SOME PROBLEMS ON TRIGONOMETRIC APPROXIMATION IN $L_p \ (p \geq 1)$ SPACES

Ph. D. THESIS

by

SHAILESH KUMAR SRIVASTAVA

DEPARTMENT OF MATHEMATICS
INDIAN INSTITUTE OF TECHNOLOGY ROORKEE
ROORKEE- 247 667 (INDIA)
AUGUST, 2014

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A THESIS

Submitted in partial fulfilment of the requirements for the award of the degree of

DOCTOR OF PHILOSOPHY
in

MATHEMATICS

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CANDIDATE'S DECLARATION

I hereby certify that the work which is being presented in the thesis entitled "**SOME PROBLEMS ON TRIGONOMETRIC APPROXIMATION IN** $L_p\ (p \geq 1)$ **SPACES"** in partial fulfillment of the requirements for the award of the Degree of **Doctor of Philosophy** and submitted in the Department of Mathematics of the Indian Institute of Technology Roorkee is an authentic record of my own work carried out during a period from January, 2009 to August, 2014 under the supervision of Dr. Uaday Singh, Assistant Professor, Department of Mathematics, Indian Institute of Technology Roorkee.

The matter presented in this thesis has not been submitted by me for the award of any other degree of this or any other Institute.

(SHAILESH KUMAR SRIVASTAVA)

This is to certify that the above statement made by the candidate is correct to the best of my knowledge.

(Uaday Singh)
Supervisor

Date: August , 2014

Acknowledgement

Through this thanks giving note, at the beginning, I express my deepest thanks to "Almighty" God for providing me this rare opportunity to pursue higher studies under the supervision of Dr. Uaday Singh, Department of Mathematics, IIT Roorkee. I feel privileged to express my sincere regards and gratitude to my supervisor for his valuable guidance and constant encouragement throughout my research work. I shall always remain indebted to him for appropriate comments rendered by him at many critical junctures during the entire course of study. Physical insight, moral support, guidance and encouragement provided by him have been the impetus for me to learn a lot in every phase of my PhD programme. Most importantly, his technical soundness along with spirituality and wonderful social behavior made him a visionary, a wonderful and inspiring teacher and an ideal supervisor for me.

I am also indebted to Prof. R. C. Mittal, Head, Department of Mathematics, IIT Roorkee, for his invaluable direction, encouragement and support and for providing me all the necessary facilities for carrying out this work till its completion. I also thankful to Ex. DRC Chairman (Prof. T. R. Gulati) and present DRC Chairman (Prof. P. N. Agrawal), all the members of Student Research Committee and faculty members of Department of Mathematics for their unstinted support and encouragement at all levels.

My sincere thanks are also due to Dr. Ashok Kumar, Dr. Rajesh Sharma, Dr. Kuldeep Sharma, Dr. Karunesh Singh, Dr. Smita Sonkar and Dr. Mohit Kumar for their valuable suggestions and discussions during my research work. My special thanks go to my buddies Puneet Rana, Manish Kumar Khandelwal, Kamlesh Kumar Sharma, Anupam Yadav, Amit Tripathi, Shoshal, Kamlesh Jangid and Pooja Raj Verma for their whole-hearted help and assistance. I am also grateful of my well-wishers Lok Priya Srivastava and Veerendra Yadav for their timely help in many ways to accomplish this arduous task.

I wish to put on records that whatever I have achieved in my life is due to the blessing of my mother Smt. Sarojani Srivastava and father Shri S.P. Srivastava. My heartfelt thanks are due to my elder brother Manish Kumar without their support this work would not have been possible. They supported me unconditionally all through my research. It is difficult to find adequate words to express my deepest appreciation for the help given by them.

Apart from this, I also wish to express my deep sense of gratitude to all persons who with their encouraging, caring words, constructive criticism and suggestions or by simply doing their duty with sincerity and smile have contributed directly or indirectly in a significant way towards completion of this work.

I shall be failing in my duty if I do not express my deep sense of gratitude to the Council of Scientific and Industrial Research, New Delhi for their financial assistance in the form of JRF and SRF.

(Shailesh Kumar Srivastava)

Abstract

The study of approximation properties of the periodic functions in $L^p(p \geq 1)$-spaces, in general and in Lipschitz classes $Lip\alpha$, $Lip(\alpha, p)$, $Lip(\alpha, p, w)$, $Lip(\xi(t), p)$, $Lip(\omega(t), p)$, $W(L^p, \omega(t), \beta)$ and $W(L^p, \Psi(t), \beta)$, $p \geq 1$, in particular, through trigonometric Fourier series, although is an old problem and known as Fourier approximation in the existing literature, has been of a growing interests over the last four decades due to its application in filters and signals. The most common methods used for the determination of the degree of approximation of periodic functions are based on the minimization of the L_p-norm of $f(x) - T_n(x)$, where $T_n(x)$ is a trigonometric polynomial of degree n and called approximant of the function f. In this thesis, we study the approximation properties of functions belonging to various function classes through trigonometric Fourier series and conjugate trigonometric series.

The present thesis is divided into six chapters and the chapterwise description is given below:

Chapter 1 is introductory in nature and gives the details of developments in research on the trigonometric Fourier approximation and some basic concepts and definitions. Current status of the field, objective of the work done and layout of the thesis are also given in this chapter.

Chapter 2 deals with the approximation properties of the periodic functions and their conjugates belonging to the Lipschitz classes $Lip\alpha$ and $W(L^p, \omega(t), \beta)$, $p \geq 1$ by a trigonometric polynomial generated by the product matrix ($C^1.T$) means of the Fourier series and conjugate Fourier series, respectively. We prove the following theorems in Chapter 2:

Theorem 2.2.1. *Let $T \equiv (a_{n,k})$ be a lower triangular regular matrix with non-negative and non-decreasing (with respect to k, for $0 \leq k \leq n$) entries which satisfies, $A_{n,0} = 1, \forall n \in \mathbb{N}_0$ and*

$$a_{n,n-k} - a_{n+1,n+1-k} \geq 0 \quad for \quad 0 \leq k \leq n. \tag{0.1}$$

Then the degree of approximation of a 2π-periodic function $f \in Lip\alpha$ by $C^1.T$ means of its Fourier series is given by

$$\|t_n^{C^1.T}(f) - f(x)\|_\infty = \begin{cases} O((n+1)^{-\alpha}), & 0 < \alpha < 1, \\ O\left(\log(n+1)/(n+1)\right), & \alpha = 1. \end{cases} \tag{0.2}$$

Theorem 2.2.2. *Let $T \equiv (a_{n,k})$ be a lower triangular regular matrix same as in Theorem* 2.2.1. *Then the degree of approximation of a 2π-periodic function $f \in W(L^p, \omega(t), \beta)$ with $p > 1$ and $0 < \beta < 1/p$ by $C^1.T$ means of its Fourier series is given by*

$$\| t_n^{C^1.T}(f;x) - f(x) \|_p = O\left((n+1)^\beta \omega\left(1/(n+1)\right)\right), \tag{0.3}$$

provided a positive increasing function $\omega(t)$ satisfies the following conditions:

$$\omega(t)/t \text{ is a decreasing function,} \tag{0.4}$$

$$\left(\int_0^{\pi/(n+1)} \left|\phi(t)\sin^\beta(t/2)/\omega(t)\right|^q dt\right)^{1/q} = O((n+1)^{-1/q}), \tag{0.5}$$

$$\left(\int_{\pi/(n+1)}^{\pi} \left(t^{-\delta}|\phi(t)|\sin^\beta(t/2)/\omega(t)\right)^p dt\right)^{1/p} = O((n+1)^{\delta-1/p}), \tag{0.6}$$

where δ is a real number such that $p^{-1} < \delta < \beta + p^{-1}$and $p^{-1} + q^{-1} = 1$. Also conditions (0.5) *and* (0.6) *hold uniformly in x.*

In the case $p = 1$, i.e., $q = \infty$; sup norm is required while using Hölder's inequality. Therefore, the above proof will not work for $p = 1$. Thus, for $p = 1$, we have the following theorem.

Theorem 2.5.1. *Let $T \equiv (a_{n,k})$ be a lower triangular regular matrix same as in Theorem* 2.2.1. *Then the degree of approximation of a 2π-periodic function f belonging to the weighted Lipschitz class $W(L^1, \omega(t), \beta)$, with $0 < \beta < 1$ by $C^1.T$ means of its Fourier series is given by*

$$\| t_n^{C^1.T}(f;x) - f(x) \|_1 = O\left((n+1)^\beta \omega(1/(n+1))\right), \tag{0.7}$$

provided a positive increasing function $\omega(t)$ satisfies (0.4) *and the following condition:*

$$\omega(t)/t^\beta \text{ is non-decreasing,} \tag{0.8}$$

$$\int_0^{\pi/(n+1)} \frac{|\ \phi(t)\ |\ \sin^\beta(t/2)}{\omega(t)} dt = O((n+1)^{-1}), \tag{0.9}$$

$$\int_{\pi/(n+1)}^{\pi} \frac{t^{-\delta}\ |\ \phi(t)\ |\ .\sin^\beta(t/2)}{\omega(t)} dt = O((n+1)^{\delta-1}), \tag{0.10}$$

where $1 < \delta < \beta + 1$. *The conditions* (0.9) *and* (0.10) *hold uniformly in* x.

If we replace matrix T with Nörlund matrix, then $C^1.T$ means of Fourier series of f reduces to $C^1.N_p$ means.

In the next section, we obtain, in Theorems 2.6.1, 2.6.2 and 2.9.1, the degree of approximation for the function $\tilde{f}$, conjugate to the function f belonging to the same classes by $C^1.T$ means of the conjugate Fourier series of f. Some corollaries and particular cases are also discussed in this chapter.

In **Chapter 3**, we determine the degree of approximation of $\widetilde{f}$, conjugate of a 2π-periodic function f belonging to the weighted $W(L^p, \omega(t), \beta)$, $p \geq 1$-class by using Hausdorff means of conjugate Fourier series of f. More precisely, we prove:

Theorem 3.2.1. *Let f be a 2π-periodic function belonging to the weighted Lipschitz class $W(L^p, \omega(t), \beta)$-class, with $p > 1$ and $0 \leq \beta \leq 1 - 1/p$. Then the degree of approximation of $\tilde{f}$ by Hausdorff means of conjugate Fourier series of f generated by $H \in H_1$,is given by*

$$\| \tilde{H}_n(f;x) - \tilde{f}(x) \|_p = O\left((n+1)^{\beta+1/p}\xi(1/(n+1))\right), \quad (0.11)$$

provided a positive increasing function $\xi(t)$ satisfies the following conditions:

$$\xi(t)/t \text{ is non-increasing,} \quad (0.12)$$

$$\left\{\int_0^{\pi/(n+1)} \left(\frac{|\psi_x(t)| \sin^\beta(t/2)}{\xi(t)}\right)^p dt\right\}^{1/p} = O((n+1)^{-1/p}), \quad (0.13)$$

$$\left\{\int_\epsilon^{\pi/(n+1)} \left(\frac{\xi(t)}{t \sin^\beta(t/2)}\right)^q dt\right\}^{1/q} = O((n+1)^{\beta+1/p}\xi(\pi/(n+1))), \quad (0.14)$$

$$\left\{\int_{\pi/(n+1)}^{\pi} \left(\frac{t^{-\delta} |\psi_x(t)|}{\xi(t)}\right)^p dt\right\}^{1/p} = O((n+1)^\delta), \quad (0.15)$$

where δ is an arbitrary number such that $0 < \delta < \beta + 1/p$ and $p^{-1} + q^{-1} = 1$ for $p > 1$. The conditions (0.13) *and* (0.15) *hold uniformly in* x.

Since $(C, 1)$, the Cesàro matrix of order 1, and (E, q), the Euler matrix of order $q > 0$, are Hausdorff matrices, and the product of two Hausdorff matrices is also a Hausdorff matrix, so the results proved by using product of $(C, 1)$ and (E, q) $(q > 0)$ matrices are particular cases of Theorem 3.2.1.

In Theorem 3.4.1, we prove the above result for $p = 1$.

In **Chapter 4**, we introduce a more general Lipschitz class $Lip(\omega(t), p)$ which includes the classical $Lip(\xi(t), p)$ class of functions i.e.,

$$\{f \in L^p[0, 2\pi] : \| f(x+t) - f(x) \|_p = O(\xi(t)), t > 0\}$$

and $\{f \in L^p[0, 2\pi] :| f(x+t) - f(x) | = O(t^{-1/p}\xi(t)), t > 0\}$ defined by Khan and Ram and compute analytically the degree of approximation of $f \in Lip(\omega(t), p)$ using matrix means of the Fourier series of f generated by the matrix $T \equiv (a_{n,k})$. We also discuss an example to show the application of the result. Also, in the corollaries of the theorems of this paper, we observe that the degree of approximation of $f \in Lip(\xi(t), p)$ is free from p and sharper than the earlier one. The main results of this chapter are:

Theorem 4.2.1. *Let $T \equiv (a_{n,k})$ be a lower triangular regular matrix with non-negative and non-decreasing* (*with respect to k, for* $0 \leq k \leq n$) *entries with $A_{n,0} = 1$. Then the degree of approximation of a 2π-periodic function $f \in Lip(\omega(t), p)$, with $p \geq 1$ by matrix means of its Fourier series is given by*

$$\| t_n(f; x) - f(x) \|_p = O\left((n+1)^{1/p}\, \omega\left(\pi/(n+1)\right)\right), \tag{0.16}$$

provided a positive increasing function $\omega(t)$ satisfies the following conditions:

$$\omega(t)/t^{\sigma} \text{ is an increasing function for } 0 < \sigma < 1, \tag{0.17}$$

$$\left(\frac{\phi(t)}{(t^{-1/p}\omega(t))}\right) \text{ is a bounded function of t}, \tag{0.18}$$

$$\left(\int_{\pi/(n+1)}^{\pi} \left(\frac{\omega(t)}{t^{1+\,1/p}}\right)^p dt\right)^{1/p} = O\left((n+1)\,\omega\left(\frac{\pi}{n+1}\right)\right), \tag{0.19}$$

where $p^{-1} + q^{-1} = 1$. *Also condition* (0.18) *holds uniformly in x.*

In **Theorem 4.2.2**, we prove (0.16) for hump matrices with the condition $(n+1)\max_k\{a_{n,k}\} = O(1)$.

In the next section of this chapter, we define $W(L^p, \Psi(t), \beta)$-class, a weighted version of $Lip(\omega(t), p)$-class, with weight function $\sin^{\beta p}(x/2)$ and determine the error of approximation of $f \in W(L^p, \Psi(t), \beta)$ using the same matrix means. More precisely, we prove:

Theorem 4.7.1. *Let $T \equiv (a_{n,k})$ be a lower triangular regular matrix with non-negative and non-decreasing* (*with respect to k, for* $0 \leq k \leq n$) *entries. Then the degree of approximation of* *a 2π-periodic function $f \in W(L^p, \Psi(t), \beta)$ with $0 \le \beta < 1/p$ and $p \ge 1$ by matrix means of its Fourier series is given by*

$$\| t_n(f;x) - f(x) \|_p = O\left((n+1)^{\beta+1/p}\,\Psi\left(\pi/(n+1)\right)\right), \tag{0.20}$$

provided a positive increasing function $\Psi(t)$ satisfies the following conditions:

$$\Psi(t)/t^{\beta+1/p} \text{ is an increasing function,} \tag{0.21}$$

$$\left(\frac{\phi(t)\sin^{\beta}(t/2)}{t^{-1/p}\,\Psi(t)}\right) \text{ is bounded function of t, hold uniformely in x,} \tag{0.22}$$

$$\left(\int_{\pi/(n+1)}^{\pi}\left(\frac{\Psi(t)}{t^{1+1/p+\beta}}\right)^p dt\right)^{1/p} = O\left((n+1)^{\beta+1}\,\Psi\left(\frac{\pi}{n+1}\right)\right), \tag{0.23}$$

where $p^{-1} + q^{-1} = 1$.

In **Theorem 4.7.2**, we prove (0.20) by using hump matrices with the condition $(n+1)\max_k\{a_{n,k}\} = O(1)$. We also derive some corollaries from our results.

In **chapter 5**, the approximation properties of the matrix means of trigonometric Fourier series of f belonging to weighted Lipschitz class $Lip(\alpha, p, w)$ with Muckenhoupt weights generated by $T \equiv (a_{n,k})$ under relaxed conditions has been investigated. Our theorem extends some of the previous results pertaining to the degree of approximation of functions in weighted Lipschitz class $Lip(\alpha, p, w)$ and the ordinary Lipschitz class $Lip(\alpha, p)$. The main theorem of this chapter is:

Theorem 5.2.1. *Let $f \in Lip(\alpha, p, w)$, $p > 1$, $w \in A_p$ and let $T \equiv (a_{n,k})$ be an infinite lower triangular regular matrix and satisfies one of the following conditions:*

(i) $0 < \alpha < 1$, $\{a_{n,k}\} \in AMIS$ in k,

(ii) $0 < \alpha < 1$, $\{a_{n,k}\} \in AMDS$ in k and $(n+1)a_{n,0} = O(1)$,

(iii) $\alpha = 1$ and $\sum_{k=0}^{n-1}(n-k)\,|\Delta_k a_{n,k}| = O(1)$,

(iv) $\alpha = 1$, $\sum_{k=0}^{n}|\Delta_k a_{n,k}| = O(a_{n,0})$ with $(n+1)a_{n,0} = O(1)$,

(v) $0 < \alpha < 1$, $\sum_{k=0}^{n-1}\left|\Delta_k\left(\frac{A_{n,0}-A_{n,k+1}}{k+1}\right)\right| = O\left(\frac{1}{n+1}\right)$.

Then

$$\|f(x) - \tau_n(f;x)\|_{p,w} = O((n+1)^{-\alpha}),\ n = 0, 1, 2... \tag{0.24}$$

In **Chapter 6**, we generalize the notion of Λ-strong convergence of numerical sequences to T-strong convergence (an intermediate notion between bounded variation and ordinary convergence), using a lower triangular matrix $T = (a_{n,k})$ with nondecreasing monotone rows of positive numbers tending to ∞ i.e., $a_{n,k} \leq a_{n,k+1} \forall n$ and $\lim_{k\to\infty} a_{n,k} = \infty \ \forall n$. We say, a sequence $U = \{u_k\}$ of complex numbers converges T-strongly to a complex number u if

$$\lim_{n\to\infty} \frac{1}{a_{n,n}} \sum_{k=0}^{n} \mid a_{n,k}(u_k - u) - a_{n,k-1}(u_{k-1} - u) \mid = 0$$

Here $a_{n,-1} = 0$ and $u_{-1} = 0$. We also establish a relationship between ordinary convergence and T-strong convergence. We denote, class all the T-strongly convergent sequences $U = \{u_k\}$ of complex numbers by $c(T)$. Obviously, $c(T)$ is a linear space. Further, we define a norm on $c(T)$ as

$$\|U\|_{c(T)} := \sup_{n\geq 0} \frac{1}{a_{n,n}} \sum_{k=0}^{n} \mid a_{n,k}u_k - a_{n,k-1}u_{k-1} \mid$$

The main result of this chapter is:

Lemma 6.1.1. *T-strong convergence of a sequence $U = \{u_k\}$ to a number u implies the following two conditions*

$$\text{(i) ordinary convergence of } U = \{u_k\} \text{ to } u, \text{and} \tag{0.25}$$

$$\text{(ii)} \lim_{n\to\infty} \frac{1}{a_{n,n}} \sum_{k=1}^{n} a_{n,k-1} \mid u_k - u_{k-1} \mid = 0, \tag{0.26}$$

and vice-versa.

We write

$$\sigma_n := \frac{1}{a_{n,n}} \sum_{k=0}^{n} (a_{n,k} - a_{n,k-1}) u_k \qquad (n = 0, 1, ...).$$

Lemma 6.1.2. *Convergence of σ_n to u in the ordinary sense together with* (0.26) *of Lemma* 6.1.1 *implies the T-strong convergence of U to number u.*

Theorem 6.2.1. *The class $c(T)$ together with the norm $\|.\|_{c(T)}$ is a Banach space.*

In Theorem 6.2.2, we show that Banach space $c(T)$ has a Schauder basis.

We also apply the notion of T-strong convergence on the trigonometric Fourier series under C-metric and L_p-metric.

List of Publications

Journal Papers

1. Singh, U. and Srivastava, S. K. Degree of approximation of functions in Lipschitz class with Muckenhoupt weights by matrix means, *IAENG International Journal of Applied Mathematics*, 2013, 43(4), 190-194, **(Scopus - indexed)**.

2. Singh, U. and Srivastava, S. K. Approximation of conjugate of functions belonging to weighted Lipchitz class $W(Lp, \xi(t))$ by Hausdorff means of conjugate Fourier series, *Journal of Computational and Applied Mathematics*, Elsevier publications, 2014, 259, 633-640, **(SCI, IF 0.989)**.

3. Srivastava, S. K. and Singh, U. Trigonometric approximation of periodic functions belonging to $Lip(\omega(t), p)$-class, *Journal of Computational and Applied Mathematics*, Elsevier publications, 2014, 270, 223-230, **(SCI, IF 0.989)**.

4. Singh, U. and Srivastava, S. K. Trigonometric approximation of functions belonging to certain Lipschitz classes by $C^1.T$ operator, accepted for publication in *Asian-European Journal of Mathematics* (*World Scientific publications*), **(Scopus - indexed)**.

Conference Papers

5. Singh, U. and Srivastava, S. K. Fourier approximation of functions conjugate to the functions belonging to weighted Lipschitz class, *Lecture Notes in Engineering and Computer Science, Proceedings of World Congress on Engineering, London* (*U.K.*), 3-5 *July*, 2013 (*pp.* 236-240), *Newswood limited, International Association of Engineers* (*ISBN:* 978-988-19251-0-7)(*ISSN:* 2078-0966 (*online*))**(Google Scholar- indexed)**.

6. Srivastava, S. K. and Singh, U. Trigonometric approximation of periodic functions belonging to weighted Lipschitz class $W(L^p, \omega(t))$ using matrix means of Fourier series, presented in *"The 7th Conference on Function Spaces"*, held during

May 20-24, 2014 at the "Department of Mathematics & Statistics, College of Arts and Sciences, **Southeren Illinois University, Edwardsville, Illinois, USA**", likely to be published in Contemporary Mathematics (AMS).

Books/Chapters/Monographs

7. Singh, U. and Srivastava, S. K. On the degree of approximation of conjugate functions in weighted Lipschitz class, *IAENG Transactions on Engineering Sciences, CRC Press/Balkema* (*Taylor & Francis Group*), 2014, 81-89.

Communicated papers

8. Srivastava, S. K. and Singh, U. On T-strong convergence of numerical sequences and Fourier series, communicated for possible Publication.

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Chapter 1

Introduction

Fourier approximation was originally evolved from the problem of approximating a typically unknown function f [i.e., f is not completely known to us, we have only some information about f for example values of f at some points, continuity/differentiability/ integrability of the function etc. This information may be encoded by the statements like $f \in W$, some function class] by a trigonometric polynomial using trigonometric Fourier series of f. We know that a function f which is not (among other things) infinitely differentiable in some interval cannot be approximated by algebraic polynomial (truncated Taylor series of f). Since sines and cosines serve as much more versatile "prime elements" than powers of t, the trigonometric polynomial (truncated Fourier series of f) can be used to approximate not only non-analytic functions; they even do good job in the wilderness of the widely discontinuities.

As we know that the Fourier series of a function f need not to be convergent to f, the summability techniques play key role to find the sum of non-convergent series which in turn approximate the function under consideration.

1.1 Summability

The notion of the sum of an infinite series $\sum_{n=0}^{\infty} u_n$ has been based on the construction of a suitable sequence (say, $s_n = \sum_{k=0}^{n} u_k$) and on the limit of this sequence. The definition of the limit of a sequence is an arbitrary one imposed by mathematicians, although it may appeal to our intuition and may appear as a natural one. Consequently, notion of the sum of an infinite series is arbitrary to some extent. These are not sums in the sense of addition in arithmetic. They may be considered as a logical outcome of an "ultimate" process of successive approximations, but at a certain stage this "ultimate" process has to be ended by an arbitrary definition. So there is no a priori reason why the definitions may not be replaced by quite different definitions, and why new processes may not be introduced whereby a sequence or a series is associated with a certain definite number. If by such a new process, a number is associated with a sequence $\{s_n\}_{n=0}^{\infty}$ or with a series $\sum_{n=0}^{\infty} u_n$, then we usually say that $\{s_n\}_{n=0}^{\infty}$ is limitable, and $\sum_{n=0}^{\infty} u_n$ is summable by this process.

Thus summability is the extended notion of the convergence of series/integrals by which we attach a value (number) to them.

Definition

Let $\sum_{n=0}^{\infty} u_n$ be an infinite series with sequence of partial sums $\{s_n\}_{n=0}^{\infty}$. Let $T \equiv (a_{n,k})$ be an infinite matrix with real or complex entries. Then the sequence-to-sequence transformation

$$t_n := \sum_{k=0}^{n} a_{n,k} s_k, \quad n \in \mathbb{N}_0, \tag{1.1}$$

defines the matrix transform of the sequence $\{s_n\}_{n=0}^{\infty}$ generated by the elements $a_{n,k}$ of the matrix T, and we call it the matrix means of $\sum_{n=0}^{\infty} u_n$. If $\lim_{n\to\infty} t_n = s$, then the sequence $\{s_n\}_{n=0}^{\infty}$ or the series $\sum_{n=0}^{\infty} u_n$ is said to be matrix summable or simply T-summable to s. Then we write

$$\sum u_n = s \quad (T).$$

If for every convergent sequence $\{s_n\}_{n=0}^{\infty}$, $\lim_{n\to\infty} s_n = s$ implies $\lim_{n\to\infty} t_n = s$, then the matrix T and corresponding summability method is said to be regular. The necessary and sufficient conditions for the regularity of matrix T, obtained by Toeplitz [121] and Silverman [111], are

(1) $\lim_{n\to\infty} \sum_{k=0}^{n} a_{n,k} = 1,$

(2) $\sum_{k=0}^{n} |a_{n,k}| < M \quad (n = 0, 1, 2...),$

(3) $\lim_{n\to\infty} a_{n,k} = 0 \ (k = 0, 1, 2...),$

where M is a finite constant independent of n.

1.1.1 Some Particular Cases of Matrix T

1. **Cesàro Matrix of Order δ :** For a positive real number δ, if

$$a_{n,k} = \begin{cases} \mathrm{E}_{n-k}^{\delta-1} \Big/ \mathrm{E}_n^{\delta}, & 0 \leq k \leq n, \\ 0\,, & k > n, \end{cases}$$

where

$$E_n^{\delta} = \frac{\Gamma(n+\delta+1)}{\Gamma(n+1)\,\Gamma(\delta+1)} = \sum_{k=0}^{n} E_k^{\delta-1},$$

is the binomial coefficient given by $\sum \mathrm{E}_n^{\delta} x^n = (1-x)^{-\delta-1}$, $\delta > -1$, $|\,x\,| < 1$, then the linear means t_n reduces to Cesàro means of order δ or symbolically t_n^{δ}.

If

$$t_n^{\delta} = \frac{1}{\mathrm{E}_n^{\delta}} \sum_{k=0}^{n} \mathrm{E}_{n-k}^{\delta-1}\, s_k \to s, \text{ as } n \to \infty,$$

then the series $\sum_{n=0}^{\infty} u_n$ is said to be Cesàro (C, δ))-summable to s.

2. **Harmonic Matrix:** If

$$a_{n,k} = \begin{cases} \dfrac{1}{(n-k+1)\log n}, & 0 \leq k \leq n, \\ 0, & k > n, \end{cases}$$

then the linear means t_n reduces to Harmonic (H1) means.

The series $\sum_{n=0}^{\infty} u_n$ is said to be Harmonic (H1) - summable to s, if

$$t_n = \frac{1}{\log n} \sum_{k=0}^{n} \frac{s_k}{(n-k+1)} \to s, \text{ as } n \to \infty.$$

3. **Nörlund Matrix:** If

$$a_{n,k} = \begin{cases} p_{n-k}/\mathrm{P}_n, & 0 \leq k \leq n, \\ 0, & k > n, \end{cases}$$

where $P_n = \sum_{k=0}^{n} p_k \neq 0$ and $P_{-1} = p_{-1} = 0$, then the linear means t_n reduces to Nörlund (N_p) means, where $\{p_n\}$ is any sequence of real or complex numbers.

If

$$t_n = \frac{1}{\mathrm{P}_n} \sum_{k=0}^{n} p_{n-k}\, \mathrm{s}_k \to s, \text{ as } n \to \infty,$$

then the series $\sum_{n=0}^{\infty} u_n$ is said to be Nörlund (N_p)-summable to s.

