APPROXIMATION BY CERTAIN POSITIVE LINEAR METHODS OF CONVERGENCE

Ph. D. THESIS

by

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INDIAN INSTITUTE OF TECHNOLOGY ROORKEE
ROORKEE- 247 667 (INDIA)
MAY, 2016

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A THESIS

Submitted in partial fulfilment of the requirements for the award of the degree
of

DOCTOR OF PHILOSOPHY

in

MATHEMATICS

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ROORKEE

CANDIDATE'S DECLARATION

I hereby certify that the work which is being presented in the thesis entitled "**APPROXIMATION BY CERTAIN POSITIVE LINEAR METHODS OF CONVERGENCE"** in partial fulfilment of the requirements for the award of the Degree of Doctor of Philosophy and submitted in the Department of Mathematics of the Indian Institute of Technology Roorkee, Roorkee is an authentic record of my own work carried out during a period from January, 2013 to May, 2016 under the supervision of Dr. P. N. Agrawal, Professor, Department of Mathematics, Indian Institute of Technology Roorkee, Roorkee.

The matter presented in this thesis has not been submitted by me for the award of any other degree of this or any other Institute.

(MEENU RANI)

This is to certify that the above statement made by the candidate is correct to the best of my knowledge.

(P. N. Agrawal)
Supervisor

The Ph.D. Viva-Voce Examination of **Ms. Meenu Rani,** Research Scholar, has been held on…………...

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Abstract

The present thesis deals with the approximation properties of some well-known linear positive operators and their new generalizations which include the Stancu type generalization, bivariate generalization, Bézier variant and $q-$variant of the well known operators. We divide the thesis into eight chapters. In Chapter 0, we mention literature survey, basic definitions and some notations of approximation techniques which we have used throughout the thesis. In Chapter 1, we define a general sequence of linear positive operators and discuss their approximation behaviour e.g. rate of convergence in ordinary and simultaneous approximation and the estimate of the rate of convergence for functions having a derivative equivalent to a function of bounded variation. Further, we illustrate the convergence of these operators and their first order derivatives by graphics using Matlab algorithms.

In Chapter 2, we consider a one parameter family of hybrid operators and study the local, weighted approximation results, simultaneous approximation of derivatives and statistical convergence. Also, we show the rate of convergence of these operators to a certain function by illustrative graphics in Matlab.
The third chapter involves the Kantorovich modification of generalized Baskakov operators. We obtain some direct results and then study weighted approximation, simultaneous approximation and statistical convergence properties for these operators. We also obtain the rate of convergence for functions having a derivative equivalent with a function of bounded variation for these operators. Further, we define the bivariate extension of the generalized Baskakov Kantorovich operators and discuss the results on the degree of approximation, asymptotic theorem, order of approximation using Peetre's $K-$functional and simultaneous approximation for first order derivatives of operators in polynomial weighted spaces. Lastly, we also show

the convergence of the bivariate operators to a certain function and demonstrate the comparison with the bivariate Szász-Kantorovich operators through graphics using Matlab algorithm. In Chapter 4, we study some approximation properties of the Bézier variant of generalized Baskakov Kantorovich operators. We obtain direct theorem by means of the first order modulus of smoothness and the rate of convergence for the functions having a derivative of bounded variation.

The fifth and sixth chapters deal with the $q-$analogues of general Gamma type operators and the Stancu generalization of Szász-Baskakov operators respectively. First, we obtain the moments of the operators by using the $q-$beta function and then prove the basic convergence theorem. The Voronovskaja type theorem, local and direct results and weighted approximation in terms of modulus of continuity have been discussed for both of these operators. Lastly, we study the King type approach in order to obtain the better approximation for both of these operators.

In the last chapter, we introduce the complex case of the Szász-Durrmeyer-Chlodowsky operators and obtain the upper estimate, Voronovskaja type result, the exact order in simultaneous approximation and asymptotic result with quantitative estimates. In this way, we show the overconvergence phenomenon for these operators, namely the extensions of approximation properties orders of these convergencies to sets in the complex plane that contain the interval $[0, \infty)$.

Acknowledgements

It is my pleasure to acknowledge the roles of several individuals who were instrumental for completion of my Ph.D research.
First of all, I would like to express my gratitude to my supervisor Dr. P. N. Agrawal, Professor, Department of Mathematics, I.I.T. Roorkee, Roorkee, for invaluable guidance, unrelenting support and encouragement throughout my research work. Without his active guidance and persistent help, it would not have been possible for me to complete this thesis.

I would like to thank my research committee members, Prof. Kusum Deep, DRC Chairperson, Prof. R. C. Mittal, SRC Chairperson, Dr. A. Swaminathan, Prof. Rajdeep Chatterjee for their guidance, cooperation and many valuable suggestions that have helped me in improving the quality of my research work.

It is a great pleasure to convey my gratitude to Prof. V. K. Katiyar, Head, Department of Mathematics, Indian Institute of Technology Roorkee for providing valuable advice, computational and other infrastructural facilities during my research work. I am indebted to the "Council of Scientific and Industrial Research" India for offering me scholarship to carry out my research work.

I am hugely indebted to Prof. Harun Karsli, Department of Mathematics, Abant Izzet Baysal University, Turkey, Prof. Nurhayat Ispir, Department of Mathematics, Gazi University, Turkey and Prof. Vijay Gupta, School of Applied Sciences, NSIT, New Delhi for finding out time to reply to my emails for being ever so kind to show interest in my research and for giving their precious and kind advice regarding the topic of my research.

Above all, my deepest appreciation belongs to my parents Mr. Satish Goel and Mrs. Raj Goel for their blessings, patience, understanding and support whom I dedicate this thesis. I would also like to thank my family members especially my uncles

Mr. Virender Goel, Mr. Hari Garg, aunts Mrs. Nishi Gupta, Mrs. Raj Bansal, Mrs. Sunita Goel, brothers Mukesh, Deepak, Gourav, Sumit, sisters Pallavi, Neha, Alisha and nephew Vivan Goel for encouraging me to achieve my goals. I am indebted to them for their help.

These Acknowledgements would not be complete without mentioning my friends and colleagues: Dr. Karunesh, Dr. Durvesh, Arun, Anu, Anamika, Arti, Kavita, Bablesh, Raman, Yogesh, Swati, Dr. Sathish, Pooja, Ruchi, Sheetal, Alka, Sourav Das, Reenu, Urvashi, Kiran, Rakesh, Priyanka and Renu. I appreciate their ideas, help and moral support.

Last but not the least, thanks to God for being my strength and guide in the writing of this thesis. Without Him, I would not have had the wisdom or the physical ability to do so.

Roorkee (Meenu Rani)

August , 2016

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Introduction

0.1 General Introduction

The theory of approximation of functions is now an extremely extensive branch of mathematical analysis. The Weierstrass approximation theorem [155] is one of the most fundamental theorems of approximation theory given by Weierstrass in the year 1885. This theorem characterizes that there exists a sequence of polynomials which is dense in space of all continuous functions on a closed interval. After that there were great mathematicians such as Runge, Lebesgue, Landau, Vall*é*e-Poussin, Fej*é*r, Jackson and Bernstein who relate their names with this most celebrated theorem. In 1912, Bernstein constructed Bernstein polynomials

$$\mathcal{B}_n(f;x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} f\left(\frac{k}{n}\right), \text{ for any } f \in C[0,1],\ 0 \leq x \leq 1.$$

The sequence of Bernstein polynomials converges uniformly to f on $[0,1]$, thus giving a constructive proof of Weierstrass's theorem. In 1940s Mirakjan [116], Favard [47] and Sz*á*sz [148] independently introduced the generalization of Bernstein polynomials to infinite intervals which is given by

$$\mathcal{G}_n(f;x) = e^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} f\left(\frac{k}{n}\right), \text{ for any } x \in [0,\infty), \text{ and } n \in \mathbb{N}.$$

If f is continuous on $(0,\infty)$ having a finite limit at infinity, then these operators named Sz*á*sz-Mirakjan operators $\mathcal{G}_n(f)$ converge uniformly to f as $n \to \infty$.
The foundation of the theory of approximation by general sequences of linear positive operators was introduced by Popoviciu [133], Bohman [29] and Korovkin [104]. Subsequently, many linear positive operators were defined and their approximation behaviour was discussed by many researchers. Now, we list some of the expert

mathematicians who involved in significant activities in development of theory of approximation by different linear positive operators and made many efforts to improve the degree of approximation of the different linear positive operators: Kantorovich [95], Phillips [130], Baskakov [25], Durrmeyer [43], Stancu [145], Rathore and Singh [137], Lupaş and Lupaş [109], Mazhar and Totik [114], Agrawal and Thamer [13], Srivastava and Gupta [143], Ranadive and Singh [134], Abel and Ivan [3], Abel and Heilmann [2] etc.

The aim of the general approximation methods concerning linear positive operators is to deal with convergence behaviour. For the convergence, the important basic concept is concerned with the study of direct results such as rate of convergence, asymptotic behaviour and order of approximation. The direct results provide the order of approximation for functions of a specified smoothness. The best approximation by direct estimates was first obtained by Jackson [92] for algebraic and trigonometric polynomials. Rate of convergence of linear positive operators infers speed at which a convergent sequence of polynomials approaches to the function. Some operators reproduce constant as well as linear functions. We can also get a better approximation by modifying the operators that do not even preserve linear functions. King [101] was the first person who has taken initiative by modifying Bernstein operators. He showed that the modified operators yield a better approximation than the operators $\mathcal{B}_n$ whenever $0 \leq x \leq \frac{1}{3}$.

Apart from earlier well known operators, several new sequences and classes of operators were introduced and studied. Here we give the series of some linear positive operators: Stancu [145] introduced the positive linear operators $\mathcal{P}_n^{(\alpha,\beta)} : C[0,1] \to C[0,1]$ by modifying the Bernstein polynomials as

$$\mathcal{P}_n^{(\alpha,\beta)}(f;x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} f\left(\frac{k+\alpha}{n+\beta}\right),$$

where α, β be any two non-negative real numbers which satisfy the condition $0 \leq \alpha \leq \beta$. If $\alpha, \beta = 0$, the above sequence of operators reduces to Bernstein polynomials. Subsequently, several authors (cf. [5], [78], [94], [150] etc.) developed such a modification for some other sequences of positive linear operators. Another modification in Bernstein polynomials consists of Kantorovich integral operators by Kantorovich [95] and Durrmeyer integral operators by Durrmeyer [43] in order to approximate

Lebesgue integrable functions on $[0, 1]$. Also, several mixed summation-integral-type operators [37], [44], [66], [76], [79], [139] have been constructed by using different basis functions and their approximation behaviours were studied. In [132], Phillips defined another alteration in Szász-Mirakjan operators by considering the value of the function at zero explicitly. After that many researchers implemented this technique on different linear positive operators [11], [13], [69] etc.

In order to study more general case of linear positive operators, many mathematicians combined different operators by using different parameters [120], [128]. Another method to generalize the operators is given by Mihesan [115] who investigated the generalized Baskakov operators and obtained the uniform convergence on closed interval and point-wise estimate for these operators. After that Wafi and Khatoon [152] studied many approximation properties i.e. rate of convergence, asymptotic formula, direct and inverse estimates. They also defined the bivariate extension of these operators and studied convergence of first derivative and Voronvskaja type results. In [45], Erençin and Başcanbaz-Tunca also found the weighted approximation theorem and estimated the order of convergence for these opeators. Later on, Erençin [44] introduced the Durrmeyer type modification of generalized Baskakov operators and obtained some local direct results. Recently, Agrawal et al. [8] studied simultaneous approximation and rate of convergence for these operators. In the present thesis, we define the Kantorovich modification of the generalized Baskakov operators.

Mazhar [113] investigated the general gamma type operators and discussed some approximation properties. In [97], Karsli reconstructed these operators and obtained rate of convergence for that modified operators. After that Mao [111] considered generalization of gamma type operators which include both operators defined by Mazhar and Karsli and studied the rate of convergence for these operators.
A Bèzier curve is a parametric curve frequently used in computer graphics (i.e. design of fonts, animation etc.) and related fields. Bèzier basis functions are very useful in computer aided design. Bojanic and Cheng [31], [30] obtained the rate of convergence for functions with derivatives of bounded variation for Bernstein and

Hermite-Fejer polynomials. Guo [64] studied it for the Bernstein-Durrmeyer polynomials by using Berry Esseen theorem. Zeng and Chen [158] initiated the study of rate of convergence for Bernstein-Bèzier-Durrmeyer operators. Zeng and Tao [159] obtained the rate of convergence for Bèzier variant of Baskaov-Durrmeyer operators for $\theta \geq 1$. They also termed these operators as integral type Lupaş -Bèzier operators. Abel and Gupta [1] introduced the Bèzier variant of the Baskakov operators and then Gupta [68] estimated the convergence of Bèzier type Baskakov-Kantorovich operators and studied the rate of convergence for $0 < \theta < 1$. Guo et al. [63] gave the direct, inverse and equivalence approximation theorems with unified Ditzian-Totik modulus $\omega_{\phi^\lambda}(f,t)(0 \leq \lambda \leq 1)$. For further research in this direction, we refer to [70], [76] and [90] etc.

0.2 Fundamental of $q-$Calculus

Let $q > 0$. For each integer $k \geq 0$, the $q-$integer $[k]_q$ and the $q-$factorial $[k]_q!$ are defined by

$$[k]_q := \begin{cases} \dfrac{1-q^k}{1-q}, & q \neq 1 \text{ for } k \in \mathbb{N} \text{ and } [0]_q = 0, \\ k, & q = 1, \end{cases}$$

$$[k]_q! := [1]_q[2]_q \cdots [k]_q \text{ for } k \in \mathbb{N} \text{ and } [0]_q! = 1.$$

The q-analogue of beta function of second kind is defined by

$$B_q(t,s) = K(A,t) \int_0^{\infty/A} \frac{x^{t-1}}{(1+x)_q^{t+s}} d_qx, \tag{0.2.1}$$

where $K(x,t) = \dfrac{1}{x+1} x^t \left(1+\dfrac{1}{x}\right)_q^t (1+x)_q^{1-t}$, and $(a+b)_q^s = \Pi_{i=0}^{s-1}\left(a+q^ib\right)$, $s \in \mathbb{Z}^+$.

In particular, for any positive integers n, m, we have

$$K(x,n) = q^{\frac{n(n-1)}{2}}, \qquad K(x,0) = 1, \tag{0.2.2}$$

and

$$B_q(m,n) = \frac{\Gamma_q(m)\Gamma_q(n)}{\Gamma_q(m+n)}.$$

$\Gamma_q(m)$ and $B_q(m,n)$ are the $q-$analogues of the Gamma and Beta functions. As $q \to 1$ the $q-$Gamma function and $q-$Beta function reduce to $\Gamma(m)$ and $B(m,n)$ respectively.
The $q-$analogue E_q^x of classical exponential function is defined as

$$E_q^x = \sum_{j=0}^{\infty} q^{j(j-1)/2} \frac{x^j}{[j]!}.$$

For further details on $q-$calculus, one can refer to [22] and [93]. The applications of $q-$calculus have been proved to be an active area of recent researches in approximation theory. It has been shown that linear positive operators investigated by $q-$numbers are quite effective as far as the rate of convergence is concerned and we can have some unexpected results, which are not observed for classical case. This type of construction was first applied to generate Bernstein operators. The generalization of Bernstein polynomials involving $q-$integers was first investigated by Lupaş [108] which provides rational functions rather than polynomials. In the discussion of uniform convergence of $q-$Bernstein operators, the Korovkin theorem is used in classical Bernstein operators. But for $0 < q < 1$, we get $[n]_q = \frac{1-q^n}{1-q} \to \frac{1}{1-q}$ as $n \to \infty$, so for uniform convergence of $q-$Bernstein polynomials we take a sequence of q_n which goes to 1 as $n \to \infty$. In the case $q = 1$, these polynomials coincide with the classical ones. Phillips [130] further generalized $q-$Bernstein polynomials and Phillips and other researchers obtained extensively studied these polynomials (see [131] and references therein). Ostrovska and other researchers [88], [100], [123], [124], [154] also derived new results about convergence properties of the $q-$Bernstein polynomials. In [38], Derriennic discussed modified Bernstein polynomials with Jacobi weights in $q-$calculus.

Many researchers introduced a similar modification on different discrete $q-$operators which include the $q-$Blemian-Butzer-Hahn operators, $q-$Szász operators, $q-$Baskakov operators [19], [18], [20]. Although, from the structural point of view the $q-$Szász-Mirakyan operators have some resemblances to the classical Szász Mirakyan operators, they have some similarities to the Bernstein-Chlodowsky operators from the properties of convergence standpoint, but $q-$Szász-Mirakyan operators with this construction are sensitive or flexible to the rate of convergence to f. In point of

$q-$Baskakov operators, Aral and Gupta [21] improved and represented the operators in terms of divided differences to study the $q-$derivatives, shape preserving properties and the applications of these operators. Integral modifications of these operators using $q-$beta functions of first and second kind such as $q-$Bernstein-Kantorovich operators, $q-$Bernstein-Durrmeyer operators, $q-$Baskakov-Durrmeyer operators, $q-$Szász-Durrmeyer operators and $q-$Phillips operators were also proposed and moment estimation, direct results, asymptotic formula, weighted approximation and the rate of convergence results have been studied for such operators. In this direction, significant contributions have been made by (cf. [6], [35], [71], [122] etc.).

0.3 Simultaneous Approximation

By the simultaneous approximation, we mean the approximation of derivatives of functions by the corresponding order derivatives of operators. Lorentz [107] pioneered the study in this direction who studied the point-wise convergence for Bernstein operators. Rathore [135], [136] studied more results of simultaneous approximation in detail and established the existence of Voronvskaja type formulae and degree of approximation by means of the higher order modulus of continuity in simultaneous approximation. Later on several researchers implemented this method on various operators [36], [77], [80], [84], [85] and [140].

0.4 Statistical Convergence

Six decades ago, the concept of statistical convergence was first introduced by Steinhaus [147]. It was further developed by Fast [46] and studied by various authors. After Gadjiev and Orhan [51], many researchers have been concerned with the Korovkin type approximation theorems for positive linear operators by using statistical convergence. The main concept of the statistical convergence of a sequence is that the majority of elements from sequence converge and we do not consider the other elements in that sequence.

Let M be a subset of the set of positive integers; then the natural density of the set M is denoted by $\delta(M)$ and defined as

$$\delta(M) := \lim_{n\to\infty} \frac{|M_n|}{n}$$

where $M_n := \{k \leq n : k \in M\}$. The statistical convergence is a generalization of ordinary convergence and was defined by Fast in the following way [46]:

- **Statistical convergence** A sequence $x := (x_n)$ is called statistically convergent to l and denoted by $st - \lim_{n\to\infty} x_n = l$, if for every $\epsilon > 0$,

$$\lim_{n\to\infty} \frac{|\{k \in [1, n] : |x_k - l| \geq \epsilon\}|}{n} = 0.$$

  After the statistical convergence, the different researchers introduced some other methods.

- **Lacunary statistical convergence :** Recall that a lacunary sequence $\theta = \{k_n\}$ is an increasing integer sequence such that $k_0 = 0$ and $h_n = k_n - k_{n-1} \to \infty$ as $n \to \infty$. A sequence $x := (x_n)$ is said to exhibit lacunary statistical convergence [49], [118] if there exists l such that, for every $\epsilon > 0$,

$$\lim_{n\to\infty} \frac{|\{k \in (k_{n-1}, k_n) : |x_k - l| \geq \epsilon\}|}{h_n} = 0.$$

- $\lambda-$**statistical convergence :** Let $\lambda = (\lambda_n)$ be a non-decreasing sequence of positive numbers satisfying $\lambda_n \to \infty\, (n \to \infty), \lambda_1 = 1, \lambda_{n+1} \leq \lambda_n + 1$; then a sequence $x := (x_n)$ is said to be $\lambda-$statistically convergent [117] if there exists l such that, for every $\epsilon > 0$,

$$\lim_{n\to\infty} \frac{|\{k \in [n - \lambda_n + 1, n] : |x_k - l| \geq \epsilon\}|}{\lambda_n} = 0.$$

- $A-$**statistical convergence :** For any non-negative regular matrix $A = (a_{nk})$, statistical convergence was extended to $A-$statistical convergence by Kolk [103].

  Let $A = (a_{nj})$ be an infinite summability matrix. For a given sequence $x := (x_k)$, the $A-$transform of x, denoted by $Ax = (Ax)_n$ is defined as $(Ax)_n := \sum_{k=1}^{\infty} a_{nk}x_k$, provided the series converges for each n. A is said to be regular if

$\lim_n (Ax)_n = l$ whenever $\lim_n x_n = l$. Suppose that A is a non-negative regular summability matrix. Then x is $A-$statistically convergent to L and denoted by $st_A - \lim_n x_n = l$, if for every $\epsilon > 0$,

$$\lim_n \sum_{k \in K(\epsilon)} a_{nk} = 0$$

where $K(\epsilon) := \{k : |x_k - l| \geq \epsilon\}$ and we write $st_A - \lim_n x_n = l$. If $A = C_1$, the Cesàro matrix of order one, then the $A-$statistical convergence reduces to the statistical convergence. Also, if $A = I$, the identity matrix, $A-$statistical convergence coincides with the ordinary convergence. Statistical convergence, lacunary statistical convergence and $\lambda-$statistical convergence are well known examples of $A-$statistical convergence.

Recently, Aktuğlu [14] introduce $\alpha\beta-$statistical convergence methods which include not only some well known regular matrix methods such as statistical convergence, lacunary statistical convergence and $\lambda-$statistical convergence methods but also some non-regular matrix methods.

0.5 Bivariate Extension

Kingsley [102] first introduced the Bernstein polynomials for functions of two variables of class $C^{(k)}$ (class of k times continuous and differentiable functions) on a closed region $R : 0 \leq x \leq 1, 0 \leq y \leq 1$. If $f(x, y)$ is a continuous function in R, then the bivariate generalization of Bernstein polynomials $V_{m,n}(f; x, y)$ is given by:

$$V_{m,n}(f; x, y) = \sum_{p=0}^{n} \sum_{q=0}^{m} f\left(\frac{p}{n}, \frac{q}{m}\right) \lambda_{n,p}(x) \lambda_{m,q}(y), \tag{0.5.1}$$

where $\lambda_{n,p}(x) = \binom{n}{p} x^p (1-x)^{n-p}$, $\lambda_{m,q}(y) = \binom{m}{q} y^q (1-y)^{n-q}$.

Butzer [32] has also proven some results for these polynomials. After that Stancu [144] defined another bivariate Bernstein polynomials on the triangle

$$\Delta := S = \{(x, y) : x + y \leq 1,\ 0 \leq x, y \leq 1\}.$$

In [146], Stancu considered new linear positive operators in two and several dimensional variables. Barbosu [23], [24] proposed the $q-$analogues of two dimensional Bernstein operators and Kantorovich-Schurer operators respectively. Many papers

were published on approximation by modified Szász-Mirakyan and Baskakov operators for univariate and bivariate cases [48], [83], [119], [138], [142], [157] which deal with convergence, degree of approximation and Voronovskaja type theorems as well as convergence of partial derivatives of these operators. Later on Doğru and Gupta [41] constructed a bivariate generalization of the $q-$Meyer-König and Zeller while in [7], Agratini proposed two-dimensional extension of some univariate positive approximation processes expressed by series.
In [151], [153] Wafi and Khatoon introduced generalized bivariate Baskakov operators in polynomial and exponential weighted spaces. They studied the basic convergence, degree of approximation, direct theorems, convergence of partial first order derivatives and also obtained a Voronovskaja type theorem.
Örkcü [121] proposed a new bivariate generalization by q_R-integral. She presented many approximation properties i.e. Voronovskaja-type theorem in polynomial weighted spaces and weighted $A-$statistical approximation properties for these operators. Also, she estimated the rate of convergence of the proposed operators in terms of modulus of continuity. In the present thesis, we also define the bivariate extension for the Kantorovich modification of generalized Baskakov operators.

0.6 Complex Extension

In the complex extension of linear positive operators, we mean to extend and preserve the convergence properties and orders of approximation to large sets in complex plane than real interval. Bernstein [107] proved that if $f : G \to \mathbb{C}$ is analytic in the open set $G \subset \mathbb{C}$, with $\overline{\mathbb{D}}_1 \subset G$ (with $\overline{\mathbb{D}}_1 = z \in \mathbb{C} : |z| < 1$), then the complex Bernstein polynomials $\mathcal{B}_n(f; z) = \sum_{k=0}^{n} \binom{n}{k} z^k (1-z)^{n-k} f\left(\frac{k}{n}\right)$, uniformly converges to f in $\overline{\mathbb{D}}_1$. This important concept of the overconvergence of Bernstein polynomials has been discussed by many researchers: Wright [156], Kantorovich [96], Bernstein ([26], [27], [28]), Tonne [149] and Lorentz [107].

Initially, Sorin Gal has done commendable work in this direction and for uniform convergence of $\mathcal{B}_n(f; z)$ to f, he estimated upper quantitative estimates in [52]. Exact quantitative estimates for different operators also studied by him in his recent papers [17], [54], [55]. Voronovskaja-type results with quantitative estimates for the different operators attached to analytic functions on compact disks and the exact order of simultaneous approximation by different complex operators were collected by Gal [53], [56].

After that similar results have been done in this direction on different linear positive operators e.g. complex Bernstein-Schurer and Kantorovich-Schurer polynomials [16], complex Bernstein-Durrmeyer polynomials [17], complex $q-$Durrmeyer type operator [60], complex Kantorovich type operators [110], complex $q-$Szász-Kantorovich operators [61] etc. Recently, Gal and Gupta [57], [58], [59], Gupta [73], Gupta and Soybaş [82] put in the overconvergence phenomenon for complex Phillips Stancu, complex beta operators, complex Szász-Durrmeyer operators, complex Baskakov-Szász-Stancu and complex genuine hybrid operators respectively.

0.7 Basic Definitions and Notations

Throughout the thesis, let $\mathbb{R}$ denote the set of all real numbers, $\mathbb{C}$ the set of all complex numbers, $\mathbb{N}$ the set of all positive integers, $\mathbb{N}^0 = \mathbb{N} \cup \{0\}$, and $C[a, b]$, the space of continuous functions on $[a, b]$.

Let $C_B[0, \infty)$ denote the space of all real valued bounded and uniformly continuous functions on $[0, \infty)$ endowed with the norm

$$\|f\| = \sup_{x\in[0,\infty)} |f(x)|$$

and $C_B^2[0, \infty) = \{g \in C_B[0, \infty) : g', g'' \in C_B[0, \infty)\}$.

Definition 1. The m^{th} order modulus of continuity $\omega_m(f; \delta, I)$ for a function continuous on I is defined by

$$\omega_m(f; \delta, I) = \sup\{|\Delta_h^m f(x)| : |h| \leq \delta, x, x + h \in I\}.$$

For $m = 1, \omega_m(f; \delta)$ is usual modulus of continuity on $[0, \infty)$.

The Peetre's K-functional is defined as

$$\mathcal{K}_2(f, \delta) = \inf\{\| f - g \| +\delta \| g'' \|; g \in C_B^2[0, \infty)\}, \tag{0.7.1}$$

where $\delta > 0$. By Devore and Lorentz [39, p.177, Theorem 2.4], there exists an absolute constant $C > 0$ such that

$$\mathcal{K}_2(f,\delta) \leq C\omega_2(f;\sqrt{\delta}), \tag{0.7.2}$$

where $\omega_2(f;\sqrt{\delta})$ is the second order modulus of continuity on $[0,\infty)$.

Further, for $\gamma > 0$, we define $C_\gamma[0,\infty) := \{f \in C[0,\infty) : |f(t)| \leq C\, e^{\gamma t}$, for some $C > 0$ and $t \in [0,\infty)\}$, endowed with the norm $\| f \|_{C_\gamma[0,\infty)} = \sup\limits_{t\in[0,\infty)} |f(t)|e^{-\gamma t}$,
and for $\vartheta > 0$, we define $D_\vartheta[0,\infty) = \{f \in C[0,\infty) : |f(t)| \leq M_f(1+t^\vartheta),$ for some $M_f > 0\}$ endowed with the norm $\|f\|_\vartheta = \sup\limits_{t\in[0,\infty)} \dfrac{|f(t)|}{(1+t^\vartheta)}$
and also $D_2^*[0,\infty) = \left\{ f \in D_2[0,\infty) : \lim\limits_{t\to\infty} |f(t)|(1+t^2)^{-1} < \infty \right\}$.

Let $f \in D_2^*[0,\infty)$. The weighted modulus of continuity is defined as :

$$\Omega_2(f,\delta) = \sup_{x\geq 0, 0<|h|\leq\delta} \frac{|f(x+h)-f(x)|}{1+(x+h)^2}.$$

Lemma 0.7.1. *[106] Let $f \in D_2^*[0,\infty)$, then:*

(i) $\Omega_2(f,\delta)$ is a monotone increasing function of δ;

(ii) $\lim\limits_{\delta\to 0^+} \Omega_2(f,\delta) = 0$;

(iii) for each $m \in \mathbb{N}, \Omega_2(f,m\delta) \leq m\Omega_2(f,\delta)$;

(iv) for each $\lambda \in [0,\infty), \Omega_2(f,\lambda\delta) \leq (1+\lambda)\Omega_2(f,\delta)$.

The Lipschitz-type maximal function of order τ introduced by Lenze [105] as

$$\widehat{\omega}_\tau(f,x) = \sup_{t\neq x,\ t\in[0,\infty)} \frac{|f(t)-f(x)|}{|t-x|^\tau}, \quad x \in [0,\infty) \text{ and } \tau \in (0,1]. \tag{0.7.3}$$

Let $f \in DBV_\gamma[0,\infty)$, $\gamma \geq 0$ be the class of all functions defined on $[0,\infty)$, having a derivative that coincides a.e. with a function of bounded variation on every finite subinterval of $[0,\infty)$ and $|f(t)| \leq Mt^\gamma$, $\forall\ \ t > 0$.
We notice that the functions $f \in DBV_\gamma(0,\infty)$ possess a representation

$$f(x) = \int_0^x h(t)dt + f(0),$$

where $h(t)$ is a function of bounded variation on each finite subinterval of $(0,\infty)$.

Definition 2. For sufficiently small $\eta > 0$, the Steklov mean $f_{\eta,2}$ of 2^{nd} order corresponding to $f \in C_\gamma[0,\infty)$ and $t \in H_i = [a_i, b_i], i = 1, 2$ is defined as follows:

$$f_{\eta,2}(t) = \eta^{-2} \int_{-\eta/2}^{\eta/2} \int_{-\eta/2}^{\eta/2} (f(t) - \Delta_h^2 f(t)) dt_1 dt_2,$$

where $h = \frac{t_1+t_2}{2}$ and Δ_h^2 is the second order forward difference operator with step length h. The following properties are satisfied (see [74], [86] and references therein):

(i) $f_{\eta,2}$ has continuous derivatives up to order 2 over H_1;

(ii) $\| f_{\eta,2}^{(r)} \|_{C(H_2)} \leq C\eta^{-r}\omega_r(f; \eta, H_2), r = 1, 2;$

(iii) $\| f - f_{\eta,2} \|_{C(H_2)} \leq C\omega_2(f; \eta, H_1);$

(iv) $\| f_{\eta,2} \|_{C(H_2)} \leq C \| f \|_{C(H_1)} \leq C||f||_{C\gamma[0,\infty)},$

where C is a constant not necessarily the same at each occurrence and is independent of f and η.

0.8 Contents of Thesis

The thesis consists of seven chapters and the description of contents in these chapters is given below:

Chapter 1. In this chapter, we introduce a general sequence of summation-integral type operators. We discuss some direct results which include Voronovskaja type asymptotic formula, point-wise convergence for derivatives, error estimations in terms of modulus of continuity and weighted approximation for these operators. Also, we study simultaneous approximation by these operators and estimate the rate of convergence for functions having a derivative that coincides a.e. with a function of bounded variation. Furthermore, the convergence of these operators and their first order derivatives to certain functions and their corresponding derivatives respectively is illustrated by graphics using Matlab algorithms for some particular values of the parameters c and ρ.
The results related to the simultaneous approximation and the rate of approximation of functions with derivatives of bounded variation are published in the **Proceedings of the International Conference on Recent Trends in Mathematical**

Analysis and its Applications (ICRTMAA-2014), (Springer).

Chapter 2. We introduce a one parameter family of hybrid operators and study quantitative convergence theorems e.g. local and weighted approximation results and simultaneous approximation of derivatives. Further, we discuss the statistical convergence of these operators. Lastly, we show the rate of convergence of these operators to a certain function by illustrative graphics in Matlab.
This chapter is published in **Applied Mathematics and Computation (Elsevier Publications)**.

Chapter 3. In the present chapter, we construct generalized Baskakov Kantorovich operators. We establish some direct results and then study weighted approximation, simultaneous approximation and statistical convergence properties for these operators. We also obtain the rate of convergence for functions having a derivative coinciding almost everywhere with a function of bounded variation for these operators.
Also, we define the bivariate extension of the generalized Baskakov Kantorovich operators and discuss the results on the degree of approximation, Voronovskaja type theorems and their first order derivatives in polynomial weighted spaces. Furthermore, we illustrate the convergence of the bivariate operators to a certain function through graphics using Matlab algorithm. We also discuss the comparison of the convergence of the bivariate generalized Baskakov Kantorovich operators and the bivariate Szász-Kantorovich operators to the function through illustrations using Matlab.
The results in the first part of this chapter are accepted in **Filomat**.

Chapter 4. In this chapter, we introduce the Bèzier variant of the generalized Baskakov Kantorovich operators. We establish a direct approximation theorem with the aid of the Ditzian-Totik modulus of smoothness and also study the rate of convergence for the functions having a derivative of bounded variation for these operators.
This chapter is published in **Bollettino dell'Unione Matematica Italiana (Springer publication)**.

Chapter 5. The purpose of this chapter is to introduce the $q-$analogue of the general Gamma type operators. Here, we establish the moments of the operators and then prove the basic convergence theorem. Next, the Voronovskaja type theorem and some direct results for the above operators are discussed. We also study the rate of convergence and weighted approximation by these operators in terms of modulus of continuity. Further, we study the A-statistical convergence of these operators. Lastly, we modify these operators by King type approach to obtain better estimates.
This chapter is published in **Applied Mathematics and Computation (Elsevier publication)**.

Chapter 6. In the present chapter, we construct the $q-$analogue of the Stancu variant of Szász-Baskakov operators. We obtain the moments of the operators and then prove the basic convergence theorem. Next, the Voronovskaja type theorem and some direct results for the above operators are studied. Also, the rate of convergence and weighted approximation by these operators in terms of modulus of continuity are discussed. Lastly, we study the $A-$statistical convergence of these operators and also in order to obtain better approximation we find a King type modification of the above operators.
The results of this chapter is published in **Journal of Inequalities and Applications (Springer publication)**.

Chapter 7. The present chapter deals with the overconvergence of the Szász-Durrmeyer-Chlodowsky operators. Here we study the approximation properties e.g. upper estimates, Voronovskaja type result for these operators attached to analytic functions in compact disks. Also, we discuss the exact order in simultaneous approximation by these operators and its derivatives and the asymptotic result with quantitative upper estimate. In this way, we put in evidence the overconvergence phenomenon for the Szász-Durrmeyer-Chlodowsky operators, namely the extensions of approximation properties with exact quantitative estimates and orders of these convergencies to larger sets in the complex plane than the real interval $[0, \infty)$.

Based on the subject matter of the thesis, the following papers have been prepared:

Published/Submitted

1. P. N. Agrawal, Harun Karsli and Meenu Goyal, Szász-Baskakov type operators based on q-integers, J. Inequal. Appl. 441 (2014) 1-18.

2. Harun Karsli, P. N. Agrawal and Meenu Goyal, General Gamma type operators based on $q-$integers, Appl. Math. Comput. 251 (2015) 564-575.

3. Meenu Goyal, Vijay Gupta and P. N. Agrawal, Quantitative convergence results for a family of hybrid operators, Appl. Math. Comput. 271 (2015) 893-904.

4. Meenu Goyal and P. N. Agrawal, Bèzier variant of the generalized Baskakov Kantorovich Operators, Boll. Unione Mat. Ital. 8 (4) (2016) 229-238.

5. Meenu Goyal and P. N. Agrawal, Degree of approximation by certain genuine hybrid operators, in the Proceedings of the International Conference on Recent Trends in Mathematical Analysis and its Applications (ICRTMAA-2014), (Springer), (2014) 131-148.

6. P. N. Agrawal and Meenu Goyal, Generalized Baskakov Kantorovich operators, Filomat, accepted.

7. Vijay Gupta, P. N. Agrawal and Meenu Goyal, Approximation by certain genuine hybrid operators, submitted.

8. Meenu Goyal and P. N. Agrawal, Approximation properties of bivariate generalized Baskakov Kantorovich operators, submitted.

9. Meenu Goyal and P. N. Agrawal, Approximation by Complex Szász-Durrmeyer-Chlodowsky operators in compact disks, submitted.

Chapter 1

Approximation by certain genuine hybrid operators

1.1 Introduction

Gupta and Rassias [81] introduced the Lupaş-Durrmeyer operators based on Polya distribution and discussed some local and global direct results. Gupta [72] studied some other hybrid operators of Durrmeyer type. Păltănea [129] (see also [128]) considered a Durrmeyer type modification of the genuine Szász-Mirakjan operators based on two parameters $\alpha, \rho > 0$. Inspired by his work, we now propose for $f \in C_\gamma[0,\infty)$, a general hybrid family of summation-integral type operators based on the parameters $\rho > 0$ and $c \in \{0,1\}$ in the following way:

$$B_\alpha^\rho(f,x) = \sum_{k=1}^{\infty} p_{\alpha,k}(x,c)\int_0^\infty \theta_{\alpha,k}^\rho(t)f(t)dt + p_{\alpha,0}(x,c)f(0), \tag{1.1.1}$$

$$= \int_0^\infty \mathcal{A}_\alpha^\rho(x,t)f(t)dt, \tag{1.1.2}$$

where

$$p_{\alpha,k}(x,c) = \frac{(-x)^k}{k!}\phi_{\alpha,c}^{(k)}(x), \theta_{\alpha,k}^\rho(t) = \frac{\alpha\rho}{\Gamma(k\rho)}e^{-\alpha\rho t}(\alpha\rho t)^{k\rho-1}$$

$$\text{and } \mathcal{A}_\alpha^\rho(x,t) = \sum_{k=1}^{\infty} p_{\alpha,k}(x,c)\theta_{\alpha,k}^\rho(t) + p_{\alpha,0}(x,c)\delta(t);\ x \in (0,\infty).$$

It is observed that the operators $B_\alpha^\rho(f,x)$ are well defined for $\alpha\rho > \gamma$. Further, we note that the operators (1.1.1) preserve the linear functions.

Special cases:

1. If $\phi_{\alpha,0}(x) = e^{-\alpha x}$, then $p_{\alpha,k}(x,0) = e^{-\alpha x}\frac{(\alpha x)^k}{k!}$, we get the operators due to Păltănea [129]. Also, for this case if $\rho = 1$, we get the Phillips operators [132].

2. If $\phi_{\alpha,1}(x) = (1+x)^{-\alpha}$ and $\alpha = n$, then $p_{\alpha,k}(x,1) = \dfrac{(\alpha)_k}{k!}\dfrac{x^k}{(1+x)^{\alpha+k}}$, with the rising factorial given by $(n)_i = n(n+1)\cdots(n+i-1)$, $(n)_0 = 1$. For $\rho = 1$, we get the operators studied in [12].

3. If $c = 0, \alpha = n$ and $\rho \to \infty$, then in view of ([128], Theorem 2.2), we get the Szász-Mirakjan operators.

4. Similarly, if $c = 1, \alpha = n,\ f \in \overline{\Pi}$, the closure of the space of algebraic polynomials in space $C[0,\infty)$ and $\rho \to \infty$, we obtain at once Baskakov operators.

The aim of the present chapter is to discuss some direct results for the generalized operators (1.1.1). We study some direct results in simultaneous approximation by these operators e.g. point-wise convergence theorem, Voronovskaja-type theorem and an error estimate in terms of the modulus of continuity. Then, we obtain error estimations by means of modulus of continuity and weighted approximation. Next, we estimate the rate of convergence for functions having a derivative that coincides a.e. with a function of bounded variation.

1.2 Basic Results

In the sequel, we need the following lemmas. For $f : [0,\infty) \to \mathbb{R}$, we define

$$S_\alpha(f;x) = \sum_{k=0}^{\infty} p_{\alpha,k}(x,c) f\left(\frac{k}{\alpha}\right) \tag{1.2.1}$$

such that (1.2.1) makes sense for all $x \geq 0$.

For $m \in \mathbb{N}^0$, the m^{th} order central moment of the operators S_α is given by

$$\upsilon_{\alpha,m}(x) := S_\alpha((t-x)^m;x) = \sum_{k=0}^{\infty} p_{\alpha,k}(x,c)\left(\frac{k}{\alpha}-x\right)^m.$$

Lemma 1.2.1. *For the function $\upsilon_{\alpha,m}(x)$, we have $\upsilon_{\alpha,0}(x) = 1,\ \upsilon_{\alpha,1}(x) = 0$*

$$\text{and}\ \ (1+cx)[\upsilon'_{\alpha,m}(x) + m\upsilon_{\alpha,m-1}(x)] = \alpha\upsilon_{\alpha,m+1}(x).$$

Thus,

(i) $\upsilon_{\alpha,m}(x)$ *is a polynomial in* x *of degree* $[m/2]$;

(ii) for each $x \in [0,\infty), \upsilon_{\alpha,m}(x) = O(\alpha^{-[(m+1)/2]})$, *where* $[\beta]$ *denotes the integral part of* β.

Proof. For the cases $c = 0$ and 1, the proof of this lemma can be found in [99] and [141] respectively. □

Lemma 1.2.2. *For the* m^{th} *order* $(m \in \mathbb{N}^0)$ *moment of the operators (1.1.1) defined as* $u^\rho_{\alpha,m}(x) := B^\rho_\alpha(t^m; x)$, *we have* $u^\rho_{\alpha,0}(x) = 1,\ u^\rho_{\alpha,1}(x) = x$,
$u^\rho_{\alpha,2}(x) = x^2 + \frac{x}{\alpha}\left(\frac{1}{\rho} + (1+cx)\right)$
and $x(1+cx)\left(u^\rho_{\alpha,m}(x)\right)' = \alpha u^\rho_{\alpha,m+1}(x) - \left(\frac{m}{\rho} + \alpha x\right) u^\rho_{\alpha,m}(x),\ m \in \mathbb{N}$.
Consequently, for each $x \in (0,\infty)$ *and* $m \in \mathbb{N}, u^\rho_{\alpha,m}(x) = x^m + \alpha^{-1}(p_m(x,c) + o(1))$, *where* $p_m(x,c)$ *is a rational function of* x *depending on the parameters* m *and* c.

Lemma 1.2.3. *For* $c = 0, 1$ *if the* m^{th} *order central moment* $\mu^\rho_{\alpha,m}(x)$ *is defined as*

$$\mu^\rho_{\alpha,m}(x) := B^\rho_\alpha((t-x)^m, x) = \sum_{k=1}^{\infty} p_{\alpha,k}(x,c) \int_0^\infty \theta^\rho_{\alpha,k}(t)(t-x)^m dt + p_{\alpha,0}(x,c)(-x)^m,$$

then, $\mu^\rho_{\alpha,0}(x) = 1,\ \mu^\rho_{\alpha,1}(x) = 0$ *and there holds the following recurrence relation:*

$$\alpha\mu^\rho_{\alpha,m+1}(x) \;=\; x(1+cx)(\mu^\rho_{\alpha,m}(x))' + mx\left(\frac{1}{\rho} + (1+cx)\right)\mu^\rho_{\alpha,m-1}(x) + \frac{m}{\rho}\mu^\rho_{\alpha,m}(x).$$

Consequently,

(i) $\mu^\rho_{\alpha,m}(x)$ *is a polynomial in* x *of degree atmost* m *depending on the parameters* c *and* α;

(ii) for every $x \in (0,\infty)$, $\mu^\rho_{\alpha,m}(x) = O\left(\alpha^{-[(m+1)/2]}\right)$, *where* $[s]$ *denotes the integer part of* s.

Proof. We shall prove the result for different values of c separately. For $c \in \{0,1\}$, using the identity $x(1+cx)p'_{\alpha,k}(x,c) = (k-\alpha x)p_{\alpha,k}(x,c)$, we may write

$$\begin{aligned}
x(1+cx)(\mu^\rho_{\alpha,m}(x))' &= \sum_{k=1}^{\infty}(k-\alpha x)p_{\alpha,k}(x,c)\int_0^\infty \theta^\rho_{\alpha,k}(t)(t-x)^m dt \\
&\quad -mx(1+cx)\sum_{k=1}^{\infty} p_{\alpha,k}(x,c)\int_0^\infty \theta^\rho_{\alpha,k}(t)(t-x)^{m-1}dt \\
&\quad +(-\alpha x)p_{\alpha,0}(x,c)(-x)^m - mx(1+cx)p_{\alpha,0}(x,c)(-x)^{m-1} \\
&= \sum_{k=1}^{\infty} p_{\alpha,k}(x,c)\int_0^\infty [(k-\alpha t) + \alpha(t-x)]\theta^\rho_{\alpha,k}(t)(t-x)^m dt \\
&\quad -mx(1+cx)\mu^\rho_{\alpha,m-1}(x) + \alpha p_{\alpha,0}(x,c)(-x)^{m+1}.
\end{aligned}$$

Next, using the identity $\dfrac{d}{dt}(t\theta^{\rho}_{\alpha,k}(t)) = \rho(k-\alpha t)\theta^{\rho}_{\alpha,k}(t)$, we have

$$\begin{aligned}
x(1+cx)(\mu^{\rho}_{\alpha,m}(x))' &= \sum_{k=1}^{\infty} p_{\alpha,k}(x,c)\int_0^{\infty}\frac{1}{\rho}(t\theta^{\rho}_{\alpha,k}(t))'(t-x)^m dt \\
&\quad -mx(1+cx)\mu^{\rho}_{\alpha,m-1}(x)+\alpha\mu^{\rho}_{\alpha,m+1}(x) \\
&= -\frac{m}{\rho}\sum_{k=1}^{\infty} p_{\alpha,k}(x,c)\int_0^{\infty} t\theta^{\rho}_{\alpha,k}(t)(t-x)^{m-1}dt \\
&\quad -mx(1+cx)\mu^{\rho}_{\alpha,m-1}(x)+\alpha\mu^{\rho}_{\alpha,m+1}(x) \\
&= -\frac{m}{\rho}\Big(\left(\mu^{\rho}_{\alpha,m}(x)-p_{\alpha,0}(x,c)(-x)^m\right)+x\left(\mu^{\rho}_{\alpha,m-1}(x)-p_{\alpha,0}(x,c)(-x)^{m-1}\right)\Big) \\
&\quad -mx(1+cx)\mu^{\rho}_{\alpha,m-1}(x)+\alpha\mu^{\rho}_{\alpha,m+1}(x) \\
&= -\frac{m}{\rho}\Big(\mu^{\rho}_{\alpha,m}(x)+x\mu^{\rho}_{\alpha,m-1}(x)\Big)-mx(1+cx)\mu^{\rho}_{\alpha,m-1}(x)+\alpha\mu^{\rho}_{\alpha,m+1}(x),
\end{aligned}$$

which is the required recurrence relation. The consequences (i) and (ii) easily follow from the recurrence relation on using mathematical induction on m. □

Remark 1. From Lemma 1.2.3, for each $x\in(0,\infty)$ and $c\in\{0,1\}$ we have

$$\mu^{\rho}_{\alpha,2}(x) = \frac{x(1+\rho(1+cx))}{\alpha\rho}; \tag{1.2.2}$$

$$\begin{aligned}
\mu^{\rho}_{\alpha,4}(x) &= \frac{x(1+cx)}{\alpha^3\rho^2}\left(3\rho(1+2cx)+\rho^2\left((1+2cx)^2+2cx(1+cx)\right)+2\right)+\frac{3x^2(1+\rho(1+cx))^2}{(\alpha\rho)^2} \\
&\quad +\frac{1}{(\alpha\rho)^3}\left(3\rho x(1+cx)(3+\rho(1+2cx))+6x\right).
\end{aligned}$$

Corollary 1. For $x\in[0,\infty)$ and $\alpha>0$, it is observed that

$$\mu^{\rho}_{\alpha,2}(x)\le\frac{\lambda x(1+cx)}{\alpha},\ \text{ where } \lambda = 1+\frac{1}{\rho}>1.$$

Corollary 2. Let γ and δ be any two positive real numbers and $[a,b]\subset(0,\infty)$ be any bounded interval. Then, for any $m>0$ there exists a constant M' depending on m only such that

$$\left\|\sum_{k=1}^{\infty}p_{\alpha,k}(x,c)\int_{|t-x|\ge\delta}\theta^{\rho}_{\alpha,k}(t)e^{\gamma t}dt\right\|\le M'\alpha^{-m},$$

where $\|.\|$ is the sup-norm over $[a,b]$.

Lemma 1.2.4. *For every $x \in (0,\infty)$ and $r \in \mathbb{N}^0$, there exist polynomials $q_{i,j,r}(x,c)$ in x independent of α and k such that*

$$\frac{d^r}{dx^r} p_{\alpha,k}(x,c) = p_{\alpha,k}(x,c) \sum_{\substack{2i+j\le r\\ i,j\ge 0}} \alpha^i (k-\alpha x)^j \frac{(q_{i,j,r}(x,c))}{(p(x,c))^r},$$

where $p(x,c) = x(1+cx)$.

Proof. For the cases $c = 0, 1$, the proof of this lemma can be seen in [99] and [141] respectively. □

1.3 Convergence Estimates

Our first main result is the basic convergence theorem for the operators defined in (1.1.1).

1.3.1 Simultaneous approximation

Throughout this section, we assume that $0 < a < b < \infty$.
In the following theorem, we show that the derivative $B_\alpha^{\rho(r)}(f;.)$ is also an approximation process for $f^{(r)}$.

Theorem 1.3.1. ***(Basic convergence theorem)*** *Let $f \in C_\gamma[0,\infty)$. If $f^{(r)}$ exists at a point $x \in (0,\infty)$, then we have*

$$\lim_{\alpha\to\infty} \left(\frac{d^r}{dw^r} B_\alpha^\rho(f;w)\right)_{w=x} = f^{(r)}(x). \tag{1.3.1}$$

Further, if $f^{(r)}$ is continuous on $(a-\eta, b+\eta), \eta > 0$, then the limit in (1.3.1) holds uniformly in $[a,b]$.

Proof. By our hypothesis, we have

$$f(t) = \sum_{\nu=0}^{r} \frac{f^{(\nu)}(x)}{\nu!}(t-x)^\nu + \psi(t,x)(t-x)^r,\ t \in [0,\infty),$$

where the function $\psi(t,x) \to 0$ as $t \to x$. From the above equation, we may write

$$\begin{aligned}
&\left(\frac{d^r}{dw^r} B_\alpha^\rho(f(t);w)\right)_{w=x}\\
&= \sum_{\nu=0}^{r} \frac{f^{(\nu)}(x)}{\nu!}\left(\frac{d^r}{dw^r} B_\alpha^\rho((t-x)^\nu;w)\right)_{w=x} + \left(\frac{d^r}{dw^r} B_\alpha^\rho(\psi(t,x)(t-x)^r;w)\right)_{w=x}\\
&:= I_1 + I_2, \text{say}.
\end{aligned}$$

First, we estimate I_1.

$$\begin{aligned} I_1 &= \sum_{\nu=0}^{r}\frac{f^{(\nu)}(x)}{\nu!}\left\{\frac{d^r}{dw^r}\left(\sum_{j=0}^{\nu}\binom{\nu}{j}(-x)^{\nu-j}B_\alpha^\rho(t^j;w)\right)_{w=x}\right\} \\ &= \sum_{\nu=0}^{r}\frac{f^{(\nu)}(x)}{\nu!}\sum_{j=0}^{\nu}\binom{\nu}{j}(-x)^{\nu-j}\left(\frac{d^r}{dw^r}B_\alpha^\rho(t^j;w)\right)_{w=x} \\ &= \sum_{\nu=0}^{r-1}\frac{f^{(\nu)}(x)}{\nu!}\sum_{j=0}^{\nu}\binom{\nu}{j}(-x)^{\nu-j}\left(\frac{d^r}{dw^r}B_\alpha^\rho(t^j;w)\right)_{w=x} \\ &\quad +\frac{f^{(r)}(x)}{r!}\sum_{j=0}^{r}\binom{r}{j}(-x)^{r-j}\left(\frac{d^r}{dw^r}B_\alpha^\rho(t^j;w)\right)_{w=x} := I_3+I_4, \text{say.} \end{aligned}$$

First, we estimate I_4.

$$\begin{aligned} I_4 &= \frac{f^{(r)}(x)}{r!}\sum_{j=0}^{r-1}\binom{r}{j}(-x)^{r-j}\left(\frac{d^r}{dw^r}B_\alpha^\rho(t^j;w)\right)_{w=x}+\frac{f^{(r)}(x)}{r!}\left(\frac{d^r}{dw^r}B_\alpha^\rho(t^r;w)\right)_{w=x} \\ &:= I_5+I_6, \text{say.} \end{aligned}$$

By using Lemma 1.2.2, we get
$I_6=f^{(r)}(x)+O\left(\frac{1}{\alpha}\right)$, $I_3=O\left(\frac{1}{\alpha}\right)$ and $I_5=O\left(\frac{1}{\alpha}\right)$, as $\alpha\to\infty$.
Combining the above estimates, $\forall x\in(0,\infty)$ we obtain $I_1\to f^{(r)}(x)$ as $\alpha\to\infty$.
Next, we estimate I_2. By making use of Lemma 1.2.4, we have

$$\begin{aligned} |I_2| &\le \sum_{k=1}^{\infty}\frac{p_{\alpha,k}(x,c)}{(p(x,c))^r}\sum_{\substack{2i+j\le r\\ i,j\ge 0}}\alpha^i|k-\alpha x|^j|q_{i,j,r}(x,c)|\int_0^\infty \theta_{\alpha,k}^\rho(t)|\psi(t,x)||(t-x)^r|dt \\ &\quad +\left|\left(\frac{d^r}{dw^r}p_{\alpha,0}(w,c)\right)_{w=x}\right||\psi(0,x)(-x)^r| := I_7+I_8,\ \text{say.} \end{aligned}$$

Since $\psi(t,x)\to 0$ as $t\to x$, for a given $\epsilon>0$ there exists a $\delta>0$ such that $|\psi(t,x)|<\epsilon$ whenever $|t-x|<\delta$. For $|t-x|\ge\delta$, $|(t-x)^r\psi(t,x)|\le Me^{\gamma t}$, for some constant $M>0$. Thus, we may write

$$\begin{aligned} |I_7| &\le \sum_{k=1}^{\infty}\sum_{\substack{2i+j\le r\\ i,j\ge 0}}\alpha^i|k-\alpha x|^j\frac{|q_{i,j,r}(x,c)|}{(p(x,c))^r}p_{\alpha,k}(x,c) \\ &\quad \times\left(\epsilon\int_{|t-x|<\delta}\theta_{\alpha,k}^\rho(t)|t-x|^r dt+M\int_{|t-x|\ge\delta}\theta_{\alpha,k}^\rho(t)e^{\gamma t}dt\right) := I_9+I_{10},\ \text{say.} \end{aligned}$$

Let $S=\sup\limits_{\substack{2i+j\le r\\ i,j\ge 0}}\frac{|q_{i,j,r}(x,c)|}{(p(x,c))^r}$. By applying the Schwarz inequality, Lemmas 1.2.1 and

1.2.3, we get

$$
\begin{aligned}
|I_9| &\leq \epsilon S\sum_{k=1}^{\infty}\sum_{\substack{2i+j\leq r\\ i,j\geq 0}}\alpha^i|k-\alpha x|^j p_{\alpha,k}(x,c)\left(\int_0^\infty \theta^{\rho}_{\alpha,k}(t)(t-x)^{2r}dt\right)^{\frac{1}{2}}\\
&\leq \epsilon S\sum_{\substack{2i+j\leq r\\ i,j\geq 0}}\alpha^{i+j}\left(\sum_{k=1}^{\infty}\left(\frac{k}{\alpha}-x\right)^{2j}p_{\alpha,k}(x,c)\right)^{\frac{1}{2}}\times\left(\sum_{k=1}^{\infty}p_{\alpha,k}(x,c)\int_0^\infty \theta^{\rho}_{\alpha,k}(t)(t-x)^{2r}dt\right)^{\frac{1}{2}}\\
&\leq \epsilon S\sum_{\substack{2i+j\leq r\\ i,j\geq 0}}\alpha^{i+j}\left(\upsilon_{\alpha,2j}(x)-x^{2j}\phi_{\alpha,c}(x)\right)^{\frac{1}{2}}\times\left(B^{\rho}_{\alpha}((t-x)^{2r};x))-x^{2r}\phi_{\alpha,c}(x)\right)^{\frac{1}{2}}\\
&= \epsilon\sum_{\substack{2i+j\leq r\\ i,j\geq 0}}\alpha^{i+j}\{O(\alpha^{-j})+O(\alpha^{-s_1})\}^{1/2}\times\{O(\alpha^{-r})+O(\alpha^{-s_2})\}^{1/2},\ \text{for any } s_1,s_2>0.
\end{aligned}
$$

Choosing s_1, s_2 such that $s_1>j$ and $s_2>r$, we have

$$
|I_9| \;=\; \epsilon\sum_{\substack{2i+j\leq r\\ i,j\geq 0}}\alpha^{i+j}O(\alpha^{-j/2})O(\alpha^{-r/2})=\epsilon.O(1).
$$

Since $\epsilon>0$ is arbitrary, $I_9\to 0$ as $\alpha\to\infty$.

