Efficient Numerical Solutions of Poisson's and Biharmonic Equations

Ph.D. THESIS

by

ALEMAYEHU SHIFERAW KIBRET

DEPARTMENT OF MATHEMATICS
INDIAN INSTITUTE OF TECHNOLOGY ROORKEE
ROORKEE-247667 (INDIA)

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INDIAN INSTITUTE OF TECHNOLOGY ROORKEE
ROORKEE

CANDIDATE'S DECLARATION

I hereby certify that the work which is being presented in the thesis entitled **EFFICIENT NUMERICAL SOLUTIONS OF POISSON'S AND BIHARMONIC EQUATIONS** in partial fulfillment of the requirements for the award of the Degree of Doctor of Philosophy and submitted in the department of Mathematics of the Indian Institute of Technology Roorkee, Roorkee is an authentic record of my own work carried out during a period from August, 2008 to December, 2013 under the supervision of Dr. R. C. Mittal, Professor, Department of Mathematics, Indian Institute of Technology Roorkee, Roorkee.

The matter presented in this thesis has not been submitted by me for the award of any other degree of this or any other Institute.

(ALEMAYEHU SHIFERAW KIBRET)

This is to certify that the above statement made by the candidate is correct to the best of my knowledge.

(R. C. Mittal)
Supervisor

Date: Dec , 2013

The Ph.D. Viva-Voce Examination of **Mr. ALEMAYEHU SHIFERAW KIBRET**,

Research Scholar, has been held on ……………………………………………………

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Head of the Department/ Chairman, ODC

ABSTRACT

Differential Equations are the language in which many laws of nature and their governing rules are expressed mathematically. Most physical phenomena can be modeled mathematically by second order and still some by fourth order partial differential equations; and these equations (PDEs) have become enormously successful as models of physical phenomena in all areas of engineering and sciences. The growing need for understanding the partial differential equations modeling of the physical problem has seen an increase in the use of mathematical theory and techniques, and has attracted the interest of many mathematicians. The elliptic type is perhaps one of the most important second order partial differential equation in applied mathematics. In engineering and many science fields one of the best known applicable theories in elliptic equations is potential equation.

These equations describe many physical problems, like the slow motion of incompressible viscous fluid; the St. Venant theory of torsion; electrostatics; in heat and mass transfer theory; elasticity; magnetism and gravitating matter at points where the charge density, pole strength or mass density are non zero; and other areas of mechanics and physics. In particular, the Poisson's equation describes stationary temperature distribution in the presence of thermal sources or sinks in the domain under consideration.

In this thesis an attempt has been made to find efficient numerical solution of Poisson's equation and biharmonic boundary value problem by considering different approximation schemes and extending the method of Hockney's in Cartesian and cylindrical coordinate systems (including when $r = 0$ is an interior or a boundary point) with respect to the given boundary conditions.

Chapter I is an introductory part and it deals with the important ideas and historical background of the development of finding the solution of Poisson's equation.

Chapter II deals with the numerical solution of the Poisson's equation in a cube with the given Dirichlet's boundary conditions. The Poisson's equation is approximated by its equivalent finite difference second order approximation scheme in order to obtain a large number of algebraic

linear equations and these equations are systematically arranged to get a block diagonal matrices structure. The obtained systems of block diagonal matrices are reduced, then, by extending the method of Hockney to a tri-diagonal matrix. Six examples have been considered in both cases and it is found that the method produce accurate results considering double precision.

Chapter III deals with the numerical solution of the three dimensional Poisson's equation approximated by a fourth order finite difference method in Cartesian coordinate systems in a cube with the Dirichlet's boundary conditions. Based on the approximation scheme we have developed 19 and 27 points stencil schemes. Both schemes result in a large algebraic system of linear equations and are treated systematically in order to get a block tri-diagonal system by extending the method of Hockney, and these systems of linear equations are solved by the use of Thomas algorithm. It is shown that the method produce accurate results and moreover 19-point formula produces comparable results with 27-point formula, though computational efforts are more in 27-point formula. Six examples are taken to show the accuracy of the method and it is shown that the method produces accurate results.

Part of this chapter has been published in the ***American Journal of Computational Mathematics* 2011, Vol 1, No. 4 pp. 285-293.**

Chapter IV deals with the numerical solution of the three dimensional Poisson's equation in cylindrical coordinate systems for $r \neq 0$ approximated by a second order finite difference method in a cylinder or portion of cylinder with the Dirichlet's boundary conditions. Based on the approximation scheme we have transformed the Poisson's equation in to a large number of algebraic systems of linear equations and these systems of linear equations are treated systematically in order to get a block tri-diagonal system, and these systems of linear equations are solved by the use of Thomas algorithm. Seven examples have been tested to verify the efficiency of the method and it is shown that this method produces good result.

Part of this work is to appear in the ***American Journal of Computational Mathematics.***

Chapter V deals with the fourth-order numerical solution of the three dimensional Poisson's equation in cylindrical coordinate systems for $r \neq 0$ with the Dirichlet's boundary conditions. The Poisson's equation is approximated by a fourth order finite difference approximation (19 points stencil scheme) to convert the equation in to a large number of system of algebraic linear equations; and the resulting large number of these algebraic system of linear equations is treated systematically in order to get a block tri-diagonal system. These systems of linear equations are solved by the use of *Thomas algorithm*, and using backward substitution we obtain the solution for the Poisson's equation. Seven examples have been considered and it is shown that this method produces good result.

Part of this work is to appear in the ***American Journal of Computational Mathematics.***

Chapter VI deals with the second and fourth-order approximation scheme for the numerical solution of the three dimensional Poisson's equation in cylindrical coordinate systems when $r = 0$ is an interior or a boundary point.

Chapter VII deals with the numerical solution of the two (three) dimensional biharmonic boundary value problem of the second kind in a rectangular region (a cube) respectively, in Cartesian coordinate systems. Using the splitting method the two/three dimensional linear biharmonic boundary value problem is replaced by a coupled Poisson's equations and these coupled Poisson's equations are solved directly by using the fourth order finite difference approximation scheme which we have developed in Chapter III. For non-linear biharmonic boundary value problem of the second kind, we use splitting and iterative method together. Eight examples have been considered to test the efficiency of the methods.

Finally**,** in Chapter VIII, based on the present study, conclusions are drawn and in this direction future research work is suggested.

LIST OF RESEARCH PAPERS

1. An Efficient Direct Method to Solve the Three Dimensional Poisson's Equation, ***American Journal of Computational Mathematics***, Vol. 1 No. 4, 2011, pp. 285-293.

2. Fast Finite Difference Solutions of the Three Dimensional Poisson's Equation in Cylindrical Coordinates, ***American Journal of Computational Mathematics.*** 2013,vol. 3, pp. 356-361

3. High Accurate Finite Difference Solutions of the Three Dimensional Poisson's Equation in Cylindrical Coordinates to appear in ***American Journal of Computational Mathematics***

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Alemayehu Shiferaw

Roorkee

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CHAPTER I

INTRODUCTION

1.1 Historical Background of PDE

Partial differential equations (PDEs) have become enormously successful as models of physical phenomena in all areas of engineering and sciences. The growing need for understanding the partial differential equations modeling of the physical problem has seen an increase in the use of mathematical theory and techniques, and has attracted the interest of many mathematicians. Many interesting progresses have been achieved in the last 60 years with the introduction of numerical methods that allow the use of modern computers to solve PDEs of every kind, in general geometries and under arbitrary external conditions (at least in theory; in practice there are still a large number of hurdles to be overcome).

Especially in recent years we have seen a dramatic increase in the use of PDEs in areas such as biology, chemistry, computer sciences (particularly in relation to image processing and graphics) and in economics (finance). The primary reason for this interest was that partial differential equations both express many fundamental laws of nature and frequently arise in the mathematical analysis of diverse problems in science and engineering. The theoretical analysis of PDEs is not merely of academic interest, but rather has many applications that originate from a model of a physical or engineering problem in real life situations [31],[103],[109].

Perhaps one of the most important of all the partial differential equations involved in applied mathematics and mathematical physics is the potential equation, also known as the *Laplace equation* $U_{xx} + U_{yy} = 0$, where subscripts denote partial derivatives. This equation was first discovered by Pierre-Simon Laplace (1749–1827) while he was involved in an extensive study of gravitational attraction of arbitrary bodies in space; and this equation arose in steady state heat conduction problems involving homogeneous solids. *James Clerk Maxwell* (1831–1879) also gave a new initiative to potential theory through his famous equations, known as *Maxwell's equations* for electromagnetic fields.

The problem of finding a solution of Laplace's equation that takes on the given boundary values is known as the *Dirichlet boundary-value problem*, after Peter Gustav Lejeune Dirichlet (1805–1859). On the other hand, if the values of the normal derivative are prescribed on the boundary, the problem is known as *Neumann boundary-value problem*, in honor of Karl Gottfried Neumann (1832–1925). Despite great efforts by many mathematicians including Gaspard Monge (1746–1818), Adrien-Marie Legendre (1752–1833), Carl Friedrich Gauss (1777–1855), Simeon-Denis Poisson (1781–1840), and Jean Victor Poncelet (1788–1867), very little was known about the general properties of the solutions of Laplace's equation until 1828, when George Green (1793–1841) and Mikhail Ostrogradsky (1801–1861) independently investigated properties of a class of solutions known as *harmonic functions*. Partial differential equations have been the subject of vigorous mathematical research for over three centuries and remain so today [3],[103].

1.2 Poisson's Equation

Poisson's equation was used by the French mathematician Simeon Poisson (1781–1840) in his studies of diverse problems in mechanics, gravitation, electricity, and magnetism. Therefore it is called *Poisson's equation*. No partial differential equation competes with Poisson's equation in regard to its importance and ubiquity in applications [103],[109].

A variety of problems in scientific computing involve the solution of the Poisson's equation

$$\nabla^2 U = f \quad \text{in } D \tag{1.1}$$

subject to appropriate boundary conditions (BC),

i) $U = f_1$ on ∂D for a given function f_1, (i.e. U specified on the boundary) is called the *Dirichlet problem*,

ii) $\frac{\partial U}{\partial n} = f_2$ on ∂D where f_2 is a given function, $\hat{n}$ denotes the unit outward normal to ∂D, and $\frac{\partial U}{\partial n}$ denotes a differentiation in the direction of $\hat{n}$ (i.e. $\frac{\partial U}{\partial n} = \hat{n} \cdot \nabla$), (i.e. gradient of U normal to the boundary is specified) is called the *Neumann problem*, and

iii) $U + \alpha \frac{\partial U}{\partial n} = f_3$ on ∂D where α and f_3 are given functions, (i.e. the BC is in terms of a mixture of the first two types – typically a linear combination) is called *a problem of the third kind* (it is also sometimes called the *Robin problem*).

iv) *mixed boundary condition* for a PDE; that is, different boundary are used on different parts of the boundary of the domain of the equation. For example, if U is a solution to a partial differential equation on D with piecewise-smooth boundary ∂D, and ∂D is divided into two parts, Γ_1 and Γ_2, one can use a Dirichlet boundary condition on Γ_1 and a Neumann boundary condition on Γ_2, i.e. $\partial D = \Gamma_1 \cup \Gamma_2$ and U is prescribed on the boundary as

$U = g_1$ on Γ_1 and

$\dfrac{\partial U}{\partial n} = g_2$ on Γ_2, where g_1 and g_2 are given functions defined on those portions of the boundary.

Equation (1.1) along with the boundary conditions i) to iv) is said to be a *boundary value problem*.

The first question what we have to address now is whether there exist a solution to each one of the problems we just defined in i) to iii) or not. This question is not at all easy, it has been considered by many great mathematicians since the middle of the nineteenth century. It was discovered that when the domain D is bounded and 'sufficiently smooth', then the Dirichlet problem, for example, does indeed have a solution.

Theorem: Let D be a smooth, bounded domain. Then there exists at most one solution $U \in C^2(D) \cap C^1(\partial D)$ of the Poisson's equation (1.1), satisfying either i) or iii) on ∂D and for case ii) on ∂D there might be more than one solution but any two solutions differ by a constant.

For the proof and some details of this part refer [54],[83],[103],[109].

The Poisson's equation in different coordinates system is expressed as;

Two Dimensional

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = f(x, y) \qquad \text{Cartesian}$$

$$\frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r} + \frac{1}{r^2}\frac{\partial^2 U}{\partial \theta^2} = f(r, \theta) \qquad \text{Polar}$$

Three Dimensional

$$\frac{\partial^2 U}{\partial x^2}+\frac{\partial^2 U}{\partial y^2}+\frac{\partial^2 U}{\partial z^2}=f(x,y,z) \qquad \text{Cartesian}$$

$$\frac{\partial^2 U}{\partial r^2}+\frac{1}{r}\frac{\partial U}{\partial r}+\frac{1}{r^2}\frac{\partial^2 U}{\partial \theta^2}+\frac{\partial^2 U}{\partial z^2}=f(r,\theta,z) \qquad \text{Cylindrical}$$

$$\frac{\partial^2 U}{\partial \rho^2}+\frac{2}{\rho}\frac{\partial U}{\partial \rho}+\frac{1}{\rho^2}\frac{\partial^2 U}{\partial \phi^2}+\frac{\cot\phi}{\rho^2}\frac{\partial U}{\partial \phi}+\frac{1}{\rho^2\sin^2\phi}\frac{\partial^2 U}{\partial \theta^2}=f(\rho,\phi,\theta) \qquad \text{Spherical}$$

Based on the nature of the geometry of D, we have to choose the appropriate Poisson's equation accordingly.

1.3 Biharmonic Equations

The biharmonic equation is a fourth-order elliptic PDE which arises in areas of continuum mechanics, including linear elasticity theory (to find the displacement of the bending of elastic plates), the solution of the stream function of incompressible Stokes flow, and other areas of engineering and sciences.

The biharmonic problem for the domain D consists of determining a function U which satisfies the partial differential equation

$$\begin{aligned} \nabla^4 U(P) &= f(P) \qquad P\in D \\ U(P) &= f_1(P) \qquad P\in \partial D \\ i)\ \frac{\partial U(P)}{\partial n} &= f_2(P) \qquad P\in \partial D \\ or\ \ ii)\ \frac{\partial^2 U(P)}{\partial n^2} &= f_2(P) \qquad P\in \partial D \end{aligned} \tag{1.2}$$

Here we assume that f, f_1 and f_2 are given, sufficiently smooth functions and that the boundary ∂D is sufficiently smooth to insure the existence of a solution to the biharmonic equation (1.2). $\frac{\partial}{\partial n}$ or $\frac{\partial^2}{\partial n^2}$ denote the derivative in the direction of the exterior normal.

A solution of (1.2) can be obtained analytically by using some theories of harmonic functions in complex analysis (See [83]), but always it is not an easy task. Thus many of the developments in this area are based on the numerical solutions of biharmonic equation.

1.4 Numerical Methods

The PDEs associated with most science and engineering applications are often impossible, or impractical, to solve using analytic methods, such as separation of variables and Fourier series. PDEs with non constant coefficients, equations in complicated domains, and nonlinear equations cannot, in general, be solved analytically. Even when we can produce an 'exact' analytical solution, it is often in the form of an infinite series. Worse than that, the computation of each term in the series, although feasible in principle, might be tedious in practice, and, in addition, the series might converge very slowly.

Numerical solution methods provide a reasonable alternative in many of these situations. The method is based on replacing the continuous variables by discrete variables, and thus, the continuum problem represented by the PDE is transformed into a discrete problem in finitely many variables. Naturally we pay a price for this simplification; we can only obtain an approximation to the exact answer, and even this approximation is only obtained at the discrete values taken by the variables [5],[10],[16],[59],[75].

The discipline of numerical solution of PDEs is rather young. The first analysis (and, in fact, also the first formulation) of a discrete approach on the solution of problems of mathematical physics by means of finite differences to a PDE was presented in 1929 by the German-American mathematicians Richard Courant (1888–1972), Kurt Otto Friedrichs (1901–1982), and Hans Lewy (1905–1988) for the special case of the wave equation. Incidentally, they were not interested in the numerical solution of the PDE (their work preceded the era of electronic computers by almost two decades), but rather they formulated the discrete problem as a means for a theoretical analysis of the wave equation. The Second World War witnessed the introduction of the first computers that were built to solve problems in continuum mechanics. Following the war and the rapid progress in the computational power of computers, it was argued by many scientists that soon people would be able to solve numerically any PDE. As a result of these renewed interest, towards finding the solution of PDEs numerically, forced scientists to develop different numerical methods and approaches. Some of the most popular numerical methods are the *Finite Difference Method (FDM),* the F*inite Elements Method (FEM), the Finite Volume Method (FVM), Fast Fourier Transform Methods, Spline Collocation Methods, Spectral Methods, Multigrid Methods, Galerkin Method, Domain Decomposition Methods,* B*oundary Element Methods, Wavelet Methods* and others [81],[94],[100],[103],[109].

Using either of these numerical methods, they have also developed different solvers for the solutions of PDEs. In Particular, to find the numerical solution of the two or three dimensional Poisson's and biharmonic equations they have introduced several fast solvers (a fast solver is an algorithm for the efficient implementation of a method for solving equation in a standardized region D) based on the geometry of the problem and the type of boundary conditions. For Poisson's equation (two or three dimensional), for instance, in simple geometries (circular or rectangular domains) with regular grids, there are well-known fast direct solvers [13],[20],[21],[22],[24],[25],[26],[62], which typically rely on the fast Fourier transform (FFT) [2],[28],[66],[88],[96],[98] and are well suited to the task. When either restriction is relaxed, however, these methods no longer apply. Since practical problems tend to involve complex geometries, highly inhomogeneous source distributions *f*, or both, there has been a lot of effort directed at developing alternative approaches [4],[7],[12], [33],[36], [64],[65],[70],[108]. Most currently available solvers rely on iterative techniques using multigrid, [47],[48],[80],[101],[110]; domain decomposition, or some other adaptive methods [55]. Unfortunately, while such multilevel strategies can achieve nearly optimal efficiency in theory, they require an appropriate hierarchy of coarse grids which is not provided in practice. Although there has been significant progress in this direction, the available solvers compare unfavorably with the fast direct solvers in terms of work per grid point.

Several attempts have been made in developing and improving different techniques and methods to solve the Poisson's equation in polar and cylindrical coordinates system both analytically [3],[5],[10],[16],[75],[94],[103],[109] and numerically [9],[14],[30],[31],[40],[54],[59],[67], [68], [69],[97] for practical and theoretical problems in many branches of engineering and physics. When solving such boundary value problems, appropriate choice of coordinates system is very useful, because most of these problems along with their solutions are mainly dependent on the geometry of the boundaries. In physical problems that involve a cylindrical surface, (for example the problem of evaluating the temperature in a cylindrical rod), it will be convenient to make use of cylindrical coordinates.

For biharmonic equation (two/ three dimensional) several solvers have been developed based on the availability of the data kind (first or second), the numerical method, and the geometry of the problem [11],[15],[18],[34],[41],[42],[45],[53],[56],[57],[58],[61],[72],[73],[78],[86],[93],[95].

1.4 Finite Difference Method

A computational solution of a partial differential equation (PDE) involves a discretization procedure by which the continuous equation is replaced by a discrete algebraic equation. The discretization procedure consists of an approximation of the derivatives in the governing PDE by differences of the dependent variables, which are computed only at discrete points (grid or mesh points) in different geometries. In general, one starts with a given PDE and uses a discretization procedure for developing a finite-difference equation (FDE) that is a linear relation between discrete values of the unknown function computed on grid point.

Thus, a finite difference solution basically involves three steps:

1. Dividing the solution into grids of nodes.
2. Approximating the given differential equation by finite difference equivalence that relates the solutions to grid points.
3. Solving the difference equations subject to the prescribed boundary and/or initial conditions.

When approximating the given PDE by its finite difference approximation, we have to consider some factors, for instance, the order of accuracy of an approximation, stability, consistency and convergence of the difference scheme having a potential impact on the approximate solution. Many works have been done in this regard (see [9],[16],[30], [40], [67],[68],[69],[75],[87],[97]).

The search for approximate solution of PDEs like Poisson's equation, researchers were forced to study and develop different approximation schemes to reduce the error with the exact solution. The most popular approximation in finite difference method is the polynomial approximation by using Taylor's series method (higher accuracy can be derived by keeping more terms in the Taylor series). Based on the geometry of the problem, particular numerical method and the nature of the accuracy of the solution, different approximation schemes have been introduced and developed using Taylor's method. (For two dimensional Poisson/biharmonic equation [1],[2],[6],[11],[12],[19],[24],[25],[26],[28],[35],[36],[45],[48],[55],[57],[58],[61],[62],[63],[65],[73],[76],[78],[85],[86],[89],[90],[96],[102],[108]).

Consider the three dimensional Poisson's equation in Cartesian coordinate system

$$\nabla^2 U = \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} = f(x, y, z) \tag{1.3}$$

Applying the Taylor series expansion of $U(x+h, y, z)$ and $U(x-h, y, z)$, where h is the mesh size in the x direction, we get

$$U(x+h, y, z) = U(x, y, z) + h\frac{\partial U(x, y, z)}{\partial x} + \frac{h^2}{2!}\frac{\partial^2 U(x, y, z)}{\partial x^2} + \frac{h^3}{3!}\frac{\partial^3 U(x, y, z)}{\partial x^3} + \ldots \tag{1.4}$$

$$U(x-h, y, z) = U(x, y, z) - h\frac{\partial U(x, y, z)}{\partial x} + \frac{h^2}{2!}\frac{\partial^2 U(x, y, z)}{\partial x^2} - \frac{h^3}{3!}\frac{\partial^3 U(x, y, z)}{\partial x^3} + \ldots \tag{1.5}$$

Adding (1.4) and (1.5), gives us

$$U(x+h, y, z) - 2U(x, y, z) + U(x-h, y, z) = 2\left(\frac{h^2}{2!}\frac{\partial^2 U(x, y, z)}{\partial x^2} + \frac{h^4}{4!}\frac{\partial^4 U(x, y, z)}{\partial x^4} + \ldots\right) \tag{1.6}$$

Dividing (1.6) by h^2, we get

$$\frac{U(x+h, y, z) - 2U(x, y, z) + U(x-h, y, z)}{h^2} = \frac{\partial^2 U(x, y, z)}{\partial x^2} + \frac{h^2}{12}\frac{\partial^4 U(x, y, z)}{\partial x^4} + \frac{h^4}{360}\frac{\partial^6 U(x, y, z)}{\partial x^6} + \ldots \tag{1.7}$$

Using (1.7), $\frac{\partial^2 U(x, y, z)}{\partial x^2}$ can be written as

$$\frac{\partial^2 U(x, y, z)}{\partial x^2} = \frac{U(x+h, y, z) - 2U(x, y, z) + U(x-h, y, z)}{h^2} + O\left(h^2\right) \tag{1.8}$$

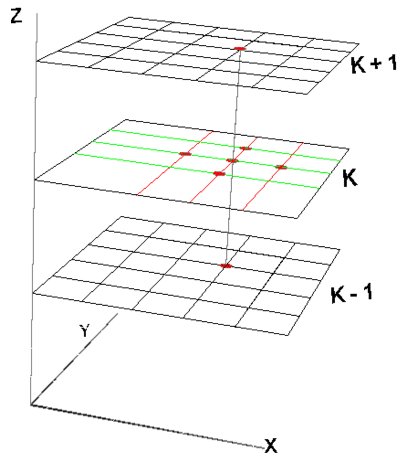

Figure 1.1

Three dimensional Cartesian grid

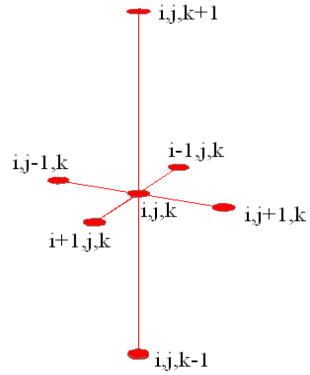

Figure 1.2

seven-point stencil

Assume that there are M, N and P mesh points along the X, Y and Z directions respectively, and let $U(x,y,z)$ be discretized at the mesh point (i,j,k) and we adopt writing $U_{i,j,k}$ for $U(x,y,z)$, where $i=1(1)M$, $j=1(1)N$ and $k=1(1)P$ and

let $x=i\Delta x \;\; \Rightarrow x+\Delta x = i\Delta x+\Delta x=(i+1)\Delta x$, and

$$\Rightarrow x-\Delta x = i\Delta x-\Delta x=(i-1)\Delta x$$

Similarly we have for $y+\Delta y=(j+1)\Delta y, \quad y-\Delta y=(j-1)\Delta y$

$$z+\Delta z=(k+1)\Delta z, \quad z-\Delta z=(k-1)\Delta z$$

Thus we write $U_{i\pm1,j,k}$ for $U(x\pm\Delta x,y,z)$, $U_{i,j\pm1,k}$ for $U(x,y\pm\Delta y,z)$ and $U_{i,j,k\pm1}$ for $U(x,y,z\pm\Delta z)$, and thus

$$\frac{U(x+\Delta x,y,z)-2U(x,y,z)+U(x-\Delta x,y,z)}{(\Delta x)^2}=\frac{U_{i+1,j,k}-2U_{i,j,k}+U_{i-1,j,k}}{(\Delta x)^2} \tag{1.9a}$$

$$\frac{U(x,y+\Delta y,z)-2U(x,y,z)+U(x,y-\Delta y,z)}{(\Delta y)^2}=\frac{U_{i,j+1,k}-2U_{i,j,k}+U_{i,j-1,k}}{(\Delta y)^2} \tag{1.9b}$$

$$\frac{U(x,y,z+\Delta z)-2U(x,y,z)+U(x,y,z-\Delta z)}{(\Delta z)^2}=\frac{U_{i,j,k+1}-2U_{i,j,k}+U_{i,j,k-1}}{(\Delta z)^2} \tag{1.9c}$$

Thus we can write $\dfrac{\partial^2 U(x,y,z)}{\partial x^2}$ by its equivalent central difference approximation as

$$\frac{\partial^2 U(x,y,z)}{\partial x^2} = \frac{U_{i+1,j,k} - 2U_{i,j,k} + U_{i-1,j,k}}{(\Delta x)^2} \tag{1.10a}$$

Similarly, $$\frac{\partial^2 U(x,y,z)}{\partial y^2} = \frac{U_{i,j+1,k} - 2U_{i,j,k} + U_{i,j-1,k}}{(\Delta y)^2} \tag{1.10b}$$

and $$\frac{\partial^2 U(x,y,z)}{\partial z^2} = \frac{U_{i,j,k+1} - 2U_{i,j,k} + U_{i,j,k-1}}{(\Delta z)^2} \tag{1.10c}$$

Now substituting (1.10a), (1.10b) and (1.10c) in (1.3), we get

$$\frac{U_{i+1,j,k} - 2U_{i,j,k} + U_{i-1,j,k}}{(\Delta x)^2} + \frac{U_{i,j+1,k} - 2U_{i,j,k} + U_{i,j-1,k}}{(\Delta y)^2} + \frac{U_{i,j,k+1} - 2U_{i,j,k} + U_{i,j,k-1}}{(\Delta z)^2} = f_{i,j,k} \tag{1.11}$$

This approximation scheme is a second order approximation one, and gives a seven point stencil form. This means that the Poisson's equation is transformed in to a large system of linear equations in terms of the functional values of U at the grid points.

We can consider the fourth order approximation of $\frac{\partial^2 U}{\partial x^2}$ by

$$h^2 \frac{\partial^2 U}{\partial x^2} = \left(\frac{\delta_x^{\,2}}{1 + \frac{1}{12}\delta_x^{\,2}} \right) U + O\left(h^4\right) \tag{1.12}$$

Similarly, we write for the operators $\frac{\partial^2 U}{\partial y^2}$ and $\frac{\partial^2 U}{\partial z^2}$ using (1.12) and substitute these in to (1.3), we get another higher order approximation scheme

$$\left(\frac{\delta_x^{\,2}}{h_1^{\,2}\left(1 + \frac{1}{12}\delta_x^{\,2}\right)} + \frac{\delta_y^{\,2}}{h_2^{\,2}\left(1 + \frac{1}{12}\delta_y^{\,2}\right)} + \frac{\delta_z^{\,2}}{h_3^{\,2}\left(1 + \frac{1}{12}\delta_z^{\,2}\right)} + O\left(h_1^{\,4}\right) + O\left(h_2^{\,4}\right) + O\left(h_3^{\,4}\right) \right) U_{i,j,k} = f_{i,j,k} \tag{1.13}$$

On further simplification of (1.13), depending on the accuracy of the approximation, we can get different systems of a large number of linear equations such as 19-point stencil scheme or 27-point scheme and other schemes.

Similarly we can establish the equivalent finite difference approximation of the three dimensional Poisson's equation in Cylindrical/Spherical coordinates system.

For biharmonic equation (1.2) we have also different approximation schemes, based on the coordinates system of the problem and its boundary kind (first or second). Efficient numerical methods for the solution of the discrete biharmonic equation on simple regions have recently received a great deal of attention.

There are essentially two approaches to solve the biharmonic problem numerically:

The first approach consists of the direct discretization of the biharmonic equation, which results either 9-points, 13-points or 25-points formula (See the derivation of the 13 or 25-points approximation schemes in [35]). The 13-points and 25-points approximation schemes lead to a system of algebraic equations with a 13-diagonal or 25-diagonal coefficient matrix which is, in general, very ill-conditioned. Some progress has been made in recent years for the direct solution of these matrix equations, (see [11], [53], [57]) because most of the iterative methods require a large number of iterations in order to obtain some satisfactory solutions (see [11],[53],[56], [57],[58] and other references therein).

For the first kind problem Gupta and Manohar [78] have considered several such schemes and have shown that the accuracy of its numerical solution depends upon the boundary approximation used.

The second approach is the splitting method where the biharmonic equation (1.2) is replaced by introducing an auxiliary variable $v(x,y)=\nabla^2 U(x,y)$ and splitting the biharmonic equation into a coupled system of Poisson equations as

$$\nabla^2 U = v$$
$$\nabla^2 v = f$$

That is, the biharmonic equation with the given boundary conditions is equivalent to the Dirichlet problems for two Poisson equations.

Thus (1.2) can be written using the splitting method for the first kind boundary problem as

$$\begin{aligned} &\nabla^2 U = v && \text{on } D \\ &U = f_1 && \text{on } \partial D \qquad \text{and} \qquad \nabla^2 v = f \\ &\frac{\partial U}{\partial n} = f_2 && \text{on } \partial D \end{aligned}$$

And one can easily see that under this formulation for the first kind boundary problem, one of these Poisson equations has no boundary conditions and we consider two classes of boundary approximations for this undefined boundary condition. The functions U and v are coupled through the boundary conditions implicitly which turns out to be the main difficulty of solving such problem.

The second kind boundary problem (1.2) can be written using the splitting method as

$$\begin{cases} \nabla^2 U = v & \text{on } D \\ U = f_1 & \text{on } \partial D \end{cases} \quad \text{and} \quad \begin{cases} \nabla^2 v = f & \text{on } D \\ v = f_2 & \text{on } \partial D \end{cases}$$

Observe that the biharmonic equation is converted in to two Poisson's equation with sufficient boundary conditions for both Poisson's equations. These two second order boundary value problems can be discretized and solved using one of the fast Poisson solvers that are available, (see [25],[35],[41],[42], [44],[73]); we first solve for v from the second Poisson's equation and use this value to solve U in the first. One of the important reasons for using the splitting method, in general, is that the accumulation of rounding errors is substantially reduced; and it is shown that the splitting of the biharmonic equation produces a numerically efficient procedure and is very successful for the second kind problem since the boundary conditions (the second derivative) do not have to be discretized at all in this case (see [18],[58],[77]).

Still now there are tremendous progresses in developing higher order approximation schemes of the finite difference method to address the accuracy of the numerical solutions of Poisson's equation. For different applications in engineering and sciences such as the computation of incompressible viscous flows, the higher-order compact (HOC) finite difference schemes have been developed ([37],[38],[39],[79],[82]). Most of these schemes were developed for equations of the convection–diffusion type and were well equipped to simulate incompressible viscous flows governed by the N–S equations as well. A compact finite difference scheme is one that utilizes grid points located only directly adjacent to the node about which the differences are taken. In addition, if the scheme has an order of accuracy greater than two, it is termed as HOC method. The higher-order accuracy of the HOC methods combined with the compactness of the difference stencils yields highly accurate numerical solutions on relatively coarser grids with greater computational efficiency.

Since the 1950s considerable contributions in developing high-order-accurate finite-difference discretization schemes for elliptic partial differential equations have been made by, for instance, O. Buneman [43], R.W Hockney [96], and others[22],[29],[35],[46],[74],[76],[90],[92].

1.5 Systems of Linear Equations

One of the most important phases in the analysis of many engineering systems is the solution of a set of linear equations. For the case of a linear boundary value problem, only one solution of equations is required. However, for initial value problems and for nonlinear systems several solutions of sets of equations may be necessary for a complete analysis. There are a number of different techniques to solve a system of linear equations. Solution methods for linear systems fall into two categories: *direct methods*, which provide the answer with finitely many operations; and *iterative methods*, construct a sequence of approximations to the exact solution of a linear system.

Many direct methods are developed for the Poisson's equation and they fall into the following categories:

-methods based on Gaussian-Elimination and reordering; such as LU factorization, Cholesky factorization, QR factorization and others.

-marching techniques [49],[50],[51]

-methods based on fast Fourier transform (FFT);

-methods based on block cyclic reduction;

-methods based on both FFT and cyclic reduction (FACR) [96].

The most well-known iterative methods are of course the Jacobi method and the Gauss-Seidel method. These methods are easy to implement but usually not efficient. More recent iterative methods, like the Conjugate Gradient (CG) method and the Generalized Minimal Residual (GMRes) method are much more efficient. Furthermore, the performance of iterative methods depends on the spectrum of the coefficient matrix. For systems with low or moderate dimensions and for large systems with a band structure, the most efficient algorithms for solving linear systems are direct [23],[43],[60].

Theoretically many of the methods are similar; however, the computer program implementation of these methods may differ significantly. During recent years it has been recognized that in most cases a direct solution of linear equations is preferable to using an iterative technique; and as a consequence of this considerable research has been devoted toward finding very efficient equation solvers by direct method.

Thus to find a numerical solution of PDEs, like Poisson's/biharmonic, first we transform the PDE using *finite difference method* in to a system of linear equations and we solve the resulting equations by one of the technique developed so far. Once we obtain these systems of linear equations, applying some points from the theory of matrix we can fully discuss about the nature of the solution. [8],[9],[10],[16],[17],[27],[30],[31],[43],[52],[54],[59],[71],[75],[84],[87],[94], [97], [99],[105],[106][109]).

A key strategy in matrix computations is to transform the matrix in question to a form that makes the problem at hand easy to solve. Major important contributions have been made to solve a large number of systems of linear equations using eigenvalues and eigenvectors. Historically, Euler first solved the eigenvalue problem when he developed a simple mathematical model for describing the 'buckling' modes of a vertical elastic beam. The general theory of eigenvalue problems for second-order differential equations, now known as the *Sturm–Liouville Theory*, originated from the study of a class of boundary-value problems due to Charles Sturm (1803–1855) and Joseph Liouville (1809–1882).

Major results of eigenvalues and eigenvectors have made the community of computations for the solution of PDE somewhat smooth and helped a lot to develop solvers. Here we present some points that are relevant for our discussion in the coming chapters.

Definition: Let A be a matrix of order n and let U be nonsingular. Then the matrices A and $B = U^{-1}AU$ are said to be *similar*. We also say that B is obtained from A by a *similarity transformation.*

The following is an important result of a similarity transformation.

Let the matrix A have a complete system of eigenpairs (λ_i, x_i) $i = 1, 2, ..., n$ where λ_i is the eigenvalue and x_i is the eigenvector of A and $X = (x_1, x_2, \cdots, x_n)$ and $\Lambda = diag(\lambda_1, \lambda_2, \cdots, \lambda_n)$.

Then, the individual relations $Ax_i = \lambda_i x_i$ can be combined in the matrix equation as $AX = X\Lambda$.

Because the eigenvectors x_i are linearly independent, and the matrix X is nonsingular.

Hence we may write $X^{-1}AX = \Lambda$.

Thus we have shown that a matrix with a complete system of eigenpairs can be reduced to diagonal form by a similarity transformation, whose columns are eigenvectors of A. Conversely, by reversing the above argument, we see that if A can be diagonalized by a similarity transformation $X^{-1}AX$, then the columns of X are eigenvectors of A, which form a complete system. The matrix X formed by the linearly independent eigenvectors of a matrix A is called a modal matrix for A and the diagonal matrix Λ having the eigenvalues of A as the diagonal elements is called a spectral matrix for A.

A nice feature of similarity transformations is that they affect the eigensystem of a matrix in a systematic way, as the following theorem shows.

<u>Theorem</u>: Let A be a matrix of order n and let $B = U^{-1}AU$ be similar to A. Then the eigenvalues of A and B are the same and have the same multiplicities.

If (λ, x) is an eigenpairs of A, then $(\lambda, U^{-1}x)$ is an eigenpairs of B.

<u>Proof</u>

Since $\det(U^{-1})\det(U) = \det(U^{-1}U) = \det(I) = 1$,

$$\det(\lambda I - A) = \det(U^{-1})\det(\lambda I - A)\det(U) = \det(\lambda I - U^{-1}AU) = \det(\lambda I - B)$$

Thus A and B have the same characteristic polynomial and hence the same eigenvalues with the same multiplicities.

If (λ, x) is an eigenpairs of A, then

$$B(U^{-1}x) = U^{-1}AU(U^{-1}x) = U^{-1}Ax = \lambda(U^{-1}x)$$

so that $(\lambda, U^{-1}x)$ is an eigenpairs of B.

In addition to preserving the eigenvalues of a matrix (and transforming eigenvectors in a predictable manner), similarity transformations preserve functions of the eigenvalues. The determinant of a matrix, which is the product of its eigenvalues, is clearly unchanged by similarity transformations.

<u>Corollary</u>: Let the matrix A of order n have distinct eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_k$ with multiplicities $m_1, m_2, \cdots, m_k$. Then there is a nonsingular matrix X such that

$X^{-1}AX = diag(L_1, L_2, \cdots, L_k)$ where L_i is of order m_i and has only the eigenvalue λ_i.

- If all the eigenvalues of A are distinct, then the blocks L_i are scalars.
- Hence, if A has distinct eigenvalues, A is diagonalizable and has a complete system of eigenvectors.

- Let $A \in \mathbb{R}^{n \times n}$ and suppose it has a real eigenvalue λ. Then in the homogeneous equation $(XI - A)v = 0$, the coefficient matrix $(XI - A)$ is real. The fact that $(XI - A)$ is singular implies that the equation $(XI - A)v = 0$ has nontrivial real solutions. We conclude that every real eigenvalue of a real matrix has a real eigenvector associated with it.

Corollary: The eigenvectors associated with distinct eigenvalues of a real symmetric matrix constitute an orthogonal set.

Corollary: An $n \times n$ real symmetric matrix A possesses an orthogonal (as well as orthonormal) set of n eigenvectors.

Theorem: The eigenvalues of a real symmetric matrix are all real.

Theorem: All the eigenvalues of a symmetric positive-definite matrix are positive.

For, suppose $AX = \lambda X \quad (X \neq 0)$

Then $0 < X^T AX = \lambda X^T X = \lambda \|X\|_2^2$

$\Rightarrow \lambda > 0$

To find the numerical solution of PDE, a great number of computer soft ware have been introduced and developed; in particular to find the numerical solution of Poisson's equation such as Fortran module *Poisson solvers.f95,* POSSOL (a two-dimensional Poisson equation solver for problems with arbitrary non-uniform gridding in Cartesian coordinates developed by *Schwarztrauber* and *Sweet*), POISSON, SUPERFISH (used to compute field quality for both magnets and fixed electric potentials and RF cavity codes that calculate resonant frequencies and field distributions of the fundamental and higher modes), Scilab, Matlab, and others have been developed in the last 60 years. This era is one of the best periods in history in developing numerical algorithms and their computer codes.

1.6 Organization of the Thesis

In this thesis an attempt has been made to solve the three dimensional Poisson's equation in Cartesian and Cylindrical coordinates system and the two and three dimensional biharmonic equation with the Dirichlet boundary problem of the second kind. Here is the chapter wise summary of the thesis.

Chapter II deals with the second order numerical solution of the three dimensional Poisson's equation in Cartesian coordinates system

$$\frac{U_{i+1,j,k}-2U_{i,j,k}+U_{i-1,j,k}}{(\Delta x)^2}+\frac{U_{i,j+1,k}-2U_{i,j,k}+U_{i,j-1,k}}{(\Delta y)^2}+\frac{U_{i,j,k+1}-2U_{i,j,k}+U_{i,j,k-1}}{(\Delta z)^2}=f_{i,j,k}$$

in a cube with the given Dirichlet's boundary conditions.

When $\Delta x=\Delta y=h_1$ and $\Delta z=h_2$ (h_1 and h_2 need not be equal), this Poisson's equation reduces to

$$U_{i+1,j,k}+U_{i-1,j,k}+U_{i,j+1,k}+U_{i,j-1,k}+r\left(U_{i,j,k+1}+U_{i,j,k-1}\right)-(4+2r)U_{i,j,k}=h^2 f_{i,j,k}$$

where $r=\dfrac{h_1^{\,2}}{h_2^{\,2}}$

A large number of linear equations are obtained and systematically arranged in order to get a block tri diagonal matrix structure. The obtained systems of linear equations are solved by extending the method of Hockney in three dimensional Cartesian coordinates system. Six examples have been considered in both cases and it is found that the method produce accurate results considering double precision.

Chapter III deals with the fourth order numerical solution of the three dimensional Poisson's equation in Cartesian coordinates system, i.e.

$$\left(\delta_x^{\,2}\left(1+\frac{1}{12}\delta_y^{\,2}\right)\left(1+\frac{1}{12}\delta_z^{\,2}\right)+\delta_y^{\,2}\left(1+\frac{1}{12}\delta_x^{\,2}\right)\left(1+\frac{1}{12}\delta_z^{\,2}\right)+r\delta_z^{\,2}\left(1+\frac{1}{12}\delta_x^{\,2}\right)\left(1+\frac{1}{12}\delta_y^{\,2}\right)\right)U_{i,j,k}$$
$$=h_1^{\,2}\left(1+\frac{1}{12}\delta_x^{\,2}\right)\left(1+\frac{1}{12}\delta_y^{\,2}\right)\left(1+\frac{1}{12}\delta_z^{\,2}\right)f_{i,j,k}$$

in a cube with the Dirichlet's boundary conditions. Based on the approximation we have considered two cases.

Case I   19 points stencil scheme

Case II   27 points stencil scheme

Both schemes results in a large algebraic system of linear equations and are treated systematically in order to get a block tri-diagonal system, and these systems of linear equations are solved by the use of Thomas algorithm. Six examples are taken to show the accuracy of the method and it is shown that the method produces accurate results.

Part of this chapter has been published in the ***American Journal of Computational Mathematics*, 2011,vol 1, No. 4 pp 285-293.**

Chapter IV deals with the second order numerical solution of the three dimensional Poisson's equation in cylindrical coordinates system given by

$$u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} + u_{zz} = f\left(r,\theta,z\right)$$

with the Dirichlet's boundary conditions for $r \neq 0$

The Poisson equation for $r \neq 0$ is approximated by second order finite difference approximation

$$f_{i,j,k} = \frac{U_{i+1,j,k} - 2U_{i,j,k} + U_{i-1,j,k}}{(\Delta r)^2} + \frac{U_{i+1,j,k} - U_{i-1,j,k}}{(2\Delta r)\ r_i} + \frac{1}{r_i^2}\left(\frac{U_{i,j+1,k} - 2U_{i,j,k} + U_{i,j-1,k}}{(\Delta\theta)^2}\right) + \frac{U_{i,j,k+1} - 2U_{i,j,k} + U_{i,j,k-1}}{(\Delta z)^2} + O\left((\Delta r)^2\right) + O\left((\Delta\theta)^2\right) + O\left((\Delta z)^2\right)$$

and truncating higher order differences and simplifying, we have

$$(1+\omega_i)U_{i+1,j,k} + (1-\omega_i)U_{i-1,j,k} + \alpha_i(U_{i,j+1,k} + U_{i,j-1,k}) + \rho(U_{i,j,k+1} + U_{i,j,k-1}) + y_i U_{i,j,k} = (\Delta r)^2\, f_{i,j,k}$$

where $\omega_i = \frac{\Delta r}{2r_i}$, $\alpha_i = \frac{(\Delta r)^2}{r_i^2(\Delta\theta)^2}$, $\rho = \frac{(\Delta r)^2}{(\Delta z)^2}$ and $y_i = -2(1 + \alpha_i + \rho)$

the resulting large algebraic system of linear equations obtained is treated systematically in order to get a block tri-diagonal system, and these systems of linear equations are solved by the use of Thomas algorithm. Seven examples have been tested to verify the efficiency of the method and it is shown that this method produces good result.

Part of this chapter has been published in the ***American Journal of Computational Mathematics*****, 2013,vol. 3, pp 356-361.**

Chapter V deals with the fourth order numerical solution of the three dimensional Poisson's equation in cylindrical coordinates system given by

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} + u_{zz} = f\left(r,\theta,z\right)$$

with the Dirichlet's boundary conditions for $r \neq 0$.

The Poisson equation for $r \neq 0$ is approximated by fourth order finite difference approximation scheme

$$\begin{aligned}(\Delta r)^2\left(24 + \delta_r^2 + \delta_\theta^2 + \delta_z^2 + \frac{3\Delta r}{2r_i}\delta_{2r}\right)f_{i,j,k} &= a_0(i)U_{i,j,k} + a_1(i)U_{i+1,j,k} + a_2(i)U_{i-1,j,k} \\ &+ a_3(i)(U_{i,j+1,k} + U_{i,j-1,k}) + a_4(i)(U_{i,j,k+1} + U_{i,j-1,k}) + a_5(i)(U_{i+1,j+1,k} + U_{i+1,j-1,k}) \\ &+ a_6(i)(U_{i-1,j+1,k} + U_{i-1,j-1,k}) + a_7(i)(U_{i+1,j,k+1} + U_{i+1,j,k-1}) + a_8(i)(U_{i-1,j,k+1} + U_{i-1,j,k-1}) \\ &+ a_9(i)(U_{i,j+1,k+1} + U_{i,j-1,k+1} + U_{i,j+1,k-1} + U_{i,j-1,k-1})\end{aligned}$$

and the resulting large algebraic system of linear equations is treated systematically in order to get a block tri-diagonal system, and these systems of linear equations are solved by the use of *Thomas algorithm*. Seven examples for both cases have been considered and it is shown that this method produces good result.

Part of this chapter is to appear in the ***American Journal of Computational Mathematics***

Chapter VI deals with the second and fourth-order approximation scheme for the numerical solution of the three dimensional Poisson's equation in cylindrical coordinates system when $r = 0$ is an interior or a boundary point where the system becomes singular. By taking an appropriate approximation scheme together with the method developed in chapters IV and V we solve the system for the given Dirichlet's boundary conditions.

Chapter VII deals with the numerical solution of the two and three dimensional biharmonic equation of the second type in a rectangular region and a cube respectively, in Cartesian coordinate systems

$$\nabla^4 U = f \quad \text{on} \quad D$$

$$U = g_1 \quad \text{on} \quad \partial D$$

$$\nabla^2 U = g_2 \quad \text{on} \quad \partial D.$$

The two or three dimensional biharmonic equation can be decoupled as two Poisson's equations with a Dirichlet boundary condition on the same domain as

$$\begin{cases} \nabla^2 U = v & \text{on } D \\ U = g_1 & \text{on } \partial D \end{cases}$$

$$\begin{cases} \nabla^2 v = f & \text{on } D \\ v = \Delta U & \text{on } \partial D \end{cases}$$

These coupled Poisson's equations are solved directly by using the fourth order finite difference approximation schemes which we have developed so far in Chapter III adapted to two or three dimensional Cartesian coordinates system. Seven examples have been taken to test the efficiency of the method, and results have shown that the method produced comparable results as shown in literatures.

Chapter VIII is about the conclusion part and the future work plan.

