring in the right hand side of (1.2.8) contain hidden information about the properties of the dynamical systems or the special functions they represent. One way to explore this hidden information is through perturbation; that is, through a study of the consequences when some disturbance is introduced in the parameters g_n, $n \geq 0$. The main objective of **Chapter 2** is to study the structural and qualitative aspects of the following two perturbations in the parameters g_n, $n \geq 0$.

(i) The first is when a finite number of parameters g_j are missing in which case we call the corresponding g-fraction a *gap-g-fraction*.

(ii) The second case is replacing $\{g_n\}_{n=0}^{\infty}$ by a new sequence $\{g_n^{(\beta_k)}\}_{n=0}^{\infty}$ in which the j^{th} term g_j is replaced by $g_j^{(\beta_k)}$.

The first case is illustrated using Gaussian hypergeometric functions, where we use the fact that many g-fractions converge to ratios of Gaussian hypergeometric functions in slit complex domains. The second case is studied by applying the technique of coefficient stripping (Simon [156]) to the sequence of Schur parameters $\{\alpha_j\}_{j=0}^{\infty}$. This follows from the fact that the Schur fraction and the g-fraction are completely determined by the related Schur parameters α_k and the g_k-parameters respectively. These parameters are related by $g_0 = 0$, $g_k = (1-\alpha_{k-1})/2$, $k \geq 1$, so that a perturbation in α_j produces a unique change in g_j and vice-versa.

The first half of **Chapter 3** draws inspiration from the recurrence relation

$$R_{n+1}(z) = [(1+ic_{n+1}) + i(1-ic_{n+1})]R_n(z) - 4d_{n+1}zR_{n-1}(z), \quad n \geq 1, \tag{1.6.1}$$

studied extensively in Castillo et al. [37], Costa et al. [40] and Bracciali et al. [25]. Both the Szegő polynomials and the Verblunsky coefficients given respectively in (1.4.3) and (1.4.4) are expressed in terms of the minimal parameter sequence $\{m_n^{(t)}\}_{n=0}^{\infty}$ of the chain sequence $\{d_n\}_{n=1}^{\infty}$. Hence, we study the case when the minimal parameter sequence $\{m_n^{(t)}\}_{n=0}^{\infty}$ of $\{d_n\}_{n=0}^{\infty}$ is replaced by the minimal parameter sequence $\{k_n\}_{n=0}^{\infty}$ of the chain sequence $\{a_n\}_{n=1}^{\infty}$, where $k_0^{(t)} := 0$ and $k_n^{(t)} := 1 - m_n^{(t)}$, $n \geq 1$. This motivates us to define the concept of complementary chain sequences.These complementary chain sequences are unique in perturbation theory of chain sequences, due to the following reason. In the theory of chains sequences, if minimal parameter sequence $\{m_n\}_{n=0}^{\infty}$ and maximal parameter sequence $\{M_n\}_{n=0}^{\infty}$ coincide, it is called a single parameter positive chain sequence (SPPCS) which is very useful in characterizing corresponding functions and polynomials. However, all functions and polynomials may not have SPPCS. The concept of complementary chain sequence provide SPPCS, even if the original chain sequence do not possess this property. Hence the concept of complementary chain sequence play an important contribution in the thesis.

The second half of **Chapter 3** explores the consequences of the perturbation $m_n \mapsto 1 - m_n$ in case of polynomials orthogonal on the real line. The perturbation is extended to *any* parameter sequence of the associated chain sequence, which is called as the generalized complementary chain sequence. The generalized perturbation yields two sequences of orthogonal polynomial on the real line that have the same kernel polynomials. Chapter 4 serves as the bridge between the two parts of the thesis. The matrix representation of the polynomial map $\mathcal{S}(\lambda) \mapsto \lambda\mathcal{S}(\lambda^2)$, where $\mathcal{S}(\lambda)$

is a Stieltjes function is studied in Derevyagin [61]. Precisely, the map is interpreted as the Darboux transformation of the Jacobi matrix associated with the orthogonal polynomials on the real line, which are in turn related to $\mathcal{S}_n(x)$. Hence, we study the map $\mathcal{F}(\lambda) \mapsto \mathcal{F}(\lambda^2)$, where $\mathcal{F}(\lambda)$ is a general T-fraction. The denominators of the approximants of a T-fraction satisfy a R_I type recurrence relation (1.5.1), with which is associated a sequence of Laurent polynomials. This yields a generalized linear pencil matrix, which further leads to biorthogonality relations of the associated Laurent polynomials.

Results on linear combination of orthogonal polynomials are abundant in literature and are studied in the context of quasi-orthogonality. A polynomial sequence $\{q_n(x)\}_{n=0}^{\infty}$ is said to be quasi-orthogonal of order r with respect to a positive weight $\omega(x)$ on $[a,b] \subseteq \mathbb{R}$ if, and only there exists another sequence of polynomials $\{p_n(x)\}_{n=0}^{\infty}$ orthogonal with respect to $\omega(x)$ on $[a,b]$ such that

$$q_n(x) = c_0 p_n(x) + c_1 p_{n-1}(x) + \cdots + c_r p_{n-r}(x),$$

where c_i depend only on n and $c_0 c_r \neq 0$. Linear combination of polynomials which are orthogonal either on the real line or on the unit circle has also been studied as an inverse problem in Alfaro et al. [3], with conditions being obtained for the orthogonality of such linear combinations.

Motivated by linear combinations of polynomials and their orthogonality as well as algebraic properties, our aim in **Chapter 5** is to study the linear combination of two successive R_I polynomials of a sequence $\{\mathcal{P}_n(\lambda)\}_{n=0}^{\infty}$ that satisfies (1.5.1), that is

$$\mathcal{Q}_n(\lambda) := \mathcal{P}_n(\lambda) + \alpha_n \mathcal{P}_{n-1}(\lambda), \quad \alpha_n \in \mathbb{R} \setminus \{0\}, \quad n \geq 0,$$

where $\beta_0 \neq 0, \pm 1$ and $\beta_n \neq 0$, $n \geq 1$. We construct a unique sequence $\{\alpha_n\}_{n=0}^{\infty}$ such that $\{\mathcal{Q}_n(\lambda)\}_{n=0}^{\infty}$ not only satisfies mixed recurrence relations of R_I and R_{II} type but also has a common zero.

The polynomials $\mathcal{Q}_n(\lambda)$, $n \geq 1$, are shown to satisfy biorthogonality relations that follow from their eigenvalue representations. With certain additional conditions, we also show that a para-orthogonal polynomial of degree n can be obtained from $\mathcal{Q}_{n+1}(\lambda)$.

Further, common zeros of an orthogonal sequence have been considered in the past, see for example, Driver and Muldoon [70] and Wong [186]. However, the novelty in our approach is that we actually construct such a sequence before studying its orthogonality properties. Such procedure of constructing the zeros before characterizing the orthogonality properties is limited in the literature. Hence this approach provides an important contribution to the thesis.

The recurrence relations (1.5.2) of R_{II} type are used to define the generalized eigenvalue problems $\mathcal{G}_n\varphi_n^R = \lambda\mathcal{H}_n\varphi_n^R$ and $\varphi_n^L\mathcal{G}_n = \lambda\varphi_n^L\mathcal{H}_n$. In this case, both $\mathcal{G}_n$ and $\mathcal{H}_n$ are tridiagonal matrices and the components of the eigenvectors φ_n^L and φ_n^R are rational functions with the numerator polynomials $\mathcal{Q}_n(\lambda)$ satisfying the recurrence relations (1.5.2) of R_{II} type. However, while the three term recurrence relation satisfied by the sequence of rational functions

$$\phi_0(\lambda) := 1, \qquad \phi_n(\lambda) := \frac{\mathcal{P}_n(\lambda)}{\prod_{j=1}^{n}(\lambda - a_j)(\lambda - b_j)}, \qquad n \geq 1,$$

is used to obtain the pencil matrix $\mathcal{G}_n - \lambda\mathcal{H}_n$, the usual process available in the literature, for example in Zhedanov [192] and Beckermann et.al. [16], is to partition the poles to form the rational functions

$$p_n^L(\lambda) = \frac{\mathcal{P}_n(\lambda)}{\prod_{k=1}^{n}(\lambda - a_k)} = \frac{\mathcal{P}_n(\lambda)}{\prod_{k=1}^{n}\sigma_k^L} \quad \text{and} \quad p_n^R(\lambda) = \frac{\mathcal{P}_n(\lambda)}{\prod_{k=1}^{n}(\lambda - b_k)} = \frac{\mathcal{P}_n(\lambda)}{\prod_{k=1}^{n}\sigma_k^R}.$$

The two sequences of rational functions $\{p_n^L(\lambda)\}_{n=0}^{\infty}$ and $\{p_n^R(\lambda)\}_{n=0}^{\infty}$ form the components of the left and right eigenvectors of the matrix pencil $\mathcal{G} - \lambda\mathcal{H}$ and hence are biorthogonal to each other. However we note that two sequences of rational functions that are biorthogonal to each other need not themselves form an orthogonal sequence.

Motivated by the procedure of proving biorthogonality given in Zhedanov [192], and other results referred to earlier in Askey [7], Sri Ranga [162] and Temme [171] the central theme of the **Chapter 6** is to study a sequence of rational functions that is both orthogonal as well as biorthogonal. Precisely, we are interested in constructing a sequence of *orthogonal* rational functions $\{\phi_n(\lambda)\}$ satisfying the following two properties

(i) The related matrix pencil has the numerator polynomials $\mathcal{P}_n(z)$ as the characteristic polynomials and $\phi_n(z)$ as components of the eigenvectors.

(ii) The orthogonal sequence $\{\phi_n(z)\}$ is also *biorthogonal* to another sequence of rational functions.

We also give a Christoffel type transformation of such rational functions illustrating the differences with the ones that are available in the literature on orthogonal rational functions.

Concluding remarks are given at the end of each chapter to provide a passage to the next chapter through existing literature. A detailed list of references related to the thesis is given at the end. There is a conscious effort to include both the seminal papers that initiated a particular direction of study as well as recent references that reflect developments in the last few years. At the same time, due to too classical nature of the references and in some cases non-availability of the original research article, we cite modern works for classical results. It is expected that the references given at the end of such citations will provide a good idea of various problems that are based on these classical results. Finally, wherever required, we include references within brackets so as not to disturb the flow of the language.

Chapter 2

Perturbed g-fractions and a class of Pick functions

The main objective of the chapter is to investigate structural and qualitative aspects of two different perturbations of the parameters of g-fractions. In this context, the concept of *gap* g-fractions is introduced. While tail sequences of a continued fraction play a significant role in the first perturbation, Schur fractions are used in the second perturbation of the g-parameters. The application of such perturbations is illustrated in geometric properties of analytic functions like subordination. Further, using a particular gap g-fraction, a class of Pick functions is identified.

2.1 Gap g-fractions

Given an arbitrary real sequence $\{g_k\}_{k=0}^{\infty}$, a continued fraction expansion of the form

$$\frac{1}{1} \underset{-}{} \frac{(1-g_0)g_1 z}{1} \underset{-}{} \frac{(1-g_1)g_2 z}{1} \underset{-}{} \frac{(1-g_2)g_3 z}{1} \underset{-}{} \cdots, \quad z \in \mathbb{C}, \tag{2.1.1}$$

is called a g-fraction if the parameters $g_j \in [0,1]$, $j \geq 0$. It terminates and equals a rational function if $g_j \in \{0,1\}$ for some $j \geq 0$. If $0 < g_j < 1$, $j \geq 0$, the g-fraction (2.1.1) still converges uniformly on every compact subset of the slit domain $\mathbb{C} \setminus [1,\infty)$ (Wall [184, Theorem 27.5] and Jones and Thron [102, Corollary 4.60]) and in this case, (2.1.1) represents an analytic function, say $\mathcal{F}(z)$.

Applications of g-fractions in number theory (Runckel's points) and dynamical systems like the ABC flow are studied in Tsygvintsev [175, 176]. The fact that many analytic functions on bounded domains possess a g-fraction expansion provides a convenient way to approach moment problems. In particular, it has been proved in Wall [184, Theorem 69.2] that the Hausdorff moment problem

$$\nu_j = \int_0^1 \sigma^j d\nu(\sigma), \quad j \geq 0,$$

has a solution if and only if (2.1.1) corresponds to a power series of the form $1 + \nu_1 z + \nu_2 z^2 + \cdots$, $z \in \mathbb{C} \setminus [1, \infty)$. Further, the g-fractions have also been used, in the methodology, to study the geometric properties of ratios of Gaussian hypergeometric functions as well as their q-analogues. See for example, the proofs of Küstner [117, Theorem 1.5] and Baricz and Swaminathan [13, Theorem 2.2]).

Among several properties of g-fractions, one of the most fundamental results is given by Wall [184, Theorem 74.1] in which holomorphic functions having positive real part in $\mathbb{C} \setminus [1, \infty)$ are characterized. Precisely, Re $(\sqrt{1+z}\, \mathcal{F}(z))$ is positive if, and only if, $\mathcal{F}(z)$ has a continued fraction expansion of the form (2.1.1). Moreover, $\mathcal{F}(z)$ has the integral representation

$$\mathcal{F}(z) = \int_0^1 \frac{d\phi(t)}{1 - zt}, \quad z \in \mathbb{C} \setminus [1, \infty),$$

where $\phi(t)$ is a bounded non-decreasing function having a total increase 1.

As the name suggests, gap-g-fractions correspond to a sequence of g-parameters, $\{g_k\}_{k=0}^{\infty}$, with certain terms missing. We study three cases in this section and in each case, the concept of tail sequences of a continued fraction plays an important role. We present only those results on the tail sequences that are required. For more information, we refer to Lisa and Waadeland [124, Ch. II].

The N^{th} tail of the continued fraction

$$b_0(z) + \frac{a_1(z)}{b_1(z)} \underset{+}{} \frac{a_2(z)}{b_2(z)} \underset{+}{} \cdots \underset{+}{} \frac{a_{N+1}(z)}{b_{N+1}(z)} \underset{+}{} \cdots, \tag{2.1.2}$$

is the continued fraction

$$\frac{a_{N+1}(z)}{b_{N+1}(z)} + \frac{a_{N+2}(z)}{b_{N+2}(z)} + \frac{a_{N+3}(z)}{b_{N+3}(z)} + \cdots . \tag{2.1.3}$$

One of the properties of tails of continued fractions is the following.

Theorem 2.1.1. [124, Theorem1, Ch. II] *The following three statements are equivalent.*

(i) The continued fraction (2.1.2) converges.

(ii) The N^{th} tail (2.1.3) converges for an $N \in \mathbb{N} \cup \{0\}$.

(iii) The N^{th} tail (2.1.3) converges for all $N \in \mathbb{N} \cup \{0\}$.

Further, if the numerator and denominator of the approximants of the tail (2.1.3) are denoted, respectively, by $\mathrm{A}_n^N(z)$ and $\mathrm{B}_n^N(z)$, then the n^{th} approximant is given by $f_n^N(z) = S_n^N(0)$, where $S_n^N(z) = s_{N+1} \circ s_{N+2} \circ \cdots \circ s_{N+n}(z)$, with $s_n(z) = a_n(z)/(b_n(z)+z)$, $n = 1, 2, \cdots$. The corresponding determinant formula is

$$\mathrm{A}_n^N(z)\mathrm{B}_{n-1}^N(z) - \mathrm{A}_{n-1}^N(z)\mathrm{B}_n^N(z) = -\prod_{j=N+1}^{N+n}(-a_j(z)). \tag{2.1.4}$$

2.1.1 Structural relations

For $z \in \mathbb{C} \setminus [1, \infty)$, let $\mathcal{F}(z)$ be the continued fraction (2.1.1) and

$$\mathcal{F}(k; z) =$$
$$\frac{1}{1} - \frac{(1-g_0)g_1 z}{1} - \cdots \frac{(1-g_{k-2})g_{k-1} z}{1} - \frac{(1-g_{k-1})g_{k+1} z}{1} - \frac{(1-g_{k+1})g_{k+2} z}{1} - \cdots . \tag{2.1.5}$$

Note that (2.1.5) is obtained from (2.1.1) by removing g_k for some arbitrary k (which is not equivalent to substituting $g_k = 0$).

Theorem 2.1.2. *Suppose $\mathcal{F}(z)$ is given. Let $\mathcal{F}(k;z)$ denote the perturbed g-fraction in which the parameter g_k is missing. Then, with $d_j = (1-g_{j-1})g_j$, $j \geq 1$,*

$$\mathcal{F}(k;z) = \mathcal{S}_k(0) - \frac{\prod_{j=1}^{k-1} d_j z^{k-1} h(k;z)}{\mathcal{Y}_{k-1}(0;z)\mathcal{Y}_k(0;z)h(k;z) - [\mathcal{Y}_k(0;z)]^2}, \tag{2.1.6}$$

where $\mathcal{Y}_k(0;z)$, $\mathcal{S}_k(0)$ and $-(1-g_{k-1})^{-1}(1-g_k)h(k;z)$ are, respectively, the k^{th} denominator, the k^{th} approximant and the $(k+1)^{th}$ tail of $\mathcal{F}(z)$.

Proof. Let $-(1-g_k)\mathcal{H}_{k+1}(z)$ be the $(k+1)^{th}$ tail of $\mathcal{F}(z)$ so that

$$\mathcal{H}_{k+1}(z) = \frac{g_{k+1}z}{1} \, \underline{\ } \, \frac{(1-g_{k+1})g_{k+2}z}{1} \, \underline{\ } \, \frac{(1-g_{k+2})g_{k+3}z}{1} \, \underline{\ } \, \cdots . \tag{2.1.7}$$

We note that Theorem 2.1.1 guarantees the existence of $\mathcal{H}_{k+1}(z)$ since $\mathcal{F}(z)$ always converges. Further, if $h(k;z) = (1-g_{k-1})\mathcal{H}_{k+1}(z)$, $\quad k \geq 1$, then, from (2.1.5) and (2.1.7) we obtain the rational function

$$\frac{\mathcal{X}_k(h(k;z);z)}{\mathcal{Y}_k(h(k;z);z)} = \frac{1}{1} \, \underline{\ } \, \frac{(1-g_0)g_1z}{1} \, \underline{\ } \, \cdots \frac{(1-g_{k-2})g_{k-1}z}{1-h(k;z)}, \tag{2.1.8}$$

where we note that

$$S_k(0) = \frac{\mathcal{X}_k(0;z)}{\mathcal{Y}_k(0;z)} = \frac{\mathrm{A}_k(0)}{\mathrm{B}_k(0)} = \frac{1}{1} \, \underline{\ } \, \frac{(1-g_0)g_1z}{1} \, \underline{\ } \, \cdots \frac{(1-g_{k-2})g_{k-1}z}{1},$$

is the k^{th} approximant of (2.1.1). From (2.1.8) it follows that

$$\frac{\mathcal{X}_k(h(k;z);z)}{\mathcal{Y}_k(h(k;z);z)} = \frac{\mathcal{X}_k(0;z) - h(k;z)\mathcal{X}_{k-1}(0;z)}{\mathcal{Y}_k(0;z) - h(k;z)\mathcal{Y}_{k-1}(0;z)}.$$

Then,

$$\begin{aligned}\frac{\mathcal{X}_k(h(k;z);z)}{\mathcal{Y}_k(h(k;z);z)} - \frac{\mathcal{X}_k(0;z)}{\mathcal{Y}_k(0;z)} &= \frac{h(k;z)[\mathcal{X}_k(0;z)\mathcal{Y}_{k-1}(0;z) - \mathcal{X}_{k-1}(0;z)\mathcal{Y}_k(0;z)]}{\mathcal{Y}_k(0;z)[\mathcal{Y}_k(0;z) - h(k;z)\mathcal{Y}_{k-1}(0;z)]} \\ &= \frac{h(k;z)z^{k-1}\prod_{j=1}^{k-1}(1-g_{j-1})g_j}{\mathcal{Y}_k(0;z)[\mathcal{Y}_k(0;z) - h(k;z)\mathcal{Y}_{k-1}(0;z)]},\end{aligned}$$

where the last equality follows from from the determinant formula for a continued fraction. Denoting $d_j = (1-g_{j-1})g_j$, $j \geq 1$, we have from (2.1.8)

$$\frac{\mathcal{X}_k(h(k;z);z)}{\mathcal{Y}_k(h(k;z);z)} = \frac{\mathcal{X}_k(0;z)}{\mathcal{Y}_k(0;z)} - \frac{\prod_{j=1}^{k-1} d_j z^{k-1} h(k;z)}{\mathcal{Y}_{k-1}(0;z)\mathcal{Y}_k(0;z)h(k;z) - [\mathcal{Y}_k(0;z)]^2},$$

which is (2.1.6). □

Note 2.1.1. *In what follows, by $\mathcal{F}(z)$ we will mean the unperturbed g-fraction as given in (2.1.1) with $g_k \in [0,1]$, $k \in \mathbb{N} \cup \{0\}$. Further, as the notation suggests, the rational function $\mathcal{S}_k(0)$ is independent of the missing parameter g_k and is known whenever $\mathcal{F}(z)$ is given. The information of the missing parameter g_k at the k^{th} position is stored in $h(k;z)$ and hence the notation $\mathcal{F}(k;z)$.*

It may also be noted that the polynomials $\mathcal{Y}_k(0;z)$ can be easily computed from the recurrence relations

$$\mathcal{Y}_j(0;z) = \mathcal{Y}_{j-1}(0;z) - (1-g_{j-2})g_{j-1}z\mathcal{Y}_{j-2}(0;z), \quad j \geq 2,$$

with the initial values $\mathcal{Y}_0(0;z) = \mathcal{Y}_1(0;z) = 1$.

It is evident that the right side of (2.1.6) is of the form

$$\frac{\mathfrak{a}(z)h(k;z) + \mathfrak{b}(z)}{\mathfrak{c}(z)h(k;z) + \mathfrak{d}(z)},$$

with $\mathfrak{a}(z), \mathfrak{b}(z), \mathfrak{c}(z), \mathfrak{d}(z)$ being well defined polynomials. Rational functions of such form are said to be rational transformations of $h(k;z)$ and occur frequently in the spectral theory of orthogonal polynomials. For more details, we refer, for example, to Garza and Marcellán [77], Zhedanov [189] and references therein.

A similar result for the perturbed g-fraction in which a finite number of consecutive parameters are missing can be obtained by an argument analogous to Theorem 2.1.2. We state this result without proof.

Theorem 2.1.3. *Let $\mathcal{F}(z)$ be given. Let $\mathcal{F}(k, k+1, \cdots, k+l-1; z)$ denote the perturbed g-fraction in which the l consecutive parameters $g_k, g_{k+1}, \cdots, g_{k+l-1}$ are missing. Then,*

$$\mathcal{F}(k, k+1, \cdots, k+l-1; z) =$$

$$\mathcal{S}_k(0) - \frac{\prod_{j=1}^{k-1} d_j z^{k-1} h(k, k+1, \cdots, k+l-1; z)}{\mathcal{Y}_{k-1}(0;z)\mathcal{Y}_k(0;z)h(k, k+1, \cdots, k+l-1; z) - [\mathcal{Y}_k(0;z)]^2}, \tag{2.1.9}$$

where $-(1-g_{k-1})^{-1}(1-g_{k+l-1})h(k, k+1, \cdots, k+l-1; z)$ *is the* $(k+l)^{th}$ *tail of* $\mathcal{F}(z)$.

The next result is about the perturbation in which only two parameters g_k and g_l are missing, where l need not be $k \pm 1$.

Theorem 2.1.4. *Let* $\mathcal{F}(z)$ *be given and* $-(1-g_{k-1})^{-1}(1-g_k)h(k, l; z)$ *be the perturbed* $(k+1)^{th}$ *tail of* $\mathcal{F}(z)$ *in which* g_l *is missing. Then*

$$\begin{aligned}\mathcal{S}_m^{(k+1)}(0) &+ \frac{(1-g_k)}{(1-g_{k-1})}h(k, k+m+1; z)\\ &= \frac{\prod_{j=k+1}^{k+m} d_j z^m h(k+m+1; z)}{[\mathcal{Y}_m^{(k+1)}(0;z)]^2 - \mathcal{Y}_{m-1}^{(k+1)}(0;z)\mathcal{Y}_m^{(k+1)}(0;z)h(k+m+1; z)}.\end{aligned}$$

where we assume $l = k+m+1$, $m \geq 1$. *Further, if* $\mathcal{F}(k, l; z)$ *denotes the perturbed g-fraction in which two parameters* g_k *and* g_l *are missing, then*

$$\mathcal{F}(k, l; z) = \mathcal{S}_k(0) - \frac{\prod_{j=1}^{k-1} d_j z^{k-1} h(k, l; z)}{\mathcal{Y}_{k-1}(0;z)\mathcal{Y}_k(0;z)h(k, l; z) - [\mathcal{Y}_k(0;z)]^2}. \tag{2.1.10}$$

Here, $\mathcal{Y}_m^{(k+1)}(0;z)$ *and* $\mathcal{S}_m^{(k+1)}(0)$ *are, respectively, the* m^{th} *denominator and* m^{th} *approximant of the* $(k+1)^{th}$ *tail of* $\mathcal{F}(z)$ *and* $-(1-g_{l-1})^{-1}(1-g_l)h(l; z)$ *is the* $(l+1)^{th}$ *tail of* $\mathcal{F}(z)$.

Proof. Let

$$\mathcal{H}_{k+1}(l; z) = \frac{g_{k+1}z}{1} \underset{-}{} \frac{(1-g_{k+1})g_{k+2}z}{1} \underset{-}{} \cdots \frac{(1-g_{l-1})g_{l+1}z}{1} \underset{-}{} \frac{(1-g_{l+1})g_{l+2}z}{1} \underset{-}{} \cdots \tag{2.1.11}$$

so that $-(1-g_k)\mathcal{H}_{k+1}(l; z)$ is the perturbed $(k+1)^{th}$ tail of $\mathcal{F}(z)$ in which g_l is missing. Then, we can write

$$\mathcal{F}(k, l; z) = \frac{\mathcal{X}_k(h(k, l; z); z)}{\mathcal{Y}_k(h(k, l; z); z)} = \frac{1}{1} \underset{-}{} \frac{(1-g_0)g_1 z}{1} \underset{-}{} \cdots \frac{(1-g_{k-2})g_{k-1}z}{1-h(k, l; z)},$$

where $h(k,l;z) = (1-g_{k-1})\mathcal{H}_{k+1}(l;z)$. Now, proceeding as in Theorem 2.1.2, we obtain

$$\mathcal{F}(k,l;z) = \mathcal{S}_k(0;z) - \frac{\prod_{j=1}^{k-1} d_j z^{k-1} h(k,l;z)}{\mathcal{Y}_{k-1}(0;z)\mathcal{Y}_k(0;z)h(k,l;z) - [\mathcal{Y}_k(0;z)]^2}.$$

Hence, all that remains is to find the expression for $h(k,l;z)$ or $\mathcal{H}_{k+1}(l;z)$. Now, let

$$\mathcal{H}_{l+1}(z) = \frac{g_{l+1}z}{1} \underset{-}{} \frac{(1-g_{l+1})g_{l+2}z}{1} \underset{-}{} \frac{(1-g_{l+2})g_{l+3}z}{1} \underset{-}{} \cdots,$$

and $h(l;z) = (1-g_{l-1})\mathcal{H}_{l+1}(z)$. From (2.1.11) and Lorentzen and Waadeland [124, eqn.(1.1.4), p.57], we have

$$\begin{aligned} -(1-g_k)\mathcal{H}_{k+1}(l;z) &= \frac{-(1-g_k)g_{k+1}z}{1} \underset{-}{} \frac{(1-g_{k+1})g_{k+2}z}{1} \underset{-}{} \cdots \frac{(1-g_{l-2})g_{l-1}z}{1-h(l;z)} \\ &= \frac{\mathcal{X}_{l-k-1}^{(k+1)}(h(l;z);z)}{\mathcal{Y}_{l-k-1}^{(k+1)}(h(l;z);z)}. \end{aligned}$$

It is clear that, the rational function $[\mathcal{X}_{l-k-1}^{(k+1)}(0;z)/\mathcal{Y}_{l-k-1}^{(k+1)}(0;z)]$ is the approximant of the $(k+1)^{th}$ tail $-(1-g_k)\mathcal{H}_{k+1}(z)$ of $\mathcal{F}(z)$. Then, using (2.1.4) we obtain

$$\frac{\mathcal{X}_{l-k-1}^{(k+1)}(h(l;z);z)}{\mathcal{Y}_{l-k-1}^{(k+1)}(h(l;z);z)} - \frac{\mathcal{X}_{l-k-1}^{(k+1)}(0;z)}{\mathcal{Y}_{l-k-1}^{(k+1)}(0;z)} = \frac{-h(l;z)\prod_{k}^{l-1}[d_j z]}{\mathcal{Y}_{l-k-1}^{(k+1)}(0;z)[\mathcal{Y}_{l-k-1}^{(k+1)}(0;z) - h(l;z)\mathcal{Y}_{l-k-2}^{(k+1)}(0;z)]}.$$

Finally, using the fact that $l = k+m+1$, we obtain

$$\begin{aligned} &-(1-g_k)\mathcal{H}_{k+1}(k+m+1;z) = \\ &\quad \mathcal{S}_m^{(k+1)}(0,z) - \frac{\prod_{j=k+1}^{k+m} d_j z^m h(k+m+1;z)}{[\mathcal{Y}_{k+m+1}^{(k+1)}(0;z)]^2 - \mathcal{Y}_{k+m}^{(k+1)}(0;z)\mathcal{Y}_{k+m+1}^{(k+1)}(0;z)h(k+m+1;z)}, \end{aligned}$$

where

$$\mathcal{S}_m^{(k+1)}(0,z) = \frac{-(1-g_k)g_{k+1}z}{1} \underset{-}{} \frac{(1-g_{k+1})g_{k+2}z}{1} \underset{-}{} \cdots \frac{(1-g_{k+m-1})g_{k+m}z}{1},$$

is the m^{th} approximant of the $(k+1)^{th}$ tail of $\mathcal{F}(z)$. □

As mentioned earlier, from (2.1.6), (2.1.9) and (2.1.10), it is clear that tail sequences play a significant role in deriving the structural relations for the gap g-fractions. We now illustrate the role of tail sequences using particular g-fraction expansions, which will be later used to derive the class of Pick functions.

2.1.2 Tail sequences using hypergeometric functions

Consider the Gauss continued fraction

$$\frac{F(a,b+1;c+1;\omega)}{F(a,b;c;\omega)} = \frac{1}{1} \mathbin{-} \frac{(1-g_0)g_1\omega}{1} \mathbin{-} \frac{(1-g_1)g_2\omega}{1} \mathbin{-} \frac{(1-g_2)g_3\omega}{1} \mathbin{-} \cdots, \quad (2.1.12)$$

where $F(a,b;c;\omega)$ is the Gauss hypergeometric function and the parameters g_j, $j \geq 0$ are given by

$$g_{2p} = \frac{c-a+p}{c+2p} \qquad \text{and} \qquad g_{2p+1} = \frac{c-b+p}{c+2p+1}, \quad p \geq 0.$$

The correspondence and convergence properties of the Gauss continued fraction is studied in Wall [184, Theorem 89.1] using the contiguous relations satisfied by the Gauss hypergeometric function. Now, shifting b and c to $b-1$ and $c-1$ respectively in (2.1.12), we obtain

$$\frac{F(a,b;c;\omega)}{F(a,b-1;c-1;\omega)} = \frac{1}{1} \mathbin{-} \frac{(1-g_0)g_1\omega}{1} \mathbin{-} \frac{(1-g_1)g_2\omega}{1} \mathbin{-} \frac{(1-g_2)g_3\omega}{1} \mathbin{-} \cdots \quad (2.1.13)$$

so that the parameters g_j, $j \geq 0$ are given by

$$g_{2p} = \frac{c-a+p-1}{c+2p-1} \qquad \text{and} \qquad g_{2p+1} = \frac{c-b+p}{c+2p}, \quad p \geq 0.$$

For the rest of the chapter, the g-fractions that will be studied have the parameters k_p where $k_p = 1 - g_p$, $p \geq 0$ with g_p appearing as in (2.1.13). That is

$$k_{2p} = \frac{a+p}{c+2p-1} \qquad \text{and} \qquad k_{2p+1} = \frac{b+p}{c+2p}, \quad p \geq 0.$$

Our first goal is to find the analytic function having the continued fraction representation

$$\frac{1}{1}\underset{-}{} \frac{k_1\omega}{1}\underset{-}{} \frac{(1-k_1)k_2\omega}{1}\underset{-}{} \frac{(1-k_2)k_3\omega}{1}\underset{-}{} \cdots .$$

For this, let $\mathrm{R}^{(a,b,c)}(\omega)$ be the analytic function obtained from (2.1.13), as

$$\begin{aligned}\mathrm{R}^{(a,b,c)}(\omega) &= 1 - \frac{1}{k_0}\left[1 - \frac{F(a,b-1;c-1;\omega)}{F(a,b;c;\omega)}\right] \\ &= 1 - \frac{(1-k_1)\omega}{1}\underset{-}{} \frac{k_1(1-k_2)\omega}{1}\underset{-}{} \frac{k_2(1-k_3)\omega}{1}\underset{-}{} \frac{k_3(1-k_4)\omega}{1}\underset{-}{} \cdots .\end{aligned} \tag{2.1.14}$$

Consider the following two contiguous relations

$$F(a,b;c;\omega) = F(a,b-1;c-1;\omega) + \frac{a(c-b)}{(c-1)c}\omega F(a+1,b;c+1;\omega), \tag{2.1.15a}$$

$$F(a,b;c;\omega) = (1-\omega)F(a+1,b;c;\omega) + \frac{c-b}{c}\omega F(a+1,b;c+1;\omega), \tag{2.1.15b}$$

that can be easily proved by comparing the coefficients of ω^k on both sides. Now, using $k_0 = a/(c-1)$ in (2.1.15a), we have

$$\begin{aligned}\mathrm{R}^{(a,b,c)}(\omega) &= 1 - \frac{c-1}{a}\left[\frac{F(a,b;c;\omega) - F(a,b-1;c-1;\omega)}{F(a,b;c;\omega)}\right] \\ &= 1 - \frac{c-b}{c}\frac{F(a+1,b;c+1;\omega)}{F(a,b;c;\omega)} \\ &= (1-\omega)\frac{F(a+1,b;c;\omega)}{F(a,b;c;\omega)},\end{aligned}$$

where the last equality follows from (2.1.15b). Using the well known identity

$$\cfrac{1}{1+\cfrac{g_1 z}{1+\cfrac{(1-g_1)g_2 z}{1+\cfrac{(1-g_2)g_3 z}{1+\ddots}}}} \cdot \cfrac{1}{1+\cfrac{(1-g_1)z}{1+\cfrac{g_1(1-g_2)z}{1+\cfrac{g_2(1-g_3)z}{1+\ddots}}}} = \frac{1}{1+z} \tag{2.1.16}$$

proved in Wall [184, (75.3), p.281], we obtain from the continued fraction representation (2.1.14) of $\mathrm{R}^{(a,b,c)}(\omega)$,

$$\frac{F(a+1,b;c;\omega)}{F(a,b;c;\omega)} = \frac{1}{1} \underset{-}{} \frac{\frac{b}{c}\omega}{1} \underset{-}{} \frac{\frac{(c-b)(a+1)}{c(c+1)}\omega}{1} \underset{-}{} \frac{\frac{(c-a)(b+1)}{(c+1)(c+2)}\omega}{1} \underset{-}{} \cdots . \tag{2.1.17}$$

Remark 2.1.1. *The continued fraction (2.1.17) has also been derived by different means in Küstner [117] and studied in the context of geometric properties of hypergeometric functions.*

For further analysis, we note that by interchanging a and b in the Gauss continued fraction (2.1.12), we obtain

$$\frac{F(a+1,b;c+1;\omega)}{F(a,b;c;\omega)} = \frac{1}{1} \underset{-}{} \frac{\frac{b(c-a)}{c(c+1)}\omega}{1} \underset{-}{} \frac{\frac{(a+1)(c-b+1)}{(c+1)(c+2)}\omega}{1} \underset{-}{} \frac{\frac{(b+1)(c-a+1)}{(c+2)(c+3)}\omega}{1} \underset{-}{} \cdots . \tag{2.1.18}$$

Denoting the analytic function in the left hand side of (2.1.18) by $\mathrm{G}_1^{(a,b,c)}(\omega)$, we have

$$\mathrm{G}_1^{(a,b,c)}(\omega) = \frac{1}{1} \underset{-}{} \frac{k_1(1-k_2)\omega}{1} \underset{-}{} \frac{k_2(1-k_3)\omega}{1} \underset{-}{} \frac{k_3(1-k_4)\omega}{1} \underset{-}{} \cdots .$$

The following result gives a kind of generalization of the continued fraction identity (2.1.18). The correspondence and convergence properties of these continued fractions can be discussed similarly to that of the Gauss continued fraction.

Proposition 2.1.1. *For $n \geq 1$, let*

$$\mathrm{G}_n^{(a,b,c)}(\omega) = \frac{1}{1} \underset{-}{} \frac{k_n(1-k_{n+1})\omega}{1} \underset{-}{} \frac{k_{n+1}(1-k_{n+2})\omega}{1} \underset{-}{} \frac{k_{n+2}(1-k_{n+3})\omega}{1} \underset{-}{} \cdots . \tag{2.1.19}$$

Then, $\mathrm{G}_n^{(a,b,c)}(\omega)$, $n \geq 1$, is given by ratios of Gaussian hypergeometric functions, where

$$\mathrm{G}_{2j}^{(a,b,c)}(\omega) = \frac{F(a+j,b+j;c+2j;\omega)}{F(a+j,b+j-1;c+2j-1;\omega)} \quad j \geq 1,$$
$$\mathrm{G}_{2j+1}^{(a,b,c)}(\omega) = \frac{F(a+j+1,b+j;c+2j+1;\omega)}{F(a+j,b+j;c+2j;\omega)} \quad j \geq 0.$$

Proof. The case $j=0$ is the identity (2.1.18). Comparing the continued fractions for $\mathrm{G}_{2j+1}^{(a,b,c)}(\omega)$ and $\mathrm{G}_{2j-1}^{(a,b,c)}(\omega)$, $j\geq 1$, it can be seen that $\mathrm{G}_{2j+1}^{(a,b,c)}(\omega)$ can be obtained from $\mathrm{G}_{2j-1}^{(a,b,c)}(\omega)$ $j\geq 1$ by shifting $a\mapsto a+1$, $b\mapsto b+1$ and $c\mapsto c+2$.

For $n=2j$, $j\geq 1$, we note that the continued fraction on the right side of (2.1.19) is nothing but the Gauss continued fraction (2.1.12) with the respective shifts $a\mapsto a+j$, $b\mapsto b+j-1$ and $c\mapsto c+2j-1$ in the parameters. □

Instead of starting with $k_n(1-k_{n+1})$, as the first partial numerator term in the continued fraction (2.1.19), a modification by inserting a new term changes the hypergeometric ratio given in Proposition 2.1.1. We state this result as follows.

Theorem 2.1.5. *For $n\geq 1$, let*

$$\mathrm{F}_n^{(a,b,c)}(\omega)=\frac{1}{1}\,_{-}\,\frac{k_n\omega}{1}\,_{-}\,\frac{(1-k_n)k_{n+1}\omega}{1}\,_{-}\,\frac{(1-k_{n+1})k_{n+2}\omega}{1}\,_{-}\,\cdots.$$

Then for $j\geq 0$,

$$\mathrm{F}_{2j+1}^{(a,b,c)}(\omega)=\frac{F(a+j+1,b+j;c+2j;\omega)}{F(a+j,b+j;c+2j;\omega)},$$
$$\mathrm{F}_{2j+2}^{(a,b,c)}(\omega)=\frac{F(a+j+1,b+j+1;c+2j+1;\omega)}{F(a+j+1,b+j;c+2j+1;\omega)}.$$

Proof. For $n\geq 1$, let us denote

$$\begin{aligned}\mathrm{E}_{n+1}^{(a,b,c)}(\omega)&:=1-\frac{1}{k_n}\left(1-\frac{1}{\mathrm{G}_n^{(a,b,c)}(\omega)}\right)\\&=1-\frac{(1-k_{n+1})\omega}{1}\,_{-}\,\frac{k_{n+1}(1-k_{n+2})\omega}{1}\,_{-}\,\frac{k_{n+2}(1-k_{n+3})\omega}{1}\,_{-}\,\cdots.\end{aligned}$$

Then, using the identity (2.1.16), we obtain

$$\mathrm{F}_{n+1}^{(a,b,c)}(\omega)=\frac{\mathrm{E}_{n+1}^{(a,b,c)}(\omega)}{1-z}=\frac{1}{1}\,_{-}\,\frac{k_{n+1}\omega}{1}\,_{-}\,\frac{(1-k_{n+1})k_{n+2}\omega}{1}\,_{-}\,\frac{(1-k_{n+2})k_{n+3}\omega}{1}\,_{-}\,\cdots.$$

Hence, we need to determine the functions $\mathrm{E}_{n+1}^{(a,b,c)}(\omega)$ in terms of hypergeometric functions. For $n = 2j$, $j \geq 1$, using $k_{2j} = (a+j)/(c+2j-1)$ in (2.1.15a) yields

$$\frac{1}{k_{2j}}\left[1 - \frac{1}{\mathrm{G}_{2j}^{(a,b,c)}(\omega)}\right] = \frac{c-b+j}{c+2j}\omega\frac{F(a+j+1,b+j;c+2j+1;\omega)}{F(a+j,b+j;c+2j;\omega)}.$$

Further, shifting $a \mapsto a+j$, $b \mapsto b+j$ and $c \mapsto c+2j$ in (2.1.15b), we find that

$$\mathrm{E}_{2j+1}^{(a,b,c)}(\omega) = (1-\omega)\frac{F(a+j+1,b+j;c+2j;\omega)}{F(a+j,b+j;c+2j;\omega)},$$

so that

$$\mathrm{F}_{2j+1}^{(a,b,c)}(\omega) = \frac{F(a+j+1,b+j;c+2j;\omega)}{F(a+j;b+j;c+2j;\omega)} \quad j \geq 1.$$

Repeating the above steps, we find that for $n = 2j+1$, $j \geq 0$ and $k_{2j+1} = (b+j)/(c+2j)$, $j \geq 0$,

$$\mathrm{E}_{2j+2}^{(a,b,c)}(\omega) = (1-\omega)\frac{F(a+j+1,b+j+1;c+2j+1;\omega)}{F(a+j+1,b+j;c+2j+1;\omega)}, \quad j \geq 0,$$

which implies

$$\mathrm{F}_{2j+2}^{(a,b,c)}(\omega) = \frac{F(a+j+1,b+j+1;c+2j+1;\omega)}{F(a+j+1;b+j;c+2j+1;\omega)}, \quad j \geq 0. \qquad \square$$

For particular values of $\mathrm{F}_n^{(a,b,c)}(\omega)$, further properties of the ratio of hypergeometric function can be discussed. One particular case and ratios of hypergeometric functions that belong to a class of Pick functions are given in Section 2.3. Before discussing such specific cases, we consider another type of perturbation in g-fractions in the next section.

2.2 Perturbed Schur parameters

The Schur fraction is defined as

$$\alpha_0 + \frac{(1-|\alpha_0|^2)z}{\bar{\alpha}_0 z} \; \underset{+}{} \; \frac{1}{\alpha_1} \; \underset{+}{} \; \frac{(1-|\alpha_1|^2)z}{\bar{\alpha}_1 z} \; \underset{+}{} \; \frac{1}{\alpha_2} \; \underset{+}{} \; \frac{(1-|\alpha_2|^2)z}{\bar{\alpha}_2 z} \; \underset{+}{} \; \cdots, \qquad (2.2.1)$$

where α_j is related to the g_j occurring in the g-fraction (1.2.8) of a Carathéodory function by $\alpha_j = 1-2g_j$, $j \geq 1$. Similar to g-fraction, the Schur fraction also terminates if $|\alpha_n| = 1$ for some $n \in \mathbb{N} \cup \{0\}$. It may be noted that such a case occurs if, and only if, (Jones et al. [99]) $f(z)$ is a finite Blaschke product.

Let $\mathrm{A}_n(z)$ and $\mathrm{B}_n(z)$ denote the n^{th} partial numerator and denominator of (2.2.1) respectively. Then, with the initial values $\mathrm{A}_0(z) = \alpha_0$, $\mathrm{B}_0(z) = 1$, $\mathrm{A}_1(z) = z$ and $\mathrm{B}_1(z) = \bar{\alpha}_0 z$, the numerators and denominators of the even approximants satisfy

$$\begin{aligned} \mathrm{A}_{2p}(z) &= \alpha_p \mathrm{A}_{2p-1}(z) + \mathrm{A}_{2p-2}(z), \\ \mathrm{B}_{2p}(z) &= \alpha_p \mathrm{B}_{2p-1}(z) + \mathrm{B}_{2p-2}(z), \quad p \geq 1, \end{aligned} \tag{2.2.2}$$

while the numerators and denominators of the odd approximants satisfy

$$\begin{aligned} \mathrm{A}_{2p+1}(z) &= \bar{\alpha}_p z \mathrm{A}_{2p}(z) + (1-|\alpha_p|^2) z \mathrm{A}_{2p-1}(z), \\ \mathrm{B}_{2p+1}(z) &= \bar{\alpha}_p z \mathrm{B}_{2p}(z) + (1-|\alpha_p|^2) z \mathrm{B}_{2p-1}(z), \quad p \geq 1. \end{aligned} \tag{2.2.3}$$

Using (2.2.2) in (2.2.3) we obtain

$$\begin{aligned} \mathrm{A}_{2p+1}(z) &= z \mathrm{A}_{2p-1}(z) + \bar{\alpha}_p z \mathrm{A}_{2p-2}(z), \\ \mathrm{B}_{2p+1}(z) &= z \mathrm{B}_{2p-1}(z) + \bar{\alpha}_p z \mathrm{B}_{2p-2}(z). \end{aligned} \tag{2.2.4}$$

The relations (2.2.2) and (2.2.4) are sometimes written in the matrix form as

$$\begin{pmatrix} \mathrm{A}_{2p+1}(z) & \mathrm{B}_{2p+1}(z) \\ \mathrm{A}_{2p}(z) & \mathrm{B}_{2p}(z) \end{pmatrix} = \begin{pmatrix} z & \bar{\alpha}_p z \\ \alpha_p & 1 \end{pmatrix} \begin{pmatrix} \mathrm{A}_{2p-1}(z) & \mathrm{B}_{2p-1}(z) \\ \mathrm{A}_{2p-2}(z) & \mathrm{B}_{2p-2}(z) \end{pmatrix}, \quad p \geq 1. \tag{2.2.5}$$

It is also known (Njåstad [137]) that

$$\begin{aligned} \mathrm{A}_{2n+1}(z) = z\mathrm{B}^*_{2n}(z) \quad &; \quad \mathrm{B}_{2n+1}(z) = z\mathrm{A}^*_{2n}(z) \\ \mathrm{A}_{2n}(z) = \mathrm{B}^*_{2n+1}(z) \quad &; \quad \mathrm{B}_{2n}(z) = \mathrm{A}^*_{2n+1}(z), \end{aligned} \tag{2.2.6}$$

where $\mathrm{P}^*_n(z) = z^n \overline{\mathrm{P}_n(1/\bar{z})}$ for any polynomial $\mathrm{P}_n(z)$ with complex coefficients and of degree n. From (2.2.6), it follows that

$$\frac{\mathrm{A}_{2n+1}(z)}{\mathrm{B}_{2n+1}(z)} = \left(\frac{\mathrm{A}^*_{2n}(z)}{\mathrm{B}^*_{2n}(z)}\right)^{-1}, \quad \frac{\mathrm{A}_{2n}(z)}{\mathrm{B}_{2n}(z)} = \left(\frac{\mathrm{A}^*_{2n+1}(z)}{\mathrm{B}^*_{2n+1}(z)}\right)^{-1}.$$

Since the parameters g_j and α_j are uniquely related, the case of a single parameter g_k being replaced by $g_k^{(\beta_k)}$ can be studied using the Schur parameters. It is obvious that this is equivalent to studying the perturbed sequence $\{\alpha_j^{(\beta_k)}\}_{j=0}^{\infty}$, where

$$\alpha_j^{(\beta_k)} = \begin{cases} \alpha_j, & j \neq k; \\ \beta_k, & j = k. \end{cases} \tag{2.2.7}$$

Hence, we start with a given Schur function and study the perturbed Carathéodory function and its corresponding g-fraction. The following theorem gives the structural relation between the Schur function and the perturbed one. The proof follows the transfer matrix approach, which has also been used in the literature, for example, in Castillo [36] to study perturbed Szegő recurrences. For details of the method, we refer to Simon [156, Sections 3.2 and 3.4]

Theorem 2.2.1. *Let* $\mathrm{A}_k(z)$ *and* $\mathrm{B}_k(z)$ *be the* n^{th} *partial numerators and denominators of the Schur fraction associated with the sequence* $\{\alpha_k\}_{k=0}^{\infty}$. *If* $\mathrm{A}_j(z;k)$ *and* $\mathrm{B}_j(z;k)$ *are the* j^{th} *partial numerators and denominators of the Schur fraction associated with the sequence* $\{\alpha_j^{(\beta_k)}\}_{j=0}^{\infty}$ *as defined in* (2.2.7), *then the following structural relations hold for* $p \geq 2k$, $k \geq 1$.

$$z^{k-1}\prod_{j=0}^{k}(1-|\alpha_j|^2)\begin{pmatrix} \mathrm{A}_{2p+1}(z;k) & \mathrm{A}_{2p}(z;k) \\ \mathrm{B}_{2p+1}(z;k) & \mathrm{B}_{2p}(z;k) \end{pmatrix} = \mathfrak{T}(z;k)\begin{pmatrix} \mathrm{A}_{2p+1}(z) & \mathrm{A}_{2p}(z) \\ \mathrm{B}_{2p+1}(z) & \mathrm{B}_{2p}(z) \end{pmatrix}, \tag{2.2.8}$$

where the entries of the transfer matrix $\mathfrak{T}(z;k)$ *are given by*

$$\begin{pmatrix} \mathfrak{T}_{(1,1)} & \mathfrak{T}_{(1,2)} \\ \mathfrak{T}_{(2,1)} & \mathfrak{T}_{(2,2)} \end{pmatrix}$$

$$= \begin{pmatrix} p_k(z,k)\mathrm{A}_{2k-1}(z) + q_k^*(z,k)\mathrm{A}_{2k-2}(z) & q_k(z,k)\mathrm{A}_{2k-1}(z) + p_k^*(z,k)\mathrm{A}_{2k-2}(z) \\ p_k(z,k)\mathrm{B}_{2k-1}(z) + q_k^*(z,k)\mathrm{B}_{2k-2}(z) & q_k(z,k)\mathrm{B}_{2k-1}(z) + p_k^*(z,k)\mathrm{B}_{2k-2}(z) \end{pmatrix},$$

with the polynomial coefficients

$$p_k(z,k) = (\alpha_k - \beta_k)\mathrm{B}_{2k-1}(z) + (1 - \beta_k\bar{\alpha}_k)\mathrm{B}_{2k-2}(z),$$
$$q_k(z;k) = (\beta_k - \alpha_k)\mathrm{A}_{2k-1}(z) - (1 - \bar{\alpha}_k\beta_k)\mathrm{A}_{2k-2}(z).$$

Proof. Let us define

$$\Omega_p(z;\alpha) := \begin{pmatrix} \mathrm{A}_{2p+1}(z) & \mathrm{B}_{2p+1}(z) \\ \mathrm{A}_{2p}(z) & \mathrm{B}_{2p}(z) \end{pmatrix} \quad \text{and}$$
$$\Omega_p(z;\alpha;k) := \begin{pmatrix} \mathrm{A}_{2p+1}(z;k) & \mathrm{B}_{2p+1}(z;k) \\ \mathrm{A}_{2p}(z;k) & \mathrm{B}_{2p}(z;k) \end{pmatrix}.$$

Then the matrix relation (2.2.5) can be written as

$$\begin{aligned} \Omega_p(z;\alpha) &= T_p(\alpha_p)\cdot\Omega_{p-1}(z;\alpha) \\ &= T_p(\alpha_p)\cdot T_{p-1}(\alpha_{p-1})\cdot\dots\cdot T_1(\alpha_1)\cdot\Omega_0(z;\alpha), \quad p \geq 1, \end{aligned} \tag{2.2.9}$$

with the transfer matrices for $p \geq 1$ given by

$$T_p(\alpha_p) := \begin{pmatrix} z & \bar{\alpha}_p z \\ \alpha_p & 1 \end{pmatrix} \quad \text{and} \quad \Omega_0(z;\alpha) := T_0(\alpha_0) = \begin{pmatrix} z & \bar{\alpha}_0 z \\ \alpha_0 & 1 \end{pmatrix}.$$

From (2.2.7), it is clear that the perturbation arises while replacing α_k by β_k and hence

$$\Omega_p(z;\alpha;k) = \underbrace{T_p(\alpha_p)\cdots T_{k+1}(\alpha_{k+1})}\cdot T_k(\beta_k)\cdot\underbrace{T_{k-1}(\alpha_{k-1})\cdots T_1(\alpha_1)\Omega_0(z;\alpha)}. \tag{2.2.10}$$

Defining the matrix product

$$\Omega_{p-(k+1)}^{(k+1)}(z;\alpha) := T_p(\alpha_p)T_{p-1}(\alpha_{p-1})\cdots T_{k+1}(\alpha_{k+1})\Omega_0(z;\alpha),$$

whose entries are called associated polynomials of order $k+1$, we have

$$T_p(\alpha_p)T_{p-1}(\alpha_{p-1})\cdots T_{k+1}(\alpha_{k+1}) = \Omega^{(k+1)}_{p-(k+1)}(z;\alpha)\cdot[\Omega_0(z;\alpha)]^{-1}, \tag{2.2.11}$$

where $[\Omega_0(z;\alpha)]^{-1}$ denotes the matrix inverse of $\Omega_0(z;\alpha)$. Now, using (2.2.9) and (2.2.11) in (2.2.10), we get

$$\Omega_p(z;\alpha;k) = \Omega^{(k+1)}_{p-(k+1)}(z;\alpha)\cdot[\Omega_0(z;\alpha)]^{-1}\cdot T_k(\beta_k)\cdot\Omega_{k-1}(z;\alpha). \tag{2.2.12}$$

Again from (2.2.9),

$$\begin{aligned}\Omega_p(z;\alpha) &= \underbrace{T_p(\alpha_p)\cdots T_{k+1}(\alpha_{k+1})}\cdot\underbrace{T_k(\alpha_k)\cdots T_1(\alpha_1)\Omega_0(z;\alpha)}\\ &= \Omega^{(k+1)}_{p-(k+1)}(z;\alpha)\Omega_0^{-1}(z;\alpha)\cdot\Omega_k(z;\alpha),\end{aligned}$$

which means

$$\Omega^{(k+1)}_{p-(k+1)}(z;\alpha) = \Omega_p(z;\alpha)\cdot[\Omega_k(z;\alpha)]^{-1}\cdot\Omega_0(z;\alpha). \tag{2.2.13}$$

Using (2.2.13) in (2.2.12), we get

$$\Omega_p(z;\alpha;k) = \Omega_p(z,\alpha)\cdot[\Omega_k(z,\alpha)]^{-1}\cdot\Omega_0(z;\alpha)\cdot[\Omega_0(z;\alpha)]^{-1}\cdot T_k(\beta_k)\cdot\Omega_{k-1}(z,\alpha),$$

which implies

$$[\Omega_p(z;\alpha;k)]^T = \underbrace{[T_k(\beta_k)\Omega_{k-1}(z,\alpha)]^T\cdot[\Omega_k(z,\alpha)]^{-T}}\cdot[\Omega_p(z,\alpha)]^T$$

where $[\Omega_p(z,\alpha)]^T$ denotes the matrix transpose of $\Omega_p(z,\alpha)$. Using the relations (2.2.6), it can be proved that the product $[T_k(\beta_k)\Omega_{k-1}(z,\alpha)]^T\cdot\Omega_k^{-T}(z,\alpha)$ gives the transfer matrix $\mathfrak{T}(z;k)$ leading to the structural relations (2.2.8). □

2.2.1 Rational transformation of Schur functions

It follows from the structural relations obtained in Theorem (2.2.1) that

$$z^{k-1}\prod_{j=0}^{k}(1-|\alpha_j|^2)\begin{pmatrix}\mathrm{A}_{2p}(z;k)\\ \mathrm{B}_{2p}(z;k)\end{pmatrix}=\mathfrak{T}(z;k)\begin{pmatrix}\mathrm{A}_{2p}(z)\\ \mathrm{B}_{2p}(z)\end{pmatrix}$$
$$=\begin{pmatrix}\mathfrak{T}_{(1,1)}\mathrm{A}_{2p}(z)+\mathfrak{T}_{(1,2)}\mathrm{B}_{2p}(z)\\ \mathfrak{T}_{(2,1)}\mathrm{A}_{2p}(z)+\mathfrak{T}_{(2,2)}\mathrm{B}_{2p}(z)\end{pmatrix},$$

which implies the sequence of even approximants of the Schur fraction is

$$\frac{\mathrm{A}_{2p}(z;k)}{\mathrm{B}_{2p}(z;k)}=\frac{\mathfrak{T}_{(1,2)}+\mathfrak{T}_{(1,1)}\left(\mathrm{A}_{2p}(z)/\mathrm{B}_{2p}(z)\right)}{\mathfrak{T}_{(2,2)}+\mathfrak{T}_{(2,1)}\left(\mathrm{A}_{2p}(z)/\mathrm{B}_{2p}(z)\right)},\quad p\geq 1. \tag{2.2.14}$$

The following result regarding the convergence of a positive ($|\alpha_n|<1$, $n\geq 0$) Schur fraction is proved in Jones et al. [99, Theorem 2.2].

