MODELLING AND ORDER REDUCTION OF GUIDED WEAPON CONTROL SYSTEM

A DISSERTATION

Submitted in partial fulfillment of the requirements for the award of the degree

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in

ELECTRICAL ENGINEERING

(With Specialization in Measurement and Instrumentation)

By

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CANDIDATE'S DECLARATION

I hereby declare that the work presented in this dissertation entitled **"Modelling and Order Reduction of Guided Weapon Control System"** submitted in partial fulfilment of the requirements for the award of the degree of Master of Technology in Electrical Engineering with specialization in **Measurement and Instrumentation** in the **Department of Electrical Engineering, Indian Institute of Technology Roorkee,** is an authentic record of my own work carried out from July 2005 to June 2006 under the guidance of **Dr. R. N. Mishra,** Professor, Department of Electrical Engineering, Indian Institute of Technology Roorkee.

I have not submitted the matter embodied in this report for the award of any other degree or diploma.

Date: 30 June 2006

Place: Roorkee

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CERTIFICATE

This is to certify that the above statement made by the candidate is correct to the best of my knowledge.

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ABSTRACT

Weapons used for self defence such as surface to air missiles or anti-ballistic missiles, should have a very high single shot kill probability. Guided weapon systems use a close loop system to reduce the miss distance and improve the single shot kill probability. The objective here is to obtain a mathematical model of the Guided missile having six degrees of freedom in a three dimensional plane; and to reduce the order of the system. Modelling of the system is useful for the purpose of analysis, design and testing for the accuracy of the control system, since testing on the real systems are too expensive and limited by political and strategic reasons. The overall model obtained is reduced to 4^{th} and 3^{rd} order using four different algorithms, and their performance is compared. Integral square error obtained for a step input to the original system and reduced order system is used as the merit index. Simulation of the missile trajectory with user configurable initial condition is also carried out on a three dimensional plane.

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Chapter - 1
INTRODUCTION

More and more sophisticated weapons and surveillance equipments are being deployed in a modern battlefield. Accuracy and effectiveness of a weapon system, in many cases, becomes more critical and desirable than the lethality of the system [20]. For example, the self-defence against a high speed new generation targets such as ballistic missiles becomes extremely difficult. When a ballistic missile re-enters the atmosphere from a very high altitude, its radar cross-section is relatively small, its speed is high and the remaining time of impact is short [21]. In such a scenario, intelligent instrumentation and control systems play a vital role.

Guided missiles employ a closed loop guidance system for engaging to a target. A target tracker, typically radar or an infrared tracker acts as the observation instrumentation, which measures the position, size, and behaviour of the target and the missile. Such instrumentation is actually contained in the missile itself or it is situated at a ground or mobile platform control centre. A guidance computer, which also can be either inside the missile or at ground, processes the available information and generates the necessary commands. These commands are communicated to the missile by wireless means such as radio signal, infrared signal, or in some cases using copper wires or optical fibre cable (OFC) [22]. A missile responds to the guidance commands either by modulating its fins controlled by a fin servo system or alters the direction of its propulsive thrust (thrust vector control) [2] which results in the desired change in the course of the missile trajectory.

As indicated earlier, the accuracy of the overall system, which consist of the target tracker and the missile guidance system is of prime concern from the perspective of a control engineer. The various sources of error in the target tracker are : (i) Error in estimation of target direction due to inertia of the target tracker (ii) Biases and disturbances such as wind, variable friction, biases in receiver etc. (iii) Thermal noise and glint (iv) Error inherent in the control loop [2]. Error in the missile guidance loop is ascribed mainly to the nonlinearities associated with the missile dynamics, maximum lateral acceleration that the missile can supply and the error inherent in the control loop. The overall accuracy also depends greatly on the guidance law that is employed.