CHAPTER II

Second Order Numerical Solution of the Three Dimensional Poisson's Equation in Cartesian Coordinates System

2.1 Introduction

The three-dimensional Poisson's equation is often encountered in heat and mass transfer theory, fluid mechanics, elasticity, electrostatics, and other areas of mechanics and physics. In particular, the Poisson's equation describes stationary temperature distribution in the presence of thermal sources or sinks in the domain under consideration. A variety of problems in computational physics require the second order numerical solution of the three dimensional Poisson's equation in Cartesian coordinates system. To solve the two or three dimensional Poisson's equation in Cartesian coordinates system, different attempts have been made with respect to developing new methods and accuracy of the solution.

For two dimensional Poisson's equation in Cartesian coordinates system, for instance, *Averbuch et al* [1] developed a direct method in rectangular regions based on a pseudospectral Fourier approximation and a polynomial subtraction technique; *McKenney and Greengard* [4] developed a fast Poisson Solver based on potential theory by combining fast algorithms for computing volume integrals and evaluating layer potentials on a grid with a fast multipole accelerated integral equation solver; *Banegas* [6] developed a Fast Poisson Solvers for Problems with Sparsity; *Buzbee et al* developed [12] the direct solution of the discrete Poisson equation on irregular regions, [13] a unified mathematical development and generalization of the method of matrix decomposition or discrete separation of variables and the block-cyclic reduction process and techniques for solving the reduced system; *Braverman et al* [19] developed a fast spectral Subtractional solver for elliptic equations based on the eigenfunction expansion of the right hand side with integration and the successive solution of the corresponding homogeneous equation using Modified Fourier Method; *Ethridge and Greengard* [24] developed an integral equation method for solving the Poisson equation in two dimensions in which they claim that their method is direct, high order accurate, insensitive to the degree of adaptive mesh refinement, and accelerated by the fast multipole method; *Skolermo* [28] developed a method in a rectangle, based on the relation between the Fourier coefficients for the solution and those for the right-hand side, and the Fast Fourier Transform is used for the computation; *Greengard* [55] developed a direct, adaptive solver for the Poisson equation based on a domain decomposition approach using local spectral approximation, as well as potential theory and the fast multipole method; *Kadalbajoo and Bharadwaj* [70] presented a survey of fast direct methods for solving elliptic boundary-value problems and the methods reviewed are based on Fourier analysis, block reduction techniques, and marching algorithms; *Swarztrauber* [84] developed approximate cycle reduction for solving Poisson's equation; *Hockney* [96] developed a technique using Fourier series for numerically approximating the solution of the Poisson equation in a rectangle.

For the three dimensional Poisson's equation, *Braverman et al* [20] have developed a fast 3D Poisson solver of arbitrary order accuracy based on the application of the discrete Fourier transform accompanied by a subtraction technique which allows reducing the errors associated with the Gibbs phenomenon; *Israeli et al* [66] a domain decomposition non-iterative solver in a 3D rectangular box based on the application of the discrete Fourier transform accompanied by a subtraction technique.

The aim of this Chapter is to derive a second order finite difference approximation scheme to solve the three dimensional Poisson's equation on Cartesian coordinates system. The resulting large algebraic system of linear equations is treated systematically in order to get a block tri-diagonal system [60] and extend the Hockney's method [30].

2.2 *Finite Difference Approximation*

Consider the three dimensional Poisson equation in Cartesian coordinate system

$$\nabla^2 U = \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} = f(x,y,z) \qquad \text{on } D \tag{2.1}$$

$$U(x,y,z) = g(x,y,z) \qquad \text{on } \partial D \tag{2.2}$$

where $D = \{(x,y,z): 0 < x < a,\ 0 < y < b,\ 0 < z < c\}$ and ∂D is the boundary of D.

Assume that there are M, N and P mesh points along the X, Y and Z directions respectively, and let $U(x,y,z)$ be discretized at the mesh point (i,j,k) and we adopt writing $U_{i,j,k}$ for $U(x_i, y_j, z_k)$, where $i = 1(1)M$, $j = 1(1)N$ and $k = 1(1)P$

Let Δx, Δy and Δz be the step sizes in the X, Y and Z directions respectively, and suppose $x_i = i\Delta x \quad \Rightarrow x + \Delta x = i\Delta x + \Delta x = (i+1)\Delta x$, and

$$\Rightarrow x_i - \Delta x = i\Delta x - \Delta x = (i-1)\Delta x$$

Similarly we have for $y_j + \Delta y = (j+1)\Delta y, \quad y_j - \Delta y = (j-1)\Delta y$

$$z_k + \Delta z = (k+1)\Delta z, \quad z_k - \Delta z = (k-1)\Delta z$$

Thus we write $U_{i\pm1,j,k}$ for $U(x_i \pm \Delta x, y_j, z_k)$, $U_{i,j\pm1,k}$ for $U(x_i, y_j \pm \Delta y, z_k)$ and $U_{i,j,k\pm1}$ for $U(x_i, y_j, z_k \pm \Delta z)$

We transform (2.1) in to its equivalent finite difference approximation by

$$\frac{U_{i+1,j,k} - 2U_{i,j,k} + U_{i-1,j,k}}{(\Delta x)^2} + \frac{U_{i,j+1,k} - 2U_{i,j,k} + U_{i,j-1,k}}{(\Delta y)^2} + \frac{U_{i,j,k+1} - 2U_{i,j,k} + U_{i,j,k-1}}{(\Delta z)^2} = f_{i,j,k} \tag{2.3}$$

Assume that $\Delta x = \Delta y = h_1$ and $\Delta z = h_2$ (h_1 and h_2 need not be equal)

Let $r = \dfrac{h_1^{\,2}}{h_2^{\,2}}$, we can write the above equation as

$$U_{i+1,j,k} + U_{i-1,j,k} + U_{i,j+1,k} + U_{i,j-1,k} + r\left(U_{i,j,k+1} + U_{i,j,k-1}\right) - (4+2r)U_{i,j,k} = h^2 f_{i,j,k} \tag{2.4}$$

$$i = 1(1)M, \qquad j = 1(1)N, \qquad k = 1(1)P$$

In (2.3) we put $k = 1$ and $j = 1$, by taking i from 1 to M we get M set of equations along the x direction. Again putting $j = 2$, still we get another M set of equations, and so on until $j = N$, getting a total of MN set of equations on the plane parallel to the XY plane. Again for $k = 2(1)P$ we follow the same pattern as $k = 1$ and finally we have P block of equations and each block has a set of MN equations. Thus, in general, (2.1) can be written in matrix form as

$$AU = \mathcal{B} \tag{2.5}$$

where

$$A = \begin{pmatrix} R & S & & & & & \\ S & R & S & & & & \\ & S & R & S & & & \\ & & & \ddots & & & \\ & & & & S & R & S \\ & & & & & S & R \end{pmatrix} \tag{2.6}$$

Matrix A has P blocks and each block is of order $MN \times MN$,

$$R = \begin{pmatrix} T & I_M & & & & & \\ I_M & T & I_M & & & & \\ & I_M & T & I_M & & & \\ & & & \ddots & & & \\ & & & & I_M & T & I_M \\ & & & & & I_M & T \end{pmatrix}, \qquad S = \begin{pmatrix} r\,I_M & & & & \\ & r\,I_M & & & \\ & & r\,I_M & & \\ & & & \ddots & \\ & & & & r\,I_M \end{pmatrix}$$

R and S have N blocks and each block is of order $M \times M$.

$$T = \begin{pmatrix} -4-2r & 1 & & & & & \\ 1 & -4-2r & 1 & & & & \\ & 1 & -4-2r & 1 & & & \\ & & & \ddots & & & \\ & & & & 1 & -4-2r & 1 \\ & & & & & 1 & -4-2r \end{pmatrix} \tag{2.7}$$

where T is a square matrix of order M, and I_M is an identity matrix of order M,

$U = \begin{pmatrix} \mathbf{U}_1 & \mathbf{U}_2 & \mathbf{U}_3 & \cdots & \mathbf{U}_{P-1} & \mathbf{U}_P \end{pmatrix}^T$, and

$\mathcal{B} = \begin{pmatrix} \mathbf{B}_1 & \mathbf{B}_2 & \mathbf{B}_3 & \ldots & \mathbf{B}_{P-1} & \mathbf{B}_P \end{pmatrix}^T$

where

$$\mathbf{U}_k = [\mathbf{u}_{1k} \ \mathbf{u}_{2k} \ldots \mathbf{u}_{Nk}]^T \quad \text{and} \quad \mathbf{u}_{jk} = [\ U_{1jk} \ \ U_{2jk} \ \ldots U_{Mjk}]^T$$

and $\mathbf{B}_k = [\mathbf{d}_{1k} \ \mathbf{d}_{2k} \ldots \mathbf{d}_{Nk}]^T$ and $\mathbf{d}_{jk} = [\ d_{1jk} \ \ d_{2jk} \ldots d_{Mjk}]^T \ k = 1, 2, 3, \ldots, P$

is the known column vector such that each d_{ijk} represents known boundary values of U and values of f .

Thus, (2.5) can be written as

$$\begin{pmatrix} R & S & & & & & \\ S & R & S & & & & \\ & S & R & S & & & \\ & & & \ddots & & & \\ & & & & S & R & S \\ & & & & & S & R \end{pmatrix} \begin{pmatrix} \mathbf{U}_1 \\ \mathbf{U}_2 \\ \mathbf{U}_3 \\ \vdots \\ \mathbf{U}_{P-1} \\ \mathbf{U}_P \end{pmatrix} = \begin{pmatrix} \mathbf{B}_1 \\ \mathbf{B}_2 \\ \mathbf{B}_3 \\ \vdots \\ \mathbf{B}_{P-1} \\ \mathbf{B}_P \end{pmatrix} \tag{2.8}$$

Equation (2.8) once again can be written as

$$\begin{aligned} R\mathbf{U}_1 + S\mathbf{U}_2 \quad &= \mathbf{B}_1 \\ S\mathbf{U}_1 + R\mathbf{U}_2 + S\mathbf{U}_3 &= \mathbf{B}_2 \\ S\mathbf{U}_2 + R\mathbf{U}_3 + S\mathbf{U}_4 &= \mathbf{B}_3 \\ &\ldots \\ S\mathbf{U}_{P-1} + R\mathbf{U}_P &= \mathbf{B}_P \end{aligned} \tag{2.9}$$

We obtain the solution of the system of linear equations (2.9) by applying extended Hockney's method to three dimensions.

2.3 Extended Hockney's Method

As we can see the matrix T is a real tridiagonal symmetric matrix and hence its eigenvalues and eigenvectors can easily be obtained (See G.D Smith [30]).

Note that the eigenvalues η_i of T are given by

$$\eta_i = -4 - 2r + 2\cos\left(\frac{i\pi}{M+1}\right) \quad i = 1, 2, \cdots, M$$

Let $\mathbf{q}_i$ be an eigenvector of T corresponding to the eigenvalue η_i and Q be the modal matrix $[\mathbf{q}_1 \; \mathbf{q}_2 \; \mathbf{q}_3 \ldots \mathbf{q}_M]$ of the matrix T of order M such that

$$Q^T Q = I \text{ , and}$$

$$Q^T T Q = diag(\eta_1, \eta_2, \eta_3, \ldots, \eta_M) = \mathrm{H} \text{ (say)} \tag{2.10}$$

The $M \times M$ modal matrix Q is defined by

$$q_{ij} = \sqrt{\frac{2}{M+1}} \sin\left(\frac{ij\pi}{M+1}\right) \quad i, j = 1, 2, \cdots, M$$

Let $\mathbb{Q} = diag\left(Q, Q, Q, \ldots, Q\right)$ be a matrix of order $MN \times MN$.

Thus $\mathbb{Q}$ satisfy $\mathbb{Q}^T \mathbb{Q} = I$, and

$$\mathbb{Q}^T R \mathbb{Q} = \begin{pmatrix} \mathrm{H} & I_M & & & & & \\ I_M & \mathrm{H} & I_M & & & & \\ & I_M & \mathrm{H} & I_M & & & \\ & & & \ddots & & & \\ & & & & I_M & \mathrm{H} & I_M \\ & & & & & I_M & \mathrm{H} \end{pmatrix} = R^*$$

and $\mathbb{Q}^T S \mathbb{Q} = S$ (say)

$$\text{Let} \quad \mathbb{Q}^T \mathbf{U}_k = \mathbf{V}_k \Rightarrow \mathbf{U}_k = \mathbb{Q}\, \mathbf{V}_k$$

$$\mathbb{Q}^T \mathbf{B}_k = B_k \Rightarrow \mathbf{B}_k = \mathbb{Q} B_k \tag{2.11}$$

where $\mathbf{V}_k = [\mathbf{v}_{1k} \; \mathbf{v}_{2k} \ldots \mathbf{v}_{Nk}]^T$ and $\mathbf{v}_{jk} = [v_{1jk} \; v_{2jk} \ldots v_{Mjk}]^T$

$$B_k = [\mathbf{b}_{1k}\ \ \mathbf{b}_{2k}\ \dots\ \mathbf{b}_{Nk}]^T \text{ and } \mathbf{b}_{jk} = [\ b_{1jk}\ \ b_{2jk}\ \dots\ b_{Mjk}]^T$$

Pre multiplying (2.9) by $\mathbb{Q}^T$ and using (2.11), we get

$$\begin{aligned}
R^*\mathbf{V}_1 + S\mathbf{V}_2 &= B_1 \\
S\mathbf{V}_1 + R^*\mathbf{V}_2 + SV_3 &= B_2 \\
S\mathbf{V}_2 + R^*\mathbf{V}_3 + S\mathbf{V}_4 &= B_3 \\
&\dots \\
S\mathbf{V}_{P-1} + R^*\mathbf{V}_P &= B_P
\end{aligned} \tag{2.12}$$

Consider the first equation of (2.12) i.e. $R^*\mathbf{V}_1 + S\mathbf{V}_2 = B_1$ which we can write it as

$$\begin{pmatrix} \mathrm{H} & I_M & & & & \\ I_M & \mathrm{H} & I_M & & & \\ & I_M & \mathrm{H} & I_M & & \\ & & & \ddots & & \\ & & & I_M & \mathrm{H} & I_M \\ & & & & I_M & \mathrm{H} \end{pmatrix} \begin{pmatrix} \mathbf{v}_{11} \\ \mathbf{v}_{21} \\ \mathbf{v}_{31} \\ \vdots \\ \mathbf{v}_{(N-1)1} \\ \mathbf{v}_{N1} \end{pmatrix} + \begin{pmatrix} rI_M & & & & \\ & rI_M & & & \\ & & rI_M & & \\ & & & \ddots & \\ & & & & rI_M \end{pmatrix} \begin{pmatrix} \mathbf{v}_{12} \\ \mathbf{v}_{22} \\ \mathbf{v}_{32} \\ \vdots \\ \mathbf{v}_{N2} \end{pmatrix} = \begin{pmatrix} \mathbf{b}_{11} \\ \mathbf{b}_{21} \\ \mathbf{b}_{31} \\ \vdots \\ \mathbf{b}_{N1} \end{pmatrix} \tag{2.13}$$

Again we write equation (2.13) as

$$\begin{aligned}
\mathrm{H}\mathbf{v}_{11} + \mathbf{v}_{21} + \qquad r\mathbf{v}_{12} &= \mathbf{b}_{11} \\
\mathbf{v}_{11} + \mathrm{H}\mathbf{v}_{21} + \mathbf{v}_{31} + r\mathbf{v}_{22} &= \mathbf{b}_{21} \\
\mathbf{v}_{21} + \mathrm{H}\mathbf{v}_{31} + \mathbf{v}_{41} + r\mathbf{v}_{32} &= \mathbf{b}_{31} \\
&\dots \\
\mathbf{v}_{(N-1)1} + \mathrm{H}\mathbf{v}_{N1} \qquad + r\mathbf{v}_{N2} &= \mathbf{b}_{N1}
\end{aligned} \tag{2.14}$$

Now collect the first equations from each of (2.14) and consider as one group of equations

$$\begin{aligned}
\eta_1 v_{111} + v_{121} + \qquad rv_{112} &= b_{111} \\
v_{111} + \eta_1 v_{121} + v_{131} + rv_{122} &= b_{121} \\
v_{121} + \eta_1 v_{131} + v_{141} + rv_{132} &= b_{131} \\
&\dots \\
v_{1(N-1)1} + \eta_1 v_{1N1} \qquad + rv_{1N2} &= b_{1N1}
\end{aligned} \tag{2.15a}$$

Again we collect the second equations from each equation of (2.14) and consider as a second group of equations

$$
\begin{aligned}
&\eta_2 v_{211} + v_{221} + \qquad r v_{212} = b_{211} \\
&v_{211} + \eta_2 v_{221} + v_{231} + r v_{222} = b_{221} \\
&v_{221} + \eta_2 v_{231} + v_{241} + r v_{232} = b_{231} \\
&\qquad\qquad \cdots \\
&v_{2(N-1)1} + \eta_2 v_{2N1} \quad + r v_{2N2} = b_{2N1}
\end{aligned}
\tag{2.15b}
$$

Lastly we collect the last equations from each equation of (2.14) and consider as a last group of equations

$$
\begin{aligned}
&\eta_M v_{M11} + v_{M21} + \qquad r v_{M12} = b_{M11} \\
&v_{M11} + \eta_M v_{M21} + v_{M31} + r v_{M22} = b_{M21} \\
&v_{M21} + \eta_M v_{M31} + v_{M41} + r v_{M32} = b_{M31} \\
&\qquad\qquad \cdots \\
&v_{M(N-1)1} + \eta_M v_{MN1} \quad + r v_{MN2} = b_{MN1}
\end{aligned}
\tag{2.15c}
$$

Equations (2.15a) to (2.15c) can be written in matrix form as

$$
\begin{pmatrix}
\eta_i & 1 & & & & \\
1 & \eta_i & 1 & & & \\
 & 1 & \eta_i & 1 & & \\
 & & & \ddots & & \\
 & & & 1 & \eta_i & 1 \\
 & & & & 1 & \eta_i
\end{pmatrix}
\begin{pmatrix} v_{i11} \\ v_{i21} \\ v_{i31} \\ \vdots \\ v_{i(N-1)1} \\ v_{iN1} \end{pmatrix}
+
\begin{pmatrix}
r & & & & \\
 & r & & & \\
 & & r & & \\
 & & & \ddots & \\
 & & & & r
\end{pmatrix}
\begin{pmatrix} v_{i12} \\ v_{i22} \\ v_{i32} \\ \vdots \\ v_{iN2} \end{pmatrix}
=
\begin{pmatrix} b_{i11} \\ b_{i21} \\ b_{i31} \\ \vdots \\ b_{iN1} \end{pmatrix}
$$

$$
i = 1, 2, \ldots, M \tag{2.16}
$$

Let $\mathcal{F}_i = \begin{pmatrix}
\eta_i & 1 & & & & \\
1 & \eta_i & 1 & & & \\
 & 1 & \eta_i & 1 & & \\
 & & & \ddots & & \\
 & & & 1 & \eta_i & 1 \\
 & & & & 1 & \eta_i
\end{pmatrix}$, $\mathbf{W}_{ik} = \begin{pmatrix} v_{i1k} \\ v_{i2k} \\ v_{i3k} \\ \vdots \\ v_{iNk} \end{pmatrix}$, $\bar{\mathbf{B}}_{ik} = \begin{pmatrix} b_{i1k} \\ b_{i2k} \\ b_{i3k} \\ \vdots \\ b_{iNk} \end{pmatrix}$ and

$\mathbf{r} = diag(r, r, r, \ldots, r)$

We can write equation (2.16) as

$$\mathcal{F}_i \mathbf{W}_{i1} + \mathbf{r}\mathbf{W}_{i2} = \bar{\mathbf{B}}_{i1} \tag{2.17}$$

Let $\mathcal{F} = \begin{pmatrix} \mathcal{F}_i & & & & \\ & \mathcal{F}_i & & & \\ & & \mathcal{F}_i & & \\ & & & \ddots & \\ & & & & \mathcal{F}_i \end{pmatrix}$ is of order MP

$\mathbf{w}_k = [\mathbf{W}_{i1}\ \mathbf{W}_{i2}\ \mathbf{W}_{i3}\ ...\ \mathbf{W}_{iP}]^T$ and $\bar{B}_k = [\bar{\mathbf{B}}_{i1}, \bar{\mathbf{B}}_{i2}, \cdots, \bar{\mathbf{B}}_{iP}]^T$

Thus the first equation of (2.12) can be written as

$$\mathcal{F}\mathbf{w}_1 + S\mathbf{w}_2 = \bar{\mathrm{B}}_1$$

Similarly we write the other equations in (2.12) using the matrices $\mathcal{F}$, $\mathbf{w}_k$ and $\bar{B}_k$.

Thus, equation (2.12) can be written, equivalently, as

$$\begin{aligned} \mathcal{F}\mathbf{w}_1 + S\mathbf{w}_2 \qquad &= \bar{\mathrm{B}}_1 \\ S\mathbf{w}_1 + \mathcal{F}\mathbf{w}_2 + S\mathbf{w}_3 &= \bar{\mathrm{B}}_2 \\ S\mathbf{w}_2 + \mathcal{F}\mathbf{w}_3 + S\mathbf{w}_4 &= \bar{\mathrm{B}}_3 \\ \cdots \\ S\mathbf{w}_{(P-1)} + \mathcal{F}\mathbf{w}_P &= \bar{\mathrm{B}}_P \end{aligned} \tag{2.18}$$

Observe that

$\mathbb{Q}^T \mathcal{F} \mathbb{Q} = diag(\Phi_1, \Phi_2, \Phi_3, ..., \Phi_N) = \Lambda$ (say) where $\Phi_j = diag(\lambda_1, \lambda_2, \lambda_3, ..., \lambda_M)$

Here $\lambda_i = \eta_i + 2\cos\left(\dfrac{i\pi}{M+1}\right)$ $\quad i = 1(1)M$

$$\text{Let}\quad \mathbb{Q}^T \mathbf{w}_k = \Psi_k \Rightarrow \mathbf{w}_k = \mathbb{Q}\,\Psi_k$$

$$\mathbb{Q}^T \bar{B}_k = \Gamma_k \Rightarrow \bar{B}_k = \mathbb{Q}\Gamma_k \tag{2.19}$$

where $\Psi_k = [\mathbf{\Psi}_{1k}\ \mathbf{\Psi}_{2k}\ \mathbf{\Psi}_{3k}\ ...\ \mathbf{\Psi}_{Nk}]^T$ and $\mathbf{\Psi}_{jk} = [\psi_{1jk}\ \psi_{2jk}\ \psi_{3jk}\ ...\ \psi_{Mjk}]^T$

$\Gamma_k = [\boldsymbol{\beta}_{1k}, \boldsymbol{\beta}_{2k}, \boldsymbol{\beta}_{3k}, ..., \boldsymbol{\beta}_{Nk}]^T$ and $\boldsymbol{\beta}_{jk} = [\beta_{1jk}, \beta_{2jk}, ..., \beta_{Mjk}]^T$

Now pre-multiplying (2.18) by $\mathbb{Q}^T$ and make use of (2.19), we get

$$\Lambda\Psi_1 + S\Psi_2 = \Gamma_1$$
$$S\Psi_1 + \Lambda\Psi_2 + S\Psi_3 = \Gamma_2$$
$$S\Psi_2 + \Lambda\Psi_3 + S\Psi_4 = \Gamma_3 \qquad (2.20)$$
$$\dots$$
$$S\Psi_{P-1} + \Lambda\Psi_P = \Gamma_P$$

Now we write these sets of equations (2.20) turn by turn starting from the first row i.e. $\Lambda\Psi_1 + S\Psi_2 = \Gamma_1$, as

$$\lambda_1\psi_{111} + r\psi_{112} = \beta_{111}$$
$$\lambda_2\psi_{211} + r\psi_{212} = \beta_{211}$$
$$\lambda_3\psi_{311} + r\psi_{312} = \beta_{311}$$
$$\dots$$
$$\lambda_M\psi_{M11} + r\psi_{M12} = \beta_{M11} \qquad (2.21a)$$
$$\lambda_1\psi_{121} + r\psi_{122} = \beta_{121}$$
$$\lambda_2\psi_{221} + r\psi_{222} = \beta_{221}$$
$$\lambda_3\psi_{321} + r\psi_{322} = \beta_{321}$$
$$\dots$$
$$\lambda_M\psi_{M21} + r\psi_{M22} = \beta_{M21}$$
$$\dots \quad \dots \quad \dots$$
$$\lambda_1\psi_{1N1} + r\psi_{1N2} = \beta_{1N1}$$
$$\lambda_2\psi_{2N1} + r\psi_{2N2} = \beta_{2N1}$$
$$\lambda_3\psi_{3N1} + r\psi_{3N2} = \beta_{3N1}$$
$$\dots$$
$$\lambda_M\psi_{MN1} + r\psi_{MN2} = \beta_{MN1}$$

For the second equation of (2.20) i.e. $S\Psi_1 + \Lambda\Psi_2 + S\Psi_3 = \Gamma_2$, we get the second group of system of equations

$$r\psi_{111} + \lambda_1\psi_{112} + r\psi_{113} = \beta_{112}$$
$$r\psi_{211} + \lambda_2\psi_{212} + r\psi_{213} = \beta_{212}$$
$$r\psi_{311} + \lambda_3\psi_{312} + r\psi_{313} = \beta_{312}$$
$$\dots$$
$$r\psi_{M11} + \lambda_M\psi_{M12} + r\psi_{M13} = \beta_{M12} \qquad (2.21b)$$

$$r\psi_{121} + \lambda_1\psi_{122} + r\psi_{123} = \beta_{122}$$
$$r\psi_{221} + \lambda_2\psi_{222} + r\psi_{223} = \beta_{222}$$
$$r\psi_{321} + \lambda_3\psi_{322} + r\psi_{323} = \beta_{322}$$
$$\dots$$
$$r\psi_{M21} + \lambda_M\psi_{M22} + r\psi_{M23} = \beta_{M22}$$
$$\dots \quad \dots \quad \dots$$
$$r\psi_{1N1} + \lambda_1\psi_{1N2} + r\psi_{1N3} = \beta_{1N2}$$
$$r\psi_{2N1} + \lambda_2\psi_{2N2} + r\psi_{2N3} = \beta_{2N2}$$
$$r\psi_{3N1} + \lambda_3\psi_{3N2} + r\psi_{3N3} = \beta_{3N2}$$
$$\dots$$
$$r\psi_{MN1} + \lambda_M\psi_{MN2} + r\psi_{MN3} = \beta_{MN2}$$

For the last equation of (2.20), i.e. $S\Psi_{P-1} + \Lambda\Psi_P = \Gamma_P$, we obtain

$$r\psi_{11(P-1)} + \lambda_1\psi_{11P} = \beta_{11P}$$
$$r\psi_{21(P-1)} + \lambda_2\psi_{21P} = \beta_{21P}$$
$$r\psi_{31(P-1)} + \lambda_3\psi_{31P} = \beta_{31P}$$
$$\dots$$
$$r\psi_{M1(P-1)} + \lambda_M\psi_{M1P} = \beta_{M1P} \tag{2.21c}$$
$$r\psi_{12(P-1)} + \lambda_1\psi_{12P} = \beta_{12P}$$
$$r\psi_{22(P-1)} + \lambda_2\psi_{22P} = \beta_{22P}$$
$$r\psi_{32(P-1)} + \lambda_3\psi_{32P} = \beta_{32P}$$
$$\dots$$
$$r\psi_{M2(P-1)} + \lambda_M\psi_{M2P} = \beta_{M2P}$$
$$\dots \quad \dots \quad \dots$$
$$r\psi_{1N(P-1)} + \lambda_1\psi_{1NP} = \beta_{1NP}$$
$$r\psi_{2N(P-1)} + \lambda_2\psi_{2NP} = \beta_{2NP}$$
$$r\psi_{3N(P-1)} + \lambda_3\psi_{3NP} = \beta_{3NP}$$
$$\dots$$
$$r\psi_{MN(P-1)} + \lambda_M\psi_{MNP} = \beta_{MNP}$$

Now from each set of equations of (2.21a) to (2.21c), we select the first equations from (2.21a), (2.21b), …, (2.21c) and put together as one group of equations; again we take the second equations from each of (2.21a), (2.21b), …, (2.21c) and put together as a second group of equations; consider the third equations and put together as a third group of equations and so on and finally we consider the last equations and put together. In doing these we obtain the following sets of equations, each set being of order P and has tri-diagonal form

$$\lambda_1\psi_{111} + r\psi_{112} = \beta_{111}$$

$$r\psi_{111} + \lambda_1\psi_{112} + r\psi_{113} = \beta_{112}$$

$$r\psi_{112} + \lambda_1\psi_{113} + r\psi_{114} = \beta_{113}$$

$$\cdots$$

$$r\psi_{11(P-1)} + \lambda_1\psi_{11P} = \beta_{11P}$$

$$\lambda_2\psi_{211} + r\psi_{212} = \beta_{211}$$

$$r\psi_{211} + \lambda_2\psi_{212} + r\psi_{213} = \beta_{212}$$

$$r\psi_{212} + \lambda_2\psi_{213} + r\psi_{214} = \beta_{213}$$

$$\cdots$$

$$r\psi_{21(P-1)} + \lambda_2\psi_{21P} = \beta_{21P} \qquad (2.22)$$

$$\cdots \quad \cdots \quad \cdots$$

$$\lambda_M\psi_{M11} + r\psi_{M12} = \beta_{M11}$$

$$r\psi_{M11} + \lambda_M\psi_{M12} + r\psi_{M13} = \beta_{M12}$$

$$r\psi_{M12} + \lambda_M\psi_{M13} + r\psi_{M14} = \beta_{M13}$$

$$\cdots$$

$$r\psi_{M1(P-1)} + \lambda_M\psi_{M1P} = \beta_{M1P}$$

Observe that the above set of equations (2.22), for $j = 1$, and for each $i = 1(1)M$ the coefficient matrix of the left hand side is a tridiagonal matrix of order P and has the form

$$\mathcal{M}_i = \begin{pmatrix} \lambda_i & r & & & & & \\ r & \lambda_i & r & & & & \\ & r & \lambda_i & r & & & \\ & & & \ddots & & & \\ & & & & r & \lambda_i & r \\ & & & & & r & \lambda_i \end{pmatrix}, \quad i = 1(1)M$$

Continuing for the other groups of equations as above for $j = 2, \dots, N$, we get

$$\begin{aligned}
\lambda_1\psi_{1j1} + r\psi_{1j2} &= \beta_{1j1} \\
r\psi_{1j1} + \lambda_1\psi_{1j2} + r\psi_{1j3} &= \beta_{1j2} \\
r\psi_{1j2} + \lambda_1\psi_{1j3} + r\psi_{1j4} &= \beta_{1j3} \\
&\dots \\
r\psi_{1j(P-1)} + \lambda_1\psi_{1jP} &= \beta_{1jP} \\
\lambda_2\psi_{2j1} + r\psi_{2j2} &= \beta_{2j1} \\
r\psi_{2j1} + \lambda_2\psi_{2j2} + r\psi_{2j3} &= \beta_{2j2} \\
r\psi_{2j2} + \lambda_2\psi_{2j3} + r\psi_{2j4} &= \beta_{2j3} \\
&\dots \\
r\psi_{2j(P-1)} + \lambda_2\psi_{2jP} &= \beta_{2jP} \qquad (2.23) \\
\dots \quad \dots \quad &\dots \\
\lambda_M\psi_{Mj1} + r\psi_{Mj2} &= \beta_{Mj1} \\
r\psi_{Mj1} + \lambda_M\psi_{Mj2} + r\psi_{Mj3} &= \beta_{Mj2} \\
r\psi_{Mj2} + \lambda_M\psi_{Mj3} + r\psi_{Mj4} &= \beta_{Mj3} \\
&\dots \\
r\psi_{Mj(P-1)} + \lambda_M\psi_{MjP} &= \beta_{MjP}
\end{aligned}$$

In this case also for each $j = 2, \dots, N$ the coefficients matrix of the left hand side of (2.23) is a tridiagonal matrix similar to that of (2.22) for $j = 1$

$$\mathcal{M}_i = \begin{pmatrix} \lambda_i & r & & & & \\ r & \lambda_i & r & & & \\ & r & \lambda_i & r & & \\ & & & \ddots & & \\ & & & r & \lambda_i & r \\ & & & & r & \lambda_i \end{pmatrix}, \quad i = 1(1)M, \quad j = 2, \dots, N \tag{2.24}$$

We can easily observe that (2.8) reduces to the matrix (2.24) which is a tridiagonal matrix for $j = 1, 2, 3, \dots, N$ and hence we solve these sets of equations (2.24) for $\psi_{i,j,k}$ by the use of Thomas Algorithm [30]. Once after getting each $\psi_{i,j,k}$ (and hence Ψ_k) by the help of (2.19) we get $\mathbf{w}_k$ and again by the help of (2.11) we obtain $\mathbf{U}_k$ and this means that each $U_{i,j,k}$ are obtained. Thus, this solves our problem.

2.4 Numerical Results

In order to test the efficiency and adaptability of this proposed method, a computational experiment is done on six examples for which the analytical solutions of U are known to us. The computed solutions are displayed in terms of maximum absolute error (i.e. the error taken between the exact value and the computed value using this method) for some grid points but results are available for all grid points. The results for these test problems are reported in Tables 2.1 to 2.6.

Example 2.1 Suppose $\nabla^2 U = 0$, $\quad 0 < x < 1, 0 < y < 1, 0 < z < 1$

with the boundary conditions

$$U(0, y, z) = U(x, 0, z) = U(x, y, 0) = U(x, y, 1) = U(1, y, z) = U(x, 1, z) = 1$$

The analytical solution is $U(x, y, z) = 1$ and its results are shown in Table 2.1

Example 2.2 Consider $\nabla^2 U = 0$, $\quad 0 < x < 1, 0 < y < 1, 0 < z < 1$

with the boundary conditions

$$U(0, y, z) = U(x, 0, z) = U(x, y, 0) = 0,$$

$$U(1, y, z) = yz, \quad U(x, 1, z) = xz, \quad U(x, y, 1) = xy.$$

The analytical solution is $U(x, y, z) = xyz$ and its results are shown in Table 2.2

Example 2.3 Suppose $\nabla^2 U = 6$, $0 < x < 1, 0 < y < 1, 0 < z < 1$

with the boundary conditions

$$U(0,y,z) = y^2 + z^2, \qquad U(x,0,z) = x^2 + z^2,$$

$$U(x,y,0) = x^2 + y^2, \qquad U(1,y,z) = 1 + y^2 + z^2,$$

$$U(x,1,z) = 1 + x^2 + z^2, \qquad U(x,y,1) = 1 + x^2 + y^2$$

The analytical solution is $U(x,y,z) = x^2 + y^2 + z^2$ and its results are shown in Table 2.3

Example 2.4 Suppose $\nabla^2 U = 2(xy + xz + yz)$, $0 < x < 1, 0 < y < 1, 0 < z < 1$

with the boundary conditions

$$U(0,y,z) = U(x,0,z) = U(x,y,0) = 0,$$

$$U(1,y,z) = yz(1 + y + z),$$

$$U(x,1,z) = xz(1 + x + z), \qquad U(x,y,1) = xy(1 + x + y)$$

The analytical solution is $U(x,y,z) = xyz(x + y + z)$ and its results are shown in Table 2.4

Example 2.5 Suppose $\nabla^2 U = -\pi^2 xy \sin(\pi z)$ $0 < x < 1, 0 < y < 1, 0 < z < 1$

with the boundary conditions

$$U(0,y,z) = U(x,0,z) = U(x,y,0) = U(x,y,1) = 0$$

$$U(1,y,z) = y\sin(\pi z), \quad \text{and}$$

$$U(x,1,z) = x\sin(\pi z)$$

The analytical solution is $U = xysin(\pi z)$ and its results are shown in Table 2.5

Example 2.6 Suppose $\nabla^2 U = -\pi^2 \sin(\pi z)$ with the boundary conditions

$$U(0,y,z) = \sin(\pi z) = U(x,0,z), \qquad U(x,y,0) = xy = U(x,y,1)$$

$$U(1,y,z) = y + \sin(\pi z), \; U(x,1,z) = x + \sin(\pi z)$$

The analytical solution is $U(x,y,z) = xy + \sin(\pi z)$ and its results are shown in Table 2.6

Table 2.1

The maximum absolute error of example 2.1

M P	Max. abs. error	M P	Max. abs. error
9 9	2.22945e-015	29 9	1.69864e-014
9 19	1.77636e-015	29 19	1.47660e-014
9 29	1.33227e-015	29 29	1.50990e-014
9 39	2.44249e-015	29 39	1.04361e-014
19 9	6.43929e-015	39 9	2.08722e-014
19 19	7.10543e-015	39 19	2.10942e-014
19 29	7.10543e-015	39 29	7.43849e-015
19 39	4.95159e-014	39 39	4.17444e-014

Table 2.2

The maximum absolute error of example 2.2

M P	Max. abs. error	M P	Max. abs. error
9 9	6.66134e-016	29 9	2.52576e-015
9 19	7.77156e-016	29 19	2.22045e-015
9 29	1.11022e-015	29 29	3.71925e-015
9 39	7.21645e-016	29 39	2.2482e-015
19 9	1.16573e-015	39 9	3.94129e-015
19 19	1.27676e-015	39 19	3.9968e-015
19 29	1.38778e-015	39 29	2.19269e-015
19 39	7.49401e-015	39 39	6.57807e-015

Table 2.3

The maximum absolute error of example 2.3

M *P*	Max. abs. error	*M* *P*	Max. abs. error
9 9	2.22045e-015	29 9	1.4877e-014
9 19	2.22045e-015	29 19	1.39888e-014
9 29	4.44089e-015	29 29	1.75415e-014
9 39	3.88578e-015	29 39	1.17684e-014
19 9	6.66134e-015	39 9	2.15383e-014
19 19	7.32747e-015	39 19	2.24265e-014
19 29	7.54952e-015	39 29	9.65894e-015
19 39	4.65183e-014	39 39	4.05231e-014

Table 2.4

The maximum absolute error of example 2.4

M *P*	Max. abs. error	*M* *P*	Max. abs. error
9 9	1.11022e-015	29 9	4.38538e-015
9 19	1.33227e-015	29 19	4.27436e-015
9 29	2.88658e-015	29 29	7.77156e-015
9 39	1.77636e-015	29 39	4.71845e-015
19 9	2.44249e-015	39 9	7.77156e-015
19 19	2.83107e-015	39 19	8.21565e-015
19 29	3.10862e-015	39 29	5.71765e-015
19 39	1.43219e-014	39 39	1.31006e-014

Table 2.5

The maximum absolute error of example 2.5

M *P*	Max. abs. error	*M* *P*	Max. abs. error
9 9	1.17276e-003	29 9	1.1972e-003
9 19	2.93418e-004	29 19	2.99503e-004
9 29	1.30427e-004	29 29	1.33129e-004
9 39	7.33687e-005	29 39	7.48883e-005
19 9	1.19084e-003	39 9	1.19791e-003
19 19	2.97895e-004	39 19	2.9969e-004
19 29	1.32413e-004	39 29	1.33213e-004
19 39	7.444852e-005	39 39	7.49356e-005

Table 2.6

The maximum absolute error of example 2.6

M *P*	Max. abs. error	*M* *P*	Max. abs. error
9 9	3.7907e-003	29 9	3.82394e-003
9 19	9.4785e-004	29 19	9.56166e-004
9 29	4.21281e-004	29 29	4.24977e-004
9 39	2.36973e-004	29 39	2.39053e-004
19 9	3.81868e-003	39 9	3.82579e-003
19 19	9.5485e-004	39 19	9.56629e-004
19 29	4.24392e-004	39 29	4.25183e-004
19 39	2.38723e-004	39 39	2.39168e-004

2.5 Conclusion

In this work, we have transformed the three dimensional Poisson’s equation in Cartesian coordinates system in to a system of algebraic linear equations using its equivalent finite difference approximation scheme. The resulting large number of algebraic equation is, then, systematically arranged in order to get a block matrix. Based on the extension of Hockney’s method we reduced the obtained matrix in to a block tridiagonal matrix, and each block is solved by the help of Thomas algorithm. We have successfully implemented this method to find the solution of the three dimensional Poisson’s equation in Cartesian coordinates system. It is found that the method can easily be applied and adapted to find a solution for large set of equations and produce accurate results considering double precision. This method is direct and allows considerable savings in computer storage as well as execution speed.

Therefore, the method is suitable to apply on any three dimensional Poisson’s equations.

CHAPTER III

Fourth Order Numerical Solution of the Three Dimensional Poisson's Equation in Cartesian Coordinate Systems

3.1. Introduction

Poisson's equation in three dimensional Cartesian coordinates system plays an important role due to its wide range of application in areas like ideal fluid flow, heat conduction, elasticity, electrostatics, gravitation etc, in physics, engineering fields and other sciences. Attempts have been made to solve Poisson's equation numerically by using higher order finite difference approximation.

For two dimensional Poisson's equation, for instance, *Averbuch et al* [2] developed a high order numerical algorithm based on the Fourier method in combination with a subtraction procedure; *Houstis and Papatheodorou* [22] developed an algorithm that uses high-order 9-point difference approximations to the Helmholtz-type (fourth-order) or Poisson (sixth-order) equations and the fast Fourier transform; *Jun Zhang* [48] developed a multigrid method and fourth order compact difference scheme; *Barad and Colella* [74] developed a fourth order accurate local refinement method for either Dirichlet, Neumann, or periodic boundary conditions, and their approach uses a conservative, finite-volume, block-structured local refinement discretization that generalizes the classical Mehrstellen methods; *Gupta* [76] developed a fourth order Poisson solver; *Gupta et al [80]* developed second and fourth order discretization for Multigrid Poisson Solvers that combine a compact high-order difference approximation with multigrid V-cycle algorithm; *Wang et al* [107] have developed a high order compact difference scheme in non uniform grid systems.

To solve the three dimensional Poisson's equations in Cartesian coordinate systems using finite difference approximations; for instance, *Braverman et al* [20] have developed an arbitrary order accuracy fast 3D Poisson Solver on a rectangular box

and their method is based on the application of the discrete Fourier transform accompanied by a subtraction technique which allows reducing the errors associated with the Gibbs phenomenon; *Sutmann and Steffen* [29] a compact approximation schemes for the Laplace operator of fourth and sixth order based on Padé approximation of the Taylor expansion for the discretized Laplace operator; *Jie Wang et al* [46] a fourth-order compact difference scheme with unrestricted general mesh sizes in different coordinates direction and used a preconditioned conjugate gradient method to solve the sparse linear systems; *Jun Zhang* [47] developed a multigrid solution for the Poisson's equation based on uniform mesh size finite difference approximation and has solved the resulting system of linear equations by a residual or multigrid method; *Israeli et al* [66] developed a hierarchical 3D Poisson modified Fourier solver by domain decomposition; *Spotz and Carey* [104] have constructed a higher order compact formulation using central difference scheme to obtain a 19-point stencil and a 27-point stencil with some modification on the right hand side terms; *Yongbin* [109] developed a multigrid and fourth-order compact difference discretization scheme with unequal mesh sizes, and other contributions have been made.
Interesting developments have been observed in recent years to solve the three dimensional Poisson's equation in Cartesian coordinates system using modern computers and different application packages.

The aim of this chapter is to develop a fourth order finite difference approximation schemes and solve the resulting large algebraic system of linear equations systematically using block tridiagonal system [60] and extend the Hockney's method [96] to solve the three dimensional Poisson's equation on Cartesian coordinates system.

3.2 Finite Difference Approximation

Consider the three dimensional Poisson equation in Cartesian coordinate system

$$\nabla^2 U = \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} = f(x,y,z) \qquad \text{on } D \tag{3.1}$$

$$U(x,y,z) = g(x,y,z) \qquad \text{on } \partial D \tag{3.2}$$

where $D = \{(x,y,z) : 0 < x < a,\ 0 < y < b,\ 0 < z < c\}$ and ∂D is the boundary of D.

Assume that there are M, N and P mesh points along the X, Y and Z directions respectively, and let $U(x,y,z)$ be discretized at the mesh point (i, j, k) and we adopt writing $U_{i,j,k}$ for $U(x_i, y_j, z_k)$, where $i = 1(1)M$, $j = 1(1)N$ and $k = 1(1)P$.

Let the mesh step size along the X -direction and Y -direction be h_1, and along the Z -direction be h_2 (h_1 and h_2 need not be equal)

We transform (3.1) in to its equivalent finite difference equation using the fourth order finite difference approximation (1.12) and get

$$\left(\frac{\delta_x^2}{h_1^2\left(1+\frac{1}{12}\delta_x^2\right)} + \frac{\delta_y^2}{h_1^2\left(1+\frac{1}{12}\delta_y^2\right)} + \frac{\delta_z^2}{h_2^2\left(1+\frac{1}{12}\delta_z^2\right)} + O\left(h_1^4\right) + O\left(h_2^4\right) \right) U_{i,j,k} = f_{i,j,k} \tag{3.3}$$

where $i = 1, 2, 3, \ldots, M$, $j = 1, 2, 3, \ldots, N$, and $k = 1, 2, 3, \ldots, P$

Letting $r = \frac{h_1^2}{h_2^2}$, neglecting the truncation error and simplifying (3.3), we get

$$\left(\frac{\delta_x^2\left(1+\frac{1}{12}\delta_y^2\right)\left(1+\frac{1}{12}\delta_z^2\right) + \delta_y^2\left(1+\frac{1}{12}\delta_x^2\right)\left(1+\frac{1}{12}\delta_z^2\right) + r\delta_z^2\left(1+\frac{1}{12}\delta_x^2\right)\left(1+\frac{1}{12}\delta_y^2\right)}{\left(1+\frac{1}{12}\delta_x^2\right)\left(1+\frac{1}{12}\delta_y^2\right)\left(1+\frac{1}{12}\delta_z^2\right)} \right) U_{i,j,k} = h_1^2 f_{i,j,k} \tag{3.4}$$

$$\left(\delta_x^2\left(1+\frac{1}{12}\delta_y^2\right)\left(1+\frac{1}{12}\delta_z^2\right)+\delta_y^2\left(1+\frac{1}{12}\delta_x^2\right)\left(1+\frac{1}{12}\delta_z^2\right)+r\delta_z^2\left(1+\frac{1}{12}\delta_x^2\right)\left(1+\frac{1}{12}\delta_y^2\right)\right)U_{i,j,k}$$
$$=h_1^2\left(1+\frac{1}{12}\delta_x^2\right)\left(1+\frac{1}{12}\delta_y^2\right)\left(1+\frac{1}{12}\delta_z^2\right)f_{i,j,k} \qquad (3.5)$$

This implies

$$\left(\left(\delta_x^2+\delta_y^2+r\delta_z^2\right)+\frac{1}{6}\delta_x^2\delta_y^2+\frac{1+r}{12}\left(\delta_x^2\delta_z^2+\delta_y^2\delta_z^2\right)+\frac{2+r}{144}\delta_x^2\delta_y^2\delta_z^2\right)U_{i,j,k}$$
$$=h_1^2\left(1+\frac{1}{12}\left(\delta_x^2+\delta_y^2+\delta_z^2\right)+\frac{1}{144}\left(\delta_x^2\delta_y^2+\delta_x^2\delta_z^2+\delta_y^2\delta_z^2\right)+\frac{1}{1728}\delta_x^2\delta_y^2\delta_z^2\right)f_{i,j,k} \qquad (3.6)$$

Now we will consider two different schemes:

I. 19 Points Stencil Scheme

Omitting the term $\delta_x^2\delta_y^2\delta_z^2$ in both sides of equation (3.6) and simplifying it further, we get

$$12h_1^2\left(1+\frac{1}{12}\left(\delta_x^2+\delta_y^2+\delta_z^2\right)\right)f_{i,j,k}=-(32+16r)U_{i,j,k}+(8r-4)\left(U_{i,j,k+1}+U_{i,j,k-1}\right)$$
$$+(6-2r)\left(U_{i+1,j,k}+U_{i-1,j,k}+U_{i,j+1,k}+U_{i,j-1,k}\right)+2\left(U_{i+1,j+1,k}+U_{i+1,j-1,k}+U_{i-1,j+1,k}+U_{i-1,j-1,k}\right)$$
$$+(1+r)\left(U_{i+1,j,k+1}+U_{i+1,j,k-1}+U_{i-1,j,k+1}+U_{i-1,j,k-1}+U_{i,j+1,k+1}+U_{i,j+1,k-1}+U_{i,j-1,k+1}+U_{i,j-1,k-1}\right)$$
$$(3.7)$$

Taking first in the X -direction, next Y -direction and lastly Z -direction in (3.7) we get a large system of equations (the number of equations actually depends on the values of M, N and P), and these systems of equations can be written in matrix form as

$$AU=\mathcal{B} \qquad (3.8)$$

where $$A=\begin{pmatrix} R & S & & & & \\ S & R & S & & & \\ & S & R & S & & \\ & & & \ddots & & \\ & & & S & R & S \\ & & & & S & R \end{pmatrix} \tag{3.9}$$

it has P blocks and each block is of order $MN \times MN$,

$$R=\begin{pmatrix} R_1 & R_2 & & & & \\ R_2 & R_1 & R_2 & & & \\ & R_2 & R_1 & R_2 & & \\ & & & \ddots & & \\ & & & R_2 & R_1 & R_2 \\ & & & & R_2 & R_1 \end{pmatrix}, \quad S=\begin{pmatrix} S_1 & S_2 & & & & \\ S_2 & S_1 & S_2 & & & \\ & S_2 & S_1 & S_2 & & \\ & & & \ddots & & \\ & & & S_2 & S_1 & S_2 \\ & & & & S_2 & S_1 \end{pmatrix}$$

R and S have N blocks and each block is of order $M \times M$.

$$R_1=\begin{pmatrix} -32-16r & 6-2r & & & & \\ 6-2r & -32-16r & 6-2r & & & \\ & 6-2r & -32-16r & 6-2r & & \\ & & & \ddots & & \\ & & & 6-2r & -32-16r & 6-2r \\ & & & & 6-2r & -32-16r \end{pmatrix}$$

$$R_2=\begin{pmatrix} 6-2r & 2 & & & & \\ 2 & 6-2r & 2 & & & \\ & 2 & 6-2r & 2 & & \\ & & & \ddots & & \\ & & & 2 & 6-2r & 2 \\ & & & & 2 & 6-2r \end{pmatrix}$$

$$S_1 = \begin{pmatrix} 8r-4 & 1+r & & & & \\ 1+r & 8r-4 & 1+r & & & \\ & 1+r & 8r-4 & 1+r & & \\ & & & \ddots & & \\ & & & 1+r & 8r-4 & 1+r \\ & & & & 1+r & 8r-4 \end{pmatrix}$$

$$S_2 = \begin{pmatrix} 1+r & & & & \\ & 1+r & & & \\ & & 1+r & & \\ & & & \ddots & \\ & & & & 1+r \end{pmatrix}$$

$$U = \begin{pmatrix} \mathbf{U}_1 & \mathbf{U}_2 & \mathbf{U}_3 & \cdots & \mathbf{U}_{P-1} & \mathbf{U}_P \end{pmatrix}^T, \text{ and}$$

$$\mathcal{B} = \begin{pmatrix} \mathbf{B}_1 & \mathbf{B}_2 & \mathbf{B}_3 & \dots & \mathbf{B}_{P-1} & \mathbf{B}_P \end{pmatrix}^T \qquad (3.10)$$

where

$$\mathbf{U}_k = [\mathbf{u}_{1k} \ \mathbf{u}_{2k} \dots \mathbf{u}_{Nk}]^T \quad \text{and } \mathbf{u}_{jk} = [\ U_{1jk} \ U_{2jk} \ \dots U_{Mjk}]^T$$

and $\mathbf{B}_k = [\mathbf{d}_{1k} \ \mathbf{d}_{2k} \dots \mathbf{d}_{Nk}]^T$ and $\mathbf{d}_{jk} = [\ d_{1jk} \ d_{2jk} \dots d_{Mjk}]^T \ k = 1, 2, 3, \dots, P$

is the known column vectors such that each d_{ijk} represents known boundary values of U and values of f.