Theorem 2.2.2. *Given a positive Schur fraction, the sequence $\{\mathrm{A}_{2m}/\mathrm{B}_{2m}\}_{m=0}^{\infty}$ converges to a function $f(z)$, analytic for $|z|<1$. Further, if $\alpha_0\in\mathbb{R}\setminus\{0\}$, then the sequence $\{\mathrm{A}_{2m+1}/\mathrm{B}_{2m+1}\}_{m=0}^{\infty}$ converges to $g(z)$, analytic for $|z|>1$. The function $f(z)$ is a Schur function and $g(z)=1/\overline{f(1/\bar{z})}$.*

The essence of Theorem 2.2.2 is that the $(2n)^{th}$ approximant of the Schur fraction (1.2.5) coincides with the n^{th} approximant of the Schur algorithm (1.2.4) so that $f(z)$ is the limit of $\mathrm{A}_{2n}(z)/\mathrm{B}_{2n}(z)$ in the unit disk $\mathbb{D}$. Hence taking limits on both sides of (2.2.14) gives the perturbed Schur function as

$$f^{(\beta_k)}(z;k)=\frac{\mathfrak{T}_{(1,2)}+\mathfrak{T}_{(1,1)}f(z)}{\mathfrak{T}_{(2,2)}+\mathfrak{T}_{(2,1)}f(z)}. \tag{2.2.15}$$

Now, we discuss these structural relations in case of a non-constant Schur function of the form $f(z)=cz+d$ with the perturbation $\alpha_1\mapsto\beta_1$. From Theorem 2.2.1, we have

$$p_1(z,1)=(\alpha_1-\beta_1)\bar{\alpha}_0z+(1-\bar{\alpha}_1\beta_1);\quad p_1^*(z,1)=(1-\alpha_1\bar{\beta}_1)z+(\bar{\alpha}_1-\bar{\beta}_1)\alpha_0;$$
$$q_1(z,1)=(\beta_1-\alpha_1)z-(1-\bar{\alpha}_1\beta_1)\alpha_0;\quad q_1^*(z,1)=(\bar{\beta}_1-\bar{\alpha}_1)-(1-\alpha_1\bar{\beta}_1)\bar{\alpha}_0z.$$

The entries of the transfer matrix $\mathfrak{T}(z;k)$ are

$$\begin{aligned}
\mathfrak{T}_{(1,1)} &= (\alpha_1-\beta_1)z^2+[(1-\beta_1\bar{\alpha}_1)-(1-\alpha_1\bar{\beta}_1)|\alpha_0|^2]z+\alpha_0(\bar{\beta}_1-\bar{\alpha}_1),\\
\mathfrak{T}_{(1,2)} &= (\beta_1-\alpha_1)z^2+[(1-\alpha_1\bar{\beta}_1)\alpha_0-(1-\bar{\alpha}_1\beta_1)\alpha_0]z+(\bar{\alpha}_1-\bar{\beta}_1)\alpha_0^2,\\
\mathfrak{T}_{(2,1)} &= (\alpha_1-\beta_1)(\bar{\alpha}_0)^2z^2+[(1-\beta_1\bar{\alpha}_1)\bar{\alpha}_0-(1-\alpha_1\bar{\beta}_1)\bar{\alpha}_0]z+(\bar{\beta}_1-\bar{\alpha}_1),\\
\mathfrak{T}_{(2,2)} &= (\beta_1-\alpha_1)\bar{\alpha}_0z^2+[(1-\alpha_1\bar{\beta}_1)-(1-\bar{\alpha}_1\beta_1)|\alpha_0|^2]z+(\bar{\alpha}_1-\bar{\beta}_1)\alpha_0.
\end{aligned}$$

Using (2.2.15), the transformed Schur function is a rational function given by

$$f^{(\beta_1)}(z,1)=\frac{Az^3+Bz^2+Cz+D}{\hat{A}z^3+\hat{B}z^2+\hat{C}z+\hat{D}},$$

where

$$\begin{aligned}
&A=(\alpha_1-\beta_1)\bar{\alpha}_0c, \quad B=(\beta_1-\alpha_1)(1-\bar{\alpha}_0d)+c(1-\beta_1\bar{\alpha}_1)-c|\alpha_0|^2(1-\alpha_1\bar{\beta}_1),\\
&C=(1-\alpha_1\bar{\beta}_1)(\alpha_0-d|\alpha_0|^2)+(1-\bar{\alpha}_1\beta_1)(d-\alpha_0)+c\alpha_0(\bar{\beta}_1-\bar{\alpha}_1),\\
&D=(\bar{\beta}_1-\bar{\alpha}_1)(d-\alpha_0)\alpha_0,
\end{aligned}$$

and

$$\begin{aligned}
&\hat{A}=(\alpha_1-\beta_1)(\bar{\alpha}_0)^2c, \quad \hat{B}=(\beta_1-\alpha_1)(1-\bar{\alpha}_0d)\bar{\alpha}_0+c(1-\beta_1\bar{\alpha}_1)\bar{\alpha}_0-c(1-\alpha_1\bar{\beta}_1)\bar{\alpha}_0,\\
&\hat{C}=(1-\alpha_1\bar{\beta}_1)(1-d\bar{\alpha}_0)+(1-\beta_1\bar{\alpha}_1)(d\bar{\alpha}_0-|\alpha_0|^2)+c(\bar{\beta}_1-\bar{\alpha}_1),\\
&\hat{D}=(\bar{\beta}_1-\bar{\alpha}_1)(d-\alpha_0).
\end{aligned}$$

Since $d=f(0)=\alpha_0$, $D=\hat{D}=0$. This leads to the following easy consequence of Theorem 2.2.1.

Corollary 2.2.1. *Let $f(z)=cz+\alpha_0$ denote the class of Schur functions. Then, with the perturbation $\alpha_1\mapsto\beta_1$, the resulting Schur function is the rational function given by*

$$f^{(\beta_1)}(z;1)=\frac{Az^2+Bz+C}{\hat{A}z^2+\hat{B}z+\hat{C}}, \quad A\neq 0, \quad \hat{A}\neq 0. \tag{2.2.16}$$

We consider an example illustrating the above results.

Example 2.2.1. *Consider the sequence of Schur parameters $\{\alpha_n\}_{n=0}^{\infty}$ given by $\alpha_0=1/2$*

and $\alpha_n = 2/(2n+1)$, $n \geq 1$. *Then, as in Jones et al.* [99, Example 6.3], *the Schur function is* $f(z) = (1+z)/2$ *with*

$$\begin{aligned} \mathrm{A}_{2m}(z) &= \frac{1}{2} + \frac{2z^{m+2} - 2(m+1)z^2 + 2mz}{(2m+1)(z-1)^2}, \\ \mathrm{B}_{2m}(z) &= 1 + \frac{z^{m+2} + z^{m+1} - (2m+1)z^2 + (2m-1)z}{(2m+1)(z-1)^2}, \\ \mathrm{A}_{2m+1}(z) &= \frac{z + z^2 - (2m+3)z^{m+2} + (2m+1)z^{m+3}}{(2m+1)(z-1)^2}, \\ \mathrm{B}_{2m+1}(z) &= \frac{z^{m+1}}{2} + 2\frac{z - (m+1)z^{m+1} + mz^{m+2}}{(2m+1)(z-1)^2}. \end{aligned}$$

We study the perturbation $\alpha_1 \mapsto \beta_1 = 1/2$. *For the transfer matrix* $\mathfrak{T}(z;k)$, *the polynomials required are*

$$p_1(z) = \frac{z}{12} + \frac{2}{3}, \quad p_1^*(z) = \frac{2}{3}z + \frac{1}{12}, \quad q_1(z) = -\frac{z}{6} - \frac{1}{3}, \quad q_1^*(z) = -\frac{z}{3} - \frac{1}{6},$$

so that the entries of $\mathfrak{T}(z;k)$ *are*

$$\begin{aligned} \mathfrak{T}_{(1,1)} &= \frac{z^2}{12} + \frac{z}{2} - \frac{1}{12}, \quad \mathfrak{T}_{(1,2)} = -\frac{z^2}{6} + \frac{1}{12}, \\ \mathfrak{T}_{(2,1)} &= \frac{z^2}{24} - \frac{1}{6}, \quad \mathfrak{T}_{(2,2)} = -\frac{z^2}{12} + \frac{z}{2} + \frac{1}{12}. \end{aligned}$$

Hence, the transformed Schur function is given by (2.2.16),

$$f^{(1/2)}(z;1) = 2\frac{z^2 - 3z + 5}{z^2 - 3z + 20}.$$

Observe that the functions $f(z)$ *and* $f^{(1/2)}(z;1)$ *are analytic in the unit disk* $\mathbb{D}$ *with* $f(0) = f^{(1/2)}(0;1)$. *Further, the analytic function* $\omega(z)$ *defined as*

$$\omega(z) := f^{-1}(f^{(1/2)}(z;1)) = \frac{3z(z-3)}{z^2 - 3z + 20},$$

is analytic in $\mathbb{D}$ *with* $|\omega(z)| < 1$. *By Schwarz lemma* $|\omega(z)| < |z|$ *for* $0 < |z| < 1$ *unless* $\omega(z)$ *is a pure rotation. In such a case the range of* $f^{(1/2)}(z;1)$ *is contained in the range of* $f(z)$. *The function* $f^{(1/2)}(z;1)$ *is said to be subordinate to* $f(z)$ *and written as* $f^{(1/2)}(z;1) \prec f(z)$ *for* $z \in \mathbb{D}$. *For more information about subordination of analytic* *functions, we refer to Duren* [72, Ch. 6].

We plot the ranges of both the Schur functions below. In Figure 2.1, the outermost

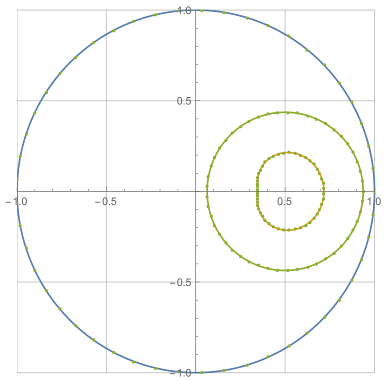


Figure 2.1: Subordinate Schur functions

circle is the unit circle while the middle one is the image of $|z| = 0.9$ *under* $f(z)$ *which is again a circle with center at* $z = 1/2$. *The innermost figure is the image of* $|z| = 0.9$ *under* $f^{(1/2)}(z;1)$.

2.2.2 The change in Carathéodory function

Let the Carathéodory function associated with the perturbed Schur function $f^{(\beta_k)}(z;k)$ be denoted by $\mathcal{C}^{(\beta_k)}(z;k)$. Then, using (2.2.15), we can write

$$\mathcal{C}^{(\beta_k)}(z;k) = \frac{1 + zf^{(\beta_k)}(z;k)}{1 - zf^{(\beta_k)}(z;k)} = \frac{(\mathfrak{T}_{2,2} + z\mathfrak{T}_{1,2}) + (\mathfrak{T}_{2,1} + z\mathfrak{T}_{1,1})f(z)}{(\mathfrak{T}_{2,2} - z\mathfrak{T}_{1,2}) + (\mathfrak{T}_{2,1} - z\mathfrak{T}_{1,1})f(z)}.$$

Further, using the relations

$$\mathcal{C}(z) = \frac{1 + zf(z)}{1 - zf(z)} \Longleftrightarrow f(z) = \frac{\mathcal{C}(z) - 1}{z(\mathcal{C}(z) + 1)},$$

the perturbed Carathéodory function is

$$\mathcal{C}^{(\beta_k)}(z;k) = \frac{\mathcal{Y}^{-}(z) + \mathcal{Y}^{+}(z)\mathcal{C}(z)}{\mathcal{W}^{-}(z) + \mathcal{W}^{+}(z)\mathcal{C}(z)}, \tag{2.2.17}$$

where the coefficients in terms of the entries of the transfer matrix are

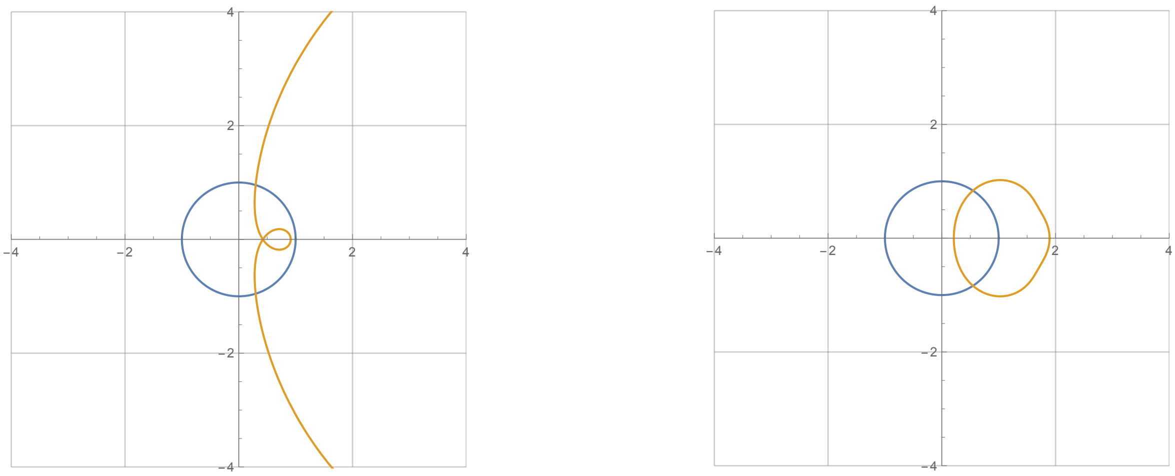
$$\mathcal{Y}^{\pm}(z) = z(\mathfrak{T}_{(2,2)} + z\mathfrak{T}_{(1,2)}) \pm (\mathfrak{T}_{(2,1)} + z\mathfrak{T}_{(1,1)}),$$

$$\mathcal{W}^{\pm}(z) = z(\mathfrak{T}_{(2,2)} - z\mathfrak{T}_{(1,2)}) \pm (\mathfrak{T}_{(2,1)} - z\mathfrak{T}_{(1,1)}).$$

As an illustration, for the Schur function $f(z) = (1+z)/2$, it can be verified from (2.2.17) that

$$\mathcal{C}(z) = \frac{2+z+z^2}{2-z-z^2} \quad \text{and} \quad \mathcal{C}^{(1/2)}(z;1) = \frac{2z^3-5z^2+7z+20}{-2z^3+7z^2-13z+20}.$$

We plot these Carathéodory functions below.



(a) The function $\mathcal{C}(z)$. (b) The function $\mathcal{C}^{(1/2)}(z;1)$.

Figure 2.2: Perturbed mapping properties of Carathéodory functions.

In Figures 2.2a and 2.2b, the ranges of the original and perturbed Carathéodory functions are respectively plotted for $|z| = 0.9$. It may be observed that the range of $\mathcal{C}(z)$ is unbounded (Figure 2.2a) which is clear as $z = 1$ is a pole of $\mathcal{C}(z)$. However $\mathcal{C}^{1/2}(z;1)$ has simple poles at $5/2$ and $(1 \pm i\sqrt{15})/2$ and hence with the use of perturbation we are able to make the range bounded (Figure 2.2b).

We recall that the sequence $\{\gamma_j\}_{j=0}^{\infty}$ satisfying the recurrence relation

$$\gamma_0 = 1, \quad \gamma_{p+1} = \frac{\gamma_p - \bar{\alpha}_p}{1-\alpha_p\gamma_p}, \quad p \geq 0. \tag{2.2.18}$$

where $\alpha_j' s$ are the Schur parameters play an important role in the g-fraction expansion for a special class of Carathéodory functions $\mathcal{C}(z)$ such that $\mathcal{C}(\mathbb{R}) \subseteq \mathbb{R}$ and $\mathcal{C}(0) = 1$.

Let the sequence $\{\gamma_j^{(\beta_k)}\}$ correspond to the perturbed Carathéodory function $\mathcal{C}^{(\beta_k)}(z;k)$. Since only α_k is perturbed, it is clear that γ_j remains unchanged for $j = 0, 1, \cdots, k$. The first change, γ_{k+1} to $\gamma_{k+1}^{(\beta_k)}$, occurs when α_k is replaced by β_k. Consequently, γ_{k+j}, $j \geq 1$, change to $\gamma_{k+j}^{(\beta_k)}$, $j \geq 1$, respectively. The next result shows that $\gamma_j^{(\beta_k)}$ can be expressed as a bilinear transformation of γ_j for $j \geq k+1$.

Theorem 2.2.3. *Let $\{\gamma_j\}_{j=0}^{\infty}$ be the sequence corresponding to $\{\alpha_j\}_{j=0}^{\infty}$ and $\{\gamma_j^{(\beta_k)}\}_{j=0}^{\infty}$ that to $\{\alpha_j^{(\beta_k)}\}_{j=0}^{\infty}$. Then,*

$$\gamma_{k+j}^{(\beta_k)} = \frac{\bar{a}_{k+j}\gamma_{k+j} - b_{k+j}}{-\bar{b}_{k+j}\gamma_{k+j} + a_{k+j}}, \quad j \geq 1, \tag{2.2.19}$$

where

(i) $a_{k+1} = \dfrac{1-\bar{\alpha}_k\beta_k}{1-|\beta_k|^2}$ *and* $b_{k+1} = \dfrac{\bar{\beta}_k - \bar{\alpha}_k}{1-|\beta_k|^2}$, *(j=1).*

(ii) For $j \geq 2$,

$$\begin{pmatrix} a_{k+j} \\ b_{k+j} \end{pmatrix} = \frac{1}{1-|\alpha_{k+j-1}|^2}\begin{pmatrix} 1 & \alpha_{k+j-1} \\ \bar{\alpha}_{k+j-1} & 1 \end{pmatrix}\begin{pmatrix} a_{k+j-1} - \bar{\alpha}_{k+j-1}\bar{b}_{k+j-1} \\ b_{k+j-1} - \bar{\alpha}_{k+j-1}\bar{a}_{k+j-1} \end{pmatrix}.$$

Proof. First, consider the expression

$$\frac{a_{k+1}\gamma_{k+1}^{(\beta_k)} + b_{k+1}}{\bar{b}_{k+1}\gamma_{k+1}^{(\beta_k)} + \bar{a}_{k+1}}.$$

Substituting $\gamma_{k+1}^{(\beta_k)} = (\gamma_k - \bar{\beta}_k)/(1-\beta_k\gamma_k)$ and the given values of a_{k+1} and b_{k+1}, it simplifies to

$$\frac{(1-\bar{\alpha}_k\beta_k)(\gamma_k - \bar{\beta}_k) + (\bar{\beta}_k - \bar{\alpha}_k)(1-\alpha_k\gamma_k)}{(\beta_k - \alpha_k)(\gamma_k - \bar{\beta}_k) + (1-\alpha_k\bar{\gamma}_k)(1-\beta_k\gamma_k)} = \frac{\gamma_k(1-|\beta_k|^2) - \bar{\alpha}_k(1-|\beta_k|^2)}{(1-|\beta_k|^2) - \alpha_k\gamma_k(1-|\beta|^2)} = \gamma_{k+1}.$$

Since $|a_{k+1}|^2 - |b_{k+1}|^2 = (1-|\alpha_k|^2)/(1-|\beta_k|^2) \neq 0$, (2.2.19) is proved for $j = 1$.

Next, let

$$\frac{a_{k+2}\gamma_{k+2}^{(\beta_k)} + b_{k+2}}{\bar{b}_{k+2}\gamma_{k+2}^{(\beta_k)} + \bar{a}_{k+2}} = \frac{(a_{k+2} - \alpha_{k+1}b_{k+2})\gamma_{k+1}^{(\beta_k)} + (b_{k+2} - \bar{\alpha}_{k+1}a_{k+2})}{(\bar{b}_{k+2} - \alpha_{k+1}\bar{a}_{k+2})\gamma_{k+1}^{(\beta_k)} + (\bar{a}_{k+2} - \bar{\alpha}_{k+1}\bar{b}_{k+2})} = \frac{NUM(\gamma_k)}{DEN(\gamma_k)}.$$

Substituting the given values of a_{k+2} and b_{k+2}, the numerator is

$$NUM(\gamma_k) = (1 - |\alpha_{k+1}|^2)[(a_{k+1} - \bar{\alpha}_{k+1}\bar{b}_{k+1})\gamma_{k+1}^{(\beta_k)} + (b_{k+1} - \bar{\alpha}_{k+1}\bar{a}_{k+1})],$$

which, by using $\gamma_{k+1}^{(\beta_k)} = (\bar{a}_{k+1} - b_{k+1})/(-\bar{b}_{k+1}\gamma_{k+1} + a_{k+1})$, can be written as

$$NUM(\gamma_k) = (1 - |\alpha_{k+1}|^2)(|\alpha_{k+1}|^2 - |b_{k+1}|^2)(\gamma_{k+1} - \bar{\alpha}_{k+1}).$$

With similar calculations, we obtain

$$DEN(\gamma_k) = (1 - |\alpha_{k+1}|^2)(|\alpha_{k+1}|^2 - |b_{k+1}|^2)(1 - \alpha_{k+1}\gamma_{k+1}).$$

This means

$$\frac{NUM(\gamma_k)}{DEN(\gamma_k)} = \frac{\gamma_{k+1} - \bar{\alpha}_{k+1}}{1 - \alpha_{k+1}\gamma_{k+1}} = \gamma_{k+2},$$

where $|a_{k+2}|^2 - |b_{k+2}|^2 = |a_{k+1}|^2 - |b_{k+1}|^2 \neq 0$, thus proving (2.2.19) for $j = 2$. The remaining part of the proof follows by a simple induction on j. □

However, we note that since γ_j and $\gamma_j^{(\beta_k)}$ are related by a bilinear transformation, the expressions for a_{k+j} and b_{k+j}, $j \geq 1$, are not unique. Finally, we note the following. $\mathcal{C}(0) = \mathcal{C}^{(\beta_k)}(0; k) = 1$ and both $\mathcal{C}(z)$ and $\mathcal{C}^{(\beta_k)}(z; k)$ are real for real z. Hence, the following g-fraction

$$\frac{1-z}{1+z}\mathcal{C}^{(\beta_k)}(z) = \frac{1}{1}\,\underline{}\,\frac{g_1\omega}{1}\,\underline{}\,\frac{(1-g_1)g_2\omega}{1}\,\underline{}\,\cdots\frac{(1-g_k)g_{k+1}^{(\beta_k)}\omega}{1}\,\underline{}\,\frac{(1-g_{k+1}^{(\beta_k)})g_{k+2}\omega}{1}\,\underline{}\,\cdots$$

is obtained for $k \geq 0$, where $g_j = (1-\alpha_{j-1})/2$, $j = 1, \cdots, k, k+2 \cdots$, $g_{k+1}^{(\beta_k)} = (1-\beta_k)/2$ and $\omega = -4z/(1-z)^2$.

Remark 2.2.1. *It is interesting to observe that study of such perturbations in Schur parameters leads to subordination results in function spaces. The Littlewood Subordination Theorem (Kumar and Singh [114]) for example leads to contractive composition operators on spaces of functions holomorphic on the unit disk. We refer to Cui et al. [44], Ghosh and Srivastava [82], Sharma and Kumar [154] and Maji and Srivas-*

tava [126] for information on such function spaces.

2.3 Two classes of special functions

Consider a function $f(\omega)$ analytic in the complex plane except for a slit along the real axis from 1 to ∞. If $f(0) = f'(0) - 1 = 0$, $f(\mathbb{R}) \subseteq \mathbb{R}$ and $\text{Im } f(\omega) \neq 0$ for $z \in \mathbb{C} \setminus \mathbb{R}$ in this slit plane, $f(\omega)$ is said to be typically-real in this slit plane. A fundamental result regarding such functions is the following lemma proved in corollaries 2.1 and 2.2 of Merkes [132].

Lemma 2.3.1. [132] *A necessary and sufficient condition for $f(\omega)$ to be a typically real function in the cut plane $\mathbb{C} \setminus [1, \infty)$ is that there exists a non-decreasing function $\nu(t) : [0, 1] \longrightarrow [0, 1]$ such that $\nu(1) - \nu(0) = 1$ and*

$$f(\omega) = \int_0^1 \frac{\omega}{1 - \omega t} d\nu(t), \quad \omega \in \mathbb{C} \setminus [1, \infty). \tag{2.3.1}$$

Let the sequence $\{\nu_j\}_{j=0}^{\infty}$ with $\nu_0 = 1$ be defined by $\nu_j = \int_0^1 t^j d\nu(t)$, $j \geq 0$, where $d\nu(t)$ is as obtained in (2.3.1). Then, $\{\nu_j\}_{j=0}^{\infty}$ becomes the Hausdorff moment sequence. By Wall [184, Theorem 69.2], the power series $F(\omega) = \sum_{j=0}^{\infty} \nu_j \omega^j = \int_0^1 \dfrac{1}{1 - \omega t} d\nu(t)$, has a continued fraction expansion of the form

$$\int_0^1 \frac{1}{1 - \omega t} d\nu(t) = \frac{1}{1} \underset{-}{} \frac{(1 - g_0) g_1 \omega}{1} \underset{-}{} \frac{(1 - g_1) g_2 \omega}{1} \underset{-}{} \frac{(1 - g_2) g_3 \omega}{1} \underset{-}{} \cdots,$$

where $0 \leq g_p \leq 1$, $p \geq 0$. Such functions $F(\omega)$ are analytic in the slit domain $\mathbb{C} \setminus [1, \infty)$ and belong to the class of Pick functions which are analytic in the upper half plane and have a positive imaginary part. For more details on Pick functions, we refer to the monograph of Donoghue [68].

2.3.1 A class of Pick functions

We characterize some members of the class of Pick functions using the gap g-fraction $\mathrm{F}_2^{(a,b,c)}(\omega)$ (which is still a g-fraction) given in Theorem 2.1.5. The procedure is similar to that of Küstner [117, Theorem 1.5]. The discussion uses the results of Küstner [117, Lemma 3.1, Remark 3.2] and Merkes [132, Theorem 3.1] which we state as the following lemma.

Lemma 2.3.2. [117,132] *Let* $\nu : [0,1] \longrightarrow [0,1]$ *be non-decreasing with* $\nu(1)-\nu(0) = 1$. *Then the function*

$$\omega \mapsto \int_0^1 \frac{\omega}{1-\omega t} d\nu(t)$$

is analytic in the cut plane $\mathbb{C} \setminus [1,\infty)$ *and maps both the unit disk* $\mathbb{D}$ *and the half-plane* $\{\omega \in \mathbb{C} : \mathrm{Re}\ \omega < 1\}$ *univalently onto domains which are convex in the direction of the imaginary axis.*

By a domain convex in the direction of imaginary axis, it is meant that every line parallel to the imaginary axis has either connected or empty intersection with the corresponding domain. For more information in this direction, we refer to Baricz and Swaminathan [13], Duren [72] and Ismail et al. [92].

Theorem 2.3.1. *If* $a, b, c \in \mathbb{R}$ *with* $-1 < a \leq c$ *and* $0 \leq b \leq c$, *then the functions*

$$\begin{array}{lll} \omega \mapsto \dfrac{F(a+1,b+1;c+1;\omega)}{F(a+1,b;c+1;\omega)} & ; & \omega \mapsto \dfrac{\omega F(a+1,b+1;c+1;\omega)}{F(a+1,b;c+1;\omega)} \\ \omega \mapsto \dfrac{F(a+2,b+1;c+2;\omega)}{F(a+1,b;c+1;\omega)} & ; & \omega \mapsto \dfrac{\omega F(a+2,b+1;c+2;\omega)}{F(a+1,b;c+1;\omega)} \\ \omega \mapsto \dfrac{F(a+2,b+1;c+2;\omega)}{F(a+1,b+1;c+1;\omega)} & ; & \omega \mapsto \dfrac{\omega F(a+2,b+1;c+2;\omega)}{F(a+1,b+1;c+1;\omega)} \end{array}$$

are analytic in $\mathbb{C} \setminus [1,\infty)$ *and each function map both the open unit disk* $\mathbb{D}$ *and the half plane* $\{\omega \in \mathbb{C} : \mathrm{Re}\ \omega < 1\}$ *univalently onto domains that are convex in the direction of the imaginary axis.*

Proof. From Theorem 2.1.5 we have

$$\mathrm{F}_2^{(a,b,c)}(\omega) = \frac{1}{1}\, _- \, \frac{k_2\omega}{1}\, _- \, \frac{(1-k_2)k_3\omega}{1}\, _- \, \frac{(1-k_3)k_4\omega}{1}\, _- \, \cdots,$$

where $\mathrm{F}_2^{(a,b,c)}(\omega) = F(a+1,b+1;c+1;\omega)/F(a+1,b;c+1;\omega)$,

$$k_{2p} = \frac{a+p}{c+2p-1} \qquad \text{and} \qquad k_{2p+1} = \frac{b+p}{c+2p}, \quad p \geq 1.$$

Then, with the given restrictions on a, b and c, $0 < k_j < 1$ and hence $\mathrm{F}_2^{(a,b,c)}(\omega)$ has a g-fraction expansion. By Wall [184, Theorem 69.2], there exists a non-decreasing function $\nu_0 : [0,1] \mapsto [0,1]$ with a total increase of 1 and

$$\frac{F(a+1,b+1;c+1;\omega)}{F(a+1,b;c+1;\omega)} = \int_0^1 \frac{1}{1-\omega t} d\nu_0(t), \quad \omega \in \mathbb{C} \setminus [1,\infty), \tag{2.3.2}$$

which implies

$$\frac{\omega F(a+1,b+1;c+1;\omega)}{F(a+1,b;c+1;\omega)} = \int_0^1 \frac{\omega}{1-\omega t} d\nu_0(t), \quad \omega \in \mathbb{C} \setminus [1,\infty). \tag{2.3.3}$$

From the power series correspondence of the g-fraction expansion of $\mathrm{F}_2^{(a,b,c)}(\omega)$, k_2 is the coefficient of w which, from (2.3.2), is also given by $\int_0^1 t d\nu_0(t)$. Hence, if we define

$$\nu_1(\sigma) = \frac{1}{k_2}\int_0^\sigma s d\nu_0(s), \quad k_2 = \frac{(a+1)}{(c+1)} > 0,$$

it follows that $\nu_1 : [0,1] \mapsto [0,1]$ is again a non-decreasing map with $\nu_1(1) - \nu_1(0) = 1$. Further, interchanging a and b in the contiguous relation

$$F(a+1,b;c;\omega) - F(a,b;c;\omega) = \frac{b}{c}\omega F(a+1,b+1;c+1;\omega), \tag{2.3.4}$$

we obtain

$$k_2\omega\frac{F(a+2,b+1;c+2;\omega)}{F(a+1,b;c+1;\omega)} = \frac{F(a+1,b+1;c+1;\omega)}{F(a+1,b;c+1;\omega)} - 1. \tag{2.3.5}$$

Now, $\int_0^1 \frac{\omega}{1-\omega t} d\nu_1(t) = \frac{1}{k_2}\int_0^1 \frac{\omega t - 1 + 1}{1-\omega t} d\nu_0(t)$ implies

$$k_2 \int_0^1 \frac{\omega}{1-\omega t} d\nu_1(t) = \frac{F(a+1,b+1;c+1;\omega)}{F(a+1,b;c+1;\omega)} - 1.$$

Comparing with (2.3.5), we obtain

$$\frac{\omega F(a+2,b+1;c+2;\omega)}{F(a+1,b;c+1;\omega)} = \int_0^1 \frac{\omega}{1-\omega t} d\nu_1(t), \quad \omega \in \mathbb{C} \setminus [1,\infty),$$

and hence

$$\frac{F(a+1,b+1;c+1;\omega)}{F(a+1,b;c+1;\omega)} = 1 + k_2 \int_0^1 \frac{\omega}{1-\omega t} d\nu_1(t), \quad \omega \in \mathbb{C} \setminus [1,\infty).$$

Further, noting that the coefficient of ω in $F(a+2,b+1;c+2;\omega)/F(a+1,b;c+1;\omega)$ is $[(b+1)(c-a)]/[(c+1)(c+2)] = k_3 + (1-k_3)k_2$, we define

$$\nu_2(\sigma) = \frac{1}{k_3 + k_2(1-k_3)} \int_0^\sigma s d\nu_1(s),$$

and find that

$$\frac{F(a+2,b+1;c+2;\omega)}{F(a+1,b;c+1;\omega)} = 1 + [k_3 + k_2(1-k_3)] \int_0^1 \frac{\omega}{1-\sigma\omega} d\nu_2(\sigma).$$

Finally from Gauss continued fraction (2.1.12), we conclude that $F(a+2,b+1;c+2;\omega)/F(a+1,b+1;c+1;\omega)$ has a g-fraction expansion and so there exists a map $\nu_3 : [0,1] \mapsto [0,1]$ which is non-decreasing, $\nu_3(1) - \nu_3(0) = 1$ and

$$\frac{\omega F(a+2,b+1;c+2;\omega)}{F(a+1;b+1;c+1;\omega)} = \int_0^1 \frac{\omega}{1-\sigma\omega} d\nu_3(\sigma), \quad \omega \in \mathbb{C} \setminus [1,\infty).$$

For $a < c$, defining

$$\nu_4(\sigma) = \frac{1}{(1-k_2)k_3} \int_0^\sigma s d\nu_3(s),$$

gives $(1-k_2)k_3 > 0$, and using the fact that the coefficient of ω in $F(a+2,b+1;c+2;\omega)/F(a+1,b+1;c+1;\omega)$ is $(1-k_2)k_3$, we obtain

$$\frac{F(a+2,b+1;c+2;\omega)}{F(a+1,b+1;c+1;\omega)} = 1 + [(1-k_2)k_3] \int_0^1 \frac{\omega}{1-\sigma\omega} d\nu_4(\sigma).$$

Thus, with ν_j, $j = 0,1,2,3,4$, satisfying the conditions of Küstner [117, Lem. 3.1] and Merkes [132, Cor. 2.1], the proof of the theorem is complete. □

Remark 2.3.1. *Ratios of Gaussian hypergeometric functions having mapping properties described in Theorem 2.3.1 are also found in Küstner* [117, Theorem 1.5] *but for the ranges* $-1 \leq a \leq c$ *and* $0 < b \leq c$*. Hence for the common range* $-1 < a \leq c$ *and*

$0 < b \leq c$, *two different ratios of hypergeometric functions belonging to the class of Pick functions can be obtained leading to the expectation of finding more such ratios for every possible range.*

In particular, it may be noted that it is proved in Küstner [117, Theorem 1.5] that the ratio of Gaussian hypergeometric functions in (2.1.17), denoted here as $\mathrm{F}(z)$, has the mapping properties given in Theorem 2.3.1.

Remark 2.3.2. *The Gaussian hypergeometric function has been generalized in several directions. One of them is the Wright type hypergeometric functions, whose basic properties like integral representions are studied, for example, in Desai and Shukla [64] and and Rao et al. [144]. In the present context, it would be interesting to associate a g-fraction expansion to ratios of Wright type hypergeometric functions and study the resulting class of Pick functions.*

We now consider the g-fraction expansion of $\mathrm{F}_3^{(a,b,c)}(\omega)$ with the parameter k_2 missing. Using the contiguous relation (2.3.4) and the notations used in Theorems 2.1.2 and 2.1.5, it is clear that $\mathrm{F}_3^{(a,b,c)}(\omega) = F(a+2,b+1;c+2;\omega)/F(a+1,b+1;c+2;\omega)$ and

$$\mathrm{H}_3(\omega) = 1 - \frac{1}{\mathrm{F}_3^{(a,b,c)}(\omega)} = \frac{b+1}{c+2}\omega\frac{F(a+2,b+2;c+3;\omega)}{F(a+2,b+1;c+2;\omega)},$$

so that

$$h(2;\omega) = (1-k_1)\mathrm{H}_3(\omega) = \frac{(c-b)(b+1)}{(c)(c+2)}\omega\frac{F(a+2,b+2;c+3;\omega)}{F(a+2,b+1;c+2;\omega)}.$$

Then, from Theorem 2.1.2,

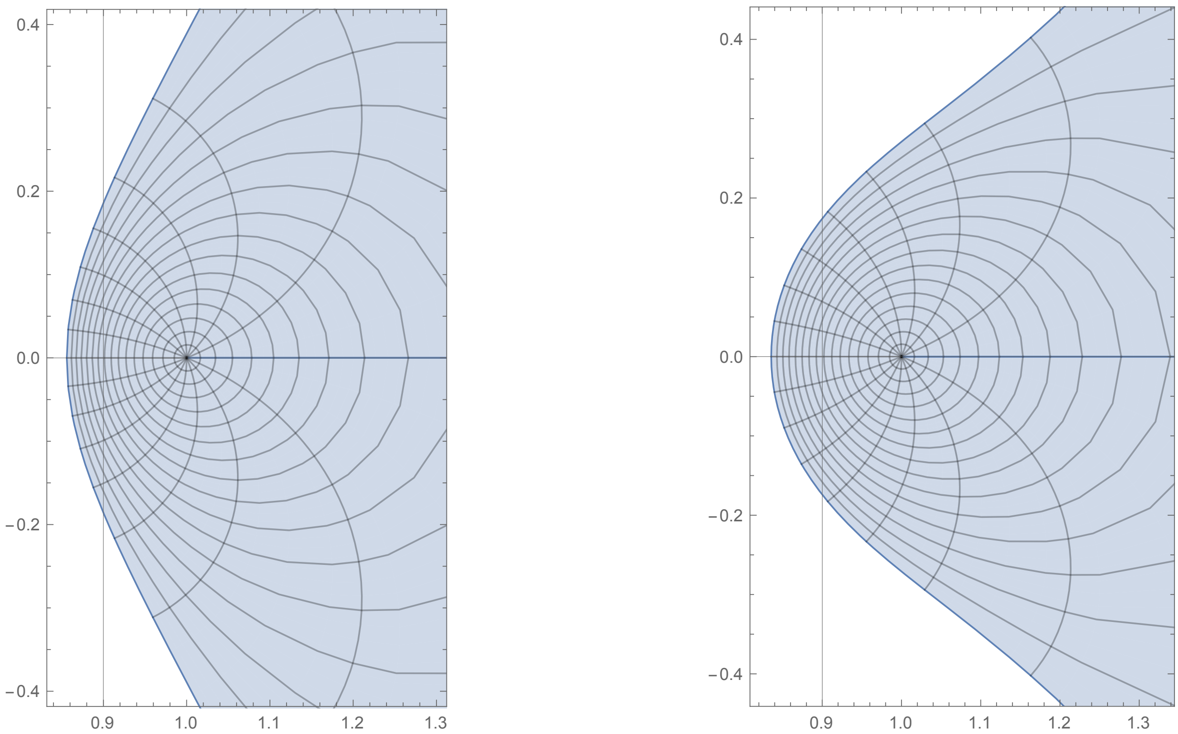
$$\begin{aligned}\mathrm{F}(2;\omega) &= \frac{1}{1-(1-k_0)k_1\omega} - \frac{(1-k_0)k_1\omega h(2;\omega)}{[1-(1-k_0)k_1\omega]h(2;\omega) - [1-(1-k_0)k_1\omega]^2}\\ &= \frac{c}{c-b\omega} - \frac{bc\omega h(2;\omega)}{c(c-b\omega)h(2;\omega) - (c-b\omega)^2},\end{aligned}$$

which implies

$$\mathrm{F}(2;\omega) = \frac{c}{c-b\omega} - \frac{\dfrac{b(b+1)(c-b)}{c+2}\omega^2\dfrac{F(a+2,b+2;c+3;\omega)}{F(a+2,b+1;c+2;\omega)}}{\dfrac{(c-b)(b+1)(c-b\omega)}{c+2}\omega\dfrac{F(a+2,b+2;c+3;\omega)}{F(a+2,b+1;c+2;\omega)} - (c-b\omega)^2}$$

that is $\mathrm{F}(2;\omega)$ is given as a rational transformation of a new ratio of hypergeometric functions. It may also be noted that for $-1 \leq a \leq c$ and $0 < b \leq c$, both $\mathrm{F}(\omega)$ and $\mathrm{F}(2;\omega)$ will map both the unit disk $\mathbb{D}$ and the half plane $\{\omega \in \mathbb{C} : \mathrm{Re}\ \omega < 1\}$ univalently onto domains that are convex in the direction of the imaginary axis.

As an illustration, we plot both these functions in Figures (2.3a) and (2.3b).



(a) The function $\mathrm{F}(\omega)$ (b) The function $\mathrm{F}(2;\omega)$

Figure 2.3: The images of the disc $|\omega| < 0.999$ under the mappings $\mathrm{F}(\omega)$ and $\mathrm{F}(2;\omega)$ for $a = 0$, $b = 0.1$, $c = 0.4$.

2.3.2 A class of Schur functions

From Theorem 2.1.5 we obtain

$$\cfrac{k_{2j+1}\omega}{1-\cfrac{(1-k_{2j+1})k_{2j+2}\omega}{1-\cfrac{(1-k_{2j+2})k_{2j+3}\omega}{1-\cdots}}} = 1-\frac{F(a+j,b+j;c+2j;\omega)}{F(a+j+1,b+j;c+2j;\omega)}$$
$$= \frac{b+j}{c+2j}\frac{\omega F(a+j+1,b+j+1;c+2j+1;\omega)}{F(a+j+1,b+j,c+2j;\omega)}$$

where the last equality follows from the contiguous relation (2.3.4). Further, using Wall [183, eqns. 3.3 and 5.1] we get

$$\frac{1-z}{2}\frac{1-f_{2j}(z)}{1+zf_{2j}(z)} = \frac{b+j}{c+2j}\frac{F(a+j+1,b+j+1;c+2j+1;\omega)}{F(a+j+1,b+j;c+2j;\omega)}, \quad j\geq 1,$$

where $f_n(z)$ is the Schur function and ω and z are related as $\omega = -4z/(1-z)^2$. Similarly, interchanging a and b in (2.3.4) we obtain

$$\frac{1-z}{2}\frac{1-f_{2j+1}(z)}{1+zf_{2j+1}(z)} = \frac{a+j+1}{c+2j+1}\frac{F(a+j+2,b+j+1;c+2j+2;\omega)}{F(a+j+1,b+j+1;c+2j+1;\omega)}, \quad j\geq 0,$$

where $\omega = -4z/(1-z)^2$.

Moreover, using the relation $\alpha_{j-1} = 1-2k_j$, $j\geq 1$, the related sequence of Schur parameters is given by

$$\alpha_j = \begin{cases} \dfrac{c-2b}{c+j}, & j=2n,\ n\geq 0; \\ \dfrac{c-2a-1}{c+j}, & j=2n+1,\ n\geq 1. \end{cases}$$

We note the following particular case. For $a=b-1/2$ and $c=b$, the resulting Schur parameters are $\alpha_j^{(b)} = -b/(b+j)$, $j\geq 0$. Such parameters have been considered in Sri Ranga [162] (when $b\in\mathbb{R}$) in the context of orthogonal polynomials on the unit circle or the Szegő polynomials.

Finally, as an illustration we note that while the Schur function associated with the parameters $\{\alpha_j^{(b)}\}_{j\geq 0}$ is $f(z) = -1$, that associated with the parameters $\{\alpha_j^{(b)}\}_{j\geq 1}$ is given by

$$\frac{1-z}{2}\frac{1-f^{(b)}(z)}{1+zf^{(b)}(z)} = \frac{b+1/2}{b+1}\frac{F(b+3/2,b+1;b+2;\omega)}{F(b+1/2,-;-;\omega)}, \qquad \omega = -4z/(1-z)^2.$$

2.4 Concluding remarks

In this chapter, certain perturbation of g-fraction and Schur fraction are considered that provide some mapping properties and admissible function to the class of Pick functions. The partial numerators of a g-fraction which are of the form $(1-g_{n-1})g_n$, arise in chain sequences which are already defined in Chapter 1. We also obtained a well known g-fraction (with parameters k_n) from the Gauss continued fraction using the transformation $k_n = 1-g_n$, $n \geq 0$. As our focus is on orthogonal polynomials pertaining to unit circle, we study the perturbation $m_n \mapsto 1-m_n$ in the context of chain sequences related to orthogonal polynomials on the unit circle in the next chapter.

Chapter 3

Orthogonal Polynomials from Complementary Chain Sequences

In this chapter, we define and study the consequences of complementary chain sequences both on the unit circle and on the real line which we view as perturbations of the minimal parameter sequences. Using the relation between these complementary chain sequences and the corresponding Verblunsky coefficients, the para-orthogonal polynomials and the associated Szegő polynomials are analyzed. On the real line, they are studied in the context of the Chihara construction of symmetric orthogonal polynomials. Three illustrations, involving Gaussian hypergeometric functions, Carathéodory functions and Laguerre polynomials are also provided.

3.1 Para-orthogonal polynomials from an Uvarov transformation

The Uvarov transformation is precisely the addition of mass points which are usually taken to be lying either on the unit circle or outside the unit circle. This is achieved through the addition of the Dirac delta functional or the Dirac measure. Huertas et al. [87] and Arceo et al. [5] respectively studied the Uvarov modification of a positive measure and a Freud like weight leading, in both the cases, to an electrostatic interpretation of the perturbed zeros. In Castillo et al. [39], perturbation by the addition of the first derivative of Dirac delta to a quasi-definite functional is introduced, which is also shown to be an iteration of the Christoffel and Geronimus transformations. In the present chapter, however, we will study perturbations in the chain sequences obtained by an Uvarov transformation in the context of para-orthogonality.

The connection between kernel polynomials $K_n(z;\omega)$ defined as

$$K_n(z,\omega) = \sum_{k=0}^{n} \phi_k(z)\overline{\phi_k(\omega)} = \frac{\phi^*_{n+1}(z)\overline{\phi^*_{n+1}(\omega)} - \phi_{n+1}(z)\overline{\phi_{n+1}(\omega)}}{1 - z\bar{\omega}}.$$

and the para-orthogonal polynomials has been observed by Cantero et al. [34], Golinskii [84] and later in the articles Bracciali et al. [25] and Costa et al. [40]. We briefly describe the results obtained in the last two references that will serve as the motivation for the definition of the complementary chain sequences.

Let $\Phi_n(z)$ be the monic Szegő polynomials, with α_{n-1} and $\phi_n(z)$, the corresponding Verblunsky coefficients and orthonormal Szegő polynomials respectively. With $\boldsymbol{t}_n = \|\Phi_n\|^2$ and $\tau_n(\omega) = \Phi_n(\omega)/\Phi^*_n(\omega)$, $|\omega| = 1$, the monic form of the kernel polynomials $K_n(z;\omega)$, given by

$$P_n(\omega; z) = \frac{\boldsymbol{t}_{n+1}\bar{\omega}}{\overline{\Phi_n(\omega)}} \frac{K_n(z;\omega)}{1 + \alpha_n \tau_{n+1}(\omega)}, \quad n \geq 1,$$

satisfy the orthogonality property

$$\int_{\partial\mathbb{D}} z^{-n+j} P_n(\omega; z)(\omega - z) d\mu(z) = \begin{cases} \check{\gamma}_n(\omega), & j = -1; \\ 0, & 0 \leq j \leq n-1; \\ \hat{\gamma}_n(\omega), & j = n, \end{cases} \quad n \geq 1, \tag{3.1.1}$$

where $\mu(z)$ is a non-trivial measure on the unit circle, $\hat{\gamma}_n(\omega) = -\tau_{n+1}(\omega)\check{\gamma}_n(\omega)$ and $\check{\gamma}_n(\omega) = -(1 - \omega\tau_n(\omega)\alpha_n)\boldsymbol{t}_n$.