4. **Riesz Matrix:** If
$$a_{n,k} \;=\; \begin{cases} p_k/\mathrm{P}_n, & 0 \le k \le n, \\ 0, & k > n, \end{cases}$$
where $P_n = \sum_{k=0}^{n} p_k \neq 0$ and $P_{-1} = p_{-1} = 0$, then the linear means t_n reduces to Riesz $(\bar{N}_p)$ means, where $\{p_n\}$ is any sequence of real or complex numbers.
The series $\sum_{n=0}^{\infty} u_n$ is said to be Riesz $\bar{N}_p$-summable to s, if
$$t_n = \frac{1}{\mathrm{P}_n} \sum_{k=0}^{n} p_k\, s_k \rightarrow s, \text{ as } n \rightarrow \infty.$$

5. **Euler Matrix of Order** q**:** For $q > 0$, if
$$a_{n,k} = \begin{cases} \binom{n}{k} q^{n-k}/(1+q)^n & 0 \le k \le n \\ 0 & k > n. \end{cases}$$
If
$$t_n = \frac{1}{(1+q)^n} \sum_{k=0}^{n} \binom{n}{k} q^{n-k} s_k \rightarrow s, \text{ as } n \rightarrow \infty,$$
then the series $\sum_{n=0}^{\infty} u_n$ is said to be Euler-summable to s.

Some more summability methods such as lag-averaged Euler scheme, Euler-Abel method can be seen in [4; 5], which work excellently in case of alternating series and give exponential convergence.

1.1.2 Composition of Two Summability Methods

Composition of two summability methods is same as the compositions of two functions. For defining this, we take two summability matrix $S = (a_{n,k})$ and $T = (b_{n,k})$. By superimposing T-summability upon S-summability, we get TS-summability defined by
$$t_n^{T.S} = \sum_{r=0}^{n} b_{n,r} \sum_{k=0}^{r} a_{r,k} s_k.$$
The following example shows that product/composite summability method is powerful than the individual summability method.
Example 1: Consider the infinite series $1 - 4\sum_{n=1}^{\infty} (-3)^{n-1}$.
Here $s_n = 1 - 4\ \sum_{k=1}^{n} (-3)^{k-1} = (-3)^n$.
The $(C, 1)$ means of given series are
$$C_n^1 = \frac{1}{n+1} \sum_{k=0}^{n} (-3)^k = \frac{(1 - (-3)^{n+1})}{4(n+1)}.$$

The $\lim_{n\to\infty} C_n^1$ does not exist, therefore given series is not $(C,1)$ summable. Also the $(E,1)$ means of given series are

$$E_n^1 = \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k} s_k = \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}(-3)^k = (-1)^n.$$

The series is not $(E,1)$ summable also. The $(C,1)(E,1)$ means of the given series are given by $(CE)_n^1 = \frac{1}{n+1}\sum_{k=0}^{n}(E_k^1) = \frac{1}{n+1}\sum_{k=0}^{n}(-1)^k \to 0$ as $n\to\infty$. Hence the given series is $(C,1)(E,1)$ summable.

We note that $ST \neq TS$.

1.2 Trigonometric Fourier Series

For a 2π-periodic function $f \in L^p := L^p[0,2\pi]$, $p \geq 1$, integrable in the sense of Lebesgue, the trigonometric Fourier series $s(f)$ of f is

$$f(x) \cong \frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k\cos kx + b_k \sin kx),$$

where

$$a_k = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos kx dx,\ \ k = 0,1,2...$$

and

$$b_k = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin kx dx,\ \ k = 0,1,2...$$

The constants a_k and b_k are known as Fourier coefficients.

The

$$s_n(f;x) := \frac{a_0}{2} + \sum_{k=1}^{n}(a_k\cos kx + b_k\sin kx),\ \ n\in\mathbb{N} \text{ and } s_0(f;x) = \frac{a_0}{2}, \tag{1.2}$$

denote the $(n+1)^{\text{th}}$ partial sums, called trigonometric polynomials of degree (or order) n, of the Fourier series of f.

Consider the power series $\sum_{k=0}^{\infty} c_k z^k$, $z\in\mathbb{C}$ and $c_k = \begin{cases} a_k - ib_k, & k\geq 1,\\ a_0/2, & k = 0\,.\end{cases}$

If $z = e^{ix}$, then

$$\begin{aligned}\sum_{k=0}^{\infty} c_k z^k = \sum_{k=0}^{\infty} c_k e^{ikx} &= a_0/2 + \sum_{k=1}^{\infty}(a_k - ib_k)(\cos x + i\sin x)^k \\ &= a_0/2 + \sum_{k=1}^{\infty}(a_k\cos kx + b_k\sin kx) + i\sum_{k=1}^{\infty}(a_k\sin kx - b_k\cos kx).\end{aligned}$$

The imaginary part of $\sum_{k=0}^{\infty} c_k\, e^{ikx}$, i.e., $\sum_{k=1}^{\infty}(a_k \sin kx - b_k \cos kx)$ is called the conjugate series of $s(f)$, and denoted by $\tilde{s}(f)$.
The n^{th} partial sum of the conjugate series $\tilde{s}(f)$ is defined as

$$\tilde{s}_n(f;x) := \sum_{k=1}^{n}(a_k \sin kx - b_k \cos kx), \quad n \in \mathbb{N} \text{ and } \tilde{s}_0(f;x) = 0. \tag{1.3}$$

The $s_n(f;x)$ and $\tilde{s}_n(f;x)$ have the following integral representations:

$$s_n(f;x) = \frac{1}{\pi}\int_0^{\pi}[f(x+t)+f(x-t)]\frac{\sin(n+1/2)t}{2\sin(t/2)}dt \tag{1.4}$$

and

$$\begin{aligned}\tilde{s}_n(f;x) &= \textstyle\sum_{k=1}^{n}(b_k \cos kx - a_k \sin kx) \\ &= -\frac{1}{\pi}\int_0^{\pi}\psi_x(t)\left\{\frac{\cos(t/2)-\cos(n+1/2)t}{2\sin(t/2)}\right\}dt,\end{aligned} \tag{1.5}$$

where

$$\psi_x(t) = f(x+t) - f(x-t), \tag{1.6}$$

and we also write

$$\phi(t) \equiv \phi(x,t) := f(x+t) + f(x-t) - 2f(x). \tag{1.7}$$

It is known that the series conjugate to a Fourier series is not necessarily itself a Fourier series, e. g., $\sum_{k=0}^{\infty}\frac{\cos k\theta}{\log(k+2)}$ is a Fourier series but the corresponding sine series is not a Fourier series [2, pp. 218-219]. Therefore, a separate study of conjugate Fourier series is required [123].
The conjugate of f, denoted by $\tilde{f}$, is defined as

$$\tilde{f}(x) = -\frac{1}{2\pi}\lim_{\epsilon\to 0}\int_{\epsilon}^{\pi}\psi_x(t)\cot(t/2)dt\ . \tag{1.8}$$

There are some other Fourier series also like Mellin, Walsh [110], Ciesliski [122], Legendre, and Bessel [88, pp. 775 & 812] etc. In this thesis, we confine ourselves to trigonometric Fourier series only.

1.3 Some Basic Definitions

1. L_p **Norm:** The L^p-norm of $f \in L^p[0,2\pi]$ is defined by

$$\|f\|_p := \left\{\frac{1}{2\pi}\int_0^{2\pi} \mid f(x) \mid^p dx\right\}^{1/p} \quad (1 \le p < \infty)$$

and

$$\|f\|_\infty := ess \sup_{x\in[0,2\pi]} \mid f(x) \mid .$$

2. **Modulus of Continuity:** Let $f(x)$ be a continuous function in the interval $[a,b]$. Then the modulus of continuity $w(\delta)$ of the function $f(x)$ is defined as

$$w(\delta) = w(f,\delta) = \max_{|x-y|\le\delta} |f(x) - f(y)|, \quad a \le x,\ y \le b.$$

Also $w(\delta)$ is a monotonically increasing function of argument δ.
Some basic properties of modulus of continuity $w(f,\delta)$ are as follows:

(a) $w(f,\delta) \ge 0$ and $w(f,\delta) = 0$, if $\delta = 0$.

(b) $w(f,\delta_1) \le w(f,\delta_2)$ for $0 < \delta_1 < \delta_2$.

(c) The function f is uniformly continuous on $[a,b]$ if and only if

$$\lim_{\delta\to 0} w(f,\delta) = 0.$$

(d) If $n \in \mathbb{N}$ and $\delta > 0$, then $w(f,n\delta) \le n\, w(f,\delta)$.

3. **Integral Modulus of Continuity:** Let $f(x)$ be a function of period 2π in $L_p(1 \le p < \infty)$. Then the integral modulus of continuity of first and second order of f in L_p-spaces are defined by

$$\omega_p(h;f) = \sup_{0<|t|\le h} \| f(x+t) - f(x) \|_p$$

and

$$\omega_p^2(h;f) = \sup_{0<|t|\le h} \| f(x+t) + f(x-t) - 2f(x) \|_p$$

respectively [11]. More generally, the integral modulus of continuity of order k of f in L_p-spaces are defined by

$$\omega_p^k(h;f) = \sup_{0<|t|\le h} \| \triangle_t^k f(x) \|_p,$$

where

$$\triangle_t^k f(x) = \sum_{\alpha=0}^{k} (-1)^{k-\alpha} \tbinom{k}{\alpha} f(x+\alpha t) \ [105].$$

4. **Function Spaces:** A function $f \in Lip\alpha$ if $|f(x+t) - f(x)| = O(t^{\alpha})$, for $0 < \alpha \leq 1$, $f \in Lip(\alpha, p)$ if $\left(\int_0^{2\pi} |f(x+t) - f(x)|^p dx\right)^{1/p} = O(t^{\alpha}), 0 < \alpha \leq 1,\ p \geq 1$, $f \in Lip(\xi(t), p)$ if $\left(\int_0^{2\pi} |f(x+t) - f(x)|^p dx\right)^{1/p} = O(\xi(t))$ and $f \in W(L^p, \omega(t), \beta)$ if $\left(\int_0^{2\pi} |(f(x+t) - f(x)) \sin^{\beta}(\frac{x}{2})|^p dx\right)^{1/p} = O(\omega(t))$, $\beta \geq 0$, $p \geq 1$, where $\xi(t)$ and $\omega(t)$ are positive increasing functions of t [35; 112; 114].
It is important to note that the increasing function $\omega(t)$ in the definition of $W(L^p, \omega(t), \beta)$-class is not the same as $\xi(t)$ in the definition of $Lip(\xi(t), p)$-class. The $\xi(t)$ in $Lip(\xi(t), p)$-class depends on t only, whereas $\omega(t)$ in $W(L^p, \omega(t), \beta)$-class depends on both t and β [35]. In particular, if we take $\omega(t) = t^{\beta}\psi(t)$ for $\beta \geq 0$ and some positive increasing function $\psi(t)$, then $W(L^p, \omega(t), \beta)$-class defined above reduces to $W'(L^p, \psi(t))$-class defined by Khan [35].
If $\beta = 0$ and $\omega(t) = \xi(t)$, then $W(L^p, \omega(t), \beta) \equiv Lip(\xi(t), p)$ and for $\xi(t) = t^{\alpha}(0 < \alpha \leq 1), Lip(\xi(t), p) \equiv Lip(\alpha, p)$. $Lip(\alpha, p) \rightarrow Lip\alpha$ as $p \rightarrow \infty$. Thus

$$Lip\alpha \subseteq Lip(\alpha, p) \subseteq Lip(\xi(t), p) \subseteq W(L^p, \omega(t), \beta).$$

In this thesis, we use notation $W(L^p, \omega(t), \beta)$ for $W(L^p, \xi(t))$ and replace $\sin(x)$ by $\sin(x/2)$ as given in [29; 112; 114].
There are many other function classes such as Lebesgue space with Muckenhoupt weights [21], homogeneous Banach space [34, p. 14], classical Lorentz space [59; 60], grand Lorentz space [30], weighted grand Lebesgue space [16] and generalized Orlicz space[31] etc.

Note 1: We note that $\phi(x, t)$ also belongs to $W(L^p, \omega(t), \beta)$.
Clearly,

$$\begin{aligned} | \phi(x+t, t) - \phi(x, t) | &\leq | f(x+2t) - f(x+t) | + 2 | f(x+t) - f(x) | \\ &+ | f(x) - f(x-t) | . \end{aligned}$$

Hence by Minkowski's inequality,

$$\left(\int_0^{2\pi} | (\phi(x+t, t) - \phi(x, t)) \sin^{\beta} (x/2)|^p dx\right)^{1/p}$$

$$
\begin{aligned}
&\leq \left(\int_0^{2\pi} |(f(x+2t)-f(x+t))\sin^{\beta}(x/2)\,|^p\,dx\right)^{1/p}\\
&+ 2\left(\int_0^{2\pi} |\,(f(x+t)-f(x))\sin^{\beta}(x/2)\,|^p\,dx\right)^{1/p}\\
&+ \left(\int_0^{2\pi} |\,(f(x)-f(x-t))\sin^{\beta}(x/2)\,|^p\,dx\right)^{1/p} = O(\omega(t)),
\end{aligned}
$$

i.e., $\phi(t) \equiv \phi(x,t) \in W(L^p, \omega(t), \beta)$.

1.4 Trigonometric Fourier Approximation

A function f in L_p-spaces is approximated by a trigonometric polynomials $T_n(x)$ of degree $\leq n$ (which is either partial sum or some summability means of the Fourier series of f), and the error of approximation $E_n(f)$ in terms of n, is given by

$$
E_n(f) = \min_{T_n} \|f(x) - T_n(x)\|_p \,.
$$

The trigonometric polynomial $T_n(x)$ is known as the Fourier-approximant of f and this method of approximation is called trigonometric Fourier approximation.

1.5 Some Important Tools

We use the following tools/inequalities for the proof of lemmas and theorems of the present thesis:

1. **The Hölder Inequality** (for $p > 1$ and q such that $\frac{1}{p} + \frac{1}{q} = 1$) : Let x and y be scalar-valued Lebesgue-measurable functions on the Lebesgue-measurable set T such that $\int_T |\, x(t)\,|^p\, dt < \infty$ and $\int_T |\, y(t)\,|^q\, dt < \infty$. Then $\int_T |\, x(t)y(t)\,|\, dt < \infty$ and
$$
\int_T |\, x(t)y(t)\,|\,dt \leq \left(\int_T |\, x(t)\,|^p\,dt\right)^{1/p}\left(\int_T |\, y(t)\,|^q\,dt\right)^{1/q}.
$$
For $p = 1$ and $q = \infty$, we have
$$
\int_T |\, x(t)y(t)\,|\,dt \leq \left(\int_T |\, x(t)\,|\,dt\right)(ess \sup_{t\in T} |\, y(t)\,|).
$$

2. **The Minkowski Inequality** (for $1 \leq p < \infty$): For scalar-valued Lebesgue-measurable functions x and y on a Lebesgue-measurable set T such that $\int_T \mid x(t) \mid^p dt < \infty$ and $\int_T \mid y(t) \mid^p dt < \infty$, we have

$$\left(\int_T \mid x(t) + y(t) \mid^P dt\right)^{1/p} \leq \left(\int_T \mid x(t) \mid^p dt\right)^{1/p} + \left(\int_T \mid y(t) \mid^p dt\right)^{1/p}.$$

For $p = \infty$, we have

$$ess \sup \mid (x+y)(T) \mid \leq (ess \sup \mid x(T) \mid) + (ess \sup \mid y(T) \mid).$$

3. **Abel's Lemma**: If $\{a_n\}_{n=0}^{\infty}$ is a sequence of real numbers whose partial sums $s_n = \sum_{k=1}^{\infty} a_k$ satisfy

$$m \leq s_n \leq M \quad n = 0, 1, 2, \ldots ,$$

for some m, $M \in R$, and if $\{b_n\}_{n=0}^{\infty}$ is a nonincreasing sequence of nonnegative numbers, then

$$mb_1 \leq \sum_{k=1}^{n} a_k b_k \leq Mb_1 \quad n = 0, 1, 2, \ldots .$$

4. **Abel's Transformation**: Let $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ be two sequence of real numbers, then

$$\sum_{k=m}^{n} a_k b_k = \sum_{k=m}^{n-1} A_k \triangle b_k + A_n b_n - A_{m-1} b_m, \tag{1.9}$$

where $A_k = \sum_{r=0}^{k} a_r$ and $\triangle b_k \equiv b_k - b_{k+1}$. For $m = 0$, (1.9) reduces to

$$\sum_{k=0}^{n} a_k b_k = \sum_{k=0}^{n-1} A_k \triangle b_k + b_n A_n.$$

1.6 Literature Review

The study of error estimates of the periodic functions in Lipschitz classes [Viz. $Lip\alpha \subseteq Lip(\alpha, p) \subseteq Lip(\xi(t), p)$] spaces through the summability means of Fourier series, referred as Fourier approximation in the literature, has been of a growing interests over the last few decades. The engineers and scientists use properties of Fourier approximation for designing digital filters. As mentioned in [99], the L_p-space in general, and L_2 and L_∞ in particular play an important role in the theory of signals and filters. In [99], Psarakis and Moustakides presented a new L_2 based method for designing the

Finite Impulse Response (FIR) digital filters and get corresponding optimum approximations having improved performance. Lubinsky and Mache [61] have discussed $(C,1)$-summability of the orthonormal expansions for exponential weights. Also, a good amount of work on L_p-boundedness of Cesàro means, a particular type of Hausdorff matrix, of orthonormal expansions for general exponential weights has been carried out in [62].

Many researchers use the Euler-Abel summation method to reduce the error exponentially rather than algebraically with n, the number of terms retained in the truncated series, by using an approximation which weights all the terms in the truncated series. But, unfortunately, as mentioned by Boyd and Moore [5, p. 59] and Boyd [4, pp. 1-2], Euler's method is an efficient accelerator for an alternating series and works best, but when applied to a series whose terms are all positive, it invariably increases the error rather than decreases. It is not necessary that all functions have alternating Fourier series. In such cases, the matrix method, a generalized form of regular summability methods, works efficiently. However, the best use of Euler-Abel summation method is in solving the problems in equatorial oceanography via series of Hermite functions.

Govil[18; 19] has discussed the convergence of derived Fourier series (series obtained from term by term differentiation of a Fourier series) and its conjugate. In [20], Govil and Mohapatra have discussed several inequalities by using trigonometric form of Markov and Bernstein type inequalities, arousen from the problem of chemist Mendeleev.

The first result in the direction of Fourier approximation is that of Bernstein [3]. He used T for $(C,1)$, the Cesàro matrix of order one with assumption $f \in Lip\,1$ and obtained the result $E_n(f) = O(n^{-1}\log n)$. Jackson [29] has shown that if $T \equiv (C,\delta)-$ the Cesàro matrix of order δ with assumption $f \in Lip\,\alpha$, then

$$E_n(f) = O(n^{-\alpha}),\ \ 0 < \alpha < \delta \le 1 \text{ and } E_n(f) = O(n^{-\alpha}\log n),\ \ 0 < \alpha \le \delta \le 1\,.$$

The problem was further extended by many investigators such as Holland et al. [25], Chandra [7; 8], by choosing T to be Nörlund and Riesz matrices with monotonic weights p_n. Leindler [51] moderated the classical monotonicity conditions on $\{p_n\}$ in four theorems of Chandra [8]. Further Szal [118] generalized the result of Leindler [51] for Mean Rest Bounded Variation sequences and extend it to the strong summability with a mediate function satisfying the standard conditions. Some authors such as Kathal et al. [33], Holland et al. [25] and Mittal and Rhoades [78] have replaced the assumption $f \in Lip\,\alpha$ with an estimate involving modulus of continuity. A very good survey paper on trigonometric approximation of continuous functions has been published by Holland, A.S.B. [24] in 1981. However, in case of $f \in Lip(\alpha, p)$, a nice work has been done by Chandra [9], Liendler [52] and Mittal et al. [80; 81] have extended these results to T by relaxing the condition on monotonicity on $(a_{n,k})$ and error of approximation in their papers is of order $n^{-\alpha}$, which is free from p. Sun [117], Mazhar and Totik [68], Mazhar and Siddiqi [67], Mazhar [63–66], Khan and Wafi [37], Kumar and Sikdar [39], Dubey and Kumar [15] and Cheney [10] have proved a number of interesting results on Fourier approximations of functions belonging to $L_p(p \geq 1)$-spaces using Cesàro, Nörlund and general matrix T.

Khan and Ram [36] introduced a new function class $Lip(\psi(t), p)$, $p > 1$ as the collection of all 2π-periodic functions such that $\mid f(x+t) - f(x) \mid \leq M(\psi(t)t^{-1/p})$, $0 < t < \pi$, where $\psi(t)$ is a positive increasing function and M is a positive number independent of x and t, and obtain the degree of approximation of functions belonging to this class by using the Euler's means. Apart from this Nigam and Sharma [95] applied the concept of $(C, 1)(E, q)$ summability method and establish a new theorem on degree of approximation of a function $f \in Lip(\xi(t), r)$ class. Further, Nigam [90] used the $(E, q)(C, 1)$ product means of the Fourier series and obtain the order $(n + 1)^{1/p}\xi(1/(n+1))$ for the functions belonging to $Lip(\xi(t), p)$ class, which clearly depends on p, even though $\xi(t)$ is free from p. In the sequel, Lal [45] has studied the degree of approximation of $f \in W(L_p, \xi(t))$ by using Cesàro-Nörlund summability, which has been extended and improved recently by Singh et al. [112] by dropping the monotonocity condition on $\{p_n\}$. Nigam [89] determined the error of functions belonging to $Lip\alpha$ and weighted $(L_p, \xi(t))$-class by $(E, 1)(C, 1)$ product summability means of Fourier series. Lal [45], Nigam and Sharma [93; 97] studied the same in $Lip(\alpha, p)$, $Lip(\xi(t), p)$ and $W(L^p, \xi(t))$ classes by using product summability. Also Rhoades [107] and Rhoades et al. [108] have used the Hausdorff matrices to determine the degree of approximation of $f \in W(L^p, \xi(t))$ and $f \in Lip\,\alpha$, respectively.

In the last four decades, many researchers have been approximated the function $\tilde{f}$, conjugate of f belonging to $Lip\alpha$, $Lip(\alpha, p)$, $Lip(\xi(t), p)$ and $W(L^p, \xi(t))$-classes with $p \geq 1$, by different summability means of the conjugate Fourier series of f and obtained the error of approximation $E_n(\tilde{f})$, which depends heavily on p [22; 74; 99; 115]. Qureshi [100; 101], Lal and Kushwaha [46], Nigam and Sharma [97] and Padhy et al. [98] have proved some interesting theorems on the degree of approximation of function conjugate of $f \in Lip\alpha(0 < \alpha \leq 1)$ using almost reisz means, lower triangular matrix means, $(E,1)(C,1)$ product means, $(E,q)(\tilde{N}, p_n)$ product means of conjugate Fourier series, respectively. Lal [42] obtained the degree of approximation of conjugates of almost Lipschitz functions by $(C,1)(E,1)$ means. Also Lal and Singh [48] determined the degree of approximation of $\tilde{f}$, conjugate of a function $f \in$ almost Lipα using $(N, p, q)(E,1)$ means of the conjugate Fourier series of f. Further Qureshi [102], Nigam and Sharma [96], Lal and Singh [49] have used Nörlund, Karmata and $(C,1)(E,1)$ summability means to obtain $E_n(\tilde{f}) = O(n^{-\alpha+1/p})$ for the function conjugate to $f \in Lip(\alpha, p)$ through conjugate Fourier series of f.
Approximation of the conjugate to functions $f \in Lip\alpha$ and $Lip(\alpha, p)$ $(p \geq 1, 0 < \alpha \leq 1)$ motivated the researchers to analyze the degree of approximation of the conjugate of the functions belonging to more general Lipschitz classes $Lip(\xi(t), p)$ and $W(L^p, \xi(t))(p \geq 1)$, where $\xi(t)$ is a positive increasing function. Lal and Nigam [47], Mittal et al. [83], Lal and Kushwaha [46], Lal and Srivastava [50], Nigam and Sharma [92] have obtained the degree of approximation $E_n(\tilde{f}) = O(n^{1/p}\xi(1/n))$ for conjugate of functions $f \in Lip(\xi(t), p)$ using different summability means such as matrix $T \equiv (a_{n,k})$ means, and $(E,q)(C,1)$ product means of the conjugate Fourier series of f. Qureshi [103], Dhakal [14], Nigam and Sharma [94] introduced new theorems concerning the degree of approximation of the conjugate of a function belonging to $W(L^p, \xi(t))-$class by Nörlund, $(N, p_n)(E,1)$ product means and $(C,1)(E,q)$ product means of conjugate Fourier series, respectively. Lal [43] used the matrix summability with increasing rows for this purpose. The degree of approximation so obtained is $E_n(\tilde{f}) = O(n^{\beta+1/p}\xi(1/n))$. Rhoades [107], Mittal et al. [79; 82] have proved the same results by using a summability matrix without monotonicity condition. Mishra [70] extended the results of Lal and Srivastava [50] for the functions $f \in W(L^p, \xi(t))$. Recently, Mishra et al. [72; 73]

and Nigam and Sharma [93] have obtained the degree of approximation of function conjugate to $f \in W(L^p, \xi(t))$ and gave interesting results using lower triangular matrix operator, $(E, q)(C, 1)$ and $(N, p_n)(C, 1)$ product summability, respectively. Nigam and Sharma [96] study the degree of approximation of function and their conjugates belonging to $W(L^r, \xi(t))$ by K^λ-summability means of its Fourier series and conjugate Fourier series, respectively and established quietly new theorems. Very recently, Łenski and Szal [57], Kranz et al. [38] and Mishra et al. [75] have studied the same problem by using matrix means and $(E, 1)(C, 1)$ means, and gave important remarks which may be useful to improve the above results.
Moreover, some investigators like Das and Mohapatra [13], Mohapatra and Chandra [84], etc. investigated the error of approximation of functions in the Hölder metric. For $0 < \alpha \leq 1$ and some positive constant K, Mohapatra and Chandra [84] defined the function space H_α given by

$$H_\alpha = \{f \in C_{2\pi} : |f(x) - f(y)| \leq K|x - y|^\alpha\}$$

and investigated the error of functions belonging to H_α-class using a lower triangular infinite matrix with non-negative and non-decreasing entries (with respect to k) and row sum 1. Sun [116] obtained the degree of approximation in generalized Hölder metric and Das et al. [12] used the matrix means of the Fourier series in generalized Hölder metric to calculate the degree of approximation . Further, Singh and Sonkar [113] have studied the degree of approximation of periodic functions in generalized Hölder metric space through matrix means of Fourier series, where matrix $T \equiv (a_{n,k})$ has almost monotone rows, which in turn generalizes most of the results of Liendler [53].

Mohapatra and Russell [85] proved some results on the convergence of a sequence of linear operators connected with the Fourier series of a function of class $L_p (p > 1)$ to that function and some inverse results in relating the convergence to the classes of functions. Łenski and Szal [54] have proved the approximation results for the functions belonging to the class $L^p(\omega)_\beta$ by linear operators and Łenski and Szal [55] have proved the some approximation results of integrable functions by general linear operators of their Fourier series at the Lebesgue points.

F. Móricz [86] gave the notion of Λ-strong convergence, an intermediate notion between bounded variation and and ordinary convergence, using a nondecreasing sequence $\Lambda = \{\lambda_k : k = 0, 1, ...\}$ of positive numbers tending to ∞. He defined that a sequence $S = \{s_k\}$ of complex numbers converges Λ-srtongly to a complex number s if

$$\lim_{n\to\infty} \frac{1}{\lambda_n} \sum_{k=0}^{n} \mid \lambda_k(s_k - s) - \lambda_{k-1}(s_{k-1} - s) \mid = 0.$$

Braha and Mansour [6] generalized the concept of Λ-strong convergence given by Móricz [86] and introduced the concept of Λ^2-strong convergence using the second difference defined as $\triangle^2(\lambda_k) = \triangle(\triangle(\lambda_k)) = \lambda_k - 2\lambda_{k-1} + \lambda_{k-2}$.
They also proved lemma [Lemma 1, p. 113] related to ordinary convergence and Λ^2-strong convergence, and a theorem [Theorem 1, p.120] showing that $c^2(\Lambda)$, the collection of all Λ^2-strong convergent sequences $S = \{s_k\}$ of complex numbers form a Banach space with the norm defined as

$$\|S\|_{c^2(\Lambda)} := \sup_{n\geq 0} \frac{1}{\lambda_n - \lambda_{n-1}} \sum_{k=0}^{n} \mid \lambda_k s_k - 2\lambda_{k-1}s_{k-1} + \lambda_{k-2}s_{k-2} \mid .$$

Also, they applied these concepts to Fourier series in C-metric and L_p-metric, and proved the corresponding results.

1.7 Objective of the Present Study (Summary of the thesis)

The objective of this thesis is to fill the gap in the Literature and also making some advancement in the direction of Trigonometric Fourier approximation. The pointwise objective is as follows:

- To study trigonometric approximation of functions and their conjugates belonging to certain Lipschitz classes such as $Lip\alpha$ and $W(L^p, \omega(t), \beta)$ by $C^1.T$ operator (Chapter 2).
- To study approximation of conjugate of functions belonging to weighted Lipschitz class $W(L^p, \omega(t), \beta)$ by Hausdorff means of conjugate Fourier series (Chapter 3).
- To introduce the function classes $Lip(\omega(t), p)$ and $W(L^p, \Psi(t), \beta)$ and determine the degree of approximation of functions belonging to them (Chapter 4).