Using (3.9) and (3.10), we write (3.8) as

$$\begin{pmatrix} R & S & & & & \\ S & R & S & & & \\ & S & R & S & & \\ & & & \ddots & & \\ & & & S & R & S \\ & & & & S & R \end{pmatrix} \begin{pmatrix} \mathbf{U}_1 \\ \mathbf{U}_2 \\ \mathbf{U}_3 \\ \vdots \\ \mathbf{U}_{P-1} \\ \mathbf{U}_P \end{pmatrix} = \begin{pmatrix} \mathbf{B}_1 \\ \mathbf{B}_2 \\ \mathbf{B}_3 \\ \vdots \\ \mathbf{B}_{P-1} \\ \mathbf{B}_P \end{pmatrix} \qquad (3.11)$$

Equation (3.11) again can be written as

$$
\begin{aligned}
R\mathbf{U}_1 + S\mathbf{U}_2 \quad &= \mathbf{B}_1 \\
S\mathbf{U}_1 + R\mathbf{U}_2 + S\mathbf{U}_3 &= \mathbf{B}_2 \\
S\mathbf{U}_2 + R\mathbf{U}_3 + S\mathbf{U}_4 &= \mathbf{B}_3 \\
&\cdots \\
S\mathbf{U}_{P-1} + R\mathbf{U}_P &= \mathbf{B}_P
\end{aligned}
\tag{3.12}
$$

Now by applying extended Hockney's method to three dimensions we obtain the solution of the system of linear equations (3.12)

3.3.1 Extended Hockney's Method for 19-Points Scheme

As we can see all the matrices R_1, R_2, S_1 and S_2 are real tridiagonal symmetric matrices and hence their eigenvalues and eigenvectors can easily be obtained. [30]
Note that the eigenvalues of R_1, R_2, and S_1 are given by

$$\eta_i = -32 - 16r + 2(6-2r)\cos\left(\frac{i\pi}{M+1}\right)$$

$$\tau_i = 6 - 2r + 4\cos\left(\frac{i\pi}{M+1}\right) \quad \text{and}$$

$$\alpha_i = 8r - 4 + 2(1+r)\cos\left(\frac{i\pi}{M+1}\right) \qquad i = 1, 2, \cdots, M$$

Let $\mathbf{q}_i$ be an eigenvector of R_1, R_2, S_1 and S_2 corresponding to the eigenvalues η_i, τ_i, α_i, and $1+r$ respectively, and Q be the modal matrix $[\mathbf{q}_1\ \mathbf{q}_2\ \mathbf{q}_3 \ldots \mathbf{q}_M]$ of the matrix R_1, R_2, S_1 and S_2 of order M such that $Q^T Q = I$,

$$Q^T R_1 Q = diag(\eta_1, \eta_2, \eta_3, \ldots, \eta_M) = \mathrm{H} \text{ (say)},$$

$$Q^T R_2 Q = diag(\tau_1, \tau_2, \tau_3, \ldots, \tau_M) = \mathrm{T} \text{ (say)}$$

$$Q^T S_1 Q = diag(\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_M) = \Phi \text{ (say)} \quad \text{and}$$

$$Q^T S_2 Q = S_2 \text{ (since } S_2 \text{ is a diagonal matrix)}$$

Note that the $M \times M$ modal matrix Q is defined by

$$q_{ij} = \sqrt{\frac{2}{M+1}}\sin\left(\frac{ij\pi}{M+1}\right) \quad i, j = 1, 2, \cdots, M$$

Let $\mathbb{Q} = diag\left(Q, Q, Q, \ldots, Q\right)$ is a matrix of order $MN \times MN$

Thus $\mathbb{Q}$ satisfy $\mathbb{Q}^T\mathbb{Q} = I$,

$$\mathbb{Q}^T R\mathbb{Q} = \begin{pmatrix} \mathrm{H} & \mathrm{T} & & & & \\ \mathrm{T} & \mathrm{H} & \mathrm{T} & & & \\ & \mathrm{T} & \mathrm{H} & \mathrm{T} & & \\ & & & \ddots & & \\ & & & \mathrm{T} & \mathrm{H} & \mathrm{T} \\ & & & & \mathrm{T} & \mathrm{H} \end{pmatrix} \text{ and}$$

$$\mathbb{Q}^T S\mathbb{Q} = \begin{pmatrix} \Phi & S_2 & & & & \\ S_2 & \Phi & S_2 & & & \\ & S_2 & \Phi & S_2 & & \\ & & & \ddots & & \\ & & & S_2 & \Phi & S_2 \\ & & & & S_2 & \Phi \end{pmatrix}$$

Let $\qquad \mathbb{Q}^T\mathbf{U}_k = \mathbf{V}_k \Rightarrow \mathbf{U}_k = \mathbb{Q}\,\mathbf{V}_k$

$$\mathbb{Q}^T\mathbf{B}_k = B_k \Rightarrow \mathbf{B}_k = \mathbb{Q}B_k \qquad (3.13)$$

where $\mathbf{V}_k = [\mathbf{v}_{1k}\ \ \mathbf{v}_{2k} \ldots \mathbf{v}_{Nk}]^T$ and $\mathbf{v}_{jk} = [v_{1jk}\ \ v_{2jk} \ldots\ v_{Mjk}]^T$

$B_k = [\mathbf{b}_{1k}\ \ \mathbf{b}_{2k} \ldots \mathbf{b}_{Nk}]^T$ and $\mathbf{b}_{jk} = [\ b_{1jk}\ \ b_{2jk} \ldots b_{Mjk}]^T$

Consider the first equation of (3.12) i.e. $R\mathbf{U}_1 + S\mathbf{U}_2 = \mathbf{B}_1$, and pre multiplying it by $\mathbb{Q}^T$ and using (3.13), we get

$$\begin{pmatrix} \mathrm{H} & \mathrm{T} & & & & \\ \mathrm{T} & \mathrm{H} & \mathrm{T} & & & \\ & \mathrm{T} & \mathrm{H} & \mathrm{T} & & \\ & & & \ddots & & \\ & & & \mathrm{T} & \mathrm{H} & \mathrm{T} \\ & & & & \mathrm{T} & \mathrm{H} \end{pmatrix} \begin{pmatrix} \mathbf{v}_{11} \\ \mathbf{v}_{21} \\ \mathbf{v}_{31} \\ \vdots \\ \mathbf{v}_{(N-1)1} \\ \mathbf{v}_{N1} \end{pmatrix} + \begin{pmatrix} \Phi & S_2 & & & & \\ S_2 & \Phi & S_2 & & & \\ & S_2 & \Phi & S_2 & & \\ & & & \ddots & & \\ & & & S_2 & \Phi & S_2 \\ & & & & S_2 & \Phi \end{pmatrix} \begin{pmatrix} \mathbf{v}_{12} \\ \mathbf{v}_{22} \\ \mathbf{v}_{32} \\ \vdots \\ \mathbf{v}_{(N-1)2} \\ \mathbf{v}_{N2} \end{pmatrix} = \begin{pmatrix} \mathbf{b}_{11} \\ \mathbf{b}_{21} \\ \mathbf{b}_{31} \\ \vdots \\ \mathbf{b}_{(N-1)1} \\ \mathbf{b}_{N1} \end{pmatrix} \tag{3.14}$$

Again we write equation (3.14) as

$$\begin{aligned} &\mathrm{H}\mathbf{v}_{11} + \mathrm{T}\mathbf{v}_{21} + \Phi\mathbf{v}_{12} + S_2\mathbf{v}_{22} = \mathbf{b}_{11} \\ &\mathrm{T}\mathbf{v}_{11} + \mathrm{H}\mathbf{v}_{21} + \mathrm{T}\mathbf{v}_{31} + S_2\mathbf{v}_{12} + \Phi\mathbf{v}_{22} + S_2\mathbf{v}_{32} = \mathbf{b}_{21} \\ &\mathrm{T}\mathbf{v}_{21} + \mathrm{H}\mathbf{v}_{31} + \mathrm{T}\mathbf{v}_{41} + S_2\mathbf{v}_{22} + \Phi\mathbf{v}_{32} + S_2\mathbf{v}_{42} = \mathbf{b}_{31} \\ &\qquad \cdots \\ &\mathrm{T}\mathbf{v}_{(N-1)1} + \mathrm{H}\mathbf{v}_{N1} + S_2\mathbf{v}_{(N-1)2} + \Phi\mathbf{v}_{N2} = \mathbf{b}_{N1} \end{aligned} \tag{3.15}$$

Now collect the first equations from each of (3.15) and consider as one group of equations

$$\begin{aligned} &\eta_1 v_{111} + \tau_1 v_{121} + \alpha_1 v_{112} + (1+r)v_{122} = b_{111} \\ &\tau_1 v_{111} + \eta_1 v_{121} + \tau_1 v_{131} + (1+r)v_{112} + \alpha_1 v_{122} + (1+r)v_{132} = b_{121} \\ &\tau_1 v_{121} + \eta_1 v_{131} + \tau_1 v_{141} + (1+r)v_{122} + \alpha_1 v_{132} + (1+r)v_{142} = b_{131} \\ &\qquad \cdots \\ &\tau_1 v_{1(N-1)1} + \eta_1 v_{1N1} + (1+r)v_{1(N-1)12} + \alpha_1 v_{1N2} = b_{1N1} \end{aligned} \tag{3.16a}$$

Again we collect the second equations from each equation of (3.15) and consider as a second group of equations

$$\begin{aligned} &\eta_2 v_{211} + \tau_2 v_{221} + \alpha_1 v_{212} + (1+r)v_{222} = b_{211} \\ &\tau_2 v_{211} + \eta_2 v_{221} + \tau_2 v_{231} + (1+r)v_{212} + \alpha_1 v_{222} + (1+r)v_{232} = b_{221} \\ &\tau_2 v_{221} + \eta_2 v_{231} + \tau_2 v_{241} + (1+r)v_{222} + \alpha_2 v_{232} + (1+r)v_{242} = b_{231} \\ &\qquad \cdots \\ &\tau_2 v_{2(N-1)1} + \eta_2 v_{2N1} + (1+r)v_{2(N-1)12} + \alpha_2 v_{1N2} = b_{2N1} \end{aligned} \tag{3.16b}$$

Lastly we collect the last equations from each equation of (3.15) and consider as a last group of equations

$$\eta_M v_{M11} + \tau_M v_{M21} + \alpha_M v_{M12} + (1+r)v_{M22} = b_{M11}$$
$$\tau_M v_{M11} + \eta_M v_{M21} + \tau_M v_{M31} + (1+r)v_{M12} + \alpha_M v_{M22} + (1+r)v_{M32} = b_{M21}$$
$$\tau_M v_{M21} + \eta_M v_{M31} + \tau_M v_{M41} + (1+r)v_{M22} + \alpha_M v_{M32} + (1+r)v_{M42} = b_{M31} \qquad (3.16c)$$
$$\cdots$$
$$\tau_M v_{M(N-1)1} + \eta_M v_{MN1} + (1+r)v_{M(N-1)12} + \alpha_M v_{MN2} = b_{MN1}$$

Now we write equations (3.16a) to (3.16c) in matrix form as

$$\begin{pmatrix} \eta_i & \tau_i & & & \\ \tau_i & \eta_i & \tau_i & & \\ & \tau_i & \eta_i & \tau_i & \\ & & & \ddots & \\ & & & \tau_i & \eta_i \end{pmatrix} \begin{pmatrix} v_{i11} \\ v_{i21} \\ v_{i31} \\ \vdots \\ v_{iN1} \end{pmatrix} + \begin{pmatrix} \alpha_i & 1+r & & & \\ 1+r & \alpha_i & 1+r & & \\ & 1+r & \alpha_i & 1+r & \\ & & & \ddots & \\ & & & 1+r & \alpha_i \end{pmatrix} \begin{pmatrix} v_{i12} \\ v_{i22} \\ v_{i32} \\ \vdots \\ v_{iN2} \end{pmatrix} = \begin{pmatrix} b_{i11} \\ b_{i21} \\ b_{i31} \\ \vdots \\ b_{iN1} \end{pmatrix}$$

$$i = 1, 2, \ldots, M \qquad (3.17)$$

Let $\mathcal{F}_i = \begin{pmatrix} \eta_i & \tau_i & & & \\ \tau_i & \eta_i & \tau_i & & \\ & \tau_i & \eta_i & \tau_i & \\ & & & \ddots & \\ & & & \tau_i & \eta_i \end{pmatrix}$, $\mathcal{L}_i = \begin{pmatrix} \alpha_i & 1+r & & & \\ 1+r & \alpha_i & 1+r & & \\ & 1+r & \alpha_i & 1+r & \\ & & & \ddots & \\ & & & 1+r & \alpha_i \end{pmatrix}$

$$\mathbf{W}_{ik} = \begin{pmatrix} v_{i1k} \\ v_{i2k} \\ v_{i3k} \\ \vdots \\ v_{iNk} \end{pmatrix} \text{ and } \bar{\mathbf{B}}_{ik} = \begin{pmatrix} b_{i1k} \\ b_{i2k} \\ b_{i3k} \\ \vdots \\ b_{iNk} \end{pmatrix}$$

Equation (3.17) which is the same as the first equation of (3.12) once again can be written as $\mathcal{F}\mathbf{w}_1 + \mathcal{L}\mathbf{w}_2 = \bar{\mathrm{B}}_1$

where $\mathcal{F} = \begin{pmatrix} \mathcal{F}_i & & & & \\ & \mathcal{F}_i & & & \\ & & \mathcal{F}_i & & \\ & & & \ddots & \\ & & & & \mathcal{F}_i \end{pmatrix}$, $\mathcal{L} = \begin{pmatrix} \mathcal{L}_i & & & & \\ & \mathcal{L}_i & & & \\ & & \mathcal{L}_i & & \\ & & & \ddots & \\ & & & & \mathcal{L}_i \end{pmatrix}$ both are of order MP

$\mathbf{w}_1 = [\mathbf{W}_{i1}\ \mathbf{W}_{i2}\ \mathbf{W}_{i3}\ ...\ \mathbf{W}_{iP}]^T$ and $\bar{B}_k = [\mathbf{B}_{i1}, \mathbf{B}_{i2}, \cdots, \mathbf{B}_{iP}]^T$

Similarly, we can write the other equations in (3.12) using the matrices $\mathcal{F}$, $\mathcal{L}$, $\mathbf{W}_k$ and $\bar{B}_k$.

Therefore, (3.12) can, equivalently, be written as

$$\begin{aligned} \mathcal{F}\mathbf{w}_1 + \mathcal{L}\mathbf{w}_2 &= \bar{\mathrm{B}}_1 \\ \mathcal{L}\mathbf{w}_1 + \mathcal{F}\mathbf{w}_2 + \mathcal{L}\mathbf{w}_3 &= \bar{\mathrm{B}}_2 \\ \mathcal{L}\mathbf{w}_2 + \mathcal{F}\mathbf{w}_3 + \mathcal{L}\mathbf{w}_4 &= \bar{\mathrm{B}}_3 \\ &\cdots \\ \mathcal{L}\mathbf{w}_{P-1} + \mathcal{F}\mathbf{w}_P &= \bar{\mathrm{B}}_P \end{aligned} \tag{3.18}$$

Observe that

$\mathbb{Q}^T \mathcal{F} \mathbb{Q} = diag(\mathrm{Z}_1, \mathrm{Z}_2, \mathrm{Z}_3, ..., \mathrm{Z}_N) = \Lambda$ (say) where $\mathrm{Z}_j = diag(\lambda_1, \lambda_2, \lambda_3, ..., \lambda_M)$

$\mathbb{Q}^T \mathcal{L} \mathbb{Q} = diag(\mathrm{E}_1, \mathrm{E}_2, \mathrm{E}_3, ..., \mathrm{E}_N) = \Upsilon$ (say) where $\mathrm{E}_j = diag(\mu_1, \mu_2, \mu_3, ..., \mu_M)$

Here $\lambda_i = \eta_i + 2\tau_i \cos\left(\dfrac{i\pi}{M+1}\right)$ and

$$\mu_i = \alpha_i + 2(1+r)\cos\left(\frac{i\pi}{M+1}\right) \qquad i = 1, 2, \ldots, M$$

Let $\mathbb{Q}^T \mathbf{w}_k = \Psi_k \Rightarrow \mathbf{w}_k = \mathbb{Q}\,\Psi_k$

$$\mathbb{Q}^T \bar{B}_k = \Gamma_k \Rightarrow \bar{B}_k = \mathbb{Q}\Gamma_k \tag{3.19}$$

where $\Psi_k = [\mathbf{\Psi}_{1k}\ \mathbf{\Psi}_{2k}\ \mathbf{\Psi}_{3k}\ ...\ \mathbf{\Psi}_{Nk}]^T$ and $\mathbf{\Psi}_{jk} = [\psi_{1jk}\ \psi_{2jk}\ \psi_{3jk}\ ...\ \psi_{Mjk}]^T$

$\Gamma_k = [\boldsymbol{\beta}_{1k}, \boldsymbol{\beta}_{2k}, \boldsymbol{\beta}_{3k}, ..., \boldsymbol{\beta}_{Nk}]^T$ and $\boldsymbol{\beta}_{jk} = [\beta_{1jk}, \beta_{2jk}, ..., \beta_{Mjk}]^T$

Now pre-multiplying (3.18) by $\mathbb{Q}^T$ and make use of (3.19), we get

$$\begin{aligned}
\Lambda\Psi_1 + \Upsilon\Psi_2 \quad &= \Gamma_1 \\
\Upsilon\Psi_1 + \Lambda\Psi_2 + \Upsilon\Psi_3 &= \Gamma_2 \\
\Upsilon\Psi_2 + \Lambda\Psi_3 + \Upsilon\Psi_4 &= \Gamma_3 \\
&\dots \\
\Upsilon\Psi_{P-1} + \Lambda\Psi_P &= \Gamma_P
\end{aligned} \tag{3.20}$$

Starting from the first row of (3.20), i.e. $\Lambda\Psi_1 + \Upsilon\Psi_2 = \Gamma_1$, we write these set of equations turn by turn as

$$\begin{aligned}
&\lambda_1\psi_{111} + \mu_1\psi_{112} = \beta_{111} \\
&\lambda_2\psi_{211} + \mu_2\psi_{212} = \beta_{211} \\
&\lambda_3\psi_{311} + \mu_3\psi_{312} = \beta_{311} \\
&\qquad\dots \\
&\lambda_M\psi_{M11} + \mu_M\psi_{M12} = \beta_{M11} \\
&\qquad\lambda_1\psi_{121} + \mu_1\psi_{122} = \beta_{121} \\
&\qquad\lambda_2\psi_{221} + \mu_2\psi_{222} = \beta_{221} \\
&\qquad\lambda_3\psi_{321} + \mu_3\psi_{322} = \beta_{321} \\
&\qquad\qquad\dots \\
&\qquad\lambda_M\psi_{M21} + \mu_M\psi_{M22} = \beta_{M21} \\
&\qquad\qquad\dots \quad \dots \quad \dots \\
&\qquad\qquad\lambda_1\psi_{1N1} + \mu_1\psi_{1N2} = \beta_{1N1} \\
&\qquad\qquad\lambda_2\psi_{2N1} + \mu_2\psi_{2N2} = \beta_{2N1} \\
&\qquad\qquad\lambda_3\psi_{3N1} + \mu_3\psi_{3N2} = \beta_{3N1} \\
&\qquad\qquad\qquad\dots \\
&\qquad\qquad\lambda_M\psi_{MN1} + \mu_M\psi_{MN2} = \beta_{MN1}
\end{aligned} \tag{3.21a}$$

Again for the second equation of (3.20) $\Upsilon\Psi_1 + \Lambda\Psi_2 + \Upsilon\Psi_3 = \Gamma_2$, we get

$$\mu_1\psi_{111} + \lambda_1\psi_{112} + \mu_1\psi_{113} = \beta_{112}$$
$$\mu_2\psi_{211} + \lambda_2\psi_{212} + \mu_2\psi_{213} = \beta_{212}$$
$$\mu_3\psi_{311} + \lambda_3\psi_{312} + \mu_3\psi_{313} = \beta_{312}$$
$$\ldots$$
$$\mu_M\psi_{M11} + \lambda_M\psi_{M12} + \mu_M\psi_{M13} = \beta_{M12} \tag{3.21b}$$
$$\mu_1\psi_{121} + \lambda_1\psi_{122} + \mu_1\psi_{123} = \beta_{122}$$
$$\mu_2\psi_{221} + \lambda_2\psi_{222} + \mu_2\psi_{223} = \beta_{222}$$
$$\mu_3\psi_{321} + \lambda_3\psi_{322} + \mu_3\psi_{323} = \beta_{322}$$
$$\ldots$$
$$\mu_M\psi_{M21} + \lambda_M\psi_{M22} + \mu_M\psi_{M23} = \beta_{M22}$$
$$\ldots \quad \ldots \quad \ldots$$
$$\mu_1\psi_{1N1} + \lambda_1\psi_{1N2} + \mu_1\psi_{1N3} = \beta_{1N2}$$
$$\mu_2\psi_{2N1} + \lambda_2\psi_{2N2} + \mu_2\psi_{2N3} = \beta_{2N2}$$
$$\mu_3\psi_{3N1} + \lambda_3\psi_{3N2} + \mu_3\psi_{3N3} = \beta_{3N2}$$
$$\ldots$$
$$\mu_M\psi_{MN1} + \lambda_M\psi_{MN2} + \mu_M\psi_{MN3} = \beta_{MN2}$$

And the last equation of (3.20), i.e. $\Upsilon\Psi_{P-1} + \Lambda\Psi_P = \Gamma_P$ is written as

$$\mu_1\psi_{11(P-1)} + \lambda_1\psi_{11P} = \beta_{11P}$$
$$\mu_2\psi_{21(P-1)} + \lambda_2\psi_{21P} = \beta_{21P}$$
$$\mu_3\psi_{31(P-1)} + \lambda_3\psi_{31P} = \beta_{31P}$$
$$\ldots$$
$$\mu_M\psi_{M1(P-1)} + \lambda_M\psi_{M1P} = \beta_{M1P} \tag{3.21c}$$
$$\mu_1\psi_{12(P-1)} + \lambda_1\psi_{12P} = \beta_{12P}$$
$$\mu_2\psi_{22(P-1)} + \lambda_2\psi_{22P} = \beta_{22P}$$
$$\mu_3\psi_{32(P-1)} + \lambda_3\psi_{32P} = \beta_{32P}$$
$$\ldots$$
$$\mu_M\psi_{M2(P-1)} + \lambda_M\psi_{M2P} = \beta_{M2P}$$
$$\ldots \quad \ldots \quad \ldots$$

$$\mu_1\psi_{1N(P-1)} + \lambda_1\psi_{1NP} = \beta_{1NP}$$
$$\mu_2\psi_{2N(P-1)} + \lambda_2\psi_{2NP} = \beta_{2NP}$$
$$\mu_3\psi_{3N(P-1)} + \lambda_3\psi_{3NP} = \beta_{3NP}$$
$$\cdots$$
$$\mu_M\psi_{MN(P-1)} + \lambda_M\psi_{MNP} = \beta_{MNP}$$

From each set of equations of (3.21), we select the first equations and put together as one group of equations; again we take the second equations and put together as a second group of equations and so on till we get the last set of equations from each of (3.21a), (3.21b), …, (3.21c). In doing these we obtain the following sets of equations

$$\begin{aligned}
\lambda_1\psi_{111} + \mu_1\psi_{112} &= \beta_{111}\\
\mu_1\psi_{111} + \lambda_1\psi_{112} + \mu_1\psi_{113} &= \beta_{112}\\
\mu_1\psi_{112} + \lambda_1\psi_{113} + \mu_1\psi_{114} &= \beta_{113}\\
&\cdots\\
\mu_1\psi_{11(P-1)} + \lambda_1\psi_{11P} &= \beta_{11P}\\
\lambda_2\psi_{211} + \mu_2\psi_{212} &= \beta_{211}\\
\mu_2\psi_{211} + \lambda_2\psi_{212} + \mu_2\psi_{213} &= \beta_{212}\\
\mu_2\psi_{212} + \lambda_2\psi_{213} + \mu_2\psi_{214} &= \beta_{213}\\
&\cdots\\
\mu_2\psi_{21(P-1)} + \lambda_2\psi_{21P} &= \beta_{21P}\\
&\cdots \quad \cdots \quad \cdots\\
\lambda_M\psi_{M11} + \mu_M\psi_{M12} &= \beta_{M11}\\
\mu_M\psi_{M11} + \lambda_M\psi_{M12} + \mu_M\psi_{M13} &= \beta_{M12}\\
\mu_M\psi_{M12} + \lambda_M\psi_{M13} + \mu_M\psi_{M14} &= \beta_{M13}\\
&\cdots\\
\mu_M\psi_{M1(P-1)} + \lambda_M\psi_{M1P} &= \beta_{M1P}
\end{aligned} \tag{3.22}$$

Observe that here in the above set of equations (3.22) $j = 1$, and for each $i = 1(1)M$ the coefficient matrix of the left hand side is a tri-diagonal matrix of order P and has the form

$$\mathcal{M} = \begin{pmatrix} \lambda_i & \mu_i & & & & & \\ \mu_i & \lambda_i & \mu_i & & & & \\ & \mu_i & \lambda_i & \mu_i & & & \\ & & & \ddots & & & \\ & & & & \mu_i & \lambda_i & \mu_i \\ & & & & & \mu_i & \lambda_i \end{pmatrix}, \ i = 1(1)M$$

Continuing for the other groups of equations as above from $j = 2, \ldots, N$, we get

$$\begin{aligned}
\lambda_1\psi_{1j1} + \mu_1\psi_{1j2} \qquad\qquad &= \beta_{1j1} \\
\mu_1\psi_{1j1} + \lambda_1\psi_{1j2} + \mu_1\psi_{1j3} &= \beta_{1j2} \\
\mu_1\psi_{1j2} + \lambda_1\psi_{1j3} + \mu_1\psi_{1j4} &= \beta_{1j3} \\
&\ldots \\
\mu_1\psi_{1j(P-1)} + \lambda_1\psi_{1jP} &= \beta_{1jP} \\
\lambda_2\psi_{2j1} + \mu_1\psi_{2j2} \qquad\qquad &= \beta_{2j1} \\
\mu_1\psi_{2j1} + \lambda_2\psi_{2j2} + \mu_1\psi_{2j3} &= \beta_{2j2} \\
\mu_1\psi_{2j2} + \lambda_2\psi_{2j3} + \mu_1\psi_{2j4} &= \beta_{2j3} \\
&\ldots \\
\mu_1\psi_{2j(P-1)} + \lambda_2\psi_{2jP} &= \beta_{2jP} \qquad (3.23) \\
\ldots \quad \ldots \quad \ldots \\
\lambda_M\psi_{Mj1} + \mu_1\psi_{Mj2} \qquad\qquad &= \beta_{Mj1} \\
\mu_1\psi_{Mj1} + \lambda_M\psi_{Mj2} + \mu_1\psi_{Mj3} &= \beta_{Mj2} \\
\mu_1\psi_{Mj2} + \lambda_M\psi_{Mj3} + \mu_1\psi_{Mj4} &= \beta_{Mj3} \\
&\ldots \\
\mu_1\,\psi_{Mj(P-1)} + \lambda_M\psi_{MjP} &= \beta_{MjP}
\end{aligned}$$

For each $j = 2, \ldots, N$ the coefficients matrix of the left hand side of (3.23) is a tridiagonal matrix similar to that of the form for $j = 1$

$$\mathcal{M}=\begin{pmatrix} \lambda_i & \mu_i & & & & & \\ \mu_i & \lambda_i & \mu_i & & & & \\ & \mu_i & \lambda_i & \mu_i & & & \\ & & & \ddots & & & \\ & & & & \mu_i & \lambda_i & \mu_i \\ & & & & & \mu_i & \lambda_i \end{pmatrix}, i = 1(1)M,\ j = 2, \dots, N \text{ is of order } P. \tag{3.24}$$

Observe that (3.12) reduces to a diagonal matrix (3.24) for $i = 1,2, \dots, M$, $j = 1,2, \dots, N$ and $k = 1,2, \dots, P$. Thus we solve these sets of equations (3.24) for $\psi_{i,j,k}$ by the use of Thomas Algorithm [30]. Once after getting each $\psi_{i,j,k}$ (and hence Ψ_k) by the help of (3.19) we get $\mathbf{w}_k$ and again by the help of (3.13) we obtain $\mathbf{U}_k$ and this means that each $U_{i,j,k}$ are obtained. Thus, this solves our problem.

II. 27 Points Stencil Scheme

In this scheme we consider all terms of equation (3.6) and simplifying

$$\begin{aligned} &-(400+200r)U_{i,j,k}+(80-20r)\left(U_{i+1,j,k}+U_{i-1,j,k}+U_{i,j+1,k}+U_{i,j-1,k}\right)+(20-2r)\left(U_{i+1,j+1,k}+U_{i-1,j+1,k}\right.\\ &\left.+U_{i+1,j-1,k}+U_{i-1,j-1,k}\right)+(100r-40)\left(U_{i+1,j,k+1}+U_{i-1,j,k+1}+U_{i+1,j,k-1}+U_{i-1,j,k-1}+U_{i,j+1,k+1}+U_{i,j-1,k+1}\right.\\ &\left.+U_{i,j+1,k+1}+U_{i,j-1,k-1}\right)+(2+r)\left(U_{i+1,j+1,k+1}+U_{i-1,j+1,k+1}+U_{i+1,j-1,k+1}+U_{i-1,j-1,k-1}+U_{i+1,j+1,k-1}+U_{i-1,j+1,k-1}\right.\\ &\left.+U_{i+1,j-1,k+1}+U_{i-1,j-1,k-1}\right)\\ &\qquad=h_1^2\left(144+12\left(\delta_x^2+\delta_y^2+\delta_z^2\right)+\left(\delta_x^2\delta_y^2+\delta_x^2\delta_z^2+\delta_y^2\delta_z^2\right)+\frac{1}{12}\delta_x^2\delta_y^2\delta_z^2\right)f_{i,j,k} \end{aligned} \tag{3.25}$$

Taking first in the X-direction, next Y-direction and lastly Z-direction in (3.25) we get a large system of equations (the number of equations actually depends on the values of M, N and P); and these systems of equations can be written in matrix form as

$$AU=\mathcal{B} \tag{3.26}$$

where

$$A=\begin{pmatrix} R & S & & & & \\ S & R & S & & & \\ & S & R & S & & \\ & & & \ddots & & \\ & & & S & R & S \\ & & & & S & R \end{pmatrix} \tag{3.27}$$

it has P blocks and each block is of order $MN \times MN$.

$$R=\begin{pmatrix} R_1 & R_2 & & & & \\ R_2 & R_1 & R_2 & & & \\ & R_2 & R_1 & R_2 & & \\ & & & \ddots & & \\ & & & R_2 & R_1 & R_2 \\ & & & & R_2 & R_1 \end{pmatrix}, \quad S=\begin{pmatrix} S_1 & S_2 & & & & \\ S_2 & S_1 & S_2 & & & \\ & S_2 & S_1 & S_2 & & \\ & & & \ddots & & \\ & & & S_2 & S_1 & S_2 \\ & & & & S_2 & S_1 \end{pmatrix}$$

R and S have N blocks and each block is of order $M \times M$.

$$R_1=\begin{pmatrix} -400-200r & 80-20r & & & & \\ 80-20r & -400-200r & 80-20r & & & \\ & 80-20r & -400-200r & 80-20r & & \\ & & & \ddots & & \\ & & & 80-20r & -400-200r & 80-20r \\ & & & & 80-20r & -400-200r \end{pmatrix}$$

$$R_2=\begin{pmatrix} 80-20r & 20-2r & & & & \\ 20-2r & 80-20r & 20-2r & & & \\ & 20-2r & 80-20r & 20-2r & & \\ & & & \ddots & & \\ & & & 20-2r & 80-20r & 20-2r \\ & & & & 20-2r & 80-20r \end{pmatrix}$$

$$S_1=\begin{pmatrix} 100r-40 & 8-10r & & & & \\ 8-10r & 100r-40 & 8-10r & & & \\ & 8-10r & 100r-40 & 8-10r & & \\ & & & \ddots & & \\ & & & 8-10r & 100r-40 & 8-10r \\ & & & & 8-10r & 100r-40 \end{pmatrix}$$

$$S_2 = \begin{pmatrix} 8-10r & 2+r & & & & & \\ 2+r & 8-10r & 2+r & & & & \\ & 2+r & 8-10r & 2+r & & & \\ & & & \ddots & & & \\ & & & & 2+r & 8-10r & 2+r \\ & & & & & 2+r & 8-10r \end{pmatrix}$$

$$U = \begin{pmatrix} \mathbf{U}_1 & \mathbf{U}_2 & \mathbf{U}_3 & \cdots & \mathbf{U}_{P-1} & \mathbf{U}_P \end{pmatrix}^T, \text{ and}$$

$$\mathcal{B} = \begin{pmatrix} \mathbf{B}_1 & \mathbf{B}_2 & \mathbf{B}_3 & \dots & \mathbf{B}_{P-1} & \mathbf{B}_P \end{pmatrix}^T \tag{3.28}$$

where

$$\mathbf{U}_k = [\mathbf{u}_{1k}\ \mathbf{u}_{2k} \dots \mathbf{u}_{Nk}]^T \quad \text{and } \mathbf{u}_{jk} = [\ U_{1jk}\ U_{2jk}\ \dots U_{Mjk}]^T$$

and $\mathbf{B}_k = [\mathbf{d}_{1k}\ \mathbf{d}_{2k} \dots \mathbf{d}_{Nk}]^T$ and $\mathbf{d}_{jk} = [\ d_{1jk}\ d_{2jk} \dots d_{Mjk}]^T\ k = 1, 2, 3, \dots, P$

is the known column vectors such that each d_{ijk} represents known boundary values of U and values of f.

Using (3.27) and (3.28), we write (3.26) as

$$\begin{pmatrix} R & S & & & & \\ S & R & S & & & \\ & S & R & S & & \\ & & & \ddots & & \\ & & & S & R & S \\ & & & & S & R \end{pmatrix} \begin{pmatrix} \mathbf{U}_1 \\ \mathbf{U}_2 \\ \mathbf{U}_3 \\ \vdots \\ \mathbf{U}_{P-1} \\ \mathbf{U}_P \end{pmatrix} = \begin{pmatrix} \mathbf{B}_1 \\ \mathbf{B}_2 \\ \mathbf{B}_3 \\ \vdots \\ \mathbf{B}_{P-1} \\ \mathbf{B}_P \end{pmatrix} \tag{3.29}$$

Equation (3.29) again can be written as

$$\begin{aligned} R\mathbf{U}_1 + S\mathbf{U}_2 \quad &= \mathbf{B}_1 \\ S\mathbf{U}_1 + R\mathbf{U}_2 + S\mathbf{U}_3 &= \mathbf{B}_2 \\ S\mathbf{U}_2 + R\mathbf{U}_3 + S\mathbf{U}_4 &= \mathbf{B}_3 \\ &\dots \\ S\mathbf{U}_{P-1} + R\mathbf{U}_P &= \mathbf{B}_P \end{aligned} \tag{3.30}$$

Now by applying extended Hockney's method to three dimensions we obtain the solution of the system of linear equations (3.30)

3.3.2 Extended Hockney's Method for 27-Points Scheme

Observe that all the matrices R_1, R_2, S_1 and S_2 are real tridiagonal symmetric matrices and hence their eigenvalues and eigenvectors can easily be obtained. [30]

The eigenvalues of R_1, R_2, S_1 and S_2 are given by

$$\eta_i = (-400 - 200r) + 2(80 - 20r)\cos\left(\frac{i\pi}{M+1}\right)$$

$$\tau_i = (80 - 20r) + 2(20 - 2r)\cos\left(\frac{i\pi}{M+1}\right)$$

$$\alpha_i = (100r - 40) + 2(8 - 10r)\cos\left(\frac{i\pi}{M+1}\right)$$

$$\omega_i = (8 - 10r) + 2(2 + r)\cos\left(\frac{i\pi}{M+1}\right) \qquad i, j = 1, 2, \cdots, M$$

Let $\mathbf{q}_i$ be an eigenvector of R_1, R_2, S_1 and S_2 corresponding to the eigenvalues η_i, τ_i, α_i, and ω_i respectively, and Q be the modal matrix $[\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3, \ldots, \mathbf{q}_M]$ of the matrix R_1, R_2, S_1 and S_2 of order M such that $Q^T Q = I$,

$$Q^T R_1 Q = diag(\eta_1, \eta_2, \eta_3, \ldots, \eta_M) = \mathrm{H} \text{ (say)},$$

$$Q^T R_2 Q = diag(\tau_1, \tau_2, \tau_3, \ldots, \tau_M) = \mathrm{T} \text{ (say)},$$

$$Q^T S_1 Q = diag(\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_M) = \Phi \text{ (say) and}$$

$$Q^T S_2 Q = diag(\omega_1, \omega_2, \omega_3, \ldots, \omega_M) = \Omega \text{ (say)}$$

The $M \times M$ modal matrix Q is defined by

$$q_{ij} = \sqrt{\frac{2}{M+1}} \sin\left(\frac{ij\pi}{M+1}\right) \qquad i, j = 1, 2, \cdots, M$$

Let $\mathbb{Q} = diag(Q, Q, Q, \ldots, Q)$ is a matrix of order $\times MN$.

Thus $\mathbb{Q}$ satisfy $\mathbb{Q}^T\mathbb{Q} = I$,

$$\mathbb{Q}^T R\mathbb{Q} = \begin{pmatrix} \mathrm{H} & \mathrm{T} & & & & \\ \mathrm{T} & \mathrm{H} & \mathrm{T} & & & \\ & \mathrm{T} & \mathrm{H} & \mathrm{T} & & \\ & & & \ddots & & \\ & & & \mathrm{T} & \mathrm{H} & \mathrm{T} \\ & & & & \mathrm{T} & \mathrm{H} \end{pmatrix} \text{ and}$$

$$\mathbb{Q}^T S\mathbb{Q} = \begin{pmatrix} \Phi & \Omega & & & & \\ \Omega & \Phi & \Omega & & & \\ & \Omega & \Phi & \Omega & & \\ & & & \ddots & & \\ & & & \Omega & \Phi & \Omega \\ & & & & \Omega & \Phi \end{pmatrix}$$

Let $\mathbb{Q}^T\mathbf{U}_k = \mathbf{V}_k \Rightarrow \mathbf{U}_k = \mathbb{Q}\,\mathbf{V}_k$

$$\mathbb{Q}^T\mathbf{B}_k = B_k \Rightarrow \mathbf{B}_k = \mathbb{Q}B_k \qquad (3.31)$$

where $\mathbf{V}_k = [\mathbf{v}_{1k}\ \mathbf{v}_{2k} \ldots \mathbf{v}_{Nk}]^T$ and $\mathbf{v}_{jk} = [v_{1jk}\ v_{2jk} \ldots v_{Mjk}]^T$

$$B_k = [\mathbf{b}_{1k}\ \mathbf{b}_{2k} \ldots \mathbf{b}_{Nk}]^T \text{ and } \mathbf{b}_{jk} = [\,b_{1jk}\ b_{2jk} \ldots b_{Mjk}]^T$$

Consider the first equation of (3.30) i.e. $R\mathbf{U}_1 + S\mathbf{U}_2 = \mathbf{B}_1$, and pre multiplying it by $\mathbb{Q}^T$ and using (3.31), we get

$$\begin{pmatrix} \mathrm{H} & \mathrm{T} & & & & \\ \mathrm{T} & \mathrm{H} & \mathrm{T} & & & \\ & \mathrm{T} & \mathrm{H} & \mathrm{T} & & \\ & & & \ddots & & \\ & & & \mathrm{T} & \mathrm{H} & \mathrm{T} \\ & & & & \mathrm{T} & \mathrm{H} \end{pmatrix} \begin{pmatrix} \mathbf{v}_{11} \\ \mathbf{v}_{21} \\ \mathbf{v}_{31} \\ \vdots \\ \mathbf{v}_{(N-1)1} \\ \mathbf{v}_{N1} \end{pmatrix} + \begin{pmatrix} \Phi & \Omega & & & & \\ \Omega & \Phi & \Omega & & & \\ & \Omega & \Phi & \Omega & & \\ & & & \ddots & & \\ & & & \Omega & \Phi & \Omega \\ & & & & \Omega & \Phi \end{pmatrix} \begin{pmatrix} \mathbf{v}_{12} \\ \mathbf{v}_{22} \\ \mathbf{v}_{32} \\ \vdots \\ \mathbf{v}_{(N-1)2} \\ \mathbf{v}_{N2} \end{pmatrix} = \begin{pmatrix} \mathbf{b}_{11} \\ \mathbf{b}_{21} \\ \mathbf{b}_{31} \\ \vdots \\ \mathbf{b}_{(N-1)1} \\ \mathbf{b}_{N1} \end{pmatrix} \qquad (3.32)$$

Again we write equation (3.32) as

$$
\begin{aligned}
&\mathrm{H}\mathbf{v}_{11}+\mathrm{T}\mathbf{v}_{21}+\Phi\mathbf{v}_{12}+\Omega\mathbf{v}_{22}=\mathbf{b}_{11}\\
&\mathrm{T}\mathbf{v}_{11}+\mathrm{H}\mathbf{v}_{21}+\mathrm{T}\mathbf{v}_{31}+\Omega\mathbf{v}_{12}+\Phi\mathbf{v}_{22}+\Omega\mathbf{v}_{32}=\mathbf{b}_{21}\\
&\mathrm{T}\mathbf{v}_{21}+\mathrm{H}\mathbf{v}_{31}+\mathrm{T}\mathbf{v}_{41}+\Omega\mathbf{v}_{22}+\Phi\mathbf{v}_{32}+\Omega\mathbf{v}_{42}=\mathbf{b}_{31}\\
&\qquad\cdots\\
&\mathrm{T}\mathbf{v}_{(N-1)1}+\mathrm{H}\mathbf{v}_{N1}+\Omega\mathbf{v}_{(N-1)2}+\Phi\mathbf{v}_{N2}=\mathbf{b}_{N1}
\end{aligned}
\tag{3.33}
$$

Now collect the first equations from each of (3.33) and consider as one group of equations

$$
\begin{aligned}
&\eta_1 v_{111}+\tau_1 v_{121}+\alpha_1 v_{112}+\omega_1 v_{122}=b_{111}\\
&\tau_1 v_{111}+\eta_1 v_{121}+\tau_1 v_{131}+\omega_1 v_{112}+\alpha_1 v_{122}+\omega_1 v_{132}=b_{121}\\
&\tau_1 v_{121}+\eta_1 v_{131}+\tau_1 v_{141}+\omega_1 v_{122}+\alpha_1 v_{132}+\omega_1 v_{142}=b_{131}\\
&\qquad\cdots\\
&\tau_1 v_{1(N-1)1}+\eta_1 v_{1N1}+\omega_1 v_{1(N-1)12}+\alpha_1 v_{1N2}=b_{1N1}
\end{aligned}
\tag{3.34a}
$$

Again we collect the second equations from each equation of (3.33) and consider as a second group of equations

$$
\begin{aligned}
&\eta_2 v_{211}+\tau_2 v_{221}+\alpha_2 v_{212}+\omega_2 v_{222}=b_{211}\\
&\tau_2 v_{211}+\eta_2 v_{221}+\tau_2 v_{231}+\omega_2 v_{212}+\alpha_2 v_{222}+\omega_2 v_{232}=b_{221}\\
&\tau_2 v_{221}+\eta_2 v_{231}+\tau_2 v_{241}+\omega_2 v_{222}+\alpha_2 v_{232}+\omega_2 v_{242}=b_{231}\\
&\qquad\cdots\\
&\tau_2 v_{2(N-1)1}+\eta_2 v_{2N1}+\omega_2 v_{2(N-1)12}+\alpha_2 v_{1N2}=b_{2N1}
\end{aligned}
\tag{3.34b}
$$

Lastly we collect the last equations from each equation of (3.33) and consider as a last group of equations

$$
\begin{aligned}
&\eta_M v_{M11}+\tau_M v_{M21}+\alpha_M v_{M12}+\omega_M v_{M22}=b_{M11}\\
&\tau_M v_{M11}+\eta_M v_{M21}+\tau_M v_{M31}+\omega_M v_{M12}+\alpha_M v_{M22}+\omega_M v_{M32}=b_{M21}\\
&\tau_M v_{M21}+\eta_M v_{M31}+\tau_M v_{M41}+\omega_M v_{M22}+\alpha_M v_{M32}+\omega_M v_{M42}=b_{M31}\\
&\qquad\cdots\\
&\tau_M v_{M(N-1)1}+\eta_M v_{MN1}+\omega_M v_{M(N-1)12}+\alpha_M v_{MN2}=b_{MN1}
\end{aligned}
\tag{3.34c}
$$

Now we write equations (3.34a) to (3.34c) in matrix form as

$$\begin{pmatrix} \eta_i & \tau_i & & & \\ \tau_i & \eta_i & \tau_i & & \\ & \tau_i & \eta_i & \tau_i & \\ & & & \ddots & \\ & & & \tau_i & \eta_i \end{pmatrix} \begin{pmatrix} v_{i11} \\ v_{i21} \\ v_{i31} \\ \vdots \\ v_{iN1} \end{pmatrix} + \begin{pmatrix} \alpha_i & \omega_i & & & \\ \omega_i & \alpha_i & \omega_i & & \\ & \omega_i & \alpha_i & \omega_i & \\ & & & \ddots & \\ & & & \omega_i & \alpha_i \end{pmatrix} \begin{pmatrix} v_{i12} \\ v_{i22} \\ v_{i32} \\ \vdots \\ v_{iN2} \end{pmatrix} = \begin{pmatrix} b_{i11} \\ b_{i21} \\ b_{i31} \\ \vdots \\ b_{iN1} \end{pmatrix}$$

$$i = 1, 2, \ldots, M \tag{3.35}$$

Let $\mathcal{F}_i = \begin{pmatrix} \eta_i & \tau_i & & & \\ \tau_i & \eta_i & \tau_i & & \\ & \tau_i & \eta_i & \tau_i & \\ & & & \ddots & \\ & & & \tau_i & \eta_i \end{pmatrix}$, $\mathcal{L}_i = \begin{pmatrix} \alpha_i & \omega_i & & & \\ \omega_i & \alpha_i & \omega_i & & \\ & \omega_i & \alpha_i & \omega_i & \\ & & & \ddots & \\ & & & \omega_i & \alpha_i \end{pmatrix}$

$$\mathbf{W}_{ik} = \begin{pmatrix} v_{i1k} \\ v_{i2k} \\ v_{i3k} \\ \vdots \\ v_{iNk} \end{pmatrix} \text{ and } \bar{\mathbf{B}}_{ik} = \begin{pmatrix} b_{i1k} \\ b_{i2k} \\ b_{i3k} \\ \vdots \\ b_{iNk} \end{pmatrix}$$

Equation (3.35) which is the same as the first equation of (3.30) once again can be written as $\mathcal{F}\mathbf{w}_1 + \mathcal{L}\mathbf{w}_2 = \bar{\mathrm{B}}_1$

where $\mathcal{F} = \begin{pmatrix} \mathcal{F}_i & & & & \\ & \mathcal{F}_i & & & \\ & & \mathcal{F}_i & & \\ & & & \ddots & \\ & & & & \mathcal{F}_i \end{pmatrix}$, $\mathcal{L} = \begin{pmatrix} \mathcal{L}_i & & & & \\ & \mathcal{L}_i & & & \\ & & \mathcal{L}_i & & \\ & & & \ddots & \\ & & & & \mathcal{L}_i \end{pmatrix}$ both are of order MP

$\mathbf{w}_1 = [\mathbf{W}_{i1}\ \mathbf{W}_{i2}\ \mathbf{W}_{i3}\ \ldots\ \mathbf{W}_{iP}]^T$ and $\bar{B}_k = [\mathbf{B}_{i1}, \mathbf{B}_{i2}, \cdots, \mathbf{B}_{iP}]^T$

Similarly, we can write the other equations in (3.12) using the matrices $\mathcal{F}$, $\mathcal{L}$, $\mathbf{W}_k$ and $\bar{B}_k$.