However, though the polynomials $P_n(\omega; z)$ are uniquely defined by the orthogonality conditions (3.1.1), the corresponding sequence of Szegő polynomials that generate $P_n(\omega; z)$, $n \geq 1$, is not unique. In fact, assuming the measure $\mu(z)$ has a jump δ,

$0 \le \delta < 1$, at $z = \omega$, which we denote as $\mu_{(\delta)}$, the following family of measures

$$\int_{\partial\mathbb{D}} f(z) d\mu^{(t,\omega)}(z) = \frac{1-t}{1-\delta} \int_{\partial\mathbb{D}} f(z) d\mu_{(\delta)}(z) + \frac{t-\delta}{1-\delta} f(\omega), \quad 0 \le t < 1,$$

was defined in Costa et al. [40] which can be seen as an Uvarov transformation of the measure $d\mu_{(\delta)}$. The measure $\mu^{(t,\omega)}$ has a jump t, $0 \le t < 1$, at $z = \omega$ with $\mu^{(\delta,\omega)} = \mu_{(\delta)}$. Further, if we denote the family of Szegő polynomials and monic kernel polynomials associated with the measure $\mu^{(t,\omega)}$ by $\Phi_n^{(t,\omega)}(z)$ and $P_n^{(t)}(\omega; z)$ respectively, then $P_n^{(t)}(\omega; z) = P_n(\omega; z)$ with $\Phi_n^{(\delta,\omega)}(z) = \Phi_n(z)$, $n \ge 0$.

Restricting the value of ω to be 1, the rest of the discussion will be for the measure $\mu^{(t,1)}$ defined as

$$\int_{\partial\mathbb{D}} f(z) d\mu^{(t,1)}(z) = \frac{1-t}{1-\delta} \int_{\partial\mathbb{D}} f(z) d\mu_{(\delta)}(z) + \frac{t-\delta}{1-\delta} f(1), \quad 0 \le t < 1, \tag{3.1.2}$$

where δ is the jump in $\mu_{(\delta)}$ at $z = 1$. In this context, the normalized monic polynomials $R_n(z)$ given by

$$R_n(z) = \frac{\prod_{j=0}^{n-1}[1 - \tau_j \alpha_j]}{\prod_{j=0}^{n-1}[1 - \text{Re } (\tau_j \alpha_j)]} P_n(1; z), \quad n \ge 1, \tag{3.1.3}$$

was introduced in Costa et al. [40, Theorem 2.2] and satisfy the recurrence relation

$$R_{n+1}(z) = [(1 + ic_{n+1})z + (1 - ic_{n+1})]R_n(z) - 4d_{n+1}R_{n-1}(z), \quad n \ge 1, \tag{3.1.4}$$

with $R_0(z) = 1$, $R_1(z) = (1 + ic_1)z + (1 - ic_1)$, where

$$c_n = \frac{-\text{Im } (\tau_{n-1}\alpha_{n-1})}{1 - \text{Re } (\tau_{n-1}\alpha_{n-1})} \quad \text{and} \quad d_{n+1} = \frac{1}{4} \frac{[1 - |\tau_{n-1}\alpha_{n-1}|^2]|1 - \tau_n\alpha_n|^2}{[1 - \text{Re } (\tau_{n-1}\alpha_{n-1})][1 - \text{Re } (\tau_n\alpha_n)]}, \quad n \ge 1.$$

Here $\tau_n := \tau_n(1) = \Phi_n(1)/\Phi_n^*(1)$. Further, while $\{c_n\}$ is a real sequence, $\{d_{n+1}\}_{n=1}^{\infty}$ is a positive chain sequence such that $d_{n+1} = (1 - g_n)g_{n+1}$, $n \ge 1$, where

$$0 < g_{n+1} = \frac{1}{2} \frac{|1 - \tau_n\alpha_n|^2}{[1 - \text{Re } (\tau_n\alpha_n)]} < 1, \quad n \ge 0.$$

We note from (3.1.3) and (3.1.4), that, $R_n(z)$ being a constant multiple of $P_n(1; z)$

is independent of the jump t at $z = \omega = 1$, and so do the sequences $\{c_n\}_{n=1}^{\infty}$, $\{d_{n+1}\}_{n=1}^{\infty}$ and $\{\tau_n\}_{n=0}^{\infty}$. Moreover, $\{g_{n+1}\}_{n=0}^{\infty}$ is only one of the parameter sequence of the chain sequence $\{d_{n+1}\}_{n=1}^{\infty}$. The following result of Costa et al. [40, Theorem 3.3] describes all the parameter sequences of $\{d_{n+1}\}_{n=1}^{\infty}$ as well as the maximal parameter sequence.

Theorem 3.1.1. [40] *For $0 \leq t < 1$, let $\phi_n^{(t,1)}(z)$ be the monic Szegő polynomials with respect to the measure $\mu^{(t,1)}$ defined in (3.1.2) with $\alpha_n^{(t,1)} = -\overline{\phi_{n+1}^{(t,1)}(0)}$ as the Verblunsky coefficients. Then the sequence $\{g_{n+1}^{(t,1)}\}_{n=0}^{\infty}$ given by*

$$g_{n+1}^{(t,1)} = \frac{1}{2}\frac{|1-\tau_n\alpha_n^{(t,1)}|^2}{[1-Re\ (\tau_n\alpha_n^{(t,1)})]}, \quad n \geq 0.$$

is a parameter sequence of the chain sequence $\{d_{n+1}\}_{n=1}^{\infty}$. Further, if $0 \leq t_1 < t_2 < 1$, then $0 < g_{n+1}^{(t_2,1)} < g_{n+1}^{(t_1,1)} < 1$, $n \geq 0$, with the initial parameters related by $g_1^{(t_2,1)} = \frac{1-t_2}{1-t_1}g_1^{(t_1,1)}$.

An important implication of Theorem 3.1.1 is that the initial parameter $g_1^{(t,1)}$ is a decreasing function of t and hence $\{g_{n+1}^{(0,1)}\}_{n=0}^{\infty}$ is the maximal parameter sequence of $\{d_{n+1}\}_{n=1}^{\infty}$. That is, we obtain the maximal parameter sequence $\{M_{n+1}\}_{n=0}^{\infty}$ of $\{d_{n+1}\}_{n=1}^{\infty}$ in case of "zero" jump at $z = 1$. Further, with $t_2 = 1$ and $t_1 = 0$, the relation

$$g_1^{(t,1)} = (1-t)g_1^{(0,1)} = (1-t)M_1, \tag{3.1.5}$$

describes the dependence of the initial parameter on the jump t. In particular, for the measure $\mu_{(\delta)}$, with respect to which $R_n(z)$ is orthogonal with the parameter sequence $\{g_{n+1}\}_{n=0}^{\infty}$, we have $g_1 = (1-\delta)M_1$.

The relation (3.1.5) also provides a way to extend the chain sequence $\{d_{n+1}\}_{n=1}^{\infty}$ to include the term d_1 so that $\{d_n\}_{n=1}^{\infty}$ is a chain sequence corresponding to the measure $\mu^{(t,1)}$. Thus, if we choose $d_1(t) = g_1^{(t,1)} = (1-t)M_1$, then, $\{d_n\}_{n=1}^{\infty}$ is a chain sequence with the minimal $\{m_n\}_{n=0}^{\infty}$ and maximal $\{M_n\}_{n=1}^{\infty}$ parameter sequences given by

$$m_0 = 0, \quad m_n = g_n^{(t,1)}, \quad n \geq 1, \quad M_0 = t \quad \text{and} \quad M_n = g_n^{(0,1)}, \quad n \geq 1.$$

This suggests that the "zero" jump is interpreted as the maximal and the minimal parameter sequences coinciding, in which case, the chain sequence is said to determine its parameters uniquely. The above extension forms the content of Costa et al. [40, Theorem 4.2].

The recurrence relation (3.1.4) is also studied independently in Castillo et al. [37]. It is proved that there exists a unique non-trivial probability measure on the unit circle such that $t = M_0$ is the jump at $z = 1$ and $\phi_n^{(t,1)}(z) = R_n(z) - 2(1-m_n)R_{n-1}(z)$, $n \geq 1$, forms a sequence of Szegő polynomials. The associated Verblunsky coefficients are given by

$$\alpha_{n-1}^{(t,1)} = \frac{1 - 2m_n - ic_n}{1 + ic_n} \prod_{k=1}^{n} \frac{1 + ic_k}{1 - ic_k}, \quad n \geq 1. \tag{3.1.6}$$

The zeros of $R_n(z)$ are proved in Dimitrov and Sri Ranga [66] to be simple and lying on the unit circle. Further, if the zeros of $R_n(z)$ are denoted as $z_{n,j} = e^{i\theta_{n,j}}$, $j = 1, 2, \cdots, n$, then

$$0 < \theta_{n+1,1} < \theta_{n,1} < \theta_{n+1,2} < \cdots < \theta_{n,n} < \theta_{n+1,n+1} < 2\pi, \quad n \geq 1,$$

that is, the zeros of $R_n(z)$ and $R_{n+1}(z)$ interlace.

Remark 3.1.1. *It can be said that the kernel polynomials $P_n(\omega; z)$, $n \geq 1$, are invariant under the addition of a dirac measure. In another direction, it is proved in Dueñas and Garza [71] that the Laguerre-Hahn class of functionals (whose corresponding Stieltjes function satisfies Riccati differential equation with polynomial coefficients) is preserved under the addition of Dirac delta derivatives.*

3.2 Complementary chain sequences

As is obvious from the definition of chain sequences, the minimal and maximal parameter sequences are uniquely defined for any given chain sequence. Also, the chain sequence for which the minimal and maximal parameter sequences coincide, that is, $M_0 = 0$, has its own importance as illustrated in the previous section. Such a chain sequence is said to determine its parameters uniquely and is referred to as a single parameter positive chain sequence (SPPCS) in Bracciali et al. [25]. By Wall's criteria [184, p. 82] for maximal parameter sequence, this is equivalent to

$$\sum_{n=1}^{\infty} \frac{m_1}{1-m_1} \cdot \frac{m_2}{1-m_2} \cdot \frac{m_3}{1-m_3} \cdots \frac{m_n}{1-m_n} = \infty. \tag{3.2.1}$$

Thus, introducing a perturbation in the minimal parameters m_n will lead to a uniquely defined change in the chain sequence.

Definition 3.2.1. *Suppose $\{d_n\}_{n=1}^{\infty}$ is a chain sequence with $\{m_n\}_{n=0}^{\infty}$ as its minimal parameter sequence. Let $\{k_n\}_{n=0}^{\infty}$ be another sequence given by $k_0 = 0$ and $k_n = 1 - m_n$ for $n \geq 1$. Then the chain sequence $\{a_n\}_{n=1}^{\infty}$ having $\{k_n\}_{n=0}^{\infty}$ as its minimal parameter sequence is called the complementary chain sequence of $\{d_n\}$.*

Such chain sequences satisfy relations like Wall [184, equation (75.3)]

$$\cfrac{1}{1+\cfrac{d_1 z}{1+\cfrac{d_2 z}{1+\cfrac{d_3 z}{1+\ddots}}}} \cdot \cfrac{1}{1+\cfrac{a_1 z}{1+\cfrac{a_2 z}{1+\cfrac{a_3 z}{1+\ddots}}}} = \frac{1}{1+z}.$$

They also satisfy

$$d_1 - a_1 = 1 - 2k_1 = 2m_1 - 1, \quad d_n - a_n = \triangle m_{n-1} = -\nabla k_n, \quad n \geq 2,$$

where $\triangle$ and ∇ are the forward and backward difference operators respectively. Further, of particular interest is the ratio of these two chain sequences given by

$$\frac{d_1}{a_1} = \frac{m_1}{1-m_1}, \qquad \frac{d_n}{a_n} = \frac{k_{n-1}}{1-k_{n-1}} \frac{m_n}{1-m_n}, \quad n \geq 2.$$

This implies

$$\frac{m_n}{1-m_n} = \frac{d_n}{a_n} \frac{m_{n-1}}{1-m_{n-1}} = \cdots = \frac{d_n d_{n-1} \cdots d_1}{a_n a_{n-1} \cdots a_1}, \quad n \geq 1. \tag{3.2.2}$$

Substituting (3.2.2) in (3.2.1), we have the following lemma.

Lemma 3.2.1. *Let $\{d_n\}_{n=1}^{\infty}$ and $\{a_n\}_{n=1}^{\infty}$ be two chain sequences complementary to* *each other. Then* $\{d_n\}_{n=1}^{\infty}$ *will be a SPPCS if and only if*

$$\sum_{n=1}^{\infty}\prod_{j=1}^{n}\frac{d_1d_2\cdots d_j}{a_1a_2\cdots a_j}=\infty.$$

Similarly, $\{k_n\}_{n=0}^{\infty}$ being the minimal parameter sequence of $\{a_n\}_{n=1}^{\infty}$, the chain sequence $\{a_n\}_{n=1}^{\infty}$ is a SPPCS if and only if

$$\sum_{n=1}^{\infty}\prod_{j=1}^{n}\frac{a_1a_2\cdots a_j}{d_1d_2\cdots d_j}=\infty.$$

Lemma 3.2.2. *Let* $\{d_n\}_{n=1}^{\infty}$ *and* $\{a_n\}_{n=1}^{\infty}$ *be two complementary chain sequences of each other. If* $\{d_n\}_{n=1}^{\infty}$ *is not a SPPCS, then* $\{a_n\}_{n=1}^{\infty}$ *is a SPPCS.*

Proof. If $\{d_n\}_{n=1}^{\infty}$ is not a SPPCS then its minimal parameter sequence $\{m_n\}_{n=0}^{\infty}$ is such that

$$\sum_{n=1}^{\infty}\frac{m_1}{1-m_1}\cdot\frac{m_2}{1-m_2}\cdot\frac{m_3}{1-m_3}\cdots\frac{m_n}{1-m_n}<\infty.$$

Hence, $\lim_{n\to\infty}\prod_{j=1}^{n} m_j/(1-m_j)=0$ and we have

$$\sum_{n=1}^{\infty}\prod_{j=1}^{n}\frac{k_j}{1-k_j}=\sum_{n=1}^{\infty}\prod_{j=1}^{n}\frac{1-m_j}{m_j}=\infty,$$

thus concluding the proof of the lemma. □

3.2.1 On unit circle: Two sequences of Szegő polynomials

The results in Lemma 3.2.1 and Lemma 3.2.2 are useful in determining whether a chain sequence or its complementary chain sequence is a SPPCS without using the corresponding minimal parameter sequences. The next lemma however imposes conditions on the minimal parameters.

Lemma 3.2.3. *Let* $\{d_n\}_{n=1}^{\infty}$ *be a chain sequence and* $\{a_n\}_{n=1}^{\infty}$ *be its complementary chain sequence with minimal parameter sequences* $\{m_n\}_{n=0}^{\infty}$ *and* $\{k_n\}_{n=0}^{\infty}$ *respectively.*

1. *If* $0<m_n<1/2$, $n\geq 1$, *then* a_n *is a SPPCS.*

2. *If* $1/2 < m_n < 1$, $n \geq 1$, *then* d_n *is a SPPCS.*

Proof. Observe that if $0 < m_n < 1/2$, $k_n/(1-k_n) > 1$ for all $n \geq 1$. Similarly, $1/2 < m_n < 1$ implies $m_n/(1-m_n) > 1$ for all $n \geq 1$. The results now follow from (3.2.1). □

It is known that (Wall [184, p. 79]) if $d_n \geq 1/4$, $n \geq 1$, every parameter sequence $\{g_n\}_{n=0}^{\infty}$, in particular the minimal parameter sequence $\{m_n\}_{n=0}^{\infty}$, of $\{d_n\}_{n=1}^{\infty}$ is non-decreasing. For the special case when $d_n = 1/4$, $n \geq 1$, $m_n \to 1/2$ as $n \to \infty$. This implies $0 < m_n < 1/2$, $n \geq 1$. By Lemma 3.2.3, $\{a_n\}_{n=1}^{\infty}$ is a SPPCS. In other words, the chain sequence complementary to the constant chain sequence $\{1/4\}$ determines its parameters g_n, $n \geq 1$, uniquely, which are further given by

$$g_0 = 0, \qquad g_n = \frac{n+2}{2(n+1)}, \qquad n \geq 1.$$

Moreover, if $d_n \geq 1/4$, there exist some $n \in \mathbb{N}$ such that $a_n < 1/4 \leq d_n$. Indeed

$$d_n = (1-m_{n-1})m_n \geq m_{n-1}(1-m_n) = a_n, \qquad n \geq 2,$$

with the sign of the difference of d_1 and a_1 depending on whether $m_1 \in (0, 1/2)$ or $(1/2, 1)$. If $a_n \in (1/4, 1)$ for $n \geq 1$, k_n has to be non-decreasing. This is a contradiction as $k_n = 1 - m_n$ for $n \geq 1$.

Let $\{c_n\}_{n=1}^{\infty}$ and $\{d_{n+1}\}_{n=1}^{\infty}$ be, respectively, the real sequence and positive chain sequence as given in (3.1.4). Let $\left\{m_n^{(t,1)}\right\}_{n=0}^{\infty}$ be the minimal parameter sequence of the augmented chain sequence $\{d_n\}_{n=1}^{\infty}$, where $d_1 = (1-t)M_1$ and $\{M_{n+1}\}_{n=0}^{\infty}$ is the maximal parameter sequence of $\{d_{n+1}\}_{n=1}^{\infty}$. Viewing complementary chain sequences as a perturbation of the Verblunsky coefficients given by (3.1.6) we give the following result.

Theorem 3.2.1. *Let* $\left\{k_n^{(t,1)}\right\}_{n=0}^{\infty}$ *be the minimal parameter sequence of the positive chain sequence* $\{a_n\}_{n=1}^{\infty}$ *obtained as complementary to* $\{d_n\}_{n=1}^{\infty}$. *Set* $\tau_n = \frac{1-ic_n}{1+ic_n}\tau_{n-1}$,

$$\alpha_{n-1}^{(t,1)} = \overline{\tau}_n \left[\frac{1-2m_n^{(t)}-ic_n}{1+ic_n}\right] \quad \textit{and} \quad \beta_{n-1}^{(t,1)} = \overline{\tau}_n \left[\frac{1-2k_n^{(t)}-ic_n}{1+ic_n}\right],$$

for $n \geq 1$, *with* $\tau_0 = 1$. *Let* $\mu^{(t,1)}(z)$ *and* $\nu^{(t,1)}(z)$ *be, respectively, the probability measures having* $\alpha_{n-1}^{(t,1)}$ *and* $\beta_{n-1}^{(t,1)}$ *as the corresponding Verblunsky coefficients. Then the following statements hold.*

1. *For* $0 < t < 1$, *the measure* $\mu^{(t,1)}(z)$ *has a jump of size* t *at* $z = 1$, *while* $\nu^{(t,1)}(z)$ *does not.*
2. $\beta_{n-1}^{(t,1)} = -\overline{\tau}_n \overline{\tau}_{n-1} \overline{\alpha}_{n-1}^{(t,1)}$, $n \geq 1$.

Proof. First we observe that $\alpha_{n-1}^{(t,1)}$, $n \geq 1$, are the generalized Verblunsky coefficients of the measure $\mu^{(t,1)}(z)$ as given by (3.1.2). Consequently, for $0 < t < 1$, the probability measure $\mu^{(t,1)}(z)$ has a jump of size t at $z = 1$. Since $d_1 = (1-t)M_n$, choosing $M_0 = t > 0$, the sequence $\{t, M_1, M_2, M_3, \ldots\}$ is the maximal parameter sequence of $\{d_n\}_{n=1}^{\infty}$. Since $t > 0$, $\{d_n\}_{n=1}^{\infty}$ is a non SPPCS and hence, by Lemma 3.2.2, the sequence $\{a_n\}_{n=1}^{\infty}$ is a SPPCS so that $\left\{k_n^{(t,1)}\right\}_{n=0}^{\infty}$ is also its maximal parameter sequence. Thus, by results established in Costa et al. [40], the measure $\nu^{(t,1)}(z)$ has a "zero" jump $(t = 0)$ at $z = 1$. This proves the first part of the theorem.

Now to prove the second part, we first have

$$\beta_{n-1}^{(t,1)} = \overline{\tau}_n \left[\frac{1 - 2k_n^{(t,1)} - ic_n}{1 + ic_n} \right] = \overline{\tau}_n \left[\frac{-1 + 2m_n^{(t,1)} - ic_n}{1 + ic_n} \right].$$

By conjugation of the expression for $\alpha_{n-1}^{(t,1)}$, we have

$$-\overline{\alpha}_{n-1}^{(t,1)} = \tau_n \left[\frac{-1 + 2m_n^{(t,1)} - ic_n}{1 - ic_n} \right],$$

which leads to the second part of the theorem. □

We note the following two particular cases of the real sequence $\{c_n\}_{n=1}^{\infty}$ in the context of complementary chain sequence.

Proposition 3.2.1. *With the notations of Theorem 3.2.1,*

1. *For* $n \geq 1$, *if* $c_n = (-1)^n c$, $c \in \mathbb{R}$, *then,* $\beta_{n-1}^{(t,1)} = -\frac{1-ic}{1+ic} \alpha_{n-1}^{(t,1)}$, $n \geq 1$.
2. *If* $c_n = 0$, $n \geq 1$ *then the Verblunsky coefficients, given to be real, are such that* $\beta_{n-1}^{(t,1)} = -\alpha_{n-1}^{(t,1)}$, $n \geq 1$.

Proof. Clearly with $c_n = (-1)^n c$, $n \geq 1$ we have $\overline{\tau}_{2n} = 1$ and $\overline{\tau}_{2n+1} = \frac{1-ic}{1+ic}$. Thus, the first part of the proposition is established. The other part follows by taking $\overline{\tau}_n \overline{\tau}_{n-1} = 1$, $n \geq 1$. This is only possible if $c_n = 0$, $n \geq 1$. □

The results of Proposition 3.2.1 are important cases of the Aleksandrov transformation (1.3.6) with $\lambda = 1$ and, the case of second part of Proposition 3.2.1 gives rise to second kind polynomials (Simon [156]) for the measure $\mu^{(t,1)}$. In this particular case, the recurrence relation (3.1.4) assumes a very simple form, similar to the one considered in Delsarte and Genin [58].

3.2.2 On real line: A variant of the Chihara construction

The theory of chain sequences is used to study many properties of a given orthogonal polynomial sequence on the real line (OPS) and its kernel polynomial sequence (KOPS). For instance, chain sequences are used to characterize the true interval of orthogonality of an OPS, which is the smallest closed interval that contains all the zeros of all the polynomials of such sequence. Let the true interval of orthogonality of the OPS $\{\mathrm{P}_n(x)\}_{n=0}^{\infty}$ satisfying the recurrence relation

$$\mathrm{P}_{n+1}(x) = (x - b_{n+1})\mathrm{P}_n(x) - a_n^2 \mathrm{P}_{n-1}(x), \quad n \geq 1, \tag{3.2.3}$$

with $\mathrm{P}_0(x) = 1$ and $\mathrm{P}_1(x) = x - b_1$ be denoted as $[\xi_1, \eta_1]$. Then, with

$$\omega_n(t) = \frac{a_n^2}{(t - b_n)(t - b_{n+1})}, \quad n \geq 1,$$

the following statements

(i) $[\xi_1, \eta_1]$ is contained in (a, b),

(ii) $b_{n+1} \in (a, b)$ for $n \geq 0$ and both $\{\omega_n(a)\}$ and $\{\omega_n(b)\}$ are chain sequences,

are proved to be equivalent in Ismail [90, Corollary 7.2.4]. In particular, $[\xi_1, \eta_1]$ is a subset of $(0, \infty)$ if and only if $b_{n+1} > 0$ for $n \geq 0$ and $\omega_n(0)$ is a chain sequence, that is, there are numbers g_n such that $0 \leq g_0 < 1$, $0 < g_n < 1$, $n \geq 1$, satisfying,

$$(1 - g_{n-1})g_n = \frac{a_n^2}{b_n b_{n+1}} = \omega_n(0), \quad n \geq 1.$$

As in Chihara [42, Chapter 1, Theorems 9.1, 9.2], the sequence $\{g_n\}_{n=0}^{\infty}$ is constructed using another sequence $\{\gamma_n\}_{n=1}^{\infty}$, where $\gamma_1 \geq 0$ and $\gamma_n > 0$, $n \geq 2$. Precisely, $g_n = \gamma_{2n+1}/b_{n+1}$, $n \geq 0$, where $b_{n+1} = \gamma_{2n+1} + \gamma_{2n+2}$, $n \geq 0$ and $a_n^2 = \gamma_{2n}\gamma_{2n+1}$, $n \geq 1$. Hence when $\gamma_1 = 0$, we obtain the minimal parameter sequence $\{m_n\}_{n=0}^{\infty}$.

Associated with the chain sequence $\{\omega_n(0)\}$, the sequence $\{\tilde{\vartheta}_n(0)\}$ arises in a very natural way. Defining $\tilde{\vartheta}_1(0) = (1-k_0)k_1 = \gamma_4/b_2$ and

$$\tilde{\vartheta}_n(0) = \frac{\gamma_{2n-1}\gamma_{2n+2}}{b_n b_{n+1}} = (1-k_{n-1})k_n, \quad n \geq 2,$$

it can be seen that $\{\tilde{\vartheta}_n(0)\}_{n=1}^{\infty}$ becomes a chain sequence with the minimal parameter sequence $\{k_n\}_{n=0}^{\infty}$ where $k_0 = 0$ and $k_n = 1 - g_n$, $n \geq 1$. Hence $\{\tilde{\vartheta}_n(0)\}_{n=1}^{\infty}$ is the *complementary chain sequence* of $\{\omega_n(0)\}_{n=1}^{\infty}$.

If $\gamma_1 > 0$, it is clear that a non-minimal parameter sequence $\{g_n\}_{n=0}^{\infty}$ is obtained for the chain sequence $\{\omega_n(0)\}_{n=1}^{\infty}$. In this case, the associated chain sequence $\{\hat{\vartheta}_n(0)\}_{n=1}^{\infty}$, is defined as

$$\hat{\vartheta}_n(0) = (1-k'_{n-1})k'_n = \frac{\gamma_{2n-1}\gamma_{2n+2}}{b_n b_{n+1}}, \quad n \geq 1,$$

where $k'_n = 1 - g_n$ for $n \geq 0$. We call $\{\hat{\vartheta}_n(0)\}_{n=1}^{\infty}$ the *generalized complementary chain sequence* of $\{\omega_n(0)\}_{n=1}^{\infty}$.

It may be noted from the above two definitions that for a fixed chain sequence, while its complementary chain sequence is unique, its generalized complementary chain sequence need not be unique. In fact, a chain sequence will have as many generalized complementary chain sequences as its non-minimal parameter sequences. However, it is obvious that the complementary chain sequence and all the generalized complementary chain sequences will coincide in case the chain sequence is a SPPCS.

We would like to mention that the chain sequences $\{\tilde{\vartheta}_n(0)\}_{n=1}^{\infty}$ and $\{\hat{\vartheta}_n(0)\}_{n=1}^{\infty}$ have definite sources in the theory of orthogonal polynomials on the real line. To see this, we first construct the OPS $\{\mathrm{P}_n(x)\}_{n=0}^{\infty}$ and the KOPS $\{\mathrm{K}_n(x)\}_{n=0}^{\infty}$ from $\{\mathcal{S}_n(x)\}_{n=0}^{\infty}$ as in (1.1.3)

$$\mathrm{P}_m(x^2) = \mathcal{S}_{2m}(x) \quad \text{and} \quad x\mathrm{K}_m(x^2) = \mathcal{S}_{2m+1}(x), \quad m \geq 0, \tag{3.2.4}$$

and note the following result of Chihara [42, Theorem 9.1].

Suppose the polynomials $\{\mathrm{R}_n^{(i)}(x)\}$, $i = 1, 2$, *satisfy the recurrence relation,*

$$\mathrm{R}_{n+1}^{(i)}(x) = (x - b_{n+1}^{(i)})\mathrm{R}_n^{(i)}(x) - (a_n^2)^{(i)}\mathrm{R}_{n-1}^{(i)}(x), \quad n \geq 0, \tag{3.2.5}$$

with $\mathrm{R}_{-1}^{(i)}(x) = 0$ *and* $\mathrm{R}_0^{(i)}(x) = 1$. *Then,*

(i) $\mathrm{R}_n^{(1)}(x) \equiv \mathrm{P}_n(x)$, $n \geq 0$, *if, and only if,*

$$b_1^{(1)} = \gamma_2, \quad b_{n+1}^{(1)} = \gamma_{2n+1} + \gamma_{2n+2}, \quad (a_n^2)^{(1)} = \gamma_{2n}\gamma_{2n+1}, \quad n \geq 1,$$

(ii) $\mathrm{R}_n^{(2)}(x) \equiv \mathrm{K}_n(x)$, $n \geq 1$, *if, and only if,*

$$b_{n+1}^{(2)} = \gamma_{2n+2} + \gamma_{2n+3}, \quad n \geq 0 \quad \textit{and} \quad (a_n^2)^{(2)} = \gamma_{2n+1}\gamma_{2n+2}, \quad n \geq 1.$$

With these notations, the parameter sequences can be denoted as $m_n = \gamma_{2n+1}/b_{n+1}^{(1)}$ and $g_n = \gamma_{2n+1}/b_{n+1}^{(1)}$, $n \geq 0$. Further, denoting $\tilde{a}_n^2 = \gamma_{2n-1}\gamma_{2n+2}$, $n \geq 1$, the following theorem shows that the polynomials $\{\tilde{\mathrm{P}}_n(x)\}$ and $\{\hat{\mathrm{P}}_n(x)\}$ associated respectively, with the complementary chain sequence $\{\tilde{\omega}_n(0)\}$ and the generalized complementary chain sequence $\{\hat{\vartheta}_n(0)\}$, can be attributed to a particular perturbation of the recurrence relation satisfied by the polynomials $\mathcal{S}_n(x)$, $n \geq 1$.

Theorem 3.2.2. *Let the symmetric polynomials* $\{\tilde{S}_n(x)\}_{n=0}^{\infty}$ *satisfy*

$$\tilde{S}_n(x) = x\tilde{S}_{n-1}(x) - \tilde{\nu}_n\tilde{S}_{n-2}(x), \quad n \geq 1, \tag{3.2.6}$$

with $\tilde{S}_{-1}(x) = 0$, $\tilde{S}_0(x) = 1$ *and where, for* $n \geq 1$,

$$\tilde{\nu}_n = \begin{cases} \gamma_{2j-1}, & n=2j, \; j=1,2,\cdots \\ \gamma_{2j+2}, & n=2j+1, \; j=0,1,\cdots. \end{cases} \tag{3.2.7}$$

Then, with $\gamma_1 \neq 0$, $\{\tilde{\mathrm{P}}_n(x)\}_{n=0}^{\infty}$, *where* $\tilde{S}_{2n}(x) = \tilde{\mathrm{P}}_n(x^2)$, *satisfy,*

$$\tilde{\mathrm{P}}_{n+1}(x) = (x - b_{n+1}^{(1)})\tilde{\mathrm{P}}_n(x) - \tilde{a}_n^2\tilde{\mathrm{P}}_{n-1}(x), \quad n \geq 1, \tag{3.2.8}$$

with the initial conditions $\tilde{\mathrm{P}}_0(x) = 1$ *and* $\tilde{\mathrm{P}}_1(x) = (x - \gamma_1)$.

Proof. First note that, the perturbation (3.2.7) implies that the sequence of coefficients $\{\gamma_1, \gamma_2, \gamma_3, \gamma_4, \cdots\}$ is replaced by $\{\gamma_2, \gamma_1, \gamma_4, \gamma_3, \cdots\}$. That is, $\{\gamma_{2k-1}, \gamma_{2k}\}$ are pair-wise interchanged to $\{\gamma_{2k}, \gamma_{2k-1}\}$, $k \geq 1$. Then, for $n = 2m$, (3.2.6) yields

$$\tilde{\mathcal{S}}_{2m}(x) = x\tilde{\mathcal{S}}_{2m-1}(x) - \gamma_{2m-1}\tilde{\mathcal{S}}_{2m-2}(x), \quad m \geq 1$$

which implies,

$$\tilde{\mathrm{P}}_m(x) = x\tilde{\mathrm{K}}_{m-1}(x) - \gamma_{2m-1}\tilde{\mathrm{P}}_{m-1}(x), \quad m \geq 1. \tag{3.2.9}$$

Similarly, for $n = 2m+1$,

$$\tilde{\mathcal{S}}_{2m+1}(x) = x\tilde{\mathcal{S}}_{2m}(x) - \gamma_{2m+2}\tilde{\mathcal{S}}_{2m-1}(x), \quad m \geq 0,$$

which implies

$$\tilde{\mathrm{K}}_m(x) = \tilde{\mathrm{P}}_m(x) - \gamma_{2m+2}\tilde{\mathrm{K}}_{m-1}(x), \quad m \geq 0. \tag{3.2.10}$$

Using (3.2.9) and (3.2.10), it can be seen that,

$$\begin{aligned} x\tilde{\mathrm{K}}_{m-1}(x) &= \tilde{\mathrm{P}}_m(x) + \gamma_{2m-1}\tilde{\mathrm{P}}_{m-1}(x), \\ \tilde{\mathrm{P}}_m(x) &= \tilde{\mathrm{K}}_m(x) + \gamma_{2m+2}\tilde{\mathrm{K}}_{m-1}(x), \quad m \geq 0. \end{aligned}$$

Using these relations in (3.2.10) and (3.2.9), respectively, yield

$$\tilde{\mathrm{K}}_m(x) = [x - (\gamma_{2m-1} + \gamma_{2m+2})]\tilde{\mathrm{K}}_{m-1}(x) - \gamma_{2m-1}\gamma_{2m}\tilde{\mathrm{K}}_{m-2}(x), \quad m \geq 1, \tag{3.2.11a}$$
$$\tilde{\mathrm{P}}_{m+1}(x) = [x - (\gamma_{2m+1} + \gamma_{2m+2})]\tilde{\mathrm{P}}_m(x) - \gamma_{2m-1}\gamma_{2m+2}\tilde{\mathrm{P}}_{m-1}(x), \quad m \geq 1, \tag{3.2.11b}$$

with the initial conditions $\tilde{\mathrm{K}}_{-1}(x) = 0$, $\tilde{\mathrm{K}}_0(x) = 1$ (using (3.2.10)), $\tilde{\mathrm{P}}_0(x) = 1$ and $\tilde{\mathrm{P}}_1(x) = x - \gamma_1$ (using (3.2.9)), thus proving the theorem. □

Proposition 3.2.2. *Consider the OPS* $\{\hat{\mathrm{P}}_n(x)\}_{n=0}^{\infty}$ *satisfying* (3.2.11b) *with the modification* $m \geq 0$. *Then* $\{\hat{\mathrm{P}}_n(x)\}_{n=0}^{\infty}$ *is associated with the generalized complementary* *chain sequence* $\{\hat{\vartheta}_n(0)\}_{n=1}^{\infty}$.

Proof. From the recurrence relation

$$\hat{\mathrm{P}}_{m+1}(x) = [x - (\gamma_{2m+1} + \gamma_{2m+2})]\hat{\mathrm{P}}_m(x) - \gamma_{2m-1}\gamma_{2m+2}\hat{\mathrm{P}}_{m-1}(x), \quad m \geq 0, \qquad (3.2.12)$$

with $\hat{\mathrm{P}}_{-1}(x) = 0$ and $\hat{\mathrm{P}}_0(x) = 1$, the chain sequence is given by

$$\left\{\frac{\gamma_{2n-1}\gamma_{2n+2}}{(\gamma_{2n-1} + \gamma_{2n})(\gamma_{2n+1} + \gamma_{2n+2})}\right\}_{n=1}^{\infty}$$

with the parameter sequence $\{k'_n\}_{n=0}^{\infty} = \{\gamma_{2n+2}/(\gamma_{2n+1} + \gamma_{2n+2})\}_{n=0}^{\infty}$. The result now follows since $k'_n = 1 - g_n$, $n \geq 0$. □

The OPS $\{\tilde{\mathrm{P}}_n(x)\}_{n=1}^{\infty}$ can be seen to be co-recursive with respect to the OPS $\{\hat{\mathrm{P}}_n(x)\}_{n=1}^{\infty}$ arising from the initial conditions $\tilde{\mathrm{P}}_0(x) = 1$ and $\tilde{\mathrm{P}}_1(x) = \hat{\mathrm{P}}_1(x) + \gamma_2$. The co-recursive polynomials have been investigated in the past, for example, in Chihara [41] and Marcellán et al. [129], in which the structure and spectrum of the generalized co-recursive polynomials have been studied.

Further, from (3.2.11b), the associated chain sequence is $\{\tilde{a}_n^2/b_n^{(1)}b_{n+1}^{(1)}\}_{n=1}^{\infty}$ with the first few terms as

$$\frac{\tilde{a}_1^2}{b_1^{(1)}b_2^{(1)}} = \frac{\gamma_4}{(\gamma_3 + \gamma_4)} = (1 - k_0)k_1; \quad \frac{\tilde{a}_2^2}{b_2^{(1)}b_3^{(1)}} = \frac{\gamma_3\gamma_6}{(\gamma_3 + \gamma_4)(\gamma_5 + \gamma_6)} = (1 - k_1)k_2;$$
$$\frac{\tilde{a}_3^2}{b_3^{(1)}b_4^{(1)}} = \frac{\gamma_5\gamma_8}{(\gamma_5 + \gamma_6)(\gamma_7 + \gamma_8)} = (1 - k_2)k_3; \ \ldots$$

Proceeding as above, we obtain the minimal parameter sequence $\{k_n\}_{n=0}^{\infty}$ where $k_0 = 0$ and $k_n = \gamma_{2n+2}/b_{n+1}^{(1)} = 1 - g_n$, $n \geq 1$, which shows that the OPS $\{\tilde{\mathrm{P}}_n(x)\}_{n=1}^{\infty}$ is associated with the complementary chain sequence $\{\tilde{\vartheta}_n(0)\}_{n=1}^{\infty}$.

Viewing the generalized complementary chain sequences as perturbations of the minimal parameters or simply a transformation of the original chain sequence, we give an important consequence of Theorem 3.2.2.

Proposition 3.2.3. *The kernel polynomial sequence* $\{\mathrm{P}_n(x)\}_{n=1}^{\infty}$ *remains invariant* *under generalized complementary chain sequence if the sequence* $\{\gamma_n\}_{n=1}^{\infty}$ *satisfies,*

$$\gamma_{2n+1} - \gamma_{2n-1} = \gamma_{2n+2} - \gamma_{2n}, \quad n \geq 1.$$

Proof. The proof follows from a comparison of (3.2.11a) and the expressions for $b_{n+1}^{(2)}$ and $a_n^{(2)}$. □

Proposition 3.2.3 is important because it is known (Chihara [42, Ex. 7.2, p. 39]), that the relation between the monic orthogonal polynomials and the kernel polynomials is not unique. That is, for fixed $t \in \mathbb{R}$, though $\{\mathrm{P}_n(x)\}$ will lead to a unique kernel polynomial system $\{\mathrm{K}_n(t,x)\}$, there are infinite number of other monic orthogonal polynomial systems which has the same $\{\mathrm{K}_n(x)\}$ as their kernel polynomial system. Hence generalized complementary chain sequences can be used to construct two orthogonal polynomials systems having the same kernel polynomial systems.

We make two observations about the consequences of complementary chain sequences regarding zeros of an OPS. Let the zeros of $\{\mathrm{P}_n(x)\}_{n=1}^{\infty}$ and $\{\tilde{\mathrm{P}}_n(x)\}_{n=1}^{\infty}$ be denoted as

$$0 < x_{n,1} < x_{n,2} < \cdots < x_{n,n-1} < x_{n,n} \quad \text{and} \quad 0 < \tilde{x}_{n,1} < \tilde{x}_{n,2} < \cdots < \tilde{x}_{n,n-1} < \tilde{x}_{n,n},$$

respectively. For fixed n, by interlacing of zeros of $\mathrm{P}_n(x)$ and $\tilde{\mathrm{P}}_n(x)$ it is understood that $x_{n,j}$ are mutually separated by $\tilde{x}_{n,j}$ for $j = 1, 2, \cdots, n$. In the present case, it is interesting to note from (3.2.5) and (3.2.11b), that the sum of the roots of $\mathrm{P}_n(x)$ is given by $\gamma_2 + \gamma_3 + \cdots + \gamma_{2n}$ while that for $\tilde{\mathrm{P}}_n(x)$ is $\gamma_1 + \gamma_3 + \cdots + \gamma_{2n}$.

Observation 3.2.1. *It is clear that if* $\gamma_1 = \gamma_2$*, interlacing of the zeros of* $\{\mathrm{P}_n(x)\}_{n=1}^{\infty}$ *and* $\{\tilde{\mathrm{P}}_n(x)\}_{n=1}^{\infty}$ *can never occur.*

Observation 3.2.2. *For* $\gamma_1 \neq \gamma_2$ *and fixed* n*, the zeros* $\{x_{n,j}\}_{j=1}^{n}$ *and* $\{\tilde{x}_{n,j}\}_{j=1}^{n}$ *cannot interlace if* $(\gamma_1 - \gamma_2)$ *and* $(x_{n,j} - \tilde{x}_{n,j})$ *have the same sign for some* $j = 1, 2, \cdots, n$*. Indeed, suppose* $\gamma_1 > \gamma_2$ *and* $x_{n,j} > \tilde{x}_{n,j}$ *for some* $j = 1, 2, \cdots, n$*. If the zeros of* $\mathrm{P}_n(x)$ *and* $\tilde{\mathrm{P}}_n(x)$ *interlace, then* $\sum_{j=1}^{n} \tilde{x}_{n,j} < \sum_{j=1}^{n} x_{n,j}$ *which is a contradiction. The case* $\gamma_1 < \gamma_2$ *and* $x_{n,j} < \tilde{x}_{n,j}$ *follows similarly.*

The next result shows that while the generalized complementary chain sequence of associated with $\{\hat{\mathrm{P}}_n(x)\}_{n=1}^{\infty}$ yields an OPS, that associated with the associated (numerator) polynomials $\{\hat{\mathrm{P}}_n^{(1)}(x)\}_{n=1}^{\infty}$ leads to a KOPS.

Theorem 3.2.3. *Consider the OPS $\{\hat{\mathrm{P}}_n^{(1)}(x)\}_{n=1}^{\infty}$. Then the generalized complementary chain sequence associated with $\hat{\mathrm{P}}_n^{(1)}(x)$ leads to a KOPS $\{\mathcal{Q}_n(x)\}_{n=1}^{\infty}$ satisfying the relation*

$$\mathcal{Q}_{n+1}(x) = (x - \gamma_{2n+3} - \gamma_{2n+4})\mathcal{Q}_n(x) - \gamma_{2n+2}\gamma_{2n+3}\mathcal{Q}_{n-1}(x), \quad n \geq 0, \tag{3.2.13}$$

with $\mathcal{Q}_{-1}(x) = 0$ and $\mathcal{Q}_0(x) = 1$.

Proof. It is clear from (3.2.12) that $\{\hat{\mathrm{P}}_n^{(1)}(x)\}_{n=1}^{\infty}$ satisfy

$$\hat{\mathrm{P}}_{n+1}^{(1)}(x) = (x - \gamma_{2n+3} - \gamma_{2n+4})\hat{\mathrm{P}}_n^{(1)}(x) - \gamma_{2n+2}\gamma_{2n+3}\hat{\mathrm{P}}_{n-1}^{(1)}(x), \quad n \geq 1$$

with $\hat{\mathrm{P}}_{-1}^{(1)}(x) = 0$ and $\hat{\mathrm{P}}_0^{(1)}(x) = 1$. The associated chain sequence is

$$\left\{\frac{\gamma_{2n+1}\gamma_{2n+4}}{(\gamma_{2n+1} + \gamma_{2n+4})(\gamma_{2n+3} + \gamma_{2n+5})}\right\}_{n=1}^{\infty}$$

with the (non-minimal) parameter sequence $\{\gamma_{2n+4}/(\gamma_{2n+3}+\gamma_{2n+4})\}_{n=0}^{\infty}$. Hence the OPS $\{\mathcal{Q}_n(x)\}_{n=1}^{\infty}$ associated with the generalized complementary chain sequence satisfy the three term recurrence relation (3.2.13). To prove that $\{\mathcal{Q}_n(x)\}_{n=1}^{\infty}$ is a KOPS, consider the polynomials $\{\mathcal{X}_n(x)\}_{n=1}^{\infty}$ given by $x\mathcal{Q}_n(x) = \mathcal{X}_{n+1}(x) + \gamma_{2n+3}\mathcal{X}_n(x)$, $n \geq 0$. The first thing we require is $\mathcal{X}_{n+1}(0) = -\gamma_{2n+3}\mathcal{X}_n(0)$, so that choosing $\mathcal{X}_1(0) = -\gamma_3$, we have $\mathcal{X}_{n+1}(0) = (-1)^{n+1}\gamma_{2n+3}\gamma_{2n+1}\cdots\gamma_5\gamma_3$.

Now, suppose that $\{\mathcal{X}_n(x)\}_{n=1}^{\infty}$ satisfy the recurrence relation

$$\mathcal{X}_{n+1}(x) = (x - \sigma_{n+1})\mathcal{X}_n(x) - \eta_n\mathcal{X}_{n-1}(x), \quad n \geq 1,$$

with $\mathcal{X}_0(x) = 1$ $\mathcal{X}_1(x) = x - \gamma_3$ and where the coefficients $\{\sigma_n\}$ and $\{\eta_n\}$ are to be determined. One way to determine them is that the equality $\mathcal{X}_{n+1}(0) = -\sigma_{n+1}\mathcal{X}_n(0) - \mu_n\mathcal{X}_{n-1}(0)$ must hold, which implies $\gamma_{2n+1}\sigma_{n+1} - \mu_n = \gamma_{2n+3}\gamma_{2n+1}$, $n \geq 1$. A possible choice for σ_{n+1} and μ_n satisfying these relations is

$$\sigma_{n+1} = \gamma_{2n+3} + \gamma_{2n+2} \quad \text{and} \quad \mu_n = \gamma_{2n+1}\gamma_{2n+2}, \quad n \geq 1.$$

Since $\mu_n > 0$ for $n \geq 1$, by Favard's Theorem (Chihara [42, Theorem 4.4, p. 21]) $\{\mathcal{X}_n(x)\}_{n=1}^{\infty}$ becomes a OPS and $\{\mathcal{Q}_n(x)\}_{n=1}^{\infty}$ its corresponding KOPS (Chihara [42, eqn. 7.3, p. 35]). □

We end this section with some information on the Jacobi matrices associated with the complementary chain sequences.

The Jacobi matrix of the polynomials $\mathcal{P}_n(x)$ and $\tilde{\mathcal{P}}_n(x)$ are given respectively by,

$$J_{\mathcal{P}} = \begin{pmatrix} \gamma_2 & 1 & 0 & \cdots \\ \gamma_2\gamma_3 & \gamma_3+\gamma_4 & 1 & \cdots \\ 0 & \gamma_4\gamma_5 & \gamma_5+\gamma_6 & \cdots \\ 0 & 0 & \gamma_6\gamma_7 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \quad J_{\tilde{\mathcal{P}}} = \begin{pmatrix} \gamma_1 & 1 & 0 & \cdots \\ \gamma_1\gamma_4 & \gamma_3+\gamma_4 & 1 & \cdots \\ 0 & \gamma_3\gamma_6 & \gamma_5+\gamma_6 & \cdots \\ 0 & 0 & \gamma_5\gamma_8 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

The respective LU decomposition of the above Jacobi matrices are then given by,

$$L_{\mathcal{P}} = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ \gamma_3 & 1 & 0 & 0 & \cdots \\ 0 & \gamma_5 & 1 & 0 & \cdots \\ 0 & 0 & \gamma_7 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad U_{\mathcal{P}} = \begin{pmatrix} \gamma_2 & 1 & 0 & 0 & \cdots \\ 0 & \gamma_4 & 1 & 0 & \cdots \\ 0 & 0 & \gamma_6 & 1 & \cdots \\ 0 & 0 & 0 & \gamma_8 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

$$L_{\tilde{\mathcal{P}}} = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ \gamma_4 & 1 & 0 & 0 & \cdots \\ 0 & \gamma_6 & 1 & 0 & \cdots \\ 0 & 0 & \gamma_8 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad U_{\tilde{\mathcal{P}}} = \begin{pmatrix} \gamma_1 & 1 & 0 & 0 & \cdots \\ 0 & \gamma_3 & 1 & 0 & \cdots \\ 0 & 0 & \gamma_5 & 1 & \cdots \\ 0 & 0 & 0 & \gamma_7 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

It may be observed that $L_{\mathcal{P}}$ and $L_{\tilde{\mathcal{P}}}$ can be obtained from the matrix products $U_{\tilde{\mathcal{P}}} \cdot \mathfrak{J}$ and $U_{\mathcal{P}} \cdot \mathfrak{J}$ respectively, where

$$\mathfrak{J} = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

This is equivalent to the removal of the first column of the matrices $U_{\tilde{\mathcal{P}}}$ and $U_{\mathcal{P}}$.

3.3 Three illustrations

In this section, starting with particular minimal parameter sequences and assuming $c_n = 0$, $n \geq 1$, we construct the para-orthogonal polynomials and the related Szegő polynomials to illustrate our results.

3.3.1 Using Carathéodory functions

Consider the sequence $\{\delta_n\}_{n=0}^{\infty}$, which satisfies $\delta_0 > 0$, $|\delta_n| < 1$ and

$$\delta_{n+1} - \delta_n = \delta_n \delta_{n+1}, \quad n \geq 1. \tag{3.3.1}$$

Our aim is to first use a chain sequence to construct the Szegő polynomials $\Phi_n^{(t,1)}(z)$, having $\delta_n \in \mathbb{R}$ and satisfying (3.3.1) as the Verblunsky coefficients. We will also use the complementary chain sequence to get another sequence of Szegő polynomials $\tilde{\Phi}_n^{(t,1)}(z)$ which has $-\delta_n$ as the Verblunsky coefficients. The associated Carathéodory function in each case is also given.

We start with the sequence $\{m_n^{(t,1)}\}_{n=0}^{\infty}$, where $m_0^{(t,1)} = 0$ and $m_n^{(t,1)} = (1 - \delta_n)/2$, $n \geq 1$. These minimal parameters are obtained by first substituting $c_k = 0$, $k \geq 1$ in the Verblunsky coefficients (3.1.6) and then equating them to δ_n. The corresponding chain sequence is

$$d_1 = \frac{1-\delta_1}{2} \quad \text{and} \quad d_n = \frac{1}{4}(1+\delta_{n-1})(1-\delta_n) = \frac{1}{4}(1-2\delta_{n-1}\delta_n), \quad n \geq 2.$$

The following two algebraic relations of δ_n, $n \geq 1$,

$$\begin{aligned} \delta_{n+1} - \delta_1 &= \delta_1\delta_2 + \delta_2\delta_3 + \delta_3\delta_4 + \cdots + \delta_n\delta_{n+1}, \\ \delta_n &= \frac{\delta_{n+1}}{1+\delta_{n+1}} = \cdots = \frac{\delta_{n+k}}{1+k\delta_{n+k}}, \quad k \in \mathbb{N}, \end{aligned} \tag{3.3.2}$$

will be needed later and can be proved by simple induction using (3.3.1).

Proposition 3.3.1. *The sequence of monic polynomial* $\{R_n(z)\}_{n=0}^{\infty}$*, where*

$$R_0(z) = 1, \quad R_n(z) = 1 + \sum_{k=1}^{n}[1 + 2k(n-k)\delta_1\delta_n]z^k, \quad n \geq 1, \tag{3.3.3}$$

satisfies the recurrence relation

$$R_{n+1}(z) = (z+1)R_n(z) - (1-2\delta_n\delta_{n+1})zR_{n-1}(z), \quad n \geq 0,$$

with $R_{-1}(z) = 0$ *and* $R_0(z) = 1$.

Proof. First, note that $R_1(z) = (z+1)$ is of the form (3.3.3). Suppose $R_n(z)$ has this form and satisfies the recurrence relation for $n = 1, 2, \ldots, j$. We shall now show

$$R_{j+1}(z) + (1-2\delta_j\delta_{j+1})zR_{j-1}(z) = (z+1)R_j(z), \quad j \geq 1. \tag{3.3.4}$$

Using (3.3.2), the coefficient of z^k in the left-hand side of (3.3.4) is

$$\begin{aligned} &1 + 2k(j-k+1)\delta_1\delta_{j+1} + (1-2\delta_j\delta_{j+1})[1 + 2(k-1)(j-k)\delta_1\delta_{j-1}] \\ &= 1 + 2\frac{k(j-k+1)}{j}(\delta_{j+1}-\delta_1) + 1 - 2(\delta_{j+1}-\delta_j) + 2\frac{(k-1)(j-k)}{j-2}(\delta_{j-1}-\delta_1) \\ &- \frac{2\cdot 2(k-1)(j-k)}{j-2}(\delta_{j-1}-\delta_1)(\delta_{j+1}-\delta_j). \end{aligned} \tag{3.3.5}$$

It is easy to verify that the coefficients of δ_{j+1} and δ_{j-1} vanish in (3.3.5). The coefficient of δ_1 is

$$-\frac{2k(j-k+1)}{j}-\frac{2(k-1)(j-k)}{j-2}-\frac{2\cdot 2(k-1)(j-k)}{j(j-2)}+\frac{2\cdot 2(k-1)(j-k)}{(j-1)(j-2)}$$
$$=-\frac{2k(j-k)}{j-1}-\frac{2(k-1)(j-k+1)}{j-1}. \tag{3.3.6}$$

Similarly, the coefficient of δ_j is

$$2+\frac{2\cdot 2(k-1)(j-k)}{j-1}=\frac{2k(j-k)}{j-1}+\frac{2(k-1)(j-k+1)}{j-1}. \tag{3.3.7}$$

Using (3.3.6) and (3.3.7) in (3.3.5), the coefficient of z^k in the left-hand side of (3.3.4) is given by

$$[1+2(k-1)(j-k+1)\delta_1\delta_j]+[1+2k(j-k)\delta_1\delta_j],$$

which is nothing but the coefficient of z^k in the right-hand side of (3.3.4). Hence, by induction the proof is complete. □

We now obtain the Szegő polynomials $\Phi_n^{(t,1)}(z)$ from the para-orthogonal polynomials $R_n(z)$ given by (3.3.3). Since $\Phi_n^{(t,1)}(z)=R_n(z)-2(1-m_n)R_{n-1}(z)$, $n\geq 1$, it can be seen that the coefficient of z^k, $1\leq k\leq n-1$, in $\Phi_n^{(t,1)}(z)$ is $-\delta_n(1-2k\delta_1)$. Hence, the Szegő polynomials are given by

$$\Phi_n^{(t,1)}(z)=z^n-\delta_n\big[(1-2(n-1)\delta_1)z^{n-1}+\cdots+(1-2\delta_1)z+1\big],\quad n\geq 1, \tag{3.3.8}$$

with $\alpha_{n-1}^{(t,1)}=-\Phi_n^{(t,1)}(0)=\delta_n$.