Now, we estimate I_{10}. By applying Cauchy-Schwarz inequality, Lemma 1.2.1 and Corollary 2, we obtain

$$
\begin{aligned}
|I_{10}| &\leq MS\sum_{k=1}^{\infty}\sum_{\substack{2i+j\leq r\\ i,j\geq 0}}\alpha^i|k-\alpha x|^j p_{\alpha,k}(x,c)\int_{|t-x|\geq\delta}\theta^{\rho}_{\alpha,k}(t)e^{\gamma t}dt\\
&\leq M_1\sum_{\substack{2i+j\leq r\\ i,j\geq 0}}\alpha^{i+j}\left(\sum_{k=1}^{\infty}\left(\frac{k}{\alpha}-x\right)^{2j}p_{\alpha,k}(x,c)\right)^{1/2}\\
&\quad\times\left(\sum_{k=1}^{\infty}p_{\alpha,k}(x,c)\int_{|t-x|\geq\delta}\theta^{\rho}_{\alpha,k}(t)e^{2\gamma t}dt\right)^{1/2}\text{, where } M_1=MS\\
&\leq M_1\sum_{\substack{2i+j\leq r\\ i,j\geq 0}}\alpha^{i+j}\left(\upsilon_{\alpha,2j}(x)-x^{2j}\phi_{\alpha,c}(x)\right)^{1/2}\left(\sum_{k=1}^{\infty}p_{\alpha,k}(x,c)\int_{|t-x|\geq\delta}\theta^{\rho}_{\alpha,k}(t)e^{2\gamma t}dt\right)^{1/2}\\
&= \sum_{\substack{2i+j\leq r\\ i,j\geq 0}}\alpha^{i+j}\{O(\alpha^{-j})+O(\alpha^{-m_1})\}^{1/2}\{O(\alpha^{-m_2})\}^{1/2},\ \text{for any } m_1,m_2>0.
\end{aligned}
$$

Choosing $m_1>j$, we get

$$
|I_{10}| \;=\; \sum_{\substack{2i+j\leq r\\ i,j\geq 0}}\alpha^{i+j}O(\alpha^{-j/2})O(\alpha^{-m_2/2}) = O(\alpha^{(r-m_2)/2}),
$$

which implies that $I_{10} = o(1)$, as $\alpha \to \infty$, on choosing $m_2 > r$.
Next, we estimate I_8. We may write

$$|I_8| = \left| \left(\frac{d^r}{dw^r} p_{\alpha,0}(w,c) \right)_{w=x} \right| |\psi(0,x)| x^r = |\phi^{(r)}_{\alpha,c}(x)| \; |\psi(0,x)| x^r.$$

Now, we observe that $\phi^{(r)}_{\alpha,0}(x) = e^{-\alpha x}(-\alpha)^r$ and $\phi^{(r)}_{\alpha,1}(x) = \dfrac{(-1)^r(\alpha)_r}{(1+x)^{\alpha+r}}$, which implies that $I_8 = O(\alpha^{-p})$ for any $p > 0$, in view of the fact that $|\psi(0,x)x^r| \leq N_1$, for some $N_1 > 0$.
By combining the estimates $I_7 - I_{10}$, we obtain $I_2 \to 0$ as $\alpha \to \infty$.
To prove the uniformity assertion, it is sufficient to remark that $\delta(\epsilon)$ in the above proof can be chosen to be independent of $x \in [a,b]$ and also that the other estimates hold uniformity in $x \in [a,b]$. This completes the proof of the theorem. □

Next, we establish an asymptotic formula.

Theorem 1.3.2. ***(Voronovskaja type result)*** *Let $f \in C_\gamma[0,\infty)$. If f admits a derivative of order $(r+2)$ at a fixed point $x \in (0,\infty)$, then we have*

$$(1.3.2) \quad \lim_{\alpha\to\infty} \alpha \left(\left(\frac{d^r}{dw^r} B^\rho_\alpha(f;w) \right)_{w=x} - f^{(r)}(x) \right) = \sum_{\nu=1}^{r+2} Q(\nu,r,c,x) f^{(\nu)}(x),$$

where $Q(\nu,r,c,x)$ are certain rational functions of x independent of α.
Further, if $f^{(r+2)}$ is continuous on $(a-\eta, b+\eta), \eta > 0$, then the limit in (1.3.2) holds uniformly in $[a,b]$.

Proof. From the Taylor's theorem, for $t \in [0,\infty)$ we may write

$$(1.3.3) \qquad f(t) = \sum_{\nu=0}^{r+2} \frac{f^{(\nu)}(x)}{\nu!}(t-x)^\nu + \psi(t,x)(t-x)^{r+2},$$

where the function $\psi(t,x) \to 0$ as $t \to x$.
Now, from equation (1.3.3), we have

$$\begin{aligned}
\left(\frac{d^r}{dw^r} B^\rho_\alpha(f(t);w) \right)_{w=x} &= \sum_{\nu=0}^{r+2} \frac{f^{(\nu)}(x)}{\nu!} \left(\frac{d^r}{dw^r} (B^\rho_\alpha((t-x)^\nu;w) \right)_{w=x} \\
&\quad + \left(\frac{d^r}{dw^r} B^\rho_\alpha(\psi(t,x)(t-x)^{r+2};w) \right)_{w=x} \\
&= \sum_{\nu=0}^{r+2} \frac{f^{(\nu)}(x)}{\nu!} \sum_{j=0}^{\nu} \binom{\nu}{j} (-x)^{\nu-j} \left(\frac{d^r}{dw^r} B^\rho_\alpha(t^j;w) \right)_{w=x} \\
&\quad + \left(\frac{d^r}{dw^r} B^\rho_\alpha(\psi(t,x))(t-x)^{r+2};w \right)_{w=x} \\
&:= J_1 + J_2, \text{ say.}
\end{aligned}$$

Proceeding in a manner similar to the estimate of I_2 in Theorem 1.3.1, for each $x \in (0,\infty)$ we get $\alpha J_2 \to 0$ as $\alpha \to \infty$.
Next, we estimate J_1.

$$\begin{aligned} J_1 &= \sum_{\nu=0}^{r-1} \frac{f^{(\nu)}(x)}{\nu!} \sum_{j=0}^{\nu} \binom{\nu}{j} (-x)^{\nu-j} \left(\frac{d^r}{dw^r} B_\alpha^\rho(t^j;w) \right)_{w=x} \\ &+ \frac{f^{(r)}(x)}{r!} \sum_{j=0}^{r} \binom{r}{j} (-x)^{r-j} \left(\frac{d^r}{dw^r} B_\alpha^\rho(t^j;w) \right)_{w=x} \\ &+ \frac{f^{(r+1)}(x)}{(r+1)!} \sum_{j=0}^{r+1} \binom{r+1}{j} (-x)^{r+1-j} \left(\frac{d^r}{dw^r} B_\alpha^\rho(t^j;w) \right)_{w=x} \\ &+ \frac{f^{(r+2)}(x)}{(r+2)!} \sum_{j=0}^{r+2} \binom{r+2}{j} (-x)^{r+2-j} \left(\frac{d^r}{dw^r} B_\alpha^\rho(t^j;w) \right)_{w=x}. \end{aligned}$$

By making use of Lemma 1.2.2, we have

$$J_1 = f^{(r)}(x) + \alpha^{-1} \left(\sum_{\nu=1}^{r+2} Q(\nu,r,c,x) f^{(\nu)}(x) + o(1) \right).$$

Thus, from the estimates of J_1 and J_2, the required result follows.
The uniformity assertion follows as in the proof of Theorem 1.3.1. This completes the proof. □

Corollary 3. From the above theorem, we have

(i) for $r = 0$
$$\lim_{\alpha\to\infty} \alpha(B_\alpha^\rho(f;x) - f(x)) = \frac{x\{1+\rho(1+cx)\}}{2\rho} f''(x);$$

(ii) for $r = 1$
$$\lim_{\alpha\to\infty} \alpha \left(\left(\frac{d}{dw} B_\alpha^\rho(f;w) - f'(x) \right)_{w=x} \right) = f''(x) \left(cx + \frac{1}{2}\left(\frac{1}{\rho}+1\right) \right) + \frac{f'''(x)}{2} \left(cx^2 + \left(\frac{1}{\rho}+1\right) x \right).$$

Example 1. For $\alpha = 20, 50, 100$, the convergence of the operators $B_\alpha^\rho(f;x,c)$ to the function $f(x) = x^8 - 6x^7 + 5x^4 - 4x^3 + 2x^2 + 3$ (blue) is illustrated for $c = 0, \rho = 1$ (green) and $c = 1, \rho = 1$ (red) in figures 1.1 – 1.3, respectively.

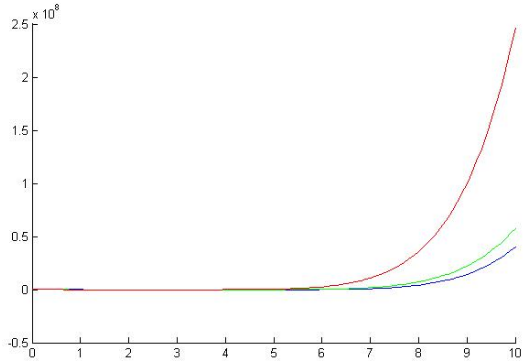

Figure 1.1

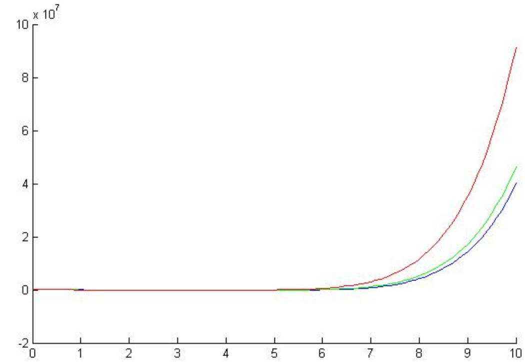

Figure 1.2

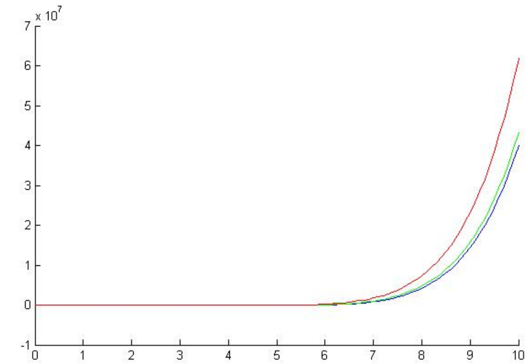

Figure 1.3

Example 2. For $\alpha = 20, 50, 100$, the convergence of the operators $B_{\alpha}^{\rho}(f; x, c)$ to the function $f(x) = x^4 e^{(-2\pi x)}$ (blue) is illustrated for $c = 0, \rho = 1$ (green) and $c = 1, \rho = 1$ (red) in figures $1.4 - 1.6$, respectively.

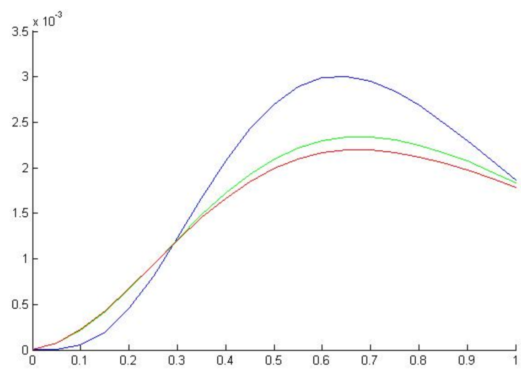

Figure 1.4

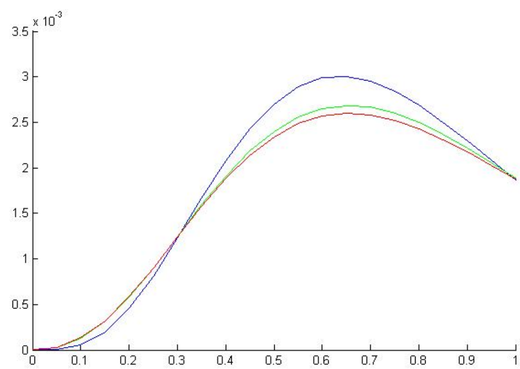

Figure 1.5

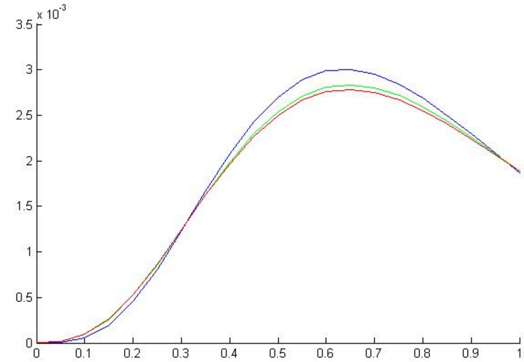

Figure 1.6

Example 3. For $\alpha = 50, 100, 140$, the convergence of the operators $\left(\frac{d}{dw}B_{\alpha}^{\rho}(f;w,c)\right)_{w=x}$ to the function $\frac{d}{dx}f(x) = \frac{d}{dx}(x^8 - 6x^7 + 5x^4 - 4x^3 + 2x^2 + 3)$ (blue) is illustrated for $c = 0, \rho = 1$ (green) and $c = 1, \rho = 1$ (red) in figures $1.7 - 1.9$, respectively.

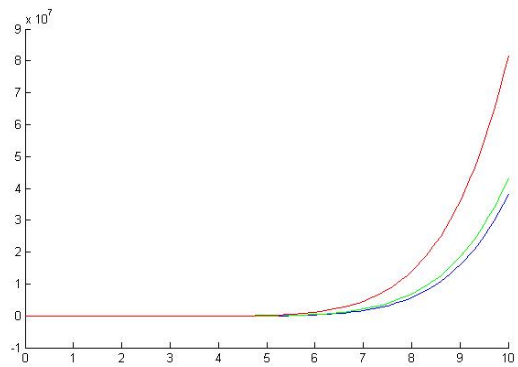

Figure 1.7

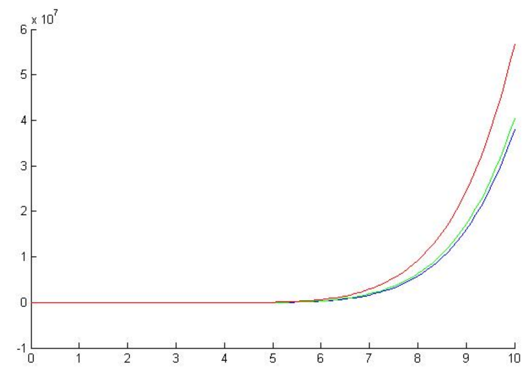

Figure 1.8

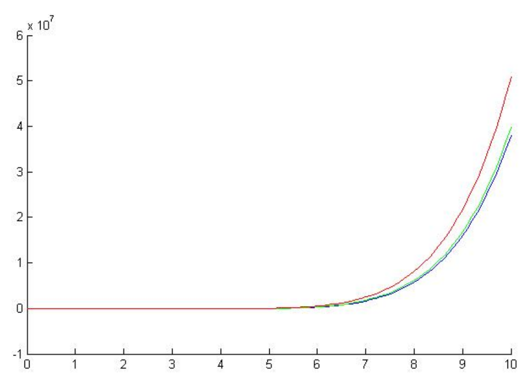

Figure 1.9

Example 4. For $\alpha = 50, 100, 140$, the convergence of the operators $\left(\frac{d}{dw}B_\alpha^\rho(f;w,c)\right)_{w=x}$ to the function $\frac{d}{dx}f(x) = \frac{d}{dx}\left(x^4 e^{(-2\pi x)}\right)$ (blue) is illustrated for $c = 0, \rho = 1$ (green) and $c = 1, \rho = 1$ (red) in figures $1.10 - 1.12$, respectively.

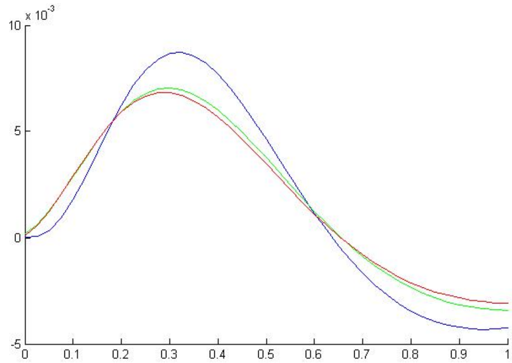

Figure 1.10

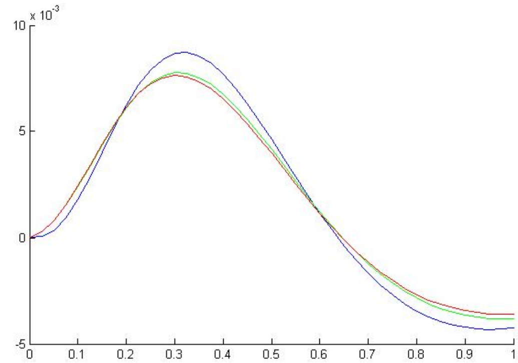

Figure 1.11

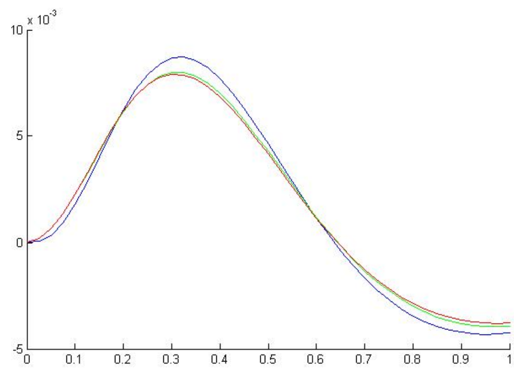

Figure 1.12

The next result provides an estimate of the degree of approximation in $B_\alpha^{\rho(r)}(f;x) \to f^{(r)}(x), r \in \mathbb{N}$.

Theorem 1.3.3. ***(Degree of approximation)*** *Let* $r \le q \le r+2, f \in C_\gamma[0,\infty)$ *and* $f^{(q)}$ *exists and be continuous on* $(a-\eta, b+\eta)$ *where* $\eta > 0$ *is sufficiently small. Then, for sufficiently large* α

$$\left\| \left(\frac{d^r}{dw^r}B_\alpha^\rho(f;w)\right)_{w=x} - f^{(r)}(x)\right\|_{C[a,b]} \le \max\{C_1\alpha^{-(q-r)/2}\omega\left(f^{(q)};\alpha^{-1/2},(a-\eta,b+\eta)\right), C_2\ \alpha^{-1}\},$$

where $C_1 = C_1(r,c)$ *and* $C_2 = C_2(r,f,c)$.

Proof. By our hypothesis we have

$$\begin{aligned} f(t) &= \sum_{i=0}^{q} \frac{f^{(i)}(x)}{i!}(t-x)^i + \frac{f^{(q)}(\xi) - f^{(q)}(x)}{q!}(t-x)^q \chi(t) \\ &\quad + \phi(t,x)(1-\chi(t)), \end{aligned} \tag{1.3.4}$$

where ξ lies between t and x and $\chi(t)$ is the characteristic function of $(a-\eta, b+\eta)$. The function $\phi(t,x)$ for $t \in (0,\infty) \setminus (a-\eta, b+\eta)$ and $x \in [a,b]$ is bounded by $Me^{\gamma t}$ for some constant $M > 0$.

We operate $\frac{d^r}{dw^r} B_\alpha^\rho(.;w)$ on the equality (1.3.4) and break the right hand side into three parts E_1, E_2 and E_3, say, corresponding to the three terms on the right hand side of equation (1.3.4).

Now, treating E_1 in a manner similar to the treatment of J_1 of Theorem 1.3.2, we get $E_1 = f^{(r)}(x) + O(\alpha^{-1})$, uniformly in $x \in [a,b]$.

By making use of the inequality

$$|f^{(q)}(\xi) - f^{(q)}(x)| \le \left(1 + \frac{|t-x|}{\delta}\right) \omega\left(f^{(q)}; \delta, (a-\eta, b+\eta)\right), \ \delta > 0,$$

and Lemma 1.2.4, we get

$$\begin{aligned} |E_2| &\le \frac{\omega\left(f^{(q)}; \delta, (a-\eta, b+\eta)\right)}{q!} \left\{ \sum_{k=1}^{\infty} \sum_{\substack{2i+j \le r \\ i,j \ge 0}} \frac{\alpha^i |k-\alpha x|^j |q_{i,j,r}(x,c)|}{(p(x,c))^r} p_{\alpha,k}(x,c) \right. \\ &\quad \left. \times \int_0^\infty \theta_{\alpha,k}^\rho(t) \left(1 + \frac{|t-x|}{\delta}\right) |t-x|^q \chi(t) dt + \left(x^q + \frac{x^{q+1}}{\delta}\right) \phi_{\alpha,c}^{(r)}(x) \right\} \\ &= E_4 + E_5. \end{aligned}$$

Finally, let

$$S^* = \sup_{x \in [a,b]} \sup_{\substack{2i+j \le r \\ i,j \ge 0}} \frac{|q_{i,j,r}(x,c)|}{(p(x,c))^r},$$

then by applying Schwarz inequality, Lemmas 1.2.1 and 1.2.3, we obtain

$$\begin{aligned}
E_4 &\leq \frac{\omega\left(f^{(q)};\delta,(a-\eta,b+\eta)\right)S^*}{q!}\sum_{\substack{2i+j\leq r\\ i,j\geq 0}}\alpha^{i+j}\left(\sum_{k=1}^{\infty}\left(\frac{k}{\alpha}-x\right)^{2j}p_{\alpha,k}(x,c)\right)^{1/2}\\
&\quad\left\{\left(\sum_{k=1}^{\infty}p_{\alpha,k}(x,c)\int_0^{\infty}\theta^{\rho}_{\alpha,k}(t)(t-x)^{2q}dt\right)^{1/2}\right.\\
&\quad\left.+\frac{1}{\delta}\left(\sum_{k=1}^{\infty}p_{\alpha,k}(x,c)\int_0^{\infty}\theta^{\rho}_{\alpha,k}(t)(t-x)^{2q+2}dt\right)^{1/2}\right\}\\
&\leq \omega\left(f^{(q)};\delta,(a-\eta,b+\eta)\right)S^*\sum_{\substack{2i+j\leq r\\ i,j\geq 0}}\alpha^{i+j}\left(\upsilon_{\alpha,2j}(x)-x^{2j}\phi_{\alpha,c}(x)\right)^{1/2}\\
&\quad\times\left\{\left(B^{\rho}_{\alpha}((t-x)^{2q};x)-x^{2q}\phi_{\alpha,c}(x)\right)^{1/2}+\frac{1}{\delta}\left(B^{\rho}_{\alpha}((t-x)^{2q+2};x)-x^{2q+2}\phi_{\alpha,c}(x)\right)^{1/2}\right\}\\
&= \omega\left(f^{(q)};\delta,(a-\eta,b+\eta)\right)\sum_{\substack{2i+j\leq r\\ i,j\geq 0}}\alpha^{i+j}\{O(\alpha^{-j})+O(\alpha^{-s_1})\}^{1/2}\\
&\quad\times\{(O(\alpha^{-q})+O(\alpha^{-s_2})\}^{1/2}+\frac{1}{\delta}\{(O(\alpha^{-(q+1)})+O(\alpha^{-s_3}))\}^{1/2},\ \text{for any } s_1,s_2,s_3>0
\end{aligned}$$

Choosing s_1,s_2,s_3 such that $s_1>j, s_2>q, s_3>q+1$, we have

$$|E_4| = \omega\left(f^{(q)};\delta,(a-\eta,b+\eta)\right)\sum_{\substack{2i+j\leq r\\ i,j\geq 0}}\alpha^{i+j}O\left(\frac{1}{\alpha^{j/2}}\right)\left\{O\left(\frac{1}{\alpha^{q/2}}\right)+\frac{1}{\delta}O\left(\frac{1}{\alpha^{(q+1)/2}}\right)\right\}.$$

Now, on choosing $\delta=\alpha^{-1/2}$, we get

$$|E_4| \leq C_1\alpha^{-(q-r)/2}\omega(f^{(q)};\alpha^{-1/2},(a-\eta,b+\eta)).$$

Next, proceeding in a manner similar to the estimate of I_8 in Theorem 1.3.1, we have $E_5=O(\alpha^{-p})$, for any $p>0$. Choosing $p>1$, we have $E_5=O(\alpha^{-1})$, as $\alpha\to\infty$. Finally, proceeding along the lines of the estimate of I_{10} of Theorem 1.3.1, we obtain $E_3=o(\alpha^{-1})$ as $\alpha\to\infty$. On combining the estimates of E_1-E_5, we get the required result. □

1.3.2 Local approximation

Theorem 1.3.4. *Let $f\in C_B[0,\infty)$ and $x\geq 0$. Then, there exists a constant $C>0$ such that*

$$|B^{\rho}_{\alpha}(f,x)-f(x)|\leq C\omega_2\left(f;\sqrt{\frac{x\{1+\rho(1+cx)\}}{\alpha\rho}}\right).$$

Proof. Let $g \in C_B^2[0,\infty)$. From the Taylor's theorem, we may write

$$g(t) = g(x) + (t-x)g'(x) + \int_x^t (t-v)g''(v)dv,$$

which implies that

$$|B_\alpha^\rho(g,x) - g(x)| \;\; = \;\; \left| B_\alpha^\rho \left(\int_x^t (t-v)g''(v)dv, x \right) \right|.$$

Since

$$\left| \int_x^t (t-v)g''(v)dv \right| \leq (t-x)^2 ||g''||,$$

by Remark 1, we have

$$|B_\alpha^\rho(g,x) - g(x)| \;\; \leq \;\; \frac{x\{1+\rho(1+cx)\}}{\alpha\rho} ||g''||.$$

From (1.1.1) it follows that

$$|B_\alpha^\rho(f,x)| \;\; \leq \;\; ||f||.$$

Hence

$$\begin{aligned} |B_\alpha^\rho(f,x)(f,x) - f(x)| \;\; &\leq \;\; |B_\alpha^\rho(f-g,x) - (f-g)(x)| + |B_\alpha^\rho(g,x) - g(x)| \\ &\leq \;\; 2||f-g|| + \frac{x\{1+\rho(1+cx)\}}{\alpha\rho} ||g''||. \end{aligned}$$

Taking infimum on the right hand side over all $g \in C_B^2[0,\infty)$ and using (1.3.5), we obtain the desired result. Hence, the proof is completed. □

Let us now consider the Lipschitz-type space [126]:

$$Lip_M^*(r) := \left\{ f \in C_B[0,\infty) : |f(t) - f(x)| \leq M \frac{|t-x|^r}{(t+x)^{\frac{r}{2}}}; x,t \in (0,\infty) \right\},$$

where M is a positive constant and $r \in (0,1]$.

Theorem 1.3.5. *Let $f \in Lip_M^*(r)$. Then, for all $x > 0$, we have*

$$|B_\alpha^\rho(f,x) - f(x)| \leq M \left(\frac{1+\rho(1+cx)}{\alpha\rho} \right)^{\frac{r}{2}}.$$

Proof. Initially for $r=1$, we may write

$$\begin{aligned}|B_\alpha^\rho(f,x)-f(x)| &\leq \sum_{k=0}^{\infty} p_{\alpha,k}(x,c)\int_0^\infty \theta_{\alpha,k}^\rho(t)|f(t)-f(x)|dt\\ &\leq M\sum_{k=0}^{\infty} p_{\alpha,k}(x,c)\int_0^\infty \theta_{\alpha,k}^\rho(t)\frac{|t-x|}{\sqrt{t+x}}dt.\end{aligned}$$

Using the fact that $\frac{1}{\sqrt{t+x}} < \frac{1}{\sqrt{x}}$ and the Cauchy-Schwarz inequality, the above inequality implies that

$$\begin{aligned}|B_\alpha^\rho(f,x)-f(x)| &\leq \frac{M}{\sqrt{x}}\sum_{k=0}^{\infty} p_{\alpha,k}(x,c)\int_0^\infty \theta_{\alpha,k}^\rho(t)|t-x|dt\\ &= \frac{M}{\sqrt{x}}B_\alpha^\rho(|t-x|,x) \leq M\left(\sqrt{\frac{1+\rho(1+cx)}{\alpha\rho}}\right),\end{aligned}$$

which proves the required result for $r=1$. Now for $r\in(0,1)$, applying the Hölder inequality with $p=\frac{1}{r}$ and $q=\frac{1}{1-r}$, we have

$$\begin{aligned}|B_\alpha^\rho(f,x)-f(x)| &\leq \sum_{k=0}^{\infty} p_{\alpha,k}(x,c)\int_0^\infty \theta_{\alpha,k}^\rho(t)|f(t)-f(x)|dt\\ &\leq \left\{\sum_{k=0}^{\infty} p_{\alpha,k}(x,c)\left(\int_0^\infty \theta_{\alpha,k}^\rho(t)|f(t)-f(x)|dt\right)^{\frac{1}{r}}\right\}^r\\ &\leq \left\{\sum_{k=0}^{\infty} p_{\alpha,k}(x,c)\int_0^\infty \theta_{\alpha,k}^\rho(t)|f(t)-f(x)|^{\frac{1}{r}}dt\right\}^r\\ &\leq M\left\{\sum_{k=0}^{\infty} p_{\alpha,k}(x,c)\int_0^\infty \theta_{\alpha,k}^\rho(t)\frac{|t-x|}{\sqrt{t+x}}dt\right\}^r\\ &\leq \frac{M}{x^{\frac{r}{2}}}\left\{\sum_{k=0}^{\infty} p_{\alpha,k}(x,c)\int_0^\infty \theta_{\alpha,k}^\rho(t)|t-x|dt\right\}^r\\ &\leq \frac{M}{x^{\frac{r}{2}}}(B_\alpha^\rho(|t-x|,x))^r \leq M\left(\frac{1+\rho(1+cx)}{\alpha\rho}\right)^{\frac{r}{2}}.\end{aligned}$$

Thus, the proof is completed. □

Theorem 1.3.6. *Let $f\in D_2[0,\infty)$ and $\omega(f;\delta,[0,b+1])$ be its modulus of continuity on the finite interval $[0,b+1]\subset[0,\infty)$. Then for any $\alpha>0$, we have*

$$\| B_\alpha^\rho(f)-f \|_{C[0,b]} \leq 4M_f(1+b^2)\mu_{\alpha,2}^\rho(b)+2\omega\left(f;\sqrt{\mu_{\alpha,2}^\rho(b)},[0,b+1]\right),$$

where $\mu_{\alpha,2}^\rho(b)=\frac{b\{1+\rho(1+cb)\}}{\alpha\rho}$.

Proof. From ([75], p.378), for $x \in [0,b]$ and $t \in [0,\infty)$, we have

$$|f(t)-f(x)| \leq 4M_f(1+b^2)(t-x)^2 + \left(1+\frac{|t-x|}{\delta}\right)\omega\left(f;\delta,[0,b+1]\right), \delta>0.$$

Applying $B_\alpha^\rho(.,x)$ and then Cauchy-Schwarz inequality to the above inequality, we get

$$\begin{aligned}|B_\alpha^\rho(f,x)-f(x)| &\leq 4M_f(1+b^2)B_\alpha^\rho((t-x)^2,x)+\omega\left(f;\delta,[0,b+1]\right)\left(1+\frac{1}{\delta}B_\alpha^\rho(|t-x|,x)\right)\\ &\leq 4M_f(1+b^2)\mu_{\alpha,2}^\rho(b)+\omega\left(f;\delta,[0,b+1]\right)\left(1+\frac{1}{\delta}\sqrt{\mu_{\alpha,2}^\rho(b)}\right).\end{aligned}$$

By choosing $\delta=\sqrt{\mu_{\alpha,2}^\rho(b)}$, we obtain the desired result. □

1.3.3 Weighted approximation

Theorem 1.3.7. *For each $f \in D_2^*[0,\infty)$, we have*

$$\lim_{\alpha\to\infty} \parallel B_\alpha^\rho(f)-f \parallel_2 = 0.$$

Proof. From the Korovkin theorem, we see that it is sufficient to verify the following three conditions

$$\lim_{\alpha\to\infty} \parallel B_\alpha^\rho(t^k;x)-x^k) \parallel_2 = 0,\ k=0,1,2. \tag{1.3.5}$$

Since $B_\alpha^\rho(1;x)=1,$ the condition in (1.3.5) holds for $k=0$. By Lemma 1.2.1, we have for $\alpha>0$

$\parallel B_\alpha^\rho(t;x)-x) \parallel_2 = 0$, which implies that the condition in (1.3.5) holds for $k=1$.
Similarly, we can write for $\alpha>0$

$$\begin{aligned}\parallel B_\alpha^\rho(t^2;x)-x^2 \parallel_2 &= \left\|\frac{\rho x(1+cx)+x}{\rho\alpha}\right\|_2 \leq \frac{\rho+1}{\rho\alpha}\sup_{x\in[0,\infty)}\frac{x}{1+x^2}+\frac{c}{\alpha}\sup_{x\in[0,\infty)}\frac{x^2}{1+x^2}\\ &\leq \left(1+\frac{1}{\rho}+c\right)\frac{1}{\alpha}\end{aligned}$$

which implies that $\lim\limits_{\alpha\to\infty} \parallel B_\alpha^\rho(t^2;x)-x^2 \parallel_2 = 0$, the equation (1.3.5) holds for $k=2$.
This completes the proof. □

Theorem 1.3.8. *Let $f \in D_2^*[0,\infty)$, then there exists a positive constant K such that*

$$\sup_{x\in[0,\infty)}\frac{|B_\alpha^\rho(f,x)-f(x)|}{(1+x^2)^{\frac{3}{2}}} \leq K\Omega_2\left(f,\sqrt{\frac{1}{\alpha}}\right).$$

Proof. For $t>0, x\in[0,\infty)$ and $\delta>0$, by definition of $\Omega_2(f,\delta)$ and Lemma 0.7.1, we get

$$\begin{aligned}|f(t)-f(x)| &\leq (1+(x+|x-t|)^2)\Omega_2(f,|t-x|)\\ &\leq (1+(2x+t)^2)\left(1+\frac{|t-x|}{\delta}\right)\Omega_2(f,\delta).\end{aligned}$$

Since B_α^ρ is linear and positive, we have
$|B_\alpha^\rho(f,x)-f(x)|$

$$\leq \Omega_2(f,\delta)\left\{B_\alpha^\rho(1+(2x+t)^2,x)+B_\alpha^\rho\left((1+(2x+t)^2)\frac{|t-x|}{\delta},x\right)\right\}. \quad (1.3.6)$$

Applying Cauchy-Schwarz inequality to the second term of equation (1.3.6), we have
$B_\alpha^\rho\left((1+(2x+t)^2)\frac{|t-x|}{\delta},x\right)$

$$\leq \frac{1}{\delta}\sqrt{B_\alpha^\rho\left(\left(1+(2x+t)^2\right)^2,x\right)}\sqrt{B_\alpha^\rho\left((t-x)^2,x\right)}.$$

Also from Lemma (1.2.2), there exist positive constant K_1 and K_2 such that

$$B_\alpha^\rho(1+(2x+t)^2,x)\leq K_1(1+x^2), \quad (1.3.7)$$

and

$$\left(B_\alpha^\rho\left((1+(2x+t)^2)^2,x\right)\right)^{1/2}\leq K_2(1+x^2). \quad (1.3.8)$$

Now, from (1.3.8) and Corollary 1, we get

$$\begin{aligned}&\frac{1}{\delta}\sqrt{B_\alpha^\rho\left(\left(1+(2x+t)^2\right)^2,x\right)}\sqrt{B_\alpha^\rho\left((t-x)^2,x\right)}\\ &\leq \frac{\sqrt{\lambda}}{\delta}K_2(1+x^2)\sqrt{\frac{x(1+cx)}{\alpha}},\ \lambda>1\\ &\leq \frac{1}{\delta\sqrt{\alpha}}K_3(1+x^2)^{3/2}, \text{ for some positive number } K_3. \quad (1.3.9)\end{aligned}$$

Combining the estimates of (1.3.6), (1.3.7), (1.3.9) and taking $K=(K_1+K_3)$, $\delta=\frac{1}{\sqrt{\alpha}}$, we obtain the required result.

□

1.3.4 Rate of convergence

In this section, we shall estimate the rate of convergence for the generalized hybrid operators B_α^ρ for functions with derivatives of bounded variation. In the recent years, several researchers have obtained results in this direction for different sequences of linear positive operators. We refer the reader to some of the related papers (cf. [4], [127], [65], [74], [89] and [97] etc).

Lemma 1.3.9. *For all $x \in (0,\infty)$, $\lambda > 1$ and α sufficiently large, we have*

(i) $\lambda_\alpha^\rho(x,t) = \int_0^t \mathcal{A}_\alpha^\rho(x,u)du \leq \frac{1}{(x-t)^2}\frac{\lambda x(1+cx)}{\alpha},\ 0 \leq t < x;$

(ii) $1-\lambda_\alpha^\rho(x,z) = \int_z^\infty \mathcal{A}_\alpha^\rho(x,u)du \leq \frac{1}{(z-x)^2}\frac{\lambda x(1+cx)}{\alpha},\ x < z < \infty.$

Proof. First we prove (i).

$$\begin{aligned} \lambda_\alpha^\rho(x,t) &= \int_0^t \mathcal{A}_\alpha^\rho(x,u)du \leq \int_0^t \left(\frac{x-u}{x-t}\right)^2 \mathcal{A}_\alpha^\rho(x,u)du \\ &\leq \frac{1}{(x-t)^2} B_\alpha^\rho((u-x)^2;x) \\ &\leq \frac{1}{(x-t)^2}\frac{\lambda x(1+cx)}{\alpha}. \end{aligned}$$

The proof of (ii) is similar. □

Theorem 1.3.10. *Let $f \in DBV_\gamma[0,\infty), \gamma \geq 0$. Then for every $x \in (0,\infty)$, $2r(\in \mathbb{N}) > \gamma$ and sufficiently large α, we have*
$|B_\alpha^\rho(f;x) - f(x)|$

$$\begin{aligned} &\leq \left|\frac{f'(x+)-f'(x-)}{2}\right|\left\{\frac{\lambda x(1+cx)}{\alpha}\right\}^{1/2} + \frac{x}{\sqrt{\alpha}}\bigvee_{x-\frac{x}{\sqrt{\alpha}}}^{x+\frac{x}{\sqrt{\alpha}}}(f_x') + \frac{\lambda(1+cx)}{\alpha}\sum_{m=1}^{\sqrt{[\alpha]}}\bigvee_{x-\frac{x}{m}}^{x+\frac{x}{m}}(f_x') \\ &\quad +|f'(x+)|\left\{\frac{\lambda x(1+cx)}{\alpha}\right\}^{1/2} + |f(2x)-f(x)-xf'(x+)|\frac{\lambda(1+cx)}{\alpha x} \\ &\quad +M'\frac{A(r,x)}{\alpha^{\gamma/2}} + |f(x)|\frac{\lambda(1+cx)}{\alpha x}, \end{aligned}$$

where

$$f_x'(t) = \begin{cases} f'(t)-f'(x+), & x < t < \infty \\ 0\ , & t = x \\ f'(t)-f'(x-), & 0 \leq t < x, \end{cases}$$

$\bigvee_a^b(f_x')$ is the total variation of f_x' on $[a,b]$, $A(r,x)$ is a constant depending on r and x and M' is a constant depending on f and γ.

Proof. By the hypothesis, we may write

$$f'(t) = \frac{1}{2}\left(f'(x+)+f'(x-)\right)+f'_x(t)+\frac{1}{2}\left(f'(x+)-f'(x-)\right)sgn(t-x)$$
$$+\delta_x(t)\left(f'(t)-\frac{1}{2}\left(f'(x+)+f'(x-)\right)\right), \tag{1.3.10}$$

where

$$\delta_x(t)=\begin{cases}1 &, \quad t=x\\ 0 &, \quad t\neq x.\end{cases}$$

From equations (1.1.2) and (1.3.10), we have

$$\begin{aligned}B_\alpha^\rho(f;x)-f(x) &= \int_0^\infty \mathcal{A}_\alpha^\rho(x,t)f(t)dt-f(x)=\int_0^\infty (f(t)-f(x))\mathcal{A}_\alpha^\rho(x,t)dt\\ &= \int_0^x (f(t)-f(x))\mathcal{A}_\alpha^\rho(x,t)dt+\int_x^\infty (f(t)-f(x))\mathcal{A}_\alpha^\rho(x,t)dt\\ &= -\int_0^x\left(\int_t^x f'(u)du\right)\mathcal{A}_\alpha^\rho(x,t)dt+\int_x^\infty\left(\int_x^t f'(u)du\right)\mathcal{A}_\alpha^\rho(x,t)dt\\ &= -I_1(x,\alpha,\rho)+I_2(x,\alpha,\rho),\ \ say.\end{aligned}$$

By using equation (1.3.10), we get

$$\begin{aligned}I_1(x,\alpha,\rho) &= \int_0^x\left\{\int_t^x \frac{1}{2}\left(f'(x+)+f'(x-)\right)+f'_x(u)+\frac{1}{2}\left(f'(x+)-f'(x-)\right)sgn(u-x)\right.\\ &\left.+\delta_x(u)\left(f'(u)-\frac{1}{2}\left(f'(x+)+f'(x-)\right)\right)du\right\}\mathcal{A}_\alpha^\rho(x,t)dt.\end{aligned}$$

Since $\int_x^t \delta_x(u)du=0$, we have

$$I_1(x,\alpha,\rho) = \frac{1}{2}\left(f'(x+)+f'(x-)\right)\int_0^x (x-t)\mathcal{A}_\alpha^\rho(x,t)dt+\int_0^x\left(\int_x^t f'_x(u)du\right)\mathcal{A}_\alpha^\rho(x,t)dt$$
$$+\frac{1}{2}\left(f'(x+)-f'(x-)\right)\int_0^x |x-t|\mathcal{A}_\alpha^\rho(x,t)dt. \tag{1.3.11}$$

Proceeding similarly, we find that

$$I_2(x,\alpha,\rho) = \frac{1}{2}\left(f'(x+)+f'(x-)\right)\int_x^\infty (t-x)\mathcal{A}_\alpha^\rho(x,t)dt+\int_x^\infty\left(\int_x^t f'_x(u)du\right)\mathcal{A}_\alpha^\rho(x,t)dt$$
$$+\frac{1}{2}\left(f'(x+)-f'(x-)\right)\int_x^\infty |t-x|\mathcal{A}_\alpha^\rho(x,t)dt. \tag{1.3.12}$$

By combining (1.3.11) and (1.3.12), we get

$$\begin{aligned}B_\alpha^\rho(f;x)-f(x) &= \frac{1}{2}\left(f'(x+)+f'(x-)\right)\int_0^\infty (t-x)\mathcal{A}_\alpha^\rho(x,t)dt\\ &+\frac{1}{2}\left(f'(x+)-f'(x-)\right)\int_0^\infty |t-x|\mathcal{A}_\alpha^\rho(x,t)dt\\ &-\int_0^x\left(\int_t^x f'_x(u)du\right)\mathcal{A}_\alpha^\rho(x,t)dt+\int_x^\infty\left(\int_x^t f'_x(u)du\right)\mathcal{A}_\alpha^\rho(x,t)dt.\end{aligned}$$

Hence

$$
\begin{aligned}
|B_\alpha^\rho(f;x)-f(x)| \;\le\;& \left|\frac{f'(x+)+f'(x-)}{2}\right| |B_\alpha^\rho(t-x;x)| + \left|\frac{f'(x+)-f'(x-)}{2}\right| B_\alpha^\rho(|t-x|;x) \\
&+\left|\int_0^x\left(\int_t^x f'_x(u)du\right)\mathcal{A}_\alpha^\rho(x,t)dt\right| + \left|\int_x^\infty\left(\int_x^t f'_x(u)du\right)\mathcal{A}_\alpha^\rho(x,t)dt\right|. \qquad (1.3.13)
\end{aligned}
$$

On an application of Lemma 1.3.9 and integration by parts, we obtain

$$
\int_0^x\left(\int_t^x f'_x(u)du\right)\mathcal{A}_\alpha^\rho(x,t)dt = \int_0^x\left(\int_t^x f'_x(u)du\right)\frac{\partial}{\partial t}\lambda_\alpha^\rho(x,t)dt = \int_0^x f'_x(t)\lambda_\alpha^\rho(x,t)dt.
$$

Thus,

$$
\begin{aligned}
\left|\int_0^x\left(\int_t^x f'_x(u)du\right)\mathcal{A}_\alpha^\rho(x,t)dt\right| \;\le\;& \int_0^x |f'_x(t)|\lambda_\alpha^\rho(x,t)dt \\
\le\;& \int_0^{x-\frac{x}{\sqrt{\alpha}}} |f'_x(t)|\lambda_\alpha^\rho(x,t)dt + \int_{x-\frac{x}{\sqrt{\alpha}}}^x |f'_x(t)|\lambda_\alpha^\rho(x,t)dt.
\end{aligned}
$$

Since $f'_x(x)=0$ and $\lambda_\alpha^\rho(x,t)\le 1$, we get

$$
\begin{aligned}
\int_{x-\frac{x}{\sqrt{\alpha}}}^x |f'_x(t)|\lambda_\alpha^\rho(x,t)dt \;=\;& \int_{x-\frac{x}{\sqrt{\alpha}}}^x |f'_x(t)-f'_x(x)|\lambda_\alpha^\rho(x,t)dt \le \int_{x-\frac{x}{\sqrt{\alpha}}}^x \bigvee_t^x (f'_x)dt \\
\le\;& \bigvee_{x-\frac{x}{\sqrt{\alpha}}}^x (f'_x)\int_{x-\frac{x}{\sqrt{\alpha}}}^x dt = \frac{x}{\sqrt{\alpha}}\bigvee_{x-\frac{x}{\sqrt{\alpha}}}^x (f'_x).
\end{aligned}
$$

Similarly, by using Lemma 1.3.9 and putting $t = x-\frac{x}{u}$, we get

$$
\begin{aligned}
\int_0^{x-\frac{x}{\sqrt{\alpha}}} |f'_x(t)|\lambda_\alpha^\rho(x,t)dt \;\le\;& \frac{\lambda x(1+cx)}{\alpha}\int_0^{x-\frac{x}{\sqrt{\alpha}}} |f'_x(t)|\frac{dt}{(x-t)^2} \\
\le\;& \frac{\lambda x(1+cx)}{\alpha}\int_0^{x-\frac{x}{\sqrt{\alpha}}} \bigvee_t^x (f'_x)\frac{dt}{(x-t)^2} \\
=\;& \frac{\lambda(1+cx)}{\alpha}\int_1^{\sqrt{\alpha}} \bigvee_{x-\frac{x}{u}}^x (f'_x)du \le \frac{\lambda(1+cx)}{\alpha}\sum_{m=1}^{[\sqrt{\alpha}]}\bigvee_{x-\frac{x}{m}}^x (f'_x).
\end{aligned}
$$

Consequently,

$$
\left|\int_0^x\left(\int_t^x f'_x(u)du\right)\mathcal{A}_\alpha^\rho(x,t)dt\right|
$$

$$
\le \frac{x}{\sqrt{\alpha}}\bigvee_{x-\frac{x}{\sqrt{\alpha}}}^x (f'_x) + \frac{\lambda(1+cx)}{\alpha}\sum_{m=1}^{[\sqrt{\alpha}]}\bigvee_{x-\frac{x}{m}}^x (f'_x). \qquad (1.3.14)
$$

Also, we have

$$\left|\int_x^\infty \left(\int_x^t f_x'(u)du\right)\mathcal{A}_\alpha^\rho(x,t)dt\right|$$

$$\begin{aligned}
&\leq \left|\int_x^{2x}\left(\int_x^t f_x'(u)du\right)\frac{\partial}{\partial t}(1-\lambda_\alpha^\rho(x,t))dt\right| + \left|\int_{2x}^\infty\left(\int_x^t f_x'(u)du\right)\mathcal{A}_\alpha^\rho(x,t)dt\right| \\
&\leq \left|\int_{2x}^\infty (f(t)-f(x))\mathcal{A}_\alpha^\rho(x,t)dt\right| + |f'(x+)|\left|\int_{2x}^\infty (t-x)\mathcal{A}_\alpha^\rho(x,t)dt\right| \\
&\quad + \left|\int_x^{2x} f_x'(u)du\right||1-\lambda_\alpha^\rho(x,2x)| + \int_x^{2x}|f_x'(t)|(1-\lambda_\alpha^\rho(x,t))dt.
\end{aligned}$$

Applying Lemma 1.3.9, we get

$$\left|\int_x^\infty \left(\int_x^t f_x'(u)du\right)\mathcal{A}_\alpha^\rho(x,t)dt\right|$$

$$\begin{aligned}
&\leq M\int_{2x}^\infty t^\gamma \mathcal{A}_\alpha^\rho(x,t)dt + |f(x)|\int_{2x}^\infty \mathcal{A}_\alpha^\rho(x,t)dt + |f'(x+)|\left\{\frac{\lambda x(1+cx)}{\alpha}\right\}^{1/2} \\
&\quad + \frac{\lambda(1+cx)}{\alpha x}|f(2x)-f(x)-xf'(x+)| \\
&\quad + \frac{x}{\sqrt{\alpha}}\bigvee_x^{x+\frac{x}{\sqrt{\alpha}}}(f_x') + \frac{\lambda(1+cx)}{\alpha}\sum_{m=1}^{[\sqrt{\alpha}]}\bigvee_x^{x+\frac{x}{m}}(f_x'). \qquad (1.3.15)
\end{aligned}$$

We note that we can choose $r \in \mathbb{N}$ such that $2r > \gamma$.

Since $t \leq 2(t-x)$ and $x \leq t-x$ when $t \geq 2x$, by using Hölder's inequality and Lemma 1.2.3, we obtain

$$M\int_{2x}^\infty t^\gamma \mathcal{A}_\alpha^\rho(x,t)dt + |f(x)|\int_{2x}^\infty \mathcal{A}_\alpha^\rho(x,t)dt$$

$$\begin{aligned}
&\leq 2^\gamma M\int_{2x}^\infty (t-x)^\gamma \mathcal{A}_\alpha^\rho(x,t)dt + \frac{|f(x)|}{x^2}\int_{2x}^\infty (t-x)^2\mathcal{A}_\alpha^\rho(x,t)dt \\
&\leq 2^\gamma M\left(\int_0^\infty (t-x)^{2r}\mathcal{A}_\alpha^\rho(x,t)dt\right)^{\gamma/2r} + |f(x)|\frac{\lambda(1+cx)}{\alpha x} \\
&\leq M'\frac{A(r,x)}{\alpha^{\gamma/2}} + |f(x)|\frac{\lambda(1+cx)}{\alpha x}, \text{ where } M' = 2^\gamma M. \qquad (1.3.16)
\end{aligned}$$

By using Lemma 1.2.3 and combining (1.3.13), (1.3.14), (1.3.15) and (1.3.16), we get the required result. □

Chapter 2

Quantitative convergence results for a family of hybrid operators

2.1 Introduction

In order to generalize the Baskakov operators, Mihesan [115] proposed the following operators based on a non-negative constant a, independent of n as

$$M_n^a(f;x) = \sum_{k=0}^{\infty} W_{n,k}^a(x) f\left(\frac{k}{n}\right), \tag{2.1.1}$$

where

$$W_{n,k}^a(x) = e^{-\frac{ax}{1+x}} \frac{\sum_{i=0}^{k} \binom{k}{i} (n)_i a^{k-i}}{k!} \frac{x^k}{(1+x)^{n+k}},$$

and the rising factorial is given by $(n)_i = n(n+1)\cdots(n+i-1)$, $(n)_0 = 1$. It was seen in [115] that $\sum_{k=0}^{\infty} W_{n,k}^a(x) = 1$. Obviously, if $a = 0$, we obtain at once the Baskakov basis function

$$W_{n,k}^0(x) = \binom{n+k-1}{k} \frac{x^k}{(1+x)^{n+k}}.$$

By considering the generalized Baskakov basis functions, Erençin [44] proposed the Durrmeyer type operators, which for $a = 0$ reduce to the modified Baskakov type operators considered in [67].

Here we propose a new kind of hybrid operators by considering the two generalized basis functions of [44] and [128].