Therefore, (3.12) can, equivalently, be written as

$$\begin{aligned}
\mathcal{F}\mathbf{w}_1 + \mathcal{L}\mathbf{w}_2 &= \bar{B}_1 \\
\mathcal{L}\mathbf{w}_1 + \mathcal{F}\mathbf{w}_2 + \mathcal{L}\mathbf{w}_3 &= \bar{B}_2 \\
\mathcal{L}\mathbf{w}_2 + \mathcal{F}\mathbf{w}_3 + \mathcal{L}\mathbf{w}_4 &= \bar{B}_3 \\
&\cdots \\
\mathcal{L}\mathbf{w}_{P-1} + \mathcal{F}\mathbf{w}_P &= \bar{B}_P
\end{aligned} \tag{3.36}$$

Observe that

$\mathbb{Q}^T\mathcal{F}\mathbb{Q} = diag(Z_1, Z_2, Z_3, ..., Z_N) = \Lambda$ (say) where $Z_j = diag(\lambda_1, \lambda_2, \lambda_3, ..., \lambda_M)$

$\mathbb{Q}^T\mathcal{L}\mathbb{Q} = diag(E_1, E_2, E_3, ..., E_N) = \Upsilon$ (say) where $E_j = diag(\mu_1, \mu_2, \mu_3, ..., \mu_M)$

Here $\lambda_i = \eta_i + 2\tau_i \cos\left(\dfrac{i\pi}{M+1}\right)$ and

$$\mu_i = \alpha_i + 2\ \beta_i \cos\left(\frac{i\pi}{M+1}\right) \qquad i, j = 1, 2, \cdots, M$$

Let $\mathbb{Q}^T\mathbf{w}_k = \Psi_k \Rightarrow \mathbf{w}_k = \mathbb{Q}\ \Psi_k$

$$\mathbb{Q}^T\bar{B}_k = \Gamma_k \Rightarrow \bar{B}_k = \mathbb{Q}\Gamma_k \tag{3.37}$$

where $\Psi_k = [\boldsymbol{\Psi}_{1k}\ \boldsymbol{\Psi}_{2k}\ \boldsymbol{\Psi}_{3k}\ ...\ \boldsymbol{\Psi}_{Nk}]^T$ and $\boldsymbol{\Psi}_{jk} = [\psi_{1jk}\ \psi_{2jk}\ \psi_{3jk}\ ...\ \psi_{Mjk}]^T$

$\Gamma_k = [\boldsymbol{\beta}_{1k}, \boldsymbol{\beta}_{2k}, \boldsymbol{\beta}_{3k}, ..., \boldsymbol{\beta}_{Nk}]^T$ and $\boldsymbol{\beta}_{jk} = [\beta_{1jk}, \beta_{2jk}, ..., \beta_{Mjk}]^T$

Now pre-multiplying (3.36) by $\mathbb{Q}^T$ and make use of (3.37), we get

$$\begin{aligned}
\Lambda\Psi_1 + \Upsilon\Psi_2 &= \Gamma_1 \\
\Upsilon\Psi_1 + \Lambda\Psi_2 + \Upsilon\Psi_3 &= \Gamma_2 \\
\Upsilon\Psi_2 + \Lambda\Psi_3 + \Upsilon\Psi_4 &= \Gamma_3 \\
&\cdots \\
\Upsilon\Psi_{P-1} + \Lambda\Psi_P &= \Gamma_P
\end{aligned} \tag{3.38}$$

Starting from the first row of (3.38), i.e. $\Lambda\Psi_1 + \Upsilon\Psi_2 = \Gamma_1$, we write these set of equations turn by turn as

$$\lambda_1\psi_{111} + \mu_1\psi_{112} = \beta_{111}$$
$$\lambda_2\psi_{211} + \mu_2\psi_{212} = \beta_{211}$$
$$\lambda_3\psi_{311} + \mu_3\psi_{312} = \beta_{311}$$
$$\cdots$$
$$\lambda_M\psi_{M11} + \mu_M\psi_{M12} = \beta_{M11} \tag{3.39a}$$
$$\lambda_1\psi_{121} + \mu_1\psi_{122} = \beta_{121}$$
$$\lambda_2\psi_{221} + \mu_2\psi_{222} = \beta_{221}$$
$$\lambda_3\psi_{321} + \mu_3\psi_{322} = \beta_{321}$$
$$\cdots$$
$$\lambda_M\psi_{M21} + \mu_M\psi_{M22} = \beta_{M21}$$
$$\cdots \quad \cdots \quad \cdots$$
$$\lambda_1\psi_{1N1} + \mu_1\psi_{1N2} = \beta_{1N1}$$
$$\lambda_2\psi_{2N1} + \mu_2\psi_{2N2} = \beta_{2N1}$$
$$\lambda_3\psi_{3N1} + \mu_3\psi_{3N2} = \beta_{3N1}$$
$$\cdots$$
$$\lambda_M\psi_{MN1} + \mu_M\psi_{MN2} = \beta_{MN1}$$

For $\Upsilon\Psi_1 + \Lambda\Psi_2 + \Upsilon\Psi_3 = \Gamma_2$, we get

$$\mu_1\psi_{111} + \lambda_1\psi_{112} + \mu_1\psi_{113} = \beta_{112}$$
$$\mu_2\psi_{211} + \lambda_2\psi_{212} + \mu_2\psi_{213} = \beta_{212}$$
$$\mu_3\psi_{311} + \lambda_3\psi_{312} + \mu_3\psi_{313} = \beta_{312}$$
$$\cdots$$
$$\mu_M\psi_{M11} + \lambda_M\psi_{M12} + \mu_M\psi_{M13} = \beta_{M12} \tag{3.39b}$$
$$\mu_1\psi_{121} + \lambda_1\psi_{122} + \mu_1\psi_{123} = \beta_{122}$$
$$\mu_2\psi_{221} + \lambda_2\psi_{222} + \mu_2\psi_{223} = \beta_{222}$$
$$\mu_3\psi_{321} + \lambda_3\psi_{322} + \mu_3\psi_{323} = \beta_{322}$$
$$\cdots$$
$$\mu_M\psi_{M21} + \lambda_M\psi_{M22} + \mu_M\psi_{M23} = \beta_{M22}$$
$$\cdots \quad \cdots \quad \cdots$$

$$\mu_1\psi_{1N1} + \lambda_1\psi_{1N2} + \mu_1\psi_{1N3} = \beta_{1N2}$$
$$\mu_2\psi_{2N1} + \lambda_2\psi_{2N2} + \mu_2\psi_{2N3} = \beta_{2N2}$$
$$\mu_3\psi_{3N1} + \lambda_3\psi_{3N2} + \mu_3\psi_{3N3} = \beta_{3N2}$$
$$\dots$$
$$\mu_M\psi_{MN1} + \lambda_M\psi_{MN2} + \mu_M\psi_{MN3} = \beta_{MN2}$$

And the last equation of (3.20), i.e. $\Upsilon\Psi_{P-1} + \Lambda\Psi_P = \Gamma_P$ is written as

$$\mu_1\psi_{11(P-1)} + \lambda_1\psi_{11P} = \beta_{11P}$$
$$\mu_2\psi_{21(P-1)} + \lambda_2\psi_{21P} = \beta_{21P}$$
$$\mu_3\psi_{31(P-1)} + \lambda_3\psi_{31P} = \beta_{31P}$$
$$\dots$$
$$\mu_M\psi_{M1(P-1)} + \lambda_M\psi_{M1P} = \beta_{M1P} \tag{3.39c}$$
$$\mu_1\psi_{12(P-1)} + \lambda_1\psi_{12P} = \beta_{12P}$$
$$\mu_2\psi_{22(P-1)} + \lambda_2\psi_{22P} = \beta_{22P}$$
$$\mu_3\psi_{32(P-1)} + \lambda_3\psi_{32P} = \beta_{32P}$$
$$\dots$$
$$\mu_M\psi_{M2(P-1)} + \lambda_M\psi_{M2P} = \beta_{M2P}$$
$$\dots \quad \dots \quad \dots$$
$$\mu_1\psi_{1N(P-1)} + \lambda_1\psi_{1NP} = \beta_{1NP}$$
$$\mu_2\psi_{2N(P-1)} + \lambda_2\psi_{2NP} = \beta_{2NP}$$
$$\mu_3\psi_{3N(P-1)} + \lambda_3\psi_{3NP} = \beta_{3NP}$$
$$\dots$$
$$\mu_M\psi_{MN(P-1)} + \lambda_M\psi_{MNP} = \beta_{MNP}$$

From each set of equations of (3.39), we select the first equations and put together as one group of equations; again we take the second equations and put together as a second group of equations and so on till we get the last set of equations from each of (3.39a), (3.39b), …, (3.39c). In doing these we obtain the following sets of equations

$$
\begin{aligned}
\lambda_1\psi_{111} + \mu_1\psi_{112} &= \beta_{111} \\
\mu_1\psi_{111} + \lambda_1\psi_{112} + \mu_1\psi_{113} &= \beta_{112} \\
\mu_1\psi_{112} + \lambda_1\psi_{113} + \mu_1\psi_{114} &= \beta_{113} \\
&\cdots \\
\mu_1\psi_{11(P-1)} + \lambda_1\psi_{11P} &= \beta_{11P} \\
\lambda_2\psi_{211} + \mu_2\psi_{212} &= \beta_{211} \\
\mu_2\psi_{211} + \lambda_2\psi_{212} + \mu_2\psi_{213} &= \beta_{212} \\
\mu_2\psi_{212} + \lambda_2\psi_{213} + \mu_2\psi_{214} &= \beta_{213} \\
&\cdots \\
\mu_2\psi_{21(P-1)} + \lambda_2\psi_{21P} &= \beta_{21P} \\
\cdots \quad \cdots \quad &\cdots \\
\lambda_M\psi_{M11} + \mu_M\psi_{M12} &= \beta_{M11} \\
\mu_M\psi_{M11} + \lambda_M\psi_{M12} + \mu_M\psi_{M13} &= \beta_{M12} \\
\mu_M\psi_{M12} + \lambda_M\psi_{M13} + \mu_M\psi_{M14} &= \beta_{M13} \\
&\cdots \\
\mu_M\psi_{M1(P-1)} + \lambda_M\psi_{M1P} &= \beta_{M1P}
\end{aligned}
\tag{3.40}
$$

Observe that here in the above set of equations (3.40) $j = 1$, and for each $i = 1(1)M$ the coefficient matrix of the left hand side is a tri-diagonal matrix of order P and has the form

$$
\mathcal{M} = \begin{pmatrix}
\lambda_i & \mu_i & & & & \\
\mu_i & \lambda_i & \mu_i & & & \\
& \mu_i & \lambda_i & \mu_i & & \\
& & & \ddots & & \\
& & & \mu_i & \lambda_i & \mu_i \\
& & & & \mu_i & \lambda_i
\end{pmatrix}, \ i = 1(1)M
$$

Continuing for the other groups of equations as above from $j = 2, \ldots, N$, we get

$$\begin{aligned}
&\lambda_1\psi_{1j1} + \mu_1\psi_{1j2} = \beta_{1j1}\\
&\mu_1\psi_{1j1} + \lambda_1\psi_{1j2} + \mu_1\psi_{1j3} = \beta_{1j2}\\
&\mu_1\psi_{1j2} + \lambda_1\psi_{1j3} + \mu_1\psi_{1j4} = \beta_{1j3}\\
&\cdots\\
&\mu_1\psi_{1j(P-1)} + \lambda_1\psi_{1jP} = \beta_{1jP}\\
&\lambda_2\psi_{2j1} + \mu_1\psi_{2j2} = \beta_{2j1}\\
&\mu_1\psi_{2j1} + \lambda_2\psi_{2j2} + \mu_1\psi_{2j3} = \beta_{2j2}\\
&\mu_1\psi_{2j2} + \lambda_2\psi_{2j3} + \mu_1\psi_{2j4} = \beta_{2j3}\\
&\cdots\\
&\mu_1\psi_{2j(P-1)} + \lambda_2\psi_{2jP} = \beta_{2jP}\\
&\cdots \quad \cdots \quad \cdots\\
&\lambda_M\psi_{Mj1} + \mu_1\psi_{Mj2} = \beta_{Mj1}\\
&\mu_1\psi_{Mj1} + \lambda_M\psi_{Mj2} + \mu_1\psi_{Mj3} = \beta_{Mj2}\\
&\mu_1\psi_{Mj2} + \lambda_M\psi_{Mj3} + \mu_1\psi_{Mj4} = \beta_{Mj3}\\
&\cdots\\
&\mu_1\,\psi_{Mj(P-1)} + \lambda_M\psi_{MjP} = \beta_{MjP}
\end{aligned} \tag{3.41}$$

For each $j = 2, \dots, N$ the coefficients matrix of the left hand side of (3.41) is a tridiagonal matrix similar to that of the form for $j = 1$

$$\mathcal{M} = \begin{pmatrix} \lambda_i & \mu_i & & & & \\ \mu_i & \lambda_i & \mu_i & & & \\ & \mu_i & \lambda_i & \mu_i & & \\ & & & \ddots & & \\ & & & \mu_i & \lambda_i & \mu_i \\ & & & & \mu_i & \lambda_i \end{pmatrix}, \; i = 1(1)M,\; j = 2, \dots, N \text{ is of order } P. \tag{3.42}$$

Observe that (3.30) reduces to a diagonal matrix (3.42) for $i = 1,2,\dots,M$, $j = 1,2,\dots,N$ and $k = 1,2,\dots,P$. Thus we solve these sets of equations (3.42) for $\psi_{i,j,k}$ by the use of Thomas Algorithm [30]. Once after getting each $\psi_{i,j,k}$ (and hence Ψ_k) by the help of (3.37) we get $\mathbf{w}_k$ and again by the help of (3.31) we obtain $\mathbf{U}_k$ and this means that each $U_{i,j,k}$ are obtained. Thus, this solves our problem.

3.4 Numerical Results

For 19 and 27 points stencil schemes, computational experiment is done on six selected examples as a test problem in order to test the efficiency and adaptability of both methods. The results are reported in terms of maximum absolute error and the computed solution is found for the entire interior grid points but results are reported only at some mesh points. The results for these test problems are reported in Tables 3.1 to 3.6.

Example 3.1 Suppose $\nabla^2 U = 0$, $\quad 0 < x < 1, 0 < y < 1, 0 < z < 1$

with the boundary conditions

$$U(0,y,z) = U(x,0,z) = U(x,y,0) = U(x,y,1) = U(1,y,z) = U(x,1,z) = 1$$

The analytical solution is $U(x,y,z) = 1$ and its results are shown in Table 3.1

Example 3.2 Consider $\nabla^2 U = 0$, $\quad 0 < x < 1, 0 < y < 1, 0 < z < 1$

with the boundary conditions

$$U(0,y,z) = U(x,0,z) = U(x,y,0) = 0,$$
$$U(1,y,z) = yz, \quad U(x,1,z) = xz, \quad U(x,y,1) = xy.$$

The analytical solution is $U(x,y,z) = xyz$ and its results are shown in Table 3.2

Example 3.3 Suppose $\nabla^2 U = 6$, $0<x<1, 0<y<1, 0<z<1$

with the boundary conditions

$$U(0,y,z) = y^2 + z^2, \qquad U(x,0,z) = x^2 + z^2,$$

$$U(x,y,0) = x^2 + y^2, \qquad U(1,y,z) = 1 + y^2 + z^2,$$

$$U(x,1,z) = 1 + x^2 + z^2, \qquad U(x,y,1) = 1 + x^2 + y^2$$

The analytical solution is $U(x,y,z) = x^2 + y^2 + z^2$ and its results are shown in Table 3.3

Example 3.4 Suppose $\nabla^2 U = 2(xy + xz + yz)$, $0<x<1, 0<y<1, 0<z<1$

with the boundary conditions

$$U(0,y,z) = U(x,0,z) = U(x,y,0) = 0,$$

$$U(1,y,z) = yz(1+y+z),$$

$$U(x,1,z) = xz(1+x+z), \qquad U(x,y,1) = xy(1+x+y)$$

The analytical solution is $U(x,y,z) = xyz(x+y+z)$ and its results are shown in Table 3.4.

Example 3.5 Suppose $\nabla^2 U = -\pi^2 xy \sin(\pi z)$ $0<x<1, 0<y<1, 0<z<1$

with the boundary conditions

$$U(0,y,z) = U(x,0,z) = U(x,y,0) = U(x,y,1) = 0$$

$$U(1,y,z) = y\sin(\pi z), \quad \text{and}$$

$$U(x,1,z) = x\sin(\pi z)$$

The analytical solution is $U = xysin(\pi z)$ and its results are shown in Table 3.5

Example 3.6 Suppose $\nabla^2 U = -3\pi^2 \sin(\pi x)\sin(\pi y)\sin(\pi z)$

with boundary conditions

$$U(0,y,z) = U(1,y,z) = U(x,0,z) = U(x,1,z) = U(x,y,0) = U(x,y,1) = 0$$

The analytical solution is $U(x,y,z) = \sin(\pi x)\sin(\pi y)\sin(\pi z)$ and its results are shown in Table 3.6

Table 3.1

The maximum absolute error for example 3.1

(M, N, P)	***19-Points Scheme***	***27-Points Scheme***
(9,9,9)	1.99840e-015	1.55431e-015
(9,9,19)	1.55431e-015	2.22045e-015
(9,9,29)	1.90958e-014	7.88258e-015
(9,9,39)	5.10703e-015	5.66214e-015
(19,19,9)	7.54952e-015	9.54792e-015
(19,19,19)	1.55431e-015	1.19904e-014
(19,19,29)	6.43929e-015	2.15383e-014
(19,19,39)	7.21645e-015	234257e-014
(29,29,9)	1.69864e-014	1.11022e-014
(29,29,19)	9.43690e-015	4.66294e-015
(29,29,29)	1.63203e-014	6.66134e-015
(29,29,39)	3.96350e-014	8.43769e-015
(39,39,9)	6.43929e-015	5.77316e-015
(39,39,19)	2.33147e-014	2.88658e-015
(39,39,29)	4.46310e-014	1.62093e-014
(39,39,39)	3.29736e-014	1.39888e-014

Table 3.2

The maximum absolute error for example 3.2

(M, N, P)	*19-Points Scheme*	*27-Points Scheme*
(9,9,9)	5.55112e-016	5.55112e-016
(9,9,19)	7.77156e-016	5.55112e-016
(9,9,29)	2.83107e-015	1.30451e-015
(9,9,39)	1.72085e-015	1. 16573e-015
(19,19,9)	1.27676e-015	1.55431e-015
(19,19,19)	2.33147e-015	1.99840e-015
(19,19,29)	1.38778e-015	3.38618e-015
(19,19,39)	1.33227e-015	3.99680e-015
(29,29,9)	2.44249e-015	1.52656e-015
(29,29,19)	1.77636e-015	2.02616e-015
(29,29,29)	2.80331e-015	1.67921e-015
(29,29,39)	6.16174e-015	2.08167e-015
(39,39,9)	1.24900e-015	1.66533e-015
(39,39,19)	3.69149e-015	1.88738e-015
(39,39,29)	7.54952e-015	3.05311e-015
(39,39,39)	6.59195e-015	4.49640e-015

Table 3.3

The maximum absolute error for example 3.3

(M, N, P)	*19-Points Scheme*	*27-Points Scheme*
(9,9,9)	1.11022e-015	1.33227e-015
(9,9,19)	1.55431e-015	1.11022.e-015
(9,9,29)	5.13478e-015	2.88658e-015
(9,9,39)	3.83027e-015	2.88658e-015
(19,19,9)	2.77556e-015	3.05311e-015
(19,19,19)	4.77396e-015	4.10783e-015
(19,19,29)	3.55271e-015	6.71685e-015
(19,19,39)	3.10862e-015	7.99361e-015
(29,29,9)	4.71845e-015	3.10862e-015
(29,29,19)	3.94129e-015	4.44089e-015
(29,29,29)	5.27356e-014	4.44089e-015
(29,29,39)	1.24623e-014	4.44089e-015
(39,39,9)	3.10862e-015	4.44089e-015
(39,39,19)	7.82707e-015	4.99600e-015
(39,39,29)	1.49880e-014	6.32827e-015
(39,39,39)	1.37113e-014	1.03251e-014

Table 3.4

The maximum absolute error for example 3.4

(M, N, P)	*19-Points Scheme*	*27-Points Scheme*
(9,9,9)	2.55351e-015	1.77636e-015
(9,9,19)	3.10862e-015	2.22045e-015
(9,9,29)	1.74305e-014	7.43849e-015
(9,9,39)	6.88338e-015	5.88418e-015
(19,19,9)	7.54952e-015	9.32587e-015
(19,19,19)	1.44329e-014	1.28786e-015
(19,19,29)	6.99441e-015	2.14273e-014
(19,19,39)	7.77156e-015	2.28706e-014
(29,29,9)	1.48770e-014	9.99201e-015
(29,29,19)	9.32587e-015	8.43769e-015
(29,29,29)	1.58762e-014	8.88178e-015
(29,29,39)	3.67484e-014	1.02141e-014
(39,39,9)	7.32747e-015	7.43849e-015
(39,39,19)	2.17604e-014	7.10543e-015
(39,39,29)	4.39648e-014	1.78746e-014
(39,39,39)	3.39728e-014	1.79856e-014

Table 3.5

The maximum absolute error for example 3.5

(M, N, P)	*19-Points Scheme*	*27-Points Scheme*
(9,9,9)	5.89952e-006	5.90013e-006
(9,9,19)	3.67647e-007	3.67666e-007
(9,9,29)	7.25822e-008	7.25853e-008
(9,9,39)	2.29611e-008	2.2962e-008
(19,19,9)	5.92320e-006	5.9233e-006
(19,19,19)	3.69122e-007	3.69124e-007
(19,19,29)	7.28734e-008	7.28737e-008
(19,19,39)	2.30532e-008	2.30533e-008
(29,29,9)	5.94508e-006	5.94513e-006
(29,29,19)	3.70486e-007	3.70487e-007
(29,29,29)	7.31427e-008	7.31428e-008
(29,29,39)	2.31384e-008	2.31384e-008
(39,39,9)	5.94513e-006	5.94515e-006
(39,39,19)	3.70489e-007	3.70489e-007
(39,39,29)	7.31432e-008	7.31433e-008
(39,39,39)	2.31386e-008	2.31386e-008

Table 3.6

The maximum absolute error for example 3.6

(M,N,P)	*19-Points Scheme*	*27-Points Scheme*
(9,9,9)	4.07466e-005	9.56601e-005
(9,9,19)	2.80105e-005	3.9912e-005
(9,9,29)	2.73312e-005	2.79092e-005
(9,9,39)	2.72169e-005	2.35847e-005
(19,19,9)	1.52747e-005	1.01614e-005
(19,19,19)	2.53918e-006	5.93387e-006
(19,19,29)	1.85988e-006	3.47172e-006
(19,19,39)	1.74565e-006	2.48652e-006
(29,29,9)	1.3916e-005	3.32432e-006
(29,29,19)	1.18059e-006	1.88497e-006
(29,29,29)	5.01294e-007	1.17049e-006
(29,29,39)	3.87057e-007	7.96897e-007
(39,39,9)	1.36876e-005	7.87013e-006
(39,39,19)	9.52114e-007	6.34406e-007
(39,39,29)	2.72819e-007	5.30167e-007
(39,39,39)	1.58582e-007	3.70168e-007

This example 3.6 was considered as a test case in [29] and [104] and we have found that our scheme shows better results as compared to results in [29] and [104].For instance, when $h = \frac{1}{8}$ our result is 9.96995e-005 but in [104] it is 3.84e-003 and in [29] it is 1.015e-004.

3.5 Conclusion

In this work, the three dimensional Poisson's equation in Cartesian coordinates system is approximated by two different fourth order finite difference approximation schemes. Here we used to approximate the Poisson's equation by 19-points and 27- points stencil schemes. In doing this, by the very nature of the finite difference method for elliptic partial differential equations, it resulted in transforming the Poisson's equation in to a large number of algebraic systems of linear equations. In both schemes these systems of linear equations, then, arranged in order to get a block matrices; these block matrices are reduced to a block tridiagonal matrix by extending Hockney's method, and by the help of Thomas Algorithm [30] we obtained the solution of the system.

It is found that both fourth order approximations methods produce accurate results for the test problems. Actually it is shown that the discussed method, in general, for 27-points scheme produces better results (though the computational cost is high) than 19-points scheme but 19-points scheme has also shown comparable results to 27-points scheme.

The main advantage of these methods is that we have used a direct method to solve the Poisson's equation for which the error in the solution arises only from rounding off errors; and the methods allow considerable savings in computer storage as well as execution speed, that is it reduces the number of computations and computational time.

Therefore, this method is suitable to find the solution of any three dimensional Poisson's equation with the given boundary conditions in Cartesian coordinates system.

CHAPTER IV

Second Order Numerical Solution of the Three Dimensional Poisson's Equation in Cylindrical Coordinates System

4.1 Introduction

The three-dimensional Poisson's equation in cylindrical coordinates (r,θ,z) is given by

$$U_{rr}+\frac{1}{r}U_{r}+\frac{1}{r^{2}}U_{\theta\theta}+U_{zz}=f\left(r,\theta,z\right) \tag{4.1}$$

is often encountered in heat and mass transfer theory, fluid mechanics, elasticity, electrostatics, and other areas of mechanics and physics. In particular, the Poisson equation describes stationary temperature distribution in the presence of thermal sources or sinks in the domain under consideration.

To solve the two dimensional Poisson's equation in polar and cylindrical coordinates geometry, different approaches and numerical methods using finite difference approximation have been developed, for instance, *Chao* [21] developed a direct solver method for the electrostatic potential in a cylindrical region; *Chen* [32] a direct spectral collocation Poisson solver for several different domains including a part of a disk, an annulus, a unit disk, and a cylinder using the weighted interpolation technique and non classical collocation points; *Christopher* [36] a solution method in an annulus using conformal mapping and Fast Fourier Transform; *Kalita and Ray* [39] have developed a high order compact scheme on a circular cylinder to solve their problem on incompressible viscous flows; *Lai and Wang* [65] a fast direct solvers for Poisson's equation on 2D polar and spherical coordinates based on FFT; *Swarztrauber and Sweet* [85] a direct solution of the discrete Poisson equation on a disk in the sense of least squares; *Mittal and Gahlaut* [89] a boundary integral formulation in polar coordinates using conformal mapping and four point Gaussian quadrature formula, and other several attempts have been made to solve the two dimensional Poisson equation in particular for physical problems that are related directly or indirectly to this equation.

The analytic solution for the three dimensional Poisson's equation in cylindrical coordinate system is much more complicated and tedious because of the complexity of the nature of the problems and their geometry, and the availability of methods. Some contributions have been done to solve the three dimensional Poisson equation, for instance, *Tan* [14] developed a spectrally accurate solution for the three dimensional Poisson's equation and Helmholtz's equation using Chebyshev series and Fourier series for a simple domain in a cylindrical coordinate system; *Iyengar and Goyal* [101] developed a multigrid method in cylindrical coordinates system; *Lai and Tseng* [64] have developed a fourth-order compact scheme, and their scheme relies on the truncated Fourier series expansion, where the partial differential equations of Fourier coefficients are solved by a formally fourth-order accurate compact difference discretization; *Xu et al* [106] developed a parallel three-dimensional Poisson solver in cylindrical coordinate system for the electrostatic potential of a charged particle beam in a circular, which uses Fourier expansions in the longitudinal and azimuthal directions, and Spectral Element discretization in the radial direction, and some other developments have also been observed. The need to obtain the best solution for the Poisson's equation is still in progress.

In this chapter, we develop a second order finite difference approximation scheme and solve the resulting large algebraic system of linear equations systematically using block tridiagonal system [60] and extend the Hockney's method [96] to solve the three dimensional Poisson's equation on Cylindrical coordinates system.

4.2 Finite Difference Approximation

Consider the three dimensional Poisson's equation in cylindrical coordinates (r, θ, z) (4.1) given by

$$\frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r} + \frac{1}{r^2}\frac{\partial^2 U}{\partial \theta^2} + \frac{\partial^2 U}{\partial z^2} = f(r,\theta,z) \quad on\ D$$

and the boundary condition

$$U(r,\theta,z) = g(r,\theta,z) \quad \text{on } C \tag{4.2}$$

where C is the boundary of D and

D is a) $D_1 = \{(r,\theta,z) : R_0 < r < R_1 \ , \ a < z < b, \theta_0 < \theta < \theta_1 \ , \ \theta_0 < \theta_1 < 2\pi\}$

and b) $D_2 = \{(r,\theta,z) : R_0 < r < R_1 \ , \ a < z < b, \ 0 \le \theta < 2\pi\}$

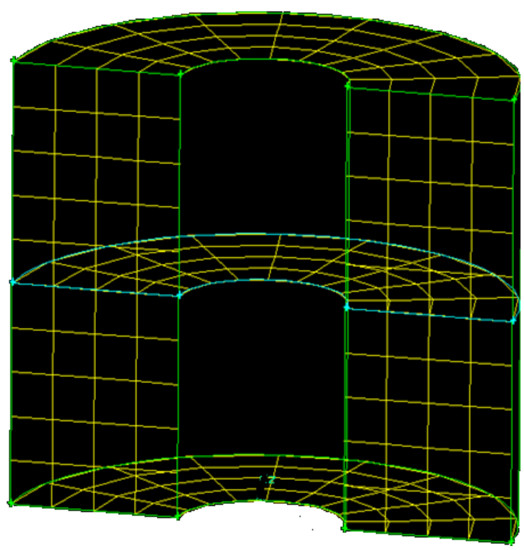

Figure 4.1
Portion of a cylindrical grid

Consider figure 4.1 as the geometry of the problem.

Let $U(r,\theta,z)$ be discretized at the point (r_i,θ_j,z_k) and for simplicity write a point (r_i,θ_j,z_k) as (i,j,k) and $U(r_i,\theta_j,z_k)$ as $U_{i,j,k}$.

Assume that there are M points along r, N points along θ and P points along the z directions to form the mesh, and let the step size along the direction of r be Δr, along the direction of θ be $\Delta\theta$ and along the direction of z be Δz.

Here $r_i = R_0 + i\Delta r,\ \theta_j = \theta_0 + j\Delta\theta$ and $z_k = a + k\Delta z$ where $i = 1,2,\dots,M$, $j = 1,2,\dots,N$ and $k = 1,2,\dots,P$.

When $r = 0$ is an interior or boundary point in (4.1), the Poisson's equation becomes singular. To avoid the singularity in (4.1) we take the condition $\frac{\partial U}{\partial r} = 0$ (this is the essential condition), thus the limiting value of $\frac{1}{r}\frac{\partial U}{\partial r}$ as $r \to 0$ (applying L'hopital's rule) is the value of $\frac{\partial^2 U}{\partial r^2}$ at $r = 0$. The second order finite difference method to solve Poisson's equation under this condition is well explained in the paper by Mittal and Gahlaut [91]. As a result of this we adopt a new approximation scheme in Chapter VI.

Thus we discuss in this chapter only for the case $r \neq 0$.

Using the central difference scheme

$$\frac{\partial U}{\partial r} = \frac{U_{i+1,j,k} - U_{i-1,j,k}}{2\Delta r} + O(\Delta r) \quad ,$$

$$\frac{\partial^2 U}{\partial r^2} = \frac{U_{i+1,j,k} - 2U_{i,j,k} + U_{i-1,j,k}}{(\Delta r)^2} + O\left((\Delta r)^2\right)$$

$$\frac{\partial^2 U}{\partial \theta^2} = \frac{U_{i,j+1,k} - 2U_{i,j,k} + U_{i,j-1,k}}{(\Delta\theta)^2} + O\left((\Delta\theta)^2\right)$$

$$\frac{\partial^2 U}{\partial z^2} = \frac{U_{i,j,k+1} - 2U_{i,j,k} + U_{i,j,k-1}}{(\Delta z)^2} + O\left((\Delta z)^2\right) \tag{4.3}$$

Substituting (4.3) in (4.1), we get

$$f_{i,j,k} = \frac{U_{i+1,j,k} - 2U_{i,j,k} + U_{i-1,j,k}}{(\Delta r)^2} + \frac{U_{i+1,j,k} - U_{i-1,j,k}}{(2\Delta r)\ r_i} + \frac{1}{r_i^2}\left(\frac{U_{i,j+1,k} - 2U_{i,j,k} + U_{i,j-1,k}}{(\Delta\theta)^2}\right)$$
$$+ \frac{U_{i,j,k+1} - 2U_{i,j,k} + U_{i,j,k-1}}{(\Delta z)^2} + O\left((\Delta r)^2\right) + O\left((\Delta\theta)^2\right) + O\left((\Delta z)^2\right) \tag{4.4}$$

and truncating higher order differences of (4.4), we have

$$f_{i,j,k} = \frac{U_{i+1,j,k} - 2U_{i,j,k} + U_{i-1,j,k}}{(\Delta r)^2} + \frac{U_{i+1,j,k} - U_{i-1,j,k}}{(2\Delta r)\, r_i} + \frac{1}{r_i^2}\left(\frac{U_{i,j+1,k} - 2U_{i,j,k} + U_{i,j-1,k}}{(\Delta \theta)^2}\right) + \frac{U_{i,j,k+1} - 2U_{i,j,k} + U_{i,j,k-1}}{(\Delta z)^2} \tag{4.5}$$

Let $\omega_i = \dfrac{\Delta r}{2r_i}$, $\alpha_i = \dfrac{(\Delta r)^2}{r_i^2(\Delta\theta)^2}$, $\rho = \dfrac{(\Delta r)^2}{(\Delta z)^2}$ and $y_i = -2(1+\alpha_i+\rho)$

Multiplying both sides of (4.5) by $(\Delta r)^2$ and rearranging and simplifying further, we get

$$(1+\omega_i)U_{i+1,j,k} + (1-\omega_i)U_{i-1,j,k} + \alpha_i(U_{i,j+1,k} + U_{i,j-1,k}) + \rho(U_{i,j,k+1} + U_{i,j,k-1}) + y_i U_{i,j,k} = (\Delta r)^2 f_{i,j,k} \tag{4.6}$$

When there are two or more space dimensions the band width is larger and the number of operations goes up and thus the computation for the solution is not such an easy task. Thus our aim now is to solve (4.6) systematically.

The system of equations in (4.6) is a linear sparse system, and if we solve this sparse system technically we save both work and storage compared with a general system of equations. Such savings are basically true of finite difference methods: they yield sparse systems because each equation involves only a few variables. Now we use these advantages.

Consider equation (4.6) first in the θ direction, next in the z direction and lastly in the r direction, and hence equation (4.5) can be put in matrix form as

$$A\mathbf{U} = \mathcal{B} \tag{4.7}$$

where $A = \begin{pmatrix} R_1 & S_1 & & & & & \\ S_2^* & R_2 & S_2 & & & & \\ & S_3^* & R_3 & S_3 & & & \\ & & & \ddots & & & \\ & & & & S_{M-1}^* & R_{M-1} & S_{M-1} \\ & & & & & S_M^* & R_M \end{pmatrix}$, it has M blocks and each block is of order NP by NP

$$R_i = \begin{pmatrix} L_i & \mathrm{T} & & & & & \\ \mathrm{T} & L_i & \mathrm{T} & & & & \\ & \mathrm{T} & L_i & \mathrm{T} & & & \\ & & & \ddots & & & \\ & & & & \mathrm{T} & L_i & \mathrm{T} \\ & & & & & \mathrm{T} & L_i \end{pmatrix} \quad \text{is of order } NP \text{ by } NP \tag{4.8}$$

For domain D_1,

$$L_i = \begin{pmatrix} y_i & \alpha_i & & & & & \\ \alpha_i & y_i & \alpha_i & & & & \\ & \alpha_i & y_i & \alpha_i & & & \\ & & & \ddots & & & \\ & & & & \alpha_i & y_i & \alpha_i \\ & & & & & \alpha_i & y_i \end{pmatrix} \tag{4.9}$$

and **for D_2**,

$$L_i = \begin{pmatrix} y_i & \alpha_i & & & & & \alpha_i \\ \alpha_i & y_i & \alpha_i & & & & \\ & \alpha_i & y_i & \alpha_i & & & \\ & & & \ddots & & & \\ & & & & \alpha_i & y_i & \alpha_i \\ \alpha_i & & & & & \alpha_i & y_i \end{pmatrix} \text{is a circulant matrix,} \tag{4.10}$$

in both (4.9) and (4.10) L_i is of order N.

and $\mathrm{T} = diag(\rho, \rho, \rho, \ldots, \rho)$ is of order N.

$S_i = diag(\boldsymbol{\omega}_i, \boldsymbol{\omega}_i, \boldsymbol{\omega}_i, \ldots, \boldsymbol{\omega}_i)$ has P blocks and $\boldsymbol{\omega}_i = diag(1+\omega_i, 1+\omega_i, ..., 1+\omega_i)$ is of order N by N

$S_i^* = diag(\boldsymbol{\varphi}_i, \boldsymbol{\varphi}_i, \boldsymbol{\varphi}_i, \ldots, \boldsymbol{\varphi}_i)$ has P blocks and $\boldsymbol{\varphi}_i = diag(1-\omega_i, 1-\omega_i, ..., 1-\omega_i)$ is of order N by N.

$\mathcal{B} = \begin{bmatrix}\mathbf{B}_1 & \mathbf{B}_2 & \mathbf{B}_3 & \dots & \mathbf{B}_M\end{bmatrix}^T$, $\mathbf{B}_k = [\mathbf{d}_{1k}\ \mathbf{d}_{2k} \dots \mathbf{d}_{Nk}]^T$ and $\mathbf{d}_{jk} = [\ d_{1jk}\ d_{2jk} \dots d_{Mjk}]^T$

such that each d_{ijk} represents a known boundary values of U and values of f, and

$\mathbf{U} = \begin{bmatrix}\mathbf{U}_1 & \mathbf{U}_2 & \mathbf{U}_3 & \dots & \mathbf{U}_M\end{bmatrix}^T$, and $\mathbf{U}_i = \begin{pmatrix}U_{i1} & U_{i2} & U_{i3} & \cdots & U_{iP}\end{pmatrix}^T$ and

$U_{ik} = \begin{pmatrix}U_{ij1} & U_{ij2} & U_{ij3} & \cdots & U_{ijP}\end{pmatrix}^T$

We write (4.7) as

$$\begin{aligned} R_1\mathbf{U}_1 + S_1\mathbf{U}_2 &= \mathbf{B}_1 \\ S^*_2\,\mathbf{U}_1 + R_2\mathbf{U}_2 + S_2\mathbf{U}_3 &= \mathbf{B}_2 \\ S^*_3\,\mathbf{U}_2 + R_3\mathbf{U}_3 + S_3\mathbf{U}_4 &= \mathbf{B}_3 \\ &\dots \\ S^*_M\,\mathbf{U}_{M-1} + R_M\mathbf{U}_M &= \mathbf{B}_M \end{aligned} \tag{4.11}$$

4.3 *Extended Hockney's Method*

Observe that the matrix L_i is a real symmetric matrix and hence its eigenvalues and eigenvectors can easily be obtained. [30]

$$\eta_{i\,j} = y_i + 2\alpha_i \cos\left(\frac{j\pi}{N+1}\right) \quad i = 1,2,3,\dots,M \text{ and } j = 1,2,3,\dots,N \text{ for } D_1$$

and $$\eta_{i\,j} = y_i + 2\alpha_i \cos\left(\frac{2\pi j}{N}\right) \quad i = 1,2,3,\dots,M \text{ and } j = 1,2,3,\dots,N \text{ for } D_2.$$

Let $\mathbf{q}_j$ be an eigenvector of L_i corresponding to the eigenvalue η_{ij} and Q be the matrix $[\mathbf{q}_1\,\mathbf{q}_2\ \mathbf{q}_3 \dots \mathbf{q}_n]^T$ be a modal matrix of L_i , $\forall i$ such that

$$QQ^T = I \text{ and}$$

$$Q^T L_i Q = diag\left(\eta_{i1}, \eta_{i2}, \eta_{i3}, \dots, \eta_{iN}\right) = \mathrm{E}_i \text{ (say)}$$

The $N \times N$ modal matrix Q is defined by

$$q_{i,j} = \sqrt{\frac{2}{N+1}} \sin\left(\frac{ij\pi}{N+1}\right) \quad i = 1,2,3,\dots,N \text{ and } j = 1,2,3,\dots,N \text{ for } D_1.$$

$$q_{i,j} = \frac{(\cos\theta + \sin\theta)}{\sqrt{N}}, \quad \text{where } \theta = \left(\frac{2\pi}{N}\right)(i-1)(j-1) ,$$

$i = 1, 2, 3, \ldots, N$ and $j = 1, 2, 3, \ldots, N$ for D_2

Let $\mathbb{Q} = diag(Q, Q, \ldots, Q)$ be a matrix of order NP by NP.

Thus $\mathbb{Q}$ satisfy the property that $\mathbb{Q}^T\mathbb{Q} = I$ since $Q^TQ = I$ and

$$\mathbb{Q}^T R_i \mathbb{Q} = \begin{pmatrix} \mathrm{E}_i & \mathrm{T} & & & \\ \mathrm{T} & \mathrm{E}_i & \mathrm{T} & & \\ & \mathrm{T} & \mathrm{E}_i & \mathrm{T} & \\ & & & \ddots & \\ & & & \mathrm{T} & \mathrm{E}_i \end{pmatrix} = R_i^* \tag{4.12}$$

$\mathbb{Q}^T S_i \mathbb{Q} = S_i$ and $\mathbb{Q}^T S_i^* \mathbb{Q} = S_i^*$ since both S_i and S^*_i are diagonal matrices.