Next, we find the Carathéodory function associated with the parameters δ_n's given by (3.3.1). For this, consider the analytic function

$$\mathcal{C}(z)=1-\frac{2(1-\sigma)z}{1+(1-2\sigma)z}=\frac{1-z}{1+(1-2\sigma)z},\qquad |z|<1,$$

where $0<\sigma<1$. That $\mathcal{C}(z)$ corresponds to a PPC-fraction with the parameter γ_n, where

$$\gamma_n = \frac{1}{n + \frac{\sigma}{1-\sigma}}, \quad n \geq 1, \tag{3.3.9}$$

can be shown by applying the algorithm in Jones et al. [99], which is similar to the Schur algorithm. With the initial values $\mathcal{C}_0(z) = (1-z)/(1+(1-2\sigma)z)$, $\gamma_0 = \mathcal{C}_0(0) = 1$, define

$$\mathcal{C}_1(z) = \frac{\gamma_0 - \mathcal{C}_0(z)}{\gamma_0 + \mathcal{C}_0(z)} = \frac{z}{1 + \frac{\sigma}{1-\sigma} - \left(1 - \frac{1-2\sigma}{1-\sigma}\right) z}, \quad \gamma_1 = \mathcal{C}_1'(0) = \frac{1}{1 + \frac{\sigma}{1-\sigma}}.$$

Assume, for $k \geq 1$, the following form of the Carathéodory function

$$\mathcal{C}_k(z) = \frac{z}{k + \frac{\sigma}{1-\sigma} - \left(k - \frac{1-2\sigma}{1-\sigma}\right) z}, \quad \gamma_k = \mathcal{C}_k'(0).$$

The form is true for $k = 1$. Now define

$$\mathcal{C}_{k+1}(z) = \frac{\gamma_k z - \mathcal{C}_k(z)}{\gamma_k \mathcal{C}_k(z) - z}, \quad n \geq 1. \tag{3.3.10}$$

It can be shown that

$$\gamma_k = \frac{1-\sigma}{k-(k-1)\sigma} = \frac{1}{k + \frac{\sigma}{1-\sigma}},$$

which is also true for $k = 1$. Simplifying (3.3.10), we obtain

$$\mathcal{C}_{k+1} = \frac{z}{\left(k + 1 + \frac{\sigma}{1-\sigma}\right) - \left(k + 1 - \frac{1-2\sigma}{1-\sigma}\right) z},$$

from which $\gamma_{k+1} = \frac{1}{k+1+\frac{\sigma}{1-\sigma}}$. Hence by induction, and because of the uniqueness of the Carathéodory function that corresponds to a given PPC-fraction, the assertion follows. Moreover, observe that $\delta_n = -\gamma_n$ satisfies (3.3.1) and so $\Phi_n^{(t)}(0) = \frac{1}{n+\frac{\sigma}{1-\sigma}}$.

Further, if $\chi_n^{-2} = \left\|\Phi_n^{(t,1)}(z)\right\|^2$ then, using the fact that the Verblunsky coefficients are all real, we have

$$\chi_n^{-2} = \mu_0(1-|\Phi_1(0)|^2)(1-|\Phi_2(0)|^2)\cdots(1-|\Phi_n(0)|^2) = \prod_{k=1}^{n}\big(1-\delta_k^2\big).$$

Moreover, the Verblunsky coefficients can be written as

$$\delta_n = \frac{1}{n+\frac{\sigma}{1-\sigma}} = \frac{1-\sigma}{n(1-\sigma)+\sigma}, \qquad n \geq 1,$$

from which we obtain

$$\begin{aligned} 1-\delta_n^2 &= \frac{[n(1-\sigma)+\sigma-1+\sigma][n(1-\sigma)+\sigma+1-\sigma]}{[n(1-\sigma)+\sigma]^2} \\ &= \frac{[(n-1)-(n-2)\sigma][(n+1)-n\sigma]}{[n-(n-1)\sigma]^2}. \end{aligned}$$

This yields the fact that

$$\chi_n^{-2} = \frac{\sigma[(n+1)-n\sigma]}{[n-(n-1)\sigma]} = \sigma\big(1+\frac{1-\sigma}{n(1-\sigma)+\sigma}\big) = \sigma(1+\delta_n).$$

Hence, $\chi_n^{-2}{=}\big\|\Phi_n^{(t)}(z)\big\|^2$ tends to $\sigma > 0$ as $n \to \infty$.

Consider now the parameter sequence $\big\{k_n^{(t,1)}\big\}_{n=0}^{\infty}$, defined by $k_0^{(t,1)} = 0$ and $k_n^{(t,1)} = 1-m_n^{(t,1)} = (1+\delta_n)/2$, $n \geq 1$. From (3.3.1), it is easy to check that $1+\delta_{n+1} = 1/(1-\delta_n)$, $n \geq 1$. In this case, the constant sequence $\{1/4\}$ becomes the complementary chain sequence so that equation (3.1.4) assumes the form

$$\tilde{R}_{n+1}(z) = (1+z)\tilde{R}_n(z) - z\tilde{R}_{n-1}(z), \quad n \geq 1.$$

The above recurrence relation is satisfied by the palindromic polynomials $z^n + r(z^{n-1} + \cdots + z) + 1$, $r \in \mathbb{R}$. (We note that a polynomial $p_n(z) = c_0 + c_1 z + \cdots + c_n z^n$ is called palindromic if $c_i = c_{n-i}$, $i = 0, 1, \cdots, n$). For $r = 1$, the para-orthogonal polynomials are the partial sums of the geometric series given by

$$\tilde{R}_n(z) = 1 + z + z^2 + \cdots + z^n = \frac{1-z^{n+1}}{1-z}, \quad n \geq 1.$$

Then the Szegő polynomials from the complementary chain sequence are given by

$$\tilde{\Phi}_n^{(t,1)}(z) = z^n + \delta_n z^{n-1} + \cdots + \delta_n z + \delta_n, \quad n \geq 1, \tag{3.3.11}$$

with $\alpha_{n-1}^{(t,1)} = -\delta_n$. The polynomials $\tilde{\Phi}_n^{(t,1)}(z)$ are also considered in Ronning [149] where it is proved that

$$\tilde{\Phi}_n^{(t,1)}(0) = \delta_n = -\frac{1}{n + \frac{\sigma}{1-\sigma}}, \qquad n \geq 1. \tag{3.3.12}$$

Further, the corresponding Carathéodory function is $\tilde{\mathcal{C}}(z) = \frac{1+(1-2\sigma)z}{1-z}$, $|z| < 1$, where $0 < \sigma < 1$.

Further, let $\mu^{(t,1)}(z)$ be the probability measure associated with the positive chain sequence $\{d_n\}_{n=1}^{\infty}$. Its complementary chain sequence $\{1/4\}$ is not a SPPCS, with $\{1/2\}$ as its maximal parameter sequence. Hence by Lemma 3.2.2, $\{d_n\}_{n=1}^{\infty}$ is a SPPCS and $\mu^{(t)}(z)$ has zero jump $(t = 0)$ at $z = 1$. This also implies that if $\nu^{(t)}(z)$ is the measure associated with $\{1/4\}$, then, $\nu^{(t)}(z)$ has a jump $t = 1/2$ at $z = 1$.

We end this illustration with two observations which we state as remarks.

Remark 3.3.1. *As $n \to \infty$, both the minimal parameter sequences approach 1/2. From the expressions (3.3.8) and (3.3.11) it follows that for fixed z, $\Phi_n^{(t)}(z)$ and $\tilde{\Phi}_n^{(t)}(z)$ approach z^n as n becomes large. The polynomials z^n are called the Szegő–Chebyshev polynomials and correspond to the standard Lebesgue measure on the unit circle.*

Remark 3.3.2. *Suppose the minimal parameters are given in terms of some variable ε. Then, the coefficients of the polynomial $R_n(z)$ satisfying (3.1.4) with $c_n = 0$, $n \geq 1$ will be given in terms of ε. Since $R_n(z)$ is palindromic for the chain sequence $\{d_n\} = \{1/4\}$, $R_n(z)$ can always be expressed as the sum of two polynomials, one of them a palindromic and the other one being such that it vanishes whenever ε is chosen so that $d_n = 1/4$.*

3.3.2 Using Gaussian hypergeometric functions

Consider the contiguous relation

$$(c-a)F(a-1, b; c; z) = (c - 2a - (b-a)z)F(a, b; c; z) + a(1-z)F(a+1, b; c; z),$$

which, as shown in Sri Ranga [162], can be transformed to the three term recurrence relation

$$\varrho_{n+1}(z) = \left(z + \frac{c-b+n}{b+n}\right)\varrho_n(z) - \frac{n(c+n-1)}{(b+n-1)(b+n)} z\varrho_{n-1}(z), \quad n \geq 1, \quad (3.3.13)$$

satisfied by the monic polynomial

$$\varrho_n(z) = \frac{(c)_n}{(b)_n} F(-n, b; c; 1-z). \quad (3.3.14)$$

It was also shown that for the specific values $b = \lambda \in \mathbb{R}$ and $c = 2\lambda - 1$, the polynomials (3.3.14) are Szegő polynomials. We note that with $b = \lambda + 1$, $\varrho_n(z)$ given by (3.3.14) are called the circular Jacobi polynomials, Ismail [90, Example 8.2.5]. For other specialized values of b and c in (3.3.13), $\varrho_n(z)$ becomes a para-orthogonal polynomial.

Let $\lambda > -1/2 \in \mathbb{R}$. Taking $b = \lambda + 1$ and $c = 2\lambda + 2$, (3.3.13) reduces to

$$\varrho_{n+1}(z) = (z+1)\varrho_n(z) - \frac{n(2\lambda+n+1)}{(\lambda+n)(\lambda+n+1)} z\varrho_{n-1}(z), \quad n \geq 1,$$

satisfied by

$$\varrho_n(z) = R_n(z) = \frac{(2\lambda+2)_n}{(\lambda+1)_n} F(-n, \lambda+1; 2\lambda+2; 1-z), \quad n \geq 1.$$

Consider now the sequence $\{d_{n+1}\}_{n=1}^{\infty}$, where

$$d_{n+1} = \frac{1}{4} \frac{n(2\lambda+n+1)}{(\lambda+n)(\lambda+n+1)}, \quad n \geq 1.$$

As established in Bracciali et al. [25, Example 3], Castillo et al. [37] and Costa et al. [40], for $\lambda > -1$, the sequence $\{d_{n+1}\}_{n=1}^{\infty}$ is a positive chain sequence and $\{\mathfrak{m}_n^{(t,1)}\}_{n=0}^{\infty}$, where

$$\mathfrak{m}_n^{(t,1)} = \frac{n}{2(\lambda+n+1)}, \quad n \geq 0,$$

is its minimal parameter sequence. When $-1/2 \geq \lambda > -1$, $\{\mathfrak{m}_n^{(t,1)}\}_{n=0}^{\infty}$ is also the maximal parameter sequence of $\{d_{n+1}\}_{n=1}^{\infty}$, which makes it a SPPCS. However, when $\lambda > -1/2$ then $\{d_{n+1}\}_{n=1}^{\infty}$ is not a SPPCS and its maximal parameter sequence

$\{M_{n+1}\}_{n=0}^{\infty}$ is such that

$$M_{n+1} = \frac{2\lambda + n + 1}{2(\lambda + n + 1)}, \quad n \geq 0.$$

The coefficients d_{n+1}, $n \geq 1$ are the same coefficients occurring in the recurrence formula for ultraspherical (or Gegenbauer) polynomials.

Further, for $\lambda > -1/2$ and $0 \leq t < 1$, if $\{m_n^{(t,1)}\}_{n=0}^{\infty}$ is the minimal parameter sequence of the positive chain sequence $\{d_n\}_{n=1}^{\infty}$, obtained by adding $d_1 = (1-t)M_1$ to $\{d_{n+1}\}_{n=1}^{\infty}$, then

$$\Phi_n^{(t,1)}(z) = R_n(z) - 2\big(1 - m_n^{(t,1)}\big)R_{n-1}(z), \quad n \geq 1$$

and are the monic OPUC with respect to the measure $\mu^{(t,1)}(z)$, where $\mu^{(t,1)}(z)$ is as defined by (3.1.2). To find $\mu^{(t,1)}(z)$, we first find the measure $\mu^{(0,1)}(z)$ arising when $\{d_n\}_{n=1}^{\infty}$ becomes a SPPCS ($t = 0$). As shown in Sri Ranga [162], the monic OPUC $\Phi_n^{(0)}(z)$, $n \geq 1$, are given by

$$R_n(z) - 2(1 - M_n)R_{n-1}(z) = \frac{(2\lambda+1)_n}{(\lambda+1)_n} F(-n, \lambda+1; 2\lambda+1; 1-z), \quad n \geq 1.$$

Further, the Verblunsky coefficients are given by

$$\alpha_{n-1}^{(0)} = -\Phi_n^{(0)}(0) = -\frac{(\lambda)_n}{(\lambda+1)_n}, \qquad n \geq 1. \tag{3.3.15}$$

It is proved in Sri Ranga [162] that the Verblunsky coefficients $\alpha_{n-1}^{(0)}$ are associated with the non-trivial probability measure

$$d\mu^{(0)}\big(e^{i\theta}\big) = \tau^{(\lambda)} \sin^{2\lambda}(\theta/2) d\theta, \quad \tau^{(\lambda)} = \frac{|\Gamma(1+\lambda)|^2}{\Gamma(2\lambda+1)} 4^{\lambda}.$$

Hence

$$\int_{\partial\mathbb{D}} f(\zeta) d\mu^{(t)}(\zeta) = (1-t)\tau^{(\lambda)} \int_0^{2\pi} f\big(e^{i\theta}\big) \sin^{2\lambda}(\theta/2) d\theta + t f(1).$$

We characterize the Szegő polynomials associated with the complementary chain sequence since it is not possible to find closed form expressions for the coefficients of the para-orthogonal polynomials and Szegő polynomials. Since $\{R_n(z)\}$, depends on the parameter b $(= \lambda+1)$, in what follows, we denote $R_n(z)$ by $R_n^{(b)}(z)$. We also denote c_n and d_n by $c_n^{(b)}$ and $d_n^{(b)}$ respectively. Now, note that if

$$Q_n^{(b)}(z) = \frac{1}{2(1-t)M_1}\int_{\mathbb{T}} \frac{R_n^{(b)}(z) - R_n^{(b)}(\zeta)}{z-\zeta}(1-\zeta)d\mu^{(t)}(\zeta), \quad n \geq 0,$$

then $\{Q_n^{(b)}(z)\}_{n=0}^{\infty}$ satisfies

$$Q_{n+1}^{(b)}(z) = \left[\left(1+ic_{n+1}^{(b)}\right)z + \left(1-ic_{n+1}^{(b)}\right)\right]Q_n^{(b)}(z) - 4d_{n+1}^{(b)}zQ_{n-1}^{(b)}(z), \quad n \geq 1,$$

with $Q_0^{(b)}(z) = 0$ and $Q_1^{(b)}(z) = 1$. That is, the three term recurrence for $\{Q_n^{(b)}(z)\}_{n=0}^{\infty}$ is the same as for $\{R_n^{(b)}(z)\}_{n=0}^{\infty}$, with the difference being only on the initial conditions. The polynomials $\{Q_n^{(b)}(z)\}$ are generally called the numerator polynomials associated with $\{R_n^{(b)}(z)\}$. Further, observe that the three term recurrence for $\{Q_n^{(b)}(z)\}_{n=0}^{\infty}$ can also be given in the shifted form

$$Q_{n+2}^{(b)}(z) = \left[\left(1+ic_{n+2}^{(b)}\right)z + \left(1-ic_{n+2}^{(b)}\right)\right]Q_{n+1}^{(b)}(z) - 4d_{n+2}^{(b)}zQ_n^{(b)}(z), \quad n \geq 1, \tag{3.3.16}$$

with $Q_1^{(b)}(z) = 1$ and $Q_2^{(b)}(z) = \left(1+ic_2^{(b)}\right)z + \left(1-ic_2^{(b)}\right)$.

Consider now the parameter sequence given by $k_n^{(t,1)} = 1 - m_n^{(0,1)} = n/[2(\lambda+n)]$ for $n \geq 1$. For sake of clarity, we would like to note that t need not be necessarily 0. It depends on whether the resulting chain sequence for $\{k_n^{(t,1)}\}$, given by

$$a_1^{(b)} = \frac{1}{2\lambda+2} \quad \text{and} \quad a_{n+1}^{(b)} = \frac{1}{4}\frac{(n+1)(2\lambda+n)}{(\lambda+n)(\lambda+n+1)}, \quad n \geq 1, \tag{3.3.17}$$

is a SPPCS or not.

Let $\nu^{(t,1)}(z)$ be the measure associated with the Verblunsky coefficients $\{\beta_{n-1}^{(t,1)}\}_{n=1}^{\infty}$ given by

$$\beta_{n-1}^{(t,1)} = \overline{\tau}_n \left[\frac{1-2k_n^{(t)} - ic_n^{(b)}}{1+ic_n^{(b)}}\right], \quad n \geq 1.$$

Following Theorem 3.2.1, the corresponding Szegő polynomials are

$$\tilde{\Phi}_n^{(t,1)}(z) = \frac{\tilde{R}_n^{(b)}(z) - 2\big(1 - k_n^{(t,1)}\big)\tilde{R}_{n-1}^{(b)}(z)}{\prod_{k=1}^{n}\big(1 + ic_k^{(b)}\big)}, \quad n \geq 1,$$

where the polynomials $\tilde{R}_n^{(b)}$ are given by

$$\tilde{R}_{n+1}^{(b)}(z) = \big[\big(1 + ic_{n+1}^{(b)}\big)z + \big(1 - ic_{n+1}^{(b)}\big)\big]\tilde{R}_n^{(b)}(z) - 4a_{n+1}^{(b)} z \tilde{R}_{n-1}^{(b)}(z), \quad n \geq 1, \quad (3.3.18)$$

with $\tilde{R}_0^{(b)}(z) = 1$ and $\tilde{R}_1^{(b)}(z) = \big(1 + ic_1^{(b)}\big)z + \big(1 - ic_1^{(b)}\big)$. Observing that $c_n^{(b)} = c_{n+1}^{(b-1)}$, $a_{n+1}^{(b)} = d_{n+2}^{(b-1)}$, $n \geq 1$, we have from (3.3.16) and (3.3.18)

$$\tilde{R}_n^{(b)}(z) = Q_{n+1}^{(b-1)}(z), \qquad n \geq 0,$$

and hence

$$\tilde{\Phi}_n^{(t)}(z) = \frac{Q_{n+1}^{(b-1)}(z) - 2\big(1 - k_n^{(t,1)}\big)Q_n^{(b-1)}(z)}{\prod_{k=1}^{n}\big(1 + ic_{k+1}^{(b-1)}\big)}, \quad n \geq 1.$$

That is, if $R_n^{(b)}(z)$ generates the Szegő polynomials $\Phi_n^{(t)}(z)$, $Q_n^{(b-1)}(z)$, which are the numerator polynomials for $R_n^{(b-1)}(z)$ generates the Szegő polynomials $\tilde{\Phi}_n^{(t)}(z)$ associated with the complementary chain sequences. We note that, in the present case too, $c_n^{(b)}(= c_n) = 0$, $n \geq 1$ and so by Theorem 3.2.1, $\beta_{n-1}^{(t)} = -\alpha_{n-1}^{(0)}$, $n \geq 1$. Hence $d\nu^{(t)}(z)$ are the Aleksandrov measures associated with $d\mu^{(0)}(z)$, Simon [156].

Further, we note that such Szegő polynomials result from perturbations of the Verblunsky coefficients obtained in the Illustration 3.3.1. Indeed, for $\sigma = \lambda/(1 + \lambda)$, $\{\lambda\delta_n\}$ corresponds to the Verblunsky coefficients given by (3.3.15), whereas by Verblunsky theorem, $\{\lambda\gamma_n\}$ corresponds to those given by the complementary chain sequence $\{a_{n+1}^{(b)}\}$ given by (3.3.17). Here $\{\delta_n\}$ and $\{\gamma_n\}$ are the ones chosen respectively by (3.3.9) and (3.3.12).

Further, when $\{a_{n+1}^{(b)}\}_{n=1}^{\infty}$ is the constant chain sequence $\{1/4\}$, $\tilde{R}_n^{(b)}(z)$ are the palindromic polynomials given by

$$\tilde{R}_n^{(b)}(z) = z^n + \nu^{(\lambda)}\big(z^{n-1} + \cdots + z\big) + 1, \quad n \geq 1,$$

where $\nu^{(\lambda)}$ is a constant depending on λ. Here we study the cases $\lambda = 0$ and $\lambda = 1$ for which the complementary chain sequence $a_{n+1}^{(b)} = 1/4$.

Case 1, $\lambda = 0$. Let

$$\tilde{R}_n^{(b)}(z) = z^n + \nu^{(0)}\big(z^{n-1} + \cdots + z\big) + 1, \quad n \geq 1.$$

The complementary chain sequence is $\{1/2, 1/4, 1/4, \dots\}$ which is known to be a SP-PCS. Hence $\{k_n^{(t)}\}_{n=0}^{\infty}$ where $k_0^{(t)} = 0$, $k_n^{(t)} = 1/2$, $n \geq 1$ is also the maximal parameter sequence implying that $t = 0$ and so

$$\tilde{\Phi}_n^{(0)}(z) = z^n + \big(\nu^{(0)} - 1\big)z^{n-1}.$$

For $\nu^{(0)} = 1$, $\tilde{\Phi}_n^{(0)}(z) = z^n$ and from Remark 3.3.1, $\lambda = 0$ can be viewed as the limiting case for the Verblunsky coefficients obtained in Section 3.3.1. Note that the Verblunsky coefficients are 0, as can be verified from (3.3.15).

Case 2, $\lambda = 1$. Let

$$\tilde{R}_n^{(b)}(z) = z^n + \nu^{(1)}\big(z^{n-1} + \cdots + z\big) + 1, \quad n \geq 1.$$

The complementary chain sequence is $\{1/4, 1/4, 1/4, \dots\}$ and $k_0^{(t)} = 0$, $k_n^{(t)} = n/2(n+1)$, $n \geq 1$. In this case, $t = 1/2$ and

$$\tilde{\Phi}_n^{(1/2)}(z) = z^n + \left(\nu^{(1)} - \frac{n+2}{n+1}\right) z^{n-1} - \frac{\nu^{(1)}}{n+1}\big(z^{n-2} + \cdots + z\big) - \frac{1}{n+1}, \quad n \geq 1,$$

so that the Verblunsky coefficients are given by $1/(n+1)$. Again it can be verified from (3.3.15) that the Verblunsky coefficients corresponding to $\lambda = 1$ are $(1)_n/(2)_n = 1/(n+1)$. Finally, for $\nu^{(1)} = 0$, $\tilde{R}_n^{(b)} = z^n + 1$, which has been considered as the first example in Bracciali et al. [25].

3.3.3 Using Laguerre polynomials

Perturbations of the Laguerre weight $x^\alpha e^{-x}$ have been studied by many authors. For example, Xu et al. [187, 188] found the Hankel determinants associated with the perturbed measure using Painlevé transcendents. Deaño et al. [54] considered the Geronimus transformation of the Laguerre weight $x^\alpha e^{-x}$ along with the addition of a mass and studied related asymptotic behavior. Such perturbation, particularly its numerical aspects, is also investigated in Branquinho et al. [26] and Beuno et al. [30].

In this illustration, we study a perturbation in the chain sequences related to these orthogonal polynomials. The Laguerre polynomials are orthogonal on $(0,\infty)$ with respect to the weight function $x^\alpha e^{-x}$ for $\alpha > -1$. Consider the three term recurrence relation satisfied by the monic Laguerre polynomials $\{L_n^{(\alpha)}(x)\}$, (Chihara [42, Page 154]),

$$\mathcal{R}_{n+1}^{(1)}(x) = [x-(2n+\alpha+1)]\mathcal{R}_n^{(1)}(x) - n(n+\alpha)\mathcal{R}_{n-1}^{(1)}(x), \quad n \geq 1, \tag{3.3.19}$$

with $\mathcal{R}_0^{(1)}(x) = 1$ and $\mathcal{R}_1^{(1)}(x) = x-(1+\alpha)$ and where $\mathcal{R}_n^{(1)}(x) \equiv L_n^{(\alpha)}(x)$, $n \geq 1$. Using the notations introduced immediately after (3.2.5), the associated chain sequence $\{d_n\}_{n=1}^{\infty}$ is,

$$d_n = \frac{(a_n^2)^{(1)}}{b_n^{(1)}b_{n+1}^{(1)}} = \frac{n(n+\alpha)}{(2n+\alpha-1)(2n+\alpha+1)}, \quad n \geq 1,$$

and as can be easily verified, the minimal parameters are given by, $m_n = n/(2n+\alpha+1)$, $n \geq 0$. It is easily seen that $0 < m_n < 1/2$, $n \geq 1$ and hence by Lemma 3.2.3, the chain sequence complementary to d_n is a chain sequence with a single parameter sequence. Moreover, for $-1 < \alpha < 0$, $m_n/(1-m_n) > n/(n-1) > 1$ and hence by Wall's criteria [184, Theorem 19.3], for SPPCS $\{d_n\}$ determines its parameters uniquely. Further, choosing $\gamma_1 = 0$, it is found that $\gamma_2 = (1+\alpha)$ and $\gamma_2\gamma_3 = 1.(1+\alpha)$ implies $\gamma_3 = 1$. Similarly, $\gamma_3+\gamma_4 = \alpha+3$ implies $\gamma_4 = \alpha+2$. Proceeding further on similar lines, it can be easily proved by induction that $\gamma_1 = 0$, $\gamma_{2n} = n+\alpha$ and $\gamma_{2n+1} = n$, $n \geq 1$. This gives the recurrence relation for the associated kernel polynomials as

$$\mathcal{R}_{n+1}^{(2)}(x) = [x-(2n+\alpha+2)]\mathcal{R}_n^{(2)}(x) - n(n+\alpha+1)\mathcal{R}_{n-1}^{(2)}(x), \quad n \geq 1, \tag{3.3.20}$$

with $\mathcal{R}_0^{(2)}(x) = 1$ and $\mathcal{R}_1^{(2)}(x) = x - (2+\alpha)$. Clearly, as is known, $\mathcal{R}_n^{(2)}(x) = L_n^{(\alpha+1)}(x)$, $n \geq 1$.

Consider now the polynomials $\{\mathcal{E}_n(x)\}_{n=0}^{\infty}$ satisfying the recurrence relation

$$\mathcal{E}_{n+1}(x) = [x - (2n+\alpha+2)]\mathcal{E}_n(x) - (n+1)(n+\alpha)\mathcal{E}_{n-1}(x), \quad n \geq 1$$

with $\mathcal{E}_0(x) = 1$ and $\mathcal{E}_1(x) = x - (\alpha+1)$. From the related chain sequence, we obtain the sequence $\{\gamma_n\}_{n=1}^{\infty}$ where $\gamma_1 = 0$, $\gamma_{2n} = n + \alpha$ and $\gamma_{2n+1} = n+1$, $n \geq 1$. The kernel polynomial sequence $\{\mathcal{K}_n(x)\}_{n=0}^{\infty}$ associated with $\{\mathcal{E}_n(x)\}_{n=0}^{\infty}$ satisfy

$$\mathcal{K}_{n+1}(x) = [x - (2n+\alpha+3)]\mathcal{K}_n(x) - (n+1)(n+\alpha+1)\mathcal{K}_n(x), \quad n \geq 0$$

with $\mathcal{K}_{-1}(x) = 0$ and $\mathcal{K}_0(x) = 1$. If we let $\gamma_1 = 1$, the resulting polynomials satisfy

$$\mathcal{P}_{n+1}(x) = [x - (2n+\alpha+2)]\mathcal{P}_n(x) - (n+1)(n+\alpha)\mathcal{P}_{n-1}(x), \quad n \geq 0$$

with $\mathcal{P}_{-1}(x) = 0$ and $\mathcal{P}_0(x) = 1$. From (3.3.19) it is clear that these polynomials are the associated generalized Laguerre polynomials of order 1 but with α shifted to $\alpha - 1$. The polynomial sequence $\{\hat{\mathcal{P}}_n(x)\}$ corresponding to the generalized complementary chain sequence satisfy

$$\hat{\mathcal{P}}_{n+1}(x) = [x - (2n+\alpha+2)]\hat{\mathcal{P}}_n(x) - n(n+\alpha+1)\hat{\mathcal{P}}_{n-1}(x), \quad n \geq 0$$

with $\hat{\mathcal{P}}_{-1}(x) = 0$ and $\hat{\mathcal{P}}_0(x) = 1$. Comparing with (3.3.20), we find that $\hat{\mathcal{P}}_n(x) \equiv L_n^{(\alpha+1)}(x)$, $n \geq 1$.

The (co-recursive) polynomials $\{\tilde{\mathcal{P}}_n(x)\}$ corresponding to the complementary chain sequence satisfy the recurrence relation

$$\tilde{\mathcal{P}}_{n+1}(x) = [x - (2n+\alpha+2)]\tilde{\mathcal{P}}_n(x) - n(n+\alpha+1)\tilde{\mathcal{P}}_{n-1}(x), \quad n \geq 1$$

with $\tilde{\mathcal{P}}_0(x) = 1$ and $\tilde{\mathcal{P}}_1(x) = x - 1$.

Moreover, since the condition in Proposition 3.2.3 is satisfied, the kernel polynomials for the OPS $\{\tilde{\mathcal{P}}_n(x)\}_{n=0}^{\infty}$ is the same (upto a constant multiple) as that for the OPS

$\{\mathcal{E}_n(x)\}_{n=0}^{\infty}$.

One of the aspects of Laguerre polynomials is that their asymptotic behavior in the complex plane has been illustrated for example, in Atia et al. [10], Dai and Wong [47], Dai et al. [46] and Deano et al [53]. It would be interesting to explore the consequences of complementary chain sequences in the context of asymptotics of such orthogonal polynomials.

3.4 Concluding remarks

In this chapter, we defined and studied consequences of the complementary chain sequences which we view as perturbations of the minimal parameter sequences. However, such a perturbation is one of many perturbations that one can have in the context of chain sequences. It would be interesting to see how these perturbations of the chain sequence affect the measure of orthogonality.

We would like to add that the motivation to study chain sequences was provided by perturbations of the g-fractions discussed in the previous chapter.We proceed further in this direction where, instead of considering the perturbation given in this chapter, we consider a polynomial map on the T-fraction in the next chapter. This provides important consequences leading to the study of different polynomial sequences in the subsequent chapters.

Chapter 4

Generalized Jacobi pencil matrix

In the previous two chapters, we studied two perturbations in the g-fractions, which served as the motivation to define complementary chain sequence. In this chapter, a general T-fraction denoted by $\mathcal{F}(\lambda)$ is studied under the transformation $\mathcal{F}(\lambda) \mapsto \mathcal{F}(\lambda^2)$. Two generalized linear matrix pencils are defined and the orthogonality properties of the associated Laurent polynomials are discussed.

4.1 The polynomial map $S(\lambda) \mapsto \lambda S(\lambda^2)$

The theory of polynomial mappings in the framework of orthogonal polynomials has received special attention in the recent past. Given two sequences $\{f_n(x)\}$ and $\{g_n(x)\}$, Carlitz [35] found conditions such that the sequence $\{h_n(x)\}$, defined as $h_{2m}(x) = f_m(x^2)$ and $h_{2m+1}(x) = xg_m(x^2)$, is also an orthogonal sequence. After this, several authors studied quadratic and cubic transformations of orthogonal polynomials, including Atia et al. [11] and Barrucand and Dickinson [14]. We also refer to Bessis and Moussa [19], de Jesus and Petronilho [51], Geronimo and Van Assche [80], Peherstorfer [140], and Petronilho [141] for various results in this direction.

Consider the Stieltjes function $S(\lambda)$ having the following asymptotic expansion $S(\lambda) = -\frac{s_0}{\lambda} - \frac{s_1}{\lambda^2} - \frac{s_2}{\lambda^3} \cdots - \frac{s_{2n}}{\lambda^{2n+1}} - \cdots$, $\lambda \in \mathbb{C}$, at infinity. The transformation

$$S(\lambda) \mapsto \lambda S(\lambda^2) \tag{4.1.1}$$

was interpreted in Derevyagin [61] in terms of a tridiagonal matrix which is a generalization of the Jacobi matrix associated with orthogonal polynomials on the real line. Precisely, it was shown (Derevyagin [61]) that the transformation (4.1.1) is equivalent to the Darboux transformation of such generalized Jacobi matrices, which proved to be an important aspect for further study in this direction. The spectral theory of these generalized Jacobi matrices is explored in Kovalyov [109] and Derevyagin and Derkach [60]. The generalized Stieltjes continued fractions associated with such generalized Jacobi matrices is studied in Derkach and Kovalyov [63].

The transformation (4.1.1) appears in the classical formula relating Hermite and Laguerre polynomials (Carlitz [35]) and is used in Dickinson and Warsi [65] to generate two different sequences of polynomials orthogonal on the real line. We recall that, given a sequence of polynomials $\{\mathrm{P}_n(\lambda)\}_{n=1}^{\infty}$, where $\mathrm{P}_n(\lambda)$ is orthogonal with respect to a measure $d\mu(t)$ on $(0,\infty)$, the kernel polynomials $\mathrm{K}_n(\lambda) := \mathrm{K}_n(0;\lambda)$ associated with $\mathrm{P}_n(\lambda)$ are orthogonal with respect to $td\mu(t)$ on $(0,\infty)$.

The symmetric polynomials $\mathcal{S}_n(\lambda)$, $n \geq 1$, defined by

$$\mathcal{S}_{2n}(\lambda) = \mathrm{P}_n(\lambda^2), \quad \mathcal{S}_{2n+1}(\lambda) = \lambda\mathrm{K}_n(\lambda^2), \quad n \geq 1, \tag{4.1.2}$$

are orthogonal with respect to the measure $d\zeta(t) = \frac{\mathrm{sgn}\,(t)}{2}d\mu(t^2)$ on the real line. Here, sgn is the signum function defined as

$$\mathrm{sgn}(x) := \begin{cases} -1, & x < 0; \\ 0, & x = 0; \\ 1, & x > 0. \end{cases}$$

The monic polynomials $\mathcal{S}_n(\lambda)$, $n \geq 0$, are also known to satisfy the recurrence relation (Chihara [42, Theorem 4.3, p.21])

$$\mathcal{S}_n(\lambda) = \lambda\mathcal{S}_{n-1}(\lambda) - \gamma_n\mathcal{S}_{n-2}(\lambda), \quad n \geq 1,$$

with $\mathcal{S}_{-1}(\lambda) = 0$ and $\mathcal{S}_0(\lambda) = 1$. Here $\gamma_n \neq 0$, $n \geq 2$ and $\gamma_1 = \int_{-\infty}^{\infty} \frac{\mathrm{sgn}\,(t)}{2}d\mu(t^2)$. Following Derevyagin [61], we call (4.1.2) the Chihara construction, whose details can be found in Chihara [42, Chapter I, Section 9].

Remark 4.1.1. *From computational point of view, it is desirable to associate matrix representations, in particular, matrix decompositions to polynomial maps. In this respect, one direction of study could be the polynomials maps arising from matrix splitting, which are special matrix decompositions used in numerical linear algebra. Such decompositions are studied in Baliarsingh and Mishra* [12]*, Jena et al.* [96] *and Kurmayya and Sivakumar* [115].

4.1.1 Extending the map to T-fractions

It can be seen that using the map (4.1.1), one can construct a sequence of polynomials orthogonal on $(-\infty, \infty)$ from a sequence of polynomials orthogonal on $(0, \infty)$. Hence, the transformation (4.1.1) is also called the unwrapping transformation in Derevyagin [61], the unwrapping happening here being that of the measure $d\mu(t)$. In terms of moment problems, the transformation (4.1.1) is used in Wall [184, Section 87] to reduce a Stieltjes moment problem to a Hamburger moment problem

The objective of this chapter is to study the transformation (4.1.1) in the context of T-fractions

$$\mathcal{F}(\lambda) = \frac{f_1\lambda}{1+g_1\lambda}\,_{+}\,\frac{f_2\lambda}{1+g_2\lambda}\,_{+}\,\frac{f_3\lambda}{1+g_3\lambda}\,_{+}\,\cdots, \quad \lambda \in \mathbb{C}, \tag{4.1.3}$$

introduced in the study of strong Stieltjes moment problems by Jones et al. [103]. However, we restrict ourselves to the transformation $\mathcal{F}(\lambda) \mapsto \mathcal{F}(\lambda^2)$, which in the process leads to two generalized linear tridiagonal matrix pencils.

Further, for computation purposes, we consider the following T-fraction

$$\mathcal{F}(\lambda) = \frac{\alpha_1\lambda}{\lambda-\beta_1}\,_{-}\,\frac{\alpha_2\lambda}{\lambda-\beta_2}\,_{-}\,\frac{\alpha_3\lambda}{\lambda-\beta_3}\,_{-}\,\cdots, \quad \lambda \in \mathbb{C}, \tag{4.1.4}$$

which is equivalent to (4.1.3). Here $\alpha_1 = -f_1/g_1$, $\alpha_{n+1} = -f_{n+1}/g_n g_{n+1}$ and $\beta_n = -1/g_n$, $n \geq 1$. Equivalent continued fractions have the same sequence of approximants and hence converge to the same analytic function. Details about the equivalent transformations of continued fractions can be found in Jones and Thron [102, Section 2.3] and Wall [184, Section 3]. The numerators $\mathrm{A}_n(\lambda)$ and denominators $\mathrm{B}_n(\lambda)$ of the approximants to (4.1.4) satisfy the recurrence relation

$$\mathrm{W}_{n+1}(\lambda) = (\lambda - \beta_{n+1})\mathrm{W}_n(\lambda) - \alpha_{n+1}\lambda \mathrm{W}_{n-1}(\lambda), \quad n \geq 1, \tag{4.1.5}$$

with the initial values $\mathrm{A}_0(\lambda) = 0$, $\mathrm{A}_1(\lambda) = -\alpha_1\lambda$, $\mathrm{B}_0(\lambda) = 1$ and $\mathrm{B}_1(\lambda) = \lambda - \beta_1$. The determinant formula for (4.1.4) is given by

$$\mathrm{A}_{n+1}(\lambda)\mathrm{B}_n(\lambda) - \mathrm{A}_n(\lambda)\mathrm{B}_{n+1}(\lambda) = (-1)^{2n+1}\prod_{j=1}^{n+1}\alpha_j\lambda^{n+1}. \tag{4.1.6}$$

The rest of the chapter is devoted to the study of the transformation $\mathcal{F}(\lambda) \mapsto \mathcal{F}(\lambda^2)$ using the recurrence relation (4.1.5). Laurent polynomials play a fundamental role, leading to generalized tridiagonal matrix pencils and biorthogonality relations.

4.1.2 Associated Laurent polynomials

First, we recall some basic facts about the general T-fraction (4.1.3) and the recurrence relation (4.1.5). We state the following theorem, proved in Jones et al. [103, Theorem 2.1], illustrating the correspondence properties of the T-fraction (4.1.3).

Theorem 4.1.1. [103, Theorem 2.1] *Let the formal Laurent series*

$$L = \sum_{n=1}^{\infty} a_n\lambda^n \quad \text{and} \quad L^* = \sum_{n=0}^{\infty} a^*_{-n}\lambda^{-n} \tag{4.1.7}$$

be given. There exists a general T-fraction which with $f_n \neq 0$ and $g_n \neq 0$ for all $n \geq 1$ corresponds to L at $z = 0$ and to L^* at $z = \infty$ if, and only if, $\Delta_n \neq 0$ and $\Phi_n \neq 0$, $n = 0, 1, \cdots$. Here $\Delta_0 = \Phi_0 = 1$ and for $n \geq 1$

$$\Delta_n = \begin{vmatrix} \delta_{-n+1} & \delta_{-n+2} & \cdots & \delta_0 \\ \delta_{-n+2} & \delta_{-n+3} & \cdots & \delta_1 \\ \vdots & \vdots & \ddots & \vdots \\ \delta_0 & \delta_1 & \cdots & \delta_{n-1} \end{vmatrix}; \quad \Phi_{n+1} = \begin{vmatrix} \delta_{-n+1} & \delta_{-n+2} & \cdots & \delta_1 \\ \delta_{-n+2} & \delta_{-n+3} & \cdots & \delta_2 \\ \vdots & \vdots & \ddots & \vdots \\ \delta_1 & \delta_2 & \cdots & \delta_{n+1} \end{vmatrix}.$$

The δ_k are defined by $\delta_k = a^*_k - a_k$ where it is understood that $a_k = 0$ for $k \leq 0$ and $a^*_k = 0$ for $k > 1$. The order of correspondence is $\{n+1\}$ at $\lambda = 0$ and $\{n\}$ at $\lambda = \infty$.

The f_n and g_n are given by

$$f_1 = -\Phi_1; \quad f_{n+1} = \frac{-\Delta_{n-1}\Phi_{n+1}}{\Delta_n \Phi_n}; \quad g_n = \frac{-\Delta_{n-1}\Phi_n}{\Delta_n \Phi_{n-1}}, \quad n \geq 1.$$

We suppose that $\Delta_n \neq 0$ and $\Phi_{n+1} \neq 0$, $n \geq 1$, so that the general T-fraction (4.1.3) exists with $f_n \neq 0$ and $g_n \neq 0$, $n \geq 1$. It can be proved from (4.1.4) and (4.1.5) that $\mathrm{B}_n(0) = (-1)^n \beta_1 \beta_2 \cdots \beta_n$. By use of the determinant formula (4.1.6), we have

$$\frac{\mathrm{A}_{n+1}(\lambda)}{\mathrm{B}_{n+1}(\lambda)} - \frac{\mathrm{A}_n(\lambda)}{\mathrm{B}_n(\lambda)} = \frac{(-1)^{2n+1} \prod_{j=1}^{n+1} \alpha_j \lambda^{n+1}}{\mathrm{B}_n(\lambda)\mathrm{B}_{n+1}(\lambda)},$$

which yields the following, for correspondence at $\lambda = 0$,

$$\frac{\mathrm{A}_{n+1}(\lambda)}{\mathrm{B}_{n+1}(\lambda)} - \frac{\mathrm{A}_n(\lambda)}{\mathrm{B}_n(\lambda)} = \frac{1}{\beta_1 \cdots \beta_n} \prod_{j=1}^{n+1} \frac{\alpha_j}{\beta_j} \lambda^{n+1} + g_{n+2}\lambda^{n+2} + \cdots . \tag{4.1.8}$$

Similarly, for correspondence at $\lambda = \infty$, we have

$$\frac{\mathrm{A}_{n+1}(\lambda)}{\mathrm{B}_{n+1}(\lambda)} - \frac{\mathrm{A}_n(\lambda)}{\mathrm{B}_n(\lambda)} = (-1)^{2n+1} \prod_{j=1}^{n+1} \alpha_j \lambda^{-n} + h_{-n-1}\lambda^{-n-1} + \cdots . \tag{4.1.9}$$

It follows that the order of correspondence of the general T-fraction (4.1.3) to L and L^* defined in (4.1.7) are $n+1$ and n respectively.

Lemma 4.1.1. *The polynomials $\mathcal{V}_n(\lambda)$, $n \geq 1$, satisfying the recurrence relation (4.1.5) have the determinant expression*

$$\mathcal{V}_n(\lambda) = \frac{\chi_{n+1}}{\Delta_{n+1}} \begin{vmatrix} \lambda^n & \lambda^{n-1} & \lambda^{n-2} & \cdots & \lambda^2 & \lambda & 1 \\ a^*_{-n+1} & a^*_{-n+2} & a^*_{-n+3} & \cdots & a^*_{-1} & a^*_0 & -a_1 \\ a^*_{-n+2} & a^*_{-n+3} & a^*_{-n+4} & \cdots & a^*_0 & -a_1 & -a_2 \\ a^*_{-n+3} & a^*_{-n+4} & a^*_{-n+5} & \cdots & -a_1 & -a_2 & -a_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ a^*_{-1} & a^*_0 & -a_1 & \cdots & -a_{n-3} & -a_{n-2} & -a_{n-1} \\ a^*_0 & -a_1 & -a_2 & \cdots & -a_{n-2} & -a_{n-1} & -a_n \end{vmatrix}, \tag{4.1.10}$$

where a_j and a^*_j are the coefficients of the formal power series L and L^* defined in

(4.1.7) *and* $\chi_{n+1} = (-1)^{2n+1} \prod_{j=1}^{n+1} \alpha_j$.

Proof. We provide an outline of the proof. Let the polynomials $\mathcal{U}_n(\lambda)$ and $\mathcal{V}_n(\lambda)$, $n \geq 1$, satisfying the recurrence relation (4.1.5) be written as

$$\mathcal{U}_n(\lambda) = u_{n1}\lambda + u_{n2}\lambda^2 + \cdots + u_{nn}\lambda^n,$$
$$\mathcal{V}_n(\lambda) = v_{n0} + v_{n1}\lambda + v_{n2}\lambda^2 + \cdots + v_{nn}\lambda^n, \quad n \geq 1.$$

From the correspondence relations (4.1.8) and (4.1.9), it can be shown that

$$L\mathcal{V}_n(\lambda) = u_{n1}\lambda + u_{n2}\lambda^2 + \cdots + u_{nn}\lambda^n + (-1)^n \prod_{j=1}^{n+1} \frac{\alpha_j}{\beta_j}\lambda^{n+1} + \mu_{n+2}\lambda^{n+2} + \cdots \quad \text{and}$$
$$L^*\mathcal{V}_n(\lambda) = u_{n1}\lambda^{-n+1} + u_{n2}\lambda^{-n+2} + \cdots + u_{nn} + (-1)^{2n+1} \prod_{j=1}^{n+1} \alpha_j\lambda^{-n} + \mu_{-n-1}\lambda^{-n-1} + \cdots$$

where L and L^* are the formal Laurent series defined in (4.1.7). Equating coefficients of λ in the above two power series expressions, we obtain two systems of equations respectively. Subtracting corresponding equations to eliminate u_{n1}, u_{n2},$\cdots$,u_{nn}, and choosing $\delta_k = a_k^* - a_k$, we obtain the system of equations

$$\begin{aligned}
&\delta_{-n}v_{nn} + \delta_{-n+1}v_{n,n-1} + \delta_{-n+2}v_{n,n-2} + \cdots + \delta_{-1}v_{n1} + \delta_0 v_{n0} = (-1)^{2n+1}\prod_{j=1}^{n+1}\alpha_j,\\
&\delta_{-n+1}v_{nn} + \delta_{-n+2}v_{n,n-1} + \delta_{-n+3}v_{n,n-2} + \cdots + \delta_0 v_{n1} + \delta_1 v_{n0} = 0,\\
&\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots\\
&\delta_{-1}v_{nn} + \delta_0 v_{n,n-1} + \delta_1 v_{n,n-2} + \cdots + \delta_{n-2}v_{n1} + \delta_{n-1}v_{n0} = 0,\\
&\delta_0 v_{nn} + \delta_1 v_{n,n-1} + \delta_2 v_{n,n-2} + \cdots + \delta_{n-1}v_{n1} + \delta_n v_{n0} = 0.
\end{aligned} \tag{4.1.11}$$

The determinant of the system of equations (4.1.11) is $\Delta_{n+1} \neq 0$. Hence an application of Cramer's rule along with the facts that $a_k = 0$ for $k \leq 0$ and $a_k^* = 0$ for $k > 1$ yield the required determinant (4.1.10). $\square$

For further discussion, we study the T-fraction (4.1.4), which under the transformation $\mathcal{F}(\lambda) \mapsto \mathcal{F}(\lambda^2) =: \mathfrak{F}(\lambda)$ is given by

$$\mathfrak{F}(\lambda) = \frac{\alpha_1\lambda^2}{\lambda^2-\beta_1}\,\underset{-}{}\,\frac{\alpha_2\lambda^2}{\lambda^2-\beta_2}\,\underset{-}{}\,\frac{\alpha_3\lambda^2}{\lambda^2-\beta_3}\,\underset{-}{}\,\frac{\alpha_4\lambda^2}{\lambda^2-\beta_4}\,\underset{-}{}\,\cdots, \quad \lambda \in \mathbb{C}. \tag{4.1.12}$$

As mentioned earlier, our objective is to associate two generalized linear matrix pencils with (4.1.12) and to discuss the orthogonality of the related Laurent polynomials. For this, we note from (4.1.7), that the asymptotic series to which $\mathfrak{F}(\lambda)$ corresponds are

$$\mathfrak{L}(\lambda) := \mathcal{L}(\lambda^2) = \sum_{n=1}^{\infty} a_n\lambda^{2n} \quad \text{and} \quad \mathfrak{L}^*(\lambda) := \mathcal{L}^*(\lambda^2) = \sum_{n=0}^{\infty} a^*_{-n}\lambda^{-2n},$$

where the correspondence is at $\lambda = 0$ and $\lambda = \infty$ respectively. However, it is constructive to deal with

$$\mathfrak{L}(\lambda) = \sum_{n=1}^{\infty} \mathfrak{a}_n\lambda^{n} \quad \text{and} \quad \mathfrak{L}^*(\lambda) = \sum_{n=0}^{\infty} \mathfrak{a}^*_{-n}\lambda^{-n} \tag{4.1.13}$$

where $\mathfrak{a}_{2n+1} := 0$, $\mathfrak{a}_{2n} := a_n$, $n \geq 1$ and $\mathfrak{a}^*_{-2n+1} := 0$, $\mathfrak{a}^*_{-2n} := a^*_{-n}$, $n \geq 0$. Further, the denominators $\mathfrak{P}_n(\lambda)$, $n \geq 1$, of the partial approximants of the continued fraction (4.1.12) satisfy the recurrence relation

$$\mathfrak{P}_{n+1}(\lambda) = (\lambda^2 - \beta_{n+1})\mathfrak{P}_n(\lambda) - \alpha_{n+1}\lambda^2\mathfrak{P}_{n-1}(\lambda), \quad n \geq 1, \tag{4.1.14}$$

with the initial conditions $\mathfrak{P}_0(\lambda) = 1$ and $\mathfrak{P}_1(\lambda) = \lambda^2 - \beta_1$. It can be easily seen that $\deg \mathfrak{P}_n(\lambda) = 2n$ and $\mathfrak{P}_n(\lambda) = \mathcal{V}_n(\lambda^2)$, $n \geq 1$, where $\mathcal{V}_n(\lambda)$ satisfies (4.1.5).

Introducing the Laurent polynomials

$$\sigma_0^R(\lambda) := \mathfrak{P}_0(\lambda) = 1, \quad \sigma_n^R(\lambda) := \lambda^{-2n}\mathfrak{P}_n(\lambda), \quad n \geq 1,$$

we obtain the recurrence relation for $\sigma_n^R(\lambda)$, $n \geq 1$, from (4.1.14) as

$$\lambda^2\sigma_{n+1}^R(\lambda) = (\lambda^2 - \beta_{n+1})\sigma_n^R(\lambda) - \alpha_{n+1}\sigma_{n-1}^R(\lambda), \quad n \geq 1, \tag{4.1.15}$$

with $\sigma_0^R(\lambda) = 1$ and $\sigma_1^R(\lambda) = \lambda^{-2}(\lambda^2 - \beta_1)$.