For $f \in C_\gamma[0,\infty)$, we propose the hybrid operators depending on two parameters a and ρ as follows:

$$(2.1.2)\, L_n^{a,\rho}(f;x) \;=\; \sum_{k=1}^{\infty} W_{n,k}^a(x) \int_0^\infty s_{n,k}^\rho(t) f(t)dt + W_{n,0}^a(x) f(0),\ x \in [0,\infty)$$

where

$$s_{n,k}^\rho(t) = n\rho e^{-n\rho t}\frac{(n\rho t)^{k\rho-1}}{\Gamma(k\rho)}.$$

It is observed that the operators (2.1.2) preserve only the constant functions.

Special cases:

1. For $a = 0$ and $\rho = 1$, these operators include the well known operators introduced in [10].

2. For $a = 0$ and $\rho \to \infty$, these operators reduce to the well known Baskakov operators.

3. For $a > 0$ and $\rho \to \infty$, these operators reduce to the generalized Baskakov operators [115].

The aim of the present chapter is to study some direct results in terms of the modulus of continuity of second order, the weighted space and the degree of approximation of $f^{(r)}$ by $L_n^{a,\rho(r)}(f;.)$. We also study the statistical convergence. The rate of convergence of the operators $L_n^{a,\rho}$ to a certain function is also illustrated through graphics in Matlab.

In what follows, let us assume that $0 < c < d < \infty$, $H = [c,d]$; $0 < c_1 < c_2 < d_2 < d_1 < \infty$ and $H_i = [c_i, d_i], i = 1, 2$.

2.2 Basic Results

For $m \in \mathbb{N}^0$, the m^{th} order central moment of the generalized Baskakov operators M_n^a is defined as

$$\wp_{n,m}^a(x) = M_n^a((t-x)^m;x) = \sum_{k=0}^{\infty} W_{n,k}^a(x)\left(\frac{k}{n} - x\right)^m.$$

Lemma 2.2.1. *[9] For the function $\wp^a_{n,m}(x)$, we have*

$$\wp^a_{n,0}(x) = 1, \quad \wp^a_{n,1}(x) = \frac{ax}{n(1+x)}$$

and

$$x(1+x)^2(\wp^a_{n,m}(x))'$$

$$(2.2.1) \quad = n(1+x)\wp^a_{n,m+1}(x) - ax\wp^a_{n,m}(x) - mx(1+x)^2\wp^a_{n,m-1}(x), \quad \textit{for} \;\; m \geq 1.$$

Consequently,
(i) $\wp^a_{n,m}(x)$ is a rational function of x depending on the parameter a;
(ii) for each $x \in (0,\infty)$ and $m \in \mathbb{N}^0$, $\wp^a_{n,m}(x) = O(n^{-[(m+1)/2]})$, where $[\alpha]$ denotes the integer part of α.

Lemma 2.2.2. *For each $x \in (0,\infty)$ and $r \in \mathbb{N}^0$, there exist polynomials $q_{i,j,r}(x)$ in x independent of n and k such that*

$$\frac{d^r}{dx^r}W^a_{n,k}(x) = W^a_{n,k}(x)\sum_{\substack{2i+j\leq r\\ i,j\geq 0}} n^i(k-nx)^j\frac{q_{i,j,r}(x)}{(p(x))^r},$$

where $p(x) = x(1+x)^2$.

Proof. The proof of this lemma easily follows on proceeding along the lines of the proof of ([141], Lemma 4). Hence the details are omitted. □

Lemma 2.2.3. *For $m \in \mathbb{N}^0$, the m^{th} order moment for the operators (2.1.2) defined as*

$$\mu^{a,\rho}_{n,m}(x) := L^{a,\rho}_n(t^m;x) = \sum_{k=1}^{\infty} W^a_{n,k}(x)\int_0^{\infty} s^\rho_{n,k}(t)t^m dt,$$

we have $\mu^{a,\rho}_{n,0}(x) = 1$ and there holds the following recurrence relation:

$$n(1+x)\mu^{a,\rho}_{n,m+1}(x) \quad = \quad x(1+x)^2(\mu^{a,\rho}_{n,m}(x))' + \left(nx(1+x) + ax + \frac{m(1+x)}{\rho}\right)\mu^{a,\rho}_{n,m}(x).$$

Proof. By using the identity

$$x(1+x)^2\left\{\frac{d}{dx}W^a_{n,k}(x)\right\} = \left\{(k-nx)(1+x) - ax\right\}W^a_{n,k}(x),$$

we may write

$$\begin{aligned} x(1+x)^2(\mu_{n,m}^{a,\rho}(x))' &= \sum_{k=1}^{\infty}((k-nx)(1+x)-ax)W_{n,k}^a(x)\int_0^{\infty} s_{n,k}^{\rho}(t)t^m dt \\ &= \sum_{k=1}^{\infty} W_{n,k}^a(x)\int_0^{\infty}(1+x)((k-nt)+nt)s_{n,k}^{\rho}(t)t^m dt \\ &\quad -(nx(1+x)+ax)\mu_{n,m}^{a,\rho}(x). \end{aligned}$$

Using the identity $(ts_{n,k}^{\rho}(t))' = \rho(k-nt)s_{n,k}^{\rho}(t)$, we have
$x(1+x)^2(\mu_{n,m}^{a,\rho}(x))' + (nx(1+x)+ax)\mu_{n,m}^{a,\rho}(x)$

$$\begin{aligned} &= (1+x)\sum_{k=1}^{\infty} W_{n,k}^a(x)\int_0^{\infty}\frac{1}{\rho}(ts_{n,k}^{\rho}(t))' t^m dt + n(1+x)\mu_{n,m+1}^{a,\rho}(x) \\ &= -\frac{m(1+x)}{\rho}\mu_{n,m}^{a,\rho}(x) + n(1+x)\mu_{n,m+1}^{a,\rho}(x), \end{aligned}$$

which is the required recurrence relation. □

Remark 2. By Lemma 2.2.3, we have

(i) $\mu_{n,1}^{a,\rho}(x) = x + \dfrac{ax}{n(1+x)}$,

(ii) $\mu_{n,2}^{a,\rho}(x) = x^2 + \dfrac{x(1+x)}{n} + \dfrac{ax}{n^2(1+x)}\left(1+\dfrac{1}{\rho}\right) + \dfrac{a^2x^2}{n^2(1+x)^2} + \dfrac{2ax^2}{n(1+x)} + \dfrac{x}{n\rho}$,

(iii) for each $x \in (0,\infty)$ $\mu_{n,m}^{a,\rho}(x) = x^m + n^{-1}(q_m(x,a)+o(1))$, where $q_m(x,a)$ is a rational function of x depending on a and m.

Lemma 2.2.4. *If the m^{th} order $(m \in \mathbb{N}^0)$ central moment for the operators (2.1.2) is defined as*

$$T_{n,m}^{a,\rho}(x) := L_n^{a,\rho}((t-x)^m;x) = \sum_{k=1}^{\infty} W_{n,k}^a(x)\int_0^{\infty} s_{n,k}^{\rho}(t)(t-x)^m dt + W_{n,0}^a(x)(-x)^m,$$

then $T_{n,0}^{a,\rho}(x) = 0,\ T_{n,1}^{a,\rho}(x) = \frac{ax}{n(1+x)}$ and there holds the following recurrence relation:

$$\begin{aligned} n(1+x)T_{n,m+1}^{a,\rho}(x) &= x(1+x)^2\left\{(T_{n,m}^{a,\rho}(x))' + m\left(1+\frac{1}{\rho(1+x)}\right)T_{n,m-1}^{a,\rho}(x)\right\} \\ &\quad + \left(ax + \frac{m(1+x)}{\rho}\right)T_{n,m}^{a,\rho}(x). \end{aligned}$$

Proof. By using the identity

$$x(1+x)^2\left\{\frac{d}{dx}W^a_{n,k}(x)\right\} = \left\{(k-nx)(1+x)-ax\right\}W^a_{n,k}(x),$$

we may write

$$\begin{aligned} x(1+x)^2(T^{a,\rho}_{n,m}(x))' &= \sum_{k=1}^{\infty}((k-nx)(1+x)-ax)W^a_{n,k}(x)\int_0^\infty s^\rho_{n,k}(t)(t-x)^m dt \\ &\quad -mx(1+x)^2\sum_{k=1}^{\infty}W^a_{n,k}(x)\int_0^\infty s^\rho_{n,k}(t)(t-x)^{m-1}dt \\ &\quad -(nx(1+x)+ax)W^a_{n,0}(x)(-x)^m - m(-x)^{m-1}x(1+x)^2W^a_{n,0}(x). \end{aligned}$$

Thus,

$$\begin{aligned} &x(1+x)^2\left\{(T^{a,\rho}_{n,m}(x))' + mT^{a,\rho}_{n,m-1}(x)\right\} \\ &= \sum_{k=1}^{\infty}((k-nx)(1+x)-ax)W^a_{n,k}(x)\int_0^\infty s^\rho_{n,k}(t)(t-x)^m dt \\ &\quad -(nx(1+x)+ax)W^a_{n,0}(x)(-x)^m \\ &= \sum_{k=1}^{\infty}W^a_{n,k}(x)\int_0^\infty (1+x)((k-nt)+n(t-x)+nx)s^\rho_{n,k}(t)(t-x)^m dt \\ &\quad -(nx(1+x)+ax)T^{a,\rho}_{n,m}(x). \end{aligned}$$

Using the identity $(ts^\rho_{n,k}(t))' = \rho(k-nt)s^\rho_{n,k}(t)$, we have

$$\begin{aligned} &x(1+x)^2\left((T^{a,\rho}_{n,m}(x))' + mT^{a,\rho}_{n,m-1}(x)\right) + (nx(1+x)+ax)T^{a,\rho}_{n,m}(x) \\ &= (1+x)\sum_{k=1}^{\infty}W^a_{n,k}(x)\int_0^\infty \frac{1}{\rho}(ts^\rho_{n,k}(t))'(t-x)^m dt + n(1+x)T^{a,\rho}_{n,m+1}(x) + nx(1+x)T^{a,\rho}_{n,m}(x) \\ &= -\frac{m(1+x)}{\rho}T^{a,\rho}_{n,m}(x) - \frac{mx(1+x)}{\rho}T^{a,\rho}_{n,m-1}(x) + n(1+x)T^{a,\rho}_{n,m+1}(x) + nx(1+x)T^{a,\rho}_{n,m}(x), \end{aligned}$$

which is the required recurrence relation. □

Corollary 4. For the function $T^{a,\rho}_{n,m}(x)$, we have

(i) $T^{a,\rho}_{n,2}(x) = \dfrac{ax}{n(1+x)} + \dfrac{x(1+x)}{n} + \dfrac{x}{n\rho} + \dfrac{a^2x^2}{n^2(1+x)^2} + \dfrac{ax}{n^2\rho(1+x)}$;

(ii) $T^{a,\rho}_{n,m}(x)$ is a rational function of x;

(iii) for every $x \in (0,\infty)$, $T^{a,\rho}_{n,m}(x) = O\left(n^{-[(m+1)/2]}\right)$, where $[\alpha]$ denotes the integer part of α.

Corollary 5. Let γ and δ be any two positive real numbers and $H \subset (0,\infty)$ be any bounded interval. Then, for any $m>0$ there exists a constant M' depending on m only such that

$$\left\| \sum_{k=1}^{\infty} W_{n,k}^{a}(x) \int_{|t-x|\geq\delta} s_{n,k}^{\rho}(t) e^{\gamma t} dt \right\| \leq M' n^{-m},$$

where $\|.\|$ is the sup-norm over H.

Lemma 2.2.5. *[62] Let $f \in C(H)$. Then,*

$$\| f_{\eta,2k}^{(i)} \|_{C(H)} \leq C_i \{ \| f_{\eta,2} \|_{C(H)} + \| f_{\eta,2}^{(2k)} \|_{C(H)} \},\ i=1,2,\cdots,2k-1,$$

where C_i's are certain constants independent of f.

2.3 Main Results

2.3.1 Local approximation

Theorem 2.3.1. *Let $f \in C_B[0,\infty)$ and $x \geq 0$. Then, there exists a constant $C>0$ such that*

$$|L_n^{a,\rho}(g;x) - g(x)| \leq C\omega_2(f; \sqrt{\zeta_n^{a,\rho}(x)}) + \omega\left(f; \frac{ax}{n(1+x)}\right),$$

where $\zeta_n^{a,\rho}(x) = \dfrac{ax}{n(1+x)} + \dfrac{x(1+x)}{n} + \dfrac{x}{n\rho} + \dfrac{2a^2x^2}{n^2(1+x)^2} + \dfrac{ax}{n^2\rho(1+x)}$.

Proof. First, we define the auxiliary operators

$$\overline{L}_n^{a,\rho}(f;x) = L_n^{a,\rho}(f;x) + f(x) - f\left(x + \frac{ax}{n(1+x)}\right). \tag{2.3.1}$$

We observe that $\overline{L}_n^{a,\rho}(1;x)=1$ and $\overline{L}_n^{a,\rho}(t;x)=x$.

Let $g \in C_B^2[0,\infty)$. From the Taylor's theorem, we may write

$$g(t) = g(x) + (t-x)g'(x) + \int_x^t (t-v)g''(v)dv,$$

which implies that

$$\begin{aligned}
\overline{L}_n^{a,\rho}(g;x) - g(x) &= g'(x)\overline{L}_n^{a,\rho}((t-x);x) + \overline{L}_n^{a,\rho}\left(\int_x^t (t-v)g''(v)dv; x\right) \\
&= \overline{L}_n^{a,\rho}\left(\int_x^t (t-v)g''(v)dv; x\right) \\
&= L_n^{a,\rho}\left(\int_x^t (t-v)g''(v)dv; x\right) - \int_x^{x+\frac{ax}{n(1+x)}} \left(x + \frac{ax}{n(1+x)} - v\right) g''(v)dv.
\end{aligned}$$

Hence,

$|\overline{L}_n^{a,\rho}(g;x)-g(x)|$

$$\leq L_n^{a,\rho}\left(\left|\int_x^t (t-v)g''(v)dv\right|;x\right)+\left|\int_x^{x+\frac{ax}{n(1+x)}}\left(x+\frac{ax}{n(1+x)}-v\right)g''(v)dv\right|.$$

Since

$$\left|\int_x^t (t-v)g''(v)dv\right|\leq (t-x)^2\|g''\|$$

and

$$\left|\int_x^{x+\frac{ax}{n(1+x)}}\left(x+\frac{ax}{n(1+x)}-v\right)g''(v)dv\right|\leq\left(\frac{ax}{n(1+x)}\right)^2\|g''\|,$$

we have

$$\begin{aligned}|\overline{L}_n^{a,\rho}(g;x)-g(x)| &\leq \left\{L_n^{a,\rho}((t-x)^2;x)+\left(\frac{ax}{n(1+x)}\right)^2\right\}\|g''\|\\ &\leq \left\{\frac{ax}{n(1+x)}+\frac{x(1+x)}{n}+\frac{x}{n\rho}+\frac{2a^2x^2}{n^2(1+x)^2}+\frac{ax}{n^2\rho(1+x)}\right\}\|g''\|\\ &\leq \zeta_n^{a,\rho}(x)\|g''\|. \qquad (2.3.2)\end{aligned}$$

In view of (2.3.1), we obtain

$|L_n^{a,\rho}(g;x)-g(x)|$

$$\leq|\overline{L}_n^{a,\rho}(f-g;x)|+|(f-g)(x)|+|\overline{L}_n^{a,\rho}(g;x)-g(x)|+\left|f\left(x+\frac{ax}{n(1+x)}\right)-f(x)\right|.$$

Since $|\overline{L}_n^{a,\rho}(f;x)|\leq 3\|f\|$, we have

$$|L_n^{a,\rho}(g;x)-g(x)|\leq 4\|f-g\|+|\overline{L}_n^{a,\rho}(g;x)-g(x)|+\left|f\left(x+\frac{ax}{n(1+x)}\right)-f(x)\right|.$$

Using (2.3.2), we get

$$|L_n^{a,\rho}(g;x)-g(x)|\leq 4\|f-g\|+\zeta_n^{a,\rho}(x)\|g''\|+\omega\left(f;\frac{ax}{n(1+x)}\right).$$

Now, taking the infimum on the right hand side over all $g\in C_B^2[0,\infty)$, we obtain

$$|L_n^{a,\rho}(g;x)-g(x)|\leq 4\mathcal{K}_2(f;\zeta_n^{a,\rho}(x))+\omega\left(f;\frac{ax}{n(1+x)}\right).$$

Thus in view of (0.7.2), we get

$$|L_n^{a,\rho}(g;x)-g(x)|\leq C\omega_2\left(f;\sqrt{\zeta_n^{a,\rho}(x)}\right)+\omega\left(f;\frac{ax}{n(1+x)}\right).$$

Hence, the proof is completed. □

2.3.2 Weighted approximation

Theorem 2.3.2. *Let $f \in D_2^*[0,\infty)$. Then, we have*

$$\lim_{n\to\infty} \|L_n^{a,\rho}(f) - f\|_2 = 0. \tag{2.3.3}$$

Proof. From [50], we know that it is sufficient to verify the following three conditions

$$\lim_{n\to\infty} \|L_n^{a,\rho}(t^k;x) - x^k\|_2 = 0, \ \ k = 0,1,2. \tag{2.3.4}$$

Since $L_n^{a,\rho}(1;x) = 1$, (2.3.4) holds true for $k = 0$.
By Remark 2, we have

$$\begin{aligned} \|L_n^{a,\rho}(t;x) - x\|_2 &= \sup_{x\in[0,\infty)} \left| x + \frac{ax}{n(1+x)} - x \right| \frac{1}{1+x^2} \\ &\leq \frac{a}{n}. \end{aligned}$$

Thus, $\lim\limits_{n\to\infty} \|L_n^{a,\rho}(t;x) - x\|_2 = 0$. Similarly, we obtain
$\|L_n^{a,\rho}(t^2;x) - x^2\|_2$

$$\begin{aligned} &= \sup_{x\in[0,\infty)} \left| \frac{x(1+x)}{n} + \frac{ax}{n^2(1+x)}\left(1+\frac{1}{\rho}\right) + \frac{a^2x^2}{n^2(1+x)^2} + \frac{2ax^2}{n(1+x)} + \frac{x}{n\rho} \right| \frac{1}{1+x^2} \\ &\leq \frac{2+2a+\rho^{-1}}{n} + \frac{a^2+a+a\rho^{-1}}{n^2}, \end{aligned}$$

which implies that $\lim\limits_{n\to\infty} \|L_n^{a,\rho}(t^2;x) - x^2\|_2 = 0$. Thus, the proof is completed. □

2.3.3 Simultaneous approximation

In the following theorem, we show that the derivative $\left(\frac{d^r}{dw^r} L_n^{a,\rho}(f;w)\right)_{w=x}$ is also an approximation process for $f^{(r)}(x)$.

Theorem 2.3.3. ***(Basic convergence theorem)*** *Let $f \in C_\gamma[0,\infty)$. If $f^{(r)}$ exists at a point $x \in (0,\infty)$, then we have*

$$\lim_{n\to\infty} \left(\frac{d^r}{dw^r} L_n^{a,\rho}(f;w)\right)_{w=x} = f^{(r)}(x). \tag{2.3.5}$$

Further, if $f^{(r)}$ is continuous on $(c-\kappa, d+\kappa), \kappa > 0$, then the limit in (2.3.5) holds uniformly in $[c,d]$.

Proof. By our hypothesis, we have

$$f(t)=\sum_{i=0}^{r}\frac{f^{(i)}(x)}{i!}(t-x)^i+\psi(t,x)(t-x)^r,\quad t\in[0,\infty), \tag{2.3.6}$$

where the function $\psi(t,x)\to 0$ as $t\to x$. From equation (2.3.6), we can write

$$\begin{aligned}\left(\frac{d^r}{dw^r}L_n^{a,\rho}(f(t);w)\right)_{w=x} &= \sum_{i=0}^{r}\frac{f^{(i)}(x)}{i!}\left(\frac{d^r}{dw^r}L_n^{a,\rho}((t-x)^i;w)\right)_{w=x}+\left(\frac{d^r}{dw^r}L_n^{a,\rho}(\psi(t,x)(t-x)^r;w)\right)_{w=x}\\ &:= I_1+I_2,\text{ say.}\end{aligned}$$

First, we estimate I_1.

$$\begin{aligned}I_1 &= \sum_{i=0}^{r}\frac{f^{(i)}(x)}{i!}\left\{\frac{d^r}{dw^r}\left(\sum_{v=0}^{i}\binom{i}{v}(-x)^{i-v}L_n^{a,\rho}(t^v;w)\right)\right\}_{w=x}\\ &= \sum_{i=0}^{r}\frac{f^{(i)}(x)}{i!}\sum_{v=0}^{i}\binom{i}{v}(-x)^{i-v}\left(\frac{d^r}{dw^r}L_n^{a,\rho}(t^v;w)\right)_{w=x}\\ &= \sum_{i=0}^{r-1}\frac{f^{(i)}(x)}{i!}\sum_{v=0}^{i}\binom{i}{v}(-x)^{i-v}\left(\frac{d^r}{dw^r}L_n^{a,\rho}(t^v;w)\right)_{w=x}\\ &\quad+\frac{f^{(r)}(x)}{r!}\sum_{v=0}^{r}\binom{r}{v}(-x)^{r-v}\left(\frac{d^r}{dw^r}L_n^{a,\rho}(t^v;w)\right)_{w=x} := I_3+I_4,\text{ say.}\end{aligned}$$

Now, we may write

$$\begin{aligned}I_4 &= \frac{f^{(r)}(x)}{r!}\sum_{v=0}^{r-1}\binom{r}{v}(-x)^{r-v}\left(\frac{d^r}{dw^r}L_n^{a,\rho}(t^v;w)\right)_{w=x}+\frac{f^{(r)}(x)}{r!}\left(\frac{d^r}{dw^r}L_n^{a,\rho}(t^r;w)\right)_{w=x}\\ &:= I_5+I_6,\text{ say.}\end{aligned}$$

Making use of Remark 2 (iii) , we obtain
$I_6=f^{(r)}(x)+O\left(\frac{1}{n}\right)$, $I_3=O\left(\frac{1}{n}\right)$ and $I_5=O\left(\frac{1}{n}\right)$, as $n\to\infty$.
From the above estimates, for each $x\in(0,\infty)$ we have $I_1\to f^{(r)}(x)$ as $n\to\infty$.
In view of Lemma 2.2.2, we have

$$\begin{aligned}|I_2| &\leq \sum_{k=1}^{\infty}\sum_{\substack{2i+j\leq r\\ i,j\geq 0}}n^i|k-nx|^j\frac{|q_{i,j,r}(x)|}{(p(x))^r}W_{n,k}^a(x)\int_0^\infty s_{n,k}^{\rho}(t)\psi(t,x)|t-x|^r dt\\ &\quad+|\psi(0,x)(-x)^r|\left(\frac{d^r}{dw^r}W_{n,0}^a(w)\right)_{w=x} := I_7+I_8.\end{aligned} \tag{2.3.7}$$

Now, we estimate I_7.

Since $\psi(t,x) \to 0$ as $t \to x$, for a given $\epsilon > 0$ there exists a $\delta > 0$ such that $|\psi(t,x)| < \epsilon$ whenever $|t-x| < \delta$. For $|t-x| \geq \delta$, we have $|(t-x)^r \psi(t,x)| \leq Me^{\gamma t}$, for some $M > 0$. Thus, from equation (2.3.7) we may write

$$\begin{aligned} |I_7| \ &\leq \ \sum_{k=1}^{\infty} \sum_{\substack{2i+j\leq r \\ i,j\geq 0}} n^i |k-nx|^j \frac{|q_{i,j,r}(x)|}{(p(x))^r} W_{n,k}^a(x) \Big(\epsilon \int_{|t-x|<\delta} s_{n,k}^{\rho}(t)|t-x|^r dt \\ &\quad + M \int_{|t-x|\geq\delta} s_{n,k}^{\rho}(t) e^{\gamma t} dt \Big) := J_1 + J_2, \text{ say.} \end{aligned}$$

Let $K = \sup\limits_{\substack{2i+j\leq r \\ i,j\geq 0}} \dfrac{|q_{i,j,r}(x)|}{(p(x))^r}$.

Using Schwarz inequality, Lemma 2.2.1 and Corollary 4 we have

$$\begin{aligned} J_1 \ &= \ \epsilon \ K \sum_{k=1}^{\infty} \sum_{\substack{2i+j\leq r \\ i,j\geq 0}} n^i |k-nx|^j W_{n,k}^a(x) \left(\int_0^{\infty} s_{n,k}^{\rho}(t) \right)^{1/2} \\ &\quad \times \left(\int_0^{\infty} s_{n,k}^{\rho}(t) |t-x|^{2r} dt \right)^{1/2} \\ &\leq \ \epsilon \ K \sum_{\substack{2i+j\leq r \\ i,j\geq 0}} n^{i+j} \left(\sum_{k=0}^{\infty} W_{n,k}^a(x) \left(\frac{k}{n} - x \right)^{2j} - x^{2j} W_{n,0}^a(x) \right)^{1/2} \\ &\quad \times \left(L_n^{a,\rho}((t-x)^{2r}; x) - x^{2r} W_{n,0}^a(x) \right)^{1/2} \\ &= \ \epsilon \sum_{\substack{2i+j\leq r \\ i,j\geq 0}} n^{i+j} \left\{ O\left(\frac{1}{n^j} \right) + O\left(\frac{1}{n^s} \right) \right\}^{1/2} \\ &\quad \times \left\{ O\left(\frac{1}{n^r} \right) + O\left(\frac{1}{n^p} \right) \right\}^{1/2}, \text{ for any } s, p > 0. \end{aligned}$$

Choose s and p such that $s > j$, and $p > r$

$$J_1 \leq \epsilon \sum_{\substack{2i+j\leq r \\ i,j\geq 0}} n^{i+j} O\left(\frac{1}{n^{j/2}} \right) O\left(\frac{1}{n^{r/2}} \right) = \epsilon . O(1).$$

Since $\epsilon > 0$ is arbitrary, $J_1 \to 0$ as $n \to \infty$.

Again, by using Schwarz inequality, Lemma 2.2.1 and Corollary 5, we obtain

$$\begin{aligned} J_2 &\leq M_1 \sum_{\substack{2i+j\leq r\\ i,j\geq 0}} n^{i+j}\left(\sum_{k=0}^{\infty}\left(\frac{k}{n}-x\right)^{2j} W_{n,k}^a(x) - x^{2j}W_{n,0}^a(x)\right)^{1/2} \\ &\quad \times \left(\sum_{k=1}^{\infty} W_{n,k}^a(x)\int_{|t-x|\geq\delta} s_{n,k}^{\rho}(t)e^{2\gamma t}dt\right)^{1/2} \\ &\leq M_1 \sum_{\substack{2i+j\leq r\\ i,j\geq 0}} n^{i+j}\left\{O\left(\frac{1}{n^j}\right)+O\left(\frac{1}{n^p}\right)\right\}^{1/2}\left\{O\left(\frac{1}{n^m}\right)\right\}^{1/2}, \text{ for any } p>0. \end{aligned}$$

Choose p such that $p > j$

$$J_2 \leq M_1 \sum_{\substack{2i+j\leq r\\ i,j\geq 0}} n^{i+j} O\left(\frac{1}{n^{j/2}}\right) O\left(\frac{1}{n^{m/2}}\right) = M_1 O\left(\frac{1}{n^{(m-r)/2}}\right)$$

which implies that $J_2 \to 0$, as $n \to \infty$ on choosing $m > r$.
From the above estimates of J_1 and J_2, $I_7 \to 0$, as $n \to \infty$.
Next, we estimate I_8.

$$|I_8| = |\psi(0,x)(-x)^r| \left(\frac{d^r}{dw^r} W_{n,0}^a(w)\right)_{w=x}.$$

Since $|\psi(0,x)(-x)^r| < N_1$ for some $N_1 > 0$, we get

$$\left(\frac{d^r}{dw^r} W_{n,0}^a(w)\right)_{w=x} = \left(\frac{d^r}{dw^r}\left(e^{\frac{-aw}{1+w}}(1+w)^{-n}\right)\right)_{w=x} \to 0 \text{ as } n\to\infty,$$

which yields that $I_8 \to 0$ as $n \to \infty$. By combining the estimates of I_7 and I_8, we obtain $I_2 \to 0$ as $n \to \infty$. Thus, from the estimates of I_1 and I_2, the required result follows.
To prove the uniformity assertion, it is sufficient to remark that $\delta(\epsilon)$ in the above proof can be chosen to be independent of $x \in [c,d]$ and also that the other estimates hold uniformity in $x \in [c,d]$. This completes the proof. □

Next, we establish a Voronovskaja type asymptotic formula in simultaneous approximation.

Theorem 2.3.4. ***(Voronovskaja type result)*** *Let $f \in C_\gamma[0,\infty)$. If $f^{(r)}$ exists at a point $x \in (0,\infty)$, then we have*

$$\text{(2.3.8)}\quad \lim_{n\to\infty} n\left(\left(\frac{d^r}{dw^r} L_n^{a,\rho}(f;w)\right)_{w=x} - f^{(r)}(x)\right) = \sum_{v=1}^{r+2} Q(v,r,a,x) f^{(v)}(x),$$

where $Q(v,r,a,x)$ are certain rational functions of x depending on the parameter a. Further, if $f^{(r+2)}$ is continuous on $(c-\kappa, d+\kappa), \kappa > 0$, then the limit in (2.3.8) holds uniformly in $[c,d]$.

Proof. From the Taylor's theorem, we may write

$$f(t) = \sum_{v=0}^{r+2} \frac{f^{(v)}(x)}{v!}(t-x)^v + \psi(t,x)(t-x)^{r+2}, \quad t \in [0,\infty), \tag{2.3.9}$$

where the function $\psi(t,x) \to 0$ as $t \to x$. From equation (2.3.9), we obtain

$$\begin{aligned}
\left(\frac{d^r}{dw^r} L_n^{a,\rho}(f(t);w)\right)_{w=x} &= \sum_{v=0}^{r+2} \frac{f^{(v)}(x)}{v!} \left(\frac{d^r}{dw^r} L_n^{a,\rho}((t-x)^v;w)\right)_{w=x} \\
&\quad + \left(\frac{d^r}{dw^r} L_n^{a,\rho}(\psi(t,x)(t-x)^{r+2};w)\right)_{w=x} \\
&= \sum_{v=0}^{r+2} \frac{f^{(v)}(x)}{v!} \sum_{j=0}^{v} \binom{v}{j}(-x)^{v-j} \left(\frac{d^r}{dw^r} L_n^{a,\rho}(t^j;w)\right)_{w=x} \\
&\quad + \left(\frac{d^r}{dw^r} L_n^{a,\rho}(\psi(t,x)(t-x)^{r+2};w)\right)_{w=x} := S_1 + S_2, \text{ say.}
\end{aligned} \tag{2.3.10}$$

Proceeding along the lines of the estimate of I_2 of Theorem 2.3.3, it follows that for each $x \in (0,\infty)$

$$\lim_{n\to\infty} n \left(\frac{d}{dw}(L_n^{a,\rho}(\psi(t,x)(t-x)^{r+2};w)\right)_{w=x} = 0.$$

Now, we estimate S_1.

$$\begin{aligned}
S_1 &= \sum_{v=0}^{r-1} \frac{f^{(v)}(x)}{v!} \sum_{j=0}^{v} \binom{v}{j}(-x)^{v-j} \left(\frac{d^r}{dw^r} L_n^{a,\rho}(t^j;w)\right)_{w=x} \\
&\quad + \frac{f^{(r)}(x)}{r!} \sum_{j=0}^{r} \binom{r}{j}(-x)^{r-j} \left(\frac{d^r}{dw^r} L_n^{a,\rho}(t^j;w)\right)_{w=x} \\
&\quad + \frac{f^{(r+1)}(x)}{(r+1)!} \sum_{j=0}^{r+1} \binom{r+1}{j}(-x)^{r+1-j} \left(\frac{d^r}{dw^r} L_n^{a,\rho}(t^j;w)\right)_{w=x} \\
&\quad + \frac{f^{(r+2)}(x)}{(r+2)!} \sum_{j=0}^{r+2} \binom{r+2}{j}(-x)^{r+2-j} \left(\frac{d^r}{dw^r} L_n^{a,\rho}(t^j;w)\right)_{w=x}.
\end{aligned}$$

In view of Remark 2 (iii), we have

$$\begin{aligned}
S_1 &= \sum_{v=1}^{r-1} f^{(v)}(x) O\left(\frac{1}{n}\right) + f^{(r)}(x)\left(1 + O\left(\frac{1}{n}\right)\right) + f^{(r+1)}(x) O\left(\frac{1}{n}\right) + f^{(r+2)}(x) O\left(\frac{1}{n}\right) \\
&= f^{(r)}(x) + n^{-1}\left(\sum_{v=1}^{r+2} Q(v,r,a,x) f^{(v)}(x) + o(1)\right).
\end{aligned}$$

Combining the estimates of S_1 and S_2, we get the required result. The uniformity assertion follows as in proof of Theorem 2.3.3. Hence the proof is completed. $\square$

Corollary 6. From the above theorem, we deduce the following Voronovskaja type asymptotic results:
(i) for $r = 0$, we have

$$\lim_{n\to\infty} n(L_n^{a,\rho}(f;x) - f(x)) = \frac{ax}{1+x} f'(x) + \frac{1}{2}\left(\frac{ax}{(1+x)} + x(1+\rho^{-1}+x)\right) f''(x);$$

and
(ii) for $r = 1$, we get

$$\begin{aligned}\lim_{n\to\infty} n\left(\left(\frac{d}{dw} L_n^{a,\rho}(f;w)\right)_{w=x} - f'(x)\right) &= \frac{a}{(1+x)^2} f'(x) + \left(x + 2 + \frac{a(x^2-x+1)}{(1+x)^2}\right) f''(x) \\ &\quad + \left(\frac{x^2}{2} + x + \frac{ax(2+x)(3x+1)}{3(1+x)^2}\right) f'''(x).\end{aligned}$$

The next result provides an estimate of the degree of approximation in $L_n^{a,\rho(r)}(f;x) \to f^{(r)}(x), r \in \mathbb{N}$.

Theorem 2.3.5. ***(Degree of approximation)*** *Let $f \in C_\gamma[0,\infty)$ for some $\gamma > 0$ and $0 < c < c_1 < d_1 < d < \infty$. Then for n sufficiently large, we have*

$$\left\| \left(L_n^{a,\rho(r)}(f;.)\right) - f^{(r)} \right\|_{C(H_1)} \leq C_1 \omega_2(f^{(r)}; n^{-1/2}, H) + C_2 n^{-1} \|f\|_{C_\gamma[0,\infty)},$$

where $C_1 = C_1(r)$ and $C_2 = C_2(r,f)$.

Proof. We can write

$$\begin{aligned}\left\| \left(L_n^{a,\rho(r)}(f;.)\right) - f^{(r)} \right\|_{C(H_1)} &\leq \left\| L_n^{a,\rho(r)}((f - f_{\eta,2});.) \right\|_{C(H_1)} + \left\| \left(L_n^{a,\rho(r)}(f_{\eta,2};.)\right) - f_{\eta,2}^{(r)} \right\|_{C(H_1)} \\ &\quad + \|f^{(r)} - f_{\eta,2}^{(r)}\|_{C(H_1)} := M_1 + M_2 + M_3.\end{aligned}$$

Since $f_{\eta,2}^{(r)} = (f^{(r)})_{\eta,2}$, hence by property (iii) of the Steklov mean, we get

$$M_3 \leq C_1 \omega_2(f^{(r)}; \eta, H).$$

Next, applying Theorem 2.3.4 and Lemma 2.2.5, we obtain

$$M_2 \leq C_2 n^{-1} \sum_{i=r}^{r+2} \| f_{\eta,2}^{(i)} \|_{C(H_1)} \leq C_3 n^{-1} \{ \| f_{\eta,2} \|_{C(H_1)} + \| f_{\eta,2}^{(r+2)} \|_{C(H_1)} \}.$$

By using properties (ii) and (iv) of Steklov mean, we get

$$M_2 \leq C_4 n^{-1}\{\| f \|_{C_\gamma[0,\infty)} + \eta^{-2}\omega_2(f^{(r)}); \eta, H)\}.$$

Let c^* and d^* be such that $0 < c < c^* < c_1 < d_1 < d^* < d < \infty$ and H^* denote the interval $[c^*, d^*]$. Now, we estimate M_1.

Let $f - f_{\eta,2} \equiv F$. By our hypothesis we can write

$$\begin{aligned} F(t) &= \sum_{m=0}^{r} \frac{F^{(m)}(x)}{m!}(t-x)^m + \frac{F^{(r)}(\xi) + F^{(r)}(x)}{r!}(t-x)^r \chi(t) \\ &\quad + \theta(t,x)(1-\chi(t)), \end{aligned} \tag{2.3.11}$$

where ξ lies in between t and x, and χ is the characteristic function of the interval H^*. For $t \in H^*$ and $x \in H_1$, we get

$$F(t) = \sum_{m=0}^{r} \frac{F^{(m)}(x)}{m!}(t-x)^m + \frac{F^{(r)}(\xi) + F^{(r)}(x)}{r!}(t-x)^r,$$

and for $t \in [0,\infty) \setminus H^*, x \in H_1$ we define

$$\theta(t,x) = F(t) - \sum_{m=0}^{r} \frac{F^{(m)}(x)}{m!}(t-x)^m.$$

Now, operating $L_n^{a,\rho(r)}$ on both sides of (2.3.11), we get three terms J_1, J_2, and J_3, corresponding to three terms in right hand side of (2.3.11). By using Theorem 2.3.4, we get

$$|J_1| = F^{(r)} + n^{-1}\left(\sum_{v=1}^{r-1} Q(v,r,a,x)F^{(v)}(x) + o(1)\right)$$

which implies that $J_1 \leq \| f^{(r)} - f_{\eta,2}^{(r)} \|_{C(H_1)}$.

Next, by applying Theorem 2.3.4, we obtain

$$\begin{aligned} |J_2| &\leq \frac{2 \| F^{(r)} \|}{r!} L_n^{a,\rho(r)}((t-x)^r \chi(t); x) \\ &\leq C_5 \| f^{(r)} - f_{\eta,2}^{(r)} \|_{C(H^*)} . \end{aligned}$$

Lastly, we can easily find that

$$|J_3| = L_n^{a,\rho(r)}(1-\chi(t)\theta(t,x); x) = O(n^{-s}), \text{ for any } s > 0.$$

Combining the estimates of $J_1 - J_3$, and from property (iii) of Steklov mean, we get

$$M_1 \leq C_6 \| f^{(r)} - f_{\eta,2}^{(r)} \|_{C(H^*)} \leq C_6\, \omega_2(f^{(r)}; \eta, H).$$

On choosing $\eta = n^{-1/2}$, the required result follows. □

2.3.4 Statistical convergence

In the following result, we prove a weighted Korovkin theorem via A-statistical convergence.

Theorem 2.3.6. *Let (a_{nk}) be a non-negative regular summability matrix and $x \in [0,\infty)$. Then, for all $f \in D_2^*[0,\infty)$ we have*

$$st_A - \lim_n \|L_n^{a,\rho}(f;.) - f\|_2 = 0.$$

Proof. From (i) of Remark 2, $st_A - \lim_n \|L_n^{a,\rho}(e_0;x) - e_0(x)\|_2 = 0$. By (ii) of Remark 2, we obtain

$$\sup_{x\in[0,\infty)} \frac{|L_n^{a,\rho}(e_1;x) - e_1(x)|}{1+x^2} = \sup_{x\in[0,\infty)} \frac{|x + \frac{ax}{n(1+x)} - x|}{1+x^2} \leq \frac{a}{n}.$$

Since $st_A - \lim_n \frac{a}{n} = 0$, $st_A - \lim_n \|L_n^{a,\rho}(e_1;.) - e_1\|_2 = 0$. Similarly, from (iii) of Remark 2, we get

$$\sup_{x\in[0,\infty)} \frac{|L_n^{a,\rho}(e_2;x) - e_2(x)|}{1+x^2}$$

$$= \sup_{x\in[0,\infty)} \frac{1}{1+x^2}\left|\frac{x(1+x)}{n} + \frac{ax}{n^2(1+x)}\left(1+\frac{1}{\rho}\right) + \frac{a^2x^2}{n^2(1+x)^2} + \frac{2ax^2}{n(1+x)} + \frac{x}{n\rho}\right|$$

$$\leq \frac{2+2a+\rho^{-1}}{n} + \frac{a+a\rho^{-1}+a^2}{n^2}. \tag{2.3.12}$$

For $\epsilon > 0$, we define the following sets

$$E := \left\{n : \|L_n^{a,\rho}(e_2;.) - e_2\|_2 \geq \epsilon\right\}$$
$$E_1 := \left\{n : \frac{2+2a+\rho^{-1}}{n} \geq \frac{\epsilon}{2}\right\}$$
$$E_2 := \left\{n : \frac{a+a\rho^{-1}+a^2}{n^2} \geq \frac{\epsilon}{2}\right\}.$$

From (2.3.12), it is clear that $E \subseteq E_1 \cup E_2$ which implies that for all $n \in \mathbb{N}$

$$\sum_{k\in E} a_{nk} \leq \sum_{k\in E_1} a_{nk} + \sum_{k\in E_2} a_{nk}.$$

Taking the limit $n \to \infty$, we have $st_A - \lim_n \|L_n^{a,\rho}(e_2;.) - e_2\|_2 = 0$. This completes the proof of the theorem. □

Now, we illustrate the rate of convergence of the operators $L_n^{a,\rho}$ to a certain function using Matlab in the following example:

Example 5. For $n = 50, 100$, the rate of convergence of the operators $L_n^{a,\rho}(f;x)$ to the function $f(x) = x^2 - \sqrt{5}x - \sqrt{3}$ (red) is illustrated for $a = 1, \rho = 2$ (blue) in figures $2.1 - 2.2$, respectively.

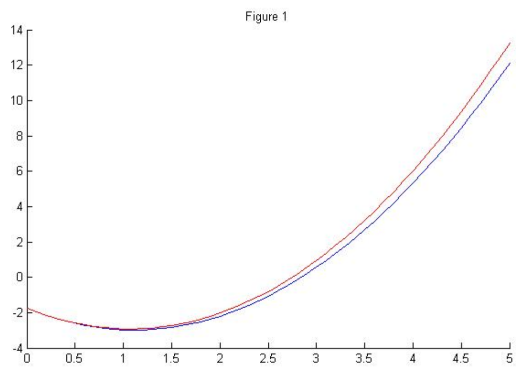

Figure 2.1 The Convergence of $L_{50}^{1,2}(f;x)$ (blue) to $f(x)$ (red).

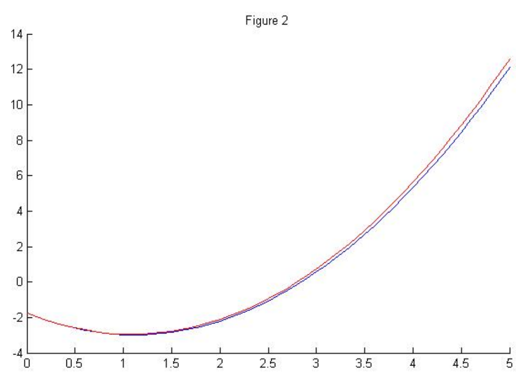

Figure 2.2 The Convergence of $L_{100}^{1,2}(f;x)$ (blue) to $f(x)$ (red).

Chapter 3

Generalized Baskakov Kantorovich operators

3.1 Introduction

The generalized Baskakov operators $\widetilde{M}_n^a$ [115] is defined as

$$\widetilde{M}_n^a(f;x) = \sum_{k=0}^{\infty} W_{n,k}^a(x) f\left(\frac{k}{n+1}\right),$$

where $W_{n,k}^a(x) = e^{\frac{-ax}{1+x}} \frac{P_k(n,a)}{k!} \frac{x^k}{(1+x)^{n+k}}$, $P_k(n,a) = \sum_{i=0}^{k} \binom{k}{i} (n)_i a^{k-i}$, and $(n)_0 = 1, (n)_i = n(n+1)\cdots(n+i-1)$ for $i \geq 1$. In [44], Erençin defined the Durrmeyer type modification of generalized Baskakov operators introduced by Mihesan [115], as

$$L_n^a(f;x) = \sum_{k=0}^{\infty} W_{n,k}^a(x) \frac{1}{B(k+1,n)} \int_0^{\infty} \frac{t^k}{(1+t)^{n+k+1}} f(t)dt, x \geq 0,$$

and discussed some approximation properties. Here, we consider the Kantorovich modification of generalized Baskakov operators for the function $f \in D_\vartheta[0,\infty)$ as follows :

$$K_n^a(f;x) = (n+1) \sum_{k=0}^{\infty} W_{n,k}^a(x) \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(t)dt, \ a \geq 0. \tag{3.1.1}$$

As a special case, for $a = 0$ these operators include the well known Baskakov-Kantorovich operators (see e.g. [160]). The purpose of this chapter is to study some local direct results, degree of approximation for functions in a Lipschitz type space, approximation of continuous functions with polynomial growth, simultaneous approximation, statistical convergence and the approximation of absolutely continuous functions having a derivative coinciding almost everywhere with a function of bounded variation by the operators defined in (3.1.1). Lastly, we also introduce the bivariate extension of these operators and obtain some approximation properties. Throughout this chapter, M denotes a constant not necessary the same at each occurrence.

3.2 Moment Estimates

For $r \in \mathbb{N}^0$, the rth order moment of the generalized Baskakov operators $\widetilde{M}_n^a$ is defined as

$$\upsilon_{n,r}^a(x) := \widetilde{M}_n^a(t^r;x) = \sum_{k=0}^{\infty} W_{n,k}^a(x)\left(\frac{k}{n+1}\right)^r$$

and the central moment of rth order for the operators $\widetilde{M}_n^a$ is defined as

$$\tilde{\wp}_{n,r}^a(x) := \widetilde{M}_n^a((t-x)^r;x) = \sum_{k=0}^{\infty} W_{n,k}^a(x)\left(\frac{k}{n+1}-x\right)^r.$$

Lemma 3.2.1. *For the function $\upsilon_{n,r}^a(x)$, we have*

$$\upsilon_{n,0}^a(x) = 1,\ \upsilon_{n,1}^a(x) = \frac{1}{n+1}\left(nx + \frac{ax}{1+x}\right)$$

and

(3.2.1) $$x(1+x)^2(\upsilon_{n,r}^a(x))' = (n+1)(1+x)\upsilon_{n,r+1}^a(x) - (a+n(1+x))x\,\upsilon_{n,r}^a(x).$$

Consequently, for each $x \in [0,\infty)$ and $r \in \mathbb{N}$,

(3.2.2) $$\upsilon_{n,r}^a(x) = x^r + n^{-1}(q_r(x,a) + o(1))$$

where $q_r(x,a)$ is a rational function of x depending on the parameters of a and r.

Proof. The values of $\upsilon_{n,r}^a(x), r = 0,1$ can be found by a simple computation. Differentiating $\upsilon_{n,r}^a(x)$ with respect to x and using the relation

$$x(1+x)^2\left(\frac{d}{dx}W_{n,k}^a(x)\right) = ((k-nx)(1+x)-ax)\,W_{n,k}^a(x),$$

we can easily get the recurrence relation (3.2.1). To prove the last assertion, we note that the equation (3.2.2) clearly holds for $r = 1$. The rest of the proof follows by using (3.2.1) and induction on r. □

Lemma 3.2.2. *For the function $\tilde{\wp}^a_{n,r}(x)$, we have*

$$\tilde{\wp}^a_{n,0}(x) = 1,\ \tilde{\wp}^a_{n,1}(x) = \frac{1}{n+1}\left(-x + \frac{ax}{(1+x)}\right)$$

and

$x(1+x)^2(\tilde{\wp}^a_{n,r}(x))'$

$$(3.2.3) = (n+1)(1+x)\tilde{\wp}^a_{n,r+1}(x) - ax\tilde{\wp}^a_{n,r}(x) - rx(1+x)^2\tilde{\wp}^a_{n,r-1}(x), r \in \mathbb{N},$$

Consequently,

(i) $\tilde{\wp}^a_{n,r}(x)$ is a rational function of x depending on the parameters a and r;

(ii) for each $x \in (0,\infty)$ and $r \in \mathbb{N}^0$, $\tilde{\wp}^a_{n,r}(x) = O(n^{-[(r+1)/2]})$, where $[\alpha]$ denotes the integer part of α.

Proof. Proof of this lemma follows along the lines similar to Lemma 3.2.1. The consequences (i) and (ii) follow from (3.2.3) by using induction on r. □

Lemma 3.2.3. *For the rth order $(r \in \mathbb{N}^0)$ moment of the operators (3.1.1), defined as $T^a_{n,r}(x) := K^a_n(t^r; x)$, we have*

$$T^a_{n,r}(x) = \frac{1}{r+1}\sum_{j=0}^{r}\binom{r+1}{j}\frac{1}{(n+1)^{r-j}}\upsilon^a_{n,j}(x),$$

where $\upsilon^a_{n,j}(x)$ is the jth order moment of the operators $\widetilde{M}^a_n$.

Consequently, $T^a_{n,0}(x) = 1$, $T^a_{n,1}(x) = \frac{1}{n+1}\left(nx + \frac{ax}{1+x} + \frac{1}{2}\right)$,

$$T^a_{n,2}(x) = \frac{1}{(n+1)^2}\left(n^2x^2 + n\left(x^2 + 2x + \frac{2ax^2}{1+x}\right) + \frac{a^2x^2}{(1+x)^2} + \frac{2ax}{1+x} + \frac{1}{3}\right),$$

and for each $x \in (0,\infty)$ and $r \in \mathbb{N}$, $T^a_{n,r}(x) = x^r + n^{-1}(p_r(x,a) + o(1))$, where $p_r(x,a)$ is a rational function of x depending on the parameters a and r.

Proof. From equation (3.1.1), we have

$$
\begin{aligned}
T^a_{n,r}(x) &= (n+1)\sum_{k=0}^{\infty} W^a_{n,k}(x)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} t^r dt \\
&= \frac{n+1}{r+1}\sum_{k=0}^{\infty} W^a_{n,k}(x)\left\{\left(\frac{k+1}{n+1}\right)^{r+1} - \left(\frac{k}{n+1}\right)^{r+1}\right\} \\
&= \frac{n+1}{r+1}\sum_{k=0}^{\infty} W^a_{n,k}(x)\left\{\sum_{j=0}^{r+1}\binom{r+1}{j}\left(\frac{k}{n+1}\right)^{j}\left(\frac{1}{n+1}\right)^{r+1-j} - \left(\frac{k}{n+1}\right)^{r+1}\right\} \\
&= \frac{n+1}{r+1}\sum_{k=0}^{\infty} W^a_{n,k}(x)\left\{\sum_{j=0}^{r}\binom{r+1}{j}\left(\frac{k}{n+1}\right)^{j}\left(\frac{1}{n+1}\right)^{r+1-j}\right\} \\
&= \frac{1}{r+1}\sum_{j=0}^{r}\binom{r+1}{j}\frac{1}{(n+1)^{r-j}}\sum_{k=0}^{\infty} W^a_{n,k}(x)\left(\frac{k}{n+1}\right)^{j} \\
(3.2.4) \quad &= \frac{1}{r+1}\sum_{j=0}^{r}\binom{r+1}{j}\frac{1}{(n+1)^{r-j}}\upsilon^a_{n,j}(x)
\end{aligned}
$$

from which the values of $T^a_{n,r}(x), r = 0,1,2$ can be found easily. The last assertion follows from equation (3.2.4) by using Lemma 3.2.1, the required result is immediate.