- To study degree of approximation of functions in Lipschitz class with Muckenhoupt weights by matrix means (Chapter 5).
- To study T-strong convergence of numerical sequences and Fourier series (Chapter 6).

□□□

Chapter 2

Approximation of Functions and their Conjugates Belonging to Certain Lipschitz Classes by $C^1.T$ Operator

2.1 Introduction

The study of approximation properties of the periodic functions in $L^p (p \geq 1)$-spaces, in general and in Lipschitz classes $Lip\alpha$, $Lip(\alpha, p)$, $Lip(\xi(t), p)$ and weighted Lipschitz class $W(L^p, \omega(t), \beta)(= W(L_p, \xi(t)))$, in particular, through trigonometric Fourier series, has attracted the researchers over the last four decades due to its application in filters and signals [Emmanouil Z. Psarakis and George V. Moustakides, An L_2-based method for the design of 1-D zero phase FIR digital filters, IEEE Transactions on Circuits and Systems-I: Fundamental Theory And Applications, 44(7) (1997), 551-601]. The most common methods used for the determination of the degree of approximation of periodic functions are based on the minimization of the L_p-norm of $f(x) - T_n(x)$, where $T_n(x)$ is a trigonometric polynomial of degree at most n and called approximant of the function f. In this chapter, we discuss the approximation properties of the periodic functions and their conjugates in the Lipschitz classes $Lip\alpha$ and $W(L^p, \omega(t), \beta)$, $p \geq 1$ by a trigonometric polynomial generated by the product matrix means of the Fourier series associated with the function. The degree of approximation obtained in our theorems of this chapter is free from p and sharper than earlier results.

Let $T \equiv (a_{n,k})$ is a lower triangular matrix with non-negative entries such that $a_{n,-1} = 0, A_{n,k} = \sum_{r=k}^{n} a_{n,r}$, $n \in \mathbb{N}_0$.

In particular, if $a_{n,k} = 1/(n+1)$ for $k \leq n$ and $a_{n,k} = 0$ for $k > n$, then $t_n(f,x)$ reduces to Cesàro means of order one defined as

$$\sigma_n(f;x) = \frac{1}{n+1}\sum_{k=0}^{n} s_k(f;x).$$

By superimposing C^1-summability upon T-summability, we get the $C^1.T$-summability. Thus the $C^1.T$ means of $\{s_n(f;x)\}$ denoted by $t_n^{C^1.T}(f;x)$ are given by

$$t_n^{C^1.T}(f;x) := (n+1)^{-1}\sum_{r=0}^{n}\left(\sum_{k=0}^{r} a_{r,k}s_k(f;x)\right), \ \ n \in \mathbb{N}_0. \tag{2.1}$$

If $t_n^{C^1.T}(f;x) \to s_1$ as $n \to \infty$, then the Fourier series of f is said to be $C^1.T$-summable to the sum s_1. The regularity of methods C^1 and T implies regularity of method $C^1.T$.

We also write

$$(C^1.T)_n(t) := \frac{1}{2\pi(n+1)}\sum_{r=0}^{n}\sum_{k=0}^{r} a_{r,r-k}\frac{\sin(r-k+1/2)t}{\sin(t/2)},$$

$b_{n,n-k} := \Delta_n a_{n,n-k} = a_{n,n-k} - a_{n+1,n+1-k}$ and $\tau := [1/t]$, the integer part of $1/t$.

Various investigators such as Qureshi and Nema [104], Rhoades [106], Lal [44; 45] and Nigam [90; 91] have determined the degree of approximation of 2π-periodic functions belonging to weighted Lipschitz class $W(L^p, \omega(t), \beta)$ with $p \geq 1$, through trigonometric polynomials using different summability methods and obtained $||t_n(f;x) - f(x)||_p = O((n+1)^{\beta+1/p}\ \omega(1/(n+1)))$ which clearly depends on p. However, Khan [35] has obtained $|s_n(f;x) - f(x)| = O((n+1)^{1/p}\ \omega(1/(n+1)))$. Since $Lip\alpha \subseteq Lip(\alpha,p) \subseteq Lip(\xi(t),p) \subseteq W(L^p,\omega(t),\beta)$, a number of corollaries have also been deduced from the above results with a separate proof for the Lipschitz class $Lip\,\alpha\ (\alpha = 1)$ [45; 108; 112]. The authors have used various auxiliary conditions on the positive increasing function $\omega(t)$. Recently, following the remarks of Rhoades et al. [108, pp. 6870-6871] about the conditions on $\omega(t)$ and definition of $W(L^p, \omega(t), \beta)$, Singh et al. [112, pp. 3-4, Remarks 2.3 & 2.4], and Singh and Sonkar [114, p. 4, Remark 2.1] have improved the results of Lal [45] and Rhoades [106], respectively and obtained the same degree of approximation. The similar type of remarks can also be seen in [57]. Very recently, Łenski and Szal [56] have generalized the $C^1.T$ method of summability introduced by Mittal [77] to the product of two general summability matrices and obtained the degree of approximation point-wise in L_p-space, which is dependent on p.

Chandra [9] was the first to obtain the degree of approximation of $f \in Lip(\alpha, p)$ as $||t_n(f;x) - f(x)||_p = O(n^{-\alpha})$ which is free from p, and the same was continued by Liendler [52] and Mittal et al. [80; 81]. Thus to obtain a degree of approximation of $f \in W(L^p, \omega(t), \beta)$ which is sharper than earlier results and free from p is still an open problem which we address in this chapter.

2.2 Main Results

The above mentioned observations motivate us to study further the problem of approximation in weighted Lipschitz class $W(L^p, \omega(t), \beta)$, particularly result of Lal [45]. In this chapter, we use the $C^1.T$ method with a set of weaker conditions on $\omega(t)$ and prove the following:

Theorem 2.2.1. *Let $T \equiv (a_{n,k})$ be a lower triangular regular matrix with non-negative and non-decreasing (with respect to k, for $0 \leq k \leq n$) entries which satisfies, $A_{n,0} = 1, \forall n \in \mathbb{N}_0$ and*

$$b_{n,n-k} \geq 0 \quad for \quad 0 \leq k \leq n. \tag{2.2}$$

Then the degree of approximation of a 2π-periodic function $f \in Lip\alpha$ by $C^1.T$ means of its Fourier series is given by

$$\|t_n^{C^1.T}(f;x) - f(x)\|_\infty = \begin{cases} O((n+1)^{-\alpha}), & 0 < \alpha < 1, \\ O\left(\log(n+1)/(n+1)\right), & \alpha = 1. \end{cases} \tag{2.3}$$

Theorem 2.2.2. *Let $T \equiv (a_{n,k})$ be a lower triangular regular matrix same as in Theorem 2.2.1. Then the degree of approximation of a 2π-periodic function $f \in W(L^p, \omega(t), \beta)$ with $p > 1$ and $0 < \beta < 1/p$ by $C^1.T$ means of its Fourier series is given by*

$$\| t_n^{C^1.T}(f;x) - f(x) \|_p = O\left((n+1)^\beta \omega\left(1/(n+1)\right)\right), \tag{2.4}$$

provided a positive increasing function $\omega(t)$ satisfies the following conditions:

$$\omega(t)/t \;\; \text{is a decreasing function,} \tag{2.5}$$

$$\left(\int_0^{\pi/(n+1)} \left|\phi(t)\sin^\beta(t/2)/\omega(t)\right|^q dt\right)^{1/q} = O((n+1)^{-1/q}), \tag{2.6}$$

$$\left(\int_{\pi/(n+1)}^{\pi} \left(t^{-\delta}|\phi(t)|\sin^\beta(t/2)/\omega(t)\right)^p dt\right)^{1/p} = O((n+1)^{\delta-1/p}), \tag{2.7}$$

where δ is a real number such that $p^{-1} < \delta < \beta + p^{-1}$and $p^{-1} + q^{-1} = 1$. Also conditions (2.6) *and* (2.7) *hold uniformly in x.*

The conditions (2.6) and (2.7) of Theorem 2.2.2 give better estimate than theorems of Lal [45, p. 347], Singh et al. [112, p. 4] and Singh and Sonkar [114, p. 40], as the degree of approximation in our Theorem 2.2.2 is free from p.

Remark 2.2.1. *If we replace matrix T with Nörlund matrix N_p, then $C^1.T$ means of Fourier series of f defined in* (2.1) *reduces to $C^1.N_p$ means. Also for non-decreasing sequence $\{a_{n,k}\}$ in k, $\{p_n\}$ is non-increasing such that*

$$b_{n,n-k} = a_{n,n-k} - a_{n+1,n+1-k} = p_k/P_n - p_k/P_{n+1} \geq 0,$$

and condition (3) *of Theorem* 1 *of Lal* [45, p. 347] *is obvious as under*

$$P_\tau \sum_{v=\tau}^{n} P_v^{-1} \leq P_\tau . \frac{1}{P_\tau} \sum_{v=\tau}^{n} (1) \leq (n+1).$$

Hence Theorems 1 *and* 2 *of Lal* [45, p. 347] *are particular and improved cases of our Theorems* 2.2.1 *and* 2.2.2, *respectively.*

For the easy understanding of the conditions of the matrix T in Theorems 2.2.1 and 2.2.2, we show an example of the lower triangular regular matrix $T \equiv (a_{n,k})$ which satisfy the mentioned assumptions . Let us consider the matrix $T \equiv (a_{n,k})$ defined by

$$a_{n,k} = \begin{cases} \dfrac{2^k}{2^{n+1}-1}, & 0 \leq k \leq n \\ 0, & k > n\,. \end{cases}$$

It is easy to check that entries of the matrix $T \equiv (a_{n,k})$ are non-negative and non-decreasing (with respect to k, for $0 \leq k \leq n$). It also satisfies the conditions

$$A_{n,0} = \sum_{r=0}^{n} a_{n,r} = \sum_{r=0}^{n} \frac{2^r}{2^{n+1}-1} = \frac{1}{2^{n+1}-1} \sum_{r=0}^{n} 2^r = 1, \forall n \in \mathbb{N}_0$$

and

$$\begin{aligned} b_{n,n-k} := \Delta_n a_{n,n-k} &= a_{n,n-k} - a_{n+1,n+1-k} \\ &= \frac{2^{n-k}}{2^{n+1}-1} - \frac{2^{n+1-k}}{2^{n+2}-1} \\ &= 2^{n-k} \left[\frac{2^{n+2} - 2^{n+1}}{(2^{n+1}-1)(2^{n+2}-1)} \right] \geq 0 \text{ for } 0 \leq k \leq n. \end{aligned}$$

2.3 Lemmas

We need the following lemmas for the proof of our theorems.

Lemma 2.3.1. *If $\{a_{n,k}\}$ satisfies the conditions of Theorem* 2.2.1, *then* $(C^1.T)_n(t) = O(n+1)$, *for* $0 < t \leq \pi/(n+1)$.

Proof. Using $\sin nt \leq nt$ and $\sin(t/2) \geq t/\pi$ for $0 < t \leq \pi/(n+1)$, we have

$$\begin{aligned}
\left|(C^1.T)_n(t)\right| &= (2\pi(n+1))^{-1}\left|\sum_{r=0}^{n}\sum_{k=0}^{r} a_{r,r-k}\left(\sin(r-k+1/2)t\right)/\sin(t/2)\right| \\
&\leq (2\pi(n+1))^{-1}\sum_{r=0}^{n}\sum_{k=0}^{r} a_{r,r-k}\left|(\sin(r-k+1/2)t)/\sin(t/2)\right| \\
&\leq (2\pi(n+1))^{-1}\sum_{r=0}^{n}\sum_{k=0}^{r} a_{r,r-k}((r-k+1/2)t)/(t/\pi) \\
&\leq (4(n+1))^{-1}\sum_{r=0}^{n}\sum_{k=0}^{r} a_{r,r-k}(2r-2k+1) \\
&\leq (4(n+1))^{-1}\sum_{r=0}^{n}(2r+1)A_{r,0} \\
&= (4(n+1))^{-1}\sum_{r=0}^{n}(2r+1) = (n+1)^2/4(n+1) = O(n+1),
\end{aligned}$$

in view of $A_{r,0} = 1$.

Lemma 2.3.2. *If $\{a_{n,k}\}$ satisfies the conditions of Theorem* 2.2.1, *then*

$$|(C^1.T)_n(t)| = O\left(t^{-2}/(n+1)\right), \text{ for } \pi/(n+1) < t \leq \pi.$$

Proof. Using $\sin(t/2) \geq t/\pi$, for $\pi/(n+1) < t \leq \pi$, we have

$$\begin{aligned}
|\ (C^1.T)_n(t)\ | &= (2\pi(n+1))^{-1}\left|\sum_{r=0}^{n}\sum_{k=0}^{r} a_{r,r-k}(\sin(r-k+1/2)t)/\sin(t/2)\right| \\
&= O(t(n+1))^{-1}\left|Im\sum_{r=0}^{n}\sum_{k=0}^{r} a_{r,r-k}e^{i(r-k+1/2)t}\right| \\
&= O(t(n+1))^{-1}\left|\sum_{r=0}^{n}\sum_{k=0}^{r} a_{r,r-k}e^{i(r-k)t}\right|.
\end{aligned}$$

Following [77, pp. 445-446], we have

$$\begin{aligned}
\left|\sum_{r=0}^{n}\sum_{k=0}^{r} a_{r,r-k}e^{i(r-k)t}\right| &\leq \left|\sum_{r=0}^{\tau}\sum_{k=0}^{r} a_{r,r-k}e^{i(r-k)t}\right| + \left|\sum_{r=\tau+1}^{n}\sum_{k=0}^{\tau} a_{r,r-k}e^{i(r-k)t}\right| \\
&\quad + \left|\sum_{r=\tau+1}^{n}\sum_{k=\tau+1}^{r} a_{r,r-k}e^{i(r-k)t}\right| \\
&\leq K_1 + K_2 + K_3, \ \ say,
\end{aligned}$$

where τ is the integer part of $1/t$.

Now

$$K_1 \leq \sum_{r=0}^{\tau}\sum_{k=0}^{r} a_{r,r-k}\left|e^{i(r-k)t}\right| \leq \sum_{r=0}^{\tau} A_{r,0} = (\tau+1) = O(t^{-1}).$$

Using Abel's transformation after changing the order of summation in K_2, we have

$$\begin{aligned}
K_2 &:= \left|\sum_{k=0}^{\tau}\sum_{r=\tau+1}^{n} a_{r,r-k}e^{i(r-k)t}\right| = \left|\sum_{k=0}^{\tau}\left[\sum_{r=\tau+1}^{n-1}\left\{b_{r,r-k}\sum_{v=0}^{r} e^{i(v-k)t}\right\}\right.\right. \\
&\qquad \left.\left. + \ a_{n,n-k}\sum_{v=0}^{n} e^{i(v-k)t} - a_{\tau+1,\tau+1-k}\sum_{v=0}^{\tau} e^{i(v-k)t}\right]\right| \\
&= O(t^{-1})\sum_{k=0}^{\tau}\left(\sum_{r=\tau+1}^{n-1} b_{r,r-k} + a_{n,n-k} + a_{\tau+1,\tau+1-k}\right) \\
&= O(t^{-1})\sum_{k=0}^{\tau}\left(\sum_{r=\tau+1}^{n-1}(a_{r,r-k} - a_{r+1,r+1-k}) + a_{n,n-k} + a_{\tau+1,\tau+1-k}\right) \\
&= O(t^{-1})\sum_{k=0}^{\tau}(2a_{\tau+1,\tau+1-k} + a_{n,n-k} + a_{n+1,n+1-k}) \\
&= O(t^{-1})\sum_{k=0}^{\tau}(a_{\tau,\tau-k} + a_{n,n-k}) = O(t^{-1})\left(\sum_{k=0}^{\tau} a_{\tau,\tau-k} + \sum_{k=0}^{n} a_{n,n-k}\right) \\
&= O(t^{-1})\,(A_{\tau,0} + A_{n,0}) = O(t^{-1}),
\end{aligned}$$

in view of $b_{r,r-k} = a_{r,r-k} - a_{r+1,r+1-k} \geq 0$ for $0 \leq k \leq r$, and $A_{\tau,0} = A_{n,0} = 1$. Again using Abel's transformation in K_3, we have

$$K_3 := \left|\sum_{r=\tau+1}^{n}\left[\sum_{k=\tau+1}^{r-1}\left\{\Delta_k a_{r,r-k}\sum_{v=0}^{k} e^{i(r-v)t}\right\}\right.\right.$$

$$+ \, a_{r,0} \sum_{v=0}^{r} e^{i(r-v)t} - a_{r,r-\tau-1} \sum_{v=0}^{\tau} e^{i(r-v)t} \Big] \Big|$$

$$\begin{aligned}
&= O(t^{-1}) \sum_{r=\tau+1}^{n} \left[\sum_{k=\tau+1}^{r-1} \left| a_{r,r-k} - a_{r,r-k+1} \right| + a_{r,0} + a_{r,r-\tau-1} \right] \\
&= O(t^{-1}) \sum_{r=\tau+1}^{n} (a_{r,r-\tau} + a_{r,1} + a_{r,0} + a_{r,r-\tau-1}) \\
&= O(t^{-1}) \sum_{r=\tau+1}^{n} (a_{r,r-\tau}) \\
&= O(t^{-1}) \left[a_{\tau+1,1} + a_{\tau+2,2} + a_{\tau+3,3} + ... + a_{n,n-\tau} \right] \\
&= O(t^{-1}) \left[a_{\tau+1,1} + a_{\tau+1,2} + a_{\tau+1,3} + ... + a_{\tau+1,n-\tau} \right] \\
&= O(t^{-1}) A_{\tau+1,0} = O(t^{-1}),
\end{aligned}$$

in view of $a_{r,r-k} \geq a_{r+1,r+1-k} \geq a_{r+1,r-k}$ for $0 \leq k \leq r$, and $A_{\tau+1,0} = 1$. Collecting K_1, K_2 and K_3, we get

$$|(C^1.T)_n(t)| = O\left(t^{-2}/(n+1)\right).$$

2.4 Proof of Theorem 2.2.1

We have

$$s_n(f;x) - f(x) = \frac{1}{2\pi} \int_0^{\pi} \phi(t)(\sin(n+1/2)t/\sin(t/2))dt.$$

Using (2.1), we can write

$$\begin{aligned}
t_n^{C^1.T}(f;x) - f(x) &= \frac{1}{n+1} \sum_{r=0}^{n} \sum_{k=0}^{r} a_{r,k} \left[s_k(f;x) - f(x) \right] \\
&= \int_0^{\pi} \phi(t)(2\pi(n+1))^{-1} \sum_{r=0}^{n} \sum_{k=0}^{r} a_{r,r-k} \frac{\sin(r-k+1/2)t}{\sin(t/2)} dt \\
&= \int_0^{\pi/(n+1)} \phi(t)(C^1.T)_n(t)dt + \int_{\pi/(n+1)}^{\pi} \phi(t)(C^1.T)_n(t)dt \\
&= I_1 + I_2, \text{ say.} \qquad (2.8)
\end{aligned}$$

Using Lemma 2.3.1 and the fact that $f \in Lip\alpha \Rightarrow \phi(t) \in Lip\alpha$, we have

$$\begin{aligned} |I_1| &\leq \int_0^{\pi/(n+1)} |\phi(t)||(C^1.T)_n(t)|dt = O(n+1)\int_0^{\pi/(n+1)} t^\alpha dt \\ &= O(n+1)((n+1)^{-\alpha-1}) = O((n+1)^{-\alpha}). \end{aligned} \tag{2.9}$$

Now, using Lemma 2.3.2 and the fact that $f \in Lip\,\alpha \Rightarrow \phi(t) \in Lip\,\alpha$,

$$\begin{aligned} |I_2| &\leq \int_{\pi/(n+1)}^{\pi} |\phi(t)| \left|(C^1.T)_n(t)\right| dt \\ &= \int_{\pi/(n+1)}^{\pi} |\phi(t)| O\left[t^{-2}/(n+1)\right] dt \\ &= O((n+1)^{-1}) \int_{\pi/(n+1)}^{\pi} t^{\alpha-2} dt \\ &= \begin{cases} O((n+1)^{-\alpha}), & 0 < \alpha < 1, \\ O\left(\log(n+1)/(n+1)\right), & \alpha = 1. \end{cases} \end{aligned} \tag{2.10}$$

Collecting (2.8)-(2.10) and using $1/(n+1) \leq \log(n+1)/(n+1)$, we get

$$|\, t_n^{C^1.T}(f;x) - f(x) \,| = \begin{cases} O((n+1)^{-\alpha}), & 0 < \alpha < 1, \\ O\left(\log(n+1)/(n+1)\right), & \alpha = 1. \end{cases}$$

Thus

$$\begin{aligned} \|t_n^{C^1.T}(f;x) - f(x)\|_\infty &= ess \sup_{0 \leq x \leq 2\pi} |t_n^{C^1.T}(f;x) - f(x)| \\ &= \begin{cases} O((n+1)^{-\alpha}), & 0 < \alpha < 1, \\ O\left(\log(n+1)/(n+1)\right), & \alpha = 1. \end{cases} \end{aligned}$$

This completes the proof of Theorem 2.2.1.

2.5 Proof of Theorem 2.2.2

Following the proof of Theorem 2.2.1, we have

$$\begin{aligned} t_n^{C^1.T}(f;x) - f(x) &= \int_0^{\pi/(n+1)} \phi(t)(C^1.T)_n(t)dt + \int_{\pi/(n+1)}^{\pi} \phi(t)(C^1.T)_n(t)dt \\ &= I_1' + I_2', \text{ say.} \end{aligned} \tag{2.11}$$

Using Hölder's inequality, $\phi(t) \in W(L^p, \omega(t), \beta)$, condition (2.6), $\sin(t/2) \geq t/\pi$, Lemma 2.3.1 and the mean value theorem for integrals, we have

$$\begin{aligned}
|I_1'| &= \left|\lim_{\epsilon\to 0}\int_\epsilon^{\pi/(n+1)} \left[(\phi(t)\sin^\beta(t/2)/\omega(t)).(\omega(t)(C^1.T)_n(t)/\sin^\beta(t/2))\right]dt\right| \\
&\leq \left[\int_0^{\pi/(n+1)} \left(|\phi(t)|\sin^\beta(t/2)/\omega(t)\right)^q dt\right]^{1/q} \times \\
&\quad \left[\lim_{\epsilon\to 0}\int_\epsilon^{\pi/(n+1)} \left(\omega(t)|(C^1.T)_n(t)|/\sin^\beta(t/2)\right)^p dt\right]^{1/p} \\
&= O((n+1)^{-1/q})\left[\lim_{\epsilon\to 0}\int_\epsilon^{\pi/(n+1)} \left|\omega(t)(n+1)/\sin^\beta(t/2)\right|^p dt\right]^{1/p} \\
&= O((n+1)^{1-1/q})(\omega(\pi/(n+1)))\left[\lim_{\epsilon\to 0}\int_\epsilon^{\pi/(n+1)} t^{-\beta p}dt\right]^{1/p} \\
&= O(\omega(\pi/(n+1))(n+1)^{\beta+1-1/q-1/p}) = O((n+1)^\beta\omega(\pi/(n+1))), \quad (2.12)
\end{aligned}$$

in view of $0 < \beta < 1/p$ and $p^{-1} + q^{-1} = 1$.

Using Lemma 2.3.2, Hölder's inequality, $|\sin t| \leq 1$, $\sin(t/2) \geq t/\pi$, the mean value theorem for integrals and condition (2.7), we have

$$\begin{aligned}
|I_2'| &= \left[\int_{\pi/(n+1)}^{\pi} |\phi(t)|\left[O\left(t^{-2}/(n+1)\right)\right]dt\right] \\
&= O\left[\int_{\pi/(n+1)}^{\pi} t^{-2}|\phi(t)|/(n+1)dt\right] \\
&= O\left[(n+1)^{-1}\int_{\pi/(n+1)}^{\pi} \left\{(t^{-\delta}|\phi(t)|\sin^\beta(t/2)/\omega(t)) \times \right.\right. \\
&\quad \left.\left.(\omega(t)/(t^{-\delta+2}\sin^\beta(t/2)))\right\}dt\right] \\
&\leq O((n+1)^{-1})\left[\int_{\pi/(n+1)}^{\pi} \left|t^{-\delta}|\phi(t)|\sin^\beta(t/2)/\omega(t)\right|^p dt\right]^{1/p} \times \\
&\quad \left[\int_{\pi/(n+1)}^{\pi} \left|\omega(t)/\left(t^{-\delta+2}\sin^\beta(t/2)\right)\right|^q dt\right]^{1/q} \\
&= O((n+1)^{-1})\left[\int_{\pi/(n+1)}^{\pi} \left|t^{-\delta}|\phi(t)|\sin^\beta(t/2)/\omega(t)\right|^p dt\right]^{1/p} \times
\end{aligned}$$

$$
\begin{aligned}
& \left[\int_{\pi/(n+1)}^{\pi} \left|\omega(t)/\left(t^{-\delta+2}\sin^{\beta}(t/2)\right)\right|^{q} dt\right]^{1/q} \\
= \;& O((n+1)^{-1})O\left((n+1)^{\delta-1/p}\right)\left[\int_{\pi/(n+1)}^{\pi} \left|\omega(t)/t^{-\delta+2+\beta}\right|^{q} dt\right]^{1/q} \\
= \;& O((n+1)^{\delta-1-1/p})((n+1)/\pi)\omega(\pi/(n+1))\left[\int_{\pi/(n+1)}^{\pi} t^{-(1-\delta+\beta)q} dt\right]^{1/q} \\
= \;& O\left((n+1)^{\delta-1/p}\omega(\pi/(n+1))(n+1)^{(1+\beta-\delta)-1/q}\right) \\
= \;& O((n+1)^{\beta}\omega(\pi/(n+1))), \qquad (2.13)
\end{aligned}
$$

in view of condition (2.5), $p^{-1} < \delta < \beta + p^{-1}$ and $p^{-1} + q^{-1} = 1$.

Collecting (2.11) - (2.13), we have

$$
|t_n^{C^1.T}(f;x) - f(x)| = O\left((n+1)^{\beta}\omega(\pi/(n+1))\right).
$$

Hence,

$$
\begin{aligned}
\|t_n^{C^1.T}(f;x) - f(x)\|_p &= \left(\frac{1}{2\pi}\int_0^{2\pi} |t_n^{C^1.T}(f;x) - f(x)|^p dx\right)^{1/p} \\
&= O\left((n+1)^{\beta}\omega\left(\pi/(n+1)\right)\right) \\
&= O\left((n+1)^{\beta}\omega\left(1/(n+1)\right)\right),
\end{aligned}
$$

in view of condition (2.5), i.e., $\omega(\pi/(n+1))/(\pi/(n+1)) \le \omega(1/(n+1))/(1/(n+1))$.

This completes the proof of Theorem 2.2.2.

Remark 2.5.1. *Most of the authors mentioned above have taken $p \ge 1$ in their theorems and applied Hölder's inequality without using L_∞-norm when $q = \infty$(i.e., $p = 1$). Thus proofs of their theorems are not valid for $p = 1$. Therefore, we have taken $p > 1$ in Theorem* 2.2.2 *stated above* [2, pp. 32-33].

