INTEGRAL MODIFICATION OF CERTAIN POSITIVE LINEAR OPERATORS

Ph.D. THESIS

by

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INDIAN INSTITUTE OF TECHNOLOGY ROORKEE
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INTEGRAL MODIFICATION OF CERTAIN POSITIVE LINEAR OPERATORS

A THESIS

Submitted in partial fulfilment of the requirements for the award of the degree

of

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in

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CANDIDATE'S DECLARATION

I hereby certify that the work which is being presented in this thesis entitled, **"INTEGRAL MODIFICATION OF CERTAIN POSITIVE LINEAR OPERATORS"** in partial fulfilment of the requirements for the award of the Degree of Doctor of Philosophy and submitted in the Department of Mathematics of the Indian Institute of Technology Roorkee, Roorkee is an authentic record of my own work carried out during a period from July, 2014 to December, 2017 under the supervision of Dr. P. N. Agrawal, Professor, Department of Mathematics, Indian Institute of Technology Roorkee, Roorkee.

The matter presented in this thesis has not been submitted by me for the award of any other degree of this or any other Institution.

(TRAPTI NEER)

This is to certify that the above statement made by the candidate is correct to the best of my knowledge.

(P. N. Agrawal)
Supervisor

The Ph.D. Viva-Voce Examination of **Ms. Trapti Neer,** Research Scholar, has been held on ....

Chairman, SRC **Signature of External Examiner**

This is to certify that the student has made all the corrections in the thesis.

Signature of Supervisor **Head of the Department**

Dated:............................

Abstract

This thesis presents approximation of functions by several well known positive linear operators, by their generalized forms and integral modifications. We divide the thesis into nine chapters. Chapter 0 is an introductory part of the thesis which deals with the upbringing of approximation theory, literature survey, some notations and basic definitions of approximation methods which are used throughout the thesis.

In the first chapter, we define a genuine family of Bernstein-Durrmeyer type operators based on Polya basis functions. We establish a global approximation theorem, local approximation theorem, Voronovskaya-type asymptotic theorem and a quantitative estimate of the same type. Lastly, we study the approximation of functions having a derivative of bounded variation.

The second chapter is a continuation of the first one in which we introduce the Bézier variant of genuine Durrmeyer type operators and give direct approximation results and a Voronovskaya type theorem by using the Ditzian-Totik modulus of smoothness. The rate of convergence for functions whose derivatives are of bounded variation is also obtained. Further, we show the rate of convergence of these operators to certain functions by illustrative graphics using the Matlab algorithms.

In the third chapter, we define the Szász-Durrmeyer type operators by means of

multiple Appell polynomials. We study a quantitative Voronovskaya type theorem and Grüss-Voronovskaya type theorem. We also establish a local approximation theorem in terms of the Steklov means and Voronovskaya type asymtotic theorem. Further, we discuss the degree of approximation by means of a weighted space. Lastly, we find the rate of approximation of functions having derivatives of bounded variation.

In the fourth chapter, we introduce the Bézier variant of Durrmeyer modification of the Bernstein operators based on a function τ. We give the rate of approximation of these operators in terms of usual modulus of continuity and the $K-$functional. Next, we establish the quantitative Voronovskaja type theorem. In the last section we obtain the rate of convergence for functions having derivatives of bounded variation.

In the fifth chapter, we define a sequence of Stancu type operators based on the same function τ as defined in the preceding chapter and show that these operators present a better degree of approximation than the original ones. We give a direct approximation theorem by means of the Ditzian-Totik modulus of smoothness and a Voronovskaya type theorem.

In the sixth chapter, we introduce the Bézier variant of modified Srivastava-Gupta operators and give a direct approximation theorem by means of the Ditzian-Totik modulus of smoothness and the rate of convergence for functions with derivatives equivalent to a function of bounded variation. Furthermore, we show the comparisons of the rate of convergence of the Srivastava-Gupta operators vis-a-vis its Bézier variant to a certain function by illustrative graphics using Maple algorithms.

In the seventh chapter, we construct the Stancu-Durrmeyer-type modification of q-Bernstein operators by means of Jackson integral. Here, we establish basic convergence theorem, local approximation theorem and an approximation result for a Lipschitz type space. Also, we establish the Korovkin type A-statistical approximation theorem and rates of A-statistical convergence in terms of the modulus of continuity.

The last chapter is an continuation of our work in chapter seven. Here, we construct a bivariate generalization of Stancu-Durrmeyer type operators and study the rate of convergence by means of the complete modulus of continuity and the partial moduli of continuity. Subsequently, we define the GBS (Generalized Boolean Sum) operators of Stancu-Durrmeyer type and give the rate of approximation by means of the mixed modulus of smoothness and the Lipschitz class of Bögel-continuous functions.

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Roorkee (Trapti Neer)

December , 2017 ——————————————————-

Introduction

0.1 General Introduction

Approximation theory is used both in pure and applied mathematics. It includes a wide area ranging from abstract problems in real, complex, and functional analysis to direct applications in engineering and industry. Therefore, approximation theory is closely related to mathematical analysis, operator theory, harmonic analysis, quantum calculus, algorithms, probability theory etc. In mathematical analysis, it deals with the approximation of some kind of complicated functions by the simpler one with desirable rate of approximation.

Approximation of functions by positive linear operators is an important research area that provides us key tools for exploring the computer-aided geometric design, numerical analysis and the solutions of ordinary and partial differential equations that arise in the mathematical modeling of real world phenomena. The foundation of approximation theory known as Weierstrass approximation theorem was introduced by Carl Weierstrass in 1885, which states that any real continuous function on a closed and bounded interval can be uniformly approximated on that interval by a sequence of polynomials to any degree of accuracy. Several proofs of this theorem have been given by great mathematicians e.g. Runge, Lebesgue, Landau, Fejér and Jackson. In 1912, S. N. Bernstein [29] gave a simpler proof by constructing a sequence of polynomials called Bernstein polynomials as

$$\mathcal{B}_n(f;x) = \sum_{k=0}^{n} p_{n,k}(x) f\left(\frac{k}{n}\right), \quad x \in [0,1],$$

where $p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}$. This sequence converges uniformly to any continuous function on $[0,1]$.

After that the fundamental theorem of uniform convergence by a general sequence of positive linear operators was established by Bohman [33] and Korovkin [107]. Szász [152] generalized the Bernstein polynomials to approximate continuous functions defined on the infinite interval $[0,\infty)$. Kantorovich [101] introduced an integral modification of Bernstein polynomials to approximate Lebesgue integrable functions defined on $[0,1]$. Durrmeyer [52] used another kind of integral modification of Bernstein polynomials to approximate these functions. Subsequently, many new sequences and classes of operators were constructed and studied for their approximation behavior by prominent researchers. Some well known operators introduced by researchers to study the approximation of functions are due to Stancu [148], Lupas and Lupas [111], Phillips [130], Baskakov [28], Gupta and Srivastava [146], Rathore and Singh [137], Abel and Heilmann[2] etc.

The approximation methods deal with the convergence behavior of the positive linear operators to the functions. The study of the convergence is carried out by some direct results, asymptotic behavior of the operators, several tools of approximation and weighted approximation. In the field of approximation theory, Jackson [95] was the first who gave the direct theorems in his classical work on algebraic and trigonometric polynomials. For more contribution on the study of direct theorems we refer to ([60], [61], [140] and [151], etc.). King [104] initiated a new kind of modification for the operators which do not reproduce the linear functions, to achieve a better degree of approximation. Motivated by this, Cardenas-Morales et al. [37] defined a sequence of Bernstein type operators by generalizing the Korovkin set from $\{1, t, t^2\}$ to $\{1, \tau, \tau^2\}$ and investigated its shape preserving and convergence properties as well as its asymptotic behavior.

The study of the rate of convergence for functions of bounded variation by linear positive operators is another interesting area of research. Cheng [40] investigated the rate of convergence of Bernstein polynomials for functions of bounded variation. Using probablistic approach, Bojanic and Cheng ([34], [35]) studied the rate of convergence of Bernstein polynomials for functions with derivatives of bounded variation. Srivastava et al. [145] estimated the rate of convergence for functions having derivative of bounded variation. Recently, Ispir et al. [91] considered the Kantorovich modification of Lupas operators based on Polya distribution and studied the rate of approximation of the functions having derivatives of bounded variation. Researchers studied these problems for several other sequences of linear positive operators (cf. [13], [14], [82], [86], [90] and [112] etc.).

It is well known that Bézier curves are the parametric curves, used in computer graphics and designs, interpolation, approximation, curve fitting etc. In graphics of vectors, these are used to model smooth curves and also used in animation designs. These curves were invented by Pierre Etienne Bézier, a French engineer at Renault. Zeng and Piriou [169] pioneered the study of Bézier variant of Bernstein operators. Later on, Chang [39] introduced Bézier variant for generalized Bernstein operators and studied some of its approximation properties. Zeng and Chen [168] introduced the Bézier Bernstein-Durrmeyer operators and studied the rate of convergence for these operators. Srivastava and Gupta [147] studied the rate of convergence for the Bézier variant of the Bleimann-Butzer-Hahn operators for the functions of bounded variation. Subsequently, Bézier variants for several sequences of operators have been introduced and studied by researchers (cf. [15], [73], [166], [162] etc.).

In 1968, Stancu [148] introduced a generalization of Bernstein operators depending on a non negative parameter α. Lupas and Lupas [111] considered a special case $\alpha = \frac{1}{n}, \ n \in \mathbb{N}$, for these operators. Gupta and Rassias [82] introduced the Durrmeyer-type integral modification of Lupas and Lupas operators and obtained local and global direct estimates and a Voronvskaya-type asymptotic formula. Later, the same authors [83] considered a Durrmeyer type modification of the Jain operators and studied the asymptotic formula, error estimation in terms of the modulus of continuity and weighted approximation. Gupta et al. [84] proposed certain Lupas-beta operators which preserve constant as well as linear functions and established some direct results and the approximation of functions having a derivative of bounded variation.

Jakimovski and Leviatan [96] proposed a generalization of Szász-Mirakjan operators by means of the Appell polynomials and gave the rate of approximation for these operators. Subsequently, generalizations of the Szász-Mirakyan operators by means of Sheffer polynomials, Brenke-type polynomials and Boas-Buck type polynomials were introduced and investigated in (cf. [88], [142], [154] and [155] etc.).

Gupta and Srivastava [146] introduced a general family of summation-integral type operators known as Srivastava-Gupta operators. Yadav [163] introduced a modification of these operators and studied a direct estimate, asymptotic formula and statistical convergence. After that, Verma and Agrawal [157] introduced the generalized form of these operators and studied some of its approximation properties. Many researchers have studied the approximation properties of Srivastava-Gupta operators and its various generalizations over the past decade (cf. [9], [45] and [93] etc.).

Some of the recently introduced sequences and classes of operators which have been extensively studied by researchers are, Bernstein-Durrmeyer type operators ([1], [67], [66], [78] etc.), Bernstein-Kantorovich type operators ([74], [125], [92], [114] etc.), Hybrid type operators ([14], [80], [84] etc.), Gamma type operators ([94], [102], [103] etc.), Chlodowsky and Stancu variants of operators ([17], [126], [156] etc.), linear positive operators constructed by means of the Chan-Chayan-Srivastava multivariable polynomials [54] and the operators defined in ([43], [44], [46], [98], [118], [136], [138], [139], [160], [161] etc.). More detailed account of such operators can be found in the books (cf. [18], [82] and [85] etc.).

0.2 Fundamentals of q-calculus

In the last decade, the application of q-calculus in the field of approximation theory has been an active area of research. More applications of q-calculus are in number theory, combinatorics, orthogonal polynomials, hypergeometric functions, mechanics, theory of relativity, quantum theory and theoretical physics. In 1987, Lupas [110] initiated the study of q-analogue of the classical Bernstein polynomials. Later, Phillips [130] proposed another q-generalization of the Bernstein polynomials and established the rate of convergence and Voronovskaja type asymptotic formula for these operators. Gupta [75] introduced the q-analogue of Bernstein-Durrmeyer operators and studied some approximation properties of these operators. Gupta and Wang [79] introduced q-Durrmeyer type operators and studied the rate of convergence in terms of modulus of continuity. Subsequently, Finta and Gupta [59] studied some local and global approximation theorems for the q-Durrmeyer operators. Acar and Aral [6] studied the pointwise convergence for q-Bernstein operators and their q-derivatives. Dalmanoglu [41] introduced Kantorovich type modification of q-Bernstein operators. For some deatails we refer the readers to (cf. [4], [53], [59], [75], [87], [113], [116], [119], [135] [164], [165] and [167] etc.)

0.3 Statistical convergence

The study of statistical convergence is another interesting area of research in the field of approximation theory. In 1951, Fast [57] introduced the concept of statistical convergence. After that, Gadjiev [62] proved Korovkin type approximation theorem via statistical convergence. Kolk [106] proved that statistical convergence is stronger than ordinary convergence. In this direction, for some related papers we refer to (cf. [51], [81], [117], [123], [150] etc.).

Statistical convergence: Any sequence $x =< x_n >$, is said to be statistically convergent to a number l if for any given $\epsilon > 0$, we get $\lim_{n\to\infty} \frac{|\{k : |x_k - l| \geq \epsilon\}|}{n} = 0$ and it is denoted by $st - \lim_{n\to\infty} x_n = l$.

A-Statistical convergence: Let $A = (a_{jn})$ be a non-negative infinite summability matrix. For a given sequence $x =< x_n >$, the A-transform of x denoted by $Ax = (Ax)_j$ is defined as

$$(Ax)_j = \sum_{n=1}^{\infty} a_{jn} x_n$$

provided the series converges for each j. A is said to be regular if $\lim_j (Ax)_j = L$ whenever $\lim_n x_n = L$. The A-density of $K, K \subseteq \mathbb{N}$(the set of the natural numbers), denoted by $\delta_A(K)$, is defined as $\delta_A(K) = \lim_j \sum_{n=1}^{\infty} a_{jn}\chi_K(n)$, provided the limit exists, where $\chi_K(n)$ is the characteristic function of K.

A sequence $x =< x_n >$ is said to be A-statistically convergent to L i.e. $st_A - \lim_n x_n = L$ if for every $\epsilon > 0, \lim_j \sum_{n:|x_n - L|\geq\epsilon} a_{jn} = 0$ or equivalently $\delta_A\{n \in K : |x_n - L| \geq \epsilon\} = 0$.

If we replace A by C_1 then A is a Cesaro matrix of order one and A-statistical convergence is reduced to the statistical convergence. Similarly, if $A = I$, the identity matrix then A-statistical convergence reduces to ordinary convergence.

Let $A = (a_{jn})$ be a non negative infinite regular summability matrix and $< b_j >$ be a positive non increasing sequence. If for every $\epsilon > 0, \lim_j \frac{1}{b_j} \sum_{n:|x_n - L| \geq \epsilon} a_{jn} = 0$, then we say that the sequence $x =< x_n >$, converges A-statistically to number L with the rate of $o(b_j)$ and this is denoted by $x_n - L = st_A - o(b_n)$, as $n \to \infty$. If for every $\epsilon > 0$, $\sup_j \frac{1}{j} \sum_{n:|x_n| \geq \epsilon} a_{jn} < \infty$, then x is called A-statistically bounded with the rate $O(b_n)$, as $n \to \infty$.

0.4 Bivariate and GBS (Generalized Boolean sum) Extension

Kingsely [105] initiated the study of Bernstein operators for the two variable case for the class of $k-$ times continuously differentiable functions on a closed and bounded rectangle region. Butzer [36] investigated some approximation properties for these operators. After that, Stancu [149] introduced another kind of generalization of Bernstein operators for the two and several variables case. Barbosu et. al [24] introduced a q-analogue of the bivariate Durrmeyer operators and studied the rate of convergence in terms of modulus of continuity. Örkcü [122] introduced a bivariate generalization of the q-Szász-Mirakyan-Kantorovich operators and established the rate of pointwise convergence and weighted A-statistical approximation properties. Bivariate generalization for several positive linear operators have been discussed in ([20], [25], [50], [144] and [159] etc.)

Dobrescu and Matei [49] introduced the GBS-Bernstein operators and obtained some convergence theorems for these operators. Subsequently, Badea and Cottin [23] obtained Korovkin theorems for GBS operators. After that, Pop [133] introduced Voronovoskaja type theorems for certain GBS operators. Recently, Sidharth et al.

[143] introduced the GBS operators of bivariate q-Bernstein-Schurer-Kantorovich type and estimated the rate of convergence in terms of mixed modulus of smoothness. We refer the readers to some of the related papers (cf. [24], [25], [26], [27], [55], [56], [134] and [132] etc.)

0.5 Notations and Basic definitions

Now, we recall some basic definitions of q-calculus. For more details we refer to books (cf. [18], [99] etc.).

Definition 0.5.1. *For a non-negative integer n, the q-integer $[n]_q$ is defined as*

$$[n]_q = \begin{cases} \dfrac{1-q^n}{1-q}, & q \neq 1, \\ n, & q = 1. \end{cases}$$

Definition 0.5.2. *The q-factorial $[n]_q!$ is defined as*

$$[n]_q! = \begin{cases} [1]_q[2]_q[3]_q.......[n]_q, & n \geq 1, \\ 1, & n = 0. \end{cases}$$

Definition 0.5.3. *The q-binomial coefficient is defined as*

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q![n-k]_q!}.$$

Definition 0.5.4. *The q-beta function is defined as*

$$B_q(k,n) = \int_0^1 t^{k-1}(1-qt)_q^{n-1} d_q t$$

or

$$B_q(k,n) = \frac{[k-1]_q![n-1]_q!}{[n+k-1]_q!}.$$

Definition 0.5.5. *Suppose that $0 < a < b$, $0 < q < 1$ and f be a real valued function. Then the q-Jackson integral of f over the interval $[0,b]$ and over the generic interval $[a,b]$ are respectively defined as*

$$\int_0^b f(x) d_q x = (1-q)b \sum_{j=0}^{\infty} f(bq^j)q^j \tag{0.5.1}$$

and
$$\int_a^b f(x)d_qx = \int_0^b f(x)d_qx - \int_0^a f(x)d_qx,$$
provided the sum in (0.5.1) converges absolutely.

Throughout this thesis we denote by C, a constant not necessarily the same at each occurrence and $[0,\infty)$ by $\mathbb{R}_0^+$.
Let $f \in C(I)$ be the space of all continuous functions on an interval I.

Definition 0.5.6. *For r being a positive integer, the r^{th} order modulus of continuity $\omega_r(f,\delta)$, for $f \in C(I)$ is defined by*
$$\omega_r(f,\delta) = \sup_{0<|h|\leqslant\delta} \big\{|\Delta_h^r f(x)| : x, x+rh \in I\big\},$$
where $\Delta_h^k f(x)$ is the k^{th} forward difference with step length h.

Let us define
$$W^r(I) = \{g \in C(I) : g^{(r)} \in C(I)\}.$$

Definition 0.5.7. *The Peetre's $K-$functional [129] is defined as*
$$K_r(f;\delta) = \inf_{g\in W^r(I)} \big\{||f-g|| + \delta||g^{(r)}|| : \delta > 0\big\},$$

From [47], it is known that there exists a constant $C > 0$, such that
$$K_r(f;\delta^r) \leq C\omega_r(f;\delta) \;\; \forall r = 1,\; 2,\; 3... \tag{0.5.2}$$
Let $C_B(I)$ be the space of all continuous and bounded functions on I with the norm $||f|| = \sup\limits_{x\in I}|f(x)|$.

Definition 0.5.8. *The r^{th} order Ditzian-Totik modulus of smoothness $\omega_{r,\phi}(f,\delta)$, for $f \in C_B(I)$ is defined by*
$$\omega_{r,\phi}(f,\delta) = \sup_{0<|h|\leqslant\delta} \big\{|\Delta_{h\phi}^r f(x)| : x, x+rh\phi \in I\big\},$$

where $\Delta^r_{h\phi} f(x)$ *is the* r^{th} *forward difference with step length* $h\phi$. *In the particular case* $r = 1$, *we denote* $\omega_{1,\phi}(f, \delta)$ *by* $\omega_\phi(f, \delta)$.

Let us define

$$W^r_\phi(I) = \{g : g^{(r-1)} \in AC_{loc}(I)\, and\, ||\phi^r g^{(r)}|| < \infty\},$$

where $g^{(r-1)} \in AC_{loc}(I)$ means $g^{(r-1)}$ is absolutely continuous on every $[a, b] \subset I$. In the particular case $r = 1$, we denote $W^1_\phi(I)$ by $W_\phi(I)$.

Definition 0.5.9. *The* $K-$*functional is defined as*

$$K_{r,\phi}(f; \delta) = \inf_{g \in W^r_\phi(I)} \left\{ ||f - g|| + \delta^r ||\phi^r g^{(r)}|| : \delta > 0 \right\}.$$

In the particular case $r = 1$, *we denote* $K_{1,\phi}(f; \delta)$ *by* $K_\phi(f; \delta)$.

From [48], it is known that there exists a constant $C > 0$, such that

$$C^{-1}\omega_{r,\phi}(f; \delta) \leq K_{r,\phi}(f; \delta^r) \leq C\omega_{r,\phi}(f; \delta) \;\; \forall r = 1,\ 2,\ 3... \tag{0.5.3}$$

Let $DBV(I)$ be the class of all absolutely continuous functions f having a derivative f' equivalent with a function of bounded variation on every finite subinterval of I. We observe that the functions $f \in DBV(I)$ possess a representation

$$f(x) = \int_0^x g(t)dt + f(0), \tag{0.5.4}$$

where $g \in BV(I)$ i.e. g is a function of bounded variation on every finite subinterval of I. Throughout the thesis $\bigvee_a^b f(x)$ denotes the total variation of $f(x)$ on $[a, b]$.

0.6 Contents of the Thesis

The present thesis consists of eight chapters and the contents of these are as given below:

Chapter 1. In this chapter, we construct a genuine family of Bernstein-Durrmeyer type operators based on Polya basis functions. We establish some moment estimates and the direct results which include global approximation theorem in terms of classical modulus of continuity and a local approximation theorem in terms of the second order Ditizian-Totik modulus of smoothness. Also, we obtain a Voronovskaya-type asymptotic theorem and a quantitative Voronovskaya-type estimate. Lastly, we study the approximation of functions having a derivative of bounded variation.

The results in this chapter are published in **Filomat (University of Niš, Serbia)**.

Chapter 2. This chapter is the study of the Bézier variant of genuine-Durrmeyer type operators having Polya basis functions. We give a global approximation theorem in terms of second order modulus of continuity, a direct approximation theorem and a Voronovskaja type theorem by using the Ditzian-Totik modulus of smoothness. Next, we establish the rate of convergence for functions whose derivatives are of bounded variation. Further, we show the rate of convergence of these operators to certain functions by illustrative graphics using the Matlab algorithms.

The results in this chapter are published in **Carpathian Journal of Mathematics (North University of Baia Mare, Romania).**

Chapter 3. In the present chapter, we establish a link between the Szász-Durrmeyer type operators and multiple Appell polynomials. We study a quantitative-Voronovskaya type theorem in terms of weighted modulus of smoothness using sixth order central moment and Grüss-Voronovskaya type theorem. We also establish a local approximation theorem by means of the Steklov means in terms of first and second order modulus of continuity and Voronovskaya type asymtotic theorem. Further, we discuss the degree of approximation for functions in polynomial weighted spaces. Lastly, we find the rate of approximation of functions having a derivative of bounded variation.

The contents of this chapter accepted for publication in **Journal of Inequalities and Applications (Springer Publications)**.

Chapter 4. In this chapter, we introduce the Bézier-variant of Durrmeyer modification of the Bernstein operators based on a function τ, which is infinite times continuously differentiable and strictly increasing function on $[0,1]$ such that $\tau(0) = 0$ and $\tau(1) = 1$. Here the Korovkin set $\{1, t, t^2\}$ is generalized to $\{1, \tau, \tau^2\}$. We give the rate of approximation of these operators in terms of usual modulus of continuity and the $K-$functional. Next, we establish the quantitative Voronovskaja type theorem. In the last section, we obtain the rate of convergence for functions having derivatives of bounded variation.

The contents of this chapter are published in **Results in Mathematics (Springer Publications).**

Chapter 5. In this chapter, we construct a sequence of Stancu-type operators that are based on the same function τ, defined in preceding chapter. We compare the new operators with classical Stancu operators and show that on a certain interval, these operators present a better degree of approximation than the original ones. Also, we give a direct approximation theorem by means of the Ditzian-Totik modulus of smoothness and a Voronovskaja type theorem.

The results of this chapter are published in **Numerical Functional Analysis and**

Optimization (Taylor and Francis Group).

Chapter 6. In the present chapter, we introduce the Bézier variant of the modified Srivastava-Gupta operators defined by Yadav [163] and give a local approximation theorem by means of the Ditzian-Totik modulus of smoothness and the rate of convergence for absolutely continuous functions having a derivative equivalent to a function of bounded variation. Furthermore, we show the comparisons of the rate of convergence of the Srivastava-Gupta operators vis-a-vis its Bézier variant to a certain function by illustrative graphics using Matlab algorithms.

The content of this chapter are published in **Revista de la Union Mathematica Argentina (Union Mathematica Argentina).**

Chapter 7. In this chapter, we propose the Stancu-Durrmeyer-type modification of q-Bernstein operators by means of q-Jackson integral. Here, we study basic convergence theorem, local approximation theorem in terms of the first and second order modulus of continuity and direct theorems by means of Lipschitz type space and Lipschitz type maximal function. Further, we establish the Korovkin type approximation theorem by using A-statistical convergence. Lastly, we give the rates of A-statistical convergence in terms of the modulus of continuity.

The contents of this chapter are published in **Applied Mathematics and Information Sciences (Natural Sciences).**

Chapter 8. This chapter is in continuation of our work in Chapter 7. Here, we construct a bivariate generalization of Stancu-Durrmeyer-type operators and study the rate of convergence by means of the complete modulus of continuity and the partial moduli of continuity and the degree of approximation with the aid of the

Peetre's $K-$functional. Also, we show the convergence of the operators to a certain function for two different values of q by illustrative graphics. Subsequently, we define the GBS(Generalized Boolean Sum) operators of Stancu-Durrmeyer type and give the rate of approximation by means of the mixed modulus of smoothness and the Lipschitz class of Bögel-continuous functions.

The results of this chapter are accepted for publication in **Mathematical Communications (Croatian Mathematical Society).**

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Published/Accepted

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Chapter 1

A genuine family of Bernstein-Durrmeyer type operators based on Polya basis functions

1.1 Introduction

In 1968, Stancu [148] introduced a sequence of positive linear operators $P_n^{(\alpha)} : C[0,1] \longrightarrow C[0,1]$, depending on a non negative parameter α as

$$P_n^{(\alpha)}(f;x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) p_{n,k}^{(\alpha)}(x), \tag{1.1.1}$$

where $p_{n,k}^{(\alpha)}(x)$ is the Polya distribution with density function given by

$$p_{n,k}^{(\alpha)}(x) = \binom{n}{k} \frac{\prod_{v=0}^{k-1}(x+v\alpha)\prod_{\mu=0}^{n-k-1}(1-x+\mu\alpha)}{\prod_{\lambda=0}^{n-1}(1+\lambda\alpha)}, \quad x \in [0,1].$$

In case $\alpha = 0$, the operators (1.1.1) reduce to the classical Bernstein polynomials. For these operators , Lupas and Lupas [111] considered a special case of the above operators for $\alpha = \dfrac{1}{n}$ which reduces to

$$P_n^{(\frac{1}{n})}(f;x) = \frac{2(n!)}{(2n)!}\sum_{k=0}^{n}\binom{n}{k} f\left(\frac{k}{n}\right)(nx)_k(n-nx)_{n-k}, \tag{1.1.2}$$

where the rising factorial $(x)_n$ is given by $(x)_n = x(x+1)(x+2).....(x+n-1)$ with $(x)_0 = 1$.

Gupta and Rassias [82] introduced the Durrmeyer-type integral modification for the operators (1.1.2) and obtained local and global direct estimates and a Voronvskaya-type asymptotic formula. Very recently, Gupta [76] defined a genuine Durrmeyer type modification of the operators given by (1.1.2) and obtained a Voronovskaya-type asymptotic theorem and a local approximation theorem. Gonska and Păltănea [67] established a very interesting link between the Bernstein polynomials and their Bernstein-Durrmeyer variants with several particular cases which preserve linear functions and gave recursion formula for moments and estimates for simultaneous approximation of derivatives. After that, the same authors [66] established quantitative Voronovskaya-type assertions in terms of the first order and second order moduli of smoothness.

Motivated by these studies, for $f \in L_B[0,1]$, the space of bounded and Lebesgue integrable functions on $[0,1]$ and a parameter $\rho > 0$, we now propose a genuine Durrmeyer type modification of the operators given by (1.1.2), which preserve linear functions, as

$$U_n^{\rho}(f;x) = \sum_{k=0}^{n} F_{n,k}^{\rho} p_{n,k}^{(\frac{1}{n})}(x), \tag{1.1.3}$$

where

$$F_{n,k}^{\rho} = \begin{cases} \int_0^1 f(t)\mu_{n,k}^{\rho} dt, & 1 \le k \le n-1 \\ f(0), & k = 0 \\ f(1), & k = n, \end{cases}$$

and

$$\mu_{n,k}^{\rho}(t) = \frac{t^{k\rho-1}(1-t)^{(n-k)\rho-1}}{B(k\rho,(n-k)\rho)},$$

$B(m,n)$ being the beta function defined as

$$B(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}, \quad m,n>0.$$

For $\rho = 1$, the operators U_n^ρ reduce to the operators defined by Gupta [76] and when $\rho \to \infty$, these operators reduce to the operators considered by Lupas and Lupas [111], in view of the fact that $F_{n,k}^\rho \to f\left(\frac{k}{n}\right)$, as shown by Gonska and Păltănea [3,Thm 2.3, p.786].

The purpose of this chapter is to establish a global approximation theorem in terms of the classical second order modulus of continuity and a local-approximation theorem in terms of the second order Ditzian-Totik modulus of smoothness, a Voronovskaya type asymptotic theorem and also a quantitative Voronovskaya type estimate. In the last section of the chapter, the approximation of functions having a derivative of bounded variation is also discussed.

1.2 Auxiliary Results

Lemma 1.2.1. *[115] For the operators defined by (1.1.2), one has*

(i) $P_n^{(\frac{1}{n})}(1;x) = 1,$

(ii) $P_n^{(\frac{1}{n})}(t;x) = x,$

(iii) $P_n^{(\frac{1}{n})}(t^2;x) = x^2 + \dfrac{2x(1-x)}{n+1},$

(iv) $P_n^{(\frac{1}{n})}(t^3;x) = x^3 + \dfrac{6nx^2(1-x)}{(n+1)(n+2)} + \dfrac{6x(1-x)}{(n+1)(n+2)},$

(v) $P_n^{(\frac{1}{n})}(t^4;x) = x^4 + \dfrac{12(n^2+1)x^3(1-x)}{(n+1)(n+2)(n+3)} + \dfrac{12(3n-1)x^2(1-x)}{(n+1)(n+2)(n+3)} + \dfrac{2(13n-1)x(1-x)}{n(n+1)(n+2)(n+3)}.$

Lemma 1.2.2. *For* $U_n^\rho(t^m;x), \ m = 0,1,2,3,4,$ *we obtain,*

(i) $U_n^\rho(1;x) = 1,$

(ii) $U_n^\rho(t;x) = x,$

(iii) $U_n^\rho(t^2;x) = \dfrac{n\rho}{n\rho+1}\left(x^2 + \dfrac{2x(1-x)}{n+1}\right) + \dfrac{x}{n\rho+1},$

(iv) $U_n^\rho(t^3;x) = \dfrac{1}{(n\rho+1)(n\rho+2)}\Big\{n^2\rho^2\left(x^3 + \dfrac{6nx^2(1-x)}{(n+1)(n+2)} + \dfrac{6x(1-x)}{(n+1)(n+2)}\right) + 3n\rho\left(x^2 + \dfrac{2x(1-x)}{n+1}\right) + 2x\Big\},$

(v) $U_n^\rho(t^4;x) = \dfrac{1}{(n\rho+1)(n\rho+2)(n\rho+3)}\Big\{n^3\rho^3\left(x^4 + \dfrac{12(n^2+1)x^3(1-x)}{(n+1)(n+2)(n+3)} + \dfrac{12(3n-1)x^2(1-x)}{(n+1)(n+2)(n+3)} + \dfrac{2(13n-1)x(1-x)}{n(n+1)(n+2)(n+3)}\right) + 6n^2\rho^2\left(x^3 + \dfrac{6nx^2(1-x)}{(n+1)(n+2)} + \dfrac{6x(1-x)}{(n+1)(n+2)}\right) + 11n\rho\left(x^2 + \dfrac{2x(1-x)}{n+1}\right) + 6x\Big\}.$

By a simple calculation and using Lemma 1.2.1, we obtain the proof of the lemma. Hence we omit the details.

In our next lemma, we find the central moment estimates required for the main results of the paper.

Lemma 1.2.3. *For* $U_n^\rho((t-x)^m;x),\quad m \in \mathbb{N}\bigcup\{0\}$, *we have,*

(i) $U_n^\rho((t-x);x) = 0,$

(ii) $U_n^\rho((t-x)^2;x) = \dfrac{(2n\rho+n+1)x(1-x)}{(n+1)(n\rho+1)},$

(iii) $U_n^\rho((t-x)^4;x) = \dfrac{x(1-x)}{(n+1)(n+2)(n+3)(1+\rho n)(2+\rho n)(3+\rho n)}\Big\{3\Big(-\rho(2\rho+1)^2n^4+2(14\rho^3+14\rho^2+9\rho+3)n^3+(120\rho^2+109\rho+36)n^2+6(23\rho+11)n+36\Big)(x(1-x))+2\Big((13\rho^3+18\rho^2+11\rho+3)n^3+(-\rho^3+54\rho^2+55\rho+18)n^2+33(2\rho+1)n+18\Big)\Big\}.$

Consequently, for every $x \in [0,1]$,

$$\lim_{n\to\infty} nU_n^\rho((t-x)^2;x) = \frac{2\rho+1}{\rho}\phi^2(x)$$

and

$$\lim_{n\to\infty} n^2U_n^\rho((t-x)^4;x) = \frac{-3(2\rho+1)^2}{\rho^2}\phi^4(x), \tag{1.2.1}$$

where $\phi^2(x) = x(1-x)$.

Remark 1.2.4. *From Lemma 1.2.3, we have*

$$\begin{aligned} U_n^{\rho}((t-x)^2;x) &\leq \frac{(2\rho+1)}{(n\rho+1)}\phi^2(x) \leq \frac{1}{4}\frac{(2\rho+1)}{(n\rho+1)}, \ \ \forall x \in [0,1] \\ &= \delta_{n,\rho}^2, \ (say), \end{aligned}$$

and for every $x \in [0,1]$,

$$U_n^{\rho}((t-x)^4;x) \leq \frac{C}{n^2}\phi^4(x),$$

where C is some positive contant dependent on ρ.

In what follows, let $\|.\|$ denote the uniform norm on $[0,1]$, $||f|| = \sup_{x\in[0,1]} |f(x)|$.

Lemma 1.2.5. *For every $f \in C[0,1]$, we have*

$$|U_n^{\rho}(f;.)\| \leq \|f\|.$$

Proof. Using Lemma 1.2.2, the proof of this Lemma is straightforward. Hence we skip the details. □

Now, we present a theorem which will be needed to obtain a quantitative Voronovskaya type theorem using the least concave majorant of the first order modulus of continuity.

Theorem 1.2.6. *[65] Let $q \in \mathbb{N}\bigcup\{0\}$ and $f \in C^q[0,1]$ (space of q-times continuously differentiable functions on $[0,1]$) and let $L : C[0,1] \to C[0,1]$ be a positive linear operator. Then*

$$\left|L(f;x) - \sum_{r=0}^{q} L\left((t-x)^r; \frac{f^{(r)}(x)}{r!}\right)\right| \leq \frac{L(|e_1-x|^q;x)}{q!}\widetilde{\omega}\left(f^{(q)}; \frac{1}{(q+1)}\frac{L(|t-x|^{q+1};x)}{L(|t-x|^q;x)}\right),$$

where $\widetilde{\omega}$ is the least concave majorant of the first-order modulus of continuity.

1.3 Main results

1.3.1 Global approximation theorem

First we will establish a global approximation theorem for the operators $U_n^{\rho}(f;x)$ using the classical modulus of continuity.

Theorem 1.3.1. *Let $f \in C[0,1]$. Then there exists a constant $C > 0$, such that*

$$\|U_n^\rho(f;.) - f(.)\| \leq C\omega_2(f;\delta_{n,\rho}),$$

where $\delta_{n,\rho}$ is as defined in Remark 1.2.4 and $C > 0$, is an absolute constant.

Proof. Let $g \in W^2[0,1]$ and $t \in [0,1]$. Then by Taylor's expansion, we have

$$g(t) - g(x) = (t-x)g'(x) + \int_x^t (t-u)g''(u)du.$$

Applying $U_n^\rho(.;x)$ to both sides of the above equation, we get

$$U_n^\rho(g;x) - g(x) = g'(x)U_n^\rho(t-x;x) + U_n^\rho\bigg(\int_x^t (t-u)g''(u)du;x\bigg).$$

Using Lemma 1.2.3 and Remark 1.2.4, we get

$$\begin{aligned} |U_n^\rho(g;x) - g(x)| &\leq U_n^\rho\bigg(\bigg|\int_x^t |(t-u)||g''(u)|du\bigg|;x\bigg) \\ &\leq \frac{\|g''\|}{2}U_n^\rho((t-x)^2;x) \\ &\leq \frac{\|g''\|}{2}\delta_{n,\rho}^2. \end{aligned} \tag{1.3.1}$$

Now, for $f \in C[0,1]$ and $g \in W^2[0,1]$, using Lemma 1.2.5 and inequality (1.3.1), we obtain

$$\begin{aligned} |U_n^\rho(f;x) - f(x)| &\leq |U_n^\rho(f-g;x)| + |U_n^\rho(g;x) - g(x)| + |f(x) - g(x)| \\ &\leq 2\|f-g\| + \frac{\|g''\|}{2}\delta_{n,\rho}^2. \end{aligned}$$

Taking infimum on the right side of the above inequality over all $g \in W^2[0,1]$, we get

$$|U_n^\rho(f;x) - f(x)| \leq 2K_2(f;\delta_{n,\rho}^2), \ \forall x \in [0,1].$$

Consequently,

$$\|U_n^\rho(.;x) - f(.)\| \leq 2K_2(f;\delta_{n,\rho}^2).$$

Using the relation (0.5.2) between K-functional and the second order modulus of continuity, we get the required result. This completes the proof. □

1.3.2 Local approximation theorem

Next, we shall prove a local approximation theorem by using the Ditzian-Totik modulus of smoothness.

Theorem 1.3.2. *Let $f \in C[0,1]$. Then for every $x \in [0,1]$, we have*

$$|U_n^\rho(f;x) - f(x)| \leq C\omega_{2,\phi}\bigg(f; \sqrt{\frac{2\rho+1}{n\rho+1}}\bigg),$$

where $C > 0$, is an absolute constant and $\phi(x) = \sqrt{x(1-x)}$.

Proof. Let $g \in W_\phi^2[0,1]$ and $t \in [0,1]$. Then by Taylor's expansion, we have

$$g(t) - g(x) = (t-x)g'(x) + \int_x^t (t-u)g''(u)du.$$

Now applying $U_n^\rho(f;x)$ to both sides of the above equation and using Lemma 1.2.3, we get

$$\begin{aligned} |U_n^\rho(g;x) - g(x)| &= |g'(x)U_n^\rho(t-x;x)| + \left|U_n^\rho\bigg(\int_x^t (t-u)g''(u)du; x\bigg)\right| \\ &\leq U_n^\rho\bigg(\left|\int_x^t |t-u||g''(u)|du\right|; x\bigg). \end{aligned} \tag{1.3.2}$$

Since $\phi^2(x)$ is a concave function on $[0,1]$, for $u = \lambda x + (1-\lambda)t$, $0 < \lambda \leq 1$ and $x \in (0,1)$, we get,

$$\frac{|t-u|}{\phi^2(u)} = \frac{|t-\lambda x-(1-\lambda)t|}{\phi^2(\lambda x+(1-\lambda)t)} \leq \frac{\lambda|t-x|}{\lambda\phi^2(x)+(1-\lambda)\phi^2(t)} \leq \frac{|t-x|}{\phi^2(x)}.$$

Combining this inequality and equation (1.3.2), in view of Remark 1.2.4 we obtain

$$\begin{aligned} |U_n^\rho(g;x) - g(x)| &\leq U_n^\rho\bigg(\left|\int_x^t \frac{|t-u|}{\phi^2(u)}\|\phi^2 g''\|du\right|; x\bigg) \\ &\leq \frac{1}{\phi^2(x)}\|\phi^2 g''\|U_n^\rho((t-x)^2;x) \\ &\leq \frac{2\rho+1}{n\rho+1}\|\phi^2 g''\|. \end{aligned}$$

Using Lemma 1.2.5 and taking infimum over all $g \in W_\phi^2[0,1]$ on the right hand side of the above inequality, we have

$$\begin{aligned}
|U_n^\rho(f;x) - f(x)| &\leq |U_n^\rho(f-g;x)| + |U_n^\rho(g;x) - g(x)| + |g(x) - f(x)| \\
&\leq 2\|f-g\| + \frac{2\rho+1}{n\rho+1}\|\phi^2 g''\| \\
&\leq 2K_{2,\phi}\left(f; \frac{2\rho+1}{n\rho+1}\right).
\end{aligned}$$

In view of the relation (0.5.3), we get the desired result. □

1.3.3 Voronovskaya theorem

Now we will establish a Voronovskaya type asymptotic for the operators $U_n^\rho(f;x)$.

Theorem 1.3.3. *Let $f \in L_B[0,1]$. If f'' exists at a point $x \in [0,1]$, then*

$$\lim_{n\to\infty} n[U_n^\rho(f;x) - f(x)] = \frac{2\rho+1}{2\rho}\phi^2(x)f''(x). \tag{1.3.3}$$

The convergence in (1.3.3) holds uniformly if $f'' \in C[0,1]$.

Proof. By Taylor's expansion for the function f, we may write

$$f(t) - f(x) = (t-x)f'(x) + \frac{(t-x)^2}{2}f''(x) + \eta(t,x)(t-x)^2,$$

where $\eta(t,x) \to 0$ as $t \to x$ and is a bounded function, $\forall t \in [0,1]$. Now, applying U_n^ρ on the above Taylor's expansion and using Lemma 1.2.3, we get

$$\begin{aligned}
U_n^\rho(f;x) - f(x) &= U_n^\rho((t-x)f'(x);x) + U_n^\rho\left(\frac{(t-x)^2}{2}f''(x);x\right) + U_n^\rho\left(\eta(t,x)(t-x)^2;x\right) \\
&= \frac{f''(x)}{2}\frac{(2n\rho+n+1)}{(n+1)(n\rho+1)}x(1-x) + U_n^\rho\left(\eta(t,x)(t-x)^2;x\right).
\end{aligned}$$

Hence,

$$\lim_{n\to\infty} n[U_n^\rho(f;x) - f(x)] = \frac{(2\rho+1)}{2\rho}\phi^2(x)f''(x) + F,$$

where $F = \lim\limits_{n\to\infty} nU_n^\rho\left(\eta(t,x)(t-x)^2;x\right)$. Now we shall show that $F=0$. Since $\eta(t,x) \to 0$ as $t \to x$, for a given $\epsilon > 0$, there exists a $\delta > 0$, such that $|\eta(t,x)| < \epsilon$

whenever $|t-x|<\delta$. For $|t-x|\geq\delta$, the boundedness of $\eta(t,x)$ on $[0,1]$ implies that $|\eta(t,x)|\leq M\dfrac{(t-x)^2}{\delta^2}$, for some $M>0$. Let $\chi_\delta(t)$ be the characteristic function of the interval $(x-\delta,x+\delta)$. Then, from Lemma 1.2.3, for every $x\in[0,1]$, we have

$$\begin{aligned}\left|U_n^\rho\Big(\eta(t,x)(t-x)^2;x\Big)\right| &\leq U_n^\rho\Big(|\eta(t,x)|(t-x)^2\chi_\delta(t);x\Big)\\ &+ U_n^\rho\Big(|\eta(t,x)|(t-x)^2(1-\chi_\delta(t));x\Big)\\ &\leq \epsilon U_n^\rho\Big((t-x)^2;x\Big)+\frac{M}{\delta^2}U_n^\rho\Big((t-x)^4;x\Big)\\ &= \epsilon O\Big(\frac{1}{n}\Big)+\frac{M}{\delta^2}O\Big(\frac{1}{n^2}\Big).\end{aligned}$$

Thus, for every $x\in[0,1]$, we get

$$n\left|U_n^\rho\Big(\eta(t,x)(t-x)^2;x\Big)\right| = \epsilon O(1)+\frac{M}{\delta^2}O\Big(\frac{1}{n}\Big).$$

Taking limit as $n\to\infty$, due to the arbitrariness of $\epsilon>0$, we get $F=0$. This completes the proof of the first assertion of the theorem.

To prove the uniformity assertion, it is sufficient to remark that $\delta(\epsilon)$ in the above proof can be chosen to be independent of $x\in[0,1]$ and all the other estimates hold uniformly on $[0,1]$. This completes the proof. □

1.3.4 Quantitative Voronovskaya type theorem

In the next result, we establish a quantitative Voronovskaya type estimate for the operators U_n^ρ.

Theorem 1.3.4. *For $f\in C^2[0,1]$ and $x\in[0,1]$, we have*

$$\left|U_n^\rho(f;x)-f(x)-\frac{f''(x)}{2!}(t-x)^2\right|\leq\frac{1}{2!}\frac{(2\rho+1)}{(n\rho+1)}\phi^2(x)\widetilde{\omega}\Big(f'';\frac{M}{3\sqrt{n}}\Big),$$

where, $M>0$ and $\widetilde{\omega}(f;.)$ is the least concave majorant of first order of the function $\omega(f;.)$(see[65], Thm 2.1), defined as

$$\widetilde{\omega}(f;\epsilon)=\begin{cases}\sup\limits_{0\leq x\leq\epsilon\leq y\leq 1}\dfrac{(\epsilon-x)\omega(f;y)+(y-x)\omega(f;x)}{y-x}, & 0\leq\epsilon\leq 1,\\ \omega(f;1), & \epsilon>1.\end{cases}$$

Proof. Using the Cauchy-Schwarz inequality, we note that

$$\frac{U_n^\rho(|t-x|^3;x)}{U_n^\rho((t-x)^2;x)} \leq \sqrt{\frac{U_n^\rho((t-x)^4;x)}{U_n^\rho((t-x)^2;x)}}. \tag{1.3.4}$$

For $q=2$, using Theorem 1.2.6 and equation (1.3.4), we get

$$\begin{aligned}\left|U_n^\rho(f;x)-f(x)-\frac{f''(x)}{2!}U_n^\rho((t-x)^2;x)\right| &\leq \frac{U_n^\rho((t-x)^2;x)}{2!}\widetilde{\omega}\left(f'';\frac{1}{3}\frac{U_n^\rho(|t-x|^3;x)}{U_n^\rho((t-x)^2;x)}\right)\\ &\leq \frac{U_n^\rho((t-x)^2;x)}{2!}\widetilde{\omega}\left(f'';\frac{1}{3}\sqrt{\frac{U_n^\rho((t-x)^4;x)}{U_n^\rho((t-x)^2;x)}}\right)\\ &\leq \frac{1}{2!}\frac{(2\rho+1)}{(n\rho+1)}\phi^2(x)\widetilde{\omega}\left(f'';\frac{M}{3\sqrt{n}}\right).\end{aligned}$$

This completes the proof. □

1.3.5 Rate of approximation

In order to discuss the approximation of functions with derivatives of bounded variation, we express the operators U_n^ρ in an integral form as follows:

$$U_n^\rho(f;x)=\int_0^1 K_n^\rho(x,t)f(t)dt, \tag{1.3.5}$$

where the kernel $K_n^\rho(x,t)$ is given by

$$K_n^\rho(x,t)=\sum_{k=1}^{n-1}p_{n,k}^{(1/n)}(x)\mu_{n,k}^\rho(t)+p_{n,0}^{(\frac{1}{n})}(x)\delta(t)+p_{n,n}^{(\frac{1}{n})}(x)\delta(1-t),$$

$\delta(u)$ being the Dirac-delta function.

Lemma 1.3.5. *For a fixed $x\in(0,1)$ and sufficiently large n, we have*

(i) $\xi_n^\rho(x,y)=\int_0^y K_n^\rho(x,t)dt \leq \dfrac{(2\rho+1)}{(n\rho+1)}\dfrac{\phi^2(x)}{(x-y)^2}$, $0\leq y<x$,

(ii) $1-\xi_n^\rho(x,z)=\int_z^1 K_n^\rho(x,t)dt \leq \dfrac{(2\rho+1)}{(n\rho+1)}\dfrac{\phi^2(x)}{(z-x)^2}$, $x<z<1$.