□

Lemma 3.2.4. *For the rth order central moment of K^a_n, defined as*

$$V^a_{n,r}(x) := K^a_n((t-x)^r; x),$$

we have

(i) $V^a_{n,0}(x) = 1, V^a_{n,1}(x) = \frac{1}{n+1}\left(-x + \frac{ax}{1+x} + \frac{1}{2}\right)$
and $V^a_{n,2}(x) = \frac{1}{(n+1)^2}\left\{nx(x+1) - x(1-x) + \frac{ax}{1+x}\left(\frac{ax}{1+x} + 2(1-x)\right) + \frac{1}{3}\right\};$

(ii) $V^a_{n,r}(x)$ *is a rational function of x depending on parameters a and r;*

(iii) for each $x \in (0,\infty), V^a_{n,r}(x) = O\left(\frac{1}{n^{[\frac{r+1}{2}]}}\right).$

Proof. Using equation (3.1.1), assertion (i) follows by a simple computation. To prove the assertions (ii) and (iii), we may write

$$\begin{aligned} V^a_{n,r}(x) &= (n+1)\sum_{k=0}^{\infty} W^a_{n,k}(x)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}}(t-x)^r dt \\ &= \frac{n+1}{r+1}\sum_{k=0}^{\infty} W^a_{n,k}(x)\Big\{\left(\frac{k+1}{n+1}-x\right)^{r+1}-\left(\frac{k}{n+1}-x\right)^{r+1}\Big\} \\ &= \frac{n+1}{r+1}\sum_{k=0}^{\infty} W^a_{n,k}(x)\left\{\sum_{\nu=0}^{r+1}\binom{r+1}{\nu}\left(\frac{k}{n+1}-x\right)^{r+1-\nu}\left(\frac{1}{n+1}\right)^{\nu}-\left(\frac{k}{n+1}-x\right)^{r+1}\right\} \\ &= \frac{1}{r+1}\sum_{\nu=1}^{r+1}\binom{r+1}{\nu}\frac{1}{(n+1)^{\nu-1}}\sum_{k=0}^{\infty} W^a_{n,k}(x)\left(\frac{k}{n+1}-x\right)^{r+1-\nu} \\ &= \frac{1}{r+1}\sum_{\nu=1}^{r+1}\binom{r+1}{\nu}\frac{1}{(n+1)^{\nu-1}}\tilde{\wp}^a_{n,r+1-\nu}(x), \end{aligned}$$

from which assertion (ii) follows in view of Lemma 3.2.2. Also, we get

$$|V^a_{n,r}(x)| \leq C\sum_{\nu=1}^{r+1}\frac{1}{n^{\nu-1}}\frac{1}{n^{[\frac{r+1-\nu}{2}]}} \leq C\frac{1}{n^{[\frac{r+1}{2}]}}.$$

This completes the proof. □

Remark 3. From Lemma 3.2.4, for $\lambda > 1, x \in (0,\infty)$ and n sufficiently large, we have

$$K^a_n((t-x)^2;x) = V^a_{n,2}(x) \leq \frac{\lambda x(1+x)}{n+1}.$$

Now, for $f \in C_B[0,\infty), x \geq 0$ the auxiliary operators are defined as

$$\widetilde{K}^a_n(f;x) = K^a_n(f;x) - f\left(\frac{1}{n+1}\left(nx+\frac{ax}{1+x}+\frac{1}{2}\right)\right)+f(x).$$

Lemma 3.2.5. *Let $f \in C^2_B[0,\infty)$. Then for all $x \geq 0$, we have*

$$|\widetilde{K}^a_n(f;x)-f(x)| \leq \frac{1}{2}\gamma^a_n(x) \parallel f'' \parallel,$$

where

$$\begin{aligned} \gamma^a_n(x) &= K^a_n((t-x)^2;x)+\frac{1}{(n+1)^2}\left(-x+\frac{ax}{1+x}+\frac{1}{2}\right)^2 \\ &= \frac{1}{(n+1)^2}\Big\{(n+2)x^2+(n-2)x+\frac{2a^2x^2}{(1+x)^2}-\frac{4ax^2}{1+x}+\frac{3ax}{1+x}+\frac{7}{12}\Big\}. \end{aligned}$$

Proof. It is clear from the definition of $\widetilde{K}_n^a$ that

$$\widetilde{K}_n^a(t-x;x)=0.$$

Let $f\in C_B^2[0,\infty)$. From the Taylor expansion of f, we have

$$f(t)-f(x)=(t-x)f'(x)+\int_x^t (t-u)f''(u)du.$$

Hence

$$\begin{aligned}
&\widetilde{K}_n^a(f;x)-f(x)\\
&= f'(x)\widetilde{K}_n^a(t-x;x)+\widetilde{K}_n^a\left(\int_x^t (t-u)f''(u)du;x\right)=\widetilde{K}_n^a\left(\int_x^t (t-u)f''(u)du;x\right)\\
&= K_n^a\left(\int_x^t (t-u)f''(u)du;x\right)-\int_x^{\frac{1}{n+1}(nx+\frac{ax}{1+x}+\frac{1}{2})}\left(\frac{1}{n+1}\left(nx+\frac{ax}{1+x}+\frac{1}{2}\right)-u\right)f''(u)du
\end{aligned}$$

and thus

$$\begin{aligned}
&|\widetilde{K}_n^a(f;x)-f(x)|\\
&\le K_n^a\left(\left|\int_x^t (t-u)f''(u)du\right|;x\right)\\
&\quad+\left|\int_x^{\frac{1}{n+1}(nx+\frac{ax}{1+x}+\frac{1}{2})}\left(\frac{1}{n+1}\left(nx+\frac{ax}{1+x}+\frac{1}{2}\right)-u\right)f''(u)du\right|. \qquad (3.2.5)
\end{aligned}$$

Since

$$\left|\int_x^t (t-u)f''(u)du\right|\le\frac{(t-x)^2}{2}\parallel f''\parallel$$

and

$$\begin{aligned}
&\left|\int_x^{\frac{1}{n+1}(nx+\frac{ax}{1+x}+\frac{1}{2})}\left(\frac{1}{n+1}\left(nx+\frac{ax}{1+x}+\frac{1}{2}\right)-u\right)f''(u)du\right|\\
&\qquad\le\frac{1}{2(n+1)^2}\left(-x+\frac{ax}{1+x}+\frac{1}{2}\right)^2\|f''\|,
\end{aligned}$$

it follows from (3.2.5) that

$$\begin{aligned}
|\widetilde{K}_n^a(f;x)-f(x)| &\le \frac{1}{2}\left\{K_n^a((t-x)^2;x)+\frac{1}{(n+1)^2}\left(-x+\frac{ax}{1+x}+\frac{1}{2}\right)^2\right\}\parallel f''\parallel\\
&= \frac{1}{2}\gamma_n^a(x)\|f''\|.
\end{aligned}$$

This completes the proof of the lemma. □

3.3 Main Results

3.3.1 Local approximation

Theorem 3.3.1. *Let $f \in C_B[0,\infty)$. Then for all $x \geq 0$, there exists a constant $C > 0$ such that*

$$|K_n^a(f;x) - f(x)| \leq C\omega_2\left(f; \sqrt{\gamma_n^a(x)}\right) + \omega\left(f; \frac{1}{n+1}\left| - x + \frac{ax}{1+x} + \frac{1}{2}\right|\right),$$

where $\gamma_n^a(x)$ is as defined in Lemma 3.2.5.

Proof. For $f \in C_B[0,\infty)$ and $g \in C_B^2[0,\infty)$, by the definition of the operators $\widetilde{K}_n^a$, we obtain

$$\begin{aligned} |K_n^a(f;x) - f(x)| &\leq |\widetilde{K}_n^a(f-g;x)| + |(f-g)(x)| + |\widetilde{K}_n^a(g;x) - g(x)| \\ &\quad + \left| f\left(\frac{1}{n+1}\left(nx + \frac{ax}{1+x} + \frac{1}{2}\right)\right) - f(x)\right| \end{aligned}$$

and

$$|\widetilde{K}_n^a(f;x)| \leq \| f \| \, K_n^a(1;x) + 2 \| f \| = 3 \| f \| .$$

Therefore, we have

$$|K_n^a(f;x) - f(x)| \leq 4 \| f - g \| + |\widetilde{K}_n^a(g;x) - g(x)| + \omega\left(f; \frac{1}{n+1}\left| - x + \frac{ax}{1+x} + \frac{1}{2}\right|\right).$$

Now, using Lemma 3.2.5, the above inequality reduces to

$$|K_n^a(f;x) - f(x)| \leq 4 \| f - g \| + \gamma_n^a(x) \| g'' \| + \omega\left(f; \frac{1}{n+1}\left| - x + \frac{ax}{1+x} + \frac{1}{2}\right|\right).$$

Thus, taking infimum over all $g \in C_B^2[0,\infty)$ on the right-hand side of the last inequality and using (0.7.2), we get the required result. □

Let us now consider the Lipschitz-type space in two parameters [125]:

$$Lip_M^{(a_1,a_2)}(\alpha) := \left\{ f \in C_B[0,\infty) : |f(t) - f(x)| \leq M \frac{|t-x|^\alpha}{(t + a_1x^2 + a_2x)^{\frac{\alpha}{2}}}; x, t \in (0,\infty)\right\},$$

for $a_1, a_2 > 0$, where M is a positive constant and $\alpha \in (0,1]$.

Theorem 3.3.2. *Let $f \in Lip_M^{(a_1,a_2)}(\alpha)$. Then, for all $x > 0$, we have*

$$|K_n^a(f;x) - f(x)| \leq M \left(\frac{V_{n,2}^a(x)}{(a_1x^2 + a_2x)}\right)^{\frac{\alpha}{2}}.$$

Proof. First we prove the theorem for the case $\alpha = 1$. We may write

$$\begin{aligned}|K_n^a(f;x) - f(x)| &\leq (n+1)\sum_{k=0}^{\infty} W_{n,k}^a(x)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} |f(t)-f(x)|dt \\ &\leq M(n+1)\sum_{k=0}^{\infty} W_{n,k}^a(x)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} \frac{|t-x|}{\sqrt{t+a_1x^2+a_2x}}dt.\end{aligned}$$

Using the fact that $\frac{1}{\sqrt{t+a_1x^2+a_2x}} < \frac{1}{\sqrt{a_1x^2+a_2x}}$ and the Cauchy-Schwarz inequality, the above inequality implies that

$$\begin{aligned}|K_n^a(f;x) - f(x)| &\leq \frac{M(n+1)}{\sqrt{a_1x^2+a_2x}}\sum_{k=0}^{\infty} W_{n,k}^a(x)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} |t-x|dt \\ &\leq \frac{M}{\sqrt{a_1x^2+a_2x}}(K_n^a((t-x)^2;x))^{1/2} \leq M\left(\sqrt{\frac{V_{n,2}^a(x)}{a_1x^2+a_2x}}\right).\end{aligned}$$

Thus the result hold for $\alpha = 1$. Now, let $0 < \alpha < 1$, then applying the Hölder inequality with $p = \frac{1}{\alpha}$ and $q = \frac{1}{1-\alpha}$, we have

$$\begin{aligned}|K_n^a(f;x) - f(x)| &\leq (n+1)\sum_{k=0}^{\infty} W_{n,k}^a(x)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} |f(t)-f(x)|dt \\ &\leq \left\{\sum_{k=0}^{\infty} W_{n,k}^a(x)\left((n+1)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} |f(t)-f(x)|dt\right)^{\frac{1}{\alpha}}\right\}^{\alpha} \\ &\leq \left\{\sum_{k=0}^{\infty} W_{n,k}^a(x)(n+1)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} |f(t)-f(x)|^{\frac{1}{\alpha}}dt\right\}^{\alpha} \\ &\leq M\left\{\sum_{k=0}^{\infty} W_{n,k}^a(x)(n+1)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} \frac{|t-x|}{\sqrt{t+a_1x^2+a_2x}}dt\right\}^{\alpha} \\ &\leq \frac{M}{(a_1x^2+a_2x)^{\frac{\alpha}{2}}}\left\{\sum_{k=0}^{\infty} W_{n,k}^a(x)(n+1)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} |t-x|dt\right\}^{\alpha} \\ &\leq \frac{M}{(a_1x^2+a_2x)^{\frac{\alpha}{2}}}(K_n^a((t-x)^2;x))^{\alpha/2} \leq M\left(\frac{V_{n,2}^a(x)}{(a_1x^2+a_2x)}\right)^{\frac{\alpha}{2}}.\end{aligned}$$

Thus, the proof is completed. □

Next, we obtain the local direct estimate of the operators defined in (3.1.1) using the Lipschitz-type maximal function of order τ.

Theorem 3.3.3. *Let $f \in C_B[0,\infty)$ and $0 < \tau \leq 1$. Then, for all $x \in [0,\infty)$ we have*

$$|K_n^a(f;x) - f(x)| \leq \widehat{\omega}_\tau(f,x)(V_{n,2}^a(x))^{\frac{\tau}{2}}.$$

Proof. From the equation (0.7.3), we have

$$|K_n^a(f;x)-f(x)|\leq \widehat{\omega}_\tau(f,x)K_n^a(|t-x|^\tau;x).$$

Applying the Hölder's inequality with $p=\dfrac{2}{\tau}$ and $\dfrac{1}{q}=1-\dfrac{1}{p}$, we get

$$|K_n^a(f;x)-f(x)|\leq \widehat{\omega}_\tau(f,x)(K_n^a(t-x)^2;x)^{\frac{\tau}{2}}=\widehat{\omega}_\tau(f,x)(V_{n,2}^a(x))^{\frac{\tau}{2}}.$$

Thus, the proof is completed. □

Theorem 3.3.4. *Let $f\in D_2[0,\infty)$ and $\omega\left(f;\delta,[0,b+1]\right)$ be its modulus of continuity on the finite interval $[0,b+1]\subset[0,\infty)$ with $b>0$. Then for every $x\in[0,b]$ and $n\in\mathbb{N}$, we have*

$$|K_n^a(f;x)-f(x)|\leq 4M_f(1+b^2)V_{n,2}^a(x)+2\omega\left(f;\sqrt{V_{n,2}^a(x)},[0,b+1]\right).$$

Proof. From [87], for $x\in[0,b]$ and $t\in[0,\infty)$, we have

$$|f(t)-f(x)|\leq 4M_f(1+b^2)(t-x)^2+\left(1+\frac{|t-x|}{\delta}\right)\omega\left(f;\delta,[0,b+1]\right),\delta>0.$$

Applying $K_n^a(.;x)$ to the above inequality and then Cauchy-Schwarz inequality to the above inequality, we obtain

$$\begin{aligned}|K_n^a(f;x)-f(x)| &\leq 4M_f(1+b^2)K_n^a((t-x)^2;x)+\omega\left(f;\delta,[0,b+1]\right)\left(1+\frac{1}{\delta}K_n^a(|t-x|;x)\right)\\ &\leq 4M_f(1+b^2)V_{n,2}^a(x)+\omega\left(f;\delta,[0,b+1]\right)\left(1+\frac{1}{\delta}\sqrt{V_{n,2}^a(x)}\right).\end{aligned}$$

By choosing $\delta=\sqrt{V_{n,2}^a(x)}$, we obtain the desired result. □

3.3.2 Weighted approximation

Theorem 3.3.5. *For each $f\in D_2^*[0,\infty)$, we have*

$$\lim_{n\to\infty}\parallel K_n^a(f)-f\parallel_2=0.$$

Proof. From [50], we observe that it is sufficient to verify the following three conditions:

$$\lim_{n\to\infty}\parallel K_n^a(t^k;x)-x^k)\parallel_2=0,\ k=0,1,2. \tag{3.3.1}$$

Since $K_n^a(1;x) = 1,$ the condition in (3.3.1) holds for $k = 0.$ Also, by Lemma 3.2.3 we have

$$\begin{aligned}
&\| K_n^a(t;x) - x) \|_2 \\
&= \left\| \frac{1}{n+1}\left(-x + \frac{ax}{1+x} + \frac{1}{2}\right) \right\|_2 \\
&\leq \frac{1}{n+1}\left(\sup_{x\in[0,\infty)} \frac{x}{1+x^2} + a \sup_{x\in[0,\infty)} \frac{x}{(1+x)(1+x^2)} + \frac{1}{2} \sup_{x\in[0,\infty)} \frac{1}{1+x^2}\right) \\
&\leq \frac{1}{n+1}\left(a + \frac{3}{2}\right),
\end{aligned}$$

which implies that the condition in (3.3.1) holds for $k = 1$. Similarly, we can write

$$\begin{aligned}
&\| K_n^a(t^2;x) - x^2 \|_2 \\
&= \left\| \frac{1}{(n+1)^2}\left(n^2x^2 + n\left(x^2 + 2x + \frac{2ax^2}{1+x}\right) + \frac{a^2x^2}{(1+x)^2} + \frac{2ax}{1+x} + \frac{1}{3} - (n+1)^2x^2\right) \right\|_2 \\
&\leq \frac{1}{(n+1)^2}\Bigg((n+1) \sup_{x\in[0,\infty)} \frac{x^2}{1+x^2} + 2n \sup_{x\in[0,\infty)} \frac{x}{1+x^2} + 2an \sup_{x\in[0,\infty)} \frac{x^2}{(1+x)(1+x^2)} \\
&\quad +2a \sup_{x\in[0,\infty)} \frac{x}{(1+x)(1+x^2)} + a^2 \sup_{x\in[0,\infty)} \frac{x^2}{(1+x)^2(1+x^2)} + \frac{1}{3} \sup_{x\in[0,\infty)} \frac{1}{1+x^2}\Bigg) \\
&\leq \frac{1}{(n+1)^2}\left((n+1)(2a+1) + \left(2n + a^2 + \frac{1}{3}\right)\right),
\end{aligned}$$

which implies that the equation (3.3.1) holds for $k = 2$. This completes the proof of theorem. □

Theorem 3.3.6. *Let $f \in D_2^*[0,\infty)$, then there exists a positive constant M_1 such that*

$$\sup_{x\in[0,\infty)} \frac{|K_n^a(f,x) - f(x)|}{(1+x^2)^{\frac{3}{2}}} \leq M_1\Omega_2\left(f, n^{-1/2}\right).$$

Proof. For $t \geq 0, x \in [0,\infty)$ and $\delta > 0$, by the definition of $\Omega_2(f,\delta)$ and Lemma 0.7.1, we get

$$\begin{aligned}
|f(t) - f(x)| &\leq (1 + (x + |x-t|)^2)\Omega_2(f, |t-x|) \\
&\leq (1 + (2x+t)^2)\left(1 + \frac{|t-x|}{\delta}\right)\Omega_2(f,\delta).
\end{aligned}$$

Since K_n^a is linear and positive, we have

$|K_n^a(f,x) - f(x)|$

$$\leq \Omega_2(f,\delta)\left\{K_n^a(1 + (2x+t)^2, x) + K_n^a\left((1+(2x+t)^2)\frac{|t-x|}{\delta}, x\right)\right\}. \tag{3.3.2}$$

From Lemma (3.2.3), there exist positive constant M_1 and M_2 such that

$$K_n^a(1+(2x+t)^2,x) \leq M_1(1+x^2), \tag{3.3.3}$$

and

$$\left(K_n^a\left((1+(2x+t)^2)^2,x\right)\right)^{1/2} \leq M_2(1+x^2). \tag{3.3.4}$$

Applying Cauchy-Schwarz inequality to the second term of equation (3.3.2), (3.3.4) and Remark 3, we get

$$\begin{aligned}
&K_n^a\left((1+(2x+t)^2)\frac{|t-x|}{\delta},x\right)\\
&\leq \frac{1}{\delta}\sqrt{K_n^a\left((1+(2x+t)^2)^2,x\right)}\sqrt{K_n^a\left((t-x)^2,x\right)}\\
&\leq \frac{\sqrt{\lambda}}{\delta}M_2(1+x^2)\sqrt{\frac{x(1+x)}{n+1}},\ \lambda>1\\
&\leq \frac{1}{\delta\sqrt{n}}M_3(1+x^2)^{3/2}, \text{ for some positive number } M_3. \qquad (3.3.5)
\end{aligned}$$

Combining the estimates of (3.3.2), (3.3.3), (3.3.5) and taking $M=(M_1+M_3)$, $\delta=\frac{1}{\sqrt{n}}$, we obtain the required result. □

3.3.3 Simultaneous approximation

Theorem 3.3.7. ***(Basic convergence theorem)*** *Let $f \in D_\vartheta[0,\infty)$. If $f^{(r)}$ exists at a point $x \in (0,\infty)$, then we have*

$$\lim_{n\to\infty}\left(\frac{d^r}{dw^r}K_n^a(f;w)\right)_{w=x} = f^{(r)}(x).$$

Proof. By our hypothesis, we have

$$f(t)=\sum_{\nu=0}^{r}\frac{f^{(\nu)}(x)}{\nu!}(t-x)^\nu+\psi(t,x)(t-x)^r,\ t\in[0,\infty),$$

where the function $\psi(t,x)\to 0$ as $t\to x$. From the above equation, we may write

$$\begin{aligned}
\left(\frac{d^r}{dw^r}K_n^a(f(t);w)\right)_{w=x} &= \sum_{\nu=0}^{r}\frac{f^{(\nu)}(x)}{\nu!}\left(\frac{d^r}{dw^r}K_n^a(t-x)^\nu;w)\right)_{w=x}\\
&\quad+\left(\frac{d^r}{dw^r}K_n^a(\psi(t,x)(t-x)^r;w)\right)_{w=x}\\
&:= I_1+I_2, \text{say}.
\end{aligned}$$

Now, we estimate I_1.

$$\begin{aligned}
I_1 &= \sum_{\nu=0}^{r} \frac{f^{(\nu)}(x)}{\nu!}\left\{\frac{d^r}{dw^r}\left(\sum_{j=0}^{\nu}\binom{\nu}{j}(-x)^{\nu-j}K_n^a(t^j;w)\right)_{w=x}\right\}\\
&= \sum_{\nu=0}^{r} \frac{f^{(\nu)}(x)}{\nu!}\sum_{j=0}^{\nu}\binom{\nu}{j}(-x)^{\nu-j}\left(\frac{d^r}{dw^r}K_n^a(t^j;w)\right)_{w=x}\\
&= \sum_{\nu=0}^{r-1} \frac{f^{(\nu)}(x)}{\nu!}\sum_{j=0}^{\nu}\binom{\nu}{j}(-x)^{\nu-j}\left(\frac{d^r}{dw^r}K_n^a(t^j;w)\right)_{w=x}\\
&\quad+\frac{f^{(r)}(x)}{r!}\sum_{j=0}^{r}\binom{r}{j}(-x)^{r-j}\left(\frac{d^r}{dw^r}K_n^a(t^j;w)\right)_{w=x}\\
&:= I_3+I_4, \text{say.}
\end{aligned}$$

First, we estimate I_4.

$$\begin{aligned}
I_4 &= \frac{f^{(r)}(x)}{r!}\sum_{j=0}^{r-1}\binom{r}{j}(-x)^{r-j}\left(\frac{d^r}{dw^r}K_n^a(t^j;w)\right)_{w=x}+\frac{f^{(r)}(x)}{r!}\left(\frac{d^r}{dw^r}K_n^a(t^r;w)\right)_{w=x}\\
&:= I_5+I_6, \text{say.}
\end{aligned}$$

By using Lemma 3.2.3, we get $I_6 = f^{(r)}(x)+O\left(\frac{1}{n}\right)$, $I_3 = O\left(\frac{1}{n}\right)$ and $I_5 = O\left(\frac{1}{n}\right)$. Combining the above estimates, for each $x \in (0,\infty)$ we obtain $I_1 \to f^{(r)}(x)$ as $n \to \infty$.

Since $\psi(t,x) \to 0$ as $t \to x$, for a given $\epsilon > 0$ there exists a $\delta > 0$ such that $|\psi(t,x)| < \epsilon$ whenever $|t-x| < \delta$. For $|t-x| \geq \delta, |\psi(t,x)| \leq M|t-x|^\beta$, for some $M, \beta > 0$.

By making use of Lemma 2.2.2, we have

$$\begin{aligned}
|I_2| &\leq (n+1)\sum_{k=0}^{\infty}\sum_{\substack{2i+j\leq r\\ i,j\geq 0}} n^i|k-nx|^j\frac{|q_{i,j,r}(x)|}{(p(x))^r}W_{n,k}^a(x)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}}\psi(t,x)(t-x)^r dt\\
&\leq (n+1)\sum_{k=0}^{\infty}\sum_{\substack{2i+j\leq r\\ i,j\geq 0}} n^i|k-nx|^j\frac{|q_{i,j,r}(x)|}{(p(x))^r}W_{n,k}^a(x)\\
&\quad\times\left(\epsilon\int_{|t-x|<\delta}|t-x|^r dt+M\int_{|t-x|\geq\delta}|t-x|^{r+\beta}dt\right) := I_7+I_8, \text{ say.}
\end{aligned}$$

Let $S = \sup\limits_{\substack{2i+j\leq r\\ i,j\geq 0}} \frac{|q_{i,j,r}(x)|}{(p(x))^r}$ and by applying the Schwarz inequality, Lemmas 2.2.1 and

3.2.4, we get

$$\begin{aligned}
|I_7| &\leq \epsilon(n+1)^{\frac{1}{2}} S \sum_{k=0}^{\infty} \sum_{\substack{2i+j\leq r\\ i,j\geq 0}} n^i |k-nx|^j W_{n,k}^a(x) \left(\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} (t-x)^{2r} dt \right)^{\frac{1}{2}} \\
&\leq \epsilon(n+1)^{\frac{1}{2}} S \sum_{\substack{2i+j\leq r\\ i,j\geq 0}} n^i \left(\sum_{k=0}^{\infty} (k-nx)^{2j} W_{n,k}^a(x) \right)^{\frac{1}{2}} \left(\sum_{k=0}^{\infty} W_{n,k}^a(x) \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} (t-x)^{2r} dt \right)^{\frac{1}{2}} \\
&= \epsilon . S \sum_{\substack{2i+j\leq r\\ i,j\geq 0}} n^i \left(\sum_{k=0}^{\infty} (k-nx)^{2j} W_{n,k}^a(x) \right)^{\frac{1}{2}} \left((n+1) \sum_{k=0}^{\infty} W_{n,k}^a(x) \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} (t-x)^{2r} dt \right)^{\frac{1}{2}} \\
&\leq \epsilon . S \sum_{\substack{2i+j\leq r\\ i,j\geq 0}} O\left(n^{\frac{2i+j}{2}}\right) O\left(n^{-r/2}\right) = \epsilon . O(1).
\end{aligned}$$

Since $\epsilon > 0$ is arbitrary, $I_7 \to 0$ as $n \to \infty$. Let $s(\in \mathbb{N}) > r + \beta$. Again, by using Schwarz inequality, Lemmas 2.2.1 and 3.2.4, we obtain

$$\begin{aligned}
I_8 &\leq MS(n+1) \sum_{k=0}^{\infty} \sum_{\substack{2i+j\leq r\\ i,j\geq 0}} n^i |k-nx|^j W_{n,k}^a(x) \int_{|t-x|\geq\delta} |t-x|^{r+\beta} dt \\
&\leq \frac{M'(n+1)}{\delta^{s-r-\beta}} \sum_{k=0}^{\infty} \sum_{\substack{2i+j\leq r\\ i,j\geq 0}} n^i |k-nx|^j W_{n,k}^a(x) \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} |t-x|^s dt, \text{ where } MS = M' \\
&\leq \frac{M'(n+1)^{1/2}}{\delta^{s-r-\beta}} \sum_{k=0}^{\infty} \sum_{\substack{2i+j\leq r\\ i,j\geq 0}} n^i |k-nx|^j W_{n,k}^a(x) \left(\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} |t-x|^{2s} dt \right)^{1/2} \\
&\leq \frac{M'}{\delta^{s-r-\beta}} \sum_{\substack{2i+j\leq r\\ i,j\geq 0}} n^i \left(\sum_{k=0}^{\infty} W_{n,k}^a(x)(k-nx)^{2j} \right)^{1/2} \left((n+1) \sum_{k=0}^{\infty} W_{n,k}^a(x) \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} (t-x)^{2s} dt \right)^{1/2} \\
&= \frac{M'}{\delta^{s-r-\beta}} \sum_{\substack{2i+j\leq r\\ i,j\geq 0}} n^i O(n^{j/2}) O(n^{-s/2}) = \frac{M'}{\delta^{s-r-\beta}} O(n^{\frac{(r-s)}{2}})
\end{aligned}$$

which implies that $I_8 \to 0$, as $n \to \infty$.

Now, by combining the estimates of I_7 and I_8, we get $I_2 \to 0$ as $n \to \infty$. Thus, from the estimates of I_1 and I_2, we obtained the required result. □

Theorem 3.3.8. ***(Voronovskaja type result)*** *Let $f \in D_{\vartheta}[0,\infty)$. If f admits a derivative of order $(r+2)$ at a fixed point $x \in (0,\infty)$, then we have*

$$\lim_{n\to\infty} n \left(\left(\frac{d^r}{dw^r} K_n^a(f;w) \right)_{w=x} - f^{(r)}(x) \right) = \sum_{\nu=1}^{r+2} Q(\nu, r, a, x) f^{(\nu)}(x),$$

where $Q(\nu,r,a,x)$ are certain rational functions of x depending on the parameters a,r,ν.

Proof. From the Taylor's theorem, we have

$$f(t)=\sum_{\nu=0}^{r+2}\frac{f^{(\nu)}(x)}{\nu!}(t-x)^{\nu}+\psi(t,x)(t-x)^{r+2},\ t\in[0,\infty) \tag{3.3.6}$$

where $\psi(t,x)\to 0$ as $t\to x$ and $\psi(t,x)=O(t-x)^{\vartheta}$.
From the equation (3.3.6), we have

$$\begin{aligned}
&\left(\frac{d^r}{dw^r}K_n^a(f(t);w)\right)_{w=x}\\
&= \sum_{\nu=0}^{r+2}\frac{f^{(\nu)}(x)}{\nu!}\left(\frac{d^r}{dw^r}(K_n^a((t-x)^{\nu};w)\right)_{w=x}+\left(\frac{d^r}{dw^r}K_n^a(\psi(t,x)(t-x)^{r+2};w)\right)_{w=x}\\
&= \sum_{\nu=0}^{r+2}\frac{f^{(\nu)}(x)}{\nu!}\sum_{j=0}^{\nu}\binom{\nu}{j}(-x)^{\nu-j}\left(\frac{d^r}{dw^r}K_n^a(t^j;w)\right)_{w=x}+\left(\frac{d^r}{dw^r}K_n^a(\psi(t,x))(t-x)^{r+2};w\right)_{w=x}\\
&:= T_1+T_2,\ \text{say.}
\end{aligned}$$

Proceeding in a manner to the estimates of I_2 in Theorem 3.3.7, for each $x\in(0,\infty)$ we get

$$\lim_{n\to\infty} n\left(\frac{d^r}{dw^r}(K_n^a(\psi(t,x)(t-x)^{r+2};w)\right)_{w=x}=0.$$

Now, we estimate T_1.

$$\begin{aligned}
T_1 &= \sum_{\nu=0}^{r-1}\frac{f^{(\nu)}(x)}{\nu!}\sum_{j=0}^{\nu}\binom{\nu}{j}(-x)^{\nu-j}\left(\frac{d^r}{dw^r}K_n^a(t^j;w)\right)_{w=x}\\
&\quad+\frac{f^{(r)}(x)}{r!}\sum_{j=0}^{r}\binom{r}{j}(-x)^{r-j}\left(\frac{d^r}{dw^r}K_n^a(t^j;w)\right)_{w=x}\\
&\quad+\frac{f^{(r+1)}(x)}{(r+1)!}\sum_{j=0}^{r+1}\binom{r+1}{j}(-x)^{r+1-j}\left(\frac{d^r}{dw^r}K_n^a(t^j;w)\right)_{w=x}\\
&\quad+\frac{f^{(r+2)}(x)}{(r+2)!}\sum_{j=0}^{r+2}\binom{r+2}{j}(-x)^{r+2-j}\left(\frac{d^r}{dw^r}K_n^a(t^j;w)\right)_{w=x}.
\end{aligned}$$

By making use of Lemma 3.2.3, we have
$T_1=f^{(r)}(x)+n^{-1}\left(\sum_{\nu=1}^{r+2}Q(\nu,r,a,x)f^{(\nu)}(x)+o(1)\right)$. Thus, from the estimates of T_1 and T_2, the required result follows. This completes the proof.

Corollary 7. From the above theorem, we have

(i) for $r=0$
$$\lim_{n\to\infty} n(K_n^a(f;x)-f(x)) = \left(\frac{ax}{1+x}+\frac{1}{2}-x\right)f'(x)+\frac{1}{2}(x+x^2)f''(x);$$

(ii) for $r=1$
$$\lim_{n\to\infty} n\left(\left(\frac{d}{dw}K_n^a(f;w)-f'(x)\right)_{w=x}\right) = \left(-1+\frac{a}{(1+x)^2}\right)f'(x)+\left(1+\frac{ax}{1+x}\right)f''(x)$$
$$+\frac{1}{2}x(1+x)f'''(x).$$

□

In this section, we obtain an estimate of the degree of approximation for rth order derivative of K_n^a for smooth functions.

Theorem 3.3.9. ***(Degree of approximation)*** *Let $r\le q\le r+2$, $f\in D_\vartheta[0,\infty)$ and $f^{(q)}$ exists and be continuous on $(a-\eta,b+\eta),\eta>0$. Then, for sufficiently large n, we have*
$$\left\|\left(\frac{d^r}{dw^r}K_n^a(f;w)\right)_{w=x}-f^{(r)}(t)\right\|_{C[a,b]} \le C_1 n^{-(q-r)/2}\omega\left(f^{(q)};n^{-1/2},(a-\eta,b+\eta)\right)+C_2\, n^{-1},$$
where $C_1=C_1(r)$ and $C_2=C_2(r,f)$.

Proof. By our hypothesis we have
$$\begin{aligned} f(t) &= \sum_{i=0}^{q}\frac{f^{(i)}(x)}{i!}(t-x)^i+\frac{f^{(q)}(\xi)-f^{(q)}(x)}{q!}(t-x)^q\chi(t) \\ &\quad +\phi(t,x)(1-\chi(t)), \end{aligned} \tag{3.3.7}$$

where ξ lies between t and x and $\chi(t)$ is the characteristic function of $(a-\eta,b+\eta)$. The function $\phi(t,x)$ for $t\in[0,\infty)\setminus(a-\eta,b+\eta)$ and $x\in[a,b]$ is bounded by $M|t-x|^\kappa$ for some constants $M,\kappa>0$.
Operating $\dfrac{d^r}{dw^r}K_n^a(.;w)$ on the equality (3.3.7) and breaking the right hand side into three parts J_1,J_2 and J_3, say, corresponding to the three terms on the right hand side of equation (3.3.7) as in the estimate of I_8 in Theorem 3.3.7, it can be easily shown that $J_3=o(n^{-1})$, uniformly in $x\in[a,b]$.
Now treating J_1 in a manner similar to the treatment of T_1 of Theorem 3.3.8, we

get $J_1 = f^{(r)}(t) + O(n^{-1})$, uniformly in $t \in [a,b]$.
Finally, let

$$S_1 = \sup_{x\in[a,b]} \sup_{\substack{2i+j\le r\\ i,j\ge 0}} \frac{q_{i,j,r}(x)}{(p(x))^r},$$

then making use of the inequality

$$|f^{(q)}(\xi) - f^{(q)}(x)| \le \left(1 + \frac{|t-x|}{\delta}\right)\omega\left(f^{(q)};\delta,(a-\eta,b+\eta)\right),\ \delta>0,$$

the Schwarz inequality, Lemmas 2.2.1 and 3.2.4, we obtain

$$\begin{aligned}
|J_2| &\le (n+1)\sum_{k=0}^{\infty}\sum_{\substack{2i+j\le r\\ i,j\ge 0}} \frac{n^i|k-nx|^j q_{i,j,r}(x)}{(p(x))^r} W^a_{n,k}(x)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} \frac{|f^{(q)}(\xi)-f^{(q)}(x)|}{q!}|t-x|^q\chi(t)dt\\
&\le \frac{\omega\left(f^{(q)};\delta,(a-\eta,b+\eta)\right)S_1}{q!}\sum_{\substack{2i+j\le r\\ i,j\ge 0}} n^i\left(\sum_{k=0}^{\infty}(k-nx)^{2j}W^a_{n,k}(x)\right)^{1/2}\\
&\quad\times\left\{\left((n+1)\sum_{k=0}^{\infty}W^a_{n,k}(x)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}}(t-x)^{2q}dt\right)^{1/2}\right.\\
&\quad\left.+\frac{1}{\delta}\left((n+1)\sum_{k=0}^{\infty}W^a_{n,k}(x)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}}(t-x)^{2q+2}dt\right)^{1/2}\right\}\\
&\le C_1\left(n^{-(q-r)/2}\right)\omega\left(f^{(q)};n^{-1/2},(a-\eta,b+\eta)\right),\ \text{on choosing } \delta = n^{-1/2}.
\end{aligned}$$

By combining the estimates of $J_1 - J_3$, we get the required result. □

3.3.4 Statistical convergence

Theorem 3.3.10. *Let (a_{nk}) be a non-negative regular summability matrix and $x \in [0,\infty)$. Then, for all $f \in D_2^*[0,\infty)$ we have*

$$st_A - \lim_n \|K_n^a(f,.) - f\|_{\zeta+2} = 0.$$

Proof. From ([42], p. 191, Th. 3), it is sufficient to show that $st_A - \lim_n \|K_n^a(e_i,.) - e_i\|_2 = 0$, where $e_i(x) = x^i$, $i = 0,1,2$.
In view of Lemma 3.2.3, it follows that

$$st_A - \lim_n \|K_n^a(e_0,.) - e_0\|_2 = 0. \tag{3.3.8}$$

Again, by using Lemma 3.2.3, we have

$$\begin{aligned}\sup_{x\in[0,\infty)}\frac{|K_n^a(e_1,x)-e_1(x)|}{1+x^2} &= \sup_{x\in[0,\infty)}\frac{\left|\frac{1}{n+1}\left(-x+\frac{ax}{1+x}+\frac{1}{2}\right)\right|}{1+x^2}\\ &\leq \frac{1}{n+1}\left(a+\frac{3}{2}\right).\end{aligned}$$

For $\epsilon>0$, we define the following sets

$$\begin{aligned}B: &= \left\{n:\|K_n^a(e_1,.)-e_1\|_2\geq\epsilon\right\}\\ B_1: &= \left\{n:\frac{1}{n+1}\left(a+\frac{3}{2}\right)\geq\epsilon\right\},\end{aligned}$$

which yields us $B\subseteq B_1$ and therefore for all n, we have $\sum_{k\in B}a_{nk}\leq\sum_{k\in B_1}a_{nk}$ and hence

$$st_A-\lim_n\|K_n^a(e_1,.)-e_1\|_2=0. \tag{3.3.9}$$

Proceeding similarly,

$\| K_n^a(e_2;.)-e_2 \|_2$

$$\begin{aligned}&= \sup_{x\in[0,\infty)}\frac{1}{1+x^2}\left|\frac{n}{(n+1)^2}\left(-x^2+2x+\frac{2ax^2}{1+x}\right)+\frac{1}{(n+1)^2}\left(\frac{a^2x^2}{(1+x)^2}+\frac{2ax}{1+x}-x^2+\frac{1}{3}\right)\right|\\ &\leq \frac{1}{n+1}(2a+3)+\frac{1}{(n+1)^2}\left(a^2+4a+\frac{13}{3}\right).\end{aligned}$$

Let us define the following sets

$$\begin{aligned}G: &= \left\{n:\|K_n^a(e_2,.)-e_2\|_2\geq\epsilon\right\}\\ G_1: &= \left\{n:\frac{1}{n+1}(2a+3)\geq\frac{\epsilon}{2}\right\}\\ G_2: &= \left\{n:\frac{1}{(n+1)^2}\left(a^2+4a+\frac{13}{3}\right)\geq\frac{\epsilon}{2}\right\}.\end{aligned}$$

Then, we obtain $G\subseteq G_1\cup G_2$, which implies that

$$\sum_{k\in G}a_{nk}\leq\sum_{k\in G_1}a_{nk}+\sum_{k\in G_2}a_{nk}$$

and hence

$$st_A-\lim_n\|K_n^a(e_2,.)-e_2\|_2=0. \tag{3.3.10}$$

This completes the proof of the theorem. □

3.3.5 Rate of approximation

The rate of convergence for functions with derivative of bounded variation is an interesting area of research in approximation theory. A pioneering work in this direction is due to Bojanic and Cheng ([31], [30]) who estimated the rate of convergence with derivatives of bounded variation for Bernstein and Hermite-Fejer polynomials by using different methods.

Now, we shall estimate the rate of convergence for the generalized Baskakov Kantorovich operators K_n^a for functions with derivatives of bounded variation defined on $(0, \infty)$ at points x where $f'(x+)$ and $f'(x-)$ exist, we shall prove that the operators (3.1.1) converge to the limit $f(x)$.

The operators $K_n^a(f; x)$ also admit the integral representation

$$K_n^a(f;x) = \int_0^\infty \mathcal{J}_n^a(x,t) f(t)dt, \tag{3.3.11}$$

where $\mathcal{J}_n^a(x,t) := (n+1)\sum_{k=0}^{\infty} W_{n,k}^a(x)\chi_{n,k}(t)$, where $\chi_{n,k}(t)$ is the characteristic function of the interval $\left[\frac{k}{n+1}, \frac{k+1}{n+1}\right]$ with respect to $[0, \infty)$.

In order to prove the main result, we need the following Lemma.

Lemma 3.3.11. *For fixed $x \in (0,\infty), \lambda > 1$ and n sufficiently large, we have*

(i) $\alpha_n^a(x,y) = \int_0^y \mathcal{J}_n^a(x,t)dt \leq \dfrac{1}{(x-y)^2}\dfrac{\lambda x(1+x)}{n+1},\ 0 \leq y < x,$

(ii) $1 - \alpha_n^a(x,z) = \int_z^\infty \mathcal{J}_n^a(x,t)dt \leq \dfrac{1}{(z-x)^2}\dfrac{\lambda x(1+x)}{n+1},\ x < z < \infty.$

Proof. First we prove (i).

$$\begin{aligned}
\alpha_n^a(x,y) &= \int_0^y \mathcal{J}_n^a(x,t)dt \leq \int_0^y \left(\frac{x-t}{x-y}\right)^2 \mathcal{J}_n^a(x,t)dt \\
&\leq \frac{1}{(x-y)^2} K_n^a((t-x)^2;x) \\
&\leq \frac{1}{(x-y)^2}\frac{\lambda x(1+x)}{n+1}.
\end{aligned}$$

The proof of (ii) is similar. □

Theorem 3.3.12. *Let $f \in DBV_\gamma(0,\infty)$. Then, for every $x \in (0,\infty)$, and n sufficiently large, we have*

$$\begin{aligned}|K_n^a(f;x)-f(x)| \;\le\; & \frac{\left|-x+\dfrac{ax}{1+x}+\dfrac{1}{2}\right|}{n+1}\frac{|f'(x+)+f'(x-)|}{2}+\sqrt{\frac{\lambda x(1+x)}{n+1}}\,\frac{|f'(x+)-f'(x-)|}{2}\\ & +\frac{\lambda(1+x)}{n+1}\sum_{k=1}^{[\sqrt{n}]}\bigvee_{x-(x/k)}^{x}(f'_x)+\frac{x}{\sqrt{n}}\bigvee_{x-(x/\sqrt{n})}^{x}(f'_x)\\ & +\frac{\lambda(1+x)}{n+1}\sum_{k=0}^{[\sqrt{n}]}\bigvee_{x}^{x+x/k}(f'_x)+\frac{x}{\sqrt{n}}\bigvee_{x}^{x+x/\sqrt{n}}(f'_x),\end{aligned}$$

where $\bigvee_c^d(f'_x)$ denotes the total variation of f'_x on $[c,d]$ and f'_x is defined by

$$f'_x(t)=\begin{cases} f'(t)-f'(x-), & 0\le t<x\\ 0\;, & t=x\\ f'(t)-f'(x+), & x<t<\infty.\end{cases}$$

Proof. For $u\in[0,\infty)$, we may write

$$\begin{aligned} f'(u) &= f'_x(u)+\frac{1}{2}(f'(x+)+f'(x-))+\frac{1}{2}(f'(x+)-f'(x-))sgn(u-x)\\ &\quad +\delta_x(u)\{f'(u)-\frac{1}{2}(f'(x+)+f'(x-))\},\end{aligned}\tag{3.3.12}$$

where

$$\delta_x(u)=\begin{cases}1\;, & u=x\\ 0\;, & u\neq x.\end{cases}$$

From (3.3.11) we get

$$\begin{aligned} K_n^a(f;x)-f(x) &= \int_0^\infty \mathcal{J}_n^a(x,t)(f(t)-f(x))dt\\ &= \int_0^\infty \mathcal{J}_n^a(x,t)\int_x^t f'(u)du dt.\end{aligned}\tag{3.3.13}$$

It is obvious that

$$\int_0^\infty\left(\int_x^t\left(f'(u)-\frac{1}{2}(f'(x+)+f'(x-))\right)\delta_x(u)du\right)\mathcal{J}_n^a(x,t)dt=0.$$

By (3.3.13), we have

$$\begin{aligned}\int_0^\infty\left(\int_x^t\frac{1}{2}(f'(x+)+f'(x-))du\right)\mathcal{J}_n^a(x,t)dt &= \frac{1}{2}(f'(x+)+f'(x-))\int_0^\infty(t-x)\mathcal{J}_n^a(x,t)dt\\ &= \frac{1}{2}(f'(x+)+f'(x-))K_n^a((t-x);x)\end{aligned}$$

and by using Schwarz inequality, we obtain

$$\begin{aligned}
&\left|\int_0^\infty \mathcal{J}_n^a(x,t)\left(\int_x^t \frac{1}{2}(f'(x+)-f'(x-))sgn(u-x)du\right)dt\right| \\
&\qquad \le \frac{1}{2}|f'(x+)-f'(x-)|\int_0^\infty |t-x|\mathcal{J}_n^a(x,t)dt \\
&\qquad = \frac{1}{2}|f'(x+)-f'(x-)|K_n^a(|t-x|;x) \\
&\qquad \le \frac{1}{2}|f'(x+)-f'(x-)|(K_n^a((t-x)^2;x))^{1/2}.
\end{aligned}$$

From Lemma 3.2.3, Remark 3 and from the above estimates, (3.3.13) becomes

$$\begin{aligned}
|K_n^a(f;x)-f(x)| \le\ & \frac{1}{2(n+1)}|f'(x+)+f'(x-)|\left|-x+\frac{ax}{1+x}+\frac{1}{2}\right| \\
&+\frac{1}{2}|f'(x+)-f'(x-)|\sqrt{\frac{\lambda x(1+x)}{n+1}}+\left|\int_0^x\left(\int_x^t f_x'(u)du\right)\mathcal{J}_n^a(x,t)dt\right. \\
&\left.+\int_x^\infty\left(\int_x^t f_x'(u)du\right)\mathcal{J}_n^a(x,t)dt\right|. \qquad (3.3.14)
\end{aligned}$$

Let

$$\mathcal{U}_n^a(f_x',x)=\int_0^x\left(\int_x^t f_x'(u)du\right)\mathcal{J}_n^a(x,t)dt,$$

and

$$\mathcal{V}_n^a(f_x',x)=\int_x^\infty\left(\int_x^t f_x'(u)du\right)\mathcal{J}_n^a(x,t)dt.$$

Now, we estimate the terms $\mathcal{U}_n^a(f_x',x)$ and $\mathcal{V}_n^a(f_x',x)$. Since $\int_c^d d_t\alpha_n^a(x,t)\le 1$ for all $[c,d]\subseteq(0,\infty)$, using integration by parts and Lemma 3.3.11 with $y=x-\frac{x}{\sqrt{n}}$ we have

$$\begin{aligned}
|\mathcal{U}_n^a(f_x',x)| &= \left|\int_0^x\left(\int_x^t f_x'(u)du\right)d_t\alpha_n^a(x,t)dt\right| \\
&= \left|\int_0^x \alpha_n^a(x,t)f_x'(t)dt\right| \\
&\le \left(\int_0^y+\int_y^x\right)|f_x'(t)||\alpha_n^a(x,t)|dt \\
&\le \frac{\lambda x(1+x)}{n+1}\int_0^y \bigvee_t^x(f_x')(x-t)^{-2}dt+\int_y^x\bigvee_t^x(f_x')dt \\
&\le \frac{\lambda x(1+x)}{n+1}\int_0^y \bigvee_t^x(f_x')(x-t)^{-2}dt+\frac{x}{\sqrt{n}}\bigvee_{x-(x/\sqrt{n})}^x(f_x').
\end{aligned}$$

By the substitution of $u = \dfrac{x}{x-t}$, we get

$$\begin{aligned}
\frac{\lambda x(1+x)}{n+1} \int_0^{x-x/\sqrt{n}} & (x-t)^{-2} \bigvee_t^x (f'_x) dt \\
&= \frac{\lambda(1+x)}{n+1} \int_1^{\sqrt{n}} \bigvee_{x-(x/u)}^{x} (f'_x) du \\
&\leq \frac{\lambda(1+x)}{n+1} \sum_{k=1}^{[\sqrt{n}]} \int_k^{k+1} \bigvee_{x-(x/u)}^{x} (f'_x) du \leq \frac{\lambda(1+x)}{n+1} \sum_{k=1}^{[\sqrt{n}]} \bigvee_{x-(x/k)}^{x} (f'_x).
\end{aligned}$$

Thus we obtain

$$|\mathcal{U}_n^a(f'_x, x)| \leq \frac{\lambda(1+x)}{n+1} \sum_{k=1}^{[\sqrt{n}]} \bigvee_{x-(x/k)}^{x} (f'_x) + \frac{x}{\sqrt{n}} \bigvee_{x-(x/\sqrt{n})}^{x} (f'_x). \tag{3.3.15}$$

Also

$$\begin{aligned}
|\mathcal{V}_n^a(f'_x, x)| &= \left| \int_x^\infty \left(\int_x^t f'_x(u) du \right) \mathcal{J}_n^a(x,t) dt \right| \\
&= \left| \int_z^\infty \left(\int_x^t f'_x(u) du \right) d_t(1-\alpha_n^a(x,t)) + \int_x^z \left(\int_x^t f'_x(u) du \right) d_t(1-\alpha_n^a(x,t)) \right| \\
&= \left| \int_x^z f'_x(u)(1-\alpha_n^a(x,z)) du - \int_x^z f'_x(t)(1-\alpha_n^a(x,t)) dt \right. \\
&\quad \left. + \left(\int_x^t f'_x(u)(1-\alpha_n^a(x,t)) du \right)_z^\infty - \int_z^\infty f'_x(t)(1-\alpha_n^a(x,t)) dt \right| \\
&\leq \left| \int_x^z f'_x(t)(1-\alpha_n^a(x,t)) dt \right| + \left| \int_z^\infty f'_x(t)(1-\alpha_n^a(x,t)) dt \right|.
\end{aligned}$$

By using Lemma 3.3.11, with $z = x + (x/\sqrt{n})$, we obtain

$$\begin{aligned}
|\mathcal{V}_n^a(f'_x, x)| &\leq \frac{\lambda x(1+x)}{n+1} \int_z^\infty \bigvee_x^t (f'_x)(t-x)^{-2} dt + \int_x^z \bigvee_x^t (f'_x) dt \\
&\leq \frac{\lambda x(1+x)}{n+1} \int_{x+(x/\sqrt{n})}^\infty \bigvee_x^t (f'_x)(t-x)^{-2} dt + \frac{x}{\sqrt{n}} \bigvee_x^{x+(x/\sqrt{n})} (f'_x).
\end{aligned}$$

By substitution of $v = \dfrac{x}{t-x}$, we get

$$\begin{aligned}
\frac{\lambda x(1+x)}{n+1} \int_{x+(x/\sqrt{n})}^\infty \bigvee_x^t (f'_x)(t-x)^{-2} dt &= \frac{\lambda(1+x)}{n+1} \int_0^{\sqrt{n}} \bigvee_x^{x+(x/v)} (f'_x) dv \\
&\leq \frac{\lambda(1+x)}{n+1} \sum_{k=0}^{[\sqrt{n}]} \int_k^{k+1} \bigvee_x^{x+(x/v)} (f'_x) dv \\
&\leq \frac{\lambda(1+x)}{n+1} \sum_{k=0}^{[\sqrt{n}]} \bigvee_x^{x+(x/k)} (f'_x).
\end{aligned}$$

Thus, we obtain

$$(3.3.16) \qquad |\mathcal{V}_n^a(f_x', x)| \leq \frac{\lambda(1+x)}{n+1} \sum_{k=0}^{[\sqrt{n}]} \bigvee_{x}^{x+(x/k)} (f_x') + \frac{x}{\sqrt{n}} \bigvee_{x}^{x+(x/\sqrt{n})} (f_x').$$

From (3.3.14)-(3.3.16), we get the required result. □

3.4 Bivariate Operators

Let $I = [0, \infty) \times [0, \infty)$ and $w_\gamma(x) = (1 + x^\gamma)^{-1}$ for $\gamma \in \mathbb{N}^0$(set of all non-negative integers). Further, for fixed $\gamma_1, \gamma_2 \in \mathbb{N}^0$, let $w_{\gamma_1,\gamma_2}(x, y) = w_{\gamma_1}(x)w_{\gamma_2}(y)$. Then, for $f \in C_{\gamma_1,\gamma_2}(I) := \{f \in C(I) : w_{\gamma_1,\gamma_2}(x, y)f(x, y)$ is bounded and uniformly continuous on I$\}$, we define a bivariate extension of the operators (3.1.1) as follows:

$$K_{n_1,n_2}^a(f; x, y)$$
$$(3.4.1) \qquad = (n_1 + 1)(n_2 + 1) \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} W_{n_1,n_2,k_1,k_2}^a(x, y) \int_{\frac{k_2}{n_2+1}}^{\frac{k_2+1}{n_2+1}} \int_{\frac{k_1}{n_1+1}}^{\frac{k_1+1}{n_1+1}} f(u, v)du\, dv,$$

where

$$W_{n_1,n_2,k_1,k_2}^a(x, y) = \frac{x^{k_1} y^{k_2} p_{k_1}(n, a) p_{k_2}(n, a) e^{\frac{-ax}{1+x}} e^{\frac{-ay}{1+y}}}{k_1! k_2! (1 + x)^{n_1+k_1} (1 + y)^{n_2+k_2}}.$$

If $f \in C_{\gamma_1,\gamma_2}(I)$ and if $f(x, y) = f_1(x)f_2(y)$ for all $(x, y) \in I$, then

$$(3.4.2) \qquad K_{n_1,n_2}^a(f(u, v); x, y) = K_{n_1}^a(f_1(u); x)K_{n_2}^a(f_2(v); y),$$

for $(x, y) \in I$ and $n_1, n_2 \in \mathbb{N}$. The sup norm on $C_{\gamma_1,\gamma_2}(I)$ is given by

$$(3.4.3) \qquad \| f \|_{\gamma_1,\gamma_2} = \sup_{(x,y)\in I} |f(x, y)| w_{\gamma_1,\gamma_2}(x, y), f \in C_{\gamma_1,\gamma_2}(I).$$

For $f \in C_{\gamma_1,\gamma_2}(I)$, we define the modulus of continuity

$$(3.4.4) \qquad \omega(f; C_{\gamma_1,\gamma_2}; t, s) := \sup_{0<h<t, 0<\delta<s} \| \Delta_{h,\delta} f(.,.) \|_{\gamma_1,\gamma_2}, \; t, s \geq 0,$$

where $\Delta_{h,\delta} f(x, y) := f(x + h, y + \delta) - f(x, y)$ for $(x, y) \in I$ and $h, \delta > 0$. Moreover, for fixed $m \in \mathbb{N}$, let $C_{\gamma_1,\gamma_2}^m(I)$ be the space of all functions $f \in C_{\gamma_1,\gamma_2}(I)$ having the partial derivatives $\dfrac{\partial^k}{\partial x^s \partial y^{k-s}} \in C_{\gamma_1,\gamma_2}(I), s = 1, 2, \cdots, k;\; k = 1, 2, \cdots, m$. The bivariate part is organized as follows:

In Section 3.5 of this part, we give some definitions and auxiliary results. In Section 3.6, we prove the main results of this section wherein we study the degree of approximation, Voronovskaja type theorems and simultaneous approximation of first order derivatives for bivariate Baskakov Kantorovich operators $K^a_{n_1,n_2}$. The section 3.7 is devoted to the illustrations of the convergence of the operators $K^a_{n_1,n_2}$ to a certain function and the comparison of the convergence with the bivariate Szász Kantorovich operators to the function using Matlab.