Theorem 2.5.1. *Let $T \equiv (a_{n,k})$ be a lower triangular regular matrix same as in Theorem* 2.2.1. *Then the degree of approximation of a 2π-periodic function f belonging to the weighted Lipschitz class $W(L^1, \omega(t), \beta)$, with $0 < \beta < 1$ by $C^1.T$ means of its Fourier series is given by*

$$
\| t_n^{C^1.T}(f;x) - f(x) \|_1 = O\left((n+1)^{\beta}\omega(1/(n+1))\right), \qquad (2.14)
$$

provided a positive increasing function $\omega(t)$ satisfies (2.5) *and the following condition:*

$$
\omega(t)/t^{\beta} \text{ is non-decreasing}, \qquad (2.15)
$$

$$\int_0^{\pi/(n+1)} \frac{\mid \phi(t) \mid \sin^{\beta}(t/2)}{\omega(t)} dt = O((n+1)^{-1}), \tag{2.16}$$

$$\int_{\pi/(n+1)}^{\pi} \frac{t^{-\delta} \mid \phi(t) \mid . \sin^{\beta}(t/2)}{\omega(t)} dt = O((n+1)^{\delta-1}), \tag{2.17}$$

where $1 < \delta < \beta + 1$. *The conditions* (2.16) *and* (2.17) *hold uniformly in x.*

2.6 Proof of Theorem 2.5.1

Following the proof of Theorem 2.2.2, for $p = 1$ *i.e.*, $q = \infty$, we have

$$\begin{aligned}
|I_1'| &\leq \int_0^{\pi/(n+1)} \left(\frac{|\phi(t)| \sin^{\beta}(t/2)}{\omega(t)} \right) dt \times ess \sup_{0<t\leq\pi/(n+1)} \left| \frac{\omega(t)|(C^1.T)_n(t)}{\sin^{\beta}(t/2)} | \right| \\
&= O(n+1)^{-1} \times ess \sup_{0<t\leq\pi/(n+1)} \left| \frac{\omega(t).(n+1)}{\sin^{\beta}(t/2)} \right| \\
&= O(1)\ ess \sup_{0<t\leq\pi/(n+1)} \left| \frac{\omega(t)}{t^{\beta}} \right| \\
&= O(1) \left\{ \frac{\omega(\pi/(n+1))}{(\pi/(n+1))^{\beta}} \right\} \\
&= O((n+1)^{\beta}\, \omega(\pi/(n+1)), \qquad (2.18)
\end{aligned}$$

in view of conditions (2.15) and (2.16).

$$\begin{aligned}
|I_2'| &= O\left\{ \frac{1}{n+1} \int_{\pi/(n+1)}^{\pi} \frac{t^{-\delta}|\phi(t)| \sin^{\beta}(t/2)}{\omega(t)} dt \right\} \\
&\quad \times ess \sup_{\pi/(n+1)\leq t\leq\pi} \left| \frac{\omega(t)}{t^{-\delta+2}. \sin^{\beta}(t/2)} \right| \\
&= O\left[(n+1)^{\delta-2} \omega\left(\frac{\pi}{n+1}\right) \left(\frac{(n+1)^{2+\beta-\delta}}{\pi^{2+\beta-\delta}} \right) \right] \\
&= O[(n+1)^{\beta} \omega(\pi/(n+1))], \qquad (2.19)
\end{aligned}$$

in view of decreasing nature of $\omega(t)/t^{\beta-\delta+2}$ and condition (2.17).

Collecting (2.18) and (2.19), we get

$$\mid t_n^{C^1.T}(f;x) - f(x) \mid = O[(n+1)^{\beta} \omega(\pi/(n+1))]. \tag{2.20}$$

The work of Theorems 2.2.1, 2.2.2 and 2.5.1 have been accepted for publication in *Asian-European Journal of Mathematics* (*World Scientific*).

Hence

$$\| t_n^{C^1.T}(f;x) - f(x) \|_1 = O\left((n+1)^{\beta}\omega(1/(n+1))\right), \tag{2.21}$$

in view of (2.5). This completes the proof of Theorem 2.5.1.

As mentioned in Chapter 1 on Page 5, a separate study of conjugate Fourier series is required. In the next theorems, we shall discuss the degree of approximation of $\tilde{f}$, conjugate of f belonging to $Lip\alpha$ and $W(L^p,\omega(t),\beta)$, $p \geq 1$ by the $C^1.T$ means of conjugate Fourier series.

The sequence-to-sequence transformation

$$\tilde{t}_n(f;x) := \sum_{k=0}^{n} a_{n,k}\tilde{s}_k(f;x), \quad n \in \mathbb{N}_0,$$

defines the matrix means of $\{\tilde{s}_n(f;x)\}$. The conjugate Fourier series of the function f is said to be T-summable to s, if $\tilde{t}_n(f;x) \to s$ as $n \to \infty$.
The $C^1.T$ means of $\{\tilde{s}_n(f;x)\}$ denoted by $\tilde{t}_n^{C^1.T}(f;x)$ are given by

$$\tilde{t}_n^{C^1.T}(f;x) := (n+1)^{-1}\sum_{r=0}^{n}\left(\sum_{k=0}^{r} a_{r,k}\tilde{s}_k(f;x)\right), \quad n \in \mathbb{N}_0. \tag{2.22}$$

If $\tilde{t}_n^{C^1.T}(f;x) \to s_1$ as $n \to \infty$, then the conjugate Fourier series of f is said to be $C^1.T$ -summable to the sum s_1.

We also write

$$\widetilde{(C^1.T)}_n(t) = \frac{1}{2\pi(n+1)}\sum_{r=0}^{n}\sum_{k=0}^{r} a_{r,r-k}\frac{\cos(r-k+1/2)t}{\sin(t/2)}.$$

Theorem 2.6.1. *Let $T \equiv (a_{n,k})$ be a lower triangular regular matrix same as in Theorem 2.2.1. Then the degree of approximation of $\tilde{f}$, conjugate of a 2π - periodic function $f \in Lip\alpha$ by $C^1.T$ means of its conjugate Fourier series is given by*

$$\|\tilde{t}_n^{C^1.T}(f;x) - \tilde{f}(x)\|_\infty = \begin{cases} O((n+1)^{-\alpha}), & 0 < \alpha < 1, \\ O\left(\log(n+1)/(n+1)\right), & \alpha = 1. \end{cases} \tag{2.23}$$

Theorem 2.6.2. *Let $T \equiv (a_{n,k})$ be a lower triangular regular matrix same as in Theorem 2.2.1. Then the degree of approximation of $\tilde{f}$, conjugate of a 2π-periodic function $f \in W(L^p,\omega(t),\beta)$, with $p > 1$ and $0 \leq \beta < 1/p$ by $C^1.T$ means of its conjugate Fourier series is given by*

$$\| \tilde{t}_n^{C^1.T}(f;x) - \tilde{f}(x) \|_p = O\left((n+1)^{\beta}\omega(1/(n+1))\right), \tag{2.24}$$

provided a positive increasing function $\omega(t)$ satisfies the following conditions:

$$\omega(t)/t^{\beta+1-\sigma} \text{ is non-decreasing}, \tag{2.25}$$

$$\left\{ \int_0^{\pi/(n+1)} \left(\frac{t^{-\sigma} \mid \psi_x(t) \mid \sin^{\beta}(t/2)}{\omega(t)} \right)^p dt \right\}^{1/p} = O((n+1)^{\sigma - 1/p}), \tag{2.26}$$

for $\beta < \sigma < 1/p$,

$$\omega(t)/t \text{ is non-increasing}, \tag{2.27}$$

$$\left\{ \int_{\pi/(n+1)}^{\pi} \left(\frac{t^{-\delta} \mid \psi_x(t) \sin^{\beta}(t/2) \mid}{\omega(t)} \right)^p dt \right\}^{1/p} = O((n+1)^{\delta - 1/p}), \tag{2.28}$$

where δ *is an arbitrary number such that* $1/p < \delta < \beta + 1/p$ *and* $p^{-1} + q^{-1} = 1$. *The conditions* (2.26) *and* (2.28) *hold uniformly in* x.

The condition (2.28) above is improved version of condition (14) of [22].

2.7 Lemmas

We need the following lemmas for the proof of Theorems 2.6.1 and 2.6.2.

Lemma 2.7.1. *If* $\{a_{n,k}\}$ *satisfies the conditions of Theorem 2.2.1, then* $(\widetilde{C^1}.T)_n(t) = O(1/t)$, *for* $0 < t \leq \pi/(n+1)$.

Proof. Using $|\cos t| \leq 1$ and $\sin(t/2) \geq t/\pi$ for $0 < t \leq \pi/(n+1)$, we have

$$\begin{aligned}
\left|(\widetilde{C^1}.T)_n(t)\right| &= (2\pi(n+1))^{-1} \left| \sum_{r=0}^{n} \sum_{k=0}^{r} a_{r,r-k} (\cos(r-k+1/2)t)/(\sin t/2) \right| \\
&\leq (2\pi(n+1))^{-1} \sum_{r=0}^{n} \sum_{k=0}^{r} a_{r,r-k} \left|(\cos(r-k+1/2)t)/(\sin t/2)\right| \\
&\leq (2\pi(n+1))^{-1} \sum_{r=0}^{n} \sum_{k=0}^{r} a_{r,r-k} 1/(t/\pi) \\
&= O((n+1)t)^{-1} \sum_{r=0}^{n} \left(\sum_{k=0}^{r} a_{r,r-k} \right) \\
&= O((n+1)t)^{-1} \sum_{r=0}^{n} (1) \\
&= O(1/t).
\end{aligned}$$

Lemma 2.7.2. *If* $\{a_{n,k}\}$ *satisfies the conditions of Theorem 2.2.1, then*

$$|(\widetilde{C^1}.T)_n(t)| = O\left(t^{-2}/(n+1)\right), \quad \pi/(n+1) < t \leq \pi.$$

Proof. Using $\sin(t/2) \geq t/\pi$, for $\pi/(n+1) < t \leq \pi$, we have

$$\begin{aligned} |\, (\widetilde{C^1}.T)_n(t)\,| &= (2\pi(n+1))^{-1} \left| \sum_{r=0}^{n} \sum_{k=0}^{r} a_{r,r-k} (\cos(r-k+1/2)t)/\sin(t/2) \right| \\ &= O(t(n+1))^{-1} \left| Re \sum_{r=0}^{n} \sum_{k=0}^{r} a_{r,r-k} e^{i(r-k+1/2)t} \right| \\ &= O(t(n+1))^{-1} \left| \sum_{r=0}^{n} \sum_{k=0}^{r} a_{r,r-k} e^{i(r-k)t} \right|. \end{aligned}$$

Following Lemma 2.3.2, we have

$$|(\widetilde{C^1}.T)_n(t)| = O\left(t^{-2}/(n+1)\right).$$

2.8 Proof of Theorem 2.6.1

Using the integral representation of $\tilde{s}_n(f;x)$ given in (1.5), we can write

$$\tilde{s}_n(f;x) - \tilde{f}(x) = \frac{1}{\pi} \int_0^{\pi} \psi_x(t) \frac{\cos(n+1/2)t}{\sin(t/2)} dt.$$

Now, using (2.1), we write

$$\begin{aligned} \tilde{t}_n^{C^1.T}(f;x) - \tilde{f}(x) &= \frac{1}{n+1} \sum_{r=0}^{n} \sum_{k=0}^{r} a_{r,k} \left[\tilde{s}_k(f;x) - \tilde{f}(x)\right] \\ &= \int_0^{\pi} \psi_x(t) (2\pi(n+1))^{-1} \sum_{r=0}^{n} \sum_{k=0}^{r} a_{r,r-k} \frac{\cos(r-k+1/2)t}{\sin(t/2)} dt \\ &= \int_0^{\pi/(n+1)} \psi_x(t) (\widetilde{C^1}.T)_n(t) dt + \int_{\pi/(n+1)}^{\pi} \psi_x(t) (\widetilde{C^1}.T)_n(t) dt \\ &= I_1 + I_2, \text{ say.} \end{aligned} \tag{2.29}$$

Using Lemma 2.7.1 and the fact that $\psi_x(t) \in Lip\alpha$, we have

$$\begin{aligned} |I_1| \leq \int_0^{\pi/(n+1)} |\psi_x(t)| |(\widetilde{C^1}.T)_n(t)| dt &= O \int_0^{\pi/(n+1)} t^{\alpha-1} dt \\ &= O((n+1)^{-\alpha}). \end{aligned} \tag{2.30}$$

Now, using Lemma 2.7.2 and the fact that $\psi_x(t) \in Lip\, \alpha$,

$$|I_2| \leq \int_{\pi/(n+1)}^{\pi} |\psi_x(t)| \left| (\widetilde{C^1}.T)_n(t) \right| dt$$

$$
\begin{aligned}
&= \int_{\pi/(n+1)}^{\pi} |\psi_x(t)| O\left[t^{-2}/(n+1)\right] dt \\
&= O((n+1)^{-1}) \int_{\pi/(n+1)}^{\pi} t^{\alpha-2} dt \\
&= \begin{cases} O((n+1)^{-\alpha}), & 0 < \alpha < 1, \\ O\left(\log(n+1)/(n+1)\right), & \alpha = 1. \end{cases} \qquad (2.31)
\end{aligned}
$$

Collecting (2.29)-(2.31) and using $1/(n+1) = O(\log(n+1)/(n+1))$, we get

$$
|\ \tilde{t}_n^{C^1.T}(f;x) - \tilde{f}(x)\ | = \begin{cases} O((n+1)^{-\alpha}), & 0 < \alpha < 1, \\ O\left(\log(n+1)/(n+1)\right), & \alpha = 1. \end{cases} \qquad (2.32)
$$

Thus

$$
\begin{aligned}
\|\tilde{t}_n^{C^1.T}(f;x) - \tilde{f}(x)\|_\infty &= ess \sup_{0 \le x \le 2\pi} \{|\tilde{t}_n^{C^1.T}(f;x) - \tilde{f}(x)|\} \\
&= \begin{cases} O((n+1)^{-\alpha}), & 0 < \alpha < 1, \\ O\left(\log(n+1)/(n+1)\right), & \alpha = 1. \end{cases} \qquad (2.33)
\end{aligned}
$$

This completes proof of the Theorem 2.6.1.

2.9 Proof of Theorem 2.6.2

Following the proof of Theorem 2.6.1, we have

$$
\begin{aligned}
\tilde{t}_n^{C^1.T}(f;x) - \tilde{f}(x) &= \int_0^{\pi/(n+1)} \psi_x(t) (\widetilde{C^1.T})_n(t) dt + \int_{\pi/(n+1)}^{\pi} \psi_x(t) (\widetilde{C^1.T})_n(t) dt \\
&= I_1' + I_2', \text{ say.} \qquad (2.34)
\end{aligned}
$$

Using Hölder's inequality, conditions (2.25), (2.26), $\sin(t/2) \geq t/\pi$, Lemma 2.7.1, the mean value theorem for integrals and $p^{-1} + q^{-1} = 1$, we have

$$
\begin{aligned}
|I_1'| &= \lim_{\epsilon \to 0} \int_\epsilon^{\pi/(n+1)} \left| \left[t^{-\sigma} (\psi_x(t) \sin^\beta(t/2)/\omega(t)) \times \right.\right. \\
&\qquad \left.\left. (\omega(t) (\widetilde{C^1.T})_n(t) / t^{-\sigma} \sin^\beta(t/2)) \right] \right| dt \\
&\leq \left[\int_0^{\pi/(n+1)} \left(t^{-\sigma} |\psi_x(t)| \sin^\beta(t/2)/\omega(t) \right)^p dt \right]^{1/p} \times
\end{aligned}
$$

$$\left[\lim_{\epsilon\to 0}\int_{\epsilon}^{\pi/(n+1)}\left(\frac{\omega(t)|\widetilde{(C^1.T)}_n(t)|}{t^{-\sigma}\sin^{\beta}(t/2)}\right)^q dt\right]^{1/q}$$

$$\begin{aligned}
&= O((n+1)^{\sigma-1/p})\left[\lim_{\epsilon\to 0}\int_{\epsilon}^{\pi/(n+1)}\left|\omega(t)/(t^{1-\sigma}.\sin^{\beta}(t/2))\right|^q dt\right]^{1/q}\\
&= O((n+1)^{\sigma-1/p})\left[\lim_{\epsilon\to 0}\int_{\epsilon}^{\pi/(n+1)}\left|\omega(t)/(t^{\beta+1-\sigma})\right|^q dt\right]^{1/q}\\
&= O((n+1)^{\sigma-1/p})(n+1)^{\beta+1-1/q-\sigma}\omega(\pi/(n+1))\\
&= O((n+1)^{\beta}\,\omega(\pi/(n+1))). \qquad (2.35)
\end{aligned}$$

Again using Lemma 2.7.2, Hölder's inequality and $(\sin(t/2))^{-1}\le \pi/t$ for $0<t\le\pi$, we have

$$\begin{aligned}
|I_2'| &= \left[\int_{\pi/(n+1)}^{\pi}|\psi_x(t)|\left[O\left(t^{-2}/(n+1)\right)\right]dt\right]\\
&= O\left[\int_{\pi/(n+1)}^{\pi}t^{-2}|\psi_x(t)|/(n+1)dt\right]\\
&= O\left(\int_{\pi/(n+1)}^{\pi}\frac{t^{-\delta}|\psi_x(t)|\sin^{\beta}(t/2)}{(n+1)\omega(t)}\;\frac{t^{-1}\omega(t)}{t^{-\delta}t\sin^{\beta}(t/2)}dt\right)\\
&= O\left\{\frac{1}{n+1}\int_{\pi/(n+1)}^{\pi}\left(\frac{t^{-\delta}|\psi_x(t)|}{\omega(t)}\right)^p dt\right\}^{1/p}\times\left\{\int_{\pi/(n+1)}^{\pi}\left(\frac{t^{-1}\omega(t)}{t^{-\delta+\beta+1}}\right)^q dt\right\}^{1/q}\\
&= O\left[(n+1)^{\delta-1-1/p}\omega\left(\frac{\pi}{n+1}\right)\left(\frac{n+1}{\pi}\right)\left(\int_{\pi/(n+1)}^{\pi}t^{-(\beta+1-\delta)q}dt\right)^{1/q}\right]\\
&= O\left[(n+1)^{\delta-1/p}\omega(\pi/(n+1))(n+1)^{\beta+1-\delta-1/q}\right]\\
&= O[(n+1)^{\beta}\omega(\pi/(n+1))], \qquad (2.36)
\end{aligned}$$

in view of (2.27), (2.28), the mean value theorem for integrals, $1/p<\delta<\beta+1/p$ and $p^{-1}+q^{-1}=1$.

Collecting (2.34)-(2.36), we get

$$|\ \tilde{t}_n^{C^1.T}(f;x)-\tilde{f}(x)\ |= O[(n+1)^{\beta}\omega(\pi/(n+1))]. \qquad (2.37)$$

Finally from (2.37), we easily get

$$\| \tilde{t}_n^{C^1.T}(f;x) - \tilde{f}(x) \|_p = O\left((n+1)^{\beta}\omega(1/(n+1))\right), \tag{2.38}$$

in view of Note 1. This completes the proof of Theorem 2.6.2.

As mentioned in Remark 2.5.1, the above proof will not work for $p = 1$. Thus, for $p = 1$, we have the following theorem.

Theorem 2.9.1. *Let $T \equiv (a_{n,k})$ be the same as in Theorem 2.6.2. Then the degree of approximation of $\tilde{f}$, conjugate of a 2π-periodic function f belonging to the weighted Lipschitz class $W(L^1, \omega(t), \beta)$, with $0 \le \beta < 1$ by $C^1.T$ means of its conjugate Fourier series is given by*

$$\| \tilde{t}_n^{C^1.T}(f;x) - \tilde{f}(x) \|_1 = O\left((n+1)^{\beta}\omega(1/(n+1))\right), \tag{2.39}$$

provided a positive increasing function $\omega(t)$ satisfies conditions (2.25) to (2.28) of Theorem 2.6.2 for $p = 1$, $\beta < \sigma < 1$ and $1 < \delta < \beta + 1$.

2.10 Proof of Theorem 2.9.1

Following the proof of Theorem 2.6.2, for $p = 1$, *i.e.*, $q = \infty$, we have

$$\begin{aligned}
I_1' &= \int_0^{\pi/(n+1)} \left(\frac{t^{-\sigma}|\psi_x(t)|\sin^{\beta}(t/2)}{\omega(t)} \right) dt \times ess \sup_{0<t\le \pi/(n+1)} \left| \frac{\omega(t)|(\widetilde{C^1.T})_n(t)|}{t^{-\sigma}\sin^{\beta}(t/2)} \right| \\
&= \int_0^{\pi/(n+1)} \left(\frac{t^{-\sigma}|\psi_x(t)|\sin^{\beta}(t/2)}{\omega(t)} \right) dt \times ess \sup_{0<t\le \pi/(n+1)} \left| \frac{\omega(t)}{t^{-\sigma+1}\sin^{\beta}(t/2)} \right| \\
&= O((n+1)^{\sigma-1})\; ess \sup_{0<t\le \pi/(n+1)} \left| \frac{\omega(t)}{t^{\beta-\sigma+1}} \right| \\
&= O((n+1)^{\sigma-1}) \left\{ \frac{\omega(\pi/(n+1))}{(\pi/(n+1))^{\beta-\sigma+1}} \right\} \\
&= O((n+1)^{\beta}\,\omega(\pi/(n+1))).
\end{aligned} \tag{2.40}$$

in view of conditions (2.25) and (2.26) for $p = 1$.

$$I_2' = O\left\{ \frac{1}{n+1} \int_{\pi/(n+1)}^{\pi} \frac{t^{-\delta}|\psi_x(t)|\sin^{\beta}(t/2)}{\omega(t)} dt \right\} \times$$

The work of Theorems 2.6.1, 2.6.2 and 2.9.1 have been published as a book chapter in IAENG Transactions on Engineering Sciences, CRC Press/Balkema (Taylor & Francis Group), (2014), 81-89.

$$\begin{aligned} & ess \sup_{\pi/(n+1)\le t\le\pi} \left| \frac{\omega(t)}{t^{-\delta+\beta+2}} \right| \\ &= O\left[(n+1)^{\delta-2}\omega\left(\frac{\pi}{n+1}\right)\left(\frac{(n+1)^{2+\beta-\delta}}{\pi^{2+\beta-\delta}}\right)\right] \\ &= O[(n+1)^{\beta}\omega(\pi/(n+1))], \end{aligned} \tag{2.41}$$

in view of (2.27), i.e., decreasing nature of $\omega(t)/t^{-\delta+\beta+2}$ and (2.28).
Collecting (2.40) and (2.41), we get

$$| \tilde{t}_n^{C^1.T}(f;x) - \tilde{f}(x) | = O[(n+1)^{\beta}\omega(\pi/(n+1))]. \tag{2.42}$$

Finally from (2.42), we easily get

$$\| \tilde{t}_n^{C^1.T}(f;x) - \tilde{f}(x) \|_1 = O\left((n+1)^{\beta}\omega(1/(n+1))\right), \tag{2.43}$$

in view of Note 1. This completes the proof of Theorem 2.9.1.

2.11 Corollaries

If we define $T \equiv (a_{n,k})$ as

$$a_{n,k} = \begin{cases} p_{n-k}/P_n & \text{for } \ 0 \le k \le n, \\ 0, & \text{for } \qquad k > n, \end{cases}$$

where $P_n = \sum_{k=0}^{n} p_k \to \infty$ as $n \to \infty$, then matrix T reduces to Nörlund matrix N_p and the $C^1.T$ means of $\{\tilde{s}_n(f;x)\}$ defined in (2.22) reduces to $C^1.N_p$ means given by

$$\tilde{t}_n^{C^1.N}(f) := \tilde{t}_n^{C^1.N}(f;x) = \frac{1}{n+1}\sum_{r=0}^{n}\sum_{k=0}^{r} p_{r-k}\tilde{s}_k(f;x)$$

Also, if $(a_{n,k})$ is non-decreasing, then $\{p_k\}$ is non-increasing so that $b_{n,n-k} = a_{n,n-k} - a_{n+1,n+1-k} = p_k/P_n - p_k/P_{n+1} \ge 0$, i.e., condition is satisfied. Thus we have the following $C^1.N_p$ analogues of Theorem 2.6.1:

Corollary 2.11.1

Let N_p be a regular Nörlund matrix generated by non-increasing and non-negative sequence $\{p_k\}$, then the degree of approximation of $\tilde{f}$, conjugate of a $2\pi-$periodic function $f \in Lip\alpha$ by $C^1.N_p$ means of its conjugate Fourier series is given by

$$\|\tilde{t}_n^{C^1.N}(f;x) - \tilde{f}(x)\|_\infty \quad = \quad \begin{cases} O((n+1)^{-\alpha}), & 0 < \alpha < 1, \\ O\left(\log(n+1)/(n+1)\right), & \alpha = 1. \end{cases} \tag{2.44}$$

In the similar way, we can get $C^1.N_p$ analogues of our Theorems 2.6.2 and 2.9.1 of this chapter. For example, Mishra et. al [76, p. 158] have proved a theorem on the degree of approximation of conjugate of functions belonging to weighted $W(L_r, \omega(t), \beta)$-class using $C^1.N_p$ means (taking semi-monotonicity on the generating sequence $\{p_n\}$) of conjugate series of Fourier series.

Remark 2.11.1. *In the light of Rhoades et al.* [108], *Singh and Sonkar* [114] *and references therein, Łenski and Szal* [57] *and Kranz et al.* [38], *we observe that in* [14; 43; 70; 72; 73; 79; 82; 93; 94; 97; 103; 107] *the authors have defined the weighted Lipschitz class by*

$$W(L^p, \omega(t), \beta) = \{f \in L^p[0, 2\pi] : \left\|(f(x+t) - f(x)) \sin^\beta x\right\|_p = O(\omega(t))\},$$

where $\omega(t)$ is a positive increasing function and $\beta \geq 0$; and used the condition of the form $\left(\int_0^{\pi/(n+1)} \left(t|\psi_x(t)| \sin^\beta(t)/\omega(t)\right)^p dt\right)^{1/p} = O\left((n+1)^{-1}\right)$, which leads to a divergent integral of the form $\int_0^{\pi/(n+1)} t^{-(\beta+1)q}\, dt$. Also they have used $\sin t \geq (2t/\pi)$ for $\pi/(n+1) < t \leq \pi$ which is not valid as $t \to \pi$ [75; 114]. *However, the authors in* [75] *have tried to resolve this problem by replacing $\sin x$ with $\sin(x/2)$ in the definition of $W(L^p, \omega(t), \beta)$, but do not resolve the problem of divergent integral $\int_0^{\pi/(n+1)} t^{-(\beta+1)q}\, dt$* [75, p.9]. *As pointed out by the authors in* [70; 72; 73; 75; 79; 82; 83; 93; 114], *many of the authors mentioned above including Łenski and Szal* [57] *and Kranz et al.* [38] *have used $\omega(t)$ as an increasing function. This condition alone is not sufficient to prove the results. One more condition, namely, $\omega(t)/t$ is non-increasing is also required. In our theorems, we have also made an attempt to resolve these issues.*

□□□

Chapter 3

Approximation by Hausdorff Means of Conjugate Fourier Series

3.1 Introduction

In this chapter, we determine the degree of approximation of $\widetilde{f}$, conjugate of a 2π-periodic function f belonging to the weighted $W(L^p, \omega(t), \beta)$-class and its subclasses such as $Lip(\xi(t), p)$, $Lip(\alpha, p)$ and $Lip\alpha$, by using Hausdorff means of conjugate Fourier series of f. Since $(C, 1)$, the Cesàro matrix of order 1, and (E, q), the Euler matrix of order $q > 0$, are Hausdorff matrices, and the product of two Hausdorff matrices is also a Hausdorff matrix [109], our theorems generalize and improve some of the previous results. Some corollaries have also been deduced from our results.

The Hausdorff matrix $H \equiv (h_{n,k})$ is an infinite lower triangular matrix defined by

$$h_{n,k} = \begin{cases} \binom{n}{k} \triangle^{n-k} \mu_k, & 0 \leq k \leq n, \\ 0, & k > n, \end{cases}$$

where $\triangle$ is the forward difference operator defined by $\triangle \mu_n = \mu_n - \mu_{n+1}$ and $\triangle^{k+1} \mu_n = \triangle^k (\triangle \mu_n)$. If H is regular, then $\{\mu_n\}$, known as moment sequence, has the representation

$$\mu_n = \int_0^1 u^n d\gamma(u), \tag{3.1}$$

where $\gamma(u)$, known as mass function, is continuous at $u = 0$ and belongs to $BV[0, 1]$ such that $\gamma(0) = 0, \gamma(1) = 1$; and for $0 < u < 1$, $\gamma(u) = [\gamma(u + 0) + \gamma(u - 0)]/2$ [17; 106].

The work of this chapter in the form of a research paper has been published in Journal of Computational and Applied Mathematics (Elsevier Publications) 259 (2014), 633–640.

The Hausdorff means of conjugate Fourier series are defined by

$$\tilde{H}_n(f;x) := \sum_{k=0}^{n} h_{n,k}\tilde{s}_k(f;x), \quad n = 0,1,2,... \tag{3.2}$$

The conjugate Fourier series is said to be summable to s by Hausdorff means, if $\tilde{H}_n(f;x) \to s$ as $n \to \infty$. For the mass function $\gamma(u)$ given by

$$\gamma(u) = \begin{cases} 0, & \text{if } 0 \leq u < a, \\ 1, & \text{if } a \leq u \leq 1, \end{cases} \tag{3.3}$$

where $a = 1/(1+q), q > 0$, we can verify that $\mu_k = 1/(1+q)^k$ and

$$h_{n,k} = \begin{cases} \binom{n}{k}\dfrac{q^{n-k}}{(1+q)^n}, & \text{if } 0 \leq k \leq n, \\ 0, & \text{if } k > n. \end{cases} \tag{3.4}$$

Thus the Hausdorff matrix $H \equiv (h_{n,k})$ reduces to (E,q), the Euler matrix of order $q > 0$, and defines the corresponding (E,q) means by

$$E_n^q(f;x) := \frac{1}{(1+q)^n}\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\tilde{s}_k(f;x). \tag{3.5}$$

One more example of Hausdorff matrix is the well known Cesàro matrix of order 1, denoted as $(C,1)$, which defines the corresponding means by

$$\sigma_n(f;x) := \frac{1}{n+1}\sum_{k=0}^{n}\tilde{s}_k(f;x). \tag{3.6}$$

To enrich the knowledge about Hausdorff matrices one can see [17; 109]. In this chapter, the class of all regular Hausdorff matrices with moment sequence $\{\mu_n\}$ associated with mass function $\gamma(u)$ having finite derivative, is denoted by H_1.