4.2 Biorthogonality from right eigenvectors

The final step before obtaining an eigenvalue representation is to linearize the system (4.1.15) in λ, which we do by introducing another sequence of Laurent polynomials as $\mathfrak{T}^R_{2n}(\lambda) := \sigma^R_n(\lambda) = \lambda^{-2n}\mathfrak{P}_n(\lambda)$, $\mathfrak{T}^R_{2n+1}(\lambda) := \lambda\sigma^R_n(\lambda) = \lambda^{-2n+1}\mathfrak{P}_n(\lambda)$, $n \geq 0$. This gives $\lambda\mathfrak{T}^R_0(\lambda) = \mathfrak{T}^R_1(\lambda)$, $\lambda[-\mathfrak{T}^R_1(\lambda) + \mathfrak{T}^R_3(\lambda)] = -\beta_1\mathfrak{T}^R_0(\lambda)$ and

$$\begin{aligned} \lambda\mathfrak{T}^R_{2n}(\lambda) &= \mathfrak{T}^R_{2n+1}(\lambda), \\ \lambda[-\mathfrak{T}^R_{2n+1}(\lambda) + \mathfrak{T}^R_{2n+3}(\lambda)] &= -\beta_{n+1}\mathfrak{T}^R_{2n}(\lambda) - \alpha_{n+1}\mathfrak{T}^R_{2n-2}(\lambda), \quad n \geq 1. \end{aligned}$$

This yields the generalized eigenvalue problem $\lambda\mathcal{G}^R\boldsymbol{\mathfrak{T}}^R = \mathcal{H}^R\boldsymbol{\mathfrak{T}}^R$ where

$$\mathcal{G}^R = \left(\begin{array}{cc|cc|cc|c} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & -1 & 0 & 1 & 0 & 0 & \cdots \\ \hline 0 & 0 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & -1 & 0 & 1 & \cdots \\ \hline 0 & 0 & 0 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & -1 & \cdots \\ \hline \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right), \quad \mathcal{H}^R = \left(\begin{array}{cc|cc|cc|c} 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\ -\beta_1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \hline 0 & 0 & 0 & 1 & 0 & 0 & \cdots \\ -\alpha_2 & 0 & -\beta_2 & 0 & 0 & 0 & \cdots \\ \hline 0 & 0 & 0 & 0 & 0 & 1 & \cdots \\ 0 & 0 & -\alpha_3 & 0 & -\beta_3 & 0 & \cdots \\ \hline \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right)$$

and hence we obtain the following generalized linear pencil matrix

$$\mathfrak{J}^R(\lambda) = \left(\begin{array}{cc|cc|cc|c} \lambda & -1 & 0 & 0 & 0 & 0 & \cdots \\ \beta_1 & -\lambda & 0 & \lambda & 0 & 0 & \cdots \\ \hline 0 & 0 & \lambda & -1 & 0 & 0 & \cdots \\ \alpha_2 & 0 & \beta_2 & -\lambda & 0 & \lambda & \cdots \\ \hline 0 & 0 & 0 & 0 & \lambda & -1 & \cdots \\ 0 & 0 & \alpha_3 & 0 & \beta_3 & -\lambda & \cdots \\ \hline \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right),$$

associated with the eigenvector

$$\boldsymbol{\mathfrak{T}}^R = \begin{pmatrix} \mathfrak{T}^R_0(\lambda) & \mathfrak{T}^R_1(\lambda) & \mathfrak{T}^R_2(\lambda) & \mathfrak{T}^R_3(\lambda) & \mathfrak{T}^R_4(\lambda) & \mathfrak{T}^R_5(\lambda) & \cdots \end{pmatrix}^T.$$

Theorem 4.2.1. *Consider the moment functional*

$$\mathfrak{M}(\lambda^n) = \begin{cases} -\mathfrak{a}_{-n}, & n = -1, -2, \cdots; \\ \mathfrak{a}^*_{-n}, & n = 0, 1, 2, \cdots. \end{cases} \tag{4.2.1}$$

where $\mathfrak{a}_j$ *and* $\mathfrak{a}^*_j$ *are the coefficients appearing in the formal Laurent series* (4.1.13)*. Then,* $\{\mathfrak{T}^R_j(\lambda)\}_{j=0}^{\infty}$ *satisfies the following orthogonality relations*

$$\mathfrak{M}(\mathfrak{T}^R_{2n}(\lambda)\ \zeta^R_j(\lambda)) = \frac{\chi_{n+1}\Delta_n}{\Delta_{n+1}}\delta_{2n,j}, \tag{4.2.2a}$$

$$\mathfrak{M}(\mathfrak{T}^R_{2n+1}(\lambda)\ \zeta^R_j(\lambda)) = \frac{\chi_{n+1}\Delta_n}{\Delta_{n+1}}\delta_{2n-1,j}, \quad n, j \geq 1, \tag{4.2.2b}$$

where $\{\zeta^R_k(\lambda)\}_{k=0}^{\infty}$ *is a sequence of functions defined as*

$$\zeta^R_{2n}(\lambda) = \mathfrak{P}_n(\lambda), \quad \zeta^R_{2n+1}(\lambda) = \lambda^{-1}\mathfrak{P}_{n+1}(\lambda), \quad n \geq 0.$$

Proof. Note that for $n \geq 1$, $\mathfrak{T}^R_{2n}(\lambda)$ contains only even powers of λ whereas $\zeta^R_{2n+1}(\lambda)$ contains only odd powers of λ. Since $\mathfrak{a}_{2n-1} = \mathfrak{a}^*_{-(2n-1)} = 0$, $n \geq 0$, it follows that

$$\mathfrak{M}(\mathfrak{T}^R_{2n}(\lambda)\ \zeta^R_{2n-1}(\lambda)) = 0, \quad n \geq 1.$$

Further, from the determinant expression (4.1.10) for $\mathcal{V}_n(\lambda)$, we get

$$\mathfrak{T}^R_{2n}(\lambda)\,\lambda^{2j} = \frac{\chi_{n+1}}{\Delta_{n+1}} \begin{vmatrix} \lambda^{2j} & \lambda^{2j-2} & \lambda^{2j-4} & \cdots & \lambda^{-2n+2j+4} & \lambda^{-2n+2j+2} & \lambda^{-2n+2j} \\ a^*_{-n+1} & a^*_{-n+2} & a^*_{-n+3} & \cdots & a^*_{-1} & a^*_0 & -a_1 \\ a^*_{-n+2} & a^*_{-n+3} & a^*_{-n+4} & \cdots & a^*_0 & -a_1 & -a_2 \\ a^*_{-n+3} & a^*_{-n+4} & a^*_{-n+5} & \cdots & -a_1 & -a_2 & -a_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ a^*_{-1} & a^*_0 & -a_1 & \cdots & -a_{n-3} & -a_{n-2} & -a_{n-1} \\ a^*_0 & -a_1 & -a_2 & \cdots & -a_{n-2} & -a_{n-1} & -a_n \end{vmatrix}. \tag{4.2.3}$$

Hence, using (4.2.1), it is clear that $\mathfrak{M}(\mathfrak{T}^R_{2n}(\lambda)\ \lambda^{2j}) = 0$ for $j = 0, 1, \cdots, n-1$, while for $j = n$, we have $\mathfrak{M}(\mathfrak{T}^R_{2n}(\lambda)\ \lambda^{2n}) = \frac{\chi_{n+1}\Delta_n}{\Delta_{n+1}} \neq 0$. This implies

$$\mathfrak{M}(\mathfrak{T}^R_{2n}(\lambda)\ \zeta^R_{2j}(\lambda)) = 0, \quad \mathfrak{M}(\mathfrak{T}^R_{2n}(\lambda)\ \zeta^R_{2n}(\lambda)) = \frac{\chi_{n+1}\Delta_n}{\Delta_{n+1}}, \quad n \geq 1,$$

which proves (4.2.2a). Similarly, for $n \geq 0$, $\mathfrak{T}^R_{2n+1}(\lambda)$ contains only odd powers of λ (including λ^{-1}) while $\zeta^R_{2n}(\lambda)$ contains only even powers of λ. Hence

$$\mathfrak{M}(\mathfrak{T}^R_{2n+1}(\lambda)\ \zeta^R_{2n}(\lambda)) = 0, \quad n \geq 0.$$

In this case, the determinant occurring in the right hand side of (4.2.3) is equal to $\mathfrak{T}^R_{2n+1}(\lambda)\,\lambda^{2j-1}$ and so $\mathfrak{M}(\mathfrak{T}^R_{2n+1}(\lambda)\ \lambda^{2j-1}) = 0$ for $j = 0, 1, \cdots, n-1$ while for $j = n$, we have $\mathfrak{M}(\mathfrak{T}^R_{2n+1}(\lambda)\ \lambda^{2n-1}) = \frac{\chi_{n+1}\Delta_n}{\Delta_{n+1}} \neq 0$. This implies

$$\mathfrak{M}(\mathfrak{T}^R_{2n+1}(\lambda)\ \zeta^R_{2j-1}(\lambda)) = 0, \quad \mathfrak{M}(\mathfrak{T}^R_{2n+1}(\lambda)\ \zeta^R_{2n-1}(\lambda)) = \frac{\chi_{n+1}\Delta_n}{\Delta_{n+1}}, \quad n \geq 1,$$

which proves (4.2.2b). □

4.3 Biorthogonality relations from left eigenvectors

Consider the normalized polynomials

$$\sigma^L_0(\lambda) := \mathfrak{P}_0(\lambda) = 1 \quad \text{and} \quad \sigma^L_n(\lambda) := \frac{\mathfrak{P}_n(\lambda)}{(\alpha_2\alpha_3\cdots\alpha_{n+1})}, \quad n \geq 1.$$

Then from (4.1.14), we obtain the recurrence relation

$$\alpha_{n+2}\sigma^L_{n+1}(\lambda) = (\lambda^2 - \beta_{n+1})\sigma^L_n(\lambda) - \lambda^2\sigma^L_{n-1}(\lambda), \quad n \geq 1, \tag{4.3.1}$$

with $\sigma^L_0(\lambda) = 1$ and $\sigma^L_1(\lambda) = \alpha_2^{-1}(\lambda^2 - \beta_1)$. To linearize (4.3.1) in λ, we introduce the sequence of polynomials $\{\mathfrak{T}^L_n(\lambda)\}_{n=0}^{\infty}$ where $\mathfrak{T}^L_0(\lambda) := \sigma^L_0(\lambda) = 1$, $\mathfrak{T}^L_{2n}(\lambda) := \sigma^L_n(\lambda)$, $\mathfrak{T}^L_{2n+1}(\lambda) := \lambda\sigma^L_n(\lambda)$, $n \geq 1$. Hence, we obtain the system $\lambda\mathfrak{T}^L_0(\lambda) = \mathfrak{T}^L_1(\lambda)$, $\lambda[-\mathfrak{T}^L_1(\lambda)] = -\beta_1\mathfrak{T}^L_0(\lambda) - \alpha_2\mathfrak{T}^L_2(\lambda)$ and

$$\lambda\mathfrak{T}^L_{2n}(\lambda) = \mathfrak{T}^L_{2n+1}(\lambda),$$

$$\lambda[\mathfrak{T}^L_{2n-1}(\lambda) - \mathfrak{T}^L_{2n+1}(\lambda)] = -\beta_{n+1}\mathfrak{T}^L_{2n}(\lambda) - \alpha_{n+2}\mathfrak{T}^L_{2n+2}(\lambda), \quad n \geq 1.$$

This gives the following generalized eigenvalue problem $\lambda\boldsymbol{\mathfrak{T}}^L\mathcal{G}^L = \boldsymbol{\mathfrak{T}}^L\mathcal{H}^L$ where

$$\mathcal{G}^L = \left(\begin{array}{cc|cc|cc|c} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & -1 & 0 & 1 & 0 & 0 & \cdots \\ \hline 0 & 0 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & -1 & 0 & 1 & \cdots \\ \hline 0 & 0 & 0 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & -1 & \cdots \\ \hline \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right), \quad \mathcal{H}^L = \left(\begin{array}{cc|cc|cc|c} 0 & -\beta_1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \hline 0 & -\alpha_2 & 0 & -\beta_2 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & 0 & 0 & \cdots \\ \hline 0 & 0 & 0 & -\alpha_3 & 0 & -\beta_3 & \cdots \\ 0 & 0 & 0 & 0 & 1 & 0 & \cdots \\ \hline \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right)$$

and

$$\boldsymbol{\mathfrak{T}}^L = \begin{pmatrix} \mathfrak{T}^L_0(\lambda) & \mathfrak{T}^L_1(\lambda) & \mathfrak{T}^L_2(\lambda) & \mathfrak{T}^L_3(\lambda) & \mathfrak{T}^L_4(\lambda) & \mathfrak{T}^L_5(\lambda) & \cdots \end{pmatrix}.$$

Thus, we obtain the following generalized linear pencil matrix

$$\mathcal{J}^L(\lambda) = \left(\begin{array}{cc|cc|cc|c} \lambda & \beta_1 & 0 & 0 & 0 & 0 & \cdots \\ -1 & -\lambda & 0 & \lambda & 0 & 0 & \cdots \\ \hline 0 & \alpha_2 & \lambda & \beta_2 & 0 & 0 & \cdots \\ 0 & 0 & -1 & -\lambda & 0 & \lambda & \cdots \\ \hline 0 & 0 & 0 & \alpha_3 & \lambda & \beta_3 & \cdots \\ 0 & 0 & 0 & 0 & -1 & -\lambda & \cdots \\ \hline \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right)$$

Note that the corresponding blocks in $\mathfrak{J}^R(\lambda)$ and $\mathfrak{J}^L(\lambda)$ are transpose of each other and lead to the respective generalized eigenvalue problems

$$\mathfrak{T}^L\mathcal{J}^L(\lambda) = 0 \quad \text{and} \quad \mathcal{J}^R(\lambda)\mathfrak{T}^R = 0.$$

We state the following result about the orthogonality of $\mathfrak{T}^L_j(\lambda)$, $j \geq 1$. The proof is similar to that of Theorem 4.2.1 and hence omitted.

Theorem 4.3.1. *Let the moment functions* $\mathfrak{M}$ *be defined as in* (4.2.1). *Then the sequence of polynomials* $\{\mathfrak{T}_j^L(\lambda)\}_{j=1}^{\infty}$ *satisfies the orthogonality property*

$$\mathfrak{M}(\mathfrak{T}_{2n}^L(\lambda)\ \zeta_j^L(\lambda)) = \frac{\chi_{n+1}\Delta_n}{\Delta_{n+1}}\delta_{2n,j},$$
$$\mathfrak{M}(\mathfrak{T}_{2n+1}^L(\lambda)\ \zeta_j^L(\lambda)) = \frac{\chi_{n+1}\Delta_n}{\Delta_{n+1}}\delta_{2n+1,j}, \quad n, j \geq 1,$$

where $\{\zeta_k^L(\lambda)\}_{k=0}^{\infty}$ *is a sequence of Laurent polynomials defined as*

$$\zeta_{2n}^L(\lambda) := \lambda^{-2n}\mathfrak{P}_n(\lambda), \quad \zeta_{2n+1}^L(\lambda) := \lambda^{-2n-1}\mathfrak{P}_n(\lambda), \quad n \geq 0.$$

Remark 4.3.1. *In continuation of Remark 4.1.1, an operator theoretic interpretation of the matrix pencils and hence the polynomials maps considered in this chapter could be an interesting direction of study. We refer to Ganesh et al.* [75] *and Veeramani and Sukumar* [166] *for information in this direction.*

4.4 Towards R_I and R_{II} polynomials

As mentioned in Chapter 1, the polynomial sequences $\{\mathcal{P}_n(\lambda)\}_{n=0}^{\infty}$ and $\{\mathcal{Q}_n(\lambda)\}_{n=0}^{\infty}$ satisfying

$$\mathcal{P}_{n+1}(\lambda) = \rho_n(\lambda - \beta_n)\mathcal{P}_n(\lambda) + \tau_n(\lambda - \gamma_n)\mathcal{P}_{n-1}(\lambda), \quad n \geq 1 \tag{4.4.1a}$$
$$\mathcal{Q}_{n+1}(\lambda) = \rho_n(\lambda - \nu_n)\mathcal{Q}_n(z) + \tau_n(\lambda - a_n)(\lambda - b_n)\mathcal{Q}_{n-1}(\lambda), \quad n \geq 1, \tag{4.4.1b}$$

with appropriately defined initial values are called R_I and R_{II} polynomials respectively. We note that the recurrence relations satisfied by the para-orthogonal polynomials $\{R_n(z)\}$ studied in Chapter 3 as well as (4.1.5) satisfied by the denominators of a general T-fraction are particular cases of the recurrence relation (4.4.1a) of R_I type.

The R_{II} polynomials are studied extensively in the context of biorthogonal rational functions. Some references in this direction are Bultheel et al. [32] and Zhedanov [192]. Precisely, $\mathcal{Q}_n(\lambda)$ satisfying (4.4.1b) appear as the numerator polynomials of the orthogonal rational functions having poles at $\{a_n\}$ and $\{b_n\}$. In case, $a_n = z_n$ and $b_n = \bar{z}_n$, (4.4.1b) is the recurrence relation satisfied by the denominators of the partial approximants of the continued fraction expansions of Nevanlinna functions (Derevyagin [62]). The particular case of $a_n = i$ and $b_n = -i$ was studied in Ismail and Ranga [94] in the context of orthogonality on the unit circle.

It is also proved in Zhedanov [192] that the polynomials $\{\mathcal{P}_n(\lambda)\}$ and $\{\mathcal{Q}_n(\lambda)\}$ arise as characteristic polynomials of the linear matrix pencil

$$\mathcal{G} - \lambda\mathcal{H} := \begin{pmatrix} \zeta_1(\lambda) & -\sigma_1^R(\lambda) & 0 & 0 & \cdots \\ -\sigma_1^L(\lambda) & \zeta_2(\lambda) & -\sigma_2^R(\lambda) & 0 & \cdots \\ 0 & -\sigma_2^L(\lambda) & \zeta_3(\lambda) & -\sigma_3^R(\lambda) & \cdots \\ 0 & 0 & -\sigma_3^L(\lambda) & \zeta_4(\lambda) & \cdots \\ \cdots & \cdots & \cdots & \cdots & \ddots \end{pmatrix}, \qquad (4.4.2)$$

where $\mathcal{G}$, $\mathcal{H}$ are tridiagonal matrices, $\zeta_j(\lambda)$ is a polynomial of degree one while $\sigma_j^L(\lambda)$ and $\sigma_1^R(\lambda)$ are polynomials of degree at most one.

It may be observed that the generalized Jacobi pencil matrices obtained in the present chapter are comparable to (4.4.2). This serves as our motivation to study the R_I and R_{II} polynomials in the context of biorthogonality, generalized eigenvalue problems and orthogonal rational functions. This forms the content of the next two chapters of the thesis.

Chapter 5

Orthogonality of linear combinations of R_I polynomials

The recurrence relations involving linear combinations of orthogonal polynomials on the real line (quasi-orthogonal) are obtained in Draux [69]. Such recurrence relations have polynomial coefficients that are either linear or quadratic. In some cases, as in Jordaan and Toókos et al. [104], the recurrence relations are of mixed type depending on the parameters involved. In this chapter, we consider one such linear combination of R_I polynomials that satisfy orthogonality properties. In addition to the mixed recurrence relations, such linear combinations also satisfy biorthogonality relations that can be derived from their eigenvalue representations. With certain additional conditions, we further show that such linear combinations lead to para-orthogonal polynomials.

5.1 Constructing a sequence with a common zero

We study the linear combination of two successive R_I polynomials of a sequence $\{\mathcal{P}_n(\lambda)\}_{n=0}^{\infty}$, where $\mathcal{P}_n(\lambda)$, $n \geq 0$, satisfies the recurrence relation

$$\mathcal{P}_{n+1}(\lambda) = \rho_n(\lambda - \beta_n)\mathcal{P}_n(\lambda) + \tau_n(\lambda - \gamma_n)\mathcal{P}_{n-1}(\lambda), \quad n \geq 1, \tag{5.1.1}$$

of R_I type with $\mathcal{P}_0(\lambda) := 1$ and $\mathcal{P}_1(\lambda) := \rho_0(\lambda - \beta_0)$. That is, we consider the sequence $\{\mathcal{Q}_n(\lambda)\}_{n=0}^{\infty}$ where

$$\mathcal{Q}_n(\lambda) := \mathcal{P}_n(\lambda) + \alpha_n \mathcal{P}_{n-1}(\lambda), \quad \alpha_n \in \mathbb{R} \setminus \{0\}, \quad n \geq 0.$$

We assume that $\beta_n \neq 0$, $n \geq 0$, so that $\mathcal{P}_n(0) \neq 0$, $n \geq 1$, in case $\gamma_n = 0$, $n \geq 1$. We will also use the condition $\beta_0 \neq \pm 1$, so that $\mathcal{Q}_2(\lambda)$ does not have a repeated root at $\lambda = 1$. The later condition will become clearer in section 5.3. We construct a unique sequence $\{\alpha_n\}_{n=0}^{\infty}$ such that $\{\mathcal{Q}_n(\lambda)\}_{n=0}^{\infty}$ not only satisfies mixed recurrence relations of R_I and R_{II} type but also all terms of the sequence have a common zero. We recall that the following recurrence relation

$$\hat{\mathcal{P}}_{n+1}(\lambda) = \hat{\rho}_n(\lambda - \hat{\beta}_n)\hat{\mathcal{P}}_n(\lambda) + \hat{\tau}_n(\lambda - \hat{\gamma}_n^{(1)})(\lambda - \hat{\gamma}_n^{(2)})\hat{\mathcal{P}}_{n-1}(\lambda), \quad n \geq 1,$$

with $\hat{\mathcal{P}}_0(\lambda) = 1$ and $\hat{\mathcal{P}}_1(\lambda) = \hat{\rho}_0(\lambda - \hat{\beta}_0)$ is called a recurrence relation of R_{II} type in Ismail and Masson [91] if $\hat{\rho}_n \neq 0$ and $\hat{\tau}_n \neq 0$ for $n \geq 0$.

The first result shows that $\mathcal{Q}_n(\lambda)$, $n \geq 1$, satisfies a three term recurrence relation with polynomial coefficients of degree at most two.

Theorem 5.1.1. *Given a sequence $\{\mathcal{P}_n(\lambda)\}_{n=1}^{\infty}$ of polynomials satisfying R_I type recurrence relations (5.1.1), consider the linear combinations of two successive such polynomials*

$$\mathcal{Q}_n(\lambda) = \mathcal{P}_n(\lambda) + \alpha_n \mathcal{P}_{n-1}(\lambda), \quad \alpha_n \in \mathbb{R} \setminus \{0\}, \quad n \geq 0, \tag{5.1.2}$$

where $\mathcal{Q}_0(\lambda) := \mathcal{P}_0(\lambda) = 1$. Then there exist constants $\{p_n, q_n, r_n, s_n, t_n, u_n, v_n, w_n\}$ such that $\{\mathcal{Q}_n(\lambda)\}_{n=1}^{\infty}$ satisfies a three term recurrence relation of the form

$$\begin{aligned}(p_n\lambda + q_n)&\mathcal{Q}_{n+1}(\lambda)\\ &= (r_n\lambda^2 + s_n\lambda + t_n)\mathcal{Q}_n(\lambda) + (u_n\lambda^2 + v_n\lambda + w_n)\mathcal{Q}_{n-1}(\lambda), \quad n \geq 1,\end{aligned} \tag{5.1.3}$$

with $\mathcal{Q}_0(\lambda) = 1$ and $\mathcal{Q}_1(\lambda) = \rho_0(\lambda + \alpha_1\rho_0^{-1} - b_0)$.

Proof. For $n \geq 1$, consider the following system

$$\begin{aligned} &\mathcal{Q}_k(\lambda) = \mathcal{P}_k(\lambda) + \alpha_k \mathcal{P}_{k-1}(\lambda), \quad k = n-1, n, n+1, \\ &\mathcal{P}_k(\lambda) = \rho_{k-1}(\lambda - \beta_{k-1})\mathcal{P}_{k-1}(\lambda) + \tau_{k-1}(\lambda - \gamma_{k-1})\mathcal{P}_{k-2}(\lambda), \quad k = n, n+1, \end{aligned} \tag{5.1.4}$$

written in the matrix form as

$$[\mathcal{C}_{n-1}] \begin{pmatrix} \mathcal{Q}_{n-1}(\lambda) \\ \mathcal{P}_{n-2}(\lambda) \\ \mathcal{P}_{n-1}(\lambda) \\ \mathcal{P}_n(\lambda) \\ \mathcal{P}_{n+1}(\lambda) \end{pmatrix} = \begin{pmatrix} \mathcal{Q}_{n+1}(\lambda) \\ \mathcal{Q}_n(\lambda) \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

where

$$[\mathcal{C}_{n-1}] = \begin{pmatrix} 0 & 0 & 0 & \alpha_{n+1} & 1 \\ 0 & 0 & \alpha_n & 1 & 0 \\ -1 & \alpha_{n-1} & 1 & 0 & 0 \\ 0 & 0 & \tau_n(\lambda - \gamma_n) & \rho_n(\lambda - \beta_n) & -1 \\ 0 & \tau_{n-1}(\lambda - \gamma_{n-1}) & \rho_{n-1}(\lambda - \beta_{n-1}) & -1 & 0 \end{pmatrix}$$

is the coefficient matrix. Using Cramer's rule, the first unknown variable $\mathcal{Q}_{n-1}(\lambda)$ is given by

$$\det[\mathcal{C}_{n-1}]\mathcal{Q}_{n-1}(\lambda) = \det[\mathcal{A}_{n+1}]\mathcal{Q}_{n+1}(\lambda) - \det[\mathcal{B}_n]\mathcal{Q}_n(\lambda),$$

where

$$[\mathcal{A}_{n+1}] = \begin{pmatrix} 0 & \alpha_n & 1 & 0 \\ \alpha_{n-1} & 1 & 0 & 0 \\ 0 & \tau_n(\lambda - \gamma_n) & \rho_n(\lambda - \beta_n) & -1 \\ \tau_{n-1}(\lambda - \gamma_{n-1}) & \rho_{n-1}(\lambda - \beta_{n-1}) & -1 & 0 \end{pmatrix},$$

$$[\mathcal{B}_n] = \begin{pmatrix} 0 & 0 & \alpha_{n+1} & 1 \\ \alpha_{n-1} & 1 & 0 & 0 \\ 0 & \tau_n(\lambda-\gamma_n) & \rho_n(\lambda-\beta_n) & -1 \\ \tau_{n-1}(\lambda-\gamma_{n-1}) & \rho_{n-1}(\lambda-\beta_{n-1}) & -1 & 0 \end{pmatrix}.$$

Expanding all the determinants by their respective last columns gives $\det[\mathcal{A}_{n+1}] = p_n\lambda + q_n$, $\det[\mathcal{B}_n] = r_n\lambda^2 + s_n\lambda + t_n$ and $\det[\mathcal{C}_{n-1}] = u_n\lambda^2 + v_n\lambda + w_n$, where, for $n \geq 1$,

$$\begin{aligned}
&p_n = \alpha_{n-1}\rho_{n-1} - \tau_{n-1},\ \ q_n = \alpha_{n-1}(\alpha_n - \rho_{n-1}\beta_{n-1}) + \tau_{n-1}\gamma_{n-1}, \\
&r_n = \rho_n p_n,\ \ s_n = \rho_n q_n + \alpha_n^{-1} p_n q_{n+1} - \alpha_{n-1} p_{n+1}, \\
&t_n = -\alpha_{n-1}\rho_{n-1}\beta_{n-1}(\alpha_{n+1} - \rho_n\beta_n) + \gamma_{n-1}(\alpha_{n-1}\alpha_{n+1} - \tau_{n-1}\beta_n\rho_n) - \alpha_{n-1}\tau_n\gamma_n, \\
&u_n = \tau_{n-1}p_{n+1},\ \ v_n = \tau_{n-1}(q_{n+1} - \gamma_{n-1}p_{n+1}),\ \ w_n = -\tau_{n-1}\gamma_{n-1}q_{n+1},
\end{aligned}$$

are the constants given explicitly in terms of the recurrence parameters used in (5.1.1) and α_n, $n \geq 0$. Further, the values of $\mathcal{Q}_k(\lambda)$, $k = 0, 1$ are obtained from (5.1.4) for $n = 0, 1$ and hence the recurrence relation (5.1.3) is well-defined. □

An immediate consequence is the following result that provides certain choices for α_n, $n \geq 0$, such that $\mathcal{Q}_n(\lambda)$, $n \geq 1$, again satisfies a recurrence relation of R_I type.

Corollary 5.1.1. *The polynomials $\mathcal{Q}_n(\lambda) = \mathcal{P}_n(\lambda) + \alpha_n\mathcal{P}_{n-1}(\lambda)$, $n \geq 0$, form a sequence of R_I polynomials if $\alpha_n = \rho_{n-1}\beta_{n-1}$ and $\gamma_n = 0$, $n \geq 0$ or $\alpha_n = \tau_n\rho_n^{-1}$, $n \geq 0$.*

Proof. Choosing $\alpha_n = \tau_n\rho_n^{-1}$, $n \geq 0$, makes p_n and hence r_n and u_n equal to zero for $n \geq 1$. Similarly, with $\alpha_n = \rho_{n-1}\beta_{n-1}$ and $\gamma_n = 0$ we have $t_n = w_n = 0$ for $n \geq 1$. Thus (5.1.3) reduces to the recurrence relations

$$\begin{aligned}
\mathcal{Q}_{n+1}(\lambda) &= q_n^{-1}(s_n\lambda + t_n)\mathcal{Q}_n(\lambda) + q_n^{-1}(v_n\lambda + w_n)\mathcal{Q}_{n-1}(\lambda), \quad n \geq 1, \\
\mathcal{Q}_{n+1}(\lambda) &= p_n^{-1}(r_n\lambda + s_n)\mathcal{Q}_n(\lambda) + p_n^{-1}(u_n\lambda + v_n)\mathcal{Q}_{n-1}(\lambda), \quad n \geq 1,
\end{aligned}$$

of R_I type respectively. □

5.1.1 Mixed recurrence relations of R_I and R_{II} type

It is clear that there are obvious ways to choose α_n, $n \geq 0$, if we require the linear combinations $\mathcal{Q}_n(\lambda)$, $n \geq 1$, to be R_I polynomials *ab initio*. We will verify this choice in Section 5.3 when we obtain a para-orthogonal polynomial from $\mathcal{Q}_n(\lambda)$. However, in the present section, we suppose $p_{n+1} \neq 0$, $q_n \neq 0$, $\gamma_n = \gamma \in \mathbb{C}$, $n \geq 1$ and consider the linear combinations

$$\mathcal{Q}_n(\lambda) = \mathcal{P}_n(\lambda+\gamma) + a_n\mathcal{P}_{n-1}(\lambda+\gamma), \quad n \geq 1,$$

where $\mathcal{P}_n(\lambda+\gamma)$ satisfies the recurrence relation (5.1.1). In such a case, we have

$$\mathcal{P}_{n+1}(\lambda+\gamma) = \rho_n(\lambda+\gamma-\beta_n)\mathcal{P}_n(\lambda+\gamma) + \tau_n\lambda\mathcal{P}_{n-1}(\lambda+\gamma), \quad n \geq 1,$$

which is nothing but the recurrence relation (5.1.1) with $\gamma_n = \gamma = 0$, $n \geq 1$. Then, $\mathcal{Q}_n(\lambda)$, $n \geq 1$, satisfies the recurrence relation (5.1.3) but with the much simplified constants

$$\begin{aligned} &p_n = \alpha_{n-1}\rho_{n-1} - \tau_{n-1}, q_n = \alpha_{n-1}(\alpha_n - \rho_{n-1}\beta_{n-1}), r_n = \rho_n p_n, \\ &s_n = \alpha_n^{-1}p_nq_{n+1} + \alpha_{n-1}(\tau_n - \rho_{n-1}\beta_{n-1}\rho_n), t_n = -\alpha_n^{-1}\alpha_{n-1}\rho_{n-1}\beta_{n-1}q_{n+1}, \\ &u_n = \tau_{n-1}p_{n+1}, v_n = \tau_{n-1}q_{n+1}, w_n = 0. \end{aligned} \tag{5.1.5}$$

which are obtained by substituting $\gamma_n = 0$ in Theorem 5.1.1. We use these simplified constants to convert the recurrence relation (5.1.3) into a form that is appropriate for further discussion.

Theorem 5.1.2. *Suppose the sequence $\{\alpha_n\}_{n=0}^{\infty}$ is constructed recursively as*

$$\alpha_n = -(\rho_{n-1} - \alpha_{n-1}^{-1}\tau_{n-1}) + \rho_{n-1}\beta_{n-1}, \quad n \geq 2, \tag{5.1.6}$$

where $\alpha_1 \neq \rho_0\beta_0$ is arbitrary. If $\alpha_0 := \tau_0\rho_0^{-1}$, then $\{\mathcal{Q}_n(\lambda)\}_{n=1}^{\infty}$ satisfies the mixed recurrence relations

$$\mathcal{Q}_2(\lambda) = \frac{s_1}{q_1}\left(\lambda + \frac{t_1}{s_1}\right)\mathcal{Q}_1(\lambda) - \frac{\tau_0 q_2}{q_1}\lambda(\lambda-1)\mathcal{Q}_0(\lambda) \qquad \text{and} \tag{5.1.7a}$$

$$\mathcal{Q}_{n+1}(\lambda) = \rho_n\left(\lambda - \frac{t_n}{r_n}\right)\mathcal{Q}_n(\lambda) + \frac{\tau_{n-1}q_{n+1}}{q_n}\lambda\mathcal{Q}_{n-1}(\lambda), \quad n \geq 2, \tag{5.1.7b}$$

with $\mathcal{Q}_0(\lambda) = 1$ *and* $\mathcal{Q}_1(\lambda) = \rho_0(\lambda + \alpha_1\rho_0^{-1} - \beta_0)$.

Proof. The recursive relation (5.1.6) can be rearranged as

$$\alpha_{n-1}\alpha_n = -\rho_{n-1}\alpha_{n-1} + \tau_{n-1} + \alpha_{n-1}\rho_{n-1}\beta_{n-1},$$

which implies that the relations (5.1.5) can be further simplified, for $n \geq 2$, as

$$q_n = -p_n, \quad r_n = \rho_n p_n, \quad s_n = -\alpha_n^{-1}p_np_{n+1} - \alpha_{n-1}p_{n+1} - \rho_np_n,$$
$$t_n = \alpha_n^{-1}\alpha_{n-1}\rho_{n-1}\beta_{n-1}p_{n+1}, \quad u_n = -v_n = \tau_{n-1}p_{n-1}, \quad w_n = 0.$$

Then, for $n \geq 2$, the recurrence relation (5.1.3) takes the form

$$(\lambda - 1)\mathcal{Q}_{n+1}(\lambda) = p_n^{-1}[r_n\lambda^2 + s_n\lambda + t_n]\mathcal{Q}_n(\lambda) + p_n^{-1}u_n\lambda(\lambda-1)\mathcal{Q}_{n-1}(\lambda). \tag{5.1.8}$$

Further, $p_n + q_n = 0$, $n \geq 2$, implies

$$\begin{aligned}
& p_n + \alpha_{n-1}\alpha_n\left(1 - \frac{\rho_{n-1}\beta_{n-1}}{\alpha_n}\right) = 0 \\
\iff\ & \frac{p_{n+1}}{\alpha_n} + \frac{\alpha_{n-1}p_{n+1}}{p_n}\left(1 - \frac{\rho_{n-1}\beta_{n-1}}{\alpha_n}\right) = 0 \\
\iff\ & \rho_n - \frac{p_{n+1}}{\alpha_n} - \frac{\alpha_{n-1}p_{n+1}}{p_n} - \rho_n + \frac{\alpha_{n-1}\rho_{n-1}\beta_{n-1}p_{n+1}}{\alpha_np_n} = 0 \\
\iff\ & p_n^{-1}(r_n + s_n + t_n) = 0, \quad n \geq 2.
\end{aligned}$$

This means that $\lambda - 1$ is a factor of the polynomial coefficient of $\mathcal{Q}_n(\lambda)$ in (5.1.8). Hence, excluding the factor $(\lambda - 1)$, (5.1.8) can now be written as

$$\mathcal{Q}_{n+1}(\lambda) = \rho_n\left(\lambda - \frac{\alpha_{n-1}\rho_{n-1}\beta_{n-1}p_{n+1}}{\alpha_n\rho_np_n}\right)\mathcal{Q}_n(\lambda) + \frac{\tau_{n-1}p_{n+1}}{p_n}\lambda\mathcal{Q}_{n-1}(\lambda),$$

which is (5.1.7b) for $n \geq 2$.

Now, with the conditions $\alpha_0 = \tau_0\rho_0^{-1}$ and $\alpha_1 \neq \rho_0\beta_0$, we observe that $p_1 = 0$ and

$q_1 \neq 0$. Further

$$r_1 = 0,\ s_1 = \rho_1 q_1 - \alpha_0 p_2,\ t_1 = a_0\rho_0 b_0 p_2 \alpha_1^{-1},\ u_1 = -v_1 = \tau_0 p_2,\ w_1 = 0.$$

Thus, (5.1.3) takes the form

$$q_1 \mathcal{Q}_2(\lambda) = (s_1\lambda + t_1)\mathcal{Q}_1(\lambda) + \tau_0 p_2 \lambda(\lambda - 1)\mathcal{Q}_0(\lambda),$$

which is (5.1.7a). □

Remark 5.1.1. *Similar to (5.1.6), one can also construct $\{\alpha_n\}_{n=0}^{\infty}$ with $q_n = p_n$, $n \geq 2$. In this case also, as in (5.1.8), $\lambda + 1$ becomes a factor of both the coefficients of $\mathcal{Q}_n(\lambda)$ and $\mathcal{Q}_{n-1}(\lambda)$ which by a simple computation leads to the same mixed recurrence relations (5.1.7a) and (5.1.7b).*

5.1.2 Uniqueness of the sequence $\{\alpha_n\}_{n=0}^{\infty}$

It is desirable to have the sequence $\{\alpha_n\}_{n=0}^{\infty}$ uniquely defined. With $\alpha_0 = \tau_0\rho_0^{-1}$, we only need to define α_1, so that (5.1.6) generates α_n, $n \geq 2$, uniquely. It is clear that there are various ways to choose α_1 subject to the only restriction that $\alpha_1 \neq \rho_0\beta_0$. We choose α_1 so that one of our objectives that is, construction of a sequence of linear combinations of R_I polynomials such that all the terms in this sequence have a common zero is achieved.

Theorem 5.1.3. *Suppose $\mathcal{Q}_1(1) = 0$. Then $\mathcal{Q}_n(\lambda)$ has a common zero at $\lambda = 1$ for $n \geq 2$, with $\mathcal{Q}_2(\lambda)$ having a double zero at $\lambda = 1$ if, and only if, $\alpha_1 \neq \rho_0 b_0$ is a root of the quadratic equation*

$$\rho_1 x^2 - \rho_0\beta_0\tau_1 = 0. \tag{5.1.9}$$

However, if $\mathcal{Q}_2(\lambda)$ does not have a double zero at $\lambda = 1$, then, $\mathcal{Q}_n(\lambda)$ and $\mathcal{Q}_{n-1}(\lambda)$, $n \geq 2$, do not have a common zero except at $\lambda = 1$.

Proof. If $\mathcal{Q}_1(\lambda)$ has a zero at $\lambda = 1$, it follows from the recurrence relations (5.1.7a) and (5.1.7b) that $\lambda = 1$ is a root of $\mathcal{Q}_n(\lambda)$, $n \geq 1$. To check whether $\mathcal{Q}_2(\lambda)$ has a double zero at $\lambda = 1$, differentiating both sides of (5.1.7a) with respect to λ, we obtain

$$q_1\mathcal{Q}_2'(\lambda) = s_1\mathcal{Q}_1(\lambda) + (s_1\lambda + t_1)\mathcal{Q}_1'(\lambda) + 2\lambda\tau_0 p_2 - \tau_0 p_2.$$

Since $q_1 \neq 0$, $\mathcal{Q}_2(\lambda)$ has a double zero at $\lambda = 1$ if, and only if, $(s_1 + t_1)\rho_0 + \tau_0 p_2 = 0$, which upon substitution of the values of s_1, t_1 and p_2 from Theorem 5.1.2, is equivalent to the condition

$$\begin{aligned}
& \rho_0\left(\rho_1 q_1 + \alpha_0\alpha_1^{-1}q_2(\alpha_1 - \rho_0\beta_0)\right) - \tau_0\alpha_1(\alpha_2 - \rho_1\beta_1) = 0 \\
\iff\quad & \rho_0 q_1(\rho_1 + \alpha_2 - \rho_1\beta_1) - \tau_0\alpha_1(\alpha_2 - \rho_1\beta_1) = 0 \\
\iff\quad & \rho_0\alpha_0\alpha_1^{-1}\tau_1(\alpha_1 - \rho_0\beta_0) + \tau_0\rho_1\alpha_1 - \tau_0\tau_1 = 0 \\
\iff\quad & \rho_1\alpha_1^2 - \rho_0\beta_0\tau_1 = 0.
\end{aligned}$$

To prove the last part of the theorem, we use the procedure given in da Silva and Sri Ranga [45, Lemma 2.1]. First, we note that $\mathcal{Q}_1(0) = \alpha_1 - \rho_0\beta_0 \neq 0$ and $\mathcal{Q}_{n+1}(0) = t_n q_n^{-1} \neq 0$, $n \geq 1$. We write (5.1.7b) as

$$\frac{\tau_{n-1}q_{n+1}}{q_n}\lambda = \frac{\mathcal{Q}_{n+1}(\lambda)}{\mathcal{Q}_{n-1}(\lambda)} - \rho_n\left(\lambda - \frac{t_n}{r_n}\right)\frac{\mathcal{Q}_n(\lambda)}{\mathcal{Q}_{n-1}(\lambda)} \tag{5.1.10}$$

and proceed to prove by induction. By hypothesis, $\mathcal{Q}_1(\lambda)$ and $\mathcal{Q}_2(\lambda)$ do not have common zeros except at $\lambda = 1$. For $n \geq 3$, suppose that $\mathcal{Q}_{n-1}(\lambda)$ and $\mathcal{Q}_n(\lambda)$ do not vanish simultaneously at any point $\omega \neq 1 \in \mathbb{C}$. This means if $\mathcal{Q}_n(\omega) = 0$, then $\mathcal{Q}_{n-1}(\omega) \neq 0$ and (5.1.10) yields $\tau_{n-1}q_{n+1}\omega/q_n = \mathcal{Q}_{n+1}(\omega)/\mathcal{Q}_{n-1}(\omega)$. Since $\mathcal{Q}_{n+1}(0) \neq 0$, $\omega \neq 0$, which further implies that $\mathcal{Q}_{n+1}(\omega) \neq 0$. □

The uniqueness of the sequence $\{\alpha_n\}_{n=0}^{\infty}$ hence follows from the observation $\mathcal{Q}_1(1) = 0$ implies choosing $\alpha_1 = \rho_0(\beta_0 - 1)$. In case we require that $\mathcal{Q}_2(\lambda)$ does not have a repeated root at $\lambda = 1$, we assign a value to α_1 such that α_1 does not satisfy (5.1.9). Hence, we choose the initial values such that

$$\tau_1\rho_1^{-1} \neq \rho_0\beta_0^{-1}(1 - \beta_0)^2 \quad \text{and} \quad \beta_0 \neq 0, \pm 1. \tag{5.1.11}$$

For the rest of the chapter, we will assume that the sequence $\{\mathcal{Q}_n(\lambda)\}_{n=0}^{\infty}$ has been constructed with ρ_0, β_0, ρ_1 and τ_1 satisfying the inequalities (5.1.11). However, the fact that such a sequence is obtained from linear combinations of R_I polynomials suggest that this sequence must itself satisfy some orthogonality properties and this forms the content of the next two sections.

5.2 Biorthogonality from linear combinations

In this section, we consider the sequence $\{\mathcal{Q}_n(\lambda)\}_{n=1}^{\infty}$ constructed from (5.1.6) with $\alpha_1 = \rho_0(\beta_0 - 1)$, so that $\mathcal{Q}_n(1) = 0$ for $n \geq 1$. We further assume that the other zeros of $\mathcal{Q}_n(\lambda)$ are distinct, that is $\mathcal{Q}_n(\lambda)$, $n \geq 1$, has no repeated zeros.

Note that from the recurrence relations (5.1.7a) and (5.1.7b), the leading coefficient of $\mathcal{Q}_n(\lambda)$ is $\kappa_{n-1} = \rho_{n-1}\rho_{n-2}\cdots\rho_0 \neq 0$, $n \geq 1$. This also follows from the fact that the leading coefficients of $\mathcal{Q}_n(\lambda)$ and $\mathcal{P}_n(\lambda)$ are equal for $n \geq 1$. Again from the recurrence relations (5.1.7a) and (5.1.7b), it can be shown that $\mathcal{Q}_n(\lambda)$, $n \geq 1$, has the determinant representation

$$\begin{vmatrix} \rho_0(\lambda-1) & \tau_0 q_2 q_1^{-1}(\lambda-1) & 0 & \cdots & 0 \\ \lambda & q_1^{-1}(s_1\lambda + t_1) & -\tau_1 q_3 q_2^{-1} & \cdots & 0 \\ 0 & \lambda & \rho_2(\lambda - t_2 r_2^{-1}) & \cdots & 0 \\ 0 & 0 & \lambda & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \rho_{n-1}(\lambda - t_{n-1}r_{n-1}^{-1}) \end{vmatrix}, \tag{5.2.1}$$

if one expands the determinant by its last column.

As the first step towards obtaining a biorthogonality relation, we express $\mathcal{Q}_n(\lambda)$ as the characteristic polynomial of a matrix.

Theorem 5.2.1. *The zeros of the monic polynomials* $\hat{\mathcal{Q}}_n(\lambda) = \kappa_{n-1}^{-1}\mathcal{Q}_n(\lambda)$ *are the* *eigenvalues of the matrix*

$$\mathcal{D}_n := \begin{pmatrix} \frac{s_1 q_1^{-1}}{\rho_1} & \frac{\tau_0\tau_1 q_2 q_1^{-1}}{\alpha_1\rho_1\rho_0} & \frac{-\tau_0\tau_1 q_3 q_1^{-1}}{\rho_1\rho_0} & 0 & \cdots & 0 \\ \frac{-1}{\rho_1} & \frac{-\vartheta_2}{\alpha_1\rho_1} & \frac{\tau_1 q_3 q_2^{-1}}{\rho_1} & 0 & \cdots & 0 \\ \frac{1}{\rho_2\rho_1} & \frac{\vartheta_2}{\alpha_1\rho_2\rho_1} & \frac{-\vartheta_3}{\rho_2\rho_1} & -\frac{\tau_2 q_3^{-1} q_4}{\rho_2} & \cdots & 0 \\ \frac{-1}{\rho_3\rho_2\rho_1} & \frac{-\vartheta_2}{\alpha_1\rho_3\rho_2\rho_1} & \frac{\vartheta_3}{\rho_3\rho_2\rho_1} & \frac{-\vartheta_4}{\rho_3\rho_2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{(-1)^{n+1}}{\rho_{n-1}\cdots\rho_1} & \frac{(-1)^{n+1}\vartheta_2}{\alpha_1\rho_{n-1}\cdots\rho_1} & \frac{(-1)^{n+2}\vartheta_3}{\rho_{n-1}\cdots\rho_1} & \frac{(-1)^{n+3}\vartheta_4}{\rho_{n-1}\cdots\rho_2} & \cdots & \frac{-\vartheta_n}{\rho_{n-1}\rho_{n-2}} \end{pmatrix}$$

where $\vartheta_2 = q_2$ and $\vartheta_n = q_n(1+\rho_{n-2}\alpha_{n-1}^{-1})$, $n \geq 3$.

The proof of Theorem 5.2.1 involves the inversion of a tridiagonal matrix. Such inverses for particular tridiagonal matrix operators are studied in Sivakumar [159]. However, we use the following lemma proved in Usmani [177] for the expression of the matrix inverse of the general tridiagonal matrix given by

$$\mathcal{J}_n = \begin{pmatrix} b_1 & c_1 & 0 & \cdots & 0 & 0 \\ a_2 & b_2 & c_2 & \cdots & 0 & 0 \\ 0 & a_3 & b_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & b_{n-1} & c_{n-1} \\ 0 & 0 & 0 & \cdots & a_n & b_n \end{pmatrix}.$$

Lemma 5.2.1. *Let $\mathcal{J}_n^{-1} = [\alpha_{i,j}]$. Then for $a_k \neq 0$ and $i \geq j$,*

$$\alpha_{i,j} = \begin{cases} (-1)^{i+j} a_{j+1}a_{j+2}\cdots a_i \frac{\theta_{j-1}\phi_{i+1}}{\theta_n}, & j = 1,2,\cdots,i-1; \\ \frac{\theta_{i-1}\phi_{i+1}}{\theta_n}, & j = i, \end{cases} \tag{5.2.2}$$

while for $r < s$,

$$\alpha_{r,s} = \prod_{k=r}^{s-1} \frac{c_k}{a_{k+1}} \alpha_{s,r}. \tag{5.2.3}$$

Here, θ_n is the determinant of the matrix $\mathcal{J}_n$, while the sequence $\{\phi_k\}_{k=1}^{n}$ is defined as

$$\begin{aligned} \phi_k &= b_k\phi_{k+1} - a_{k+1}c_k\phi_{k+2}, \quad k = 1, 2, \cdots, n, \\ \phi_{n+1} &= 1, \quad \phi_{n+2} = 0. \end{aligned} \tag{5.2.4}$$

Proof of Theorem 5.2.1. The determinant expression (5.2.1) for $\mathcal{Q}_n(\lambda)$, $n \geq 1$, can be expressed as the determinant of a linear matrix pencil $\lambda\mathcal{G}_n - \mathcal{H}_n$, where

$$\mathcal{G}_n = \begin{pmatrix} \rho_0 & \tau_0 q_2 q_1^{-1} & 0 & \cdots & 0 \\ 1 & s_1 q_1^{-1} & 0 & \cdots & 0 \\ 0 & 1 & \rho_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \rho_{n-1} \end{pmatrix} \quad \text{and}$$

$$\mathcal{H}_n = \begin{pmatrix} \rho_0 & \tau_0 q_2 q_1^{-1} & 0 & \cdots & 0 \\ 0 & -t_1 q_1^{-1} & \tau_1 q_3 q_2^{-1} & \cdots & 0 \\ 0 & 0 & -t_2 q_2^{-1} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -t_{n-1} q_{n-1}^{-1} \end{pmatrix}.$$

Here we used the relation $t_j\rho_j r_j^{-1} = t_j p_j^{-1} = -t_j q_j^{-1}$, $j \geq 2$. Then, for $n \geq 1$,

$$\mathcal{Q}_n(\lambda) = \det(\lambda\mathcal{G}_n - \mathcal{H}_n) = \det(\mathcal{G}_n) \cdot \det(\lambda\mathcal{I}_n - \mathcal{G}_n^{-1}\mathcal{H}_n),$$

where $\mathcal{I}_n$ is the $n \times n$ identity matrix. However $\det(\mathcal{G}_n) = \kappa_{n-1} \neq 0$ implies that $\mathcal{G}_n$ is invertible and $\kappa_{n-1}^{-1}\mathcal{Q}_n(\lambda) = \det(\lambda\mathcal{I}_n - \mathcal{G}_n^{-1}\mathcal{H}_n)$. Hence, $\hat{\mathcal{Q}}_n(\lambda)$ is the characteristic polynomial of the matrix product $\mathcal{D}_n = \mathcal{G}_n^{-1}\mathcal{H}_n$. The proof will be complete once we find the matrix inverse $\mathcal{G}_n^{-1}$.