Let $\quad \mathbb{Q}^T \mathbf{U}_i = \boldsymbol{V}_i \Rightarrow \mathbf{U}_i = \mathbb{Q}\boldsymbol{V}_i$

$$\mathbb{Q}^T \mathbf{B}_i = B_i \Rightarrow \mathbf{B}_i = \mathbb{Q}B_i \tag{4.13}$$

where $\boldsymbol{V}_i = [V_{i1}\ V_{i2}\ \ V_{i3} \ldots V_{iP}]^T$ and $V_{ik} = [v_{i1k}\ \ v_{i2k}\ \ v_{i3k} \ldots v_{iNk}]^T$

$\mathbf{B}_i = [B_{i1}\ B_{i2} \ldots \mathrm{B}_{ik}]^T$ and $B_{ik} = [\, b_{i1k}\ b_{i2k} \ldots b_{iNk}]^T$

Pre-multiplying equation (4.11) by $\mathbb{Q}^T$ and applying (4.13), we get

$$\begin{aligned} R^*_1\boldsymbol{V}_1 + S_1\boldsymbol{V}_2 &= B_1 \\ S^*_2\boldsymbol{V}_1 + R^*_2\boldsymbol{V}_2 + S_2\boldsymbol{V}_3 &= B_2 \\ S^*_3\boldsymbol{V}_2 + R^*_3\boldsymbol{V}_3 + S_3\boldsymbol{V}_4 &= B_3 \\ &\ldots \\ S^*_M\boldsymbol{V}_{M-1} + R^*_{\mathrm{M}}\boldsymbol{V}_M &= B_M \end{aligned} \tag{4.14}$$

Consider the first equation of (4.14) i.e. $R^*_1\boldsymbol{V}_1 + S_1\boldsymbol{V}_2 = B_1$ and write this equation as

$$\begin{pmatrix} E_1 & T & & & \\ T & E_1 & T & & \\ & T & E_1 & T & \\ & & & \ddots & \\ & & & T & E_1 \end{pmatrix} \begin{pmatrix} \mathbf{v}_{11} \\ \mathbf{v}_{12} \\ \mathbf{v}_{13} \\ \vdots \\ \mathbf{v}_{1P} \end{pmatrix} + \begin{pmatrix} \boldsymbol{\omega}_1 & & & \\ & \boldsymbol{\omega}_1 & & \\ & & \boldsymbol{\omega}_1 & \\ & & & \ddots & \\ & & & & \boldsymbol{\omega}_1 \end{pmatrix} \begin{pmatrix} \mathbf{v}_{21} \\ \mathbf{v}_{22} \\ \mathbf{v}_{23} \\ \vdots \\ \mathbf{v}_{2P} \end{pmatrix} = \begin{pmatrix} \mathbf{b}_{11} \\ \mathbf{b}_{12} \\ \mathbf{b}_{13} \\ \vdots \\ \mathbf{b}_{1P} \end{pmatrix} \quad (4.15)$$

Again we write (4.15) as

$$\begin{aligned} E_1\mathbf{v}_{11} + T\mathbf{v}_{12} + \quad \boldsymbol{\omega}_1\mathbf{v}_{21} &= \mathbf{b}_{11} \\ T\mathbf{v}_{11} + E_1\mathbf{v}_{12} + T\mathbf{v}_{13} + \boldsymbol{\omega}_1\mathbf{v}_{22} &= \mathbf{b}_{12} \\ T\mathbf{v}_{12} + E_1\mathbf{v}_{13} + T\mathbf{v}_{14} + \boldsymbol{\omega}_1\mathbf{v}_{23} &= \mathbf{b}_{13} \\ \cdots \\ T\mathbf{v}_{1(P-1)} + E_1\mathbf{v}_{1P} + \boldsymbol{\omega}_1\mathbf{v}_{2P} &= \mathbf{b}_{1P} \end{aligned} \quad (4.16)$$

Now from each equation of (4.16) we collect the first equations and put as one group of equation

$$\begin{aligned} \eta_{11}v_{111} + \rho v_{112} + (1+\omega_1)v_{211} &= b_{111} \\ \rho v_{111} + \eta_{11}v_{112} + \rho v_{113} + (1+\omega_1)v_{212} &= b_{112} \\ \rho v_{112} + \eta_{11}v_{113} + \rho v_{114} + (1+\omega_1)v_{213} &= b_{113} \\ \cdots \\ \rho v_{11(P-1)} + \eta_{11}v_{11P} + (1+\omega_1)v_{21P} &= b_{11P} \end{aligned} \quad (4.17a)$$

Once again collect the second equations of (4.16) and put them as other group of equation

$$\begin{aligned} \eta_{12}v_{121} + \rho v_{122} + (1+\omega_1)v_{221} &= b_{121} \\ \rho v_{121} + \eta_{12}v_{122} + \rho v_{123} + (1+\omega_1)v_{222} &= b_{122} \\ \rho v_{122} + \eta_{12}v_{123} + \rho v_{124} + (1+\omega_1)v_{223} &= b_{123} \\ \cdots \\ \rho v_{12(P-1)} + \eta_{12}v_{12P} + (1+\omega_1)v_{22P} &= b_{12P} \end{aligned} \quad (4.17b)$$

And lastly collect the last equations of (4.16) and put them altogether to form one group of equation

$$\eta_{1N}v_{1N1}+\rho v_{1N2}+(1+\omega_1)v_{2N1}=b_{1N1}$$

$$\rho v_{1N1}+\eta_{1N}v_{1N2}+\rho v_{1N3}+(1+\omega_1)v_{2N2}=b_{1N2}$$

$$\rho v_{1N2}+\eta_{1N}v_{1N3}+\rho v_{1N4}+(1+\omega_1)v_{2N3}=b_{1N3} \tag{4.17c}$$

$$\cdots$$

$$\rho v_{1N(P-1)}+\eta_{1N}v_{1NP}+(1+\omega_1)v_{2NP}=b_{1NP}$$

The equations (4.17a) to (4.17c) can again be written as

$$\begin{pmatrix} \eta_{1j} & \rho & & & \\ \rho & \eta_{1j} & \rho & & \\ & \rho & \eta_{1j} & \rho & \\ & & & \ddots & \\ & & & \rho & \eta_{1j} \end{pmatrix}\begin{pmatrix} v_{1j1} \\ v_{1j2} \\ v_{1j3} \\ \vdots \\ v_{1jP} \end{pmatrix}+\begin{pmatrix} 1+\omega_1 & & & & \\ & 1+\omega_1 & & & \\ & & 1+\omega_1 & & \\ & & & \ddots & \\ & & & & 1+\omega_1 \end{pmatrix}\begin{pmatrix} v_{2j1} \\ v_{2j2} \\ v_{2j3} \\ \vdots \\ v_{2jP} \end{pmatrix}=\begin{pmatrix} b_{1j1} \\ b_{1j2} \\ b_{1j3} \\ \vdots \\ b_{1jP} \end{pmatrix}$$

$$j=1,2,3,\ldots,N \tag{4.18}$$

Let $\mathcal{F}_{1j}=\begin{pmatrix} \eta_{1j} & \rho & & & & \\ \rho & \eta_{1j} & \rho & & & \\ & \rho & \eta_{1j} & \rho & & \\ & & & \ddots & & \\ & & & \rho & \eta_{1j} & \rho \\ & & & & \rho & \eta_{1j} \end{pmatrix}$, $\mathbf{W}_{ij}=\begin{pmatrix} v_{ij1} \\ v_{ij2} \\ v_{ij3} \\ \vdots \\ v_{ijP} \end{pmatrix}$ and $\bar{\mathbf{B}}_{ij}=\begin{pmatrix} b_{ij1} \\ b_{ij2} \\ b_{ij3} \\ \vdots \\ b_{ijP} \end{pmatrix}$

Thus the equations (4.17a) to (4.17c) can be written as $\mathcal{F}_{1j}\mathbf{W}_{1j}+\boldsymbol{\omega}_1\mathbf{W}_{2j}=\bar{\mathbf{B}}_{1j}$.

Let $\mathcal{F}=\begin{pmatrix} \mathcal{F}_{ij} & & & & \\ & \mathcal{F}_{ij} & & & \\ & & \mathcal{F}_{ij} & & \\ & & & \ddots & \\ & & & & \mathcal{F}_{ij} \end{pmatrix}$ is of order NP

$\mathbf{w}_i=[\mathbf{W}_{i1}\ \mathbf{W}_{i2}\ \mathbf{W}_{i3}\ \ldots\ \mathbf{W}_{iN}]^T$ and $\bar{B}_i=[\bar{\mathbf{B}}_{i1},\bar{\mathbf{B}}_{i2},\cdots,\bar{\mathbf{B}}_{iN}]^T$

Thus the first equation of (4.14) can be written as

$$\mathcal{F}\mathbf{w}_1+S_1\mathbf{w}_2=\bar{\mathrm{B}}_1$$

By a similar procedure we can write for the second, third and last equations of (4.14) as the first equation, for instance the i[th] equation of (4.14) can be written as

$$S^*_i\,\mathbf{w}_{i-1} + \mathcal{F}\mathbf{w}_i + S_i\mathbf{w}_{i+1} = \bar{\mathrm{B}}_i$$

Hence we can write (4.14) as

$$\begin{aligned}
\mathcal{F}\mathbf{w}_1 + S_1\mathbf{w}_2 &= \bar{\mathrm{B}}_1 \\
S^*_2\mathbf{w}_1 + \mathcal{F}\mathbf{w}_2 + S_2\mathbf{w}_3 &= \bar{\mathrm{B}}_2 \\
S^*_3\,\mathbf{w}_2 + \mathcal{F}\mathbf{w}_3 + S\mathbf{w}_4 &= \bar{\mathrm{B}}_3 \\
&\dots \\
S^*_M\,\mathbf{w}_{M-1} + \mathcal{F}\mathbf{w}_M &= \bar{\mathrm{B}}_M
\end{aligned} \tag{4.19}$$

Observe that

$\mathbb{Q}^T\mathcal{F}\mathbb{Q} = diag(\Phi_{i1}, \Phi_{i2}, \Phi_{i3}, ..., \Phi_{iP}) = \Lambda_i$ (say) where $\Phi_{ik} = diag(\lambda_{i1k}, \lambda_{i2k}, \lambda_{i3k}, ..., \lambda_{iNk})$

Here $\lambda_{ijk} = \eta_{ij} + 2\rho\cos\left(\dfrac{k\pi}{P+1}\right), i = 1, 2, 3, \dots, M,\ j = 1, 2, 3, \dots N$ and $k = 1, 2, 3, \dots,, P$

Let $\mathbb{Q}^T\mathbf{w}_i = \Psi_i \Rightarrow \mathbf{w}_i = \mathbb{Q}\,\Psi_i$

$$\mathbb{Q}^T\bar{B}_i = \Gamma_i \Rightarrow \bar{B}_i = \mathbb{Q}\Gamma_i \tag{4.20}$$

where $\Psi_i = [\mathbf{\Psi}_{i1}\ \mathbf{\Psi}_{i2}\ \mathbf{\Psi}_{i3} ...\ \mathbf{\Psi}_{iP}]^T$ and $\mathbf{\Psi}_{ik} = [\psi_{i1k}\ \ \psi_{i2k}\ \ \psi_{i3k}\ \dots\ \psi_{iNk}]^T$

$\Gamma_i = [\boldsymbol{\beta}_{i1}, \boldsymbol{\beta}_{i2}, \boldsymbol{\beta}_{i3}, ..., \boldsymbol{\beta}_{iP}]^T$ and $\boldsymbol{\beta}_{ik} = [\beta_{i1k}, \beta_{i2k}, ..., \beta_{iNk}]^T$

Pre-multiplying (4.19) by $\mathbb{Q}^T$ and make use of (4.20), we get

$$\begin{aligned}
\Lambda_1\Psi_1 + S_1\Psi_2 &= \Gamma_1 \\
S^*_2\,\Psi_1 + \Lambda_2\Psi_2 + S_2\Psi_3 &= \Gamma_2 \\
S^*_3\,\Psi_2 + \Lambda_3\Psi_3 + S_3\Psi_4 &= \Gamma_3 \\
&\dots \\
S^*_M\,\Psi_{M-1} + \Lambda_M\Psi_M &= \Gamma_M
\end{aligned} \tag{4.21}$$

Now we write (4.21) turn by turn starting from the first equation i.e. $\Lambda_1\Psi_1 + S_1\Psi_2 = \Gamma_1$ as

$$\lambda_{111}\psi_{111} + (1+\omega_1)\psi_{211} = \beta_{111}$$
$$\lambda_{121}\psi_{121} + (1+\omega_1)\psi_{221} = \beta_{121}$$
$$\lambda_{131}\psi_{131} + (1+\omega_1)\psi_{231} = \beta_{131}$$
$$\cdots$$
$$\lambda_{1N1}\psi_{1N1} + (1+\omega_1)\psi_{2N1} = \beta_{1N1}$$
$$\lambda_{112}\psi_{112} + (1+\omega_1)\psi_{212} = \beta_{112} \qquad (4.22a)$$
$$\lambda_{122}\psi_{122} + (1+\omega_1)\psi_{222} = \beta_{122}$$
$$\lambda_{132}\psi_{132} + (1+\omega_1)\psi_{232} = \beta_{132}$$
$$\cdots$$
$$\lambda_{1N2}\psi_{1N2} + (1+\omega_1)\psi_{2N2} = \beta_{1N2}$$
$$\cdots \qquad \cdots \qquad \cdots$$
$$\lambda_{11P}\psi_{11P} + (1+\omega_1)\psi_{21P} = \beta_{11P}$$
$$\lambda_{12P}\psi_{12P} + (1+\omega_1)\psi_{22P} = \beta_{12P}$$
$$\lambda_{13P}\psi_{13P} + (1+\omega_1)\psi_{23P} = \beta_{13P}$$
$$\cdots$$
$$\lambda_{1NP}\psi_{1NP} + (1+\omega_1)\psi_{2NP} = \beta_{1NP}$$

The second equation of (4.21) i.e. $S^{*}_{2}\Psi_1 + \Lambda_2\Psi_2 + S_2\Psi_3 = \Gamma_2$ is written as

$$(1-\omega_2)\psi_{111} + \lambda_{211}\psi_{211} + (1+\omega_2)\psi_{311} = \beta_{211}$$
$$(1-\omega_2)\psi_{121} + \lambda_{221}\psi_{221} + (1+\omega_2)\psi_{321} = \beta_{221}$$
$$(1-\omega_2)\psi_{131} + \lambda_{231}\psi_{231} + (1+\omega_2)\psi_{331} = \beta_{231}$$
$$\cdots$$
$$(1-\omega_2)\psi_{1N1} + \lambda_{2N1}\psi_{2N1} + (1+\omega_2)\psi_{3N1} = \beta_{2N1}$$
$$(1-\omega_2)\psi_{112} + \lambda_{212}\psi_{212} + (1+\omega_2)\psi_{312} = \beta_{212} \qquad (4.22b)$$
$$(1-\omega_2)\psi_{122} + \lambda_{222}\psi_{222} + (1+\omega_2)\psi_{322} = \beta_{222}$$
$$(1-\omega_2)\psi_{132} + \lambda_{232}\psi_{232} + (1+\omega_2)\psi_{332} = \beta_{232}$$
$$\cdots$$
$$(1-\omega_2)\psi_{1N2} + \lambda_{2N2}\psi_{2N2} + (1+\omega_2)\psi_{3N2} = \beta_{2N2}$$
$$\cdots \qquad \cdots \qquad \cdots$$

$$(1-\omega_2)\psi_{11P} + \lambda_{21P}\psi_{21P} + (1+\omega_2)\psi_{31P} = \beta_{21P}$$
$$(1-\omega_2)\psi_{12P} + \lambda_{22P}\psi_{22P} + (1+\omega_2)\psi_{32P} = \beta_{22P}$$
$$(1-\omega_2)\psi_{13P} + \lambda_{23P}\psi_{23P} + (1+\omega_2)\psi_{33P} = \beta_{23P}$$
$$\cdots$$
$$(1-\omega_2)\psi_{1NP} + \lambda_{2NP}\psi_{2NP} + (1+\omega_2)\psi_{3NP} = \beta_{2NP}$$

And the last equation of (4.21) i.e. $S^*_M \Psi_{M-1} + \Lambda_M \Psi_M = \Gamma_M$ is written as

$$(1-\omega_M)\psi_{(M-1)11} + \lambda_{M11}\psi_{M11} = \beta_{M11}$$
$$(1-\omega_M)\psi_{(M-1)21} + \lambda_{M21}\psi_{M21} = \beta_{M21}$$
$$(1-\omega_M)\psi_{(M-1)31} + \lambda_{M31}\psi_{M11} = \beta_{M31}$$
$$\cdots$$
$$(1-\omega_M)\psi_{(M-1)N1} + \lambda_{MN1}\psi_{MN1} = \beta_{MN1}$$
$$(1-\omega_M)\psi_{(M-1)12} + \lambda_{M12}\psi_{M12} = \beta_{M12} \tag{4.22c}$$
$$(1-\omega_M)\psi_{(M-1)22} + \lambda_{M22}\psi_{M22} = \beta_{M22}$$
$$(1-\omega_M)\psi_{(M-1)32} + \lambda_{M32}\psi_{M12} = \beta_{M32}$$
$$\cdots$$
$$(1-\omega_M)\psi_{(M-1)N2} + \lambda_{MN2}\psi_{MN2} = \beta_{MN2}$$
$$\cdots \quad \cdots \quad \cdots$$
$$(1-\omega_M)\psi_{(M-1)1P} + \lambda_{M1P}\psi_{M1P} = \beta_{M1P}$$
$$(1-\omega_M)\psi_{(M-1)2P} + \lambda_{M2P}\psi_{M1P} = \beta_{M2P}$$
$$(1-\omega_M)\psi_{(M-1)3P} + \lambda_{M3P}\psi_{M1P} = \beta_{M3P}$$
$$\cdots$$
$$(1-\omega_M)\psi_{(M-1)NP} + \lambda_{MNP}\psi_{MNP} = \beta_{MNP}$$

Now take the first equations of (4.22a) to (4.22c) , and get

$$\lambda_{111}\psi_{111} + (1+\omega_1)\psi_{211} = \beta_{111}$$
$$(1-\omega_2)\psi_{111} + \lambda_{211}\psi_{211} + (1+\omega_2)\psi_{311} = \beta_{211}$$
$$(1-\omega_3)\psi_{211} + \lambda_{311}\psi_{311} + (1+\omega_3)\psi_{411} = \beta_{311}$$
$$\cdots$$
$$(1-\omega_M)\psi_{(M-1)11} + \lambda_{M11}\psi_{M11} = \beta_{M11}$$

$$\lambda_{112}\psi_{112} + (1+\omega_1)\psi_{212} = \beta_{112} \tag{4.23a}$$
$$(1-\omega_2)\psi_{112} + \lambda_{212}\psi_{212} + (1+\omega_2)\psi_{312} = \beta_{212}$$
$$(1-\omega_3)\psi_{212} + \lambda_{312}\psi_{312} + (1+\omega_3)\psi_{412} = \beta_{312}$$
$$\dots$$
$$(1-\omega_M)\psi_{(M-1)12} + \lambda_{M12}\psi_{M12} = \beta_{M12}$$
$$\dots \quad \dots \quad \dots$$
$$\lambda_{11P}\psi_{11P} + (1+\omega_1)\psi_{21P} = \beta_{11P}$$
$$(1-\omega_2)\psi_{11P} + \lambda_{21P}\psi_{21P} + (1+\omega_2)\psi_{31P} = \beta_{21P}$$
$$(1-\omega_3)\psi_{21P} + \lambda_{31P}\psi_{31P} + (1+\omega_3)\psi_{41P} = \beta_{31P}$$
$$\dots$$
$$(1-\omega_M)\psi_{(M-1)1P} + \lambda_{M1P}v_{M1P} = \beta_{M1P}$$

Observe that here in the above set of equations $j = 1$, and for each $k = 1(1)P$ the coefficient matrix of the left hand side is a tri-diagonal matrix of order M and has the form

$$\mathcal{M}_{1k} = \begin{pmatrix} \lambda_{11k} & 1+\omega_1 & & & & & \\ 1-\omega_2 & \lambda_{21k} & 1+\omega_2 & & & & \\ & 1-\omega_3 & \lambda_{31k} & 1+\omega_3 & & & \\ & & & \ddots & & & \\ & & & & 1-\omega_{M-1} & \lambda_{(M-1)1k} & 1+\omega_{M-1} \\ & & & & & 1-\omega_M & \lambda_{M1k} \end{pmatrix}$$

Now collect again other groups of equations of (4.11) for $j = 2(1)N$, and get

$$\lambda_{1j1}\psi_{1j1} + (1+\omega_1)\psi_{2j1} = \beta_{1j1}$$
$$(1-\omega_2)\psi_{1j1} + \lambda_{2j1}\psi_{2j1} + (1+\omega_2)\psi_{3j1} = \beta_{2j1}$$
$$(1-\omega_3)\psi_{2j1} + \lambda_{3j1}\psi_{3j1} + (1+\omega_3)\psi_{4j1} = \beta_{3j1}$$
$$\dots$$
$$(1-\omega_M)\psi_{(M-1)j1} + \lambda_{Mj1}\psi_{Mj1} = \beta_{Mj1}$$
$$\lambda_{1j2}\psi_{1j2} + (1+\omega_1)\psi_{2j2} = \beta_{1j2} \tag{4.23b}$$
$$(1-\omega_2)\psi_{1j2} + \lambda_{2j2}\psi_{2j2} + (1+\omega_2)\psi_{3j2} = \beta_{2j2}$$
$$(1-\omega_3)\psi_{2j2} + \lambda_{3j2}\psi_{3j2} + (1+\omega_3)\psi_{4j2} = \beta_{3j2}$$
$$\dots$$
$$(1-\omega_M)\psi_{(M-1)j2} + \lambda_{Mj2}\psi_{Mj2} = \beta_{Mj2}$$

$$\cdots \qquad \cdots \qquad \cdots$$

$$\lambda_{1jP}\psi_{1jP} + (1+\omega_1)\psi_{2jP} = \beta_{1jP}$$

$$(1-\omega_2)\psi_{1jP} + \lambda_{2jP}v_{2jP} + (1+\omega_2)\psi_{3jP} = \beta_{2jP}$$

$$(1-\omega_3)\psi_{2jP} + \lambda_{3jP}\psi_{3jP} + (1+\omega_3)\psi_{4jP} = \beta_{3jP}$$

$$\cdots$$

$$(1-\omega_M)\psi_{(M-1)jP} + \lambda_{MjP}\psi_{MjP} = \beta_{MjP}$$

As we can easily see that these set of equations (4.23b), have tridiagonal matrix structure for $j = 2(1)N$ and observe that (4.11) reduces to equation (4.23b) which is a tridiagonal matrix

$$\mathcal{M} = \begin{pmatrix} \lambda_{1jk} & 1+\omega_1 & & & & & \\ 1-\omega_2 & \lambda_{2jk} & 1+\omega_2 & & & & \\ & 1-\omega_3 & \lambda_{3jk} & 1+\omega_3 & & & \\ & & & \ddots & & & \\ & & & & 1-\omega_{M-1} & \lambda_{(M-1)jk} & 1+\omega_{M-1} \\ & & & & & 1-\omega_M & \lambda_{Mjk} \end{pmatrix}$$

for $j = 1, 2, 3, \ldots, N$ and hence we solve these sets of equations (2.23b) for $\psi_{i,j,k}$ by the use of Thomas Algorithm [30]. Once after getting each $\psi_{i,j,k}$ (and hence Ψ_i) by the help of (4.20) we get $\mathbf{w}_i$ and again by the help of (4.13) we obtain $\mathbf{U}_k$ and this means that each $U_{i,j,k}$ are obtained. Thus, this solves our problem.

4.4 Numerical Results

In order to test the efficiency and adaptability of the method, computational experiments are done on seven test problems that may arise in practice, for which the analytical solutions of U are known to us. The computed solutions are found for all grid points but results are reported here only for some points in terms of the absolute maximum error and are shown from table 4.1 to 4.7.

Example 4.1 Consider $\nabla^2 U = -3cos\theta$ with the boundary conditions

$$U(0,\theta,z) = U(1,\theta,z) = -2z\ ,\ U(r,0,z) = r(1-r) - 2z,\ U\left(r,\frac{\pi}{2},z\right) = -2z$$

$$U(r,\theta,0) = r(1-r)cos\theta\ ,\quad U(r,\theta,1) = r(1-r)\,cos\theta - 2$$

The analytical solution is $U(r,\theta,z) = r(1-r)\cos\theta - 2z$ and the results of this example are shown in Table 4.1.

Example 4. 2 Consider $\nabla^2 U = -\pi^2 rcos\theta\sin(\pi z)$ with the boundary conditions

$$U(1,\theta,z) = cos\theta\sin(\pi z),\quad U(2,\theta,z) = 2cos\theta\sin(\pi z)$$

$U(r,0,z) = r\sin(\pi z)$, $U(r,\pi,z) = -r\sin(\pi z)$, and $U(r,\theta,0) = U(r,\theta,1) = 0$

The analytical solution is $U(r,\theta,z) = r\,cos\theta\sin(\pi z)$ and the results of this example are shown in Table 4.2.

Example 4.3 Consider $\nabla^2 U = -\pi^2\left(r^2 - \frac{1}{r^2}\right)\sin(2\theta)\sin(\pi z)$ with the boundary

conditions $U(1,\theta,z) = 0,\quad U(2,\theta,z) = \frac{15}{4}\sin(2\theta)\sin(\pi z)$

$$U(r,0,z) = 0 = U\left(r,\frac{\pi}{2},z\right) \quad \text{and} \quad U(r,\theta,0) = 0 = U(r,\theta,1)$$

The analytical solution is $U = \left(r^2 - \frac{1}{r^2}\right)\sin(2\theta)\sin(\pi z)$ and the results of this example are shown in Table 4.3.

Example 4.4 (See [101]) Consider

$$\nabla^2 U = \left(\left(\frac{\pi}{2r} - \frac{\pi^2}{2} - 4\frac{\pi^2}{r^2}\right)\cos r^* - \left(\frac{\pi}{2r} + \frac{\pi^2}{2} + 4\frac{\pi^2}{r^2}\right)\sin r^*\right)(cos\theta_1 + \sin\theta_1)\ (cosz_1 + sinz_1)$$

where $r^* = \frac{\pi}{2}(r-4)$, $\theta_1 = \pi(2\theta - 1)$, $z_1 = \frac{\pi}{2}(z-1)$, $2 \le r \le 4$, $0 \le \theta \le 0.5$,

and $-1 \le z \le 1$.

The analytical solution is $U(r,\theta,z) = (\cos r^* + \sin r^*)(\cos\theta_1 + \sin\theta_1)\ (\cos z_1 + \sin z_1)$

This problem was considered as one test problem by Iyengar and Goyal [101]. Their results and ours are found to be the same for $h = \frac{1}{8}$, but their method is restricted only to the same values of h_1, h_2 and h_3. We have shown the results of this example in Table 4.4.

Example 4.5 Consider $\nabla^2 U = -3\cos\theta$ where $0 \le \theta < 2\pi$, with the boundary conditions

$$U(0,\theta,z) = z = U(1,\theta,z),\ \ U(r,\theta,0) = r(1-r)\cos\theta, \text{ and}$$

$$U(r,\theta,1) = 1 + r(1-r)\cos\theta$$

The analytical solution is $U = r(1-r)\cos\theta + z$ and the results of this example are shown in Table 4.5.

Example 4.6 Consider $\nabla^2 U = 6rz\ \cos\theta$ where $0 \le \theta < 2\pi$, with the boundary conditions $U(0,\theta,z) = 0$, $U(1,\theta,z) = z\cos^3\theta$, $U(r,\theta,0) = 0$ and $(r,\theta,1) = r^3\cos^3\theta$

The analytical solution is $U = r^3 z\cos^3\theta$ and the results of this example are shown in Table 4.6.

Example 4.7 Consider $\nabla^2 U = -\pi^2\left(r^2 - \frac{1}{r^2}\right)\sin(2\theta)\sin(\pi z)\ \nabla^2 U = 6rz\ \cos\theta$ with the boundary conditions

$$U(1,\theta,z) = 0, \quad U(2,\theta,z) = \frac{15}{4}\sin(2\theta)\sin(\pi z)\ U(r,\theta,0) = 0 = U(r,\theta,1)$$

The analytical solution is $U = \left(r^2 - \frac{1}{r^2}\right)\sin(2\theta)\sin(\pi z)$ and the results of this example are shown in Table 4.7.

Table 4.1

The maximum absolute error of example 4.1

N,P,M	Max. abs. error		N,P,M	Max. abs. error		N,P,M	Max. abs. error
$(9,9,9)$	6.83901e-005		$(19,19,19)$	1.71675e-005		$(29,39,19)$	7.63846e-006
$(9,9,29)$	6.85359e-005		$(19,29,39)$	1.72164e-005		$(29,39,29)$	7.65884e-006
$(9,19,9)$	6.85882e-005		$(19,39,9)$	1.71752e-005		$(39,9,19)$	4.28351e-006
$(9,19,19)$	6.85881e-005		$(19,39,39)$	1.72198e-005		$(39,9,39)$	4.29603e-006
$(9,29,39)$	6.87927e-005		$(29,9,39)$	7.63105e-006		$(39,19,29)$	4.30845e-006
$(9,39,29)$	6.88027e-005		$(29,19,9)$	7.64055e-006		$(39,29,19)$	4.29998e-006
$(19,9,9)$	1.71132e-005		$(29,29,19)$	7.63684e-006		$(39,39,9)$	4.30369e-006
$(19,9,19)$	1.71127e-005		$(29,29,29)$	7.65726e-006		$(39,39,39)$	4.31261e-006

Table 4.2

The maximum absolute error of example 4.2

N,P,M	Max. abs. error		$N,P,M)$	Max. abs. error		N,P,M	Max. abs. error
$(9,9,9)$	5.97565e-003		$(19,19,19)$	1.6095e-003		$(29,39,19)$	4.18664e-004
$(9,9,29)$	6.04232e-003		$(19,29,39)$	7.53504e-004		$(29,39,29)$	4.19068e-004
$(9,19,9)$	1.68198e-003		$(19,39,9)$	4.50526e-004		$(39,9,19)$	6.23836e-003
$(9,19,19)$	1.69672e-003		$(19,39,39)$	4.53562e-004		$(39,9,39)$	6.24401e-003
$(9,29,39)$	8.9486e-004		$(29,9,39)$	6.25865e-003		$(39,19,29)$	1.5717e-003
$(9,39,29)$	6.14273e-004		$(29,19,9)$	1.56933e-003		$(39,29,19)$	7.07837e-004
$(19,9,9)$	6.18349e-003		$(29,29,19)$	7.21066e-004		$(39,39,9)$	4.01493e-004
$(19,9,19)$	6.24022e-003		$(29,29,29)$	7.21498e-004		$(39,39,39)$	4.05867e-004

Table 4.3

The maximum absolute error of example 4.3

N,P,M	Max. abs. error		N,P,M	Max. abs. error		N,P,M	Max. abs. error
$(9,9,9)$	1.0041e-002		$(19,19,19)$	2.51841e-003		$(29,39,19)$	6.45366e-004
$(9,9,29)$	1.03659e-002		$(19,29,39)$	1.33191e-003		$(29,39,29)$	6.85093e-004
$(9,19,9)$	3.33415e-003		$(19,39,9)$	6.40825e-004		$(39,9,19)$	8.98428e-003
$(9,19,19)$	3.5776e-003		$(19,39,39)$	8.95344e-004		$(39,9,39)$	9.07348e-003
$(9,29,39)$	2.39279e-003		$(29,9,39)$	9.14135e-003		$(39,19,29)$	2.30437e-003
$(9,39,29)$	1.94206e-003		$(29,19,9)$	2.09239e-003		$(39,29,19)$	1.01451e-003
$(19,9,9)$	8.97807e-003		$(29,29,19)$	1.08113e-003		$(39,39,9)$	3.98681e-004
$(19,9,19)$	9.24935e-003		$(29,29,29)$	1.12241e-003		$(39,39,39)$	6.31752e-004

Table 4.4

The maximum absolute error of example 4.4

N,P,M	Max. abs. error		N,P,M	Max. abs. error		N,P,M	Max. abs. error
$(9,9,9)$	1.52153e-002		$(19,19,19)$	3.87647e-003		$(29,39,19)$	2.13081e-003
$(9,9,29)$	1.18196e-002		$(19,29,39)$	2.66641e-003		$(29,39,29)$	1.558e-003
$(9,19,9)$	1.26939e-002		$(19,39,9)$	6.34865e-003		$(39,9,19)$	4.98915e-003
$(9,19,19)$	9.82575e-003		$(19,39,39)$	2.49938e-003		$(39,9,39)$	4.21468e-003
$(9,29,39)$	8.67411e-003		$(29,9,39)$	4.59667e-003		$(39,19,29)$	1.82246e-003
$(9,39,29)$	8.68314e-003		$(29,19,9)$	5.9446e-003		$(39,29,19)$	1.91856e-003
$(19,9,9)$	9.48954e-003		$(29,29,19)$	2.29605e-003		$(39,39,9)$	4.97913e-003
$(19,9,19)$	6.4572e-003		$(29,29,29)$	1.72475e-003		$(39,39,39)$	9.7357e-004

Table 4.5

The maximum absolute error of example 4.5

N,P,M	Max. abs. error		N,P,M	Max. abs. error		N,P,M	Max. abs. error
(10,9,9)	2.91054e-003		(20,19,19)	7.34274e-004		(30,39,19)	3.26426e-004
(10,9,29)	2.94732e-003		(20,29,39)	7.39078e-004		(30,39,29)	3.28008e-004
(10,19,9)	2.91942e-003		(20,39,9)	7.28843e-004		(40,9,19)	1.82891e-004
(10,19,19)	2.94503e-003		(20,39,39)	7.3923e-004		(40,9,39)	1.83977e-004
(10,29,39)	2.96297e-003		(30,9,39)	3.27118e-004		(40,19,29)	1.84332e-004
(10,39,29)	2.95912e-003		(30,19,9)	3.2355e-004		(40,29,19)	1.83546e-004
(20,9,9)	7.26075e-004		(30,29,19)	3.26362e-004		(40,39,9)	1.8211e-004
(20,9,19)	7.32065e-004		(30,29,29)	3.27939e-004		(40,39,39)	1.84702e-004

Table 4.6

The maximum absolute error of example 4.6

N,P,M	Max. abs. error		N,P,M	Max. abs. error		N,P,M	Max. abs. error
(10,9,9)	8.08303e-003		(20,19,19)	2.06074e-003		(30,39,19)	9.74094e-004
(10,9,29)	7.78161e-003		(20,29,39)	1.99772e-003		(30,39,29)	9.19684e-004
(10,19,9)	8.13051e-003		(20,39,9)	2.35229e-003		(40,9,19)	5.8771e-004
(10,19,19)	7.87559e-003		(20,39,39)	1.99854e-003		(40,9,39)	5.1461e-004
(10,29,39)	7.82057e-003		(30,9,39)	8.96012e-004		(40,19,29)	5.35984e-004
(10,39,29)	7.84024e-003		(30,19,9)	1.26723e-003		(40,29,19)	5.9185e-004
(20,9,9)	2.33516e-003		(30,29,19)	9.73704e-004		(40,39,9)	9.27532e-004
(20,9,19)	2.0482e-003		(30,29,29)	9.19304e-004		(40,39,39)	5.18641e-004

Table 4.7

The maximum absolute error of example 4.7

N,P,M	Max. abs. error		N,P,M	Max. abs. error		N,P,M	Max. abs. error
$(10,9,9)$	2.8935e-002		$(20,19,19)$	7.39905e-003		$(30,39,19)$	2.97592e-003
$(10,9,29)$	2.9406e-002		$(20,29,39)$	6.2994e-003		$(30,39,29)$	3.02879e-003
$(10,19,9)$	2.2444e-002		$(20,39,9)$	5.55157e-003		$(40,9,19)$	8.42863e-002
$(10,19,19)$	2.29089e-002		$(20,39,39)$	5.89069e-003		$(40,19,29)$	7.43613e-002
$(10,29,39)$	2.18002e-002		$(30,9,39)$	1.14219e-002		$(40,29,19)$	3.30954e-002
$(10,39,29)$	2.13515e-002		$(30,19,9)$	4.40139e-003		$(40,29,39)$	6.92702e-002
$(20,9,9)$	1.35531e-002		$(30,29,19)$	3.40452e-003		$(40,39,9)$	7.68463e--003
$(20,9,19)$	1.38225e-002		$(30,29,29)$	3.4587e-003		$(40,39,39)$	5.36082e-002

4.5 Conclusion

In this work, we have transformed the three dimensional Poisson's equation in cylindrical coordinates system in to a system of algebraic linear equations using its equivalent second order finite difference approximation scheme. The resulting large number of algebraic equation is, then, systematically arranged in order to get a block matrix. Based on the extension of Hockney's method we reduced the obtained matrix in to a block tridiagonal matrix, and each block is solved by the help of Thomas algorithm.[30] We have successfully implemented this method to find the solution of the three dimensional Poisson's equation in cylindrical coordinates system and it is found that the method can easily be applied and adapted to find a solution of some related applied problems. The method produced accurate results considering double precision. This method is direct and allows considerable savings in computer storage as well as execution speed.

Therefore, the method is suitable to apply on some three dimensional Poisson's equations.

CHAPTER V

Fourth Order Numerical Solution of the Three Dimensional Poisson's Equation in Cylindrical Coordinates System

5.1 Introduction

The three dimensional Poisson's equation in cylindrical coordinates system (r, θ, z) is given by

$$\frac{\partial^2 U}{\partial r^2}+\frac{1}{r}\frac{\partial U}{\partial r}+\frac{1}{r^2}\frac{\partial^2 U}{\partial \theta^2}+\frac{\partial^2 U}{\partial z^2}=f(r,\theta,z) \tag{5.1}$$

has a wide range of application in engineering and physics.

In physical problems that involve a cylindrical surface, (for example the problem of evaluating the temperature in a cylindrical rod), it will be convenient to make use of cylindrical coordinates. For the numerical solution of the three dimensional Poisson's equation in cylindrical coordinates system several attempts have been made in particular for physical problems that are related directly or indirectly to this equation. For instance, *Lai* [62] a simple compact fourth-order Poisson solver on polar geometry based on the truncated Fourier series expansion, where the differential equations of the Fourier coefficients are solved by the compact fourth order finite difference scheme; *Mittal and Gahlaut* [90] have developed high order finite difference schemes of second and fourth order in polar coordinates using a direct method similar to Hockney's method; Mittal and Gahlaut [91] have developed a second and fourth order finite difference scheme to solve Poisson's equation in the case of cylindrical symmetry; *Iyengar and Manohar* [102] derived fourth-order difference schemes for the solution of the Poisson equation which occurs in problems of heat transfer. The need to obtain the best solution for the three dimensional Poisson's equation in cylindrical coordinates system is still in progress.

In this chapter, we develop a fourth order finite difference approximation scheme and solve the resulting large algebraic system of linear equations systematically using block tridiagonal system [60] and extend the Hockney's method [96] to solve the three dimensional Poisson's equation on Cylindrical coordinates system.

5.2 Finite Difference Approximation

Consider the three dimensional Poisson's equation in cylindrical coordinates (r,θ,z) (5.1) given by

$$\frac{\partial^2 U}{\partial r^2}+\frac{1}{r}\frac{\partial U}{\partial r}+\frac{1}{r^2}\frac{\partial^2 U}{\partial \theta^2}+\frac{\partial^2 U}{\partial z^2}=f(r,\theta,z) \quad on\ D \qquad \text{and}$$

$$U(r,\theta,z)=g(r,\theta,z) \quad \text{on } C \tag{5.2}$$

C is the boundary of D where D is

a) $D_1=\{(r,\theta,z): R_0<r<R_1\ ,\ a<z<b, \theta_0<\theta<\theta_1\ ,\ \theta_0<\theta_1<2\pi\}$ and

b) $D_2=\{(r,\theta,z): R_0<r<R_1\ ,\ a<z<b,\ 0\le\theta<2\pi\}$

Consider figure 4.1 in chapter IV as the geometry of the problem.

Let $U(r,\theta,z)$ be discretized at the point (r_i,θ_j,z_k) and write the point (r_i,θ_j,z_k) as (i,j,k) and $U(r_i,\theta_j,z_k)$ as $U_{i,j,k}$. Assume that there are M points along the r direction, N points along θ and P points along the z directions to form the mesh, and let the step size along the direction of r be Δr, along the direction of θ be $\Delta\theta$ and along the direction of z be Δz.

Here $r_i=R_0+i\Delta r, \theta_j=\theta_0+j\Delta\theta$ and $z_k=a+k\Delta z,\qquad i=1(1)M,\ j=1(1)N$ and $k=1(1)P$

When $r=0$ is an interior or a boundary point in (4.1), then the Poisson's equation becomes singular and to take care of the singularity we consider a different approach will be taken in the next chapter.

Thus we discuss in this chapter the fourth-order approximation scheme only for the case $r\neq 0$.

Using the approximations that

$$\left(\frac{\partial^2 U}{\partial r^2}\right)_{i,j,k} = \frac{1}{(\Delta r)^2}\left(1+\frac{1}{12}\delta_r^2\right)^{-1}\delta_r^2 U_{i,j,k} + O\left((\Delta r)^4\right) \tag{5.3}$$

$$\left(\frac{\partial^2 U}{\partial \theta^2}\right)_{i,j,k} = \frac{1}{(\Delta \theta)^2}\left(\frac{\delta_\theta^2}{1+\frac{1}{12}\delta_\theta^2}\right)U_{i,j,k} + O\left((\Delta \theta)^4\right) \tag{5.4}$$

and $$\left(\frac{\partial^2 U}{\partial z^2}\right)_{i,j,k} = \frac{1}{(\Delta z)^2}\left(\frac{\delta_z^2}{1+\frac{1}{12}\delta_z^2}\right)U_{i,j,k} + O\left((\Delta z)^4\right) \tag{5.5}$$

Now using (5.3), (5.4) and (5.5), we get the following approximations (Refer the work of Mittal and Ghalaut in [91])

i) In the three dimensional Poisson's equation in cylindrical coordinates system (5.1) consider only the approximation of the sum of the first term and the third term, that is, the sum of $\frac{\partial^2 U}{\partial r^2}$ and $\frac{1}{r_i^2}\frac{\partial^2 U}{\partial \theta^2}$

$$\begin{aligned}\left(\frac{\partial^2 U}{\partial r^2}+\frac{1}{r_i^2}\frac{\partial^2 U}{\partial \theta^2}\right)_{i,j,k} &= \frac{1}{12(\Delta r)^2}\Bigg[\left(1+\frac{\omega}{r_i^2}\right)\left(U_{i+1,j+1,k}+U_{i+1,j-1,k}+U_{i-1,j+1,k}+U_{i-1,j-1,k}\right)\\ &+2\left(5-\frac{\omega}{r_i^2}\right)\left(U_{i+1,j,k}+U_{i-1,j,k}\right)+2\left(\frac{5\omega}{r_i^2}-1\right)\left(U_{i,j+1,k}+U_{i,j-1,k}\right)-20\left(1+\frac{\omega}{r_i^2}\right)U_{i,j,k}\Bigg]\\ &-\frac{1}{12}\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r_i^2}\frac{\partial^2}{\partial \theta^2}\right)\left((\Delta r)^2\frac{\partial^2}{\partial r^2}+(\Delta\theta)^2\frac{\partial^2}{\partial \theta^2}\right)U_{i,j,k}+O\left((\Delta r)^4+(\Delta\theta)^4\right)\end{aligned} \tag{5.6}$$

where $\omega = \frac{(\Delta r)^2}{(\Delta \theta)^2}$

ii) Again in (5.1) consider only the approximation of the sum of the first term and the fourth term, that is, the sum of $\frac{\partial^2 U}{\partial r^2}$ and $\frac{\partial^2 U}{\partial z^2}$.

$$\left(\frac{\partial^2 U}{\partial r^2}+\frac{\partial^2 U}{\partial z^2}\right)_{i,j,k}=\frac{1}{12(\Delta r)^2}\left[\left(1+\frac{(\Delta r)^2}{(\Delta z)^2}\right)\left(U_{i+1,j,k+1}+U_{i+1,j,k-1}+U_{i-1,j,k+1}+U_{i-1,j,k-1}\right)\right.$$
$$+2\left(5-\frac{(\Delta r)^2}{(\Delta z)^2}\right)\left(U_{i+1,j,k}+U_{i-1,j,k}\right)+2\left(5\frac{(\Delta r)^2}{(\Delta z)^2}-1\right)\left(U_{i,j+1,k}+U_{i,j-1,k}\right)-20\left.\left(1+\frac{(\Delta r)^2}{(\Delta z)^2}\right)U_{i,j,k}\right]$$
$$-\frac{1}{12}\left(\frac{\partial^2}{\partial r^2}+\frac{\partial^2}{\partial z^2}\right)\left((\Delta r)^2\frac{\partial^2}{\partial r^2}+(\Delta z)^2\frac{\partial^2}{\partial z^2}\right)U_{i,j,k}+O\left((\Delta r)^4+(\Delta z)^4\right)$$

(5.7)

iii) Again in (5.1) consider only the approximation of the sum of the second term and the fourth term, that is, the sum of $\frac{1}{r_i^2}\frac{\partial^2 U}{\partial \theta^2}$ and $\frac{\partial^2 U}{\partial z^2}$

$$\left(\frac{1}{r_i^2}\frac{\partial^2 U}{\partial \theta^2}+\frac{\partial^2 U}{\partial z^2}\right)_{i,j,k}=\frac{1}{12}\left[\left(\frac{1}{(r_i\Delta\theta)^2}+\frac{1}{(\Delta z)^2}\right)\left(U_{i,j+1,k+1}+U_{i,j+1,k-1}+U_{i,j-1,k+1}+U_{i,j-1,k-1}\right)\right.$$
$$+2\left(\frac{5}{(r_i\Delta\theta)^2}-\frac{1}{(\Delta z)^2}\right)\left(U_{i,j+1,k}+U_{i,j-1,k}\right)+2\left(\frac{5}{(\Delta z)^2}-\frac{1}{(r_i\Delta\theta)^2}\right)\left(U_{i,j,k+1}+U_{i,j,k-1}\right)$$
$$-20\left.\left(\frac{1}{(r_i\Delta\theta)^2}+\frac{1}{(\Delta z)^2}\right)U_{i,j,k}\right]-\frac{1}{12}\left(\frac{1}{r_i^2}\frac{\partial^2}{\partial\theta^2}+\frac{\partial^2}{\partial z^2}\right)\left((\Delta\theta)^2\frac{\partial^2}{\partial r^2}+(\Delta z)^2\frac{\partial^2}{\partial z^2}\right)U_{i,j,k} \qquad (5.8)$$
$$+O\left((\Delta\theta)^4+(\Delta z)^4\right)$$

Once again approximating the term $\frac{\partial U}{\partial r}$ by

$$\left(\frac{\partial U}{\partial r}\right)_{i,j,k}=\frac{\phi\delta_{2r}(U_{i,j+1,k}+U_{i,j-1,k}+U_{i,j,k+1}+U_{i,j,k-1})+(1-4\phi)\delta_{2r}U_{i,j,k}}{2\Delta r}-\frac{1}{3}(\Delta r)^2\frac{\partial^3 U_{i,j,k}}{\partial r^3}$$
$$-\phi(\Delta\theta)^2\frac{\partial^3 U_{i,j,k}}{\partial r\partial\theta^2}-\phi(\Delta z)^2\frac{\partial^3 U_{i,j,k}}{\partial r\partial z^2}+O\left((\Delta r)^4+(\Delta\theta)^4+(\Delta z)^4\right),\quad 0\le\phi\le 1$$

(5.9)

Equation (5.9) implying that

$$\frac{1}{r_i}\left(\frac{\partial U}{\partial r}\right)_{i,j,k} = \frac{\phi\delta_{2r}(U_{i,j+1,k}+U_{i,j-1,k}+U_{i,j,k+1}+U_{i,j,k-1})+(1-4\phi)\delta_{2r}U_{i,j,k}}{2r_i\Delta r} - \frac{1}{3r_i}(\Delta r)^2\frac{\partial^3 U_{i,j,k}}{\partial r^3}$$
$$-\phi(\Delta\theta)^2\frac{1}{r_i}\frac{\partial^3 U_{i,j,k}}{\partial r\partial\theta^2} - \phi(\Delta z)^2\frac{1}{r_i}\frac{\partial^3 U_{i,j,k}}{\partial r\partial z^2} + O\left((\Delta r)^4+(\Delta\theta)^4+(\Delta z)^4\right) \tag{5.10}$$

Now letting $\alpha = \frac{(\Delta r)^2}{(\Delta z)^2}$ and adding (5.6), (5.7), (5.8) and twice of (5.10), we get

$$2\left(\frac{\partial^2 U}{\partial r^2}+\frac{1}{r_i}\frac{\partial U}{\partial r}+\frac{1}{r_i^2}\frac{\partial^2 U}{\partial\theta^2}+\frac{\partial^2 U}{\partial z^2}\right)_{i,j,k}$$
$$= \frac{1}{12(\Delta r)^2}\Bigg[\left(1+\frac{\omega}{r_i^2}\right)\left(U_{i+1,j+1,k}+U_{i+1,j-1,k}+U_{i-1,j+1,k}+U_{i-1,j-1,k}\right)+2\left(5-\frac{\omega}{r_i^2}\right)\left(U_{i+1,j,k}+U_{i-1,j,k}\right)$$
$$+2\left(\frac{5\omega}{r_i^2}-1\right)\left(U_{i,j+1,k}+U_{i,j-1,k}\right)+\left(1+\alpha\right)\left(U_{i+1,j,k+1}+U_{i+1,j,k-1}+U_{i-1,j,k+1}+U_{i-1,j,k-1}\right)$$
$$+2\left(5-\alpha\right)\left(U_{i+1,j,k}+U_{i-1,j,k}\right)+2\left(5\alpha-1\right)\left(U_{i,j+1,k}+U_{i,j-1,k}\right)-20\left(2+\alpha+\frac{\omega}{r_i^2}\right)U_{i,j,k}\Bigg]$$
$$+\frac{1}{12}\Bigg[\left(\frac{1}{(r_i\Delta\theta)^2}+\frac{1}{(\Delta z)^2}\right)\left(U_{i,j+1,k+1}+U_{i,j+1,k-1}+U_{i,j-1,k+1}+U_{i,j-1,k-1}\right)-20\left(\frac{1}{(r_i\Delta\theta)^2}+\frac{1}{(\Delta z)^2}\right)U_{i,j,k}$$
$$+2\left(\frac{5}{(r_i\Delta\theta)^2}-\frac{1}{(\Delta z)^2}\right)\left(U_{i,j+1,k}+U_{i,j-1,k}\right)+2\left(\frac{5}{(\Delta z)^2}-\frac{1}{(r_i\Delta\theta)^2}\right)\left(U_{i,j,k+1}+U_{i,j,k-1}\right)$$
$$-\frac{1}{12}\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r_i^2}\frac{\partial^2}{\partial\theta^2}\right)\left((\Delta r)^2\frac{\partial^2}{\partial r^2}+(\Delta\theta)^2\frac{\partial^2}{\partial\theta^2}\right)U_{i,j,k}-\frac{1}{12}\left(\frac{\partial^2}{\partial r^2}+\frac{\partial^2}{\partial z^2}\right)\left((\Delta r)^2\frac{\partial^2}{\partial r^2}+(\Delta z)^2\frac{\partial^2}{\partial z^2}\right)U_{i,j,k}$$
$$-\frac{1}{12}\left(\frac{1}{r_i^2}\frac{\partial^2}{\partial\theta^2}+\frac{\partial^2}{\partial z^2}\right)\left((\Delta\theta)^2\frac{\partial^2}{\partial r^2}+(\Delta z)^2\frac{\partial^2}{\partial z^2}\right)U_{i,j,k}+\frac{\phi\delta_{2r}(U_{i,j+1,k}+U_{i,j-1,k}+U_{i,j,k+1}+U_{i,j,k-1})}{\Delta r}$$
$$+\frac{(1-4\phi)\delta_{2r}U_{i,j,k}}{\Delta r}-\frac{1}{3}(\Delta r)^2\frac{\partial^3 U_{i,j,k}}{\partial r^3}-\phi(\Delta\theta)^2\frac{\partial^3 U_{i,j,k}}{\partial r\partial\theta^2}-\phi(\Delta z)^2\frac{\partial^3 U_{i,j,k}}{\partial r\partial z^2}+O\left((\Delta r)^4+(\Delta\theta)^4+(\Delta z)^4\right) \tag{5.11}$$