We also write

$$g(u,t) := Re\left[\textstyle\sum_{k=0}^{n}\binom{n}{k}u^k(1-u)^{n-k}e^{i(k+1/2)t}\right].$$

We note that the series, conjugate to a Fourier series, is not necessarily a Fourier series [2; 123]. *Hence a separate study of conjugate series is desirable and attracted the attention of researchers.*

3.2 Main Results

Taking into account of the observations in Remark 2.11.1 of Chapter 2, in this chapter, we prove the followings theorems:

Theorem 3.2.1. *Let f be a 2π-periodic function belonging to the weighted Lipschitz class $W(L^p, \omega(t), \beta)$, with $p > 1$ and $0 \leq \beta \leq 1 - 1/p$. Then the degree of approximation of $\tilde{f}$ by Hausdorff means of conjugate Fourier series of f generated by $H \in H_1$, is given by*

$$\| \tilde{H}_n(f; x) - \tilde{f}(x) \|_p = O\left((n+1)^{\beta+1/p}\omega(1/(n+1))\right), \tag{3.7}$$

provided a positive increasing function $\omega(t)$ satisfies the following conditions:

$$\omega(t)/t \text{ is non-increasing}, \tag{3.8}$$

$$\left\{\int_0^{\pi/(n+1)} \left(\frac{|\psi_x(t)| \sin^\beta(t/2)}{\omega(t)}\right)^p dt\right\}^{1/p} = O((n+1)^{-1/p}), \tag{3.9}$$

$$\left\{\int_\epsilon^{\pi/(n+1)} \left(\frac{\omega(t)}{t\sin^\beta(t/2)}\right)^q dt\right\}^{1/q} = O((n+1)^{\beta+1/p}\omega(\pi/(n+1))), \tag{3.10}$$

$$\left\{\int_{\pi/(n+1)}^{\pi} \left(\frac{t^{-\delta} |\psi_x(t)|}{\omega(t)}\right)^p dt\right\}^{1/p} = O((n+1)^\delta), \tag{3.11}$$

where δ is an arbitrary number such that $0 < \delta < \beta + 1/p$ and $p^{-1} + q^{-1} = 1$ for $p > 1$. The conditions (3.9) *and* (3.11) *hold uniformly in x.*

3.3 Lemma

For the proof of theorem 3.2.1, we need the following lemma:

Lemma 3.3.1. *Let $g(u,t) = \mathrm{Re}[\sum_{k=0}^{n} \binom{n}{k} u^k (1-u)^{n-k} e^{i(k+1/2)t}]$ for $0 < u < 1$ and $0 \leq t \leq \pi$. Then*

$$\left|\int_0^1 g(u,t) d\gamma(u)\right| = \begin{cases} O(1), & 0 < t \leq \pi/(n+1) \\ O(t^{-1}/(n+1)), & \pi/(n+1) \leq t \leq \pi. \end{cases}$$

Proof.

We can write

$$\begin{aligned} g(u,t) &= \mathrm{Re} \textstyle\sum_{k=0}^{n} \binom{n}{k} u^k (1-u)^{n-k} e^{i(k+1/2)t} \\ &= (1-u)^n \mathrm{Re}\left\{e^{it/2} \textstyle\sum_{k=0}^{n} \binom{n}{k} \left(\frac{ue^{it}}{1-u}\right)^k\right\} \\ &= \mathrm{Re}\left\{e^{it/2}\left(1-u+ue^{it}\right)^n\right\}. \end{aligned}$$

Now, since $\gamma'(u) \leq M$ (a constant), for $0 < t \leq \pi/(n+1)$ we have

$$\left|\int_0^1 g(u,t)\, d\gamma(u)\right| \leq \left|M \int_0^1 \mathrm{Re}\{e^{it/2}(1-u+ue^{it})^n\} du\right|$$

$$
\begin{aligned}
&= \left| M\mathrm{Re}\left\{ \int_0^1 \frac{e^{it/2}(1-u+ue^{it})^n}{(-1+e^{it})}(-1+e^{it})du \right\} \right| \\
&= \left| M\mathrm{Re}\frac{e^{i(n+1)t}-1}{(n+1)(e^{it/2}-e^{-it/2})} \right| = \left| M\frac{\sin(n+1)t}{2(n+1)\sin(t/2)} \right| \\
&\le \frac{(n+1)t}{(n+1)}(\pi/t) = O(1), \qquad (3.12)
\end{aligned}
$$

in view of $(\sin t)^{-1} \le \pi/2t$ for $0 < t \le \pi/2$ and $\sin t \le t$ for $t \ge 0$ [2, p.247].
Similarly, for $\pi/(n+1) \le t \le \pi$ we have

$$
\left| \int_0^1 g(u,t)\, d\gamma(u) \right| = O\left(\frac{t^{-1}}{n+1} \right), \qquad (3.13)
$$

in view of $|\sin t| \le 1$ for all t [2, p.247].
Collecting (3.12) and (3.13), we get Lemma 3.3.1.

3.4 Proof of Theorem 3.2.1

Using (1.5), we can write

$$
\tilde{s}_n(f;x) - \tilde{f}(x) = \frac{1}{2\pi}\int_0^\pi \psi_x(t)\frac{\cos(n+1/2)t}{\sin(t/2)}dt.
$$

Now,

$$
\begin{aligned}
\tilde{H}_n(f;x) - \tilde{f}(x) &= \textstyle\sum_{k=0}^n h_{n,k}\{\tilde{s}_k(f;x) - \tilde{f}(x)\} \\
&= \frac{1}{2\pi}\int_0^\pi \left(\frac{\psi_x(t)}{sin(t/2)} \textstyle\sum_{k=0}^n h_{n,k}\cos(k+1/2)t \right) dt \\
&= \frac{1}{2\pi}\int_0^\pi \left(\frac{\psi_x(t)}{\sin(t/2)} \textstyle\sum_{k=0}^n \binom{n}{k}\Delta^{n-k}\mu_k\cos(k+1/2)t \right) dt \\
&= \frac{1}{2\pi}\int_0^\pi \left(\frac{\psi_x(t)}{\sin(t/2)} \times \right. \\
&\qquad \left. \textstyle\sum_{k=0}^n \binom{n}{k} \displaystyle\int_0^1 u^k(1-u)^{n-k}d\gamma(u)\mathrm{Re}\{e^{i(k+1/2)t}\} \right) dt \\
&= \frac{1}{2\pi}\int_0^\pi \left(\frac{\psi_x(t)}{\sin(t/2)} \times \right. \\
&\qquad \left. \int_0^1 Re\left[\textstyle\sum_{k=0}^n \binom{n}{k} u^k(1-u)^{n-k}e^{i(k+1/2)t} \right] d\gamma(u) \right) dt
\end{aligned}
$$

$$= \frac{1}{2\pi}\int_0^{\pi}\left(\frac{\psi_x(t)}{\sin(t/2)}\int_0^1 g(u,t)d\gamma(u)\right)dt. \tag{3.14}$$

Using $(\sin(t/2))^{-1} \leq \pi/t$ for $0 < t \leq \pi$, we have

$$\begin{aligned}
|\tilde{H}_n(f;x) - \tilde{f}(x)| &\leq \int_0^{\pi}\frac{|\psi_x(t)|}{t}\left|\int_0^1 g(u,t)d\gamma(u)\right|dt \\
&= \left(\int_0^{\pi/(n+1)} + \int_{\pi/(n+1)}^{\pi}\right)\frac{|\psi_x(t)|}{t}\left|\int_0^1 g(u,t)d\gamma(u)\right|dt \\
&= I_1 + I_2, \text{say.}
\end{aligned} \tag{3.15}$$

Now using Lemma 3.3.1 and Hölder's inequality, we have

$$\begin{aligned}
I_1 &= O(1)\left\{\int_0^{\pi/(n+1)}\frac{\psi_x(t)\sin^{\beta}(t/2)}{\omega(t)}\frac{\omega(t)}{t\sin^{\beta}(t/2)}dt\right\} \\
&= O(1)\left\{\int_0^{\pi/(n+1)}\left(\frac{|\psi_x(t)|\sin^{\beta}(t/2)}{\omega(t)}\right)^p dt\right\}^{1/p} \times \\
&\qquad \left\{\lim_{\epsilon\to 0}\int_{\epsilon}^{\pi/(n+1)}\left(\frac{\omega(t)}{t\sin^{\beta}(t/2)}\right)^q dt\right\}^{1/q} \\
&= O[(n+1)^{-1/p}(n+1)^{\beta+1/p}\,\omega(\pi/(n+1))] \\
&= O((n+1)^{\beta}\,\omega(\pi/(n+1)),
\end{aligned} \tag{3.16}$$

in view of (3.9), (3.10) and $p^{-1} + q^{-1} = 1$.

Again using Lemma 3.3.1, Hölder's inequality and $(\sin(t/2))^{-1} \leq \pi/t$ for $0 < t \leq \pi$, we have

$$\begin{aligned}
I_2 &= O\left(\int_{\pi/(n+1)}^{\pi}\frac{t^{-\delta}|\psi_x(t)|\sin^{\beta}(t/2)}{(n+1)\omega(t)}\frac{t^{-1}\omega(t)}{t^{-\delta}t\sin^{\beta}(t/2)}dt\right) \\
&= O\left\{\frac{1}{n+1}\int_{\pi/(n+1)}^{\pi}\left(\frac{t^{-\delta}|\psi_x(t)|\sin^{\beta}(t/2)}{\omega(t)}\right)^p dt\right\}^{1/p} \times \\
&\qquad \left\{\int_{\pi/(n+1)}^{\pi}\left(\frac{t^{-1}\omega(t)}{t^{-\delta+\beta+1}}\right)^q dt\right\}^{1/q} \\
&= O\left[(n+1)^{\delta-1}\omega\left(\frac{\pi}{n+1}\right)\left(\frac{n+1}{\pi}\right)\left(\int_{\pi/(n+1)}^{\pi} t^{-(\beta+1-\delta)q}dt\right)^{1/q}\right]
\end{aligned}$$

$$= O\left[(n+1)^{\delta}\omega(\pi/(n+1))(n+1)^{\beta+1-\delta-1/q}\right]$$

$$= O[(n+1)^{\beta+1/p}\omega(\pi/(n+1))], \tag{3.17}$$

in view of (3.11), the mean value theorem for integrals, $0 < \delta < \beta + 1/p$ and $p^{-1} + q^{-1} = 1$.

Collecting (3.15)-(3.17), we get

$$|\ \tilde{H}_n(f;x) - \tilde{f}(x)\ | = O[(n+1)^{\beta+1/p}\omega(\pi/(n+1))]. \tag{3.18}$$

Finally, from (3.18) we easily get

$$\|\ \tilde{H}_n(f;x) - \tilde{f}(x)\ \|_p = O\left((n+1)^{\beta+1/p}\omega(1/(n+1))\right), \tag{3.19}$$

in view of condition (3.8), i.e., $\omega(\pi/(n+1)) = O(\omega(1/(n+1)))$.

This completes the proof of Theorem 3.2.1.

Remark 3.4.1. *In the case $p = 1$, i.e., $q = \infty$; sup norm is required while using Hölder's inequality. Therefore, the above proof will not work for $p = 1$. Thus, for $p = 1$, we have the following theorem.*

Theorem 3.4.1. *Let f be a 2π-periodic function belonging to the weighted Lipschitz class $W(L^1, \omega(t), \beta)$, with $0 \leq \beta < 1$. Then the degree of approximation of $\tilde{f}$ by Hausdorff means of conjugate Fourier series of f is given by*

$$\|\ \tilde{H}_n(f;x) - \tilde{f}(x)\ \|_1 = O\left((n+1)^{\beta+1}\omega(1/(n+1))\right), \tag{3.20}$$

provided a positive increasing function $\omega(t)$ satisfies (3.8) *and the following conditions:*

$$\omega(t)/t^{\beta+\sigma} \text{ is non-decreasing }, \tag{3.21}$$

$$\int_0^{\pi/(n+1)} \frac{t^{\sigma-1}\ |\ \psi_x(t)\ |\ \sin^{\beta}(t/2)}{\omega(t)} dt = O((n+1)^{-\sigma}), \tag{3.22}$$

for some $\sigma > 0$ such that $\sigma + \beta < 1$,

$$\int_{\pi/(n+1)}^{\pi} \frac{t^{-\delta}\ |\ \psi_x(t)\ |}{\omega(t)} dt = O((n+1)^{\delta}), \tag{3.23}$$

and

$$\frac{\omega(t)}{t^{-\delta+\beta+2}} \text{ is non-increasing }, \tag{3.24}$$

where $0 < \delta < \beta + 1$. The conditions (3.22) *and* (3.23) *hold uniformly in x.*

3.5 Proof of Theorem 3.4.1

Following the proof of Theorem 3.2.1, for $p = 1, i.e., q = \infty$, we have

$$
\begin{aligned}
I_1 &= \int_0^{\pi/(n+1)} \left(\frac{t^{\sigma-1}|\psi_x(t)| \sin^\beta(t/2)}{\omega(t)} \right) dt \times ess \sup_{0<t\leq\pi/(n+1)} \left| \frac{\omega(t)}{t^\sigma \sin^\beta(t/2)} \right| \\
&= O((n+1)^{-\sigma} \times ess \sup_{0<t\leq\pi/(n+1)} \left| \frac{\omega(t)}{t^{\sigma+\beta}} \right| \\
&= O((n+1)^{-\sigma}) \left\{ \frac{\omega(\pi/(n+1))}{(\pi/(n+1))^{\sigma+\beta}} \right\} \\
&= O((n+1)^\beta \, \omega(\pi/(n+1)). \qquad (3.25)
\end{aligned}
$$

and

$$
\begin{aligned}
I_2 &= O\left\{ \frac{1}{n+1} \int_{\pi/(n+1)}^{\pi} \frac{t^{-\delta}|\psi_x(t)| \sin^\beta(t/2)}{\omega(t)} dt \right\} \times ess \sup_{\pi/(n+1)\leq t\leq\pi} \left| \frac{\omega(t)}{t^{-\delta+\beta+2}} \right| \\
&= O\left[(n+1)^{\delta-1} \omega\left(\frac{\pi}{n+1}\right) \left(\frac{(n+1)^{2+\beta-\delta}}{\pi^{2+\beta-\delta}} \right) \right] \\
&= O[(n+1)^{\beta+1} \omega(\pi/(n+1))]. \qquad (3.26)
\end{aligned}
$$

Collecting (3.25) and (3.26), we get

$$
| \tilde{H}_n(f;x) - \tilde{f}(x) | = O[(n+1)^{\beta+1} \omega(\pi/(n+1))]. \qquad (3.27)
$$

Finally, from (3.27) we easily get

$$
\| \tilde{H}_n(f;x) - \tilde{f}(x) \|_1 = O\left((n+1)^{\beta+1} \omega(1/(n+1)) \right), \qquad (3.28)
$$

in view of condition (3.8).

This completes the proof of Theorem 3.4.1.

3.6 Corollaries

The following corollaries can be derived from our theorems:

1. If $\beta = 0$, then for $f \in Lip(\xi(t), p)$ with $p \geq 1$,

$$
\|\tilde{H}_n(f;x) - \tilde{f}(x)\|_p = O\left((n+1)^{1/p} \xi(1/(n+1)) \right).
$$

2. If $\beta = 0, \xi(t) = t^{\alpha} (0 < \alpha \leq 1)$, then for $f \in Lip(\alpha, p) (\alpha > 1/p)$,

$$\|\tilde{H}_n(f;x) - \tilde{f}(x)\|_p = O\left((n+1)^{1/p-\alpha}\right). \tag{3.29}$$

3. If $p \to \infty$ in Corollary 2, then for $f \in Lip\alpha (0 < \alpha < 1)$, (3.29) gives

$$\|\tilde{H}_n(f;x) - \tilde{f}(x)\|_\infty = O((n+1)^{-\alpha}).$$

For $\alpha = 1$, we can write an independent proof to obtain

$$\|\tilde{H}_n(f;x) - \tilde{f}(x)\|_\infty = O(\log(n+1)/(n+1)).$$

Since the product of two Hausdorff matrices is a Hausdorff matrix ([106, Lemma 1] and [109, Theorem 1]), the results proved by Nigam and Sharma [92–94], Lal and Singh [49], Nigam and Sharma [97], Mishra et al. [73; 75] and Sonkar and Singh [115] pertaining to the product of $(C, 1)$ and (E, q) $(q > 0)$ matrices, which are Hausdorff matrices, are also particular cases of Theorems 3.2.1 and 3.3.1.

□□□

Chapter 4

Approximation of Periodic Functions Belonging to $Lip(\omega(t),p)$ and $W(L^p,\Psi(t),\beta)$ -Classes

4.1 Introduction

In this chapter, we introduce a more general Lipschitz class $Lip(\omega(t),p)$ which includes the classical $Lip(\xi(t),p)$ class of functions and the function class $\{f\in L^p[0,2\pi] :\mid f(x+t)-f(x)\mid = O(t^{-1/p}\xi(t)), t>0\}$ defined by Khan and Ram [36] and compute analytically the degree of approximation of $f\in Lip(\omega(t),p)$ using matrix means of the Fourier series of f generated by the matrix $T\equiv(a_{n,k})$. We also discuss an example to show the application of the result. In the corollaries of the theorems of this chapter, we observe that the degree of approximation of $f\in Lip(\xi(t),p)$ is free from p and sharper than the earlier one.

Various investigators such as Nigam [90], Nigam and Sharma [92; 94], Lal and Srivastava [50], Nigam and Sharma [95–97], Mishra et al. [76], Dhakal [14], Lal and Nigam [47], Mishra et al. [71], Lal and Kushwaha [46] and Rhoades [107] have defined the $Lip(\xi(t),p)$-class by

$$Lip(\xi(t),p)=\left\{f\in L^p[0,2\pi] :\| f(x+t)-f(x)\|_p = O(\xi(t))\right\}, t>0, p\geq 1, \tag{4.1}$$

where $\xi(t)$ is a positive increasing function of t. On the other hand for positive increasing function $\xi(t)$, Khan and Ram [36] have defined

$$Lip(\xi(t),p)=\left\{f\in L^p[0,2\pi] :\mid f(x+t)-f(x)\mid = O(t^{-1/p}\xi(t))\right\}, t>0, p>1. \tag{4.2}$$

Here we generalize the definition of $Lip(\xi(t),p)$-classes given in (4.1) and (4.2) by introducing a new Lipschitz class $Lip(\omega(t),p)$ defined as

$$Lip(\omega(t),p) = \left\{ f \in L^p[0,2\pi] : \| f(x+t) - f(x) \|_p = O(t^{-1/p}\,\omega(t)) \right\}, t > 0, \tag{4.3}$$

where $p \geq 1$ and $\omega(t)$ is a positive increasing function. If we take $\omega(t) = t^{1/p}\xi(t)$, $Lip(\omega(t),p)$ coincides with $Lip(\xi(t),p)$ defined in (4.1). Also $Lip(\xi(t),p)$ defined in (4.2) is a subset of $Lip(\omega(t),p)$ for $\omega(t) = \xi(t)$, since $\|.\|_p = O(\|.\|_\infty)$. For $\omega(t) = t^{\alpha+1/p}, 0 < \alpha \leq 1$, $Lip(\omega(t),p)$ reduces to $Lip(\alpha,p)$.

We also write

$$K(n,t) := \frac{1}{2\pi} \sum_{k=0}^{n} a_{n,n-k} \frac{\sin(n-k+1/2)t}{\sin(t/2)},$$

and $\tau := [1/t]$, the integer part of $1/t$.

For the functions $f \in Lip\alpha$, Alexits [1] proved a theorem [36, Theorem A, p. 48] concerning the degree of approximation using the (C,δ) means of its Fourier series. Later Hölland et al. [25] extended results of Alexits [1] to functions belonging to $C^*[0,2\pi]$, the class of 2π-periodic continuous functions on $[0,2\pi]$, using Nörlund means of Fourier series. Khan and Ram [36] defined another class, the so called $Lip(\xi(t),p)$-class, which does include the $Lip\alpha$ class discussed by Alexits [1] and obtain the degree of approximation [36, Theorem 1.1, p. 49] using the Euler's means of functions belonging to this class . He also proved that the order of approximation arrived at is best possible and is free from the means generating sequences.

4.2 Main Results

In this chapter, we determine the degree of approximation of $f \in Lip(\omega(t),p)$ for $p \geq 1$, through trigonometric polynomials of the form given in (1.1). More precisely, we prove:

Theorem 4.2.1. *Let $T \equiv (a_{n,k})$ be a lower triangular regular matrix with non-negative and non-decreasing (with respect to k, for $0 \leq k \leq n$) entries with $A_{n,0} = 1$. Then the degree of approximation of a 2π-periodic function $f \in Lip(\omega(t),p)$, with $p \geq 1$ by matrix means of its Fourier series is given by*

$$\| t_n(f;x) - f(x) \|_p = O\left((n+1)^{1/p}\, \omega\left(\pi/(n+1)\right) \right), \tag{4.4}$$

provided a positive increasing function $\omega(t)$ satisfies the following conditions:

$$\omega(t)/t^{\sigma} \text{ is an increasing function for some } 0 < \sigma < 1, \tag{4.5}$$

$$\left(\frac{\phi(t)}{(t^{-1/p}\omega(t))}\right) \text{ is a bounded function of } t, \tag{4.6}$$

$$\left(\int_{\pi/(n+1)}^{\pi} \left(\frac{\omega(t)}{t^{1+1/p}}\right)^p dt\right)^{1/p} = O\left((n+1)\,\omega\left(\frac{\pi}{n+1}\right)\right), \tag{4.7}$$

where $p^{-1} + q^{-1} = 1$. *Also condition* (4.6) *holds uniformly in* x.

As mentioned in [80, p. 674], it is not necessary that all matrices have monotonic rows; for example, the hump matrices, defined as: A lower triangular matrix T is called a hump matrix if, for each n, there exists an integer $k_0 = k_0(n)$, such that $a_{n,k} \le a_{n,k+1}$ for $0 \le k < k_0$, and $a_{n,k} \ge a_{n,k+1}$ for $k_0 \le k < n$.

For hump matrices, we have the following Theorem.

Theorem 4.2.2. *Let* $T \equiv (a_{n,k})$ *be a hump matrix with non-negative entries and satisfies* $(n+1)\max_k\{a_{n,k}\} = O(1)$. *Then the degree of approximation of a* 2π*-periodic function* $f \in Lip(\omega(t), p)$, *with* $p \ge 1$ *by matrix means of its Fourier series is given by*

$$\| t_n(f;x) - f(x) \|_p = O\left((n+1)^{1/p}\,\omega\left(\pi/(n+1)\right)\right), \tag{4.8}$$

provided a positive increasing function $\omega(t)$ *satisfies the conditions* (4.5), (4.6) *and* (4.7).

4.3 Lemmas

For the proof of our theorems, we need the following lemmas:

Lemma 4.3.1. *Let* $T \equiv (a_{n,k})$ *be a lower triangular regular matrix. Then for* $0 < t \le \pi/(n+1)$, $K(n,t) = O(n+1)$.

Proof. Using $\sin nt \le nt$ and $\sin(t/2) \ge t/\pi$ for $0 < t \le \pi/(n+1)$, we have

$$\begin{aligned} |K(n,t)| &= (2\pi)^{-1}\left|\sum_{k=0}^{n} a_{n,n-k}\,(\sin(n-k+1/2)t)/(\sin t/2)\right| \\ &\le (2\pi)^{-1}\sum_{k=0}^{n} a_{n,n-k}\,|(\sin(n-k+1/2)t)/(\sin t/2)| \\ &\le (2\pi)^{-1}\sum_{k=0}^{n} a_{n,n-k}((n-k+1/2)t)/(t/\pi) \end{aligned}$$

The work of Theorems 4.2.1 and 4.2.2 have been published in Journal of Computational and Applied Mathematics (Elsevier Publications) 270 (2014), 223–230.

$$
\begin{aligned}
&\le (4)^{-1}\sum_{k=0}^{n} a_{n,n-k}(2n-2k+1)\\
&\le (4)^{-1}(2n+1)\sum_{k=0}^{n} a_{n,n-k}\\
&\le (4)^{-1}(2n+1)A_{n,0} = (2n+1)/4 = O(n+1).
\end{aligned}
$$

Lemma 4.3.2. *Let $T \equiv (a_{n,k})$ be a lower triangular regular matrix with non-negative and non-decreasing entries (with respect to k, for $0 \le k \le n$). Then for $\pi/(n+1) < t \le \pi$, $K(n,t) = O(A_{n,n-\tau}/t)$.*

Proof. Using $\sin(t/2) \ge t/\pi$, for $\pi/(n+1) < t \le \pi$, we have

$$
\begin{aligned}
\mid K(n,t) \mid &= (2\pi)^{-1}\left|\sum_{k=0}^{n} a_{n,n-k}(\sin(n-k+1/2)t)/\sin(t/2)\right|\\
&\le O(t^{-1})\left|Im\sum_{k=0}^{n} a_{n,n-k}e^{i(n-k+1/2)t}\right|\\
&\le O(t^{-1})\left|\sum_{k=0}^{n} a_{n,n-k}e^{i(n-k)t}\right|. \qquad (4.9)
\end{aligned}
$$

Following McFadden [69, p.8, Lemma 5.11], we have

$$
\begin{aligned}
\left|\sum_{k=0}^{n} a_{n,n-k}e^{i(n-k)t}\right| &= \left|e^{int}\sum_{k=0}^{n} a_{n,n-k}e^{-ikt}\right|\\
&\le \left|\sum_{k=0}^{\tau-1} a_{n,n-k}e^{-ikt}\right| + \left|\sum_{k=\tau}^{n} a_{n,n-k}e^{-ikt}\right|\\
&\le \sum_{k=0}^{\tau-1} a_{n,n-k} + 2a_{n,n-\tau}\max_{\tau\le k\le n}\left|\frac{1-e^{-i(k+1)t}}{1-e^{-it}}\right|\\
&\le A_{n,n-\tau+1} + 2a_{n,n-\tau}(1/\sin(t/2))\\
&\le A_{n,n-\tau} + 2(\tau+1)a_{n,n-\tau} = O(A_{n,n-\tau}), \qquad (4.10)
\end{aligned}
$$

in view of increasing nature of $a_{n,k}$, i.e., $2(\tau+1)a_{n,n-\tau} = O(A_{n,n-\tau})$.

Thus from (4.9) and (4.10), we have

$$
\mid K(n,t) \mid = O(A_{n,n-\tau}/t).
$$

Lemma 4.3.3. *For a hump matrix $T \equiv (a_{n,k})$ with $(n+1)\max_k\{a_{n,k}\} = O(1)$, $K(n,t) = O(t^{-2}/(n+1))$ for $\pi/(n+1) < t \le \pi$.*

Proof. Following Lemma 4.3.2, we have

$$\begin{aligned} \mid K(n,t) \mid &\leq O(t^{-1}) \left| \sum_{k=0}^{n} a_{n,n-k} e^{i(n-k)t} \right| = O(t^{-1}) \left| \sum_{k=0}^{n} a_{n,k} e^{ikt} \right| \\ &= O(t^{-1}).a_{n,k_0} \left| \frac{1 - e^{i(n+1)t}}{1 - e^{it}} \right| \\ &= O(t^{-1}).a_{n,k_0} (1/\sin(t/2)) = O\left(\frac{t^{-2}}{n+1} \right), \end{aligned} \tag{4.11}$$

in view of $a_{n,k_0} = \max\{a_{n,0}, a_{n,1}, ..., a_{n,n}\}$ and condition $(n+1)\max_k\{a_{n,k}\} = O(1)$.

We note that the authors in [90; 94–96] have taken $p \geq 1$ and used Hölder's inequality for $p > 1$. On the other hand the author in [36] has proved their result for $p > 1$. In this paper, we shall prove our theorems for $p \geq 1$ by using proper form of Hölder's inequality for $p = 1$.