Proof. (i) Using Lemma 1.2.3, we get

$$\begin{aligned}
\xi_n^\rho(x,y) &= \int\limits_0^y K_n^\rho(x,t)dt \le \int_0^y \left(\frac{x-t}{x-y}\right)^2 K_n^\rho(x,t)dt \\
&\le U_n^\rho((t-x)^2;x)(x-y)^{-2} \\
&\le \frac{(2\rho+1)}{(n\rho+1)}\frac{\phi^2(x)}{(x-y)^2}.
\end{aligned}$$

The proof of (ii) is similar hence the details are omitted. □

Theorem 1.3.6. *Let $f \in DBV([0,1])$. Then, for every $x \in (0,1)$ and sufficiently large n, we have*

$$\begin{aligned}
|U_n^\rho(f;x) - f(x)| \le\ & \sqrt{\frac{2\rho+1}{(n\rho+1)}}\phi(x)\left|\frac{f'(x+)-f'(x-)}{2}\right| + \frac{2\rho+1}{(n\rho+1)}\phi^2(x)x^{-1} \\
& \sum_{k=1}^{[\sqrt{n}]} \bigvee_{x-x/k}^{x} ((f^{'})_x) + \frac{x}{\sqrt{n}} \bigvee_{x-x/\sqrt{n}}^{x} ((f^{'})_x) + \frac{2\rho+1}{(n\rho+1)}\frac{\phi^2(x)}{(1-x)} \\
& \sum_{k=1}^{[\sqrt{n}]} \bigvee_{x}^{x+(1-x)/k} ((f^{'})_x) + \frac{(1-x)}{\sqrt{n}} \bigvee_{x}^{x+(1-x)/\sqrt{n}} ((f^{'})_x),
\end{aligned}$$

where f'_x is defined by

$$f_x^{'}(t) = \begin{cases} f^{'}(t) - f^{'}(x-), & 0 \le t < x \\ 0, & t = x \\ f^{'}(t) - f^{'}(x+) & x < t < 1. \end{cases} \tag{1.3.6}$$

Proof. Since $U_n^\rho(1;x) = 1$, using (1.3.5), for every $x \in (0,1)$ we get

$$\begin{aligned}
U_n^\rho(f;x) - f(x) &= \int_0^1 K_n^\rho(x,t)(f(t)-f(x))dt \\
&= \int_0^1 K_n^\rho(x,t)\int_x^t f^{'}(u)dudt.
\end{aligned} \tag{1.3.7}$$

For any $f \in DBV[0,1]$, from (1.3.6) we may write

$$\begin{aligned}
f'(u) =\ & (f')_x(u) + \frac{1}{2}(f'(x+)+f'(x-)) + \frac{1}{2}(f'(x+)-f'(x-))sgn(u-x) \\
& +\delta_x(u)[f'(u) - \frac{1}{2}(f'(x+)+f'(x-))],
\end{aligned} \tag{1.3.8}$$

where

$$\delta_x(u) = \begin{cases} 1 \ , \ u = x \\ 0 \ , \ u \neq x \end{cases} .$$

Obviously,

$$\int_0^1 \left(\int_x^t \left(f'(u) - \frac{1}{2}(f'(x+) + f'(x-)) \right) \delta_x(u) du \right) K_n^\rho(x,t) dt = 0. \qquad (1.3.9)$$

Using Lemma 1.2.3, we get

$$\begin{aligned}
&\int_0^1 \left(\int_x^t \frac{1}{2}(f'(x+) + f'(x-)) du \right) K_n^\rho(x,t) dt \\
&= \frac{1}{2}(f'(x+) + f'(x-)) \int_0^1 (t-x) K_n^\rho(x,t) dt \\
&= \frac{1}{2}(f'(x+) + f'(x-)) U_n^\rho((t-x); x) \\
&= 0. \qquad (1.3.10)
\end{aligned}$$

Applying Cauchy-Schwarz inequality, we have

$$\begin{aligned}
&\int_0^1 K_n^\rho(x,t) \bigg(\int_x^t \frac{1}{2}(f'(x+) - f'(x-)) sgn(u-x) du \bigg) dt \\
&\leq \frac{1}{2} \mid f'(x+) - f'(x-) \mid \int_0^1 |t-x| K_n^\rho(x,t) dt \\
&= \frac{1}{2} \mid f'(x+) - f'(x-) \mid U_n^\rho(|t-x|; x) \\
&\leq \frac{1}{2} \mid f'(x+) - f'(x-) \mid \left(U_n^\rho((t-x)^2; x) \right)^{1/2}. \qquad (1.3.11)
\end{aligned}$$

Using Lemma 1.2.3 and equations (1.3.7-1.3.11) , we obtain

$$\begin{aligned}
&|U_n^\rho(f;x) - f(x)| \leq \frac{1}{2}|f'(x+) - f'(x-)| \sqrt{\frac{2\rho+1}{(n\rho+1)}} \phi(x) \\
&+ \left| \int_0^x \int_x^t ((f^{'})_x(u) du) K_n^\rho(x,t) dt + \int_x^1 \int_x^t \left((f^{'})_x(u) du \right) K_n^\rho(x,t) dt \right|. \\
&\qquad (1.3.12)
\end{aligned}$$

Now, let

$$A_n^\rho(f', x) = \int_0^x \int_x^t ((f')_x(u) du) K_n^\rho(x,t) dt,$$

and

$$B_n^\rho(f',x)=\int_x^1\int_x^t((f')_x(u)du)K_n^\rho(x,t)dt.$$

Thus our problem is reduced to calculate the estimates of the terms $A_n^\rho(f',x)$ and $B_n^\rho(f',x)$. Since $\int_a^b d_t\xi_n^\rho(x,t)\leq 1$ for all $[a,b]\subseteq[0,1]$, using integration by parts and applying Lemma 1.3.5 with $y=x-x/\sqrt{n}$, we have

$$\begin{aligned}
|A_n^\rho(f',x)| &= \left|\int_0^x\int_x^t\left((f')_x(u)du\right)d_t\xi_n^\rho(x,t)\right| \\
&= \left|\int_0^x \xi_n^\rho(x,t)(f')_x(t)dt\right| \\
&\leq \int_0^y |(f')_x(t)|\ |\xi_n^\rho(x,t)|dt+\int_y^x |(f')_x(t)|\ |\xi_n^\rho(x,t)|dt \\
&\leq \frac{2\rho+1}{n\rho+1}\phi^2(x)\int_0^y\bigvee_t^x((f')_x)(x-t)^{-2}dt+\int_y^x\bigvee_t^x((f')_x)dt \\
&\leq \frac{2\rho+1}{n\rho+1}\phi^2(x)\int_0^y\bigvee_t^x((f^{'})_x)(x-t)^{-2}dt+\frac{x}{\sqrt{n}}\bigvee_{x-x/\sqrt{n}}^x((f')_x) \\
&= \frac{2\rho+1}{n\rho+1}\phi^2(x)\int_0^{x-x/\sqrt{n}}\bigvee_t^x((f^{'})_x)(x-t)^{-2}dt+\frac{x}{\sqrt{n}}\bigvee_{x-x/\sqrt{n}}^x((f')_x).
\end{aligned}$$

Substituting $u=x/(x-t)$, we get

$$\begin{aligned}
\int_0^{x-x/\sqrt{n}}(x-t)^{-2}\bigvee_t^x(f^{'})_x)dt &= \int_1^{\sqrt{n}}\bigvee_{x-x/u}^x((f^{'})_x)du \\
&\leq x^{-1}\sum_{k=1}^{[\sqrt{n}]}\int_k^{k+1}\bigvee_{x-x/k}^x((f^{'})_x)du \\
&\leq x^{-1}\sum_{k=1}^{[\sqrt{n}]}\bigvee_{x-x/k}^x((f')_x).
\end{aligned}$$

Thus,

$$|A_n^\rho(f',x)|\leq\frac{2\rho+1}{n\rho+1}\phi^2(x)x^{-1}\sum_{k=1}^{[\sqrt{n}]}\bigvee_{x-x/k}^x((f^{'})_x)+\frac{x}{\sqrt{n}}\bigvee_{x-x/\sqrt{n}}^x((f^{'})_x). \tag{1.3.13}$$

Again, using integration by parts in $B_n^\rho(f^{'},x)$ and applying Lemma 1.3.5 with

$z = x + (1-x)/\sqrt{n}$, we have

$$\begin{aligned}
|B_n^\rho(f^{'},x)| &= \Big|\int_x^1 \int_x^t ((f')_x(u)du) K_n^\rho(x,t)dt\Big| \\
&= \Big|\int_x^z \int_x^t ((f')_x(u)du) d_t(1-\xi_n^\rho(x,t)) + \int_z^1 \int_x^t ((f')_x(u)du) d_t(1-\xi_n^\rho(x,t))\Big| \\
&= \Big|\Big[\int_x^t ((f')_x(u)du)(1-\xi_n^\rho(x,t))\Big]_x^z - \int_x^z (f')_x(t)(1-\xi_n^\rho(x,t))dt \\
&\quad + \int_z^1 \int_x^t ((f')_x(u)du) d_t(1-\xi_n^\rho(x,t))\Big| \\
&= \Big|\int_x^z ((f')_x(u)du)(1-\xi_n^\rho(x,z)) - \int_x^z (f')_x(t)(1-\xi_n^\rho(x,t))dt \\
&\quad + \Big[\int_x^t ((f')_x(u)du)(1-\xi_n^\rho(x,t))\Big]_z^1 - \int_z^1 (f')_x(t)(1-\xi_n^\rho(x,t))dt\Big| \\
&= \Big|\int_x^z (f')_x(t)(1-\xi_n^\rho(x,t))dt + \int_z^1 (f')_x(t)(1-\xi_n^\rho(x,t))dt\Big| \\
&\leq \frac{2\rho+1}{n\rho+1}\phi^2(x)\int_z^1 \bigvee_x^t (f')_x (t-x)^{-2}dt + \int_x^z \bigvee_x^t (f')_x dt \\
&= \frac{2\rho+1}{n\rho+1}\phi^2(x)\int_{x+(1-x)/\sqrt{n}}^1 \bigvee_x^t (f')_x (t-x)^{-2}dt + \frac{(1-x)}{\sqrt{n}} \bigvee_x^{x+(1-x)/\sqrt{n}} (f')_x.
\end{aligned}$$

By substituting $u = (1-x)/(t-x)$, we get

$$\begin{aligned}
|B_n^\rho(f^{'},x)| &\leq \frac{2\rho+1}{n\rho+1}\phi^2(x)\int_1^{\sqrt{n}} \bigvee_x^{x+(1-x)/u} (f')_x (1-x)^{-1}du + \frac{(1-x)}{\sqrt{n}} \bigvee_x^{x+(1-x)/\sqrt{n}} (f')_x \\
&\leq \frac{2\rho+1}{n\rho+1}\frac{\phi^2(x)}{(1-x)}\sum_{k=1}^{[\sqrt{n}]}\int_k^{k+1} \bigvee_x^{x+(1-x)/k} (f')_x du + \frac{(1-x)}{\sqrt{n}} \bigvee_x^{x+(1-x)/\sqrt{n}} (f')_x \\
&= \frac{2\rho+1}{n\rho+1}\frac{\phi^2(x)}{(1-x)}\sum_{k=1}^{[\sqrt{n}]} \bigvee_x^{x+(1-x)/k} (f')_x + \frac{(1-x)}{\sqrt{n}} \bigvee_x^{x+(1-x)/\sqrt{n}} (f')_x.
\end{aligned} \tag{1.3.14}$$

Collecting the estimates (1.3.12-1.3.14), we get the required result. This completes the proof. □

Chapter 2

Bézier variant of genuine-Durrmeyer type operators based on Polya distribution

2.1 Construction of Operator

In this chapter, we define the Bézier variant of the genuine Bernstein-Durrmeyer operators given by (1.1.3) and study some approximation properties.

Păltănea [127] defined a class of operators $\overline{U}_n^{\rho} : C[0,1] \to \prod_n$(the class of all polynomials of degree at most n) as follows:

$$\overline{U}_n^{\rho}(f;x) := \sum_{k=1}^{n-1} \left(\int_0^1 \frac{t^{k\rho-1}(1-t)^{(n-k)\rho-1}}{\beta(k\rho,(n-k)\rho)} f(t)dt \right) \overline{p}_{n,k}(x) + f(0)(1-x)^n + f(1)x^n, \tag{2.1.1}$$

where $\rho > 0$, $x \in [0,1]$ and $\overline{p}_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}$.

Remark 2.1.1. *Let us consider a function* $f : [0,1] \to \mathbb{R}$,

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x},\ x \neq 0, \\ 0,\ x = 0. \end{cases}$$

For $\rho = 1$ and $n = 20$, we computed the error of approximation for U_n^ρ given by (1.1.3) and $\overline{U}_n^\rho$ given by (2.1.1) at certain points from $[0.6, 0.7]$ in the Table 1.

Table 1. *Error of approximation for U_n^ρ and $\overline{U}_n^\rho$*

x	$\lvert U_n^\rho(f;x) - f(x)\rvert$	$\lvert \overline{U}_n^\rho(f;x) - f(x)\rvert$
0.60	0.001239490900	0.001509158700
0.61	0.000351701000	0.001971924300
0.62	0.0004378093000	0.002365328800
0.63	0.001135096600	0.002695352000
0.64	0.001746098000	0.002967653800
0.65	0.002276601100	0.003187565800
0.66	0.002732224700	0.003360091000
0.67	0.003118397000	0.003489901500
0.68	0.003440343200	0.003581349400

From the above results it follows that the error of approximation for U_n^ρ is better than $\overline{U}_n^\rho$ to the function f at the points $x_i = 0.6 + 0.01 \cdot i$, $i = \overline{0, 8}$.

Here, we propose a Bézier variant of the operators given by (1.1.3) as

$$U_{n,\alpha}^\rho(f;x) = \sum_{k=0}^{n} F_{n,k}^\rho Q_{n,k}^{(\alpha)}(x), \tag{2.1.2}$$

where, $Q_{n,k}^{(\alpha)}(x) = [J_{n,k}(x)]^\alpha - [J_{n,k+1}(x)]^\alpha$, $\alpha \geq 1$ with $J_{n,k}(x) = \sum_{j=k}^{n} p_{n,j}^{(1/n)}(x)$, when $k \leq n$ and 0 otherwise. Clearly, $U_{n,\alpha}^\rho$ is a linear positive operator. If $\alpha = 1$, then the operators $U_{n,\alpha}^\rho$ reduce to the operators U_n^ρ.

The aim of this chapter is to investigate a global approximation theorem, a direct approximation result, a quantitative Voronovskaya type theorem and the rate of convergence for differentiable functions having derivatives of bounded variation on $[0, 1]$ for the operators (2.1.2). Lastly, we show the rate of convergence of these operators to certain functions by some illustrative graphics.

2.2 Auxiliary Results

In what follows let $\|.\|$ denotes uniform norm on $C[0,1]$.

Lemma 2.2.1. *Let $f \in C[0,1]$. Then, we have*

$$\| U_{n,\alpha}^{\rho}(f;.) \| \leq \alpha \| f \| .$$

Proof. Using the inequality $| a^{\alpha} - b^{\alpha} | \leq \alpha \mid a - b \mid$ with $0 \leq a, b \leq 1, \alpha \geq 1$ and from the definition of $Q_{n,k}^{(\alpha)}$, we have

$$0 \;<\; [J_{n,k}(x)]^{\alpha} - [J_{n,k+1}(x)]^{\alpha} \leq \alpha(J_{n,k}(x) - J_{n,k+1}(x) = \alpha p_{n,k}^{(1/n)}(x).$$

Hence from the definition of $U_{n,\alpha}^{\rho}$ and Lemma 1.2.5, we obtain

$$\| U_{n,\alpha}^{\rho}(f) \| \leq \alpha \| U_{n}^{\rho}(f) \| \leq \alpha \| f \| .$$

This completes the proof. □

Remark 2.2.2. *We have*

$$\begin{aligned} U_{n,\alpha}^{\rho}(e_0;x) &= \sum_{k=0}^{n} Q_{n,k}^{(\alpha)}(x) = [J_{n,0}(x)]^{\alpha} \\ &= \left[\sum_{j=0}^{n} p_{n,j}^{(1/n)}(x) \right]^{\alpha} = 1, \; \textit{since} \; \sum_{j=0}^{n} p_{n,j}^{(1/n)}(x) = 1. \end{aligned}$$

The operators $U_{n,\alpha}^{\rho}$ can be expressed in an integral form as follows:

$$U_{n,\alpha}^{\rho}(f;x) = \int_0^1 K_{n,\alpha}^{\rho}(x,t)f(t)dt, \tag{2.2.1}$$

where

$$K_{n,\alpha}^{\rho}(x,t) = \sum_{k=1}^{n-1} Q_{n,k}^{\alpha}(x)p_{n,k}^{\rho}(t) + Q_{n,0}^{\alpha}(x)\delta(t) + Q_{n,n}^{\alpha}(x)\delta(1-t),$$

$\delta(u)$ being the Dirac-delta function.

Lemma 2.2.3. *For a fixed $x \in (0,1)$ and sufficiently large n, we have*

(i) $\xi^{\rho}_{n,\alpha}(x,y) = \int_0^y K^{\rho}_{n,\alpha}(x,t)dt \leq \alpha \dfrac{2\rho+1}{n\rho+1}\dfrac{\phi^2(x)}{(x-y)^2}$, $0 \leq y < x$,

(ii) $1 - \xi^{\rho}_{n,\alpha}(x,z) = \int_z^1 K^{\rho}_{n,\alpha}(x,t)dt \leq \alpha \dfrac{2\rho+1}{n\rho+1}\dfrac{\phi^2(x)}{(z-x)^2}$, $x < z < 1$,

Proof. (i) Using Lemma 2.2.1 and Remark 1.2.4, we get

$$\begin{aligned}
\xi^{\rho}_{n,\alpha}(x,y) &= \int\limits_0^y K^{\rho}_{n,\alpha}(x,t)dt \leq \int_0^y \left(\frac{x-t}{x-y}\right)^2 K^{\rho}_{n,\alpha}(x,t)dt \\
&\leq \frac{U^{\rho}_{n,\alpha}((t-x)^2;x)}{(x-y)^2} \leq \alpha \frac{U^{\rho}_{n}((t-x)^2;x)}{(x-y)^2} \leq \alpha \frac{2\rho+1}{n\rho+1}\frac{\phi^2(x)}{(x-y)^2}.
\end{aligned}$$

The proof of (ii) is similar hence the details are omitted. □

2.3 Main Results

2.3.1 Direct results

For $f \in C[0,1]$ and $\delta > 0$, the appropriate Peetre's K-functional [129] is defined by

$$\overline{K}_2(f;\delta) = \inf_{g \in W^2[0,1]} \left\{ \|f-g\| + \delta\|g'\| + \delta^2\|g''\| \right\}. \tag{2.3.1}$$

From [47], there exists an absolute constant $C > 0$, such that

$$\overline{K}_2(f;\gamma) \leq C\omega_2(f;\sqrt{\gamma}), \tag{2.3.2}$$

where $\omega_2(f;\sqrt{\gamma})$ is the second order modulus of continuity of $f \in C[0,1]$.

First, we establish a global approximation theorem for the operators $U^{\rho}_{n,\alpha}$ using the classical modulus of continuity.

Theorem 2.3.1. *Let $f \in C[0,1]$. Then there exists an absolute constant $C > 0$, such that*

$$||U^{\rho}_{n}(f;.) - f(.)|| \leq C\omega_2(f;\sqrt{\alpha^{1/2}\delta_{n,\rho}}),$$

where $\delta_{n,\rho}$ is the same as defined in Remark 1.2.4.

Proof. Let $g \in W^2[0,1]$ and $t \in [0,1]$. Then by Taylor's expansion, we have

$$g(t) - g(x) = (t-x)g'(x) + \int_x^t (t-u)g''(u)du.$$

Now applying $U_{n,\alpha}^{\rho}$ to both sides of the above equation, we get

$$U_{n,\alpha}^{\rho}(g;x) - g(x) = g'(x)U_{n,\alpha}^{\rho}(t-x;x) + U_{n,\alpha}^{\rho}\bigg(\int_x^t (t-u)g''(u)du;x\bigg).$$

Applying Cauchy-Schwarz inequality and Lemma 2.2.1 and Remark 1.2.4, we get

$$\begin{aligned} |U_{n,\alpha}^{\rho}(g;x) - g(x)| &\leq |g'(x)||U_{n,\alpha}^{\rho}(t-x;x)| + \left|U_{n,\alpha}^{\rho}\bigg(\int_x^t (t-u)g''(u)du;x\bigg)\right| \\ &\leq \|g'\|U_{n,\alpha}^{\rho}(|t-x|;x) + \frac{\|g''\|}{2}U_{n,\alpha}^{\rho}((t-x)^2;x) \\ &\leq \|g'\|\bigg(U_{n,\alpha}^{\rho}((t-x)^2;x)\bigg)^{\frac{1}{2}} + \alpha\frac{\|g''\|}{2}U_n^{\rho}((t-x)^2;x) \\ &\leq \sqrt{\alpha}\|g'\|\delta_{n,\rho} + \alpha\frac{\|g''\|}{2}\delta_{n,\rho}^2. \end{aligned} \tag{2.3.3}$$

Now, for $f \in C[0,1]$ and $g \in W^2[0,1]$, using Lemma 2.2.1 and (2.3.3), we obtain

$$\begin{aligned} |U_{n,\alpha}^{\rho}(f;x) - f(x)| &\leq |U_{n,\alpha}^{\rho}(f-g;x)| + |U_{n,\alpha}^{\rho}(g;x) - g(x)| + |f(x) - g(x)| \\ &\leq (\alpha+1)\|f-g\| + \sqrt{\alpha}\|g'\|\delta_{n,\rho} + \alpha\frac{\|g''\|}{2}\delta_{n,\rho}^2. \end{aligned}$$

Taking infimum on the right side of the above inequality over all $g \in W^2[0,1]$, we get

$$|U_{n,\alpha}^{\rho}(f;x) - f(x)| \leq (\alpha+1)K_2(f;\alpha^{1/2}\delta_{n,\rho}), \ \forall x \in [0,1].$$

Using the relation (2.3.2) between the K-functional and the second order modulus of continuity, we get the required result. This completes the proof. □

Now, we establish a direct approximation theorem by means of Ditzian-Totik modulus of smoothness.

Theorem 2.3.2. *Let f be in $C[0,1]$ and $\phi(x) = \sqrt{x(1-x)}$ then for every $x \in (0,1)$, we have*

$$| U_{n,\alpha}^{\rho}(f;x) - f(x) | < C\omega_{\phi}\bigg(f;\sqrt{\frac{2\rho+1}{n\rho+1}}\bigg), \tag{2.3.4}$$

where C is a constant independent of n and x.

Proof. Using the representation $g(t) = g(x) + \int_x^t g'(u)du$, we get

$$\left|U^{\rho}_{n,\alpha}(g;x) - g(x)\right| = \left|U^{\rho}_{n,\alpha}\left(\int_x^t g'(u)du; x\right)\right|. \tag{2.3.5}$$

For any $x, t \in (0,1)$, we find that

$$\left|\int_x^t g'(u)du\right| \leq ||\phi g'||\left|\int_x^t \frac{1}{\phi(u)}du\right|. \tag{2.3.6}$$

But,

$$\begin{aligned}
\left|\int_x^t \frac{1}{\phi(u)}du\right| &= \left|\int_x^t \frac{1}{\sqrt{u(1-u)}}du\right| \leq \left|\int_x^t \left(\frac{1}{\sqrt{u}} + \frac{1}{\sqrt{1-u}}\right)du\right| \\
&\leq 2\Big(\mid \sqrt{t} - \sqrt{x} \mid + \mid \sqrt{1-t} - \sqrt{1-x} \mid\Big) \\
&= 2|t-x|\left(\frac{1}{\sqrt{t}+\sqrt{x}} + \frac{1}{\sqrt{1-t}+\sqrt{1-x}}\right) \\
&< 2|t-x|\left(\frac{1}{\sqrt{x}} + \frac{1}{\sqrt{1-x}}\right) \leq \frac{2\sqrt{2}\ |t-x|}{\phi(x)}.
\end{aligned} \tag{2.3.7}$$

Combining (2.3.5)-(2.3.7) and using Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}
|U^{\rho}_{n,\alpha}(g;x) - g(x)| &< 2\sqrt{2}||\phi g'||\phi^{-1}(x)U^{\rho}_{n,\alpha}(|t-x|;x) \\
&\leq 2\sqrt{2}||\phi g'||\phi^{-1}(x)\Big(U^{\rho}_{n,\alpha}((t-x)^2;x)\Big)^{1/2} \\
&\leq 2\sqrt{2}||\phi g'||\phi^{-1}(x)\Big(\alpha\ U^{\rho}_{n}((t-x)^2;x)\Big)^{1/2}.
\end{aligned}$$

Now, using Remark 1.2.4, we get

$$|U^{\rho}_{n,\alpha}(g;x) - g(x)| < C\sqrt{\frac{2\rho+1}{n\rho+1}}||\phi g'||. \tag{2.3.8}$$

Using Lemma 2.2.1 and (2.3.8), we can write

$$\begin{aligned}
\mid U^{\rho}_{n,\alpha}(f:x) - f(x) \mid &\leq \mid U^{\rho}_{n,\alpha}(f-g;x) \mid + |f(x) - g(x)| + \mid U^{\rho}_{n,\alpha}(g;x) - g(x) \mid \\
&\leq C\left(||f-g|| + \left(\sqrt{\frac{2\rho+1}{n\rho+1}}||\phi g'||\right)\right).
\end{aligned} \tag{2.3.9}$$

Taking infimum on the right hand side of the above inequality over all $g \in W_{\phi}[0,1]$, we get

$$\mid U^{\rho}_{n,\alpha}(f;x) - f(x) \mid < CK_{\phi}\left(f; \sqrt{\frac{2\rho+1}{n\rho+1}}\right).$$

Using the relation (0.5.3), this theorem is proven. □

2.3.2 Quantitative Voronovskaja type theorem

In the following we prove a quantitative Voronovskaja type theorem for the operator $U^{\rho}_{n,\alpha}$. This result is established using the first order Ditzian-Totik modulus of smoothness.

Theorem 2.3.3. *For any $f \in C^2[0,1]$ the following inequalities hold*

i) $\left|n\left\{U^{\rho}_{n,\alpha}(f;x)-f(x)-f'(x)U^{\rho}_{n,\alpha}(t-x;x)-\dfrac{1}{2}f''(x)U^{\rho}_{n,\alpha}\left((t-x)^2;x\right)\right\}\right|$
$\leq C\omega_{\phi}\left(f'',\phi(x)n^{-1/2}\right),$

ii) $\left|n\left\{U^{\rho}_{n,\alpha}(f;x)-f(x)-f'(x)U^{\rho}_{n,\alpha}(t-x;x)-\dfrac{1}{2}f''(x)U^{\rho}_{n,\alpha}\left((t-x)^2;x\right)\right\}\right|$
$\leq C\phi(x)\omega_{\phi}\left(f'',n^{-1/2}\right).$

Proof. Let $f \in C^2[0,1]$ be given and $t, x \in [0,1]$. Then by Taylor's expansion, we have

$$f(t)-f(x)=(t-x)f'(x)+\int_x^t (t-u)f''(u)du.$$

Hence

$$\begin{aligned} f(t)-f(x)-(t-x)f'(x)-\frac{1}{2}(t-x)^2f''(x) &= \int_x^t (t-u)f''(u)du-\int_x^t (t-u)f''(x)du \\ &= \int_x^t (t-u)[f''(u)-f''(x)]du. \end{aligned}$$

Applying $U^{\rho}_{n,\alpha}(\cdot;x)$ to both sides of the above relation, we get

$$\begin{aligned} &\left|U^{\rho}_{n,\alpha}(f;x)-f(x)-f'(x)U^{\rho}_{n,\alpha}(t-x;x)-\frac{1}{2}f''(x)U^{\rho}_{n,\alpha}\left((t-x)^2;x\right)\right| \\ &\leq U^{\rho}_{n,\alpha}\left(\left|\int_x^t |t-u||f''(u)-f''(x)|du\right|;x\right). \end{aligned} \tag{2.3.10}$$

The quantity $\left|\int_x^t |f''(u)-f''(x)|\,|t-u|du\right|$ was estimated in [58, p. 337] as follows:

$$\left|\int_x^t |f''(u)-f''(x)||t-u|du\right| \leq 2\|f''-g\|(t-x)^2+2\|\phi g'\|\phi^{-1}(x)|t-x|^3, \tag{2.3.11}$$

where $g \in W_{\phi}[0,1]$.

Now combining (2.3.10)-(2.3.11), applying Cauchy-Schwarz inequality and using Remark 1.2.4, we get

$$\begin{aligned}
&\left|U_{n,\alpha}^{\rho}(f;x)-f(x)-f'(x)U_{n,\alpha}^{\rho}(t-x;x)-\frac{1}{2}f''(x)U_{n,\alpha}^{\rho}\left((t-x)^2;x\right)\right|\\
&\leq 2\|f''-g\|U_{n,\alpha}^{\rho}\left((t-x)^2;x\right)+2\|\phi g'\|\phi^{-1}(x)U_{n,\alpha}^{\rho}\left(|t-x|^3;x\right)\\
&\leq 2\|f''-g\|\alpha\frac{2\rho+1}{n\rho+1}\phi^2(x)+2\alpha\|\phi g'\|\phi^{-1}(x)\left\{U_n^{\rho}(t-x)^2;x\right\}^{1/2}\left\{U_n^{\rho}\left((t-x)^4;x\right)\right\}^{1/2}\\
&\leq 2\|f''-g\|\alpha\frac{2\rho+1}{n\rho+1}\phi^2(x)+2\alpha\frac{C}{n}\|\phi g'\|\sqrt{\frac{2\rho+1}{n\rho+1}}\phi^2(x)\\
&\leq C\left\{\frac{2\rho+1}{n\rho+1}\phi^2(x)\|f''-g\|+\frac{1}{n}\sqrt{\frac{2\rho+1}{n\rho+1}}\phi^2(x)\|\phi g'\|\right\}\\
&\leq \frac{C}{n}\left\{\phi^2(x)\|f''-g\|+n^{-1/2}\phi^2(x)\|\phi g'\|\right\}.
\end{aligned}$$

Since $\phi^2(x)\leq\phi(x)\leq 1, x\in[0,1]$, we obtain

$$\begin{aligned}
&\left|U_{n,\alpha}^{\rho}(f;x)-f(x)-f'(x)U_{n,\alpha}^{\rho}(t-x;x)-\frac{1}{2}f''(x)U_{n,\alpha}^{\rho}\left((t-x)^2;x\right)\right|\\
&\leq\frac{C}{n}\left\{\|f''-g\|+n^{-1/2}\phi(x)\|\phi g'\|\right\}.
\end{aligned}$$

and

$$\begin{aligned}
&\left|U_{n,\alpha}^{\rho}(f;x)-f(x)-f'(x)U_{n,\alpha}^{\rho}(t-x;x)-\frac{1}{2}f''(x)U_{n,\alpha}^{\rho}\left((t-x)^2;x\right)\right|\\
&\leq\frac{C}{n}\phi(x)\left\{\|f''-g\|+n^{-1/2}\|\phi g'\|\right\}.
\end{aligned}$$

Taking the infimum on the right hand side of the above relations over $g\in W_\phi[0,1]$, we get

$$\begin{aligned}
&\left|n\left\{U_{n,\alpha}^{\rho}(f;x)-f(x)-f'(x)U_{n,\alpha}^{\rho}(t-x;x)-\frac{1}{2}f''(x)U_{n,\alpha}^{\rho}\left((t-x)^2;x\right)\right\}\right|\\
&\leq\begin{cases} CK_\phi\left(f'';\phi(x)n^{-1/2}\right),\\ C\phi(x)K_\phi\left(f'';n^{-1/2}\right).\end{cases}
\end{aligned}$$

Using relation (0.5.3), the theorem is proved. □

2.3.3 Degree of approximation

Lastly, we discuss the approximation of functions with a derivative of bounded variation on $[0,1]$.

Theorem 2.3.4. *Let $f \in DBV[0,1]$. Then, for every $x \in (0,1)$ and sufficiently large n, we have*

$$\begin{aligned} |U^{\rho}_{n,\alpha}(f;x)-f(x)| &\le \{|f'(x+)+\alpha f'(x-)|+\alpha|f'(x+)-f'(x-)|\}\frac{\sqrt{\alpha}}{\alpha+1}\frac{2\rho+1}{n\rho+1}\phi^2(x) \\ &+\frac{2\rho+1}{n\rho+1}\frac{\alpha\,\phi(x)}{x}\sum_{k=1}^{[\sqrt{n}]}\left(\bigvee_{x-x/k}^{x} f'_x\right)+\frac{x}{\sqrt{n}}\left(\bigvee_{x-x/\sqrt{n}}^{x} f'_x\right) \\ &+\frac{2\rho+1}{n\rho+1}\frac{\alpha\,\phi(x)}{1-x}\sum_{k=1}^{[\sqrt{n}]}\left(\bigvee_{x}^{x+(1-x)/k} f'_x\right)+\frac{1-x}{\sqrt{n}}\left(\bigvee_{x}^{x+(1-x)/\sqrt{n}} f'_x\right), \end{aligned}$$

where f'_x is defined by

$$f'_x(t)=\begin{cases} f'(t)-f'(x-), & 0\le t<x \\ 0, & t=x \\ f'(t)-f'(x+) & x<t<1. \end{cases} \tag{2.3.12}$$

Proof. Since $U^{\rho}_{n,\alpha}(1;x)=1$, using (2.2.1), for every $x\in(0,1)$ we get

$$\begin{aligned} U^{\rho}_{n,\alpha}(f;x)-f(x) &= \int_0^1 K^{\rho}_{n,\alpha}(x,t)(f(t)-f(x))dt \\ &= \int_0^1 K^{\rho}_{n,\alpha}(x,t)\int_x^t f'(u)du dt. \end{aligned} \tag{2.3.13}$$

For any $f\in DBV[0,1]$, from (2.3.12) we may write

$$\begin{aligned} f'(u) &= f'_x(u)+\frac{1}{\alpha+1}(f'(x+)+\alpha f'(x-))+\frac{1}{2}(f'(x+)-f'(x-))\Big(sgn(u-x) \\ &+\frac{\alpha-1}{\alpha+1}\Big)+\gamma_x(u)[f'(u)-\frac{1}{2}(f'(x+)+f'(x-))], \end{aligned} \tag{2.3.14}$$

where

$$\gamma_x(u)=\begin{cases} 1\ , & u=x \\ 0\ , & u\neq x \end{cases}.$$

Obviously,

$$\int_0^1 \left(\int_x^t \left(f'(u) - \frac{1}{2}(f'(x+) + f'(x-)) \right) \gamma_x(u) du \right) K_{n,\alpha}^{\rho}(x,t) dt = 0. \quad (2.3.15)$$

Using Lemma 2.2.1, we get

$$\begin{aligned} A_1 &= \int_0^1 \left(\int_x^t \frac{1}{\alpha+1}(f'(x+) + \alpha f'(x-)) du \right) K_{n,\alpha}^{\rho}(x,t) dt \\ &= \frac{1}{\alpha+1}(f'(x+) + \alpha f'(x-)) \int_0^1 (t-x) K_{n,\alpha}^{\rho}(x,t) dt \\ &= \frac{1}{\alpha+1}(f'(x+) + \alpha f'(x-)) U_{n,\alpha}^{\rho}((t-x); x) \qquad (2.3.16) \end{aligned}$$

and

$$\begin{aligned} |A_2| &= \left| \int_0^1 K_{n,\alpha}^{\rho}(x,t) \left(\int_x^t \frac{1}{2}(f'(x+) - f'(x-)) \left(sgn(u-x) + \frac{\alpha-1}{\alpha+1} \right) du \right) dt \right| \\ &= \left| \frac{1}{2} \left(f'(x+) - f'(x-) \right) \left[- \int_0^x \left(\int_t^x \left(sgn(u-x) + \frac{\alpha-1}{\alpha+1} \right) du \right) K_{n,\alpha}^{\rho}(x,t) dt \right. \right| \\ &\quad + \int_x^1 \left(\int_x^t \left(sgn(u-x) + \frac{\alpha-1}{\alpha+1} \right) du \right) K_{n,\alpha}^{\rho}(x,t) dt \Bigg] \\ &\leq \frac{\alpha}{\alpha+1} |f'(x+) - f'(x-)| \int_0^1 |t-x| \, K_{n,\alpha}^{\rho}(x,t) dt \\ &= \frac{\alpha}{\alpha+1} |f'(x+) - f'(x-)| U_{n,\alpha}^{\rho} \left(|t-x| \, ; x \right). \qquad (2.3.17) \end{aligned}$$

Using Lemma 2.2.1 and equations (2.3.13-2.3.17) and applying Cauchy-Schwarz inequality, we obtain

$$\begin{aligned} |U_{n,\alpha}^{\rho}(f;x) - f(x)| \leq\ & \frac{1}{\alpha+1} |f'(x+) + \alpha f'(x-)| \sqrt{\alpha} \sqrt{\frac{2\rho+1}{n\rho+1}} \phi(x) \\ & + \frac{\alpha}{\alpha+1} |f'(x+) - f'(x-)| \sqrt{\alpha} \sqrt{\frac{2\rho+1}{n\rho+1}} \phi(x) \\ & + \left| \int_0^x \left(\int_x^t f_x'(u) du \right) K_{n,\alpha}^{\rho}(x,t) dt \right| \\ & + \left| \int_x^1 \left(\int_x^t f_x'(u) du \right) K_{n,\alpha}^{\rho}(x,t) dt \right|. \qquad (2.3.18) \end{aligned}$$

Now, let

$$A_{n,\alpha}^{\rho}(f_x', x) = \int_0^x \left(\int_x^t f_x'(u) du \right) K_{n,\alpha}^{\rho}(x,t) dt,$$

and

$$B^{\rho}_{n,\alpha}(f'_x,x)=\int_x^1\left(\int_x^t f'_x(u)du\right)K^{\rho}_{n,\alpha}(x,t)dt.$$

Thus our problem is reduced to calculate the estimates of the terms $A^{\rho}_{n,\alpha}(f'_x,x)$ and $B^{\rho}_{n,\alpha}(f'_x,x)$. From the definition of $\xi^{\rho}_{n,\alpha}$ given in Lemma 2.2.3, we can write

$$A^{\rho}_{n,\alpha}(f'_x,x)=\int_0^x\left(\int_x^t f'_x(u)du\right)\frac{\partial}{\partial t}\xi^{\rho}_{n,\alpha}(x,t)dt.$$

Applying the integration by parts, we get

$$\begin{aligned}\left|A^{\rho}_{n,\alpha}(f'_x,x)\right| &\le \int_0^x |f'_x(t)|\xi^{\rho}_{n,\alpha}(x,t)dt\\ &\le \int_0^{x-\frac{x}{\sqrt{n}}} |f'_x(t)|\xi^{\rho}_{n,\alpha}(x,t)dt+\int_{x-\frac{x}{\sqrt{n}}}^x |f'_x(t)|\xi^{\rho}_{n,\alpha}(x,t)dt := I_1+I_2.\end{aligned}$$

Since $f'_x(x)=0$ and $\xi^{\rho}_{n,\alpha}(x,t)\le 1$, we have

$$\begin{aligned}I_2 &:= \int_{x-\frac{x}{\sqrt{n}}}^x \left|f'_x(t)-f'_x(x)\right|\xi^{\rho}_{n,\alpha}(x,t)dt \le \int_{x-\frac{x}{\sqrt{n}}}^x\left(\bigvee_t^x f'_x\right)dt\\ &\le \left(\bigvee_{x-\frac{x}{\sqrt{n}}}^x f'_x\right)\int_{x-\frac{x}{\sqrt{n}}}^x dt=\frac{x}{\sqrt{n}}\left(\bigvee_{x-\frac{x}{\sqrt{n}}}^x f'_x\right).\end{aligned}$$

By applying Lemma 2.2.3 and considering $t=x-\dfrac{x}{u}$, we get

$$\begin{aligned}I_1 &\le \alpha\frac{2\rho+1}{n\rho+1}\phi^2(x)\int_0^{x-\frac{x}{\sqrt{n}}}\left|f'_x(t)-f'_x(x)\right|\frac{dt}{(x-t)^2}\\ &\le \alpha\frac{2\rho+1}{n\rho+1}\phi^2(x)\int_0^{x-\frac{x}{\sqrt{n}}}\left(\bigvee_t^x f'_x\right)\frac{dt}{(x-t)^2}\\ &= \alpha\frac{2\rho+1}{n\rho+1}\frac{\phi^2(x)}{x}\int_1^{\sqrt{n}}\left(\bigvee_{x-\frac{x}{u}}^x f'_x\right)du\le \alpha\frac{2\rho+1}{n\rho+1}\frac{\phi^2(x)}{x}\sum_{k=1}^{[\sqrt{n}]}\left(\bigvee_{x-\frac{x}{k}}^x f'_x\right).\end{aligned}$$

Therefore,

$$|A^{\rho}_{n,\alpha}(f'_x,x)|\le\alpha\frac{2\rho+1}{n\rho+1}\frac{\phi^2(x)}{x}\sum_{k=1}^{[\sqrt{n}]}\left(\bigvee_{x-\frac{x}{k}}^x f'_x\right)+\frac{x}{\sqrt{n}}\left(\bigvee_{x-\frac{x}{\sqrt{n}}}^x f'_x\right). \qquad (2.3.19)$$

Also, using integration by parts in $B_n^\rho(f_x', x)$ and applying Lemma 2.2.3 with $z = x + (1-x)/\sqrt{n}$, we have

$$\begin{aligned}
|B^\rho_{n,\alpha}(f_x^{'}, x)| &= \left| \int_x^1 \left(\int_x^t f_x'(u)du \right) K^\rho_{n,\alpha}(x,t)dt \right| \\
&= \left| \int_x^z \left(\int_x^t f_x'(u)du \right) \frac{\partial}{\partial t}(1-\xi^\rho_{n,\alpha}(x,t))dt \right. \\
&+ \left. \int_z^1 \left(\int_x^t f_x'(u)du \right) \frac{\partial}{\partial t}(1-\xi^\rho_{n,\alpha}(x,t))dt \right| \\
&= \left| \left[\int_x^t (f_x'(u)du)(1-\xi^\rho_{n,\alpha}(x,t)) \right]_x^z - \int_x^z f_x'(t)(1-\xi^\rho_{n,\alpha}(x,t))dt \right. \\
&+ \left. \int_z^1 \int_x^t (f_x'(u)du) \frac{\partial}{\partial t}(1-\xi^\rho_{n,\alpha}(x,t))dt \right| \\
&= \left| \int_x^z (f_x'(u)du)(1-\xi^\rho_{n,\alpha}(x,z)) - \int_x^z f_x'(t)(1-\xi^\rho_{n,\alpha}(x,t))dt \right. \\
&+ \left. \left[\int_x^t (f_x'(u)du)(1-\xi^\rho_{n,\alpha}(x,t)) \right]_z^1 - \int_z^1 f_x'(t)(1-\xi^\rho_{n,\alpha}(x,t))dt \right| \\
&= \left| \int_x^z f_x'(t)(1-\xi^\rho_{n,\alpha}(x,t))dt + \int_z^1 f_x'(t)(1-\xi^\rho_{n,\alpha}(x,t))dt \right| \\
&\leq \alpha \frac{2\rho+1}{n\rho+1}\phi^2(x) \int_z^1 \left(\bigvee_x^t f_x' \right) (t-x)^{-2}dt + \int_x^z \bigvee_x^t f_x' dt \\
&\leq \alpha \frac{2\rho+1}{n\rho+1}\phi^2(x) \int_{x+(1-x)/\sqrt{n}}^1 \left(\bigvee_x^t f_x' \right) (t-x)^{-2}dt + \frac{1-x}{\sqrt{n}} \left(\bigvee_x^{x+(1-x)/\sqrt{n}} f_x' \right).
\end{aligned}$$

By substituting $u = (1-x)/(t-x)$, we get

$$\begin{aligned}
|B^\rho_{n,\alpha} f_x', x)| &\leq \alpha \frac{2\rho+1}{n\rho+1}\phi^2(x) \int_1^{\sqrt{n}} \left(\bigvee_x^{x+(1-x)/u} f_x' \right) (1-x)^{-1}du + \frac{1-x}{\sqrt{n}} \left(\bigvee_x^{x+(1-x)/\sqrt{n}} f_x' \right) \\
&\leq \alpha \frac{2\rho+1}{n\rho+1} \frac{\phi^2(x)}{(1-x)} \sum_{k=1}^{[\sqrt{n}]} \left(\bigvee_x^{x+(1-x)/k} f_x' \right) + \frac{1-x}{\sqrt{n}} \left(\bigvee_x^{x+(1-x)/\sqrt{n}} f_x' \right). \qquad (2.3.20)
\end{aligned}$$

Collecting the estimates (2.3.18 - 2.3.20), we get the required result. This completes the proof. □

2.4 Numerical examples

Example 2.4.1. *Let us consider the following two functions* $f,\ g:[0,1]\to\mathbb{R}$

$$f(x)=\begin{cases} x^2\sin\dfrac{1}{x},\ x\neq 0\\ 0,\ x=0\end{cases}$$

and

$$g(x)=\begin{cases} (1-x)\cos\dfrac{\pi}{1-x},\ x\neq 1\\ 0,\ x=1\end{cases}$$

The function f *is differentiable and of bounded variation on* $[0,1]$*, while* g *is continuous but is not of bounded variation on* $[0,1]$*.*

For $n=20$*,* $\rho=1$ *and* $\alpha\in\left\{\dfrac{1}{2},1,\dfrac{3}{2}\right\}$*, the convergence of* $U^{\rho}_{n,\alpha}$ *to* f *and* g *is illustrated in Figure 2.1 and Figure 2.2, respectively.*

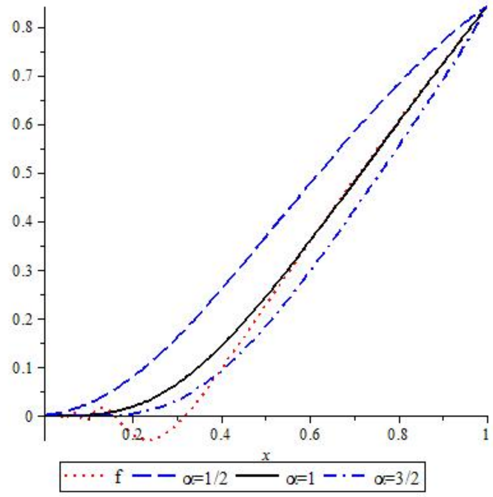

Figure 2.1: The convergence of $U^{\rho}_{n,\alpha}(f;x)$ *to* $f(x)$

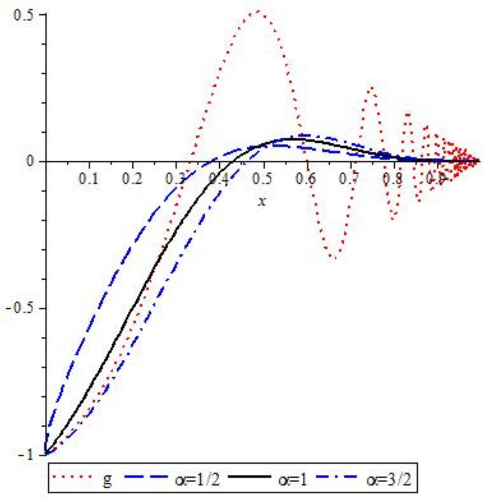

Figure 2.2: The convergence of $U^{\rho}_{n,\alpha}(g;x)$ to $g(x)$

Chapter 3

Quantitative Voronovskaya and Grüss-Voronovskaya type theorems for Szász-Durrmeyer type operators blended with multiple Appell polynomials

3.1 Introduction

For $f \in C(\mathbb{R}_0^+)$ and $x \in \mathbb{R}_0^+$, Szász [152] introduced the well-known operators as

$$S_n(f;x) = e^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} f\left(\frac{k}{n}\right), \tag{3.1.1}$$

such that $S_n(|f|;x) < \infty$. Aral et al. [19] proposed a generalization of Szász-Mirakyan operators defined in (3.1.1) by introducing a function ρ and studied some shape-preserving properties such as the ρ-convexity and the monotonicity for these operators. Several generalizations of Szász operators have been introduced in the literature and authors have studied their approximation properties.

Jakimovski and Leviatan [96] proposed a generalization of Szász-Mirakjan operators by means of the Appell polynomials as follows:

$$P_n(f;x) = \frac{e^{-nx}}{g(1)} \sum_{k=0}^{\infty} p_k(nx) f\left(\frac{k}{n}\right), \tag{3.1.2}$$

where $g(z) = \sum_{n=0}^{\infty} a_n z^n$, $g(1) \neq 0$ is an analytic function in the disc $|z| \leq r$, $r > 1$ and $p_k(x)$ denote the Appell polynomials having the generating function

$$g(u)e^{ux} = \sum_{k=0}^{\infty} p_k(x)u^k, \;\; p_k(x) \geq 0, \;\; \forall x \in \mathbb{R}_0^+.$$

For $g(u) = 1$, the operators defined by (3.1.2) reduce to Szász-Mirakjan operators given by (3.1.1).

Now let us recall the definition of multiple Appell polynomials [108]. A set of polynomials $\{p_{k_1,k_2}(x)\}_{k_1,k_2=0}^{\infty}$ with degree (k_1+k_2) for $k_1, k_2 \geq 0$, is called multiple polynomial system (multiple PS) and a multiple PS is called multiple Appell if it is generated by the relation

$$A(t_1,t_2)e^{x(t_1+t_2)} = \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \frac{p_{k_1,k_2}(x)}{k_1!k_2!} t_1^{k_1} t_2^{k_2}, \tag{3.1.3}$$

where A is given by

$$A(t_1,t_2) = \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \frac{a_{k_1,k_2}}{k_1!k_2!} t_1^{k_1} t_2^{k_2}, \tag{3.1.4}$$

with $A(0,0) = a_{0,0} \neq 0$.