3.5 Auxiliary Results

Lemma 3.5.1. *Let $e_{i,j} : I \to I, e_{i,j} = x^i y^j, 0 \leq i, j \leq 2$ be two-dimensional test functions. Then the bivariate operators defined in (3.4.1) satisfy the following results:*

(i) $K^a_{n_1,n_2}(e_{0,0}; x, y) = 1;$

(ii) $K^a_{n_1,n_2}(e_{1,0}; x, y) = \dfrac{1}{n_1+1}\left(n_1 x + \frac{ax}{1+x} + \frac{1}{2}\right);$

(iii) $K^a_{n_1,n_2}(e_{0,1}; x, y) = \dfrac{1}{n_2+1}\left(n_2 y + \frac{ay}{1+y} + \frac{1}{2}\right);$

(iv) $K^a_{n_1,n_2}(e_{2,0}; x, y) = \dfrac{1}{(n_1+1)^2}\left(n_1^2 x^2 + n_1 x^2 + 2n_1 x + \dfrac{2an_1 x^2}{1+x} + \dfrac{a^2x^2}{(1+x)^2} + \dfrac{2ax}{1+x} + \frac{1}{3}\right);$

(v) $K^a_{n_1,n_2}(e_{0,2}; x, y) = \dfrac{1}{(n_2+1)^2}\left(n_2^2 y^2 + n_2 y^2 + 2n_2 y + \dfrac{2an_2 y^2}{1+y} + \frac{a^2y^2}{(1+y)^2} + \frac{2ay}{1+y} + \frac{1}{3}\right);$

(vi)
$$K^a_{n_1,n_2}(e_{3,0}; x, y) = \frac{1}{(n_1+1)^3}\left(n_1^3 x^3 + \frac{3n_1^2 x^2}{2}(3+2x) + \frac{3n_1 x^2}{2} + \frac{n_1 x}{2}(4x^2+6x+5)\right.$$
$$\left. + \frac{3ax^3n_1^2}{1+x} + \frac{3an_1x^2}{1+x}\left\{(3+x) + \frac{ax}{1+x}\right\} + \frac{ax}{1+x}\left\{\frac{7}{2} + \frac{7}{2}\frac{ax}{(1+x)} + \frac{a^2x^2}{(1+x)^2}\right\} + \frac{1}{4}\right);$$

(vii)
$$K^a_{n_1,n_2}(e_{0,3}; x, y) = \frac{1}{(n_2+1)^3}\left(n_2^3 y^3 + \frac{3n_2^2 y^2}{2}(3+2y) + \frac{3n_2 y^2}{2} + \frac{n_2 y}{2}(4y^2+6y+5)\right.$$
$$\left. + \frac{3ay^3n_2^2}{1+y} + \frac{3an_2y^2}{1+y}\left\{(3+y) + \frac{ay}{1+y}\right\} + \frac{ay}{1+y}\left\{\frac{7}{2} + \frac{7}{2}\frac{ay}{(1+y)} + \frac{a^2y^2}{(1+y)^2}\right\} + \frac{1}{4}\right).$$

Lemma 3.5.2. *For $n_1, n_2 \in \mathbb{N}$, we have*

(i) $K^a_{n_1,n_2}(u-x; x, y) = \dfrac{1}{n_1+1}\left(-x + \dfrac{ax}{1+x} + \dfrac{1}{2}\right);$

(ii) $K^a_{n_1,n_2}(v-y;x,y) = \frac{1}{n_2+1}\left(-y+\frac{ay}{1+y}+\frac{1}{2}\right);$

(iii) $K^a_{n_1,n_2}((u-x)^2;x,y) = \frac{1}{(n_1+1)^2}\left((n_1+1)x^2+(n_1-1)x+\frac{a^2x^2}{(1+x)^2}+2ax\left(\frac{1-x}{1+x}\right)+\frac{1}{3}\right);$

(iv) $K^a_{n_1,n_2}((v-y)^2;x,y) = \frac{1}{(n_2+1)^2}\left((n_2+1)y^2+(n_2-1)y+\frac{a^2y^2}{(1+y)^2}+2ay\left(\frac{1-y}{1+y}\right)+\frac{1}{3}\right).$

Remark 4. For every $x \in [0,\infty)$ and $n \in \mathbb{N}$, we have

$$K^a_{n_1,n_2}((u-x)^2;x,y) \leq \frac{\{\xi^a_{n_1}(x)\}^2}{n_1+1},$$

where $\{\xi^a_{n_1}(x)\}^2 = \phi^2(x) + \frac{(1+a)^2}{n_1+1}$ and $\phi(x) = \sqrt{x(1+x)}$.

Proof. From Lemma 3.5.2 (iii), we have

$$\begin{aligned} K^a_{n_1,n_2}((u-x)^2;x,y) &\leq \frac{(n_1+1)x^2+n_1x}{(n_1+1)^2} + \frac{1}{(n_1+1)^2}\left(\frac{a^2x^2}{(1+x)^2}+\frac{2ax}{1+x}+\frac{1}{3}\right) \\ &\leq \frac{1}{n_1+1}\left(x(1+x)+\frac{(1+a)^2}{n_1+1}\right) = \frac{\{\xi^a_{n_1}(x)\}^2}{n_1+1}. \end{aligned}$$

□

Lemma 3.5.3. *For every $\gamma_1 \in \mathbb{N}^0$ there exist positive constants $M_k(\gamma_1), k = 1,2$ such that*

(i) $w_{\gamma_1}(x)K^a_n\left(\frac{1}{w_{\gamma_1}(t)};x\right) \leq M_1(\gamma_1),$

(ii) $w_{\gamma_1}(x)K^a_n\left(\frac{(t-x)^2}{w_{\gamma_1}(t)};x\right) \leq M_2(\gamma_1)\frac{\{\xi^a_n(x)\}^2}{n+1},$

for all $x \in \mathbb{R}^0 = \mathbb{R}_+ \cup \{0\}, \mathbb{R}_+ = (0,\infty)$ and $n \in \mathbb{N}$.

Lemma 3.5.4. *For every $\gamma_1, \gamma_2 \in \mathbb{N}^0$ there exist positive constant $M_3(\gamma_1,\gamma_2)$, such that*

$$\| K^a_{n_1,n_2}(f;.,.) \|_{\gamma_1,\gamma_2} \leq M_3(\gamma_1,\gamma_2) \| f \|_{\gamma_1,\gamma_2} \tag{3.5.1}$$

for every $f \in C_{\gamma_1,\gamma_2}(I)$ and for all $n_1,n_2 \in \mathbb{N}$.

Proof. From equation (3.4.2) and Lemma 3.5.3, we get

$$\begin{aligned} w_{\gamma_1,\gamma_2}(x,y)K^a_{n_1,n_2}\left(\frac{1}{w_{\gamma_1,\gamma_2}(u,v)};x,y\right) &= \left(w_{\gamma_1}(x)K^a_{n_1}\left(\frac{1}{w_{\gamma_1}(u)};x\right)\right)\left(w_{\gamma_2}(y)K^a_{n_2}\left(\frac{1}{w_{\gamma_2}(v)};y\right)\right) \\ &\leq M_1(\gamma_1)\ M_2(\gamma_2), \end{aligned} \tag{3.5.2}$$

for all $(x,y) \in I$ and $n_1, n_2 \in \mathbb{N}$.

Taking supremum on the left side of the inequality of (3.5.2) and using (3.4.3), we obtain

$$\left\| K^a_{n_1,n_2}\left(\frac{1}{w_{\gamma_1,\gamma_2}(u,v)};.,.\right)\right\|_{\gamma_1,\gamma_2} \leq M_4(\gamma_1,\gamma_2). \tag{3.5.3}$$

Now, $\| K^a_{n_1,n_2}(f;.,.) \|_{\gamma_1,\gamma_2} \leq \| f \|_{\gamma_1,\gamma_2} \left\| K^a_{n_1,n_2}\left(\frac{1}{w_{\gamma_1,\gamma_2}(u,v)};.,.\right)\right\|_{\gamma_1,\gamma_2}$. From (3.5.3), we get the desired result. □

3.6 Main Results

3.6.1 Local approximation

For $f \in C_B(I)$ (the space of all bounded and uniformly continuous functions on I, let $C_B^2(I) = \{f \in C_B(I) : f^{(p,q)} \in C_B(I),\ 0 \leq p+q \leq 2\}$, where $f^{(p,q)}$ is (p,q)th-order partial derivative with respect to x, y of f, equipped with the norm

$$||f||_{C_B^2(I)} = ||f||_{C_B(I)} + \sum_{i=1}^{2}\left\|\frac{\partial^i f}{\partial x^i}\right\|_{C_B(I)} + \sum_{i=1}^{2}\left\|\frac{\partial^i f}{\partial y^i}\right\|_{C_B(I)}.$$

The Peetre's $K-$functional of the function $f \in C_B(I)$ is given by

$$\mathcal{K}(f;\delta) = \inf_{g \in C_B^2(I)}\{||f-g||_{C_B(I)} + \delta||g||_{C_B^2(I)}, \delta > 0\}.$$

It is also known that the following inequality

$$\mathcal{K}(f;\delta) \leq M_1\{\overline{\omega}_2(f;\sqrt{\delta}) + \min(1,\delta)||f||_{C_B(I)}\} \tag{3.6.1}$$

holds for all $\delta > 0$ ([33], page 192). The constant M_1 is independent of δ and f and $\overline{\omega}_2(f;\sqrt{\delta})$ is the second order modulus of continuity.

For $f \in C_B(I)$, the complete modulus of continuity for bivariate case is defined as follows:

$\overline{\omega}(f;\delta) = \sup\left\{|f(u,v)-f(x,y)| : (u,v),(x,y) \in I \text{ and } \sqrt{(u-x)^2+(v-y)^2} \leq \delta\right\}$.

The details of the modulus of continuity for the bivariate case can be found in [15].

Now, we find the order of approximation of the sequence $K^a_{n_1,n_2}(f;x,y)$ to the function $f(x,y) \in C_B(I)$ by Peetre's $K-$functional.

Theorem 3.6.1. *For the function $f \in C_B(I)$, the following inequality*
$|K^a_{n_1,n_2}(f;x,y) - f(x,y)|$

$$\begin{aligned}
&\leq 4\mathcal{K}(f; M^a_{n_1,n_2}(x,y)) \\
&\quad + \overline{\omega}\left(f; \sqrt{\left(\frac{1}{n_1+1}\left(-x+\frac{ax}{1+x}+\frac{1}{2}\right)\right)^2 + \left(\frac{1}{n_2+1}\left(-y+\frac{ay}{1+y}+\frac{1}{2}\right)\right)^2}\right) \\
&\leq M\left\{\overline{\omega}_2\left(f; \sqrt{M^a_{n_1,n_2}(x,y)}\right) + min\{1, M^a_{n_1,n_2}(x,y)\}||f||_{C^2_B(I)}\right\} \\
&\quad + \overline{\omega}\left(f; \sqrt{\left(\frac{1}{n_1+1}\left(-x+\frac{ax}{1+x}+\frac{1}{2}\right)\right)^2 + \left(\frac{1}{n_2+1}\left(-y+\frac{ay}{1+y}+\frac{1}{2}\right)\right)^2}\right)
\end{aligned}$$

holds. The constant $M > 0$ is independent of f and $M^a_{n_1,n_2}(x,y)$,
where $M^a_{n_1,n_2}(x,y) = \dfrac{\{\xi^a_{n_1}(x)\}^2}{n_1+1} + \dfrac{\{\xi^a_{n_2}(y)\}^2}{n_2+1}$.

Proof. We define the auxiliary operators as follows:

$$\begin{aligned}
\overline{K}^a_{n_1,n_2}(f;x,y) &= K^a_{n_1,n_2}(f;x,y) \\
&\quad -f\left(\frac{1}{n_1+1}\left(n_1x+\frac{ax}{1+x}+\frac{1}{2}\right), \frac{1}{n_2+1}\left(n_2y+\frac{ay}{1+y}+\frac{1}{2}\right)\right) + f(x,y). \qquad (3.6.2)
\end{aligned}$$

Then, from Lemma 3.5.2, we have

$$\overline{K}^a_{n_1,n_2}((u-x);x,y) = 0 \text{ and } \overline{K}^a_{n_1,n_2}((v-y);x,y) = 0.$$

Let $g \in C^2_B(I)$ and $(u,v) \in I$. Using the Taylor's theorem, we have

$$\begin{aligned}
g(u,v) - g(x,y) &= \frac{\partial g(x,y)}{\partial x}(u-x) + \int_x^u (u-\alpha)\frac{\partial^2 g(\alpha,y)}{\partial \alpha^2}d\alpha + \frac{\partial g(x,y)}{\partial y}(v-y) \\
&\quad + \int_y^v (v-\beta)\frac{\partial^2 g(x,\beta)}{\partial \beta^2}d\beta. \qquad (3.6.3)
\end{aligned}$$

Operating $\overline{K}^a_{n_1,n_2}$ on both sides of (3.6.3), we get

$$
\begin{aligned}
&\overline{K}^a_{n_1,n_2}(g;x,y)-g(x,y)\\
&\quad= \overline{K}^a_{n_1,n_2}\left(\int_x^u (u-\alpha)\frac{\partial^2 g(\alpha,y)}{\partial\alpha^2}d\alpha;x,y\right)+\overline{K}^a_{n_1,n_2}\left(\int_y^v (v-\beta)\frac{\partial^2 g(x,\beta)}{\partial\beta^2}d\beta;x,y\right)\\
&\quad= K^a_{n_1,n_2}\left(\int_x^u (u-\alpha)\frac{\partial^2 g(\alpha,y)}{\partial\alpha^2}d\alpha;x,y\right)\\
&\qquad -\int_x^{\frac{1}{n_1+1}\left(n_1x+\frac{ax}{1+x}+\frac{1}{2}\right)}\left(\frac{1}{n_1+1}\left(n_1x+\frac{ax}{1+x}+\frac{1}{2}\right)-\alpha\right)\frac{\partial^2 g(\alpha,y)}{\partial\alpha^2}d\alpha\\
&\qquad +K^a_{n_1,n_2}\left(\int_y^v (v-\beta)\frac{\partial^2 g(x,\beta)}{\partial\beta^2}d\beta;x,y\right)\\
&\qquad -\int_y^{\frac{1}{n_2+1}\left(n_2y+\frac{ay}{1+y}+\frac{1}{2}\right)}\left(\frac{1}{n_2+1}\left(n_2y+\frac{ay}{1+y}+\frac{1}{2}\right)-\beta\right)\frac{\partial^2 g(x,\beta)}{\partial\beta^2}d\beta.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&|\overline{K}^a_{n_1,n_2}(g;x,y)-g(x,y)|\\
&\quad\le K^a_{n_1,n_2}\left(\left|\int_x^u |u-\alpha|\left|\frac{\partial^2 g(\alpha,y)}{\partial\alpha^2}\right|d\alpha\right|;x,y\right)\\
&\qquad +\left|\int_x^{\frac{1}{n_1+1}\left(n_1x+\frac{ax}{1+x}+\frac{1}{2}\right)}\left|\frac{1}{n_1+1}\left(n_1x+\frac{ax}{1+x}+\frac{1}{2}\right)-\alpha\right|\left|\frac{\partial^2 g(\alpha,y)}{\partial\alpha^2}\right|d\alpha\right|\\
&\qquad +K^a_{n_1,n_2}\left(\left|\int_y^v |v-\beta|\left|\frac{\partial^2 g(x,\beta)}{\partial\beta^2}\right|d\beta\right|;x,y\right)\\
&\qquad +\left|\int_y^{\frac{1}{n_2+1}\left(n_2y+\frac{ay}{1+y}+\frac{1}{2}\right)}\left|\frac{1}{n_2+1}\left(n_2y+\frac{ay}{1+y}+\frac{1}{2}\right)-\beta\right|\left|\frac{\partial^2 g(x,\beta)}{\partial\beta^2}\right|d\beta\right|\\
&\quad\le \left\{K^a_{n_1,n_2}((u-x)^2;x,y)+\left(\frac{1}{n_1+1}\left(n_1x+\frac{ax}{1+x}+\frac{1}{2}\right)-x\right)^2\right\}||g||_{C^2_B(I)}\\
&\qquad +\left\{K^a_{n_1,n_2}((v-y)^2;x,y)+\left(\frac{1}{n_2+1}\left(n_2y+\frac{ay}{1+y}+\frac{1}{2}\right)-y\right)^2\right\}||g||_{C^2_B(I)}\\
&\quad\le \left\{\frac{1}{n_1+1}\{\xi^a_{n_1}(x)\}^2+\left(\frac{1}{n_1+1}\left(-x+\frac{ax}{1+x}+\frac{1}{2}\right)\right)^2\right.\\
&\qquad \left.+\frac{1}{n_2+1}\{\xi^a_{n_2}(y)\}^2+\left(\frac{1}{n_2+1}\left(-y+\frac{ay}{1+y}+\frac{1}{2}\right)\right)^2\right\}||g||_{C^2_B(I)}.
\end{aligned}
$$

Thus, we get

$$
|\overline{K}^a_{n_1,n_2}(g;x,y)-g(x,y)| \le \left\{\frac{2}{n_1+1}\{\xi^a_{n_1}(x)\}^2+\frac{2}{n_2+1}\{\xi^a_{n_2}(y)\}^2\right\}||g||_{C^2_B(I)}.
$$

Also,

$$
\begin{aligned}
|\overline{K}^a_{n_1,n_2}(f;x,y)| &\le |K^a_{n_1,n_2}(f;x,y)|+|f(x,y)|\\
&\quad +\left|f\left(\frac{1}{n_1+1}\left(n_1x+\frac{ax}{1+x}+\frac{1}{2}\right),\frac{1}{n_2+1}\left(n_2y+\frac{ay}{1+y}+\frac{1}{2}\right)\right)\right|\\
&\le 3||f||_{C_B(I)}. \qquad (3.6.4)
\end{aligned}
$$

Now, from equation (3.6.4), we get

$$
\begin{aligned}
&|K^a_{n_1,n_2}(f;x,y)-f(x,y)|\\
\leq\ & |\overline{K}^a_{n_1,n_2}(f-g;x,y)|+|\overline{K}^a_{n_1,n_2}(g;x,y)-g(x,y)|+|g(x,y)-f(x,y)|\\
&+\left|f\left(\frac{1}{n_1+1}\left(n_1x+\frac{ax}{1+x}+\frac{1}{2}\right),\frac{1}{n_2+1}\left(n_2y+\frac{ay}{1+y}+\frac{1}{2}\right)\right)-f(x,y)\right|\\
\leq\ & 3||f-g||_{C_B(I)}+||f-g||_{C_B(I)}+|\overline{K}^a_{n_1,n_2}(g;x,y)-g(x,y)|\\
&+\left|f\left(\frac{1}{n_1+1}\left(n_1x+\frac{ax}{1+x}+\frac{1}{2}\right),\frac{1}{n_2+1}\left(n_2y+\frac{ay}{1+y}+\frac{1}{2}\right)\right)-f(x,y)\right|\\
\leq\ & 4||f-g||_{C_B(I)}+\left\{\frac{2}{n_1+1}\{\xi^a_{n_1}(x)\}^2+\frac{2}{n_2+1}\{\xi^a_{n_2}(y)\}^2\right\}||g||_{C^2_B(I)}\\
&+\left|f\left(\frac{1}{n_1+1}\left(n_1x+\frac{ax}{1+x}+\frac{1}{2}\right),\frac{1}{n_2+1}\left(n_2y+\frac{ay}{1+y}+\frac{1}{2}\right)\right)-f(x,y)\right|\\
\leq\ & \Big(4||f-g||_{C_B(I)}+2M^a_{n_1,n_2}(x,y)||g||_{C^2_B(I)}\Big)\\
&+\overline{\omega}\left(f;\sqrt{\left(\frac{1}{n_1+1}\left(-x+\frac{ax}{1+x}+\frac{1}{2}\right)\right)^2+\left(\frac{1}{n_2+1}\left(-y+\frac{ay}{1+y}+\frac{1}{2}\right)\right)^2}\right).
\end{aligned}
$$

Taking the infimum on the right hand side over all $g\in C^2_B(I)$ and using (3.6.1), we obtain

$$
\begin{aligned}
&|K^a_{n_1,n_2}(f;x,y)-f(x,y)|\\
\leq\ & 4\mathcal{K}(f;M^a_{n_1,n_2}(x,y))\\
&+\overline{\omega}\left(f;\sqrt{\left(\frac{1}{n_1+1}\left(-x+\frac{ax}{1+x}+\frac{1}{2}\right)\right)^2+\left(\frac{1}{n_2+1}\left(-y+\frac{ay}{1+y}+\frac{1}{2}\right)\right)^2}\right)\\
\leq\ & M\left\{\overline{\omega}_2\left(f;\sqrt{M^a_{n_1,n_2}(x,y)}\right)+\min\{1,M^a_{n_1,n_2}(x,y)\}||f||_{C^2_B(I)}\right\}\\
&+\overline{\omega}\left(f;\sqrt{\left(\frac{1}{n_1+1}\left(-x+\frac{ax}{1+x}+\frac{1}{2}\right)\right)^2+\left(\frac{1}{n_2+1}\left(-y+\frac{ay}{1+y}+\frac{1}{2}\right)\right)^2}\right).
\end{aligned}
$$

Hence, the proof is completed. □

3.6.2 Rate of convergence of bivariate operators

Theorem 3.6.2. *Suppose that $f\in C^1_{\gamma_1,\gamma_2}(I)$ with $\gamma_1,\gamma_2\in\mathbb{N}^0$ then there exist a positive constant $M_5(\gamma_1,\gamma_2)$ such that for all $(x,y)\in I$ and $n_1,n_2\in\mathbb{N}$*

$$w_{\gamma_1,\gamma_2}(x,y)|K^a_{n_1,n_2}(f;x,y)-f(x,y)|\leq M_5(\gamma_1,\gamma_2)\left\{\parallel f_x\parallel_{\gamma_1,\gamma_2}\frac{\xi^a_{n_1}(x)}{\sqrt{n_1+1}}+\parallel f_y\parallel_{\gamma_1,\gamma_2}\frac{\xi^a_{n_2}(y)}{\sqrt{n_2+1}}\right\}.$$

Proof. Let $(x,y)\in I$ be a fixed point. Then, we have

$$f(t,z)-f(x,y) = \int_x^t f_u(u,z)du+\int_y^z f_v(x,v)dv \text{ for } (t,z)\in I$$

$$K^a_{n_1,n_2}(f(t,z);x,y)-f(x,y)$$

$$= K^a_{n_1,n_2}\left(\int_x^t f_u(u,z)du;x,y\right)+K^a_{n_1,n_2}\left(\int_y^z f_v(x,v)dv;x,y\right). \tag{3.6.5}$$

Now, by using (3.4.3), we get

$$\left|\int_x^t f_u(u,z)du\right|\leq \| f_x \|_{\gamma_1,\gamma_2}\left|\int_x^t \frac{du}{w_{\gamma_1,\gamma_2}(u,z)}\right|\leq \| f_x \|_{\gamma_1,\gamma_2}\left(\frac{1}{w_{\gamma_1,\gamma_2}(t,z)}+\frac{1}{w_{\gamma_1,\gamma_2}(x,z)}\right)|t-x|,$$

and analogously

$$\left|\int_y^z f_v(x,v)dv\right|\leq \| f_y \|_{\gamma_1,\gamma_2}\left(\frac{1}{w_{\gamma_1,\gamma_2}(x,z)}+\frac{1}{w_{\gamma_1,\gamma_2}(x,y)}\right)|z-y|.$$

By using these inequalities and from (3.4.2), we obtain for $n_1,n_2\in\mathbb{N}$

$$\begin{aligned}
&w_{\gamma_1,\gamma_2}(x,y)\left|K^a_{n_1,n_2}\left(\int_x^t f_u(u,z)du;x,y\right)\right|\\
&\leq w_{\gamma_1,\gamma_2}(x,y)K^a_{n_1,n_2}\left(\left|\int_x^t f_u(u,z)du\right|;x,y\right)\\
&\leq \| f_x \|_{\gamma_1,\gamma_2} w_{\gamma_1,\gamma_2}(x,y)\left\{K^a_{n_1,n_2}\left(\frac{|t-x|}{w_{\gamma_1,\gamma_2}(t,z)};x,y\right)+K^a_{n_1,n_2}\left(\frac{|t-x|}{w_{\gamma_1,\gamma_2}(x,z)};x,y\right)\right\}\\
&= \| f_x \|_{\gamma_1,\gamma_2} w_{\gamma_2}(y)K^a_{n_2}\left(\frac{1}{w_{\gamma_2}(z)};y\right)\\
&\quad\times\left\{w_{\gamma_1}(x)K^a_{n_1}\left(\frac{|t-x|}{w_{\gamma_1}(t)};x\right)+K^a_{n_1}(|t-x|;x)\right\},
\end{aligned}\tag{3.6.6}$$

and analogously

$$\begin{aligned}
&w_{\gamma_1,\gamma_2}(x,y)\left|K^a_{n_1,n_2}\left(\int_y^z f_v(x,v)dv;x,y\right)\right|\\
&\leq \| f_y \|_{\gamma_1,\gamma_2}\left\{w_{\gamma_2}(y)K^a_{n_2}\left(\frac{|z-y|}{w_{\gamma_2}(z)};y\right)+K^a_{n_2}(|z-y|;y)\right\}.
\end{aligned}\tag{3.6.7}$$

By the Cauchy Schwarz inequality and Remark 4, we get for $n_1\in\mathbb{N}$

$$K^a_{n_1}(|t-x|;x)\leq\left(K^a_{n_1}((t-x)^2;x)\right)^{1/2}\left(K^a_{n_1}(1;x)\right)^{1/2}\leq\frac{\xi^a_{n_1}(x)}{\sqrt{n_1+1}} \tag{3.6.8}$$

and

$$\begin{aligned}
&w_{\gamma_1}(x)K^a_{n_1}\left(\frac{|t-x|}{w_{\gamma_1}(t)};x\right)\\
&\leq w_{\gamma_1}(x)\left(K^a_{n_1}\left(\frac{(t-x)^2}{w_{\gamma_1}(t)};x\right)\right)^{1/2}\left(K^a_{n_1}\left(\frac{1}{w_{\gamma_1}(t)};x\right)\right)^{1/2}\\
&\leq M_6(\gamma_1)\frac{\xi^a_{n_1}(x)}{\sqrt{n_1+1}}, \text{ in view of Lemma 3.5.3.}
\end{aligned}\tag{3.6.9}$$

Analogously for $n_2 \in \mathbb{N}$, we have

$$K^a_{n_2}(|z-y|;y) \le \frac{\xi^a_{n_2}(y)}{\sqrt{n_2+1}}, \tag{3.6.10}$$

and

$$w_{\gamma_2}(y)K^a_{n_2}\left(\frac{|z-y|}{w_{\gamma_2}(z)};y\right) \le M_7(\gamma_2)\frac{\xi^a_{n_2}(y)}{\sqrt{n_2+1}}. \tag{3.6.11}$$

From equations (3.6.5)-(3.6.11), we obtain
$w_{\gamma_1,\gamma_2}(x,y) \mid K^a_{n_1,n_2}(f(t,z);x,y) - f(x,y) \mid$

$$\le M_8(\gamma_1,\gamma_2)\left\{\| f_x \|_{\gamma_1,\gamma_2} \frac{\xi^a_{n_1}(x)}{\sqrt{n_1+1}} + \| f_y \|_{\gamma_1,\gamma_2} \frac{\xi^a_{n_2}(y)}{\sqrt{n_2+1}}\right\}, \text{ for all } n_1,n_2 \in \mathbb{N}.$$

Thus the proof is completed. □

Theorem 3.6.3. *Suppose that $f \in C_{\gamma_1,\gamma_2}(I)$ with some $\gamma_1,\gamma_2 \in \mathbb{N}^0$. Then there exists a positive constant $M_9(\gamma_1,\gamma_2)$ such that*

$$w_{\gamma_1,\gamma_2}(x,y) \mid K^a_{n_1,n_2}(f(t,z);x,y) - f(x,y) \mid \le M_9(\gamma_1,\gamma_2)\omega\left(f;C_{\gamma_1,\gamma_2};\frac{\xi^a_{n_1}(x)}{\sqrt{n_1+1}},\frac{\xi^a_{n_2}(y)}{\sqrt{n_2+1}}\right),$$

for all $(x,y) \in I$ and $n_1,n_2 \in \mathbb{N}$.

Proof. Let $f_{h,\delta}$ be the Steklov function of $f \in C_{\gamma_1,\gamma_2}(I)$, defined by the formula

$$f_{h,\delta}(x,y) := \frac{1}{h\delta}\int_0^h du \int_0^\delta f(x+u,y+v)dv, \tag{3.6.12}$$

$(x,y) \in I$ and $h,\delta \in \mathbb{R}_+$. From (3.6.12) it follows that

$$\begin{aligned}
f_{h,\delta}(x,y) - f(x,y) &= \frac{1}{h\delta}\int_0^h du \int_0^\delta \Delta_{u,v}f(x,y)dv, \\
\frac{\partial}{\partial x} f_{h,\delta}(x,y) &= \frac{1}{h\delta}\int_0^\delta \Delta_{h,0}f(x,y+v)dv \\
&= \frac{1}{h\delta}\int_0^\delta (\Delta_{h,v}f(x,y) - \Delta_{0,v}f(x,y))dv, \\
\frac{\partial}{\partial y} f_{h,\delta}(x,y) &= \frac{1}{h\delta}\int_0^h \Delta_{0,\delta}f(x+u,y)du \\
&= \frac{1}{h\delta}\int_0^\delta (\Delta_{u,\delta}f(x,y) - \Delta_{u,0}f(x,y))du.
\end{aligned}$$

Thus, from (3.4.3) and (3.4.4) we obtain

$$\| f_{h,\delta} - f \|_{\gamma_1,\gamma_2} \le \omega(f;C_{\gamma_1,\gamma_2};h,\delta), \tag{3.6.13}$$

$$\text{(3.6.14)} \qquad \left\| \frac{\partial f_{h,\delta}}{\partial x} \right\|_{\gamma_1,\gamma_2} \leq 2h^{-1}\omega(f; C_{\gamma_1,\gamma_2}; h, \delta),$$

$$\text{(3.6.15)} \qquad \left\| \frac{\partial f_{h,\delta}}{\partial y} \right\|_{\gamma_1,\gamma_2} \leq 2\delta^{-1}\omega(f; C_{\gamma_1,\gamma_2}; h, \delta).$$

For $h, \delta \in \mathbb{R}_+$, we can write
$w_{\gamma_1,\gamma_2}(x, y) \mid K^a_{n_1,n_2}(f(t, z); x, y) - f(x, y) \mid$

$$\leq w_{\gamma_1,\gamma_2}(x, y)\{K^a_{n_1,n_2}(f(t, z) - f_{h,\delta}(t, z); x, y) + \mid K^a_{n_1,n_2}(f_{h,\delta}(t, z); x, y) - f_{h,\delta}(x, y) \mid$$
$$\text{(3.6.16)} \qquad + \mid f_{h,\delta}(x, y) - f(x, y) \mid\} := R_1 + R_2 + R_3.$$

By (3.4.3), Lemma 3.5.4 and (3.6.13) it follows that

$$\begin{aligned} R_1 \leq \| K^a_{n_1,n_2}(f - f_{h,\delta;\cdot,\cdot}) \|_{\gamma_1,\gamma_2} &\leq M_{10}(\gamma_1, \gamma_2)\|f - f_{h,\delta}\|_{\gamma_1,\gamma_2} \\ &\leq M_{10}(\gamma_1, \gamma_2)\omega(f; C_{\gamma_1,\gamma_2}; h, \delta), \end{aligned}$$

and

$$R_3 \leq \omega(f; C_{\gamma_1,\gamma_2}; h, \delta).$$

By using Theorem 3.6.2 and (3.6.14) and (3.6.15), we get

$$\begin{aligned} R_2 &\leq M_{11}(\gamma_1, \gamma_2)\left\{ \left\| \frac{\partial f_{h,\delta}}{\partial x} \right\|_{\gamma_1,\gamma_2} \frac{\xi^a_{n_1}(x)}{\sqrt{n_1 + 1}} + \left\| \frac{\partial f_{h,\delta}}{\partial y} \right\|_{\gamma_1,\gamma_2} \frac{\xi^a_{n_2}(y)}{\sqrt{n_2 + 1}} \right\} \\ &\leq 2M_{11}(\gamma_1, \gamma_2)\omega(f; C_{\gamma_1,\gamma_2}; h, \delta)\left\{ h^{-1}\frac{\xi^a_{n_1}(x)}{\sqrt{n_1 + 1}} + \delta^{-1}\frac{\xi^a_{n_2}(y)}{\sqrt{n_2 + 1}} \right\}. \end{aligned}$$

Consequently, we drive from (3.6.16)
$w_{\gamma_1,\gamma_2}(x, y) \mid K^a_{n_1,n_2}(f(t, z); x, y) - f(x, y) \mid$

$$\leq M_{12}(\gamma_1, \gamma_2)\omega(f; C_{\gamma_1,\gamma_2}; h, \delta)\left\{ 1 + h^{-1}\frac{\xi^a_{n_1}(x)}{\sqrt{n_1 + 1}} + \delta^{-1}\frac{\xi^a_{n_2}(y)}{\sqrt{n_2 + 1}} \right\},$$

for all $(x, y) \in I, n_1, n_2 \in \mathbb{N}$ and $h, \delta \in \mathbb{R}_+$.
On choosing $h = \dfrac{\xi^a_{n_1}(x)}{\sqrt{n_1 + 1}}$ and $\delta = \dfrac{\xi^a_{n_2}(y)}{\sqrt{n_2 + 1}}$, we immediately obtain the required result. □

As a consequence of Theorem 3.6.3, we have

Theorem 3.6.4. *Let $f \in C_{\gamma_1,\gamma_2}(I)$ with some $\gamma_1, \gamma_2 \in \mathbb{N}^0$. Then for every $(x, y) \in I$,*

$$\lim_{n_1,n_2\to\infty} K^a_{n_1,n_2}(f; x, y) = f(x, y).$$

Theorem 3.6.5. ***(Voronovskaja type theorem)*** *Let $f \in C^2_{\gamma_1,\gamma_2}(I)$. Then for every $(x,y) \in I$, we have*

$$\begin{aligned}\lim_{n\to\infty} n\{K^a_{n,n}(f;x,y) - f(x,y)\} &= \left(-x + \frac{ax}{1+x} + \frac{1}{2}\right) f_x(x,y) + \left(-y + \frac{ay}{1+y} + \frac{1}{2}\right) f_y(x,y) \\ &\quad + \frac{x}{2}(1+x) f_{xx}(x,y) + \frac{y}{2}(1+y) f_{yy}(x,y).\end{aligned}$$

Proof. Let $(x,y) \in I$ be fixed. By Taylor formula, we may write

$$\begin{aligned} f(u,v) &= f(x,y) + f_x(x,y)(u-x) + f_y(x,y)(v-y) \\ &\quad + \frac{1}{2}\{f_{xx}(x,y)(u-x)^2 + 2f_{xy}(x,y)(u-x)(v-y) + f_{yy}(x,y)(v-y)^2\} \\ &\quad + \psi(u,v;x,y)\sqrt{(u-x)^4 + (v-y)^4},\end{aligned}$$

where $\psi(.,.;x,y) \equiv \psi(.,.) \in C_{\gamma_1,\gamma_2}(I)$ and $\psi(x,y) = 0$. Thus, we get

$$\begin{aligned} K^a_{n,n}(f(u,v);x,y) &= f(x,y) + f_x(x,y)K^a_n(u-x;x) + f_y(x,y)K^a_n(v-y;y) \\ &\quad + \frac{1}{2}\{f_{xx}(x,y)K^a_n((u-x)^2;x) + 2f_{xy}(x,y)K^a_n(u-x;x)K^a_n(v-y;y) \\ &\quad + f_{yy}(x,y)K^a_n((v-y)^2;y)\} + K^a_{n,n}(\psi(u,v)\sqrt{(u-x)^4+(v-y)^4};x,y).\end{aligned}$$

Hence, using Lemma 3.2.4, we have

$$\begin{aligned} &\lim_{n\to\infty} n\{K^a_{n,n}(f(u,v);x,y) - f(x,y)\} \\ &= f_x(x,y)\left(-x + \frac{ax}{1+x} + \frac{1}{2}\right) + f_y(x,y)\left(-y + \frac{ay}{1+y} + \frac{1}{2}\right) \\ &\quad + \frac{1}{2}\{x(1+x)f_{xx}(x,y) + y(1+y)f_{yy}(x,y)\} \\ &\quad + \lim_{n\to\infty} nK^a_{n,n}\left(\psi(u,v)\sqrt{(u-x)^4+(v-y)^4};x,y\right). \qquad (3.6.17)\end{aligned}$$

Applying the Hölder's inequality, we have

$$\begin{aligned} &|K^a_{n,n}(\psi(u,v)\sqrt{(u-x)^4+(v-y)^4};x,y)| \\ &\le \{K^a_{n,n}(\psi^2(u,v);x,y)\}^{1/2}\left\{K^a_{n,n}\left(((u-x)^4+(v-y)^4);x,y\right)\right\}^{1/2} \\ &\le \{K^a_{n,n}(\psi^2(u,v);x,y)\}^{1/2}\{K^a_n((u-x)^4;x) + K^a_n((v-y)^4;y)\}^{1/2}.\end{aligned}$$

By Theorem 3.6.4

$$\lim_{n\to\infty} K^a_{n,n}(\psi^2(u,v);x,y) = \psi^2(x,y) = 0,$$

and from Lemma 3.2.4 (iii), for each $(x,y) \in I, K_n^a((u-x)^4;x) = O\left(\frac{1}{n^2}\right)$ and $K_n^a((v-y)^4;y) = O\left(\frac{1}{n^2}\right)$. Hence

$$\lim_{n\to\infty} nK_{n,n}^a\left(\psi(u,v)\sqrt{(u-x)^4+(v-y)^4};x,y\right) = 0. \tag{3.6.18}$$

By combining (3.6.17) and (3.6.18), we obtain the desired result. □

3.6.3 Simultaneous approximation

Theorem 3.6.6. *Let $f \in C^1_{\gamma_1,\gamma_2}(I)$. Then for every $(x,y) \in \mathbb{R}^2_+ = \mathbb{R}_+ \times \mathbb{R}_+$,*

$$\lim_{n\to\infty}\left(\frac{\partial}{\partial w}K_{n,n}^a(f;w,y)\right)_{w=x} = \frac{\partial f}{\partial x}(x,y), \tag{3.6.19}$$

$$\lim_{n\to\infty}\left(\frac{\partial}{\partial \nu}K_{n,n}^a(f;x,\nu)\right)_{\nu=y} = \frac{\partial f}{\partial y}(x,y). \tag{3.6.20}$$

Proof. We shall prove only (3.6.19) because the proof of (3.6.20) is similar. By the Taylor formula for $f \in C^1_{\gamma_1,\gamma_2}(I)$, we have

$$\begin{aligned} f(u,v) &= f(x,y) + f_x(x,y)(u-x) + f_y(x,y)(v-y) \\ &\quad + \psi(u,v;x,y)\sqrt{(u-x)^2+(v-y)^2} \text{ for } (u,v) \in I, \end{aligned}$$

where $\psi(u,v;x,y) \equiv \psi(.,.) \in C_{\gamma_1,\gamma_2}(I)$ and $\psi(x,y) = 0$.
Operating $K_{n,n}^a(.;.,y)$ to the above inequality and then by using Lemma 3.5.2, we get

$$\begin{aligned} &\left(\frac{\partial}{\partial w}K_{n,n}^a(f(u,v);w,y)\right)_{w=x} \\ &= f(x,y)\left(\frac{\partial}{\partial w}K_{n,n}^a(1;w,y)\right)_{w=x} + f_x(x,y)\left(\frac{\partial}{\partial w}K_{n,n}^a(u-x;w,y)\right)_{w=x} \\ &\quad + f_y(x,y)\left(\frac{\partial}{\partial w}K_{n,n}^a(v-y;w,y)\right)_{w=x} \\ &\quad + \left(\frac{\partial}{\partial w}K_{n,n}^a(\psi(u,v;x,y)\sqrt{(u-x)^2+(v-y)^2};w,y)\right)_{w=x}, \text{ for } (u,v) \in I \\ &= f_x(x,y)\left\{\frac{\partial}{\partial w}\left(\frac{1}{n+1}\left(nw+\frac{aw}{1+w}+\frac{1}{2}\right)\right)\right\}_{w=x} \\ &\quad + f_y(x,y)\left\{\frac{\partial}{\partial w}\left(\frac{1}{n+1}\left(ny+\frac{ay}{1+y}+\frac{1}{2}\right)\right)\right\}_{w=x} + E, \text{ say.} \end{aligned}$$

It is sufficient to prove that $E \to 0$, as $n \to \infty$.

$$\begin{aligned}
E &= (n+1)^2 \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\left(\frac{\partial}{\partial w} W^a_{n,n,k_1,k_2}(w,y)\right)_{w=x} \\
&\quad \times \int_{\frac{k_2}{n+1}}^{\frac{k_2+1}{n+1}}\int_{\frac{k_1}{n+1}}^{\frac{k_1+1}{n+1}} \psi(u,v)\sqrt{(u-x)^2+(v-y)^2}\,du\,dv \\
&= (n+1)^2 \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\frac{\{(k_1-nx)(1+x)-ax\}}{x(1+x)^2} W^a_{n,n,k_1,k_2}(x,y) \\
&\quad \times \int_{\frac{k_2}{n+1}}^{\frac{k_2+1}{n+1}}\int_{\frac{k_1}{n+1}}^{\frac{k_1+1}{n+1}} \psi(u,v)\sqrt{(u-x)^2+(v-y)^2}\,du\,dv \\
&= \frac{n(n+1)^2}{x(1+x)} \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\left(\frac{k_1}{n}-x\right) W^a_{n,n,k_1,k_2}(x,y) \\
&\quad \times \int_{\frac{k_2}{n+1}}^{\frac{k_2+1}{n+1}}\int_{\frac{k_1}{n+1}}^{\frac{k_1+1}{n+1}} \psi(u,v)\sqrt{(u-x)^2+(v-y)^2}\,du\,dv \\
&\quad -\frac{a}{(1+x)^2} K^a_{n,n}(\psi(u,v)\sqrt{(u-x)^2+(v-y)^2};x,y) := E_1+E_2, \text{ say.}
\end{aligned}$$

First, we estimate E_1 by using Schwarz inequality.

$$\begin{aligned}
E_1 &\leq \frac{n}{x(1+x)}\left(\sum_{k_1=0}^{\infty} W^a_{n,k_1}(x)\left(\frac{k_1}{n}-x\right)^2\right)^{1/2} \\
&\quad \times \left((n+1)^2 \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty} W^a_{n,n,k_1,k_2}(x,y)\int_{\frac{k_2}{n+1}}^{\frac{k_2+1}{n+1}}\int_{\frac{k_1}{n+1}}^{\frac{k_1+1}{n+1}} \psi^2(u,v)((u-x)^2+(v-y)^2)\,du\,dv\right)^{1/2} \\
&\leq \frac{n}{x(1+x)}\left(\sum_{k_1=0}^{\infty} W^a_{n,k_1}(x)\left(\frac{k_1}{n}-x\right)^2\right)^{1/2} \{K^a_{n,n}(\psi^4(u,v);x,y)(K^a_n((u-x)^4;x) \\
&\quad +2K^a_n((u-x)^2;x)(K^a_n((v-y)^2;y)+K^a_n((v-y)^4;y))\}^{1/4} \\
|E_1| &\leq M_{12}(x,y)\{K^a_{n,n}(\psi^4(u,v);x,y)\}^{1/4}, \text{ in view of Lemma 3.2.4.}
\end{aligned}$$

From Theorem 3.6.4, we obtain

$$\lim_{n\to\infty} K^a_{n,n}(\psi^4(u,v);x,y) = \psi^4(x,y) = 0, \text{ for } (x,y)\in\mathbb{R}^2_+.$$

To estimate E_2, proceeding in a manner similar to the estimate of E_1, we get $E_2 \to 0$, as $n \to \infty$. Combining the estimates of E_1 and E_2, it follows that $E \to 0$, as $n \to \infty$. This completes the proof. □

Similarly, we can prove the following theorem:

Theorem 3.6.7. *Let $f \in C^3_{\gamma_1,\gamma_2}(I)$. Then for every $(x,y) \in \mathbb{R}^2_+$, we have*

$$\lim_{n\to\infty} n\left\{ \left(\frac{\partial}{\partial w} K^a_{n,n}(f;w,y)\right)_{w=x} - \frac{\partial f}{\partial x}(x,y)\right\}$$

$$= \left(-1+\frac{a}{(1+x)^2}\right) f_x(x,y) + \left(1+\frac{ax}{1+x}\right) f_{xx}(x,y)$$
$$+\left(-y+\frac{ay}{1+y}+\frac{1}{2}\right) f_{xy}(x,y) + \frac{y}{2}(1+y) f_{xyy}(x,y) + \frac{x}{2}(1+x) f_{xxx}(x,y)$$

and

$$\lim_{n\to\infty} n\left\{ \left(\frac{\partial}{\partial \nu} K^a_{n,n}(f;x,\nu)\right)_{\nu=y} - \frac{\partial f}{\partial y}(x,y)\right\}$$

$$= \left(-1+\frac{a}{(1+y)^2}\right) f_y(x,y) + \left(1+\frac{ay}{1+y}\right) f_{yy}(x,y)$$
$$+\left(-x+\frac{ax}{1+x}+\frac{1}{2}\right) f_{xy}(x,y) + \frac{x}{2}(1+x) f_{xxy}(x,y) + \frac{y}{2}(1+y) f_{yyy}(x,y).$$

3.7 Numerical Examples

In the following, we give some numerical results regarding the approximation properties of bivariate generalized Baskakov-Kantorovich operators $K^a_{n_1,n_2}(f;x,y)$ using Matlab algorithms for construction of operators.

Let us consider the function $f : I \to \mathbb{R}^0, f(x,y) = x^2y^2 - 9xy^2 + 4x^2$. The convergence of the bivariate generalized Baskakov-Kantorovich operators to the function f is illustrated in Example 6.

Example 6. For $n_1 = n_2 = 100$; $n_1 = n_2 = 500$ and $a = 10$, the convergence of the bivariate generalized Baskakov-Kantorovich operators $K^a_{n_1,n_2}(f;x,y)$ (red) to the function $f(x,y) = x^2y^2 - 9xy^2 + 4x^2$ (yellow) is illustrated in figures 3.1 and 3.2 respectively. We observe that as the values of n_1 and n_2 increase, the error in the approximation of the function by the operators becomes smaller.

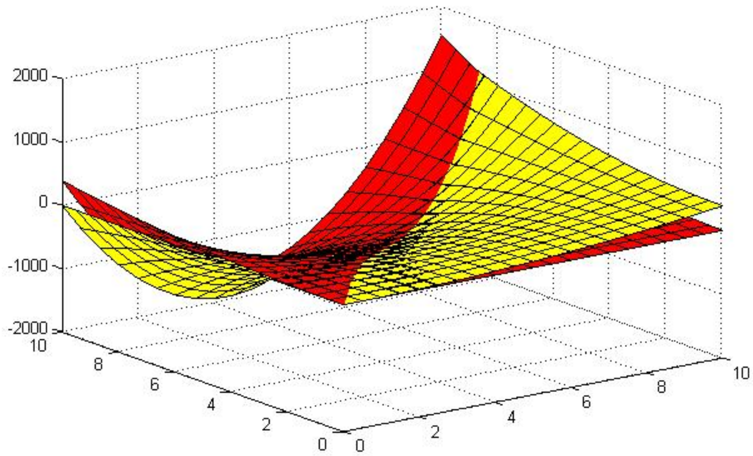

Figure 3.1 The Convergence of $K^{10}_{100,100}(f;x,y)$ (red) to $f(x,y)$ (yellow).

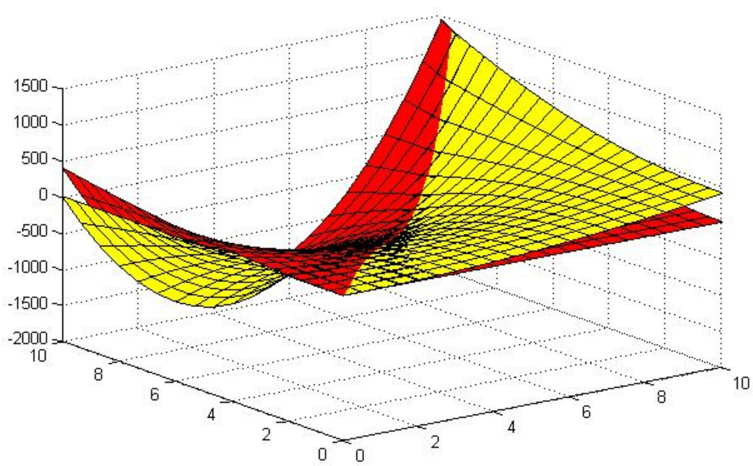

Figure 3.2 The Convergence of $K^{10}_{500,500}(f;x,y)$ (red) to $f(x,y)$ (yellow).

Example 7. For $n_1 = n_2 = 100$; $n_1 = n_2 = 500$ and $a = 10$, the comparison of the bivariate generalized Baskakov-Kantorovich operators $K^a_{n_1,n_2}(f;x,y)$ (red) and the bivariate Szász-Kantorovich operators (blue) to the function $f(x,y) = x^2y^2 - 9xy^2 + 4x^2$ (yellow) is illustrated in figures 3.3 and 3.4 respectively. It is observed that the error in the approximation of f by the bivariate Szász-Kantorovich operators is smaller than the bivariate generalized Baskakov-Kantorovich operators.

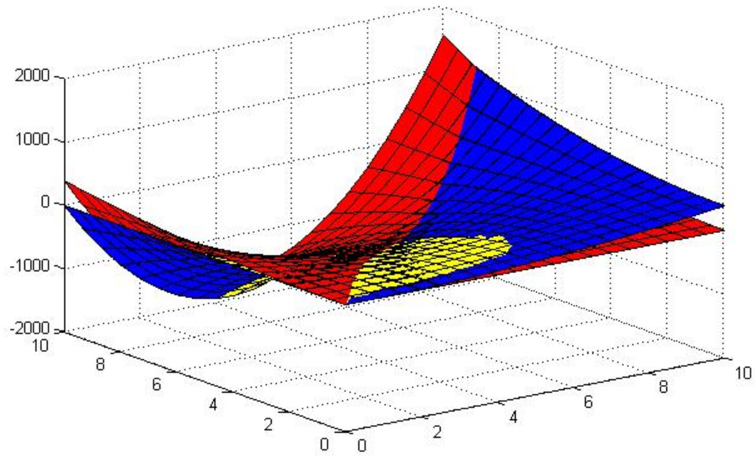

Figure 3.3 The Comparison of bivariate Szász-Kantorovich (blue) and bivariate generalized Baskakov-Kantorovich $K^{10}_{100,100}(f;x,y)$ (red) to $f(x,y)$ (yellow).

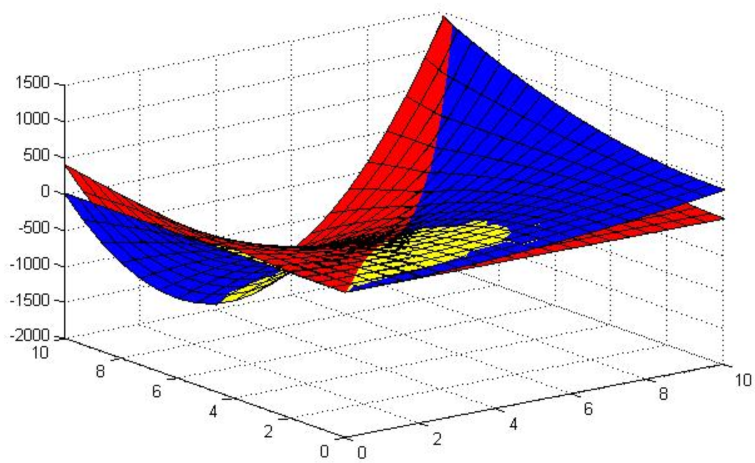

Figure 3.4 The Comparison of bivariate Szász-Kantorovich (blue) and bivariate generalized Baskakov-Kantorovich $K^{10}_{500,500}(f;x,y)$ (red) to $f(x,y)$ (yellow).

Chapter 4

Bèzier variant of the generalized Baskakov Kantorovich operators

4.1 Introduction

For $\theta \geq 1$, we now define the Bèzier variant of the operators (3.1.1) on $[0,\infty)$ as:

$$K^a_{n,\theta}(f;x) = (n+1)\sum_{k=0}^{\infty} F^{(\theta)}_{n,k,a}(x)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(t)dt, \tag{4.1.1}$$

where $F^{(\theta)}_{n,k,a}(x) = [J^a_{n,k}(x)]^{\theta} - [J^a_{n,k+1}(x)]^{\theta}$ and $J^a_{n,k}(x) = \sum_{j=k}^{\infty} W^a_{n,j}(x)$, when $k \leq n$ and 0 otherwise.

Some important properties of $J^a_{n,k}(x)$ are as follows:

- $J^a_{n,k}(x) - J^a_{n,k+1}(x) = W^a_{n,k}(x), k = 0,1,2,3\cdots;$
- $J^a_{n,0}(x) > J^a_{n,1}(x) > J^a_{n,2}(x) > \cdots > J^a_{n,n}(x) > 0, x \in [0,\infty).$

The operators $K^a_{n,\theta}(f;x)$ also admit the integral representation

$$K^a_{n,\theta}(f;x) = \int_0^{\infty} \mathcal{J}^a_{n,\theta}(x,t)f(t)dt, \tag{4.1.2}$$

where $\mathcal{J}^a_{n,\theta}(x,t) := (n+1)\sum_{k=0}^{\infty} F^{(\theta)}_{n,k,a}(x)\chi_{n,k}(t)$, where $\chi_{n,k}(t)$ is the characteristic function of the interval $\left[\frac{k}{n+1},\frac{k+1}{n+1}\right]$ with respect to $[0,\infty)$.

It is easily verified that for $\theta = 1$, the operators (4.1.1) reduce to (3.1.1), i.e.

$K^a_{n,1}(f;x) = K^a_n(f;x)$.

The purpose of this chapter is to introduce the Bèzier variant of the operators (3.1.1) and investigate a direct approximation theorem with the aid of the Ditzian-Totik modulus of smoothness and the rate of convergence for functions with derivatives of bounded variation.