Now choose $\phi = \frac{1}{12}$ and consider the following terms in (5.11)

$$-\frac{1}{12}\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r_i^2}\frac{\partial^2}{\partial \theta^2}\right)\left((\Delta r)^2\frac{\partial^2}{\partial r^2}+(\Delta\theta)^2\frac{\partial^2}{\partial\theta^2}\right)U_{i,j,k}$$
$$-\frac{1}{12}\left(\frac{\partial^2}{\partial r^2}+\frac{\partial^2}{\partial z^2}\right)\left((\Delta r)^2\frac{\partial^2}{\partial r^2}+(\Delta z)^2\frac{\partial^2}{\partial z^2}\right)U_{i,j,k}-\frac{1}{3r_i}(\Delta r)^2\frac{\partial^3 U_{i,j,k}}{\partial r^3}$$
$$-\frac{1}{12}\left(\frac{1}{r_i^2}\frac{\partial^2}{\partial \theta^2}+\frac{\partial^2}{\partial z^2}\right)\left((\Delta\theta)^2\frac{\partial^2}{\partial r^2}+(\Delta z)^2\frac{\partial^2}{\partial z^2}\right)U_{i,j,k}-\frac{1}{12r_i}\left((\Delta\theta)^2\frac{\partial^3 U_{i,j,k}}{\partial r\partial\theta^2}+(\Delta z)^2\frac{\partial^3 U_{i,j,k}}{\partial r\partial z^2}\right)$$

$$=-\frac{1}{3r_i}(\Delta r)^2\frac{\partial^3 U}{\partial r^3}-\frac{1}{12}\left((\Delta r)^2\frac{\partial^2}{\partial r^2}+(\Delta\theta)^2\frac{\partial^2}{\partial\theta^2}+(\Delta z)^2\frac{\partial^2}{\partial z^2}\right)\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r_i^2}\frac{\partial^2}{\partial\theta^2}+\frac{\partial^2}{\partial z^2}\right)U_{i,j,k}$$
$$-\frac{1}{12}\left((\Delta r)^2\frac{\partial^2}{\partial r^2}\left(\frac{\partial^2}{\partial r^2}\right)+(\Delta\theta)^2\frac{\partial^2}{\partial\theta^2}\left(\frac{\partial^2}{\partial\theta^2}\right)+(\Delta z)^2\frac{\partial^2}{\partial z^2}\left(\frac{\partial^2}{\partial z^2}\right)\right)U_{i,j,k}$$
$$-\frac{1}{12r_i}\left((\Delta\theta)^2\frac{\partial^3 U_{i,j,k}}{\partial r\partial\theta^2}+(\Delta z)^2\frac{\partial^3 U_{i,j,k}}{\partial r\partial z^2}\right)$$

$$=-\frac{1}{12}\left((\Delta r)^2\frac{\partial^2}{\partial r^2}+(\Delta\theta)^2\frac{\partial^2}{\partial\theta^2}+(\Delta z)^2\frac{\partial^2}{\partial z^2}\right)\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r_i^2}\frac{\partial^2}{\partial\theta^2}+\frac{\partial^2}{\partial z^2}\right)U_{i,j,k}$$
$$-\frac{1}{12}\left(\frac{1}{r_i}\frac{\partial}{\partial r}\left((\Delta r)^2\frac{\partial^2}{\partial r^2}+(\Delta\theta)^2\frac{\partial^2}{\partial\theta^2}+(\Delta z)^2\frac{\partial^2}{\partial z^2}\right)U_{i,j,k}\right)-\frac{1}{4r_i}(\Delta r)^2\frac{\partial^3 U_{i,j,k}}{\partial r^3}$$
$$=-\frac{1}{12}\left((\Delta r)^2\frac{\partial^2}{\partial r^2}+(\Delta\theta)^2\frac{\partial^2}{\partial\theta^2}+(\Delta z)^2\frac{\partial^2}{\partial z^2}\right)\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r_i}\frac{\partial}{\partial r}+\frac{1}{r_i^2}\frac{\partial^2}{\partial\theta^2}+\frac{\partial^2}{\partial z^2}\right)U_{i,j,k}-\frac{1}{4r_i}(\Delta r)^2\frac{\partial^3 U_{i,j,k}}{\partial r^3}$$

(5.12)

Again consider the term $-\frac{1}{4r_i}(\Delta r)^2\frac{\partial^3 U_{i,j,k}}{\partial r^3}$ in (5.12)

$$-\frac{(\Delta r)^2}{4r_i}\frac{\partial^3 U_{i,j,k}}{\partial r^3} = -\frac{(\Delta r)^2}{4r_i}\frac{\partial}{\partial r}\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r_i}\frac{\partial}{\partial r}+\frac{1}{r_i^2}\frac{\partial^2}{\partial \theta^2}+\frac{\partial^2}{\partial z^2}\right)U_{i,j,k} + \frac{(\Delta r)^2}{4r_i}\frac{\partial}{\partial r}\left(\frac{1}{r_i}\frac{\partial U_{i,j,k}}{\partial r}\right)$$
$$+\frac{(\Delta r)^2}{4r_i}\frac{\partial}{\partial r}\left(\frac{1}{r_i^2}\frac{\partial^2 U_{i,j,k}}{\partial \theta^2}\right)+\frac{(\Delta r)^2}{4r_i}\frac{\partial}{\partial r}\left(\frac{\partial^2 U_{i,j,k}}{\partial z^2}\right)$$
$$= -\frac{(\Delta r)^2}{4r_i}\frac{\partial f}{\partial r} - \frac{(\Delta r)^2}{4r_i^3}\frac{\partial U_{i,j,k}}{\partial r} + \frac{(\Delta r)^2}{4r_i^2}\frac{\partial^2 U_{i,j,k}}{\partial r^2} - \frac{1}{2}\frac{(\Delta r)^2}{r_i^4}\frac{\partial^2 U_{i,j,k}}{\partial \theta^2}$$
$$+\frac{(\Delta r)^2}{4r_i^3}\frac{\partial}{\partial r}\left(\frac{\partial^2 U_{i,j,k}}{\partial \theta^2}\right)+\frac{(\Delta r)^2}{4r_i}\frac{\partial}{\partial r}\left(\frac{\partial^2 U_{i,j,k}}{\partial z^2}\right)$$

(5.13)

Using (5.12), (5.13), and multiplying both sides of (5.11) by $12(\Delta r)^2$ and rearranging and simplifying further, we get

$$(\Delta r)^2\left(24+\delta_r^2+\delta_\theta^2+\delta_z^2+\frac{3\Delta r}{2r_i}\delta_{2r}\right)f_{i,j,k} = a_0(i)U_{i,j,k}+a_1(i)U_{i+1,j,k}+a_2(i)U_{i-1,j,k}$$
$$+a_3(i)(U_{i,j+1,k}+U_{i,j-1,k})+a_4(i)(U_{i,j,k+1}+U_{i,j,k-1})+a_5(i)(U_{i+1,j+1,k}+U_{i+1,j-1,k})$$
$$+a_6(i)(U_{i-1,j+1,k}+U_{i-1,j-1,k})+a_7(i)(U_{i+1,j,k+1}+U_{i+1,j,k-1})+a_8(i)(U_{i-1,j,k+1}+U_{i-1,j,k-1}) \quad (5.14)$$
$$+a_9(i)(U_{i,j+1,k+1}+U_{i,j-1,k+1}+U_{i,j+1,k-1}+U_{i,j-1,k-1})$$

where

$$a_0(i) = -40\left(1+\alpha+\frac{\omega}{r_i^2}\right)-6\frac{(\Delta r)^2}{r_i^2}+12\frac{\omega}{r_i^2}\frac{(\Delta r)^2}{r_i^2}$$

$$a_1(i) = 20-2\alpha-\frac{2\omega}{r_i^2}+8\frac{\Delta r}{r_i}-\frac{3}{2}\left(\frac{\Delta r}{r_i}\right)^3+3\left(\frac{\Delta r}{r_i}\right)^2-3\frac{\omega}{r_i^2}\frac{\Delta r}{r_i}-3\alpha\frac{\Delta r}{r_i}$$

$$a_2(i) = 20-2\alpha-\frac{2\omega}{r_i^2}-8\frac{\Delta r}{r_i}+\frac{3}{2}\left(\frac{\Delta r}{r_i}\right)^3+3\left(\frac{\Delta r}{r_i}\right)^2+3\frac{\omega}{r_i^2}\frac{\Delta r}{r_i}+3\alpha\frac{\Delta r}{r_i}$$

$$a_3(i) = -2\alpha+12\frac{\omega}{r_i^2}-2$$

$$a_4(i) = 20\alpha-\frac{2\omega}{r_i^2}-2$$

$$a_5(i) = 1 + \frac{\omega}{r_i^2} + \frac{\Delta r}{r_i} + \frac{3}{2}\frac{\omega}{r_i^2}\frac{\Delta r}{r_i}$$

$$a_6(i) = 1 + \frac{\omega}{r_i^2} - \frac{\Delta r}{r_i} - \frac{3}{2}\frac{\omega}{r_i^2}\frac{\Delta r}{r_i}$$

$$a_7(i) = 1 + \alpha + \frac{\Delta r}{r_i} + \frac{3}{2}\alpha\frac{\Delta r}{r_i}$$

$$a_8(i) = 1 + \alpha - \frac{\Delta r}{r_i} - \frac{3}{2}\alpha\frac{\Delta r}{r_i}$$

$$a_9(i) = \alpha + \frac{\omega}{r_i^2}$$

The system of equations in (5.14) is a linear sparse system, and thereby when solving we save both work and storage compared with a general system of equations. Such savings are basically true of finite difference methods: they yield sparse systems because each equation involves only few variables.

To solve equation (5.14), consider first in the θ direction, next in the Z direction and lastly in the r direction, and as a result of this (5.14) can be written in matrix form as

$$AU = \mathcal{B} \tag{5.15}$$

where $A = \begin{pmatrix} R_1 & S_1 & & & & & \\ T_2 & R_2 & S_2 & & & & \\ & T_3 & R_3 & S_3 & & & \\ & & & \ddots & & & \\ & & & & T_{M-1} & R_{M-1} & S_{M-1} \\ & & & & & T_M & R_M \end{pmatrix}$, it has M blocks and each is of order NP

$$R_i = \begin{pmatrix} R_i^{'} & R_i^{''} & & & & & \\ R_i^{''} & R_i^{'} & R_i^{''} & & & & \\ & R_i^{''} & R_i^{'} & R_i^{''} & & & \\ & & & \ddots & & & \\ & & & & R_i^{''} & R_i^{'} & R_i^{''} \\ & & & & & R_i^{''} & R_i^{'} \end{pmatrix},$$

$$S_i = \begin{pmatrix} S_i' & S_i'' & & & & & \\ S_i'' & S' & S_i'' & & & & \\ & S_i'' & S' & S_i'' & & & \\ & & & & \ddots & & \\ & & & & S_i'' & S' & S_i'' \\ & & & & & S_i'' & S' \end{pmatrix}$$

$$T_i = \begin{pmatrix} T_i' & T_i'' & & & & & \\ T_i'' & T_i' & T_i'' & & & & \\ & T_i'' & T_i' & T_i'' & & & \\ & & & & \ddots & & \\ & & & & T_i'' & T_i' & T_i'' \\ & & & & & T_i'' & T_i' \end{pmatrix}$$

$R_i, S_i,$ and T_i are of order NP

for the domain D_1

$$R_i' = \begin{pmatrix} a_0(i) & a_3(i) & & & & & \\ a_3(i) & a_0(i) & a_3(i) & & & & \\ & a_3(i) & a_0(i) & a_3(i) & & & \\ & & & & \ddots & & \\ & & & & a_3(i) & a_0(i) & a_3(i) \\ & & & & & a_3(i) & a_0(i) \end{pmatrix}$$

$$R_i'' = \begin{pmatrix} a_4(i) & a_9(i) & & & & & \\ a_9(i) & a_4(i) & a_9(i) & & & & \\ & a_9(i) & a_4(i) & a_9(i) & & & \\ & & & & \ddots & & \\ & & & & a_9(i) & a_4(i) & a_9(i) \\ & & & & & a_9(i) & a_4(i) \end{pmatrix}$$

$$S_i^{'} = \begin{pmatrix} a_1(i) & a_5(i) & & & & & \\ a_5(i) & a_1(i) & a_5(i) & & & & \\ & a_5(i) & a_1(i) & a_5(i) & & & \\ & & & \ddots & & & \\ & & & & a_5(i) & a_1(i) & a_5(i) \\ & & & & & a_5(i) & a_1(i) \end{pmatrix},$$

$$S_i^{''} = \begin{pmatrix} a_7(i) & & & & \\ & a_7(i) & & & \\ & & a_7(i) & & \\ & & & \ddots & \\ & & & & a_7(i) \end{pmatrix}$$

$$T_i^{'} = \begin{pmatrix} a_2(i) & a_6(i) & & & & & \\ a_6(i) & a_2(i) & a_6(i) & & & & \\ & a_6(i) & a_2(i) & a_6(i) & & & \\ & & & \ddots & & & \\ & & & & a_6(i) & a_2(i) & a_6(i) \\ & & & & & a_6(i) & a_2(i) \end{pmatrix}, \quad T_i^{''} = \begin{pmatrix} a_8(i) & & & & \\ & a_8(i) & & & \\ & & a_8(i) & & \\ & & & \ddots & \\ & & & & a_8(i) \end{pmatrix}$$

for the domain D_2**,**

$$R_i^{'} = \begin{pmatrix} a_0(i) & a_3(i) & & & & & a_3(i) \\ a_3(i) & a_0(i) & a_3(i) & & & & \\ & a_3(i) & a_0(i) & a_3(i) & & & \\ & & & \ddots & & & \\ & & & & a_3(i) & a_0(i) & a_3(i) \\ a_3(i) & & & & & a_3(i) & a_0(i) \end{pmatrix}$$

$$R_i^{''} = \begin{pmatrix} a_4(i) & a_9(i) & & & & & a_9(i) \\ a_9(i) & a_4(i) & a_9(i) & & & & \\ & a_9(i) & a_4(i) & a_9(i) & & & \\ & & & & \ddots & & \\ & & & & a_9(i) & a_4(i) & a_9(i) \\ a_9(i) & & & & & a_9(i) & a_4(i) \end{pmatrix}$$

$$S_i^{'} = \begin{pmatrix} a_1(i) & a_5(i) & & & & & a_5(i) \\ a_5(i) & a_1(i) & a_5(i) & & & & \\ & a_5(i) & a_1(i) & a_5(i) & & & \\ & & & & \ddots & & \\ & & & & a_5(i) & a_1(i) & a_5(i) \\ a_5(i) & & & & & a_5(i) & a_1(i) \end{pmatrix},$$

$$S_i^{''} = \begin{pmatrix} a_7(i) & & & & \\ & a_7(i) & & & \\ & & a_7(i) & & \\ & & & \ddots & \\ & & & & a_7(i) \end{pmatrix}$$

$$T_i^{'} = \begin{pmatrix} a_2(i) & a_6(i) & & & & & a_6(i) \\ a_6(i) & a_2(i) & a_6(i) & & & & \\ & a_6(i) & a_2(i) & a_6(i) & & & \\ & & & & \ddots & & \\ & & & & a_6(i) & a_2(i) & a_6(i) \\ a_6(i) & & & & & a_6(i) & a_2(i) \end{pmatrix},$$

$$T_i^{''} = \begin{pmatrix} a_8(i) & & & & \\ & a_8(i) & & & \\ & & a_8(i) & & \\ & & & \ddots & \\ & & & & a_8(i) \end{pmatrix}$$

Here in D_2, the matrices $R_i', R_i'', S_i', S_i'', T_i'$, and T_i'' are circulant matrices of order N; and

$\mathcal{B} = \left[\mathbf{B}_0 \;\; \mathbf{B}_1 \;\; \mathbf{B}_2 \cdots \mathbf{B}_M \right]^T$, $\mathbf{B}_i = [\mathbf{d}_{i1} \;\; \mathbf{d}_{i2} \;\; \mathbf{d}_{i3} \ldots \;\; \mathbf{d}_{iP}]^T$ and $\mathbf{d}_{ik} = [\; d_{ij1} \;\; d_{ij2} \ldots \; d_{ijP}]^T$ such that each d_{ijk} represents a known boundary values of U and values of f, and

$\mathbf{U} = \left[\mathbf{U}_1 \quad \mathbf{U}_2 \quad \mathbf{U}_3 \quad \ldots \quad \mathbf{U}_M\right]^T$, $\mathbf{U}_i = \left(U_{i1} \quad U_{i2} \quad U_{i3} \quad \cdots \quad U_{iP}\right)^T$ and

$U_{ij} = \left(U_{ij1} \quad U_{ij2} \quad U_{ij3} \quad \cdots \quad U_{ijP}\right)^T$

We write (5.15) as

$$\begin{aligned} R_1\mathbf{U}_1 + S_1\mathbf{U}_2 &= \mathbf{B}_1 \\ T_2\mathbf{U}_1 + R_2\mathbf{U}_2 + S_2\mathbf{U}_3 &= \mathbf{B}_2 \\ T_3\mathbf{U}_2 + R_3\mathbf{U}_3 + S_3\mathbf{U}_4 &= \mathbf{B}_3 \\ &\cdots \\ T_M\mathbf{U}_{M-1} + R_M\mathbf{U}_M &= \mathbf{B}_M \end{aligned} \tag{5.16}$$

5.3 Extended Hockney's Method

Observe that the matrices R_i', R_i'', S_i' and T_i' are real symmetric matrices and hence their eigenvalues and eigenvectors can easily be obtained. [30]

For D_1 $\quad \lambda_{ij} = a_0(i) + 2a_3(i)\cos\left(\dfrac{j\pi}{N+1}\right)$

$$\beta_{ij} = a_4(i) + 2a_9(i)\cos\left(\frac{j\pi}{N+1}\right)$$

$$\eta_{ij} = a_1(i) + 2a_5(i)\cos\left(\frac{j\pi}{N+1}\right)$$

$$\zeta_{ij} = a_2(i) + 2a_6(i)\cos\left(\frac{j\pi}{N+1}\right) \qquad i = 1(1)M \text{ and } j = 1(1)N$$

and for D_2 $\lambda_{ij} = a_0(i) + 2a_3(i)\cos\left(\frac{2\pi j}{N}\right)$

$$\beta_{ij} = a_4(i) + 2a_9(i)\cos\left(\frac{2\pi j}{N}\right)$$

$$\eta_{ij} = a_1(i) + 2a_5(i)\cos\left(\frac{2\pi j}{N}\right)$$

$$\zeta_{ij} = a_2(i) + 2a_6(i)\cos\left(\frac{2\pi j}{N}\right) \qquad i = 1(1)M \text{ and } j = 1(1)N$$

Let $\mathbf{q}_j$ be an eigenvector of R_i', R_i'', S_i' and T_i' corresponding to the eigenvalue $\lambda_{ij}, \beta_{ij}, \eta_{ij}$ and ξ_{ij}; and $Q = [\mathbf{q}_1\ \mathbf{q}_2\ \mathbf{q}_3\ ...\ \mathbf{q}_N]^T$ be a modal matrix of R_i', R_i'', S_i' and T_i', $\forall i$ such that

$Q^T Q = I$.

The $N \times N$ modal matrix Q is defined by

For D_1 $q_{ij} = \sqrt{\frac{2}{N+1}}\sin\left(\frac{ij\pi}{N+1}\right) \quad i, j = 1(1)N$

For D_2 $q_{ij} = \left(\frac{\cos\theta + \sin\theta}{\sqrt{N}}\right)$ where $\theta = \frac{2\pi}{N}(i-1)(j-1), \quad i, j = 1(1)N$.

Since R_i', R_i'', S_i' and T_i' are symmetric matrices, we have

$$Q^T R_i' Q = diag(\delta_{i1}, \delta_{i2}, \delta_{i3}, \cdots, \delta_{iN}) = \mathrm{X}_i \text{ (say)}$$

$$Q^T R_i'' Q = diag(\varphi_{i1}, \varphi_{i2}, \varphi_{i3}, \cdots, \varphi_{iN}) = Y_i \text{ (say)}$$

$$Q^T S_i' Q = diag(\eta_{i1}, \eta_{i2}, \eta_{i3}, \cdots, \eta_{iN}) = \mathrm{H}_i \text{ (say)}$$

$$Q^T T_i' Q = diag(\zeta_{i1}, \zeta_{i2}, \zeta_{i3}, \cdots, \zeta_{iN}) = \mathrm{Z}_i \text{ (say)}$$

Let $\mathbb{Q} = diag(Q, Q, Q, ..., Q)$ be a matrix of order NP by NP.

Thus $\mathbb{Q}$ satisfy the property that $\mathbb{Q}^T\mathbb{Q} = I$ since $Q^TQ = I$ and

$$\mathbb{Q}^T R_i \mathbb{Q} = \begin{pmatrix} \mathrm{X}_i & Y_i & & & & \\ Y_i & \mathrm{X}_i & Y_i & & & \\ & Y_i & \mathrm{X}_i & Y_i & & \\ & & & \ddots & & \\ & & & Y_i & \mathrm{X}_i & Y_i \\ & & & & Y_i & \mathrm{X}_i \end{pmatrix} \tag{5.17}$$

$$\mathbb{Q}^T S_i \mathbb{Q} = \begin{pmatrix} \mathrm{H}_i & S_i'' & & & & \\ S_i'' & \mathrm{H}_i & S_i'' & & & \\ & S_i'' & \mathrm{H}_i & S_i'' & & \\ & & & \ddots & & \\ & & & S_i'' & \mathrm{H}_i & S_i'' \\ & & & & S_i'' & \mathrm{H}_i \end{pmatrix} \tag{5.18}$$

$$\mathbb{Q}^T T_i \mathbb{Q} = \begin{pmatrix} Z_i & T_i'' & & & & \\ T_i'' & Z_i & T_i'' & & & \\ & T_i'' & Z_i & T_i'' & & \\ & & & \ddots & & \\ & & & T_i'' & Z_i & T_i'' \\ & & & & T_i'' & Z_i \end{pmatrix} \tag{5.19}$$

$\mathbb{Q}^T S_i'' \mathbb{Q} = S_i''$ and $\mathbb{Q}^T T_i'' \mathbb{Q} = T_i''$ since both S_i'' and T_i'' are diagonal matrices.

Let $\quad \mathbb{Q}^T \mathbf{U}_i = \boldsymbol{V}_i \Rightarrow \mathbf{U}_i = \mathbb{Q}\boldsymbol{V}_i$

$$\mathbb{Q}^T \mathbf{B}_i = B_i \Rightarrow \mathbf{B}_i = \mathbb{Q}B_i \tag{5.20}$$

where $\boldsymbol{V}_i = [\mathbf{v}_{i1}\ \mathbf{v}_{i2}\ \ \mathbf{v}_{i3} \ldots \mathbf{v}_{iP}]^T$ and $\mathbf{v}_{ik} = [v_{i1k}\ \ v_{i2k}\ \ v_{i3k} \ldots v_{iNk}]^T$ and

$\mathbf{B}_i = [\mathbf{b}_{i1}\ \mathbf{b}_{i2} \ldots \mathbf{b}_{ik}]^T$ and $\mathbf{b}_{ik} = [\ b_{i1k}\ \ b_{i2k} \ldots b_{iNk}]^T$

Consider the first equation of (5.16) i.e. $R_1\mathbf{U}_1 + S_1\mathbf{U}_2 \;\; = \mathbf{B}_1$ and pre-multiplying it by $\mathbb{Q}^T$ and further by using (5.20), we get

$$\begin{pmatrix} \mathrm{X}_1 & Y_1 & & & & \\ Y_1 & \mathrm{X}_1 & Y_1 & & & \\ & Y_1 & \mathrm{X}_1 & Y_1 & & \\ & & & \ddots & & \\ & & & Y_1 & \mathrm{X}_1 & Y_1 \\ & & & & Y_1 & \mathrm{X}_1 \end{pmatrix} \begin{pmatrix} \mathbf{v}_{11} \\ \mathbf{v}_{12} \\ \mathbf{v}_{13} \\ \vdots \\ \mathbf{v}_{1(P-1)} \\ \mathbf{v}_{1P} \end{pmatrix} + \begin{pmatrix} H_1 & S_1'' & & & & \\ S_1'' & H_1 & S_1'' & & & \\ & S_1'' & H_1 & S_1'' & & \\ & & & \ddots & & \\ & & & S_1'' & H_1 & S_1'' \\ & & & & S_1'' & H_1 \end{pmatrix} \begin{pmatrix} \mathbf{v}_{21} \\ \mathbf{v}_{22} \\ \mathbf{v}_{23} \\ \vdots \\ \mathbf{v}_{2(P-1)} \\ \mathbf{v}_{2P} \end{pmatrix} = \begin{pmatrix} \mathbf{b}_{11} \\ \mathbf{b}_{12} \\ \mathbf{b}_{13} \\ \vdots \\ \mathbf{b}_{1(P-1)} \\ \mathbf{b}_{1P} \end{pmatrix} \tag{5.21}$$

We write (5.21) again as

$$\begin{aligned} &\mathrm{X}_1\mathbf{v}_{11} + Y_1\mathbf{v}_{12} + \mathrm{H}_1\mathbf{v}_{21} + S_1''\mathbf{v}_{22} = \mathbf{b}_{11} \\ &Y_1\mathbf{v}_{11} + \mathrm{X}_1\mathbf{v}_{12} + Y_1\mathbf{v}_{13} + S_1''\mathbf{v}_{21} + \mathrm{H}_1\mathbf{v}_{22} + S_1''\mathbf{v}_{23} = \mathbf{b}_{12} \\ &Y_1\mathbf{v}_{12} + \mathrm{X}_1\mathbf{v}_{13} + Y_1\mathbf{v}_{14} + S_1''\mathbf{v}_{22} + \mathrm{H}_1\mathbf{v}_{23} + S_1''\mathbf{v}_{24} = \mathbf{b}_{13} \\ &\cdots \\ &Y_1\mathbf{v}_{1(P-1)} + \mathrm{X}_1\mathbf{v}_{1P} + S_1''\mathbf{v}_{2(P-1)} + \mathrm{H}_1\mathbf{v}_{2P} = \mathbf{b}_{1P} \end{aligned} \tag{5.22}$$

We write (5.22) turn by turn as

$$\begin{aligned} &\delta_{11}v_{111} + \varphi_{11}v_{112} + \eta_{11}v_{211} + a_7(1)v_{212} = b_{111} \\ &\delta_{12}v_{121} + \varphi_{12}v_{122} + \eta_{12}v_{221} + a_7(1)v_{222} = b_{121} \\ &\delta_{13}v_{131} + \varphi_{13}v_{132} + \eta_{13}v_{231} + a_7(1)v_{232} = b_{131} \\ &\cdots \\ &\delta_{1N}v_{1N1} + \varphi_{1N}v_{1N2} + \eta_{1N}v_{2N1} + a_7(1)v_{2N2} = b_{1N1} \end{aligned} \tag{5.23}$$

$$\begin{aligned} &\varphi_{11}v_{111} + \delta_{11}v_{112} + \varphi_{11}v_{113} + a_7(1)v_{211} + \eta_{11}v_{212} + a_7(1)v_{213} = b_{112} \\ &\varphi_{12}v_{121} + \delta_{12}v_{122} + \varphi_{12}v_{123} + a_7(1)v_{221} + \eta_{12}v_{222} + a_7(1)v_{223} = b_{122} \\ &\varphi_{13}v_{131} + \delta_{13}v_{132} + \varphi_{13}v_{133} + a_7(1)v_{231} + \eta_{13}v_{232} + a_7(1)v_{233} = b_{132} \\ &\cdots \\ &\varphi_{1N}v_{1N1} + \delta_{1N}v_{1N2} + \varphi_{1N}v_{1N3} + a_7(1)v_{2N1} + \eta_{1N}v_{2N2} + a_7(1)v_{2N3} = b_{1N2} \end{aligned}$$

$$\dots \quad \dots \quad \dots$$

$$\varphi_{11}v_{11(P-1)} + \delta_{11}v_{11P} + a_7(1)v_{21(P-1)} + \eta_{11}v_{21P} = b_{11P}$$

$$\varphi_{12}v_{12(P-1)} + \delta_{12}v_{12P} + a_7(1)v_{22(P-1)} + \eta_{12}v_{22P} = b_{12P}$$

$$\varphi_{13}v_{13(P-1)} + \delta_{13}v_{13P} + a_7(1)v_{23(P-1)} + \eta_{13}v_{23P} = b_{13P}$$

$$\dots$$

$$\varphi_{1N}v_{1N(P-1)} + \delta_{1N}v_{1NP} + a_7(1)v_{2N(P-1)} + \eta_{1N}v_{2NP} = b_{1NP}$$

Now collect the first equations from each of (5.23) and consider as a first group of equations

$$
\begin{aligned}
\delta_{11}v_{111} + \varphi_{11}v_{112} + \eta_{11}v_{211} + a_7(1)v_{212} &= b_{111} \\
\varphi_{11}v_{111} + \delta_{11}v_{112} + \varphi_{11}v_{113} + a_7(1)v_{211} + \eta_{11}v_{212} + a_7(1)v_{213} &= b_{112} \\
\varphi_{11}v_{112} + \delta_{11}v_{113} + \varphi_{11}v_{114} + a_7(1)v_{212} + \eta_{11}v_{213} + a_7(1)v_{214} &= b_{113} \\
\dots \\
\varphi_{11}v_{11(P-1)} + \delta_{11}v_{11P} + a_7(1)v_{21(P-1)} + \eta_{11}v_{21P} &= b_{11P}
\end{aligned}
\tag{5.24a}
$$

Now once again collect the second equations from each of (5.23) and consider as a second group of equations

$$
\begin{aligned}
\delta_{12}v_{121} + \varphi_{12}v_{122} + \eta_{12}v_{221} + a_7(1)v_{222} &= b_{121} \\
\varphi_{12}v_{121} + \delta_{12}v_{122} + \varphi_{12}v_{123} + a_7(1)v_{221} + \eta_{12}v_{222} + a_7(1)v_{223} &= b_{122} \\
\varphi_{12}v_{122} + \delta_{12}v_{123} + \varphi_{12}v_{124} + a_7(1)v_{222} + \eta_{12}v_{223} + a_7(1)v_{224} &= b_{123} \\
\dots \\
\varphi_{12}v_{12(P-1)} + \delta_{12}v_{12P} + a_7(1)v_{22(P-1)} + \eta_{12}v_{22P} &= b_{12P}
\end{aligned}
\tag{5.24b}
$$

Similarly collect the last equations of (5.23) and considering as a last group of equations

$$
\begin{aligned}
\delta_{1N}v_{1N1} + \varphi_{1N}v_{1N2} + \eta_{1N}v_{2N1} + a_7(1)v_{2N2} &= b_{1N1} \\
\varphi_{1N}v_{1N1} + \delta_{1N}v_{1N2} + \varphi_{1N}v_{1N3} + a_7(1)v_{2N1} + \eta_{1N}v_{2N2} + a_7(1)v_{2N3} &= b_{1N2} \\
\varphi_{1N}v_{1N2} + \delta_{1N}v_{1N3} + \varphi_{1N}v_{1N4} + a_7(1)v_{2N2} + \eta_{1N}v_{2N3} + a_7(1)v_{2N4} &= b_{1N3} \\
\dots \\
\varphi_{1N}v_{1N(P-1)} + \delta_{1N}v_{1NP} + a_7(1)v_{2N(P-1)} + \eta_{1N}v_{2NP} &= b_{1NP}
\end{aligned}
\tag{5.24c}
$$

Now we write these set of equations (5.24a) to (5.24c) in matrix form as

$$\begin{pmatrix} \delta_{1j} & \varphi_{1j} & & & \\ \varphi_{1j} & \delta_{1j} & \varphi_{1j} & & \\ & \varphi_{1j} & \delta_{1j} & \varphi_{1j} & \\ & & & \ddots & \\ & & & \varphi_{1j} & \delta_{1j} \end{pmatrix} \begin{pmatrix} v_{1j1} \\ v_{1j2} \\ v_{1j3} \\ \vdots \\ v_{1jP} \end{pmatrix} + \begin{pmatrix} \eta_{1j} & a_7(1) & & & \\ a_7(1) & \eta_{1j} & a_7(1) & & \\ & a_7(1) & \eta_{1j} & a_7(1) & \\ & & & \ddots & \\ & & & a_7(1) & \eta_{1j} \end{pmatrix} \begin{pmatrix} v_{2j1} \\ v_{2j2} \\ v_{2j3} \\ \vdots \\ v_{2jP} \end{pmatrix} = \begin{pmatrix} b_{1j1} \\ b_{1j2} \\ b_{1j3} \\ \vdots \\ b_{1jP} \end{pmatrix}$$

$$j = 1, 2, \ldots, N \qquad (5.25)$$

Let $\mathcal{F}_{ij} = \begin{pmatrix} \delta_{ij} & \varphi_{ij} & & & \\ \varphi_{ij} & \delta_{ij} & \varphi_{ij} & & \\ & \varphi_{ij} & \delta_{ij} & \varphi_{ij} & \\ & & & \ddots & \\ & & & \varphi_{ij} & \delta_{ij} \end{pmatrix}$, $\mathbf{W}_{ij} = \begin{pmatrix} v_{ij1} \\ v_{ij2} \\ v_{ij3} \\ \vdots \\ v_{ijP} \end{pmatrix}$ and $\bar{\mathbf{B}}_{ij} = \begin{pmatrix} b_{ij1} \\ b_{ij2} \\ b_{ij3} \\ \vdots \\ b_{ijP} \end{pmatrix}$

and $C_{ij} = \begin{pmatrix} \eta_{ij} & a_7(1) & & & \\ a_7(1) & \eta_{ij} & a_7(1) & & \\ & a_7(1) & \eta_{ij} & a_7(1) & \\ & & & \ddots & \\ & & & a_7(1) & \eta_{ij} \end{pmatrix}$

Thus we write the jth equation of (5.22) as

$$\mathcal{F}_{1j}\mathbf{W}_{1j} + C_{1j}\mathbf{W}_{2j} = \bar{\mathbf{B}}_{1j}$$

Let $F_i = \begin{pmatrix} \mathcal{F}_{ij} & & & & \\ & \mathcal{F}_{ij} & & & \\ & & \mathcal{F}_{ij} & & \\ & & & \ddots & \\ & & & & \mathcal{F}_{ij} \end{pmatrix}$, $\mathbf{C}_i = \begin{pmatrix} \mathcal{C}_{ij} & & & & \\ & C_{ij} & & & \\ & & \mathcal{C}_{ij} & & \\ & & & \ddots & \\ & & & & \mathcal{C}_{ij} \end{pmatrix}$ both are of order NP

$\mathbf{w}_i = [\mathbf{W}_{i1}\ \mathbf{W}_{i2}\ \mathbf{W}_{i3}\ \ldots\ \mathbf{W}_{iN}]^T$ and $\bar{B}_i = [\bar{\mathbf{B}}_{i1}, \bar{\mathbf{B}}_{i2}, \cdots, \bar{\mathbf{B}}_{iN}]^T$

Thus the first equation of (5.16) can be written as

$$F_1\mathbf{w}_1 + C_1\mathbf{w}_2 = \bar{\mathrm{B}}_1$$

By the same procedure as above after pre-multiplying by $\mathbb{Q}^T$ and make use of (5.20) we can write the second equation of (5.16) i.e. $T_2\mathbf{U}_1 + R_2\mathbf{U}_2 + S_2\mathbf{U}_3 = \mathbf{B}_2$ as

$$\begin{pmatrix} Z_2 & T_2'' & & & \\ T_2'' & Z_2 & T_2'' & & \\ & T_2'' & Z_2 & T_2'' & \\ & & & \ddots & \\ & & & T_2'' & Z_2 \end{pmatrix} \begin{pmatrix} \mathbf{v}_{11} \\ \mathbf{v}_{12} \\ \mathbf{v}_{13} \\ \vdots \\ \mathbf{v}_{1(P-1)} \\ \mathbf{v}_{1P} \end{pmatrix} + \begin{pmatrix} \mathrm{X}_2 & Y_2 & & & & \\ Y_2 & \mathrm{X}_2 & Y_2 & & & \\ & Y_2 & \mathrm{X}_2 & Y_2 & & \\ & & & \ddots & & \\ & & & Y_2 & \mathrm{X}_2 & Y_2 \\ & & & & Y_2 & \mathrm{X}_2 \end{pmatrix} \begin{pmatrix} \mathbf{v}_{21} \\ \mathbf{v}_{22} \\ \mathbf{v}_{23} \\ \vdots \\ \mathbf{v}_{2(P-1)} \\ \mathbf{v}_{2P} \end{pmatrix} +$$

$$\begin{pmatrix} H_1 & S_1'' & & & & \\ S_1'' & H_1 & S_1'' & & & \\ & S_1'' & H_1 & S_1'' & & \\ & & & \ddots & & \\ & & & S_1'' & H_1 & S_1'' \\ & & & & S_1'' & H_1 \end{pmatrix} \begin{pmatrix} \mathbf{v}_{31} \\ \mathbf{v}_{32} \\ \mathbf{v}_{33} \\ \vdots \\ \mathbf{v}_{3(P-1)} \\ \mathbf{v}_{3P} \end{pmatrix} = \begin{pmatrix} \mathbf{b}_{21} \\ \mathbf{b}_{22} \\ \mathbf{b}_{23} \\ \vdots \\ \mathbf{b}_{2(P-1)} \\ \mathbf{b}_{2P} \end{pmatrix} \qquad (5.26)$$

After rearranging and applying the same process as in the first case, equation (5.26) can be written as

$$E_2\mathbf{w}_1 + F_2\mathbf{w}_2 + C_2\mathbf{w}_3 = \bar{\mathrm{B}}_2$$

where $$E_i = \begin{pmatrix} E_{ij} & & & & \\ & E_{ij} & & & \\ & & E_{ij} & & \\ & & & \ddots & \\ & & & & E_{ij} \end{pmatrix}$$

and $E_{ij} = \begin{pmatrix} \zeta_{ij} & a_8(i) & & & & \\ a_8(i) & \zeta_{ij} & a_8(i) & & & \\ & a_8(i) & \zeta_{ij} & a_8(i) & & \\ & & & & \ddots & \\ & & & & a_8(i) & \zeta_{ij} \end{pmatrix}$

Thus, in general, pre-multiplying (5.16) by $\mathbb{Q}^T$ and make use of (5.20), and rearranging the equations, we obtain

$$\begin{aligned} F_1\mathbf{w}_1 + C_1\mathbf{w}_2 &= \bar{\mathrm{B}}_1 \\ E_2\mathbf{w}_1 + F_2\mathbf{w}_2 + C_2\mathbf{w}_3 &= \bar{\mathrm{B}}_2 \\ E_3\mathbf{w}_2 + F_3\mathbf{w}_3 + C_3\mathbf{w}_4 &= \bar{\mathrm{B}}_3 \\ &\cdots \\ E_M\mathbf{w}_{M-1} + F_M\mathbf{w}_M &= \bar{\mathrm{B}}_M \end{aligned} \tag{5.27}$$

Observe that

$\mathbb{Q}^T \mathrm{F}_i \mathbb{Q} = diag(\Phi_{i1}, \Phi_{i2}, \Phi_{i3}, ..., \Phi_{iP}) = \Lambda_i$ (say) where $\Phi_{ik} = diag(\lambda_{i1k}, \lambda_{i2k}, \lambda_{i3k}, ..., \lambda_{iNk})$

$\mathbb{Q}^T \mathrm{C}_i \mathbb{Q} = diag(L_{i1}, L_{i2}, L_{i3}, ..., L_{iP}) = \mathcal{L}_i$ (say) where $L_{ik} = diag(\tau_{i1k}, \tau_{i2k}, ..., \tau_{iNk})$

$\mathbb{Q}^T E_i \mathbb{Q} = diag(G_{i1}, G_{i2}, G_{i3}, ..., G_{iP}) = \mathcal{E}_i$ (say) where $G_{ik} = diag(\mu_{i1k}, \mu_{i2k}, ..., \mu_{iNk})$

Note that

$$\lambda_{ijk} = \delta_{ij} + 2\varphi_{ij}\cos\left(\frac{k\pi}{P+1}\right)$$

$$\tau_{ijk} = \eta_{ij} + 2a_7(i)\cos\left(\frac{k\pi}{P+1}\right) \text{ and}$$

$$\mu_{ijk} = \zeta_{ij} + 2a_8(i)\cos\left(\frac{k\pi}{P+1}\right)$$

$i = 1, 2, 3, \ldots, M$, $j = 1, 2, 3, \ldots N$ and $k = 1, 2, 3, \ldots,, P$

$$\text{Let}\quad \mathbb{Q}^T \mathbf{w}_i = \Psi_i \Rightarrow \mathbf{w}_i = \mathbb{Q}\,\Psi_i$$

$$\mathbb{Q}^T \bar{B}_i = \Gamma_i \Rightarrow \bar{B}_i = \mathbb{Q}\Gamma_i \tag{5.28}$$

where $\Psi_i = [\boldsymbol{\Psi}_{i1}\ \boldsymbol{\Psi}_{i2}\ \boldsymbol{\Psi}_{i3} \ldots \boldsymbol{\Psi}_{iP}]^T$ and $\boldsymbol{\Psi}_{ik} = [\psi_{i1k}\ \ \psi_{i2k}\ \ \psi_{i3k}\ \ldots\ \psi_{iNk}]^T$

$\Gamma_i = [\boldsymbol{\beta}_{i1}, \boldsymbol{\beta}_{i2}, \boldsymbol{\beta}_{i3}, \ldots, \boldsymbol{\beta}_{iP}]^T$ and $\boldsymbol{\beta}_{ik} = [\beta_{i1k}, \beta_{i2k}, \ldots, \beta_{iNk}]^T$

Pre-multiplying (5.27) by $\mathbb{Q}^T$ and make use of (5.28), we get

$$\begin{aligned}
\Lambda_1\Psi_1 + \mathcal{L}_1\Psi_2 &= \Gamma_1 \\
\mathcal{E}_2\Psi_1 + \Lambda_2\Psi_2 + \mathcal{L}_2\Psi_3 &= \Gamma_2 \\
\mathcal{E}_3\Psi_2 + \Lambda_3\Psi_3 + \mathcal{L}_3\Psi_4 &= \Gamma_3 \\
&\ldots \\
\mathcal{E}_M\Psi_{M-1} + \Lambda_M\Psi_M &= \Gamma_M
\end{aligned} \tag{5.29}$$

Now we write (5.29) turn by turn starting from the first equation i.e. $\Lambda_1\Psi_1 + \mathcal{L}_1\Psi_2 = \Gamma_1$ as

$$\begin{aligned}
\lambda_{111}\psi_{111} + \tau_{111}\psi_{211} &= \beta_{111} \\
\lambda_{121}\psi_{121} + \tau_{121}\psi_{221} &= \beta_{121} \\
\lambda_{131}\psi_{131} + \tau_{131}\psi_{231} &= \beta_{131} \\
&\ldots \\
\lambda_{1N1}\psi_{1N1} + \tau_{1N1}\psi_{2N1} &= \beta_{1N1}
\end{aligned}$$

$$\begin{aligned}
\lambda_{112}\psi_{112} + \tau_{112}\psi_{212} &= \beta_{112} \\
\lambda_{122}\psi_{122} + \tau_{122}\psi_{222} &= \beta_{122} \\
\lambda_{132}\psi_{132} + \tau_{132}\psi_{232} &= \beta_{132} \\
&\ldots \\
\lambda_{1N2}\psi_{1N2} + \tau_{1N2}\psi_{2N2} &= \beta_{1N2} \\
\ldots \qquad \ldots &\qquad \ldots
\end{aligned} \tag{5.30a}$$

$$\lambda_{11P}\psi_{11P} + \tau_{11P}\psi_{21P} = \beta_{11P}$$
$$\lambda_{12P}\psi_{12P} + \tau_{12P}\psi_{22P} = \beta_{12P}$$
$$\lambda_{13P}\psi_{13P} + \tau_{13P}\psi_{23P} = \beta_{13P}$$
$$\cdots$$
$$\lambda_{1NP}\psi_{1NP} + \tau_{1NP}\psi_{2NP} = \beta_{1NP}$$

The second equation of (5.29) i.e. $\mathcal{E}_2\Psi_1 + \Lambda_2\Psi_2 + \mathcal{L}_2\Psi_3 = \Gamma_2$ is written as

$$\mu_{211}\psi_{111} + \lambda_{211}\psi_{211} + \tau_{211}\psi_{311} = \beta_{211}$$
$$\mu_{221}\psi_{121} + \lambda_{221}\psi_{221} + \tau_{221}\psi_{321} = \beta_{221}$$
$$\mu_{231}\psi_{131} + \lambda_{231}\psi_{231} + \tau_{231}\psi_{331} = \beta_{231}$$
$$\cdots$$
$$\mu_{2N1}\psi_{1N1} + \lambda_{2N1}\psi_{2N1} + \tau_{2N1}\psi_{3N1} = \beta_{2N1}$$

$$\mu_{212}\psi_{112} + \lambda_{212}\psi_{212} + \tau_{212}\psi_{312} = \beta_{212} \qquad (5.30b)$$
$$\mu_{222}\psi_{122} + \lambda_{222}\psi_{222} + \tau_{222}\psi_{322} = \beta_{222}$$
$$\mu_{232}\psi_{132} + \lambda_{232}\psi_{232} + \tau_{232}\psi_{332} = \beta_{232}$$
$$\cdots$$
$$\mu_{2n2}\psi_{1N2} + \lambda_{2N2}\psi_{2N2} + \tau_{2N2}\psi_{3N2} = \beta_{2N2}$$

$$\cdots \qquad \cdots \qquad \cdots$$

$$\mu_{21P}\psi_{11P} + \lambda_{21P}\psi_{21P} + \tau_{21P}\psi_{31P} = \beta_{21P}$$
$$\mu_{22P}\psi_{12P} + \lambda_{22P}\psi_{22P} + \tau_{22P}\psi_{32P} = \beta_{22P}$$
$$\mu_{23P}\psi_{13P} + \lambda_{23P}\psi_{23P} + \tau_{23P}\psi_{33P} = \beta_{23P}$$
$$\cdots$$
$$\mu_{2NP}\psi_{1NP} + \lambda_{2NP}\psi_{2NP} + \tau_{2NP}\psi_{3NP} = \beta_{2NP}$$

And the last equation of (5.29) i.e. $\mathcal{E}_M \Psi_{M-1} + \Lambda_M \Psi_M = \Gamma_M$ is written as

$$\mu_{M11}\psi_{(M-1)11} + \lambda_{M11}\psi_{M11} = \beta_{M11}$$
$$\mu_{M21}\psi_{(M-1)21} + \lambda_{M21}\psi_{M21} = \beta_{M21}$$
$$\mu_{M31}\psi_{(M-1)31} + \lambda_{M31}\psi_{M11} = \beta_{M31}$$
$$\ldots$$
$$\mu_{MN1}\psi_{(M-1)N1} + \lambda_{MN1}\psi_{MN1} = \beta_{MN1}$$
$$\mu_{M12}\psi_{(M-1)12} + \lambda_{M12}\psi_{M12} = \beta_{M12} \qquad (5.30c)$$
$$\mu_{M22}\psi_{(M-1)22} + \lambda_{M22}\psi_{M22} = \beta_{M22}$$
$$\mu_{M32}\psi_{(M-1)32} + \lambda_{M32}\psi_{M12} = \beta_{M32}$$
$$\ldots$$
$$\mu_{MN2}\psi_{(M-1)N2} + \lambda_{MN2}\psi_{MN2} = \beta_{MN2}$$
$$\ldots \quad \ldots \quad \ldots$$
$$\mu_{M1P}\psi_{(M-1)1P} + \lambda_{M1P}\psi_{M1P} = \beta_{M1P}$$
$$\mu_{M2P}\psi_{(M-1)2P} + \lambda_{M2P}\psi_{M1P} = \beta_{M2P}$$
$$\mu_{M3P}\psi_{(M-1)3P} + \lambda_{M3P}\psi_{M1P} = \beta_{M3P}$$
$$\ldots$$
$$\mu_{MNP}\psi_{(M-1)NP} + \lambda_{MNP}\psi_{MNP} = \beta_{MNP}$$

Now we collect the first equations of (5.30a) to (5.30c) , and get

$$\lambda_{111}\psi_{111} + \tau_{111}\psi_{211} = \beta_{111}$$
$$\mu_{211}\psi_{111} + \lambda_{211}\psi_{211} + \tau_{211}\psi_{311} = \beta_{211}$$
$$\mu_{311}\psi_{211} + \lambda_{311}\psi_{311} + \tau_{311}\psi_{411} = \beta_{311}$$
$$\ldots$$
$$\mu_{M11}\psi_{(M-1)11} + \lambda_{M11}\psi_{M11} = \beta_{M11}$$

$$\lambda_{112}\psi_{112} + \tau_{112}\psi_{212} = \beta_{112} \tag{5.31a}$$

$$\mu_{212}\psi_{112} + \lambda_{212}\psi_{212} + \tau_{212}\psi_{312} = \beta_{212}$$

$$\mu_{312}\psi_{212} + \lambda_{312}\psi_{312} + \mu_{312}\psi_{412} = \beta_{312}$$

$$\cdots$$

$$\mu_{M12}\psi_{(M-1)12} + \lambda_{M12}\psi_{M12} = \beta_{M12}$$

$$\cdots \qquad \cdots \qquad \cdots$$

$$\lambda_{11P}\psi_{11P} + \tau_{11P}\psi_{21P} = \beta_{11P}$$

$$\mu_{21P}\psi_{11P} + \lambda_{21P}\psi_{21P} + \tau_{21P}\psi_{31P} = \beta_{21P}$$

$$\mu_{31P}\psi_{21P} + \lambda_{31P}\psi_{31P} + \tau_{31P}\psi_{41P} = \beta_{31P}$$

$$\cdots$$

$$\mu_{M1P}\psi_{(M-1)1P} + \lambda_{M1P}\psi_{M1P} = \beta_{M1P}$$

Observe that here in the above set of equations $j = 1$, and for each $k = 1(1)P$ the coefficient matrix of the left hand side is a tri-diagonal matrix of order M and has the form

$$\mathcal{M}_{1k} = \begin{pmatrix} \lambda_{11k} & \tau_{11k} & & & & & \\ \mu_{21k} & \lambda_{21k} & \tau_{21k} & & & & \\ & \mu_{31k} & \lambda_{31k} & \tau_{31k} & & & \\ & & & & \ddots & & \\ & & & & \mu_{(M-1)1k} & \lambda_{(M-1)1k} & \tau_{(M-1)1k} \\ & & & & & \mu_{M1k} & \lambda_{M1k} \end{pmatrix}$$

Now collect again other groups of equations of (5.30) for $j = 2(1)N$, and get

$$\lambda_{1j1}\psi_{1j1} + \tau_{1j1}\psi_{2j1} = \beta_{1j1}$$

$$\mu_{2j1}\psi_{1j1} + \lambda_{2j1}\psi_{2j1} + \tau_{2j1}\psi_{3j1} = \beta_{2j1}$$

$$\mu_{3j1}\psi_{2j1} + \lambda_{3j1}\psi_{3j1} + \tau_{3j1}\psi_{4j1} = \beta_{3j1}$$

$$\cdots$$

$$\mu_{Mj1}\psi_{(M-1)j1} + \lambda_{Mj1}\psi_{Mj1} = \beta_{Mj1}$$

$$\lambda_{1j2}\psi_{1j2} + \tau_{1j2}\psi_{2j2} = \beta_{1j2} \tag{5.31b}$$
$$\mu_{2j2}\psi_{1j2} + \lambda_{2j2}\psi_{2j2} + \tau_{2j2}\psi_{3j2} = \beta_{2j2}$$
$$\mu_{3j2}\psi_{2j2} + \lambda_{3j2}\psi_{3j2} + \mu_{3j2}\psi_{4j2} = \beta_{3j2}$$
$$\cdots$$
$$\mu_{Mj2}\psi_{(M-1)j2} + \lambda_{Mj2}\psi_{Mj2} = \beta_{Mj2}$$
$$\cdots \quad \cdots \quad \cdots$$
$$\lambda_{1jP}\psi_{1jP} + \tau_{1jP}\psi_{2jP} = \beta_{1jP}$$
$$\mu_{2jP}\psi_{1jP} + \lambda_{2jP}\psi_{2jP} + \tau_{2jP}\psi_{3jP} = \beta_{2jP}$$
$$\mu_{3jP}\psi_{2jP} + \lambda_{3jP}\psi_{3jP} + \tau_{3jP}\psi_{4jP} = \beta_{3jP}$$
$$\cdots$$
$$\mu_{MjP}\psi_{(M-1)jP} + \lambda_{MjP}\psi_{MjP} = \beta_{MjP}$$

As we can see from these set of equations (5.31b), the matrix structure for $j = 2(1)N$ has the form

$$\mathcal{M}_{jk} = \begin{pmatrix} \lambda_{1jk} & \tau_{1jk} & & & & \\ \mu_{2jk} & \lambda_{2jk} & \tau_{2jk} & & & \\ & \mu_{3jk} & \lambda_{3jk} & \tau_{3jk} & & \\ & & & \ddots & & \\ & & & \mu_{(M-1)jk} & \lambda_{(M-1)jk} & \tau_{(M-1)jk} \\ & & & & \mu_{Mjk} & \lambda_{Mjk} \end{pmatrix} \tag{5.32}$$

This Matrix (5.32) is a tri-diagonal matrix and hence we solve for ψ_{ijk} by using Thomas algorithm [30]. Once we get ψ_{ijk} by the help (5.28) again we get all v_{ijk} and by the help of (5.20) again we get U_{ijk} and this solves our problem as desired. By doing this we generally reduce the number of computations and computational time.