Theorem 3.1.1. *For multiple PS, $\{p_{k_1,k_2}(x)\}_{k_1,k_2=0}^{\infty}$, the following statements are equivalent:*

(a) $\{p_{k_1,k_2}(x)\}_{k_1,k_2=0}^{\infty}$ is a set of multiple Appell polynomials.

(b) There exists a sequence $\{a_{k_1,k_2}\}_{k_1,k_2=0}^{\infty}$ with $a_{0,0} \neq 0$ such that

$$p_{k_1,k_2}(x) = \sum_{r_1=0}^{\infty} \sum_{r_2=0}^{\infty} \binom{k_1}{r_1} \binom{k_2}{r_2} a_{k_1-r_1,k_2-r_2} x^{r_1+r_2}.$$

(c) For every $k_1+k_2 \geq 1$, we have

$$p'_{k_1,k_2}(x) = k_1 p_{k_1-1,k_2}(x) + k_2 p_{k_1,k_2-1}(x).$$

For any $f \in C(\mathbb{R}_0^+)$, Varma [154] defined a sequence of linear positive operators as

$$K_n(f;x) = \frac{e^{-nx}}{A(1,1)} \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \frac{p_{k_1,k_2}\left(\frac{nx}{2}\right)}{k_1!k_2!} f\left(\frac{k_1+k_2}{n}\right), \tag{3.1.5}$$

provided $A(1,1) \neq 0$, $\frac{a_{k_1,k_2}}{A(1,1)} \geq 0$ for $k_1, k_2 \in \mathbb{N}$, and series (3.1.3) and(3.1.4) converge for $|t_1| < R_1,\ |t_2| < R_2\,(R_1, R_2 > 1)$ respectively.

For $\alpha > 0,\ \rho > 0,\ x \in \mathbb{R}_0^+$ and $f : \mathbb{R}_0^+ \to \mathbb{R}$, being integrable function, Păltănea [128] defined a modification of Szász operators as

$$L_\alpha^\rho(f;x) = \sum_{k=1}^{\infty} e^{-\alpha x} \frac{(\alpha x)^k}{k!} \int_0^\infty \frac{\alpha\rho e^{-\alpha\rho t}(\alpha\rho t)^{k\rho-1}}{\Gamma(k\rho)} f(t)dt + e^{-\alpha x} f(0), \tag{3.1.6}$$

which reproduce linear functions and established the rate of convergence of these operators for continuous functions by means of moduli of continuity.

Motivated by the above research work for $f \in C_E(\mathbb{R}_0^+)$, the space of all continuous functions satisfying $|f(t)| \leq Ke^{at},\ (t \geq 0)$ for some positive constants K and a, we propose an approximation method by linking the operators (3.1.6) and the multiple Appell polynomials as

$$\begin{aligned}\mathcal{L}_n^\rho(f;x) &= \frac{e^{-nx}}{A(1,1)} \sum_{k_1} \sum_{\substack{k_2 \\ k_1+k_2\geq 1}} \frac{p_{k_1,k_2}\left(\frac{nx}{2}\right)}{k_1!k_2!} \int_0^\infty \frac{n\rho e^{-n\rho t}(n\rho t)^{(k_1+k_2)\rho-1}}{\Gamma(k_1+k_2)\rho} f(t)dt \\ &\quad + \frac{e^{-nx}}{A(1,1)} p_{0,0}\left(\frac{nx}{2}\right) f(0),\end{aligned}$$

and establish a quantitative Voronovskaya type theorem, Grüss Voronovskaya type theorem, a local approximation theorem by means of Steklov mean, a Voronovskaya type asymptotic theorem and error estimates for a space. Lastly, we study the rate of convergence of functions having derivatives of bounded variation.

3.2 Basic Results

In order to prove the main results of the chapter, we shall need the following auxiliary results:

Lemma 3.2.1. *For* $K_n(t^i;x),\ i=0,1,2,3,4,$ *we have*

(i) $K_n(1;x)=1,$

(ii) $K_n(t;x)=x+\dfrac{A_{t_1}(1,1)+A_{t_2}(1,1)}{nA(1,1)},$

(iii) $K_n(t^2;x)=x^2+\dfrac{x}{n}\Bigg\{1+\dfrac{2}{A(1,1)}\Bigg(A_{t_1}(1,1)+A_{t_2}(1,1)\Bigg)\Bigg\}+\dfrac{1}{n^2A(1,1)}\Bigg\{A_{t_1}(1,1)+A_{t_2}(1,1)+A_{t_1t_1}(1,1)+2A_{t_1t_2}(1,1)+A_{t_2t_2}(1,1)\Bigg\},$

(iv) $K_n(t^3;x)=x^3+\dfrac{3x^2}{n}\Bigg\{1+\dfrac{1}{A(1,1)}\Bigg(A_{t_1}(1,1)+A_{t_2}(1,1)\Bigg)\Bigg\}+\dfrac{x}{n^2}\Bigg\{1+\dfrac{3}{A(1,1)}\Bigg(2A_{t_1}(1,1)+2A_{t_2}(1,1)+A_{t_1t_1}(1,1)+2A_{t_1t_2}(1,1)+A_{t_2t_2}(1,1)\Bigg)\Bigg\}+\dfrac{1}{n^3A(1,1)}\Bigg\{A_{t_1}(1,1)+A_{t_2}(1,1)+3A_{t_1t_1}(1,1)+6A_{t_1t_2}(1,1)+3A_{t_2t_2}(1,1)+A_{t_1t_1t_1}(1,1)+A_{t_2t_2t_2}(1,1)+3A_{t_1t_1t_2}(1,1)+3A_{t_2t_2t_1}(1,1)\Bigg\},$

(v) $K_n(t^4;x)=x^4+\dfrac{x^3}{n}\Bigg\{6+\dfrac{4}{A(1,1)}\Bigg(A_{t_1}(1,1)+A_{t_2}(1,1)\Bigg)\Bigg\}+\dfrac{x^2}{n^2}\Bigg\{7+\dfrac{6}{A(1,1)}\Bigg(3A_{t_1}(1,1)+3A_{t_2}(1,1)+A_{t_1t_1}(1,1)+2A_{t_1t_2}(1,1)+A_{t_2t_2}(1,1)\Bigg)\Bigg\}+\dfrac{x}{n^3}\Bigg\{1+\dfrac{1}{A(1,1)}\Bigg(14A_{t_1}(1,1)+14A_{t_2}(1,1)+18A_{t_1t_1}(1,1)+36A_{t_1t_2}(1,1)+18A_{t_2t_2}(1,1)+4A_{t_1t_1t_1}(1,1)+4A_{t_2t_2t_2}(1,1)+12A_{t_1t_1t_2}(1,1)+12A_{t_2t_2t_1}(1,1)\Bigg)\Bigg\}+\dfrac{1}{n^4A(1,1)}\Bigg\{A_{t_1}(1,1)+A_{t_2}(1,1)+7A_{t_1t_1}(1,1)+14A_{t_1t_2}(1,1)+7A_{t_2t_2}(1,1)+6A_{t_1t_1t_1}(1,1)+6A_{t_2t_2t_2}(1,1)+18A_{t_1t_1t_2}(1,1)+18A_{t_2t_2t_1}(1,1)+A_{t_1t_1t_1t_1}(1,1)+A_{t_2t_2t_2t_2}(1,1)+4A_{t_1t_1t_1t_2}(1,1)+4A_{t_2t_2t_2t_1}(1,1)+6A_{t_1t_1t_2t_2}(1,1)\Bigg\}.$

The values of the moments $K_n(t^i;x)$ for $i=0,1,2$ are given in [154] while the values of $K_n(t^i;x)$ for $i=3,4$ have been obtained by us after simple calculations and hence the details are omitted.

Lemma 3.2.2. *For the sequence of linear positive operators* $\mathcal{L}_n^{\rho}(t^i;x),\ i=0,1,2,3,4,$ *we find*

(i) $\mathcal{L}_n^{\rho}(1;x)=1,$

(ii) $\mathcal{L}_n^{\rho}(t;x)=x+\dfrac{A_{t_1}(1,1)+A_{t_2}(1,1)}{nA(1,1)},$

(iii) $\mathcal{L}_n^{\rho}(t^2;x) = x^2+\frac{x}{n}\Bigg\{\Bigg(1+\frac{1}{\rho}\Bigg)+\frac{2}{A(1,1)}\Bigg(A_{t_1}(1,1)+A_{t_2}(1,1)\Bigg)\Bigg\}+\frac{1}{n^2A(1,1)}\Bigg\{\Bigg(1+\frac{1}{\rho}\Bigg)\Bigg(A_{t_1}(1,1)+A_{t_2}(1,1)\Bigg)+A_{t_1t_1}(1,1)+2A_{t_1t_2}(1,1)+A_{t_2t_2}(1,1)\Bigg\}$,

(iv) $\mathcal{L}_n^{\rho}(t^3;x) = x^3+\frac{3x^2}{n}\Bigg\{\Bigg(1+\frac{1}{\rho}\Bigg)+\frac{1}{A(1,1)}\Bigg(A_{t_1}(1,1)+A_{t_2}(1,1)\Bigg)\Bigg\}+\frac{x}{n^2}\Bigg\{\Bigg(1+\frac{3}{\rho}+\frac{2}{\rho^2}\Bigg)+\frac{3}{A(1,1)}\Bigg(2\Bigg(1+\frac{1}{\rho}\Bigg)\Bigg(A_{t_1}(1,1)+A_{t_2}(1,1)\Bigg)+A_{t_1t_1}(1,1)+2A_{t_1t_2}(1,1)+A_{t_2t_2}(1,1)\Bigg)\Bigg\}+\frac{1}{n^3A(1,1)}\Bigg\{\Bigg(1+\frac{3}{\rho}+\frac{2}{\rho^2}\Bigg)\Bigg(A_{t_1}(1,1)+A_{t_2}(1,1)\Bigg)+3\Bigg(1+\frac{1}{\rho}\Bigg)\Bigg(A_{t_1t_1}(1,1)+2A_{t_1t_2}(1,1)+A_{t_2t_2}(1,1)\Bigg)+A_{t_1t_1t_1}(1,1)+A_{t_2t_2t_2}(1,1)+3A_{t_1t_1t_2}(1,1)+3A_{t_2t_2t_1}(1,1)\Bigg\}$,

(v) $\mathcal{L}_n^{\rho}(t^4;x) = x^4+\frac{x^3}{n}\Bigg\{6\Bigg(1+\frac{1}{\rho}\Bigg)+\frac{4}{A(1,1)}\Bigg(A_{t_1}(1,1)+A_{t_2}(1,1)\Bigg)\Bigg\}+\frac{x^2}{n^2}\Bigg\{\Bigg(\Bigg(7+\frac{18}{\rho}+\frac{11}{\rho^2}\Bigg)+\frac{6}{A(1,1)}\Bigg(3\Bigg(1+\frac{1}{\rho}\Bigg)\Bigg(A_{t_1}(1,1)+A_{t_2}(1,1)\Bigg)+A_{t_1t_1}(1,1)+2A_{t_1t_2}(1,1)+A_{t_2t_2}(1,1)\Bigg)\Bigg\}+\frac{x}{n^3}\Bigg\{\Bigg(1+\frac{6}{\rho}+\frac{11}{\rho^2}+\frac{6}{\rho^3}\Bigg)+\frac{1}{A(1,1)}\Bigg(\Bigg(14+\frac{36}{\rho}+\frac{22}{\rho^2}\Bigg)\Bigg(A_{t_1}(1,1)+A_{t_2}(1,1)\Bigg)+18\Bigg(1+\frac{1}{\rho}\Bigg)\Bigg(A_{t_1t_1}(1,1)+2A_{t_1t_2}(1,1)+A_{t_2t_2}(1,1)\Bigg)+4A_{t_1t_1t_1}(1,1)+4A_{t_2t_2t_2}(1,1)+12A_{t_1t_1t_2}(1,1)+12A_{t_2t_2t_1}(1,1)\Bigg)\Bigg\}+\frac{1}{n^4A(1,1)}\Bigg\{\Bigg(1+\frac{6}{\rho}+\frac{11}{\rho^2}+\frac{6}{\rho^3}\Bigg)\Bigg(A_{t_1}(1,1)+A_{t_2}(1,1)\Bigg)+\Bigg(7+\frac{18}{\rho}+\frac{11}{\rho^2}\Bigg)\Bigg(A_{t_1t_1}(1,1)+2A_{t_1t_2}(1,1)+A_{t_2t_2}(1,1)\Bigg)+6\Bigg(1+\frac{1}{\rho}\Bigg)\Bigg(A_{t_1t_1t_1}(1,1)+A_{t_2t_2t_2}(1,1)+3A_{t_1t_1t_2}(1,1)+3A_{t_2t_2t_1}(1,1)\Bigg)+A_{t_1t_1t_1t_1}(1,1)+A_{t_2t_2t_2t_2}(1,1)+4A_{t_1t_1t_1t_2}(1,1)+4A_{t_2t_2t_2t_1}(1,1)+6A_{t_1t_1t_2t_2}(1,1)\Bigg\}$.

Consequently,

$$\begin{aligned}\mathcal{L}_n^{\rho}((t-x)^2;x) &= \frac{x}{n}\Bigg(1+\frac{1}{\rho}\Bigg)+\frac{1}{n^2A(1,1)}\Bigg\{\Bigg(1+\frac{1}{\rho}\Bigg)\Bigg(A_{t_1}(1,1)+A_{t_2}(1,1)\Bigg)\\ &\quad +A_{t_1t_1}(1,1)+2A_{t_1t_2}(1,1)+A_{t_2t_2}(1,1)\Bigg\}\\ &\le \frac{C}{n}\Bigg(1+\frac{1}{\rho}\Bigg)(1+x)=\delta^2_{n,\rho}(x), \text{ (say)}. \qquad (3.2.1)\end{aligned}$$

where,

$$C=\max\Bigg(1,\ \ \frac{|A_{t_1}(1,1)|+|A_{t_2}(1,1)|+|A_{t_1,t_1}(1,1)|+2|A_{t_1,t_2}(1,1)|+|A_{t_2,t_2}(1,1)|}{|A(1,1)|}\Bigg)$$

and

$$\mathcal{L}_n^{\rho}((t-x)^4;x)=\frac{3x^2}{n^2}\left\{1+\frac{2}{\rho}+\frac{1}{\rho^2}\right\}+\frac{x}{n^3}\Bigg[\left(1+\frac{6}{\rho}+\frac{11}{\rho^2}+\frac{6}{\rho^3}\right)+\frac{1}{A(1,1)}\Bigg\{$$
$$\left(13+\frac{33}{\rho}+\frac{20}{\rho^2}\right)\Big(A_{t_1}(1,1)+A_{t_2}(1,1)\Big)+6\left(1+\frac{1}{\rho}\right)\Big(A_{t_1t_1}(1,1)$$
$$+2A_{t_1t_2}(1,1)+A_{t_2t_2}(1,1)\Big)-6A_{t_1t_1t_1}(1,1)-6A_{t_2t_2t_2}(1,1)\Bigg\}\Bigg].$$

Using Lemma 3.2.2, after simple calculations, the proof of the lemma easily follows. So, we omit the details. The expression for $\mathcal{L}_n^{\rho}((t-x)^6;x)$ has not been included in Lemma 3.2.2 because it is very lengthy and complicated however we found its order of convergence in the following remark as it will be required to prove the quantitative Voronovskaya type theorem.

Remark 3.2.3. *From Lemma 3.2.2, we obtain*

$$\lim_{n\to\infty} n\mathcal{L}_n^{\rho}((t-x);x) = \frac{A_{t_1}(1,1)+A_{t_2}(1,1)}{A(1,1)}, \tag{3.2.2}$$

$$\lim_{n\to\infty} n\mathcal{L}_n^{\rho}((t-x)^2;x) = x\left(1+\frac{1}{\rho}\right), \tag{3.2.3}$$

$$\lim_{n\to\infty} n^2\mathcal{L}_n^{\rho}((t-x)^4;x) = 3x^2\left(1+\frac{2}{\rho}+\frac{1}{\rho^2}\right) \tag{3.2.4}$$

$$\lim_{n\to\infty} n^3\mathcal{L}_n^{\rho}((t-x)^6;x) = 15x^3\left(1+\frac{3}{\rho}+\frac{3}{\rho^2}+\frac{1}{\rho^3}\right). \tag{3.2.5}$$

3.3 Approximation theorems

Theorem 3.3.1. *Let $f\in C_E(\mathbb{R}_0^+)$. Then $\lim_{n\to\infty}\mathcal{L}_n^{\rho}(f;x)=f(x)$, uniformly on each compact subset of $\mathbb{R}_0^+$.*

Proof. Considering Lemma 3.2.2, it follows that $\lim_{n\to\infty}\mathcal{L}_n^{\rho}(t^i;x) = x^i,\ \ i=0,1,2,$ uniformly on every compact subset of $\mathbb{R}_0^+$. Applying Bohman Korovkin theorem, we obtain the desired result. □

For $f\in C_B(\mathbb{R}_0^+)$, the Steklov mean of second order [77] is defined as

$$f_h(x)=\frac{4}{h^2}\int_0^{\frac{h}{2}}\int_0^{\frac{h}{2}}[2f(x+u+v)-f(x+2(u+v))]dudv,\ \ h>0. \tag{3.3.1}$$

Hence

$$\begin{aligned} f(x) - f_h(x) &= \frac{4}{h^2}\int_0^{h/2}\int_0^{h/2} \triangle_{u+v}^2 f(x)dudv, \text{and} \\ f_h''(x) &= \frac{1}{h^2}(8\triangle_{h/2}^2 f(x) - \triangle_h^2 f(x)). \end{aligned}$$

Thus, it follows that

$$||f_h - f|| \leq \omega_2(f,h). \tag{3.3.2}$$

Further, $f_h', f_h'' \in C_B(\mathbb{R}_0^+)$ and

$$||f_h'|| \leq \frac{5}{h}\omega(f,h), \quad ||f_h''|| \leq \frac{9}{h^2}\omega_2(f,h), \tag{3.3.3}$$

where $\delta_{n,\rho}(x)$ is defined by equation (3.2.1).

Theorem 3.3.2. *For $f \in C_B(\mathbb{R}_0^+)$ and $x \in \mathbb{R}_0^+$, we have*

$$|\mathcal{L}_n^\rho(f;x) - f(x)| \leq 5\omega(f;\delta_{n,\rho}(x)) + \frac{13}{2}\omega_2(f;\delta_{n,\rho}(x)).$$

Proof. Using the Steklov mean f_h defined by (3.3.1), we may write

$$|\mathcal{L}_n^\rho(f;x) - f(x)| \;\leq\; |\mathcal{L}_n^\rho((f-f_h);x)| + |\mathcal{L}_n^\rho(f_h(t) - f_h(x);x)| + |f_h(x) - f(x)|.$$

Applying Lemma 3.2.2, we have

$$||\mathcal{L}_n^\rho(f)|| \leq ||f||, \tag{3.3.4}$$

Using inequality (3.3.4) and relation (3.3.2), we have

$$|\mathcal{L}_n^\rho((f-f_h);x)| \leq ||f - f_h|| \leq \omega_2(f,h).$$

Now by Taylor's expansion and applying Cauchy-Schwarz inequality, we have

$$\begin{aligned} |\mathcal{L}_n^\rho(f_h(t) - f_h(x);x)| &\leq |\mathcal{L}_n^\rho((t-x)f_h'(x);x)| + \left|\mathcal{L}_n^\rho\left(\int_x^t (t-u)f_h''(u)du;x\right)\right| \\ &\leq ||f_h'|||\mathcal{L}_n^\rho(|t-x|;x)| + ||f_h''||\mathcal{L}_n^\rho\left(\left|\int_x^t |t-u|du\right|;x\right) \\ &= ||f_h'||\sqrt{\mathcal{L}_n^\rho((t-x)^2;x)} + \frac{1}{2}||f_h''||\mathcal{L}_n^\rho((t-x)^2;x). \end{aligned}$$

Applying Lemma 3.2.2 and using (3.3.2)-(3.3.3) in (3.3.4), on choosing h as $\delta_{n,\rho}(x)$ we get the required result. □

Theorem 3.3.3. *For $f \in C_E^2(\mathbb{R}_0^+)$, we obtain*

$$\lim_{n\to\infty} n[\mathcal{L}_n^\rho(f;x) - f(x)] = \frac{A_{t_1}(1,1) + A_{t_2}(1,1)}{A(1,1)} f'(x) + \frac{x}{2}\left(1 + \frac{1}{\rho}\right) f''(x),$$

uniformly in $x \in [0,a], a > 0$.

Proof. By Taylor's expansion of f, for some fixed $x \in [0,a]$ we obtain

$$f(t) - f(x) = (t-x)f'(x) + \frac{1}{2}(t-x)^2 f''(x) + \xi(t,x)(t-x)^2, \tag{3.3.5}$$

where $\xi(t,x) \in C_E(\mathbb{R}_0^+)$ and $\lim_{t\to x} \xi(t,x) = 0$.

Hence by linearity of the operators $\mathcal{L}_n^\rho$, from relation (3.3.5), we get

$$\begin{aligned} n[\mathcal{L}_n^\rho(f;x) - f(x)] &= n\mathcal{L}_n^\rho(t-x;x)f'(x) + \frac{1}{2}n\mathcal{L}_n^\rho((t-x)^2;x)f''(x) \\ &+ n\mathcal{L}_n^\rho(\xi(t,x)(t-x)^2;x). \end{aligned} \tag{3.3.6}$$

Applying Cauchy-Schwarz inequality in the last term of the equation (3.3.6), we have

$$|n\mathcal{L}_n^\rho(\xi(t,x)(t-x)^2;x)| \leq \sqrt{n^2\mathcal{L}_n^\rho((t-x)^4;x)\mathcal{L}_n^\rho(\xi^2(t,x);x)}. \tag{3.3.7}$$

From Remark 3.2.3, it follows that

$$\lim_{n\to\infty} n^2\mathcal{L}_n^\rho((t-x)^4;x) = 3x^2\left(1 + \frac{2}{\rho} + \frac{1}{\rho^2}\right),$$

uniformly in $x \in [0,a]$.

Further, let $\xi^2(t,x) = \nu(t,x),\ x \geq 0$, then $\nu(t,x) \in C_E(\mathbb{R}_0^+)$ and hence from Theorem 3.3.1, we get

$$\lim_{n\to\infty} \mathcal{L}_n^\rho(\xi^2(t,x);x) = \lim_{n\to\infty} \mathcal{L}_n^\rho(\nu(t,x);x) = \nu(x,x) = 0,$$

Hence from equation(3.3.7), we obtain

$$\lim_{n\to\infty}\left(n\mathcal{L}_n^\rho(\xi(t,x)(t-x)^2;x)\right) = 0,$$

uniformly in $x \in [0,a]$. Now, taking limit as $n \to \infty$ in (3.3.6) and using Remark 3.2.3, we get the desired result. This completes the proof. □

3.4 Weighted approximation

Let $\theta(x) \geq 1$ be a weight function on $\mathbb{R}_0^+$. We consider the following weighted space defined on $\mathbb{R}_0^+$ as

$$B_\theta(\mathbb{R}_0^+) := \{f : |f(x)| \leq M_f \theta(x), \ \forall x \in \mathbb{R}_0^+ \ and \ M_f > 0\}.$$

Further, let

$$C_\theta(\mathbb{R}_0^+) := \{f \in B_\theta(\mathbb{R}_0^+) : f \text{ is a continuous function on } \mathbb{R}_0^+\},$$

and

$$C_\theta^*(\mathbb{R}_0^+) := \left\{ f \in C_\theta(\mathbb{R}_0^+) : \lim_{x\to\infty} \frac{f(x)}{\theta(x)} = K_f < \infty \right\}.$$

We define the norm in the space $B_\theta(\mathbb{R}_0^+)$ as

$$||f||_\theta = \sup_{x\in\mathbb{R}_0^+} \frac{|f(x)|}{\theta(x)}.$$

The usual modulus of continuity of the function f on $[0,p]$ is defined as

$$\omega_p(f;\delta) = \sup_{|t-x|\leq\delta} \sup_{t,x\in[0,p]} |f(t) - f(x)|. \tag{3.4.1}$$

Let us denote $||.||_{C[a,b]}$ as supremum norm on $[a,b]$. Throughout the chapter we have taken $\theta(x) = 1 + x^2$.

Theorem 3.4.1. *For $x \in [0,c]$ and $f \in C_\theta(\mathbb{R}_0^+)$, we have*

$$||\mathcal{L}_n^\rho(f;\cdot) - f||_{C[0,c]} \ \leq \ 4M_f(1+c^2)\eta_{n,\rho}^2 + 2\omega_{c+1}(f;\eta_{n,\rho}),$$

where $\eta_{n,\rho}^2 = \max\limits_{x\in[0,c]} \left(\mathcal{L}_n^\rho((t-x)^2;x) \right)$.

Proof. Let $x \in [0,c]$ and $t > c+1$ then $t - x > 1$. Hence for $f \in C_\theta(\mathbb{R}_0^+)$, we have

$$\begin{aligned} |f(t) - f(x)| &\leq M_f(2 + t^2 + x^2) \\ &= M_f(2 + 2x^2 + (t-x)^2 + 2x(t-x)) \\ &\leq M_f(t-x)^2(3 + 2x^2 + 2x) \leq 4M_f(1+x^2)(t-x)^2. \end{aligned} \tag{3.4.2}$$

For $x \in [0,c]$ and $t \in [0,c+1]$, we have

$$|f(t)-f(x)| \quad \leq \omega_{c+1}(f;|t-x|) \leq \left(1+\frac{|t-x|}{\delta}\right)\omega_{c+1}(f;\delta). \tag{3.4.3}$$

From equations (3.4.2) and (3.4.3), for $x \in [0,c]$ and $t \geq 0$, we have

$$|f(t)-f(x)| \leq 4M_f(1+x^2)(t-x)^2+\left(1+\frac{|t-x|}{\delta}\right)\omega_{c+1}(f;\delta).$$

Applying Cauchy-Schwarz inequality and choosing $\delta = \eta_{n,\rho}$, we get

$$\begin{aligned} |\mathcal{L}_n^\rho(f;x)-f(x)| &\leq 4M_f(1+x^2)\mathcal{L}_n^\rho((t-x)^2;x)+\left(1+\frac{1}{\delta}\mathcal{L}_n^\rho(|t-x|;x)\right)\omega_{c+1}(f;\delta) \\ &\leq 4M_f(1+c^2)\eta_{n,\rho}^2(c)+2\omega_{c+1}\left(f;\eta_{n,\rho}(c)\right). \end{aligned}$$

This completes the proof. □

Theorem 3.4.2. *For $f \in C_\theta(\mathbb{R}_0^+)$, we have*

$$\lim_{n\to\infty}\sup_{x\in\mathbb{R}_0^+}\frac{|\mathcal{L}_n^\rho(f;x)-f(x)|}{(1+x^2)^{1+\eta}}=0,$$

where η is some positive constant.

Proof. Since $|f(x)| \leq ||f||_\theta(1+x^2)$, then for a fixed $y>0$, we may write

$$\begin{aligned} \sup_{x\in R_0^+}\frac{|\mathcal{L}_n^\rho(f;x)-f(x)|}{(1+x^2)^{1+\eta}} &\leq \sup_{x\in[0,y]}\frac{|\mathcal{L}_n^\rho(f;x)-f(x)|}{(1+x^2)^{1+\eta}}+\sup_{x\in(y,\infty)}\frac{|\mathcal{L}_n^\rho(f;x)-f(x)|}{(1+x^2)^{1+\eta}} \\ &\leq ||\mathcal{L}_n^\rho(f;.)-f||_{C[0,y]}+\frac{||f||_\theta}{(1+y^2)^\eta} \\ &+ ||f||_\theta\sup_{x\in(y,\infty)}\frac{|\mathcal{L}_n^\rho(1+t^2;x)|}{(1+x^2)^{1+\eta}}. \end{aligned} \tag{3.4.4}$$

Using Theorem 3.3.1, for a given $\epsilon>0$, there exists $k \in \mathbb{N}$ such that

$$|\mathcal{L}_n^\rho(1+t^2;x)-1+x^2| < \frac{\epsilon}{3||f||_\theta},\ \forall n \geq k.$$

Also, we can write

$$\mathcal{L}_n^\rho(1+t^2;x) < 1+x^2+\frac{\epsilon}{3||f||_\theta},\ \forall n \geq k.$$

Hence,

$$\begin{aligned}||f||_\theta\frac{\mathcal{L}_n^\rho(1+t^2;x)}{(1+x^2)^{1+\eta}} &< \frac{||f||_\theta}{(1+x^2)^{1+\eta}}\left(1+x^2+\frac{\epsilon}{3||f||_\theta}\right)\\ &< \frac{||f||_\theta}{(1+y^2)^\eta}+\frac{\epsilon}{3},\ \forall n\geq k.\end{aligned}$$

Therefore,

$$||f||_\theta \sup_{x\in[y,\infty)}\frac{\mathcal{L}_n^\rho(1+t^2;x)}{(1+x^2)^{1+\eta}}\leq \frac{||f||_\theta}{(1+y^2)^\eta}+\frac{\epsilon}{3},\quad \forall n\geq k. \tag{3.4.5}$$

Let us choose y to be so large that

$$\frac{||f||_\theta}{(1+y^2)^\eta}\leq\frac{\epsilon}{6}. \tag{3.4.6}$$

In view of Theorem 3.4.1, for $\epsilon>0$ there exists a $n\geq l$ such that

$$||\mathcal{L}_n^\rho(f;\cdot)-f||_{C[0,y]}<\frac{\epsilon}{3},\quad n\geq l. \tag{3.4.7}$$

Taking $m=\max(k,l)$ and combining equations (3.4.4)-(3.4.7), we get

$$\sup_{x\in\mathbb{R}_0^+}\frac{|\mathcal{L}_n^\rho(f;x)-f(x)|}{(1+x^2)^{1+\eta}} \quad < \quad \epsilon,\ \ n\geq m.$$

This completes the proof. □

Following [89], the weighted modulus of continuity $\overline{\omega}(g;\delta)$ for $g\in C_\theta(\mathbb{R}_0^+)$ is defined as

$$\overline{\omega}(g;\delta)=\sup_{0<|h|\leq\delta,\ x\in\mathbb{R}_0^+}\frac{|g(x+h)-g(x)|}{(1+h^2)(1+x^2)}. \tag{3.4.8}$$

Also for $g\in C_\theta^*(\mathbb{R}_0^+)$, the weighted modulus of continuity has the following properties

(i) $\lim\limits_{\delta\to0}\overline{\omega}(g;\delta)=0$,

(ii) $\overline{\omega}(g;\lambda\delta)\leq 2(1+\lambda)(1+\delta^2)\overline{\omega}(g;\delta),\ \ \lambda>0.$

For $g\in C_\theta(\mathbb{R}_0^+)$, from equations (3.4.8) and property (ii) of $\overline{\omega}(g;\delta)$, we get

$$\begin{aligned}|g(t)-g(x)| &\leq \left(1+(t-x)^2\right)(1+x^2)\overline{\omega}(g;|t-x|)\\ &\leq 2\left(1+\frac{|t-x|}{\delta}\right)(1+\delta^2)\overline{\omega}(g;\delta)\left(1+(t-x)^2\right)(1+x^2).\end{aligned} \tag{3.4.9}$$

Theorem 3.4.3. *For $f \in C^*_\theta(\mathbb{R}^+_0)$, we have*

$$\sup_{x\in\mathbb{R}^+_0} \frac{|\mathcal{L}^\rho_n(f;x) - f(x)|}{(1+x^2)^2} \leq C\overline{\omega}\Big(f;\sqrt{\frac{1}{n}}\Big),$$

where C is a positive constant independent of n.

Proof. By the linearity and positivity of the operators $\mathcal{L}^\rho_n$, we get

$$|\mathcal{L}^\rho_n(f;x) - f(x)| \leq \mathcal{L}^\rho_n(|f(t) - f(x)|;x)$$

Using equation (3.4.9) and the Cauchy-Schwarz inequality, we get

$$\begin{aligned} |\mathcal{L}^\rho_n(f;x) - f(x)| &\leq 2(1+\delta^2)\overline{\omega}(f;\delta)(1+x^2)\mathcal{L}^\rho_n\bigg(\Big(1+\frac{|t-x|}{\delta}\Big)\Big(1+(t-x)^2\Big);x\bigg) \\ &\leq 2(1+\delta^2)\overline{\omega}(f;\delta)(1+x^2)\bigg\{\mathcal{L}^\rho_n(1;x) + \mathcal{L}^\rho_n\Big((t-x)^2;x\Big) \\ &+ \frac{1}{\delta}\mathcal{L}^\rho_n\Big(|t-x|;x\Big) + \frac{1}{\delta}\mathcal{L}^\rho_n\Big(|t-x|(t-x)^2;x\Big)\bigg\} \\ &\leq 2(1+\delta^2)\overline{\omega}(f;\delta)(1+x^2)\bigg\{\mathcal{L}^\rho_n(1;x) + \mathcal{L}^\rho_n\Big((t-x)^2;x\Big) \\ &+ \frac{1}{\delta}\sqrt{\mathcal{L}^\rho_n\Big((t-x)^2;x\Big)} \\ &+ \frac{1}{\delta}\sqrt{\mathcal{L}^\rho_n\Big((t-x)^2;x\Big)}\sqrt{\mathcal{L}^\rho_n\Big((t-x)^4;x\Big)}\bigg\}. \end{aligned} \tag{3.4.10}$$

Using Lemma 3.2.2, we obtain

$$\mathcal{L}^\rho_n\Big((t-x)^2;x\Big) \leq C_1\frac{1}{n}(1+x^2) \tag{3.4.11}$$

and

$$\mathcal{L}^\rho_n\Big((t-x)^4;x\Big) \leq C_2\frac{1}{n^2}(1+x^2), \tag{3.4.12}$$

for some positive constants C_1 and C_2 dependent on ρ and $A(t_1, t_2)$.
Now combining equations (3.4.10)-(3.4.12) and taking $\delta = \sqrt{\frac{1}{n}}$, we have

$$\begin{aligned} |\mathcal{L}^\rho_n(f;x) - f(x)| &\leq 2\Big(1+\frac{1}{n}\Big)\overline{\omega}\Big(f;\sqrt{\frac{1}{n}}\Big)(1+x^2)\bigg\{1 + C_1\frac{1}{n}(1+x^2) \\ &+ \sqrt{C_1}\sqrt{(1+x^2)} + \sqrt{C_1}\sqrt{(1+x^2)}\sqrt{C_2}\sqrt{(1+x^2)}\bigg\}. \end{aligned}$$

Hence, we get

$$\sup_{x\in\mathbb{R}_0^+}\frac{|\mathcal{L}_n^\rho(f;x)-f(x)|}{(1+x^2)^2}\leq C\overline{\omega}\left(f;\sqrt{\frac{1}{n}}\right),$$

where $C=2(1+C_1+\sqrt{C_1}+\sqrt{C_1}\sqrt{C_2})$. This completes the proof. □

3.5 Quantitative Voronovskaya theorem

In the following result, we discuss a quantitative Voronovskaja type theorem by using the weighted modulus of smoothness $\overline{\omega}(f;\delta)$. Recently, many researchers (cf. [5], [8] and [10] etc.) have made remarkable contribution in this area.

Theorem 3.5.1. *For $f,f',f''\in C_\theta^*(\mathbb{R}_0^+)$ and any $x\in\mathbb{R}_0^+$, we have*

$$\Bigg|n\left(\mathcal{L}_n^\rho(f;x)-f(x)\right)-f'(x)\left(\frac{A_{t_1}(1,1)+A_{t_2}(1,1)}{A(1,1)}\right)-\frac{f''(x)}{2!}\Bigg[x\left(1+\frac{1}{\rho}\right)+\frac{1}{nA(1,1)}$$
$$\left\{\left(1+\frac{1}{\rho}\right)\left(A_{t_1}(1,1)+A_{t_2}(1,1)\right)+A_{t_1t_1}(1,1)+2A_{t_1t_2}(1,1)+A_{t_2t_2}(1,1)\right\}\Bigg]\Bigg|$$
$$=O(1)\overline{\omega}\left(f'';\frac{1}{\sqrt{n}}\right),\ as\ \ n\to\infty.$$

Proof. Let $x,t\in\mathbb{R}_0^+$, then by Taylor's expansion, we have

$$f(t)=f(x)+f'(x)(t-x)+\frac{f''(x)}{2!}(t-x)^2+E(t,x),$$

where $E(t,x)=\dfrac{f''(\varphi)-f''(x)}{2!}(t-x)^2$ and φ lies between t and x.
Now, we get

$$\left|\mathcal{L}_n^\rho(f;x)-f(x)-f'(x)\mathcal{L}_n^\rho((t-x);x)-\frac{f''(x)}{2!}\mathcal{L}_n^\rho((t-x)^2;x)\right|\ \ \leq\mathcal{L}_n^\rho(|E(t,x)|;x).$$

Multiplying by n on both sides of above inequality and using Lemma 3.2.2, we obtain

$$\Bigg|n\left(\mathcal{L}_n^\rho(f;x)-f(x)\right)-f'(x)\left(\frac{A_{t_1}(1,1)+A_{t_2}(1,1)}{A(1,1)}\right)-\frac{f''(x)}{2!}\Bigg[\frac{x}{n}\left(1+\frac{1}{\rho}\right)+\frac{1}{n^2A(1,1)}$$
$$\left\{\left(1+\frac{1}{\rho}\right)\left(A_{t_1}(1,1)+A_{t_2}(1,1)\right)+A_{t_1t_1}(1,1)+2A_{t_1t_2}(1,1)+A_{t_2t_2}(1,1)\right\}\Bigg]\Bigg|$$
$$\leq n\mathcal{L}_n^\rho(|E(t,x)|;x). \tag{3.5.1}$$

Using the property of weighted modulus of smoothness given by (3.4.9), we get

$$\begin{aligned}\left|\frac{f''(\varphi)-f''(x)}{2!}\right| &\leq \frac{1}{2}\overline{\omega}(f'';|\varphi-x|)(1+(\varphi-x)^2)(1+x^2)\\ &\leq \frac{1}{2}\overline{\omega}(f'';|t-x|)(1+(t-x)^2)(1+x^2)\\ &\leq \left(1+\frac{(|t-x|)}{\delta}\right)(1+\delta^2)\overline{\omega}(f'';\delta)\\ &\quad \times(1+(t-x)^2)(1+x^2),\ \delta>0.\end{aligned}$$

Also,

$$\left|\frac{f''(\varphi)-f''(x)}{2!}\right| \leq \begin{cases} 2(1+\delta^2)^2(1+x^2)\overline{\omega}(f'';\delta), & |t-x|\leq\delta,, \\ 2(1+\delta^2)^2(1+x^2)\dfrac{(t-x)^2}{\delta^4}\overline{\omega}(f'';\delta), & |t-x|\geq\delta. \end{cases}$$

Now for $0<\delta<1$, we obtain

$$\left|\frac{f''(\varphi)-f''(x)}{2!}\right| \leq 8(1+x^2)\overline{\omega}(f'';\delta)\left(1+\frac{(t-x)^4}{\delta^4}\right).$$

Therefore, we get

$$|E(t,x)| \leq 8(1+x^2)\overline{\omega}(f'';\delta)\left((t-x)^2+\frac{(t-x)^6}{\delta^4}\right).$$

Now by the linearity and positivity of the operator $\mathcal{L}_n^\rho$ and using Remark 3.2.3, for any $x\in\mathbb{R}_0^+$, we obtain

$$\begin{aligned}\mathcal{L}_n^\rho(|E(t,x)|;x) &\leq 8(1+x^2)\overline{\omega}(f'';\delta)\left\{\mathcal{L}_n^\rho((t-x)^2;x)+\frac{1}{\delta^4}\mathcal{L}_n^\rho((t-x)^6;x)\right\}\\ &= 8(1+x^2)\overline{\omega}(f'';\delta)\left\{O\left(\frac{1}{n}\right)+O\left(\frac{1}{n^3}\right)\right\},\ as\, n\to\infty.\end{aligned}$$

Choosing $\delta=\frac{1}{\sqrt{n}}$, we obtain

$$\mathcal{L}_n^\rho(|E(t,x)|;x) = 8(1+x^2)\overline{\omega}\left(f'';n^{-\frac{1}{2}}\right)O\left(\frac{1}{n}\right),\ as\, n\to\infty. \tag{3.5.2}$$

Hence combining (3.5.1) and (3.5.2), we reach the required result. □

3.6 Grüss-Voronovskaya-type theorem

Grüss [72] first established an inequality which shows the error estimate of the integral of product of two functions with the product of integrals of the two functions. Acu et al. [10] determined some applications of Grüss inequality for Bernstein, Hermite-Fejer operator, interpolation operator and convolution-type operators by using least concave majorant. After that Gonska and Tachev [69] discussed Grüss-type inequality using second order modulus of smoothness. For the first time Gal and Gonska [64], studied the Grüss Voronovskaya type theorem for the Bernstein, Păltănea and Bernstein-Faber operators by means of the Grüss inequality which concerns with the non-multiplicavity of these operators. For more papers in this direction we refer the readers to (cf. [5], [11], [42] [153] etc.) In the following theorem, we study the non-multiplicativity of the positive linear operator $\mathcal{L}_n^\rho$.

Theorem 3.6.1. *For $f'(x),\ g'(x),\ f''(x),\ g''(x),\ (fg)'(x),\ (fg)''(x) \in C_\theta^*(\mathbb{R}_0^+)$, there holds the following equality:*

$$\lim_{n\to\infty} n\Big\{\mathcal{L}_n^\rho(fg;x) - \mathcal{L}_n^\rho(f;x)\mathcal{L}_n^\rho(g;x)\Big\} = x\left(1+\frac{1}{\rho}\right)f'(x)g'(x).$$

Proof. We have

$$(fg)''(x) = f''(x)g(x) + 2f'(x)g'(x) + g''(x)f(x).$$

By making an appropriate arrangement, we get

$$\begin{aligned}
& n\{\mathcal{L}_n^\rho((fg);x) - \mathcal{L}_n^\rho(f;x)\mathcal{L}_n^\rho(g;x)\} \\
= \;& n\Big\{\mathcal{L}_n^\rho((fg);x) - f(x)g(x) - (fg)^{'}(x)\mathcal{L}_n^\rho(t-x;x) - \frac{(fg)^{''}(x)}{2!}\mathcal{L}_n^\rho((t-x)^2;x) \\
& -g(x)\left(\mathcal{L}_n^\rho(f;x) - f(x) - f'(x)\mathcal{L}_n^\rho(t-x;x) - \frac{f''(x)}{2!}\mathcal{L}_n^\rho((t-x)^2;x)\right) \\
& -\mathcal{L}_n^\rho(f;x)\left(\mathcal{L}_n^\rho(g;x) - g(x) - g'(x)\mathcal{L}_n^\rho(t-x;x) - \frac{g''(x)}{2!}\mathcal{L}_n^\rho((t-x)^2;x)\right) \\
& +2\frac{\mathcal{L}_n^\rho((t-x)^2;x)}{2!}f'(x)g'(x) + g''(x))\frac{\mathcal{L}_n^\rho((t-x)^2;x)}{2!}\left(f(x) - \mathcal{L}_n^\rho(f;x)\right) \\
& +(g)'(x))\mathcal{L}_n^\rho(t-x;x)\left(f(x) - \mathcal{L}_n^\rho(f;x)\right)\Big\}.
\end{aligned}$$

Applying Theorem 3.3.1, for each $x \in \mathbb{R}_0^+$, $L_n^\rho(f;x) \to f(x)$ as $n \to \infty$ and for $f'' \in C_\theta^*(\mathbb{R}_0^+)$, $x \in \mathbb{R}_0^+$, by Theorem 3.5.1, we have

$$\lim_{n\to\infty}\left(\mathcal{L}_n^\rho(f;x) - f(x) - f'(x)\mathcal{L}_n^\rho((t-x);x) - \frac{f''(x)}{2!}\mathcal{L}_n^\rho((t-x)^2;x)\right) = 0.$$

Therefore, using Remark 3.2.3, we get the desired result. □

3.7 Rate of approximation

In order to discuss the approximation of functions with derivatives of bounded variation, we express the operators $\mathcal{L}_n^\rho$ in an integral form as follows:

$$\mathcal{L}_n^\rho(f;x) = \int_0^\infty K_n^\rho(x,t)f(t)dt, \tag{3.7.1}$$

where the kernel $K_n^\rho(x,t)$ is given by

$$K_n^\rho(x,t) = \frac{e^{-nx}}{A(1,1)}\sum_{k_1}\sum_{k_1+k_2\geq 1}\sum_{k_2}\frac{p_{k_1,k_2}\left(\frac{nx}{2}\right)}{k_1!k_2!}\frac{n\rho e^{-n\rho t}(n\rho t)^{(k_1+k_2)\rho-1}}{\Gamma(k_1+k_2)\rho} + \frac{e^{-nx}}{A(1,1)}p_{0,0}\left(\frac{nx}{2}\right)\delta(t),$$

$\delta(t)$ being the Dirac-delta function.

In the sequel, we shall need the following lemma:

Lemma 3.7.1. *For a fixed $x \in \mathbb{R}_0^+$ and sufficiently large n, we have*

(i) $\xi_n^\rho(x,y) = \int_0^y K_n^\rho(x,t)dt \leq \dfrac{C(1+\frac{1}{\rho})(1+x)}{n}\dfrac{1}{(x-y)^2}$ $0 \leq y < x$,

(ii) $1 - \xi_n^\rho(x,z) = \int_z^\infty K_n^\rho(x,t)dt \leq \dfrac{C(1+\frac{1}{\rho})(1+x)}{n}\dfrac{1}{(z-x)^2}$, $x < z < \infty$.