Comparing the matrix $\mathcal{G}_n$ with the matrix $\mathcal{J}_n$, we have $c_1 = \tau_0 q_2 q_1^{-1}$ and $c_i = 0$, $a_i = 1$ for $i \geq 2$. Further, $\theta_n = \det(\mathcal{G}_n) = \kappa_{n-1}$ and hence from (5.2.4), $\phi_k = \rho_{n-1}\rho_{n-2}\cdots\rho_{k-1}$, $k = n, n-1, \cdots, 3, 1$ with $\phi_2 = \rho_{n-1}\rho_{n-2}\cdots\rho_2 s_1 q_1^{-1}$. Using (5.2.2), the diagonal elements of $\mathcal{G}_n^{-1}$ are

$$\alpha_{1,1} = \frac{\theta_0\phi_2}{\theta_n} = \frac{\rho_{n-1}\rho_{n-2}\cdots\rho_2 s_1 q_1^{-1}}{\rho_{n-1}\rho_{n-2}\cdots\rho_0} = \frac{s_1 q_1^{-1}}{\rho_0\rho_1},$$
$$\alpha_{i,i} = \frac{\theta_{i-1}\phi_{i+1}}{\theta_n} = \frac{\rho_0\rho_1\cdots\rho_{i-2}\cdot\rho_{n-1}\rho_{n-2}\cdots\rho_i}{\rho_0\rho_1\cdots\rho_{n-1}} = \frac{1}{\rho_{i-1}}, \quad i \geq 2,$$

while for $i \geq j$, $j = 1, 2, \cdots, i-1$, $i = 1, 2, \cdots, n$,

$$\alpha_{i,j} = \frac{(-1)^{i+j}}{\rho_{i-1}\rho_{i-2}\cdots\rho_{j-1}}.$$

Thus the elements on and below the diagonal of $\mathcal{G}_n^{-1}$ are estimated. For $i \leq j$, we note that the right hand side of (5.2.3) is non-zero only for $r = 1$ and $s = 2$ since $c_i = 0$ for $i \geq 2$. Hence

$$\alpha_{1,2} = \tau_0 q_2 q_1^{-1}\alpha_{2,1} = -\frac{\tau_0 q_2 q_1^{-1}}{\rho_1\rho_0}.$$

Explicitly, the matrix inverse $\mathcal{G}_n^{-1}$ is given by

$$\begin{pmatrix} \frac{s_1 q_1^{-1}}{\rho_1\rho_0} & -\frac{\tau_0 q_2 q_1^{-1}}{\rho_1\rho_0} & 0 & \cdots & 0 \\ -\frac{1}{\rho_1\rho_0} & \frac{1}{\rho_1} & 0 & \cdots & 0 \\ \frac{1}{\rho_2\rho_1\rho_0} & -\frac{1}{\rho_2\rho_1} & \frac{1}{\rho_2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{(-1)^{n+1}}{\rho_{n-1}\cdots\rho_0} & \frac{(-1)^{n+2}}{\rho_{n-1}\cdots\rho_1} & \frac{(-1)^{n+3}}{\rho_{n-1}\cdots\rho_2} & \cdots & \frac{1}{\rho_{n-1}} \end{pmatrix}$$

and the proof is complete by calculating the matrix product $\mathcal{D}_n = \mathcal{G}_n^{-1}\mathcal{H}_n$. □

With the assumptions made at the beginning of this section, the eigenvalues of the matrix $\mathcal{D}_n$ are distinct and the intersection of the spectrum of $\mathcal{D}_n$ with that of $\mathcal{D}_{n+1}$, $n \geq 1$, is non-empty, consisting only of the point $\lambda = 1$. However, note that both $\mathcal{G}_n^{-1}$ and $\mathcal{D}_n$ can be viewed as the matrix forms $\mathcal{E}_{1\mathcal{G}} + \epsilon_{\mathcal{G}}\mathcal{E}_{2\mathcal{G}}$ and $\mathcal{E}_{1\mathcal{D}} + \epsilon_{\mathcal{D}}\mathcal{E}_{2\mathcal{D}}$ respectively, where $\mathcal{E}_{1\mathcal{G}}$, $\mathcal{E}_{2\mathcal{G}}$, $\mathcal{E}_{1\mathcal{D}}$ and $\mathcal{E}_{2\mathcal{D}}$ are $n \times n$ matrices while $\epsilon_{\mathcal{G}}$ and $\epsilon_{\mathcal{D}}$ are constants. Precisely,

$\mathcal{E}_{1\mathcal{G}}$ is a lower triangular matrix, $\mathcal{E}_{1\mathcal{D}}$ is a lower Hessenberg matrix,

$$\mathcal{E}_{2\mathcal{G}} = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \quad \mathcal{E}_{2\mathcal{D}} = \begin{pmatrix} 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

$\epsilon_{\mathcal{G}} = -\tau_0 q_2 q_1^{-1}/\rho_1\rho_0$, and $\epsilon_{\mathcal{D}} = -\tau_0\tau_1 q_3 q_1^{-1}/\rho_1\rho_0$. The effect of such linear perturbation of the spectra of the matrix are investigated in Alam and Bora [1, 2].

5.2.1 Eigenvectors from $\mathcal{Q}_n(\lambda)$

We construct the left and right eigenvectors of the pencil matrix $\lambda\mathcal{G}_n - \mathcal{H}_n$. The superscript $L(R)$ corresponds to the left (right) eigenvectors respectively. With $\sigma_0^L(\lambda) := \sigma_0^R(\lambda) := \mathcal{Q}_0(\lambda) = 1$, consider the following rational functions

$$\sigma_k^L(\lambda) := \frac{\mathcal{Q}_k(\lambda)}{(-\lambda)^k}, \quad \sigma_k^R(\lambda) := -\frac{\mathcal{Q}_k(\lambda)}{\prod_{j=0}^{k-1} \tau_j q_{k+1} q_1^{-1}(\lambda - 1)}, \tag{5.2.5}$$

for $k = 1, 2, \cdots, n$. Then, (5.1.7a) and (5.1.7b) written in terms of $\sigma_n^L(\lambda)$ and $\sigma_n^R(\lambda)$ will yield respectively

$$\begin{aligned} \lambda[\rho_0\sigma_0^L(\lambda) + \sigma_1^L(\lambda)] &= \rho_0\sigma_0^L(\lambda), \\ \lambda[\tau_0 q_2 q_1^{-1}\sigma_0^L(\lambda) + s_1 q_1^{-1}\sigma_1^L(\lambda) + \sigma_2^L(\lambda)] &= \tau_0 q_2 q_1^{-1}\sigma_0^L(\lambda) - t_1 q_1^{-1}\sigma_1^L(\lambda), \\ \lambda[\rho_n\sigma_n^L(\lambda) + \sigma_{n+1}^L(\lambda)] &= \tau_{n-1} q_{n+1} q_n^{-1}\sigma_{n-1}^L(\lambda) - t_n q_n^{-1}\sigma_n^L(\lambda) \end{aligned} \tag{5.2.6}$$

and

$$\begin{aligned} \lambda[\rho_0\sigma_0^R(\lambda) + \tau_0 q_2 q_1^{-1}\sigma_1^R(\lambda)] &= \tau_0 q_2 q_1^{-1}\sigma_1^R(\lambda) + \rho_0\sigma_0^R(\lambda), \\ \lambda[\sigma_0^R(\lambda) + t_1 q_1^{-1}\sigma_1^R(\lambda)] &= \tau_1 q_3 q_2^{-1}\sigma_2^R(\lambda) - t_1 q_1^{-1}\sigma_1^R(\lambda), \\ \lambda[\sigma_{n-1}^R(\lambda) + \rho_n\sigma_n^R(\lambda)] &= \tau_n q_{n+2} q_{n+1}^{-1}\sigma_{n+1}^R(\lambda) - t_n q_n^{-1}\sigma_n^R(\lambda). \end{aligned} \tag{5.2.7}$$

Defining the sequences $\{\chi_n^L(\lambda)\}_{n=0}^{\infty}$ and $\{\chi_n^R(\lambda)\}_{n=0}^{\infty}$ where

$$\begin{aligned}&\chi_0^L(\lambda)=\rho_0\sigma_0^L(\lambda)+\sigma_1^L(\lambda), \quad \chi_1^L(\lambda)=\tau_0 q_2 q_1^{-1}\sigma_0^L(\lambda)+s_1 q_1^{-1}\sigma_1^L(\lambda)+\sigma_2^L(\lambda),\\ &\chi_n^L(\lambda)=\rho_n\sigma_n^L(\lambda)+\sigma_{n+1}^L(\lambda), \quad n\geq 2,\end{aligned} \tag{5.2.8}$$

and

$$\begin{aligned}&\chi_0^R(\lambda)=\rho_0\sigma_0^R(\lambda)+\tau_0 q_2 q_1^{-1}\sigma_1^R(\lambda), \quad \chi_1^R(\lambda)=\sigma_0^R(\lambda)+t_1 q_1^{-1}\sigma_1^R(\lambda),\\ &\chi_n^R(\lambda)=\sigma_{n-1}^R(\lambda)+\rho_n\sigma_n^R(\lambda), \quad n\geq 2.\end{aligned} \tag{5.2.9}$$

(5.2.6) can be written as

$$\begin{aligned}&\lambda\chi_0^L(\lambda)=\rho_0\sigma_0^L(\lambda), \quad \lambda\chi_1^L(\lambda)=\tau_0 q_2 q_1^{-1}\sigma_0^L(\lambda)-t_1 q_1^{-1}\sigma_1^L(\lambda),\\ &\lambda\chi_n^L(\lambda)=\tau_{n-1}q_{n+1}q_n^{-1}\sigma_{n-1}^L(\lambda)-t_n q_n^{-1}\sigma_n^L(\lambda), \quad n\geq 2.\end{aligned} \tag{5.2.10}$$

while (5.2.7) can be written as

$$\begin{aligned}&\lambda\chi_0^R(\lambda)=\tau_0 q_2 q_1^{-1}\sigma_1^R(\lambda)+\rho_0\sigma_0^R(\lambda), \quad \lambda\chi_1^R(\lambda)=\tau_1 q_3 q_2^{-1}\sigma_2^R(\lambda)-t_1 q_1^{-1}\sigma_1^R(\lambda),\\ &\lambda\chi_n^R(\lambda)=\tau_n q_{n+2}q_{n+1}^{-1}\sigma_{n+1}^R(\lambda)-t_n q_n^{-1}\sigma_n^R(\lambda), \quad n\geq 2.\end{aligned} \tag{5.2.11}$$

For $n\geq 1$, the matrix equations corresponding to (5.2.6) and (5.2.7) are respectively

$$\lambda\boldsymbol{\sigma}^L(\lambda)\mathcal{G}_n=\boldsymbol{\sigma}^L(\lambda)\mathcal{H}_n+\lambda\sigma_n^L(\lambda)\boldsymbol{e}_n^T \quad \text{and} \tag{5.2.12a}$$

$$\lambda\mathcal{G}_n\boldsymbol{\sigma}^R(\lambda)=\mathcal{H}_n\boldsymbol{\sigma}^R(\lambda)+\tau_{n-1}q_{n+1}q_n^{-1}\sigma_n^R(\lambda)\boldsymbol{e}_n, \tag{5.2.12b}$$

where $\boldsymbol{e}_n$ is the n^{th} column of the $n\times n$ identity matrix I_n,

$$\begin{aligned}\boldsymbol{\sigma}^L(\lambda)&=\begin{pmatrix}\sigma_0^L(\lambda) & \sigma_1^L(\lambda) & \cdots & \sigma_{n-1}^L(\lambda)\end{pmatrix} \qquad \text{and}\\ \boldsymbol{\sigma}^R(\lambda)&=\begin{pmatrix}\sigma_0^R(\lambda) & \sigma_1^R(\lambda) & \cdots & \sigma_{n-1}^R(\lambda)\end{pmatrix}^T.\end{aligned}$$

Here $\mathcal{G}_n$ and $\mathcal{H}_n$ are the matrices defined in Theorem 5.2.1. Let the zeros of $\mathcal{Q}_n(\lambda)$, $n\geq 1$, be denoted as $\lambda_{n,j}$, $j=1,2,\cdots,n$. Since both $\sigma_n^L(\lambda)$ and $\sigma_n^R(\lambda)$ vanish at λ_{nj}, it follows from (5.2.12a) and (5.2.12b) that $\boldsymbol{\sigma}_n^L(\lambda_{nj})$ and $\boldsymbol{\sigma}_n^R(\lambda_{nj})$ are the left and right eigenvectors of the linear pencil matrix $\mathcal{G}_n-\lambda\mathcal{H}_n$ respectively.

5.2.2 The associated measure

Using the matrix representations (5.2.12a) and (5.2.12b) we obtain a discrete measure of biorthogonality which requires the removal of the common zero $\lambda = 1$. First, we recall the following standard result and outline a proof to provide further clarity.

Lemma 5.2.2. *The trace of the matrix product $\boldsymbol{\sigma}^R(\omega)\boldsymbol{\sigma}^L(\lambda)\mathcal{G}_n$ which is also equal to the trace of $\mathcal{G}_n\boldsymbol{\sigma}^R(\omega)\boldsymbol{\sigma}^L(\lambda)$ is same as the matrix product $\boldsymbol{\sigma}^L(\lambda)\mathcal{G}_n\boldsymbol{\sigma}^R(\omega)$.*

Proof. The matrix product $\boldsymbol{\sigma}^R(\omega)\boldsymbol{\sigma}^L(\lambda)\mathcal{G}_n$ is

$$\begin{pmatrix} \sigma_0^R(\omega)\sigma_0^L(\lambda) & \sigma_0^R(\omega)\sigma_1^L(\lambda) & \cdots & \sigma_0^R(\omega)\sigma_{n-2}^L(\lambda) & \sigma_0^R(\omega)\sigma_{n-1}^L(\lambda) \\ \sigma_1^R(\omega)\sigma_0^L(\lambda) & \sigma_1^R(\omega)\sigma_1^L(\lambda) & \cdots & \sigma_1^R(\omega)\sigma_{n-2}^L(\lambda) & \sigma_1^R(\omega)\sigma_{n-1}^L(\lambda) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \sigma_{n-2}^R(\omega)\sigma_0^L(\lambda) & \sigma_{n-2}^R(\omega)\sigma_1^L(\lambda) & \cdots & \sigma_{n-2}^R(\omega)\sigma_{n-2}^L(\lambda) & \sigma_{n-2}^R(\omega)\sigma_{n-1}^L(\lambda) \\ \sigma_{n-1}^R(\omega)\sigma_0^L(\lambda) & \sigma_{n-1}^R(\omega)\sigma_1^L(\lambda) & \cdots & \sigma_{n-1}^R(\omega)\sigma_{n-2}^L(\lambda) & \sigma_{n-1}^R(\omega)\sigma_{n-1}^L(\lambda) \end{pmatrix}_{n\times n}$$

$$\times \begin{pmatrix} \rho_0 & \tau_0 q_2 q_1^{-1} & 0 & 0 & \cdots & 0 & 0 \\ 1 & s_1 q_1^{-1} & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \rho_2 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \rho_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \rho_{n-2} & 0 \\ 0 & 0 & 0 & 0 & \cdots & 1 & \rho_{n-1} \end{pmatrix}_{n\times n}$$

so that the trace is

$$\begin{aligned} &[\rho_0\sigma_0^L(\lambda) + \sigma_1^L(\lambda)]\sigma_0^R(\omega) + [\tau_0 q_2 q_1^{-1}\sigma_0^L(\lambda) + s_1 q_1^{-1}\sigma_1^L(\lambda) + \sigma_2^L(\lambda)]\sigma_1^R(\omega) \\ &+ [\rho_2\sigma_2^L(\lambda) + (\omega)\sigma_3^L(\lambda)]\sigma_2^R(\omega) + [\rho_3\sigma_3^L(\lambda) + \sigma_4^L(\lambda)]\sigma_3^R(\omega) + \cdots \\ &+ [\rho_{n-2}\sigma_{n-2}^L(\lambda) + \sigma_{n-1}^L(\lambda)]\sigma_{n-2}^R(\omega) + \rho_{n-1}\sigma_{n-1}^R(\omega)\sigma_{n-1}^L(\lambda). \end{aligned} \tag{5.2.13}$$

The sum (5.2.13) can also be represented as the matrix product

$$\boldsymbol{\sigma}^L(\lambda)\mathcal{G}_n\boldsymbol{\sigma}^R(\omega) = \begin{pmatrix} \rho_0\sigma_0^L(\lambda)+\sigma_1^L(\lambda) \\ \tau_0 q_2 q_1^{-1}\sigma_0^L(\lambda)+s_1q_1^{-1}\sigma_1^L(\lambda)+\sigma_2^L(\lambda) \\ \rho_2\sigma_2^L(\lambda)+\sigma_3^L(\lambda) \\ \rho_3\sigma_3^L(\lambda)+\sigma_4^L(\lambda) \\ \vdots \\ \rho_{n-2}\sigma_{n-2}^L(\lambda)+\sigma_{n-1}^L(\lambda) \\ \rho_{n-1}\sigma_{n-1}^L(\lambda) \end{pmatrix}_{n\times 1}^T \begin{pmatrix} \sigma_0^R(\omega) \\ \sigma_1^R(\omega) \\ \sigma_2^R(\omega) \\ \sigma_3^R(\omega) \\ \vdots \\ \sigma_{n-2}^R(\omega) \\ \sigma_{n-1}^R(\omega) \end{pmatrix}_{n\times 1}$$

thus proving the lemma. □

The following result gives an expression for the measure of biorthogonality. The proof is motivated by the analysis in da Silva and Sri Ranga [45, Section 2].

Theorem 5.2.2. *Let the zeros of $\mathcal{Q}_n(\lambda)$ be denoted as $\lambda_{n,j}$, $j=1,2,\cdots,n$, with $\lambda_{n,n}=1$. Then, the following biorthogonality relation*

$$\sum_{i=0}^{n-1}\sigma_i^R(\lambda_{n,j})\tilde{\chi}_i^L(\lambda_{n,k})\mu_{n,j,k}=\delta_{j,k}, \quad j,k=0,1,\cdots,n-2, \tag{5.2.14}$$

holds, where $\tilde{\chi}_i^L(\lambda_{n,k})=\chi_i^L(\lambda_{n,k})$, $i=1,2,\cdots,n-2$ and $\tilde{\chi}_{n-1}(\lambda)=\rho_{n-1}\sigma_{n-1}^L(\lambda)$. The weight function $\mu_{n,j,k}$ has the expression

$$\mu_{n,j,k}=[\tau_{n-1}q_{n+1}q_n^{-1}[\sigma_n^R(\lambda_{n,j})]'\sigma_{n-1}^L(\lambda_{n,k})]^{-1}, \quad j,k=1,\cdots,n-1.$$

Proof. Post-multiplying (5.2.12b) by $\boldsymbol{\sigma}^L(\lambda)$ and pre-multiplying (5.2.12a) by $\boldsymbol{\sigma}^R(\omega)$ after evaluating (5.2.12b) at ω, we obtain the systems

$$\lambda\boldsymbol{\sigma}^R(\omega)\boldsymbol{\sigma}^L(\lambda)\mathcal{G}_n=\boldsymbol{\sigma}^R(\omega)\boldsymbol{\sigma}^L(\lambda)\mathcal{H}_n+\lambda\boldsymbol{\sigma}^R(\omega)\sigma_n^L(\lambda)\boldsymbol{e}_n^T \quad \text{and} \tag{5.2.15a}$$

$$\omega\mathcal{G}_n\boldsymbol{\sigma}^R(\omega)\boldsymbol{\sigma}^L(\lambda)=\mathcal{H}_n\boldsymbol{\sigma}^R(\omega)\boldsymbol{\sigma}^L(\lambda)+\tau_{n-1}q_{n+1}q_n^{-1}\sigma_n^R(\omega)\boldsymbol{e}_n\boldsymbol{\sigma}^L(\lambda). \tag{5.2.15b}$$

The first step is to subtract the matrix traces of the corresponding sides of (5.2.15b) from (5.2.15a). By Lemma 5.2.2, the left hand side after subtraction of the matrix traces gives $(\lambda-\omega)\boldsymbol{\sigma}^L(\lambda)\mathcal{G}_n\boldsymbol{\sigma}^R(\lambda)$.

We now find the traces of the matrix products appearing in the right hand sides of (5.2.15a) and (5.2.15b). Since $\boldsymbol{e}_n$ is the n^{th} column of the $n \times n$ identity matrix, the matrix product $\tau_{n-1}q_{n+1}q_n^{-1}\sigma_n^R(\omega)\boldsymbol{e}_n\boldsymbol{\sigma}^L(\lambda)$ is

$$\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \tau_{n-1}q_{n+1}q_n^{-1}\sigma_n^R(\omega) \end{pmatrix}_{n\times 1} \begin{pmatrix} \sigma_0^L(\lambda) & \sigma_1^L(\lambda) & \cdots & \sigma_{n-2}^L(\lambda) & \sigma_{n-1}^L(\lambda) \end{pmatrix}_{1\times n}$$

which implies the trace of $\tau_{n-1}q_{n+1}q_n^{-1}\sigma_n^R(\omega)\boldsymbol{e}_n\boldsymbol{\sigma}^L(\lambda)$ is $\tau_{n-1}q_{n+1}q_n^{-1}\sigma_n^R(\omega)\sigma_{n-1}^L(\lambda)$. The trace of the matrix product $\lambda\boldsymbol{\sigma}^R(\omega)\sigma_n^L(\lambda)\boldsymbol{e}_n^T$ is $\lambda\sigma_{n-1}^R(\omega)\sigma_n^L(\lambda)$ which follows from

$$\lambda \begin{pmatrix} \sigma_0^R(\omega) \\ \sigma_1^R(\omega) \\ \vdots \\ \sigma_{n-2}^R(\omega) \\ \sigma_{n-1}^R(\omega) \end{pmatrix}_{n\times 1} \begin{pmatrix} 0 & 0 & \cdots & 0 & \sigma_n^L(\lambda) \end{pmatrix}_{1\times n}.$$

Hence, subtracting the matrix trace of (5.2.15b) from the matrix trace of (5.2.15a) yields

$$\boldsymbol{\sigma}^L(\lambda)\mathcal{G}_n\boldsymbol{\sigma}^R(\omega) = \frac{\lambda\sigma_{n-1}^R(\omega)\sigma_n^L(\lambda) - \tau_{n-1}q_{n+1}q_n^{-1}\sigma_n^R(\omega)\sigma_{n-1}^L(\lambda)}{\lambda - \omega}. \tag{5.2.16}$$

Since, $\sigma_n^L(\lambda_{n,j})$ and $\sigma_n^R(\lambda_{n,k})$ vanish at $\lambda = \lambda_{n,j}$ and $\omega = \lambda_{n,k}$, $j = 1, 2, \cdots, n$ with $k = 1, 2, \cdots, n$ but $j \neq k$, (5.2.16) yields

$$\boldsymbol{\sigma}^L(\lambda_{nj})\mathcal{G}_n\boldsymbol{\sigma}^R(\lambda_{nk}) = 0, \quad j, k = 1, 2, \cdots, n, \quad j \neq k.$$

Now, we proceed for the case $j = k$. The idea is to let $\omega \longrightarrow \lambda$ and see the behavior of the right hand side of (5.2.16). However, for this purpose we need to exclude the common point $\lambda_{n,n} = \omega_{n,n} = 1$. Then, as $\omega \longrightarrow \lambda$ in (5.2.16), we obtain

$$\boldsymbol{\sigma}^L(\lambda)\mathcal{G}_n\boldsymbol{\sigma}^R(\lambda) = \tau_{n-1}q_{n+1}q_n^{-1}[\sigma_n^R(\lambda)]'\sigma_{n-1}^L(\lambda) - \lambda[\sigma_{n-1}^R(\lambda)]'\sigma_n^L(\lambda).$$

Hence at the zeros $\lambda = \lambda_{n,j}$, $j = 1, 2, \cdots, n-1$,

$$\boldsymbol{\sigma}^L(\lambda_{n,j})\mathcal{G}_n\boldsymbol{\sigma}^R(\lambda_{n,j}) = \tau_{n-1}q_{n+1}q_n^{-1}[\sigma_n^R(\lambda_{n,j})]'\sigma_{n-1}^L(\lambda_{n,j}).$$

Denoting $\mu_{n,j}^{-1} := \tau_{n-1}q_{n+1}q_n^{-1}[\sigma_n^R(\lambda_{n,j})]'\sigma_{n-1}^L(\lambda_{n,j})$, we note the following points.

- By Theorem 5.1.3, $\mathcal{Q}_{n-1}(\lambda)$ and $\mathcal{Q}_n(\lambda)$ $n \geq 1$, do not have common zeros which implies that $\sigma_{n-1}^L(\lambda_{nj}) \neq 0$.
- $[\sigma_n^R(\lambda)]' = \dfrac{-1}{\prod_{j=1}^{k-1}\tau_j q_{k+1}q_1^{-1}} \dfrac{\mathcal{Q}_n'(\lambda)(\lambda-1) - \mathcal{Q}_n(\lambda)\lambda}{(\lambda-1)^2}$ leads to
$[\sigma_n^R(\lambda_{nj})]' = \dfrac{-1}{\prod_{j=1}^{k-1}\tau_j q_{k+1}q_1^{-1}} \dfrac{\mathcal{Q}_n'(\lambda_{nj})(\lambda_{nj}-1)}{(\lambda_{nj}-1)^2} \neq 0$, because $\lambda_{nj} \neq 1$,
for $j = 1, \cdots, n-1$ and $\mathcal{Q}_n'(\lambda_{nj}) \neq 0$ (as $\mathcal{Q}_n(\lambda)$ is assumed to have simple zeros).

Hence $\mu_{n,j}^{-1} \neq 0$, $j = 1, \cdots, n-1$ so that from (5.2.16), the orthogonality relations

$$\left[\mathcal{G}_n^T[\boldsymbol{\sigma}^L(\lambda_{n,j})]^T\right]^T \boldsymbol{\sigma}^R(\lambda_{n,k}) = \mu_{n,j}^{-1}\delta_{j,k}, \quad j, k = 1, 2, \cdots, n-1, \tag{5.2.17}$$

hold. This shows that the two finite sequences $\{\mathcal{G}_n^T[\boldsymbol{\sigma}^L(\lambda_{n,j})]^T\}_{j=1}^{n-1}$ and $\{\boldsymbol{\sigma}^R(\lambda_{n,k})\}_{k=1}^{n-1}$ are biorthogonal to each other.

To proceed further in the proof, we define

$$\mu_{n,j,k}^{-1} := \tau_{n-1}q_{n+1}q_n^{-1}[\sigma_n^R(\lambda_{n,j})]'\sigma_{n-1}^L(\lambda_{n,k}),$$

where we note that for $k = j$, $\mu_{n,j,j}^{-1} = \mu_{n,j}^{-1}$. Further, from (5.2.17) the relation

$$\boldsymbol{\sigma}^L(\lambda_{nj})\mathcal{G}_n\boldsymbol{\sigma}^R(\lambda_{nk}) = \tau_{n-1}q_{n+1}q_n^{-1}[\sigma_n^R(\lambda_{nj})]'\sigma_{n-1}^L(\lambda_{nj})\delta_{j,k},$$

can also be written as the matrix product

$$\frac{\boldsymbol{\sigma}^L(\lambda_{nj})}{\sigma_{n-1}^L(\lambda_{nj})}\mathcal{G}_n \cdot \frac{\boldsymbol{\sigma}^R(\lambda_{nk})}{\tau_{n-1}q_{n+1}q_n^{-1}[\sigma_n^R(\lambda_{nj})]'} = \delta_{j,k}, \qquad j, k = 1, 2, \cdots, n-1. \tag{5.2.18}$$

Note that for $j,k=1,2,\cdots,n-1$, the RHS of (5.2.18) is the $(n-1)\times(n-1)$ identity matrix $\mathcal{I}_{n-1}$. The first matrix product in the left hand side of (5.2.18) gives the row vector

$$\frac{\boldsymbol{\sigma}^L(\lambda_{nj})}{\sigma^L_{n-1}(\lambda_{nj})}\mathcal{G}_n=\begin{pmatrix}\frac{\chi^L_0(\lambda_{n,j})}{\sigma^L_{n-1}(\lambda_{n,j})} & \frac{\chi^L_1(\lambda_{n,j})}{\sigma^L_{n-1}(\lambda_{n,j})} & \cdots & \frac{\chi^L_{n-2}(\lambda_{n,j})}{\sigma^L_{n-1}(\lambda_{n,j})} & \frac{\rho_{n-1}\sigma^L_{n-1}(\lambda_{n,j})}{\sigma^L_{n-1}(\lambda_{n,j})}\end{pmatrix},$$

which when expanded for $j=1,2,\cdots,n-1$, gives the $(n-1)\times n$ matrix

$$\mathcal{B}_{n-1\times n}=\begin{pmatrix}\frac{\chi^L_0(\lambda_{n,1})}{\sigma^L_{n-1}(\lambda_{n,1})} & \frac{\chi^L_1(\lambda_{n,1})}{\sigma^L_{n-1}(\lambda_{n,1})} & \cdots & \frac{\chi^L_{n-2}(\lambda_{n,1})}{\sigma^L_{n-1}(\lambda_{n,1})} & \frac{\rho_{n-1}\sigma^L_{n-1}(\lambda_{n,1})}{\sigma^L_{n-1}(\lambda_{n,1})}\\ \frac{\chi^L_0(\lambda_{n,2})}{\sigma^L_{n-1}(\lambda_{n,2})} & \frac{\chi^L_1(\lambda_{n,2})}{\sigma^L_{n-1}(\lambda_{n,2})} & \cdots & \frac{\chi^L_{n-2}(\lambda_{n,2})}{\sigma^L_{n-1}(\lambda_{n,2})} & \frac{\rho_{n-1}\sigma^L_{n-1}(\lambda_{n,2})}{\sigma^L_{n-1}(\lambda_{n,2})}\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ \frac{\chi^L_0(\lambda_{n,j})}{\sigma^L_{n-1}(\lambda_{n,j})} & \frac{\chi^L_1(\lambda_{n,j})}{\sigma^L_{n-1}(\lambda_{n,j})} & \cdots & \frac{\chi^L_{n-2}(\lambda_{n,n-2})}{\sigma^L_{n-1}(\lambda_{n,n-2})} & \frac{\rho_{n-1}\sigma^L_{n-1}(\lambda_{n,n-2})}{\sigma^L_{n-1}(\lambda_{n,n-2})}\\ \frac{\chi^L_0(\lambda_{n,n-1})}{\sigma^L_{n-1}(\lambda_{n,n-1})} & \frac{\chi^L_1(\lambda_{n,n-1})}{\sigma^L_{n-1}(\lambda_{n,n-1})} & \cdots & \frac{\chi^L_{n-2}(\lambda_{n,j})}{\sigma^L_{n-1}(\lambda_{n,n-1})} & \frac{\rho_{n-1}\sigma^L_{n-1}(\lambda_{n,n-1})}{\sigma^L_{n-1}(\lambda_{n,n-1})}\end{pmatrix}.$$

Similarly, the second matrix product in the left hand side of (5.2.18) gives the following $n\times(n-1)$ matrix

$$\mathcal{A}_{n\times(n-1)}=\begin{pmatrix}\frac{\sigma^R_0(\lambda_{n1})}{\mathfrak{c}_{n1}} & \frac{\sigma^R_0(\lambda_{n2})}{\mathfrak{c}_{n2}} & \cdots & \frac{\sigma^R_1(\lambda_{n,n-2})}{\mathfrak{c}_{n,n-2}} & \frac{\sigma^R_1(\lambda_{n,n-1})}{\mathfrak{c}_{n,n-1}}\\ \frac{\sigma^R_1(\lambda_{n1})}{\mathfrak{c}_{n1}} & \frac{\sigma^R_1(\lambda_{n2})}{\mathfrak{c}_{n2}} & \cdots & \frac{\sigma^R_1(\lambda_{n,n-2})}{\mathfrak{c}_{n,n-2}} & \frac{\sigma^R_1(\lambda_{n,n-1})}{\mathfrak{c}_{n,n-1}}\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ \frac{\sigma^R_{n-2}(\lambda_{n1})}{\mathfrak{c}_{n1}} & \frac{\sigma^R_{n-2}(\lambda_{n2})}{\mathfrak{c}_{n2}} & \cdots & \frac{\sigma^R_{n-2}(\lambda_{n,n-2})}{\mathfrak{c}_{n,n-2}} & \frac{\sigma^R_{n-2}(\lambda_{n,n-1})}{\mathfrak{c}_{n,n-1}}\\ \frac{\sigma^R_{n-1}(\lambda_{n1})}{\mathfrak{c}_{n1}} & \frac{\sigma^R_{n-1}(\lambda_{n2})}{\mathfrak{c}_{n2}} & \cdots & \frac{\sigma^R_{n-1}(\lambda_{n,n-2})}{\mathfrak{c}_{n,n-2}} & \frac{\sigma^R_{n-1}(\lambda_{n,n-1})}{\mathfrak{c}_{n,n-1}}\end{pmatrix},$$

where $\mathfrak{c}_{nj}=\tau_{n-1}q_{n+1}q_n^{-1}[\sigma^R_n(\lambda_{nj})]'$. Note that $\chi^L_i(\lambda)$, $i=1,2,\cdots,n-2$ is as defined in (5.2.8) while $\sigma^R_i(\lambda)$ is as defined in (5.2.5). It follows from (5.2.17) that the matrix relation $\mathcal{B}_{n-1\times n}\cdot\mathcal{A}_{n\times n-1}=I_{n-1}$ holds. The system of equations that results from $\mathcal{A}^T_{n-1\times n}\cdot\mathcal{B}^T_{n\times n-1}=I_{n-1}$ can be written as

$$\sum_{i=0}^{n-1}\sigma^R_i(\lambda_{n,j})\tilde{\chi}^L_i(\lambda_{n,k})\mu_{n,j,k}=\delta_{j,k},\quad j,k=1,\cdots,n-1,$$

which is (5.2.14), thus proving that the two finite sequences $\{\sigma_i^R(\lambda)\}_{i=0}^{n-1}$ and $\{\chi_k^L(\lambda)\}_{k=0}^{n-1}$ are biorthogonal to each other on the point set $\lambda = \lambda_{n,1}, \lambda_{n,2}, \cdots, \lambda_{n,n-1}$, which is the set of zeros of $\mathcal{Q}_n(\lambda)$ excluding the point $\lambda = 1$. □

If $\bigwedge$ denotes the space of Laurent polynomials, define a linear functional $\mathcal{N}$ on $\bigwedge \times \bigwedge$ as

$$\mathcal{N}_{n-1}^{(k,j)}[h_i(\lambda)\, g_i(\lambda)] = \sum_{i=0}^{n-1} h_i(\lambda_{n,j}) g_i(\lambda_{n,k}) \mu_{n,j,k}.$$

Then, we have the following result.

Theorem 5.2.3. *The sequence $\{\mathcal{Q}_n(\lambda)\}_{n=1}^{\infty}$ satisfy*

$$\mathcal{N}_{n-1}^{(k,j)}\left[\lambda_{n,k}^{-n+m}(\lambda-1)^{-1}\mathcal{Q}_i(\lambda)\right] = 0, \quad \mathcal{N}_{n-1}^{(k,k)}\left[\lambda_{n,k}^{-n+m}(\lambda-1)^{-1}\mathcal{Q}_i(\lambda)\right] \neq 0,$$

for $k = 1, \cdots, n-1$, and $m = 1, 2, \cdots, n$.

Proof. From Theorem 5.2.2, we have

$$\mathcal{N}_{n-1}^{(k,j)}[\sigma_i^R(\lambda)\tilde{\chi}_i(\lambda)] = \delta_{j,k}, \quad 0 \le i \le n-1, \quad 1 \le j,k \le n-1.$$

Using the definitions (5.2.8) of $\chi_k^L(\lambda)$, it is clear that

$$\{\tilde{\chi}_0^L(\lambda_{n,k}), \tilde{\chi}_1^L(\lambda_{n,k}), \cdots, \tilde{\chi}_{n-1}^L(\lambda_{n,k})\},$$

(where $\tilde{\chi}_j^L(\lambda) = \chi_j^L(\lambda)$, $j = 0 \cdots, n-2$ and $\tilde{\chi}_{n=1}(\lambda) = \rho_{n-1}\sigma_{n-1}^L(\lambda)$), forms a basis for the subspace of Laurent polynomials spanned by $\{1, \lambda_{n,k}^{-1}, \lambda_{n,k}^{-2}, \cdots, \lambda_{n,k}^{-n+1}\}$. Note that each fixed k yields $n-1$ such subspaces. Further, this implies

$$\mathcal{N}_{n-1}^{(k,j)}[\lambda_{n,k}^{-n+m}\sigma_i^R(\lambda)] = 0, \quad j,k = 0, 1, \cdots, n-1, \quad m = 1, 2, \cdots, n,$$

while $\mathcal{N}_{n-1}^{(k,j)}[\lambda_{n,k}^{-n+m}\sigma_i^R(\lambda)] \neq 0$, whenever $\sigma_i^R(\lambda)$ is evaluated at $\lambda = \lambda_{n,k}$. The theorem now follows from the fact that $\sigma_i^R(\lambda) = -[\prod_{j=0}^{i-1}\tau_j q_{i+1} q_1^{-1}(\lambda-1)]^{-1}\mathcal{Q}_i(\lambda)$. □

It may be observed that biorthogonality relations satisfied only up to a finite number of polynomials in a sequence are obtained in the present section. Such cases are considered in da Silva and Sri Ranga [45] and Zhedanov [191]. In general, the matrices $\mathcal{A}$ and $\mathcal{B}$ in such cases are non-singular $n \times n$ matrices and the biorthogonality relations are obtained through the matrix relation $\mathcal{AB} = \mathcal{I} \implies \mathcal{B}^{-1}\mathcal{A}^{-1} = \mathcal{I}$. In the present case, this is not applicable since the matrices involved are rectangular.

One way of representation is to find the matrix transpose as illustrated above. Another way can be the use of generalized inverses, for example, the Moore-Penrose inverse which always exists and is unique. We refer, for example, to Mishra and Sivakumar [133,134] and Kulkarni and Ramesh [110,111] for information on the Moore-Penrose inverse of a matrix.

5.3 Para-orthogonality from linear combinations

The orthogonality condition in Theorem 5.2.2 which required the removal of any common zeros of consecutive polynomials motivates us to study the sequence of polynomials in which the common zero $\lambda = 1$ has been removed. Hence, we consider the sequence $\{\mathcal{R}_n(\lambda)\}_{n=1}^{\infty}$ where $\mathcal{R}_0(\lambda) :=$ and $\mathcal{R}_n(\lambda) = \kappa_n^{-1}(\lambda-1)^{-1}\mathcal{Q}_{n+1}, n \geq 1$. We recall that κ_n is the leading coefficient of $\mathcal{Q}_{n+1}(\lambda)$, thus making $\mathcal{R}_n(\lambda)$ monic.

For para-orthogonality, we impose conditions on the parameters used in the recurrence relation (5.1.1) of R_I type. First, $-2 < \tau_n\rho_n^{-1} < 0$, $n \geq 0$. By a direct computation from (5.1.7a), we obtain $\mathcal{Q}_2(\lambda) = \kappa_2(\lambda-1)(\lambda + t_1\rho_1^{-1}q_1^{-1})$. Then, $\mathcal{Q}_2(\lambda)$ has the second zero at $\lambda = -1$ if $t_1 = \rho_1 q_1$ which is equivalent to $\alpha_1 = \tau_1\rho_1^{-1}\beta_0(1+\beta_0)^{-1}$. Since $\alpha_1 = \rho_0(\beta_0 - 1)$, this narrows down the choice of the initial values to satisfy $\tau_1\rho_1^{-1} = \rho_0\beta_0^{-1}(\beta_0^2 - 1)$. We note that such a choice does not contradict the inequalities (5.1.11) given at the end of Section 5.1 since $\beta_0 \neq \pm 1$.

Hence from (5.1.7b), $\mathcal{R}_n(\lambda)$, $n \geq 1$, satisfies the recurrence relation

$$\mathcal{R}_{n+1}(\lambda) = \left(\lambda - \frac{t_{n+1}}{r_{n+1}}\right)\mathcal{R}_n(\lambda) + \frac{\tau_n p_{n+2}}{\rho_n r_{n+1}}\lambda\mathcal{R}_{n-1}(\lambda), \quad n \geq 1, \tag{5.3.1}$$

with the initial conditions $\mathcal{R}_0(\lambda) = 1$ and $\mathcal{R}_1(\lambda) = \lambda + 1$. The following result imposes further restrictions on the choice of α_n for $\mathcal{R}_n(\lambda)$ to be a para-orthogonal polynomial.

Theorem 5.3.1. *Let the sequence $\{\alpha_n\}_{n=1}^{\infty}$ be so constructed from the recursive relation*

(5.1.6) *that, the following conditions*

$$\alpha_n\rho_n = (2\rho_n + \tau_n)(\rho_{n-1}\alpha_{n-1} - \tau_{n-1}) + \tau_n \quad \textit{and} \tag{5.3.2a}$$

$$\alpha_n\rho_n = -\rho_{n-1}\beta_{n-1}(2\rho_n + \tau_n)\alpha_{n-1}, \quad n \geq 2, \tag{5.3.2b}$$

where $\alpha_0 = \rho_0\tau_0^{-1}$ and $\alpha_1 = \rho_0(\beta_0 - 1)$ are also satisfied. Then $\{\mathcal{R}_n(\lambda)\}_{n=1}^{\infty}$ is a sequence of para-orthogonal polynomials.

Proof. The condition (5.3.2a) implies $(\alpha_n\rho_n - \tau_n)/(\alpha_{n-1}\rho_{n-1} - \tau_{n-1}) = (p_{n+1})/(p_n) = 2\rho_n + \tau_n$, $n \geq 2$, while (5.3.2b) implies $(\alpha_{n-1}\rho_{n-1}\beta_{n-1}(2\rho_n + \tau_n))/(\alpha_n\rho_n) = t_n/r_n = -1$, $n \geq 2$. Further, with $r_n = \rho_n p_n$, we obtain

$$\frac{p_{n+1}}{p_n} = 2\rho_n + \tau_n \Longrightarrow \frac{p_{n+1}}{r_n} - \frac{\tau_n}{\rho_n} = 2, \quad n \geq 2.$$

We, choose $\tau_n\rho_n^{-1} = -2(1 - m_n)$, $n \geq 1$, so that $2m_n = p_{n+1}r_n^{-1}$, $n \geq 2$, with $2m_1 = 2 + \tau_n\rho_n^{-1}$. Further, $0 < m_n < 1$ since $-2 < \tau_n\rho_n^{-1} < 0$, $n \geq 1$ and hence the recurrence relation (5.3.1) for $\mathcal{R}_n$ reduces to

$$\mathcal{R}_{n+1}(\lambda) = (\lambda + 1)\mathcal{R}_n(\lambda) - 4d_{n+1}\lambda\mathcal{R}_{n-1}(\lambda), \quad n \geq 1, \tag{5.3.3}$$

with $\mathcal{R}_0(\lambda) = 1$ and $\mathcal{R}_1(\lambda) = \lambda + 1$. Moreover

$$\frac{\tau_n p_{n+2}}{\rho_n r_{n+1}} = -4(1 - m_n)m_{n+1} \quad n \geq 1,$$

implies $d_{n+1} = (1 - m_n)m_{n+1}$ is a positive chain sequence (Ismail [90, Section 7.2]). The recurrence relation (5.3.3) is a particular case studied in Castillo et al. [37] and hence it follows that $\{\mathcal{R}_n(\lambda)\}_{n=1}^{\infty}$ is a sequence of para-orthogonal polynomials. □

The recurrence relation (5.3.3) is also considered in Chapter 3 in the context of complementary chain sequences. Further, from Castillo et al. [37, Theorems 4.1 and 5.2], we can state the following. There exists a non-trivial probability measure on the unit circle such that for $n \geq 1$,

$$\int_{\mathcal{C}} \zeta^{-n+k}\mathcal{R}_n(\zeta)(1 - \zeta)d\mu(\zeta) = 0, \quad k = 0, 1, \cdots, n - 1.$$

This implies that the linear combinations of R_I polynomials, $\mathcal{Q}_n(\lambda)$, $n \geq 1$, satisfy

$$\int_{\mathcal{C}} \zeta^{-n+k} \mathcal{Q}_n(\zeta) d\mu(\zeta) = 0, \quad k = 0, 1, \cdots, n-1.$$

Moreover, the monic polynomials

$$\Phi_n(\lambda) = \mathcal{R}_n(\lambda) - 2(1 - m_n)\mathcal{R}_{n-1}(\lambda) = \mathcal{R}_n(\lambda) + \tau_n \rho_n^{-1} \mathcal{R}_{n-1}(\lambda), \quad n \geq 1, \qquad (5.3.4)$$

are orthogonal polynomials on the unit circle with respect to the probability measure μ and $\alpha_{n-1} = -\overline{\Phi_n(0)} = -(1+\tau_n\rho_n^{-1})$, $n \geq 1$. The parameters α_{n-1} are called Verblunsky coefficients (Simon [156]) and, since $-2 < \tau_n\rho_n^{-1} < 0$, lie in the real interval $[-1, 1]$ in the present case.

The polynomials $\Phi_n(\lambda)$, $n \geq 1$, called as Szegő polynomials, also satisfy

$$\Phi_{n+1}(\lambda) = \left(\frac{\Phi_{n+1}(0)}{\Phi_n(0)} + \lambda \right) \Phi_n(\lambda) - \frac{(1 - |\Phi_n(0)|^2)\Phi_{n+1}(0)}{\Phi_n(0)} \lambda \Phi_{n-1}(\lambda), \quad n \geq 1,$$

which is a recurrence relation of R_I type. With the results obtained in this chapter, they can be viewed as R_I polynomials, that can further be expressed as a linear combination of consecutive polynomials of another class of R_I polynomials. Note that we have chosen $\alpha_n = \tau_n\rho_n^{-1}$, $n \geq 1$, thus verifying Corollary 5.1.1.

Remark 5.3.1. *Since Szegő polynomials can also be viewed as R_I polynomials, by definition (Jones et al. [101]), a para-orthogonal polynomial is always obtained as a linear combination of R_I polynomials satisfying appropriate three term recurrence relations. In fact, expressing the parameters ρ_n, β_n and τ_n in terms of α_n can lead to the identification of the class of R_I polynomials whose linear combinations with constant coefficients are para-orthogonal polynomials.*

5.4 A hypergeometric function of R_I type

Consider the recurrence relation of R_I type

$$\mathcal{P}_{n+1}(\lambda) = \frac{b+n}{c+n} \left(\lambda - \frac{b-c-n}{b+n} \right) \mathcal{P}_n(\lambda) - \frac{n}{c+n} \lambda \mathcal{P}_{n-1}(\lambda), \quad n \geq 1,$$

where $\mathcal{P}_0(\lambda) := 1$ and $\mathcal{P}_1(\lambda) := \frac{b}{c}(\lambda - \frac{b-c}{b})$. Comparing with the general recurrence relation (5.1.1) of R_I type, the parameters are given by

$$\rho_n = \frac{b+n}{c+n}, \quad \beta_n = \frac{b-c-n}{b+n}, \quad \tau_{n+1} = -\frac{n+1}{c+n+1}, \quad \gamma_n = 0, \quad n \geq 0,$$

with $\tau_0 \neq 0$. Further, $\mathcal{P}_n(\lambda) = F(-n, b; c; 1-\lambda)$, $n \geq 0$, which follows (Sri Ranga [162]) by substituting $a = -n$ in the contiguous relation

$$(a-c+1)F(a,b;c;\lambda) = (2a-c+2+(b-a-1)\lambda)F(a+1,b;c;\lambda) \\ + (a+1)(\lambda-1)F(a+2,b;c;\lambda).$$

We consider the linear combinations $\mathcal{Q}_n(\lambda) = \mathcal{P}_n(\lambda) + \alpha_n \mathcal{P}_{n-1}(\lambda)$, $n \geq 0$, with $\mathcal{Q}_0(\lambda) := 1$ and our first aim is to construct the sequence $\{\alpha_n\}_{n=0}^{\infty}$ so that $\mathcal{Q}_n(\lambda)$, $n \geq 1$, has a common zero at $\lambda = 1$. We start with the initial verifications (5.1.11) given at the end of Section 5.1 to check if $\mathcal{Q}_2(\lambda)$ has a double root at $\lambda = 1$.

Clearly, $c \neq 0$ in $F(-n, b; c; 1-\lambda)$ and so $\beta_0 \neq 1$. Since we also require $\beta_0 \neq 0$, and $\beta_0 \neq -1$, we exclude the relations $c = b$ and $c = 2b$ respectively. We further verify that $c \neq -1$ implies

$$-(b+1)^{-1} = \tau_1 \rho_1^{-1} \neq \rho_0 \beta_0^{-1}(1-\beta_0)^2 = c(b-c)^{-1},$$

which by (5.1.11) further implies that $\mathcal{Q}_2(\lambda)$ does not have a double root at $\lambda = 1$. This can also be verified by the fact that we choose

$$\alpha_1 = \rho_0(\beta_0 - 1) = \frac{b}{c}\left(\frac{b-c}{b} - 1\right) = -1,$$

so that α_1 is not a root of the quadratic equation (5.1.9). This follows from the inequality $\rho_1 \alpha_1^2 - \rho_0 \beta_0 \tau_1 = b/c \neq 0$. Further, from the recursive relation (5.1.6), we have $\alpha_n = -1$, $n \geq 1$. Since $\alpha_1 = \rho_0(\beta_0 - 1)$, this also means that $\mathcal{Q}_n(\lambda)$, $n \geq 1$, has a common zero at $\lambda = 1$ for the unique sequence $\alpha_n = -1$, $n \geq 1$. Moreover, as $b \neq 0$ and $c \neq 2b$,

$$-n(b+n)^{-1} = \tau_n \rho_n^{-1} \neq \alpha_n = -1 \qquad \text{and}$$

$$(b-c-n+1)(c+n-1)^{-1} = \rho_{n-1}\beta_{n-1} \neq \alpha_n = -1, \quad n \geq 1,$$

respectively. Hence, by Corollary 5.1.1, $\mathcal{Q}_n(\lambda)$, $n \geq 1$, is not a R_I polynomial. In fact, the linear combination

$$\mathcal{Q}_n(\lambda) = F(-n, b; c; 1-\lambda) - F(-n+1, b; c; 1-\lambda), \quad n \geq 1,$$

satisfies the mixed recurrence relations (5.1.7a) and (5.1.7b). With $\alpha_{n+1} = -1$, $\gamma_n = 0$, $n \geq 0$, the relations in Theorem 5.1.1 yield

$$\begin{aligned} -p_n = q_n &= \frac{b}{c+n-1}, \; r_n = -\frac{b(b+n)}{(c+n)(c+n-1)}, \; s_n = \frac{b(2b-c+1)}{(c+n-1)(c+n)}, \\ t_n &= -\frac{b(b-c-n+1)}{(c+n-1)(c+n)}, \; u_n = -v_n = \frac{(n-1)b}{(c+n-1)(c+n)}, \; w_n = 0, \quad n \geq 2. \end{aligned}$$

For $n = 1$, we have

$$\begin{aligned} p_1 &= -\left(\frac{b}{c} + \tau_0\right), \; q_1 = \frac{b}{c}, \; r_1 = -\frac{b+1}{c+1}\left(\frac{b}{c} + \tau_0\right), \; s_1 = \frac{\tau_0 b}{c+1} + \frac{b(2b-c+1)}{c(c+1)}, \\ t_1 &= -\frac{b(b-c)}{c(c+1)}, \; -u_1 = v_1 = \frac{\tau_0 b}{c+1}, \; w_1 = 0. \end{aligned}$$

Hence we need to choose $\tau_0 = -b/c$ so that $p_1 = r_1 = 0$. Then, from (5.1.7a) and (5.1.7b), we obtain the mixed recurrence relations

$$\begin{aligned} \mathcal{Q}_2(\lambda) &= \frac{b-c+1}{c+1}\left(\lambda - \frac{b-c}{b-c+1}\right)\mathcal{Q}_1(\lambda) + \frac{b}{c+1}\lambda(\lambda-1)\mathcal{Q}_0(\lambda), \\ \mathcal{Q}_{n+1}(\lambda) &= \frac{b+n}{c+n}\left(\lambda - \frac{b-c-n+1}{b+n}\right)\mathcal{Q}_n(\lambda) - \frac{n-1}{c+n}\lambda\mathcal{Q}_{n-1}(\lambda), \quad n \geq 2. \end{aligned}$$

The initial conditions are $\mathcal{Q}_0(\lambda) := 1$ and $\mathcal{Q}_1(\lambda) := \frac{b}{c}(\lambda - 1)$. Note that, since $\alpha_0 = \tau_0\rho_0^{-1}$, $\rho_0 = \frac{b}{c}$ gives $\alpha_0 = -1$. Moreover, from the power series representations for the hypergeometric functions, it can be easily proved that

$$\mathcal{Q}_0(\lambda) = 1, \quad \mathcal{Q}_n(\lambda) = \frac{b}{c}(\lambda-1)F(-n+1, b+1; c+1; 1-\lambda), \quad n \geq 1.$$

5.4.1 Biorthogonal hypergeometric functions

To discuss the biorthogonality relations obtained in Section 5.2 from $\mathcal{Q}_n(\lambda)$ in the hypergeometric settings, we first note that the rational functions defined in (5.2.5) are

$$\begin{aligned}\sigma_i^R(\lambda) &= -\left[(\lambda-1)\prod_{j=0}^{i-1}\tau_j q_{i+1}q_1^{-1}\right]^{-1}\mathcal{Q}_i(\lambda)\\ &= \frac{c+i}{c}\frac{(c+1)_{i-1}}{(-i+1)_{i-1}}F(-i+1,b+1;c+1;1-\lambda), \quad i\geq 2,\end{aligned}$$

with $\sigma_0^R(\lambda) = 1$ and $\sigma_1^R(\lambda) = (c+1)/c$. Further, from (5.2.8), we have $\chi_k^L(\lambda) = \rho_k\sigma_k^L(\lambda)+\sigma_{k+1}^L(\lambda)$, $k =\geq 2$, where $\sigma_k^L(\lambda) = (-\lambda)^{-k}\mathcal{Q}_k(\lambda)$, $k\geq 1$. Hence

$$\begin{aligned}\chi_k^L(\lambda) = (-1)^{k+1}\frac{b}{c}\frac{(\lambda-1)}{\lambda^{k+1}}[F(-k,&b+1;c+1;1-\lambda)\\ &-\frac{b+k}{c+k}\lambda F(-k+1,b+1;c+1;1-\lambda)].\end{aligned}$$

We find a closed form expression for $\chi_k^L(\lambda)$ using another contiguous relation

$$(a-c)F(a-1,b;c,\lambda)+(c-b)F(a,b-1;c;\lambda)+(\lambda-1)(a-b)F(a,b;c;\lambda) = 0,$$

which can be written as

$$\frac{c-b}{c-a}F(a+1,b;c+1;\lambda) = F(a,b+1;c+1;\lambda)-\frac{b-a}{c-a}(1-\lambda)F(a+1,b+1;c+1;\lambda).$$

Hence, with $a=-k$ and $\lambda\mapsto(1-\lambda)$, we obtain

$$\chi_k^L(\lambda) = (-1)^{k+1}\frac{b(c-b)}{c(c+k)}(\lambda-1)\lambda^{-k-1}F(-k+1,b;c+1;1-\lambda), \quad k\geq 2.$$

We also have $\chi_0^L(\lambda) = \rho_0\sigma_0^L(\lambda)+\sigma_1^L(\lambda) = \dfrac{b}{c}\lambda^{-1}$ and

$$\begin{aligned}\chi_1^L(\lambda) &= \tau_0q_2q_1^{-1}\sigma_0^L(\lambda)+s_1q_1^{-1}\sigma_1^L(\lambda)+\sigma_2^L(\lambda)\\ &= -\frac{b}{c(c+1)\lambda}[(b+1)\lambda+c-b-1]+\sigma_2^L(\lambda)\\ &= \frac{b}{c}\frac{\lambda-1}{\lambda^2}F(-1,b+1;c+1;1-\lambda)-\frac{b(b+1)}{c(c+1)}\frac{\lambda-1}{\lambda}-\frac{b}{(c+1)\lambda}.\end{aligned}$$

Note that we have written $\chi_1^L(\lambda)$ in the form $\chi_1^L(\lambda) = \sigma_2^L(\lambda) + \rho_1\sigma_1^L(\lambda) - \frac{b\lambda^{-1}}{(c+1)}$ and this can be further simplified to $\chi_1^L(\lambda) = \frac{-b\lambda^{-2}}{(c+1)}F(-1, b; c; 1-\lambda)$.