Testing for accuracy on a real system is too expensive; also, it is limited by strategic and political reasons. Hence, modelling and simulation of the system is highly desirable for the performance evaluation. While modelling a complex nonlinear dynamic system, one may need to use two stages of approximation. First is the approximation based on physical characteristics; for instance, linearising the non-linear system at some specific operating point, or neglecting some environmental effects such as temperature pressure humidity etc. When the model of the system is still complex with higher system order, a second stage of mathematical approximation, namely model order reduction or model simplification can be used to obtain a simple model [8].

Reduced model order is highly appreciated in terms of computational efficiency when it comes to real time applications such as model predictive adaptive control [19] and while using genetic algorithms. For example, when the state space model is used to represent a system, the number of multiplications required is increased exponentially as the order of the system matrix increases. Reduced order model is also preferred for design of controller by using methods such as pole-zero cancellation technique [6]. This results in a lower order of the controller and subsequently, reduced computation time and a better comprehensive system.

The present work involves modelling of a typical guided missile with six degrees of freedom, having two axes of symmetry; and the target tracking system in a three dimensional plane, represented in a polar coordinate system. Model order reduction algorithms by using Ruth stability criteria and Optimal order reduction were implemented in MATLAB-7 and a new algorithm by using Genetic algorithm is also developed. Algorithm for simulation of trajectory of a surface to air missile, targeting at an aircraft is developed. It is carried out in a three dimensional polar coordinate frame, by using the state space model of the system. Trajectory simulation with different initial condition of the target such as velocity, altitude, and starting point is possible and the miss-distance, which is the measure of the accuracy of the system, can be obtained at the end of simulation.

Chapter - 2
OVERVIEW OF GUIDED MISSILES

2.1 Guided Missiles

Although the principles by which rockets operated were not well understood until late 1800s, the use of rocket dates back to as early as 1212 AD [4]. These rockets, used by the Chinese in warfare were essentially arrows powered by gunpowder as the solid fuel.

Military rockets that can be directed in flight to change its flight path are called guided missiles. The first missiles to be used operationally were a series of German missiles. They contained simple mechanical controls and autopilot to keep the missile flying along a pre-programmed trajectory. Since accuracy and timing are very critical in a battlefield, it is highly desirable that the probability of missing the target or 'miss-distance' should be as low as possible. The miss-distance in an unguided missile is caused by the reasons such as (i) incorrect direction of take off when the missile is launched, (ii) deflection of the missile by wind or weather, (iii) unpredictable movement of the target after launch of the missile and (iv) manufacturing defects [2][9]. Miss-distances in the guided missiles, on the contrary are greatly reduced by the action of feedback control system operating inside the missile, whose objective is to reduce the distance between the missile and the target. As opposed to the unguided weapons, guided missiles have assumed much importance over the past 50 years.

2.1.1 Types of Guided Missiles [4][1][18]

Guided missiles can be broadly classified into two, depending on the nature of mobility of the target. They are 'Guide Onto Location In Space' (GOLIS) systems and 'Guide Onto Target' (GOT) systems [1]. In GOLIS systems the target is fixed, such as a place or building and the objective of the control system is to steer the missile through a predefined trajectory with minimum time and energy. On the other hand GOT systems are used against fast or slow moving targets such as enemy aircraft, ballistic missiles or tanks. The trajectory of a GOT system is not pre-defined but is highly depend on enemy initiatives and target movement. Hence, they are fast response systems with dedicated surveillance equipment specifications.

Ballistic Missiles

Ballistic missiles follow a pre-determined trajectory that cannot be significantly altered after the missile has burned out its fuel, but its course is governed by the laws of ballistics. In order to cover large distances, ballistic missiles are usually launched very high into the air and when no more thrust is provided, the missiles follow a freefall or ballistic trajectory. Due to air-friction and other disturbances, the missile cannot keep a ballistic path without the necessary correction by the guidance system. The guidance will be of correcting relatively small deviations from that of the ballistic trajectory. Advanced ballistic missiles have several rocket stages and their course can be slightly adjusted from one stage to the next. They can be launched from fixed sites or mobile launchers, including vehicles, aircraft, ships and submarines.