Proof. (i) Using Lemma 3.2.2, we get

$$\begin{aligned}\xi_n^\rho(x,y) &= \int_0^y K_n^\rho(x,t)dt \leq \int_0^y\left(\frac{x-t}{x-y}\right)^2 K_n^\rho(x,t)dt \\ &\leq \mathcal{L}_n^\rho((t-x)^2;x)(x-y)^{-2} \\ &\leq \frac{C(1+\frac{1}{\rho})(1+x)}{n}\frac{1}{(x-y)^2}.\end{aligned}$$

The proof of (ii) is similar hence the details are omitted. □

Theorem 3.7.2. *Let $f \in DBV(\mathbb{R}_0^+)$. Then, for every $x \in \mathbb{R}_0^+$ and sufficiently large n, we have*

$$|\mathcal{L}_n^\rho(f;x) - f(x)| \leq \frac{1}{2}(f'(x+) + f'(x-))\left(\frac{A_{t_1}(1,1) + A_{t_2}(1,1)}{nA(1,1)}\right) + \frac{1}{2} \mid f'(x+) - f'(x-) \mid$$
$$\sqrt{\frac{C}{n}\left(1 + \frac{1}{\rho}\right)(1+x)} + \frac{C}{n}\left(1 + \frac{1}{\rho}\right)\frac{(1+x)}{x^2}|f(2x) - f(x) - xf'(x+)|$$
$$+\frac{x}{\sqrt{n}} \bigvee_{x}^{x+x/\sqrt{n}} (f_x') + \frac{C}{n}\left(1 + \frac{1}{\rho}\right)\left(1 + \frac{1}{x}\right)\sum_{k=1}^{[\sqrt{n}]} \bigvee_{x}^{x+x/\sqrt{k}} f_x'$$
$$+\frac{C}{n}\left(1 + \frac{1}{\rho}\right)(1+x)\left(\frac{M + |f(x)|}{x^2} + 4M\right) + |f'(x+)|\sqrt{\frac{C}{n}\left(1 + \frac{1}{\rho}\right)(1+x)}.$$

where f_x' is defined by

$$f_x'(t) = \begin{cases} f'(t) - f'(x-), & 0 \leq t < x \\ 0, & t = x \\ f'(t) - f'(x+) & x < t < \infty. \end{cases} \tag{3.7.2}$$

Proof. Since $\mathcal{L}_n^\rho(1;x) = 1$, using (3.7.1), for every $x \in (0,1)$ we get

$$\begin{aligned} \mathcal{L}_n^\rho(f;x) - f(x) &= \int_0^\infty K_n^\rho(x,t)(f(t) - f(x))dt \\ &= \int_0^\infty K_n^\rho(x,t) \int_x^t f'(u)dudt. \end{aligned} \tag{3.7.3}$$

For any $f \in DBV(\mathbb{R}_0^+)$, from (3.7.2) we may write

$$\begin{aligned} f'(v) &= (f')_x(v) + \frac{1}{2}(f'(x+) + f'(x-)) + \frac{1}{2}(f'(x+) - f'(x-))sgn(v-x) \\ &\quad + E_x(v)[f'(v) - \frac{1}{2}(f'(x+) + f'(x-))], \end{aligned} \tag{3.7.4}$$

where

$$E_x(u) = \begin{cases} 1\ ,\ v = x \\ 0\ ,\ v \neq x \end{cases}.$$

We get,

$$\int_0^\infty \left(\int_x^t \left(f'(u) - \frac{1}{2}(f'(x+) + f'(x-))\right)E_x(v)dv\right)K_n^\rho(x,t)dt = 0. \tag{3.7.5}$$

We have

$$\begin{aligned}
&\int_0^\infty \left(\int_x^t \frac{1}{2}(f'(x+)+f'(x-))dv\right)K_n^\rho(x,t)dt\\
&=\frac{1}{2}(f'(x+)+f'(x-))\int_0^\infty (t-x)K_n^\rho(x,t)dt\\
&=\frac{1}{2}(f'(x+)+f'(x-))\mathcal{L}_n^\rho((t-x);x)\\
&=\frac{1}{2}(f'(x+)+f'(x-))\left(\frac{A_{t_1}(1,1)+A_{t_2}(1,1)}{nA(1,1)}\right).
\end{aligned}$$

Using Lemma 3.2.2 and applying Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}
&\left|\int_0^\infty K_n^\rho(x,t)\left(\int_x^t \frac{1}{2}(f'(x+)-f'(x-))sgn(v-x)dv\right)dt\right|\\
&\leq \frac{1}{2} \mid f'(x+)-f'(x-) \mid \left(\mathcal{L}_n^\rho((t-x)^2;x)\right)^{1/2}\\
&\leq \frac{1}{2} \mid f'(x+)-f'(x-) \mid \sqrt{\frac{C}{n}\left(1+\frac{1}{\rho}\right)(1+x)}.
\end{aligned}$$

Combining equations (3.7.3 -3.7.6) , we obtain

$$\begin{aligned}
|\mathcal{L}_n^\rho(f;x)-f(x)| \;&\leq\; \frac{1}{2}(f'(x+)+f'(x-))\left(\frac{A_{t_1}(1,1)+A_{t_2}(1,1)}{nA(1,1)}\right)\\
&+\; \frac{1}{2} \mid f'(x+)-f'(x-) \mid \sqrt{\frac{C}{n}\left(1+\frac{1}{\rho}\right)(1+x)}+|I_1|+|I_2|,
\end{aligned}$$

where

$$I_1=\int_0^x\int_x^t ((f')_x(v)dv)K_n^\rho(x,t)dt$$

and

$$I_2=\int_x^1\int_x^t ((f')_x(v)dv)K_n^\rho(x,t)dt.$$

Since $\int_a^b d_t\xi_n^\rho(x,t)\leq 1$ for all $[a,b]\subseteq \mathbb{R}_0^+$, using integration by parts and applying

Lemma 3.7.1, on substituting $y = x - x/\sqrt{n}$, we get

$$\begin{aligned}
I_1 &= \left| \int_0^x \int_x^t \left((f')_x(v)dv \right) d_t \xi_n^\rho(x,t) \right| = \left| \int_0^x \xi_n^\rho(x,t)(f')_x(t)dt \right| \\
&\leq \int_0^y |(f')_x(t)|\ |\xi_n^\rho(x,t)|dt + \int_y^x |(f')_x(t)|\ |\xi_n^\rho(x,t)|dt \\
&\leq \frac{C}{n}\left(1+\frac{1}{\rho}\right)(1+x)\int_0^y \bigvee_t^x ((f')_x)(x-t)^{-2}dt + \int_y^x \bigvee_t^x ((f')_x)dt \\
&\leq \frac{C}{n}\left(1+\frac{1}{\rho}\right)(1+x)\int_0^{x-x/\sqrt{n}} \bigvee_t^x ((f')_x)(x-t)^{-2}dt + \frac{x}{\sqrt{n}} \bigvee_{x-x/\sqrt{n}}^x ((f')_x).
\end{aligned}$$

Substituting $v = x/(x-t)$, we get

$$(1+x)\int_0^{x-x/\sqrt{n}} (x-t)^{-2} \bigvee_t^x ((f')_x)dt = (1+x)x^{-1}\int_1^{\sqrt{n}} \bigvee_{x-x/u}^x ((f')_x)dv$$

$$\leq (1+x)x^{-1}\sum_{k=1}^{[\sqrt{n}]}\int_k^{k+1} \bigvee_{x-x/k}^x ((f')_x)dv \leq \left(1+\frac{1}{x}\right)\sum_{k=1}^{[\sqrt{n}]} \bigvee_{x-x/k}^x ((f')_x).$$

Thus,

$$|I_1| \leq \frac{C}{n}\left(1+\frac{1}{\rho}\right)\left(1+\frac{1}{x}\right)\sum_{k=1}^{[\sqrt{n}]} \bigvee_{x-x/k}^x ((f')_x) + \frac{x}{\sqrt{n}} \bigvee_{x-x/\sqrt{n}}^x ((f')_x). \tag{3.7.6}$$

Again, using integration by parts, we get

$$\begin{aligned}
|I_2| &= \left| \int_x^\infty \int_x^t ((f')_x(v)dv)K_n^\rho(x,t)dt \right| \\
&= \left| \int_x^{2x} \int_x^t ((f')_x(v)dv)d_t(1-\xi_n^\rho(x,t)) + \int_{2x}^\infty \int_x^t ((f')_x(v)dv)K_n^\rho(x,t)dt \right| \\
&= \left| \left[\int_x^t ((f')_x(v)dv)(1-\xi_n^\rho(x,t)) \right]_x^{2x} \right| + \left| \int_x^{2x} (f')_x(t)(1-\xi_n^\rho(x,t))dt \right| \\
&\quad + \left| \int_{2x}^\infty \int_x^t \left((f'(v)-f'(x+))dv \right) K_n^\rho(x,t)dt \right| \\
&\leq \left| \int_x^{2x} \left((f')_x(v)dv \right)(1-\xi_n^\rho(x,2x)) + \left| \int_x^{2x} (f')_x(t)(1-\xi_n^\rho(x,t))dt \right| \right. \\
&\quad + \left| \int_{2x}^\infty f(t)K_n^\rho(x,t)dt \right| + |f(x)| \left| \int_{2x}^\infty K_n^\rho(x,t)dt \right| + |f'(x+)| \left| \int_{2x}^\infty ((t-x))K_n^\rho(x,t)dt \right|.
\end{aligned}$$

Applying Cauchy-Schwarz inequality, Lemma 3.7.1 and substituting

$z = x + (1-x)/\sqrt{n}$, we obtain

$$\begin{aligned} |I_2| &\leq \frac{C}{n}\left(1+\frac{1}{\rho}\right)\frac{(1+x)}{x^2}\left|\int_x^{2x}((f'(v)-f(x+))dv\right| + \left|\int_x^{x+x/\sqrt{n}} f'_x(t)dt\right| \\ &+\frac{C}{n}\left(1+\frac{1}{\rho}\right)(1+x)\left|\int_{x+x/\sqrt{n}}^{2x}(t-x)^{-2}f'_x(t)dt\right| + \left|\int_{2x}^{\infty} f(t)K_n^{\rho}(x,t)dt\right| \\ &+|f(x)|\left|\int_{2x}^{\infty}K_n^{\rho}(x,t)dt\right| + |f'(x+)|\left(\int_{2x}^{\infty}(t-x)^2K_n^{\rho}(x,t)dt\right)^{1/2}. \end{aligned}$$

By substituting $t = x + \frac{x}{u}$ and proceeding in a similar way as in the estimate of I_1, we get

$$\begin{aligned} |I_2| &\leq \frac{C}{n}\left(1+\frac{1}{\rho}\right)\frac{(1+x)}{x^2}|f(2x)-f(x)-xf'(x+)| + \frac{x}{\sqrt{n}}\bigvee_x^{x+x/\sqrt{n}}(f'_x) \\ &+\sum_{k=1}^{[\sqrt{n}]}\frac{C}{n}\left(1+\frac{1}{\rho}\right)\left(1+\frac{1}{x}\right)\bigvee_x^{x+x/\sqrt{n}} f'_x + \int_{2x}^{\infty}M(1+t^2)K_n^{\rho}(x,t)dt \\ &+|f(x)|\left|\int_{2x}^{\infty}K_n^{\rho}(x,t)dt\right| + |f'(x+)|\sqrt{\frac{C}{n}\left(1+\frac{1}{\rho}\right)(1+x)}. \end{aligned} \tag{3.7.7}$$

For $t \geq 2x$, it follows that $t \leq 2(t-x)$ and $x \leq t-x$. Now using Lemma 3.2.2 , we obtain

$$\begin{aligned} &\int_{2x}^{\infty}M(1+t^2)K_n^{\rho}(x,t)dt + |f(x)|\int_{2x}^{\infty}K_n^{\rho}(x,t)dt \\ &\leq \frac{M}{x^2}\int_{2x}^{\infty}(t-x)^2K_n^{\rho}(x,t)dt + 4M\int_{2x}^{\infty}(t-x)^2K_n^{\rho}(x,t)dt \\ &+\frac{|f(x)|}{x^2}\int_{2x}^{\infty}(t-x)^2K_n^{\rho}(x,t)dt \\ &\leq \frac{C}{n}\left(1+\frac{1}{\rho}\right)(1+x)\left(\frac{M+|f(x)|}{x^2}+4M\right). \end{aligned} \tag{3.7.8}$$

Collecting the estimates (3.7.6-3.7.8), we get the required result. □

Chapter 4

Bézier variant of the Bernstein-Durrmeyer type operators

4.1 Introduction

In 1912, Bernstein [29] defined a sequence of positive linear operators for $f \in C[0,1]$, as

$$B_n(f;x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} f\left(\frac{k}{n}\right), x \in [0,1]$$

which preserves linear functions.

To make convergence faster, King [104] introduced a modification of these operators as

$$\left((B_n f) \circ r_n\right)(x) = \sum_{k=0}^{n} \binom{n}{k} \left(r_n\left(x\right)\right)^k \left(1 - r_n\left(x\right)\right)^{n-k} f\left(\frac{k}{n}\right),$$

which depends on a sequence $r_n(x)$ of continuous functions on $[0,1]$ with $0 \leq r_n(x) \leq 1$, for each $x \in [0,1]$ and considered a particular case for the sequence $r_n(x)$ such that the corresponding operators preserve the test functions e_0 and $e_2 (e_i = t^i,\ i = 0,\ 1,\ 2)$ of the Bohman-Korovkin theorem. Cárdenas-Morales et al. [37] extended this result considering a family of sequences of operators $B_{n,\alpha}$ that preserve e_0 and $e_2 + \alpha e_1$

with $\alpha \in [0,\infty)$. Gonska et al. [68] constructed sequences of King-type operators which are based on a strictly increasing continuous function τ such that $\tau(0) = 0$ and $\tau(1) = 1$. These operators are defined by $V_n^\tau : C[0,1] \to C[0,1]$

$$V_n^\tau f = (B_n f) \circ \tau_n = (B_n f) \circ (B_n \tau)^{-1} \circ \tau,$$

and preserve the test functions e_0 and e_1. Inspired by the above ideas, for any function τ being infinite times continuously differentiable on $[0,1]$, such that $\tau(0) = 0$, $\tau(1) = 1$ and $\tau'(x) > 0$ for $x \in [0,1]$, Cardenas-Morales et al. [38] defined a sequence of linear Bernstein type operators for $f \in C[0,1]$ as

$$B_n^\tau(f;x) = \sum_{k=0}^{n} \binom{n}{k} \tau^k(x)\,(1-\tau(x))^{n-k} \left(f \circ \tau^{-1}\right)\left(\frac{k}{n}\right), \qquad (4.1.1)$$

and investigated its shape preserving and convergence properties as well as its asymptotic behavior and saturation. This type of approximation generalizes the Korovkin set from $\{e_0,\ e_1,\ e_2\}$ to $\{1,\ \tau,\ \tau^2\}$ and also presents a better degree of approximation depending on τ. To approximate the Lebesgue integrable functions on $[0,1]$, Acar et al. [7] defined the Durrmeyer type modification for the operators (4.1.1) as

$$D_n^\tau(f;x) = (n+1)\sum_{k=0}^{n} p_{n,k}^\tau(x) \int_0^1 \left(f \circ \tau^{-1}\right)(t)\, p_{n,k}(t)\,dt, \qquad (4.1.2)$$

where, $p_{n,k}^\tau(x) := \binom{n}{k}\tau^k(x)\,(1-\tau(x))^{n-k}$, $p_{n,k}(x) := \binom{n}{k}x^k(1-x)^{n-k}$ and studied Voronovskaya type asymptotic formula as well as its quantitative version and the local approximation properties of D_n^τ in quantitative form in terms of K-functional and Ditzian-Totik moduli of smoothness.

Motivated by the above work, we introduce the Bézier-variant of the operators given by (4.1.2) as

$$D_n^{\tau,\theta}(f;x) = (n+1)\sum_{k=0}^{n} Q_{n,k}^{\tau,\theta}(x) \int_0^1 \left(f \circ \tau^{-1}\right)(t)\, p_{n,k}(t)\,dt, \qquad (4.1.3)$$

where $Q_{n,k}^{\tau,\theta}(x) = \left[I_{n,k}^{\tau}(x)\right]^{\theta} - \left[I_{n,k+1}^{\tau}(x)\right]^{\theta}$, $\theta \geq 1$ with $I_{n,k}^{\tau}(x) = \sum_{j=k}^{n} p_{n,k}^{\tau}(x)$, when $k \leq n$ and 0 otherwise and study the degree of approximation in terms of the modulus of continuity and the K-functional for the operators given by (4.1.3). The quantitative Voronoskaya type theorem and the rate of convergence of the functions having derivatives of bounded variation for these operators is also investigated.

4.2 Auxiliary Results

In the sequel, we shall require the following lemmas to prove the main results of this chapter.

Lemma 4.2.1. *[7] For the operators D_n^{τ}, one has*

$$D_n^{\tau}(1) = 1, \quad D_n^{\tau}(\tau) = \frac{1+\tau n}{n+2}, \quad D_n^{\tau}(\tau^2) = \frac{\tau^2 n\,(n-1) + 4n\tau + 2}{(n+2)\,(n+3)}.$$

Consequently, for the m-th order central moment of the operators D_n^{τ} defined as

$$\mu_{n,m}^{\tau}(x) \;=\; D_n^{\tau}\left((\tau(t) - \tau(x))^m ; x\right), \; m \in \mathbb{N},$$

for all $n \in \mathbb{N}$, there follows

$$\begin{aligned} \mu_{n,0}^{\tau}(x) &= 1, \; \mu_{n,1}^{\tau}(x) = \frac{1-2\tau(x)}{n+2}, \\ \mu_{n,2}^{\tau}(x) &= \frac{\tau(x)\,(1-\tau(x))\,(2n-6)+2}{(n+2)\,(n+3)}, \end{aligned} \tag{4.2.1}$$

By a simple calculation, we have

$$\mu_{n,4}^{\tau}(x) = \frac{4\varphi_{\tau}^2(x)\left\{(3n^2+25n-210)\varphi_{\tau}^2(x) + (6n+12)\right\} + 24}{(n+2)(n+3)(n+4)(n+5)}.$$

Remark 4.2.2. *[7] For all $n \in \mathbb{N}$, we have*

$$\mu_{n,2}^{\tau}(x) \leq \frac{2}{n+2}\delta_{n,\tau}^2(x), \tag{4.2.2}$$

where $\delta_{n,\tau}^2(x) := \varphi_{\tau}^2(x) + \dfrac{1}{n+3}$, $\varphi_{\tau}^2(x) := \tau(x)\,(1-\tau(x))$, $x \in [0,1]$.

Lemma 4.2.3. *[7] For every $f \in C[0,1]$,*

$$\|D_n^{\tau}(f;.)\| \leq \|f\|.$$

Applying Lemma 4.2.1 , the proof of this lemma easily follows. Hence the details are omitted.

Lemma 4.2.4. *Let $f \in C[0,1]$. Then, we have*

$$\| D_n^{\tau,\theta}(f;.) \| \leq \theta \| f \| .$$

Proof. Using the inequality $| a^{\theta} - b^{\theta} | \leq \theta | a - b |$ with $0 \leq a, b \leq 1, \theta \geq 1$ and from the definition of $Q_{n,k}^{\tau,\theta}$, we have

$$0 \quad < \quad \left[I_{n,k}^{\tau}(x)\right]^{\theta} - \left[I_{n,k+1}^{\tau}(x)\right]^{\theta} \leq \theta(I_{n,k}^{\tau}(x) - I_{n,k+1}^{\tau}(x) = \theta p_{n,k}^{\tau}(x).$$

Hence from the definition of $D_n^{\tau,\theta}$ and Lemma 4.2.3, we obtain

$$\| D_n^{\tau,\theta}(f) \| \leq \theta \| D_n^{\tau}(f) \| \leq \theta \| f \| .$$

This completes the proof. □

Remark 4.2.5. *We have*

$$\begin{aligned} D_n^{\tau,\theta}(e_0;x) &= \sum_{k=0}^{n} Q_{n,k}^{(\theta)}(x) = [J_{n,0}(x)]^{\theta} \\ &= \left[\sum_{j=0}^{n} p_{n,j}^{\tau}(x)\right]^{\theta} = 1, \ \textit{since} \ \sum_{j=0}^{n} p_{n,k}^{\tau}(x) = 1. \end{aligned}$$

4.3 Main Results

4.3.1 Direct results

Throughout this chapter we assume that $\inf\limits_{x\in[0,1]} \tau'(x) \geq a, a \in (0,\infty)$.

Theorem 4.3.1. *For $f \in C[0,1]$ and $x \in [0,1]$, there holds*

$$\left|D_n^{\tau,\theta}(f;x) - f(x)\right| \leq \left\{1 + \sqrt{2\theta\left(\varphi_{\tau}^2(x) + \frac{1}{n+3}\right)}\right\}\omega\left(\left(f \circ \tau^{-1}\right); \sqrt{\frac{1}{n}}\right),$$

where $\omega((f \circ \tau^{-1}); \delta)$ is the usual modulus of continuity.

Proof. By linearity of the operators $D_n^{\tau,\theta}$, we get

$$\begin{aligned}\left|D_n^{\tau,\theta}(f;x)-f(x)\right| &\le (n+1)\sum_{k=0}^{n} Q_{n,k}^{\tau,\theta}(x)\int_0^1 p_{n,k}(t)\left|\left(f\circ\tau^{-1}\right)(t)-f(x)\right|dt\\ &\le (n+1)\sum_{k=0}^{n} Q_{n,k}^{\tau,\theta}(x)\int_0^1 p_{n,k}(t)\left(1+\frac{|t-\tau(x)|}{\delta}\right)dt\,\omega(\left(f\circ\tau^{-1}\right);\delta).\end{aligned}$$

Applying Hölder's inequality and Lemma 4.2.3, we obtain

$$\begin{aligned}\left|D_n^{\tau,\theta}(f;x)-f(x)\right| &\le \left\{1+\frac{1}{\delta}\left(D_n^{\tau,\theta}((\tau(t)-\tau(x))^2;x)\right)^{1/2}\right\}\omega(\left(f\circ\tau^{-1}\right);\delta)\\ &\le \left\{1+\frac{1}{\delta}\left(\theta D_n^{\tau}((\tau(t)-\tau(x))^2;x)\right)^{1/2}\right\}\omega(\left(f\circ\tau^{-1}\right);\delta)\\ &\le \left\{1+\frac{1}{\delta}\sqrt{\frac{2\theta}{n+2}\left(\varphi_\tau^2(x)+\frac{1}{n+3}\right)}\right\}\omega\Big(\left(f\circ\tau^{-1}\right);\delta\Big).\end{aligned}$$

Taking $\delta=\sqrt{\dfrac{1}{n}}$, we get the desired result. □

Next, we establish a direct result using the Ditzian-Totik modulus of smoothness. Let us take $\phi(x)=\varphi_\tau(x):=\sqrt{\tau(x)(1-\tau(x))}$

Theorem 4.3.2. *Let $f\in C[0,1]$. Then for every $x\in(0,1)$, we have*

$$\left|D_n^{\tau,\theta}(f;x)-f(x)\right|\le C(\theta)\omega_{\varphi_\tau}\left(f;\frac{1}{a}\sqrt{\frac{\theta}{n+2}\left(1+\frac{1}{(n+3)\varphi_\tau^2(x)}\right)}\right).$$

Proof. Using the representation

$$h(t)=\left(h\circ\tau^{-1}\right)(\tau(t))=\left(h\circ\tau^{-1}\right)(\tau(x))+\int_{\tau(x)}^{\tau(t)}\left(h\circ\tau^{-1}\right)'(u)du,$$

we get

$$\left|D_n^{\tau,\theta}(h;x)-h(x)\right|=\left|D_n^{\tau,\theta}\left(\int_{\tau(x)}^{\tau(t)}\left(h\circ\tau^{-1}\right)'(u)du\right)\right|. \tag{4.3.1}$$

But,

$$\begin{aligned}\left|\int_{\tau(x)}^{\tau(t)}\left(h\circ\tau^{-1}\right)'(u)du\right| &= \left|\int_x^t\frac{h'(y)}{\tau'(y)}\tau'(y)dy\right|=\left|\int_x^t\frac{\varphi_\tau(y)}{\varphi_\tau(y)}\cdot\frac{h'(y)}{\tau'(y)}\tau'(y)dy\right|\\ &\le \frac{\|\varphi_\tau h'\|}{a}\left|\int_x^t\frac{\tau'(y)}{\varphi_\tau(y)}dy\right|,\end{aligned} \tag{4.3.2}$$

and

$$\left|\int_x^t \frac{\tau'(y)}{\varphi_\tau(y)}dy\right| \leq \left|\int_x^t \left(\frac{1}{\sqrt{\tau(y)}} + \frac{1}{\sqrt{1-\tau(y)}}\right)\tau'(y)dy\right|$$
$$\leq 2\left(\left|\sqrt{\tau(t)} - \sqrt{\tau(x)}\right| + \left|\sqrt{1-\tau(t)} - \sqrt{1-\tau(x)}\right|\right)$$
$$= 2\left|\tau(t) - \tau(x)\right|\left(\frac{1}{\sqrt{\tau(t)} + \sqrt{\tau(x)}} + \frac{1}{\sqrt{1-\tau(t)} + \sqrt{1-\tau(x)}}\right)$$
$$< 2|\tau(t) - \tau(x)|\left(\frac{1}{\sqrt{\tau(x)}} + \frac{1}{\sqrt{1-\tau(x)}}\right) \leq \frac{2\sqrt{2}|\tau(t) - \tau(x)|}{\varphi_\tau(x)}. \tag{4.3.3}$$

Hence from relations (4.3.1)-(4.3.3) and using Cauchy-Schwarz inequality, we obtain

$$|D_n^{\tau,\theta}(h;x) - h(x)| \leq 2\sqrt{2}\frac{\|\varphi_\tau h'\|}{a\varphi_\tau(x)}D_n^{\tau,\theta}\left(|\tau(t) - \tau(x)|;x\right)$$
$$\leq 2\sqrt{2}\frac{\|\varphi_\tau h'\|}{a\varphi_\tau(x)}\left[D_n^{\tau,\theta}\left((\tau(t) - \tau(x))^2;x\right)\right]^{1/2}$$
$$\leq 2\sqrt{2}\frac{\|\varphi_\tau h'\|}{a\varphi_\tau(x)}\left[\theta D_n^{\tau}\left((\tau(t) - \tau(x))^2;x\right)\right]^{1/2}$$
$$\leq \frac{4}{a}\|\varphi_\tau h'\|\sqrt{\frac{\theta}{n+2}\left(1 + \frac{1}{(n+3)\varphi_\tau^2(x)}\right)}. \tag{4.3.4}$$

Using Lemma 4.2.4 and (4.3.4) it follows that

$$\left|D_n^{\tau,\theta}(f;x) - f(x)\right| \leq \left|D_n^{\tau,\theta}(f-h;x)\right| + |f(x) - h(x)| + \left|D_n^{\tau,\theta}(h;x) - h(x)\right|$$
$$\leq \left\{(\theta+1)\|f-h\| + \frac{4}{a}\|\varphi_\tau h'\|\sqrt{\frac{\theta}{n+2}\left(1 + \frac{1}{(n+3)\varphi_\tau^2(x)}\right)}\right\}$$
$$\leq C_1(\theta)\left\{\|f-h\| + \frac{1}{a}\|\varphi_\tau h'\|\sqrt{\frac{\theta}{n+2}\left(1 + \frac{1}{(n+3)\varphi_\tau^2(x)}\right)}\right\},$$

where $C_1(\theta) = \max\left\{(\theta+1), 4\right\}$.

Taking the infimum on the right hand side of the above inequality over all $g \in W_{\varphi_\tau}[0,1]$, we get

$$\left|D_n^{\tau,\theta}(f;x) - f(x)\right| \leq C_1(\theta)K_{\varphi_\tau}\left(f;\frac{1}{a}\sqrt{\frac{\theta}{n+2}\left(1 + \frac{1}{(n+3)\varphi_\tau^2(x)}\right)}\right).$$

Using the relation (0.5.3), the theorem is proved. □

4.3.2 Quantitative Voronovskaya type theorem

In this section we prove a quantitative Voronovskaja type theorem for the operator $D_n^{\tau,\theta}$ in terms of the first order Ditzian-Totik modulus of smoothness. In the recent years, several researchers have made significant contributions in this direction (cf. [3], [58], [71], [100], [115], [131], [153] etc.).

Theorem 4.3.3. *For any $f \in C^2[0,1]$ and $x \in [0,1]$, the following inequalities hold*

$$\begin{aligned}
|\sqrt{n}\left[D_n^{\tau,\theta}(f;x)-f(x)\right]| &\leq \sqrt{2\theta\left\{\varphi_\tau^2(x)+\frac{1}{n+3}\right\}}||\left(f\circ\tau^{-1}\right)'|| \\
&+||\left(f\circ\tau^{-1}\right)''||\frac{\theta}{\sqrt{n}}\varphi_\tau^2(x)+\frac{C}{\sqrt{n}}\omega_{\varphi_\tau}\left((f\circ\tau^{-1})''; \frac{2\sqrt{6}}{an^{1/2}}\varphi_\tau(x)\right)+o(n^{-1}),\ as\, n\to\infty; \\
|\sqrt{n}\left[D_n^{\tau,\theta}(f;x)-f(x)\right]| &\leq \sqrt{2\theta\left\{\varphi_\tau^2(x)+\frac{1}{n+3}\right\}}||\left(f\circ\tau^{-1}\right)'|| \\
&+||\left(f\circ\tau^{-1}\right)''||\frac{\theta}{\sqrt{n}}\varphi_\tau^2(x)+\frac{C}{\sqrt{n}}\varphi_\tau(x)\omega_{\varphi_\tau}\left((f\circ\tau^{-1})''; \frac{2\sqrt{6}}{an^{1/2}}\right)+o(n^{-1}),\ as\, n\to\infty,
\end{aligned}$$

where C is a constant depending on θ.

Proof. Let $f \in C^2[0,1]$ and $x, t \in [0,1]$. Then by Taylor's expansion, we have

$$\begin{aligned}
f(t) &= \left(f\circ\tau^{-1}\right)(\tau(t)) = \left(f\circ\tau^{-1}\right)(\tau(x)) + \left(f\circ\tau^{-1}\right)'(\tau(x))\left(\tau(t)-\tau(x)\right) \\
&+ \int_{\tau(x)}^{\tau(t)}\left(\tau(t)-u\right)\left(f\circ\tau^{-1}\right)''(u)du.
\end{aligned}$$

Hence,

$$\begin{aligned}
f(t)-f(x) &= \left(f\circ\tau^{-1}\right)'(\tau(x))\left(\tau(t)-\tau(x)\right) - \frac{1}{2}\left(f\circ\tau^{-1}\right)''(\tau(x))\left(\tau(t)-\tau(x)\right)^2 \\
&+ \int_{\tau(x)}^{\tau(t)}\left(\tau(t)-u\right)\left(f\circ\tau^{-1}\right)''(u)du - \int_{\tau(x)}^{\tau(t)}\left(\tau(t)-u\right)\left(f\circ\tau^{-1}\right)''(\tau(x))du \\
&= \left(f\circ\tau^{-1}\right)'(\tau(x))\left(\tau(t)-\tau(x)\right) - \frac{1}{2}\left(f\circ\tau^{-1}\right)''(\tau(x))\left(\tau(t)-\tau(x)\right)^2 \\
&+ \int_{\tau(x)}^{\tau(t)}\left(\tau(t)-u\right)\left[\left(f\circ\tau^{-1}\right)''(u) - \left(f\circ\tau^{-1}\right)''(\tau(x))\right]du.
\end{aligned}$$

Applying $D_n^{\tau,\theta}$ to both sides of the above relation, we get

$$\begin{aligned}
&\left|D_n^{\tau,\theta}(f;x)-f(x)\right| \\
&\leq \left|\left(f\circ\tau^{-1}\right)'(\tau(x))D_n^{\tau,\theta}((\tau(t)-\tau(x))\,;x)\right| - \frac{1}{2}\left(f\circ\tau^{-1}\right)''(\tau(x))D_n^{\tau,\theta}((\tau(t)-\tau(x))^2\,;x) \\
&+\left|D_n^{\tau,\theta}\left(\int_{\tau(x)}^{\tau(t)}(\tau(t)-u)\left[\left(f\circ\tau^{-1}\right)''(u)-\left(f\circ\tau^{-1}\right)''(\tau(x))\right]du;x\right)\right| \qquad (4.3.5)
\end{aligned}$$

For $g\in W_{\phi^\tau}[0,1]$, we have

$$\begin{aligned}
&\left|\int_{\tau(x)}^{\tau(t)}|\tau(t)-u|\left|\left(f\circ\tau^{-1}\right)''(u)-\left(f\circ\tau^{-1}\right)''(\tau(x))\right|du\right| \\
&\leq\left|\int_{\tau(x)}^{\tau(t)}|\tau(t)-u|\left|\left(f\circ\tau^{-1}\right)''(u)-\left(g\circ\tau^{-1}\right)(u)\right|du\right| \\
&+\left|\int_{\tau(x)}^{\tau(t)}|\tau(t)-u|\left|\left(g\circ\tau^{-1}\right)(u)-\left(g\circ\tau^{-1}\right)(\tau(x))\right|du\right| \\
&+\left|\int_{\tau(x)}^{\tau(t)}|\tau(t)-u|\left|\left(g\circ\tau^{-1}\right)(\tau(x))-\left(f\circ\tau^{-1}\right)''(\tau(x))\right|du\right| \\
&=\left|\int_x^t\left|\left(f\circ\tau^{-1}\right)''(\tau(y))-g(y)\right||\tau(t)-\tau(y)|\,\tau'(y)dy\right| \\
&+\left|\int_x^t|g(y)-g(x)|\,|\tau(t)-\tau(y)|\,\tau'(y)dy\right| \\
&+\left|\int_x^t\left|g(x)-\left(f\circ\tau^{-1}\right)''(\tau(x))\right||\tau(t)-\tau(y)|\,\tau'(y)dy\right| \\
&\leq 2\|\left(f\circ\tau^{-1}\right)''-g\|\left|\int_x^t|\tau(t)-\tau(y)|\tau'(y)dy\right| \\
&+\left|\int_x^t\left|\int_x^y|g'(v)|dv\right||\tau(t)-\tau(y)|\tau'(y)dy\right| \\
&\leq\|\left(f\circ\tau^{-1}\right)''-g\|(\tau(t)-\tau(x))^2 \\
&+\|\varphi_\tau g'\|\left|\int_x^t\left|\int_x^y\frac{dv}{\varphi_\tau(v)}\right||\tau(t)-\tau(y)|\tau'(y)dy\right|. \qquad (4.3.6)
\end{aligned}$$

Using the inequality [48, p. 141]

$$\frac{|y-v|}{v(1-v)}\leq\frac{|y-x|}{x(1-x)},\ v \text{ is between } y \text{ and x },$$

we can write

$$\frac{|\tau(y)-\tau(v)|}{\tau(v)(1-\tau(v))}\leq\frac{|\tau(y)-\tau(x)|}{\tau(x)(1-\tau(x))}.$$

Also,

$$\begin{aligned}
&\|\varphi_\tau g'\|\left|\int_x^t \left|\int_x^y \frac{dv}{\varphi_\tau(v)}\right| |\tau(t)-\tau(y)|\tau'(y)dy\right| \\
&\leq \|\varphi_\tau g'\|\left|\int_x^t \left|\int_x^y \frac{|\tau(y)-\tau(x)|^{1/2}}{\tau'(v)\varphi_\tau(x)}\cdot\frac{\tau'(v)}{|\tau(y)-\tau(v)|^{1/2}}dv\right| |\tau(t)-\tau(y)|\tau'(y)dy\right| \\
&\leq 2\frac{\|\varphi_\tau g'\|}{a}\varphi_\tau^{-1}(x)\left|\int_x^t |\tau(y)-\tau(x)|\,|\tau(t)-\tau(y)|\tau'(y)dy\right| \\
&\leq 2\frac{\|\varphi_\tau g'\|}{a}\varphi_\tau^{-1}(x)\left|\int_x^t (\tau(t)-\tau(x))^2\tau'(y)dy\right| \\
&\leq 2\frac{\|\varphi_\tau g'\|}{a}\varphi_\tau^{-1}(x)|\tau(t)-\tau(x)|^3. \qquad (4.3.7)
\end{aligned}$$

Now combining the relations (4.3.5)-(4.3.7) and applying the Cauchy-Schwarz inequality, we get

$$\begin{aligned}
&\left|D_n^{\tau,\theta}(f;x)-f(x)\right| \\
&\leq |\left(f\circ\tau^{-1}\right)'(\tau(x))|D_n^{\tau,\theta}(|\tau(t)-\tau(x)|\,;x)+\frac{1}{2}|\left(f\circ\tau^{-1}\right)''(\tau(x)|D_n^{\tau,\theta}((\tau(t)-\tau(x))^2\,;x) \\
&+\|(f\circ\tau^{-1})''-g\|D_n^{\tau,\theta}((\tau(t)-\tau(x))^2;x)+2\frac{\|\varphi_\tau g'\|}{a}\varphi_\tau^{-1}(x)D_n^{\tau,\theta}(|\tau(t)-\tau(x)|^3;x) \\
&\leq ||\left(f\circ\tau^{-1}\right)'||\left(D_n^{\tau,\theta}\left(((\tau(t)-\tau(x))^2;x\right)\right)^{1/2}+\frac{1}{2}||\left(f\circ\tau^{-1}\right)''||D_n^{\tau,\theta}((\tau(t)-\tau(x))^2\,;x) \\
&+\|(f\circ\tau^{-1})''-g\|D_n^{\tau,\theta}\left((\tau(t)-\tau(x))^2;x\right)+\frac{2\|\varphi_\tau g'\|}{a}\varphi_\tau^{-1}(x) \\
&\left(D_n^{\tau,\theta}\left((\tau(t)-\tau(x))^2;x\right)\right)^{1/2}\left(D_n^{\tau,\theta}\left((\tau(t)-\tau(x))^4;x\right)\right)^{1/2}.
\end{aligned}$$

Applying Lemma 4.2.4, we have

$$\begin{aligned}
&\left|D_n^{\tau,\theta}(f;x)-f(x)\right| \leq ||\left(f\circ\tau^{-1}\right)'||\left(\theta D_n^{\tau}(\left((\tau(t)-\tau(x))^2\right);x)\right)^{1/2} \\
&+\frac{1}{2}||\left(f\circ\tau^{-1}\right)''||\left(\theta D_n^{\tau}((\tau(t)-\tau(x))^2\,;x)\right) \\
&+\|(f\circ\tau^{-1})''-g\|\left(\theta D_n^{\tau}\left((\tau(t)-\tau(x))^2;x\right)\right)+\frac{2\|\varphi_\tau g'\|}{a}\varphi_\tau^{-1}(x) \\
&\left(\theta D_n^{\tau}\left((\tau(t)-\tau(x))^2;x\right)\right)^{1/2}\left(\theta D_n^{\tau}\left((\tau(t)-\tau(x))^4;x\right)\right)^{1/2}.
\end{aligned}$$

Hence using Lemmas 4.2.1 and 4.2.3, we get

$$\begin{aligned}
&\left|D_n^{\tau,\theta}(f;x)-f(x)\right| \\
&\leq \sqrt{\frac{2\theta}{n+2}\left\{\varphi_\tau^2(x)+\frac{1}{n+3}\right\}}\,||\left(f\circ\tau^{-1}\right)'||+||\left(f\circ\tau^{-1}\right)''||\frac{\theta}{n+2}\left\{\varphi_\tau^2(x)+\frac{1}{n+3}\right\} \\
&+\frac{2\theta}{n+2}\left\{\varphi_\tau^2(x)+\frac{1}{n+3}\right\}\|\left(f\circ\tau^{-1}\right)''-g\|+\varphi_\tau^{-1}(x)\frac{2\|\varphi_\tau g'\|}{a}.
\end{aligned}$$

$$\sqrt{\frac{2\theta}{n+2}\left\{\varphi_\tau^2(x)+\frac{1}{n+3}\right\}}\sqrt{\theta\left[\frac{4\varphi_\tau^2(x)\left\{(3n^2+25n-210)\varphi_\tau^2(x)+(6n+12)\right\}+24}{(n+2)(n+3)(n+4)(n+5)}\right]}$$

$$\begin{aligned}
&\leq \sqrt{\frac{2\theta}{n+2}\left\{\varphi_\tau^2(x)+\frac{1}{n+3}\right\}}\,||\left(f\circ\tau^{-1}\right)'||+||\left(f\circ\tau^{-1}\right)''||\frac{\theta}{n+2}\varphi_\tau^2(x) \\
&+\frac{2\theta}{n+2}\left\{\varphi_\tau^2(x)\|\left(f\circ\tau^{-1}\right)''-g\|+\frac{\|\varphi_\tau g'\|}{a}\varphi_\tau(x)\frac{2\sqrt{6}}{n^{1/2}}\right\}+o(n^{-3/2}).
\end{aligned}$$

Because $\varphi_\tau^2(x)\leq\varphi_\tau(x)\leq 1$, $x\in[0,1]$ we obtain

$$\begin{aligned}
&\left|D_n^{\tau,\theta}(f;x)-f(x)\right| \leq \sqrt{\frac{2\theta}{n+2}\left\{\varphi_\tau^2(x)+\frac{1}{n+3}\right\}}\,||\left(f\circ\tau^{-1}\right)'||+||\left(f\circ\tau^{-1}\right)''|| \\
&\frac{\theta}{n+2}\varphi_\tau^2(x)+\frac{2\theta}{n+2}\left\{\|(f\circ\tau^{-1})''-g\|+\frac{2\sqrt{6}}{an^{1/2}}\varphi_\tau(x)\|\varphi_\tau g'\|\right\}+o(n^{3/2}), \quad (4.3.8)
\end{aligned}$$

$$\begin{aligned}
&\left|D_n^{\tau,\theta}(f;x)-f(x)\right| \leq \sqrt{\frac{2\theta}{n+2}\left\{\varphi_\tau^2(x)+\frac{1}{n+3}\right\}}\,||\left(f\circ\tau^{-1}\right)'||+||\left(f\circ\tau^{-1}\right)''|| \\
&\frac{\theta}{n+2}\varphi_\tau^2(x)+\frac{2\theta}{n+2}\varphi_\tau(x)\left\{\|(f\circ\tau^{-1})''-g\|+\frac{2\sqrt{6}}{an^{1/2}}\|\varphi_\tau g'\|\right\}+o(n^{3/2}). \quad (4.3.9)
\end{aligned}$$

Taking the infimum on the right hand side of (4.3.8) and (4.3.9) over all $g\in W_{\varphi_\tau}[0,1]$, we get

$$\begin{aligned}
&|\sqrt{n}\left[D_n^{\tau,\theta}(f;x)-f(x)\right]| \leq \sqrt{2\theta\left\{\varphi_\tau^2(x)+\frac{1}{n+3}\right\}}\,||\left(f\circ\tau^{-1}\right)'|| \\
&+||\left(f\circ\tau^{-1}\right)''||\frac{\theta}{\sqrt{n}}\varphi_\tau^2(x)+\frac{C}{\sqrt{n}}K_{\varphi_\tau}\left((f\circ\tau^{-1})'';\frac{2\sqrt{6}}{an^{1/2}}\varphi_\tau(x)\right)+o(n^{-1}); \\
&|\sqrt{n}\left[D_n^{\tau,\theta}(f;x)-f(x)\right]| \leq \sqrt{2\theta\left\{\varphi_\tau^2(x)+\frac{1}{n+3}\right\}}\,||\left(f\circ\tau^{-1}\right)'|| \\
&+||\left(f\circ\tau^{-1}\right)''||\frac{\theta}{\sqrt{n}}\varphi_\tau^2(x)+\frac{C}{\sqrt{n}}\varphi_\tau(x)K_{\varphi_\tau}\left((f\circ\tau^{-1})'';\frac{2\sqrt{6}}{an^{1/2}}\right)+o(n^{-1}).
\end{aligned}$$

Using relation (0.5.3), we reach the desired result. □

4.3.3 Rate of approximation

Lastly we discuss the approximation of functions with a derivative of bounded variation on $[0,1]$. The operators $D_n^{\tau,\theta}$ can be expressed in an integral form as follows:

$$D_n^{\tau,\theta}(f;x) = \int_0^1 K_n^{\tau,\theta}(x,t)(f\circ\tau^{-1})(t)dt, \tag{4.3.10}$$

where the kernel $K_n^{\tau,\theta}$ is given by

$$K_n^{\tau,\theta}(x,t) = (n+1)\sum_{k=0}^{n} Q_{n,k}^{\tau,\theta}(x)\, p_{n,k}(t).$$

Lemma 4.3.4. *For a fixed $x \in (0,1)$ and sufficiently large n, we have*

(i) $\xi_n^{\tau,\theta}(x,y) = \int_0^y K_n^{\tau,\theta}(x,t)dt \le \dfrac{2\theta}{n+2}\dfrac{\delta_{n,\tau}^2(x)}{(\tau(x)-y)^2}$, $0 \le y < \tau(x)$,

(ii) $1-\xi_n^{\tau,\theta}(x,z) = \int_z^1 K_n^{\tau,\theta}(x,t)dt \le \dfrac{2\theta}{n+2}\dfrac{\delta_{n,\tau}^2(x)}{(z-\tau(x))^2}$, $\tau(x) < z < 1$,

where $\delta_{n,\tau}^2(x)$ is defined in Remark 4.2.2.