4.4 Proof of Theorem 4.2.1

Case 1 ($p > 1$): We have

$$s_n(f;x) - f(x) = \frac{1}{2\pi} \int_0^{\pi} \phi(t)(\sin(n+1/2)t/\sin(t/2))dt,$$

and

$$\begin{aligned} t_n(f;x) - f(x) &= \sum_{k=0}^{n} a_{n,k} \left[s_k(f;x) - f(x) \right] \\ &= \int_0^{\pi} \phi(t)(2\pi)^{-1} \sum_{k=0}^{n} a_{n,k} \frac{\sin(k+1/2)t}{\sin(t/2)} dt \\ &= \int_0^{\pi} \phi(t)(2\pi)^{-1} \sum_{k=0}^{n} a_{n,n-k} \frac{\sin(n-k+1/2)t}{\sin(t/2)} dt \\ &= \int_0^{\pi} \phi(t) K(n,t) dt \\ &= \int_0^{\pi/(n+1)} \phi(t) K(n,t) dt + \int_{\pi/(n+1)}^{\pi} \phi(t) K(n,t) dt \\ &= I_1 + I_2, \text{ say.} \end{aligned} \tag{4.12}$$

Using Hölder's inequality, $\phi(t) \in Lip(\omega(t), p)$, Lemma 4.3.1, conditions (4.5), (4.6) and the mean value theorem for integrals, we have

$$\begin{aligned}
|I_1| &\leq \int_0^{\pi/(n+1)} \left(\frac{|\phi(t)|}{t^{-1/p}\omega(t)} . \frac{\omega(t).|K(n,t)|}{t^{1/p}} \right) dt \\
&\leq \left[\int_0^{\pi/(n+1)} \left(\frac{t^{1/p}|\phi(t)|}{\omega(t)} \right)^p dt \right]^{1/p} \times \\
&\quad \left[\int_0^{\pi/(n+1)} \left(\frac{\omega(t)}{t^{1/p}} .|K(n,t)| \right)^q dt \right]^{1/q} \\
&= O(n+1)^{-1/p}(n+1)(\omega(\pi/(n+1))).(n+1)^{1/p}(n+1)^{-1/q} \\
&= O((n+1)^{1/p}\, \omega(\pi/(n+1))), \qquad (4.13)
\end{aligned}$$

in view of $p^{-1} + q^{-1} = 1$ and decreasing nature of $\omega(t)/t^{1/p}$.

Using Hölder's inequality, Lemma 4.3.2, boundedness of $\phi(t)/(t^{-1/p}\omega(t))$ and condition (4.7), we have

$$\begin{aligned}
|I_2| &\leq \int_{\pi/(n+1)}^{\pi} \left(\frac{|\phi(t)|}{\omega(t)} \times |K(n,t)|.\omega(t) \right) dt \\
&= O \int_{\pi/(n+1)}^{\pi} \left(t^{-1/p} \times \frac{\omega(t)}{t} A_{n,n-\tau} \right) dt \\
&= O \left[\int_{\pi/(n+1)}^{\pi} \left(\frac{\omega(t)}{t^{1+1/p}} \right)^p dt \right]^{1/p} \cdot \left[\int_{\pi/(n+1)}^{\pi} (A_{n,n-\tau})^q \, dt \right]^{1/q} \\
&= O \left[\int_{\pi/(n+1)}^{\pi} \left(\frac{\omega(t)}{t^{1+1/p}} \right)^p dt \right]^{1/p} \cdot \left[\int_{\pi/(n+1)}^{\pi} \left\{ \frac{\pi}{(n+1)t} \right\}^q dt \right]^{1/q} \\
&= O(n+1)\omega(\pi/(n+1))(n+1)^{-1} \left[\int_{\pi/(n+1)}^{\pi} t^{-q} dt \right]^{1/q} \\
&= O\; \omega(\pi/(n+1))(n+1)^{1-1/q} \\
&= O((n+1)^{1/p}\omega(\pi/(n+1))). \qquad (4.14)
\end{aligned}$$

in view of $A_{n,n-\tau} = O\ (\pi/(n+1)t)$ (from the regularity condition of $(a_{n,k})$) and $p^{-1} + q^{-1} = 1$.

Collecting (4.12) - (4.14), we have

$$|t_n(f;x) - f(x)| = O\left((n+1)^{1/p}\omega(\pi/(n+1)) \right).$$

Hence,

$$\|t_n(f;x)-f(x)\|_p = \left(\frac{1}{2\pi}\int_0^{2\pi}|t_n(f;x)-f(x)|^p dx\right)^{1/p}$$
$$= O\left((n+1)^{1/p}\omega\left(\pi/(n+1)\right)\right). \quad (4.15)$$

Case 2 ($p=1$): Following the above proof and using Hölder's inequality for $p=1$, we have

$$|I_1| \leq \int_0^{\pi/(n+1)}\left(\frac{|\phi(t)|}{t^{-1}\omega(t)}.\frac{\omega(t)t^{\sigma}|K(n,t)|}{t^{\sigma+1}}\right)dt$$
$$= O(n+1)\ ess \sup_{0<t\leq\pi/(n+1)}\left(\left|\frac{\phi(t)}{t^{-1}\omega(t)}.\frac{\omega(t)}{t^{\sigma}}\right|\right)\int_0^{\pi/(n+1)}|t^{\sigma-1}|dt$$
$$= O(n+1)(n+1)^{\sigma}\ \omega(\pi/(n+1))(n+1)^{-\sigma}$$
$$= O((n+1)\ \omega(\pi/(n+1)))\ , \quad (4.16)$$

in view of Lemma 4.3.1, conditions (4.5) and (4.6).
Using Hölder's inequality for $p=1$, Lemma 4.3.2, boundedness of $\phi(t)/(t^{-1}\omega(t))$ and condition (4.7), we have

$$|I_2| \leq \int_{\pi/(n+1)}^{\pi}\left(\frac{|\phi(t)|}{\omega(t)}\times|K(n,t)|.\omega(t)\right)dt$$
$$\leq O\int_{\pi/(n+1)}^{\pi}\left(t^{-1}\times\frac{\omega(t)}{t}A_{n,n-\tau}\right)dt$$
$$\leq O\ \left[\int_{\pi/(n+1)}^{\pi}\frac{\omega(t)}{t^2}dt\right]\ ess \sup_{\pi/(n+1)\leq t\leq\pi}|\ A_{n,n-\tau}\ |$$
$$= O\ ((n+1)\omega(\pi/(n+1))\ ess \sup_{\pi/(n+1)\leq t\leq\pi}\left|\frac{\pi}{(n+1)t}\right|$$
$$= O(1)\ O((n+1)\ \omega(\pi/(n+1))), \quad (4.17)$$

in view of $A_{n,n-\tau} = O\ (\pi/(n+1)t)$.
Collecting (4.12), (4.16) and (4.17), we get

$$|\ t_n(f;x)-f(x)\ | = O[(n+1)\omega(\pi/(n+1))].$$

Hence

$$\|\ t_n(f;x)-f(x)\ \|_1 = O\left((n+1)\omega(\pi/(n+1))\right). \quad (4.18)$$

This completes the proof of Theorem 4.2.1.

4.5 Proof of Theorem 4.2.2

Case 1 ($p > 1$): Following the proof of Case I of Theorem 4.2.1, we have

$$t_n(f;x) - f(x) = I_1^{'} + I_2^{'}, \text{ say,} \tag{4.19}$$

where

$$|I_1^{'}| = O((n+1)^{1/p}\,\omega(\pi/(n+1))), \tag{4.20}$$

and

$$\begin{aligned}
|I_2^{'}| &\leq O\left\{\int_{\pi/(n+1)}^{\pi}\left(t^{-1/p}\frac{\omega(t)}{t^2(n+1)}\right)dt\right\} \\
&\leq O\,(n+1)^{-1}\left[\int_{\pi/(n+1)}^{\pi}\left(\frac{\omega(t)}{t^{1+1/p}}\right)^p dt\right]^{1/p}\cdot\left[\int_{\pi/(n+1)}^{\pi}\left(\frac{1}{t}\right)^q dt\right]^{1/q} \\
&= O\,(n+1)^{-1}(n+1)\,\omega(\pi/(n+1))(n+1)^{1-1/q} \\
&= O((n+1)^{1/p}\omega(\pi/(n+1)))\,,
\end{aligned} \tag{4.21}$$

in view of Hölder's inequality, Lemma 4.3.3, condition (4.7) and $p^{-1} + q^{-1} = 1$. Collecting (4.19)-(4.21), we have

$$|t_n(f;x) - f(x)| = O\left((n+1)^{1/p}\,\omega(\pi/(n+1))\right).$$

Hence

$$\|t_n(f;x) - f(x)\|_p = O\left((n+1)^{1/p}\,\omega\left(\pi/(n+1)\right)\right). \tag{4.22}$$

Case 2 ($p = 1$): Following the proof of Case II of Theorem 4.2.1, we have

$$|I_1^{'}| = O((n+1)\omega(\pi/(n+1))), \tag{4.23}$$

and

$$\begin{aligned}
|I_2^{'}| &\leq O\int_{\pi/(n+1)}^{\pi}\left(t^{-1}\frac{\omega(t)}{t^2(n+1)}\right)dt \\
&\leq O\,(n+1)^{-1}\left[\int_{\pi/(n+1)}^{\pi}\frac{\omega(t)}{t^2}dt\right]\ ess\sup_{\pi/(n+1)\leq t\leq\pi}\left|\frac{1}{t}\right|
\end{aligned}$$

$$= O\ (n+1)^{-1}(n+1)\ \omega(\pi/(n+1))(n+1)$$

$$= O\ ((n+1)\ \omega(\pi/(n+1))), \tag{4.24}$$

in view of Hölder's inequality for $p = 1$, Lemma 4.3.3 and condition (4.7).
Collecting (4.23) and (4.24), we get

$$|\ t_n(f;x) - f(x)\ |= O[(n+1)\omega(\pi/(n+1))].$$

Hence

$$\|\ t_n(f;x) - f(x)\ \|_1 = O\left((n+1)\omega(\pi/(n+1))\right). \tag{4.25}$$

This completes the proof of Theorem 4.2.2.

4.6 Example

In this section we show that how $t_n(f;x)$ is better approximant than $s_n(f;x)$. Let

$$f(x) = \begin{cases} 1 & \text{if } 0 < x < \pi \\ -1 & \text{if } \pi \le x \le 2\pi, \end{cases} \tag{4.26}$$

with $f(x+2\pi) = f(x)$. For this function Fourier coefficients are

$$a_n = 0, \forall n \in \mathbb{N}_0\,, \quad b_n = \begin{cases} 4/(n\pi) & \text{if n is odd} \\ 0 & \text{if n is even,} \end{cases}$$

therefore, $s_n(f;x)$ is defined as

$$s_{2n-1}(f;x) = \frac{4}{\pi}\sum_{k=1}^{n} \frac{\sin(2k-1)x}{2k-1}, \quad \text{and} \quad s_{2n}(f;x) = s_{2n-1}(f;x), \;\; n \in \mathbb{N}.$$

It is easy to verify that $f(x) \in Lip(\omega(t), p)$, for $\omega(t) = t^{2/p}$ for $p > 2$. Let $T = (a_{n,k})$ be such that $a_{n,k} = 2k/(n(n+1))$ for $0 \le k \le n$ and $a_{n,k} = 0$ for $k > n$. For this lower triangular matrix, we may write $t_n(f;x)$ as

$$t_n(f;x) := \sum_{k=0}^{n} \frac{2k}{n(n+1)} s_k(f;x), \;\; n \in \mathbb{N}_0.$$

Also from our Theorem 4.2.1, $\|\ t_n(f;x) - f(x)\ \|_p = O((n+1)^{-1/p}) \to 0$ as $n \to \infty$.

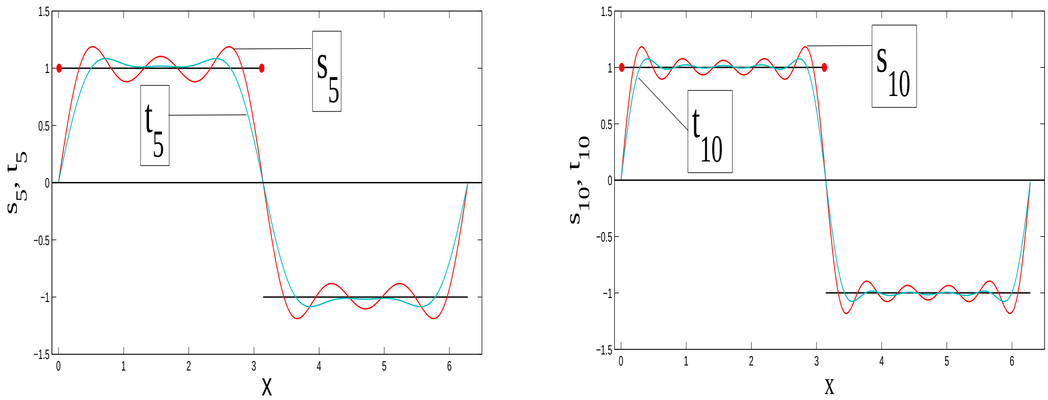

Figure 4.1: Graph of function $f(x)$, $s_n(x)$ and $t_n(x)$

Figure 4.1 represents graph of the function $f(x)$, s_5 and t_5 in $[0, 2\pi]$. It is clear from the figure that t_5 gives better approximation than s_5. Also t_{10} is a better approximant than t_5. This implies that error decreases as n increases, which verifies our theorems. We further note that near the points of discontinuities i.e., $x = 0, \pi$ and 2π, the graphs of s_5 and s_{10} show peaks and move closer to the line passing through points of discontinuity as n increases (Gibbs Phenomenon), but in the graphs of t_5 and t_{10} the peaks become flatten. Thus the matrix means of Fourier series of $f(x)$ overshoot the Gibbs Phenomenon and shows the smoothing effect of the method. We also note that a discontinuous function has been approximated by a trigonometric polynomial which is differentiable throughout the interval $[0, 2\pi]$.

4.7 Corollaries

The following corollaries can be derived from our theorem:

1. If $\omega(t) = t^{1/p}\xi(t)$, then for $f \in Lip(\xi(t), p)$

$$\| t_n(f;x) - f(x) \|_p = O(\xi(\pi/(n+1))), p \geq 1,$$

where $\xi(t)$ is a positive increasing function satisfying following conditions:

(i) $\dfrac{t^{1/p}\xi(t)}{t^\sigma}$ is an increasing function for some $0 < \sigma < 1$,

(ii) $\dfrac{\phi(t)}{\xi(t)}$ is a bounded function of t,

(iii) $$\left(\int_{\pi/(n+1)}^{\pi} \left(\frac{\xi(t)}{t}\right)^p dt\right)^{1/p} = O\left((n+1)^{1/q}\, \xi\left(\frac{\pi}{n+1}\right)\right).$$

2. If $\omega(t) = t^{\alpha+1/p}(0 < \alpha < 1/q)$, then for $f \in Lip(\alpha, p)$,

$$\|t_n(f;x) - f(x)\|_p = O\left((n+1)^{-\alpha}\right), p > 1, \sigma - 1/p < \alpha < 1/q.$$

3. If $p \to \infty$ in Corollary 2, then for $f \in Lip\alpha(0 < \alpha < 1)$,

$$\|t_n(f;x) - f(x)\|_\infty = O((n+1)^{-\alpha}).$$

A separate proof can be written for the case $\alpha = 1$, and for this case we get degree of approximation as

$$\|t_n(f;x) - f(x)\|_\infty = O(\log(n+1)/(n+1)).$$

[Please see Theorem 2.2.1 of Chapter 2].

The above corollaries are valid for matrix $T \equiv (a_{n,k})$ having non-decreasing rows as well as for hump matrix. The Nörlund version of the above theorems and corollaries can also be derived by replacing matrix T with N_p.

In the next section, we generalize the definition of Lipschitz class $Lip(\omega(t), p)$ to weighted version and introduce the weighted Lipschitz class $W(L^p, \Psi(t), \beta)$, with weight function $\sin^{\beta p}(x/2)$ and determine the error of approximation of $f \in W(L^p, \Psi(t), \beta)$. We also derive some corollaries from our results.

We define Lipschitz class $W(L^p, \Psi(t), \beta)$ as

$$W(L^p, \Psi(t), \beta) = \left\{ f \in L^p[0, 2\pi] : \| (f(x+t) - f(x)) \cdot \sin^\beta(x/2) \|_p = O(t^{-1/p}\, \Psi(t)) \right\}, \quad (4.27)$$

where $t > 0$, $\beta \geq 0$, $p \geq 1$ and $\Psi(t)$ is a positive increasing function of t and depends on β also.

For $\beta = 0$ and $\Psi(t) = \omega(t)$, an increasing function, $W(L^p, \Psi(t), \beta)$ reduces to $Lip(\omega(t), p)$ defined in (4.3). Also, if we take $\Psi(t) = t^{1/p}\xi(t)$, then $W(L^p, \Psi(t), \beta)$ coincides with classical weighted Lipschitz class $W(L^p, \xi(t)) = \{f \in L^p[0, 2\pi] : \| (f(x+t) - f(x)) \sin^\beta(x/2) \|_p = O(\xi(t))\Big\}$, where $t > 0$, $\beta \geq 0$, $p \geq 1$ and $\xi(t)$ is a positive increasing function of t, as defined in [112; 114], and the reference therein.

Also

$Lip(\xi(t), p) = \{f \in L^p[0, 2\pi] : | f(x+t) - f(x) | = O(t^{-1/p}\xi(t))\}$, $t > 0$, $p > 1$ defined by Khan and Ram [36] is a subset of $W(L^p, \Psi(t), \beta)$ for $\Psi(t) = \xi(t)$ and $\beta = 0$, since $\|.\|_p = O(\|.\|_\infty)$. For $\beta = 0$ and $\Psi(t) = t^{\alpha+1/p}, 0 < \alpha \leq 1, W(L^p, \Psi(t), \beta)$ reduces to $Lip(\alpha, p)$.

Theorem 4.7.1. *Let $T \equiv (a_{n,k})$ be a lower triangular regular matrix with non-negative and non-decreasing (with respect to k, for $0 \leq k \leq n$) entries. Then the degree of approximation of a 2π-periodic function $f \in W(L^p, \Psi(t), \beta)$ with $0 \leq \beta < 1/p$ and $p \geq 1$ by matrix means of its Fourier series is given by*

$$\| t_n(f;x) - f(x) \|_p = O\left((n+1)^{\beta+1/p}\, \Psi\left(\pi/(n+1)\right)\right), \tag{4.28}$$

provided a positive increasing function $\Psi(t)$ satisfies the following conditions:

$$\Psi(t)/t^{\beta+1/p} \text{ is an increasing function,} \tag{4.29}$$

$$\left(\frac{\phi(t)\sin^{\beta}(t/2)}{t^{-1/p}\,\Psi(t)}\right) \text{ is bounded function of } t \text{, hold uniformely in } x, \tag{4.30}$$

$$\left(\int_{\pi/(n+1)}^{\pi} \left(\frac{\Psi(t)}{t^{1+1/p+\beta}}\right)^p dt\right)^{1/p} = O\left((n+1)^{\beta+1}\, \Psi\left(\frac{\pi}{n+1}\right)\right), \tag{4.31}$$

where $p^{-1} + q^{-1} = 1$.

Remark 4.7.1. *Here $\Psi(t)$ is not an arbitrary increasing function. It must satisfy conditions* (4.29), (4.30) *and* (4.31) *simultaneously. For example, we can take a function class for which $\Psi(t) = t^{2/p}$ for $p > 2$.*

For hump matrices, we have the following theorem.

Theorem 4.7.2. *Let $T \equiv (a_{n,k})$ be a hump matrix with non-negative entries and satisfies $(n+1)\max_k\{a_{n,k}\} = O(1)$. Then the degree of approximation of a 2π-periodic function $f \in W(L^p, \Psi(t), \beta)$ with $0 \leq \beta < 1/p$ and $p \geq 1$ by matrix means of its Fourier series is given by*

$$\| t_n(f;x) - f(x) \|_p = O\left((n+1)^{\beta+1/p}\, \Psi\left(\pi/(n+1)\right)\right), \tag{4.32}$$

provided a positive increasing function $\Psi(t)$ satisfies the conditions (4.29), (4.30) *and* (4.31).

Remark 4.7.2. *If $\Psi(t) = \omega(t)$ and $\beta = 0$, then Theorem* 4.7.1 *and Theorem* 4.7.2 *coincides with Theorem* 4.2.1 *and Theorem* 4.2.1, *respectively.*

The work of Theorems 4.7.1 and 4.7.2 have been presented at "The 7th Conference on Function Spaces", held during May 20-24, 2014 at the Department of Mathematics and Statistics, College of Arts and Sciences, Southeren Illinois University, Edwardsville, Illinois, USA, likely to be published in Contemporary Mathematics (AMS).

4.8 Proof of Theorem 4.7.1

Case 1 ($p > 1$): We have

$$s_n(f;x) - f(x) = \frac{1}{2\pi}\int_0^{\pi} \phi(t)(\sin(n+1/2)t/\sin(t/2))dt,$$

and

$$\begin{aligned}
t_n(f;x) - f(x) &= \sum_{k=0}^{n} a_{n,k}\,[s_k(f;x) - f(x)] \\
&= \int_0^{\pi} \phi(t)(2\pi)^{-1}\sum_{k=0}^{n} a_{n,k}\frac{\sin(k+1/2)t}{\sin(t/2)}dt \\
&= \int_0^{\pi} \phi(t)(2\pi)^{-1}\sum_{k=0}^{n} a_{n,n-k}\frac{\sin(n-k+1/2)t}{\sin(t/2)}dt \\
&= \int_0^{\pi} \phi(t)K(n,t)dt \\
&= \int_0^{\pi/(n+1)} \phi(t)K(n,t)dt + \int_{\pi/(n+1)}^{\pi} \phi(t)K(n,t)dt \\
&= I_1 + I_2, \text{ say.} \qquad (4.33)
\end{aligned}$$

Using Hölder's inequality, $\phi(t) \in W(L^p, \Psi(t), \beta)$, Lemma 4.3.1, $(\sin(t/2))^{-1} \leq \pi/t$ for $0 < t \leq \pi$, conditions (4.29), (4.30) and the mean value theorem for integrals, we have

$$\begin{aligned}
|I_1| &\leq \int_0^{\pi/(n+1)} \left(\frac{|\phi(t)| \cdot \sin^{\beta}(t/2)}{t^{-1/p}\ \Psi(t)} . \frac{\Psi(t).|K(n,t)|}{t^{1/p} \cdot \sin^{\beta}(t/2)} \right) dt \\
&\leq \left[\int_0^{\pi/(n+1)} \left(\frac{t^{1/p}|\phi(t)| \cdot \sin^{\beta}(t/2)}{\Psi(t)} \right)^p dt \right]^{1/p} \times \\
&\qquad \left[\int_0^{\pi/(n+1)} \left(\frac{\Psi(t)}{t^{1/p} \cdot t^{\beta}} .|K(n,t)| \right)^q dt \right]^{1/q} \\
&= O(n+1)^{-1/p}(n+1)(\Psi(\pi/(n+1))).(n+1)^{\beta+1/p}(n+1)^{-1/q} \\
&= O((n+1)^{\beta+1/p}\ \Psi(\pi/(n+1))), \qquad (4.34)
\end{aligned}$$

in view of $p^{-1} + q^{-1} = 1$.

Using Hölder's inequality, Lemma 4.3.2, boundedness of $(\phi(t)\sin^{\beta}(t/2))/(t^{-1/p}\Psi(t))$,

$(\sin(t/2))^{-1} \leq \pi/t$ for $0 < t \leq \pi$ and condition (4.31), we have

$$
\begin{aligned}
|I_2| &\leq \int_{\pi/(n+1)}^{\pi} \left(\frac{|\phi(t)| \sin^{\beta}(t/2)}{\Psi(t)} \times \frac{|K(n,t)|.\Psi(t)}{\sin^{\beta}(t/2)} \right) dt \\
&= O \int_{\pi/(n+1)}^{\pi} \left(t^{-1/p} \times \frac{\Psi(t)}{t} \frac{A_{n,n-\tau}}{t^{\beta}} \right) dt \\
&= O \left[\int_{\pi/(n+1)}^{\pi} \left(\frac{\Psi(t)}{t^{1+1/p+\beta}} \right)^{p} dt \right]^{1/p} \cdot \left[\int_{\pi/(n+1)}^{\pi} (A_{n,n-\tau})^{q}\, dt \right]^{1/q} \\
&= O \left[\int_{\pi/(n+1)}^{\pi} \left(\frac{\Psi(t)}{t^{1+1/p+\beta}} \right)^{p} dt \right]^{1/p} \cdot \left[\int_{\pi/(n+1)}^{\pi} \left\{ \frac{\pi}{(n+1)t} \right\}^{q} dt \right]^{1/q} \\
&= O(n+1)^{\beta+1} \Psi(\pi/(n+1))(n+1)^{-1} \left[\int_{\pi/(n+1)}^{\pi} t^{-q} dt \right]^{1/q} \\
&= O\ (n+1)^{\beta} \Psi(\pi/(n+1))(n+1)^{1-1/q} \\
&= O((n+1)^{\beta+1/p} \Psi(\pi/(n+1))). \qquad (4.35)
\end{aligned}
$$

in view of $A_{n,n-\tau} = O\ (\pi/(n+1)t)$ (from the regularity condition of $(a_{n,k})$) and $p^{-1} + q^{-1} = 1$.

Collecting (4.33) - (4.35), we have

$$
|t_n(f;x) - f(x)| = O\left((n+1)^{\beta+1/p} \Psi(\pi/(n+1)) \right).
$$

Case 2 ($p = 1$): Following the above proof and using Hölder's inequality for $p = 1$, we have

$$
\begin{aligned}
|I_1| &\leq \int_{0}^{\pi/(n+1)} \left(\frac{|\phi(t)| \sin^{\beta}(t/2)}{t^{-1}\Psi(t)} . \frac{\Psi(t)|K(n,t)|}{t \sin^{\beta}(t/2)} \right) dt \\
&\leq O(n+1)\ ess \sup_{0<t\leq\pi/(n+1)} \left(\left| \frac{\phi(t)}{t^{-1}\Psi(t)} . \frac{\Psi(t)}{t^{\beta+1}} \right| \right) \int_{0}^{\pi/(n+1)} (1) dt \\
&= O(n+1)(n+1)^{\beta+1}\ \Psi(\pi/(n+1))(n+1)^{-1} \\
&= O((n+1)^{\beta+1}\ \Psi(\pi/(n+1))), \qquad (4.36)
\end{aligned}
$$

in view of Lemma 4.3.1, $(\sin(t/2))^{-1} \leq \pi/t$ for $0 < t \leq \pi$ and conditions (4.29), (4.30). Using Hölder's inequality for $p = 1$, Lemma 4.3.2, boundedness of $\frac{(\phi(t) \sin^{\beta}(t/2))}{(t^{-1}\Psi(t))}$ and conditions (4.31), we have

$$\begin{aligned}
|I_2| &\le \int_{\pi/(n+1)}^{\pi}\left(\frac{|\phi(t)|\sin^{\beta}(t/2)}{\Psi(t)}\times\frac{|K(n,t)|.\Psi(t)}{\sin^{\beta}(t/2)}\right)dt\\
&\le O\int_{\pi/(n+1)}^{\pi}\left(t^{-1}\times\frac{\Psi(t)}{t}\frac{A_{n,n-\tau}}{t^{\beta}}\right)dt\\
&\le O\left[\int_{\pi/(n+1)}^{\pi}\frac{\Psi(t)}{t^{2+\beta}}dt\right] ess \sup_{\pi/(n+1)\le t\le\pi} \mid A_{n,n-\tau}\mid\\
&= O\ ((n+1)^{\beta+1}\Psi(\pi/(n+1))\ ess \sup_{\pi/(n+1)\le t\le\pi}\left|\frac{\pi}{(n+1)t}\right|\\
&= O(1)\ O((n+1)^{\beta+1}\ \Psi(\pi/(n+1))), \qquad (4.37)
\end{aligned}$$

in view of $A_{n,n-\tau} = O\ (\pi/(n+1)t)$ and $(\sin(t/2))^{-1} \le \pi/t$ for $0 < t \le \pi$. Collecting (4.33), (4.36) and (4.37), we get

$$\mid t_n(f;x) - f(x) \mid = O[(n+1)^{\beta+1}\Psi(\pi/(n+1))].$$

Hence, for $p \ge 1$, we have

$$\parallel t_n(f;x) - f(x) \parallel_p = O\left((n+1)^{\beta+1/p}\Psi(\pi/(n+1))\right). \qquad (4.38)$$

This completes the proof of the Theorem 4.7.1.

4.9 Proof of Theorem 4.7.2

Case 1 ($p > 1$): Following the proof of Case I of Theorem 4.7.1, we have

$$t_n(f;x) - f(x) = I_1' + I_2', \text{ say}, \qquad (4.39)$$

where

$$|I_1'| = O((n+1)^{\beta+1/p}\ \Psi(\pi/(n+1))), \qquad (4.40)$$

and

$$\begin{aligned}
|I_2'| &\le O\left\{\int_{\pi/(n+1)}^{\pi}\left(t^{-1/p}\frac{\Psi(t)}{t^2(n+1)t^{\beta}}\right)dt\right\}\\
&\le O\ (n+1)^{-1}\left[\int_{\pi/(n+1)}^{\pi}\left(\frac{\Psi(t)}{t^{1+1/p+\beta}}\right)^p dt\right]^{1/p}\cdot\left[\int_{\pi/(n+1)}^{\pi}\left(\frac{1}{t}\right)^q dt\right]^{1/q}
\end{aligned}$$

$$= O\,(n+1)^{-1}(n+1)^{\beta+1}\,\Psi(\pi/(n+1))(n+1)^{1-1/q}$$

$$= O((n+1)^{\beta+1/p}\Psi(\pi/(n+1)))\,, \tag{4.41}$$

in view of Hölder's inequality, Lemma 4.3.3, $(\sin(t/2))^{-1} \leq \pi/t$ for $0 < t \leq \pi$, condition (4.31) and $p^{-1} + q^{-1} = 1$.