Ballistic missiles can vary widely in range and use. Short-range ballistic missile (SRBM) with range less than 1000 km; Medium-range ballistic missile (MRBM), with range between 1000 and 2500 km; Intermediate-range ballistic missile (IRBM) with range between 2500 and 3500 km; Sub-continental ballistic missile (SCBM); Intercontinental ballistic missile (ICBM) with range greater than 3500 km are some of its categories. The first ballistic missile was the V-2 rocket, developed by Nazi Germany in the 1940s. Agni (range: 4000-6000 km), Prithvi (range: 350 km) are ballistic missiles developed by India as a part of 'Integrated Guided Missile Development Program' (IGMDP) launched in 1983 by Government of India and managed by DRDO.

Cruise Missiles

Fig. 2.1 Tomahawk cruise missile (US)

Cruise missiles maintain a constant height from the surface during most of its flight path. It uses a lifting wing and jet propulsion system to have a sustained flight. They can be considered as unmanned aircraft, which are designed to carry a heavy warhead for very long range with excellent accuracy. Modern cruise missiles normally travel at subsonic speeds, are self-navigating, and fly low in order to avoid radar detection. It is commonly targeted at relatively high value targets such as ships, command bunkers, bridges and dams. Tomahawk missile of US, BrahMos developed jointly by India and Russia are some examples.

Anti-Shipping Missiles

Anti-ship missiles are designed for use against naval surface ships. Most of the anti-ship missiles fly at subsonic speed and can be launched from warships, submarines, aircrafts, helicopters etc.

Fig. 2.2 Anti-shipping missile

Surface-to-Air Missiles (SAM)

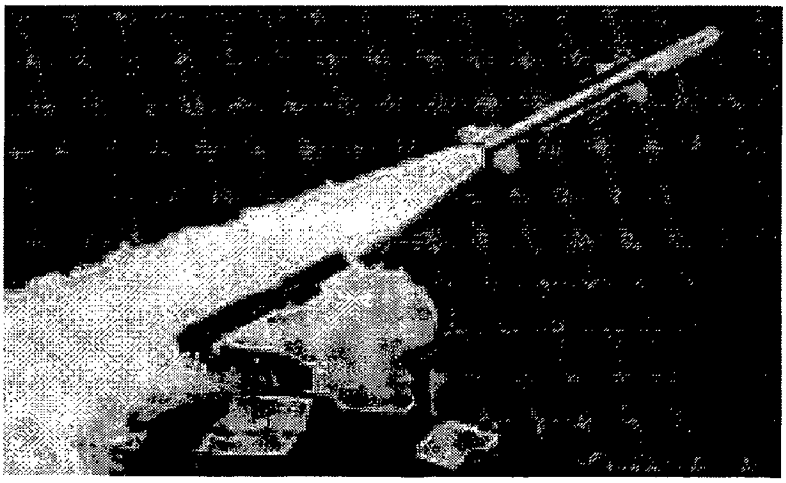

Fig. 2.3 Akash– a surface to air missile (India)

Surface-to-air missiles (SAM) are targeted at aircraft and is launched from the ground. SAMs can be launched from fixed installations or mobile launchers. SAMs, that are capable of being carried and launched by a single person is called Man-Portable Air Defence Systems (MANPADS). SAMs can also be deployed on warships. Targets for non-MANPAD SAMs are identified and tracked by an air-search radar, which then is "locked-on" to the target. Targets are identified as friend or foe by the Identification Fiend or Foe (IFF) systems. The short range Trisul (12 km, warhead : 15 kg) and medium range Akash (30 km, warhead : 60 kg) with a lethal radius of 20m are developed in India under the IGMDP.