5.4 Numerical Results

In order to test the efficiency and adaptability of the proposed method, computational experiments are done on seven selected problems that may arise in practice, for which the analytical solutions of U are known to us; and in some of the examples even we have considered for $r = 0$ regardless of the finite difference approximation when $r = 0$ is an interior or a boundary point. The computed solutions are found for all grid points incl. But here results are reported at some mesh points in terms of the absolute maximum error and are shown from table 5.1 to 5.7.

Example 5.1 Consider $\nabla^2 U = 0$ with the boundary conditions

$$U(0,\theta,z) = 0 \qquad U(1,\theta,z) = z\sin\theta$$

$$U(r,0,z) = 0 = U(r,\pi,z)\text{ , and}$$

$$U(r,\theta,0) = 0, \quad U(r,\theta,1) = r\sin\theta$$

The analytical solution is $U(r,\theta,z) = r\,z\sin\theta$ and the computed results of this example are shown in Table 5.1.

Example 5.2 Consider $\nabla^2 U = -\pi^2 r cos\theta \sin(\pi z)$ with the boundary conditions

$$U(1,\theta,z) = cos\theta\sin(\pi z), \quad U(2,\theta,z) = 2cos\theta\sin(\pi z)$$

$$U(r,0,z) = r\sin(\pi z),\ U\left(r,\frac{\pi}{2},z\right) = 0\text{ , and}$$

$$U(r,\theta,0) = U(r,\theta,1) = 0$$

The analytical solution is $U(r,\theta,z) = r\,cos\theta\sin(\pi z)$ and the computed results of this example are shown in Table 5.2.

Example 5.3 Consider $\nabla^2 U = -3cos\theta$ with the boundary conditions

$$U(0,\theta,z) = U(1,\theta,z) = -2z\text{ ,}$$

$$U(r,0,z) = r(1-r) - 2z, \quad U\left(r,\frac{\pi}{2},z\right) = -2z$$

$$U(r,\theta,0) = r(1-r)cos\theta\text{ ,} \quad U(r,\theta,1) = r(1-r)cos\theta - 2$$

The analytical solution is $U(r,\theta,z) = r(1-r)\cos\theta - 2z$ and the computed results of this example are shown in Table 5.3.

Example 5.4 Consider $\nabla^2 U = -\pi^2\left(r^2 - \frac{1}{r^2}\right)\sin(2\theta)\sin(\pi z)$

with the boundary conditions

$$U(1,\theta,z) = 0,\ U(2,\theta,z) = \frac{15}{4}\sin(2\theta)\sin(\pi z),$$

$$U(r,0,z) = 0 = U\left(r,\frac{\pi}{2},z\right) \text{ and } U(r,\theta,0) = 0 = U(r,\theta,1)$$

The analytical solution is $U = \left(r^2 - \frac{1}{r^2}\right)\sin(2\theta)\sin(\pi z)$ and the computed results of this example are shown in Table 5.4.

Example 5.5 Consider $\nabla^2 U = (8rz(1-z) - 2r^3)(\sin\theta + \cos\theta)$, where $0 \le \theta < 2\pi$ with the boundary conditions

$$U(0,\theta,z) = 0,\ \ U(1,\theta,z) = z(1-z)(\sin\theta + \cos\theta)$$

$$U(r,\theta,0) = 0,\ \ U(r,\theta,1) = 0$$

The analytical solution is $U = r^3 z(1-z)(\sin\theta + cos\theta)$ and the computed results of this example are shown in Table V.

This example was considered by M.C. Lai [64] as a test problem and our results are better than their results in terms of accuracy. For instance, for (8,16,16) the maximum absolute error in their result is 9.1438e-004 and while ours is 3.28689e-004.

Example 5.6 Consider $\nabla^2 U = 6rz\,cos\theta$, where $0 \le \theta < 2\pi$ with the boundary conditions

$$U(0,\theta,z) = 0,\ \ U(1,\theta,z) = zcos^3\theta\ ,$$

$$U(r,\theta,0) = 0 \text{ and } (r,\theta,1) = r^3cos^3\theta$$

The analytical solution is $U = r^3 zcos^3\theta$ and the computed results of this example are shown in Table 5.6.

Example 5.7 Consider $\nabla^2 U = -\pi^2\left(r^2 - \frac{1}{r^2}\right)\sin(2\theta)\sin(\pi z)$, where $0 \le \theta < 2\pi$

with the boundary conditions

$$U(1,\theta,z) = 0, U(2,\theta,z) = \frac{15}{4}\sin(2\theta)\sin(\pi z),\ \ U(r,\theta,0) = 0 = U(r,\theta,1)$$

The analytical solution is $U = \left(r^2 - \frac{1}{r^2}\right)\sin(2\theta)\sin(\pi z)$ and the computed results of this example are shown in Table 5.7.

Table 5.1

The maximum absolute error of example 5.1

(N, P, M)	Max. abs. error		(N, P, M)	Max. abs. error
(9,9,9)	3.51670e-005		$(29,9,39)$	1.37257e-006
(9,9,29)	1.46565e-005		$(29,19,9)$	4.15180e-006
(9,19,9)	3.53325e-005		$(29,29,19)$	2.45633e-006
(9,19,19)	2.06578e-005		$(29,29,29)$	1.74383e-006
(9,29,39)	1.13280e-005		(29,39,19)	2.45924e-006
$(9,39,29)$	1.46438e-005		(29,39,29)	1.74829e-006
$(19,9,9)$	9.21838e-006		(39,9,19)	1.35171e-006
$(19,9,19)$	5.32850e-006		(39,9,39)	7.75143e-007
(19,19,19)	5.46733e-006		$(39,19,29)$	9.82456e-007
(19,29,39)	3.02425e-006		$(39,29,19)$	1.38647e-006
(19,39,9)	9.27536e-006		$(39,39,9)$	2.34568e-006
(19,39,39)	3.02636e-006		$(39,39,39)$	7.68613e-007

Table 5.2

The maximum absolute error of example 5.2

(N, P, M)	Max. abs. error		(N, P, M)	Max. abs. error
$(9,9,9)$	2.93159e-003		$(29,9,39)$	2.98714e-003
$(9,9,29)$	2.95649e-003		$(29,19,9)$	7.39877e-004
$(9,19,9)$	7.32025e-004		$(29,29,19)$	3.31950e-004
$(9,19,19)$	7.38648e-004		$(29,29,29)$	3.31771e-004
$(9,29,39)$	3.27574e-004		$(29,39,19)$	1.86718e-004
$(9,39,29)$	1.83450e-004		$(29,39,29)$	1.86618e-004
$(19,9,9)$	2.95328e-003		$(39,9,19)$	2.98618e-003
$(19,9,19)$	2.97861e-003		$(39,9,39)$	2.98710e-003
$(19,19,19)$	7.44907e-004		$(39,19,29)$	7.46353e-004
$(19,29,39)$	3.31145e-004		$(39,29,19)$	3.31953e-004
$(19,39,9)$	1.84585e-004		$(39,39,9)$	1.84916e-004
$(19,39,39)$	1.86232e-004		$(39,39,39)$	1.86784e-004

Table 5.3

The maximum absolute error of example 5.3

(N,P,M)	Max. abs. error		(N,P,M)	Max. abs. error
$(9,9,9)$	1.81124e-004		$(29,9,39)$	1.16544e-005
$(9,9,29)$	4.45263e-005		$(29,19,9)$	1.82484e-004
$(9,19,9)$	1.81185e-004		$(29,29,19)$	4.61297e-005
$(9,19,19)$	6.02480e-005		$(29,29,29)$	2.04978e-005
$(9,29,39)$	3.97430e-005		$(29,39,19)$	4.61300e-005
$(9,39,29)$	4.46327e-005		$(29,39,29)$	2.04979e-005
$(19,9,9)$	1.81939e-004		$(39,9,19)$	4.61828e-005
$(19,9,19)$	4.59426e-005		$(39,9,39)$	1.17058e-005
$(19,19,19)$	4.59583e-005		$(39,19,29)$	2.05467e-005
$(19,29,39)$	1.50833e-005		$(39,29,19)$	4.61879e-005
$(19,39,9)$	1.82013e-004		$(39,39,9)$	1.82652e-004
$(19,39,39)$	1.50852e-005		$(39,39,39)$	1.15493e-005

Table 5.4

The maximum absolute error of example 5.4

(N,P,M)	Max. abs. error		(N,P,M)	Max. abs. error
$(9,9,9)$	3.68396e-003		$(29,9,39)$	3.98135e-003
$(9,9,29)$	4.07400e-003		$(29,19,9)$	6.33780e-004
$(9,19,9)$	7.68229e-004		$(29,29,19)$	3.64070e-004
$(9,19,19)$	1.04366e-003		$(29,29,29)$	4.17368e-004
$(9,29,39)$	5.73867e-004		$(29,39,19)$	1.75928e-004
$(9,39,29)$	3.62888e-004		$(29,39,29)$	2.24720e-004
$(19,9,9)$	3.58663e-003		$(39,9,19)$	3.89251e-003
$(19,9,19)$	3.92179e-003		$(39,9,39)$	3.97355e-003
$(19,19,19)$	9.34774e-004		$(39,19,29)$	9.60868e-004
$(19,29,39)$	4.61633e-004		$(39,29,19)$	3.55183e-004
$(19,39,9)$	7.29565e-004		$(39,39,9)$	7.23913e-004
$(19,39,39)$	2.68695e-004		$(39,39,39)$	2.34933e-004

Table 5.5

The maximum absolute error of example 5.5

(N, P, M)	Max. abs. error		(N, P, M)	Max. abs. error
$(10,9,9)$	5.97062e-004		$(30,9,39)$	1.65910e-004
$(10,9,29)$	4.42157e-004		$(30,19,9)$	4.11093e-004
$(10,19,9)$	5.09956e-004		$(30,29,19)$	1.01380e-004
$(10,19,19)$	3.72361e-004		$(30,29,29)$	6.92680e-005
$(10,29,39)$	3.26827e-004		$(30,39,19)$	1.03392e-004
$(10,39,29)$	3.27891e-004		$(30,39,29)$	6.38312e-005
$(20,9,9)$	3.72181e-004		$(40,9,19)$	1.80739e-004
$(20,9,19)$	2.39220e-004		$(40,9,39)$	1.49613e-004
$(20,19,19)$	1.52973e-004		$(40,19,29)$	6.95506e-005
$(20,29,39)$	1.04227e-004		$(40,29,19)$	1.06985e-004
$(20,39,9)$	3.96923e-004		$(40,39,9)$	4.28673e-004
$(20,39,39)$	9.84850e-005		$(40,39,39)$	3.91182e-005

Table 5.6

The maximum absolute error of example 5.6

(N, P, M)	Max. abs. error		(N, P, M)	Max. abs. error
$(10,9,9)$	3.04648e-003		$(30,9,39)$	3.06543e-004
$(10,9,29)$	3.18297e-003		$(30,19,9)$	2.00777e-004
$(10,19,9)$	3.05549e-003		$(30,29,19)$	2.85659e-004
$(10,19,19)$	3.16606e-003		$(30,29,29)$	3.02059e-004
$(10,29,39)$	3.19893e-003		$(30,39,19)$	2.85718e-004
$(10,39,29)$	3.19459e-003		$(30,39,29)$	3.02122e-004
$(20,9,9)$	6.03143e-004		$(40,9,19)$	1.42033e-004
$(20,9,19)$	6.87721e-004		$(40,9,39)$	1.62951e-004
$(20,19,19)$	6.89766e-004		$(40,19,29)$	1.58004e-004
$(20,29,39)$	7.13568e-004		$(40,29,19)$	1.42553e-004
$(20,39,9)$	6.05428e-004		$(40,39,9)$	1.58596e-004
$(20,39,39)$	7.13712e-004		$(40,39,39)$	1.63583e-004

Table 5.7

The maximum absolute error of example 5.7

(N,P,M)	Max. abs. error		(N,P,M)	Max. abs. error
(10,9,9)	3.00418e-003		(30,9,39)	4.13706e-003
(10,9,29)	2.36262e-003		(30,19,9)	8.41886e-004
(10,19,9)	4.54924e-003		(30,29,19)	5.78646e-004
(10,19,19)	4.13088e-003		(30,29,29)	6.30456e-004
(10,29,39)	4.49099e-003		(30,39,19)	3.91287e-004
(10,39,29)	4.67590e-003		(30,39,29)	4.41870e-004
(20,9,9)	3.54731e-003		(40,9,19)	4.01710e-003
(20,9,19)	3.84938e-003		(40,9,39)	4.09806e-003
(20,19,19)	1.08325e-003		(40,19,29)	1.10354e-003
(20,29,39)	6.43373e-004		(40,29,19)	5.01257e-004
(20,39,9)	9.83307e-004		(40,39,9)	7.61254e-004
(20,39,39)	4.66590e-004		(40,39,39)	3.82100e-004

5.5 Conclusion

In this work, we have transformed the three dimensional Poisson's equation in cylindrical coordinates system in to a system of algebraic linear equations using its equivalent fourth order finite difference approximation scheme. The resulting large number of algebraic equation is, then, systematically arranged in order to get a block matrix. Based on the extension of Hockney's method we reduced the obtained matrix in to a block tridiagonal matrix, and each block is solved by the help of Thomas algorithm.[30] We have successfully implemented this method to find the solution of the three dimensional Poisson's equation in cylindrical coordinates system and it is found that the method can easily be applied and adapted to find a solution of some related applied problems. The method produced accurate results considering double precision. This method is direct and allows considerable savings in computer storage as well as execution speed.

Therefore, the method is suitable to apply on some three dimensional Poisson's equations.

CHAPTER VI

Numerical Solution of the Three Dimensional Poisson's Equation in Cylindrical Coordinates System when r=0 is an interior or a boundary point

6.1 Introduction

In cylindrical coordinates system (r, θ, z) the three dimensional Poisson's equation given by

$$\frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r} + \frac{1}{r^2}\frac{\partial^2 U}{\partial \theta^2} + \frac{\partial^2 U}{\partial z^2} = f(r, \theta, z) \tag{6.1}$$

has a wide range of application in engineering and physics especially when $r = 0$.

For the numerical solution of the three dimensional Poisson's equation in cylindrical coordinates system several attempts have been made for $r \neq 0$. But in particular if $r = 0$ is an interior point or a boundary point, the numerical solution of this equation because of the factors $\frac{1}{r}$ and $\frac{1}{r^2}$ in (6.1) requires special attention.

In this regard, for $r = 0$ *Iyengar and Manohar* [102] derived fourth-order difference schemes for the solution of the Poisson equation which occurs in problems of heat transfer; *Iyengar* and *Goyal* [101] implement and compare S- and V-cycles in the multigrid context for the fourth-order method derived in[102]; *Mittal and Gahlaut* [90] have developed high order finite difference schemes of second and fourth order in polar coordinates using a direct method similar to Hockney's method; Mittal and Gahlaut [91] have developed a second and fourth order finite difference scheme to solve Poisson's equation in the case of cylindrical symmetry. To obtain a good approximate solution for the three dimensional Poisson's equation in cylindrical coordinates system is not an easy task.

In this chapter, we develop a second and fourth order finite difference approximation scheme when $r = 0$ is an interior point or a boundary point, and solve the resulting large algebraic system of linear equations.

6.2 Finite Difference Approximation

Consider the three dimensional Poisson's equation in cylindrical coordinates (r, θ, z) (6.1) given by

$$\frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r} + \frac{1}{r^2}\frac{\partial^2 U}{\partial \theta^2} + \frac{\partial^2 U}{\partial z^2} = f(r, \theta, z) \quad on\ D \quad \text{and}$$

$$U(r, \theta, z) = g(r, \theta, z) \quad \text{on } C \qquad (6.2)$$

where C is the boundary of D; and D is

a) $D_1 = \{(r, \theta, z) : R_0 \le r < R_1\ ,\ a < z < b, \theta_0 < \theta < \theta_1\ ,\ \theta_0 < \theta_1 < 2\pi\}$ and

b) $D_2 = \{(r, \theta, z) : R_0 \le r < R_1\ ,\ a < z < b,\ 0 \le \theta < 2\pi\}$

Consider figure 4.1 in chapter IV as the geometry of the problem.

For the discretization of (6.2) we consider $r_i = R_0 + i\Delta r,\ \ \theta_j = \theta_0 + j\Delta\theta$ and

$z_k = a + k\Delta z, \qquad i = 1(1)M,\ j = 1(1)N$ and $k = 1(1)P$.

When $r = 0$ in (6.2), the Poisson's equation becomes singular and to obtain the solution we need a difference equation which is valid at this point.

As $r \to 0$ from equation (6.2), we get

$$4U_{rr} + U_{rr\theta\theta} + 2U_{zz} = 2f(0, \theta, z) \qquad (6.3)$$

Now using the idea what we have discussed in chapters IV and V for $r \ne 0$ and the approximation scheme (6.3) when $r = 0$ is an interior point or a boundary point, we develop first a second-order and then a fourth-order numerical scheme.

CASE I SECOND-ORDER APPROXIMATION SCHEME

Consider the approximations

$$\frac{\partial^2 U}{\partial r^2}=\frac{U_{i+1,j,k}-2U_{i,j,k}+U_{i-1,j,k}}{(\Delta r)^2}+O\left((\Delta r)^2\right)$$

$$\frac{\partial^2 U}{\partial \theta^2}=\frac{U_{i,j+1,k}-2U_{i,j,k}+U_{i,j-1,k}}{(\Delta \theta)^2}+O\left((\Delta \theta)^2\right) \tag{6.4}$$

$$\frac{\partial^2 U}{\partial z^2}=\frac{U_{i,j,k+1}-2U_{i,j,k}+U_{i,j,k-1}}{(\Delta z)^2}+O\left((\Delta z)^2\right)$$

and substituting (6.4) in (6.3) when $r=0$, we get

$$4\left(\frac{U_{i+1,j,k}-2U_{i,j,k}+U_{i-1,j,k}}{(\Delta r)^2}+O\left((\Delta r)^2\right)\right)+\left(\frac{U_{i+1,j,k}-2U_{i,j,k}+U_{i-1,j,k}}{(\Delta r)^2}+O\left((\Delta r)^2\right)\right)_{\theta\theta}$$
$$+2\left(\frac{U_{i,j,k+1}-2U_{i,j,k}+U_{i,j,k-1}}{(\Delta z)^2}+O\left((\Delta z)^2\right)\right)=2f(0,\theta,z) \tag{6.5}$$

But for $r=0$, $U_r=U_{\theta\theta}=U_{r\theta\theta}=0$ and since

$$\frac{U_{i+1,j,k}-2U_{i,j,k}+U_{i-1,j,k}}{(\Delta r)^2}+O\left((\Delta r)^2\right)=\frac{2U_{i+1,j,k}-2U_{i,j,k}}{(\Delta r)^2}+O\left((\Delta r)^2\right),$$

equation (6.5) becomes

$$4\left(\frac{2U_{i+1,j,k}-2U_{i,j,k}}{(\Delta r)^2}+O\left((\Delta r)^2\right)\right)+\left(\frac{2U_{i+1,j,k}-2U_{i,j,k}}{(\Delta r)^2}+O\left((\Delta r)^2\right)\right)_{\theta\theta}$$
$$+2\left(\frac{U_{i,j,k+1}-2U_{i,j,k}+U_{i,j,k-1}}{(\Delta z)^2}+O\left((\Delta z)^2\right)\right)=2f\left(0,\theta,z\right)$$

$$8\left(\frac{U_{1,j,k}-U_{0,j,k}}{(\Delta r)^2}+O\left((\Delta r)^2\right)\right)+2\left(\frac{U_{1,j+1,k}-2U_{1,j,k}+U_{1,j-1,k}-U_{0,j+1,k}+2U_{0,j,k}-U_{0,j-1,k}}{(\Delta r)^2(\Delta\theta)^2}\right.$$
$$\left.+O\left((\Delta r)^2\right)+O\left((\Delta r)^2\right)\right)+2\left(\frac{U_{i,j,k+1}-2U_{i,j,k}+U_{i,j,k-1}}{(\Delta z)^2}+O\left((\Delta z)^2\right)\right)=2f(0,\theta,z) \tag{6.6}$$

Let $\omega_i = \dfrac{\Delta r}{2r_i}$, $\alpha_i = \dfrac{(\Delta r)^2}{r_i^2(\Delta\theta)^2}$, $\rho = \dfrac{(\Delta r)^2}{(\Delta z)^2}$ and $b = \dfrac{1}{(\Delta\theta)^2}$;

truncating higher order differences of (6.6), multiplying both sides of (6.6) by $(\Delta r)^2$ and rearranging and simplifying further, we have

$$\begin{aligned}(\Delta r)^2 f_{0,j,k} &= (4-2b)U_{1,j,k} + (4-2\rho+2b)U_{0,j,k} \\ &+\rho\left(U_{0,j,k+1}+U_{0,j,k-1}\right)+b\left(U_{1,j+1,k}+U_{1,j-1,k}-U_{0,j+1,k}-U_{0,j-1,k}\right)\end{aligned} \tag{6.7}$$

Since $U_{0,j,k}$, $U_{0,j+1,k}$ and $U_{0,j-1,k}$ are the same point and thus we write equation (6.7) as

$$(\Delta r)^2 f_{0,j,k} = (4-2b)U_{1,j,k} + (4-2\rho)U_{0,j,k} + \rho(U_{0,j,k+1}+U_{0,j,k-1}) + b(U_{1,j+1,k}+U_{1,j-1,k}) \tag{6.8}$$

and for $r \neq 0$, we have from chapter IV

$$(1+\omega_i)U_{i+1,j,k} + (1-\omega_i)U_{i-1,j,k} + \alpha_i(U_{i,j+1,k}+U_{i,j-1,k}) + \rho(U_{i,j,k+1}+U_{i,j,k-1}) + y_i U_{i,j,k} = (\Delta r)^2 f_{i,j,k} \tag{6.9}$$

Now by combining (6.8) and (6.9), we get

$$\begin{cases}(\Delta r)^2 f_{0,j,k} = (4-2b)U_{1,j,k} + (4-2\rho)U_{0,j,k} + \rho(U_{0,j,k+1}+U_{0,j,k-1}) + b(U_{1,j+1,k}+U_{1,j-1,k}), \\ (\Delta r)^2 f_{i,j,k} = (1+\omega_i)U_{i+1,j,k} + (1-\omega_i)U_{i-1,j,k} + \alpha_i(U_{i,j+1,k}+U_{i,j-1,k}) + \rho(U_{i,j,k+1}+U_{i,j,k-1}) \\ \qquad\qquad + y_i U_{i,j,k}\end{cases} \tag{6.10}$$

Considering equation (6.10) first in the θ direction, next z direction and lastly r direction, equation (6.2) can be put in matrix form as

$$A\mathbf{U} = \mathcal{B} \tag{6.11}$$

$$\text{where } A = \begin{pmatrix} R_0 & S_0 & & & & & \\ T_1 & R_1 & S_1 & & & & \\ & T_2 & R_2 & S_2 & & & \\ & & & \ddots & & & \\ & & & & T_{M-1} & R_{M-1} & S_{M-1} \\ & & & & & T_M & R_M \end{pmatrix},$$

it has $M+1$ blocks and each block is of order NP by NP

$$\text{where } R_0 = \begin{pmatrix} B & C & & & & & \\ C & B & C & & & & \\ & C & B & C & & & \\ & & & \ddots & & & \\ & & & & C & B & C \\ & & & & & C & B \end{pmatrix}$$

and $B = diag(4-2\rho, 4-2\rho, \cdots, 4-2\rho)$,

$C = diag(\rho, \rho, \cdots, \rho)$

$$S_0 = diag(D, D, \cdots, D), \text{ where } D = \begin{pmatrix} 4-2b & b & & & & & \\ b & 4-2b & b & & & & \\ & b & 4-2b & b & & & \\ & & & \ddots & & & \\ & & & & b & 4-2b & b \\ & & & & & b & 4-2b \end{pmatrix}$$

For $i = 1, 2, 3, \cdots, M$

$$R_i = \begin{pmatrix} L_i & \mathrm{T} & & & & & \\ \mathrm{T} & L_i & \mathrm{T} & & & & \\ & \mathrm{T} & L_i & \mathrm{T} & & & \\ & & & \ddots & & & \\ & & & & \mathrm{T} & L_i & \mathrm{T} \\ & & & & & \mathrm{T} & L_i \end{pmatrix} \text{ is of order } NP \text{ by } NP$$

For domain D_1,

$$L_i = \begin{pmatrix} y_i & \alpha_i & & & & & \\ \alpha_i & y_i & \alpha_i & & & & \\ & \alpha_i & y_i & \alpha_i & & & \\ & & & \ddots & & & \\ & & & & \alpha_i & y_i & \alpha_i \\ & & & & & \alpha_i & y_i \end{pmatrix} \tag{6.12}$$

and **for D_2**,

$$L_i = \begin{pmatrix} y_i & \alpha_i & & & & & \alpha_i \\ \alpha_i & y_i & \alpha_i & & & & \\ & \alpha_i & y_i & \alpha_i & & & \\ & & & \ddots & & & \\ & & & & \alpha_i & y_i & \alpha_i \\ \alpha_i & & & & & \alpha_i & y_i \end{pmatrix} \text{is a circulant matrix,} \tag{6.13}$$

in both (6.12) and (6.13) L_i is of order N.

and $\mathrm{T} = diag(\rho, \rho, \rho, \ldots, \rho)$ is of order N.

$S_i = diag(\boldsymbol{\omega}_i, \boldsymbol{\omega}_i, \boldsymbol{\omega}_i, \ldots, \boldsymbol{\omega}_i)$ has P blocks and $\boldsymbol{\omega}_i = diag(1+\omega_i, 1+\omega_i, ..., 1+\omega_i)$ is of order N by N

$T_i = diag(\boldsymbol{\varphi}_i, \boldsymbol{\varphi}_i, \boldsymbol{\varphi}_i, \ldots, \boldsymbol{\varphi}_i)$ has P blocks and $\boldsymbol{\varphi}_i = diag(1-\omega_i, 1-\omega_i, ..., 1-\omega_i)$ is of order N by N.

$\mathcal{B} = \left[\mathbf{B}_0 \quad \mathbf{B}_1 \;\; \mathbf{B}_2 \cdots \mathbf{B}_M \;\right]^T$, $\mathbf{B}_i = [\mathbf{d}_{i1} \;\; \mathbf{d}_{i2} \;\; \mathbf{d}_{i3} \ldots \; \mathbf{d}_{iN} \,]^T$ and $\mathbf{d}_{ij} = [\; d_{ij1} \;\; d_{ij2} \ldots d_{ijP}]^T$ such that each d_{ijk} represents a known boundary values of U and values of f, and

$\mathbf{U} = \left[\mathbf{U}_1 \quad \mathbf{U}_2 \quad \mathbf{U}_3 \quad \ldots \quad \mathbf{U}_M\right]^T$, $\mathbf{U}_i = \left(U_{i1} \quad U_{i2} \quad U_{i3} \quad \cdots \quad U_{iP}\right)^T$ and

$U_{ij} = \left(U_{ij1} \quad U_{ij2} \quad U_{ij3} \quad \cdots \quad U_{ijP}\right)^T$

We write (6.11) as

$$
\begin{aligned}
R_0\mathbf{U}_0 + S_0\mathbf{U}_1 \quad &= \mathbf{B}_0 \\
T_1\mathbf{U}_0 + R_1\mathbf{U}_1 + S_1\mathbf{U}_2 \quad &= \mathbf{B}_1 \\
T_2\,\mathbf{U}_1 + R_2\mathbf{U}_2 + S_2\mathbf{U}_3 &= \mathbf{B}_2 \\
T_3\mathbf{U}_2 + R_3\mathbf{U}_3 + S_3\mathbf{U}_4 &= \mathbf{B}_3 \\
\cdots \\
T_M\,\mathbf{U}_{M-1} + R_M\mathbf{U}_M &= \mathbf{B}_M
\end{aligned}
\tag{6.14}
$$

Now we follow the same procedure what we have done so far as in chapter IV.

CASE II FOURTH-ORDER SCHEME

Consider the fourth-order approximations that

$$
\left(\frac{\partial^2 U}{\partial r^2}\right)_{i,j,k} = \frac{1}{(\Delta r)^2}\left(\frac{\delta_r^2}{1+\frac{1}{12}\delta_r^2}\right)U_{i,j,k} + O\left((\Delta r)^4\right)
$$

$$
\left(\frac{\partial^2 U}{\partial \theta^2}\right)_{i,j,k} = \frac{1}{(\Delta \theta)^2}\left(\frac{\delta_\theta^2}{1+\frac{1}{12}\delta_\theta^2}\right)U_{i,j,k} + O\left((\Delta \theta)^4\right) \tag{6.15}
$$

$$
\text{and} \quad \left(\frac{\partial^2 U}{\partial z^2}\right)_{i,j,k} = \frac{1}{(\Delta z)^2}\left(\frac{\delta_z^2}{1+\frac{1}{12}\delta_z^2}\right)U_{i,j,k} + O\left((\Delta z)^4\right)
$$

Now we write (6.3) as

$$
4\left(\frac{1}{(\Delta r)^2}\left(1+\frac{1}{12}\delta_r^2\right)^{-1}\delta_r^2 U_{i,j,k} + O\left((\Delta r)^4\right)\right) + \left(\frac{1}{(\Delta r)^2}\left(1+\frac{1}{12}\delta_r^2\right)^{-1}\delta_r^2 U_{i,j,k} + O\left((\Delta r)^4\right)\right)_{\theta\theta}
$$
$$
+2\left(\frac{1}{(\Delta z)^2}\left(1+\frac{1}{12}\delta_z^2\right)^{-1}\delta_z^2 U_{i,j,k} + O\left((\Delta z)^4\right)\right) = 2\,f(0,\theta,z)
$$

$$4\left(\frac{1}{(\Delta r)^2}\left(1+\frac{1}{12}\delta_r^2\right)^{-1}\delta_r^2 U_{0,j,k}+O\left((\Delta r)^4\right)\right)+2\left(\frac{1}{(\Delta z)^2}\left(1+\frac{1}{12}\delta_z^2\right)^{-1}\delta_z^2 U_{0,j,k}+O\left((\Delta z)^4\right)\right)$$
$$+\left(\frac{1}{(\Delta r)^2(\Delta\theta)^2}\left(1+\frac{1}{12}\delta_r^2\right)^{-1}\left(1+\frac{1}{12}\delta_r^2\right)^{-1}\delta_\theta^2\delta_r^2 U_{0,j,k}+O\left((\Delta r)^4\right)+O\left((\Delta\theta)^4\right)\right)=2f(0,\theta,z)$$

(6.16)

Truncating higher order differences of (6.16), multiplying both sides of equation (6.16) by $(\Delta r)^2\left(1+\frac{1}{12}\delta_r^2\right)\left(1+\frac{1}{12}\delta_\theta^2\right)\left(1+\frac{1}{12}\delta_z^2\right)$, rearranging and simplifying, we get

$$144(1+\frac{1}{12}(\delta_r^2+\delta_\theta^2+\delta_z^2)f(0,\theta,z))=b_0U_{0,j,k}+b_1(U_{0,j+1,k}+U_{0,j-1,k})+b_2(U_{0,j,k+1}+U_{0,j,k-1})$$
$$+b_3(U_{0,j+1,k+1}+U_{0,j+1,k-1}+U_{0,j-1,k+1}+U_{0,j-1,k-1})+b_4U_{1,j,k}+b_5(U_{1,j+1,k}+U_{1,j-1,k})+b_6(U_{1,j,k+1}+U_{1,j,k-1})$$
$$+b_7(U_{1,j+1,k+1}+U_{1,j+1,k-1}+U_{1,j-1,k+1}+U_{1,j-1,k-1})$$

(6.17)

$$b_0=-400-240b-200\rho$$
$$b_1=-32+120b-20\rho$$
$$b_2=-40-12b+52\rho$$
$$b_3=-4+12b+10\rho$$
$$b_4=400-240b-40\rho$$
$$b_5=40+120b-4\rho$$
$$b_6=-8-24b+20\rho$$
$$b_7=4+12b+\rho$$

Since $U_{0,j,k}, U_{0,j+1,k}$ and $U_{0,j-1,k}$ are the same point; $U_{0,j,k+1}, U_{0,j+1,k+1}$ and $U_{0,j-1,k+1}$ are again same point; $U_{0,j,k-1}, U_{0,j+1,k-1}$ and $U_{0,j-1,k-1}$ are also same point thus we write equation (6.17) as

$$144(1+\frac{1}{12}(\delta_r^2+\delta_\theta^2+\delta_z^2)f(0,\theta,z))=(b_0+2b_1)U_{0,j,k}+(b_2+2b_3)(U_{0,j,k+1}+U_{0,j,k-1})+b_4U_{1,j,k}$$
$$+b_5(U_{1,j+1,k}+U_{1,j-1,k})+b_6(U_{1,j,k+1}+U_{1,j,k-1})+b_7(U_{1,j+1,k+1}+U_{1,j+1,k-1}+U_{1,j-1,k+1}+U_{1,j-1,k-1})$$

(6.18)

and for $r \neq 0$, we have from chapter V that

$$(\Delta r)^2\left(24+\delta_r^2+\delta_\theta^2+\delta_z^2+\frac{3\Delta r}{2r_i}\delta_{2r}\right)f_{i,j,k}=a_0(i)U_{i,j,k}+a_1(i)U_{i+1,j,k}+a_2(i)U_{i-1,j,k}$$
$$+a_3(i)(U_{i,j+1,k}+U_{i,j-1,k})+a_4(i)(U_{i,j,k+1}+U_{i,j,k-1})+a_5(i)(U_{i+1,j+1,k}+U_{i+1,j-1,k})$$
$$+a_6(i)(U_{i-1,j+1,k}+U_{i-1,j-1,k})+a_7(i)(U_{i+1,j,k+1}+U_{i+1,j,k-1})+a_8(i)(U_{i-1,j,k+1}+U_{i-1,j,k-1}) \tag{6.19}$$
$$+a_9(i)(U_{i,j+1,k+1}+U_{i,j-1,k+1}+U_{i,j+1,k-1}+U_{i,j-1,k-1})$$

Now by combining (6.18) and (6.19), we get

$$\begin{cases} 144(1+\frac{1}{12}(\delta_r^2+\delta_\theta^2+\delta_z^2)f(0,\theta,z))=(b_0+2b_1)U_{0,j,k}+(b_2+2b_3)(U_{0,j,k+1}+U_{0,j,k-1})+b_4U_{1,j,k} \\ \quad +b_5(U_{1,j+1,k}+U_{1,j-1,k})+b_6(U_{1,j,k+1}+U_{1,j,k-1})+b_7(U_{1,j+1,k+1}+U_{1,j+1,k-1}+U_{1,j-1,k+1}+U_{1,j-1,k-1}) \\ \\ (\Delta r)^2\left(24+\delta_r^2+\delta_\theta^2+\delta_z^2+\frac{3\Delta r}{2r_i}\delta_{2r}\right)f_{i,j,k}=a_0(i)U_{i,j,k}+a_1(i)U_{i+1,j,k}+a_2(i)U_{i-1,j,k} \\ \quad +a_3(i)(U_{i,j+1,k}+U_{i,j-1,k})+a_4(i)(U_{i,j,k+1}+U_{i,j,k-1})+a_5(i)(U_{i+1,j+1,k}+U_{i+1,j-1,k}) \\ \quad +a_6(i)(U_{i-1,j+1,k}+U_{i-1,j-1,k})+a_7(i)(U_{i+1,j,k+1}+U_{i+1,j,k-1})+a_8(i)(U_{i-1,j,k+1}+U_{i-1,j,k-1}) \\ \quad +a_9(i)(U_{i,j+1,k+1}+U_{i,j-1,k+1}+U_{i,j+1,k-1}+U_{i,j-1,k-1}) \end{cases} \tag{6.20}$$

Considering equation (6.20) first in the θ direction, next Z direction and lastly r direction, and hence equation (6.2) can be put in matrix form as

$$A\mathbf{U}=\mathcal{B} \tag{6.21}$$

where $A=\begin{pmatrix} R_0 & S_0 & & & & & \\ T_1 & R_1 & S_1 & & & & \\ & T_2 & R_2 & S_2 & & & \\ & & & \ddots & & & \\ & & & & T_{M-1} & R_{M-1} & S_{M-1} \\ & & & & & T_M & R_M \end{pmatrix}$, it has $M+1$blocks and each is of order NP

$$R_0 = \begin{pmatrix} B_0 & C_0 & & & & & \\ C_0 & B_0 & C_0 & & & & \\ & C_0 & B_0 & C_0 & & & \\ & & & \ddots & & & \\ & & & & C_0 & B_0 & C_0 \\ & & & & & C_0 & B_0 \end{pmatrix} \text{is of order } NP$$

$B_0 = diag\left(b_0 + 2b_1, b_0 + 2b_1, \cdots, b_0 + 2b_1\right)$,

$C_0 = diag\left(b_2 + 2b_3, b_2 + 2b_3, \cdots, b_2 + 2b_3\right)$ both are order of N

$$S_0 = \begin{pmatrix} B_1 & C_1 & & & & & \\ C_1 & B_1 & C_1 & & & & \\ & C_1 & B_1 & C_1 & & & \\ & & & \ddots & & & \\ & & & & C_1 & B_1 & C_1 \\ & & & & & C_1 & B_1 \end{pmatrix} \text{is of order } NP$$

$$B_1 = \begin{pmatrix} b_4 & b_5 & & & & & \\ b_5 & b_4 & b_5 & & & & \\ & b_5 & b_4 & b_5 & & & \\ & & & \ddots & & & \\ & & & & b_5 & b_4 & b_5 \\ & & & & & b_5 & b_4 \end{pmatrix} \text{is of order } N$$

$$C_1 = \begin{pmatrix} b_6 & b_7 & & & & & \\ b_7 & b_6 & b_7 & & & & \\ & b_7 & b_6 & b_7 & & & \\ & & & \ddots & & & \\ & & & & b_7 & b_6 & b_7 \\ & & & & & b_7 & b_6 \end{pmatrix} \text{is of order } N$$

For $i = 1, 2, 3, \cdots, M$

$$R_i = \begin{pmatrix} R_i^{'} & R_i^{''} & & & & & \\ R_i^{''} & R_i^{'} & R_i^{''} & & & & \\ & R_i^{''} & R_i^{'} & R_i^{''} & & & \\ & & & & \ddots & & \\ & & & & R_i^{''} & R_i^{'} & R_i^{''} \\ & & & & & R_i^{''} & R_i^{'} \end{pmatrix},$$

$$S_i = \begin{pmatrix} S_i' & S_i^{''} & & & & & \\ S_i^{''} & S' & S_i^{''} & & & & \\ & S_i^{''} & S' & S_i^{''} & & & \\ & & & & \ddots & & \\ & & & & S_i^{''} & S' & S_i^{''} \\ & & & & & S_i^{''} & S' \end{pmatrix}$$

$$T_i = \begin{pmatrix} T_i^{'} & T_i^{''} & & & & & \\ T_i^{''} & T_i^{'} & T_i^{''} & & & & \\ & T_i^{''} & T_i^{'} & T_i^{''} & & & \\ & & & & \ddots & & \\ & & & & T_i^{''} & T_i^{'} & T_i^{''} \\ & & & & & T_i^{''} & T_i^{'} \end{pmatrix}$$

$R_i, S_i,$ and T_i are of order NP

For the domain D_1

$$R_i^{'} = \begin{pmatrix} a_0(i) & a_3(i) & & & & & \\ a_3(i) & a_0(i) & a_3(i) & & & & \\ & a_3(i) & a_0(i) & a_3(i) & & & \\ & & & & \ddots & & \\ & & & & a_3(i) & a_0(i) & a_3(i) \\ & & & & & a_3(i) & a_0(i) \end{pmatrix}$$

$$R_i^{''} = \begin{pmatrix} a_4(i) & a_9(i) & & & & & \\ a_9(i) & a_4(i) & a_9(i) & & & & \\ & a_9(i) & a_4(i) & a_9(i) & & & \\ & & & & \ddots & & \\ & & & & a_9(i) & a_4(i) & a_9(i) \\ & & & & & a_9(i) & a_4(i) \end{pmatrix}$$

$$S_i^{'} = \begin{pmatrix} a_1(i) & a_5(i) & & & & & \\ a_5(i) & a_1(i) & a_5(i) & & & & \\ & a_5(i) & a_1(i) & a_5(i) & & & \\ & & & & \ddots & & \\ & & & & a_5(i) & a_1(i) & a_5(i) \\ & & & & & a_5(i) & a_1(i) \end{pmatrix},$$

$$S_i^{''} = \begin{pmatrix} a_7(i) & & & & \\ & a_7(i) & & & \\ & & a_7(i) & & \\ & & & \ddots & \\ & & & & a_7(i) \end{pmatrix}$$

$$T_i^{'} = \begin{pmatrix} a_2(i) & a_6(i) & & & & & \\ a_6(i) & a_2(i) & a_6(i) & & & & \\ & a_6(i) & a_2(i) & a_6(i) & & & \\ & & & & \ddots & & \\ & & & & a_6(i) & a_2(i) & a_6(i) \\ & & & & & a_6(i) & a_2(i) \end{pmatrix},$$

$$T_i^{''} = \begin{pmatrix} a_8(i) & & & & \\ & a_8(i) & & & \\ & & a_8(i) & & \\ & & & \ddots & \\ & & & & a_8(i) \end{pmatrix}$$

For the domain D_2,

$$R_i^{'} = \begin{pmatrix} a_0(i) & a_3(i) & & & & & a_3(i) \\ a_3(i) & a_0(i) & a_3(i) & & & & \\ & a_3(i) & a_0(i) & a_3(i) & & & \\ & & & & \ddots & & \\ & & & & a_3(i) & a_0(i) & a_3(i) \\ a_3(i) & & & & & a_3(i) & a_0(i) \end{pmatrix}$$

$$R_i^{''} = \begin{pmatrix} a_4(i) & a_9(i) & & & & & a_9(i) \\ a_9(i) & a_4(i) & a_9(i) & & & & \\ & a_9(i) & a_4(i) & a_9(i) & & & \\ & & & & \ddots & & \\ & & & & a_9(i) & a_4(i) & a_9(i) \\ a_9(i) & & & & & a_9(i) & a_4(i) \end{pmatrix}$$

$$S_i^{'} = \begin{pmatrix} a_1(i) & a_5(i) & & & & & a_5(i) \\ a_5(i) & a_1(i) & a_5(i) & & & & \\ & a_5(i) & a_1(i) & a_5(i) & & & \\ & & & & \ddots & & \\ & & & & a_5(i) & a_1(i) & a_5(i) \\ a_5(i) & & & & & a_5(i) & a_1(i) \end{pmatrix},$$

$$S_i^{''} = \begin{pmatrix} a_7(i) & & & & \\ & a_7(i) & & & \\ & & a_7(i) & & \\ & & & \ddots & \\ & & & & a_7(i) \end{pmatrix}$$

$$T_i' = \begin{pmatrix} a_2(i) & a_6(i) & & & & & a_6(i) \\ a_6(i) & a_2(i) & a_6(i) & & & & \\ & a_6(i) & a_2(i) & a_6(i) & & & \\ & & & \ddots & & & \\ & & & & a_6(i) & a_2(i) & a_6(i) \\ a_6(i) & & & & & a_6(i) & a_2(i) \end{pmatrix},$$

$$T_i'' = \begin{pmatrix} a_8(i) & & & & \\ & a_8(i) & & & \\ & & a_8(i) & & \\ & & & \ddots & \\ & & & & a_8(i) \end{pmatrix}$$

Here in D_2, the matrices $R_i', R_i'', S_i', S_i'', T_i'$ and T_i'' are circulant matrices of order N; and $\mathcal{B} = \left[\mathbf{B}_0 \quad \mathbf{B}_1 \;\; \mathbf{B}_2 \cdots \mathbf{B}_M \;\right]^T$, $\mathbf{B}_i = [\mathbf{d}_{i1} \;\; \mathbf{d}_{i2} \;\; \mathbf{d}_{i3} \ldots \; \mathbf{d}_{iN}]^T$ and $\mathbf{d}_{ij} = [\; d_{ij1} \;\; d_{ij2} \ldots d_{ijP}]^T$ such that each d_{ijk} represents a known boundary values of U and values of f, and

$\mathbf{U} = \left[\mathbf{U}_0 \quad \mathbf{U}_1 \quad \mathbf{U}_2 \quad \ldots \quad \mathbf{U}_M\right]^T$, $\mathbf{U}_i = \left(U_{i1} \quad U_{i2} \quad U_{i3} \quad \cdots \quad U_{iP}\right)^T$ and

$U_{ij} = \left(U_{ij1} \quad U_{ij2} \quad U_{ij3} \quad \cdots \quad U_{ijP}\right)^T$

We write (6.21) as

$$\begin{aligned} R_0\mathbf{U}_0 + S_0\mathbf{U}_1 &= \mathbf{B}_0 \\ T_1\mathbf{U}_0 + R_1\mathbf{U}_1 + S_1\mathbf{U}_2 &= \mathbf{B}_1 \\ T_2\mathbf{U}_1 + R_2\mathbf{U}_2 + S_2\mathbf{U}_3 &= \mathbf{B}_2 \\ T_3\mathbf{U}_2 + R_3\mathbf{U}_3 + S_3\mathbf{U}_4 &= \mathbf{B}_3 \\ &\cdots \\ T_M\mathbf{U}_{M-1} + R_M\mathbf{U}_M &= \mathbf{B}_M \end{aligned} \tag{6.22}$$

Now we follow all the procedure what has been done in Chapter V.