Proof. (i) Using Remark 4.2.2 and Lemma 4.2.4, we get

$$\begin{aligned}
\xi_n^{\tau,\theta}(x,y) &= \int_0^y K_n^{\tau,\theta}(x,t)dt \le \int_0^y \left(\frac{\tau(x)-t}{\tau(x)-y}\right)^2 K_n^{\tau,\theta}(x,t)dt \\
&\le \frac{D_n^{\tau,\theta}((\tau(t)-\tau(x))^2;x)}{(\tau(x)-y)^2} \le \theta\frac{D_n^{\tau}((\tau(t)-\tau(x))^2;x)}{(\tau(x)-y)^2} \le \theta\frac{\delta_{n,\tau}^2(x)}{(\tau(x)-y)^2}.
\end{aligned}$$

The proof of (ii) is similar hence the details are omitted. □

Theorem 4.3.5. *Let $f \in DBV[0,1]$. Then, for every $x \in (0,1)$ and sufficiently* *large n, we have*

$$
\begin{aligned}
|D_n^{\tau,\theta}(f;x)-f(x)| &\le \left\{\frac{1}{\theta+1}\left|(f\circ\tau^{-1})'(\tau(x+))+\theta(f\circ\tau^{-1})'(\tau(x-))\right|\right.\\
&+\left.\left|(f\circ\tau^{-1})'(\tau(x+))-(f\circ\tau^{-1})'(\tau(x-))\right|\right\}\sqrt{\frac{2\theta}{n+2}}\delta_{n,\tau}(x)\\
&+\frac{2\theta}{n+2}\frac{\delta_{n,\tau}^2(x)}{\tau(x)}\sum_{k=1}^{[\sqrt{n}]}\left(\bigvee_{\tau(x)-\frac{\tau(x)}{k}}^{\tau(x)}(f\circ\tau^{-1})'_x\right)+\frac{\tau(x)}{\sqrt{n}}\left(\bigvee_{\tau(x)-\frac{\tau(x)}{\sqrt{n}}}^{\tau(x)}(f\circ\tau^{-1})'_x\right)\\
&+\frac{2\theta}{n+2}\frac{\delta_{n,\tau}^2(x)}{(1-\tau(x))}\sum_{k=1}^{[\sqrt{n}]}\left(\bigvee_{\tau(x)}^{\tau(x)+\frac{(1-\tau(x))}{k}}(f\circ\tau^{-1})'_x\right)\\
&+\frac{(1-\tau(x))}{\sqrt{n}}\left(\bigvee_{\tau(x)}^{\tau(x)+\frac{(1-\tau(x))}{\sqrt{n}}}(f\circ\tau^{-1})'_x\right),
\end{aligned}
$$

where $(f\circ\tau^{-1})'_x$ *is defined by*

$$
(f\circ\tau^{-1})'_x(t)=\begin{cases}(f\circ\tau^{-1})'(t)-(f\circ\tau^{-1})'(\tau(x-)), & 0\le t<\tau(x)\\ 0, & t=\tau(x)\\ (f\circ\tau^{-1})'(t)-(f\circ\tau^{-1})'(\tau(x+)) & \tau(x)<t<1.\end{cases} \tag{4.3.11}
$$

Proof. Since $D_n^{\tau,\theta}(1;x)=1$, using (4.3.10), for every $x\in(0,1)$ we get

$$
\begin{aligned}
D_n^{\tau,\theta}(f;x)-f(x) &= \int_0^1 K_n^{\tau,\theta}(x,t)((f\circ\tau^{-1})(t)-(f\circ\tau^{-1})(\tau(x)))dt\\
&= \int_0^1 K_n^{\tau,\theta}(x,t)\int_{\tau(x)}^t (f\circ\tau^{-1})'(u)dudt. \qquad (4.3.12)
\end{aligned}
$$

For any $f\in DBV[0,1]$, from (4.3.11) we may write

$$
\begin{aligned}
(f\circ\tau^{-1})'(u) &= (f\circ\tau^{-1})'_x(u)+\frac{1}{\theta+1}\Big((f\circ\tau^{-1})'(\tau(x+))+\theta(f\circ\tau^{-1})'(\tau(x-))\Big)\\
&+\frac{1}{2}\Big((f\circ\tau^{-1})'(\tau(x+))-(f\circ\tau^{-1})'(\tau(x-))\Big)\Big(sgn(u-\tau(x))+\frac{\theta-1}{\theta+1}\Big)\\
&+\delta_x(u)\Big[(f\circ\tau^{-1})'(u)-\frac{1}{2}\Big((f\circ\tau^{-1})'(\tau(x+))+(f\circ\tau^{-1})'(\tau(x-))\Big)\Big],
\end{aligned} \tag{4.3.13}
$$

where

$$
\delta_x(u)=\begin{cases}1\ , & u=\tau(x)\\ 0\ , & u\ne\tau(x)\end{cases}
$$

Obviously,

$$\int_0^1 \Bigg[\int_{\tau(x)}^t \Bigg\{(f\circ\tau^{-1})'(u) - \frac{1}{2}\Bigg((f\circ\tau^{-1})'(\tau(x+)) + (f\circ\tau^{-1})'(\tau(x-))\Bigg)\Bigg\} \delta_x(u)du\Bigg] K_n^{\tau,\theta}(x,t)dt = 0. \tag{4.3.14}$$

Let us define

$$\begin{aligned} A_1 &= \int_0^1 \Bigg(\int_{\tau(x)}^t \frac{1}{\theta+1}((f\circ\tau^{-1})'(\tau(x+)) + \theta(f\circ\tau^{-1})'(\tau(x-)))du\Bigg) K_n^{\tau,\theta}(x,t)dt \\ &= \frac{1}{\theta+1}\Bigg((f\circ\tau^{-1})'(\tau(x+)) + \theta(f\circ\tau^{-1})'(\tau(x-))\Bigg)\int_0^1 (t-\tau(x))K_n^{\tau,\theta}(x,t)dt \\ &= \frac{1}{\theta+1}\Bigg((f\circ\tau^{-1})'(\tau(x+)) + \theta(f\circ\tau^{-1})'(\tau(x-))\Bigg) D_n^{\tau,\theta}((\tau(t)-\tau(x));x), \end{aligned} \tag{4.3.15}$$

and

$$\begin{aligned} A_2 &= \int_0^1 K_n^{\tau,\theta}(x,t)\Bigg(\int_{\tau(x)}^t \frac{1}{2}\Bigg((f\circ\tau^{-1})'(\tau(x+)) - (f\circ\tau^{-1})'(\tau(x-))\Bigg). \\ &\Bigg(sgn(u-\tau(x)) + \frac{\theta-1}{\theta+1}\Bigg)du\Bigg)dt = \frac{1}{2}\Bigg((f\circ\tau^{-1})'(\tau(x+)) - (f\circ\tau^{-1})'(\tau(x-))\Bigg) \\ &\Bigg[-\int_0^{\tau(x)}\Bigg\{\int_t^{\tau(x)}\Bigg(sgn(u-\tau(x)) + \frac{\theta-1}{\theta+1}\Bigg)du\Bigg\}K_n^{\tau,\theta}(x,t)dt \\ &+ \int_{\tau(x)}^1 \Bigg(\int_{\tau(x)}^{t}\Bigg(sgn(u-\tau(x)) + \frac{\theta-1}{\theta+1}\Bigg)du\Bigg)K_n^{\tau,\theta}(x,t)dt\Bigg]. \end{aligned}$$

Then,

$$\begin{aligned} |A_2| &\le \Bigg|(f\circ\tau^{-1})'(\tau(x+)) - (f\circ\tau^{-1})'(\tau(x-))\Bigg| \int_0^1 |t-\tau(x)|\, K_n^{\tau,\theta}(x,t)dt \\ &= \Bigg|(f\circ\tau^{-1})'(\tau(x+)) - (f\circ\tau^{-1})'(\tau(x-))\Bigg| D_n^{\tau,\theta}\Bigg(|\tau(t)-\tau(x)|\,; x\Bigg). \end{aligned} \tag{4.3.16}$$

Combining equations (4.3.12 -4.3.16), on an application of Cauchy-Schwarz inequality and Remark 4.2.2, we obtain

$$\begin{aligned}
|D_n^{\tau,\theta}(f;x)-f(x)| \;\leq\;& \frac{1}{\theta+1}\left|(f\circ\tau^{-1})'(\tau(x+))+\theta(f\circ\tau^{-1})'(\tau(x-))\right|\sqrt{\frac{2\theta}{n+2}}\delta_{n,\tau}(x)\\
&+\left|(f\circ\tau^{-1})'(\tau(x+))-(f\circ\tau^{-1})'(\tau(x-))\right|\sqrt{\frac{2\theta}{n+2}}\delta_{n,\tau}(x)\\
&+\left|\int_0^{\tau(x)}\left(\int_{\tau(x)}^t (f\circ\tau^{-1})'_x(u)du\right)K_n^{\tau,\theta}(x,t)dt\right|\\
&+\left|\int_{\tau(x)}^1\left(\int_{\tau(x)}^t (f\circ\tau^{-1})'_x(u)du\right)K_n^{\tau,\theta}(x,t)dt\right|. \qquad (4.3.17)
\end{aligned}$$

Now, let

$$A_n^{\tau,\theta}((f\circ\tau^{-1})'_x,x)=\int_0^{\tau(x)}\left(\int_{\tau(x)}^t (f\circ\tau^{-1})'_x(u)du\right)K_n^{\tau,\theta}(x,t)dt,$$

and

$$B_n^{\tau,\theta}((f\circ\tau^{-1})'_x,x)=\int_{\tau(x)}^1\left(\int_{\tau(x)}^t (f\circ\tau^{-1})'_x(u)du\right)K_n^{\tau,\theta}(x,t)dt.$$

In order to calculate the estimates of the terms $A_n^{\tau,\theta}((f\circ\tau^{-1})'_x,x)$, using the definition of $\xi_n^{\tau,\theta}$ given in Lemma 4.3.4, we can write

$$A_n^{\tau,\theta}((f\circ\tau^{-1})'_x,x)=\int_0^{\tau(x)}\left(\int_{\tau(x)}^t (f\circ\tau^{-1})'_x(u)du\right)\frac{\partial}{\partial t}\xi_n^{\tau,\theta}(x,t)dt.$$

Applying integration by parts, we get

$$\begin{aligned}
&\left|A_n^{\tau,\theta}((f\circ\tau^{-1})'_x,x)\right| \leq \int_0^{\tau(x)} |(f\circ\tau^{-1})'_x(t)|\xi_n^{\tau,\theta}(x,t)dt\\
&\leq \int_0^{\tau(x)-\frac{\tau(x)}{\sqrt{n}}} |(f\circ\tau^{-1})'_x(t)|\xi_n^{\tau,\theta}(x,t)dt+\int_{\tau(x)-\frac{\tau(x)}{\sqrt{n}}}^{\tau(x)} |(f\circ\tau^{-1})'_x(t)|\xi_n^{\tau,\theta}(x,t)dt\\
&:= I_1+I_2.
\end{aligned}$$

Since $(f\circ\tau^{-1})'_x(\tau(x))=0$ and $\xi_n^{\tau,\theta}(x,t)\le 1$, we have

$$I_2 := \int_{\tau(x)-\frac{\tau(x)}{\sqrt{n}}}^{\tau(x)} \left|(f\circ\tau^{-1})'_x(t)-(f\circ\tau^{-1})'_x(\tau(x))\right| \xi_n^{\tau,\theta}(x,t)dt$$

$$\le \int_{\tau(x)-\frac{\tau(x)}{\sqrt{n}}}^{\tau(x)} \left(\bigvee_t^{\tau(x)} (f\circ\tau^{-1})'_x\right) dt \le \left(\bigvee_{\tau(x)-\frac{\tau(x)}{\sqrt{n}}}^{\tau(x)} (f\circ\tau^{-1})'_x\right) \int_{\tau(x)-\frac{\tau(x)}{\sqrt{n}}}^{\tau(x)} dt$$

$$= \frac{\tau(x)}{\sqrt{n}} \left(\bigvee_{\tau(x)-\frac{\tau(x)}{\sqrt{n}}}^{\tau(x)} f'_x\right).$$

Using Lemma 4.3.4 and considering $t=\tau(x)-\dfrac{\tau(x)}{u}$, we get

$$\begin{aligned}
I_1 &\le \frac{2\theta}{n+2}\delta^2_{n,\tau}(x)\int_0^{\tau(x)-\frac{\tau(x)}{\sqrt{n}}} \left|(f\circ\tau^{-1})'_x(t)-(f\circ\tau^{-1})'_{\tau(x)}(x)\right| \frac{dt}{(\tau(x)-t)^2}\\
&\le \frac{2\theta}{n+2}\delta^2_{n,\tau}(x)\int_0^{\tau(x)-\frac{\tau(x)}{\sqrt{n}}} \left(\bigvee_t^{\tau(x)} (f\circ\tau^{-1})'_x\right)\frac{dt}{(\tau(x)-t)^2}\\
&= \frac{2\theta}{n+2}\frac{\delta^2_{n,\tau}(x)}{\tau(x)}\int_1^{\sqrt{n}} \left(\bigvee_{\tau(x)-\frac{\tau(x)}{u}}^{\tau(x)} (f\circ\tau^{-1})'_x\right) du\\
&\le \frac{2\theta}{n+2}\frac{\delta^2_{n,\tau}(x)}{\tau(x)}\sum_{k=1}^{[\sqrt{n}]} \left(\bigvee_{\tau(x)-\frac{\tau(x)}{k}}^{\tau(x)} (f\circ\tau^{-1})'_x\right).
\end{aligned}$$

Therefore,

$$\begin{aligned}
|A_n^{\tau,\theta}((f\circ\tau^{-1})'_x,x)| &\le \frac{2\theta}{n+2}\frac{\delta^2_{n,\tau}(x)}{\tau(x)}\sum_{k=1}^{[\sqrt{n}]} \left(\bigvee_{\tau(x)-\frac{\tau(x)}{k}}^{\tau(x)} (f\circ\tau^{-1})'_x\right)\\
&\quad + \frac{\tau(x)}{\sqrt{n}}\left(\bigvee_{\tau(x)-\frac{\tau(x)}{\sqrt{n}}}^{\tau(x)} (f\circ\tau^{-1})'_x\right). \qquad (4.3.18)
\end{aligned}$$

Also, using integration by parts in $B_n^{\tau,\theta}(f'_x,x)$ and applying Lemma 4.3.4

with $z = \tau(x) + (1-\tau(x))/\sqrt{n}$, we have

$$|B_n^{\tau,\theta}((f\circ\tau^{-1})'_x, x)| = \left| \int_{\tau(x)}^{1} \left(\int_{\tau(x)}^{t} (f\circ\tau^{-1})'_x(u)du \right) K_n^{\tau,\theta}(x,t)dt \right|$$

$$= \left| \int_{\tau(x)}^{z} \left(\int_{\tau(x)}^{t} (f\circ\tau^{-1})'_x(u)du \right) \frac{\partial}{\partial t}(1-\xi_n^{\tau,\theta}(x,t))dt \right.$$

$$\left. + \int_{z}^{1} \left(\int_{\tau(x)}^{t} (f\circ\tau^{-1})'_x(u)du \right) \frac{\partial}{\partial t}(1-\xi_n^{\tau,\theta}(x,t))dt \right|$$

$$= \left| \left[\int_{\tau(x)}^{t} ((f\circ\tau^{-1})'_x(u)du)(1-\xi_n^{\tau,\theta}(x,t)) \right]_{\tau(x)}^{z} - \int_{\tau(x)}^{z} (f\circ\tau^{-1})'_x(t)(1-\xi_n^{\tau,\theta}(x,t))dt \right.$$

$$\left. + \left[\int_{\tau(x)}^{t} ((f\circ\tau^{-1})'_x(u)du)(1-\xi_n^{\tau,\theta}(x,t)) \right]_{z}^{1} - \int_{z}^{1} (f\circ\tau^{-1})'_x(t)(1-\xi_n^{\tau,\theta}(x,t))dt \right|$$

$$= \left| \int_{\tau(x)}^{z} (f\circ\tau^{-1})'_x(t)(1-\xi_n^{\tau,\theta}(x,t))dt + \int_{z}^{1} (f\circ\tau^{-1})'_x(t)(1-\xi_n^{\tau,\theta}(x,t))dt \right|$$

$$\leq \frac{2\theta}{n+2}\delta_{n,\tau}^2(x) \int_{z}^{1} \left(\bigvee_{\tau(x)}^{t} (f\circ\tau^{-1})'_x \right) (t-\tau(x))^{-2}dt + \int_{\tau(x)}^{z} \bigvee_{\tau(x)}^{t} (f\circ\tau^{-1})'_x dt$$

$$\leq \frac{2\theta}{n+2}\delta_{n,\tau}^2(x) \int_{\tau(x)+\frac{(1-\tau(x))}{\sqrt{n}}}^{1} \left(\bigvee_{\tau(x)}^{t} (f\circ\tau^{-1})'_x \right) (t-\tau(x))^{-2}dt$$

$$+ \frac{1-\tau(x)}{\sqrt{n}} \left(\bigvee_{\tau(x)}^{\tau(x)+\frac{(1-\tau(x))}{\sqrt{n}}} (f\circ\tau^{-1})'_x \right).$$

By substituting $u = (1-\tau(x))/(t-\tau(x))$, we get

$$\begin{aligned}
|B_n^{\tau,\theta} f'_x, x)| \quad & \leq \frac{2\theta}{n+2}\delta_{n,\tau}^2(x) \int_{1}^{\sqrt{n}} \left(\bigvee_{\tau(x)}^{\tau(x)+\frac{(1-\tau(x))}{u}} (f\circ\tau^{-1})'_x \right) (1-\tau(x))^{-1}du \\
& + \frac{1-\tau(x)}{\sqrt{n}} \left(\bigvee_{\tau(x)}^{\tau(x)+\frac{(1-\tau(x))}{\sqrt{n}}} (f\circ\tau^{-1})'_x \right) \\
& \leq \frac{2\theta}{n+2} \frac{\delta_{n,\tau}^2(x)}{1-\tau(x)} \sum_{k=1}^{[\sqrt{n}]} \left(\bigvee_{\tau(x)}^{\tau(x)+\frac{(1-\tau(x))}{k}} (f\circ\tau^{-1})'_x \right) \\
& + \frac{1-\tau(x)}{\sqrt{n}} \left(\bigvee_{\tau(x)}^{\tau(x)+\frac{(1-\tau(x))}{k}} (f\circ\tau^{-1})'_x \right). \qquad (4.3.19)
\end{aligned}$$

Collecting the estimates (4.3.17 - 4.3.19), we get the required result. □

Chapter 5

Approximation properties of the modified Stancu operators

5.1 Introduction

In the recent years there has been an increasing interest in modifying linear positive operators so that the new versions present a better degree of approximation than the original ones.

In the present chapter, we deal with the modified Stancu operator. We compare the new operators with classical Stancu operators (1.1.2) and observe that on a certain interval, these operators present a better degree of approximation than the original ones. Also, a Voronovskaja type theorem by using the Ditzian-Totik modulus of smoothness is proved.

5.2 Modified Stancu operators

In this section, we introduce a modification of the Stancu operators (1.1.2). The main properties of this new approximation process are studied.

For an infinite times continuously differentiable function τ on $[0, 1]$, such that $\tau(0) = 0$, $\tau(1) = 1$ and $\tau'(x) > 0$ for $x \in [0, 1]$, we introduce the sequence of Stancu type operators for $f \in C[0,1]$ as

$$P_n^{<\frac{1}{n},\tau>}(f;x) = \sum_{k=0}^{n} p_{n,k}^{<\frac{1}{n},\tau>}(x)(f \circ \tau^{-1})\left(\frac{k}{n}\right), \; x \in [0,1], \tag{5.2.1}$$

where

$$p_{n,k}^{<\frac{1}{n},\tau>}(x) = \frac{2n!}{(2n)!}\binom{n}{k}\left(n\tau(x)\right)_k\left(n - n\tau(x)\right)_{n-k}.$$

In the following example, we show that if the function $\tau(x)$ is chosen suitably then the operators defined by (5.2.1) provide a better rate of convergence than the operators (1.1.2).

Example 5.2.1. *Let* $\tau_1(x) = \dfrac{x^2+x}{2}$, $\tau_2(x) = \sin\dfrac{\pi}{2}x$ *and* $f(x) = \cos(10x)$, $x \in [0,1]$. *For* $n = 40$, *the approximation to the function* f *by the modified Stancu operators* $P_n^{<\frac{1}{n},\tau_1>}$ *defined by (5.2.1)and the Stancu operators* $P_n^{(\frac{1}{n})}$ *defined by (1.1.2) is illustrated in the Figure 5.1.*

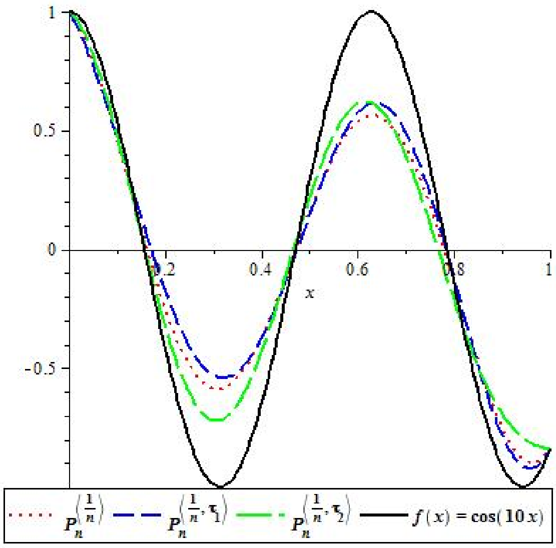

Figure 5.1

The error of approximation for $P_n^{(\frac{1}{n})}$, $P_n^{<\frac{1}{n},\tau_1>}$ and $P_n^{<\frac{1}{n},\tau_2>}$ at certain points of $[0,1]$ is computed in the following Table.

Table 2.

x	$\lvert P_n^{(\frac{1}{n})}(f;x)-f(x)\rvert$	$\lvert P_n^{<\frac{1}{n},\tau_1>}(f;x)-f(x)\rvert$	$\lvert P_n^{<\frac{1}{n},\tau_2>}(f;x)-f(x)\rvert$
0.04	0.0719078440	0.0918804542	0.0491775787
0.08	0.0899902045	0.0949187558	0.0656722344
0.12	0.0472944810	0.0215916952	0.0431323566
0.16	0.0464266575	0.1024165160	0.0144204290
0.20	0.1684314306	0.2437174913	0.0943043728
0.24	0.2889352546	0.3677734502	0.1783441641
0.28	0.3780208231	0.4453240720	0.2468493195
0.32	0.4121574493	0.4575642665	0.2826616675
0.36	0.3791213446	0.3992005780	0.2745554027
0.40	0.2804560850	0.2789088716	0.2194175892
0.44	0.1310964673	0.1171877712	0.1228721695
0.48	0.0436865547	0.0579307255	0.0017044307
0.52	0.2133696908	0.2164713841	0.1355396341
0.56	0.3482733387	0.3323367044	0.2577115929
0.60	0.4254746884	0.3882544972	0.3486510319
0.64	0.4333817144	0.3789143428	0.3932519814
0.68	0.3740607726	0.3116949702	0.3831315495
0.72	0.2628264094	0.2048387962	0.3179584077
0.76	0.1251131158	0.0834187551	0.2061763439
0.80	0.0088333903	0.0261223418	0.0655133299
0.84	0.1104739425	0.1017791431	0.0769549422
0.88	0.1594595824	0.1309512530	0.1865335751
0.92	0.1487803128	0.1135004422	0.2263598837
0.96	0.0874521271	0.0621541233	0.1673068131

Therefore, we notice that depending on choice of function τ, the modified operator $P_n^{<\frac{1}{n},\tau>}$ presents a better degree of approximation than $P_n^{(\frac{1}{n})}$ on a certain interval.

In order to prove our main results, we shall need some auxiliary results. The proofs are similar to the corresponding results for the Stancu operators, therefore the details are omitted.

Lemma 5.2.2. *For modified Stancu operator* $P_n^{<\frac{1}{n},\tau>}$, *we have*

$$P_n^{<\frac{1}{n},\tau>}1 = 1,\ P_n^{<\frac{1}{n},\tau>}\tau = \tau,\ P_n^{<\frac{1}{n},\tau>}\tau^2 = \tau^2 + \frac{2\tau(1-\tau)}{n+1}.$$

Let $\mu_{n,m}^{\tau}(x) = P_n^{<\frac{1}{n},\tau>}\left((\tau(t)-\tau(x))^m; x\right) = \sum_{k=0}^{n} p_{n,k}^{<\frac{1}{n},\tau>}(x)\left(\frac{k}{n} - \tau(x)\right)^m$ be the central moment operator.

Lemma 5.2.3. *The central moment operator verifies:*

i) $\mu_{n,2}^{\tau}(x) = \frac{2}{n+1}\varphi_\tau^2(x)$;

ii) $\mu_{n,4}^{\tau}(x) = \frac{12(n^2-7n)\varphi_\tau^2(x) + (26n-2)}{n(n+1)(n+2)(n+3)}\varphi_\tau^2(x)$,

where $\varphi_\tau^2(x) := \tau(x)(1-\tau(x))$.

Lemma 5.2.4. *If* $f \in C[0,1]$, *then* $\|P_n^{<\frac{1}{n},\tau>}f\| \le \|f\|$, *where* $\|\cdot\|$ *is the uniform norm on* $C[0,1]$.

Proof. By the definition of the modified Stancu operators (5.2.1) and using Lemma 5.2.2 we have

$$\left|P_n^{<\frac{1}{n},\tau>}(f;x)\right| \le \sum_{k=0}^{n} p_{n,k}^{<\frac{1}{n},\tau>}(x)\left|\left(f\circ\tau^{-1}\right)\left(\frac{k}{n}\right)\right| \le \|f\circ\tau^{-1}\| P_n^{<\frac{1}{n},\tau>}(e_0;x) = \|f\|.$$

□

Theorem 5.2.5. *If* $f \in C[0,1]$, *then* $P_n^{<\frac{1}{n},\tau>}f$ *converges to* f *as* n *tends to infinity, uniformly on* $[0,1]$.

Proof. Using Lemma 5.2.2, the Korovkin theorem and the fact that $\{1,\tau,\tau^2\}$ is an extended complete Tchebychev system on $[0,1]$, we obtain that the modified Stancu operator $P_n^{<\frac{1}{n},\tau>}f$ converges uniformly to $f \in C[0,1]$. □

Example: We consider $f : [0,1] \to \mathbb{R}$, $f(x) = \cos(10x)$ and $\tau(x) = \dfrac{x^2+x}{2}$. The convergence of the modified Stancu operator $P_n^{<\frac{1}{n},\tau>}$ to the function f is illustrated in Figure 5.2 for $n \in \{20, 50, 100\}$.

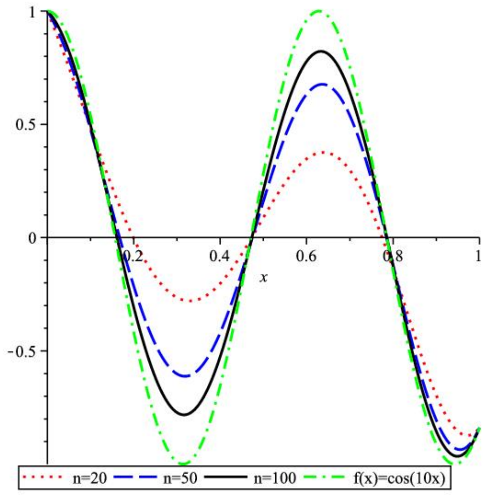

Figure 5.2

We remark that as the values of n increase, the error in the approximation of the operator to the function becomes smaller.

Using the result of Shisha and Mond [141] we have

$$\left| P_n^{<\frac{1}{n},\tau>}(f;x) - f(x) \right| \leq \left(1 + \frac{\mu_{n,2}^{\tau}(x)}{\delta^2} \right) \omega(f,\delta), \text{ for } \delta > 0,$$

where $\omega(f,\delta)$ is the usual modulus of continuity of $f \in C[0,1]$.

Example 5.2.6. *The rates of convergence of the modified operators and the original ones depend on the selection of the function τ. If we choose $\tau(x) = \left(\sin \frac{\pi x}{2}\right)^2$, we have $\tau(x)\,(1-\tau(x)) \leq x(1-x)$, for all $x \in [0,1]$ and this inequality leads to $\mu_{n,2}^{\tau}(x) \leq \mu_{n,2}(x)$. Therefore, the modified operators $P_n^{<\frac{1}{n},\tau>}$ presents a better order of approximation than $P_n^{(\frac{1}{n})}$.*

5.3 Approximation properties

In what follows, we present approximation properties for modified Stancu operators in terms of modulus of smoothness. In order to give our main results we will use the following result.

Lemma 5.3.1. *[70] If $f \in C^q[0,1]$, then for all $0 < h \leq \dfrac{1}{2}$ there are functions $g \in C^{q+2}[0,1]$, such that*

i) $\|f^{(q)} - g^{(q)}\| \leq \dfrac{3}{4}\omega_2(f^{(q)}; h)$,

ii) $\|g^{(q+1)}\| \leq \dfrac{5}{h}\omega_1(f^{(q)}; h)$,

iii) $\|g^{(q+2)}\| \leq \dfrac{3}{2h^2}\omega_2(f^{(q)}; h)$,

where ω_k is the classical k^{th} order modulus of smoothness on $[0,1]$.

Throughout this chapter, we assume that $\inf\limits_{x\in[0,1]} \tau'(x) \geq a, a \in \mathbb{R}^+$.

Theorem 5.3.2. *If $f \in C[0,1]$, then the operators $P_n^{<\frac{1}{n},\tau>}$ verify the following inequality*

$$\left|P_n^{<\frac{1}{n},\tau>}(f;x) - f(x)\right| \leq \frac{3}{2}\left(1+\frac{1}{a^2}\right)\omega_2\left(f; \frac{\varphi_\tau(x)}{\sqrt{n+1}}\right) + \frac{5\varphi_\tau(x)\|\tau''\|}{a^3\sqrt{n+1}}\omega_1\left(f; \frac{\varphi_\tau(x)}{\sqrt{n+1}}\right),$$

where $\omega_1(f;\delta)$ and $\omega_2(f;\delta)$ are the first and second order modulus of continuity respectively.

Proof. Let $g \in W^2[0,1]$ and $t \in [0,1]$. Then by Taylor's expansion, we get

$$\begin{aligned} g(t) &= \left(g\circ\tau^{-1}\right)(\tau(t)) \\ &= \left(g\circ\tau^{-1}\right)(\tau(x)) + \left(g\circ\tau^{-1}\right)'(\tau(x))\left(\tau(t)-\tau(x)\right) \\ &\quad + \int_{\tau(x)}^{\tau(t)} (\tau(t)-u)\left(g\circ\tau^{-1}\right)''(u)du. \end{aligned} \tag{5.3.1}$$

The quantity $\int_{\tau(x)}^{\tau(t)} (\tau(t)-u)\left(g\circ\tau^{-1}\right)''(u)du$ was estimated in [7, p. 35] as follows:

$$\begin{aligned} &\int_{\tau(x)}^{\tau(t)} (\tau(t)-u)\left(g\circ\tau^{-1}\right)''(u)du \\ &= \int_{\tau(x)}^{\tau(t)} (\tau(t)-u)\frac{g''(\tau^{-1}(u))}{\left[\tau'(\tau^{-1}(u))\right]^2}du - \int_{\tau(x)}^{\tau(t)} (\tau(t)-u)\frac{g'(\tau^{-1}(u))\tau''(\tau^{-1}(u))}{\left[\tau'(\tau^{-1}(u))\right]^3}du. \end{aligned} \tag{5.3.2}$$

From (5.3.1) and (5.3.2) we can write

$$g(t) = g(x) + \left(g \circ \tau^{-1}\right)' (\tau(x)) (\tau(t) - \tau(x)) + \int_{\tau(x)}^{\tau(t)} (\tau(t) - u) \frac{g''(\tau^{-1}(u))}{[\tau'(\tau^{-1}(u))]^2} du$$
$$- \int_{\tau(x)}^{\tau(t)} (\tau(t) - u) \frac{g'(\tau^{-1}(u))\tau''(\tau^{-1}(u))}{[\tau'(\tau^{-1}(u))]^3} du. \tag{5.3.3}$$

Now applying $P_n^{<\frac{1}{n},\tau>}$ to both sides of the relation (5.3.3), we can write

$$P_n^{<\frac{1}{n},\tau>}(g;x) = g(x) + P_n^{<\frac{1}{n},\tau>}\left(\int_{\tau(x)}^{\tau(t)} (\tau(t) - u) \frac{g''(\tau^{-1}(u))}{[\tau'(\tau^{-1}(u))]^2} du; x\right)$$
$$- P_n^{<\frac{1}{n},\tau>}\left(\int_{\tau(x)}^{\tau(t)} (\tau(t) - u) \frac{g'(\tau^{-1}(u))\tau''(\tau^{-1}(u))}{[\tau'(\tau^{-1}(u))]^3} du; x\right).$$

Therefore,

$$\left|P_n^{<\frac{1}{n},\tau>}(g;x) - g(x)\right| \le \frac{1}{2}\mu_{n,2}^{\tau}(x)\left(\frac{\|g''\|}{a^2} + \frac{\|g'\| \cdot \|\tau''\|}{a^3}\right) = \frac{\varphi_\tau^2(x)}{n+1}\left(\frac{\|g''\|}{a^2} + \frac{\|g'\| \cdot \|\tau''\|}{a^3}\right).$$

By Lemma 5.2.4, it follows

$$\left|P_n^{<\frac{1}{n},\tau>}(f;x) - f(x)\right| = \left|P_n^{<\frac{1}{n},\tau>}(f-g;x)\right| + \left|P_n^{<\frac{1}{n},\tau>}(g;x) - g(x)\right| + |g(x) - f(x)|$$
$$\le 2\|f-g\| + \frac{\varphi_\tau^2(x)}{n+1}\left(\frac{\|g''\|}{a^2} + \frac{\|g'\| \cdot \|\tau''\|}{a^3}\right).$$

According to Lemma 5.3.1, for the given $0 < h \le \frac{1}{2}$, there exists $g \in C^2[0,1]$ such that

$$\|f-g\| \le \frac{3}{4}\omega_2(f;h), \quad \|g'\| \le \frac{5}{h}\omega_1(f;h), \quad \|g''\| \le \frac{3}{2h^2}\omega_2(f';h).$$

Consequently,

$$\left|P_n^{<\frac{1}{n},\tau>}(f;x) - f(x)\right| \le \frac{3}{2}\omega_2(f;h) + \frac{\varphi_\tau^2(x)}{n+1}\left(\frac{3}{2h^2a^2}\omega_2(f,h) + \frac{5\|\tau''\|}{ha^3}\omega_1(f;h\right).$$

Taking $h = \dfrac{\varphi_\tau(x)}{\sqrt{n+1}}$, the theorem is proved. □

Remark 5.3.3. *If we chose $\tau(x) = x$ in Theorem 5.3.2, we get*

$$\left|P_n^{<\frac{1}{n}>}(f;x) - f(x)\right| \le 3\omega_2\left(f; \frac{\varphi(x)}{\sqrt{n+1}}\right).$$

Regarding the estimates of $P_n^{<\frac{1}{n}>}$, better constant in front of ω_2 were obtained in ([71], [100], [131]).

5.4 Voronovskaja type theorems

The Voronovskaja type theorem for Stancu operators dealing with error estimates for continuous functions and twice continuously differentiable functions, were obtained in [115] as follows:

Theorem 5.4.1. *For $x \in [0,1]$, the following inequalities hold*

i) $\left|P_n^{(\frac{1}{n})}(f;x) - f(x)\right| \leq \frac{3}{2}\omega_1\left(f;\frac{1}{\sqrt{n}}\right),\ f \in C[0,1],$

ii) $n\left|P_n^{(\frac{1}{n})}(f;x) - f(x) - \frac{x(1-x)}{n+1}f''(x)\right| \leq \frac{5}{8}\omega_1\left(f'';\frac{1}{\sqrt{n}}\right)\ f \in C^2[0,1].$

Now, we establish a local approximation theorem for the operators $P_n^{<\frac{1}{n},\tau>}$ by means of Ditzian-Totik modulus of smoothness.

Theorem 5.4.2. *Let $f \in C[0,1]$. Then for every $x \in (0,1)$, we have*

$$\left|P_n^{<\frac{1}{n},\tau>}(f;x) - f(x)\right| \leq C\omega_{\varphi_\tau}\left(f;\frac{2}{a\sqrt{n+1}}\right),$$

where $C > 0$ is a constant.

Proof. Using the representation

$$g(t) = \left(g \circ \tau^{-1}\right)(\tau(t)) = \left(g \circ \tau^{-1}\right)(\tau(x)) + \int_{\tau(x)}^{\tau(t)} \left(g \circ \tau^{-1}\right)'(u)du$$

we get

$$\left|P_n^{<\frac{1}{n},\tau>}(g;x) - g(x)\right| = \left|P_n^{<\frac{1}{n},\tau>}\left(\int_{\tau(x)}^{\tau(t)} \left(g \circ \tau^{-1}\right)'(u)du\right)\right|. \tag{5.4.1}$$

But,

$$\left|\int_{\tau(x)}^{\tau(t)} \left(g \circ \tau^{-1}\right)'(u)du\right| = \left|\int_x^t \frac{g'(y)}{\tau'(y)}\tau'(y)dy\right| = \left|\int_x^t \frac{\varphi_\tau(y)}{\varphi_\tau(y)} \cdot \frac{g'(y)}{\tau'(y)}\tau'(y)dy\right|$$
$$\leq \frac{\|\varphi_\tau g'\|}{a}\left|\int_x^t \frac{\tau'(y)}{\varphi_\tau(y)}dy\right|, \tag{5.4.2}$$

and

$$\begin{aligned}
\left|\int_x^t \frac{\tau'(y)}{\varphi_\tau(y)}dy\right| &\le \left|\int_x^t \left(\frac{1}{\sqrt{\tau(y)}}+\frac{1}{\sqrt{1-\tau(y)}}\right)\tau'(y)dy\right| \\
&\le 2\left(\left|\sqrt{\tau(t)}-\sqrt{\tau(x)}\right|+\left|\sqrt{1-\tau(t)}-\sqrt{1-\tau(x)}\right|\right) \\
&= 2\left|\tau(t)-\tau(x)\right|\left(\frac{1}{\sqrt{\tau(t)}+\sqrt{\tau(x)}}+\frac{1}{\sqrt{1-\tau(t)}+\sqrt{1-\tau(x)}}\right) \\
&< 2|\tau(t)-\tau(x)|\left(\frac{1}{\sqrt{\tau(x)}}+\frac{1}{\sqrt{1-\tau(x)}}\right) \le \frac{2\sqrt{2}|\tau(t)-\tau(x)|}{\varphi_\tau(x)}.
\end{aligned} \tag{5.4.3}$$

Hence from relations (5.4.1)-(5.4.3) and using Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}
|P_n^{<\frac{1}{n},\tau>}(g;x)-g(x)| &\le 2\sqrt{2}\frac{\|\varphi_\tau g'\|}{a\varphi_\tau(x)}P_n^{<\frac{1}{n},\tau>}\left(|\tau(t)-\tau(x)|;x\right) \\
&\le 2\sqrt{2}\frac{\|\varphi_\tau g'\|}{a\varphi_\tau(x)}\left[P_n^{<\frac{1}{n},\tau>}\left((\tau(t)-\tau(x))^2;x\right)\right]^{1/2} \\
&\le \frac{4}{a\sqrt{n+1}}\|\varphi_\tau g'\|.
\end{aligned} \tag{5.4.4}$$

Using Lemma 5.2.4 and (5.4.4), it follows that

$$\begin{aligned}
\left|P_n^{<\frac{1}{n},\tau>}(f;x)-f(x)\right| &\le \left|P_n^{<\frac{1}{n},\tau>}(f-g;x)\right|+|f(x)-g(x)|+\left|P_n^{<\frac{1}{n},\tau>}(g;x)-g(x)\right| \\
&\le 2\left\{\|f-g\|+\frac{2}{a\sqrt{n+1}}\|\varphi_\tau g'\|\right\}.
\end{aligned}$$

Taking infimum on the right hand side of the above inequality over all $g \in W_{\varphi_\tau}[0,1]$, we get

$$\left|P_n^{<\frac{1}{n},\tau>}(f;x)-f(x)\right| \le 2K_{\varphi_\tau}\left(f;\frac{2}{a\sqrt{n+1}}\right).$$

Using the relation (0.5.3) this theorem is proved. □

In this section we prove a quantitative Voronovskaja type theorem for the operator $P_n^{<\frac{1}{n},\tau>}$ in terms of the first order Ditzian-Totik modulus of smoothness.

Theorem 5.4.3. *For any $f \in C^2[0,1]$ and $x \in (0,1)$, the following inequalities: hold*

i) $n\left|P_n^{<\frac{1}{n},\tau>}(f;x)-f(x)-\frac{1}{n+1}\cdot\frac{1}{[\tau'(x)]^2}\left[f''(x)-f'(x)\frac{\tau''(x)}{\tau'(x)}\right]\varphi_\tau^2(x)\right|$
$\leq C\omega_{\varphi_\tau}\left((f\circ\tau^{-1})'';\frac{\varphi_\tau(x)}{a}u_n^\tau(x)\right),$

ii) $n\left|P_n^{<\frac{1}{n},\tau>}(f;x)-f(x)-\frac{1}{n+1}\cdot\frac{1}{[\tau'(x)]^2}\left[f''(x)-f'(x)\frac{\tau''(x)}{\tau'(x)}\right]\varphi_\tau^2(x)\right|$
$\leq C\varphi_\tau(x)\omega_{\varphi_\tau}\left((f\circ\tau^{-1})'';\frac{u_n^\tau(x)}{a}\right),$

where $C>0$ *is a constant and* $u_n^\tau(x)=2\sqrt{\frac{2(n^2-7n)\varphi_\tau^2(x)+13n-1}{n(n+2)(n+3)}}$.

Proof. Let $f\in C^2[0,1]$ and $x\in(0,1)$. Then by Taylor's expansion for $t\in(0,1)$, we have

$$\begin{aligned}f(t)=\left(f\circ\tau^{-1}\right)(\tau(t)) &= \left(f\circ\tau^{-1}\right)(\tau(x))+\left(f\circ\tau^{-1}\right)'(\tau(x))\,(\tau(t)-\tau(x))\\ &\quad+\int_{\tau(x)}^{\tau(t)}(\tau(t)-u)\left(f\circ\tau^{-1}\right)''(u)du.\end{aligned}$$

Hence, we may write

$$\begin{aligned}&f(t)-f(x)-\left(f\circ\tau^{-1}\right)'(\tau(x))\,(\tau(t)-\tau(x))-\frac{1}{2}\left(f\circ\tau^{-1}\right)''(\tau(x))\,(\tau(t)-\tau(x))^2\\ &=\int_{\tau(x)}^{\tau(t)}(\tau(t)-u)\left(f\circ\tau^{-1}\right)''(u)du-\int_{\tau(x)}^{\tau(t)}(\tau(t)-u)\left(f\circ\tau^{-1}\right)''(\tau(x))du\\ &=\int_{\tau(x)}^{\tau(t)}(\tau(t)-u)\left[\left(f\circ\tau^{-1}\right)''(u)-\left(f\circ\tau^{-1}\right)''(\tau(x))\right]du.\end{aligned}$$

Applying $P_n^{<\frac{1}{n},\tau>}$ to both sides of the above relation, we get

$$\begin{aligned}&\left|P_n^{<\frac{1}{n},\tau>}(f;x)-f(x)-\frac{1}{2}\left[\frac{f''(x)}{(\tau'(x))^2}-f'(x)\frac{\tau''(x)}{(\tau'(x))^3}\right]\mu_{n,2}^\tau(x)\right|\\ &=\left|P_n^{<\frac{1}{n},\tau>}\left(\int_{\tau(x)}^{\tau(t)}(\tau(t)-u)\left[\left(f\circ\tau^{-1}\right)''(u)-\left(f\circ\tau^{-1}\right)''(\tau(x))\right]du;x\right)\right|\\ &\leq P_n^{<\frac{1}{n},\tau>}\left(\left|\int_{\tau(x)}^{\tau(t)}|\tau(t)-u|\left|\left(f\circ\tau^{-1}\right)''(u)-\left(f\circ\tau^{-1}\right)''(\tau(x))\right|du\right|;x\right).\end{aligned} \tag{5.4.5}$$

For $g \in W_{\varphi_\tau}[0,1]$, we have

$$\begin{aligned}
&\left|\int_{\tau(x)}^{\tau(t)} |\tau(t)-u| \left|\left(f\circ\tau^{-1}\right)''(u) - \left(f\circ\tau^{-1}\right)''(\tau(x))\right| du\right| \\
&\le \left|\int_{\tau(x)}^{\tau(t)} |\tau(t)-u| \left|\left(f\circ\tau^{-1}\right)''(u) - \left(g\circ\tau^{-1}\right)(u)\right| du\right| \\
&+\left|\int_{\tau(x)}^{\tau(t)} |\tau(t)-u| \left|\left(g\circ\tau^{-1}\right)(u) - \left(g\circ\tau^{-1}\right)(\tau(x))\right| du\right| \\
&+\left|\int_{\tau(x)}^{\tau(t)} |\tau(t)-u| \left|\left(g\circ\tau^{-1}\right)(\tau(x)) - \left(f\circ\tau^{-1}\right)''(\tau(x))\right| du\right| \\
&= \left|\int_x^t \left|\left(f\circ\tau^{-1}\right)''(\tau(y)) - g(y)\right| |\tau(t)-\tau(y)|\,\tau'(y)dy\right| \\
&+\left|\int_x^t |g(y)-g(x)|\,|\tau(t)-\tau(y)|\,\tau'(y)dy\right| \\
&+\left|\int_x^t \left|g(x) - \left(f\circ\tau^{-1}\right)''(\tau(x))\right| |\tau(t)-\tau(y)|\,\tau'(y)dy\right| \\
&\le 2\|\left(f\circ\tau^{-1}\right)'' - g\| \left|\int_x^t |\tau(t)-\tau(y)|\tau'(y)dy\right| \\
&+\left|\int_x^t \left|\int_x^y |g'(v)|dv\right| |\tau(t)-\tau(y)|\tau'(y)dy\right| \\
&\le \|\left(f\circ\tau^{-1}\right)'' - g\|(\tau(t)-\tau(x))^2 \\
&+\|\varphi_\tau g'\| \left|\int_x^t \left|\int_x^y \frac{dv}{\varphi_\tau(v)}\right| |\tau(t)-\tau(y)|\tau'(y)dy\right|. \qquad (5.4.6)
\end{aligned}$$

Using the inequality [48, p. 141] $\dfrac{|y-v|}{v(1-v)} \le \dfrac{|y-x|}{x(1-x)}$, v is between y and x , we can write $\dfrac{|\tau(y)-\tau(v)|}{\tau(v)(1-\tau(v))} \le \dfrac{|\tau(y)-\tau(x)|}{\tau(x)(1-\tau(x))}$. Therefore,

$$\begin{aligned}
&\|\varphi_\tau g'\| \left|\int_x^t \left|\int_x^y \frac{dv}{\varphi_\tau(v)}\right| |\tau(t)-\tau(y)|\tau'(y)dy\right| \qquad (5.4.7) \\
&\le \|\varphi_\tau g'\| \left|\int_x^t \left|\int_x^y \frac{|\tau(y)-\tau(x)|^{1/2}}{\tau'(v)\varphi_\tau(x)} \cdot \frac{\tau'(v)}{|\tau(y)-\tau(v)|^{1/2}} dv\right| |\tau(t)-\tau(y)|\tau'(y)dy\right| \\
&\le 2\frac{\|\varphi_\tau g'\|}{a}\varphi_\tau^{-1}(x) \left|\int_x^t |\tau(y)-\tau(x)|\,|\tau(t)-\tau(y)|\tau'(y)dy\right| \\
&\le 2\frac{\|\varphi_\tau g'\|}{a}\varphi_\tau^{-1}(x) \left|\int_x^t (\tau(t)-\tau(x))^2\tau'(y)dy\right| \le 2\frac{\|\varphi_\tau g'\|}{a}\varphi_\tau^{-1}(x)|\tau(t)-\tau(x)|^3. \qquad (5.4.8)
\end{aligned}$$

Now combining the relations (5.4.5)-(5.4.8), applying Lemma 5.2.3 and the Cauchy-Schwarz inequality, we get

$$\left|P_n^{<\frac{1}{n},\tau>}(f;x) - f(x) - \frac{1}{n+1}\left[\frac{f''(x)}{(\tau'(x))^2} - f'(x)\frac{\tau''(x)}{(\tau'(x))^3}\right]\varphi_\tau^2(x)\right|$$
$$\leq \|(f\circ\tau^{-1})'' - g\| P_n^{<\frac{1}{n},\tau>}\left((\tau(t)-\tau(x))^2; x\right)$$
$$+2\frac{\|\varphi_\tau g'\|}{a}\varphi_\tau^{-1}(x) P_n^{<\frac{1}{n},\tau>}\left(|\tau(t)-\tau(x)|^3; x\right)$$
$$\leq \frac{2}{n+1}\varphi_\tau^2(x)\|\left(f\circ\tau^{-1}\right)'' - g\| + \frac{2\|\varphi_\tau g'\|}{a}\varphi_\tau^{-1}(x)$$
$$\left[P_n^{<\frac{1}{n},\tau>}\left((\tau(t)-\tau(x))^2; x\right)\right]^{1/2}\left[P_n^{<\frac{1}{n},\tau>}\left((\tau(t)-\tau(x))^4; x\right)\right]^{1/2}$$
$$= \frac{2}{n+1}\varphi_\tau^2(x)\|\left(f\circ\tau^{-1}\right)'' - g\|$$
$$+\frac{4}{n+1}\frac{\|\varphi_\tau g'\|}{a}\varphi_\tau(x)\sqrt{\frac{6(n^2-7n)\varphi_\tau^2(x) + (13n-1)}{n(n+2)(n+3)}}$$
$$= \frac{2}{n+1}\left\{\varphi_\tau^2(x)\|\left(f\circ\tau^{-1}\right)'' - g\|\right.$$
$$\left.+\frac{2\|\varphi_\tau g'\|}{a}\varphi_\tau(x)\sqrt{\frac{6(n^2-7n)\varphi_\tau^2(x) + (13n-1)}{n(n+2)(n+3)}}\right\}.$$

Because $\varphi_\tau^2(x) \leq \varphi_\tau(x) \leq 1$, $x \in (0,1)$ we obtain

$$\left|P_n^{<\frac{1}{n},\tau>}(f;x) - f(x) - \frac{1}{(n+1)[\tau'(x)]^2}\left[f''(x) - f'(x)\frac{\tau''(x)}{\tau'(x)}\right]\varphi_\tau^2(x)\right|$$
$$\leq \frac{2}{n+1}\left\{\|(f\circ\tau^{-1})'' - g\| + \frac{\varphi_\tau(x)}{a}u_n^\tau(x)\|\varphi_\tau g'\|\right\}; \tag{5.4.9}$$

$$\left|P_n^{<\frac{1}{n},\tau>}(f;x) - f(x) - \frac{1}{(n+1)[\tau'(x)]^2}\left[f''(x) - f'(x)\frac{\tau''(x)}{\tau'(x)}\right]\varphi_\tau^2(x)\right|$$
$$\leq \frac{2}{n+1}\varphi_\tau(x)\left\{\|(f\circ\tau^{-1})'' - g\| + \frac{u_n^\tau(x)}{a}\|\varphi_\tau g'\|\right\}. \tag{5.4.10}$$

Taking the infimum on the right hand side of (5.4.9) and (5.4.10) over all $g \in W_{\varphi_\tau}[0,1]$, we get

$$n\left|P_n^{<\frac{1}{n},\tau>}(f;x) - f(x) - \frac{1}{n+1}\frac{\varphi_\tau^2(x)}{[\tau'(x)]^2}\left[f''(x) - f'(x)\frac{\tau''(x)}{\tau'(x)}\right]\right|$$
$$\leq 2K_{\varphi_\tau}\left((f\circ\tau^{-1})''; \frac{u_n^\tau(x)}{a}\varphi_\tau(x)\right)$$

and

$$n\left|P_n^{<\frac{1}{n},\tau>}(f;x)-f(x)-\frac{1}{n+1}\frac{\varphi_\tau^2(x)}{[\tau'(x)]^2}\left[f''(x)-f'(x)\frac{\tau''(x)}{\tau'(x)}\right]\right|$$
$$\leq 2\varphi_\tau(x)K_{\varphi_\tau}\left((f\circ\tau^{-1})'';\frac{u_n^\tau(x)}{a}\right).$$

Using (0.5.3), the theorem is proved. □

Remark 5.4.4. *If we choose $\tau(x)=x$ in Theorem 5.4.2 and Theorem 5.4.3 we get*

i) $\left|P_n^{<\frac{1}{n}>}(f;x)-f(x)\right|\leq C_1\omega_\varphi\left(f;\frac{2}{\sqrt{n+1}}\right)$, *for* $f\in C[0,1]$

ii) $n\left|P_n^{<\frac{1}{n}>}(f;x)-f(x)-\frac{\varphi^2(x)}{n+1}f''\right|\leq C_2\omega_\varphi\left(f'';\varphi(x)u_n(x)\right)$ *for* $f\in C^2[0,1]$,

where C_1, C_2 *are positive constants and* $u_n(x)=2\sqrt{\dfrac{2(n^2-7n)\varphi^2(x)+13n-1}{n(n+2)(n+3)}}$,
$\varphi^2(x)=x(1-x)$.

Chapter 6

Bézier variant of modified Srivastava-Gupta operators

6.1 Introduction

In order to approximate Lebesgue integrable functions on $\mathbb{R}_0^+$, Srivastava and Gupta [146] introduced a general family of summation-integral type operators as

$$L_{n,c}(f,x) = n\sum_{k=1}^{\infty} p_{n,k}(x,c)\int_0^{\infty} p_{n+c,k-1}(t,c)f(t)dt + p_{n,0}(x,c)f(0), \qquad (6.1.1)$$

where $p_{n,k}(x,c) = \dfrac{(-x)^k}{k!}\phi_{n,c}^{(k)}(x)$ and

$$\phi_{n,c}(x) = \begin{cases} e^{-nx}, & c = 0, \\ (1+cx)^{-n/c}, & c \in \mathbb{N} := \{1,2,3....\}. \end{cases}$$

Ispir and Yuksel [93] introduced the Bézier variant of the operators (6.1.1) and studied the estimate of the rate of convergence of these operators for functions of bounded variation. Deo [45] gave a modification of these operators and established the rate of convergence and a Voronovskaya type result. Recently, Acar et al. [3] introduced Stancu type generalization of the operators (6.1.1) and obtained an estimate of the rate of convergence for functions having derivatives of bounded variation and also studied the simultaneous approximation for these operators.

Yadav [163] introduced a modification of the operators (6.1.1) as

$$\begin{aligned} G_{n,c}(f,x) &= n\sum_{k=1}^{\infty} p_{n,k}(x,c)\int_0^{\infty} p_{n+c,k-1}(t,c) f\left(\frac{(n-c)t}{n}\right)dt \\ &\quad + p_{n,0}(x,c)f(0) \end{aligned} \tag{6.1.2}$$

and studied its moment estimates, direct estimate, asymptotic formula and statistical convergence. Very recently, Maheshwari [112] studied the rate of approximation for the functions having derivative of bounded variation on every finite subinterval of $\mathbb{R}_0^+$ for the operators (6.1.2).

We propose a Bézier variant of the operators given by (6.1.2) as

$$\begin{aligned} G^{\alpha}_{n,c}(f,x) &= n\sum_{k=1}^{\infty} Q^{(\alpha)}_{n,k}(x,c)\int_0^{\infty} p_{n+c,k-1}(t,c) f\left(\frac{(n-c)t}{n}\right)dt \\ &\quad + Q^{(\alpha)}_{n,0}(x,c)f(0), \end{aligned} \tag{6.1.3}$$

where, $Q^{(\alpha)}_{n,k}(x,c) = [J_{n,k}(x,c)]^{\alpha} - [J_{n,k+1}(x,c)]^{\alpha}$, $\alpha \geq 1$ with $J_{n,k}(x,c) = \sum_{j=k}^{\infty} p_{n,j}(x,c)$, when $k < \infty$ and 0 otherwise. Clearly, $G^{\alpha}_{n,c}(f,x)$ is a linear positive operator. If $\alpha = 1$, then the operators $G^{\alpha}_{n,c}(f,x)$ reduce to the operators $G_{n,c}(f,x)$.