Let the weight function $\mu_{n,j,k}$ be the function μ_λ evaluated at the zeros $\lambda_{n,j}$ and $\lambda_{n,k}$, $j, k = 1, \cdots, n-1$ of $\mathcal{Q}_n(\lambda)$, that is, at the $n-1$ zeros of $F(-n+1, b+1; c+1; 1-\lambda)$ which clearly does not include the point $\lambda = 1$. We recall that

$$\mu_\lambda = \tau_{n-1}q_{n+1}q_n^{-1}[\sigma_n^R(\lambda)]'\sigma_{n-1}^L(\lambda), \quad n \geq 2.$$

Since

$$\begin{aligned}
\sigma_{n-1}^L(\lambda) &= (-1)^{n-1}\frac{b}{c}\lambda^{-n+1}(\lambda-1)F(-n+2, b+1; c+1; 1-\lambda) \quad \text{and} \\
[\sigma_n^R(\lambda)]' &= \frac{(c+n)(c+1)_{n-1}}{c(-n+1)_{n-1}}F'(-n+1, b+1; c+1; 1-\lambda) \\
&= -\frac{(b+1)(c+n)(c+2)_{n-2}}{c(-n+2)_{n-2}}F(-n+2, b+2; c+2; 1-\lambda),
\end{aligned}$$

we have

$$\mu_\lambda = \mathcal{Y}_{n-2}(\lambda)[F(-n+2, b+1; c+1; 1-\lambda) \times F(-n+2, b+2; c+2; 1-\lambda)]^{-1},$$

where

$$\mathcal{Y}_{n-2}(\lambda) = \frac{c^2(-n+2)_{n-2}\lambda^{n-1}}{(-1)^{n-1}(n-1)b(b+1)(c+2)_{n-2}(\lambda-1)}, \quad n \geq 2.$$

By Chu-Vandermonde formula (Ismail [90, p.12, (1.4.3)])

$$F(-n+1, b+1; c+1; 1) = \frac{(c-b)_{n-1}}{(c+1)_{n-1}} \neq 0.$$

This implies $\lambda = 0$ is not a zero of $F(-n+1, b+1; c+1; 1-\lambda)$. Hence $\mathcal{Y}_{n-2}(\lambda)$ never vanishes.

Hence by Theorem 5.2.2, we obtain two finite sequences $\{\sigma_i^R(\lambda)\}_{i=0}^{n-1}$ and $\{\tilde{\chi}_k^L(\lambda)\}_{k=0}^{n-1}$ of hypergeometric functions that are biorthogonal to each other with the weight function μ_λ, all quantities being evaluated at the $n-1$ zeros of $F(-n+1, b+1; c+1; 1-\lambda)$.

Here, $\tilde{\chi}_k^L(\lambda) = \chi_k^L(\lambda)$, $k = 0, 1, \cdots, n-2$ and

$$\begin{aligned}\tilde{\chi}_{n-1}^L(\lambda) &= \rho_{n-1}\sigma_{n-1}^L(\lambda)\\ &= (-1)^{n-1}\frac{b(b+n-1)}{c(c+n-1)}\lambda^{-n+1}(\lambda-1)F(-n+2, b+1; c+1; 1-\lambda).\end{aligned}$$

5.4.2 Para orthogonal polynomials with two representations

Now, we discuss the para-orthogonality of polynomials obtained from $\mathcal{Q}_n(\lambda)$ in the hypergeometric settings. With $\rho_n = \frac{b+n}{c+n}$, the leading coefficient of $\mathcal{Q}_{n+1}(\lambda)$ is $\kappa_n = \rho_n \cdots \rho_0 = \frac{(b)_{n+1}}{(c)_{n+1}}$. Hence, we consider the monic polynomials

$$\mathcal{R}_n(\lambda) = \frac{(c+1)_n}{(b+1)_n}F(-n, b+1, c+1, 1-\lambda), \quad n \geq 0,$$

and proceed to find conditions on b and c such that $\mathcal{R}_n(\lambda)$ is a para-orthogonal polynomial. The conditions obtained in the beginning of Section 5.3 are

$$-2 < \tau_n\rho_n^{-1} < 0 \quad \text{and} \quad \tau_1\rho_1^{-1} = \rho_0\beta_0^{-1}(\beta_0^2 - 1).$$

The first one requires $b > -n/2$ for $n \geq 1$ so that we have $b > -1/2$. The second one requires $c - b = (c - 2b)(b+1)$ which implies $c = 2b + 1$. It can be verified that the other conditions (5.3.2a) and (5.3.2b) in Theorem 5.3.1, that is

$$\begin{aligned}\alpha_n\rho_n &= (2\rho_n + \tau_n)(\rho_{n-1}\alpha_{n-1} - \tau_{n-1}) + \tau_n \quad \text{and}\\ \alpha_n\rho_n &= -\rho_{n-1}\beta_{n-1}(2\rho_n + \tau_n)\alpha_{n-1}, \quad n \geq 2,\end{aligned}$$

are also satisfied for $\alpha_n = -1$ and $c = 2b + 1$. Hence $\mathcal{R}_n(\lambda)$, $n \geq 1$, satisfies the recurrence relation

$$\mathcal{R}_{n+1}(\lambda) = (\lambda + 1)\mathcal{R}_n(\lambda) - 4d_{n+1}\lambda\mathcal{R}_{n-1}(\lambda), \quad n \geq 1,$$

which is (5.3.3) and when using (5.3.1), we have

$$d_{n+1} = -\frac{1}{4}\frac{\tau_n}{\rho_n}\frac{p_{n+2}}{r_{n+1}} = \frac{1}{4}\frac{n(2b+n+1)}{(b+n)(b+n+1)}, \quad n \geq 1.$$

The parameter sequence is given by $m_n = \frac{p_{n+1}}{2r_n} = \frac{2b+n}{2(b+n)}$, $n \geq 2$, with $m_1 = 1 + \frac{\tau_1}{2\rho_1} = \frac{2b+1}{2(b+1)}$. Further, the Szegő polynomials are given by

$$\begin{aligned}\Phi_n(\lambda) &= \mathcal{R}_n(\lambda) - 2(1-m_n)\mathcal{R}_{n-1}(\lambda) \\ &= \frac{(2b+2)_n}{(b+1)_n} F(-n, b; 2b+1; 1-\lambda) \\ &\quad - \frac{n}{(b+n)} \frac{(2b+2)_{n-1}}{(b+1)_{n-1}} F(-n+1, b; 2b+1; 1-\lambda), \quad n \geq 1,\end{aligned}$$

with the Verblunsky coefficients

$$\alpha_{n-1} = -(1 + \tau_n \rho_n^{-1}) = -\frac{b}{b+n}, \quad n \geq 1.$$

We make an observation. From results illustrated above, for $c = 2b+1$, the linear combination $\mathcal{Q}_n(\lambda)$ yields

$$\begin{aligned}&\frac{b}{2b+1}(\lambda-1)F(-n+1, b+1; 2b+2; 1-\lambda) \\ &\quad = F(-n, b; 2b+1; 1-\lambda) - F(-n+1, b; 2b+1; 1-\lambda), \quad n \geq 1. \qquad (5.4.1)\end{aligned}$$

Denoting the monic polynomials

$$\begin{aligned}\mathcal{R}_n(b;\lambda) &= \frac{(2b)_n}{(b)_n} F(-n, b, 2b, 1-\lambda), \qquad n \geq 1 \qquad \text{and} \\ \phi_n(b;\lambda) &= \frac{(2b+1)_n}{(b+1)_n} F(-n, b+1, 2b+1, 1-\lambda), \qquad n \geq 1,\end{aligned}$$

it has been proved (the concluding remarks in Sri Ranga [162]) that for $b > -1/2$, $\phi_n(b;\lambda)$ is a Szegő polynomial with respect to the weight function $[\sin\theta/2]^{2b}$ and in fact, is obtained from Gegengauer polynomials using the Szegő transformation. Further, the reversed Szegő polynomials $\phi_n^*(b;\lambda) = \lambda^n \overline{\phi_n(b; 1/\bar{\lambda})}$ are given by

$$\phi_n^*(b;\lambda) = \frac{(2b+1)_n}{(b+1)_n} F(-n, b; 2b+1; 1-\lambda).$$

Thus, multiplying both sides of (5.4.1) by $\frac{(2b+1)_{n+1}}{(b)_{n+1}}$, we observe that

$$\mathcal{R}_n(b+1;\lambda) = \frac{2b+n+2}{b}\left[\frac{\phi^*_{n+1}(b;\lambda) - \frac{2b+n+3}{b+n+1}\phi^*_n(b;\lambda)}{\lambda-1}\right], \quad n \geq 1. \tag{5.4.2}$$

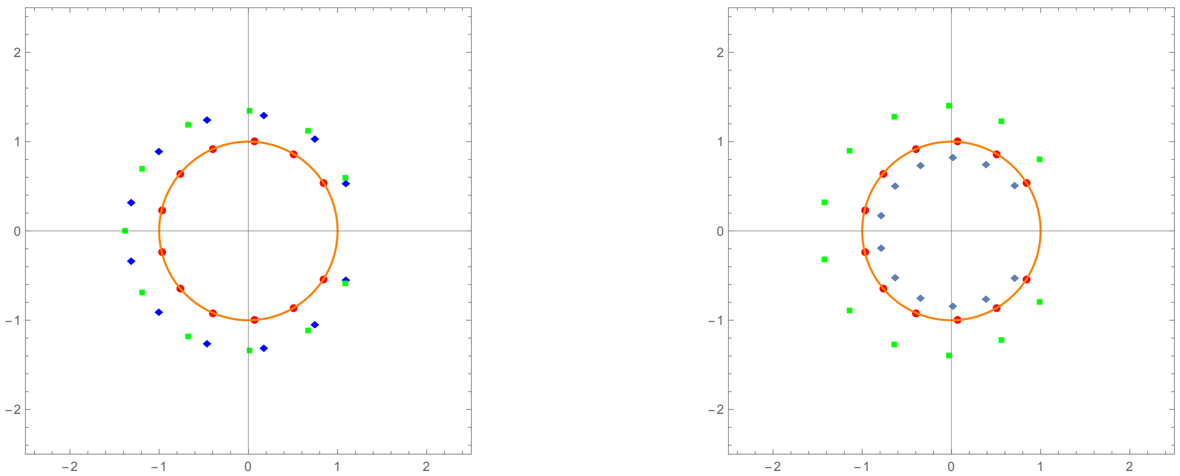
On the other hand, we obtain from Sri Ranga [162, Theorem 5.1]

$$\mathcal{R}_n(b+1;\lambda) = \left(\frac{2b+n}{b+n}\right)^{-1}[\phi_n(b+1;\lambda) + \phi^*_n(b+1;\lambda)], \quad n \geq 1, \tag{5.4.3}$$

which follows the usual definition of a para-orthogonal polynomial given in Jones et al. [101].

5.4.3 Graphical illustration

For the purpose of illustration, we plot the distribution of the zeros of $\mathcal{R}_n(b+1;\lambda)$, $\phi^*_n(b;\lambda)$, $\phi^*_{n-1}(b;\lambda)$, $\phi_n(b+1;\lambda)$ and $\phi^*_n(b+1;\lambda)$ for $n = 12$ and $b = 0.5$.



(a) The case (5.4.2) (b) The case (5.4.3)

Figure 5.1: The zeros of $\mathcal{R}_n(b+1;\lambda)$, $\phi_n(b+1;\lambda)$, $\phi^*_n(b+1;\lambda)$, $\phi^*_n(b;\lambda)$, $\phi^*_{n-1}(b;\lambda)$ for $n = 12$ and $b = 0.5$.

In figures (5.1a) and (5.1b), the red circular dots (●) are the zeros of $\mathcal{R}_{12}(1.5,\lambda)$ lying on the unit circle. The green squares (■) are the zeros of $\phi^*_{11}(0.5;\lambda)$ and $\phi^*_{12}(1.5;\lambda)$ respectively, while the blue diamonds (◆) are the zeros of $\phi^*_{12}(0.5;\lambda)$ and $\phi_{12}(1.5;\lambda)$ respectively.

5.5 Concluding remarks

Linear combinations of some particular cases of parameterized Gaussian hypergeometric polynomials like the Jacobi polynomials and Meixner-Pollaczek polynomials are studied in Johnston et al. [97]. In particular, the orthogonality and location of the zeros of these polynomials are studied for special values of the parameters. However, these are polynomials that are orthogonal on the real line and do not satisfy a recurrence relation of R_I type. The central idea of the chapter is to consider the linear combinations of polynomials that satisfy a recurrence relation of R_I type.

Further, it is interesting to note that a recurrence relation of R_{II} type arises in this discussion. This suggests that there is some sort of interplay between linear combinations of R_I polynomials and the R_{II} polynomials and the respective generalized eigenvalue problems. Expressing a R_{II} polynomial as a linear combination of two or more R_I polynomials can be of further research interest.

This motivates us to study the R_{II} polynomials and the related biorthogonality. Similar to the present chapter, we give an abstract construction of a sequence of orthogonal rational functions whose numerators satisfy recurrence relations of R_{II} type and study other consequences in the next chapter.

Chapter 6

Biorthogonal rational functions of R_{II} type

In the previous chapter, a sequence $\{\mathcal{Q}_n(z)\}$ of polynomials was constructed that satisfied biorthogonality properties. It was shown that $\mathcal{Q}_n(z)$, $n \geq 1$, satisfied mixed recurrence relations of R_I and R_{II} type. The R_{II} polynomials also appear as numerators of orthogonal rational functions, which are studied in the present chapter. Precisely, our aim is to construct a sequence $\{\varphi_n(z)\}_{n=0}^{\infty}$ of orthogonal rational functions that is biorthogonal to another sequence of rational functions. We obtain a generalized eigenvalue problem such that the numerators $r_n(z)$ of $\varphi_n(z)$, $n \geq 1$, are the characteristic polynomials, while $\varphi_n(z)$ form the components of the corresponding eigenvector. We also find a Christoffel type transform of the rational functions constructed, illustrating the differences with the available literature.

6.1 Fundamental spaces

Let $\{\alpha_j\}_{j=1}^{\infty}$ and $\{\beta_j\}_{j=1}^{\infty}$ be two given sequences where,

$$\alpha_j, \beta_j \in \mathbb{C} \setminus \{0\}, \quad j \geq 1. \tag{6.1.1}$$

We define

$$u_{2j}(z) := \frac{1}{1 - z\bar{\beta}_j}, \quad u_{2j+1}(z) := \frac{1}{z - \alpha_{j+1}}, \quad j \geq 0,$$

where $\beta_0 := 0$. The basis $\{u_j\}_{j=0}^{n}$, $n \geq 1$, is used to generate the complex linear spaces $\mathcal{L}_n$=span$\{u_0, u_1, \cdots, u_n\}$ and $\mathcal{L} = \cup_{n=0}^{\infty}\mathcal{L}_n$. Equivalently, $\mathcal{L}_n$= span$\{\mathfrak{u}_0, \mathfrak{u}_1, \cdots, \mathfrak{u}_n\}$, where

$$\mathfrak{u}_{2j}(z) = \frac{z^{2j}}{\prod_{k=1}^{j}(z - \alpha_k)\prod_{k=1}^{j}(1 - z\bar{\beta}_k)} \quad j \geq 0,$$
$$\mathfrak{u}_{2j+1}(z) = \frac{z^{2j+1}}{\prod_{k=1}^{j+1}(z - \alpha_k)\prod_{k=1}^{j}(1 - z\bar{\beta}_k)}, \quad j \geq 0.$$

Further, the product spaces $\mathcal{L}_m \cdot \mathcal{L}_n$ and $\mathcal{L} \cdot \mathcal{L}$ consist of functions of the form $h_{m,n}(z) = f_m(z)g_n(z)$ and $h(z) = f(z)g(z)$ respectively, where $f_m(z) \in \mathcal{L}_m$, $g_n(z) \in \mathcal{L}_n$ and $f(z), g(z) \in \mathcal{L}$.

The substar transform $h_*(z)$ of a function $h(z)$ is defined as $h_*(z) = \overline{h(1/\bar{z})}$. Let $\mathfrak{L}$ be a linear functional defined on $\mathcal{L} \cdot \mathcal{L}$ such that

$$\langle f(z), g(z) \rangle := \mathfrak{L}(f(z)g_*(z)), \tag{6.1.2}$$

is Hermitian and positive-definite, and hence defines an inner product on the space $\mathcal{L}$. We note that $\mathfrak{L}$ is said to be Hermitian if it satisfies $\mathfrak{L}(h_*) = \overline{\mathfrak{L}(\bar{h})}$ for every $h \in \mathcal{L} \cdot \mathcal{L}$ and positive definite if $\mathfrak{L}(hh_*) > 0$ for every $h \neq 0 \in \mathcal{L}$.

6.1.1 Associated rational functions

Let $\varphi_j(z)$, $j \geq 0$, be the sequence of functions that are orthonormal with respect to $\mathfrak{L}$ and obtained from the Gram-Schmidt process of the basis $\{\mathfrak{u}_j\}_{j=0}^{n}$, $n \geq 1$. That is $\varphi_j(z)$, $j \geq 0$, satisfy the orthogonality property

$$\langle \varphi_m(z), \varphi_n(z) \rangle = \mathfrak{L}(\varphi_m(z)\varphi_{n*}(z)) = \delta_{m,n}, \quad m, n = 0, 1, \cdots.$$

Further, it is clear that $\varphi_n(z)$ are rational functions of the form $\varphi_0(z) = 1$,

$$\begin{aligned} \varphi_{2j+2}(z) &= \frac{r_{2j+2}(z)}{\prod_{k=1}^{j+1}(z-\alpha_k)\prod_{k=1}^{j+1}(1-z\bar{\beta}_k)}, \quad j \geq 0, \\ \varphi_{2j+1}(z) &= \frac{r_{2j+1}(z)}{\prod_{k=1}^{j+1}(z-\alpha_k)\prod_{k=1}^{j}(1-z\bar{\beta}_k)}, \quad j \geq 0, \end{aligned} \tag{6.1.3}$$

where $r_n(z) \in \Pi_n$, the linear space of polynomials of degree at most n. Moreover, $\mathcal{L}_{2n}$ can now be interpreted as the space of rational functions having poles belonging to the set $\{\alpha_1, \cdots, \alpha_n, 1/\bar{\beta}_1, \cdots, 1/\bar{\beta}_n\}$ with the order of the pole at α_j or $1/\bar{\beta}_j$ depending on its multiplicity. The rational function $\varphi_{2n}(z) \in \mathcal{L}_{2n}$ has a simple pole at each of the points $\alpha_1, \cdots, \alpha_n, 1/\bar{\beta}_1, \cdots, 1/\bar{\beta}_n$. Here, α_j and β_j are as defined in (6.1.1). A similar interpretation for $\mathcal{L}_{2n+1}$ follows.

In fact, the regularity conditions in the present case can be obtained as follows. The expansion in terms of the basis elements gives

$$\varphi_{2n}(z) = A_0 + \frac{A_1 z}{z-\alpha_1} + \frac{A_2 z^2}{(z-\alpha_1)(1-z\bar{\beta}_1)} + \cdots + \frac{A_{2n} z^{2n}}{\prod_{i=1}^{n}(z-\alpha_i)\prod_{i=1}^{n}(1-z\bar{\beta}_i)},$$

so that $r_{2n}(z) = A_0 \prod_{i=1}^{n}(z-\alpha_i)\prod_{i=1}^{n}(1-z\bar{\beta}_i) + \cdots + A_{2n}$. Then $A_{2n} \neq 0$ if

$$r_{2n}(\alpha_n) \neq 0 \quad \text{and} \quad r_{2n}(1/\bar{\beta}_n) \neq 0. \tag{6.1.4}$$

Similarly, for $\varphi_{2n+1}(z)$, we obtain

$$r_{2n+1}(\alpha_{n+1}) \neq 0 \quad \text{and} \quad r_{2n+1}(1/\bar{\beta}_n) \neq 0. \tag{6.1.5}$$

The regularity conditions (6.1.4) and (6.1.5) are required to guarantee that $\varphi_{2n}(z) \in \mathcal{L}_{2n} \setminus \mathcal{L}_{2n-1}$ and $\varphi_{2n+1}(z) \in \mathcal{L}_{2n+1} \setminus \mathcal{L}_{2n}$ respectively.

6.1.2 Three term recurrence relations

Using the definition (6.1.2) and the properties of the inner product $\langle \cdot, \cdot \rangle$, the following result is immediate and will be used in deriving the recurrence relations for the orthogonal rational functions $\varphi_j(z)$.

Lemma 6.1.1. *Let $\gamma_n \in \mathbb{C}\setminus\{0\}$, $n = 1, 2, \cdots$. The following equality holds for the rational functions $f := f(z)$ and $g := g(z)$ in $\mathcal{L}$*

$$\left\langle \frac{1-z\bar{\gamma}_n}{z-\gamma_{n-1}} f, g \right\rangle = \left\langle f, \frac{z-\gamma_n}{1-z\gamma_{n-1}} g \right\rangle; \qquad \left\langle \frac{z-\gamma_{n+1}}{1-z\bar{\gamma}_n} f, g \right\rangle = \left\langle f, \frac{1-z\bar{\gamma}_{n+1}}{z-\gamma_n} g \right\rangle.$$

Proof. The first relation follows from the equality

$$\begin{aligned}\left\langle f(z), \frac{z-\gamma_n}{1-z\bar{\gamma}_{n-1}} g(z) \right\rangle &= \mathfrak{L}\left(f(z)\cdot \frac{1/z-\bar{\gamma}_n}{1-\gamma_{n-1}/z} \overline{g(1/\bar{z})}\right) = \mathfrak{L}\left(\frac{1-z\bar{\gamma}_n}{z-\gamma_{n-1}} f(z)\cdot \overline{g(1/\bar{z})}\right)\\ &= \left\langle \frac{1-z\bar{\gamma}_n}{z-\gamma_{n-1}} f(z), g(z) \right\rangle.\end{aligned}$$

Similarly, the second relation follows from

$$\begin{aligned}\left\langle f(z), \frac{1-z\bar{\gamma}_n}{z-\gamma_{n-1}} g(z) \right\rangle &= \mathfrak{L}\left(f(z)\cdot \frac{1/z-\bar{\gamma}_n}{1-\gamma_{n-1}/z} \overline{g(1/\bar{z})}\right) = \mathfrak{L}\left(\frac{1-z\bar{\gamma}_n}{z-\gamma_{n-1}} f(z)\cdot \overline{g(1/\bar{z})}\right)\\ &= \left\langle \frac{1-z\bar{\gamma}_n}{z-\gamma_{n-1}} f(z), g(z) \right\rangle,\end{aligned}$$

and the proof is complete. □

In addition to the regularity conditions (6.1.4) and (6.1.5) we also assume that the following

$$r_{2n}(\beta_{n-1}) \neq 0, \quad r_{2n}(1/\bar{\alpha}_n) \neq 0, \quad r_{2n+1}(\beta_n) \neq 0, \quad r_{2n+1}(1/\bar{\alpha}_n) \neq 0,$$

hold. Here and in what follows, we consider the sequences $\{\alpha_j\}$ and $\{\beta_j\}$ as defined in (6.1.1), unless specified otherwise.

Theorem 6.1.1. *The orthonormal rational functions $\{\boldsymbol{\phi}_n(\lambda)\}_{n=0}^{\infty}$, with $\boldsymbol{\phi}_{-1}(\lambda) := 0$ and $\boldsymbol{\phi}_0(\lambda) := 1$ satisfy the recurrence relations,*

$$\varphi_{2n+1}(z) = \left[\frac{e_{2n+1}}{z-\alpha_{n+1}} + \frac{d_{2n+1}(z-\beta_n)}{z-\alpha_{n+1}}\right]\varphi_{2n}(z) + c_{2n+1}\frac{1-z\bar{\alpha}_n}{z-\alpha_{n+1}}\varphi_{2n-1}(z), \tag{6.1.6a}$$

$$\varphi_{2n+2}(z) = \left[\frac{e_{2n+2}}{1-z\bar{\beta}_{n+1}} + \frac{d_{2n+2}(1-z\bar{\alpha}_{n+1})}{1-z\bar{\beta}_{n+1}}\right]\varphi_{2n+1}(z) + c_{2n+2}\frac{z-\beta_n}{1-z\bar{\beta}_{n+1}}\varphi_{2n}(z), \tag{6.1.6b}$$

for $n \geq 0$, where $\beta_0 := 0$, the constants $e_j, d_j \in \mathbb{C}$ and $c_j \in \mathbb{C}\setminus\{0\}$, $j \geq 0$.

Proof. Consider the function

$$\mathcal{W}_{2n}(z) = \frac{1 - z\bar{\beta}_n}{z - \beta_{n-1}}\varphi_{2n}(z) - \frac{a_{2n}}{z - \beta_{n-1}}\varphi_{2n-1}(z), \quad n \geq 1.$$

We first find the appropriate choice of a_{2n} for which $\mathcal{W}_{2n}(z) \in \mathcal{L}_{2n-1} \setminus \mathcal{L}_{2n-2}$. Using the rational forms (6.1.3) of $\varphi_{2n}(z)$ and $\varphi_{2n-1}(z)$, we have

$$\frac{1 - z\bar{\beta}_n}{z - \beta_{n-1}}\varphi_{2n}(z) = \frac{r_{2n}(z)}{(z - \beta_{n-1})\prod_{i=1}^{n}(z - \alpha_i)\prod_{i=1}^{n-1}(1 - z\bar{\beta}_i)} \qquad \text{and}$$

$$\frac{\varphi_{2n-1}(z)}{z - \beta_{n-1}} = \frac{r_{2n-1}(z)}{(z - \beta_{n-1})\prod_{i=1}^{n}(z - \alpha_i)\prod_{i=1}^{n-1}(1 - z\bar{\beta}_i)}.$$

Further, we can write the numerator polynomials $r_j(z)$, $j \geq 1$, as

$$r_{2n}(z) = (z - \beta_{n-1})q_{2n-1}(z) + r_{2n}(\beta_{n-1}),$$
$$r_{2n-1}(z) = (z - \beta_{n-1})q_{2n-2}(z) + r_{2n-1}(\beta_{n-1}),$$

which yields for $a_{2n} = r_{2n}(\beta_{n-1})/r_{2n-1}(\beta_{n-1}) \neq 0$, $n \geq 1$,

$$\mathcal{W}_{2n}(z) = \frac{q_{2n-1}(z)}{\prod_{i=1}^{n}(z - \alpha_i)\prod_{i=1}^{n-1}(1 - z\bar{\beta}_i)} - \frac{r_{2n}(\beta_{n-1})}{r_{2n-1}(\beta_{n-1})}\frac{q_{2n-2}(z)}{\prod_{i=1}^{n}(z - \alpha_i)\prod_{i=1}^{n-1}(1 - z\bar{\beta}_i)}.$$

This implies $\mathcal{W}_{2n}(z) \in \mathcal{L}_{2n-1} \setminus \mathcal{L}_{2n-2}$, so that we have

$$\mathcal{W}_{2n}(z) = b_{2n}\varphi_{2n-1}(z) + c_{2n}\varphi_{2n-2}(z) + \sum_{j=0}^{2n-3}\mathfrak{a}_j^{(2n)}\varphi_j(z),$$

where $\mathfrak{a}_j^{(2n)} = \langle \mathcal{W}_{2n}(z), \varphi_j(z)\rangle$, $j = 0, 1, \cdots, 2n-3$. We now proceed to prove that $\mathfrak{a}_j^{(2n)} = 0$ for $j = 0, 1, \cdots, 2n-3$, and $c_{2n} \neq 0$ which will lead to the required three term recurrence relation for $\varphi_{2n}(z)$. For this, we note that

$$\mathfrak{a}_j^{(2n)} = \left\langle \frac{1 - z\bar{\beta}_n}{z - \beta_{n-1}}\varphi_{2n}(z), \varphi_j(z)\right\rangle - \frac{r_{2n}(\beta_{n-1})}{r_{2n-1}(\beta_{n-1})}\left\langle \frac{\varphi_{2n-1}(z)}{z - \beta_{n-1}}, \varphi_j\right\rangle, \tag{6.1.7}$$

for $j = 0, 1, \cdots, 2n-3$. Since

$$\frac{z - \beta_n}{1 - z\bar{\beta}_{n-1}}\varphi_j \in \mathcal{L}_{2n-2} \quad \text{and} \quad \frac{z}{1 - z\bar{\beta}_{n-1}}\varphi_j \in \mathcal{L}_{2n-2},$$

for $j = 0, 1, \cdots, 2n-3$, using Lemma 6.1.1, we have

$$\left\langle \frac{1-z\bar{\beta}_n}{z-\beta_{n-1}}\varphi_{2n}(z), \varphi_j(z) \right\rangle = \left\langle \varphi_{2n}(z), \frac{z-\beta_n}{1-z\bar{\beta}_{n-1}}\varphi_j(z) \right\rangle = 0 \qquad \text{and}$$

$$\left\langle \frac{\varphi_{2n-1}(z)}{z-\beta_{n-1}}, \varphi_j \right\rangle = \left\langle \varphi_{2n-1}, \frac{z}{1-z\bar{\beta}_{n-1}}\varphi_j(z) \right\rangle = 0.$$

We conclude from (6.1.7) that $\mathfrak{a}_j^{(2n)} = 0$ for $j = 0, 1, \cdots, 2n-3$ and hence

$$\varphi_{2n}(z) = \left[\frac{a_{2n}}{1-z\bar{\beta}_n} + b_{2n}\frac{z-\beta_{n-1}}{1-z\bar{\beta}_n}\right]\varphi_{2n-1}(z) + c_{2n}\frac{z-\beta_{n-1}}{1-z\bar{\beta}_n}\varphi_{2n-2}(z), \quad n \geq 1.$$

However, we note that both $\{1, z-\beta_{n-1}\}$ and $\{1, 1-z\bar{\alpha}_n\}$ form a basis for Π_1 and hence writing $a_{2n} + b_{2n}(z-\beta_{n-1}) = e_{2n} + d_{2n}(1-z\bar{\alpha}_n)$, the recurrence relation (6.1.6b) follows.

To prove $c_{2n} \neq 0$, we multiply both sides of (6.1.6b) by

$$\frac{1-z\bar{\beta}_n}{\prod_{i=1}^{n}(1-z\bar{\alpha}_i)\prod_{i=1}^{n-1}(z-\beta_i)}.$$

The definition of the inner product (6.1.2) gives the left hand side as

$$\begin{aligned}
&\mathfrak{L}\left(\varphi_{2n}(z)\cdot\frac{1-z\bar{\beta}_n}{\prod_{i=1}^{n}(1-z\bar{\alpha}_i)\prod_{i=1}^{n-1}(z-\beta_i)}\right)\\
&= \mathfrak{L}\left(\varphi_{2n}(z)\cdot\frac{z(1/z-\bar{\beta}_n)}{z^{2n-1}\prod_{i=1}^{n}(1/z-\bar{\alpha}_i)\prod_{i=1}^{n-1}(1-\beta_i/z)}\right)\\
&= \left\langle \varphi_{2n}(z), \frac{(z-\beta_n)z^{2n-2}}{\prod_{i=1}^{n}(z-\alpha_i)\prod_{i=1}^{n-1}(1-z\bar{\beta}_i)} \right\rangle = 0,
\end{aligned}$$

since the second term in the last inner product belongs to $\mathcal{L}_{2n-1}$. With similar calculations, we obtain from (6.1.6b)

$$c_{2n}\left\langle \varphi_{2n-2}(z), \frac{z^{2n-2}}{\prod_{i=1}^{n}(z-\alpha_i)\prod_{i=1}^{n-2}(1-z\bar{\beta}_i)} \right\rangle + e_{2n}\left\langle \varphi_{2n-1}(z), \mathfrak{u}_{2n-1}(z) \right\rangle = 0,$$

which proves $c_{2n} \neq 0$, $n \geq 1$.

To derive the recurrence relation for $\varphi_{2n+1}(z)$, consider

$$\mathcal{W}_{2n+1}(z) = \frac{z-\alpha_{n+1}}{1-z\bar{\alpha}_n}\varphi_{2n+1}(z) - \frac{a_{2n+1}}{1-z\bar{\alpha}_n}\varphi_{2n}(z), \quad n \geq 0.$$

We find the appropriate value of a_{2n+1} so that $\mathcal{W}_{2n+1} \in \mathcal{L}_{2n} \setminus \mathcal{L}_{2n-1}$. Using the rational forms (6.1.3) of $\varphi_{2n}(z)$ and $\varphi_{2n+1}(z)$, we have

$$\frac{z-\alpha_{n+1}}{1-z\bar{\alpha}_n}\varphi_{2n+1}(z) = \frac{r_{2n+1}(z)}{(1-z\bar{\alpha}_n)\prod_{i=1}^{n}(z-\alpha_i)\prod_{i=1}^{n}(1-z\bar{\beta}_i)} \qquad \text{and}$$
$$\frac{\varphi_{2n}(z)}{1-z\bar{\alpha}_n} = \frac{r_{2n}(z)}{(1-z\bar{\alpha}_n)\prod_{i=1}^{n}(z-\alpha_i)\prod_{i=1}^{n}(1-z\bar{\beta}_i)}.$$

Writing the numerator polynomials $r_j(z)$ as

$$r_{2n+1}(z) = q_{2n}(z)(1-z\bar{\alpha}_n) + r_{2n+1}(1/\bar{\alpha}_n)$$
$$r_{2n}(z) = q_{2n-1}(z)(1-z\bar{\alpha}_n) + r_{2n}(1/\bar{\alpha}_n), \qquad n \geq 0,$$

we obtain for $a_{2n+1} = r_{2n+1}(1/\bar{\alpha}_n)/r_{2n}(1/\bar{\alpha}_n) \neq 0$,

$$\mathcal{W}_{2n+1}(z) = \frac{q_{2n}(z)}{\prod_{i=1}^{n}(z-\alpha_i)\prod_{i=1}^{n}(1-z\bar{\beta}_i)} - \frac{r_{2n+1}(1/\bar{\alpha}_n)}{p_{2n}(1/\bar{\alpha}_n)}\frac{r_{2n-1}(z)}{\prod_{i=1}^{n}(z-\alpha_i)\prod_{i=1}^{n}(1-z\bar{\beta}_i)},$$

so that $\mathcal{W} \in \mathcal{L}_{2n} \setminus \mathcal{L}_{2n-1}$, $n \geq 0$. Hence, we can write

$$\mathcal{W}_{2n+1}(z) = b_{2n+1}\varphi_{2n}(z) + c_{2n+1}\varphi_{2n-1}(z) + \sum_{j=0}^{2n-2}\mathfrak{a}_j^{(2n+1)}\varphi_j(z),$$

where $\mathfrak{a}_j^{(2n+1)} = \langle \mathcal{W}_{2n+1}(z), \varphi_j(z)\rangle$, $j = 0, 1, \cdots, 2n-2$. As in the case of $\varphi_{2n+1}(z)$, we show that $c_{2n+1} \neq 0$ and $\mathfrak{a}_j^{(2n+1)} = 0$ for $j = 0, 1, \cdots, 2n-2$ to obtain the required three term recurrence relation for $\varphi_{2n}(z)$. Using Lemma 6.1.1, we have

$$\begin{aligned}\mathfrak{a}_j^{(2n+1)} &= \left\langle \frac{z-\alpha_{n+1}}{1-z\bar{\alpha}_n}\varphi_{2n+1}(z), \phi_j(z)\right\rangle - \frac{r_{2n+1}(1/\bar{\alpha}_n)}{r_{2n}(1/\bar{\alpha}_n)}\left\langle \frac{\varphi_{2n}(z)}{1-z\bar{\alpha}_n}, \varphi_j(z)\right\rangle \\ &= \left\langle \varphi_{2n+1}, \frac{1-z\bar{\alpha}_{n+1}}{z-\alpha_n}\varphi_j(z)\right\rangle - \frac{r_{2n+1}(1/\bar{\alpha}_n)}{r_{2n}(1/\bar{\alpha}_n)}\left\langle \varphi_{2n}(z), \frac{z}{z-\alpha_n}\varphi_j(z)\right\rangle \\ &= 0, \qquad j = 0, 1, \cdots, 2n-2,\end{aligned}$$

where the last equality follows from the fact that

$$\frac{1 - z\bar{\alpha}_{n+1}}{z - \alpha_n}\varphi_j(z) \in \mathcal{L}_{2n-1} \qquad \text{and} \qquad \frac{z}{z - \alpha_n}\varphi_j(z) \in \mathcal{L}_{2n-1},$$

for $j = 0, 1, \cdots, 2n - 2$. Hence we obtain the recurrence relation

$$\varphi_{2n+1}(z) = \left[\frac{a_{2n+1}}{z - \alpha_{n+1}} + b_{2n+1}\frac{1 - z\bar{\alpha}_n}{z - \alpha_{n+1}}\right]\varphi_{2n}(z) + c_{2n+1}\frac{1 - z\bar{\alpha}_n}{z - \alpha_{n+1}}\varphi_{2n-1}(z), \quad n \geq 0,$$

which can also be written as (6.1.6a) since $\{1, 1 - z\bar{\alpha}_n\}$ and $\{1, z - \beta_n\}$ both span the linear space Π_1.

To prove $c_{2n+1} \neq 0$, we multiply both sides of the recurrence relation (6.1.6a) by

$$\frac{(z - \alpha_{n+1})}{\prod_{i=1}^{n}(1 - z\bar{\alpha}_i)\prod_{i=1}^{n}(z - \beta_i)}.$$

As in the case for c_{2n}, the inner product (6.1.2) and Lemma 6.1.1 gives

$$c_{2n+1}\left\langle \varphi_{2n-1}(z), \frac{z^{2n-1}}{\prod_{i=1}^{n-1}(z - \alpha_i)\prod_{i=1}^{n}(1 - z\bar{\beta}_i)}\right\rangle + e_{2n+1}\langle\varphi_{2n}(z), \mathfrak{u}_{2n}(z)\rangle = 0,$$

from which it follows that $c_{2n+1} \neq 0$, $n \geq 1$. □

The numerator polynomials of orthogonal rational functions satisfy the recurrence relations of R_{II} type. Indeed, from (6.1.6a) and (6.1.6b), it can be shown that

$$r_{2n+1}(z) = [e_{2n+1} + d_{2n+1}(z - \beta_n)]r_{2n}(z) + c_{2n+1}(1 - z\bar{\alpha}_n)(1 - z\bar{\beta}_n)r_{2n-1}(z), \quad (6.1.8a)$$

$$r_{2n+2}(z) = [e_{2n+2} + d_{2n+2}(1 - z\bar{\alpha}_{n+1})]r_{2n+1}(z) + c_{2n+2}(z - \alpha_{n+1})(z - \beta_n)r_{2n}(z), \quad (6.1.8b)$$

for $n \geq 0$, where we define $r_0(z) := 1$ and $\beta_0 := 0$. We use (6.1.8a) and (6.1.8b) to obtain a generalized eigenvalue problem such that the zeros of $r_j(z)$, $j \geq 1$, are the eigenvalues (that is, $r_j(z)$ is the characteristic polynomial) while the corresponding rational functions are the components of the corresponding eigenvector.

6.1.3 The rational functions as components of an eigenvector

Consider two infinite matrices $\mathcal{H} = (h_{i,k})_{i,k\geq 0}^{\infty}$ and $\mathcal{G} = (g_{i,k})_{i,k\geq 0}^{\infty}$, where

$$\mathcal{H} = \begin{pmatrix} d_1 & g_1 & 0 & 0 & \cdots \\ h_{1,0} & -d_2\bar{\alpha}_1 & g_2 & 0 & \cdots \\ 0 & h_{2,1} & d_3 & g_3 & \cdots \\ 0 & 0 & h_{3,2} & -d_4\bar{\alpha}_2 & \cdots \\ 0 & 0 & 0 & h_{4,3} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

$$\mathcal{G} = \begin{pmatrix} -e_1+\beta_0 d_1 & \alpha_1 g_1 & 0 & 0 & \cdots \\ h_{1,0}\beta_0 & -e_2-d_2 & \bar{\alpha}_1 g_2 & 0 & \cdots \\ 0 & h_{2,1}/\bar{\alpha}_1 & -e_3+\beta_1 d_3 & \alpha_2 g_3 & \cdots \\ 0 & 0 & h_{3,2}\beta_1 & -e_4-d_4 & \cdots \\ 0 & 0 & 0 & h_{4,3}/\bar{\alpha}_2 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

with $g_{2k+2} = -c_{2k+3}\bar{\beta}_{k+1}/h_{2k+2,2k+1}$, $g_{2k+1} = -c_{2k+2}/h_{2k+1,2k}$, $k \geq 0$. Here, α_j, β_j, e_j, d_j and c_j are the constants appearing in the recurrence relations (6.1.8a) and (6.1.8b) while $\{h_{i,i-1}\}_{i=1}^{\infty}$ is a sequence of arbitrary non-vanishing complex numbers. The following result is well-known, for example, see Ismail and Sri Ranga [94, Theorem 1.1] and Zhedanov [192].

Proposition 6.1.1. [94, 192] *Let $\mathcal{H}_j$ and $\mathcal{G}_j$ denote the j^{th} principal minors of $\mathcal{H}$ and $\mathcal{G}$ respectively. Then $(-1)^j r_j(\lambda)$, $j \geq 1$, is the characteristic polynomial of the generalized eigenvalue problem*

$$\mathcal{G}_j \boldsymbol{\varrho}_j = \lambda \mathcal{H}_j \boldsymbol{\varrho}_j, \tag{6.1.9}$$

where $\{r_j\}$ satisfies (6.1.8a) *and* (6.1.8b).

The generalized eigenvalue problem (6.1.9) has $j-1$ free variables $h_{i,i-1}$ which shows that the matrix pencil associated with the recurrence relations of R_{II} type is not unique. We now assign appropriate values to these free variables to obtain an eigenvector $\boldsymbol{\varrho}_j$.

Theorem 6.1.2. *Let the terms of the sequence* $\{h_{i,i-1}\}_{i=1}^{\infty}$ *be assigned the values*

$$h_{2i,2i-1} = -c_{2i+1}\bar{\alpha}_i, \quad h_{2i-1,2i-2} = c_{2i}, \quad i \geq 1.$$

Then, $\boldsymbol{\varrho}_j = \begin{pmatrix} \varphi_0 & \varphi_1 & \cdots & \varphi_j \end{pmatrix}^T$ *is the eigenvector of the generalized eigenvalue problem* (6.1.9) *corresponding to the eigenvalue which is a zero of* $r_j(\lambda)$.

Proof. Upon substitution of the values of $h_{i,i-1}$, the recurrence relations (6.1.6a) and (6.1.6b) can be written as $(-e_1 + d_1\beta_0)\varphi_0 - \alpha_1\varphi_1 = z[d_1\varphi_0 - \varphi_1]$ and

$$\begin{aligned} -c_{2k+3}\varphi_{2k+1} - (e_{2k+3} - d_{2k+3}\beta_{k+1})\varphi_{2k+2} - \alpha_{k+2}\varphi_{2k+3} \\ = z[-c_{2k+3}\bar{\alpha}_{k+1}\varphi_{2k+1} + d_{2k+3}\varphi_{2k+2} - \varphi_{2k+3}], \\ \beta_k c_{2k+2}\varphi_{2k} - (e_{2k+2} + d_{2k+2})\varphi_{2k+1} + \varphi_{2k+2} \\ = z[c_{2k+2}\varphi_{2k} - d_{2k+2}\bar{\alpha}_{k+1}\varphi_{2k+1} + \bar{\beta}_{k+1}\varphi_{2k+2}], \end{aligned}$$

for $k \geq 0$, which can be rearranged to yield the matrix equations

$$\begin{aligned} \mathcal{G}_{2n}\boldsymbol{\varrho}_{2n} &= z\mathcal{H}_{2n}\boldsymbol{\varrho}_{2n} - (z - \beta_n)\varphi_{2n}\boldsymbol{e}_{2n}, \\ \mathcal{G}_{2n+1}\boldsymbol{\varrho}_{2n+1} &= z\mathcal{H}_{2n+1}\boldsymbol{\varrho}_{2n+1} - (z - \alpha_{n+1})\varphi_{2n+1}\boldsymbol{e}_{2n+1}, \end{aligned}$$

where $\boldsymbol{e}_j$ is the j^{th} column of the unit matrix. Observing the fact that $(z - \beta_n)\varphi_{2n}$ does not vanish for $z = \beta_n$, $\boldsymbol{\varrho}_{2j}$ becomes an eigenvector for the generalized eigenvalue problem (6.1.9) with the zeros of $r_{2n}(z)$ as eigenvalues. Similarly, $\boldsymbol{\varrho}_{2j+1}$ becomes an eigenvector with the zeros of $r_{2n+1}(z)$ as eigenvalues and the proof is complete. □

Theorems 6.1.1 and 6.1.2 serve the first step of our construction. That is, we have obtained a sequence of rational functions that is orthogonal with respect to the linear functional $\mathfrak{L}$. These rational functions are also the components of the eigenvector of a matrix pencil whose characteristic polynomials are the numerator polynomials of such rational functions. In the next section, we will discuss the biorthogonality properties of $\{\varphi_n(z)\}$. At this point, we make the following remark about the recurrence relations (6.1.8a) and (6.1.8b) satisfied by these rational functions.

6.2 A biorthogonality relation for the rational functions

In the present section, we use the recurrence relations (6.1.8a) and (6.1.8b) obtained in Section 6.1 to define biorthogonality relations involving the orthogonal rational functions $\{\varphi_j\}$. To start with, we introduce the rational functions $\mathcal{O}_0(z) = 1$ and

$$\begin{aligned}\mathcal{O}_{2n+1}(z) &= \frac{r_{2n+1}(z)}{\prod_{j=1}^{n+1}(z-\alpha_j)\prod_{j=1}^{n}(1-z\bar{\alpha}_j)\prod_{j=0}^{n}(z-\beta_j)\prod_{j=1}^{n}(1-z\bar{\beta}_j)},\\ \mathcal{O}_{2n+2}(z) &= \frac{r_{2n+2}(z)}{\prod_{j=1}^{n+1}(z-\alpha_j)\prod_{j=1}^{n+1}(1-z\bar{\alpha}_j)\prod_{j=0}^{n}(z-\beta_j)\prod_{j=1}^{n+1}(1-z\bar{\beta}_j)}.\end{aligned} \tag{6.2.1}$$

for $n \geq 0$. Here $\{r_j\}$ satisfies (6.1.8a) and (6.1.8b) so that the sequence $\{\mathcal{O}_j(z)\}$ satisfies

$$(z-\alpha_{n+1})(z-\beta_n)\mathcal{O}_{2n+1}(z) = [e_{2n+1} + d_{2n+1}(z-\beta_n]\mathcal{O}_{2n}(z) + c_{2n+1}\mathcal{O}_{2n-1}(z),$$
$$(1-z\bar{\alpha}_n)(1-z\bar{\beta}_n)\mathcal{O}_{2n}(z) = [e_{2n} + d_{2n}(1-z\bar{\alpha}_n)]\mathcal{O}_{2n-1}(z) + c_{2n}\mathcal{O}_{2n-2}(z),$$

for $n \geq 1$. Then, similar to Theorem 3.5 and its following corollary of Ismail and Masson [91], we have

Theorem 6.2.1. *Consider the rational functions given by (6.2.1). Then there exists a linear functional $\mathfrak{N}$ on the span of rational functions $\{z\mathcal{O}_n(z)\}$ such that the orthogonality relation*

$$\mathfrak{N}(z^k\mathcal{O}_n(z)) = 0, \quad k = 0, 1, \cdots, n-1,$$

holds. Further, if $\mathfrak{N}(1) = m_0$, $\mathfrak{N}(z^n\mathcal{O}_n(z)) = m_n$, $n \geq 1$, then

$$\begin{aligned}\bar{\alpha}_n\bar{\beta}_n m_{2n} + d_{2n}\bar{\alpha}_n m_{2n-1} - c_{2n}m_{2n-2} &= 0, \quad n \geq 1\\ m_{2n+1} - d_{2n+1}m_{2n} - c_{2n+1}m_{2n-1} &= 0, \quad n \geq 1.\end{aligned} \tag{6.2.2}$$

We will also need the following relations among the leading coefficients of the polynomials $\{r_j(z)\}$, $j \geq 1$. If $r_j = \kappa_j z^j$ + lower order terms, then from (6.1.8a) and (6.1.8b),

$$\begin{aligned} \kappa_{2n} + d_{2n}\bar{\alpha}_n\kappa_{2n-1} - c_{2n}\kappa_{2n-2} &= 0 \quad n \geq 1, \\ \kappa_{2n+1} - d_{2n+1}\kappa_{2n} - \bar{\alpha}_n\bar{\beta}_n c_{2n+1}\kappa_{2n-1} &= 0 \quad n \geq 1. \end{aligned} \tag{6.2.3}$$

It is clear that each of the the recurrence relations (6.2.2) and (6.2.3) involve two arbitrary initial values. We choose m_0 and m_1 such that $m_1 \neq d_1 m_0$. Since $\kappa_0 = 1$ and $\kappa_1 = d_1$, this implies $\kappa_0 m_1 - \kappa_1 m_0 \neq 0$.

Consider another sequence of rational functions $\{\tilde{\varphi}_j(z)\}_{j=0}^{\infty}$ where $\tilde{\varphi}_0(z) := 1$,

$$\begin{aligned} \tilde{\varphi}_{2n+1}(z) &= \frac{r_{2n+1}(z)}{\prod_{j=1}^{n}(1 - z\bar{\alpha}_j)\prod_{j=0}^{n}(z - \beta_j)} \quad \text{and} \\ \tilde{\varphi}_{2n+2}(z) &= \frac{r_{2n+2}(z)}{\prod_{j=1}^{n+1}(1 - z\bar{\alpha}_j)\prod_{j=0}^{n}(z - \beta_j)}, \end{aligned} \tag{6.2.4}$$

for $n \geq 0$. Here $\{r_j(z)\}$ satisfy (6.1.8a) and (6.1.8b). Let $\tilde{\mathcal{J}}_m(z) = \chi_m^{-1}\tilde{\varphi}_m(z)$, where $\chi_{2m} = \bar{\alpha}_1(\bar{\beta}_1)^{-1}\cdots\bar{\alpha}_m(\bar{\beta}_m)^{-1}$ and $\chi_{2m+1} = \bar{\alpha}_1(\bar{\beta}_1)^{-1}\cdots\bar{\alpha}_m(\bar{\beta}_m)^{-1}\bar{\alpha}_{m+1}$. Define

$$\begin{aligned} \tilde{\psi}_{2j}(z) &:= \frac{c_{2j+1}(\bar{\beta}_j)^2}{\bar{\alpha}_{j+1}}\tilde{\mathcal{J}}_{2j-1}(z) - \frac{d_{2j+1}}{\bar{\alpha}_{j+1}}\tilde{\mathcal{J}}_{2j}(z) + \tilde{\mathcal{J}}_{2n+1}(z), \quad n \geq 1, \\ \tilde{\psi}_{2j+1}(z) &:= \frac{c_{2j+2}\bar{\beta}_{j+1}}{\bar{\alpha}_{j+1}}\tilde{\mathcal{J}}_{2j}(z) - d_{2j+2}\bar{\alpha}_{j+1}\bar{\beta}_{j+1}\tilde{\mathcal{J}}_{2j+1}(z) + \bar{\alpha}_{j+1}\tilde{\mathcal{J}}_{2j+2}(z), \quad n \geq 0, \end{aligned}$$

with $\tilde{\psi}_0(z) := 1$. The following theorem gives the biorthogonality relations for $\varphi(z)$ constructed in the previous section.