6.3 Numerical Results

In order to test the efficiency and adaptability of the proposed method, computational experiments are done on two selected problems that may arise in practice, for which the analytical solutions of U are known to us. The computed solutions are found for all grid points in both second order and fourth-order schemes. But here results are reported at some mesh points in terms of the absolute maximum error and are shown in table 6.1 and 6.3 for second order approximations and in tables 6.2 and 6.4 for fourth order approximation .

Example 6.1 Consider $\nabla^2 U = 3z \sin\theta$ with the boundary conditions

$$U(0,\theta,z) = 0 \qquad\qquad U(1,\theta,z) = z\sin\theta$$

$$U(r,0,z) = 0 = U(r,\pi,z) \text{ , and}$$

$$U(r,\theta,0) = 0, \quad U(r,\theta,1) = r^2 \sin\theta$$

The analytical solution is $U(r,\theta,z) = r^2\, z\sin\theta$ and the computed results of this example are shown in Table 6.1 and 6.2

Example 6.2 Consider $\nabla^2 U = -r^2\cos(z)\sin(2\theta)$ with the boundary conditions

$$U(0,\theta,z) = 0, \quad U(1,\theta,z) = \cos(z)\sin(2\theta)$$

$$U(r,0,z) = 0, \; U\left(r,\frac{\pi}{2},z\right) = 0 \text{ , and}$$

$$U(r,\theta,0) = r^2\sin(2\theta) \qquad U\left(r,\theta,\frac{\pi}{2}\right) = 0$$

The analytical solution is $U(r,\theta,z) = r^2\cos(z)\sin(2\theta)$ and the computed results of this example are shown in Table 6.3 and 6.4.

Table 6.1

The maximum absolute error of example 6.1 for second order scheme

(N, P, M)	Max. abs. error		(N, P, M)	Max. abs. error
(9,9,9)	2.51924e-004		(29,9,29)	2.83122e-005
(9,9,29)	2.57e-004		(29,19,9)	2.91765e-005
(9,19,9)	2.6086e-004		(29,29,19)	2.87145e-005
(9,19,19)	2.59295e-004		(29,29,29)	2.86698e-005
(9,29,29)	2. 59459e-004		(29,39,19)	2.82675e-005
(9,39,29)	2.60338e-004		(29,39,29)	2.96831e-005
(19,9,9)	6.44475e-005		(39,9,19)	1.55121e-005
(19,9,19)	6.42538e-005		(39,9,29)	1.61678e-005
(19,19,19)	6.44773e-005		(39,19,29)	1.54823e-005
(19,29,29)	6.44177e-005		(39,29,19)	1.67638e-005
(19,39,9)	6.42091e-005		(39,39,9)	1.56909e-005
(19,39,29)	6.53267e-005		(39,39,29)	1.65403e-005

Table 6.2

The maximum absolute error of example 6.1 for fourth-order scheme

(N, P, M)	Max. abs. error		(N, P, M)	Max. abs. error
(9,9,9)	1.32542e-004		(29,9,29)	2.83122e-005
(9,9,29)	1.67253e-004		(29,19,9)	1.48821e-005
(9,19,9)	1.82314e-004		(29,29,19)	1.43577e-005
(9,19,19)	1.34261e-004		(29,29,29)	1.43349e-005
(9,29,29)	1.17898e-004		(29,39,19)	1.41388e-005
(9,39,29)	1.06752e-004		(29,39,29)	1.48151e-005
(19,9,9)	3.21189e-005		(39,9,19)	1.22061e-005
(19,9,19)	3.36108e-005		(39,9,29)	1.31834e-005
(19,19,19)	3.37823e-005		(39,19,29)	1.27411e-005
(19,29,29)	3.84921e-005		(39,29,19)	1.45328e-005
(19,39,9)	3.90121e-005		(39,39,9)	1.43206e-005
(19,39,29)	3.95478e-005		(39,39,29)	1.44301e-005

Table 6.3

The maximum absolute error of example 6.2 for second-order scheme

(N,P,M)	Max. abs. error		(N,P,M)	Max. abs. error
$(9,9,9)$	1.17818e-003		$(29,9,39)$	1.58485e-004
$(9,9,29)$	1.18044e-003		$(29,19,9)$	1.36197e-004
$(9,19,9)$	1.16232e-003		$(29,29,19)$	1.32054e-004
$(9,19,19)$	1.1647e-003		$(29,29,29)$	1.33336e-004
$(9,29,39)$	1.16095e-003		$(29,39,19)$	1.38773e-004
$(9,39,29)$	1.16116e-003		$(29,39,29)$	1.31696e-004
$(19,9,9)$	3.16888e-004		$(39,9,19)$	1.0258e-004
$(19,9,19)$	3.17872e-004		$(39,9,39)$	1.01089e-004
$(19,19,19)$	2.96772e-004		$(39,19,29)$	7.88867e-005
$(19,29,39)$	2.9245e-004		$(39,29,19)$	7.39098e-005
$(19,39,9)$	2.9096e-004		$(39,39,9)$	7.24792e-005
$(19,39,39)$	2.90751e-004		$(39,39,39)$	7.14064e-005

Table 6.4

The maximum absolute error of example 6.2 for fourth-order scheme

(N, P, M)	Max. abs. error		(N, P, M)	Max. abs. error
$(9,9,9)$	1.1213e-004		$(29,9,39)$	1.78325e-005
$(9,9,29)$	1.13739e-004		$(29,19,9)$	1.76276e-005
$(9,19,9)$	1.13923e-004		$(29,29,19)$	1.72107e-005
$(9,19,19)$	1.13947e-004		$(29,29,29)$	1.69578e-005
$(9,29,39)$	1.14609e-004		$(29,39,19)$	1.65938e-005
$(9,39,29)$	1.14821e-004		$(29,39,29)$	1.60254e-005
$(19,9,9)$	6.43336e-005		$(39,9,19)$	1.57034e-005
$(19,9,19)$	6.67782e-005		$(39,9,39)$	1.50069e-005
$(19,19,19)$	6.79586e-005		$(39,19,29)$	1.20564e-005
$(19,29,39)$	6.85401e-005		$(39,29,19)$	1.17830e-005
$(19,39,9)$	6.90615e-005		$(39,39,9)$	1.0412e-005
$(19,39,39)$	6.90758e-005		$(39,39,39)$	1.00231e-005

6.5 Conclusion

In this work, we have transformed the three dimensional Poisson's equation in cylindrical coordinates system when $r = 0$ is an interior point or a boundary point in to a system of algebraic linear equations using its equivalent second and fourth-order finite difference approximation scheme. The resulting large number of algebraic equation is, then, can be solved by following the same procedure as in chapters IV and V.

We have successfully implemented this method to find the solution of the three dimensional Poisson's equation in cylindrical coordinates system when $r = 0$ is an interior point or a boundary point and it is found that the method can easily be applied and adapted to find a solution of some related applied problems. The method produced accurate results considering double precision.

This method is direct and allows considerable savings in computer storage as well as execution speed.

CHAPTER VII

Efficient Numerical Solution of the Biharmonic Equation of the Second Kind in Cartesian Coordinates System

7.1 Introduction

Consider the biharmonic equation on a domain D

$$\nabla^4 U = f(P), \tag{7.1}$$

where P is a point in the domain D, with two types of boundary conditions.

In the first case with the Dirichlet boundary conditions

$$\begin{aligned} U(P) &= f_1(P) \qquad && \mathrm{P} \in \partial D \\ \frac{\partial U(P)}{\partial n} &= f_2(P) \qquad && \mathrm{P} \in \partial D \end{aligned} \tag{7.2}$$

(where $\frac{\partial U(P)}{\partial n}$ is the normal to the boundary derivative), which we call the first kind problem and in the second case with the Dirichlet boundary conditions

$$\begin{aligned} U(P) &= f_1(P) \qquad && \mathrm{P} \in \partial D \\ \frac{\partial^2 U(P)}{\partial n^2} &= f_2(P) \qquad && \mathrm{P} \in \partial D \end{aligned} \tag{7.3}$$

which we will refer to as the second kind problem.

Attempts are made by different workers to solve (7.1) with the boundary conditions (7.2) or (7.3) in two or three dimensional Cartesian coordinates system.

For instance, in two dimensional Cartesian coordinates system *Buzbee and Dorr* [12] have developed a direct solution method based on matrix decomposition; *Smith* [12], [13] has developed a numerical solution method based on coupled equation approach using finite difference method; *Altas et al* [35] have derived a family of finite difference approximation using a symbolic algebra package; *McLaurin* [44] have developed a method based on reducing to a coupled system of Poisson equations, which depends upon an arbitrary, positive coupling constant $c \neq 0$; *Stephenson* [45] has established a second and a fourth order approximation schemes; *Ehrlich* [56], has described a semi-direct method for solving the biharmonic equation in a square using a coupled pair of finite difference equations based on a general block SOR method; *Ehrlich and Gupta* [58] have developed several finite difference schemes for solving the coupled Poisson equations obtained by splitting the biharmonic equation; *Arad et al* [61] have derived the high-order accuracy discretization scheme using the symbolic operator procedure ; *Lai and Liu* [63] developed a simple and efficient FFT-based fast direct solver for the biharmonic equation on a disk by splitting into a coupled system of harmonic problems using the truncated Fourier series expansion to derive a set of coupled singular ODEs, and solving those singular equations by second-order finite difference discretization; *Dehghan and Mohebbi* [73] have derived a method by combining compact finite difference schemes with multigrid method and Krylov iteration methods preconditioned by multigrid for the second biharmonic equation; *Gupta* [77] has examined the discretization error of the finite difference scheme by splitting the first biharmonic boundary value problems; *Gupta and Manohar* [78] have tried to solve the problem directly without splitting it in to two Poisson's equation and much more progresses has been observed in this dimension.

For the three dimensional biharmonic equation, for instance *Khattar et al* [18] derive a fourth order finite difference approximation based on arithmetic average discretization for the solution of three-dimensional non-linear biharmonic partial differential equations on a 19-point compact stencil using coupled approach; *Altas et al* [34] considered several finite-difference approximations using a symbolic algebra package to derive a family of finite difference approximations for the biharmonic equation on a 27 point compact stencil; *Dehghan and Mohebbi* [72] have developed a multigrid solution of high order (second and fourth-order approximations on a 27 point compact stencil) discretization; *Mohanty et al* [95] developed a higher order compact difference scheme, which is $O\left(h^4\right)$, using coupled approach on the 19 point 3D stencil for the solution of non-linear biharmonic equation; and other contributions have been made in developing schemes and methods.

In this chapter we examine a finite difference scheme for solving the biharmonic equation (7.1) along with the Dirichlet boundary condition of the second kind in two and three dimensions.
In the first part we consider the two dimensional biharmonic equation and in the second part the three dimensional one.

7.2 Linear Biharmonic Equation

<u>Part 1</u> : Two Dimensional Biharmonic Equation

Consider the two dimensional biharmonic equation on a rectangular domain D

$$\frac{\partial^4 U}{\partial x^4}+2\frac{\partial^4 U}{\partial x^2 \partial y^2}+\frac{\partial^4 U}{\partial y^4}=f(x,y), \quad (x,y)\in D \tag{7.1a}$$

where $D=\{(x,y):0\le x,y\le 1\}$ with the second Dirichlet boundary conditions

$$U(x,y)=f_1(x,y) \qquad (x,y)\in\partial D$$

$$\frac{\partial^2 U(x,y)}{\partial n^2}=f_2(x,y) \qquad (x,y)\in\partial D \tag{7.3a}$$

Using the splitting method where the biharmonic equation (7.1a) is replaced by introducing an auxiliary variable $v(x,y)=\nabla^2 U(x,y)$ and splitting the biharmonic equation into a coupled system of two dimensional Poisson equations,

$$\begin{aligned}\nabla^2 v(x,y)&=f(x,y)\\ \nabla^2 U(x,y)&=v(x,y)\end{aligned} \tag{7.4}$$

This means that we are transferring the two dimensional biharmonic equation with the given boundary conditions in to its equivalent coupled Poisson equations.
Since the grid lines are parallel to coordinate axes and the values of U are exactly known on the boundary, and this implies, the successive tangential partial derivatives of U are known exactly on the boundary. For example, on the side $y=0$, the values of $U(x,0)$ and $U_{yy}(x,0)$ are known, i.e. the values of $U_x(x,0)$, $U_{xx}(x,0)$ are known on the side $y=0$.

This implies, the values of $U(x,0)$ and

$\nabla^2 U(x,0) = U_{xx}(x,0) + U_{yy}(x,0)$ are known on the side $y = 0$.

Similarly, the values of U and $\nabla^2 U$ are known on all sides of the rectangular region D.

Therefore, (7.1a) is reformulated as

$$\nabla^2 U(x,y) = \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = v(x,y) \qquad (x,y) \in D \tag{7.5a}$$

$$\nabla^2 v(x,y) = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = f(x,y,U,v,U_x,v_x,U_y,v_y) \qquad (x,y) \in D \tag{7.5b}$$

with the Dirichlet boundary conditions

$$U(x,y) = f_1(x,y) \qquad (x,y) \in \partial D$$

$$v(x,y) = g(x,y) \qquad (x,y) \in \partial D \tag{7.5c}$$

7.3 Finite Difference Approximation

Let h be the mesh step size along the X- and Y-directions, and let δ_x be the central difference operator, and we know that

$$h^2 \frac{\partial^2}{\partial x^2} = \left(1 + \frac{1}{12}\delta_x^{\,2}\right)^{-1} \delta_x^{\,2} + O\left(h^4\right) \tag{7.6}$$

Using (7.6) in (7.5b), we have

$$\left(\frac{1}{h^2}\left(1 + \frac{1}{12}\delta_x^{\,2}\right)^{-1} \delta_x^{\,2} + \frac{1}{h^2}\left(1 + \frac{1}{12}\delta_y^{\,2}\right)^{-1} \delta_y^{\,2} + O\left(h^4\right)\right) v_{i,j} = f_{i,j} \tag{7.7}$$

where $i = 1,2,3,\ldots,M,\ \ j = 1,2,3,\ldots,N$

$$\left(1 + \frac{1}{12}\delta_x^{\,2}\right)^{-1}\left(1 + \frac{1}{12}\delta_y^{\,2}\right)^{-1}\left(\delta_x^{\,2}\left(1 + \frac{1}{12}\delta_y^{\,2}\right) + \delta_y^{\,2}\left(1 + \frac{1}{12}\delta_x^{\,2}\right) + O\left(h^4\right)\right) v_{i,j} = h^2 f_{i,j} \tag{7.8}$$

Simplifying (7.8) further, we get

$$\left(\delta_x^{\,2}+\delta_y^{\,2}+\frac{1}{6}\delta_x^{\,2}\delta_y^{\,2}\right)v_{i,j}=h^2\left(1+\frac{1}{12}\left(\delta_x^{\,2}+\delta_y^{\,2}\right)+\frac{1}{144}\delta_x^{\,2}\delta_y^{\,2}\right)f_{i,j} \tag{7.9}$$

Using the central difference approximation scheme, omitting the last term in the right hand side and simplifying (7.9) further we get,

$$4\left(v_{i+1,j}+v_{i-1,j}+v_{i,j+1}+v_{i,j-1}\right)+v_{i+1,j-1}+v_{i-1,j-1}+v_{i+1,j+1}+v_{i-1,j+1}-20v_{i,j}=F \tag{7.10a}$$

$$\text{where } F=\frac{h^2}{2}\left(8f_{i,j}+f_{i+1,j}+f_{i-1,j}+f_{i,j+1}+f_{i,j-1}\right)$$

Similarly, by applying the same principle as above to (7.5a), we get

$$4\left(U_{i+1,j}+U_{i-1,j}+U_{i,j+1}+U_{i,j-1}\right)+U_{i+1,j-1}+U_{i-1,j-1}+U_{i+1,j+1}+U_{i-1,j+1}-20U_{i,j}=G \tag{7.10b}$$

$$\text{where } G=\frac{h^2}{2}\left(8v_{i,j}+v_{i+1,j}+v_{i-1,j}+v_{i,j+1}+v_{i,j-1}\right)$$

Taking first the X-direction and next Y-direction in (7.10) we get a large system of equations (the number of equations actually depend on the values of M and N), and these systems of equations can be written in matrix form as

$$AV=\mathcal{B} \tag{7.11}$$

$$\text{where} \quad A=\begin{pmatrix} R & S & & & & \\ S & R & S & & & \\ & S & R & S & & \\ & & & \ddots & & \\ & & & S & R & S \\ & & & & S & R \end{pmatrix}$$

it has N blocks and each block is of order M, and

$$R=\begin{pmatrix} -20 & 4 & & & & \\ 4 & -20 & 4 & & & \\ & 4 & -20 & 4 & & \\ & & & \ddots & & \\ & & & 4 & -20 & 4 \\ & & & & 4 & -20 \end{pmatrix}, \quad S=\begin{pmatrix} 4 & 1 & & & & \\ 1 & 4 & 1 & & & \\ & 1 & 4 & 1 & & \\ & & & \ddots & & \\ & & & 1 & 4 & 1 \\ & & & & 1 & 4 \end{pmatrix}$$

R and S are each of order M,

$V=\left(V_1\ \ V_2\ \ V_3\ \ldots\ V_N\right)^T$ where $V_i=\left(v_{i1}\ \ v_{i2}\ \ v_{i3}\ \ldots\ v_{iN}\right)^T$ and

$\mathcal{B}=\left(B_1\ \ B_2\ \ B_3\ \ldots\ B_N\right)^T$ where $B_i=\left(b_{i1}\ \ b_{i2}\ \ b_{i3}\ \ldots\ b_{iN}\right)^T$.

Now by applying Hockney's method we solve (6.11).

7.4 Hockney's Method

Observe that the matrices R and S are real symmetric matrices and hence their eigenvalues and eigenvectors can easily be obtained. [30]

Note that the eigenvalues of R and S are given by

$$\lambda_i=-20+8\cos\left(\frac{i\pi}{M+1}\right) \quad \text{and}$$

$$\alpha_i=4+2\cos\left(\frac{i\pi}{M+1}\right)$$

Let $\mathbf{q}_i$ be an eigenvector of R and S corresponding to the eigenvalues λ_i and α_i, $i=1,2,\ldots,M$ respectively, and Q be the modal matrix $\left[\mathbf{q}_1\ \mathbf{q}_2\ \mathbf{q}_3\ \ldots\ \mathbf{q}_M\right]^T$ of the matrix R and S of order M such that $Q^TQ=I$,

$$Q^TRQ=diag(\lambda_1,\lambda_2,\lambda_3,\ldots,\lambda_M)=\Lambda \text{ (say), and}$$

$$Q^TSQ=diag(\alpha_1,\alpha_2,\alpha_3,\ldots,\alpha_M)=\Phi \text{ (say)}$$

Let $\quad Q^T\mathrm{V}_j=T_j \Rightarrow \mathrm{V}_j=QT_j$

$$Q^TB_j=\mathbf{b}_j \Rightarrow B_j=Q\mathbf{b}_j \tag{7.12}$$

where $T_j=[w_{1j}\ \ w_{2j}\ \ldots w_{Mj}]^T$, and $\mathbf{b}_j=[\beta_{1j}\ \ \beta_{2j}\ \ldots\beta_{Mj}]^T$ $j=1,2,\ldots,N$.

Equation (7.11) can be written in terms of the matrices R and S as

$$
\begin{aligned}
RV_1 + SV_2 \qquad &= B_1 \\
SV_1 + RV_2 + SV_3 &= B_2 \\
SV_2 + RV_3 + SV_4 &= B_3 \qquad (7.13) \\
\ldots \\
SV_{N-1} + RV_N &= B_N
\end{aligned}
$$

Pre multiplying (7.13) by Q^T and using (7.12), we get

$$
\begin{aligned}
\Lambda T_1 + \Phi T_2 &= \mathbf{b}_1 \\
\Phi T_1 + \Lambda T_2 + \Phi T_3 &= \mathbf{b}_2 \\
\Phi T_2 + \Lambda T_3 + \Phi T_4 &= \mathbf{b}_3 \qquad (7.14) \\
\ldots \\
\Phi T_{\mathrm{N}-1} + \Lambda T_N &= \mathbf{b}_N
\end{aligned}
$$

Now we write (7.14) starting from the first equation i.e. $\Lambda T_1 + \Phi T_2 = \mathbf{b}_1$ as

$$
\begin{aligned}
\lambda_1 w_{11} + \alpha_1 w_{12} &= \beta_{11} \\
\lambda_2 w_{21} + \alpha_2 w_{22} &= \beta_{21} \\
\lambda_3 w_{31} + \alpha_3 v_{32} &= \beta_{31} \qquad (7.\ 15a) \\
\ldots \\
\lambda_M w_{M1} + \alpha_M v_{M2} &= \beta_{M1}
\end{aligned}
$$

The second equation i.e. $\Phi T_1 + \Lambda T_2 + \Phi T_3 = \mathbf{b}_2$ again can be written as

$$
\begin{aligned}
\alpha_1 w_{11} + \lambda_1 w_{12} + \alpha_1 w_{13} &= \beta_{12} \\
\alpha_2 w_{21} + \lambda_2 w_{22} + \alpha_2 w_{23} &= \beta_{22} \\
\alpha_3 w_{31} + \lambda_3 w_{32} + \alpha_3 w_{33} &= \beta_{32} \qquad (7.\ 15b) \\
\ldots \\
\alpha_M w_{M1} + \lambda_M w_{M2} + \alpha_M w_{M3} &= \beta_{M2}
\end{aligned}
$$

And the last equations i.e. $\Phi T_{N-1} + \Lambda T_N = \mathbf{b}_N$ be written as

$$\alpha_1 w_{1(N-1)} + \lambda_1 w_{1N} = \beta_{1N}$$
$$\alpha_2 w_{2(N-1)} + \lambda_2 w_{2N} = \beta_{2N}$$
$$\alpha_3 w_{3(N-1)} + \lambda_3 w_{3N} = \beta_{3N} \qquad (7.\ 15c)$$
$$\ldots$$
$$\alpha_M w_{M(N-1)} + \lambda_M w_{MN} = \beta_{MN}$$

Now we arrange (7.15) as follows:

Firstly we take the first equations from each of (7.15a), (7.15b) and (7.15c) and write as one set of group of equations

$$\lambda_1 w_{11} + \alpha_1 w_{12} = \beta_{11}$$
$$\alpha_1 w_{11} + \lambda_1 w_{12} + \alpha_1 w_{13} = \beta_{12}$$
$$\alpha_1 w_{12} + \lambda_1 w_{13} + \alpha_1 w_{14} = \beta_{13} \qquad (7.16a)$$
$$\ldots$$
$$\alpha_1 w_{1(N-1)} + \lambda_1 w_{1N} = \beta_{1N}$$

Again collect the second equations from each of (7.15a), (7.15b) and (7.15c)

$$\lambda_2 w_{21} + \alpha_2 w_{22} = \beta_{21}$$
$$\alpha_2 w_{21} + \lambda_2 w_{22} + \alpha_2 w_{23} = \beta_{22}$$
$$\alpha_2 w_{22} + \lambda_2 w_{23} + \alpha_2 w_{24} = \beta_{23} \qquad (7.16b)$$
$$\ldots$$
$$\alpha_2 w_{2(N-1)} + \lambda_2 w_{2N} = \beta_{2N}$$

Keep on doing for all groups of equations and for the last equations, we get

$$\lambda_M w_{M1} + \alpha_M w_{M2} = \beta_{M1}$$
$$\alpha_M w_{M1} + \lambda_M w_{M2} + \alpha_M w_{M3} = \beta_{M2}$$
$$\alpha_M w_{M2} + \lambda_M w_{M3} + \alpha_M w_{M4} = \beta_{M3} \qquad (7.16c)$$
$$\ldots$$
$$\alpha_2 w_{2(N-1)} + \lambda_2 w_{2N} = \beta_{2N}$$

As can be seen from the above set of equations, all groups have the same matrix form

$$\mathcal{M}_i = \begin{pmatrix} \lambda_i & \alpha_i & & & & & \\ \alpha_i & \lambda_i & \alpha_i & & & & \\ & \alpha_i & \lambda_i & \alpha_i & & & \\ & & & \ddots & & & \\ & & & & \alpha_i & \lambda_i & \alpha_i \\ & & & & & \alpha_i & \lambda_i \end{pmatrix} \quad i = 1, 2, \ldots, M \text{ of order } N .$$

Thus, equation (7.13) reduces to

$$\begin{pmatrix} \mathcal{M}_i & & & & \\ & \mathcal{M}_i & & & \\ & & \mathcal{M}_i & & \\ & & & \ddots & \\ & & & & \mathcal{M}_i \end{pmatrix} \begin{pmatrix} T_1 \\ T_2 \\ T_3 \\ \vdots \\ T_N \end{pmatrix} = \begin{pmatrix} \bar{B}_1 \\ \bar{B}_2 \\ \bar{B}_3 \\ \vdots \\ \bar{B}_N \end{pmatrix} \tag{7.17}$$

We can easily observe that (7.13) reduces to the matrix (7.17) which is a tridiagonal matrix and we solve this system for $w_{i,j}$ by the use of Thomas Algorithm [30], and once after getting $w_{i,j}$ by the backward substitution and make use of (7.12) we get the solution for v_{ij}.

Now we use these values of v_{ij} as the right hand side of (7.10b) and we apply the same procedure as previous to solve the Poisson's equation in terms of U.

This solves the given two dimensional biharmonic equation as desired.

7.5.1 Numerical Results for Two Dimensional Biharmonic Equations

In order to test the efficiency and adaptability of this method in two dimensional biharmonic problems, a computational experiment is done on three linear biharmonic problems of the second kind for which the analytical solutions of U are known to us. The computed solutions are displayed in terms of maximum absolute error for some grid points but results are available for all grid points. It is shown that the method produces accurate results.

Example 7.1 Consider the biharmonic equation of the second kind on the unit square where the exact solution $U(x,y)=1$. The forcing term f and boundary data f_1 and f_2 can be obtained from exact solution.

Table 7.1

The maximum absolute error for U and $\nabla^2 U$ of example 7.1

h	U	$\nabla^2 U$
0.01	1.76081e-013	0
0.02	2.77556e-014	0
0.04	1.66533e-014	0
0.05	4.21885e-015	0
0.1	1.55431e-015	0

Example 7.2 Consider the biharmonic equation of the second kind on the unit square where the exact solution $U(x,y)=(1-\cos 2\pi x)(1-\cos 2\pi y)$. The forcing term f and boundary data f_1 and f_2 can be obtained from the exact solution.

Table 7.2

The maximum absolute error for U and $\nabla^2 U$ of example 7.2

h	U	$\nabla^2 U$
0.01	6.40764e-004	2.75741e-006
0.02	2.50084e-003	4.41543e-005
0.04	9.54848e-003	7.02282e-004
0.05	0.0146972	1.72793e-003
0.1	0.0554526	0.0281114

Example 7.3 Consider the biharmonic equation of the second kind on the unit square where the exact solution $U(x,y)=\sin\pi x\sin\pi y$. The forcing term f and boundary data f_1 and f_2 can be obtained from the exact solution.

Table 7.3

The maximum absolute error for U and $\nabla^2 U$ of example 7.3

h	U	$\nabla^2 U$
0.01	5.41107e-009	5.34052e-008
0.02	8.65655e-008	8.54367e-007
0.04	1.37861e-006	1.36063e-005
0.05	3.37726e-006	3.33323e-005
0.1	5.37909e-005	5.30902e-004

<u>Part II</u> : Three Dimensional Biharmonic Equation

Consider the three dimensional biharmonic equation on a cube D given by

$$\frac{\partial^4 U}{\partial x^4}+\frac{\partial^4 U}{\partial y^4}+\frac{\partial^4 U}{\partial z^4}+2\frac{\partial^4 U}{\partial x^2\partial y^2}+2\frac{\partial^4 U}{\partial x^2\partial z^2}+2\frac{\partial^4 U}{\partial z^2\partial y^2}=f(x,y,z),\quad (x,y,z)\in D \qquad (7.18)$$

with the second kind Dirichlet boundary condition

$$U(x,y,z)=f_1(x,y,z) \qquad (x,y,z)\in\partial D$$

$$\frac{\partial^2 U(x,y,z)}{\partial n^2}=f_2(x,y,z) \qquad (x,y,z)\in\partial D \qquad (7.19)$$

Using the splitting method where the biharmonic equation (7.18) is replaced by introducing an auxiliary variable $v(x,y,z)=\Delta U(x,y,z)$ and splitting the biharmonic equation into a coupled system of three dimensional Poisson's equations,

$$\nabla^2 v(x,y,z)=f(x,y,z) \qquad (7.20)$$

$$\nabla^2 U(x,y,z)=v(x,y,z)$$

i.e. the biharmonic equation with the given boundary conditions of the second kind is equivalent to the Dirichlet problems for two Poisson's equations of three dimensions.

Since the grid lines are parallel to coordinate axes and the values of U are exactly known on the boundary, and this implies, the successive tangential partial derivatives of U are known exactly on the boundary. For example, on the plane $y = 0$, the values of $U(x,0,z)$ and $U_{yy}(x,0,z)$ are known, i.e. the values of $U_x(x,0,z)$, $U_{xx}(x,0,z)$, $U_z(x,0,z)$, $U_{zz}(x,0,z)$, … are known on the plane $y = 0$. This implies, the values of $U(x,0,z)$ and $\nabla^2 U(x,0,z) = U_{xx}(x,0,z) + U_{yy}(x,0,z) + U_{zz}(x,0,z)$ are known on the plane $y = 0$.

Similarly, the values of U and $\nabla^2 U$ are known on all plane sides of the cubic region D.

Therefore, the biharmonic boundary value problem is reformulated as

$$\nabla^2 v(x,y,z) = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} = f(x,y,z) \qquad (x,y,z) \in D \qquad (7.21)$$

$$\nabla^2 U(x,y,z) = \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} = v(x,y,z) \qquad (x,y,z) \in D \qquad (7.22)$$

with the Dirichlet boundary conditions

$$U(x,y,z) = f_1(x,y,z) \qquad (x,y,z) \in \partial D$$

$$v(x,y,z) = g(x,y,z) \qquad (x,y,z) \in \partial D \qquad (7.23)$$

Now the given biharmonic equation is transformed to coupled Poisson's equation and we find the solution of the coupled equation first for the auxiliary variable v in (7.21) and using this value of v then for U in (7.22); this solves the given biharmonic equation (7.18).

Here to solve for v in (7.21) we directly use the fourth order finite difference approximation method especially the 19-point stencil scheme developed in chapter III with the boundary conditions (7.23).

Thus the Poisson's equation (7.21) be written using the 19-point stencil approximation scheme

$$
\begin{aligned}
&-(32+16r)v_{i,j,k}+(6-2r)\left(v_{i+1,j,k}+v_{i-1,j,k}+v_{i,j+1,k}+v_{i,j-1,k}\right)+(8r-4)\left(v_{i,j,k+1}+v_{i,j,k-1}\right)\\
&+(1+r)\left(v_{i+1,j,k+1}+v_{i+1,j,k-1}+v_{i-1,j,k+1}+v_{i-1,j,k-1}+v_{i,j+1,k+1}+v_{i,j+1,k-1}+v_{i,j-1,k+1}+v_{i,j-1,k-1}\right)\\
&+2\left(v_{i+1,j+1,k}+v_{i+1,j-1,k}+v_{i-1,j+1,k}+v_{i-1,j-1,k}\right)\\
&\qquad\qquad=12h_1^2\left(1+\frac{1}{12}\left(\delta_x^{\,2}+\delta_y^{\,2}+\delta_z^{\,2}\right)\right)f_{i,j,k}
\end{aligned}
\tag{7.24}
$$

Now we solve these systems of linear equations (7.24) for v by the method developed there.

Once if we get v then we consider the values of v as the right hand side of (7.22), write this Poisson's equation (7.22) using the 19-point stencil approximation scheme and we go to find the solution of U in (7.22).

Thus, we will have the Poisson's equation as

$$
\begin{aligned}
&-(32+16r)U_{i,j,k}+(6-2r)\left(U_{i+1,j,k}+U_{i-1,j,k}+U_{i,j+1,k}+U_{i,j-1,k}\right)+(8r-4)\left(U_{i,j,k+1}+U_{i,j,k-1}\right)\\
&+(1+r)\left(U_{i+1,j,k+1}+U_{i+1,j,k-1}+U_{i-1,j,k+1}+U_{i-1,j,k-1}+U_{i,j+1,k+1}+U_{i,j+1,k-1}+U_{i,j-1,k+1}+U_{i,j-1,k-1}\right)\\
&+2\left(U_{i+1,j+1,k}+U_{i+1,j-1,k}+U_{i-1,j+1,k}+U_{i-1,j-1,k}\right)\\
&\qquad\qquad=12h_1^2\left(1+\frac{1}{12}\left(\delta_x^{\,2}+\delta_y^{\,2}+\delta_z^{\,2}\right)\right)v_{i,j,k}
\end{aligned}
\tag{7.25}
$$

We solve these systems of equations (7.25) for U as we have solved v in the case above. Thus, the boundary value problem (7.18) with the given Dirichlet boundary condition of the second kind is solved as desired.

7.5.2 Numerical Results for Three Dimensional Biharmonic Equations

In order to test the efficiency and adaptability of this method in three dimensional biharmonic problems, a computational experiment is done on two linear biharmonic problems of the second kind for which the analytical solutions of U are known to us. The computed solutions are displayed in terms of maximum absolute error for some grid points but results are available for all grid points. It is shown that the method produces accurate results.

Example 7.4 Consider the biharmonic equation of the second kind on a cube where the exact solution $U(x,y,z) = \sin \pi x \sin \pi y \sin \pi z$. The forcing term f and boundary data f_1 and f_2 can be obtained from the exact solution.

Table 7.4

The maximum absolute error for U of example 7.4

M *P*	*Maximum Absolute error*	*M* *P*	*Maximum Absolute error*
9 9	*9.49352e-003*	29 9	7.78734e-004
9 19	9.30127e-003	29 19	2.62716e-004
9 29	9.28032e-003	29 29	1.22990e-004
9 39	9.27483e-003	29 39	1.22952e-004
19 9	2.10435e-003	39 9	3.69283e-004
19 19	6.16428e-004	39 19	1.35733e-004
19 29	6.15533e-004	39 29	6.76903e-005
19 39	6.15286-004	39 39	3.90621e-005

Example 7.5 Consider the three dimensional biharmonic equation with exact solution $U(x,y,z) = (1-\cos 2\pi x)(1-\cos 2\pi y)(1-\cos 2\pi z)$ on a unit cube. The forcing term f and boundary data f_1 can be obtained from the exact solution and the second boundary data f_2 is as follows:

$$\frac{\partial^2 U}{\partial n^2} = -U_{xx} = -4\pi^2 \cos 2\pi x(1-\cos 2\pi y)(1-\cos 2\pi z) \qquad x=0, \quad (y,z)\in \partial D$$

$$\frac{\partial^2 U}{\partial n^2} = U_{xx} = 4\pi^2 \cos 2\pi x(1-\cos 2\pi y)(1-\cos 2\pi z) \qquad x=1, \quad (y,z)\in \partial D$$

$$\frac{\partial^2 U}{\partial n^2} = -U_{yy} = -4\pi^2 \cos 2\pi y(1-\cos 2\pi x)(1-\cos 2\pi z) \qquad \mathrm{y}=0, \quad (x,z)\in \partial D$$

$$\frac{\partial^2 U}{\partial n^2} = U_{yy} = 4\pi^2 \cos 2\pi y(1-\cos 2\pi x)(1-\cos 2\pi z) \qquad \mathrm{y}=1, \quad (x,z)\in \partial D$$

$$\frac{\partial^2 U}{\partial n^2} = -U_{zz} = -4\pi^2 \cos 2\pi z(1-\cos 2\pi x)(1-\cos 2\pi y) \qquad z=0, \quad (x,y)\in \partial D$$

$$\frac{\partial^2 U}{\partial n^2} = U_{zz} = 4\pi^2 \cos 2\pi z(1-\cos 2\pi x)(1-\cos 2\pi y) \qquad z=1, \quad (x,y)\in \partial D$$

Table 7.5

The maximum absolute error of example 7.5

M *P*	*Maximum Absolute error*	*M* *P*	*Maximum Absolute error*
9 9	0.0521407	29 9	0.0654526
9 19	0.0730145	29 19	0.0526271
9 29	0.0911132	29 29	0.0516799
9 39	0.0987464	29 39	0.0512763
19 9	0.0643510	39 9	0.0438912
19 19	0.0716329	39 19	0.0413573
19 29	0.0617234	39 29	0.0406761
19 39	0.0635286	39 39	0.040124

7.6 Non-Linear Biharmonic Equation

Consider the two or three dimensional non-linear biharmonic boundary problem given by

$$\nabla^4 U = f(U) \qquad \text{on } D \tag{7.26}$$

where $f(U)$ is a non-linear function, with the second Dirichlet boundary conditions

$$\begin{aligned} U &= f_1 && \text{on } \partial D \\ \frac{\partial^2 U}{\partial n^2} &= f_2 && \text{on } \partial D \end{aligned} \tag{7.27}$$

To solve (7.26) with the given boundary conditions (7.27) first we reformulate (7.26) as coupled Poisson's equations, and write equation (7.26) as

$$\begin{aligned} \nabla^2 v &= f(U) \\ \nabla^2 U &= v \end{aligned} \tag{7.28}$$

To find the solution of the biharmonic problem we assume an initial guess for U say $U^{(0)}$ in (7.28).

Now we iterate as follows:

$$\begin{aligned} \nabla^2 v^{(n)} &= f(U^{(n)}) \\ \nabla^2 U^{(n+1)} &= v^{(n)} \qquad n = 0, 1, 2, \ldots \end{aligned} \tag{7.29}$$

We start with $n = 0$.

By using the method developed in the previous part we solve for the first equation of (7.29) and get say $v^{(0)}$, again we solve by the same method for U in the second equation of (7.29) and get say $U^{(1)}$. Therefore, equation (7.29) becomes

$$\begin{aligned} \nabla^2 v^{(1)} &= f(U^{(1)}) \\ \nabla^2 U^{(2)} &= v^{(1)} \end{aligned} \tag{7.30}$$

In the second iteration again we repeat this process to solve for $v^{(1)}$ in the first equation of (7.30) using the new $U^{(1)}$ and get say $v^{(2)}$; with this $v^{(2)}$ once again using the second equation of (7.30) we solve for $U^{(2)}$. Therefore (7.29) becomes

$$\begin{aligned} \nabla^2 v^{(2)} &= f(U^{(2)}) \\ \nabla^2 U^{(3)} &= v^{(2)} \end{aligned} \tag{7.31}$$

We keep on doing up to some iteration steps say it and get U_{it}; at each stage of the iteration we check whether the convergence condition is satisfied or not.

i.e. $|U_{it+1} - U_{it}| < \varepsilon$ where ε is the measure of the convergence.

If the convergence criterion is satisfied, then we stop the process and U_{it} will be a solution to the biharmonic problem.

Example 7.6 Consider the non-linear biharmonic equation $\nabla^4 U = 1 - U^2$ in a unit square region i.e. $D = \{(x, y) : 0 < x, y < 1\}$ with the exact solution $U(x, y) = 1$. The convergence condition here is 1.0e-010.

Table 7.6

The maximum absolute error of example 7.6

h	U
0.01	1.82675e-007
0.02	2.44147e-006
0.04	9.73538e-006
0.05	1.52481e-005
0.1	6.08285e-005

Example 7.7 Consider the non-linear biharmonic equation $\nabla^4 U = 1 - U^2$ in a unit cube i.e. $D = \{(x, y) : 0 < x, y, z < 1\}$ with the exact solution $U(x, y) = 1$. The convergence condition here is 1.0e-010.

Table 7.7

The maximum absolute error of example 7.7

h	U
0.01	2.49637e-007
0.02	7.44147e-006
0.04	3.52438e-005
0.05	5.33341e-006
0.1	9.14263e-005

Example 7.8 Consider the non-linear biharmonic equation $\nabla^4 U = U\left(4\pi^4 + U_x + U_y\right)$ in a unit square region i.e. $D = \{(x, y) : 0 < x, y < 1\}$ with the exact solution $U(x, y) = \sin \pi x \sin \pi y$. The convergence condition here is 1.0e-010.

Table 7.8

The maximum absolute error of example 7.8

h	U
0.01	4.72314e-004
0.02	5.93628e-004
0.04	8.02736e-004
0.05	9.39247e-004
0.1	9.72369e-004

7.7 Conclusion

In this work, by using splitting method we have transformed the two or three dimensional biharmonic equation of the second kind in Cartesian coordinates system in to two Poisson's equation in two or three dimensions. These two Poisson's equations by using their equivalent fourth order finite difference approximation schemes converted in to a system of algebraic linear equations. The resulting large number of linear algebraic equation is, then, systematically arranged in order to get a block matrix. Based on Hockney's method we reduced the obtained matrix in to a block tridiagonal matrix, and each block is solved by the help of Thomas algorithm [30]. We have successfully implemented this method to find the solution of the two or three dimensional linear or non-linear biharmonic equation in Cartesian coordinates system. The method produces good results and comparable with the results obtained in the literature. This method is a direct one and it allows considerable savings in computer storage as well as execution speed.

CHAPTER VIII

Conclusion and Future Plan

The aim of this work is to find efficient numerical solutions for the three dimensional Poisson's equation in Cartesian/ Cylindrical coordinates system and for the two/three dimensional biharmonic equation in Cartesian coordinates system.

In Chapter II we have transformed the three dimensional Poisson's equation in Cartesian coordinates system in to a system of algebraic linear equations using its equivalent second order finite difference approximation scheme. The resulting large number of algebraic equation is, then, systematically arranged in order to get a block matrices. Based on the extension of Hockney's method we reduced the obtained matrices in to block tridiagonal matrices, and each block is solved by the help of Thomas algorithm [30] and we use backward substitution to get the solution for $U_{i,j,k}$.

We have successfully implemented this method to find the solution of the three dimensional Poisson's equation in Cartesian coordinates system. It is found that the method can easily be applied and adapted to find a solution for problems of this kind and produce accurate results considering double precision.

In Chapter III the three dimensional Poisson's equation in Cartesian coordinates system is approximated by two different fourth order finite difference approximation schemes, the 19-points and 27- points stencil schemes. In both schemes the systems of linear equations, then, arranged in order to get a block matrices; these block matrices are reduced to block tridiagonal matrices by extending Hockney's method, and by the help of Thomas Algorithm we obtained the solution of the system and we go back to get the solution for $U_{i,j,k}$.

It is found that both fourth order approximation methods produced accurate results for the test problems. Actually it is shown that the discussed method, in general, for 27-points scheme produces better results (though the computational cost is high) than 19-points scheme but 19-points scheme has also shown comparable results to 27-points scheme. Therefore, this method is suitable to find the solution of any three dimensional Poisson's equation with the given boundary conditions in Cartesian coordinates system.

In Chapter IV the three dimensional Poisson's equation in cylindrical coordinates system is approximated by its equivalent second order finite difference approximation scheme. Here also we have applied the same technique as in the previous chapters, and we have successfully implemented the method to find the solution of the three dimensional Poisson's equation in cylindrical coordinates system. The method produced accurate results considering double precision and it is found that the method can easily be applied and adapted to find a solution of some related applied problems.

In Chapter V, the three dimensional Poisson's equation in cylindrical coordinates system is approximated by its equivalent fourth order finite difference approximation scheme, 19-point stencil scheme derived directly. Here we have also applied the same technique as in the previous chapters, and we have successfully implemented the method to find the solution of the three dimensional Poisson's equation in cylindrical coordinates system. The method produced accurate results considering double precision and it is found that the method can easily be applied and adapted to find a solution of some related applied problems.

In Chapter VI, the three dimensional Poisson's equation in cylindrical coordinates system when $r = 0$ is an interior or a boundary point is considered. We approximated by its equivalent second and fourth-order appropriate finite difference approximation scheme and by combining the difference scheme obtained in chapters IV and V we have successfully implemented the method to find the solution of the three dimensional Poisson's equation in cylindrical coordinates system. The results obtained are accurate.

In Chapter VII by using splitting method we have transformed the two or three dimensional biharmonic equation of the second kind in Cartesian coordinates system in to two Poisson's equation in two or three dimensions. For the two dimensional biharmonic equation we have established how to apply Hockney's method while for the three dimensional case when the forcing function f is linear we have solved the Poisson's equations by using the method developed in Chapter III directly; and when f is non-linear we have used the iterative method by assuming an initial guess for U. The method produced good results as in the literature.

In general, the main advantage of these methods is that we have used a direct method to solve the Poisson's equation and the biharmonic boundary value problem of the second kind for which the error in the solution arises only from rounding off errors; and the methods allow considerable savings in computer storage as well as execution speed, that is, it reduces the number of computations and computational time.

For future work,

- To work on more practical applied problems that use biharmonic equations in solid mechanics, fluid mechanics and other engineering problems.

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