Air-to-Air Missiles (AAM)

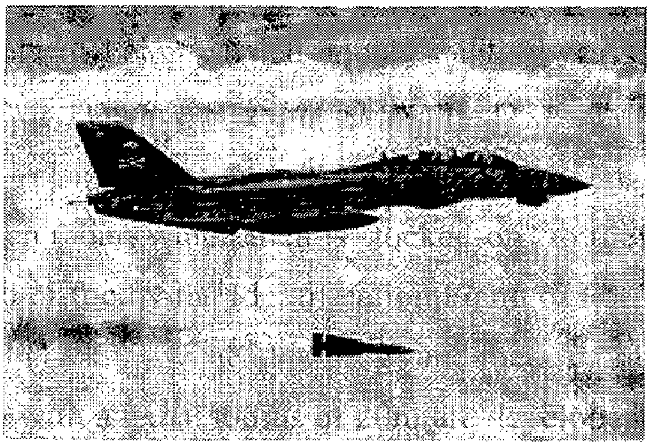

Fig. 2.4 Air-to-Air missile launched from an aircraft

Air-to-air missiles (AAM) are guided missiles fired from an aircraft targeting another aircraft. They can be either radar guided, infrared guided or laser guided (beam-riding). Air-to-air missiles are typically long, thin cylinders in order to reduce their cross section and thus minimize the drag at the high speeds at which they travel.

Anti-Tank Missiles

Anti-Tank Guided Missiles (ATGM) or Anti-Tank Guided Weapons (ATGW) are targeted to hit and destroy heavily armoured tanks and other armoured fighting vehicles. ATGMs range in size from shoulder-launched weapons which can be transported by a single soldier, to larger tripod mounted weapons. Manual Command guided MCLOS missiles are guided manually, requiring input from an operator using a joystick or similar device.

Fig. 2.5 Anti-tank missile

Semi Automatic Command guided SACLOS missiles requires an operator to keep the sights on the target until impact. Guidance commands are sent to the missile through wires or radio links Advanced guidance systems rely on laser marking or a TV camera view from the nose of the missile which does not require an external tracking system. Nag is India's all weather 'fire and forget' anti-tank missile with range of 4 to 6 km with a warhead capacity of 8 kg. It uses Imaging Infra-Red (IIR) guidance with day and night capability and is locked to the target before launch (Lock On Before Launch).

Anti-Ballistic Missiles

Anti-ballistic missiles (ABM) are used to counter long range, nuclear-armed Intercontinental ballistic missiles (ICBMs). Short range tactical ABMs cannot intercept ICBMs, even if within range since an incoming ICBM warhead moves much faster than a tactical missile warhead. Patriot PAC-3 of US and Arrow of Israel are short range ABMs.

Homing Missiles

The homing missiles do not require an external command to guide them all along their way; instead, the missile itself contains the target tracker. Three types of homing missiles are classified depending on the source of energy used for tracking the target, as shown in figure 2.6. They are Active homing, passive homing, and semi-active homing missiles. Homing missiles at their terminal phase, adopt the proportional navigation (PN) trajectory which is described in the next section.