4.2 Auxiliary Results

Lemma 4.2.1. *For $f \in C_B[0,\infty)$, $\| K^a_n(f) \| \leq \| f \|$.*

Proof. From (3.1.1) and Lemma 3.2.4, the proof of this lemma is immediate. Hence the details are omitted. □

Lemma 4.2.2. *Let $f \in C_B[0,\infty)$. Then, $\| K^a_{n,\theta}(f) \| \leq \theta \| f \|$.*

Proof. Using the well known inequality $|a^\beta - b^\beta| \leq \beta|a-b|$ with $0 \leq a, b \leq 1, \theta \geq 1$ and the definition of $F^{(\theta)}_{n,k,a}(x)$, we have

(4.2.1) $0 < [J^a_{n,k}(x)]^\theta - [J^a_{n,k+1}(x)]^\theta \leq \theta[J^a_{n,k}(x) - J^a_{n,k+1}(x)] = \theta\, W^a_{n,k}(x).$

Hence, from the definition of the operator $K^a_{n,\theta}(f;x)$ and Lemma 4.2.1, we get

$$\| K^a_{n,\theta}(f;x) \| \leq \theta \| K^a_n(f) \| \leq \theta \| f \| .$$

□

4.3 Main Results

4.3.1 Direct approximation theorem

In this section, first we recall the definitions of the Ditizian-Totik modulus of smoothness $\omega_{\phi^\tau}(f,t)$ and Peetre's $K-$functional [40]. Let $\phi(x) = \sqrt{x(1+x)}$ and $f \in C_B[0,\infty)$. Here, we use moduli $\omega_{\phi^\tau}(f,t)$ which unify the classical modulus $\omega(f,t), \tau = 0$ and the Ditzian-Totik modulus $\omega_\phi(f,t), \tau = 1$. For $0 \leq \tau \leq 1$, we define

$$\omega_{\phi^\tau}(f,t) = \sup_{0\leq h\leq t}\ \sup_{x\pm\frac{h\phi^\tau(x)}{2}\in[0,\infty)} \left| f\left(x + \frac{h\phi^\tau(x)}{2}\right) - f\left(x - \frac{h\phi^\tau(x)}{2}\right)\right|$$

and the $K-$functional

$$\mathcal{K}_{\phi^\tau}(f,t)=\inf_{g\in W_\tau}\{\| f-g \| +t \| \phi^\tau g' \|\},$$

where $W_\tau=\{g: g\in AC_{loc}; \| \phi^\tau g' \|<\infty\}$ and $\| \ . \ \|$ is the uniform norm on $C_B[0,\infty)$. It is proved that [40], $\omega_{\phi^\tau}(f,t)\sim \mathcal{K}_{\phi^\tau}(f,t)$, i.e. there exists a constant $M>0$ such that

$$M^{-1}\omega_{\phi^\tau}(f,t)\leq \mathcal{K}_{\phi^\tau}(f,t)\leq M\omega_{\phi^\tau}(f,t). \tag{4.3.1}$$

Lemma 4.3.1. *For $f\in W_\tau, \phi(x)=\sqrt{x(1+x)}, 0\leq\tau\leq 1$ and $t,x>0$, we have*

$$\left|\int_x^t f'(u)du\right|\leq 2^\tau\left(x^{-\tau/2}(1+t)^{-\tau/2}+\phi^{-\tau}(x)\right)|t-x|\,\|\phi^\tau f'\|.$$

Proof. By applying Hölder's inequality, we get

$$\left|\int_x^t f'(u)du\right| \ \leq \ \|\phi^\tau f'\|\left|\int_x^t\frac{du}{\phi^\tau(u)}\right|\leq \|\phi^\tau f'\|\,|t-x|^{1-\tau}\left|\int_x^t\frac{du}{\phi(u)}\right|^\tau. \tag{4.3.2}$$

Now,

$$\left|\int_x^t\frac{du}{\phi(u)}\right|\leq\left|\int_x^t\frac{du}{\sqrt{u}}\right|\left(\frac{1}{\sqrt{1+x}}+\frac{1}{\sqrt{1+t}}\right)$$

and

$$\left|\int_x^t\frac{du}{\sqrt{u}}\right|\leq\frac{2|t-x|}{\sqrt{x}}.$$

On using above estimates in (4.3.2) and then the inequality $|a+b|^r\leq|a|^r+|b|^r, 0\leq r\leq 1$, we obtain

$$\begin{aligned}\left|\int_x^t f'(u)du\right| \ &\leq \ \|\phi^\tau f'\|\,|t-x|\frac{2^\tau}{x^{\tau/2}}\left|\frac{1}{\sqrt{1+x}}+\frac{1}{\sqrt{1+t}}\right|^\tau\\ &\leq \ \|\phi^\tau f'\|\,|t-x|\frac{2^\tau}{x^{\tau/2}}\left((1+t)^{-\tau/2}+(1+x)^{-\tau/2}\right).\end{aligned}$$

□

Lemma 4.3.2. *For any $s\geq 0$ and each $x\in[0,\infty)$, there holds the inequality*

$$K_n^a((1+t)^{-s};x)\leq C(s)(1+x)^{-s},$$

where $C(s)$ is a constant dependent on s.

Proof. For $x = 0$, the result holds from (3.1.1). For $x \in (0, \infty)$, using (4.2.1) we have

$$K_n^a((1+t)^{-s};x) = (n+1)\sum_{k=0}^{\infty} W_{n,k}^a(x)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} \frac{1}{(1+t)^s}dt.$$

We first observe that

$$(n+1)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} \frac{1}{(1+t)^s}dt \leq \left(1+\frac{k}{n+1}\right)^{-s}.$$

Thus, we get

$$K_n^a((1+t)^{-s};x) \leq \frac{1}{(1+x)^s}\sum_{k=0}^{\infty} \frac{e^{\frac{-ax}{1+x}} p_k(n,a)x^k}{k!(1+x)^{n+k-s}}\left(1+\frac{k}{n+1}\right)^{-s}. \tag{4.3.3}$$

On using the ratio test, we note that for each $x > 0$, the series on the right hand side (4.3.3) is convergent. This proves the desired result. □

Theorem 4.3.3. *For $f \in C_B[0,\infty)$, we have*

$$\left|K_{n,\theta}^a(f;x) - f(x)\right| \leq C\omega_{\phi^\tau}\left(f, \frac{\phi^{1-\tau}(x)}{\sqrt{n+1}}\right). \tag{4.3.4}$$

Proof. By the definition of $\mathcal{K}_{\phi^\tau}(f,t)$, for fixed n, x, τ we can choose $g = g_{n,x,\tau} \in W_\tau$ such that

$$||f-g|| + \frac{\phi^{1-\tau}(x)}{\sqrt{n+1}}||\phi^\tau g'|| \leq 2\mathcal{K}_{\phi^\tau}\left(f, \frac{\phi^{1-\tau}(x)}{\sqrt{n+1}}\right). \tag{4.3.5}$$

Applying Lemma 4.2.1, we may write

$$\left|K_{n,\theta}^a(f;x) - f(x)\right| \leq 2||f-g|| + |K_{n,\theta}^a(g;x) - g(x)|. \tag{4.3.6}$$

Using the representation $g(t) = g(x) + \int_x^t g'(u)du$ and Lemma 4.3.1, we obtain

$$\begin{aligned}\left|K_{n,\theta}^a(g;x) - g(x)\right| &= \left|K_{n,\theta}^a\left(\int_x^t g'(u)du;x\right)\right| \\ &\leq 2^\tau||\phi^\tau g'||\left\{\phi^{-\tau}(x)K_{n,\theta}^a(|t-x|;x) + x^{-\tau/2}K_{n,\theta}^a\left(\frac{|t-x|}{(1+t)^{\tau/2}};x\right)\right\}. \qquad (4.3.7)\end{aligned}$$

By using Cauchy-Schwarz inequality, (4.2.1) and Remark 3, we have

$$\begin{aligned}K_{n,\theta}^a(|t-x|;x) &\leq \left(K_{n,\theta}^a((t-x)^2;x)\right)^{1/2} \\ &\leq \frac{\sqrt{\theta\lambda}\phi(x)}{\sqrt{n+1}}. \qquad (4.3.8)\end{aligned}$$

Similarly, from Lemma 4.3.2, we get

$$\begin{aligned} K_{n,\theta}^a\left(\frac{|t-x|}{(1+t)^{\tau/2}};x\right) &\leq \theta K_n^a\left(\frac{|t-x|}{(1+t)^{\tau/2}};x\right) \\ &\leq \theta\left(K_n^a((t-x)^2;x)\right)^{1/2}\left(K_n^a((1+t)^{-\tau};x)\right)^{1/2} \\ &\leq C_1\theta\frac{\sqrt{\lambda}\phi(x)}{\sqrt{n+1}}(1+x)^{-\tau/2}. \qquad (4.3.9) \end{aligned}$$

By combining (4.3.7)-(4.3.9), we get

$$\left|K_{n,\theta}^a(g;x)-g(x)\right| \leq C_2||\phi^\tau g'||\frac{\phi^{1-\tau}(x)}{\sqrt{n+1}}. \qquad (4.3.10)$$

Using (4.3.1), (4.3.5)-(4.3.6) and (4.3.10), we obtain the required relation (4.3.4). □

4.3.2 Rate of convergence

Lemma 4.3.4. *Let $x \in (0,\infty)$, then for $\theta \geq 1, \lambda > 2$ and sufficiently large n, we have*

(i) $\alpha_{n,\theta}^a(x,y) = \int_0^y \mathcal{J}_{n,\theta}^a(x,t)dt \leq \dfrac{\theta\lambda}{n+1}\dfrac{\phi^2(x)}{(x-y)^2},\ 0 \leq y < x;$

(ii) $1-\alpha_{n,\theta}^a(x,z) = \int_z^\infty \mathcal{J}_{n,\theta}^a(x,t)dt \leq \dfrac{\theta\lambda}{n+1}\dfrac{\phi^2(x)}{(z-x)^2},\ x < z < \infty.$

Proof. (i) From (4.2.1) and Remark 3, we get

$$\begin{aligned} \alpha_{n,\theta}^a(x,y) &= \int_0^y \mathcal{J}_{n,\theta}^a(x,t)dt \leq \int_0^y\left(\frac{x-t}{x-y}\right)^2\mathcal{J}_{n,\theta}^a(x,t)dt \\ &\leq K_{n,\theta}^a((t-x)^2;x)\ (x-y)^{-2} \leq \theta K_n^a((t-x)^2;x)(x-y)^{-2} \\ &\leq \theta\frac{\lambda}{n+1}\frac{\phi^2(x)}{(x-y)^2}. \end{aligned}$$

The proof of (ii) is similar, hence it is omitted.

Theorem 4.3.5. *Let $f \in DBV_\gamma(0,\infty), \theta \geq 1$ and let $\bigvee_c^d(f_x')$ be the total variation of f_x' on $[c,d] \subset (0,\infty)$. Then, for every $x \in (0,\infty)$ and sufficiently large n, we have*

$$\begin{aligned} |K_{n,\theta}^a(f;x)-f(x)| &\leq \frac{\theta^{1/2}}{\theta+1}\sqrt{\frac{\lambda x(1+x)}{n+1}}|f'(x+)+\theta f'(x-)| \\ &+\frac{\theta^{3/2}}{\theta+1}\sqrt{\frac{\lambda x(1+x)}{n+1}}|f'(x+)-f'(x-)| \\ &+\theta\frac{\lambda(1+x)}{n+1}\sum_{k=1}^{[\sqrt{n}]}\bigvee_{x-(x/k)}^{x}(f_x')+\frac{x}{\sqrt{n}}\bigvee_{x-(x/\sqrt{n})}^{x}(f_x') \\ &+\theta\frac{\lambda(1+x)}{n+1}\sum_{k=0}^{[\sqrt{n}]}\bigvee_{x}^{x+(x/k)}(f_x')+\frac{x}{\sqrt{n}}\bigvee_{x}^{x+(x/\sqrt{n})}(f_x'), \end{aligned}$$

where $\lambda > 2$, *and the auxiliary function* f'_x *is defined by*

$$f'_x(t) = \begin{cases} f'(t) - f'(x-), & 0 \leq t < x \\ 0, & t = x \\ f'(t) - f'(x+), & x < t < \infty. \end{cases}$$

Proof. From the definition of the function $f'_x(t)$, for any $f \in DBV_\gamma(0,\infty)$, we may write

$$\begin{aligned} f'(t) &= \frac{1}{\theta+1}\Big(f'(x+) + \theta f'(x-)\Big) + f'_x(t) \\ &+\frac{1}{2}\Big(f'(x+) - f'(x-)\Big)\Big(sgn(t-x) + \frac{\theta-1}{\theta+1}\Big) \\ &+\delta_x(t)\Big(f'(x) - \frac{1}{2}\Big(f'(x+) + f'(x-)\Big)\Big), \end{aligned} \tag{4.3.11}$$

where

$$\delta_x(t) = \begin{cases} 1\ , & x = t \\ 0\ , & x \neq t. \end{cases}$$

From (4.1.2) and the fact that $\int_0^\infty \mathcal{J}^a_{n,\theta}(x,t)dt = K^a_{n,\theta}(e_0;x) = 1$, we get

$$\begin{aligned} K^a_{n,\theta}(f;x) - f(x) &= \int_0^\infty [f(t) - f(x)]\mathcal{J}^a_{n,\theta}(x,t)dt \\ &= \int_0^\infty \Big(\int_x^t f'(u)du\Big)\mathcal{J}^a_{n,\theta}(x,t)dt. \end{aligned} \tag{4.3.12}$$

It is clear that

$$\int_0^\infty \mathcal{J}^a_{n,\theta}(x,t)\int_x^t \Big[f'(x) - \frac{1}{2}\Big(f'(x+) + f'(x-)\Big)\Big]\delta_x(u)dudt = 0.$$

Thus, from (4.3.11) and (4.3.12), for sufficiently large n, we have

$$\begin{aligned} &\Big|\int_0^\infty \Big(\int_x^t \frac{1}{\theta+1}\Big(f'(x+) + \theta f'(x-)\Big)du\Big)\mathcal{J}^a_{n,\theta}(x,t)dt\Big| \\ &\leq \frac{1}{\theta+1}\Big|f'(x+) + \theta f'(x-)\Big|K^a_{n,\theta}(|t-x|;x) \\ &= \frac{\sqrt{\theta}}{\theta+1}\Big|f'(x+) + \theta f'(x-)\Big|\sqrt{\frac{\lambda}{n+1}}\phi(x) \end{aligned} \tag{4.3.13}$$

and by applying Cauchy-Schwarz inequality, we obtain

$$\left|\int_0^\infty \left(\int_x^t \frac{1}{2}\Big(f'(x+)-f'(x-)\Big)\left(sgn(u-x)+\frac{\theta-1}{\theta+1}\right)du\right)\mathcal{J}_{n,\theta}^a(x,t)dt\right|$$
$$\begin{aligned}
&\leq \frac{\theta}{\theta+1}\left|f'(x+)-f'(x-)\right|\int_0^\infty |t-x|\,\mathcal{J}_{n,\theta}^a(x,t)dt\\
&= \frac{\theta}{\theta+1}\left|f'(x+)-f'(x-)\right| K_{n,\theta}^a\left(|t-x|\,;x\right)\\
&\leq \frac{\theta}{\theta+1}\left|f'(x+)-f'(x-)\right|\left(K_{n,\theta}^a\left((t-x)^2;x\right)\right)^{1/2}\\
&\leq \frac{\theta^{3/2}}{\theta+1}\left|f'(x+)-f'(x-)\right|\sqrt{\frac{\lambda}{n+1}}\,\phi(x). \qquad (4.3.14)
\end{aligned}$$

By using Lemma 3.2.4, Remark 3 and considering (4.3.12)-(4.3.14) we obtain the following estimate

$$\begin{aligned}
\left|K_{n,\theta}^a(f;x)-f(x)\right| \leq\ & \left|U_{n,\theta}^a(f'_x,x)+V_{n,\theta}^a(f'_x,x)\right|\\
&+\frac{\sqrt{\theta}}{\theta+1}\left|f'(x+)+\theta f'(x-)\right|\sqrt{\frac{\lambda}{n+1}}\,\phi(x)\\
&+\frac{\theta^{3/2}}{\theta+1}\left|f'(x+)-f'(x-)\right|\sqrt{\frac{\lambda}{n+1}}\,\phi(x), \qquad (4.3.15)
\end{aligned}$$

where

$$U_{n,\theta}^a(f'_x,x)=\int_0^x\left(\int_x^t f'_x(u)du\right)\mathcal{J}_{n,\theta}^a(x,t)dt,$$

and

$$V_{n,\theta}^a(f'_x,x)=\int_x^\infty\left(\int_x^t f'_x(u)du\right)\mathcal{J}_{n,\theta}^a(x,t)dt.$$

Now, let us estimate the terms $U_{n,\theta}^a(f'_x,x)$ and $V_{n,\theta}^a(f'_x,x)$. Since $\int_c^d d_t\alpha_{n,\theta}^a(x,t)\leq 1$, for all $[c,d]\subseteq(0,\infty)$, using integration by parts and applying Lemma 4.3.4 with $y=x-(x/\sqrt{n})$, we have

$$\begin{aligned}
\left|U_{n,\theta}^a(f'_x,x)\right| &= \left|\int_0^x\int_x^t\left(f'_x(u)du\right)d_t\alpha_{n,\theta}^a(x,t)\right|\\
&= \left|\int_0^x \alpha_{n,\theta}^a(x,t)f'_x(t)dt\right|\\
&\leq \left(\int_0^y+\int_y^x\right)|f'_x(t)|\left|\alpha_{n,\theta}^a(x,t)\right|dt\\
&\leq \theta\frac{\lambda\phi^2(x)}{n+1}\int_0^y\bigvee_t^x(f'_x)(x-t)^{-2}dt+\int_y^x\bigvee_t^x(f'_x)dt\\
&\leq \theta\frac{\lambda\phi^2(x)}{n+1}\int_0^y\bigvee_t^x(f'_x)(x-t)^{-2}dt+\frac{x}{\sqrt{n}}\bigvee_{x-(x/\sqrt{n})}^{x}(f'_x).
\end{aligned}$$

By the substitution of $u = x/(x-t)$, we obtain

$$\begin{aligned}
\theta\frac{\lambda\phi^2(x)}{n+1}\int_0^{x-(x/\sqrt{n})}(x-t)^{-2}\bigvee_t^x(f'_x)dt &= \theta\frac{\lambda(1+x)}{n+1}\int_1^{\sqrt{n}}\bigvee_{x-x/u}^{x}(f'_x)du \\
&\le \theta\frac{\lambda(1+x)}{n+1}\sum_{k=1}^{[\sqrt{n}]}\int_k^{k+1}\bigvee_{x-x/u}^{x}(f'_x)du \\
&\le \theta\frac{\lambda(1+x)}{n+1}\sum_{k=1}^{[\sqrt{n}]}\bigvee_{x-x/k}^{x}(f'_x).
\end{aligned}$$

Hence we reach the following result

$$\left|U^a_{n,\theta}(f'_x,x)\right| \le \theta\frac{\lambda(1+x)}{n+1}\sum_{k=1}^{[\sqrt{n}]}\bigvee_{x-x/k}^{x}(f'_x)+\frac{x}{\sqrt{n}}\bigvee_{x-(x/\sqrt{n})}^{x}(f'_x). \tag{4.3.16}$$

Again, using integration by parts and applying Lemma 4.3.4 with $z = x+(x/\sqrt{n})$, we have

$$\begin{aligned}
|V^a_{n,\theta}(f'_x,x)| &= \left|\int_x^\infty\left(\int_x^t f'_x(u)du\right)d_t(1-\alpha^a_{n,\theta}(x,t))dt\right| \\
&= \left|\int_x^z f'_x(t)(1-\alpha^a_{n,\theta}(x,t))dt+\int_z^\infty f'_x(t)(1-\alpha^a_{n,\theta}(x,t))dt\right| \\
&< \theta\frac{\lambda\phi^2(x)}{n+1}\int_z^\infty\bigvee_x^t(f'_x)(t-x)^{-2}dt+\int_x^z\bigvee_x^t(f'_x)dt \\
&\le \theta\frac{\lambda\phi^2(x)}{n+1}\int_{x+(x/\sqrt{n})}^\infty\bigvee_x^t(f'_x)(t-x)^{-2}dt+\frac{x}{\sqrt{n}}\bigvee_x^{x+x/\sqrt{n}}(f'_x). \qquad (4.3.17)
\end{aligned}$$

By the substitution of $u = x/(t-x)$, we get

$$\begin{aligned}
\theta\frac{\lambda\phi^2(x)}{n+1}\int_{x+(x/\sqrt{n})}^\infty\bigvee_x^t(f'_x)(t-x)^{-2}dt &= \theta\frac{\lambda\phi^2(x)}{x(n+1)}\int_0^{\sqrt{n}}\bigvee_x^{x+x/u}(f'_x)du \\
&\le \theta\frac{\lambda(1+x)}{n+1}\sum_{k=1}^{[\sqrt{n}]}\int_k^{k+1}\bigvee_x^{x+x/u}(f'_x)du \\
&\le \theta\frac{\lambda(1+x)}{n+1}\sum_{k=1}^{[\sqrt{n}]}\bigvee_x^{x+x/k}(f'_x). \qquad (4.3.18)
\end{aligned}$$

Now, combining (4.3.17)-(4.3.18), we obtain

$$|V^a_{n,\theta}(f'_x,x)| \le \theta\frac{\lambda(1+x)}{n+1}\sum_{k=1}^{[\sqrt{n}]}\bigvee_x^{x+x/k}(f'_x)+\frac{x}{\sqrt{n}}\bigvee_x^{x+(x/\sqrt{n})}(f'_x). \tag{4.3.19}$$

By collecting the estimates (4.3.15), (4.3.16) and (4.3.19), we get the required result. This completes the proof of theorem. □

Chapter 5

General Gamma type operators based on q-integers

5.1 Introduction

In [113], Mazhar investigated and studied some approximation properties of the following sequence of linear positive operators

$$\begin{aligned} H_n(f;x) &= \int_0^\infty \int_0^\infty g_n(x,u) g_{n-1}(u,t) f(t) du\, dt \\ &= \frac{(2n)! x^{n+1}}{n!(n-1)!} \int_0^\infty \frac{t^{n-1}}{(x+t)^{2n+1}} f(t) dt,\ n > 1,\ x > 0, \end{aligned}$$

where $g_n(x,u) = \frac{x^{n+1}}{n!} e^{-xu} u^n$. Recently, Karsli [97] considered a modification and studied the rate of convergence of these operators for the functions with derivatives of bounded variation.

$$\begin{aligned} L_n(f;x) &= \int_0^\infty \int_0^\infty g_{n+2}(x,u) g_n(u,t) f(t) du\, dt \\ &= \frac{(2n+3)! x^{n+3}}{n!(n+2)!} \int_0^\infty \frac{t^n}{(x+t)^{2n+4}} f(t) dt,\ x > 0. \end{aligned}$$

Later on, Karsli and Özarslan [98] established some local and global approximation results for the operators $L_n(f;x)$.

In 2007, Mao [111] defined the following generalized Gamma type operators

$$
\begin{aligned}
(M_{n,k}f)(x) &= \int_0^\infty\int_0^\infty g_n(x,u)g_{n-k}(u,t)f(t)du\,dt \\
&= \frac{(2n-k+1)!x^{n+1}}{n!(n-k)!}\int_0^\infty \frac{t^{n-k}}{(x+t)^{2n-k+2}}f(t)\,dt,\ x>0, \qquad (5.1.1)
\end{aligned}
$$

which include the operators $H_n(f;x)$ for $k=1$ and the operators $L_{n-2}(f;x)$ for $k=2$.

For $f\in D_\vartheta[0,\infty)$, $0<q<1$, $0\le\alpha\le\beta$ and each positive integer n, we introduce the following Stancu type modification of the operators (5.1.1) based on $q-$integers:

$$
\begin{aligned}
\left(M_{n,k,q}^{(\alpha,\beta)}f\right)(x) &= \frac{[2n-k+1]_q!\left(q^{\frac{2n-k+1}{2}}x\right)^{n+1}}{[n]_q![n-k]_q!}q^{\frac{(n-k)(n-k+1)}{2}} \\
&\times\int_0^{\infty/A}\frac{t^{n-k}}{(q^{\frac{2n-k+1}{2}}x+t)^{2n-k+2}}f\left(\frac{[n]_qt+\alpha}{[n]_q+\beta}\right)d_qt. \qquad (5.1.2)
\end{aligned}
$$

For $\alpha=\beta=0$, we denote $\left(M_{n,k,q}^{(\alpha,\beta)}f\right)(x)$ by $(M_{n,k,q}f)(x)$. Clearly, if $q\to1^-$ and $\alpha=\beta=0$, the operators defined by (5.1.2) reduce to the operators given by (5.1.1).

Very recently, the case $k=2$, namely $(M_{n,2,q}f)(x)$ was introduced and studied by Cai and Zeng [34]. Subsequently, Zhao et al. [161] discussed the Stancu type generalization of $(M_{n,2,q}f)(x)$, i.e. $\left(M_{n,2,q}^{(\alpha,\beta)}f\right)(x)$.

In the present chapter, we study the basic convergence theorem, Voronovskaja type asymptotic formula, local approximation, rate of convergence, weighted approximation, point-wise estimation and $A-$statistical convergence of the operators (5.1.2). Further, to obtain better approximation we also modify the operators (5.1.2) by using King type approach.

5.2 Moment Estimates

Lemma 5.2.1. *For any $m,k\in\mathbb{N}^0$ satisfying $m,k\le n$ and $0<q<1$, one has*

$$(M_{n,k,q}t^m)(x)=q^{\frac{m}{2}(k-m)}\frac{[n-k+m]_q![n-m]_q!}{[n]_q![n-k]_q!}x^m.$$

Proof. We observe that for every $x \in [0,\infty)$, using (0.2.1) and (0.2.2), we obtain
$\left(M_{n,k,q}t^m\right)(x)$

$$= \frac{[2n-k+1]_q!\left(q^{\frac{2n-k+1}{2}}x\right)^{n+1}}{[n]_q![n-k]_q!}q^{(n-k)(n-k+1)/2}\int_0^{\infty/A}\frac{t^{n-k}}{\left(q^{\frac{2n-k+1}{2}}x+t\right)^{2n-k+2}}t^m d_qt$$

$$= \frac{[2n-k+1]_q!\left(q^{\frac{2n-k+1}{2}}x\right)^{n+1}}{[n]_q![n-k]_q!}q^{\frac{(n-k)(n-k+1)}{2}}\int_0^{\infty/A}\frac{t^{n-k+m}}{\left(q^{\frac{2n-k+1}{2}}x\right)^{2n-k+2}\left(1+\frac{t}{q^{\frac{2n-k+1}{2}}x}\right)^{2n-k+2}}d_qt$$

$$= \frac{[2n-k+1]_q!}{[n]_q![n-k]_q!\left(q^{n+1}x\right)^{n-k+1}}\int_0^{\infty/A}\frac{t^{n-k+m}}{\left(1+\frac{t}{q^{\frac{2n-k+1}{2}}x}\right)^{2n-k+2}}d_qt$$

$$= \frac{[2n-k+1]_q!q^{(2n-k+1)(n-k+m+1)/2}}{[n]_q![n-k]_q!q^{(n-k+1)(n+1)/2}}x^m\int_0^{\infty/A}\frac{\left(\frac{t}{q^{\frac{2n-k+1}{2}}x}\right)^{n-k+m}}{\left(1+\frac{t}{q^{\frac{2n-k+1}{2}}x}\right)^{2n-k+2}}d_q\left(\frac{t}{q^{\frac{2n-k+1}{2}}x}\right)$$

$$= \frac{[2n-k+1]_q!q^{(2n-k+1)(n-k+m+1)/2}}{[n]_q![n-k]_q!q^{(n-k+1)(n+1)/2}}\frac{B_q(n-k+m+1,n-m+1)}{K(A,n-k+m+1)}x^m$$

$$= q^{\frac{m}{2}(k-m)}\frac{[n-k+m]_q![n-m]_q!}{[n]_q![n-k]_q!}x^m.$$

□

Lemma 5.2.2. *For the operators* $\left(M_{n,k,q}f\right)(x)$ *and* $\left(M_{n,k,q}^{(\alpha,\beta)}f\right)(x)$ *as defined in (5.1.2), the following equalities hold:*

1. $\left(M_{n,k,q}t^0\right)(x) = 1;$
2. $\left(M_{n,k,q}t\right)(x) = q^{\frac{(k-1)}{2}}\frac{[n-k+1]_q}{[n]_q}x;$
3. $\left(M_{n,k,q}t^2\right)(x) = q^{k-2}\frac{[n-k+2]_q[n-k+1]_q}{[n-1]_q[n]_q}x^2$, *for* $n>1$;
4. $\left(M_{n,k,q}^{(\alpha,\beta)}t^0\right)(x) = 1;$
5. $\left(M_{n,k,q}^{(\alpha,\beta)}t\right)(x) = q^{\frac{(k-1)}{2}}\frac{[n-k+1]_q}{[n]_q+\beta}x + \frac{\alpha}{[n]_q+\beta};$
6. $\left(M_{n,k,q}^{(\alpha,\beta)}t^2\right)(x) = \left(\frac{[n]_q}{[n]_q+\beta}\right)^2\left\{\frac{[n-k+2]_q[n-k+1]_q}{[n]_q[n-1]_q}q^{k-2}x^2 + \frac{2\alpha[n-k+1]_q}{[n]_q^2}q^{\frac{k-1}{2}}x + \frac{\alpha^2}{[n]_q^2}\right\}$, *for* $n>1$.

Proof. The proof of this lemma is an immediate consequence of Lemma 5.2.1. Hence the details are omitted. □

Remark 5. For every $q \in (0,1)$, we have

$$(M_{n,k,q}(t-x))(x) = x\left\{\frac{[n-k+1]_q}{[n]_q} q^{\frac{k-1}{2}} - 1\right\},$$

$$\begin{aligned}\left(M_{n,k,q}(t-x)^2\right)(x) &= x^2\left\{\frac{[n-k+2]_q[n-k+1]_q}{[n]_q[n-1]_q} q^{k-2} - 2\frac{[n-k+1]_q}{[n]_q} q^{\frac{k-1}{2}} + 1\right\}, \text{ for } n>1 \\ &:= \gamma_{n,k,q}(x), \text{ say}\end{aligned}$$

and

$$\begin{aligned}\left(M_{n,k,q}^{(\alpha,\beta)}(t-x)\right)(x) &= \left(q^{\frac{k-1}{2}}\frac{[n-k+1]_q}{[n]_q+\beta} - 1\right)x + \frac{\alpha}{[n]_q+\beta}, \\ \left(M_{n,k,q}^{(\alpha,\beta)}(t-x)^2\right)(x) &= \left(q^{k-2}\frac{[n-k+2]_q[n-k+1]_q[n]_q}{[n-1]_q([n]_q+\beta)^2} - q^{\frac{(k-1)}{2}}\frac{2[n-k+1]_q}{([n]_q+\beta)} + 1\right)x^2 \\ &\quad +2\alpha\left(q^{\frac{(k-1)}{2}}\frac{[n-k+1]_q}{([n]_q+\beta)^2} - \frac{1}{[n]_q+\beta}\right)x + \frac{\alpha^2}{([n]_q+\beta)^2}, \text{ for } n>1 \\ &:= \gamma_{n,k,q}^{(\alpha,\beta)}(x), \text{ say.}\end{aligned}$$

Lemma 5.2.3. *For $f \in C_B[0,\infty)$, one has*

$$\left\|M_{n,k,q}^{(\alpha,\beta)} f\right\| \leq \| f \| .$$

Proof. In view of (5.1.2) and Lemma 5.2.2, the proof of this lemma easily follows. □

5.3 Main Results

Theorem 5.3.1. *Let $q_n \in (0,1)$, such that $q_n^n \to 0$ as $n \to \infty$ and $J > 0$. Then for each $f \in D_\vartheta[0,\infty)$, the sequence $\left\{\left(M_{n,k,q}^{(\alpha,\beta)} f\right)(x)\right\}$ converges to f uniformly on $[0,J]$ if and only if $\lim\limits_{n\to\infty} q_n = 1$.*

Proof. First, we assume that $\lim\limits_{n\to\infty} q_n = 1$. We have to show that $\left\{\left(M_{n,k,q}^{(\alpha,\beta)} f\right)(x)\right\}$ converges to f uniformly on $[0,J]$. From Lemma 5.2.1, we see that $\left(M_{n,k,q}^{(\alpha,\beta)} t^0\right)(x) \to 1$, $\left(M_{n,k,q}^{(\alpha,\beta)} t^1\right)(x) \to x$, $\left(M_{n,k,q}^{(\alpha,\beta)} t^2\right)(x) \to x^2$, uniformly on $[0,J]$ as $n \to \infty$. Therefore, the well known property of Korovkin theorem implies that $\left\{\left(M_{n,k,q}^{(\alpha,\beta)} f\right)(x)\right\}$ converges to f uniformly on $[0,J]$ provided $f \in D_\vartheta[0,\infty)$.

We show the converse part by contradiction. Assume that q_n does not converge to 1. Then, it must contain a subsequence $\{q_{n_k}\}$ such that $q_{n_k} \in (0,1)$, $q_{n_k} \to a \in [0,1)$ as $k \to \infty$. Thus, $\dfrac{1}{[n_k+s]_{q_{n_k}}} = \dfrac{1-q_{n_k}}{1-(q_{n_k})^{n_k+s}} \to (1-a)$ as $k \to \infty$. Choosing $n = n_k$, $q = q_{n_k}$ in $\left(M_{n,k,q}^{(\alpha,\beta)} t^2\right)(x)$, from Lemma 5.2.1, we have $\left(M_{n,k,q}^{(\alpha,\beta)} t^2\right)(x) \nrightarrow x^2$ as $k \to \infty$, which leads us to a contradiction. Hence, $\lim\limits_{n\to\infty} q_n = 1$. This completes the proof. □

Theorem 5.3.2. ***(Voronovskaja type theorem)*** *Let $f \in D_\vartheta[0,\infty)$ and $q_n \in (0,1)$ be a sequence such that $q_n \to 1$ and $q_n^n \to 0$ as $n \to \infty$. Suppose that $f''(x)$ exists at a point $x \in [0,\infty)$, then we have*

$$\lim_{n\to\infty} [n]_{q_n} \left(\left(M_{n,k,q_n}^{(\alpha,\beta)} f \right)(x) - f(x) \right) = \left(\alpha - \left(\beta + \frac{k-1}{2} \right) x \right) f'(x) + \frac{x^2}{2} f''(x).$$

Proof. By the Taylor's formula, we may write

$$f(t) = f(x) + (t-x)f'(x) + \frac{1}{2} f''(x)(t-x)^2 + r(t,x)(t-x)^2, \tag{5.3.1}$$

where $r(t,x)$ is the Peano form of the remainder and $\lim\limits_{t\to x} r(t,x) = 0$.
Applying $\left(M_{n,k,q}^{(\alpha,\beta)} f\right)(x)$ to the both sides of (5.3.1), we get

$$\begin{aligned}
&[n]_{q_n} \left(\left(M_{n,k,q_n}^{(\alpha,\beta)} f \right)(x) - f(x) \right) \\
&\quad = [n]_{q_n} f'(x) \left(M_{n,k,q_n}^{(\alpha,\beta)} (t-x) \right)(x) + \frac{1}{2} [n]_{q_n} f''(x) \left(M_{n,k,q_n}^{(\alpha,\beta)} (t-x)^2 \right)(x) \\
&\qquad + [n]_{q_n} \left(M_{n,k,q_n}^{(\alpha,\beta)} (t-x)^2 r(t,x) \right)(x).
\end{aligned}$$

In view of Remark 5, we have

$$\lim_{n\to\infty} [n]_{q_n} \left(M_{n,k,q_n}^{(\alpha,\beta)} (t-x) \right)(x) = \alpha - \left(\beta + \frac{k-1}{2} \right) x \tag{5.3.2}$$

and

$$\lim_{n\to\infty} [n]_{q_n} \left(M_{n,k,q_n}^{(\alpha,\beta)} (t-x)^2 \right)(x) = x^2. \tag{5.3.3}$$

Now, we shall show that

$$[n]_{q_n} \left(M_{n,k,q_n}^{(\alpha,\beta)} (t-x)^2 r(t,x) \right)(x) \to 0$$

when $n \to \infty$. By using Cauchy-Schwarz inequality, we have

$$\begin{aligned}
&\left(M_{n,k,q_n}^{(\alpha,\beta)} (t-x)^2 r(t,x) \right)(x) \\
&\quad \le \sqrt{\left(M_{n,k,q_n}^{(\alpha,\beta)} r^2(t,x) \right)(x)} \sqrt{\left(M_{n,k,q_n}^{(\alpha,\beta)} (t-x)^4 \right)(x)}.
\end{aligned} \tag{5.3.4}$$

We observe that $r^2(x,x)=0$ and $r^2(\cdot,x)\in D_\vartheta[0,\infty)$. Then, it follows from Theorem 5.3.1 that

$$\lim_{n\to\infty}\left(M_{n,k,q_n}^{(\alpha,\beta)}r^2(t,x)\right)(x)=r^2(x,x)=0, \tag{5.3.5}$$

in view of the fact that $\left(M_{n,k,q_n}^{(\alpha,\beta)}(t-x)^4\right)(x)=O\left(\frac{1}{[n]_{q_n}^2}\right)$. Now, from (5.3.4) and (5.3.5), we get

$$\lim_{n\to\infty}[n]_{q_n}\left(M_{n,k,q_n}^{(\alpha,\beta)}(t-x)^2r(t,x)\right)(x)=0. \tag{5.3.6}$$

From (5.3.2), (5.3.3) and (5.3.6), we get the required result. □

Theorem 5.3.3. ***(Voronovskaja type theorem)*** *Let $f\in D_\vartheta[0,\infty)$ and $q_n\in(0,1)$ be a sequence such that $q_n\to 1$ and $q_n^n\to 0$ as $n\to\infty$. If $f''(x)$ exists on $[0,\infty)$, then*

$$\lim_{n\to\infty}[n]_{q_n}\left(\left(M_{n,k,q_n}^{(\alpha,\beta)}f\right)(x)-f(x)\right)=\left(\alpha-\left(\beta+\frac{k-1}{2}\right)x\right)f'(x)+\frac{x^2}{2}f''(x)$$

holds uniformly on $[0,J]$, where $J>0$.

Proof. Let $x\in[0,J]$. The remainder part of the proof of this theorem is similar to that of the proof of the previous Theorem. So we omit it. □

5.3.1 Local approximation

Theorem 5.3.4. *Let $f\in C_B[0,\infty)$ and $q\in(0,1)$. Then, for every $x\in[0,\infty)$ and $n\geq 2$, we have*

$$\left|\left(M_{n,k,q}^{(\alpha,\beta)}f\right)(x)-f(x)\right|\leq C\omega_2(f;\delta_{n,k,q}^{(\alpha,\beta)}(x))+\omega\left(f;\left|\frac{[n-k+1]_q}{[n]_q+\beta}q^{\frac{k-1}{2}}x+\frac{\alpha}{[n]_q+\beta}-x\right|\right),$$

where C is an absolute constant and

$$\delta_{n,k,q}^{(\alpha,\beta)}(x)=\left(\left(M_{n,k,q}^{(\alpha,\beta)}(t-x)^2\right)(x)+\left(\frac{[n-k+1]_q}{[n]_q+\beta}q^{\frac{k-1}{2}}x+\frac{\alpha}{[n]_q+\beta}-x\right)^2\right)^{1/2}.$$

Proof. For $x\in[0,\infty)$, we consider the auxiliary operators $\overline{M}_{n,k,q}^{(\alpha,\beta)}$ defined by $\left(\overline{M}_{n,k,q}^{(\alpha,\beta)}f\right)(x)$

$$= \left(M_{n,k,q}^{(\alpha,\beta)}f\right)(x)-f\left(\frac{[n-k+1]_q}{[n]_q+\beta}q^{\frac{k-1}{2}}x+\frac{\alpha}{[n]_q+\beta}\right)+f(x). \tag{5.3.7}$$

From Lemma 5.2.2, we observe that the operators $\overline{M}_{n,k,q}^{(\alpha,\beta)}$ are linear and reproduce the linear functions. Hence

$$\left(\overline{M}_{n,k,q}^{(\alpha,\beta)}(t-x)\right)(x)=0. \tag{5.3.8}$$

Let $g\in C_B^2[0,\infty)$. By Taylor's theorem, we have

$$g(t)=g(x)+g'(x)(t-x)+\int_x^t (t-u)g''(u)du,\ \ t\in[0,\infty).$$

Applying $\overline{M}_{n,k,q}^{(\alpha,\beta)}$ to both sides of the above equation and using (5.3.8), we have

$$\left(\overline{M}_{n,k,q}^{(\alpha,\beta)}g\right)(x)=g(x)+\left(\overline{M}_{n,k,q}^{(\alpha,\beta)}\int_x^t (t-u)g''(u)du\right)(x).$$

Thus, by (5.3.7) we get

$$\begin{aligned}
&\left|\left(\overline{M}_{n,k,q}^{(\alpha,\beta)}g\right)(x)-g(x)\right|\\
&\leq \left(M_{n,k,q}^{(\alpha,\beta)}\int_x^t |t-u||g''(u)|du\right)(x)\\
&\quad+\int_x^{\frac{[n-k+1]_q}{[n]_q+\beta}q^{\frac{k-1}{2}}x+\frac{\alpha}{[n]_q+\beta}}\left|\frac{[n-k+1]_q}{[n]_q+\beta}q^{\frac{k-1}{2}}x+\frac{\alpha}{[n]_q+\beta}-u\right||g''(u)|du\\
&\leq \left(\left(M_{n,k,q}^{(\alpha,\beta)}(t-x)^2\right)(x)+\left(\frac{[n-k+1]_q}{[n]_q+\beta}q^{\frac{k-1}{2}}x+\frac{\alpha}{[n]_q+\beta}-x\right)^2\right)\parallel g''\parallel\\
&\leq \left(\delta_{n,k,q}^{(\alpha,\beta)}(x)\right)^2\parallel g''\parallel. \qquad (5.3.9)
\end{aligned}$$

On the other hand, by (5.3.7) and Lemma 5.2.3, we have

$$\left|\left(\overline{M}_{n,k,q}^{(\alpha,\beta)}f\right)(x)\right|\leq\left|\left(M_{n,k,q}^{(\alpha,\beta)}f\right)(x)\right|+2\parallel f\parallel\leq 3\parallel f\parallel. \tag{5.3.10}$$

Using (5.3.9) and (5.3.10) in (5.3.7), we obtain

$$\begin{aligned}
\left|\left(M_{n,k,q}^{(\alpha,\beta)}f\right)(x)-f(x)\right| &\leq \left|\left(\overline{M}_{n,k,q}^{(\alpha,\beta)}f-g\right)(x)\right|+|(f-g)(x)|+\left|\left(\overline{M}_{n,k,q}^{(\alpha,\beta)}g\right)(x)-(g)(x)\right|\\
&\quad+\left(f\left(\frac{[n-k+1]_q}{[n]_q+\beta}q^{\frac{k-1}{2}}x+\frac{\alpha}{[n]_q+\beta}\right)-f(x)\right)\\
&\leq 4\parallel f-g\parallel+\left(\delta_{n,k,q}^{(\alpha,\beta)}(x)\right)^2\parallel g''\parallel\\
&\quad+\left(f\left(\frac{[n-k+1]_q}{[n]_q+\beta}q^{\frac{k-1}{2}}x+\frac{\alpha}{[n]_q+\beta}\right)-f(x)\right).
\end{aligned}$$

Hence, taking infimum on the right hand side over all $g\in C_B^2[0,\infty)$ and using (0.7.2), we get the required result. □

5.3.2 Rate of convergence

Theorem 5.3.5. *Let $f \in D_2[0,\infty), q \in (0,1)$ and $\omega(f;\delta,[0,b+1])$ be its modulus of continuity on the finite interval $[0,b+1] \subset [0,\infty)$, where $b>0$. Then, for every $n \geq 2$,*

$$\left|\left(M_{n,k,q}^{(\alpha,\beta)} f\right)(x) - f(x)\right| \leq 4M_f(1+b^2)\gamma_{n,k,q}^{(\alpha,\beta)}(x) + 2\omega\left(f; \sqrt{\gamma_{n,k,q}^{(\alpha,\beta)}(x)}, [0,b+1]\right),$$

where $\gamma_{n,k,q}^{(\alpha,\beta)}(x)$ is as defined in Remark 5.

Proof. From ([75], p.378), for $x \in [0,b]$ and $t \in [0,\infty)$, we get

$$|f(t)-f(x)| \leq 4M_f(1+b^2)(t-x)^2 + \left(1+\frac{|t-x|}{\delta}\right)\omega(f;\delta,[0,b+1]),\ \delta>0.$$

Thus, by applying Cauchy-Schwarz inequality, we have

$$\begin{aligned}
&\left|\left(M_{n,k,q}^{(\alpha,\beta)} f\right)(x) - f(x)\right| \\
\leq\ & 4M_f(1+b^2)\left(M_{n,k,q}^{(\alpha,\beta)}(t-x)^2\right)(x) + \omega(f;\delta,[0,b+1])\left(1+\frac{1}{\delta}\left(\left(M_{n,k,q}^{(\alpha,\beta)}(t-x)^2\right)(x)\right)^{\frac{1}{2}}\right) \\
=\ & 4M_f(1+b^2)\gamma_{n,k,q}^{(\alpha,\beta)}(x) + 2\omega\left(f; \sqrt{\gamma_{n,k,q}^{(\alpha,\beta)}(x)}, [0,b+1]\right),
\end{aligned}$$

on choosing $\delta = \sqrt{\gamma_{n,k,q}^{(\alpha,\beta)}(x)}$. This completes the proof of the theorem. □

5.3.3 Weighted approximation.

In this section, we shall discuss about the weighted approximation theorems for the operators (5.1.2). Throughout the section, we assume that $\{q_n\}$ is a sequence in $(0,1)$ such that $q_n \to 1$ and $q_n^n \to 0$ as $n \to \infty$.

Theorem 5.3.6. *For each $f \in D_2^*[0,\infty)$, we have*

$$\lim_{n\to\infty} \left\|M_{n,k,q_n}^{(\alpha,\beta)} f - f\right\|_2 = 0.$$

Proof. From the Korovkin theorem [50], we see that it is sufficient to verify the following three conditions

$$\lim_{n\to\infty} \left\|\left(M_{n,k,q_n}^{(\alpha,\beta)} t^\nu\right)(x) - x^\nu\right\|_2 = 0,\ \nu = 0,1,2. \tag{5.3.11}$$

Since $\left(M_{n,k,q_n}^{(\alpha,\beta)}1\right)(x)=1$, the condition holds for $\nu=0$.
By Lemma 5.2.2, we have

$$\begin{aligned}\left\|\left(M_{n,k,q_n}^{(\alpha,\beta)}t\right)(x)-x\right\|_2 &\leq \left\|\left(\frac{q_n^{\frac{k-1}{2}}[n-k+1]_{q_n}}{[n]_{q_n}+\beta}-1\right)x+\frac{\alpha}{[n]_{q_n}+\beta}\right\|_2\\ &\leq \left(\frac{q_n^{\frac{k-1}{2}}[n-k+1]_{q_n}}{[n]_{q_n}+\beta}-1\right)\sup_{x\in[0,\infty)}\frac{x}{1+x^2}+\frac{\alpha}{[n]_{q_n}+\beta}\sup_{x\in[0,\infty)}\frac{1}{1+x^2}\\ &\leq \left|\frac{q_n^{\frac{k-1}{2}}[n-k+1]_{q_n}}{[n]_{q_n}+\beta}-1\right|+\frac{\alpha}{[n]_{q_n}+\beta}\to 0 \text{ as } n\to\infty.\end{aligned}$$

Hence the equation (5.3.11) holds for $\nu=1$. Similarly, we can write for $n>1$
$\left\|\left(M_{n,k,q_n}^{(\alpha,\beta)}t^2\right)(x)-x^2\right\|_2$

$$\leq \left|\frac{q_n^{k-2}[n-k+2]_{q_n}[n-k+1]_{q_n}[n]_{q_n}}{[n-1]_{q_n}([n]_{q_n}+\beta)^2}-1\right|+\left|\frac{2q_n^{\frac{k-1}{2}}\alpha[n-k+1]_{q_n}}{([n]_{q_n}+\beta)^2}\right|+\frac{\alpha^2}{([n]_{q_n}+\beta)^2}$$

which implies that $\lim_{n\to\infty}\left\|\left(M_{n,k,q_n}^{(\alpha,\beta)}t^2\right)(x)-x^2\right\|_2=0$, the equation (5.3.11) holds for $\nu=2$. Thus, the proof is completed. □

Now, we present a weighted approximation theorems for functions in $D_2^*[0,\infty)$.

Theorem 5.3.7. *For each $f\in D_2^*[0,\infty)$ and $d>0$, we have*

$$\lim_{n\to\infty}\sup_{x\in[0,\infty)}\frac{\left|\left(M_{n,k,q_n}^{(\alpha,\beta)}f\right)(x)-f(x)\right|}{(1+x^2)^{1+d}}=0.$$

Proof. Let $x_0\in[0,\infty)$ be arbitrary but fixed. Then

$$\begin{aligned}\sup_{x\in[0,\infty)}\frac{\left|\left(M_{n,k,q_n}^{(\alpha,\beta)}f\right)(x)-f(x)\right|}{(1+x^2)^{1+d}} &= \sup_{x\leq x_0}\frac{\left|\left(M_{n,k,q_n}^{(\alpha,\beta)}f\right)(x)-f(x)\right|}{(1+x^2)^{1+d}}+\sup_{x>x_0}\frac{\left|\left(M_{n,k,q_n}^{(\alpha,\beta)}f\right)(x)-f(x)\right|}{(1+x^2)^{1+d}}\\ &\leq \parallel M_{n,k,q_n}^{(\alpha,\beta)}f-f\parallel_{C[0,x_0]}+\parallel f\parallel_2\sup_{x>x_0}\frac{\left|\left(M_{n,k,q_n}^{(\alpha,\beta)}(1+t^2)\right)(x)\right|}{(1+x^2)^{1+d}}\\ &\quad+\sup_{x>x_0}\frac{|f(x)|}{(1+x^2)^{1+d}}.\end{aligned} \tag{5.3.12}$$

Since $|f(x)|\leq\parallel f\parallel_2(1+x^2)$, we have $\sup_{x>x_0}\frac{|f(x)|}{(1+x^2)^{1+d}}\leq\frac{\parallel f\parallel_2}{(1+x_0^2)^d}$.
Let $\epsilon>0$ be arbitrary. We can choose x_0 to be so large that

$$\frac{\parallel f\parallel_2}{(1+x_0^2)^d}<\frac{\epsilon}{3}. \tag{5.3.13}$$

In view of Theorem 5.3.1, we obtain

$$\| f \|_2 \lim_{n\to\infty} \frac{\left|\left(M^{(\alpha,\beta)}_{n,k,q_n}(1+t^2)\right)(x)\right|}{(1+x^2)^{1+d}} = \frac{(1+x^2)}{(1+x^2)^{1+d}} \| f \|_2 = \frac{\| f \|_2}{(1+x^2)^d} \leq \frac{\| f \|_2}{(1+x_0^2)^d} < \frac{\epsilon}{3}.$$

Using Theorem 5.3.5, we can see that the first term of the inequality (5.3.12), implies that

$$\| M^{(\alpha,\beta)}_{n,k,q_n} f - f \|_{C[0,x_0]} < \frac{\epsilon}{3},\ as\ n \to \infty. \tag{5.3.14}$$

Combining (5.3.12)-(5.3.14), we get the desired result. □

Theorem 5.3.8. *If* $f \in D_2^*[0,\infty)$, *then we have*

$$\left|\left(M^{(\alpha,\beta)}_{n,k,q_n} f\right)(x) - f(x)\right| \leq C(1+x^{2+\lambda})\Omega_2(f,\delta_n), x \in [0,\infty),$$

where $\lambda \geq 1, \delta_n^2 = \max\{a_n, b_n, c_n\}, a_n, b_n, c_n$ *being* $\left(q_n^{k-2}\frac{[n-k+2]_{q_n}[n-k+1]_{q_n}[n]_{q_n}}{[n-1]_{q_n}([n]_{q_n}+\beta)^2}+1\right)$, $2\alpha\left(q_n^{\frac{(k-1)}{2}}\frac{[n-k+1]_{q_n}}{([n]_{q_n}+\beta)^2}\right)$ *and* $\frac{\alpha^2}{([n]_{q_n}+\beta)^2}$ *respectively and* C *is a positive constant independent of* f *and* n.

Proof. From the definition of $\Omega_2(f,\delta)$ and Lemma 0.7.1, we have

$$\begin{aligned} |f(t)-f(x)| &\leq (1+x+|t-x|^2)\left(1+\frac{|t-x|}{\delta}\right)\Omega_2(f,\delta) \\ &\leq (1+(2x+t)^2)\left(1+\frac{|t-x|}{\delta}\right)\Omega_2(f,\delta) \\ &:= \phi_x(t)\left(1+\frac{1}{\delta}\psi_x(t)\right)\Omega_2(f,\delta), \end{aligned}$$

where $\phi_x(t) = 1+(2x+t)^2$ and $\psi_x(t) = |t-x|$. Then, we obtain

$$\left|M^{(\alpha,\beta)}_{n,k,q_n}(f;x) - f(x)\right| \leq \left(M^{(\alpha,\beta)}_{n,k,q_n}(\phi_x;x) + \frac{1}{\delta_n}M^{(\alpha,\beta)}_{n,k,q_n}(\phi_x\psi_x;x)\right)\Omega_2(f,\delta_n).$$

Now, applying the Cauchy-Schwarz inequality to the second term on the righthand side, we get $\left|\left(M^{(\alpha,\beta)}_{n,k,q_n} f\right)(x) - f(x)\right|$

$$\leq \left(\left(M^{(\alpha,\beta)}_{n,k,q_n}\phi_x\right)(x) + \frac{1}{\delta_n}\sqrt{\left(M^{(\alpha,\beta)}_{n,k,q_n}\phi_x^2\right)(x)}\sqrt{\left(M^{(\alpha,\beta)}_{n,k,q_n}\psi_x^2\right)(x)}\right)\Omega_2(f,\delta_n). \tag{5.3.15}$$

From Lemma 5.2.2,

$$\begin{aligned} \frac{1}{1+x^2}\left(M^{(\alpha,\beta)}_{n,k,q_n}\left(1+t^2\right)\right)(x) &= \frac{1}{1+x^2} + \left(\frac{q_n^{k-2}[n]_{q_n}[n-k+1]_{q_n}[n-k+2]_{q_n}}{[n-1]_{q_n}([n]_{q_n}+\beta)^2}\right)\frac{x^2}{1+x^2} \\ &\quad + \frac{2\alpha q_n^{\frac{k-1}{2}}[n-k+1]_{q_n}}{([n]_{q_n}+\beta)^2}\frac{x}{1+x^2} + \frac{\alpha^2}{([n]_{q_n}+\beta)^2}\frac{1}{1+x^2} \\ &\leq 1+C_1. \end{aligned} \tag{5.3.16}$$

For each $t \geq 0$ and $x \in [0, \infty)$, we get

$$\phi_x(t) = 1 + (2x + t)^2 \leq 1 + 2(4x^2 + 2t^2).$$

From (5.3.15) and (5.3.16), there exists a positive constant C_2 such that

$$M^{\alpha,\beta}_{n,k,q_n}(\phi_x; x) \leq C_2(1 + x^2).$$

Proceeding similarly, $\frac{1}{1+x^4}\left(M^{(\alpha,\beta)}_{n,k,q_n}(1+t^4)\right)(x) \leq 1 + C_3$, so there exists a positive constant C_4, such that $\sqrt{\left(M^{(\alpha,\beta)}_{n,k,q_n}\phi_x^2\right)(x)} \leq C_4(1 + x^2)$, where $x \in [0, \infty)$.
Also, we get

$$\begin{aligned}
\left(M^{(\alpha,\beta)}_{n,k,q_n}\psi_x^2\right)(x) &= \left(q_n^{k-2}\frac{[n-k+2]_{q_n}[n-k+1]_{q_n}[n]_{q_n}}{[n-1]_{q_n}([n]_{q_n}+\beta)^2} - q_n^{\frac{(k-1)}{2}}\frac{2[n-k+1]_{q_n}}{([n]_{q_n}+\beta)} + 1\right)x^2 \\
&\quad + 2\alpha\left(q_n^{\frac{(k-1)}{2}}\frac{[n-k+1]_{q_n}}{([n]_{q_n}+\beta)^2} - \frac{1}{[n]_{q_n}+\beta}\right)x + \frac{\alpha^2}{([n]_{q_n}+\beta)^2} \\
&\leq a_n x^2 + b_n x + c_n.
\end{aligned}$$

Hence, from (5.3.15), we have

$$\left|\left(M^{(\alpha,\beta)}_{n,k,q_n}f\right)(x) - f(x)\right| \leq (1+x^2)\left(C_2 + \frac{1}{\delta_n}C_4\sqrt{a_n x^2 + b_n x + c_n}\right)\Omega_2(f, \delta_n).$$

If we take $\delta_n^2 = \max\{a_n, b_n, c_n\}$, then we get

$$\begin{aligned}
\left|\left(M^{(\alpha,\beta)}_{n,k,q_n}f\right)(x) - f(x)\right| &\leq (1+x^2)\left(C_2 + C_4\sqrt{x^2+x+1}\right)\Omega_2(f, \delta_n) \\
&\leq C_5(1 + x^{2+\lambda})\Omega_2(f, \delta_n),\ x \in [0, \infty).
\end{aligned}$$

Hence, the proof is completed. □

Next, we obtain the local direct estimate of the operators defined in (5.1.2), using the Lipschitz-type maximal function of order τ.