Collecting (4.39)-(4.41), we have

$$|t_n(f;x) - f(x)| = O\left((n+1)^{\beta+1/p}\,\Psi(\pi/(n+1))\right).$$

Case 2 ($p = 1$): Following the proof of Case II of Theorem 4.7.1, we have

$$|I_1^{'}| = O((n+1)^{\beta+1}\Psi(\pi/(n+1))), \tag{4.42}$$

and

$$\begin{aligned}
|I_2^{'}| &\leq O\int_{\pi/(n+1)}^{\pi}\left(t^{-1}\frac{\Psi(t)}{t^2(n+1)t^{\beta}}\right)dt \\
&\leq O\,(n+1)^{-1}\left[\int_{\pi/(n+1)}^{\pi}\frac{\Psi(t)}{t^{2+\beta}}\,dt\right] ess \sup_{\pi/(n+1)\leq t\leq\pi}\left|\frac{1}{t}\right| \\
&= O\,(n+1)^{-1}(n+1)^{\beta+1}\,\Psi(\pi/(n+1))(n+1) \\
&= O\,((n+1)^{\beta+1}\,\Psi(\pi/(n+1))),
\end{aligned} \tag{4.43}$$

in view of Hölder's inequality for $p = 1$, Lemma 4.3.3, $(\sin(t/2))^{-1} \leq \pi/t$ for $0 < t \leq \pi$ and condition (4.31).

Collecting (4.42) and (4.43), we get

$$\mid t_n(f;x) - f(x) \mid = O[(n+1)^{\beta+1}\Psi(\pi/(n+1))].$$

Hence, for $p \geq 1$, we have

$$\| t_n(f;x) - f(x) \|_p = O\left((n+1)^{\beta+1/p}\Psi(\pi/(n+1))\right), \tag{4.44}$$

This completes the proof of the Theorem 4.7.2.

4.10 Corollaries

The following corollaries can be derived from our theorem:
1. If $\Psi(t) = t^{1/p}\xi(t)$, then for $f \in W(L^p, \xi(t))$

$$\| t_n(f;x) - f(x) \|_p = O((n+1)^{\beta}\xi(\pi/(n+1))), p \geq 1,$$

where $\xi(t)$ is a positive increasing function satisfying following conditions:

(i) $\xi(t)/t^{\beta}$ is an increasing function,

(ii) $\left(\dfrac{\phi(t)\sin^{\beta}(t/2)}{\xi(t)}\right)$ is a bounded function of t,

(iii) $\left(\displaystyle\int_{\pi/(n+1)}^{\pi}\left(\frac{\xi(t)}{t^{1+\beta}}\right)^p dt\right)^{1/p} = O\left((n+1)^{\beta+1/q}\,\xi\left(\frac{\pi}{n+1}\right)\right),$

where $p^{-1} + q^{-1} = 1$. Also condition (ii) holds uniformly in x.
2. If $\beta = 0$ and $\Psi(t) = t^{1/p}\xi(t)$, then for $f \in Lip(\xi(t), p)$

$$\| t_n(f;x) - f(x) \|_p = O(\xi(\pi/(n+1))), p \geq 1,$$

where $\xi(t)$ is a positive increasing function satisfying following conditions:

(i) $\dfrac{\phi(t)}{\xi(t)}$ is a bounded function of t, uniformely in x,

(ii) $\left(\displaystyle\int_{\pi/(n+1)}^{\pi}\left(\frac{\xi(t)}{t}\right)^p dt\right)^{1/p} = O\left((n+1)^{1/q}\,\xi\left(\frac{\pi}{n+1}\right)\right).$

3. If $\beta = 0$ and $\Psi(t) = t^{\alpha+1/p}(0 < \alpha < 1/q)$, then for $f \in Lip(\alpha, p)$,

$$\|t_n(f;x) - f(x)\|_p = O\left((n+1)^{-\alpha}\right), p > 1, \sigma - 1/p < \alpha < 1/q.$$

4. If $p \to \infty$ in Corollary 2, then for $f \in Lip\alpha(0 < \alpha < 1)$,

$$\|t_n(f;x) - f(x)\|_\infty = O((n+1)^{-\alpha}).$$

As mentioned earlier, a separate proof can be written for the case $\alpha = 1$, and for this case we get degree of approximation as

$$\|t_n(f;x) - f(x)\|_\infty = O(\log(n+1)/(n+1)).$$

The above corollaries are valid for matrix $T \equiv (a_{n,k})$ having non-decreasing rows as well as for hump matrix. If $a_{n,k} = p_{n-k}/P_n$ for $0 \leq k \leq n$ and $a_{n,k} = 0$ for $k > n$, where $P_n = \sum_{k=0}^{n} p_n \neq 0 \to \infty$ as $n \to \infty$, then the matrix T reduces to Nörlund matrix N_p. The Nörlund version of the above theorems and corollaries can also be derived by replacing matrix T with N_p.

□□□

Chapter 5

Approximation of Functions in Lipschitz Class with Muckenhoupt Weights by Matrix Means

5.1 Introduction

In this chapter, we investigate the approximation properties of the matrix means of trigonometric Fourier series of f belonging to weighted Lipschitz class $Lip(\alpha, p, w)$ with Muckenhoupt weights generated by $T \equiv (a_{n,k})$ under relaxed conditions. Our theorem extends some of the previous results pertaining to the degree of approximation of functions in weighted Lipschitz class $Lip(\alpha, p, w)$ and the ordinary Lipschitz class $Lip(\alpha, p)$.

A measurable 2π-periodic function $w : [0, 2\pi] \to [0, \infty]$ is said to be a weight function if the set $w^{-1}(\{0, \infty\})$ has the Lebesgue measure zero. We say that $f \in L^p_w[0, 2\pi] (= L^p_w)$, the weighted Lebesgue space of all measurable 2π-periodic functions if

$$\|f\|_{p,w} = \left(\int_0^{2\pi} |f(x)|^p \, w(x) dx \right)^{1/p} < \infty, \; 1 \leq p < \infty.$$

Let $1 < p < \infty$. A weight function w belongs to the Muckenhoupt class A_p if

$$\sup_I \left(\frac{1}{|I|} \int_I w(x) dx \right) \left(\frac{1}{|I|} \int_I [w(x)]^{-1/(p-1)} dx \right)^{p-1} < \infty,$$

where the supremum is taken over all intervals I with length $|I| \leq 2\pi$. The weight functions belonging to A_p class, introduced by Hunt et al. [26], play an important role

The work of this chapter in the form of a research paper has been published in IAENG International Journal of Applied mathematics, 43(4) (2013), 190–194.

in different fields of mathematical analysis.
Let $w \in A_p$ and $f \in L_w^p$. The modulus of continuity of the function f is defined by

$$\Omega(f,\delta)_{p,w} = \sup_{|h|\leq\delta} \|\Delta_h(f)\|_{p,w}, \;\; \delta > 0,$$

where $\qquad (\Delta_h f)(x) = \frac{1}{h}\int_0^h |f(x+t) - f(x)|\,dt.$
The existence of the modulus of continuity of $f \in L_w^p$ follows from the boundedness of the Hardy-Littlewood maximal function in the space L_w^p [87]. The modulus of continuity $\Omega(f,\ \cdot)_{p,w}$ defined by Ky [41] is non-decreasing, non-negative, continuous function such that

$$\lim_{\delta\to 0}\Omega(f,\delta)_{p,w} = 0$$

and $\qquad \Omega(f_1+f_2,\ \cdot)_{p,w} \leq \Omega(f_1,\ \cdot)_{p,w} + \Omega(f_2,\ \cdot)_{p,w}.$

The modulus of continuity $\Omega(f,\ \cdot)_{p,w}$ is defined in this way, since the space L_w^p is non-invariant, in general, under the usual shift $f(x) \to f(x+h)$. We note that in the case $w \equiv 1$, the modulus of continuity $\Omega(f,\ \cdot)_{p,w}$ and the classical integral modulus of continuity $w_p(f,\ \cdot)$ are equivalent [41]. The weighted Lipschitz class $Lip(\alpha,p,w)$ for $0 < \alpha \leq 1$ is defined by

$$Lip(\alpha,p,w) = \{f \in L_w^p : \Omega(f,\delta)_{p,w} = O(\delta^\alpha), \delta > 0\}.$$

For $w(x) = 1\,\forall x \in [0,2\pi]$ the weighted $Lip\,(\alpha,p,w)$ class reduces to well known Lipschitz class $Lip\,(\alpha,p)$, $p > 1$.
Let $f \in L_p[0,2\pi]$ $(p \geq 1)$ be a 2π-periodic function. Then, for $n \in \mathbb{N}\cup\{0\}$ we write

$$\begin{aligned} s_n(f;x) = &\;\; a_0/2 + \sum_{k=1}^{n}(a_k \cos kx + b_k \sin kx) \\ = &\;\; \sum_{k=0}^{n} u_k(f;x) \text{ with } u_0(f;x) = s_0(f;x) = a_0/2, \end{aligned}$$

the $(n+1)^{th}$ partial sum of Fourier series of f at point x, which is a trigonometric polynomial of order (or degree) n. In this chapter, we use the matrix means of the Fourier series of f defined by the sequence to sequence transformation

$$\tau_n(f;x) = \tau_n(x) = \sum_{k=0}^{n} a_{n,k} s_k(f;x), \;\; n \in \mathbb{N}\cup\{0\}, \tag{5.1}$$

where $T \equiv (a_{n,k})$ is a lower triangular matrix with non-negative entries such that $a_{n,-1} = 0$, $A_{n,k} = \sum_{r=k}^{n} a_{n,r}$ and $A_{n,0} = 1\ \forall n \geq 0$.

A positive sequence $a = \{a_{n,k}\}$ is called almost monotonically decreasing with respect to k, if there exists a constant $K = K(a)$, depending on the sequence a only, such that $a_{n,p} \leq K a_{n,m}$ for all $p \geq m$ and we write that $a \in AMDS$. Similarly $a = \{a_{n,k}\}$ is called almost monotonically increasing with respect to k, if $a_{n,p} \leq K a_{n,m}$ for all $p \leq m$ and we write that $a \in AMIS$ [52; 81]. We note that every monotone sequence is an almost monotone sequence.

The Fourier series and trigonometric polynomials play an important role in various scientific and engineering fields, e.g., Lo and Hui [58] use the Fourier series expansion in a very nice way. Based upon the Fourier series expansion, they propose a simple and easy-to-use approach for computing accurate estimates of Black-Scholes double barrier option prices with time-dependent parameters.

Chandra [9] has studied the approximation properties of the means $N_n(f;x)$ and $R_n(f;x)$ in $Lip(\alpha, p)$, $1 \leq p < \infty$, $0 < \alpha \leq 1$ with monotonicity conditions on the means generating sequence $\{p_n\}$ and proved $\|N_n(f;x) - f(x)\|_p = O(n^{-\alpha}) = \|R_n(f;x) - f(x)\|_p$, $n = 1, 2, 3....$ Mittal et al. [80] generalized the paper of Chandra [9] partially, and extended its Theorem 1 and Theorem 2 (ii) to matrix means with $|\sum_{k=0}^{n} a_{n,k} - 1| = O(n^{-\alpha})$. On the other hand, Leindler [52] has relaxed the condition of monotonicity on $\{p_n\}$ and proved some of the results of Chandra [9] for almost monotone weights $\{p_n\}$. Mittal et al. [81] extended the results of Leindler [52] to matrix means with almost monotone sequence $\{a_{n,k}\}$ and row sums 1. Following [23], [28] and [40], recently Guven [21] has extended some of the results of Chandra [9] in another direction. He has extended Lipschitz class $Lip(\alpha, p)$ to $Lip(\alpha, p, w)$, and proved the weighted version of the Theorem 1 and Theorem 2 of Chandra [9] for $1 < p < \infty$, $0 < \alpha \leq 1$ i.e., for $f \in Lip(\alpha, p, w)$ and monotone $\{p_n\}$ he proved

$$\|N_n(f;x) - f(x)\|_{p,w} = O(n^{-\alpha}) = \|R_n(f;x) - f(x)\|_{p,w}, \ n = 1, 2, 3, \dots \ .$$

Following Mittal et al. [81], very recently, Singh and Sonkar [113] have studied the degree of approximation of periodic functions in generalized Hölder metric space through matrix means of Fourier series, where matrix $T \equiv (a_{n,k})$ has almost monotone rows, which in turn generalizes most of the results of Liendler [53]. On the other hand, Guven [22, Theorem 1 and Theorem 2] has given the weighted version of the results of Leindler [52] and some of the results of Mittal et al. [80] by assuming $T \equiv (a_{n,k})$ almost monotone with $|\sum_{k=0}^{n} a_{n,k} - 1| = O(n^{-\alpha})$. We note that condition $|\sum_{k=0}^{n} a_{n,k} - 1| = O(n^{-\alpha})$ was not used by Leindler [52] as $\sum_{k=0}^{n} a_{n,k} = 1$ for Nörlund matrix. It also appears that the author in [22] have followed many conditions and calculations as given in [81] and [113].

5.2 Main Result

In the present chapter, we continue the work of Mittal et al. [81] and Singh and Sonkar [113] and prove weighted version of the theorem of [81] for $p > 1$ which extend the result of Leindler [52] to weighted version as well as matrix version for $p > 1$. Our theorem also extends theorems of Guven [21] to matrix means $\tau_n(f;x)$ under the relaxed conditions of monotonicity and replaces the two theorems of Guven [22] by a single theorem for $\sum_{k=0}^{n} a_{n,k} = 1$. More precisely, we prove:

Theorem 5.2.1. *Let $f \in Lip(\alpha, p, w)$, $p > 1$, $w \in A_p$ and let $T \equiv (a_{n,k})$ be an infinite lower triangular regular matrix and satisfies one of the following conditions:*

(i) $0 < \alpha < 1$, $\{a_{n,k}\} \in AMIS$ in k,

(ii) $0 < \alpha < 1$, $\{a_{n,k}\} \in AMDS$ in k and $(n+1)a_{n,0} = O(1)$,

(iii) $\alpha = 1$ and $\sum_{k=0}^{n-1} (n-k) |\Delta_k a_{n,k}| = O(1)$,

(iv) $\alpha = 1$, $\sum_{k=0}^{n} |\Delta_k a_{n,k}| = O(a_{n,0})$ with $(n+1)a_{n,0} = O(1)$,

(v) $0 < \alpha < 1$, $\sum_{k=0}^{n-1} \left|\Delta_k \left(\frac{A_{n,0} - A_{n,k+1}}{k+1}\right)\right| = O\left(\frac{1}{n+1}\right)$.

Then

$$\|f(x) - \tau_n(f;x)\|_{p,w} = O((n+1)^{-\alpha}),\ n = 0, 1, 2... \tag{5.2}$$

5.3 Lemmas

To prove our theorem, we need the following lemmas.

Lemma 5.3.1 ([21]). *Let $1 < p < \infty$, $w \in A_p$ and $0 < \alpha \leq 1$. Then the estimate*

$$\|f(x) - s_n(f;x)\|_{p,w} = O((n+1)^{-\alpha}),\ n = 0, 1, 2, ..., \tag{5.3}$$

holds for every $f \in Lip(\alpha, p, w)$.

Lemma 5.3.2 ([21]). *Let* $1 < p < \infty$ *and* $w \in A_p$. *Then, for* $f \in Lip(1, p, w)$ *the estimate*

$$\|s_n(f;x) - \sigma_n(f;x)\|_{p,w} = O((n+1)^{-1}),\ n = 0, 1, 2, \dots, \tag{5.4}$$

holds.

Lemma 5.3.3 ([81]). *Let either* $\{a_{n,k}\} \in AMIS$ *or* $\{a_{n,k}\} \in AMDS$ *with* $(n+1)a_{n,0}{=}O(1)$. *Then, for* $0 < \alpha < 1$,

$$\sum_{k=0}^{n} (k+1)^{-\alpha} a_{n,k} = O((n+1)^{-\alpha}).$$

Proof. Let $r = [n/2]$ and $\{a_{n,k}\} \in AMIS$, then

$$\begin{aligned}
\sum_{k=0}^{n} (k+1)^{-\alpha} a_{n,k} &\leq Ka_{n,r}\sum_{k=0}^{r} (k+1)^{-\alpha} + (r+1)^{-\alpha} \sum_{k=r+1}^{n} a_{n,k} \\
&\leq Ka_{n,r}(r+1)^{1-\alpha} + (r+1)^{-\alpha} \sum_{k=0}^{n} a_{n,k} \\
&\leq K(r+1)^{-\alpha}(r+1)a_{n,r} + (r+1)^{-\alpha} A_{n,0} \\
&= O(r+1)^{-\alpha} = O((n+1)^{-\alpha}),
\end{aligned}$$

in view of $(r+1)a_{n,r} \leq (n-r+1)a_{n,r} \leq K(a_{n,r} + a_{n,r+1} + \dots + a_{n,n}) \leq A_{n,0}$ and $(r+1)^{-\alpha} = O((n+1)^{-\alpha})$.

If $\{a_{n,k}\} \in AMDS$ and $(n+1)a_{n,0} = O(1)$, then

$$\begin{aligned}
\sum_{k=0}^{n} (k+1)^{-\alpha} a_{n,k} &\leq Ka_{n,0} \sum_{k=0}^{n} (k+1)^{-\alpha} \\
&= O((n+1)^{-\alpha}).
\end{aligned}$$

This completes the proof of Lemma 5.3.3.

A different proof of this lemma can also be seen in [22; 81].

5.4 Proof of Theorem 5.2.1

We prove the cases (i) and (ii) together by using Lemma 5.3.1 and Lemma 5.3.3. Since

$$\tau_n(f;x) - f(x) = \sum_{k=0}^{n} a_{n,k}\{s_k(f;x) - f(x)\},$$

$$\|\tau_n(f;x) - f(x)\|_{p,w} \leq \sum_{k=0}^{n} a_{n,k} \|s_k(f;x) - f(x)\|_{p,w}$$

$$= \sum_{k=0}^{n} a_{n,k} O(k+1)^{-\alpha}$$

$$= O((n+1)^{-\alpha}).$$

Next, we consider the case (iv). By Abel's transformation and $a_{n,n+1} = 0$,

$$\begin{aligned} \tau_n(f;x) &= \sum_{k=0}^{n} a_{n,k} s_k(f;x) \\ &= \sum_{k=0}^{n} a_{n,k} \left(\sum_{i=0}^{k} u_i(f;x) \right) \\ &= \sum_{k=0}^{n} A_{n,k} u_k(f;x), \end{aligned}$$

and thus

$$\begin{aligned} s_n(f;x) - \tau_n(f;x) &= \sum_{k=0}^{n} (1 - A_{n,k}) u_k(f;x) \\ &= \sum_{k=1}^{n} k^{-1} (A_{n,0} - A_{n,k})\, k u_k(f;x). \end{aligned}$$

Hence, again by Abel's transformation and $A_{n,n+1} = 0$, we get

$$s_n(f;x) - \tau_n(f;x) = \sum_{k=1}^{n} (\Delta_k k^{-1}(A_{n,0} - A_{n,k})) \times$$

$$\sum_{i=1}^{k} i u_i(f;x) + (n+1)^{-1} \sum_{k=1}^{n} k u_k(f;x).$$

Therefore,

$$\begin{aligned} \|s_n(f;x) - \tau_n(f;x)\|_{p,w} \leq & \sum_{k=1}^{n} \left| \Delta_k k^{-1}(A_{n,0} - A_{n,k}) \right| \left\| \sum_{i=1}^{k} i u_i(f;x) \right\|_{p,w} \\ & + (n+1)^{-1} \left\| \sum_{k=1}^{n} k u_k(f;x) \right\|_{p,w}. \end{aligned} \tag{5.5}$$

Also

$$\begin{aligned} s_n(f;x) - \sigma_n(f;x) &= (n+1)^{-1} \sum_{k=0}^{n} ((n+1) u_k(f;x) - s_k(f;x)) \\ &= (n+1)^{-1} \sum_{k=1}^{n} k u_k(f;x), \end{aligned}$$

which implies

$$\left\|\sum_{k=1}^{n} k u_k(f;x)\right\|_{p,w} = (n+1)\left\|\sigma_n(f;x) - s_n(f;x)\right\|_{p,w} = O(1), \tag{5.6}$$

in view of Lemma 5.3.2.

Using (5.6) in (5.5), we get

$$\left\|s_n(f;x) - \tau_n(f;x)\right\|_{p,w} \leq \sum_{k=1}^{n}\left|\Delta_k k^{-1}(A_{n,0} - A_{n,k})\right| + (n+1)^{-1}. \tag{5.7}$$

Now,

$$\begin{aligned}
\Delta_k k^{-1}(A_{n,0} - A_{n,k}) &= \frac{A_{n,0} - A_{n,k}}{k} - \frac{A_{n,0} - A_{n,k+1}}{k+1} \\
&= k^{-1}(k+1)^{-1}(A_{n,0} - A_{n,k} - k a_{n,k}) \\
&= k^{-1}(k+1)^{-1}(\sum_{i=0}^{k-1} a_{n,i} - k a_{n,k}).
\end{aligned} \tag{5.8}$$

Next, we shall verify by induction that,

$$\left|\sum_{i=0}^{k-1} a_{n,i} - k a_{n,k}\right| \leq \sum_{i=1}^{k} i\,|\,a_{n,i-1} - a_{n,i}|. \tag{5.9}$$

For $k = 1$, we have

$$\left|\sum_{i=0}^{k-1} a_{n,i} - k a_{n,k}\right| = |a_{n,0} - a_{n,1}| = 1.\,|a_{n,0} - a_{n,1}|\,,$$

i.e., (5.9) is true for $k = 1$.

Let us assume that (5.9) is true for $k = m$, then for $k = m + 1$,

$$\begin{aligned}
|\sum_{i=0}^{m} a_{n,i} - (m+1)a_{n,m+1}| &= \left|\sum_{i=0}^{m-1} a_{n,i} + a_{n,m} + m a_{n,m} - m a_{n,m} - (m+1)a_{n,m+1}\right| \\
&\leq \left|\sum_{i=0}^{m-1} a_{n,i} - m a_{n,m}\right| + (m+1)\,|a_{n,m} - a_{n,m+1}| \\
&= \sum_{i=1}^{m} i\,|a_{n,i-1} - a_{n,i}| + (m+1)\,|a_{n,m} - a_{n,m+1}| \\
&= \sum_{i=1}^{m+1} i\,|a_{n,i-1} - a_{n,i}|.
\end{aligned}$$

Thus (5.9) is true for $k = m + 1$, hence (5.9) is true for $1 \le k \le n$.

Using (5.8) and (5.9), we get

$$\begin{aligned}
\sum_{k=1}^{n} \left|\Delta_k k^{-1}(A_{n,0} - A_{n,k})\right| &\le \sum_{k=1}^{n} k^{-1}(k+1)^{-1} \sum_{i=1}^{k} i\,|a_{n,i-1} - a_{n,i}| \\
&\le \sum_{i=1}^{n} i\,|a_{n,i-1} - a_{n,i}| \sum_{k=i}^{\infty} k^{-1}(k+1)^{-1} \\
&= \sum_{i=1}^{n} |a_{n,i-1} - a_{n,i}| = \sum_{k=0}^{n-1} |\Delta_k a_{n,n-k}| \\
&= O(a_{n,0}) = O((n+1)^{-1}). \qquad (5.10)
\end{aligned}$$

Combining (5.7) and (5.10), we get

$$\|s_n(f;x) - \tau_n(f;x)\|_{p,w} = O((n+1)^{-1}). \qquad (5.11)$$

Using Lemma 5.3.1 and (5.11), we have for $\alpha = 1$

$$\begin{aligned}
\|f(x) - \tau_n(f;x)\|_{p,w} &\le \|f(x) - s_n(f;x)\|_{p,w} + \|s_n(f;x) - \tau_n(f;x)\|_{p,w} \\
&= O((n+1)^{-1}).
\end{aligned}$$

Herewith the case (iv) is proved.

For the proof of case (iii), we first verify that the condition $\sum_{k=0}^{n-1} (n-k)\,|\Delta_k a_{n,k}| = O(1)$, implies that

$$\sum_{k=1}^{n} \left|\Delta_k k^{-1}(A_{n,0} - A_{n,k})\right| = O((n+1)^{-1}). \qquad (5.12)$$

From (5.8), we can write

$$\begin{aligned}
\sum_{k=1}^{n} |\Delta_k k^{-1}(A_{n,0} - A_{n,k})| &\le \sum_{k=1}^{n} k^{-1}(k+1)^{-1} \sum_{i=1}^{k} i\,|a_{n,i-1} - a_{n,i}| \\
&= \sum_{k=1}^{n} \Delta(k^{-1}) \sum_{i=1}^{k} i\,|a_{n,i-1} - a_{n,i}|.
\end{aligned}$$

By Abel's transformation, we have

$$\begin{aligned}
\sum_{k=1}^{n} |\Delta_k k^{-1}(A_{n,0} - A_{n,k})| &\le \sum_{k=1}^{n+1} k^{-1}.k\,|a_{n,k-1} - a_{n,k}| - \frac{1}{n+1} \sum_{k=1}^{n+1} k\,|a_{n,k-1} - a_{n,k}| \\
&= \sum_{k=1}^{n+1} \left(\frac{1}{k} - \frac{1}{n+1}\right) k\,|a_{n,k-1} - a_{n,k}|
\end{aligned}$$

$$= \sum_{k=1}^{n+1} \left(\frac{n-k+1}{n+1} \right) |a_{n,k-1} - a_{n,k}|$$

$$= \sum_{k=0}^{n} \left(\frac{n-k}{n+1} \right) |a_{n,k+1} - a_{n,k}|$$

$$\leq \frac{1}{n+1} \sum_{k=0}^{n-1} (n-k) \, |\Delta_k a_{n,k}| = O((n+1)^{-1}),$$

which verifies (5.12).

Combining (5.7), (5.12) and Lemma 5.3.2, we get (5.2) for $\alpha = 1$.

Finally, we prove the case (v). Using Lemma 5.3.1 and Abel's transform, we have

$$\begin{aligned}
\|\tau_n(f;x) - f(x)\|_{p,w} &\leq \sum_{k=0}^{n} a_{n,k} \, \|s_k(f;x) - f(x)\|_{p,w} \\
&= O\{\sum_{k=0}^{n} (k+1)^{-\alpha} a_{n,k}\} \\
&= O\{\sum_{k=0}^{n-1} \Delta_k (k+1)^{-\alpha} (\sum_{i=0}^{k} a_{n,i}) + (n+1)^{-\alpha} \sum_{i=0}^{n} a_{n,i}\} \\
&= O[\sum_{k=0}^{n-1} (A_{n,0} - A_{n,k+1}) \{(k+1)^{-\alpha} - (k+2)^{-\alpha}\} \\
&\quad + (n+1)^{-\alpha} A_{n,0}] \\
&= O\{\sum_{k=0}^{n} (k+1)^{-\alpha} (A_{n,0} - A_{n,k+1})/(k+1)\} \\
&\quad + O(n+1)^{-\alpha},
\end{aligned} \tag{5.13}$$

where by Abel's transformation

$$\begin{aligned}
\sum_{k=0}^{n} (k+1)^{-\alpha} \frac{A_{n,0} - A_{n,k+1}}{k+1} &= \sum_{k=0}^{n-1} \Delta_k \left\{ \frac{A_{n,0} - A_{n,k+1}}{k+1} \right\} \sum_{i=0}^{k} (i+1)^{-\alpha} \\
&\quad + \frac{A_{n,0} - A_{n,n+1}}{n+1} \sum_{i=0}^{n} (i+1)^{-\alpha} \\
&\leq \sum_{k=0}^{n-1} \Delta_k \left\{ \frac{A_{n,0} - A_{n,k+1}}{k+1} \right\} (k+1)^{1-\alpha} + \frac{(n+1)^{1-\alpha}}{n+1} \\
&\leq (n+1)^{1-\alpha} \sum_{k=0}^{n-1} \Delta_k \left\{ \frac{A_{n,0} - A_{n,k+1}}{k+1} \right\} + (n+1)^{-\alpha}
\end{aligned}$$

$$= O((n+1)^{-\alpha}), \tag{5.14}$$

in view of $A_{n,n+1} = 0$ and condition (v) of Theorem 5.2.1.
Collecting (5.13) and (5.14), we get (5.2). Thus proof of Theorem 5.2.1 is complete.

5.5 Corollaries

In order to justify the significance of our result, we prove that the following results are the particular cases of Theorem 5.2.1 for $p > 1$. We also drive an analogous result of Theorem 5.2.1 for monotone $\{a_{n,k}\}$.

1. If we take $a_{n,k} = p_{n-k}/P_n$ for $k \leq n$ and $a_{n,k} = 0$ for $k > n$, then conditions (i) to (iv) of Theorem 5.2.1 reduce to conditions (i) to (iv) of Theorem 1 of Leindler [52, p. 131], respectively, and $\tau_n(f;x)$ means reduces to $\mathrm{N}_n(f;x)$ means. Further, we note that $Lip(\alpha,p,1) \equiv Lip(\alpha,p)$, $p > 1$. Thus our theorem generalizes Theorem 5.2.1 of [52], except for the case $p = 1$ in two directions.