The aim of this chapter is to investigate a direct approximation result and the rate of convergence for functions having a derivative equivalent with a function of bounded variation on every finite subinterval of $\mathbb{R}_0^+$ for the operators (6.1.2). Lastly, a comparison of the rate of convergence of the operators (6.1.2) vis-a-vis operators (6.1.3) to a certain function is illustrated by some graphics.

6.2 Primary Results

Lemma 6.2.1. *[163] For $G_{n,c}(t^m,x)$, $m = 0,1,2$, one has*

(i) $G_{n,c}(1,x) = 1$

(ii) $G_{n,c}(t,x) = x$

(iii) $G_{n,c}(t^2,x) = \dfrac{(n-c)(x^2(n+c)+2x)}{n(n-2c)}$.

Consequently,

$G_{n,c}((t-x)^2,x) = \dfrac{x^2c(2n-c)+2(n-c)x}{n(n-2c)}$ and

$G_{n,c}((t-x)^2,x) \leq \dfrac{\lambda x(1+cx)}{n}$ for sufficiently large n and $\lambda > 2$.

From [112], one has

$$G_{n,c}((t-x)^{2r},x) = O(n^{-r}),\ as\, n \to \infty. \tag{6.2.1}$$

Remark 6.2.2. *We have*

$$\begin{aligned} G^{\alpha}_{n,c}(1;x) &= \sum_{k=0}^{\infty} Q^{(\alpha)}_{n,k}(x,c) = [J_{n,0}(x,c)]^{\alpha} \\ &= \left[\sum_{j=0}^{\infty} p_{n,k}(x,c)\right]^{\alpha} = 1,\ since\ \sum_{j=0}^{\infty} p_{n,k}(x,c) = 1. \end{aligned}$$

Lemma 6.2.3. *For every $f \in C_B(\mathbb{R}_0^+)$, we have*

$$\|G^{\alpha}_{n,c}(f;.)\| \leq \|f\|.$$

Applying Remark 6.2.2 , the proof of this lemma easily follows. Hence the details are omitted.

Remark 6.2.4. *For $0 \leq a, b \leq 1, \alpha \geq 1$, using the inequality*

$$| a^{\alpha} - b^{\alpha} | \leq \alpha | a - b |$$

and from the definition of $Q^{(\alpha)}_{n,k}(x,c)$, $\forall k = 0,1,2.....$, we have

$$\begin{aligned} 0 &< [J_{n,k}(x,c)]^{\alpha} - [J_{n,k+1}(x,c)]^{\alpha} \leq \alpha(J_{n,k}(x,c) - J_{n,k+1}(x,c)) \\ &= \alpha p_{n,k}(x,c). \end{aligned}$$

Hence from the definition of $G^{\alpha}_{n,c}(f;x)$, we get

$$|G^{\alpha}_{n,c}(f,x)| \leq \alpha G_{n,c}(|f|,x).$$

6.3 Main Results

To describe our first result, let us take $\phi(x) = \sqrt{x(1+cx)}$. Here the appropriate Petree's K-functional is defined by

$$\overline{K}_\phi(f,\delta) = \inf_{g\in W_\phi(\mathbb{R}_0^+)} \{||f-g|| + \delta||\phi g'|| + \delta^2||g'||,\ \delta > 0\}. \tag{6.3.1}$$

It is well known ([48] Thm. 3.1.2,) that $\overline{K}_\phi(f,t) \sim \omega_\phi(f,t)$ which means that there exists a constant $C > 0$ such that

$$C^{-1}\omega_\phi(f,t) \le \overline{K}_\phi(f,t) \le C\omega_\phi(f,t). \tag{6.3.2}$$

6.3.1 Local approximation

Theorem 6.3.1. *Let $f \in C_B(\mathbb{R}_0^+)$, then for every $x \in \mathbb{R}_0^+$ we have*

$$|G_{n,c}^\alpha(f;x) - f(x)| \le C\omega_\phi\bigg(f;\frac{1}{\sqrt{n}}\bigg), \tag{6.3.3}$$

where C is a constant independent of n and x.

Proof. For fixed $n,\ x$, choosing $g = g_{n,x} \in W_\phi(\mathbb{R}_0^+)$ and using the representation

$$g(t) = g(x) + \int_x^t g'(u)du,$$

we get

$$|G_{n,c}^\alpha(g;x) - g(x)| \ = \ \bigg|G_{n,c}^\alpha\bigg(\int_x^t g'(u)du;x\bigg)\bigg|. \tag{6.3.4}$$

Now to find the estimate of $\int_x^t g'(u)du$, we split the domain into two parts i.e. $F_n^c = [0,1/n]$ and $F_n = (1/n,\infty)$. First, if $x \in (1/n,\infty)$ then $G_{n,c}^\alpha((t-x)^2;x) \sim \frac{2\alpha}{n}\phi^2(x)$. We have

$$\bigg|\int_x^t g'(u)du\bigg| \le ||\phi g'||\bigg|\int_x^t \frac{1}{\phi(u)}du\bigg|. \tag{6.3.5}$$

For any $x, t \in (0, \infty)$, we find that

$$\begin{aligned}
\left| \int_x^t \frac{1}{\phi(u)} du \right| &= \left| \int_x^t \frac{1}{\sqrt{u(1+cu)}} du \right| \\
&\leq \left| \int_x^t \left(\frac{1}{\sqrt{u}} + \frac{1}{\sqrt{1+cu}} \right) du \right| \\
&\leq 2\left(\sqrt{t} - \sqrt{x} + \frac{\sqrt{1+ct} - \sqrt{1+cx}}{c} \right) \\
&= 2|t-x| \left(\frac{1}{\sqrt{t}+\sqrt{x}} + \frac{1}{\sqrt{1+ct}+\sqrt{1+cx}} \right) \\
&< 2|t-x| \left(\frac{1}{\sqrt{x}} + \frac{1}{\sqrt{1+cx}} \right) \\
&\leq \frac{2(c+1)}{\sqrt{c(c-1)}} \frac{|t-x|}{\phi(x)}. \qquad (6.3.6)
\end{aligned}$$

Combining (6.3.4)-(6.3.6) and using Cauchy-Schwarz inequality for $x \in (\frac{1}{n}, \infty)$, we obtain

$$\begin{aligned}
|G_{n,c}^{\alpha}(g;x) - g(x)| &< \frac{2(c+1)}{\sqrt{c(c-1)}} ||\phi g'|| \phi^{-1}(x) G_{n,c}^{\alpha}(|t-x|;x) \\
&\leq \frac{2(c+1)}{\sqrt{c(c-1)}} ||\phi g'|| \phi^{-1}(x) \left(G_{n,c}^{\alpha}((t-x)^2;x) \right)^{1/2} \\
&\leq \frac{2(c+1)}{\sqrt{c(c-1)}} ||\phi g'|| \left(\frac{2\alpha}{n} \right)^{1/2} \\
&\leq C||\phi g'|| \frac{1}{\sqrt{n}}. \qquad (6.3.7)
\end{aligned}$$

For $x \in F_n^c = [0, 1/n]$, $G_{n,c}^{\alpha}((t-x)^2;x) \sim \dfrac{2\alpha}{n^2}$ and

$$\left| \int_x^t g'(u) du \right| \leq ||g'||\ |t-x|.$$

Therefore for $x \in [1/n, \infty)$, using Cauchy-Schwarz inequality we have

$$\begin{aligned}
|G_{n,c}^{\alpha}(g;x) - g(x)| &\leq ||g'|| G_{n,c}^{\alpha}(|t-x|;x) \leq ||g'|| \left(G_{n,c}^{\alpha}((t-x)^2;x) \right)^{1/2} \\
&\leq ||g'|| \frac{\sqrt{2\alpha}}{n} \leq C||g'|| \frac{1}{n}. \qquad (6.3.8)
\end{aligned}$$

From (6.3.7) and (6.3.8), we obtain

$$|G_{n,c}^{\alpha}(g;x) - g(x)| < C\left(||\phi g'|| \frac{1}{\sqrt{n}} + ||g'|| \frac{1}{n} \right). \qquad (6.3.9)$$

Using Lemma 6.2.3 and (6.3.9), we can write

$$\begin{aligned} |G^{\alpha}_{n,c}(f;x)-f(x)| &\leq |G^{\alpha}_{n,c}(f-g;x)|+|f(x)-g(x)|+|G^{\alpha}_{n,c}(g;x)-g(x)| \\ &\leq C\bigg(||f-g||+||\phi g'||\frac{1}{\sqrt{n}}+||g'||\frac{1}{n}\bigg). \end{aligned} \tag{6.3.10}$$

Taking the infimum on the right hand side of the above inequality over all $g \in W_{\phi}(\mathbb{R}_0^+)$, we get

$$|G^{\alpha}_{n,c}(f;x)-f(x)| = CK_{\phi}\bigg(f;\frac{1}{\sqrt{n}}\bigg).$$

Using (6.3.2), we get the desired relation (6.3.3). □

6.3.2 Rate of Approximation

Lastly, we shall discuss the rate of approximation of functions with a derivative of bounded variation on $\mathbb{R}_0^+$. Let $DBV_{\gamma}(\mathbb{R}_0^+) \subset DBV(\mathbb{R}_0^+),\ \ \gamma \geq 0$ with $|f(t)| \leq Mt^{\gamma}$. In order to discuss the approximation of functions with derivatives of bounded variation, we express the operators $G^{\alpha}_{n,c}$ in an integral form as follows:

$$G^{\alpha}_{n,c}(f;x) = \int_0^{\infty} K^{\alpha}_{n,c}(x,t) f\bigg(\frac{(n-c)t}{n}\bigg)dt, \tag{6.3.11}$$

where the kernel $K^{\alpha}_{n,c}(x,t)$ is given by

$$K^{\alpha}_{n,c}(x,t) = \sum_{k=1}^{\infty} Q^{(\alpha)}_{n,k}(x,c) p_{n+c,k-1}(t,c) + Q^{(\alpha)}_{n,0}(x,c)\delta(t),$$

$\delta(u)$ being the Dirac-delta function.

Lemma 6.3.2. *For a fixed $x \in \mathbb{R}_0^+$ and sufficiently large n, we have*

(i) $\xi^{\alpha}_{n,c}(x,y) = \int_0^y K^{\alpha}_{n,c}(x,t)dt \leq \alpha\dfrac{\lambda x(1+cx)}{n}\dfrac{1}{(x-y)^2}\ \ 0 \leq y < x,$

(ii) $1-\xi^{\alpha}_{n,c}(x,z) = \int_z^{\infty} K^{\alpha}_{n,c}(x,t)dt \leq \alpha\dfrac{\lambda x(1+cx)}{n}\dfrac{1}{(z-x)^2},\ \ x < z < \infty.$

Proof. (i) Using Lemma 6.2.1 and Remark 6.2.4, we get

$$\begin{aligned}\xi_{n,c}^{\alpha}(x,y) &= \int_0^y K_{n,c}^{\alpha}(x,t)dt \leq \int_0^y \left(\frac{x-t}{x-y}\right)^2 K_{n,c}^{\alpha}(x,t)dt \\ &\leq G_{n,c}^{\alpha}((t-x)^2;x)(x-y)^{-2} \\ &\leq \alpha G_{n,c}((t-x)^2;x)(x-y)^{-2} \\ &\leq \frac{\lambda\alpha x(1+cx)}{n}\frac{1}{(x-y)^2}.\end{aligned}$$

The proof of (ii) is similar hence we skip the details. □

Theorem 6.3.3. *Let $f \in DBV_{\gamma}(\mathbb{R}_0^+)$. Then, for every $x \in (0,\infty)$ and sufficiently large n, we have*

$$\begin{aligned}&|G_{n,c}^{\alpha}(f;x)-f(x)| \\ &\leq \frac{1}{\alpha+1}|f'(x+)+\alpha f'(x-)|\sqrt{\frac{2\alpha x(1+cx)}{n}} + \frac{\alpha}{\alpha+1}|f'(x+)-f'(x-)| \\ &\sqrt{\frac{2\alpha x(1+cx)}{n}} + \frac{\lambda\alpha(1+cx)}{n}\sum_{k=1}^{[\sqrt{n}]}\bigvee_{x-x/k}^{x+x/k} f_x^{'} + \frac{x}{\sqrt{n}}\bigvee_{x-x/\sqrt{n}}^{x+x/\sqrt{n}} f_x^{'} + \frac{\lambda\alpha(1+cx)}{nx} \\ &|f(2x)-f(x)-xf'(x+)| + \frac{\alpha C(n,c,r,x)}{n^r} + \frac{|f(x)|}{x}\frac{\lambda\alpha(1+cx)}{n},\end{aligned}$$

where f_x' is defined by

$$f_x^{'}(t) = \begin{cases} f^{'}(t)-f^{'}(x-), & 0 \leq t < x \\ 0, & t = x \\ f^{'}(t)-f^{'}(x+) & x < t < \infty. \end{cases} \tag{6.3.12}$$

Proof. Since $G_{n,c}^{\alpha}(1;x)=1$, using (6.3.11), for every $x \in (0,\infty)$ we get

$$\begin{aligned}G_{n,c}^{\alpha}(f;x)-f(x) &= \int_0^{\infty} K_{n,c}^{\alpha}(x,t)(f(t)-f(x))dt \\ &= \int_0^{\infty} K_{n,c}^{\alpha}(x,t)\int_x^t f^{'}(u)dudt. \end{aligned} \tag{6.3.13}$$

For any $f \in DBV_{\gamma}(\mathbb{R}_0^+)$, from (6.3.12) we may write

$$\begin{aligned}f'(u) &= f_x'(u) + \frac{1}{\alpha+1}(f'(x+)+\alpha f'(x-)) + \frac{1}{2}(f'(x+)-f'(x-)) \\ &\left(sgn(u-x)+\frac{\alpha-1}{\alpha+1}\right) + \delta_x(u)[f'(u)-\frac{1}{2}(f'(x+)+f'(x-))],\end{aligned} \tag{6.3.14}$$

where

$$\delta_x(u) = \begin{cases} 1 \ , \ u = x \\ 0 \ , \ u \neq x \end{cases} .$$

From equations (6.3.13) and (6.3.14), we get

$$\begin{aligned}
&G_{n,c}^{\alpha}(f;x) - f(x) \\
&= \int_0^{\infty} K_{n,c}^{\alpha}(x,t) \int_x^t \Bigg[f_x'(u) + \frac{1}{\alpha+1}(f'(x+) + \alpha f'(x-)) + \frac{1}{2}(f'(x+) - f'(x-)) \\
&\left(sgn(u-x) + \frac{\alpha-1}{\alpha+1} \right) + \delta_x(u)[f'(u) - \frac{1}{2}(f'(x+) + f'(x-))]du \Bigg] dt \\
&= A_1 + A_2 + A_3 + A_{n,c}^{\alpha}(f_x', x) + B_{n,c}^{\alpha}(f_x', x),
\end{aligned}$$

where

$$\begin{aligned}
A_1 &= \int_0^{\infty} \left(\int_x^t \frac{1}{\alpha+1}(f'(x+) + \alpha f'(x-))du \right) K_{n,c}^{\alpha}(x,t)dt, \\
A_2 &= \int_0^{\infty} K_{n,c}^{\alpha}(x,t) \left(\int_x^t \frac{1}{2}(f'(x+) - f'(x-)) \left(sgn(u-x) + \frac{\alpha-1}{\alpha+1} \right) du \right) dt, \\
A_3 &= \int_0^{\infty} \left(\int_x^t \left(f'(u) - \frac{1}{2}(f'(x+) + f'(x-)) \right) \delta_x(u)du \right) K_{n,c}^{\alpha}(x,t)dt, \\
A_{n,c}^{\alpha}(f_x', x) &= \int_0^{x} \left(\int_x^t f_x'(u)du \right) K_{n,c}^{\alpha}(x,t)dt \text{ and} \\
B_{n,c}^{\alpha}(f_x', x) &= \int_x^{\infty} \left(\int_x^t f_x'(u)du \right) K_{n,c}^{\alpha}(x,t)dt.
\end{aligned}$$

Obviously,

$$A_3 = \int_0^{\infty} \left(\int_x^t \left(f'(u) - \frac{1}{2}(f'(x+) + f'(x-)) \right) \delta_x(u)du \right) K_{n,c}^{\alpha}(x,t)dt = 0.$$

Further,

$$\begin{aligned}
A_1 &= \int_0^{\infty} \left(\int_x^t \frac{1}{\alpha+1}(f'(x+) + \alpha f'(x-))du \right) K_{n,c}^{\alpha}(x,t)dt \\
&= \frac{1}{\alpha+1}(f'(x+) + \alpha f'(x-)) \int_0^{\infty} (t-x) K_{n,c}^{\alpha}(x,t)dt \\
&= \frac{1}{\alpha+1}(f'(x+) + \alpha f'(x-)) G_{n,c}^{\alpha}((t-x);x), \qquad (6.3.15)
\end{aligned}$$

and

$$
\begin{aligned}
|A_2| &= \left|\int_0^\infty K_{n,c}^\alpha(x,t)\left(\int_x^t \frac{1}{2}(f'(x+)-f'(x-))\left(sgn(u-x)+\frac{\alpha-1}{\alpha+1}\right)du\right)dt\right| \\
&= \left|\frac{1}{2}\left(f'(x+)-f'(x-)\right)\left[-\int_0^x\left(\int_t^x\left(sgn(u-x)+\frac{\alpha-1}{\alpha+1}\right)du\right)K_{n,c}^\alpha dt\right.\right. \\
&+ \left.\left.\int_x^\infty\left(\int\limits_x^t\left(sgn(u-x)+\frac{\alpha-1}{\alpha+1}\right)du\right)K_{n,c}^\alpha(x,t)dt\right]\right| \\
&\leq \frac{\alpha}{\alpha+1}\left|f'(x+)-f'(x-)\right|\int_0^\infty |t-x|\,K_{n,c}^\alpha(x,t)dt \\
&= \frac{\alpha}{\alpha+1}\left|f'(x+)-f'(x-)\right|G_{n,c}^\alpha\Big(|t-x|\,;x\Big). \qquad (6.3.16)
\end{aligned}
$$

Using Remark 6.2.4 and equations (6.3.13 -6.3.16) and applying Cauchy-Schwarz inequality we obtain

$$
\begin{aligned}
&|G_{n,c}^\alpha(f;x)-f(x)| \\
&\leq \frac{1}{\alpha+1}|f'(x+)+\alpha f'(x-)|(\alpha G_{n,c}((t-x)^2;x))^{1/2} \\
&+\frac{\alpha}{\alpha+1}|f'(x+)-f'(x-)|(\alpha G_{n,c}((t-x)^2;x))^{1/2}+|A_{n,c}^\alpha(f'_x,x)|+|B_{n,c}^\alpha(f'_x,x)| \\
&\leq \frac{1}{\alpha+1}|f'(x+)+\alpha f'(x-)|\sqrt{\frac{\lambda\alpha x(1+cx)}{n}}+\frac{\alpha}{\alpha+1}|f'(x+)-f'(x-)| \\
&\sqrt{\frac{\lambda\alpha x(1+cx)}{n}}+|A_{n,c}^\alpha(f'_x,x)|+|B_{n,c}^\alpha(f'_x,x)|. \qquad (6.3.17)
\end{aligned}
$$

Thus our problem is reduced to calculate the estimates of the terms $A_{n,c}^\alpha(f'_x,x)$ and $B_{n,c}^\alpha(f'_x,x)$. Since $\int_a^b d_t\xi_{n,c}^\alpha(x,t)\leq 1$ for all $[a,b]\subseteq\mathbb{R}_0^+$, using integration by parts we get

$$
\begin{aligned}
|A_{n,c}^\alpha(f'_x,x)| &= \left|\int_0^x\left(\int_x^t f'_x(u)du\right)d_t\xi_{n,c}^\alpha(x,t)\right| \\
&= \left|\int_0^x \xi_{n,c}^\alpha(x,t)f'_x(t)dt\right| \\
&\leq \int_0^y |f'_x(t)|\;|\xi_{n,c}^\alpha(x,t)|dt+\int_y^x |f'_x(t)|\;|\xi_{n,c}^\alpha(x,t)|dt.
\end{aligned}
$$

Applying Lemma 6.3.2 with $y = x - x/\sqrt{n}$, we have

$$\begin{aligned}
|A_{n,c}^{\alpha}(f_x', x)| &\leq \frac{\lambda\alpha x(1+cx)}{n} \int_0^y \left(\bigvee_t^x f_x'\right)(x-t)^{-2}dt + \int_y^x \left(\bigvee_t^x f_x'\right)dt \\
&\leq \frac{\lambda\alpha x(1+cx)}{n} \int_0^y \left(\bigvee_t^x f_x'\right)(x-t)^{-2}dt + \frac{x}{\sqrt{n}}\left(\bigvee_{x-x/\sqrt{n}}^x f_x'\right) \\
&= \frac{\lambda\alpha x(1+cx)}{n} \int_0^{x-\frac{x}{\sqrt{n}}} \left(\bigvee_t^x f_x'\right)(x-t)^{-2}dt + \frac{x}{\sqrt{n}}\left(\bigvee_{x-x/\sqrt{n}}^x f_x'\right).
\end{aligned}$$

Substituting $u = x/(x-t)$, we get

$$\begin{aligned}
&\int_0^{x-x/\sqrt{n}} (x-t)^{-2}\left(\bigvee_t^x f_x'\right)dt = x^{-1}\int_1^{\sqrt{n}} \left(\bigvee_{x-x/u}^x f_x'\right)du \\
&\leq x^{-1}\sum_{k=1}^{[\sqrt{n}]}\int_k^{k+1} \left(\bigvee_{x-x/k}^x f_x'\right)du \leq x^{-1}\sum_{k=1}^{[\sqrt{n}]} \left(\bigvee_{x-x/k}^x f_x'\right).
\end{aligned}$$

Thus,

$$|A_{n,c}^{\alpha}(f_x', x)| \leq \frac{\lambda(1+cx)}{n}\sum_{k=1}^{[\sqrt{n}]} \left(\bigvee_{x-x/k}^x f_x'\right) + \frac{x}{\sqrt{n}}\left(\bigvee_{x-x/\sqrt{n}}^x f_x'\right). \tag{6.3.18}$$

Again, using integration by parts in $B_n^{\rho}(f_x', x)$, applying Lemma 6.3.2 and Cauchy-Schwarz inequality, we have

$$\begin{aligned}
&|B_{n,c}^{\alpha}(f_x', x)| \\
&\leq \left|\int_{2x}^{\infty}\left(\int_x^t f_x'(u)du\right)d_tK_{n,c}^{\alpha}(x,t)\right| + \left|\int_x^{2x}\left(\int_x^t f_x'(u)du\right)d_t(1-\xi_{n,c}^{\alpha}(x,t)\right| \\
&\leq \left|\int_{2x}^{\infty}(f(t)-f(x))K_{n,c}^{\alpha}(x,t)\right| + |f'(x+)|\left|\int_{2x}^{\infty}(t-x)K_{n,c}^{\alpha}(x;t)dt\right| \\
&+\left|\int_x^{2x} f_x'(u)du\right||1-\xi_{n,c}^{\alpha}(x,2x)| + \left|\int_x^{2x} f_x'(t)(1-\xi_{n,c}^{\alpha}(x,t))dt\right| \\
&\leq \left|\int_{2x}^{\infty} f(t)K_{n,c}^{\alpha}(x,t)\right| + |f(x)|\left|\int_{2x}^{\infty} K_{n,c}^{\alpha}(x,t)\right| \\
&+|f'(x+)|\left(\int_{2x}^{\infty}(t-x)^2K_{n,c}^{\alpha}(x;t)dt\right)^{1/2} + \frac{\lambda\alpha(1+cx)}{nx}\left|\int_x^{2x}((f'(u)-f'(x+))du\right| \\
&+\left|\int_x^{x+x/\sqrt{n}} f_x'(t)dt\right| + \frac{\lambda\alpha x(1+cx)}{n}\left|\int_{x+x/\sqrt{n}}^{2x}(t-x)^{-2}f_x'(t)dt\right|.
\end{aligned}$$

We see that there exists an integer $r(2r \geq \gamma)$, such that $f(t) = O(t^{2r})$, as $t \to \infty$. Now proceeding in a manner similar to the estimate of $A^{\alpha}_{n,c}(f'_x; x)$, on substituting $t = x + \frac{x}{u}$, we get

$$\begin{aligned}
|B^{\alpha}_{n,c}(f', x)| \quad & \leq M \int_{2x}^{\infty} t^{2r} K^{\alpha}_{n,c}(x,t)dt + |f(x)| \int_{2x}^{\infty} K^{\alpha}_{n,c}(x,t)dt \\
& + |f'(x+)| \sqrt{\frac{\lambda\alpha x(1+cx)}{n}} + \frac{\lambda\alpha(1+cx)}{nx} |f(2x) - f(x) - xf'(x+)| \\
& + \frac{x}{\sqrt{n}} \bigvee_{x}^{x+x/\sqrt{n}} (f'_x) + \frac{\lambda\alpha x(1+cx)}{n} \left| \int_{x+x/\sqrt{n}}^{2x} (t-x)^{-2} f'_x(t)dt \right| \\
& \leq M \int_{2x}^{\infty} t^{2r} K^{\alpha}_{n,c}(x,t)dt + |f(x)| \int_{2x}^{\infty} K^{\alpha}_{n,c}(x,t)dt \\
& + |f'(x+)| \sqrt{\frac{\lambda\alpha x(1+cx)}{n}} + \frac{\lambda\alpha(1+cx)}{nx} |f(2x) - f(x) - xf'(x+)| \\
& + \frac{x}{\sqrt{n}} \left(\bigvee_{x}^{x+x/\sqrt{n}} f'_x \right) + \frac{\lambda\alpha(1+cx)}{n} \sum_{k=1}^{[\sqrt{n}]} \left(\bigvee_{x}^{x+x/\sqrt{n}} f'_x \right). \qquad (6.3.19)
\end{aligned}$$

For $t \geq 2x$, we get $t \leq 2(t-x)$ and $x \leq t - x$. Now using the equation (6.2.1) and Lemma 6.2.1 , we obtain

$$\begin{aligned}
& \int_{2x}^{\infty} t^{2r} K^{\alpha}_{n,c}(x,t)dt + |f(x)| \int_{2x}^{\infty} K^{\alpha}_{n,c}(x,t)dt \\
& \leq 2^{2r} \int_{2x}^{\infty} (t-x)^{2r} K^{\alpha}_{n,c}(x,t)dt + \frac{|f(x)|}{x^2} \int_{2x}^{\infty} (t-x)^2 K^{\alpha}_{n,c}(x,t)dt \\
& \leq \frac{\alpha C(n,c,r,x)}{n^r} + \frac{|f(x)|}{x^2} \frac{\lambda\alpha x(1+cx)}{n}. \qquad (6.3.20)
\end{aligned}$$

Collecting the estimates (6.3.17- 6.3.20), we get the required result. □

6.4 Numerical examples

Next, we illustrate the comparison of the rate of convergence of the operators (6.1.1), (6.1.2) and (6.1.3) to a certain function by some graphics using Matlab algorithm. Let us consider the function

$$f(x) = \begin{cases} 0, & x = 0 \\ x^{1/3}\sin(\pi/x), & x \neq 0 \end{cases}. \tag{6.4.1}$$

Then, f is of bounded variation on $[0, 1]$.

Example 1: In case $c = 150, \alpha = 10$, the convergence of the Bézier-Srivastava-Gupta (named as BzGS in Figures) operators given by (6.1.3) for $n = 160$(green) and $n = 200$(red) to function $f(x)$ (blue) given by (6.4.1), for $x \in [0, 1]$, $x \in [0, 2/\pi]$, $x \in [0, 5]$ and $x \in [0, 10]$ is shown in Figures 6.1, 6.2, 6.3 and 6.4 respectively.

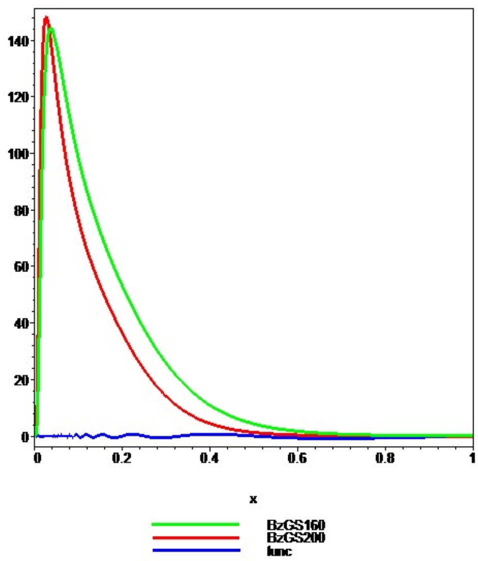

Figure 6.1

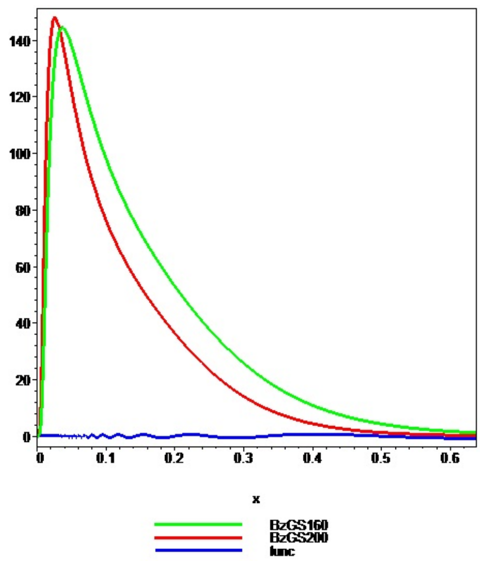

Figure 6.2

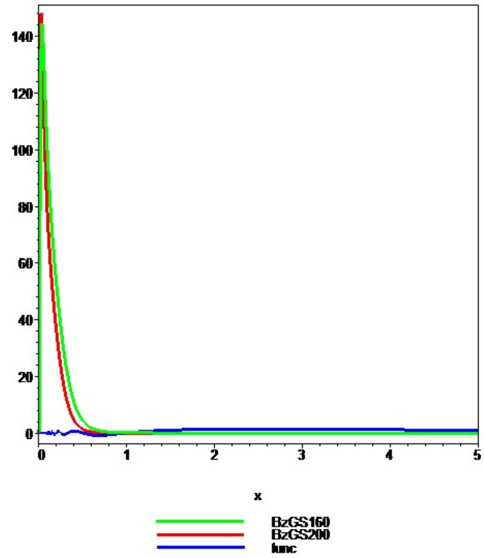

Figure 6.3

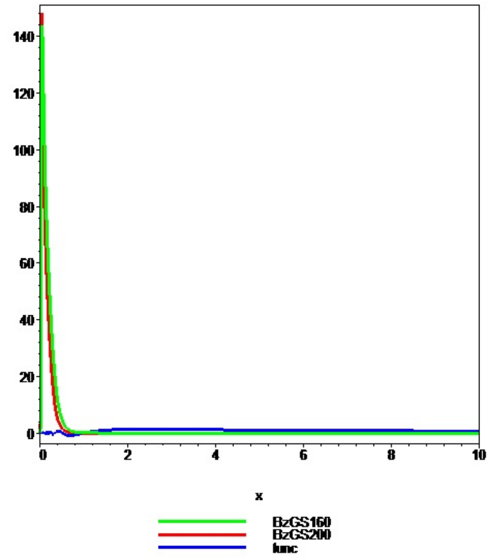

Figure 6.4

It is observed that as the interval become bigger, the approximation of operators to the function becomes better.

Example 2: In case $c = 150, \alpha = 10$ and $n = 160$, the convergence of the Srivastava-Gupta (named as GS in Figures) operators given by (6.1.1) and the Bézier-Srivastava-Gupta operators to function $f(x)$, for $x \in [0, 2/\pi]$ and $x \in [0, 10]$ is shown in Figures 6.5 and 6.6 respectively.

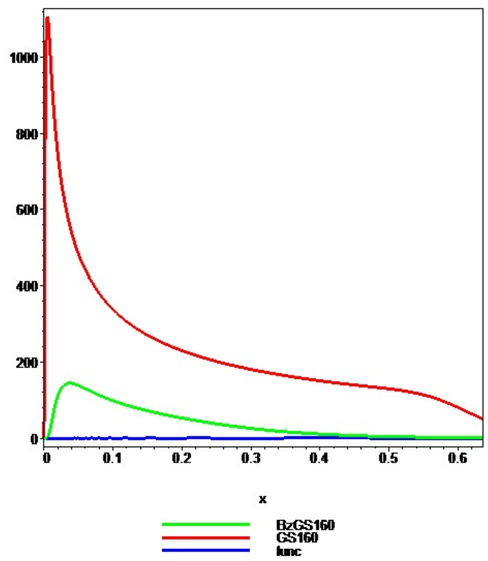

Figure 6.5

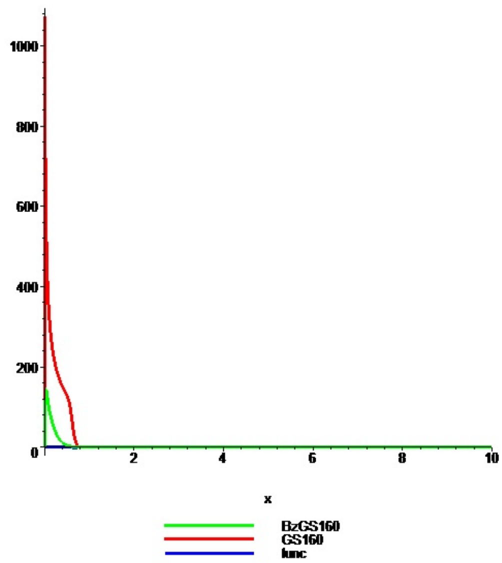

Figure 6.6

It is clearly seen that the Bézier-Srivastava-Gupta operators yield a better rate of convergence to the function than the Srivastava-Gupta operators.

Example 3: In case $c = 0$, $\alpha = 10$ and $x \in [0, 1]$, the convergence of Bézier-Srivastava-Gupta operators to function $f(x)$, for $n = 160$, $n = 200$ and $n = 50$, $n = 100$, is shown in Figures 6.7 and 6.8 respectively.

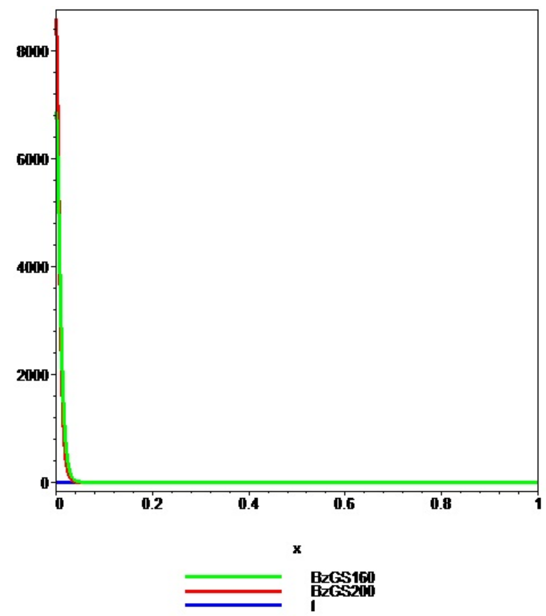

Figure 6.7

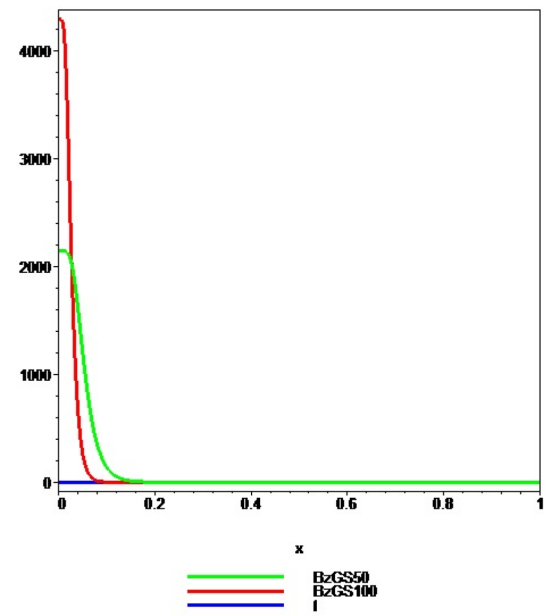

Figure 6.8

It is evident that the approximation of the function by the Bézier-Srivastava-Gupta operators becomes better as n increases.

Now let's compare the convergence of the Yadav operators given by (6.1.2) and Bézier-Srivastava-Gupta operators given by (6.1.3).

Example 4: For $c = 0$, $n = 50$ and $\alpha = 10$, the convergence of the Yadav operators(named as Ydv in Figures) and the Bézier-Srivastava-Gupta operators to function $f(x)$ is shown in Figure 6.9.

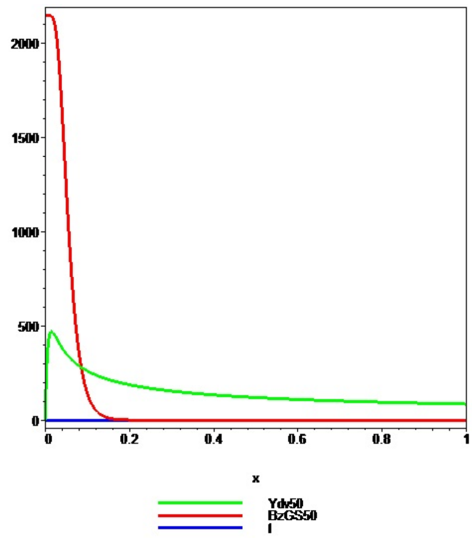

Figure 6.9

It is observed that the operator Bézier-Srivastava-Gupta operators (6.1.3) yield a better approximation to the function $f(x)$ than the Yadav operator (6.1.2).

Example 5: In case $\alpha = 10$ and $n = 160$, the convergence of the Bézier-Srivastava-Gupta operators and Yadav operators to function $f(x)$, for $c = 1$ and $c = 150$, is illustrated in Figures 6.10 and 6.11 respectively.

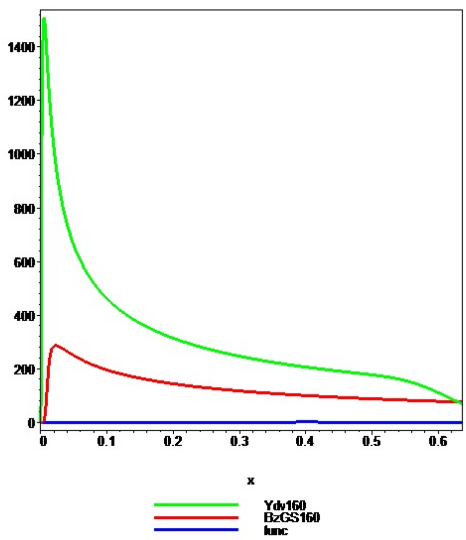

Figure 6.10

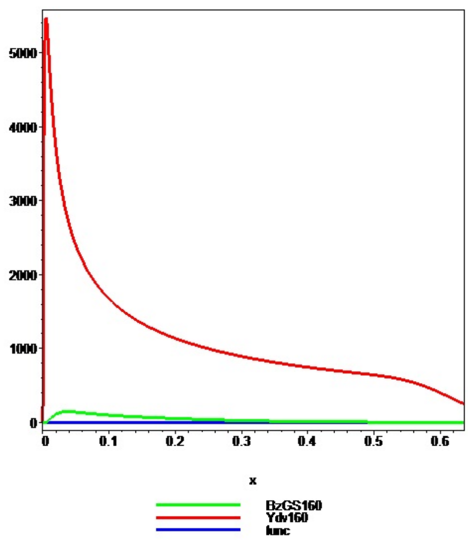

Figure 6.11

We notice that Bézier-Srivastava-Gupta operators (6.1.3) provides a better convergence to the function $f(x)$ than the Yadav operator (6.1.2).

Chapter 7

Stancu-Durrmeyer type operators based on q-integers

7.1 Introduction

In 2009, Nowak [120] defined the q-analogue for the operators defined by (1.1.1) for any function $f \in C[0,1], q > 0, \alpha \geq 0$ and each $n \in \mathbb{N}$ as

$$B_n^{q,\alpha}(f;x) = \sum_{k=0}^{n} p_{n,k}^{q,\alpha}(x) f\left(\frac{[k]_q}{[n]_q}\right), x \in [0,1], \tag{7.1.1}$$

where,

$$p_{n,k}^{q,\alpha}(x) = \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{\prod_{\nu=0}^{k-1}(x+\alpha[\nu]_q)\prod_{\mu=0}^{n-k-1}(1-q^{\mu}x+\alpha[\mu]_q)}{\prod_{\lambda=0}^{n-1}(1+\alpha[\lambda]_q)}$$

and investigated the Korovkin type approximation properties for these operators. For $\alpha = 0$, operators defined by (7.1.1) reduce to q-Bernstein polynomials and for $q \to 1-$, these operators reduce to Bernstein-Stancu operators. For $\alpha = 0$ and $q \to 1-$, these operators reduce to the classical Bernstein polynomials. Jiyang et al. [97] studied the rate of convergence and Voronovskaya type theorem for these operators defined by (7.1.1). After that "Agratini [12] introduced some estimates for the rate of convergence for the operators by means of modulus of continuity and Lipschitz type maximal function and also gave a probabilistic approach."

For $f \in C[0,1]$, $0 < q < 1$, Erencin et al. [53] introduced the Kantorovich type generalization of these operators by means of the Riemann type q-integral and investigated some approximation properties and also established a local approximation theorem.

Motivated by these studies for $f \in C[0,1]$, we propose the Durrmeyer type integral modification for the operators defined by (7.1.1) as

$$D_n^{\alpha}(f;q;x) = [n+1]_q \sum_{k=0}^{n} p_{n,k}^{q,\alpha}(x) \int_0^1 p_{n,k}^q(t) f(t) d_q t, \tag{7.1.2}$$

where

$$p_{n,k}^{q,\alpha}(x) = \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{\prod_{\nu=0}^{k-1}(x+\alpha[\nu]_q)\prod_{\mu=0}^{n-k-1}(1-q^{\mu}x+\alpha[\mu]_q)}{\prod_{\lambda=0}^{n-1}(1+\alpha[\lambda]_q)},$$

and

$$p_{n,k}^q(t) = \begin{bmatrix} n \\ k \end{bmatrix}_q t^k (1-qt)_q^{n-k}.$$

Clearly, when $\alpha = 0$, we have $p_{n,k}^q(t) = q^{-k} p_{n,k}^{q,\alpha}(qt)$.

In this chapter, we obtain moments for the operators (7.1.2), basic convergence theorem, local approximation theorem, A-statistical convergence theorem and the rate of A-statistical convergence.

7.2 Preliminaries

In what follows, $\|.\|$ will denote the uniform norm on $[0,1]$.

Lemma 7.2.1. *[120] For $B_n^{q,\alpha}(t^m, x)$, $m = 0,1,2$, one has*

(i) $B_n^{q,\alpha}(1;x) = 1$

(ii) $B_n^{q,\alpha}(t;x) = x$

(iii) $B_n^{q,\alpha}(t^2;x) = \frac{1}{(1+\alpha)}\left(x(x+\alpha) + \frac{x(1-x)}{[n]_q}\right).$

Lemma 7.2.2. *For $D_n^{\alpha}(t^m;q;x)$, $m = 0,1,2$, we have* *(i)* $D_n^{\alpha}(1;q;x) = 1$

(ii) $D_n^{\alpha}(t;q;x) = \dfrac{1}{[n+2]_q}(1+q[n]_q x)$

(iii) $D_n^{\alpha}(t^2;q;x) = \dfrac{1}{[n+2]_q[n+3]_q}\Bigg\{(1+q)+q(1+2q)[n]_q x + \dfrac{q^3[n]_q^2}{1+\alpha}\Bigg(x(x+\alpha)+ \dfrac{x(1-x)}{[n]_q}\Bigg)\Bigg\}.$

Consequently,

(i) $D_n^{\alpha}(t-x;q;x) = \dfrac{1}{[n+2]_q} + \dfrac{1}{[n+2]_q}(q[n]_q - [n+2]_q)x$

(ii) $D_n^{\alpha}((t-x)^2;q;x) = \dfrac{1+q}{[n+2]_q[n+3]_q} + \Bigg\{\dfrac{1}{[n+2]_q[n+3]_q}\Bigg(q(1+2q)[n]_q + \dfrac{q^3[n]_q([n]_q\alpha+1)}{(1+\alpha)}\Bigg) - \dfrac{2}{[n+2]_q}\Bigg\}x + \Bigg\{\dfrac{q^3[n]_q([n]_q-1)}{[n+2]_q[n+3]_q(1+\alpha)} - \dfrac{2q[n]_q}{[n+2]_q}+1\Bigg\}x^2.$

Lemma 7.2.3. *For $f \in C[0,1]$ there holds* $\|D_n^{\alpha}(f;q;.)\| \leq \|f\|$.

Proof. From (7.1.2)

$$\begin{aligned} |D_n^{\alpha}(f;q;x)| &\leq [n+1]_q \sum_{k=0}^{n} p_{n,k}^{q,\alpha}(x) \int_0^1 p_{n,k}^q(t)|f(t)|d_q t \\ &\leq \|f\|\ [n+1]_q \sum_{k=0}^{n} p_{n,k}^{q,\alpha}(x) \int_0^1 p_{n,k}^q(t) d_q t \\ &= \|f\|, \end{aligned}$$

which implies that $\|D_n^{\alpha}(f;q;.)\| \leq \|f\|$. This completes the proof. □

7.3 Main results

First we will establish the basic convergence theorem for the operators $D_n^{\alpha_n}$.

Theorem 7.3.1. *Let $< q_n >$ and $< \alpha_n >$ be the sequences such that $0 < q_n < 1$, $\alpha_n \geq 0$ and $\dfrac{1}{[n]_{q_n}} \to 0$, as $n \to \infty$. Then for any $f \in C[0,1]$, $D_n^{\alpha_n}(f;q_n;x)$ converges to f uniformly in $x \in [0,1]$ iff $\lim_n q_n = 1$ and $\lim_n \alpha_n = 0$.*

Proof. First we assume that $q_n \to 1$ *and* $\alpha_n \to 0$, as $n \to \infty$. From Lemma 7.2.2, we observe that

$$\begin{aligned}
&D_n^{\alpha_n}(1; q_n; x) = 1,\\
&D_n^{\alpha_n}(t; q_n; x) = \frac{1}{[n+2]_{q_n}}(1 + q_n[n]_{q_n}x) \to x, \text{and}\\
&D_n^{\alpha_n}(t^2; q_n; x) = \frac{1}{[n+2]_{q_n}[n+3]_{q_n}}\Bigg\{(1+q_n) + q_n(1+2q_n)[n]_{q_n}x\\
&+\frac{q_n^3[n]_{q_n}^2}{1+\alpha_n}\left(x(x+\alpha_n) + \frac{x(1-x)}{[n]_{q_n}}\right)\Bigg\} \to x^2,
\end{aligned}$$

uniformly in $x \in [0,1]$, as $n \to \infty$. Hence, by Bohman-Korovkin theorem $D_n^{\alpha_n}(f; q_n; x)$ converges to f uniformly in $x \in [0,1]$, as $n \to \infty$

Conversely, suppose that $D_n^{\alpha_n}(f; q_n; x)$ converges to f uniformly in $x \in [0,1]$, as $n \to \infty$ then

$$D_n^{\alpha_n}(t; q_n; x) = \frac{1}{[n+2]_{q_n}}(1 + q_n[n]_{q_n}x) \to x$$

uniformly in $x \in [0,1]$, as $n \to \infty$, which implies that $q_n \to 1$, as $n \to \infty$. Next,

$$\begin{aligned}
D_n^{\alpha_n}(t^2; q_n; x) &= \frac{1}{[n+2]_{q_n}[n+3]_{q_n}}\Bigg\{(1+q_n) + q_n(1+2q_n)[n]_{q_n}x\\
&+ \frac{{q_n}^3[n]_{q_n}^2}{1+\alpha_n}\left(x(x+\alpha_n) + \frac{x(1-x)}{[n]_{q_n}}\right)\Bigg\} \to x^2,
\end{aligned}$$

uniformly in $x \in [0,1]$, as $n \to \infty$.

Hence,

$$\frac{1+q_n}{[n+3]_{q_n}[n+2]_{q_n}} \to 0,$$

$$\frac{1}{[n+2]_{q_n}[n+3]_{q_n}}\left\{q_n(1+2q_n)[n]_{q_n} + \frac{q_n^3[n]_{q_n}^2}{1+\alpha_n}\left(\alpha_n + \frac{1}{[n]_{q_n}}\right)\right\} \to 0$$

and

$$\frac{q_n^3[n]_{q_n}^2}{(1+\alpha_n)[n+2]_{q_n}[n+3]_{q_n}}\left(1 - \frac{1}{[n]_{q_n}}\right) \to 1, \text{ as } n \to \infty. \tag{7.3.1}$$

From (7.3.1) it follows that, $\alpha_n \to 0$, *as* $n \to \infty$. This completes the proof. □

Now, we will prove a local approximation theorem for the operators $D_n^{\alpha}(f;q;x)$. Let us define an auxiliary operator as

$$\widetilde{D}_n^{\alpha}(f;q;x) = D_n^{\alpha}(f,q,x) + f(x) - f\left(\frac{1}{[n+2]_q}(1+q[n]_q x)\right). \qquad (7.3.2)$$

Hence applying lemma (7.2.3), it follows that

$$\|\widetilde{D}_n^{\alpha}(f;q;.)\| \leq 3\|f\|. \qquad (7.3.3)$$

In what follows, let q_n be a sequence in $(0,1)$ such that $q_n \to 1$, *as* $n \to \infty$ and $\frac{1}{[n]_{q_n}} \to 0$. Further, let α_n be a sequence of non-negative real numbers such that $\alpha_n \to 0$, *as* $n \to \infty$.