Theorem 5.3.9. *Let $f \in C_B[0, \infty)$ and $0 < \tau \leq 1$. Then, for all $x \in [0, \infty)$ we have*

$$\left|\left(M^{(\alpha,\beta)}_{n,k,q}f\right)(x) - f(x)\right| \leq \widehat{\omega}_\tau(f, x)\left(\gamma^{(\alpha,\beta)}_{n,k,q}(x)\right)^{\tau/2}.$$

Proof. From the equation (0.7.3), we have

$$\left|\left(M^{(\alpha,\beta)}_{n,k,q}f\right)(x) - f(x)\right| \leq \widehat{\omega}_\tau(f, x)\left(M^{(\alpha,\beta)}_{n,k,q}|t-x|^\tau\right)(x).$$

Applying the Hölder's inequality with $p = \frac{2}{\tau}$ and $\frac{1}{q} = 1 - \frac{1}{p}$, we get

$$\left|\left(M^{(\alpha,\beta)}_{n,k,q}f\right)(x) - f(x)\right| \leq \widehat{\omega}_\tau(f, x)\left(M^{(\alpha,\beta)}_{n,k,q}(t-x)^2\right)^{\frac{\tau}{2}}(x) = \widehat{\omega}_\tau(f, x)\left(\gamma^{(\alpha,\beta)}_{n,k,q}(x)\right)^{\tau/2}.$$

Thus, the proof is completed. □

5.3.4 Statistical convergence

Let $q_n \in (0,1)$ be a sequence such that

$$(5.3.17) \quad st_A - \lim_n q_n = 1, st_A - \lim_n q_n^n = a(a<1) \text{ and } st_A - \lim_n \frac{1}{[n]_{q_n}} = 0.$$

Theorem 5.3.10. *Let $A=(a_{nk})$ be a non-negative regular summability matrix and (q_n) be a sequence satisfying (5.3.17). Then, for any compact set $K \subset [0,\infty)$ and for each function $f \in C(K)$, we have*

$$st_A - \lim_n \left\| \left(M_{n,k,q_n}^{(\alpha,\beta)} f \right)(.) - f \right\| = 0.$$

Proof. Let $x_0 = \max_{x \in K} x$. From Lemma 5.2.2, $st_A - \lim_n \left\| \left(M_{n,k,q_n}^{(\alpha,\beta)} e_0 \right)(.) - e_0 \right\| = 0$. Again, by Lemma 5.2.2, we have

$$\sup_{x \in K} \left| \left(M_{n,k,q_n}^{(\alpha,\beta)} e_1 \right)(x) - e_1(x) \right| \leq \left| \frac{q_n^{\frac{k-1}{2}}[n-k+1]_{q_n}}{([n]_{q_n}+\beta)} - 1 \right| x_0 + \frac{\alpha}{[n]_{q_n}+\beta}.$$

For $\epsilon > 0$, let us define the following sets:

$$\begin{aligned} E &:= \left\{ j : \left\| \left(M_{j,k,q_j}^{(\alpha,\beta)} e_1 \right)(.) - e_1 \right\| \geq \epsilon \right\} \\ E_1 &:= \left\{ j : \left| \frac{q_j^{\frac{k-1}{2}}[j-k+1]_{q_j}}{([j]_{q_j}+\beta)} - 1 \right| \geq \frac{\epsilon}{2} \right\} \\ E_2 &:= \left\{ j : \frac{\alpha}{[j]_{q_j}+\beta} \geq \frac{\epsilon}{2} \right\}, \end{aligned}$$

which implies that $E \subseteq E_1 \cup E_2$ and hence for all $n \in \mathbb{N}$, we obtain

$$\sum_{j \in E} a_{nj} \leq \sum_{j \in E_1} a_{nj} + \sum_{j \in E_2} a_{nj}.$$

Hence, taking limit as $n \to \infty$, we have $st_A - \lim_n \left\| \left(M_{n,k,q_n}^{(\alpha,\beta)} e_1 \right)(.) - e_1 \right\| = 0$. Similarly, by using Lemma 5.2.2, we have

$$\begin{aligned} \sup_{x \in K} \left| \left(M_{n,k,q_n}^{(\alpha,\beta)} e_2 \right)(x) - e_2(x) \right| &\leq \left| \frac{q_n^{k-2}[n]_{q_n}[n-k+1]_{q_n}[n-k+2]_{q_n}}{[n-1]_{q_n}([n]_{q_n}+\beta)^2} - 1 \right| x_0^2 \\ &\quad + \left| \frac{2\alpha q_n^{\frac{k-1}{2}}[n-k+1]_{q_n}}{([n]_{q_n}+\beta)^2} \right| x_0 + \frac{\alpha^2}{([n]_{q_n}+\beta)^2}. \end{aligned}$$

Now, let us define the following sets:

$$\begin{aligned} F &:= \left\{ j : \left\| \left(M^{(\alpha,\beta)}_{j,k,q_j} e_2 \right)(.) - e_2 \right\| \geq \epsilon \right\} \\ F_1 &:= \left\{ j : \left| \frac{q_j^{k-2}[j]_{q_j}[j-k+1]_{q_j}[j-k+2]_{q_j}}{[j-1]_{q_j}([j]_{q_j}+\beta)^2} - 1 \right| \geq \frac{\epsilon}{3} \right\} \\ F_2 &:= \left\{ j : \left| \frac{2\alpha q_j^{\frac{k-1}{2}}[j-k+1]_{q_j}}{([j]_{q_j}+\beta)^2} \right| \geq \frac{\epsilon}{3} \right\} \\ F_3 &:= \left\{ j : \frac{\alpha^2}{([j]_{q_j}+\beta)^2} \geq \frac{\epsilon}{3} \right\}. \end{aligned}$$

Then, we obtain $F \subseteq F_1 \cup F_2 \cup F_3$, which implies that

$$\sum_{j \in F} a_{nj} \leq \sum_{j \in F_1} a_{nj} + \sum_{j \in F_2} a_{nj} + \sum_{j \in F_3} a_{nj}.$$

Thus, as $n \to \infty$ we get $st_A - \lim_n \left\| \left(M^{(\alpha,\beta)}_{n,k,q_n} e_2 \right)(.) - e_2 \right\| = 0$. This completes the proof. □

Theorem 5.3.11. *Let $A = (a_{nk})$ be a nonnegative regular summability matrix and (q_n) be a sequence in $(0, 1)$ satisfying (5.3.17). Let the operators $M^{(\alpha,\beta)}_{n,k,q_n}$, $n \in \mathbb{N}$, be defined as in (5.1.2). Then, for each function $f \in D_2^*[0, \infty)$, we have*

$$st_A - \lim_n \left\| \left(M^{(\alpha,\beta)}_{n,k,q_n} f \right)(.) - f \right\|_{\zeta+2} = 0, \ \zeta > 0.$$

Proof. From ([42], p. 191, Th. 3), it is sufficient to prove that

$$st_A - \lim_n \left\| \left(M^{(\alpha,\beta)}_{n,k,q_n} e_i \right)(.) - e_i \right\|_2 = 0, \text{ where } e_i(x) = x^i, \quad i = 0, 1, 2.$$

From Lemma 5.2.2, $st_A - \lim_n \left\| \left(M^{(\alpha,\beta)}_{n,k,q_n} e_0 \right)(.) - e_0 \right\|_2 = 0$ holds.
Again using Lemma 5.2.2, we have

$$\begin{aligned} \left\| \left(M^{(\alpha,\beta)}_{n,k,q_n} e_1 \right)(.) - e_1 \right\|_2 &\leq \sup_{x \in [0,\infty)} \left\{ \frac{x}{(1+x^2)} \left| \frac{q_n^{\frac{k-1}{2}}[n-k+1]_{q_n}}{([n]_{q_n}+\beta)} - 1 \right| + \frac{1}{(1+x^2)} \frac{\alpha}{([n]_{q_n}+\beta)} \right\} \\ &= \left| \frac{q_n^{\frac{k-1}{2}}[n-k+1]_{q_n}}{([n]_{q_n}+\beta)} - 1 \right| + \frac{\alpha}{([n]_{q_n}+\beta)}. \qquad (5.3.18) \end{aligned}$$

For each $\epsilon > 0$, let us define the following sets:

$$\begin{aligned} G &:= \left\{ j : \left\| \left(M^{(\alpha,\beta)}_{j,k,q_j} e_1 \right)(.) - e_1 \right\|_2 \geq \epsilon \right\} \\ G_1 &:= \left\{ j : \left| \frac{q_j^{\frac{k-1}{2}}[j-k+1]_{q_j}}{([j]_{q_j}+\beta)} - 1 \right| \geq \frac{\epsilon}{2} \right\} \\ G_2 &:= \left\{ j : \frac{\alpha}{[j]_{q_j}+\beta} \geq \frac{\epsilon}{2} \right\}, \end{aligned}$$

which yields us $G \subseteq G_1 \cup G_2$ in view of (5.3.18) and therefore for all $n \in \mathbb{N}$, we have

$$\sum_{j\in G} a_{nj} \leq \sum_{j\in G_1} a_{nj} + \sum_{j\in G_2} a_{nj}.$$

Hence, on taking limit as $n \to \infty$, $st_A - \lim_n \left\| \left(M_{j,k,q_j}^{(\alpha,\beta)} e_1 \right)(.) - e_1 \right\|_2 = 0$. Proceeding similarly,

$$\begin{aligned}\left\| \left(M_{n,k,q_n}^{(\alpha,\beta)} e_2 \right)(.) - e_2 \right\|_2 \leq & \left| \frac{q_n^{k-2}[n]_{q_n}[n-k+1]_{q_n}[n-k+2]_{q_n}}{[n-1]_{q_n}([n]_{q_n}+\beta)^2} - 1 \right| \\ & + \left| \frac{2\alpha q_n^{\frac{k-1}{2}}[n-k+1]_{q_n}}{([n]_{q_n}+\beta)^2} \right| + \frac{\alpha^2}{([n]_{q_n}+\beta)^2}.\end{aligned}$$

Now, let us define the following sets:

$$\begin{aligned} R := & \left\{ j : \left\| \left(M_{j,k,q_j}^{(\alpha,\beta)} e_2 \right)(.) - e_2 \right\|_2 \geq \epsilon \right\} \\ R_1 := & \left\{ j : \left| \frac{q_j^{k-2}[j]_{q_j}[j-k+1]_{q_j}[j-k+2]_{q_j}}{[j-1]_{q_j}([j]_{q_j}+\beta)^2} - 1 \right| \geq \frac{\epsilon}{3} \right\} \\ R_2 := & \left\{ j : \left| \frac{2\alpha q_j^{\frac{k-1}{2}}[j-k+1]_{q_j}}{([j]_{q_j}+\beta)^2} \right| \geq \frac{\epsilon}{3} \right\} \\ R_3 := & \left\{ j : \frac{\alpha^2}{([j]_{q_j}+\beta)^2} \geq \frac{\epsilon}{3} \right\}. \end{aligned}$$

Then, we obtain $R \subseteq R_1 \cup R_2 \cup R_3$, which implies that

$$\sum_{j\in R} a_{nj} \leq \sum_{j\in R_1} a_{nj} + \sum_{j\in R_2} a_{nj} + \sum_{j\in R_3} a_{nj}.$$

Hence, taking limit as $n \to \infty$ we get $st_A - \lim_n \left\| \left(M_{n,k,q_n}^{(\alpha,\beta)} e_2 \right)(.) - e_2 \right\|_2 = 0$. This completes the proof of the theorem. □

5.4 Better Estimates

It is well known that the classical Bernstein polynomials preserve constant as well as linear functions. To make the convergence faster, King [101] proposed an approach to modify the Bernstein polynomials, so that the sequence preserves test functions e_0 and e_2, where $e_i(t) = t^i, i = 0, 1, 2$. As the operator $M_{n,k,q}^{(\alpha,\beta)}(f; x)$ defined in (5.1.2) reproduces only constant functions, this motivated us to propose the modification of this operator, so that it can preserve constant as well as linear functions. The modification of the operators given in (5.1.2) is defined as

$$\left(\widetilde{M}_{n,k,q}^{(\alpha,\beta)}f\right)(x) = \frac{[2n-k+1]_q!\left(q^{\frac{2n-k+1}{2}}s_n^q(x)\right)^{n+1}}{[n]_q![n-k]_q!}q^{(n-k)(n-k+1)/2} \times \int_0^{\infty/A}\frac{t^{n-k}}{(q^{\frac{2n-k+1}{2}}s_n^q(x)+t)^{2n-k+2}}f\left(\frac{[n]_qt+\alpha}{[n]_q+\beta}\right)d_qt,$$

where $s_n^q(x)=\dfrac{([n]_q+\beta)x-\alpha}{q^{\frac{k-1}{2}}[n-k+1]_q}$ for $x\in I_n=\left[\frac{\alpha}{[n]_q+\beta},\infty\right)$ and $n\in\mathbb{N}$.

Lemma 5.4.1. *For each $x\in I_n$, by simple computations, we have*

1. $\left(\widetilde{M}_{n,k,q}^{(\alpha,\beta)}1\right)(x)=1;$
2. $\left(\widetilde{M}_{n,k,q}^{(\alpha,\beta)}t\right)(x)=x;$
3. $\left(\widetilde{M}_{n,k,q}^{(\alpha,\beta)}t^2\right)(x)=\dfrac{[n-k+2]_q[n]_q}{q[n-k+1]_q[n-1]_q}x^2+\dfrac{2\alpha}{[n]_q+\beta}\left(1-\dfrac{[n-k+2]_q[n]_q}{q[n-k+1]_q[n-1]_q}\right)x + \dfrac{\alpha^2}{([n]_q+\beta)^2}\left(\dfrac{[n-k+2]_q[n]_q}{q[n-k+1]_q[n-1]_q}-1\right)$, *for* $n>1$.

Consequently, for each $x\in I_n$, we have the following equalities:

$$\left(\widetilde{M}_{n,k,q}^{(\alpha,\beta)}(t-x)\right)(x)=0;$$

$$\begin{aligned}\left(\widetilde{M}_{n,k,q}^{(\alpha,\beta)}(t-x)^2\right)(x) &= \left(\frac{[n-k+2]_q[n]_q}{q[n-k+1]_q[n-1]_q}-1\right)x^2+\frac{2\alpha}{[n]_q+\beta}\left(1-\frac{[n-k+2]_q[n]_q}{q[n-k+1]_q[n-1]_q}\right)x \\ &\quad +\frac{\alpha^2}{([n]_q+\beta)^2}\left(\frac{[n-k+2]_q[n]_q}{q[n-k+1]_q[n-1]_q}-1\right),\ \textit{for } n>1 \\ (5.4.1)\quad &= \xi_{n,k,q}^{(\alpha,\beta)}(x),\ \textit{say.}\end{aligned}$$

Theorem 5.4.2. *Let $f\in C_B[0,\infty)$ and $x\in I_n$. Then for every $x\in[0,\infty)$, there exists a positive constant C such that*

$$\left|\left(\widetilde{M}_{n,k,q}^{(\alpha,\beta)}f\right)(x)-f(x)\right|\le C\omega_2\left(f;\sqrt{\xi_{n,k,q}^{(\alpha,\beta)}(x)}\right),$$

where $\xi_{n,k,q}^{(\alpha,\beta)}(x)$ is given by (5.4.1).

Proof. Let $g \in C_B^2[0,\infty)$, $x \in I_n$ and $t \in [0,\infty)$. Using the Taylor's expansion we have

$$g(t) = g(x) + (t-x)g'(x) + \int_x^t (t-u)g''(u)du.$$

Applying $\widetilde{M}_{n,k,q}^{(\alpha,\beta)}$ on both sides and using Lemma 5.4.1, we get

$$\left(\widetilde{M}_{n,k,q}^{(\alpha,\beta)} g\right)(x) - g(x) = \left(\widetilde{M}_{n,k,q}^{(\alpha,\beta)}\left(\int_x^t (t-u)g''(u)du\right)\right)(x).$$

Obviously, we have $\left|\int_x^t (t-u)g''(u)du\right| \leq (t-x)^2\|g''\|$. Therefore

$$\left|\left(\widetilde{M}_{n,k,q}^{(\alpha,\beta)} g\right)(x) - g(x)\right| \leq \left(\widetilde{M}_{n,k,q}^{(\alpha,\beta)}(t-x)^2\right)(x) \parallel g'' \parallel = \xi_{n,k,q}^{(\alpha,\beta)}(x) \parallel g'' \parallel .$$

Since $\left|\left(\widetilde{M}_{n,k,q}^{(\alpha,\beta)} f\right)(x)\right| \leq \|f\|$, we get

$$\begin{aligned}\left|\left(\widetilde{M}_{n,k,q}^{(\alpha,\beta)} f\right)(x) - f(x)\right| &\leq \left|\left(\widetilde{M}_{n,k,q}^{(\alpha,\beta)}(f-g)\right)(x)\right| + \mid (f-g)(x) \mid + \left|\left(\widetilde{M}_{n,k,q}^{(\alpha,\beta)} g\right)(x) - g(x)\right| \\ &\leq 2\|f-g\| + \xi_{n,k,q}^{(\alpha,\beta)}(x)\|g''\|.\end{aligned}$$

Finally, taking the infimum over all $g \in C_B^2[0,\infty)$ and using (0.7.1), (0.7.2) we obtain

$$\left|\left(\widetilde{M}_{n,k,q}^{(\alpha,\beta)} f\right)(x) - f(x)\right| \leq C\omega_2\left(f; \sqrt{\xi_{n,k,q}^{(\alpha,\beta)}(x)}\right),$$

which proves the theorem. □

Theorem 5.4.3. *Let $f \in D_\vartheta[0,\infty)$ and $q_n \in (0,1)$ be a sequence such that $q_n \to 1$ and $q_n^n \to 0$ as $n \to \infty$. Suppose that $f''(x)$ exists at a point $x \in I_n$, then we have*

$$\lim_{n\to\infty} [n]_{q_n}\left(\left(\widetilde{M}_{n,k,q}^{(\alpha,\beta)} f\right)(x) - f(x)\right) = \frac{1}{2}xf''(x).$$

Proof. The proof follows along the lines of Theorem 5.3.2. □

Chapter 6

Sz$\acute{a}$sz-Baskakov type operators based on q-integers

6.1 Introduction

For $f \in D_\vartheta[0,\infty)$, Gupta [69] introduced the following operators

$$\mathrm{T}_n(f;x) = \sum_{\nu=1}^{\infty} q_{n,\nu}(x) \int_0^\infty b_{n,\nu-1}(t) f(t) dt + e^{-nx} f(0), \tag{6.1.1}$$

where $q_{n,\nu}(x) = e^{-nx}\frac{(nx)^\nu}{\nu!}$, $b_{n,\nu}(x) = \dfrac{1}{B(\nu+1,n)}\dfrac{x^\nu}{(1+x)^{n+\nu+1}}$, $x \in [0,\infty)$ by considering the value of the function at zero explicitly and studied an estimate of error in terms of the higher order modulus of continuity in simultaneous approximation for a linear combination of the operators (6.1.1), introduced by May [112]. Later on, Gupta and Noor [79] discussed some direct results in simultaneous approximation for the operators (6.1.1).

For $f \in D_\vartheta[0,\infty)$, $0 < q < 1$, $0 \le \alpha \le \beta$ and each positive integer n, we introduce the following Stancu type modification of the operators (6.1.1) based on $q-$integers:

$$\begin{aligned} B_{n,q}^{(\alpha,\beta)}(f;x) &= \sum_{\nu=1}^{\infty} q_{n,\nu}(q,x) q^{\nu-1} \int_0^{\infty/A} b_{n,\nu-1}(q,t) f\left(\frac{q^\nu [n]_q t + \alpha}{[n]_q + \beta}\right) d_q t \\ &\quad + E_q\left(-[n]_q x\right) f\left(\frac{\alpha}{[n]_q + \beta}\right), \end{aligned} \tag{6.1.2}$$

where $q_{n,\nu}(q,x) = \dfrac{E_q\left(-[n]_q x\right)[n]_q^\nu x^\nu}{[\nu]_q!}$ and $b_{n,\nu}(q,t) = \dfrac{t^\nu q^{\nu(\nu-1)/2}}{(1+t)^{n+\nu+1} B_q(\nu+1,n)}$.

For $\alpha = \beta = 0$, we denote $B_{n,q}^{(\alpha,\beta)}(f;x)$ by $B_{n,q}(f;x)$.

Clearly, if $q \to 1^-$ and $\alpha = \beta = 0$, the operators defined by (6.1.2) reduce to the operators given by (6.1.1).
The purpose of the present chapter is to study the basic convergence theorem, Voronovskaja type asymptotic formula, local approximation, rate of convergence, weighted approximation, point-wise estimation and $A-$statistical convergence of the operators (6.1.2). Further, to obtain better approximation we also propose a modification of these operators by using a King type approach.

6.2 Moment Estimates

Lemma 6.2.1. *For* $B_{n,q}(t^m;x)$, $m = 0,1,2,$ *one has*

1. $B_{n,q}(1;x) = 1;$
2. $B_{n,q}(t;x) = \dfrac{[n]_q x}{[n-1]_q}$, *for* $n > 1;$
3. $B_{n,q}(t^2;x) = \dfrac{q[n]_q^2}{[n-1]_q[n-2]_q}x^2 + \dfrac{[2]_q[n]_q}{q[n-1]_q[n-2]_q}x$, *for* $n > 2.$

Proof. We observe that $B_{n,q}$ are well defined for the functions $1, t, t^2$. Thus, for every $x \in [0,\infty)$, using (0.2.1) and (0.2.2), we obtain

$$\begin{aligned} B_{n,q}(1;x) &= \sum_{\nu=1}^{\infty} q_{n,\nu}(q,x) q^{\nu-1} \int_0^{\infty/A} b_{n,\nu-1}(q,t) d_q t + E_q\left(-[n]_q x\right) \\ &= \sum_{\nu=0}^{\infty} q_{n,\nu}(q,x) = 1. \end{aligned}$$

Next, for $f(t) = t$, again applying (0.2.1) and (0.2.2), we get

$$\begin{aligned} B_{n,q}(t;x) &= \sum_{\nu=1}^{\infty} q_{n,\nu}(q,x) q^{\nu-1} \int_0^{\infty/A} b_{n,\nu-1}(q,t) q^{\nu} t \, d_q t \\ &= \frac{[n]_q}{[n-1]_q} x. \end{aligned}$$

Proceeding similarly, we have

$$B_{n,q}(t^2;x) = \frac{[2]_q[n]_q}{q[n-1]_q[n-2]_q} x + \frac{q[n]_q^2}{[n-1]_q[n-2]_q} x^2,$$

by using $[\nu+2]_q = [2]_q + q^2[\nu]_q$. $\square$

Lemma 6.2.2. *For the operators $B_{n,q}^{(\alpha,\beta)}(f;x)$ as defined in (6.1.2), the following equalities hold:*

1. $B_{n,q}^{(\alpha,\beta)}(1;x)=1;$

2. $B_{n,q}^{(\alpha,\beta)}(t;x)=\dfrac{[n]_q^2}{([n]_q+\beta)[n-1]_q}x+\dfrac{\alpha}{[n]_q+\beta},$ *for* $n>1;$

3. $B_{n,q}^{(\alpha,\beta)}(t^2;x)=\left(\dfrac{[n]_q}{[n]_q+\beta}\right)^2\left\{\dfrac{(q[n]_q^2+2\alpha[n-2]_q)}{[n-1]_q[n-2]_q}x^2+\dfrac{[2]_q[n]_q}{q[n-1]_q[n-2]_q}x\right\}$ $+\left(\dfrac{\alpha}{[n]_q+\beta}\right)^2,$ *for* $n>2.$

Proof. This Lemma is an immediate consequence of Lemma 6.2.1. Hence the details of its proof are omitted. □

Lemma 6.2.3. *For $f\in C_B[0,\infty)$, one has*

$$\left\|B_{n,q}^{(\alpha,\beta)}(f)\right\|\leq\|f\|.$$

Proof. In view of (6.1.2) and Lemma 6.2.2, the proof of this lemma easily follows. □

Remark 6. For every $q\in(0,1)$, we have

$$B_{n,q}^{(\alpha,\beta)}((t-x);x)=\frac{([n]_q^2-([n]_q+\beta)[n-1]_q)x+\alpha[n-1]_q}{([n]_q+\beta)[n-1]_q},\ \ n>1$$

and

$$\begin{aligned}B_{n,q}^{(\alpha,\beta)}((t-x)^2;x) &= \left\{1+\frac{q[n]_q^4}{([n]_q+\beta)^2[n-1]_q[n-2]_q}+(\alpha-1)\frac{2[n]_q^2}{([n]_q+\beta)^2[n-1]_q}\right\}x^2\\ &\quad+\left\{\frac{[2]_q[n]_q^3}{q([n]_q+\beta)^2[n-1]_q[n-2]_q}-\frac{2\alpha}{[n]_q+\beta}\right\}x+\frac{\alpha^2}{([n]_q+\beta)^2},\ \ n>2\\ &:= \gamma_{n,q}^{(\alpha,\beta)}(x),\ \text{say}.\end{aligned}$$

6.3 Main Results

Theorem 6.3.1. *Let $0<q_n<1$ and $J>0$. Then for each $f\in D_\vartheta[0,\infty)$, the sequence $\{B_{n,q_n}^{(\alpha,\beta)}(f;x)\}$ converges to f uniformly on $[0,J]$ if and only if $\lim_{n\to\infty}q_n=1$.*

Proof. First, we assume that $\lim_{n\to\infty} q_n = 1$. We have to show that $\{B_{n,q_n}^{(\alpha,\beta)}(f;x)\}$ converges to f uniformly on $[0,J]$. From Lemma 6.2.2, we see that $B_{n,q_n}^{(\alpha,\beta)}(1;x) \to 1$, $B_{n,q_n}^{(\alpha,\beta)}(t;x) \to x$, $B_{n,q_n}^{(\alpha,\beta)}(t^2;x) \to x^2$, uniformly on $[0,J]$ as $n\to\infty$.
Therefore, the well-known property of the Korovkin theorem implies that $\{B_{n,q_n}^{(\alpha,\beta)}(f;x)\}$ converges to f uniformly on $[0,J]$ provided $f \in D_\vartheta[0,\infty)$.
We show the converse part by contradiction. Assume that q_n does not converge to 1. Then, it must contain a subsequence $\{q_{n_k}\}$ such that $q_{n_k} \in (0,1)$, $q_{n_k} \to a \in [0,1)$ as $k\to\infty$.
Thus, $\frac{1}{[n_k+s]_{q_{n_k}}} = \frac{1-q_{n_k}}{1-(q_{n_k})^{n_k+s}} \to (1-a)$ as $k\to\infty$. Choosing $n=n_k$, $q=q_{n_k}$ in $B_{n,q_n}^{(\alpha,\beta)}(t^2;x)$, from Lemma 6.2.2, we have
$B_{n,q_n}^{(\alpha,\beta)}(t^2;x) \to \frac{(a+2\alpha(1-a))x^2}{(1+(1-a)\beta)^2} + \frac{(1-a^2)x}{a(1+(1-a)\beta)^2} + \frac{(1-a)^2\alpha^2}{(1+(1-a)\beta)^2} \nrightarrow x^2$ as $k\to\infty$,
which leads us to a contradiction. Hence, $\lim_{n\to\infty} q_n = 1$. This completes the proof. □

Theorem 6.3.2. ***(Voronovskaja type theorem)*** *Let $f \in D_\vartheta[0,\infty)$ and $q_n \in (0,1)$ be a sequence such that $q_n \to 1$ and $q_n^n \to 0$ as $n\to\infty$. Suppose that $f''(x)$ exists at a point $x\in[0,\infty)$, then we have*

$$\lim_{n\to\infty}[n]_{q_n}(B_{n,q_n}^{(\alpha,\beta)}(f;x)-f(x)) = (\alpha-\beta x)f'(x) + (1-\alpha)x(1-x)f''(x).$$

Proof. By the Taylor's formula, we may write

$$f(t) = f(x) + (t-x)f'(x) + \frac{1}{2}f''(x)(t-x)^2 + r(t,x)(t-x)^2, \tag{6.3.1}$$

where $r(t,x)$ is the Peano form of the remainder and $\lim_{t\to x} r(t,x) = 0$.
Applying $B_{n,q_n}^{(\alpha,\beta)}(f;x)$ to the both sides of (6.3.1), we get

$$\begin{aligned}[n]_{q_n}(B_{n,q_n}^{(\alpha,\beta)}(f;x)-f(x)) &= [n]_{q_n}f'(x)B_{n,q_n}^{(\alpha,\beta)}((t-x);x) + \frac{1}{2}[n]_{q_n}f''(x)B_{n,q_n}^{(\alpha,\beta)}((t-x)^2;x)\\ &\quad +[n]_{q_n}B_{n,q}^{(\alpha,\beta)}((t-x)^2 r(t,x);x).\end{aligned}$$

In view of Remark 6, we have

$$\lim_{n\to\infty}[n]_{q_n}B_{n,q_n}^{(\alpha,\beta)}((t-x);x) = \alpha-\beta x \tag{6.3.2}$$

and

$$\lim_{n\to\infty}[n]_{q_n}B_{n,q_n}^{(\alpha,\beta)}((t-x)^2;x) = 2x(1-x)(1-\alpha). \tag{6.3.3}$$

Now, we shall show that

$$[n]_{q_n} B_{n,q_n}^{(\alpha,\beta)}\left(r(t,x)(t-x)^2;x\right) \to 0,\ \text{when}\ n\to\infty.$$

By using the Cauchy-Schwarz inequality, we have

$$B_{n,q_n}^{(\alpha,\beta)}\left(r(t,x)(t-x)^2;x\right) \leq \sqrt{B_{n,q_n}^{(\alpha,\beta)}(r^2(t,x);x)}\sqrt{B_{n,q_n}^{(\alpha,\beta)}((t-x)^4;x)}. \tag{6.3.4}$$

We observe that $r^2(x,x)=0$ and $r^2(\cdot,x)\in D_\vartheta[0,\infty)$. Then, it follows from Theorem 6.3.1 that

$$\lim_{n\to\infty} B_{n,q_n}^{(\alpha,\beta)}(r^2(t,x);x) = r^2(x,x) = 0, \tag{6.3.5}$$

in view of the fact that $B_{n,q_n}^{(\alpha,\beta)}((t-x)^4;x) = O\left(\frac{1}{[n]_{q_n}^2}\right)$. Now, from (6.3.4) and (6.3.5), we get

$$\lim_{n\to\infty} B_{n,q_n}^{(\alpha,\beta)}(r(t,x)(t-x)^2;x) = 0, \tag{6.3.6}$$

and from (6.3.2), (6.3.3) and (6.3.6), we get the required result. □

Theorem 6.3.3. ***(Voronovskaja type theorem)*** *Let $f\in D_\vartheta[0,\infty)$ and $q_n\in(0,1)$ be a sequence such that $q_n\to 1$ and $q_n^n\to 0$ as $n\to\infty$. If $f''(x)$ exists on $[0,\infty)$, then*

$$\lim_{n\to\infty}[n]_{q_n}(B_{n,q_n}^{(\alpha,\beta)}(f;x)-f(x)) = (\alpha-\beta x)f'(x)+(1-\alpha)x(1-x)f''(x)$$

holds uniformly on $[0,J]$, where $J>0$.

Proof. Let $x\in[0,J]$. The remainder part of the proof of this theorem is similar to that of the proof of the previous theorem. So we omit it. □

6.3.1 Local approximation

Theorem 6.3.4. *Let $f\in C_B[0,\infty)$ and $q\in(0,1)$. Then, for every $x\in[0,\infty)$ and $n\geq 2$, we have*

$$\left|B_{n,q}^{(\alpha,\beta)}(f(t);x)-f(x)\right| \leq C\omega_2(f;\delta_{n,q}^{(\alpha,\beta)}(x))+\omega\left(f;\frac{([n]_q q^{n-1}-\beta[n-1]_q)x+\alpha[n-1]_q}{([n]_q+\beta)[n-1]_q}\right),$$

where C is an absolute constant and

$$\delta_{n,q}^{(\alpha,\beta)} = \left(B_{n,q}^{(\alpha,\beta)}((t-x)^2;x)+\left(\frac{([n]_q q^{n-1}-\beta[n-1]_q)\,x+\alpha[n-1]_q}{([n]_q+\beta)[n-1]_q}\right)^2\right)^{1/2}.$$

Proof. For $x \in [0,\infty)$, we consider the auxiliary operators $\overline{B}_{n,q}^{(\alpha,\beta)}$ defined by

$$\overline{B}_{n,q}^{(\alpha,\beta)}(f;x) = B_{n,q}^{(\alpha,\beta)}(f;x) - f\left(\frac{[n]_q^2 x}{([n]_q+\beta)[n-1]_q} + \frac{\alpha}{[n]_q+\beta}\right) + f(x). \tag{6.3.7}$$

From Lemma 6.2.2, we observe that the operators $\overline{B}_{n,q}^{(\alpha,\beta)}$ are linear and reproduce the linear functions. Hence

$$\overline{B}_{n,q}^{(\alpha,\beta)}((t-x);x) = 0. \tag{6.3.8}$$

Let $g \in C_B^2[0,\infty)$. By Taylor's theorem, we have

$$g(t) = g(x) + g'(x)(t-x) + \int_x^t (t-u)g''(u)du, \quad t \in [0,\infty).$$

Applying $\overline{B}_{n,q}^{(\alpha,\beta)}$ to the both sides of the above equation and using (6.3.8), we have

$$\overline{B}_{n,q}^{(\alpha,\beta)}(g;x) = g(x) + \overline{B}_{n,q}^{(\alpha,\beta)}\left(\int_x^t (t-u)g''(u)du;x\right).$$

Thus, by (6.3.7) we get

$$\begin{aligned}
|\overline{B}_{n,q}^{(\alpha,\beta)}(g;x) - g(x)| &\leq B_{n,q}^{(\alpha,\beta)}\left(\int_x^t |t-u||g''(u)|du;x\right)\\
&\quad + \int_x^{\frac{[n]_q^2 x}{([n]_q+\beta)[n-1]_q}+\frac{\alpha}{[n]_q+\beta}} \left|\frac{[n]_q^2 x}{([n]_q+\beta)[n-1]_q} + \frac{\alpha}{[n]_q+\beta} - u\right||g''(u)|du\\
&\leq \left(B_{n,q}^{(\alpha,\beta)}((t-x)^2;x) + \left(\frac{[n]_q^2 x}{([n]_q+\beta)[n-1]_q} + \frac{\alpha}{[n]_q+\beta} - x\right)^2\right)\|g''\|\\
&\leq \left(\delta_{n,q}^{(\alpha,\beta)}\right)^2 \|g''\|. \qquad (6.3.9)
\end{aligned}$$

On other hand, by (6.3.7) and Lemma 6.2.3, we have

$$|\overline{B}_{n,q}^{(\alpha,\beta)}(f;x)| \leq |B_{n,q}^{(\alpha,\beta)}(f;x)| + 2\|f\| \leq 3\|f\|. \tag{6.3.10}$$

Using (6.3.9) and (6.3.10) in (6.3.7), we obtain

$$\begin{aligned}
|B_{n,q}^{(\alpha,\beta)}(f;x) - f(x)| &\leq |\overline{B}_{n,q}^{(\alpha,\beta)}(f-g;x)| + |(f-g)(x)| + |\overline{B}_{n,q}^{(\alpha,\beta)}(g;x) - g(x)|\\
&\quad + \left|f\left(\frac{[n]_q^2 x}{([n]_q+\beta)[n-1]_q} + \frac{\alpha}{[n]_q+\beta}\right) - f(x)\right|\\
&\leq 4\|f-g\| + \left(\delta_{n,q}^{(\alpha,\beta)}\right)^2\|g''\| + \left|f\left(\frac{[n]_q^2 x + \alpha[n-1]_q}{([n]_q+\beta)[n-1]_q}\right) - f(x)\right|.
\end{aligned}$$

Hence, taking infimum on the right hand side over all $g \in C_B^2[0,\infty)$ and using (0.7.2), we get the required result. $\square$

Theorem 6.3.5. *Let $f \in D_2[0,\infty), q_n \in (0,1)$ and $\omega(f;\delta,[0,b+1])$ be its modulus of continuity on the finite interval $[0,b+1] \subset [0,\infty)$, where $b>0$. Then, for every $n>2$,*

$$|B_{n,q_n}^{(\alpha,\beta)}(f;x)-f(x)| \leq 4M_f(1+b^2)\gamma_{n,q_n}^{(\alpha,\beta)}(x)+2\omega\left(f;\sqrt{\gamma_{n,q_n}^{(\alpha,\beta)}(x)},[0,b+1]\right),$$

where $\gamma_{n,q_n}^{(\alpha,\beta)}(x)$ is defined in Remark 6.

Proof. From [87], for $x\in[0,b]$ and $t\in[0,\infty)$, we get

$$|f(t)-f(x)| \leq 4M_f(1+b^2)(t-x)^2+\left(1+\frac{|t-x|}{\delta}\right)\omega(f;\delta,[0,b+1]),\ \ \delta>0.$$

Thus, by applying the Cauchy-Schwarz inequality, we have

$$\begin{aligned}
&|B_{n,q_n}^{(\alpha,\beta)}(f;x)-f(x)|\\
&\leq 4M_f(1+b^2)(B_{n,q_n}^{(\alpha,\beta)}(t-x)^2;x)+\omega(f;\delta,[0,b+1])\left(1+\frac{1}{\delta}\left(B_{n,q_n}^{(\alpha,\beta)}(t-x)^2;x\right)^{\frac{1}{2}}\right)\\
&= 4M_f(1+b^2)\gamma_{n,q_n}^{(\alpha,\beta)}(x)+2\omega\left(f;\sqrt{\gamma_{n,q_n}^{(\alpha,\beta)}(x)},[0,b+1]\right),
\end{aligned}$$

on choosing $\delta=\sqrt{\gamma_{n,q_n}^{(\alpha,\beta)}(x)}$. This completes the proof of the theorem. □

6.3.2 Weighted approximation.

Throughout the section, we assume that $\{q_n\}$ is a sequence in $(0,1)$ such that $q_n\to1$ and $q_n^n\to0$ as $n\to\infty$.

Theorem 6.3.6. *For each $f\in D_2^*[0,\infty)$, we have*

$$\lim_{n\to\infty}\left\|B_{n,q_n}^{(\alpha,\beta)}(f)-f\right\|_2=0.$$

Proof. Making use of the Korovkin type theorem on weighted approximation [50], we see that it is sufficient to verify the following three conditions

$$\lim_{n\to\infty}\left\|B_{n,q_n}^{(\alpha,\beta)}(t^k;x)-x^k)\right\|_2=0,\ \ k=0,1,2. \tag{6.3.11}$$

Since $B_{n,q_n}^{(\alpha,\beta)}(1;x)=1$, the condition in (6.3.11) holds for $k=0$.
By Lemma 6.2.2, we have for $n>1$

$$\begin{aligned}
\left\|B_{n,q_n}^{(\alpha,\beta)}(t;x)-x\right\|_2 &\leq \left|\frac{[n]_{q_n}q_n^{n-1}-\beta[n-1]_{q_n}}{([n]_{q_n}+\beta)[n-1]_{q_n}}\right|\sup_{x\in[0,\infty)}\frac{x}{1+x^2}+\frac{\alpha}{[n]_{q_n}+\beta}\sup_{x\in[0,\infty)}\frac{1}{1+x^2}\\
&\leq \left|\frac{[n]_{q_n}q_n^{n-1}-\beta[n-1]_{q_n}}{([n]_{q_n}+\beta)[n-1]_{q_n}}\right|+\frac{\alpha}{[n]_{q_n}+\beta},
\end{aligned}$$

which implies that the condition in (6.3.11) holds for $k = 1$.

Similarly, we can write for $n > 2$

$$\left\|B_{n,q_n}^{(\alpha,\beta)}(t^2;x) - x^2\right\|_2$$

$$\leq \left|\frac{q_n[n]_{q_n}^2}{[n-1]_{q_n}[n-2]_{q_n}} + \frac{2\alpha}{[n-1]_{q_n}} - 1\right| + \frac{[2]_{q_n}[n]_{q_n}}{q_n[n-1]_{q_n}[n-2]_{q_n}} + \frac{\alpha^2}{([n]_{q_n}+\beta)^2},$$

which implies that $\lim_{n\to\infty}\left\|B_{n,q_n}^{(\alpha,\beta)}(t^2;x) - x^2\right\|_2 = 0$, (6.3.11) holds for $k = 2$. □

Now, we present a weighted approximation theorem for functions in $D_2^*[0,\infty)$.

Theorem 6.3.7. *For each $f \in D_2^*[0,\infty)$ and $d > 0$, we have*

$$\lim_{n\to\infty}\sup_{x\in[0,\infty)}\frac{|B_{n,q_n}^{(\alpha,\beta)}(f;x) - f(x)|}{(1+x^2)^{1+d}} = 0.$$

Proof. Let $x_0 \in [0,\infty)$ be arbitrary but fixed. Then

$$\begin{aligned}\sup_{x\in[0,\infty)}\frac{|B_{n,q_n}^{(\alpha,\beta)}(f;x) - f(x)|}{(1+x^2)^{1+d}} &\leq \sup_{x\leq x_0}\frac{|B_{n,q_n}^{(\alpha,\beta)}(f;x) - f(x)|}{(1+x^2)^{1+d}} + \sup_{x>x_0}\frac{|B_{n,q_n}^{(\alpha,\beta)}(f;x) - f(x)|}{(1+x^2)^{1+d}} \\ &\leq \left\|B_{n,q_n}^{(\alpha,\beta)}(f) - f\right\|_{C[0,x_0]} + \|f\|_2\sup_{x>x_0}\frac{|B_{n,q_n}^{(\alpha,\beta)}(1+t^2;x)|}{(1+x^2)^{1+d}} \\ &\quad + \sup_{x>x_0}\frac{|f(x)|}{(1+x^2)^{1+d}}. \qquad (6.3.12)\end{aligned}$$

Since $|f(x)| \leq \|f\|_2(1+x^2)$, we have $\sup_{x>x_0}\frac{|f(x)|}{(1+x^2)^{1+d}} \leq \frac{\|f\|_2}{(1+x_0^2)^d}$.

Let $\epsilon > 0$ be arbitrary. We can choose x_0 to be so large that

$$\frac{\|f\|_2}{(1+x_0^2)^d} < \frac{\epsilon}{3}. \qquad (6.3.13)$$

In view of Theorem 6.3.1, we obtain

$$\begin{aligned}\|f\|_2\lim_{n\to\infty}\frac{|B_{n,q_n}^{(\alpha,\beta)}(1+t^2;x)|}{(1+x^2)^{1+d}} &= \frac{1+x^2}{(1+x^2)^{1+d}}\|f\|_2 \\ &= \frac{\|f\|_2}{(1+x^2)^d} \leq \frac{\|f\|_2}{(1+x_0^2)^d} < \frac{\epsilon}{3}. \qquad (6.3.14)\end{aligned}$$

Using Theorem 6.3.5, we can see that the first term of the inequality (6.3.12), implies that

$$\left\|B_{n,q_n}^{(\alpha,\beta)}(f) - f\right\|_{C[0,x_0]} < \frac{\epsilon}{3}, \ \textit{as } n\to\infty. \qquad (6.3.15)$$

Combining (6.3.13)-(6.3.15), we get the desired result. □

Theorem 6.3.8. *If $f \in D_2^*[0,\infty)$, then we have*

$$|B_{n,q_n}^{(\alpha,\beta)}(f;x)-f(x)| \leq C(1+x^{2+\lambda})\Omega_2(f,\delta_n), x\in[0,\infty),$$

where $\lambda \geq 1, \delta_n^2 = \max\{\alpha_n,\beta_n,\gamma_n\}, \alpha_n,\beta_n,\gamma_n$ being $\left(\frac{q_n[n]_{q_n}^4+2\alpha[n-2]_{q_n}[n]_{q_n}^2}{[n-1]_{q_n}[n-2]_{q_n}([n]_{q_n}+\beta)^2}+1\right), \left(\frac{2[n]_{q_n}^3}{q_n[n-1]_{q_n}[n-2]_{q_n}([n]_{q_n}+\beta)^2}\right)$ and $\frac{\alpha^2}{([n]_{q_n}+\beta)^2}$, respectively and C is a positive constant independent of f and n.

Proof. From the definition of $\Omega_2(f,\delta)$ and Lemma 0.7.1, we have

$$\begin{aligned}|f(t)-f(x)| &\leq (1+(x+|t-x|)^2)\left(1+\frac{|t-x|}{\delta}\right)\Omega_2(f,\delta)\\ &\leq (1+(2x+t)^2)\left(1+\frac{|t-x|}{\delta}\right)\Omega_2(f,\delta)\\ &:= \phi_x(t)\left(1+\frac{1}{\delta}\psi_x(t)\right)\Omega_2(f,\delta),\end{aligned}$$

where $\phi_x(t)=1+(2x+t)^2$ and $\psi_x(t)=|t-x|$. Then we obtain

$$|B_{n,q_n}^{(\alpha,\beta)}(f;x)-f(x)| \leq \left(B_{n,q_n}^{(\alpha,\beta)}(\phi_x;x)+\frac{1}{\delta_n}B_{n,q_n}^{(\alpha,\beta)}(\phi_x\psi_x;x)\right)\Omega_2(f,\delta_n).$$

Now, applying the Cauchy-Schwarz inequality to the second term on the right hand side, we get $|B_{n,q_n}^{(\alpha,\beta)}(f;x)-f(x)|$

$$(6.3.16)\ \leq \left(B_{n,q_n}^{(\alpha,\beta)}(\phi_x;x)+\frac{1}{\delta_n}\sqrt{B_{n,q_n}^{(\alpha,\beta)}(\phi_x^2;x)}\sqrt{B_{n,q_n}^{(\alpha,\beta)}(\psi_x^2;x)}\right)\Omega_2(f,\delta_n).$$

From Lemma 6.2.2,

$$\begin{aligned}\frac{1}{1+x^2}B_{n,q_n}^{(\alpha,\beta)}(1+t^2;x) &= \frac{1}{1+x^2}+\left(\frac{q_n[n]_{q_n}^4+2\alpha[n-2]_{q_n}[n]_{q_n}^2}{[n-1]_{q_n}[n-2]_{q_n}([n]_{q_n}+\beta)^2}\right)\frac{x^2}{1+x^2}\\ &\quad+\frac{[2]_{q_n}[n]_{q_n}^3}{q_n[n-1]_{q_n}[n-2]_{q_n}([n]_{q_n}+\beta)^2}\frac{x}{1+x^2}+\frac{\alpha^2}{([n]_{q_n}+\beta)^2}\frac{1}{1+x^2}\\ (6.3.17)\qquad &\leq 1+C_1,\ \text{where } C_1 \text{ is a positive constant.}\end{aligned}$$

From (6.3.17), there exists a positive constant C_2 such that

$$B_{n,q_n}^{(\alpha,\beta)}(\phi_x;x) \leq C_2(1+x^2).$$

Proceeding similarly, $\frac{1}{1+x^4}B_{n,q_n}^{(\alpha,\beta)}(1+t^4;x) \leq 1+C_3$, where C_3 is a positive constant. So there exists a positive constant C_4, such that $\sqrt{B_{n,q_n}^{(\alpha,\beta)}(\phi_x^2;x)} \leq C_4(1+x^2)$, where $x \in [0,\infty)$. Also, we get

$$\begin{aligned} B_{n,q_n}^{(\alpha,\beta)}(\psi_x^2;x) &= \left\{\frac{q_n[n]_{q_n}^4+2\alpha[n-2]_{q_n}[n]_{q_n}^2}{[n-1]_{q_n}[n-2]_{q_n}([n]_{q_n}+\beta)^2}+1-\frac{2[n]_{q_n}^2}{([n]_{q_n}+\beta)[n-1]_{q_n}}\right\}x^2 \\ &\quad +\left\{\frac{2[n]_{q_n}^3}{q_n[n-1]_{q_n}[n-2]_{q_n}([n]_{q_n}+\beta)^2}-\frac{2\alpha}{[n]_{q_n}+\beta}\right\}x+\frac{\alpha^2}{([n]_{q_n}+\beta)^2} \\ &\le \alpha_n x^2+\beta_n x+\gamma_n. \end{aligned}$$

Hence, from (6.3.16), we have

$$|B_{n,q_n}^{(\alpha,\beta)}(f;x)-f(x)| \le (1+x^2)\left(C_2+\frac{1}{\delta_n}C_4\sqrt{\alpha_n x^2+\beta_n x+\gamma_n}\right)\Omega_2(f,\delta_n).$$

If we take $\delta_n^2=\max\{\alpha_n,\beta_n,\gamma_n\}$, then we get

$$\begin{aligned} |B_{n,q_n}^{(\alpha,\beta)}(f;x)-f(x)| &\le (1+x^2)\left(C_2+C_4\sqrt{x^2+x+1}\right)\Omega_2(f,\delta_n) \\ &\le C_5(1+x^{2+\lambda})\Omega_2(f,\delta_n),\ x\in[0,\infty). \end{aligned}$$

Hence, the proof is completed. □

Next, we obtain the local direct estimate of the operators defined in (6.1.2), using the Lipschitz-type maximal function of order τ.