2. Since $Lip(\alpha,p,1) \equiv Lip(\alpha,p)$, $p > 1$ and conditions of Theorem 1 of Mittal et al. [81, p. 4485] for $p > 1$ are included in the conditions (i) to (iv) of Theorem 5.2.1, so Theorem 5.2.1 includes weighted version of Theorem 1 of [81] for $p > 1$.

3. Since every monotone sequence is almost monotone, the conditions (i) and (ii) of Theorem 5.2.1 are satisfied in case of monotonic $\{a_{n,k}\}$. Further, every sequence $\{a_{n,k}\}$ non-decreasing with respect to k always satisfies condition (iii) of Theorem 5.2.1, e. g.,

$$\begin{aligned}\sum_{k=0}^{n-1} (n-k)\,|\Delta_k a_{n,k}| &= \sum_{k=0}^{n-1} (n-k)(a_{n,k+1} - a_{n,k}) \\ &= A_{n,0} - (n+1)a_{n,0} = O(1).\end{aligned}$$

   If $\{a_{n,k}\}$ is non-increasing with respect to k, then (iv) of Theorem 5.2.1 is also true, e. g.,

$$\sum_{k=0}^{n-1} |\Delta_k a_{n,k}| = \sum_{k=0}^{n-1} (a_{n,k} - a_{n,k+1}) = a_{n,0} - a_{n,n} \leq a_{n,0}.$$

   Thus, we have the following analogous result of Theorem 5.2.1 for monotone $\{a_{n,k}\}$:

Corollary 5.5.1. *Let $f \in Lip(\alpha, p, w)$, $p > 1$, $w \in A_p$ and let $T \equiv (a_{n,k})$ be an infinite regular triangular matrix and satisfies one of the following conditions:*

(i) $\{a_{n,k}\}$ is non-decreasing in k,

(ii) $\{a_{n,k}\}$ is non-increasing in k and $(n+1)a_{n,0} = O(1)$.

Then 5.2 holds.

4. If we take $a_{n,k} = p_{n-k}/P_n$ for $k \leq n$ and $a_{n,k} = 0$ for $k > n$, then Corollary 5.5.1 reduces to Theorem 1 of Guven [21, p. 101].

5. Finally, if we take $a_{n,k} = p_k/P_n$ for $k \leq n$ and $a_{n,k} = 0$ for $k > n$, then $\tau_n(f;x)$ means reduces to $R_n(f;x)$ means; and

$$\begin{aligned} A_{n,0} - A_{n,k+1} &= \sum_{i=0}^{n} a_{n,i} - \sum_{i=k+1}^{n} a_{n,i} \\ &= \left(\sum_{i=0}^{n} p_i - \sum_{i=k+1}^{n} p_i \right) / P_n \\ &= \sum_{i=0}^{k} p_i / P_n = P_k / P_n, \end{aligned}$$

so that

$$\begin{aligned} \Delta_k \left(\frac{A_{n,0} - A_{n,k+1}}{k+1} \right) &= \frac{A_{n,0} - A_{n,k+2}}{k+2} - \frac{A_{n,0} - A_{n,k+1}}{k+1} \\ &= \frac{1}{P_n} \left(\frac{P_{k+1}}{k+2} - \frac{P_k}{k+1} \right), \end{aligned}$$

i. e., condition (v) of Theorem 5.2.1 reduces to condition (3) of Guven [21, Theorem 2]. Thus Theorem 5.2.1 under condition (v) extends Theorem 2 of Guven [21] to matrix means.

6. If we take $a_{n,k} = A_{n-k}^{\beta-1} / A_n^{\beta}$ for $k \leq n$ and $a_{n,k} = 0$ for $k > n (\beta > 0)$, where

$$A_0^{\beta} = 1, \ A_k^{\beta} = \frac{\beta(\beta+1)...(\beta+k)}{k!}, \ k \geq 1,$$

then matrix means $\tau_n(f;x)$ reduces to Cesàro means of order $\beta > 0$ denoted by $\sigma_n^{\beta}(f;x)$ and defined as

$$\sigma_n^{\beta}(f;x) = \frac{1}{A_n^{\beta}} \sum_{k=0}^{n} A_{n-k}^{\beta-1} s_k(f;x).$$

Hence, Corollary 3 of Guven [21, p. 102] can also be derived from Theorem 5.2.1.

7. In the light of remark of Guven [21, p. 102], we note that Theorems 5.2.1 and Corollary 5.5.1 also hold in reflexive weighted Orlicz spaces L_w^M, which are discussed in [28] in detail.

□□□

Chapter 6

T-Strong Convergence of Numerical Sequences and Fourier Series

6.1 Introduction

The notion of Λ-strong convergence was first given by Móricz [86] using a nondecreasing sequence $\Lambda = \{\lambda_k : k = 0, 1, ...\}$ of positive numbers tending to ∞. In this chapter, we generalize the notion of Λ-strong convergence of numerical sequences defined by Móricz [86] to T-strong convergence, using a lower triangular matrix $T = (a_{n,k})$ with nondecreasing monotone rows of positive numbers tending to ∞ i.e., $a_{n,k} \leq a_{n,k+1} \forall n$ and $\lim_{k\to\infty} a_{n,k} = \infty \ \forall n$. We also establish a relationship between ordinary convergence and T-strong convergence. We further show that this concept can also be applied to the strong convergence of Fourier series under C-metric and L_p-metric.

Let $T = (a_{n,k})$ be a lower triangular matrix with nondecreasing monotone rows of positive numbers tending to ∞ i.e., $a_{n,k} \leq a_{n,k+1} \forall n$ and $\lim_{k\to\infty} a_{n,k} = \infty \ \forall n$. A sequence $U = \{u_k\}$ of complex numbers converges T-strongly to a complex number u if

$$\lim_{n\to\infty} \frac{1}{a_{n,n}} \sum_{k=0}^{n} \mid a_{n,k}(u_k - u) - a_{n,k-1}(u_{k-1} - u) \mid = 0$$

Here $a_{n,-1} = 0$ and $u_{-1} = 0$. Since

$$u_n - u = \frac{1}{a_{n,n}} \sum_{k=0}^{n} [a_{n,k}(u_k - u) - a_{n,k-1}(u_{k-1} - u)]. \tag{6.1}$$

From above relation, we can easily say that if a sequence converges T-strongly to a number u, then it will converge to u in ordinary sense also.

The work of this chapter in the form of a research paper has been communicated for possible Publication.

Further, we can write two inequalities

$$\frac{1}{a_{n,n}}\sum_{k=0}^{n} \mid a_{n,k}(u_k-u)-a_{n,k-1}(u_{k-1}-u)\mid$$

$$\leq \frac{1}{a_{n,n}}\sum_{k=0}^{n}(a_{n,k}-a_{n,k-1})\mid u_k-u\mid+\frac{1}{a_{n,n}}\sum_{k=1}^{n}a_{n,k-1}\mid u_k-u_{k-1}\mid, \tag{6.2}$$

and

$$\frac{1}{a_{n,n}}\sum_{k=1}^{n}a_{n,k-1}\mid u_k-u_{k-1}\mid\leq \frac{1}{a_{n,n}}\sum_{k=0}^{n}(a_{n,k}-a_{n,k-1})\mid u_k-u\mid$$

$$+\frac{1}{a_{n,n}}\sum_{k=0}^{n}\mid a_{n,k}(u_k-u)-a_{n,k-1}(u_{k-1}-u)\mid, \tag{6.3}$$

by using the representation

$$a_{n,k}(u_k-u)-a_{n,k-1}(u_{k-1}-u)=(a_{n,k}-a_{n,k-1}(u_k-u)+a_{n,k-1}(u_k-u_{k-1}).$$

It is well-known that if $u_n\to u$ as $n\to\infty$ in ordinary sense, then

$$\lim_{n\to\infty}\frac{1}{a_{n,n}}\sum_{k=0}^{n}(a_{n,k}-a_{n,k-1})\mid u_k-u\mid=0.$$

It is clear that ordinary convergence of a sequence does not imply T-strong convergence. Below, we will show the condition that is necessary for an ordinary convergent sequence to converge T-strongly. We write

$$\sigma_n:=\frac{1}{a_{n,n}}\sum_{k=0}^{n}(a_{n,k}-a_{n,k-1})u_k \qquad (n=0,1,...).$$

Lemma 6.1.1. *T-strong convergence of a sequence $U=\{u_k\}$ to a number u implies following two conditions*

(i) *ordinary convergence of* $U=\{u_k\}$ *to* u, *and* (6.4)

$$\text{(ii)}\ \lim_{n\to\infty}\frac{1}{a_{n,n}}\sum_{k=1}^{n}a_{n,k-1}\mid u_k-u_{k-1}\mid=0, \tag{6.5}$$

and vice-versa.

Proof. First we suppose T-strong convergence of sequence U, then (i) is obvious from the definition and both the quantity on the right side of equation (6.3) will be zero as $n\to\infty$. Hence, we get (ii) of Lemma 6.1.1.

On the other hand, assume both the conditions of Lemma 6.1.1 are satisfied, then both the quantity on the right side of equation (6.2) will be zero as $n \to \infty$, which imply T-strong convergence of sequence U.

Further, if $\sum_{k=0}^{\infty} |u_k - u_{k-1}| < \infty$ for a sequence $U = \{u_k\}$, then the sequence U is said to be of bounded variation.

Obviously, $\sum_{k=0}^{\infty} |u_k - u_{k-1}| < \infty$ implies ordinary convergence of U with a suitable u. Also for a sequence of bounded variation, we always find n_0 such that

$$\sum_{k=n_0+1}^{\infty} |u_k - u_{k-1}| \leq \epsilon,$$

for a given $\epsilon > 0$.

Now

$$\frac{1}{a_{n,n}} \sum_{k=1}^{n} a_{n,k-1} \mid u_k - u_{k-1} \mid \leq \frac{1}{a_{n,n}} \sum_{k=1}^{n_0} a_{n,k-1} \mid u_k - u_{k-1} \mid + \sum_{k=n_0+1}^{\infty} \mid u_k - u_{k-1} \mid \leq 2\epsilon,$$

provided n is large enough, $a_{n,k} \leq a_{n,k+1}$ and $a_{n,k} \to \infty$ as $k \to \infty$.

This means that bounded variation also implies condition (6.5) of Lemma 6.1.1. Hence, if a sequence is of bounded variation, then it converses T-strongly. Thus, the T-strong convergence is an intermediate notion between bounded variation and ordinary convergence.

If we define $T = (a_{n,k})$ by

$$a_{n,k} = \begin{cases} k+1, & 0 \leq k \leq n\,, \\ 0, & k > n\,, \end{cases}$$

then the notion of T-strong convergence and Lemma 6.1.1 is the same as given by Hyslop [27] and Tanović-Miller [120].

Lemma 6.1.2. *Convergence of σ_n to u in the ordinary sense together with* (6.5) *of Lemma* 6.1.1 *implies the T-strong convergence of U to number u.*

Proof. Clearly,

$$\begin{aligned} u_n - \sigma_n &= \frac{1}{a_{n,n}} \sum_{k=0}^{n} (a_{n,k} - a_{n,k-1})(u_n - u_k) \\ &= \frac{1}{a_{n,n}} \sum_{k=0}^{n} (a_{n,k} - a_{n,k-1}) \sum_{j=k+1}^{n} (u_j - u_{j-1}) \\ &= \frac{1}{a_{n,n}} \sum_{j=1}^{n} (u_j - u_{j-1}) \sum_{k=0}^{j-1} (a_{n,k} - a_{n,k-1}) \end{aligned}$$

$$= \quad \frac{1}{a_{n,n}} \sum_{j=1}^{n} a_{n,j-1}(u_j - u_{j-1}).$$

Using equation (6.5) of Lemma 6.1.1, we have

$$\lim_{n\to\infty} (u_n - \sigma_n) = 0 \Longrightarrow \lim_{n\to\infty} u_n = \lim_{n\to\infty} \sigma_n = u. \tag{6.6}$$

Thus, we arrive at T-strong convergence of U to u in view of Lemma 6.1.1.

6.2 Results on Numerical Sequences

Let us collect all the T-strongly convergent sequences $U = \{u_k\}$ of complex numbers and denote it by $c(T)$. Obviously, $c(T)$ is a linear space. Further,

$$\|U\|_{c(T)} := \sup_{n\geq 0} \frac{1}{a_{n,n}} \sum_{k=0}^{n} \mid a_{n,k}u_k - a_{n,k-1}u_{k-1} \mid$$

is finite for every $U \in c(T)$ and $\|.\|_{c(T)}$ is a norm on $c(T)$. Denote by $\|.\|_\infty$ and $\|.\|_{bv}$ the usual l_∞ and bv-norms, respectively; that is

$$\|U\|_\infty := \sup_{k\geq 0} \mid u_k \mid \qquad \text{and} \qquad \|U\|_{bv} := \sum_{k=0}^{\infty} \mid u_k - u_{k-1} \mid .$$

Using inequalities (6.1) and (6.2) with $u = 0$ in each, we obtain instantaneously that for any sequence U,

$$\|U\|_\infty \leq \|U\|_{c(T)} \leq 2\|U\|_{bv}. \tag{6.7}$$

As a result, we have $bv \subset c(T) \subset c$, where bv and c denote the well-familiar Banach spaces of the sequences of complex numbers that are of bounded variation and the sequence convergent in the ordinary sense, respectively.

Another trivial appraisal is

$$\|U\|_{c(T)} \leq \|U\|_\infty \sup_{n\geq 0} \frac{1}{a_{n,n}} \sum_{k=0}^{n} (a_{n,k} + a_{n,k-1}). \tag{6.8}$$

Thus, if one can find a constant K such that $\frac{a_{n,n+1}}{a_{n,n}} \geq K > 1$ for all n large enough, then $\|U\|_{c(T)} = O(\|U\|_\infty)$. Thus in this particular case, $c(T) = c$, in the light of (6.7).

Now we prove the following:

Theorem 6.2.1. *The class $c(T)$ together with the norm $\|.\|_{c(T)}$ is a Banach space.*

Proof. It is sufficient to show that $c(T)$ is complete with respect to the norm $\|.\|_{c(T)}$. Let $\{U_j : j = 1, 2, ...\}$ be a Cauchy sequence in the norm $\|.\|_{c(T)}$. In the light of inequality (6.7), $\{U_j\}$ will be a Cauchy sequence in the norm $\|.\|_\infty$. So, there exist a sequence $U \in c$ such that

$$\lim_{j\to\infty} \|U_j - U\|_\infty = 0. \tag{6.9}$$

Now our aim is to show that $U \in c(T)$ and

$$\lim_{j\to\infty} \|U_j - U\|_{c(T)} = 0. \tag{6.10}$$

Let ϵ be a positive number. Then, by presumption, there exist a positive integer $r = r(\epsilon)$ such that

$$\|U_j - U_l\|_{c(T)} \le \epsilon \quad \text{for} \quad j, l \ge r. \tag{6.11}$$

Let $U_j := \{u_{jk} : k = 0, 1, ...\}$ and $U := \{u_k : k = 0, 1, ...\}$. We shall fix l and n. Likewise equation (6.8), we have

$$\begin{aligned} &\frac{1}{a_{n,n}} \sum_{k=0}^{n} |a_{n,k}(u_{lk} - u_k) - a_{n,k-1}(u_{l,k-1} - u_{k-1})| \\ &\le \|U_l - U\|_\infty \frac{1}{a_{n,n}} \sum_{k=0}^{n} (a_{n,k} + a_{n,k-1}) \le \epsilon\,, \end{aligned} \tag{6.12}$$

provided l is large enough, due to (6.9). Here l depends on both n and ϵ, and assume $l \ge r$.

Applying the triangle inequality, using (6.11) and (6.12), we have

$$\begin{aligned} &\frac{1}{a_{n,n}} \sum_{k=0}^{n} |a_{n,k}(u_{jk} - u_k) - a_{n,k-1}(u_{j,k-1} - u_{k-1})| \\ &\le \quad \frac{1}{a_{n,n}} \sum_{k=0}^{n} |a_{n,k}(u_{jk} - u_{lk}) - a_{n,k-1}(u_{j,k-1} - u_{l,k-1})| \\ &\quad + \frac{1}{a_{n,n}} \sum_{k=0}^{n} |a_{n,k}(u_{lk} - u_k) - a_{n,k-1}(u_{l,k-1} - u_{k-1})| \\ &\le \quad \|U_j - U_l\|_{c(T)} + \epsilon \le 2\epsilon \quad \text{if} \quad j \ge r. \end{aligned}$$

Since this holds for any $n = 0, 1, ...$, by definition,

$$\|U_j - U\|_{c(T)} \le 2\epsilon \quad \text{if} \quad j \ge r,$$

thereby proving (6.10) and $U \in c(T)$. This completes the proof of Theorem 6.2.1.

Further, one more fascinating result is that Banach space $c(T)$ has a Schauder basis.

Let

$$F^{(j)} := (0, 0, ..., \overset{j-1}{\breve{0}}, \overset{j}{\breve{1}}, 1, 1, ...), \quad (j = 0, 1, ...),$$

clearly each $F^j \in c(T)$.

Theorem 6.2.2. $\{F^{(j)}$*: j=0, 1, ...\} is a basis in c(T).*

Proof. *Existence.* If $U = \{u_k\}$ is a T-strongly convergent sequence, then we will show that

$$\lim_{m\to\infty} \left\| U - \sum_{j=0}^{m} (u_j - u_{j-1}) F^{(j)} \right\|_{c(T)} = 0. \tag{6.13}$$

Since

$$U - \sum_{j=0}^{m} (u_j - u_{j-1}) F^{(j)} = (0, 0, ..., \overset{m}{\breve{0}}, \overset{m+1}{\overbrace{u_{m+1} - u_m}}, u_{m+2} - u_m, ...), \tag{6.14}$$

by definition,

$$\begin{aligned} \|U - \sum_{j=0}^{m} (u_j - u_{j-1}) F^{(j)}\|_{c(T)} &= \sup_{n\geq m+1} \frac{1}{a_{n,n}} [a_{n,m+1}|u_{m+1} - u_m| \\ &\quad + \sum_{k=m+2}^{n} |a_{n,k}(u_k - u_m) - a_{n,k-1}(u_{k-1} - u_m)|] \\ &\leq \sup_{n\geq m+1} \frac{1}{a_{n,n}} [\sum_{k=m+1}^{n} (a_{n,k} - a_{n,k-1})|u_k - u_m| \\ &\quad + \sum_{k=m+1}^{n} a_{n,k-1}|u_k - u_{k-1}|] \to 0 \\ &as \quad m \to \infty, \end{aligned}$$

in view of Lemma 6.1.1.

Linear Independence. For linear independence, we refer [86, p. 323].

This completes the proof of Theorem 6.2.2.

6.3 Results on Fourier Series: C-metric

In this section, we show that the notion of T-strong convergence, applied to sequence of complex valued functions, is also applicable to Fourier series. Let C denote the well-known Banach space of the 2π periodic complex-valued continuous functions endowed with the norm $\|f\|_c := \max_t |f(t)|$.

Suppose the Fourier series of a function $f \in C$ is

$$\frac{1}{2}a_0(f) + \sum_{n=1}^{\infty}(a_n(f)\cos nt + b_n(f)\sin nt), \tag{6.15}$$

and let

$$u_k(f) = u_k(f;x) := \frac{a_0}{2} + \sum_{n=1}^{k}(a_n(f)\cos nx + b_n(f)\sin nx),\ \ k \in \mathbb{N} \text{ with } u_0(f;x) = \frac{a_0}{2},$$

denote the $(k+1)^{\text{th}}$ partial sums of the Fourier series (6.15).

Let us denote the classes of functions $f \in C$ whose Fourier series converges uniformly, converges absolutely, and converges T-strongly on $[0, 2\pi)$ by P, A, and $U(T)$, respectively. In other words, a function $f \in C$ belongs to $U(T)$ if

$$\lim_{n\to\infty}\left\|\frac{1}{a_{n,n}}\sum_{k=0}^{n} \mid a_{n,k}(u_k(f) - f) - a_{n,k-1}(u_{k-1}(f) - f) \mid\right\|_c = 0. \tag{6.16}$$

It is well-familiar that P is a Banach space with the norm

$$\|f\|_P := \sup_{k\geq 0}\|u_k(f)\|_c,$$

and A is also a Banach space with the norm

$$\|f\|_A := \frac{1}{2}|a_0(f)| + \sum_{n=1}^{\infty}(|a_n(f)| + |b_n(f)|) \tag{6.17}$$

(see, e.g., [32, pp. 6-8]).

Here we introduce the norm

$$\|f\|_{U(T)} := \sup_{n\geq 0}\left\|\frac{1}{a_{n,n}}\sum_{k=0}^{n}|a_{n,k}u_k(f) - a_{n,k-1}u_{k-1}(f)|\right\|_c, \tag{6.18}$$

which will be finite for every $f \in U(T)$. Also, using triangle inequality, we have

$$\|f\|_{U(T)} \leq \|f\|_c + \sup_{n\geq 0}\left\|\frac{1}{a_{n,n}}\sum_{k=0}^{n}|a_{n,k}(u_k(f) - f) - a_{n,k-1}(u_{k-1}(f) - f)|\right\|_c, \tag{6.19}$$

which is finite due to (6.16).

The norm inequalities corresponding to (6.7) are

$$\|f\|_P \leq \|f\|_{U(T)} \leq 2\|f\|_A, \tag{6.20}$$

which implies that $A \subseteq U(T) \subseteq P$.

The following outcomes are the similitude to Lemmas 6.1.1, 6.1.2, Theorem 6.2.1 and Theorem6.2.2, respectively.

Lemma 6.3.1. *The Fourier series of a function $f \in C$ converges T-strongly i.e., $f \in U(T)$ if and only if the following two conditions are satisfied:*

(i) $\lim_{k\to\infty} \|u_k(f) - f\|_c = 0,$ and

(ii) $\lim_{n\to\infty} \left\| \frac{1}{a_{n,n}} \sum_{k=1}^{n} a_{n,k-1} |a_k(f)\cos kt + b_k(f)\sin kt| \right\|_c = 0.$

Denote

$$\sigma_n(f) = \sigma_n(f,t) := \frac{1}{a_{n,n}} \sum_{k=0}^{n} (a_{n,k} - a_{n,k-1}) u_k(f,t), \quad n = 0, 1, \ldots \ . \tag{6.21}$$

Lemma 6.3.2. *If condition* (ii) *of Lemma* 6.3.1 *is satisfied and* $\lim_{n\to\infty} \|\sigma_n(f) - f\|_c = 0$, *then Fourier series of a function $f \in C$ converges T-strongly i.e., $f \in U(T)$.*

Theorem 6.3.1. *The set $U(T)$ form a Banach space with the norm defined in* (6.18).

Denote $Z_0 = a_0/2$ and $Z_j = a_n(f)\cos nx + b_n(f)\sin nx$ for $j = 1, 2, \ldots$

Theorem 6.3.2. $\{Z_j : j = 0, 1, 2, \ldots\}$ *is a basis in $U(T)$, i.e., if $f \in U(T)$, then*

$$\lim_{m\to\infty} \| u_m(f) - f \|_{U(T)} = 0.$$

Proof. For the proof of Theorem 6.3.2, first we write sequence of partial sums of the Fourier series of the difference $f - u_m(f)$ which is

$$(0, 0, \ldots, \underset{\smile}{\overset{m}{0}}, u_{m+1}(f) \overset{m+1}{\smile} - u_m(f), u_{m+2}(f) - u_m(f), \ldots).$$

This sequence is same as in (6.14) occuring in the proof of Theorem 6.2.2. Using the similar calculation as in Theorem 6.2.2, we easily get $\lim_{m\to\infty} \| u_m(f) - f \|_{U(T)} = 0$, in view of Lemma 6.3.1.

For $T = (a_{n,k})$ defined by

$$a_{n,k} = \begin{cases} k+1, & 0 \le k \le n \\ 0, & k > n \end{cases},$$

Szalay [119] obtained Lemma 6.3.1, Theorem 6.3.1 and Theorem 6.3.2.

6.4 Results on Fourier Series: L_p-metric

In this section, we develop the results of Section 6.3 for a L_p-metric. We know that L_p-space endowed with the norm

$$\|f\|_p = \left\{ \int_{-\pi}^{\pi} \mid f(x) \mid^p dx \right\}^{1/p} \quad (1 \le p < \infty) \text{ and } \|f\|_\infty = \sup_{x \in [0,2\pi]} \mid f(x) \mid .$$

is a Banach space.

Let us denote the classes of functions $f \in L_p[-\pi, \pi]$ whose Fourier series converges uniformly and T-strongly by P_p and $U_p(T)$, respectively, in the L_p-metric. In other words, a function $f \in L_p$ belongs to $U_p(T)$ if

$$\lim_{n\to\infty} \left\| \frac{1}{a_{n,n}} \sum_{k=0}^{n} \mid a_{n,k}(u_k(f) - f) - a_{n,k-1}(u_{k-1}(f) - f) \mid \right\|_p = 0. \tag{6.22}$$

It is well-known that P_p is a Banach space with the norm

$$\|f\|_{P_p} := \sup_{k\ge 0} \|u_k(f)\|_p.$$

Here we introduce the norm

$$\|f\|_{U_p(T)} := \sup_{n\ge 0} \left\| \frac{1}{a_{n,n}} \sum_{k=0}^{n} |a_{n,k}u_k(f) - a_{n,k-1}u_{k-1}(f)| \right\|_p , \tag{6.23}$$

which will be finite for every $f \in U_p(T)$.

The norm inequalities corresponding to (6.20) are

$$\|f\|_{P_p} \le \|f\|_{U_p(T)} \le 2\|f\|_A, \tag{6.24}$$

where $\|.\|_A$ is the same as defined in (6.17). These relations implies that $A \subset U_p(T) \subset P_p$.

The following outcomes are the similitude to Lemma 6.3.1, Lemma 6.3.2, Theorem 6.3.1 and Theorem 6.3.2, respectively.

Lemma 6.4.1. *The Fourier series of a function $f \in L_p$ converges T-strongly i.e., $f \in U_p(T)$ if and only if the following two conditions are satisfied:*

(i) $\lim_{k\to\infty} \|u_k(f) - f\|_p = 0,$ and

(ii) $\lim_{n\to\infty} \left\| \frac{1}{a_{n,n}} \sum_{k=1}^{n} a_{n,k-1} |a_k(f) \cos kt + b_k(f) \sin kt| \right\|_p = 0.$

Lemma 6.4.2. *If condition* (ii) *of Lemma* 6.4.1 *is satisfied for $p = 1$ and $\lim_{n\to\infty} \|\sigma_n(f) - f\|_1 = 0$, then $f \in U_1(T)$.*

Theorem 6.4.1. *The set $U_p(T)$ form a Banach space with the norm defined in* (6.23).

Theorem 6.4.2. *If $f \in U_p(T)$, then*

$$\lim_{m\to\infty} \| u_m(f) - f \|_{U_p(T)} = 0.$$

Proof. By using Lemma 6.4.1, we can easily prove the above theorem as it is similar to the proof of Theorem 6.3.2.

Remark 6.4.1. *If we define $T = (a_{n,k})$ by*

$$a_{n,k} = \begin{cases} \lambda_k/\lambda_n, & 0 \le k \le n \\ 0, & k > n \end{cases}$$

then most of the results of F. Móricz [86] *will be particular cases of our theorems.*

□□□

Conclusions and Future Scope

The work carried out in the present thesis is aimed to determine the error (or order/degree) of approximation of functions and their conjugates belonging to different Lipschitz classes viz., $Lip\alpha$, $Lip(\alpha, p)$, $Lip(\alpha, p, w)$, $Lip(\xi(t), p)$, $Lip(\omega(t), p)$, $W(L^p, \omega(t), \beta)$ and $W(L^p, \Psi(t), \beta)$, $p \geq 1$ using different summability means of their trigonometric Fourier series and its conjugate, respectively. Also the case $p = 1$ is discussed separately. Some corollaries and examples are also given to show the utility of the summation methods. There is sufficient scope to extend this work in multi-directions. Some of possible options are as listed below:

- To study the degree of approximation of functions belonging to Homogeneous Banach spaces [34, p. 14].
- Recently, Jain and Kumari [30] generalized the notion of classical Lorentz space $\wedge_{p,w}$ introduced by Lorentz [59; 60] to grand Lorentz space $\wedge_{p),w}$. Our work can be extended to determine the degree of approximation of functions belonging to other function spaces namely, weighted grand Lebesgue spaces [16], grand Lorentz spaces [30] and generalized Orlicz spaces[31].
- We can generalized the notion of T-strong convergence into T^2-strong convergence using the second difference defined as $\triangle^2(\lambda_k) = \triangle(\triangle(\lambda_k)) = \lambda_k - 2\lambda_{k-1} + \lambda_{k-2}$.
- The work of this thesis can be extended to other Fourier series such as Mellin, Walsh [110], Legendre, and Bessel [88, pp. 775 & 812] Fourier series also.

□□□

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