Theorem 7.3.2. *Let $f \in C[0,1]$. Then for each $x \in [0,1]$, we have*

$$|D_n^{\alpha_n}(f;q_n;x) - f(x)| \;\leq\; 2C\omega_2\left(f; \frac{\delta_{n,q_n}^{\alpha_n}(x)}{\sqrt{2}}\right) + \omega\left(f, \left|\frac{1}{[n+2]_{q_n}} + \left(\frac{q_n[n]_{q_n}}{[n+2]_{q_n}} - 1\right)x\right|\right),$$

where $\delta_{n,q_n}^{\alpha_n}(x) = \left\{D_n^{\alpha_n}((t-x)^2;q_n;x) + \left(\frac{1+q_n[n]_{q_n}x}{[n+2]_{q_n}} - x\right)^2\right\}$

Proof. From (7.3.2), using Lemma 7.2.2 we get $\widetilde{D}_n^{\alpha_n}(1;q_n;x) = 1$ and $\widetilde{D}_n^{\alpha_n}(t;q_n;x) = x$. Hence

$$\widetilde{D}_n^{\alpha_n}(t-x;q_n;x) = 0. \qquad (7.3.4)$$

For $g \in W^2[0,1]$ and $x \in [0,1]$, by Taylor's theorem, we have

$$g(t) - g(x) = (t-x)g'(x) + \int_x^t (t-u)g''(u)du.$$

Then using (7.3.4) and (7.3.3), we get

$$\begin{aligned}
\widetilde{D}_n^{\alpha_n}(g(t);q_n;x) - g(x) &= \widetilde{D}_n^{\alpha_n}\left((t-x)g'(x);q_n;x\right) + \widetilde{D}_n^{\alpha_n}\left(\int_x^t (t-u)g''(u)du;q_n;x\right) \\
&= g'(x)\widetilde{D}_n^{\alpha_n}(t-x;q_n;x) + \widetilde{D}_n^{\alpha_n}\left(\int_x^t (t-u)g''(u)du;q_n;x\right) \\
&= \widetilde{D}_n^{\alpha_n}\left(\int_x^t (t-u)g''(u)du;q_n;x\right) \\
&= D_n^{\alpha_n}\left(\int_x^t (t-u)g''(u)du;q_n;x\right) \\
&\quad - \int_x^{\frac{1+q_n[n]_{q_n}x}{[n+2]_{q_n}}} \left(\frac{1+q_n[n]_{q_n}x}{[n+2]_{q_n}} - u\right)g''(u)du,
\end{aligned}$$

which implies that

$$\begin{aligned}
|\widetilde{D}_n^{\alpha_n}(g(t);q_n;x)-g(x)| &\leq D_n^{\alpha_n}\left(\left|\int_x^t |t-u||g''(u)|du\right|;q_n;x\right)\\
&+ \left|\int_x^{\frac{1+q_n[n]_{q_n}x}{[n+2]_{q_n}}} \left|\frac{1+q_n[n]_{q_n}x}{[n+2]_{q_n}}-u\right| g''(u)du\right|\\
&\leq \|g''\|D_n^{\alpha_n}\left(\left|\int_x^t |t-u|du\right|;q_n;x\right)\\
&+ \|g''\|\int_x^{\frac{1+q_n[n]_{q_n}x}{[n+2]_{q_n}}} \left|\frac{1+q_n[n]_{q_n}x}{[n+2]_{q_n}}-u\right|du\\
&= I_1+I_2, \text{ (say)}.
\end{aligned}$$

Now,

$$\begin{aligned}
I_1 &= \|g''\|D_n^{\alpha_n}\left(\left|\int_x^t |t-u|du\right|;q_n;x\right)=\|g''\|D_n^{\alpha_n}\left(\frac{(t-x)^2}{2}:x\right),\\
I_2 &= \|g''\|\int_x^{\frac{1+q_n[n]_{q_n}x}{[n+2]_{q_n}}} \left|\frac{1+q_n[n]_{q_n}x}{[n+2]_{q_n}}-u\right|du=\frac{\left(\frac{1+q_n[n]_{q_n}x}{[n+2]_{q_n}}-x\right)^2}{2}\|g''\|.
\end{aligned}$$

So, we have

$$\begin{aligned}
|\widetilde{D}_n^{\alpha_n}(g(t);q_n;x)-g(x)| &\leq \frac{\|g''\|}{2}\left\{D_n^{\alpha_n}((t-x)^2;q_n;x)+\left(\frac{1+q_n[n]_{q_n}x}{[n+2]_{q_n}}-x\right)^2\right\}\\
&= \frac{\|g''\|}{2}(\delta_{n,q_n}^{\alpha_n}(x))^2, \text{ (say)}. \qquad (7.3.5)
\end{aligned}$$

Now using (7.3.2), we obtain

$$\begin{aligned}
\left|D_n^{\alpha_n}(f;q_n;x)-f(x)\right| &= \left|\widetilde{D}_n^{\alpha_n}(f;q_n;x)-f(x)+f\left(\frac{1+q_n[n]_{q_n}x}{[n+2]_{q_n}}\right)-f(x)\right|\\
&\leq \left|\widetilde{D}_n^{\alpha_n}(f-g;q_n;x)\right|+\left|\widetilde{D}_n^{\alpha_n}(g;q_n;x)-g(x)\right|\\
&+ \left|f(x)-g(x)\right|+\left|f\left(\frac{1+q_n[n]_{q_n}x}{[n+2]_{q_n}}\right)-f(x)\right|.
\end{aligned}$$

Using equations (7.3.3) and (7.3.5), we get

$$\begin{aligned}
\left|D_n^{\alpha_n}(f;q_n;x)-f(x)\right| &\leq 4\|f-g\|+4\frac{\|g''\|}{2}(\delta_{n,q_n}^{\alpha_n}(x))^2\\
&+ \omega\left(f,\left|\frac{1}{[n+2]_{q_n}}+\left(\frac{q_n[n]_{q_n}}{[n+2]_{q_n}}-1\right)x\right|\right).
\end{aligned}$$

Taking infimum on the right hand side of the above inequality over all $g \in W^2[0,1]$, we obtain

$$\left|D_n^{\alpha_n}(f;q_n;x)-f(x)\right| \leq 4K_2\left(f,\frac{(\delta_{n,q_n}^{\alpha_n}(x))^2}{2}\right)+\omega\left(f,\left|\frac{1}{[n+2]_{q_n}}+\left(\frac{q_n[n]_{q_n}}{[n+2]_{q_n}}-1\right)x\right|\right).$$

Using the relation between K-functional and second modulus of smoothness from equation (0.5.2), we get

$$\left|D_n^{\alpha_n}(f;q_n;x)-f(x)\right| \leq C\omega_2\left(f,\frac{\delta_{n,q_n}^{\alpha_n}(x)}{\sqrt{2}}\right)+\omega\left(f,\left|\frac{1}{[n+2]_{q_n}}+\left(\frac{q_n[n]_{q_n}}{[n+2]_{q_n}}-1\right)x\right|\right).$$

This completes the proof. □

For $0 < r \leq 1$, we consider the following Lipschitz-type space [124]:

$$Lip_M^*(r)=\left\{f \in C[0,1]:|f(t)-f(x)| \leq M\frac{|t-x|^r}{(t+x)^{\frac{r}{2}}}; M>0, t\in[0,1] \; and \; x \in (0,1]\right\}.$$

In our next theorem, we estimate the error in the approximation for a function in $Lip_M^*(r)$.

Theorem 7.3.3. *For $f \in Lip_M^*(r)$, $0<r\leq 1$, $x \in (0,1]$ and $n \in \mathbb{N}$, we have*

$$\left|D_n^{\alpha_n}(f;q_n;x)-f(x)\right| \leq M\left(\frac{\mu_{n,2}^{q_n,\alpha_n}(x)}{x}\right)^{\frac{r}{2}},$$

where $\mu_{n,2}^{q_n,\alpha_n}(x)=D_n^{\alpha_n}((t-x)^2;q_n;x)$.

Proof. We may write

$$|D_n^{\alpha_n}(f,q_n,x)-f(x)| \leq [n+1]_{q_n}\sum_{k=0}^{n}p_{n,k}^{q_n,\alpha_n}(x)\int_0^1 p_{n,k}^{q_n}(t)|f(t)-f(x)|d_{q_n}t.$$

Applying Hölder's inequality in the integral form for $p=\dfrac{2}{r}$ and $q=\dfrac{2}{2-r}$

$$\begin{aligned}&|D_n^{\alpha_n}(f,q_n,x)-f(x)|\\ &\leq [n+1]_{q_n}\sum_{k=0}^{n}p_{n,k}^{q_n,\alpha_n}(x)\left(\int_0^1 p_{n,k}^{q_n}(t)|f(t)-f(x)|^{\frac{2}{r}}d_{q_n}t\right)^{\frac{r}{2}}\left(\int_0^1 p_{n,k}^{q_n}d_{q_n}t\right)^{\frac{2-r}{2}}\\ &= [n+1]_{q_n}\sum_{k=0}^{n}p_{n,k}^{q_n,\alpha_n}(x)\left(\int_0^1 p_{n,k}^{q_n}(t)|(f(t)-f(x))|^{\frac{2}{r}}d_{q_n}t\right)^{\frac{r}{2}}\left(\frac{1}{[n+1]_{q_n}}\right)^{\frac{2-r}{r}}.\end{aligned}$$

Again, applying Hölder's inequality for the summation for $p = \dfrac{2}{r}$ and $q = \dfrac{2}{2-r}$

$$\begin{aligned}
&|D_n^{\alpha_n}(f, q_n, x) - f(x)| \\
&\leq \left([n+1]_{q_n} \sum_{k=0}^{n} p_{n,k}^{q_n,\alpha_n}(x) \int_0^1 p_{n,k}^{q_n}(t) |f(t) - f(x)|^{\frac{2}{r}} d_{q_n} t \right)^{\frac{r}{2}} \left(\sum_{k=0}^{n} p_{n,k}^{q_n,\alpha_n}(x) \right)^{\frac{2-r}{2}} \\
&\leq \left\{ [n+1]_{q_n} \sum_{k=0}^{n} p_{n,k}^{q_n}, \alpha_n(x) \int_0^1 p_{n,k}^{q_n}(t) \left(M \frac{|t-x|^r}{(t+x)^{\frac{r}{2}}} \right)^{\frac{2}{r}} d_{q_n} t \right\}^{\frac{r}{2}} .1 \\
&\leq \frac{M}{(x)^{\frac{r}{2}}} \left([n+1]_{q_n} \sum_{k=0}^{n} p_{n,k}^{q_n,\alpha_n}(x) \int_0^1 p_{n,k}^{q_n}(t) |t-x|^2 d_{q_n} t \right)^{\frac{r}{2}} \\
&\leq \frac{M}{(x)^{\frac{r}{2}}} \left(D_n^{\alpha_n}((t-x)^2, q_n, x) \right)^{\frac{r}{2}} \\
&= M \left(\frac{\mu_{n,2}^{q_n,\alpha_n}(x)}{x} \right)^{\frac{r}{2}} .
\end{aligned}$$

This completes the proof. □

Lipschitz type maximal function of order β introduced by Lenze [109], is defined as

$$\widetilde{\omega}_\beta(f, x) = \sup_{t \neq x, t \in [0,1]} \frac{|f(t) - f(x)|}{|t-x|^\beta}, \quad x \in [0,1] \ and \ \beta \in (0,1].$$

Now, we will obtain a local direct estimate for the operator $D_n^{\alpha_n}(f, q_n, x)$ in terms of $\widetilde{\omega}_\beta(f, x)$.

Theorem 7.3.4. *Let $f \in C[0,1], 0 < \beta \leq 1$. Then $\forall\ x \in [0,1]$, we have*

$$|D_n^{\alpha_n}(f, q_n, x) - f(x)| \leq \widetilde{\omega_\beta}(f, x) \left(\mu_{n,2}^{q_n,\alpha_n}(x) \right)^{\frac{\beta}{2}},$$

where $\mu_{n,2}^{q_n,\alpha_n}(x)$ is as defined in equation (7.3.5).

Proof. Applying Hölder's inequality twice first for the integration and then for the summation with $p = \dfrac{2}{\beta}$ and $q = \dfrac{2}{2-\beta}$, we have

$$|D_n^{\alpha_n}(f, q_n, x) - f(x)| \ \leq \ \left([n+1]_{q_n} \sum_{k=0}^{n} p_{n,k}^{q_n,\alpha_n}(x) \int_0^1 p_{n,k}^{q_n}(t) |f(t) - f(x)|^{\frac{2}{\beta}} d_{q_n} t \right)^{\frac{\beta}{2}} .$$

Using the definition of the Lipschitz-type maximal function, we obtain

$$
\begin{aligned}
|D_n^{\alpha_n}(f,q_n,x)-f(x)| &\leq \widetilde{\omega_\beta}(f,x)\Bigg([n+1]_{q_n}\sum_{k=0}^{n} p_{n,k}^{q_n,\alpha_n}(x)\int_0^1 p_{n,k}^{q_n}(t)|t-x|^2 d_{q_n}t\Bigg)^{\frac{\beta}{2}}\\
&= \widetilde{\omega_\beta}(f,x)\Bigg(D_n^{\alpha_n}((t-x)^2;q_n;x)\Bigg)^{\frac{\beta}{2}}\\
&= \widetilde{\omega_\beta}(f,x)\Bigg(\mu_{n,2}^{q_n,\alpha_n}(x)\Bigg)^{\frac{\beta}{2}}.
\end{aligned}
$$

This completes the proof. □

7.4 A-statistical convergence

First, we recall the following Korovkin type theorem in the case of A-statistical convergence:

Theorem 7.4.1. *[63] If the sequence of positive linear operators $L_n : C[a,b] \to C[a,b]$ satisfies the conditions $st-\lim_n \|L_n(e_i;q;.)-e_i\| = 0$ where $e_i(t)=t^i, i = 0,1,2,$ then for any $f \in C[a,b]$, we have $st-\lim_n \|L_n(f;q;.)-f\| = 0$.*

The result given above also works for A-statistical convergence. Now we will establish the following A-statistical approximation theorem for the operators $D_n^{\alpha}(f,q,x)$.

Theorem 7.4.2. *Let $A=(a_{jn})$be a non-negative infinite regular summability matrix and $q=<q_n>$, $0<q_n<1$ and $\alpha=<\alpha_n>$ be the sequences satisfying the following conditions:*

$$
\begin{aligned}
& st_A-\lim_n q_n = 1,\;\; st_A-\lim_n q_n^n = a, a<1\\
& st_A-\lim_n \alpha_n = 0 \textit{ and } st_A-\lim_n \frac{1}{[n]_{q_n}} = 0,
\end{aligned}
\tag{7.4.1}
$$

then for $f \in C[0,1]$, we have

$$st_A-\lim_n \|D_n^{\alpha_n}(f,q_n,.)-f\| = 0.$$

Proof. From Theorem 7.4.1, it is enough to prove that

$$st_A - \lim_n \|D_n^{\alpha_n}(e_i, q_n, .) - e_i\|_{C[0,1]} = 0, \;\; i = 0, 1, 2.$$

In view of Lemma 7.2.2, we have $D_n^{\alpha_n}(e_0, q_n, x) = 1$, hence

$$st - \lim_n \|D_n^{\alpha_n}(e_0, q_n, .) - e_0\| = 0.$$

Now, again using Lemma 7.2.2 we have

$$\begin{aligned} \|D_n^{\alpha_n}(e_1, q_n, .) - e_1\| &= \sup_{x\in[0,1]} \left| \frac{1}{[n+2]_{q_n}}(1 + q_n[n]_{q_n}x) - x \right| \\ &\le \left| \frac{q_n[n]_{q_n}}{[n+2]_{q_n}} - 1 \right| + \frac{1}{[n+2]_{q_n}}. \end{aligned} \tag{7.4.2}$$

For any given $\epsilon > 0$, let us define the following sets

$$U = \left\{ n : \|D_n^{\alpha_n}(e_1, q_n, .) - e_1\| \ge \epsilon \right\},$$

$$U_1 = \left\{ n : \left| \frac{q_n[n]_{q_n}}{[n+2]_{q_n}} - 1 \right| \ge \frac{\epsilon}{2} \right\},$$

and

$$U_2 = \left\{ n : \frac{1}{[n+2]_{q_n}} \ge \frac{\epsilon}{2} \right\}.$$

From (7.4.2) it is easy to see that $U \subseteq U_1 \bigcup U_2$, so we have

$$\sum_{n\in U} a_{jn} \le \sum_{n\in U_1} a_{jn} + \sum_{n\in U_2} a_{jn}. \tag{7.4.3}$$

From equation (7.4.1), we obtain

$$st_A - \lim_n \left(\frac{q_n[n]_{q_n}}{[n+2]_{q_n}} - 1 \right) = 0$$

and

$$st_A - \lim_n \left(\frac{1}{[n+2]_{q_n}} \right) = 0.$$

Hence taking limit on both sides of (7.4.3), as $j \to \infty$, we get

$$st_A - \lim_n \|D_n^{\alpha_n}(e_1, q_n, .) - e_1\| = 0. \tag{7.4.4}$$

Similarly, using Lemma 7.2.2 , we have

$$\|D_n^{\alpha_n}(e_2,q_n,.)-e_2\|$$
$$=\sup_{x\in[0,1]}\Bigg|\frac{1}{[n+2]_{q_n}[n+3]_{q_n}}\Bigg\{(1+q_n)+q_n(1+2q_n)[n]_{q_n}x$$
$$+\frac{q_n^3[n]_{q_n}^2}{1+\alpha_n}\bigg(x(x+\alpha_n)+\frac{x(1-x)}{[n]_{q_n}}\bigg)\Bigg\}-x^2\Bigg|$$
$$\leq\frac{1+q_n}{[n+2]_{q_n}[n+3]_{q_n}}+\Bigg|\frac{1}{[n+2]_{q_n}[n+3]_{q_n}}\bigg(q_n(1+2q_n)[n]_{q_n}+\frac{q_n^3[n]_{q_n}^2}{1+\alpha_n}$$
$$\bigg(\alpha_n+\frac{1}{[n]_{q_n}}\bigg)\Bigg|+\Bigg|\frac{1}{[n+2]_{q_n}[n+3]_{q_n}}\frac{q_n^3[n]_{q_n}^2}{1+\alpha_n}\bigg(1-\frac{1}{[n]_{q_n}}\bigg)-1\Bigg|. \tag{7.4.5}$$

For $\epsilon>0$, let us define the following sets:

$$U=\Big\{n:\|D_n^{\alpha_n}(e_2,q_n,.)-e_2\|\geq\epsilon\Big\},$$

$$U_1=\Big\{n:\frac{1+q_n}{[n+2]_{q_n}[n+3]_{q_n}}\geq\frac{\epsilon}{3}\Big\},$$

$$U_2=\Bigg\{n:\Bigg|\frac{1}{[n+2]_{q_n}[n+3]_{q_n}}\bigg(q_n(1+2q_n)[n]_{q_n}+\frac{q_n^3[n]_{q_n}^2}{1+\alpha_n}\bigg(\alpha_n+\frac{1}{[n]_{q_n}}\bigg)\Bigg|\geq\frac{\epsilon}{3}\Bigg\},$$

and

$$U_3=\Bigg\{n:\Bigg|\frac{1}{[n+2]_{q_n}[n+3]_{q_n}}\frac{q_n^3[n]_{q_n}^2}{1+\alpha_n}\bigg(1-\frac{1}{[n]_{q_n}}\bigg)-1\Bigg|\geq\frac{\epsilon}{3}\Bigg\}.$$

From (7.5.2) it follows that $U\subseteq U_1\bigcup U_2\bigcup U_3$, hence

$$\sum_{n\in U}a_{jn}\leq\sum_{n\in U_1}a_{jn}+\sum_{n\in U_2}a_{jn}+\sum_{n\in U_3}a_{jn}. \tag{7.4.6}$$

Now, using (7.4.1) we find

$$st-\lim_n\frac{1+q_n}{[n+2]_{q_k}[n+3]_{q_k}}=0,$$

$$st-\lim_n\frac{1}{[n+2]_{q_n}[n+3]_{q_n}}\Bigg\{q_n(1+2q_n)[n]_{q_n}+\frac{q_n^3[n]_{q_n}^2}{1+\alpha_n}\bigg(\alpha_n+\frac{1}{[n]_{q_n}}\bigg)\Bigg\}=0,\text{ and}$$

$$st-\lim_n\Bigg\{\frac{1}{[n+2]_{q_n}[n+3]_{q_n}}\frac{q_n^3[n]_{q_n}^2}{1+\alpha_n}\bigg(1-\frac{1}{[n]_{q_n}}\bigg)-1\Bigg\}=0.$$

Hence taking limit on both sides of (7.4.6), as $j \to \infty$ we get

$$st_A - \lim_n \|D_n^{\alpha_n}(e_2, q_n, .) - e_2\| = 0. \tag{7.4.7}$$

This completes the proof. □

7.5 Rate of A-statistical convergence

Let $f \in C[0,1]$. Then for any $x, t \in [0,1]$, we have $|f(t) - f(x)| \leq \omega(f, |t-x|)$, which implies that

$$|f(t) - f(x)| \leq (1 + \delta^{-2}(t-x)^2)\omega(f, \delta), \ \ \delta > 0. \tag{7.5.1}$$

In our next theorem we give the rate of A-statistical convergence for the operators $D_n^{\alpha}(f; q; x)$ in terms of modulus of continuity.

Theorem 7.5.1. *Let $A = (a_{jn})$ be a non negative regular summability matrix and for each $x \in [0,1]$, $< b_n(x) >$ be a positive non-increasing sequence and let $q =< q_n >, 0 < q_n < 1$ and $\alpha =< \alpha_n >$ be sequences satisfying equation (7.4.1) and $\omega(f; \mu_{n,2}^{q_n,\alpha_n}) = st_A - o(b_n(x))$ with $\mu_{n,2}^{q_n,\alpha_n}(x) = D_n^{\alpha_n}((t-x)^2; q_n; x)$, then for any function $f \in C[0,1]$ and $x \in [0,1]$, we have $D_n^{\alpha_n}(f; q_n; x) - f(x) = st_A - o(b_n(x))$.*

Proof. By monotonicity and linearity of the operators $D_n^{\alpha_n}(f; q_n; x)$, we have

$$\begin{aligned} |D_n^{\alpha_n}(f; q_n; x) - f(x)| &\leq D_n^{\alpha_n}(|f(t) - f(x)|; q_n; x) \\ &\leq \Big(1 + \delta^{-2} D_n^{\alpha_n}((t-x)^2; q_n; x)\Big)\omega(f; \delta), \text{for any } \delta > 0. \end{aligned}$$

Taking δ as $\sqrt{\mu_{n,2}^{q_n,\alpha_n}(x)}$, we get

$$|D_n^{\alpha_n}(f; q_n; x) - f(x)| \leq 2\omega(f; \sqrt{\mu_{n,2}^{q_n,\alpha_n}}). \tag{7.5.2}$$

For $\epsilon > 0$, let us define the following sets:
$U = \Big\{n : |D_n^{\alpha_n}(f; q_n; x) - f(x)| \geq \epsilon\Big\}$ and $U_1 = \Big\{n : 2\omega\Big(f, \sqrt{\mu_{n,2}^{q_n,\alpha_n}}(x)\Big) \geq \epsilon\Big\}$.

From (7.5.2), we have

$$\frac{1}{b_n(x)}\sum_{n\in U} a_{jn} \leq \frac{1}{b_n(x)}\sum_{n\in U_1} a_{jn}.$$

Taking limit on the above inequality as $j \to \infty$ and using $\omega(f;\mu_{n,2}^{q_n,\alpha_n}) = st_A - o(b_n(x))$, we obtain the required result. This completes the proof. $\square$

Theorem 7.5.2. *Let $A = (a_{jn})$, $< b_n(x) >$, $q =< q_n >$, and $\alpha =< \alpha_n >$ be all same as in Theorem 7.5.1. Assume that the operators $D_n^{\alpha_n}(f;q_n;x)$ satisfy the condition $\omega(f;\mu_{n,2}^{q_n,\alpha_n}(x)) = st_A - o_\eta(b_n(x))$ with $\mu_{n,2}^{q_n,\alpha_n}(x) = D_n^{\alpha_n}((t-x)^2;q_n;x)$. Then for all $f \in C[0,1]$, we have $D_n^{\alpha}(f;q;x) - f(x) = st_A - o_\eta(b_n(x))$.*

Similar results hold when little "o_η" is replaced by the big "O_η".

Let us define

$$||f||_{W^2[0,1]} := ||f|| + ||f'|| + ||f''||. \tag{7.5.3}$$

The Peetre's $K-$functional [129] is defined as

$$K(f;\delta) = \inf_{g\in W^2[0,1]} \left\{||f-g|| + \delta||g||_{W^2[0,1]} : \delta > 0\right\}.$$

We know that for the $K-$functional and the modulus of smoothness [47], there exists a constant $C > 0$, such that

$$K(f;\delta) \leq C\omega_2(f;\sqrt{\delta}). \tag{7.5.4}$$

Theorem 7.5.3. *Let $A = (a_{jn})$ be a non negative regular summability matrix and let $q =< q_n >, 0 < q_n < 1$ and $\alpha =< \alpha_n >$ be sequences satisfying equation (7.4.1). For each $f \in C[0,1]$, we have*

$$\|D_n^{\alpha_n}(f;q_n;.) - f\| \leq C\omega_2(f;\sqrt{\delta_n^{q_n,\alpha_n}}),$$

where

$$\delta_n^{q_n,\alpha_n} = \|D_n^{\alpha_n}((e_1 - .);q_n;.)\| + \left\|D_n^{\alpha_n}((e_1 - .)^2;q_n;.)\right\|.$$

Proof. For $g \in W^2[0,1]$, applying Taylor's expansion , we have

$$D_n^{\alpha_n}(g;q_n;x) - g(x) = g'(x)D_n^{\alpha_n}(e_1 - x;q_n;x) + \frac{1}{2}g''(\xi)D_n^{\alpha_n}((e_1-x)^2;q_n;x),$$

where ξ lies between t and x.

By using (7.5.3), we get

$$\begin{aligned} \|D_n^{\alpha_n}(g;q_n;.) - g\| &\leq \|g'\|\|D_n^{\alpha_n}((e_1 - .);q_n;.)\| \\ &+ \frac{1}{2}\|g''\|\left\|D_n^{\alpha_n}((e_1-.)^2;q_n;.)\right\| \\ &\leq \delta_n^{q_n,\alpha_n}\|g\|_{W^2[0,1]}, \text{ (say)}. \end{aligned}$$

For $f \in C[0,1]$ and $g \in W^2[0,1]$, we have

$$\begin{aligned} \|D_n^{\alpha_n}(f;q_n;.) - f\| &\leq \|D_n^{\alpha_n}(f;q_n;.) - D_n^{\alpha_n}(g;q_n;.)\| \\ &+ \|D_n^{\alpha_n}(g;q_n;.) - g\| + \|f-g\| \\ &\leq 2\|f-g\| + \|D_n^{\alpha_n}(g;q_n;.) - g\| \\ &\leq 2\|f-g\| + \delta_n^{q_n,\alpha_n}\|g\|_{W^2[0,1]} \\ &\leq 2\bigg(\|f-g\| + \delta_n^{q_n,\alpha_n}\|g\|_{W^2[0,1]}\bigg). \end{aligned}$$

Taking infimum on the right hand side of the above inequality over all $g \in W^2[0,1]$ and using equation (7.5.4), we get

$$\|D_n^{\alpha_n}(f;q_n;.) - f\| \leq 2K(f;\delta_n^{q_n,\alpha_n}) \leq C\omega_2(f;\sqrt{\delta_n^{q_n,\alpha_n}}).$$

Using equations (7.4.4) and (7.4.7), we get $st_A - \lim_n \delta_n^{q_n,\alpha_n} = 0$, hence $st_A - \lim_n \omega_2(f;\sqrt{\delta_n^{q_n,\alpha_n}}) = 0$, which gives the required result. □

Chapter 8

Approximation of functions by bivariate q-Stancu-Durrmeyer type operators

8.1 Construction of bivariate operators

This chapter is concerned with the bivariate generalization of the q-analogue of the Stancu operators given by (7.1.2) in the previous chapter.

Let $I = [0, 1]$, $I^2 = I \times I$ and $C(I^2)$ denote the class of all real valued continuous functions on I^2 endowed with the norm $||f|| = \sup_{(x,y)\in I^2} |f(x, y)|$. Then, for $f \in C(I^2)$, the bivariate generalization of q-Stancu-Durrmeyer type operators (7.1.2) is defined as

$$\begin{aligned} D^{\alpha_{n_1},\alpha_{n_2}}_{n_1,n_2}(f; q_{n_1}, q_{n_2}, x, y) &= [n_1 + 1]_{q_{n_1}} [n_2 + 1]_{q_{n_2}} \sum_{k_1=0}^{n_1} \sum_{k_2=0}^{n_2} p^{q_{n_1},\alpha_{n_1}}_{n_1,k_1}(x) p^{q_{n_2},\alpha_{n_2}}_{n_2,k_2}(y) \\ & \int_0^1 \int_0^1 p^{q_{n_1}}_{n_1,k_1}(t) p^{q_{n_2}}_{n_2,k_2}(s) f(t, s) d_{q_{n_1}} t \, d_{q_{n_2}} s, \qquad (8.1.1) \end{aligned}$$

$\forall (x, y) \in I^2$. The aim of this chapter is to study the rate of convergence of the operators given by (8.1.1) by means of the complete modulus of continuity, partial modulus of continuity and the Peetre's K-functional. A bivariate Voronovskaya type theorem for the bivariate q-Stancu-Durrmeyer operators is established. We also introduce the associated GBS operators and determine the degree of approximation with the aid of mixed modulus of smoothness for B-continuous and B-differentiable functions.

8.2 Moments

Lemma 8.2.1. *For* $D_{n_1,n_2}^{\alpha_{n_1},\alpha_{n_2}}(e_{ij};q_1,q_2,x,y),\ e_{ij}=x^iy^j,\ i,j\in\mathbb{N}\cup\{0\},\ x,y\in[0,1],$ *we have*

(i) $D_{n_1,n_2}^{\alpha_{n_1},\alpha_{n_2}}(e_{00};q_{n_1},q_{n_2},x,y)=1;$

(ii) $D_{n_1,n_2}^{\alpha_{n_1},\alpha_{n_2}}(e_{10};q_{n_1},q_{n_2},x,y)=\dfrac{1}{[n_1+2]_{q_{n_1}}}(1+q_{n_1}[n_1]_{q_{n_1}}x);$

(iii) $D_{n_1,n_2}^{\alpha_{n_1},\alpha_{n_2}}(e_{01};q_{n_1},q_{n_2},x,y)=\dfrac{1}{[n_2+2]_{q_{n_2}}}(1+q_{n_2}[n_2]_{q_{n_2}}y);$

(iv) $D_{n_1,n_2}^{\alpha_{n_1},\alpha_{n_2}}(e_{11};q_{n_1},q_{n_2},x,y)=\dfrac{1}{[n_1+2]_{q_{n_1}}}(1+q_{n_1}[n_1]_{q_{n_1}}x)\dfrac{1}{[n_2+2]_{q_{n_2}}}(1+q_{n_2}[n_2]_{q_{n_2}}y);$

(v) $D_{n_1,n_2}^{\alpha_{n_1},\alpha_{n_2}}(e_{20};q_{n_1},q_{n_2},x,y)=\dfrac{1}{[n_1+2]_{q_{n_1}}[n_1+3]_{q_{n_1}}}\Bigg\{[2]_{q_{n_1}}+q_{n_1}(1+2q_{n_1})[n_1]_{q_{n_1}}x+\dfrac{q_{n_1}^3[n_1]_{q_{n_1}}^2}{1+\alpha_{n_1}}\left(x(x+\alpha_{n_1})+\dfrac{x(1-x)}{[n_1]_{q_{n_1}}}\right)\Bigg\};$

(vi) $D_{n_1,n_2}^{\alpha_{n_1},\alpha_{n_2}}(e_{02};q_{n_1},q_{n_2},x,y)=\dfrac{1}{[n_2+2]_{q_{n_2}}[n_2+3]_{q_{n_2}}}\Bigg\{[2]_{q_{n_2}}+q_{n_2}(1+2q_{n_2})[n_2]_{q_{n_2}}y+\dfrac{q_{n_2}^3[n_2]_{q_{n_2}}^2}{1+\alpha_{n_2}}\left(y(y+\alpha_{n_2})+\dfrac{y(1-y)}{[n_2]_{q_{n_2}}}\right)\Bigg\};$

(vii) $D_{n_1,n_2}^{\alpha_{n_1},\alpha_{n_2}}(e_{20}+e_{02};q_1,q_2,x,y)=\dfrac{1}{[n_1+2]_{q_{n_1}}[n_1+3]_{q_{n_1}}}\Bigg\{[2]_{q_{n_1}}+q_{n_1}(1+2q_{n_1})[n_1]_{q_{n_1}}x+\dfrac{q_{n_1}^3[n_1]_{q_{n_1}}^2}{1+\alpha_{n_1}}\left(x(x+\alpha_{n_1})+\dfrac{x(1-x)}{[n_1]_{q_{n_1}}}\right)\Bigg\}+\dfrac{1}{[n_2+2]_{q_{n_2}}[n_2+3]_{q_{n_2}}}\Bigg\{[2]_{q_{n_2}}+q_{n_2}(1+2q_{n_2})[n_2]_{q_{n_2}}y+\dfrac{q_{n_2}^3[n_2]_{q_{n_2}}^2}{1+\alpha_{n_2}}\left(y(y+\alpha_{n_2})+\dfrac{y(1-y)}{[n_2]_{q_{n_2}}}\right)\Bigg\}.$

Lemma 8.2.2. *For* $D_{n_1,n_2}^{\alpha_{n_1},\alpha_{n_2}}((t-x)^i(s-y)^j;q_{n_1},q_{n_2},x,y),\ i,\ j=1,2,$ *we have*

(i) $D_{n_1,n_2}^{\alpha_{n_1},\alpha_{n_2}}(t-x;q_{n_1},q_{n_2},x,y)=\dfrac{1}{[n_1+2]_{q_{n_1}}}+\dfrac{1}{[n_1+2]_{q_{n_1}}}\Big(q_{n_1}[n_1]_{q_{n_1}}-[n_1+2]_{q_{n_1}}\Big)x;$

(ii) $D_{n_1,n_2}^{\alpha_{n_1},\alpha_{n_2}}(s-y;q_{n_1},q_{n_2},x,y) = \dfrac{1}{[n_2+2]_{q_{n_2}}} + \dfrac{1}{[n_2+2]_{q_{n_2}}}\Big(q_{n_2}[n_2]_{q_{n_2}} - [n_2+2]_{q_{n_2}}\Big)y;$

(iii) $$D_{n_1,n_2}^{\alpha_{n_1},\alpha_{n_2}}((t-x)^2;q_{n_1},q_{n_2},x,y) = \frac{[2]_{q_{n_1}}}{[n_1+2]_{q_{n_1}}[n_1+3]_{q_{n_1}}} + \Bigg\{\frac{1}{[n_1+2]_{q_{n_1}}[n_1+3]_{q_{n_1}}} \left(q_{n_1}(1+2q_{n_1})[n_1]_{q_{n_1}} + \frac{q_{n_1}^3[n_1]_{q_{n_1}}([n_1]_{q_{n_1}}\alpha_{n_1}+1)}{(1+\alpha_{n_1})}\right) - \frac{2}{[n_1+2]_{q_{n_1}}}\Bigg\}x + \left\{\frac{q_{n_1}^3[n_1]_{q_{n_1}}([n_1]_{q_{n_1}}-1)}{[n_1+2]_{q_{n_1}}[n_1+3]_{q_{n_1}}(1+\alpha_{n_1})} - \frac{2q_{n_1}[n_1]_{q_{n_1}}}{[n_1+2]_{q_{n_1}}} + 1\right\}x^2;$$

(iv) $$D_{n_1,n_2}^{\alpha_{n_1},\alpha_{n_2}}((s-y)^2;q_{n_1},q_{n_2},x,y) = \frac{[2]_{q_{n_2}}}{[n_2+2]_{q_{n_2}}[n_2+3]_{q_{n_2}}} + \Bigg\{\frac{1}{[n_2+2]_{q_{n_2}}[n_2+3]_{q_{n_2}}} \left(q_{n_2}(1+2q_{n_2})[n_2]_{q_{n_2}} + \frac{q_{n_2}^3[n_2]_{q_{n_2}}([n_2]_{q_{n_2}}\alpha_{n_2}+1)}{(1+\alpha_{n_2})}\right) - \frac{2}{[n_2+2]_{q_{n_2}}}\Bigg\}y + \left\{\frac{q_{n_2}^3[n_2]_{q_{n_2}}([n_2]_{q_{n_2}}-1)}{[n_2+2]_{q_{n_2}}[n_2+3]_{q_{n_2}}(1+\alpha_{n_2})} - \frac{2q_{n_2}[n_2]_{q_{n_2}}}{[n_2+2]_{q_{n_2}}} + 1\right\}y^2.$$

8.3 Direct results

In what follows, let $0 < q_{n_i} < 1$ and $\alpha_{n_i} \geq 0$ be sequences such that $\lim\limits_{n_i\to\infty} q_{n_i} = 1$, $\lim\limits_{n_i\to\infty} q_{n_i}^{n_i} = a_i (0 \leq a_i < 1)$ and $\lim\limits_{n_i\to\infty} \alpha_{n_i} = 0,\ i = 1,2$. Also, assume that

$$\delta_{n_1,q_{n_1}}^{\alpha_{n_1}}(x) = \sqrt{D_{n_1}^{\alpha_{n_1}}((t-x)^2;q_{n_1},x)} \text{ and } \delta_{n_2,q_{n_2}}^{\alpha_{n_2}}(y) = \sqrt{D_{n_2}^{\alpha_{n_2}}((s-y)^2;q_{n_2},y)}. \tag{8.3.1}$$

Theorem 8.3.1. *[158] Let I_1 and I_2 be two compact intervals of the real line. Let T_{n_1,n_2} with $(n_1,n_2) \in \mathbb{N}\times\mathbb{N}$ be the linear positive operators on $C(I_1\times I_2)$ such that*

$$\lim_{n_1,n_2\to\infty} T_{n_1,n_2}(e_{ij}) = e_{ij},\ (i,j) \in \{(0,0),(1,0),(0,1)\}$$

and

$$\lim_{n_1,n_2\to\infty} T_{n_1,n_2}(e_{20}+e_{02}) = e_{20}+e_{02},$$

uniformly on $I_1\times I_2$, then the sequence $(T_{n_1,n_2}f)$ converges uniformly to f on $I_1\times I_2$, for any $f \in C(I_1\times I_2)$.

Theorem 8.3.2. *The sequence of bivariate q-Stancu Durrmeyer operators $D^{\alpha_{n_1},\alpha_{n_2}}_{n_1,n_2}(f;q_{n_1},q_{n_2},x,y)$ converges uniformly to $f(x,y)$, for any $f \in C(I^2)$.*

Proof. Using Theorem 8.3.1 and Lemma 8.2.1, the proof easily follows. Hence the details are omitted. □

In the following we give some numerical results which show the rate of convergence of the operator $D^{\alpha_{n_1},\alpha_{n_2}}_{n_1,n_2}$ to certain functions using Matlab algorithms.

Example 8.3.3. *Let us consider $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = x^2y^2 + x^3y - 2x^4$. The convergence of the operator $D^{\alpha_{n_1},\alpha_{n_2}}_{n_1,n_2}$ to the function f is illustrated in Figure 8.1 and Figure 8.2, respectively for $n_1 = n_2 = 10, q_{n_1} = q_{n_2} = 0.5, \alpha_{n_1} = \alpha_{n_2} = 0.2$ and $n_1 = n_2 = 100, q_{n_1} = q_{n_2} = 0.9, \alpha_{n_1} = \alpha_{n_2} = 0.2$, respectively.*

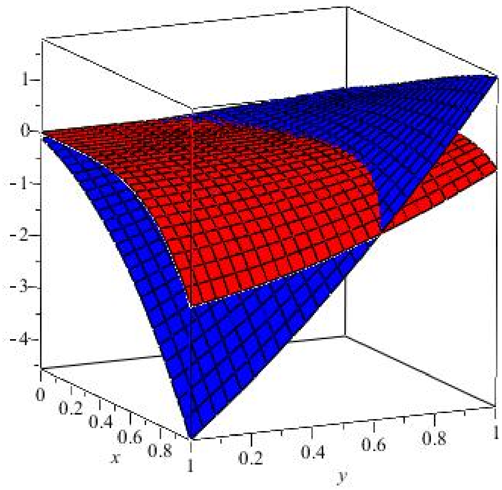

Figure 8.1: The convergence of $D^{\alpha_{n_1},\alpha_{n_2}}_{n_1,n_2}(f;q_{n_1},q_{n_2},x,y)$ to $f(x,y)$, for $q_{n_1} = q_{n_2} =$ 0.5 (red f, blue $D^{0.2,0.2}_{10,10}$)

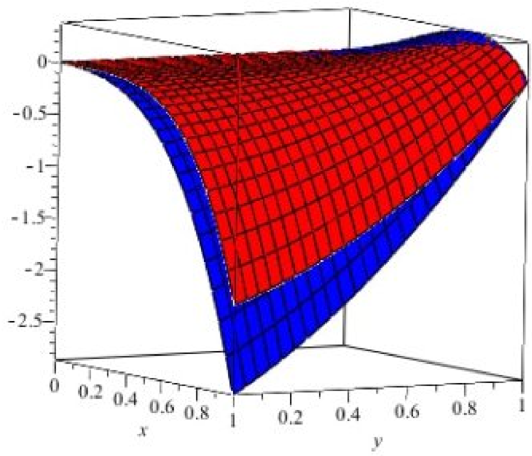

Figure 8.2: The convergence of $D^{\alpha_{n_1},\alpha_{n_2}}_{n_1,n_2}(f;q_{n_1},q_{n_2},x,y)$ to $f(x,y)$, for $q_{n_1}=q_{n_2}=0.9$ (red f, blue $D^{0.2,0.2}_{100,100}$)

We remark that as the values of n_1 and n_2 increase, the error in the approximation of the function by the operator becomes smaller.

Theorem 8.3.4. *Let $f\in C^1(I^2)$ and $(x,y)\in I^2$. Then, we have*

$$|D^{\alpha_{n_1},\alpha_{n_2}}_{n_1,n_2}(f;q_{n_1},q_{n_2},x,y)-f(x,y)|\leq ||f'_x||\delta^{\alpha_{n_1}}_{n_1,q_{n_1}}(x)+||f'_y||\delta^{\alpha_{n_2}}_{n_2,q_{n_2}}(y). \tag{8.3.2}$$

Proof. For a fixed point $(x,y)\in I^2$, we may write

$$f(t,s)-f(x,y)=\int_x^t f'_u(u,s)du+\int_y^s f'_v(x,v)dv \tag{8.3.3}$$

Applying $D^{\alpha_{n_1},\alpha_{n_2}}_{n_1,n_2}$ on the above equation (8.3.3), we get

$$\begin{aligned}|D^{\alpha_{n_1},\alpha_{n_2}}_{n_1,n_2}(f;q_{n_1},q_{n_2},x,y)-f(x,y)| &\leq D^{\alpha_{n_1},\alpha_{n_2}}_{n_1,n_2}\left(\left|\int_x^t|f'_u(u,s)|du\right|;q_{n_1},q_{n_2},x,y\right)\\ &+ D^{\alpha_{n_1},\alpha_{n_2}}_{n_1,n_2}\left(\left|\int_y^s|f'_v(x,v)|dv\right|;q_{n_1},q_{n_2},x,y\right).\end{aligned}$$

Since $\left|\int_x^t |f'_u(u,s)|du\right| \leq ||f'_x|||t-x|$ and $\left|\int_y^s |f'_v(x,v)|dv\right| \leq ||f'_y|||s-y|$, we have

$$\begin{aligned}
&|D_{n_1,n_2}^{\alpha_{n_1},\alpha_{n_2}}(f;q_{n_1},q_{n_2},x,y)-f(x,y)| \\
&\leq ||f'_x||D_{n_1,n_2}^{\alpha_{n_1},\alpha_{n_2}}(|t-x|;q_{n_1},q_{n_2},x,y)+||f'_y||D_{n_1,n_2}^{\alpha_{n_1},\alpha_{n_2}}(|s-y|;q_{n_1},q_{n_2},x,y) \\
&= ||f'_x||D_{n_1}^{\alpha_{n_1}}(|t-x|;q_{n_1},x)+||f'_y||D_{n_2}^{\alpha_{n_2}}(|s-y|;q_{n_2},y).
\end{aligned}$$

Now applying Cauchy Schwarz inequality and Lemma 7.2.2 of the previous chapter, we have

$$\begin{aligned}
|D_{n_1,n_2}^{\alpha_{n_1},\alpha_{n_2}}(f;q_{n_1},q_{n_2},x,y)-f(x,y)| &\leq ||f'_x|||\sqrt{D_{n_1}^{\alpha_{n_1}}((t-x)^2;q_{n_1},x)}\sqrt{D_{n_1}^{\alpha_{n_1}}(1;q_{n_1},x)} \\
&+ ||f'_y||\sqrt{D_{n_2}^{\alpha_{n_2}}((s-y)^2;q_{n_2},y)}\sqrt{D_{n_2}^{\alpha_{n_2}}(1;q_{n_2},y)} \\
&= ||f'_x||\delta_{n_1,q_{n_1}}^{\alpha_{n_1}}(x)+||f'_y||\delta_{n_2,q_{n_2}}^{\alpha_{n_2}}(y).
\end{aligned}$$

This completes the proof. □

Example 8.3.5. *Let $f \in C^1(I^2)$. Considering $n_1 = n_2 = 10$ and $\alpha_{n_1} = \alpha_{n_2} = 0.2$, in the Table 3 we compute the error of approximation of $f(x,y) = x^2y^2 + x^3y - 2x^4$ by using the relation (8.3.2). We observe that the error of approximation becomes smaller as $q_{n_i} \to 1$, as $n_i \to \infty$, $i = 1, 2$.*

Table 3. *Error of approximation for $D_{n_1,n_2}^{\alpha_{n_1},\alpha_{n_2}}$*

$q_{n_1} = q_{n_2}$	*Error of approximation*
0.4	2.172377390
0.5	1.880657031
0.6	1.606452189
0.7	1.366444984
0.8	1.170958744
0.9	1.022298931

For $f \in C(I^2)$, the complete modulus of continuity for the bivariate case is defined as follows:

$$\bar{\omega}(f;\delta_1,\delta_2) = \sup\left\{|f(t,s)-f(x,y)| : (t,s),(x,y) \in I^2 \text{ and } |t-x| \leq \delta_1,\ |s-y| \leq \delta_2\right\}.$$

Equivalently,

$$\bar{\omega}(f;\delta_1,\delta_2) = \sup\left\{|f(t,s)-f(x,y)| : (t,s),(x,y)\in I^2 \text{ and } \sqrt{(t-x)^2+(s-y)^2}\leq\delta\right\},$$

where $\bar{\omega}(f;\delta_1,\delta_2)$ satisfies the following properties:

(i) $\bar{\omega}(f;\delta_1,\delta_2)\to 0$, if $\delta_1\to 0$ and $\delta_2\to 0$;

(ii) $|f(t,s)-f(x,y)|\leq\bar{\omega}(f;\delta_1,\delta_2)\left(1+\dfrac{|t-x|}{\delta_1}\right)\left(1+\dfrac{|s-y|}{\delta_2}\right)$.

The details of the complete modulus of continuity for the bivariate case can be found in [16]. Further, the partial moduli of continuity with respect to x and y are given by

$$\omega^1(f;\delta) = \sup\left\{|f(x_1,y)-f(x_2,y)| : y\in I \text{ and } |x_1-x_2|\leq\delta\right\},$$

and

$$\omega^2(f;\delta) = \sup\left\{|f(x,y_1)-f(x,y_2)| : x\in I \text{ and } |y_1-y_2|\leq\delta\right\}.$$

Evidently, they satisfy the properties of the usual modulus of continuity. Let $C^2(I^2) := \left\{f\in C(I^2) : f_{xx}, f_{xy}, f_{yx}, f_{yy}\in C(I^2)\right\}$.
The norm on the space $C^2(I^2)$ is defined as

$$||f||_{C^2(I^2)} = ||f|| + \sum_{i=1}^{2}\left(\left\|\frac{\partial^i f}{\partial x^i}\right\| + \left\|\frac{\partial^i f}{\partial y^i}\right\|\right).$$

The Peetre's K-functional of the function $f\in C(I^2)$ is defined by

$$\mathcal{K}(f;\delta) = \inf_{g\in C^2(I^2)}\{||f-g||+\delta||g||\}, \delta>0.$$

Also,

$$\mathcal{K}(f;\delta)\leq M\left\{\bar{\omega}_2(f;\sqrt{\delta})+\min(1,\delta)||f||\right\},$$

holds for all $\delta>0$. The constant M in the above inequality is independent of δ and f and $\bar{\omega}_2(f;\sqrt{\delta})$ is the second order modulus of continuity for the bivariate case given as

$$\bar{\omega}_2(f;\sqrt{\delta})$$
$$= \sup\left\{\left|\sum_{\nu=0}^{2}(-1)^{2-\nu} f(x+\nu h, y+\nu k)\right| : (x,y),(x+2h,y+2k)\in J^2, |h|\le\delta, |k|\le\delta\right\}.$$

The following theorem provides the degree of approximation in terms of the mixed modulus of smoothness.