Theorem 6.2.2. *The sequences of rational functions $\{\varphi_j(z)\}$ and $\{\tilde{\psi}_j(z)\}$ satisfy the following biorthogonality relations*

$$\mathfrak{N}(\varphi_{2n}(z)\cdot\tilde{\psi}_m(z)) = \frac{c_2c_3\cdots c_{2n+1}(m_1\kappa_0 - m_0\kappa_1)}{\chi_{2n+1}}\delta_{2n,m}, \tag{6.2.5}$$

$$\mathfrak{N}(\varphi_{2n+1}(z)\cdot\tilde{\psi}_m(z)) = \frac{c_2c_3\cdots c_{2n+2}(m_1\kappa_0 - m_0\kappa_1)}{\chi_{2n+2}}\delta_{2n+1,m}, \tag{6.2.6}$$

where $m_j = \mathfrak{N}(z^jO_j(z))$ and κ_j is the leading coefficient of $r_j(z)$.

Proof. For simplicity, we write $\varphi_j := \varphi_j(z)$ and similar notations follow for others. We divide the proof into the following cases. First, let $m < 2n$ and m has even value, say $m = 2j$. Then

$$\begin{aligned}
&\mathfrak{N}(\varphi_{2n} \cdot \tilde{\psi}_m) = \mathfrak{N}(\varphi_{2n} \cdot \tilde{\psi}_{2j}) \\
&\qquad = \frac{c_{2j+1}\bar{\beta}_j}{\bar{\alpha}_{j+1}}\mathfrak{N}(\varphi_{2n} \cdot \tilde{J}_{2j-1}) - \frac{d_{2j+1}}{\bar{\alpha}_{j+1}}\mathfrak{N}(\varphi_{2n} \cdot \tilde{J}_{2j}) + \mathfrak{N}(\varphi_{2n} \cdot \tilde{J}_{2j+1}).
\end{aligned}$$

We evaluate the first term. We have $\mathfrak{N}(\varphi_{2n} \cdot \tilde{J}_{2j-1})$

$$\begin{aligned}
&= \frac{1}{\chi_{2j-1}}\mathfrak{N}\left(\frac{r_{2n}}{\prod_{k=1}^{n}(z-\alpha_k)\prod_{k=1}^{n}(1-z\bar{\beta}_k)} \cdot \frac{r_{2j-1}}{\prod_{k=1}^{j-1}(1-z\bar{\alpha}_k)\prod_{k=0}^{j-1}(z-\beta_k)}\right) \\
&= \frac{1}{\chi_{2j-1}}\mathfrak{N}(\mathcal{O}_{2n} \cdot r_{2j-1}(1-z\bar{\alpha}_j)\cdots(1-z\bar{\alpha}_n)(z-\beta_j)\cdots(z-\beta_{n-1})) \\
&= \frac{(-\bar{\alpha}_j)\cdots(-\bar{\alpha}_n)\kappa_{2j-1}}{\chi_{2j-1}}m_{2n}.
\end{aligned}$$

A similar evaluation of the remaining two terms yields

$$\begin{aligned}
\mathfrak{N}(\varphi_{2n} \cdot \tilde{J}_{2j}) &= \frac{(-\bar{\alpha}_{j+1})\cdots(-\bar{\alpha}_n)\kappa_{2j}}{\chi_{2j}}m_{2n}, \\
\mathfrak{N}(\varphi_{2n} \cdot \tilde{J}_{2j+1}) &= \frac{(-\bar{\alpha}_{j+1})\cdots(-\bar{\alpha}_n)\kappa_{2j+1}}{\chi_{2j+1}}m_{2n}.
\end{aligned}$$

Using the relations (6.2.3), we obtain $\mathfrak{N}(\varphi_{2n}(z) \cdot \tilde{\psi}_m(z)) = 0$ for $m = 2j < 2n$.

In the second case, let $m > 2n$ and m has odd value, say $m = 2j+1$. Then

$$\begin{aligned}
&\mathfrak{N}(\varphi_{2n} \cdot \tilde{\psi}_m) \\
&= \frac{c_{2j+2}\bar{\beta}_{j+1}}{\bar{\alpha}_{j+1}}\mathfrak{N}(\varphi_{2n} \cdot \tilde{\mathcal{J}}_{2j}) - d_{2j+2}\bar{\alpha}_{j+1}\bar{\beta}_{j+1}\mathfrak{N}(\varphi_{2n} \cdot \tilde{\mathcal{J}}_{2j+1}) + \bar{\alpha}_{j+1}\mathfrak{N}(\varphi_{2n} \cdot \tilde{\mathcal{J}}_{2j+2}),
\end{aligned}$$

so that, as in the case of $\tilde{\psi}_{2j}(z)$, we have

$$\begin{aligned}
\mathfrak{N}(\varphi_{2n} \cdot \tilde{\mathcal{J}}_{2j+2}) &= \frac{\kappa_{2n}m_{2j+2}}{\chi_{2j+2}}, \quad \mathfrak{N}(\varphi_{2n}(z) \cdot \tilde{\mathcal{J}}_{2j}(z)) = \frac{\kappa_{2n}m_{2j}}{\chi_{2j}}, \\
\mathfrak{N}(\varphi_{2n}(z) \cdot \tilde{\mathcal{J}}_{2j+1}(z)) &= \frac{\kappa_{2n}m_{2j+1}}{\chi_{2j+1}}.
\end{aligned}$$

Hence, using (6.2.2) we have $\mathfrak{N}(\varphi_{2n}(z) \cdot \tilde{\psi}_m(z)) = 0$ for $m = 2j+1 > 2n$.

In the third case, we prove the biorthogonality relations (6.2.5) and (6.2.6). For

$m = 2n$, we obtain

$$\mathfrak{N}(\varphi_{2n}(z) \cdot \tilde{\psi}_{2n}(z)) = \frac{1}{\chi_{2n+1}}(\kappa_{2n}m_{2n+1} - d_{2n+1}\kappa_{2n}m_{2n} - c_{2n+1}\bar{\beta}_n\bar{\alpha}_n\kappa_{2n-1}m_{2n}).$$

From (6.2.2), we find that $m_{2n+1}\kappa_{2n} - d_{2n+1}\kappa_{2n}m_{2n} = c_{2n+1}m_{2n-1}\kappa_{2n}$, so that

$$\mathfrak{M}(\varphi_{2n}(z) \cdot \tilde{\psi}_{2n}(z)) = \frac{c_{2n+1}}{\chi_{2n+1}}(\kappa_{2n}m_{2n-1} - \bar{\alpha}_n\bar{\beta}_n\kappa_{2n-1}m_{2n}).$$

To simplify the numerator in the right hand side above, we note from (6.2.2) and (6.2.3) that the following relations

$$\begin{aligned} \kappa_{2n}m_{2n-1} - \bar{\alpha}_n\bar{\beta}_n\kappa_{2n-1}m_{2n} &= c_{2n}(m_{2n-1}\kappa_{2n-2} - m_{2n-2}\kappa_{2n-1}), \\ \kappa_{2n-2}m_{2n-1} - \kappa_{2n-1}m_{2n-2} &= c_{2n-1}(m_{2n-3}\kappa_{2n-2} - \bar{\alpha}_{n-1}\bar{\beta}_{n-1}m_{2n-2}\kappa_{2n-3}), \end{aligned} \tag{6.2.7}$$

hold which further imply that

$$\kappa_{2n}m_{2n-1} - \bar{\alpha}_n\bar{\beta}_n\kappa_{2n-1}m_{2n} = c_{2n}c_{2n-1}\cdots c_2(m_1\kappa_0 - m_0\kappa_1) \neq 0.$$

The proof of (6.2.6) follows the exact techniques and line of argument as in the proof of (6.2.5). Indeed, proceeding as above we obtain, for $m = 2n + 1$,

$$\mathfrak{N}(\varphi_{2n+1}(z) \cdot \tilde{\psi}_{2n+1}(z)) = \frac{c_{2n+2}(\kappa_{2n}m_{2n+2} - \kappa_{2n+1}m_{2n})}{\chi_{2n+2}}.$$

Simplifying the numerator in the right hand side above, we note from (6.2.7) that

$$m_{2n+1}\kappa_{2n} - \kappa_{2n+1}m_{2n} = c_{2n+1}c_{2n}\cdots c_2(\kappa_0 m_1 - m_0\kappa_1) \neq 0.$$

The proof of the biorthogonality relations (6.2.5) and (6.2.6) for the remaining cases, that is, $m > 2n$, $m = 2j$ and $m < 2n$, $m = 2j + 1$, can be obtained with similar arguments, thus completing the proof. □

Remark 6.2.1. *The technique of using the leading coefficients κ_n and the normalization constants m_n to prove biorthogonality, as is evident in the present section, is available in the literature, for example, in Zhedanov [192]. However, the difference between the present work and Zhedanov [192] is our second objective of proving biorthogonal-*

ity for exactly the same rational functions that were used to arrive at the recurrence relations of R_{II} type for the numerator polynomials $r_j(z)$ which is also evident from Remark 6.3.1.

As mentioned in Chapter 1, while the sequence of rational functions

$$\phi_0(\lambda) := 1, \qquad \phi_n(\lambda) := \frac{\mathcal{P}_n(\lambda)}{\prod_{j=1}^{n}(\lambda - a_j)(\lambda - b_j)}, \qquad n \geq 1,$$

is used to obtain the pencil matrix $\mathcal{G}_n - \lambda\mathcal{H}_n$, the usual process available in the literature, for example in Zhedanov [192] and Beckermann et.al. [16], is to partition the poles to form the rational functions

$$p_n^L(\lambda) = \frac{\mathcal{P}_n(\lambda)}{\prod_{k=1}^{n}(\lambda - a_k)} \quad \text{and} \quad p_n^R(\lambda) = \frac{\mathcal{P}_n(\lambda)}{\prod_{k=1}^{n}(\lambda - b_k)}.$$

The two sequences of rational functions $\{p_n^L(\lambda)\}_{n=0}^{\infty}$ and $\{p_n^R(\lambda)\}_{n=0}^{\infty}$ form the components of the left and right eigenvectors of the matrix pencil $\mathcal{G}_n - \lambda\mathcal{H}_n$. It is clear that instead of partitioning the poles, we choose a basis of rational functions with alternating poles (in α_j and β_j) so that the degree (even or odd) of the numerator polynomials play an important role in our analysis. This is the main difference between the orthogonal rational functions constructed in the present chapter and the ones available in the literature.

6.3 Spectral transformation of Christoffel type

The Christoffel transformation (Chihara [42, p. 35]) of well-known orthogonal polynomials is abundant in the literature. In the present section, we find a Christoffel type transformation of the orthogonal rational functions given in (6.1.3) for the special case $|\beta_j| = 1$ and $\alpha_j = \alpha \in \mathbb{C} \setminus \{0\}$, $j \geq 1$. We begin with the recurrence relations (6.1.8a) and (6.1.8b) of R_{II} type for the numerator polynomials $\{r_n(z)\}_{n=0}^{\infty}$ which are now written, for $n \geq 0$, as

$$r_{2n+1}(z) = \rho_{2n}(z - \nu_{2n})r_{2n}(z) - \tau_{2n}(z - 1/\bar{\alpha})(z - \beta_n)r_{2n-1}(z), \tag{6.3.1a}$$

$$r_{2n+2}(z) = \rho_{2n+1}(z - \nu_{2n+1})r_{2n+1}(z) - \tau_{2n+1}(z - \alpha)(z - \beta_n)r_{2n}(z), \tag{6.3.1b}$$

where the new parameters $\{\rho_n\}$ and $\{\nu_n\}$ are given by

$$\rho_{2n} = d_{2n+1}, \quad \nu_{2n} = \frac{d_{2n+1}\beta_n - e_{2n+1}}{d_{2n+1}}, \quad \tau_{2n} = -c_{2n+1}\bar{\alpha}\bar{\beta}_n,$$
$$\rho_{2n+1} = -d_{2n+2}\bar{\alpha}, \quad \nu_{2n+1} = \frac{e_{2n+2} + d_{2n+2}}{d_{2n+2}\bar{\alpha}}, \quad \tau_{2n+1} = c_{2n+2}.$$

The recurrence relations (6.3.1b) and (6.3.1a) written in terms of the rational functions $\varphi_j(\lambda)$, $j \geq 0$ (as defined in (6.1.3)) yield

$$\begin{aligned}(z-\alpha)\varphi_{2n+1}(\lambda) &= u_{2n}(z-\nu_{2n})\varphi_{2n}(z) + \lambda_{2n}(z-1/\bar{\alpha})\varphi_{2n-1}(z),\\ (z-\beta_{n+1})\varphi_{2n+2}(z) &= u_{2n+1}(z-\nu_{2n+1})\varphi_{2n+1}(z) + \lambda_{2n+1}(z-\alpha_n)\varphi_{2n}(z),\end{aligned} \tag{6.3.2}$$

for $n \geq 0$, where $u_{2n} = \rho_{2n}$, $u_{2n+1} = -\rho_{2n+1}/\beta_{n+1}$, $\lambda_{2n+2} = \tau_{2n+2}\beta_{n+1}$ and $\lambda_{2n+1} = \tau_{2n+1}/\beta_{n+1}$. The recurrence relations (6.3.2) can be further arranged in the following form of an *eigenvalue problem* as

$$\begin{aligned}\alpha\varphi_1 - u_0\nu_0\varphi_0 &= z[\varphi_1 - u_0\varphi_0],\\ \alpha\varphi_{2n+1} - u_{2n}\nu_{2n}\varphi_{2n} - \frac{\lambda_{2n}}{\bar{\alpha}}\varphi_{2n-1} &= z[\varphi_{2n+1} - u_{2n}\varphi_{2n} - \lambda_{2n}\varphi_{2n-1}],\\ \beta_n\varphi_{2n} - u_{2n-1}\nu_{2n-1}\varphi_{2n-1} - \beta_{n-1}\lambda_{2n-1}\varphi_{2n-2} &= z[\phi_{2n} - u_{2n-1}\phi_{2n-1} - \lambda_{2n-1}\varphi_{2n-2}].\end{aligned} \tag{6.3.3}$$

Moreover, for $n \geq 0$, if we define the shift operators Γ and Λ as

$$\begin{aligned}\Gamma\varphi_{2n+1} &:= \beta_{n+1}\varphi_{2n+2} - u_{2n+1}\nu_{2n+1}\varphi_{2n+1} - \lambda_{2n+1}\beta_n\varphi_{2n},\\ \Gamma\varphi_{2n} &:= \alpha\varphi_{2n+1} - u_{2n}\nu_{2n}\varphi_{2n} - \lambda_{2n}/\bar{\alpha}\varphi_{2n-1},\\ \Lambda\varphi_{2n+1} &:= \varphi_{2n+2} - u_{2n+1}\varphi_{2n+1} - \lambda_{2n+1}\varphi_{2n},\\ \Lambda\varphi_{2n} &:= \varphi_{2n+1} - u_{2n}\varphi_{2n} - \lambda_{2n}\varphi_{2n-1},\end{aligned} \tag{6.3.4}$$

then (6.3.3) leads to the generalized eigenvalue problem $\Gamma\boldsymbol{\varrho} = z\Lambda\boldsymbol{\varrho}$ with the eigenvalue z and the eigenvector $\boldsymbol{\varrho} = \begin{pmatrix} \varphi_0 & \varphi_1 & \varphi_2 & \cdots \end{pmatrix}^T$.

6.3.1 The case $\varphi_{2n+1}(\lambda)$

Let $\hat{\varphi}_{2n+1}(z)$ denote the Christoffel type transform of $\varphi_{2n+1}(z)$, $n \geq 0$, obtained under the action of the operator $\mathfrak{D}$, where $\mathfrak{D}\varphi_j(z) = \hat{\varphi}_j(z)$. We note that $\hat{\varphi}_{2n}(z)$, is an arbitrary rational function in the present case. Further, we suppose that

$$\hat{\boldsymbol{\varrho}} = \begin{pmatrix} \hat{\varphi}_0 & \hat{\varphi}_1 & \hat{\varphi}_2 & \cdots \end{pmatrix}^T,$$

where $\mathfrak{D}\boldsymbol{\varrho} := \hat{\boldsymbol{\varrho}}$, is the eigenvector of some generalized eigenvalue problem with the same eigenvalue z.

The following lemma gives information on the action of the operator $\mathfrak{D}$ on an arbitrary rational function $\mathcal{Y}_k := \mathcal{Y}_k(\lambda)$ which belongs to the space $\mathcal{L}_j$. We recall that $\mathcal{L}_n = \text{span}\{\mathfrak{u}_0, \mathfrak{u}_1, \cdots, \mathfrak{u}_n\}$, where $\mathfrak{u}_j$, $j \geq 0$, are basis elements defined in Section 6.1.

Lemma 6.3.1. *Let*

$$\mathfrak{D}\mathcal{Y}_k := \Omega(z)(\mathcal{Y}_{k+1} + \zeta_j\mathcal{Y}_k), \qquad \mathcal{Y}_k \in \mathcal{L}_j, \qquad j \geq 0,$$

where $\Omega(z)$ is a function of z but independent of k and hence, is a constant with respect to $\mathfrak{D}$. Then

$$\zeta_{2j+1} = -\frac{\theta_{2j+2}}{\theta_{2j+1}} \qquad \text{and} \qquad \zeta_{2j} = -\frac{\theta_{2j+1}}{\theta_{2j}}, \qquad j \geq 0,$$

where θ_j is any function satisfying the recurrence relations (6.3.3).

Proof. Define another operator $\mathfrak{K}$ as

$$\mathfrak{K}\Gamma = \Gamma^o\mathfrak{D} \quad \text{and} \quad \mathfrak{K}\Lambda = \Lambda^o\mathfrak{D}. \tag{6.3.5}$$

Then, the effect of $\mathfrak{K}$ on the generalized eigenvalue problem $\Gamma\boldsymbol{\varrho} = z\Lambda\boldsymbol{\varrho}$ gives

$$\mathfrak{K}\Gamma\boldsymbol{\varrho} = z\mathfrak{K}\Lambda\boldsymbol{\varrho} \Longrightarrow \Gamma^o\mathfrak{D}\boldsymbol{\varrho} = z\Lambda^o\mathfrak{D}\boldsymbol{\varrho} \Longrightarrow \Gamma^o\hat{\boldsymbol{\varrho}} = z\Lambda^o\hat{\boldsymbol{\varrho}},$$

which gives the generalized eigenvalue problem for $\hat{\boldsymbol{\varrho}}$. Further, similar to (6.3.4), we define the shift operators Γ^o and Λ^o by

$$\begin{aligned}\Gamma^o\mathcal{Y}_{2n} &:= \hat{\beta}_n\mathcal{Y}_{2n+1} - \hat{u}_{2n+1}\hat{\nu}_{2n+1}\mathcal{Y}_{2n} - \hat{\beta}_{n-2}\hat{\lambda}_{2n+1}\mathcal{Y}_{2n-1},\\ \Gamma^o\mathcal{Y}_{2n+1} &:= \hat{\alpha}\mathcal{Y}_{2n+2} - \hat{u}_{2n}\hat{\nu}_{2n}\mathcal{Y}_{2n+1} - \hat{\lambda}_{2n}/\hat{\bar{\alpha}}\mathcal{Y}_{2n}\end{aligned} \tag{6.3.6}$$

and

$$\begin{aligned}\Lambda^o\mathcal{Y}_{2n} &:= \mathcal{Y}_{2n+1} - \hat{u}_{2n+1}\mathcal{Y}_{2n} - \hat{\lambda}_{2n+1}\mathcal{Y}_{2n-1},\\ \Lambda^o\mathcal{Y}_{2n+1} &:= \mathcal{Y}_{2n+2} - \hat{u}_{2n}\mathcal{Y}_{2n+1} - \hat{\lambda}_{2n}\mathcal{Y}_{2n},\end{aligned} \tag{6.3.7}$$

respectively. We proceed to find the parameters used in (6.3.6) and (6.3.7) in terms of the parameters used in the recurrence relations (6.3.1a) and (6.3.1b). For this, we use the operator relations defined in (6.3.5) for φ_{2n} and φ_{2n+1}. Similar to $\mathfrak{D}$, let the operator $\mathfrak{K}$ be defined as

$$\mathfrak{K}\mathcal{Y}_k := \Omega(z)(\mathcal{Y}_{k+1} + \eta_j\mathcal{Y}_k), \qquad \mathcal{Y}_k \in \mathcal{L}_j, \qquad j \geq 0, \tag{6.3.8}$$

where $\Omega(z)$ is a function of z and independent of k and hence, constant with respect to $\mathfrak{K}$. Then, we have the following four cases.

Case I. Using the definitions of $\mathfrak{D}$ and Γ, the relation $\Gamma^0\mathfrak{D}\varphi_{2n} = \mathfrak{K}\Gamma\varphi_{2n}$, $n \geq 0$, can be written as

$$\Gamma^o[\varphi_{2n+1} + \zeta_{2n}\varphi_{2n}] = \mathfrak{K}[\alpha\varphi_{2n+1} - u_{2n}\nu_{2n}\varphi_{2n} - (\lambda_{2n}/\bar{\alpha})\varphi_{2n-1}], \quad n \geq 0,$$

which implies

$$\begin{aligned}\hat{\alpha}[\varphi_{2n+2} + \zeta_{2n+1}\varphi_{2n+1}] - \hat{u}_{2n}\hat{\nu}_{2n}[\varphi_{2n+1} + \zeta_{2n}\varphi_{2n}] - (\hat{\lambda}_{2n}/\hat{\bar{\alpha}})[\varphi_{2n} + \zeta_{2n-1}\varphi_{2n-1}]&\\ =[\alpha\varphi_{2n+2} - u_{2n+1}\nu_{2n+1}\varphi_{2n+1} - (\lambda_{2n+1}/\bar{\alpha})\varphi_{2n}]&\\ +\eta_{2n+1}[\alpha\varphi_{2n+1} - u_{2n}\nu_{2n}\varphi_{2n} - (\lambda_{2n}/\bar{\alpha})\varphi_{2n-1}].&\end{aligned}$$

Comparing the coefficients of φ_{2n+2}, φ_{2n+1} , φ_{2n} and φ_{2n-1}, we obtain

$$\hat{\alpha} = \alpha, \quad \hat{\pi}_{2n} = \pi_{2n+1} + \alpha(\zeta_{2n+1} - \eta_{2n+1}),$$

$$\hat{u}_{2n}\hat{\nu}_{2n}\zeta_{2n} + \frac{\hat{\lambda}_{2n}}{\hat{\bar{\alpha}}} = u_{2n}\nu_{2n}\eta_{2n+1} + \frac{\lambda_{2n+1}}{\bar{\alpha}}, \quad \hat{\lambda}_{2n} = \frac{\eta_{2n+1}}{\zeta_{2n-1}}\lambda_{2n}, \quad n \geq 1,$$

where $\pi_j = u_j\nu_j$ and $\hat{\pi}_j = \hat{u}_j\hat{\nu}_j$.

Case II. Similar to Case I, comparing the coefficients of φ_{2n+1}, φ_{2n} and φ_{2n-2}, in the relation $\Gamma^0\mathfrak{D}\varphi_{2n-1} = \mathfrak{K}\Gamma\varphi_{2n-1}$ gives

$$\hat{\beta}_n = \beta_{n+1}, \quad \hat{\pi}_{2n+1} = \pi_{2n} + \beta_{n+1}\zeta_{2n} - \beta_n\eta_{2n} \quad \text{and} \quad \hat{\lambda}_{2n+1} = \frac{\eta_{2n}}{\zeta_{2n-2}}\lambda_{2n-1},$$

for $n \geq 1$, where we define $\hat{\alpha}_{-1} := 0$.

Case III. The comparison of the coefficient of φ_{2n+1} in the relation $\Lambda^o\mathfrak{D}\varphi_{2n} = \mathfrak{K}\Lambda\varphi_{2n}$, $n \geq 0$, gives

$$\hat{u}_{2n} = u_{2n+1} - \eta_{2n+1} + \zeta_{2n+1}, \quad n \geq 0.$$

Case IV. The comparison of the coefficient of φ_{2n+2} in $\Lambda^o\mathfrak{D}\varphi_{2n+1} = \mathfrak{K}\Lambda\varphi_{2n+1}$, $n \geq 0$, gives

$$\hat{u}_{2n+1} = u_{2n} - \eta_{2n} + \zeta_{2n}, \quad n \geq 0.$$

This implies that the operators Γ^o and Λ^o defined in terms of the parameters $\hat{\beta}_n$ etc. in (6.3.6) and (6.3.7) are well-defined. Now, using (6.3.5), we note

$$(\Gamma^o - z\Lambda^o)\mathfrak{D}\boldsymbol{\varrho}_{2n+1} = (\mathfrak{K}\Gamma - \lambda\mathfrak{K}\Lambda)\boldsymbol{\varrho}_{2n+1},$$

which implies that $\boldsymbol{\varrho}_{2n+1}$ is an eigenvector with respect to the operators Γ and Λ if, and only if, $\hat{\boldsymbol{\varrho}}_{2n+1}$ is an eigenvector with respect to the operators Γ^o and Λ^o. Let θ_j be an eigenvector of the generalized eigenvalue problem $\Gamma\theta_j = \hat{z}\Lambda\theta_j$, with the eigenvalue $\hat{z}$, which is equivalent to θ_j being a solution of the recurrence relation (6.3.3) with z replaced by $\hat{z}$. Then, we have $(\Gamma^o - \hat{z}\Lambda^o)\mathfrak{D}\theta_{2n+1} = 0$. This is satisfied, in particular, by $\mathfrak{D}\theta_{2n+1} = 0$, which gives

$$\theta_{2n+2} + \zeta_{2n+1}\theta_{2n+1} \Longrightarrow \zeta_{2n+1} = -\frac{\theta_{2n+2}}{\theta_{2n+1}}, \quad n \geq 0.$$

A similar argument for θ_{2n} gives $\mathfrak{D}\theta_{2n} = 0$, which implies $\zeta_{2n} = -\theta_{2n+1}/\theta_{2n}$, thus completing the proof. □

The expressions for η_j are obtained from the operator relations $\Lambda^o\mathfrak{D}\mathcal{Y}_k = \hat{z}\mathfrak{K}\Lambda\mathcal{Y}_k$ for $\mathcal{Y}_k = \theta_{2n}$ and θ_{2n+1} as

$$\begin{aligned} \eta_{2n} &= -\frac{\theta_{2n+1} - u_{2n}\theta_{2n} - \lambda_{2n}\theta_{2n-1}}{\theta_{2n} - u_{2n-1}\theta_{2n-1} - \lambda_{2n-1}\theta_{2n-2}} \quad \text{and} \\ \eta_{2n+1} &= -\frac{\theta_{2n+2} - u_{2n+1}\theta_{2n+1} - \lambda_{2n+1}\theta_{2n}}{\theta_{2n+1} - u_{2n}\theta_{2n} - \lambda_{2n}\theta_{2n-1}}. \end{aligned} \tag{6.3.9}$$

In particular, from Cases I and III, the following relations

$$\hat{u}_0\hat{\nu}_0\zeta_0 + \frac{\hat{\lambda}_0}{\bar{\alpha}} = u_0\nu_0\eta_1 + \frac{\lambda_1}{\bar{\alpha}} \quad \text{and} \quad \hat{u}_0 = u_1 + \zeta_1 - \eta_1. \tag{6.3.10}$$

hold for $n = 0$. We use the relations (6.3.10) to find the (constant) $\Omega(z)$ occurring in the definitions of both the operators $\mathfrak{D}$ and $\mathfrak{K}$ leading to the Christoffel type transform of $\varphi_{2n+1}(z)$. We also remark here that though $\beta_0 = 0$, we continue using β_0 in the expressions that follow. The reason is to show explicitly, the role played by β_0 in the calculations

Theorem 6.3.1. *The Christoffel type transform of $\varphi_{2n+1}(z)$ is given by*

$$\hat{\varphi}_{2n+1}(z) = \sigma\frac{z - \alpha_1}{z - \hat{z}}\left[\varphi_{2n+2}(z) - \frac{\varphi_{2n+2}(\hat{z})}{\varphi_{2n+1}(\hat{z})}\varphi_{2n+1}(z)\right]$$

for some constant σ. Further if $\boldsymbol{\varrho} = \begin{pmatrix} \varphi_0 & \varphi_1 & \cdots \end{pmatrix}^T$ is the eigenvector for the generalized eigenvalue problem $\Gamma\boldsymbol{\varrho} = z\Lambda\boldsymbol{\varrho}$, there exists another generalized eigenvalue problem $\Gamma^o\hat{\boldsymbol{\varrho}} = z\Lambda^o\hat{\boldsymbol{\varrho}}$, with the same eigenvalue z for which $\hat{\boldsymbol{\varrho}} = \begin{pmatrix} \hat{\varphi}_0 & \hat{\varphi}_1 & \cdots \end{pmatrix}^T$ is the eigenvector.

Proof. The last part of the theorem is about the existence of generalized eigenvalue problems for the column vectors $\boldsymbol{\varrho}$ and $\hat{\boldsymbol{\varrho}}$ which follows from the proof of Lemma 6.3.1. It is also clear that the Christoffel type transform is given by the shift operator $\mathfrak{D}$ and hence to find the expression for $\hat{\varphi}_{2n+1}(z)$, we need to find $\Omega(z)$ which is independent of n. Further, we obtained the functions θ_j, $j \geq 0$, with $\theta_{-1} = 0$, that satisfy the recurrence relations (6.3.3) with z replaced by $\hat{z}$. These equations for $n \geq 0$, written explicitly as

$$\alpha\theta_{2n+1} - u_{2n}\nu_{2n}\theta_{2n} - (\lambda_{2n}/\bar{\alpha})\theta_{2n-1} = \hat{z}[\theta_{2n+1} - u_{2n}\theta_{2n} - \lambda_{2n}\theta_{2n-1}], \quad (6.3.11a)$$

$$\beta_{n+1}\theta_{2n+2} - u_{2n+1}\nu_{2n+1}\theta_{2n+1} - \beta_n\lambda_{2n+1}\theta_{2n} = \hat{z}[\theta_{2n+2} - u_{2n+1}\theta_{2n+1} - \lambda_{2n+1}\theta_{2n}]. \quad (6.3.11b)$$

Let the Christoffel type transform of $\varphi_{2n+1}(z)$ be the rational function

$$\hat{\varphi}_{2n+1}(z) = \frac{\hat{r}_{2n+1}(z)}{(z-\hat{\alpha})^{n+1}\prod_{j=1}^{n}(1-z\hat{\beta}_j)} = \frac{\hat{r}_{2n+1}(z)}{(z-\alpha)^{n+1}\prod_{j=2}^{n+1}(1-z\hat{\beta}_j)},$$

where $\{\hat{r}_j(\lambda)\}$ satisfies the recurrence relations (6.3.1a) and (6.3.1b), but with the coefficients u replaced by $\hat{u}$ etc. To determine the constant $\Omega(z)$, we note that the implication

$$\hat{\varphi}_{2n+1} = \Omega(z)(\varphi_{2n+2} + \zeta_{2n+1}\varphi_{2n+1}) \Longrightarrow \Omega(z) = \frac{(z-\beta_1)\hat{r}_1(z)}{r_2(z) + \zeta_1(z-\beta_1)r_1(z)}$$

follows from the values for $n = 0$. Further, from (6.3.11a), we obtain for $n = 0$,

$$\alpha\theta_1 - u_0\nu_0\theta_0 = \hat{z}(\theta_1 - u_0\theta_0) \Longrightarrow \frac{\theta_1}{\theta_0} = \frac{u_0(\hat{z}-\nu_0)}{\hat{z}-\alpha}.$$

Similarly, from (6.3.11b) for $n = 0$,, we obtain

$$\beta_1\theta_2 - u_1\nu_1\theta_1 - \lambda_1\beta_0\theta_0 = \hat{z}[\theta_2 - u_1\theta_1 - \lambda_1\theta_0],$$

which gives

$$(\hat{z}-\beta_1)\frac{\theta_2}{\theta_1} = u_1(\hat{z}-\nu_1) + \lambda_1(\hat{z}-\beta_0)\frac{\theta_0}{\theta_1}.$$

Then, from Lemma 6.3.1, we obtain

$$-\zeta_1 = \frac{\theta_2}{\theta_1} = \frac{u_1(\hat{z}-\nu_1)}{\hat{z}-\beta_1} + \frac{\lambda_1(\hat{z}-\beta_0)(\hat{z}-\alpha)}{u_0(\hat{z}-\beta_1)(\hat{z}-\nu_0)},$$

so that the denominator of $\Omega(z)$ has the expression

$$r_2(\lambda) + \zeta_1(z-\beta_1)r_1(z) = u_0u_1(z-\nu_0)(z-\nu_1) + \lambda_1(z-\beta_0)(z-\alpha) \\ - \frac{z-\beta_1}{\hat{z}-\beta_1}u_0(z-\nu_0)\left[u_1(\hat{z}-\nu_1) + \frac{\lambda_1(\hat{z}-\beta_0)(\hat{z}-\alpha)}{u_0(\hat{z}-\nu_0)}\right].$$

This implies that

$$\begin{aligned}
&(\hat{z}-\nu_0)(\hat{z}-\beta_1)[r_2(z) + \zeta_1(z-\beta_1)r_1(z)] \\
&= u_0u_1[(z-\nu_0)(z-\nu_1)(\hat{z}-\nu_0)(\hat{z}-\beta_1) - (\hat{z}-\nu_0)(\hat{z}-\nu_1)(z-\nu_0)(z-\beta_1)] \\
&+ \lambda_1[(z-\beta_0)(z-\alpha)(\hat{z}-\nu_0)(\hat{z}-\beta_1) - (\hat{z}-\beta_0)(\hat{z}-\alpha)(z-\nu_0)(z-\beta_1)].
\end{aligned}$$

Further simplification yields

$$\Omega(z) = \frac{z-\beta_1}{z-\hat{z}}\frac{(\hat{z}-\nu_0)(\hat{z}-\beta_1)\hat{r}_1(z)}{\Upsilon(z)},$$

where $\Upsilon(z) = \Upsilon_1 z + \Upsilon_0$, with

$$\begin{aligned}
\Upsilon_1 &= u_0u_1(\hat{z}-\nu_0)(\nu_1-\beta_1) + \lambda_1(\beta_1\nu_0 + \beta_0\hat{z} + \alpha\hat{z} - \beta_1\hat{z} - \nu_0\hat{z} - \alpha\beta_0), \\
\Upsilon_0 &= -u_0u_1(\hat{z}-\nu_0)(\nu_1-\beta_1)\nu_0 + \lambda_1[\nu_0(\beta_1\hat{z} - \alpha\beta_1 - \beta_0\beta_1 + \alpha\beta_0) - \alpha\beta_0\hat{z} + \alpha\beta_0\beta_1].
\end{aligned}$$

Next, using the relations (6.3.9) and (6.3.10), we have $\hat{r}_1(z) = \hat{u}_0(z-\hat{\nu}_0)$, where $\hat{u}_0 = u_1 + \zeta_1 - \eta_1$ with

$$\zeta_1 = -\frac{\theta_2}{\theta_1} \quad \text{and} \quad \eta_1 = -\frac{(\theta_2 - u_1\theta_1 - \lambda_1\theta_0)}{(\theta_1 - u_0\theta_0)}.$$

Further, we have $\hat{u}_0\hat{\nu}_0 = u_1 + \alpha(\zeta_1 - \eta_1)$. This implies,

$$\frac{\hat{u}_0(\alpha-\nu_0)}{\hat{z}-\nu_0} \\ = \frac{u_1(\nu_1-\beta_1)(\alpha-\hat{z})}{(\hat{z}-\beta_1)(\hat{z}-\nu_0)} + \frac{\lambda_1(\hat{z}-\alpha)}{u_0(\hat{z}-\nu_0)^2(\hat{z}-\beta_1)}\left[\beta_1\hat{z} - \beta_1\nu_0 - \beta_0\hat{z} - \alpha\hat{z} + \nu_0\hat{z} + \alpha\beta_0\right],$$

which on further simplification yields

$$\begin{aligned}&\zeta_0\hat{u}_0(\alpha-\nu_0)(\hat{z}-\beta_1)\\&\qquad=u_0u_1(\nu_1-\beta_1)(\hat{z}-\nu_0)+\lambda_1(\beta_1\nu_0+\beta_0\hat{z}+\alpha\hat{z}-\beta_1\hat{z}-\nu_0\hat{z}-\alpha\beta_0).\end{aligned}$$

Using the fact that $-\zeta_0=u_0(\hat{z}-\nu_0)/(\hat{z}-\alpha)$, we finally have

$$\zeta_0\hat{u}_0(\alpha-\nu_0)(\hat{z}-\beta_1)=\Upsilon_1.$$

Further, substituting the value of η_1, we have from the first relation in (6.3.10)

$$\begin{aligned}u_0u_1(\beta_1-\nu_1)(\hat{z}-\nu_0)+\lambda_1\left[\nu_0(\hat{z}-\alpha)(\beta_1-\beta_0)-\frac{1}{\bar{\alpha}}(\alpha-\nu_0)(\hat{z}-\beta_1)\right]\\+\frac{\hat{\lambda}_0}{\bar{\alpha}}(\alpha-\nu_0)(\hat{z}-\beta_1)=-\zeta_0(\alpha-\nu_0)(\hat{z}-\beta_1)\hat{\rho}_0\hat{\nu}_0.\end{aligned}$$

Defining $\hat{\lambda}_0:=\lambda_0-\beta_0\bar{\alpha}$ (since $\beta_0=0$, $\hat{\lambda}_0:=\lambda_0$), finally yields

$$-\zeta_0(\alpha-\nu_0)(\hat{z}-\beta_1)\hat{u}_0\hat{\nu}_0=\Upsilon_0.$$

Hence, we have $\zeta_0(\alpha-\nu_0)(\hat{z}-\beta_1)\hat{r}_1(z)=\Upsilon(z)$, which means

$$\Omega(z)=\frac{\hat{z}-\nu_0}{\zeta_0(\alpha-\nu_0)}\frac{z-\beta_1}{z-\hat{z}}=\sigma\frac{z-\beta_1}{z-\hat{z}},$$

where $\sigma=(\hat{z}-\alpha)/(u_0(\nu_0-\alpha))$. Finally, we note that since θ_j satisfies (6.3.11a) and (6.3.11b), θ_j must necessarily be equal to $\varphi_j(\hat{z})$. □

Remark 6.3.1. *We would like to emphasize here the use of the relations* (6.3.10) *and the second degree polynomial $r_2(z)$ in deriving the above expressions. This is different from the one given in Zhedanov* [192]*, where the linear polynomial $r_1(\lambda)$ was used.*

Remark 6.3.2. *The operators used in deriving the Christoffel type transformation are basically matrix multiplication operators. A study of such operators, from a measure theoretic point of view can be found in Hudzik et al.* [86].

6.3.2 The case $\phi_{2n}(\lambda)$

Let $\hat{\varphi}_{2n}(\lambda)$ denote the Christoffel type transform of $\varphi_{2n}(\lambda)$, $n \geq 0$. In the present case, we use the shift operators Γ^e and Λ^e where, for $n \geq 0$, Γ^e is given by

$$\begin{aligned} \Gamma^e \mathcal{Y}_{2n} &:= \hat{\beta}_{n+1}\mathcal{Y}_{2n+1} - \hat{u}_{2n+1}\hat{\nu}_{2n+1}\mathcal{Y}_{2n} - \hat{\beta}_{n-1}\hat{\lambda}_{2n+1}\mathcal{Y}_{2n-1}, \\ \Gamma^e \mathcal{Y}_{2n+1} &:= \hat{\alpha}\mathcal{Y}_{2n+2} - \hat{u}_{2n}\hat{\nu}_{2n}\mathcal{Y}_{2n+1} - \hat{\lambda}_{2n}/\hat{\bar{\alpha}}\mathcal{Y}_{2n}, \end{aligned} \tag{6.3.12}$$

and Λ^e is same as Λ^o, which was defined in the case of $\varphi_{2n+1}(\lambda)$ as that is

$$\begin{aligned} \Lambda^e \mathcal{Y}_{2n} &= \mathcal{Y}_{2n+1} - \hat{u}_{2n+1}\mathcal{Y}_{2n} - \hat{\lambda}_{2n+1}\mathcal{Y}_{2n-1}, \\ \Lambda^e \mathcal{Y}_{2n+1} &= \mathcal{Y}_{2n+2} - \hat{u}_{2n}\mathcal{Y}_{2n+1} - \hat{\lambda}_{2n}\mathcal{Y}_{2n}. \end{aligned}$$

The derivation of the expression for $\hat{\varphi}_{2n}(z)$ follows the same technique as in the case of $\hat{\varphi}_{2n+1}(z)$. In fact, this technique is used to find the Christoffel type transforms of orthogonal rational functions with arbitrary poles. However, as remarked earlier, only the polynomial $r_1(z)$ is used which makes the calculations easier. We present the proof of the following theorem illustrating the difference in the calculations involved in deriving the expressions for $\hat{\varphi}_{2n+1}(\lambda)$ and $\hat{\varphi}_{2n}(\lambda)$.

Theorem 6.3.2. *The Christoffel type transform of $\varphi_{2n}(z)$ is given by*

$$\hat{\varphi}_{2n}(z) = \sigma\frac{z-\alpha}{z-\hat{z}}\left[\varphi_{2n+1}(z) - \frac{\varphi_{2n+1}(\hat{z})}{\varphi_{2n}(\hat{z})}\varphi_{2n}(z)\right],$$

for some constant $\sigma = (\hat{z}-\alpha)/(u_0(\nu_0-\alpha))$. Moreover, if $\boldsymbol{\varrho} = \begin{pmatrix} \varphi_0 & \varphi_1 & \cdots \end{pmatrix}^T$ is the eigenvector for the generalized eigenvalue problem $\Gamma\boldsymbol{\varrho} = z\Lambda\boldsymbol{\varrho}$, there exists another generalized eigenvalue problem $\Gamma^e\hat{\boldsymbol{\varrho}} = z\Lambda^e\hat{\boldsymbol{\varrho}}$, with the same eigenvalue z for which $\hat{\boldsymbol{\varrho}} = \begin{pmatrix} \hat{\varphi}_0 & \hat{\varphi}_1 & \cdots \end{pmatrix}^T$ is the eigenvector.

Proof. As in Lemma 6.3.1, we define the two shift operators $\mathfrak{D}$ and $\mathfrak{K}$ in the present case as

$$\mathfrak{D}\mathcal{Y}_k := \Psi(z)(\mathcal{Y}_{k+1} + \zeta_j\mathcal{Y}_k), \quad \mathfrak{K}\mathcal{Y}_k := \Psi(z)(\mathcal{Y}_{k+1} + \zeta_j\mathcal{Y}_k), \quad \mathcal{Y}_k \in \mathcal{L}_j, \quad j \geq 0,$$

which also satisfy the operator relations $\mathfrak{K}\Gamma = \Gamma^e\mathfrak{D}$ and $\mathfrak{K}\Gamma = \Lambda^e\mathfrak{D}$. and where $\Psi(z)$ is independent of k. It follows that

$$\Gamma\boldsymbol{\varrho} = z\Lambda\boldsymbol{\varrho} \Longleftrightarrow \Gamma^e\hat{\boldsymbol{\varrho}} = z\Lambda^e\hat{\boldsymbol{\varrho}}, \tag{6.3.13}$$

which proves the existence of a generalized eigenvalue problem for the column vector $\hat{\boldsymbol{\varrho}}$ with the eigenvalue z. Further, as in the four cases in the proof of Lemma 6.3.1, the parameters used in the definitions (6.3.7) and (6.3.12) of $\hat{\Lambda}$ and $\hat{\Gamma}^e$ can be found from

$$\Gamma^e\mathfrak{D}\mathcal{Y}_j = \mathfrak{K}\Gamma\mathcal{Y}_j \quad \text{and} \quad \Lambda^e\mathfrak{D}\mathcal{Y}_j = \mathfrak{K}\Lambda\mathcal{Y}_j,$$

first for $\mathcal{Y}_j = \varphi_{2n}$ and then for $\mathcal{Y}_j = \varphi_{2n+1}$. These parameters are given by

$$\begin{aligned} \hat{\beta}_n = \beta_n, \quad \hat{u}_{2n+1} = u_{2n} + \zeta_{2n} - \eta_{2n}, \quad \hat{\lambda}_{2n+1} = \frac{\eta_{2n}}{\zeta_{2n-2}}\lambda_{2n-1}, \\ \hat{\alpha} = \alpha, \quad \hat{\pi}_2 n = \pi_{2n+1} + \beta(\zeta_{2n+1} - \eta_{2n+1}), \quad \hat{\lambda}_{2n} = \frac{\eta_{2n+1}}{\zeta_{2n-1}}\lambda_{2n}, \\ \hat{u}_{2n} = u_{2n+1} + \zeta_{2n+1} - \eta_{2n+1}. \end{aligned}$$

We note again from (6.3.13) that

$$(\Gamma^e - z\Lambda^e)\mathfrak{D}\boldsymbol{\varrho} = (\mathfrak{K}\Gamma - z\mathfrak{K}\Lambda)\boldsymbol{\varrho}$$

implies $\boldsymbol{\varrho}$ is an eigenvector with respect to the operators Γ and Λ if, and only if, $\mathfrak{D}\boldsymbol{\varrho} = \hat{\boldsymbol{\varrho}}$ is an eigenvector with respect to the operators Γ^e and Λ^e.

As in the proof of Theorem 6.3.1, let θ_j be an eigenvector of the generalized eigenvalue problem $\Gamma\theta_j = \hat{z}\Lambda\theta_j$, with the eigenvalue $\hat{\lambda}$, which is equivalent to θ_j being a solution of the recurrence relation (6.3.3) with z replaced by $\hat{z}$. Then, we have $(\Gamma^e - \hat{z}\Lambda^e)\mathfrak{D}\theta_{2n+1} = 0$. This is satisfied, in particular, by $\mathfrak{D}\theta_{2n+1} = 0$, which gives

$$\theta_{2n+2} + \zeta_{2n+1}\theta_{2n+1} \Longrightarrow \zeta_{2n+1} = -\frac{\theta_{2n+2}}{\theta_{2n+1}}, \quad n \geq 0.$$

A similar argument for θ_{2n} gives $\mathfrak{D}\theta_{2n} = 0$, which implies $\zeta_{2n} = -\theta_{2n+1}/\theta_{2n}$. It then follows that for these values for ζ_j, η_j satisfies the equations (6.3.9).

Let the Christoffel type transform of $\phi_{2n}(\lambda)$ be the rational function

$$\hat{\varphi}_{2n}(z) = \frac{\hat{r}_{2n}(z)}{(z-\hat{\alpha})^n \prod_{j=1}^{n}(1-z\bar{\hat{\beta}}_j)} = \frac{\hat{r}_{2n}(z)}{(\lambda-\alpha)^n \prod_{j=1}^{n}(1-z\bar{\beta}_j)},$$

where $\hat{r}_{2n}(z)$ satisfies the recurrence (6.3.1b) but with the coefficients u_n replaced by $\hat{u}_n$ etc. To determine the constant $\Psi(z)$, we note that the implication

$$\hat{\varphi}_{2n}(z) = \Psi(z)(\varphi_{2n+1}(z) + \zeta_{2n}\varphi_{2n}(z)) \Longrightarrow \Psi(z) = \frac{(z-\alpha)}{r_1(z) + \zeta_0(z-\alpha)},$$

which follows from the values for φ_1 and φ_0. Using the facts that $r_1(z) = u_0(z-\nu_0)$ and $-\zeta_0 = \theta_1/\theta_0 = u_0(\hat{z}-\nu_0)/(\hat{z}-\alpha)$, we have

$$\Psi(z) = \frac{z-\alpha}{u_0(z-\nu_0) - \dfrac{u_0(\hat{z}-\nu_0)}{\hat{z}-\alpha}(z-\alpha)} = \frac{(\hat{z}-\alpha)(z-\alpha)}{u_0(\nu_0-\alpha)(z-\hat{z})}.$$

Choosing $\sigma = (\hat{z}-\alpha)/u_0(\nu_0-\alpha)$, we arrive at the required expression for the Christoffel type transform $\hat{\varphi}_{2n}(z)$. □

Note 6.3.1. *The constant σ obtained in the case of Christoffel type transform of $\varphi_{2n}(z)$ is the same as that obtained in the case of Christoffel type transform of $\varphi_{2n+1}(z)$.*

We conclude this section with the moment functional associated with the Christoffel type transforms. Define the following two linear functionals as

$$\mathfrak{N}_0 := \frac{z-\hat{z}}{z-\beta_1}\mathfrak{N} \quad \text{and} \quad \mathfrak{N}_e := \frac{z-\hat{z}}{z-\alpha}\mathfrak{N}, \tag{6.3.14}$$

where $\mathfrak{N}$ is as defined in Theorem 6.2.1. Further, by multiplication of a functional by a function $\mathfrak{f}(z)\mathfrak{N}$ it is understood that $\mathfrak{N}$ acts on the space of the space of functions $\mathfrak{g}(z)$ as $\mathfrak{N}(\mathfrak{f}(z)\mathfrak{g}(z))$. Then we have

Theorem 6.3.3. *The following orthogonality relations hold*

$$\mathfrak{N}_o\left(\frac{z^j}{(1-z\bar{\alpha})^n \prod_{k=0}^{n}(z-\beta_k)}\hat{\varphi}_{2n+1}(\lambda)\right) = 0, \quad j = 0, 1, \cdots, 2n,$$
$$\mathfrak{N}_e\left(\frac{z^j}{(1-z\bar{\alpha})^n \prod_{k=0}^{n-1}(z-\beta_k)}\hat{\varphi}_{2n}(\lambda)\right) = 0, \quad j = 0, 1, \cdots, 2n-1,$$

where $\mathfrak{N}_0$ *and* $\mathfrak{N}_e$ *are defined in* (6.3.14).

Proof. Using Theorem 6.2.1, it is easy to see that

$$\begin{aligned}\mathfrak{N}_o\left(\frac{z^j\hat{\varphi}_{2n+1}(z)}{(1-z\bar{\alpha})^n\prod_{k=0}^{n}(z-\beta_k)}\right) &= \sigma\mathfrak{N}\left(\frac{z^j(\varphi_{2n+2}(z)+\zeta_{2n+1}\varphi_{2n+1}(z))}{(1-z\bar{\alpha})^n\prod_{k=0}^{n}(z-\beta_k)}\right)\\ &= \sigma\mathfrak{N}(z^j\{(1-z\bar{\alpha})\mathcal{O}_{2n+2}(z)+\zeta_{2n+1}\mathcal{O}_{2n+1}(z)\})\\ &= 0, \quad j=0,1,2,\cdots,2n,\end{aligned}$$

where $\mathcal{O}_j(z)$ are the rational functions defined in (6.2.1). The proof for the case of $\hat{\phi}_{2n}(z)$ is similar and hence omitted. □

6.4 Concluding remarks

The orthogonal rational functions obtained in this chapter can lead to a variety of research problems. For instance, the class of Pick functions obtained in Chapter 2 consists of ratios of Gaussian hypergeometric functions which can always be brought to rational function forms under certain restrictions. Hence, it can be expected that this class of Pick functions satisfy some sort of orthogonality as well as biorthogonality properties.

Moreover, the R_{II} recurrence relations (6.1.8a) and (6.1.8b) are similar to the recurrence relations satisfied by the orthogonal Laurent polynomials (Bultheel et al. [33, Theorem 11.14, p.263]). Hence, further study of such R_{II} recurrence relations and the related eigenvalue problems can also be made. Since no conditions are imposed on the poles except that they are non-vanishing to derive biorthogonality, interesting particular cases can be obtained when the poles satisfy special conditions.

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List of Publications

1. K. K. Behera, A. Sri Ranga and A. Swaminathan, Orthogonal polynomials associated with complementary chain sequences, SIGMA Symmetry Integrability Geom. Methods Appl. **12** (2016), Paper No. 075, 17 pp.

2. K. K. Behera and A. Swaminathan, Orthogonal polynomials related to g-fractions with missing terms, Comput. Methods Funct. Theory (2017). https://doi.org/10.1007/s40315-017-0218-y

3. K. K. Behera and A. Swaminathan, Biorthogonal rational functions of R_{II} type, Communicated, available at arXiv:1712.00567 [math.CA].

4. K. K. Behera and A. Swaminathan, Biorthogonality and para-orthogonality of R_I polynomials, Communicated, available at arXiv:1801.05625 [math.CA].

5. K. K. Behera and A. Swaminathan, Generalized Jacobi pencil matrix, To be communicated