Theorem 6.3.9. *Let $f \in C_B[0,\infty)$ and $0<\tau\le 1$. Then, for all $x\in[0,\infty)$ we have*

$$|B_{n,q}^{(\alpha,\beta)}(f;x)-f(x)| \le \widehat{\omega}_\tau(f,x)\left(\gamma_{n,q}^{(\alpha,\beta)}(x)\right)^{\tau/2}.$$

Proof. From (0.7.3), we have

$$|B_{n,q}^{(\alpha,\beta)}(f;x)-f(x)| \le \widehat{\omega}_\tau(f,x)B_{n,q}^{(\alpha,\beta)}(|t-x|^\tau;x).$$

Applying Hölder's inequality with $p=\frac{2}{\tau}$ and $\frac{1}{q}=1-\frac{1}{p}$, we get

$$|B_{n,q}^{(\alpha,\beta)}(f;x)-f(x)| \le \widehat{\omega}_\tau(f,x)(B_{n,q}^{(\alpha,\beta)}(t-x)^2;x)^{\frac{\tau}{2}}=\widehat{\omega}_\tau(f,x)\left(\gamma_{n,q}^{(\alpha,\beta)}(x)\right)^{\tau/2}.$$

Thus, the proof is completed. □

6.3.3 Statistical convergence

Let $q_n\in(0,1)$ be a sequence such that

(6.3.18) $$st_A-\lim_n q_n=1, st_A-\lim_n q_n^n=a(a<1) \text{ and } st_A-\lim_n\frac{1}{[n]_{q_n}}=0.$$

Theorem 6.3.10. *Let $A = (a_{nk})$ be a non-negative regular summability matrix and (q_n) be a sequence satisfying (6.3.18). Then, for any compact set $K \subset [0, \infty)$ and for each function $f \in C(K)$, we have*

$$st_A - \lim_n \left\| B_{n,q_n}^{(\alpha,\beta)}(f;.) - f \right\| = 0.$$

Proof. Let $x_0 = \max_{x \in K} x$. From Lemma 6.2.2, $st_A - \lim_n \left\| B_{n,q_n}^{(\alpha,\beta)}(e_0;.) - e_0 \right\| = 0$. Again, by Lemma 6.2.2, we have

$$\sup_{x \in K} |B_{n,q_n}^{(\alpha,\beta)}(e_1;x) - e_1(x)| \leq \left| \frac{[n]_{q_n}^2}{([n]_{q_n} + \beta)[n-1]_{q_n}} - 1 \right| x_0 + \frac{\alpha}{[n]_{q_n} + \beta}.$$

For $\epsilon > 0$, let us define the following sets:

$$\begin{aligned} F &:= \left\{ k : \left\| B_{k,q_k}^{(\alpha,\beta)}(e_1;.) - e_1 \right\| \geq \epsilon \right\} \\ F_1 &:= \left\{ k : \left| \frac{[k]_{q_k}^2}{([k]_{q_k} + \beta)[k-1]_{q_k}} - 1 \right| \geq \frac{\epsilon}{2} \right\} \\ F_2 &:= \left\{ k : \frac{\alpha}{[k]_{q_k} + \beta} \geq \frac{\epsilon}{2} \right\}, \end{aligned}$$

which implies that $F \subseteq F_1 \cup F_2$ and hence for all $n \in \mathbb{N}$, we obtain

$$\sum_{k \in F} a_{nk} \leq \sum_{k \in F_1} a_{nk} + \sum_{k \in F_2} a_{nk}.$$

Hence, taking the limit as $n \to \infty$, we have $st_A - \lim_n \left\| B_{n,q_n}^{(\alpha,\beta)}(e_1;.) - e_1 \right\| = 0$. Similarly, by using Lemma 6.2.2, we have

$$\begin{aligned} \sup_{x \in K} |B_{n,q_n}^{(\alpha,\beta)}(e_2;x) - e_2(x)| &\leq \left| \frac{q_n[n]_{q_n}^4 + 2\alpha[n-2]_{q_n}}{[n-1]_{q_n}[n-2]_{q_n}([n]_{q_n} + \beta)^2} - 1 \right| x_0^2 \\ &\quad + \frac{[2]_{q_n}[n]_{q_n}^3}{q_n[n-1]_{q_n}[n-1]_{q_n}([n]_{q_n} + \beta)^2} x_0 + \frac{\alpha^2}{([n]_{q_n} + \beta)^2}. \end{aligned}$$

Now, let us define the following sets:

$$\begin{aligned} G &:= \left\{ k : \left\| B_{k,q_k}^{(\alpha,\beta)}(e_2;.) - e_2 \right\| \geq \epsilon \right\} \\ G_1 &:= \left\{ k : \left| \frac{q_k[k]_{q_k}^4 + 2\alpha[k-2]_{q_k}}{[k-1]_{q_k}[k-2]_{q_k}([k]_{q_k} + \beta)^2} - 1 \right| \geq \frac{\epsilon}{3} \right\} \\ G_2 &:= \left\{ k : \frac{[2]_{q_k}[k]_{q_k}^3}{q_k[k-1]_{q_k}[k-1]_{q_k}([k]_{q_k} + \beta)^2} \geq \frac{\epsilon}{3} \right\} \\ G_3 &:= \left\{ k : \frac{\alpha^2}{([k]_{q_k} + \beta)^2} \geq \frac{\epsilon}{3} \right\}. \end{aligned}$$

Then, we obtain $G \subseteq G_1 \cup G_2 \cup G_3$, which implies that

$$\sum_{k\in G} a_{nk} \leq \sum_{k\in G_1} a_{nk} + \sum_{k\in G_2} a_{nk} + \sum_{k\in G_3} a_{nk}.$$

Thus, as $n \to \infty$ we get $st_A - \lim_n \left\|B^{(\alpha,\beta)}_{n,q_n}(e_{2;}.) - e_2\right\| = 0$. This completes the proof. $\square$

Theorem 6.3.11. *Let $A = (a_{nk})$ be a non-negative regular summability matrix and (q_n) be a sequence in $(0,1)$ satisfying (6.3.18). Let the operators $B^{(\alpha,\beta)}_{n,q_n}, n \in \mathbb{N}$, be defined as in (6.1.2). Then, for each function $f \in D^*_2[0,\infty)$, we have*

$$st_A - \lim_n \left\|B^{(\alpha,\beta)}_{n,q_n}(f;.) - f\right\|_{\zeta+2} = 0,\ \zeta > 0.$$

Proof. From ([42], p. 191, Th. 3), it is sufficient to prove that $st_A - \lim_n \left\|B^{(\alpha,\beta)}_{n,q_n}(e_i;.) - e_i\right\|_2 = 0$, where $e_i(x) = x^i, \quad i = 0,1,2.$

From Lemma 6.2.2, $st_A - \lim_n \left\|B^{(\alpha,\beta)}_{n,q_n}(e_0;.) - e_0\right\|_2 = 0$ holds.
Again using Lemma 6.2.2, we have

$$\begin{aligned}\left\|B^{(\alpha,\beta)}_{n,q_n}(e_1;.) - e_1\right\|_2 &\leq \sup_{x\in[0,\infty)} \left\{\frac{x}{(1+x^2)}\left|\frac{[n]^2_{q_n}}{([n]_{q_n}+\beta)[n-1]_{q_n}} - 1\right| + \frac{1}{(1+x^2)}\frac{\alpha}{([n]_{q_n}+\beta)}\right\} \\ &\leq \left|\frac{[n]^2_{q_n}}{([n]_{q_n}+\beta)[n-1]_{q_n}} - 1\right| + \frac{\alpha}{([n]_{q_n}+\beta)}. \qquad (6.3.19)\end{aligned}$$

For each $\epsilon > 0$, let us define the following sets:

$$\begin{aligned} R &:= \left\{k : \left\|B^{(\alpha,\beta)}_{k,q_k}(e_1;.) - e_1\right\|_2 \geq \epsilon\right\} \\ R_1 &:= \left\{k : \left|\frac{[k]^2_{q_k}}{([k]_{q_k}+\beta)[k-1]_{q_k}} - 1\right| \geq \frac{\epsilon}{2}\right\} \\ R_2 &:= \left\{k : \frac{\alpha}{[k]_{q_k}+\beta} \geq \frac{\epsilon}{2}\right\},\end{aligned}$$

which yields $R \subseteq R_1 \cup R_2$ in view of (6.3.19), we have

$$\sum_{k\in R} a_{nk} \leq \sum_{k\in R_1} a_{nk} + \sum_{k\in R_2} a_{nk}.$$

Hence, on taking the limit as $n \to \infty$, $st_A - \lim_n \left\|B^{(\alpha,\beta)}_{n,q_n}(e_1;.) - e_1\right\|_2 = 0$.
Proceeding similarly,

$$\begin{aligned}\left\|B^{(\alpha,\beta)}_{n,q_n}(e_2;.) - e_2\right\|_2 &\leq \left|\frac{q_n[n]^4_{q_n} + 2\alpha[n-2]_{q_n}}{[n-1]_{q_n}[n-2]_{q_n}([n]_{q_n}+\beta)^2} - 1\right| \\ &\quad + \frac{[2]_{q_n}[n]^3_{q_n}}{q_n[n-1]_{q_n}[n-1]_{q_n}([n]_{q_n}+\beta)^2} + \frac{\alpha^2}{([n]_{q_n}+\beta)^2}.\end{aligned}$$

Now, let us define the following sets:

$$\begin{aligned} M &:= \left\{k : \left\| B_{k,q_k}^{(\alpha,\beta)}(e_2;.) - e_2 \right\|_2 \geq \epsilon \right\} \\ M_1 &:= \left\{k : \left| \frac{q_k[k]_{q_k}^4 + 2\alpha[k-2]_{q_k}}{[k-1]_{q_k}[k-2]_{q_k}([k]_{q_k}+\beta)^2} - 1 \right| \geq \frac{\epsilon}{3} \right\} \\ M_2 &:= \left\{k : \frac{[2]_{q_k}[k]_{q_k}^3}{q_k[k-1]_{q_k}[k-1]_{q_k}([k]_{q_k}+\beta)^2} \geq \frac{\epsilon}{3} \right\} \\ M_3 &:= \left\{k : \frac{\alpha^2}{([k]_{q_k}+\beta)^2} \geq \frac{\epsilon}{3} \right\}. \end{aligned}$$

Then, we obtain $M \subseteq M_1 \cup M_2 \cup M_3$, which implies that

$$\sum_{k\in M} a_{nk} \leq \sum_{k\in M_1} a_{nk} + \sum_{k\in M_2} a_{nk} + \sum_{k\in M_3} a_{nk}.$$

Hence, taking the limit as $n \to \infty$ we get $st_A - \lim_n \left\| B_{n,q_n}^{(\alpha,\beta)}(e_{2;.}) - e_2 \right\|_2 = 0$. This completes the proof of the theorem. □

6.4 Better Estimates

We propose a modification of the operators $B_{n,q}^{(\alpha,\beta)}(f;x)$ defined in (6.1.2), so that it can preserve constant as well as linear functions.

The modification of the operators given in (6.1.2) is defined as

$$\begin{aligned} \widetilde{B}_{n,q}^{(\alpha,\beta)}(f;x) &= \sum_{\nu=1}^{\infty} q_{n,\nu}(r_n^q(x))q^{\nu-1} \int_0^{\infty/A} b_{n,\nu-1}(q,t) f\left(\frac{q^{\nu}[n]_q t + \alpha}{[n]_q + \beta}\right) d_q t \\ &\quad + E_q\left(-[n]_q r_n^q(x)\right) f\left(\frac{\alpha}{[n]_q + \beta}\right), \end{aligned}$$

where $r_n^q(x) = \dfrac{([n]_q + \beta)[n-1]_q x - \alpha[n-1]_q}{[n]_q^2}$ for $x \in I_n = \left[\frac{\alpha}{[n]_q+\beta}, \infty\right)$ and $n > 1$.

Lemma 6.4.1. *For each $x \in I_n$, by simple computations, we have*

(i) $\widetilde{B}_{n,q}^{(\alpha,\beta)}(1;x) = 1$;

(ii) $\widetilde{B}_{n,q}^{(\alpha,\beta)}(t;x) = x$;

(iii) $$\widetilde{B}_{n,q}^{(\alpha,\beta)}(t^2;x) = \frac{[n-1]_q(q[n]_q^2 + 2\alpha[n-2]_q)}{[n-2]_q[n]_q^2}x^2 + \frac{[2]_q[n]_q^3 - 2q\alpha[n-1]_q(q[n]_q^2 + 2\alpha[n-2]_q)}{q([n]_q+\beta)[n-2]_q[n]_q^2}x + \frac{\alpha^2[n]_q^2(q[n-1]_q + [n-2]_q) + 2\alpha^3[n-1]_q[n-2]_q}{([n]_q+\beta)^2[n]_q^2[n-2]_q}, \quad \text{for } n > 2.$$

Consequently, for each $x \in I_n$, we have the following equalities

$$\begin{aligned}
\widetilde{B}_{n,q}^{(\alpha,\beta)}(t-x;x) &= 0;\\
\widetilde{B}_{n,q}^{(\alpha,\beta)}((t-x)^2;x) &= \frac{[n]_q^2(q[n-1]_q-[n-2]_q)+2\alpha[n-1]_q[n-2]_q}{[n-2]_q[n]_q^2}x^2\\
&\quad+\frac{[2]_q[n]_q^3-2q\alpha[n-1]_q(q[n]_q^2-2\alpha[n-2]_q)}{q([n]_q+\beta)[n-2]_q[n]_q^2}x\\
&\quad+\frac{\alpha^2[n]_q^2(q[n-1]_q+[n-2]_q)+2\alpha[n-1]_q[n-2]_q}{([n]_q+\beta)^2[n]_q^2[n-2]_q},\ \textit{for } n>2.\\
(6.4.1)\qquad &:= \xi_{n,q}^{(\alpha,\beta)}(x),\ \textit{say.}
\end{aligned}$$

Theorem 6.4.2. *Let $f \in C_B[0,\infty)$ and $x \in I_n$. Then, there exists a positive constant C such that*

$$|\widetilde{B}_{n,q}^{(\alpha,\beta)}(f;x)-f(x)| \le C\omega_2\left(f;\sqrt{\xi_{n,q}^{(\alpha,\beta)}(x)}\right),$$

where $\xi_{n,q}^{(\alpha,\beta)}(x)$ is given by (6.4.1).

Proof. Let $g \in C_B^2[0,\infty)$, $x \in I_n$ and $t \in [0,\infty)$. Using Taylor's expansion we have

$$g(t) = g(x)+(t-x)g'(x)+\int_x^t (t-u)g''(u)du.$$

Applying $\widetilde{B}_{n,q}^{(\alpha,\beta)}$ on both sides and using Lemma 6.4.1, we get

$$\widetilde{B}_{n,q}^{(\alpha,\beta)}(g;x)-g(x) = \widetilde{B}_{n,q}^{(\alpha,\beta)}((t-x);x)g'(x)+\widetilde{B}_{n,q}^{(\alpha,\beta)}\left(\int_x^t (t-u)g''(u)du;x\right).$$

Obviously, we have $\left|\int_x^t (t-u)^2 g''(u)du\right| \le (t-x)^2\|g''\|$. Therefore

$$|\ \widetilde{B}_{n,q}^{(\alpha,\beta)}(g;x)-g(x)\ | \le \widetilde{B}_{n,q}^{(\alpha,\beta)}((t-x)^2;x)\,\|g''\| = \xi_{n,q}^{(\alpha,\beta)}(x)\,\|g''\|\,.$$

Since $|\ \widetilde{B}_{n,q}^{(\alpha,\beta)}(f;x)\ | \le \|f\|$, we get

$$\begin{aligned}
|\ \widetilde{B}_{n,q}^{(\alpha,\beta)}(f;x)-f(x)\ | &\le |\ \widetilde{B}_{n,q}^{(\alpha,\beta)}(f-g;x)\ |+|\ (f-g)(x)\ |+|\ \widetilde{B}_{n,q}^{(\alpha,\beta)}(g;x)-g(x)\ |\\
&\le 2\|f-g\|+\xi_{n,q}^{(\alpha,\beta)}(x)\|g''\|.
\end{aligned}$$

Finally, taking the infimum over all $g \in C_B^2[0,\infty)$ and using (0.7.1), (0.7.2) we obtain

$$|\ \widetilde{B}_{n,q}^{(\alpha,\beta)}(f;x)-f(x)\ | \le C\omega_2\left(f;\sqrt{\xi_{n,q}^{(\alpha,\beta)}(x)}\right),$$

which proves the theorem. □

Chapter 7

Approximation by complex Szász-Durrmeyer-Chlodowsky operators in compact disks

7.1 Introduction

For a real function of real variable $f:[0,\infty)\to\mathbb{R}$, İzgi [91] introduced the following composition of Szász-Mirakjan operators by taking the weight function of Chlodowsky-Durrmeyer operators on $C[0,\infty)$ as

$$(7.1.1)\ (\mathcal{F}_n f)(x) = \frac{n+1}{b_n}\sum_{k=0}^{\infty} p_{n,k}\left(\frac{x}{b_n}\right)\int_0^{b_n}\phi_{n,k}\left(\frac{t}{b_n}\right)f(t)dt,\ 0\le x\le b_n,$$

where $p_{n,k}(x) = e^{-nx}\dfrac{(nx)^k}{k!}$, $\phi_{n,k}(t) = \dbinom{n}{k}t^k(1-t)^{n-k}$ and b_n is a sequence of positive real numbers which satisfy $\lim\limits_{n\to\infty} b_n=\infty$, $\lim\limits_{n\to\infty}\dfrac{b_n}{n}=0$.

In the present chapter, we extend some overconvergence properties of the Szász-Durrmeyer-Chlodowsky operators to complex domain. The complex Szász-Durrmeyer-Chlodowsky operators are obtained from the real version, simply by replacing the real variable x by the complex variable z in the operators defined by (7.1.1), which is given below:

$$(7.1.2)\qquad (\mathcal{F}_n f)(z) = \frac{n+1}{b_n}\sum_{k=0}^{\infty} p_{n,k}\left(\frac{z}{b_n}\right)\int_0^{b_n}\phi_{n,k}\left(\frac{t}{b_n}\right)f(t)dt,$$

where $z \in \mathbb{C}$ is such that $0 \le Re(z) \le b_n$.

Throughout the chapter, we consider $\mathbb{D}_R := \{z \in \mathbb{C} : |z| < R\}$, $R > 1$. By H_R, we mean the class of all functions satisfying: $f : [R, b_n] \cup \overline{\mathbb{D}}_R \to \mathbb{C}$ is continuous in $[R, b_n] \cup \overline{\mathbb{D}}_R$, analytic in $\mathbb{D}_R$ i.e. $f(z) = \sum_{p=0}^{\infty} c_p z^p$ for all $z \in \mathbb{D}_R$. Let $1 \le r < R$ and $\| f \|_r = \sup_{|z| \le r} |f(z)|$. In this chapter, we present rate of convergence, Voronovskaja type result for the Szász-Durrmeyer-Chlodowsky operators $\mathcal{F}_n(f; z)$ for analytic functions on compact disks and also study the exact order of approximation for these operators.

7.2 Auxiliary Results

In order to obtain the main results, we first prove basic lemmas:

Lemma 7.2.1. *Denoting* $e_p(z) = z^p$ *and* $\Pi_{n,p}(z) = \mathcal{F}_n(e_p; z)$, *for all* $e_p = t^p$, $p \in \mathbb{N}^0, n \in \mathbb{N}$ *and* $z \in \mathbb{C}$, *we have* $\mathcal{F}_n(e_0; z) = 1$ *and*

$$\Pi_{n,p+1}(z) = \frac{b_n z}{n+p+2}\,\Pi'_{n,p}(z) + \frac{nz + (p+1)b_n}{n+p+2}\,\Pi_{n,p}(z).$$

Also, $\Pi_{n,p}(z)$ *is a polynomial of degree* p.

Proof. Using $b_n z\ p'_{n,k}\left(\frac{z}{b_n}\right) = (kb_n - nz)\ p_{n,k}\left(\frac{z}{b_n}\right)$, we have

$$\begin{aligned} b_n z\ \Pi'_{n,p}(z) &= \frac{n+1}{b_n}\sum_{k=0}^{\infty} b_n z\ p'_{n,k}\left(\frac{z}{b_n}\right)\int_0^{b_n} \phi_{n,k}\left(\frac{t}{b_n}\right) t^p\,dt \\ &= \frac{n+1}{b_n}\sum_{k=0}^{\infty}(kb_n - nz)\ p_{n,k}\left(\frac{z}{b_n}\right)\int_0^{b_n} \phi_{n,k}\left(\frac{t}{b_n}\right) t^p\,dt \\ &= \frac{n+1}{b_n}\sum_{k=0}^{\infty} p_{n,k}\left(\frac{z}{b_n}\right)\int_0^{b_n} \{(k+1)b_n - (n+1)t + (n+1)t - b_n - nz\}\ \phi_{n,k}\left(\frac{t}{b_n}\right) t^p\,dt. \end{aligned}$$

Using the identity $(b_n - t)\left(t\phi_{n,k}\left(\frac{t}{b_n}\right)\right)' = \{(k+1)b_n - (n+1)t\}\left(\phi_{n,k}\left(\frac{t}{b_n}\right)\right)$, we obtain

$$\begin{aligned} b_n z\,\Pi'_{n,p}(z) &= \frac{n+1}{b_n}\sum_{k=0}^{\infty} p_{n,k}\left(\frac{z}{b_n}\right)\int_0^{b_n}\{(k+1)b_n-(n+1)t\}\left(\phi_{n,k}\left(\frac{t}{b_n}\right)\right)t^p\,dt \\ &\quad +(n+1)\,\Pi_{n,p+1}(z)-(nz+b_n)\,\Pi_{n,p}(z) \\ &= \frac{n+1}{b_n}\sum_{k=0}^{\infty} p_{n,k}\left(\frac{z}{b_n}\right)\int_0^{b_n}(b_n-t)\left(t\phi_{n,k}\left(\frac{t}{b_n}\right)\right)' t^p\,dt \\ &\quad +(n+1)\,\Pi_{n,p+1}(z)-(nz+b_n)\,\Pi_{n,p}(z). \end{aligned}$$

Thus integrating by parts on the right side, we get

$$\begin{aligned} b_n z\,\Pi'_{n,p}(z) &= -pb_n\,\Pi_{n,p}(z)+(p+1)\,\Pi_{n,p+1}(z)+(n+1)\,\Pi_{n,p+1}(z)-(nz+b_n)\,\Pi_{n,p}(z) \\ &= (n+p+2)\,\Pi_{n,p+1}(z)-(nz+(p+1)b_n)\,\Pi_{n,p}(z), \end{aligned}$$

which completes the proof of the recurrence relation. Further, by mathematical induction on p, we easily get that $\Pi_{n,p}(z)$ is a polynomial of degree p. □

Lemma 7.2.2. *Let $f \in H_R$ and be bounded and integrable on $[0, b_n]$. Suppose that $f(z) = \sum_{p=0}^{\infty} c_p z^p$ for all $z \in \mathbb{D}_R$ and $1 \le r < R$. Then for all $|z| \le r$ and $n \in \mathbb{N}$, we have*

$$\mathcal{F}_n(f;z) = \sum_{p=0}^{\infty} c_p \mathcal{F}_n(e_p;z).$$

Proof. For any $m \in \mathbb{N}$ and $r < R$, we define $f_m(z) = \sum_{p=0}^{m} c_p z^p$ if $|z| \le r$ and $f_m(x) = f(x)$ if $x \in (r, b_n]$. Since $|f_m(z)| \le \sum_{p=0}^{\infty} |c_p| r^p = C_r$ for $|z| \le r$, $m \in \mathbb{N}$ and f_m is bounded and integrable on $[0, b_n]$

$$|\mathcal{F}_n(f_m;z)| \;\le\; \frac{n+1}{b_n}\sum_{k=0}^{\infty}\left|p_{n,k}\left(\frac{z}{b_n}\right)\right|\int_0^{b_n}\phi_{n,k}\left(\frac{t}{b_n}\right)|f_m(t)|\,dt < \infty.$$

Thus, $\mathcal{F}_n(f_m;z)$ is well defined and an analytic function of z.
Similarly, for the function f, it follows that $\mathcal{F}_n(f;z)$ is also well defined and it is an analytic function of z.
Further, we assume that $f_{m,p}(z) = c_p e_p(z)$ if $|z| \le r$ and $f_{m,p}(x) = \frac{f(x)}{m+1}$ if $x \in (r, b_n]$.
Let $|z| \le r$ and $1 \le r < R$.

Defining $f_m(z) = \sum_{p=0}^{m} f_{m,p}(z)$, by the linearity of $\mathcal{F}_n$, it follows that

$$\mathcal{F}_n(f_m; z) = \sum_{p=0}^{m} c_p \mathcal{F}_n(e_p; z) \text{ for all } |z| \leq r \text{ and } m, n \in \mathbb{N}.$$

It is sufficient to prove that for any fixed $n \in \mathbb{N}$

$$\lim_{m\to\infty} \mathcal{F}_n(f_m; z) = \mathcal{F}_n(f; z),$$

uniformly in compact disk $|z| \leq r$.
But this is immediate from $\lim_{m\to\infty} \|f_m - f\|_r = 0$, from $\|f_m - f\|_{B[0,\infty)} \leq \|f_m - f\|_r$ and from the inequality
$|\mathcal{F}_n(f_m; z) - \mathcal{F}_n(f; z)|$

$$\begin{aligned}
&\leq \frac{n+1}{b_n} \sum_{k=0}^{\infty} \left| p_{n,k}\left(\frac{z}{b_n}\right) \right| \int_0^{b_n} \phi_{n,k}\left(\frac{t}{b_n}\right) |f_m(t) - f(t)| dt \\
&\leq \| f_m - f \|_r \sum_{k=0}^{\infty} \left| p_{n,k}\left(\frac{z}{b_n}\right) \right| \\
&\leq \| f_m - f \|_r \sum_{k=0}^{\infty} \left| \frac{e^{\frac{-nz}{b_n}} \left(\frac{nz}{b_n}\right)^k}{k!} \right| \\
&\leq \| f_m - f \|_r \left| e^{\frac{-nz}{b_n}} \right| \sum_{k=0}^{\infty} \frac{(n|z|)^k}{k!(b_n)^k} \\
&\leq \| f_m - f \|_r \left| e^{\frac{-nz}{b_n}} \right| e^{\frac{n|z|}{b_n}} \leq M_{r,n} \| f_m - f \|_r
\end{aligned}$$

valid for all $|z| \leq r$, where $\| \,.\, \|_{B[0,\infty)}$ denotes the uniform norm on $[0, \infty)$. Thus, as $m \to \infty$, we get the required result. □

7.3 Main Results

7.3.1 Upper estimates

In the following theorem, we obtain an upper estimate of the error in the approximation of an analytic function by the operators (7.1.2) in a compact disk.

Theorem 7.3.1. *Let $f : [R, b_n] \cup \overline{\mathbb{D}}_R \to \mathbb{C}$ be continuous in $[R, b_n] \cup \overline{\mathbb{D}}_R$ and analytic in $\mathbb{D}_R$. Further, let f be bounded and integrable in $[0, b_n]$. Suppose that there exists*

$M > 0$ and $A \in \left(\frac{1}{R}, 1\right)$ with the property $|c_p| \leq \dfrac{MA^p}{(2p)!}$, $\forall\, p \in \mathbb{N}^0$. Let $1 \leq r < \frac{1}{A}$ be arbitrary but fixed then for all $|z| \leq r$ and $n \geq n_0$, $n_0 \in \mathbb{N}$, we have

$$|\mathcal{F}_n(f;z) - f(z)| \leq C_{r,A}(f)\frac{b_n+1}{n+2}, \quad \textit{where} \ \ C_{r,A}(f) = M\sum_{p=1}^{\infty}(Ar)^p < \infty.$$

Proof. By using the recurrence relation of Lemma 7.2.1, we have

$$\Pi_{n,p+1}(z) = \frac{b_n z}{n+p+2}\,\Pi'_{n,p}(z) + \frac{nz+(p+1)b_n}{n+p+2}\,\Pi_{n,p}(z),\ \forall z \in \mathbb{C}, p \in \mathbb{N}^0, n \in \mathbb{N}.$$

From this we immediately get the recurrence formula

$$\begin{aligned}\Pi_{n,p}(z) - z^p \;\; &= \;\; \frac{b_n z}{n+p+1}\,\left(\Pi_{n,p-1}(z) - z^{p-1}\right)' + \frac{nz+pb_n}{n+p+1}\,\left(\Pi_{n,p-1}(z) - z^{p-1}\right)\\ &\quad + \frac{(2p-1)b_n - (p+1)z}{n+p+1} z^{p-1},\ \forall z \in \mathbb{C}, p, n \in \mathbb{N}.\end{aligned}$$

Now, for $1 \leq r < R$, by linear transformation the Bernstein's inequality in the closed unit disk becomes $P_p'(z) \leq \frac{p}{r}\|P_p\|_r$, for all $|z| \leq r$, where $P_p(z)$ is a polynomial of degree $\leq p$. Thus, from the above recurrence relation, we get

$$\begin{aligned}\|\Pi_{n,p} - e_p\|_r \;\; &\leq \;\; \frac{b_n r}{n+p+1}\,\|\Pi_{n,p-1} - e_{p-1}\|_r \frac{p-1}{r} + \frac{nr+pb_n}{n+p+1}\,\|\Pi_{n,p-1} - e_{p-1}\|_r\\ &\quad + \frac{(2p-1)b_n + (p+1)r}{n+p+1} r^{p-1}\\ &\leq \left(r + \frac{(2p-1)b_n}{n+2}\right)\|\Pi_{n,p-1} - e_{p-1}\|_r + \frac{(2p-1)b_n + (p+1)r}{n+2} r^{p-1}\\ &\leq \left(r + \frac{(2p-1)b_n}{n+2}\right)\|\Pi_{n,p-1} - e_{p-1}\|_r + \frac{2p(b_n+1)r^p}{n+2}.\end{aligned}$$

In what follows we prove the result by mathematical induction with respect to p, that this recurrence relation implies

$$\|\Pi_{n,p} - e_p\|_r \;\; \leq \;\; \frac{(2p)!r^p(b_n+1)}{n+2}, \text{ for all } p \in \mathbb{N},\ n \geq n_0, n_0 \in \mathbb{N}.$$

Indeed for $p = 1$ and $n \geq n_0, n_0 \in \mathbb{N}$, the left hand side is $\dfrac{b_n+2r}{n+2}$ and the right hand side is $\dfrac{2r(b_n+2)}{n+2}$. Suppose that it is valid for p, the above recurrence relation implies that

$$\|\Pi_{n,p+1} - e_{p+1}\|_r \;\; \leq \;\; \left(r + \frac{(2p+1)b_n}{n+2}\right)\frac{(2p)!r^p(b_n+1)}{n+2} + (2p+2)r^{p+1}\frac{b_n+1}{n+2},$$

it remains to prove that

$$\left(r+\frac{(2p+1)b_n}{n+2}\right)\frac{(2p)!r^p(b_n+1)}{n+2}+(2p+2)r^{p+1}\frac{b_n+1}{n+2}\leq\frac{(2p+2)!r^{p+1}(b_n+1)}{n+2}$$

or

$$\left(r+\frac{(2p+1)b_n}{n+2}\right)(2p)!+(2p+2)r \quad\leq\quad (2p+2)!r.$$

It is easy to see by mathematical induction that this last inequality holds true for all $p\geq 1$ and $n\geq n_0, n_0\in\mathbb{N}$. From the hypothesis on f, by Lemma 7.2.2 we can write

$$\mathcal{F}_n(f;z)=\sum_{p=0}^{\infty}c_p\mathcal{F}_n(e_p;z)=\sum_{p=0}^{\infty}c_p\Pi_{n,p}(z),\ \text{for all } z\in\mathbb{D}_R, n\in\mathbb{N},$$

which from the hypothesis on c_p immediately implies for all $|z|\leq r$ with $Re(z)\leq b_n, n\in\mathbb{N}$ with $n\geq n_0, n_0\in\mathbb{N}$

$$|\mathcal{F}_n(f;z)-f(z)|\leq\sum_{p=1}^{\infty}|c_p||\Pi_{n,p}(z)-e_p(z)|\leq\sum_{p=1}^{\infty}\frac{M(Ar)^p(b_n+1)}{n+2}=C_{r,A}(f)\frac{(b_n+1)}{n+2},$$

where $C_{r,A}(f)=M\sum_{p=1}^{\infty}(Ar)^p<\infty$ for all $1\leq r<\frac{1}{A}$, by ratio test. Thus, the proof is completed. □

7.3.2 Voronovskaja-type result

In the following theorem we obtain a quantitative Voronovskaja-type result:

Theorem 7.3.2. *Let $f\in H_R$ and be bounded and integrable on $[0,b_n]$ and suppose that there exists $M>0$ and $A\in\left(\frac{1}{R},1\right)$ with the property $|c_p|\leq\frac{MA^p}{(2p)!}$. Let $1\leq r<\frac{1}{A}$ be arbitrary but fixed then for all $|z|\leq r$ and $p\in\mathbb{N}^0$, $n\geq n_0,\ n_0\in\mathbb{N}$, we have*

$$\left|\mathcal{F}_n(f;z)-f(z)-\frac{b_n}{n+2}\left(\left(1-\frac{2z}{b_n}\right)f'(z)+z\left(1-\frac{z}{2b_n}\right)f''(z)\right)\right|\leq L_{r,A}(f)\frac{(b_n+1)^2}{(n+2)^2},$$

where $L_{r,A}(f)=\dfrac{2M}{(1-Ar)\log\frac{1}{Ar}}+4M\sum_{p=1}^{[\alpha]}p(Ar)^p<\infty.$

Proof. By using Lemma 7.2.2, we may write $\mathcal{F}_n(f;z) = \sum_{p=0}^{\infty} c_p \mathcal{F}_n(e_p;z)$ and

$$\frac{b_n}{n+2}\left(\left(1-\frac{2z}{b_n}\right)f'(z)+z\left(1-\frac{z}{2b_n}\right)f''(z)\right) = \frac{b_n}{n+2}\sum_{p=1}^{\infty} c_p\left(p^2z^{p-1}-\frac{p^2+3p}{2b_n}z^p\right).$$

Defining $\Pi_{n,p}(z) = \mathcal{F}_n(e_p)(z)$, we get

$$\left|\mathcal{F}_n(f;z)-f(z)-\frac{b_n}{n+2}\left(\left(1-\frac{2z}{b_n}\right)f'(z)+z\left(1-\frac{z}{2b_n}\right)f''(z)\right)\right|$$
$$\leq \sum_{p=1}^{\infty}|c_p|\left|\Pi_{n,p}(z)-e_p(z)-\frac{b_n}{n+2}\left(p^2e_{p-1}(z)-\frac{p^2+3p}{2b_n}e_p(z)\right)\right|,$$

for all $z \in \mathbb{D}_R$, $n \in \mathbb{N}$. Now, by applying Lemma 7.2.1, we get the following recurrence relation

$$\Pi_{n,p}(z) = \frac{b_n z}{n+p+1}\,\Pi'_{n,p-1}(z) + \frac{nz+pb_n}{n+p+1}\,\Pi_{n,p-1}(z).$$

Let us denote

$$\mathcal{E}_{n,p}(z) = \Pi_{n,p}(z)-e_p(z)-\frac{b_n}{n+2}\left(p^2e_{p-1}(z)-\frac{p^2+3p}{2b_n}e_p(z)\right).$$

Then,

$$\mathcal{E}_{n,p}(z) = \frac{b_n z}{n+p+1}\mathcal{E}'_{n,p-1}(z)+\frac{nz+pb_n}{n+p+1}\mathcal{E}_{n,p-1}(z)+\mathcal{X}_{n,p}(z), \tag{7.3.1}$$

where

$$\begin{aligned}
&\mathcal{X}_{n,p}(z)\\
&= \frac{1}{2(n+p+1)(n+2)}\Big(z^{p-2}b_n^2\{2(p-1)^2(p-2)+2p(p-1)^2\}\\
&\quad +z^{p-1}b_n\{(n+2)(4p-2)+2n(p-1)^2-2(n+p+1)p^2-(p-1)(2p^2+p-2)\}\\
&\quad +z^p\{2n(n+2)-2(n+p+1)(n+2)-n(p-1)^2-3n(p-1)+(p^2+3p)(n+p+1)\}\Big)\\
&= \frac{1}{2(n+p+1)(n+2)}\Big(z^{p-2}b_n^2\{4(p-1)^3\}-z^{p-1}b_n\{4p^3+p^2-10p+6\}\\
&\quad +z^p\{(p+1)(p^2+3p-4)\}\Big).
\end{aligned}$$

Hence

$$|\mathcal{X}_{n,p}(z)| \leq \frac{2(b_n+1)^2(p+1)^3}{(n+2)^2}r^p,\ \forall\, n \in \mathbb{N}. \tag{7.3.2}$$

It is immediate that $\mathcal{E}_{n,p}(z)$ is a polynomial in z of degree $\le p$ and that $\mathcal{E}_{n,0}(z) = 0$. Combining (7.3.1) and (7.3.2), we have

$$|\mathcal{E}_{n,p}(z)| \le \frac{b_n r}{n+2}|\mathcal{E}'_{n,p-1}(z)| + \left(r + \frac{pb_n}{n+2}\right)|\mathcal{E}_{n,p-1}(z)| + \frac{2(b_n+1)^2(p+1)^3}{(n+2)^2}r^p.$$

Now, we shall find the estimate of $\mathcal{E}'_{n,p-1}(z)$ for $p \ge 1$. Taking into account the fact that $\mathcal{E}_{n,p-1}(z)$ is a polynomial of degree $\le p-1$, we have

$$\begin{aligned} |\mathcal{E}'_{n,p-1}(z)| &\le \frac{p-1}{r}\|\mathcal{E}_{n,p-1}\|_r \le \frac{p-1}{r}\Bigg(\|\Pi_{n,p-1} - e_{p-1}\|_r \\ &+ \left\| \frac{b_n}{n+2}\left((p-1)^2 e_{p-2}(z) - \frac{(p-1)^2 + 3(p-1)}{2b_n} e_{p-1}(z)\right)\right\|_r \Bigg) \\ &\le \frac{p-1}{r}\left(\frac{(2p-2)!r^{p-1}(b_n+1)}{n+2} + \frac{2(p-1)^2 b_n + (p-1)(p+2)}{2(n+2)} r^{p-1}\right) \\ &\le \frac{2(2p-2)!(p-1)r^{p-2}(b_n+1)}{n+2}, \ \forall\, n \in \mathbb{N}. \end{aligned}$$

Thus

$$\frac{rb_n}{n+2}|\mathcal{E}'_{n,p-1}(z)| \le \frac{2(2p-2)!(p-1)r^{p-1}(b_n+1)^2}{(n+2)^2}, \ \forall\, n \in \mathbb{N}$$

and

$$\begin{aligned} &|\mathcal{E}_{n,p}(z)| \\ &\le \left(r + \frac{pb_n}{n+2}\right)|\mathcal{E}_{n,p-1}(z)| + \frac{2(2p-2)!(p-1)r^{p-1}(b_n+1)^2}{(n+2)^2} + \frac{2(b_n+1)^2(p+1)^3}{(n+2)^2}r^p \\ &\le \left(r + \frac{pb_n}{n+2}\right)|\mathcal{E}_{n,p-1}(z)| + \frac{4(2p)!r^p(b_n+1)^2}{(n+2)^2}, \text{ for all } |z| \le r \text{ and } n \in \mathbb{N}. \end{aligned}$$

For $1 \le p \le \dfrac{n+2}{b_n} = \alpha\,(say)$ and $|z| \le r$, taking into account that $(r + p\alpha) \le (r+1)$, we have

$$|\mathcal{E}_{n,p}(z)| \le (r+1)|\mathcal{E}_{n,p-1}(z)| + \frac{4(2p)!r^p(b_n+1)^2}{(n+2)^2}.$$

But $\mathcal{E}_{n,0}(z) = 0$, for any $z \in \mathbb{C}$, therefore by writing the inequality for $1 \le p \le \alpha$, we easily obtain step by step the following

$$|\mathcal{E}_{n,p}(z)| \le \frac{4r^p(b_n+1)^2}{(n+2)^2}\sum_{j=1}^{p}(2j)! \le \frac{4p(2p)!r^p(b_n+1)^2}{(n+2)^2}.$$

Denoting by $[\alpha]$ the integral part of α, it follows that

$$\left|\mathcal{F}_n(f;z) - f(z) - \frac{b_n}{n+2}\left(\left(1-\frac{2z}{b_n}\right)f'(z) + z\left(1-\frac{z}{2b_n}\right)f''(z)\right)\right|$$

$$\begin{aligned}
&\leq \sum_{p=1}^{[\alpha]} |c_p||\mathcal{E}_{n,p}(z)| + \sum_{p=[\alpha]+1}^{\infty} |c_p||\mathcal{E}_{n,p}(z)| \\
&\leq \sum_{p=1}^{[\alpha]} |c_p| \frac{4p(2p)!r^p(b_n+1)^2}{(n+2)^2} + \sum_{p=[\alpha]+1}^{\infty} |c_p||\mathcal{E}_{n,p}(z)| \\
&\leq \frac{4M(b_n+1)^2}{(n+2)^2} \sum_{p=1}^{[\alpha]} p(Ar)^p + \sum_{p=[\alpha]+1}^{\infty} |c_p||\mathcal{E}_{n,p}(z)|.
\end{aligned}$$

But

$$\begin{aligned}
\sum_{p=[\alpha]+1}^{\infty} |c_p||\mathcal{E}_{n,p}(z)| &\leq \sum_{p=[\alpha]+1}^{\infty} |c_p| \left(|\Pi_{n,p}(z) - e_p(z)| + \frac{b_n}{n+2}\left|p^2 e_{p-1}(z) - \frac{p^2+3p}{2b_n} e_p(z)\right|\right) \\
&\leq \sum_{p=[\alpha]+1}^{\infty} |c_p| \left(\left|\frac{(2p)!r^p(b_n+1)}{n+2}\right| + \frac{r^p}{n+2}\left|p^2 b_n + \frac{p(p+3)}{2}\right|\right) \\
&\leq 2\sum_{p=[\alpha]+1}^{\infty} |c_p| \frac{(2p)!r^p(b_n+1)}{n+2} \\
&\leq \frac{2M(b_n+1)}{(n+2)} \sum_{p=[\alpha]+1}^{\infty} (Ar)^p \leq \frac{2M(b_n+1)}{(n+2)} \frac{(Ar)^\alpha}{(1-Ar)}, \forall n \geq n_0,\ n_0 \in \mathbb{N}.
\end{aligned}$$

Also, by $e^t = 1 + t + \frac{t^2}{2} + \cdots$, we get $e^t \geq t\ \ \forall\ t \geq 0$, which combined with $\dfrac{1}{(Ar)^\alpha} = e^{\alpha \log\left(\frac{1}{Ar}\right)} \Rightarrow \dfrac{1}{(Ar)^\alpha} \geq \alpha \log \frac{1}{Ar}$, for all $\alpha > 0$. So, $(Ar)^\alpha \leq \frac{1}{\alpha \log \frac{1}{Ar}}$.
Therefore, we get

$$\sum_{p=[\alpha]+1}^{\infty} |c_p||\mathcal{E}_{n,p}(z)| \leq \frac{2M(b_n+1)^2}{(n+2)^2(1-Ar)\log\frac{1}{Ar}}, \text{ for all } |z| \leq r \text{ and } n \geq n_0,\ n_0 \in \mathbb{N}.$$

Finally, we obtain

$$\left|\mathcal{F}_n(f;z) - f(z) - \frac{b_n}{n+2}\left(\left(1-\frac{2z}{b_n}\right)f'(z) + z\left(1-\frac{z}{2b_n}\right)f''(z)\right)\right|$$

$$\leq \frac{4M(b_n+1)^2}{(n+2)^2} \sum_{p=1}^{[\alpha]} p(Ar)^p + \frac{2M(b_n+1)^2}{(n+2)^2(1-Ar)\log\frac{1}{Ar}},$$

where for $rA < 1$, by ratio test the above series is convergent. This completes the proof of the theorem. □

7.3.3 Exact order of approximation

To obtain the exact degree of approximation by $\mathcal{F}_n(f;z)$, in the following theorem we get a lower estimate of the error in the approximation of f by $\mathcal{F}_n(f;z)$:

Theorem 7.3.3. *In the hypothesis of Theorem 7.3.2, if f is not a polynomial of degree ≤ 0, then for any $1 \leq r < R$, we have*

$$\| \mathcal{F}_n(f) - f \|_r \geq \frac{b_n+1}{n+2} P_r(f), \quad n \geq n_0, \; n_0 \in \mathbb{N},$$

where the constants in the equivalence $P_r(f) > 0$, depends on f, r.

Proof. For all $|z| \leq r$, and $n \in \mathbb{N}$, we can write the following equality

$$\begin{aligned} &\mathcal{F}_n(f;z) - f(z) \\ &= \frac{b_n+1}{n+2}\Bigg\{ \left(z\left(1-\frac{z}{2b_n}\right) f''(z) + \left(1-\frac{2z}{b_n}\right) f'(z) \right) \\ &\quad + \frac{b_n+1}{n+2}\left(\frac{(n+2)^2}{(b_n+1)^2}\left(\mathcal{F}_n(f;z) - f(z) - \frac{b_n+1}{n+2}\left(\left(1-\frac{2z}{b_n}\right) f'(z) + z\left(1-\frac{z}{2b_n}\right) f''(z) \right)\right)\right)\Bigg\}. \end{aligned}$$

By applying the property

$$\| F+G \|_r \geq | \, \| F \|_r - \| G \|_r \, | \geq \| F \|_r - \| G \|_r,$$

it follows that

$$\begin{aligned} &\|\mathcal{F}_n(f;.) - f\|_r \\ &\geq \frac{b_n+1}{n+2}\Bigg\{ \left\| \left(e_1\left(1-\frac{e_1}{2b_n}\right) f'' + \left(1-\frac{2e_1}{b_n}\right) f' \right) \right\|_r \\ &\quad - \frac{b_n+1}{n+2}\left(\frac{(n+2)^2}{(b_n+1)^2}\left\| (\mathcal{F}_n(f;.) - f - \frac{b_n+1}{n+2}\left(e_1\left(1-\frac{e_1}{2b_n}\right) f'' + \left(1-\frac{2e_1}{b_n}\right) f' \right) \right\|_r \right) \Bigg\}. \end{aligned}$$

Taking into account that by hypothesis f is not a polynomial of degree ≤ 0 in $\mathbb{D}_R$, we get

$$\left\| \left(e_1\left(1-\frac{e_1}{2b_n}\right) f'' + \left(1-\frac{2e_1}{b_n}\right) f' \right) \right\|_r > 0.$$

Indeed, supposing the contrary, it follows that

$$z\left(1-\frac{z}{2b_n}\right) f''(z) + \left(1-\frac{2z}{b_n}\right) f'(z) = 0 \text{ for all } |z| \leq r. \tag{7.3.3}$$

Let us take $f(z) = \sum_{p=0}^{\infty} c_p z^p$, where $c_p, 0 \leq p < \infty$, are certain constants.

Then

$$f'(z) = \sum_{p=1}^{\infty} p c_p z^{p-1}, \; f''(z) = \sum_{p=2}^{\infty} p(p-1) c_p z^{p-2}.$$

By substituting these values in (7.3.3), we obtain

$$\left(1-\frac{z}{2b_n}\right)\sum_{p=2}^{\infty}p(p-1)c_pz^{p-1}+\left(1-\frac{2z}{b_n}\right)\sum_{p=1}^{\infty}pc_pz^{p-1}=0,$$

or

$$c_1+\left(4c_2-\frac{2}{b_n}c_1\right)z+\sum_{p=2}^{\infty}\left((p+1)^2c_{p+1}-\frac{p(p+3)}{2b_n}c_p\right)z^p=0,\ |z|\leq r.$$

From the above series, we easily get $c_p=0,\ \forall p\in\mathbb{N}$ and $f(z)=c_0$, a contradiction to hypothesis.

Now, from Theorem 7.3.2, we have

$$\frac{(n+2)^2}{(b_n+1)^2}\left\|(\mathcal{F}_n(f;.)-f-\frac{b_n+1}{n+2}\left(e_1\left(1-\frac{e_1}{2b_n}\right)f''+\left(1-\frac{2e_1}{b_n}\right)f'\right)\right\|_r$$
$$\leq\ L_{r,A}(f),\ \forall\ n\geq n_0,\ n_0\in\mathbb{N}.$$

Therefore there exists an index $n^*>n_0$ depending only on f,r such that $n\geq n^*$ we have

$$\left\|e_1\left(1-\frac{e_1}{2b_n}\right)f''+\left(1-\frac{2e_1}{b_n}\right)f'\right\|_r$$
$$-\frac{b_n+1}{n+2}\left(\frac{(n+2)^2}{(b_n+1)^2}\left\|\mathcal{F}_n(f;.)-f-\frac{b_n+1}{n+2}\left(e_1\left(1-\frac{e_1}{2b_n}\right)f''+\left(1-\frac{2e_1}{b_n}\right)f'\right)\right\|_r\right)$$
$$\geq\frac{1}{2}\left\|\left(e_1\left(1-\frac{e_1}{2b_n}\right)f''+\left(1-\frac{2z}{b_n}\right)f'\right)\right\|_r,$$

which immediately implies that

$$\|\mathcal{F}_n(f;.)-f\|_r\geq\frac{b_n+1}{2(n+2)}\left\|e_1\left(1-\frac{e_1}{2b_n}\right)f''+\left(1-\frac{2e_1}{b_n}\right)f'\right\|_r.$$

For $n_0\leq n<n^*$, we have

$$\|\mathcal{F}_n(f;.)-f\|_r\geq\frac{b_n+1}{(n+2)}J_{r,n}(f)$$

with $J_{r,n}(f)=\dfrac{n+2}{b_n+1}\|\mathcal{F}_n(f;.)-f\|_r>0$. Indeed, if we would have $\|\mathcal{F}_n(f;.)-f\|_r=0$, it would follow that $\mathcal{F}_n(f;z)=f(z)$ for all $|z|\leq r$, which is valid only for f, a constant function, contradicting the hypothesis on f. Therefore, finally we get

$$\|\mathcal{F}_n(f;.)-f\|_r\geq\frac{b_n+1}{(n+2)}P_r(f),\ \text{ for all }\ n\geq n_0,\ n_0\in\mathbb{N},$$

where

$$P_r(f) = \min\left\{J_{r,n_0}(f), J_{r,n_0+1}(f), \cdots J_{r,n^*-1}(f), \frac{1}{2}\left\|e_1\left(1-\frac{e_1}{2b_n}\right)f'' + \left(1-\frac{2e_1}{b_n}\right)f'\right\|_r\right\}$$

which completes the proof. □

Now, combining Theorem 7.3.1 and Theorem 7.3.3, we immediately get the following:

Corollary 8. In the hypothesis of Theorem 7.3.2, if f is not a polynomial of degree ≤ 0, then for any $1 \leq r < R$, we have

$$\| \mathcal{F}_n(f;.) - f \|_r \sim \frac{b_n+1}{n+2},\ n \geq n_0,\ n_0 \in \mathbb{N}$$

holds, where the constant in the equivalence $\sim$ depends on f, r.

7.3.4 Simultaneous approximation

Concerning the derivatives of complex Szász-Durrmeyer-Chlodowsky operators, we can prove the following results:

Theorem 7.3.4. *In the hypothesis of Theorem 7.3.2, let $1 \leq r < r_1 < R$ and $p \in \mathbb{N}$, then for all $|z| \leq r$ and $n \geq n_0$, $n_0 \in \mathbb{N}$, we have*

$$|\mathcal{F}_n^{(p)}(f;z) - f^{(p)}(z)| \leq \frac{(b_n+1)\, C_{r_1,A}(f)\ p!\ r_1}{(n+2)(r_1-r)^{p+1}},$$

where $C_{r_1,A}(f)$ is defined as in Theorem 7.3.1.

Proof. Denoting by Γ, the circle of radius $r_1 > 1$ and center 0, since for any $|z| \leq r$ and $\nu \in \Gamma$ we have $|\nu - z| \geq r_1 - r$, by Cauchy's formula, it follows that for all $n \in \mathbb{N}$, we get

$$|\mathcal{F}_n^{(p)}(f;z) - f^{(p)}(z)| \ = \ \frac{p!}{2\pi}\left|\int_\Gamma \frac{\mathcal{F}_n(f;\nu) - f(\nu)}{(\nu - z)^{p+1}} d\nu\right|.$$

For all $\nu \in \Gamma$ and $n \in \mathbb{N}$ with $n \geq n_0$, $n_0 \in \mathbb{N}$, we get

$$|\mathcal{F}_n^{(p)}(f;z) - f^{(p)}(z)| \ \leq \ \frac{p!\, r_1\, (b_n+1)\, C_{r_1,A}(f)}{(n+2)\,(r_1-r)^{p+1}},$$

which proves the theorem. □

Theorem 7.3.5. *In the hypothesis of Theorem 7.3.2, let $1 \le r < r_1 < R$ and f be not a polynomial of degree $\le p-1$, $(p \ge 1)$ then we have*

$$\| \mathcal{F}_n^{(p)}(f;.) - f^{(p)} \|_r \sim \frac{b_n+1}{n+2}, \text{ for all } n \ge n_0,\ n_0 \in \mathbb{N},$$

where the constant in the equivalence $\sim$ depends only on f, r, r_1, p.

Proof. Let Γ be a circle of radius $r_1 > r \ge 1$ and center 0, we have

$$\left\| \mathcal{F}_n^{(p)}(f;.) - f^{(p)} \right\|_r$$

$$= \left\| \frac{b_n+1}{(n+2)} \left\{ \frac{p!}{2\pi i} \int_\Gamma \frac{\left[\nu \left(1 - \frac{\nu}{2b_n}\right) f''(\nu) + \left(1 - \frac{2\nu}{b_n}\right) f'(\nu) \right]}{(\nu - z)^{p+1}} d\nu + \frac{b_n+1}{(n+2)} \frac{p!}{2\pi i} \right.\right.$$

$$\left.\left. \times \int_\Gamma \frac{(n+2)^2}{(b_n+1)^2} \frac{\left[\mathcal{F}_n(f;\nu) - f(\nu) - \frac{b_n+1}{(n+2)} \left(\left(1 - \frac{2\nu}{b_n}\right) f'(\nu) + \nu \left(1 - \frac{\nu}{2b_n}\right) f''(\nu) \right) \right]}{(\nu - z)^{p+1}} d\nu \right\} \right\|_r$$

$$\ge \frac{b_n+1}{(n+2)} \left\{ \left\| \left(\frac{e_1 \left(1 - \frac{e_1}{2b_n}\right) f'' + \left(1 - \frac{2e_1}{b_n}\right) f'}{2} \right)^{(p)} \right\|_r - \frac{b_n+1}{(n+2)} \left\| \frac{p!}{2\pi} \right.\right.$$

$$\left.\left. \times \frac{(n+2)^2}{(b_n+1)^2} \int_\Gamma \frac{\left(\mathcal{F}_n(f;\nu) - f(\nu) - \frac{b_n+1}{(n+2)} \left(e_1 \left(1 - \frac{e_1}{2b_n}\right) f'' + \left(1 - \frac{2e_1}{b_n}\right) f' \right) \right)}{(\nu - z)^{p+1}} d\nu \right\|_r \right\}.$$

Now, applying Theorem 7.3.2

$$\left\| \frac{p!}{2\pi} \frac{(n+2)^2}{(b_n+1)^2} \int_\Gamma \frac{\left(\mathcal{F}_n(f;\nu) - f(\nu) - \frac{b_n+1}{(n+2)} \left(e_1 \left(1 - \frac{e_1}{2b_n}\right) f'' + \left(1 - \frac{2e_1}{b_n}\right) f' \right) \right)}{(\nu - z)^{p+1}} d\nu \right\|_r \Bigg\}$$

$$\le \frac{p!}{2\pi} \frac{2\pi r_1 (n+2)^2}{(b_n+1)^2 (r_1 - r)^{p+1}} \left\| \mathcal{F}_n(f;\cdot) - f - \frac{b_n+1}{(n+2)} \left(\left(1 - \frac{2e_1}{b_n}\right) f' + e_1 \left(1 - \frac{e_1}{2b_n}\right) f'' \right) \right\|_{r_1}$$

$$\le \frac{p!\ r_1\ L_{r_1,A}(f)}{(r_1 - r)^{p+1}},$$

but by hypothesis on f, we have $\left\| \left(\left(1 - \frac{2e_1}{b_n}\right) f' + e_1 \left(1 - \frac{e_1}{2b_n}\right) f'' \right)^{(p)} \right\|_r > 0$. Indeed if we suppose the contrary that

$$\left\| \left(\left(1 - \frac{2e_1}{b_n}\right) f' + e_1 \left(1 - \frac{e_1}{2b_n}\right) f'' \right)^{(p)} \right\|_r = 0,$$

then

$$\left(1 - \frac{2z}{b_n}\right) f'(z) + z \left(1 - \frac{z}{2b_n}\right) f''(z) = Q_{p-1}(z),$$

where $Q_{p-1}(z)$ is a polynomial of degree $\leq p-1$, thus f satisfies the differential equation

$$\left(1-\frac{2z}{b_n}\right) f'(z) + z\left(1-\frac{z}{2b_n}\right) f''(z) = Q_{p-1}(z),\ \forall\ |z| \leq r.$$

Now, denoting $f'(z) = y(z)$, the above differential equation reduces to

$$\left(1-\frac{2z}{b_n}\right) y(z) + z\left(1-\frac{z}{2b_n}\right) y'(z) = Q_{p-1}(z),\ \forall\ |z| \leq r.$$

In what follows, let us define $y(x) = y_1(x) + iy_2(x)$, where $y_1(x)$ and $y_2(x)$ are real functions of the real variable and $i^2 = -1$. The functions $y_j(x), j = 1,2$ satisfy the differential equations

$$(7.3.4)\ \left(1-\frac{2x}{b_n}\right) y_j(x) + x\left(1-\frac{x}{2b_n}\right) y_j'(x) = Q_{p-1}(x),\ \forall\ x \in [-1,1],\ j = 1,2,$$

which is a non-homogeneous differential equation. By a similar reasoning as in the proof of Theorem 7.3.3, the unique solution of homogeneous differential equation corresponding to equation (7.3.4) is $y_j(x) = 0$, $\forall x \in [-1,1]$. To find the particular solution of non-homogeneous differential equation (7.3.4) of the form $y_j(x) = \sum_{k=0}^{p-1} c_k x^k$ with $c_k \in \mathbb{R}$, by simple calculations we easily obtain that

$$\left(1-\frac{2x}{b_n}\right)\sum_{k=0}^{p-1} c_k x^k + \left(1-\frac{x}{2b_n}\right)\sum_{k=1}^{p-1} k c_k x^k = Q_{p-1}(x) = \sum_{k=0}^{p-1} d_k x^k \text{ (say)},$$

which implies that

$$c_0 = d_0,\ (k+1)c_k - \left(\frac{k-1}{2b_n} + \frac{2}{b_n}\right) c_{k-1} = d_k,\ k \in \{1, 2, \cdots p-1\}.$$

Thus, c_k's can be uniquely determined. Hence, it follows that $y_1(x)$ and $y_2(x)$ are polynomials of degree $\leq p-1$ in x. Now, because $y(z)$ is the analytic continuation of $y(x)$, from the identity theorem on analytic functions, it follows that $y(z)$ is a polynomial of degree $\leq p-1$ in z, a contradiction to the hypothesis.

So,

$$\left\|\left(e_1\left(1-\frac{e_1}{2b_n}\right) f'' + \left(1-\frac{2e_1}{b_n}\right) f'\right)^{(p)}\right\|_r > 0.$$

In continuation, reasoning exactly as in the proof of Theorem 7.3.3 and using Theorem 7.3.4, we get the desired conclusion. □

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