Theorem 8.3.6. *Let $f \in C(I^2)$ and $(x,y) \in I^2$. Then, we have*

$$|D_{n_1,n_2}^{\alpha_{n_1},\alpha_{n_2}}(f;q_{n_1},q_{n_2},x,y) - f(x,y)| \le 4\bar{\omega}\Big(f, \delta_{n_1,q_{n_1}}^{\alpha_{n_1}}(x), \delta_{n_2,q_{n_2}}^{\alpha_{n_2}}(y)\Big).$$

Proof. Using the linearity and positivity of the operators $D_{n_1,n_2}^{\alpha_{n_1},\alpha_{n_2}}$ and in view of property (ii) of the complete modulus of continuity, we have

$$\begin{aligned}
&|D_{n_1,n_2}^{\alpha_{n_1},\alpha_{n_2}}(f;q_{n_1},q_{n_2},x,y) - f(x,y)| \le D_{n_1,n_2}^{\alpha_{n_1},\alpha_{n_2}}(|f(t,s)-f(x,y)|;q_{n_1},q_{n_2},x,y)\\
&\le \bar{\omega}\Big(f, \delta_{n_1,q_{n_1}}^{\alpha_{n_1}}(x), \delta_{n_2,q_{n_2}}^{\alpha_{n_2}}(y)\Big)\Big(1+\frac{1}{\delta_{n_1,q_{n_1}}^{\alpha_{n_1}}(x)} D_{n_1}^{\alpha_{n_1}}(|t-x|;q_{n_1},x)\Big)\\
&\times\Big(1+\frac{1}{\delta_{n_2,q_{n_2}}^{\alpha_{n_2}}(y)} D_{n_2}^{\alpha_{n_2}}(|s-y|;q_{n_2},y)\Big).
\end{aligned}$$

Now applying the Cauchy-Schwarz inequality and Lemma 7.2.2, we get

$$\begin{aligned}
&|D_{n_1,n_2}^{\alpha_{n_1},\alpha_{n_2}}(f;q_{n_1},q_{n_2},x,y) - f(x,y)| \le D_{n_1,n_2}^{\alpha_{n_1},\alpha_{n_2}}(|f(t,s)-f(x,y)|;q_{n_1},q_{n_2},x,y)\\
&\le \bar{\omega}\Big(f, \delta_{n_1,q_{n_1}}^{\alpha_{n_1}}(x), \delta_{n_2,q_{n_2}}^{\alpha_{n_2}}(y)\Big)\Big(1+\frac{1}{\delta_{n_1,q_{n_1}}^{\alpha_{n_1}}(x)} \sqrt{D_{n_1}^{\alpha_{n_1}}((t-x)^2;q_{n_1},x)}\Big)\\
&\times\Big(1+\frac{1}{\delta_{n_2,q_{n_2}}^{\alpha_{n_2}}(y)} \sqrt{D_{n_2}^{\alpha_{n_2}}((s-y)^2;q_{n_2},y)}\Big).
\end{aligned} \tag{8.3.4}$$

Considering (8.3.1), from the above inequality, the desired result is immediate. □

In our nest result we obtain the rate of convergence by means of the partial moduli of continuity.

Theorem 8.3.7. *Let $f \in C(I^2)$ and $(x,y) \in I^2$. Then, we have*

$$|D_{n_1,n_2}^{\alpha_{n_1},\alpha_{n_2}}(f;q_{n_1},q_{n_2},x,y) - f(x,y)| \le 2\omega^1\Big(f;\delta_{n_1,q_{n_1}}^{\alpha_{n_1}}(x)\Big) + 2\omega^2\Big(f;\delta_{n_2,q_{n_2}}^{\alpha_{n_2}}(y)\Big).$$

Proof. Using the definition of the partial modului of continuity and applying the Cauchy-Schwarz inequality and Lemma 7.2.2, we may write

$$|D_{n_1,n_2}^{\alpha_{n_1},\alpha_{n_2}}(f;q_{n_1},q_{n_2},x,y)-f(x,y)| \leq D_{n_1,n_2}^{\alpha_{n_1},\alpha_{n_2}}(|f(t,s)-f(x,y)|;q_{n_1},q_{n_2},x,y)$$
$$\leq D_{n_1,n_2}^{\alpha_{n_1},\alpha_{n_2}}(|f(t,s)-f(t,y)|;q_{n_1},q_{n_2},x,y)+D_{n_1,n_2}^{\alpha_{n_1},\alpha_{n_2}}(|f(t,y)-f(x,y)|;q_{n_1},q_{n_2},x,y)$$
$$\leq D_{n_1,n_2}^{\alpha_{n_1},\alpha_{n_2}}\bigg(\omega^1(f;|t-x|);q_{n_1},q_{n_2},x,y\bigg)+D_{n_1,n_2}^{\alpha_{n_1},\alpha_{n_2}}\bigg(\omega^2(f;|s-y|);q_{n_1},q_{n_2},x,y\bigg)$$
$$\leq \omega^1(f,\delta_{n_1,q_{n_1}}^{\alpha_{n_1}}(x))\bigg\{1+\frac{1}{\delta_{n_1,q_{n_1}}^{\alpha_{n_1}}(x)}D_{n_1}^{\alpha_{n_1}}(|t-x|;q_{n_1},x)\bigg\}$$
$$+\omega^2(f,\delta_{n_2,q_{n_2}}^{\alpha_{n_2}}(y))\bigg\{1+\frac{1}{\delta_{n_2,q_{n_2}}^{\alpha_{n_2}}(y)}D_{n_2}^{\alpha_{n_2}}(|s-y|;q_{n_2},y)\bigg\}$$
$$\leq \omega^1(f,\delta_{n_1,q_{n_1}}^{\alpha_{n_1}}(x))\bigg\{1+\frac{1}{\delta_{n_1,q_{n_1}}^{\alpha_{n_1}}(x)}\bigg(D_{n_1}^{\alpha_{n_1}}((t-x)^2;q_{n_1},x)\bigg)^{1/2}\bigg\}$$
$$+\omega^2(f,\delta_{n_2,q_{n_2}}^{\alpha_{n_2}}(y))\bigg\{1+\frac{1}{\delta_{n_2,q_{n_2}}^{\alpha_{n_2}}(y)}\bigg(D_{n_2}^{\alpha_{n_2}}((s-y)^2;q_{n_2},y)\bigg)^{1/2}\bigg\}.$$

Replacing $\delta_{n_1,q_{n_1}}^{\alpha_{n_1}}(x)$ and $\delta_{n_2,q_{n_2}}^{\alpha_{n_2}}(y)$ from (8.3.1) in the above relation, we get the desired result. □

In our next result we establish the rate of approximation of the Stancu-Durrmeyer type operators to the function $f \in C(I^2)$ by means of Peetre's K-functional.

Theorem 8.3.8. *For the function $f \in C(I^2)$, we have the following inequality*

$$|D_{n_1,n_2}^{\alpha_{n_1},\alpha_{n_2}}(f;q_{n_1},q_{n_2},x,y)-f(x,y)|$$
$$\leq M\bigg\{\bar{\omega}_2\bigg(f;\sqrt{A_{n_1,n_2}^{\alpha_{n_1},\alpha_{n_2}}(q_{n_1},q_{n_2},x,y)}\bigg)+\min\{1,A_{n_1,n_2}^{\alpha_{n_1},\alpha_{n_2}}(q_{n_1},q_{n_2},x,y)\}||f||\bigg\}$$
$$+\overline{\omega}\Bigg(f;\sqrt{\bigg(\frac{1}{[n_1+2]_{q_{n_1}}}(1+q_{n_1}[n_1]_{q_{n_1}}x)-x\bigg)^2+\bigg(\frac{1}{[n_2+2]_{q_{n_2}}}(1+q_{n_2}[n_2]_{q_{n_2}}y)-y\bigg)^2}\Bigg),$$

where

$$A_{n_1,n_2}^{\alpha_{n_1},\alpha_{n_2}}(q_{n_1},q_{n_2},x,y)$$
$$=\bigg\{(\delta_{n_1,q_{n_1}}^{\alpha_{n_1}}(x))^2+\bigg(\frac{1}{[n_1+2]_{q_{n_1}}}(1+q_{n_1}[n_1]_{q_{n_1}}x)-x\bigg)^2$$
$$+(\delta_{n_2,q_{n_2}}^{\alpha_{n_2}}(y))^2+\bigg(\frac{1}{[n_2+2]_{q_{n_2}}}(1+q_{n_2}[n_2]_{q_{n_2}}y)-y\bigg)^2\bigg\},$$

and the constant $M>0$, is independent of f and $A_{n_1,n_2}^{\alpha_{n_1},\alpha_{n_2}}(q_{n_1},q_{n_2},x,y)$.

Proof. We introduce the auxiliary operators as follows:

$$D^{*\alpha_{n_1},\alpha_{n_2}}_{n_1,n_2}(f;q_{n_1},q_{n_2},x,y)=D^{\alpha_{n_1},\alpha_{n_2}}_{n_1,n_2}(f;q_{n_1},q_{n_2},x,y)+f(x,y)$$
$$-f\left(\frac{1}{[n_1+2]_{q_{n_1}}}(1+q_{n_1}[n_1]_{q_{n_1}}x),\frac{1}{[n_2+2]_{q_{n_2}}}(1+q_{n_2}[n_2]_{q_{n_2}}y)\right). \qquad (8.3.5)$$

Then using Lemma 8.2.2, we have

$$D^{*\alpha_{n_1},\alpha_{n_2}}_{n_1,n_2}(1;q_{n_1},q_{n_2},x,y)=1,\quad D^{*\alpha_{n_1},\alpha_{n_2}}_{n_1,n_2}(t-x;q_{n_1},q_{n_2},x,y)=0$$
$$D^{*\alpha_{n_1},\alpha_{n_2}}_{n_1,n_2}(s-y;q_{n_1},q_{n_2},x,y)=0. \qquad (8.3.6)$$

Next, using Lemma 8.2.1

$$|D^{*\alpha_{n_1},\alpha_{n_2}}_{n_1,n_2}(f;q_{n_1},q_{n_2},x,y)|\leq|D^{\alpha_{n_1},\alpha_{n_2}}_{n_1,n_2}(f;q_{n_1},q_{n_2},x,y)|$$
$$+\left|f\left(\frac{1}{[n_1+2]_{q_{n_1}}}(1+q_{n_1}[n_1]_{q_{n_1}}x),\frac{1}{[n_2+2]_{q_{n_2}}}(1+q_{n_2}[n_2]_{q_{n_2}}y)\right)\right|+|f(x,y)|$$
$$\leq 3||f||. \qquad (8.3.7)$$

Let $g\in C^2(I^2)$ and $t,s\in I$. Using the Taylor's theorem, we may write

$$\begin{aligned} g(t,s)-g(x,y) &= \frac{\partial g(x,y)}{\partial x}(t-x)+\int_x^t(t-u)\frac{\partial^2 g(u,y)}{\partial u^2}du \\ &+ \frac{\partial g(x,y)}{\partial y}(s-y)+\int_y^s(s-v)\frac{\partial^2 g(t,v)}{\partial v^2}dv. \end{aligned}$$

Applying the operator $D^{*\alpha_{n_1},\alpha_{n_2}}_{n_1,n_2}(f;q_{n_1},q_{n_2},x,y)$ on both sides of the above equation and using (8.3.6) and relation (8.3.5), we get

$$\begin{aligned} D^{*\alpha_{n_1},\alpha_{n_2}}_{n_1,n_2}(g;q_{n_1},q_{n_2},x,y)-g(x,y) &= D^{*\alpha_{n_1},\alpha_{n_2}}_{n_1,n_2}\left(\int_x^t(t-u)\frac{\partial^2 g(u,y)}{\partial u^2}du;q_{n_1},q_{n_2},x,y\right) \\ &+D^{*\alpha_{n_1},\alpha_{n_2}}_{n_1,n_2}\left(\int_y^s(s-v)\frac{\partial^2 g(x,v)}{\partial v^2}dv;q_{n_1},q_{n_2},x,y\right) \\ &= D^{\alpha_{n_1},\alpha_{n_2}}_{n_1,n_2}\left(\int_x^t(t-u)\frac{\partial^2 g(u,y)}{\partial u^2}du;q_{n_1},q_{n_2},x,y\right) \\ -\int_x^{\frac{1}{[n_1+2]_{q_{n_1}}}(1+q_{n_1}[n_1]_{q_{n_1}}x)}&\left(\frac{1}{[n_1+2]_{q_{n_1}}}(1+q_{n_1}[n_1]_{q_{n_1}}x)-u\right)\frac{\partial^2 g(u,y)}{\partial u^2}du \\ &+D^{\alpha_{n_1},\alpha_{n_2}}_{n_1,n_2}\left(\int_y^s(s-v)\frac{\partial^2 g(x,v)}{\partial v^2}dv;q_{n_1},q_{n_2},x,y\right) \\ -\int_y^{\frac{1}{[n_2+2]_{q_{n_2}}}(1+q_{n_2}[n_2]_{q_{n_2}}y)}&\left(\frac{1}{[n_2+2]_{q_{n_2}}}(1+q_{n_2}[n_2]_{q_{n_2}}y)-v\right)\frac{\partial^2 g(x,v)}{\partial v^2}dv. \end{aligned}$$

Hence using (8.3.1), we get

$$
\begin{aligned}
&|D^{*\alpha_{n_1},\alpha_{n_2}}_{n_1,n_2}(g;q_{n_1},q_{n_2},x,y)-g(x,y)|\\
&\leq D^{\alpha_{n_1},\alpha_{n_2}}_{n_1,n_2}\left(\left|\int_x^t |t-u|\left|\frac{\partial^2 g(u,y)}{\partial u^2}\right|du\right|;x,y\right)\\
&+\left|\int_x^{\frac{1}{[n_1+2]_{q_{n_1}}}(1+q_{n_1}[n_1]_{q_{n_1}}x)}\left|\frac{1}{[n_1+2]_{q_{n_1}}}(1+q_{n_1}[n_1]_{q_{n_1}}x)-u\right|\left|\frac{\partial^2 g(u,y)}{\partial u^2}\right|du\right|\\
&+D^{\alpha_{n_1},\alpha_{n_2}}_{n_1,n_2}\left(\left|\int_y^s |s-v|\left|\frac{\partial^2 g(x,v)}{\partial v^2}\right|dv\right|;x,y\right)\\
&+\left|\int_y^{\frac{1}{[n_2+2]_{q_{n_2}}}(1+q_{n_2}[n_2]_{q_{n_2}}y)}\left|\frac{1}{[n_2+2]_{q_{n_2}}}(1+q_{n_2}[n_2]_{q_{n_2}}y)-v\right|\left|\frac{\partial^2 g(x,v)}{\partial v^2}\right|dv\right|\\
&\leq\left\{D^{\alpha_{n_1},\alpha_{n_2}}_{n_1,n_2}((t-x)^2;q_{n_1},q_{n_2},x,y)+\left(\frac{1}{[n_1+2]_{q_{n_1}}}(1+q_{n_1}[n_1]_{q_{n_1}}x)-x\right)^2\right\}||g||_{C^2(I^2)}\\
&+\left\{D^{\alpha_{n_1},\alpha_{n_2}}_{n_1,n_2}((s-y)^2;q_{n_1},q_{n_2},x,y)+\left(\frac{1}{[n_2+2]_{q_{n_2}}}(1+q_{n_2}[n_2]_{q_{n_2}}y)-y\right)^2\right\}||g||_{C^2(I^2)}\\
&\leq\left\{(\delta^{\alpha_{n_1}}_{n_1,q_{n_1}}(x))^2+\left(\frac{1}{[n_1+2]_{q_{n_1}}}(1+q_{n_1}[n_1]_{q_{n_1}}x)-x\right)^2\right\}||g||_{C^2(I^2)}\\
&+\left\{(\delta^{\alpha_{n_2}}_{n_2,q_{n_2}}(y))^2+\left(\frac{1}{[n_2+2]_{q_{n_2}}}(1+q_{n_2}[n_2]_{q_{n_2}}y)-y\right)^2\right\}||g||_{C^2(I^2)}\\
&=A^{\alpha_{n_1},\alpha_{n_2}}_{n_1,n_2}(q_{n_1},q_{n_2},x,y)||g||_{C^2(I^2)}. \qquad (8.3.8)
\end{aligned}
$$

Hence in view of (8.3.7) and (8.3.8), we get

$$
\begin{aligned}
&|D^{\alpha_{n_1},\alpha_{n_2}}_{n_1,n_2}(f;q_{n_1},q_{n_2},x,y)-f(x,y)|=\Bigg|D^{*\alpha_{n_1},\alpha_{n_2}}_{n_1,n_2}(f;q_{n_1},q_{n_2},x,y)-f(x,y)\\
&+f\left(\frac{1}{[n_1+2]_{q_{n_1}}}(1+q_{n_1}[n_1]_{q_{n_1}}x),\frac{1}{[n_2+2]_{q_{n_2}}}(1+q_{n_2}[n_2]_{q_{n_2}}y)\right)-f(x,y)\Bigg|\\
&\leq|D^{*\alpha_{n_1},\alpha_{n_2}}_{n_1,n_2}(f-g;q_{n_1},q_{n_2},x,y)|+|D^{*\alpha_{n_1},\alpha_{n_2}}_{n_1,n_2}(g;q_{n_1},q_{n_2},x,y)-g(x,y)|\\
&+|g(x,y)-f(x,y)|\\
&+\left|f\left(\frac{1}{[n_1+2]_{q_{n_1}}}(1+q_{n_1}[n_1]_{q_{n_1}}x),\frac{1}{[n_2+2]_{q_{n_2}}}(1+q_{n_2}[n_2]_{q_{n_2}}y)\right)-f(x,y)\right|\\
&\leq 4||f-g||+A^{\alpha_{n_1},\alpha_{n_2}}_{n_1,n_2}(q_{n_1},q_{n_2},x,y)||g||_{C^2(I^2)}\\
&+\overline{\omega}\left(f;\sqrt{\left(\frac{1}{[n_1+2]_{q_{n_1}}}(1+q_{n_1}[n_1]_{q_{n_1}}x)-x\right)^2+\left(\frac{1}{[n_2+2]_{q_{n_2}}}(1+q_{n_2}[n_2]_{q_{n_2}}y)-y\right)^2}\right).
\end{aligned}
$$

Taking the infimum on the right hand side over all $g \in C^2(I^2)$, it follows that

$$
\begin{aligned}
&|D_{n_1,n_2}^{\alpha_{n_1},\alpha_{n_2}}(f;q_{n_1},q_{n_2},x,y)-f(x,y)| \leq 4\mathcal{K}(f;A_{n_1,n_2}^{\alpha_{n_1},\alpha_{n_2}}(q_{n_1},q_{n_2},x,y)) \\
&+\overline{\omega}\left(f;\sqrt{\left(\frac{1}{[n_1+2]_{q_{n_1}}}(1+q_{n_1}[n_1]_{q_{n_1}}x)-x\right)^2+\left(\frac{1}{[n_2+2]_{q_{n_2}}}(1+q_{n_2}[n_2]_{q_{n_2}}y)-y\right)^2}\right) \\
&\leq M\left\{\overline{\omega}_2\left(f;\sqrt{A_{n_1,n_2}^{\alpha_{n_1},\alpha_{n_2}}(q_{n_1},q_{n_2},x,y)}\right)+\min\{1,A_{n_1,n_2}^{\alpha_{n_1},\alpha_{n_2}}(q_{n_1},q_{n_2},x,y)\}||f||_{C(I^2)}\right\} \\
&+\overline{\omega}\left(f;\sqrt{\left(\frac{1}{[n_1+2]_{q_{n_1}}}(1+q_{n_1}[n_1]_{q_{n_1}}x)-x\right)^2+\left(\frac{1}{[n_2+2]_{q_{n_2}}}(1+q_{n_2}[n_2]_{q_{n_2}}y)-y\right)^2}\right),
\end{aligned}
$$

which is the desired conclusion. □

8.4 Voronovskaya type theorem

In this section we shall establish a Voronovskaya type theorem for the operators $D_{n_1,n_2}^{\alpha_{n_1},\alpha_{n_2}}$. First, we need an auxiliary result contained in the following lemma.

Lemma 8.4.1. *Assume that $0 < q_n < 1$, $q_n \to 1$ and $q_n^n \to a$, $a \in [0,1)$ as $n \to \infty$. If $\lim_{n\to\infty} \alpha_n = 0$ and $\lim_{n\to\infty} n\alpha_n = l \in \mathbb{R}$, then*

$$
\begin{aligned}
&\lim_{n\to\infty}[n]_{q_n} D_n^{\alpha_n}(t-x;q_n;x) = 1-(1+a)x, \\
&\lim_{n\to\infty}[n]_{q_n} D_n^{\alpha_n}\left((t-x)^2;q_n;x\right) = (l+2)x(1-x), \qquad (8.4.1) \\
&\lim_{n\to\infty}[n]_{q_n}^2 D_n^{\alpha_n}\left((t-x)^4;q_n;x\right) = 3x^2(1-x)^2 l(l+4)+x^2(1-x)(7x^2-7x+5). \qquad (8.4.2)
\end{aligned}
$$

Proof. Using Lemma 7.2.2, we get

$$
\lim_{n\to\infty}[n]_{q_n} D_n^{\alpha_n}(t-x;q_n;x) = \frac{[n]_{q_n}}{[n+2]_{q_n}} - \frac{[n]_{q_n}}{[n+2]_{q_n}}(1+q_n^{n+1}) = 1-(1+a)x,
$$

and

$$\lim_{n\to\infty}[n]_{q_n} D_n^{\alpha_n}\left((t-x)^2;q_n;x\right) = (2+l)x - x^2 + \lim_{n\to\infty}\frac{[n]_{q_n}}{[n+2]_{q_n}[n+3]_{q_n}(1+\alpha_n)}$$
$$\times\{q^3[n]_{q_n}^2 - 2q_n[n]_{q_n}[n+2]_{q_n}[n+3]_{q_n} + [n+2]_{q_n}[n+3]_{q_n}$$
$$+[n+3]_{q_n}\alpha_n([n+2]_{q_n} - 2q_n[n]_{q_n})\}x^2$$
$$= (l+2)x - (l+1)x^2 + \lim_{n\to\infty}\frac{[n]_{q_n}}{[n+2]_{q_n}[n+3]_{q_n}(1+\alpha_n)}$$
$$\left\{q_n^3(q_n^n)^2 + 2q_n(1+q_n)q_n^n - q_n^2[n]_{q_n} + 1\right\}x^2$$
$$= (l+2)x(1-x).$$

The relation (8.4.2) is obtained in a similar way using Lemma 7.2.2 of the previous chapter and Lemma 3.2 ([121]). □

The main result of this section is the following Voronovskaja type theorem:

Theorem 8.4.2. *Let $f \in C^2(I^2)$ and $(q_n)_n$ be a sequence in the interval $(0,1)$ such that $q_n \to 1$ and $q_n^n \to a$, $a \in [0,1)$ as $n \to \infty$. If $\lim_{n\to\infty}\alpha_n = 0$ and $\lim_{n\to\infty} n\alpha_n = l \in \mathbb{R}$, then for every $(x_0,y_0) \in I^2$, we have*

$$\lim_{n\to\infty}[n]_{q_n}\left\{D_{n,n}^{\alpha_n,\alpha_n}(f;q_n,q_n,x_0,y_0) - f(x_0,y_0)\right\} = [1-(a+1)x_0]f'_x(x_0,y_0)$$
$$+[1-(a+1)y_0]f'_y(x_0,y_0) + \frac{l+2}{2}\left\{x_0(1-x_0)f''_{xx}(x_0,y_0) + y_0(1-y_0)f''_{yy}(x_0,y_))\right\}.$$

Proof. Let $(x_0,y_0) \in I^2$ be a fixed point. By the Taylor formula, it follows that

$$f(t,s) = f(x_0,y_0) + f'_x(x_0,y_0)(t-x_0) + f'_y(x_0,y_0)(s-y_0)$$
$$+\frac{1}{2}\left\{f''_{xx}(x_0,y_0)(t-x_0)^2 + 2f''_{xy}(x_0,y_0)(t-x_0)(s-y_0) + f''_{yy}(x_0,y_0)(s-y_0)^2\right\}$$
$$+\varphi(t,s,x_0,y_0)\left((t-x_0)^2 + (s-y_0)^2\right),$$

where $(t,s) \in I^2$ and $\lim_{(t,s)\to(x_0,y_0)}\varphi(s,t,x_0,y_0) = 0$. From the linearity of $D_{n,n}^{\alpha_n,\alpha_n}$, we have

$$D_{n,n}^{\alpha_n,\alpha_n}\left(f(t,s);q_n,q_n,x_0,y_0\right) = f(x_0,y_0) + f'_x(x_0,y_0)D_{n,n}^{\alpha_n,\alpha_n}(t-x_0;q_n,q_n,x_0,y_0)$$
$$+f'_y(x_0,y_0)D_{n,n}^{\alpha_n,\alpha_n}(s-y_0;q_n,q_n,x_0,y_0) + \frac{1}{2}\left\{f''_{xx}(x_0,y_0)D_{n,n}^{\alpha_n,\alpha_n}((t-x_0)^2;q_n,q_n,x_0,y_0)\right.$$
$$+2f''_{xy}(x_0,y_0)D_{n,n}^{\alpha_n,\alpha_n}((t-x_0)(s-y_0);q_n,q_n,x_0,y_0)$$
$$\left.+\ f''_{yy}(x_0,y_0)D_{n,n}^{\alpha_n,\alpha_n}((s-y_0)^2;q_n,q_n,x_0,y_0)\right\}$$
$$+D_{n,n}^{\alpha_n,\alpha_n}\left(\varphi(t,s)\left((t-x_0)^2+(s-y_0)^2\right);q_n,q_n,x_0,y_0\right)$$
$$= f(x_0,y_0) + f'_x(x_0,y_0)D_n^{\alpha_n}(t-x_0;q_n,x_0) + f'_y(x_0,y_0)D_n^{\alpha_n}(s-y_0;q_n,y_0)$$
$$+\frac{1}{2}\left\{f''_{xx}(x_0,y_0)D_n^{\alpha_n}((t-x_0)^2;q_n,x_0) + f''_{yy}(x_0,y_0)D_n^{\alpha_n}((s-y_0)^2;q_n,y_0)\right.$$
$$\left.+2f''_{xy}(x_0,y_0)D_n^{\alpha_n}((t-x_0);q_n,x_0)D_n^{\alpha_n}((s-y_0);q_n,y_0)\right\}$$
$$+D_{n,n}^{\alpha_n,\alpha_n}\left(\varphi(s,t)\left((t-x_0)^2+(s-y_0)^2\right);q_n,q_n,x_0,y_0\right). \tag{8.4.3}$$

Applying Cauchy-Schwarz inequality, we have

$$\left|D_{n,n}^{\alpha_n,\alpha_n}\left(\varphi(t,s,x_0,y_0)\left((t-x_0)^2+(s-y_0)^2\right);q_n,q_n,x_0,y_0\right)\right|$$
$$\leq\left|D_{n,n}^{\alpha_n,\alpha_n}\left(\varphi(t,s,x_0,y_0)(t-x_0)^2;q_n,q_n,x_0,y_0\right)\right|$$
$$+\left|D_{n,n}^{\alpha_n,\alpha_n}\left(\varphi(t,s,x_0,y_0)(s-y_0)^2;q_n,q_n,x_0,y_0\right)\right|$$
$$\leq\left\{D_{n,n}^{\alpha_n,\alpha_n}\left(\varphi^2(t,s,x_0,y_0);q_n,q_n,x_0,y_0\right)\right\}^{1/2}\left\{D_{n,n}^{\alpha_n,\alpha_n}\left((t-x_0)^4;q_n,q_n,x_0,y_0\right)\right\}^{1/2}$$
$$+\left\{D_{n,n}^{\alpha_n,\alpha_n}\left(\varphi^2(t,s,x_0,y_0)^2;q_n,q_n,x_0,y_0\right)\right\}^{1/2}\left\{D_{n,n}^{\alpha_n,\alpha_n}\left((s-y_0)^4;q_n,q_n,x_0,y_0\right)\right\}^{1/2}$$
$$=\left\{D_{n,n}^{\alpha_n,\alpha_n}\left(\varphi^2(t,s,x_0,y_0);q_n,q_n,x_0,y_0\right)\right\}^{1/2}\left[\left\{D_{n,n}^{\alpha_n,\alpha_n}\left((t-x_0)^4;q_n,q_n,x_0,y_0\right)\right\}^{1/2}\right.$$
$$\left.+\left\{D_{n,n}^{\alpha_n,\alpha_n}\left((s-y_0)^4;q_n,q_n,x_0,y_0\right)\right\}^{1/2}\right]$$

Using Theorem 8.3.2, we get

$$\lim_{n\to\infty} D_{n,n}^{\alpha_n,\alpha_n}\left(\varphi^2(t,s,x_0,y_0);q_n,q_n,x_0,y_0\right) = \varphi^2(x_0,y_0) = 0,$$

and hence using Lemma 8.4.1 we have

$$\lim_{n\to\infty}[n]_{q_n} D_{n,n}^{\alpha_n,\alpha_n}\left(\varphi(t,s)\left((t-x_0)^2+(s-y_0)^2\right);q_n,q_n,x_0,y_0\right) = 0.$$

Finally applying Lemma 8.4.1 in (8.4.3), the theorem is proved. □

8.5 GBS generalization

8.5.1 Introduction

In [30] and [31], Bogel introduced a new concept of Bögel-continuous (B-continuous) and Bögel-differentiable (B-differentiable) functions and also established some important theorems using these concepts.
We give some basic definitions and notations, for further details, one can refer to [32].

A function $f : X \times Y \to R$ is called B-continuous at $(x_0, y_0) \in X \times Y$, where X and Y are compact subsets of $\mathbb{R}$ if

$$\lim_{(x,y)\to(x_0,y_0)} \Delta f[(x, y); (x_0, y_0)] = 0,$$

where, $\Delta f[(x, y); (x_0, y_0)] = f(x, y)-f(x_0, y)-f(x, y_0)+f(x_0, y_0)$ denotes the mixed difference of f.

A function $f : X \times Y \to \mathbb{R}$ is called B-differentiable at $(x_0, y_0) \in X \times Y$ if,

$$\lim_{(x,y)\to(x_0,y_0)} \frac{\Delta f[(x, y); (x_0, y_0)]}{(x - x_0)(y - y_0)}$$

exists and is finite. This limit is named as B-differential of f at the point (x_0, y_0) and is denoted by $D_b f(x_0, y_0)$.

The function $f : A \subset X \times Y \to \mathbb{R}$ is called B-bounded on A if there exists $M > 0$ such that $|\Delta f [t, s; x, y]| \leq M$, for every $(x, y),(t, s) \in A$. Here, if A is a compact subset of $\mathbb{R}^2$ then each B-continuous function is a B-bounded function on A.

Throughout this chapter, we denote by $C_b (A)$ and $D_b (A)$, the space of all B-continuous and B-differentiable functions on A respectively and $B_b (A)$ denote the class of B-bounded functions on A endowed with the norm

$$\|f\|_B = \sup_{(x,y),(t,s)\in A} |\Delta f [t, s; x, y]| .$$

It is known that $C (A) \subset C_b (A)$ ([32], page 52). Badea et al. [21] proved the following Korovkin-type theorem in order to approximate B-continuous functions by using GBS-operators.

Theorem 8.5.1. *Let (L_{n_1,n_2}), $L_{n_1,n_2} : C_b(A) \to B(A)$, $n_1, n_2 \in \mathbb{N}$ be a sequence of bivariate linear positive operators, G_{n_1,n_2} be the GBS-operators associated to L_{n_1,n_2} and the following conditions are satisfied:*

(i) $L_{n_1,n_2}(e_{00}; x, y) = 1$

(ii) $L_{n_1,n_2}(e_{10}; x, y) = x + u_{n_1,n_2}(x, y)$

(iii) $L_{n_1,n_2}(e_{01}; x, y) = y + v_{n_1,n_2}(x, y)$

(iv) $L_{n_1,n_2}(e_{20} + e_{02}; x, y) = x^2 + y^2 + w_{n_1,n_2}(x, y)$

for all $(x, y) \in A$. If the sequences (u_{n_1,n_2}), (v_{n_1,n_2}) and (w_{n_1,n_2}) converge to zero uniformly on A, then the sequence $(G_{n_1,n_2} f)$ converge to f uniformly on A for all $f \in C_b(A)$.

8.5.2 Construction of operators

We define the GBS operator of the operator $D^{\alpha_{n_1},\alpha_{n_2}}_{n_1,n_2}$ given by (8.1.1) for any $f \in C_b(I^2)$ and $n_1, n_2 \in \mathbb{N}$, by

$$G^{\alpha_{n_1},\alpha_{n_2}}_{n_1,n_2}(f; q_{n_1}, q_{n_2}, x, y) = D^{\alpha_{n_1},\alpha_{n_2}}_{n_1,n_2}(f(t, y) + f(x, s) - f(t, s); q_{n_1}, q_{n_2}, x, y),$$

for all $(x, y) \in I^2$. More precisely for any $f \in C_b(I^2)$, the GBS operators of q-Stancu-Durrmeyer type operators is given by

$$G^{\alpha_{n_1},\alpha_{n_2}}_{n_1,n_2}(f; q_{n_1}, q_{n_2}, x, y) = [n_1 + 1]_{q_{n_1}} [n_2 + 1]_{q_{n_2}} \sum_{k_1=0}^{n_1} \sum_{k_2=0}^{n_2} p^{q_{n_1},\alpha_{n_1}}_{n_1,k_1}(x) p^{q_{n_2},\alpha_{n_2}}_{n_2,k_2}(y)$$
$$\int_0^1 \int_0^1 p^{q_{n_1}}_{n_1,k_1}(t) p^{q_{n_2}}_{n_2,k_2}(s)[f(t, y) + f(x, s) - f(t, s)] d_{q_{n_1}} t d_{q_{n_2}} s. \tag{8.5.1}$$

Evidently, the operators $G^{\alpha_{n_1},\alpha_{n_2}}_{n_1,n_2}$ are linear operators.

8.5.3 Approximation theorems

We define the mixed modulus of smoothness for $f \in C_b(I^2)$ as follows:

$$\omega_{mixed}(f;\delta_1,\delta_2) := \sup\{|\Delta f[t,s;x,y]| : |t-x| < \delta_1, |s-y| < \delta_2\},$$

for all $(x,y),(t,s) \in I^2$ and for any $(\delta_1,\delta_2) \in (0,\infty)\times(0,\infty)$ with $\omega_{mixed} : [0,\infty)\times[0,\infty) \to \mathbb{R}$.

The basic properties of ω_{mixed} were obtained by Badea et al. in [23] and [22] which are similar to the properties of the usual modulus of continuity.

Badea et al. [22] established the following Shisha-Mond type theorem to obtain the degree of approximation for B-continuous functions using GBS operators:

Theorem 8.5.2. *Let $L : C(A) \to C(A)$ be a bivariate linear positive operator and $G : C_b(A) \to C(A)$ be the associated GBS-operator. The following inequality*

$$\begin{aligned}|G(f;x,y)-f(x,y)| &\leq |f(x,y)||L(1;x,y)-1| + \{L(1;x,y) + \\ &\quad \delta_1^{-1}\sqrt{L((t-x)^2;x,y)} + \delta_2^{-1}\sqrt{L((s-y)^2;x,y)} \\ &+ \delta_1^{-1}\sqrt{L((t-x)^2;x,y)}\delta_2^{-1}\sqrt{L((s-y)^2;x,y)}\}\omega_{mixed}(f;\delta_1,\delta_2),\end{aligned}$$

holds for all $f \in C_b(A)$, $(x,y) \in A$ and $\delta_1,\delta_2 > 0$.

Theorem 8.5.3. *For every $f \in C_b(I^2)$ and $(x,y) \in I^2$, we have*

$$|G_{n_1,n_2}^{\alpha_{n_1},\alpha_{n_2}}(f;q_{n_1},q_{n_2},x,y) - f(x,y)| \leq 4\omega_{mixed}(f,\delta_{n_1,q_{n_1}}^{\alpha_{n_1}}(x),\delta_{n_2,q_{n_2}}^{\alpha_{n_2}}(y)).$$

Proof. Applying Theorem 8.5.2, we have

$$\begin{aligned}&|G_{n_1,n_2}^{\alpha_{n_1},\alpha_{n_2}}(f;q_{n_1},q_{n_2},x,y) - f(x,y)| \\ &\leq |f(x,y)||D_{n_1,n_2}^{\alpha_{n_1},\alpha_{n_2}}(1;q_{n_1},q_{n_2},x,y)-1| + \{D_{n_1,n_2}^{\alpha_{n_1},\alpha_{n_2}}(1;q_{n_1},q_{n_2},x,y) \\ &+\delta_1^{-1}\sqrt{D_{n_1,n_2}^{\alpha_{n_1},\alpha_{n_2}}((t-x)^2;q_{n_1},q_{n_2},x,y)} + \delta_2^{-1}\sqrt{D_{n_1,n_2}^{\alpha_{n_1},\alpha_{n_2}}((s-y)^2;q_{n_1},q_{n_2},x,y)} \\ &+\delta_1^{-1}\delta_2^{-1}\sqrt{D_{n_1,n_2}^{\alpha_{n_1},\alpha_{n_2}}((t-x)^2;q_{n_1},q_{n_2},x,y)}\sqrt{D_{n_1,n_2}^{\alpha_{n_1},\alpha_{n_2}}((s-y)^2;q_{n_1},q_{n_2},x,y)}\}. \\ &\ \omega_{mixed}(f;\delta_1,\delta_2),\end{aligned}$$

Applying Lemmas 8.2.1 and 8.2.2, on choosing $\delta_1 = \delta^{\alpha_{n_1}}_{n_1,q_{n_1}}(x)$ and $\delta_2 = \delta^{\alpha_{n_2}}_{n_2,q_{n_2}}(y)$, we obtain the desired result. $\square$

Theorem 8.5.4. *Let $f \in D_b(I^2)$ with $D_B f \in B(I^2)$. Then, for each $(x,y) \in I$, we have*

$$|G^{\alpha_{n_1},\alpha_{n_2}}_{n_1,n_2} f; q_{n_1}, q_{n_2}, x, y - f(x,y)|$$
$$\leq \frac{M}{[n_1]^{1/2}_{q_{n_1}}[n_2]^{1/2}_{q_{n_2}}}\Bigg(||D_B f||_\infty + \omega_{mixed}(D_B f; [n_1]^{-1/2}_{q_{n_1}}, [n_2]^{-1/2}_{q_{n_2}})\Bigg).$$

Proof. Since $f \in D_b(I^2)$ and $D_B f \in B(I^2)$, then from

$$D_B f(x,y) = \lim_{(x,y)\to(x_0,y_0)} \frac{\Delta f[(t,s);(x,y)]}{(t-x)(s-y)}$$

it follows that $\Delta f[t,s;x,y] = (t-x)(s-y)D_B f(\xi,\eta)$, where ξ, η lie between t and x and s and y respectively.

Since $D_B f \in B(I^2)$, using the following relation

$$D_B f(\xi,\eta) = \Delta D_B f(\xi,\eta) + D_B f(\xi,y) + D_B f(x,\eta) - D_B f(x,y),$$

we obtain

$$|D^{\alpha_{n_1},\alpha_{n_2}}_{n_1,n_2}(\Delta f[t,s;x,y]; q_{n_1}, q_{n_2}, x, y)|$$
$$= |D^{\alpha_{n_1},\alpha_{n_2}}_{n_1,n_2}((t-x)(s-y)D_B f(\xi,\eta); q_{n_1}, q_{n_2}, x, y)|$$
$$\leq D^{\alpha_{n_1},\alpha_{n_2}}_{n_1,n_2}(|t-x||s-y||\Delta D_B f(\xi,\eta)|; q_{n_1}, q_{n_2}, x, y)$$
$$+D^{\alpha_{n_1},\alpha_{n_2}}_{n_1,n_2}(|t-x||s-y|(|D_B f(\xi,y)|$$
$$+|D_B f(x,\eta)| + |D_B f(x,y)|); q_{n_1}, q_{n_2}, x, y)$$
$$\leq D^{\alpha_{n_1},\alpha_{n_2}}_{n_1,n_2}(|t-x||s-y|\omega_{mixed}(D_B f; |\xi-x|, |\eta-y|); q_{n_1}, q_{n_2}, x, y)$$
$$+3\ ||D_B f||_\infty\ D^{\alpha_{n_1},\alpha_{n_2}}_{n_1,n_2}(|t-x||s-y|; q_{n_1}, q_{n_2}, x, y).$$

Using the basic properties of ω_{mixed}, we have

$$\omega_{mixed}(D_B f; |\xi-x|, |\eta-y|) \leq \omega_{mixed}(D_B f; |t-x|, |s-y|)$$
$$\leq (1 + \delta^{-1}_{n_1}|t-x|)(1 + \delta^{-1}_{n_2}|s-y|)\ \omega_{mixed}(D_B f; \delta_{n_1}, \delta_{n_2}).$$

Hence applying the Cauchy-Schwarz inequality, we get

$$
\begin{aligned}
&|G_{n_1,n_2}^{\alpha_{n_1},\alpha_{n_2}} f; q_{n_1}, q_{n_2}, x, y - f(x,y)| = |D_{n_1,n_2}^{\alpha_{n_1},\alpha_{n_2}} \Delta f[t,s;x,y]; q_{n_1}, q_{n_2}, x, y| \\
&\leq 3||D_B f||_\infty \sqrt{D_{n_1,n_2}^{\alpha_{n_1},\alpha_{n_2}}((t-x)^2(s-y)^2; q_{n_1}, q_{n_2}, x, y)} \\
&+\Bigg(D_{n_1,n_2}^{\alpha_{n_1},\alpha_{n_2}}(|t-x||s-y|; q_{n_1}, q_{n_2}, x, y) \\
&+\delta_{n_1}^{-1} D_{n_1,n_2}^{\alpha_{n_1},\alpha_{n_2}}((t-x)^2|s-y|; q_{n_1}, q_{n_2}, x, y) \\
&+\delta_{n_2}^{-1} D_{n_1,n_2}^{\alpha_{n_1},\alpha_{n_2}}(|t-x|(s-y)^2; q_{n_1}, q_{n_2}, x, y) \\
&+\delta_{n_1}^{-1}\delta_{n_2}^{-1} D_{n_1,n_2}^{\alpha_{n_1},\alpha_{n_2}}((t-x)^2(s-y)^2; q_{n_1}, q_{n_2}, x, y)\Bigg)\omega_{mixed}(D_B f; \delta_{n_1}, \delta_{n_2}) \\
&\leq 3||D_B f||_\infty \sqrt{D_{n_1,n_2}^{\alpha_{n_1},\alpha_{n_2}}((t-x)^2(s-y)^2; q_{n_1}, q_{n_2}, x, y)} \\
&+\Bigg(\sqrt{D_{n_1,n_2}^{\alpha_{n_1},\alpha_{n_2}}((t-x)^2(s-y)^2; q_{n_1}, q_{n_2}, x, y)} \\
&+\delta_{n_1}^{-1} \sqrt{D_{n_1,n_2}^{\alpha_{n_1},\alpha_{n_2}}((t-x)^4(s-y)^2; q_{n_1}, q_{n_2}, x, y)} \\
&+\delta_{n_2}^{-1} \sqrt{D_{n_1,n_2}^{\alpha_{n_1},\alpha_{n_2}}((t-x)^2(s-y)^4; q_{n_1}, q_{n_2}, x, y)} \\
&+\delta_{n_1}^{-1}\delta_{n_2}^{-1} D_{n_1,n_2}^{\alpha_{n_1},\alpha_{n_2}}((t-x)^2(s-y)^2; q_{n_1}, q_{n_2}, x, y)\Bigg)\omega_{mixed}(D_B f; \delta_{n_1}, \delta_{n_2}).
\end{aligned}
$$

We observe that for $(x,y),(t,s) \in I$ and $i,j \in \{1,2\}$

$$D_{n_1,n_2}^{\alpha_{n_1},\alpha_{n_2}}((t-x)^{2i}(s-y)^{2j}; q_m, q_n, x, y) = D_{n_1}^{\alpha_1}((t-x)^{2i}; q_m, x, y) D_{n_2}^{\alpha_2}((s-y)^{2j}; q_n, x, y).$$

Now taking $\delta_{n_1} = \delta_{n_1,q_{n_1}}^{\alpha_{n_1}}(x) \leq \dfrac{C_1}{[n_1]_{q_{n_1}}^{1/2}}$, $\delta_{n_2} = \delta_{n_2,q_{n_2}}^{\alpha_{n_2}}(y) \leq \dfrac{C_2}{[n_2]_{q_{n_2}}^{1/2}}$, and using (8.4.2), we obtain the desired result. □

Let $\beta, \gamma \in (0,1]$, then

$$Lip_M(\beta,\gamma) = \left\{ f \in C_b(I^2) : |\Delta f[t,s;x,y]| \leq M |t-x|^\beta |s-y|^\gamma, \text{for } (t,s),(x,y) \in I^2 \right\}$$

is the Lipschitz class for B-continuous functions. Our next theorem gives the rate of convergence for the operators $G_{n_1,n_2}^{\alpha_{n_1},\alpha_{n_2}}$ by means of the class $Lip_M(\beta,\gamma)$.

Theorem 8.5.5. *Let $f \in Lip_M(\beta,\gamma)$ and $(x,y) \in I^2$. Then for $M > 0$, $\beta, \gamma \in (0,1]$, we have*

$$|G_{n_1,n_2}^{\alpha_{n_1},\alpha_{n_2}}(f; q_{n_1}, q_{n_2}, x, y) - f(x,y)| \leq M \left(\delta_{n_1,q_{n_1}}^{\alpha_{n_1}}(x)\right)^\beta \left(\delta_{n_2,q_{n_2}}^{\alpha_{n_2}}(y)\right)^\gamma.$$

Proof. By the definition of $G_{n_1,n_2}^{\alpha_{n_1},\alpha_{n_2}}$, we may write

$$\begin{aligned}
&G_{n_1,n_2}^{\alpha_{n_1},\alpha_{n_2}}(f;q_{n_1},q_{n_2},x,y)\\
&= D_{n_1,n_2}^{\alpha_{n_1},\alpha_{n_2}}(f(x,s)+f(t,y)-f(t,s);q_{n_1},q_{n_2},x,y)\\
&= D_{n_1,n_2}^{\alpha_{n_1},\alpha_{n_2}}(f(x,y)-\Delta f[t,s;x,y];q_{n_1},q_{n_2},x,y)\\
&= f(x,y)D_{n_1,n_2}^{\alpha_{n_1},\alpha_{n_2}}(e_{00};q_{n_1},q_{n_2},x,y)-D_{n_1,n_2}^{\alpha_{n_1},\alpha_{n_2}}(\Delta f[t,s;x,y];q_{n_1},q_{n_2},x,y).
\end{aligned}$$

Hence, using Lemma 8.2.1 and the definition of the Lipschitz class

$$\begin{aligned}
|G_{n_1,n_2}^{\alpha_{n_1},\alpha_{n_2}}(f;q_{n_1},q_{n_2},x,y)-f(x,y)| &\le D_{n_1,n_2}^{\alpha_{n_1},\alpha_{n_2}}(|\Delta f[t,s;x,y]|;q_{n_1},q_{n_2},x,y)\\
&\le MD_{n_1,n_2}^{\alpha_{n_1},\alpha_{n_2}}(|t-x|^{\beta}|s-y|^{\gamma};q_{n_1},q_{n_2},x,y)\\
&= MD_{n_1}^{\alpha_{n_1}}(|t-x|^{\beta};q_{n_1},x,)D_{n_2}^{\alpha_{n_2}}(|s-y|^{\gamma};q_{n_2},x).
\end{aligned}$$

Applying the Hölder's inequality with $p_1=2/\beta, q_1=2/(2-\beta)$ and $p_2=2/\gamma$, $q_2=2/(2-\gamma)$ and Lemma 7.2.2, we have

$$\begin{aligned}
&|G_{n_1,n_2}^{\alpha_{n_1},\alpha_{n_2}}(f;q_{n_1},q_{n_2},x,y)-f(x,y)|\\
&\le M\bigg(D_{n_1}^{\alpha_{n_1}}((t-x)^2;q_{n_1},x,)\bigg)^{\beta/2}\times\bigg(D_{n_2}^{\alpha_{n_2}}((s-y)^2;q_{n_2},y)\bigg)^{\gamma/2}\\
&\le M\bigg(\delta_{n_1,q_{n_1}}^{\alpha_{n_1}}(x)\bigg)^{\beta}\bigg(\delta_{n_2,q_{n_2}}^{\alpha_{n_2}}(y)\bigg)^{\gamma}.
\end{aligned}$$

□

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