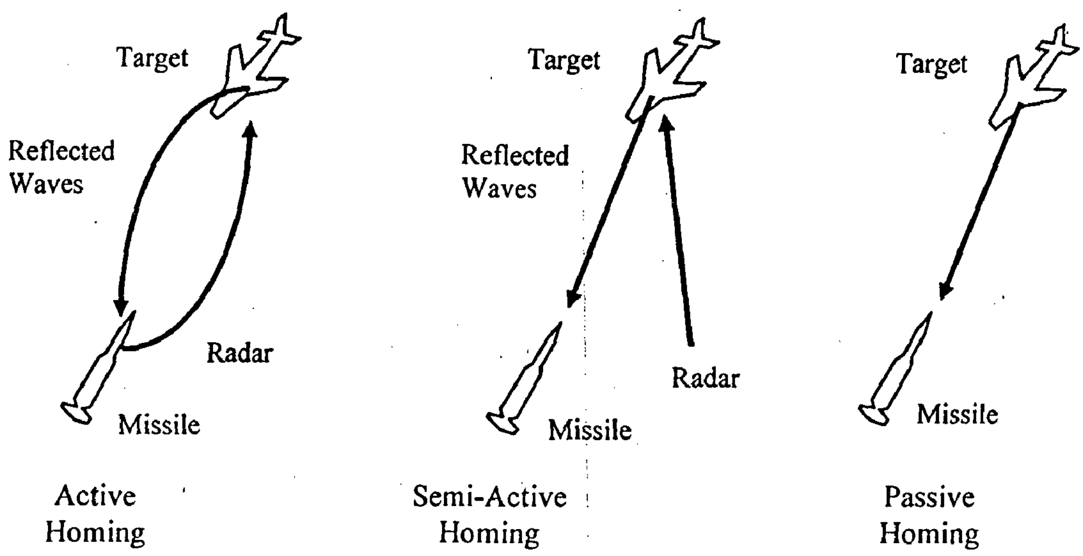

Fig 2.6 Homing missiles

Active Homing

A missile that is in active homing uses an illuminator, which beams energy at the target. The energy reflected by the target is detected by the homing head, which is relatively heavy and expensive. The missile is totally independent, and referred to as autonomous, after it has locked onto the target. Active homing is employed for the terminal phase of guidance after some other form of guidance has brought the missile to within a short distance of the target.

Semi-Active Homing

In semi-active homing systems target is illuminated by directing a beam of light, infrared or radio energy. Purely passive sensor in the missile then tracks the target using the energy reflected from it. The illumination beam should not spill over onto other objects and hence the illumination station needs to track the target. Thus, the system as a whole needs an additional tracker as well as one in each missile. Early systems used radar for the illuminating beam but recently lasers, mainly in the infra red part of the spectrum, are used with small missiles. The advantage of semi-active homing is that the missile need not carry power source for illumination, thus reducing weight and size of the missile.

Passive Homing

Passive homing relies on natural energy which is emitted or reflected from the target. The sensor in the missile makes use of this energy to track the target. Examples are anti-radar missiles, missiles targeting communication satellites, infra-red passive homing missiles targeting tail pipes of jet engines, smoke stacks of ships etc. Two verities of passive homing missiles are, lock on before launch (LOBL) and lock on after launch (LOAL).

2.2 Missile Guidance System

There is some observation instrumentation, which measures the behaviour of the missile. Such instrumentation is actually contained in the missile itself or it is situated at a ground or mobile platform control centre, such as a ship or aircraft. The missile data is fed to a guidance computer, which will also contain information on the desired path of the missile. The computer is able to determine what manoeuvres the missile should then execute in order to improve its chances of hitting the target. The computer passes steering instructions, such as desired lateral acceleration (latax) in the pitch and yaw

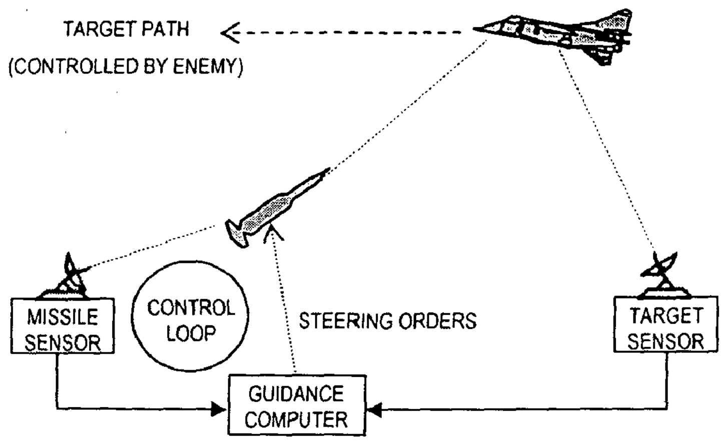

Fig 2.7 Missile guidance system

planes, to the control system, which manipulates the aerodynamic control surfaces of the missile or adjusts the direction of the propulsive thrust. The missile guidance system and its mathematical modelling is discussed in Chapter - 3. The important functional modules of a typical missile are as shown in the figure 2.8 [14].

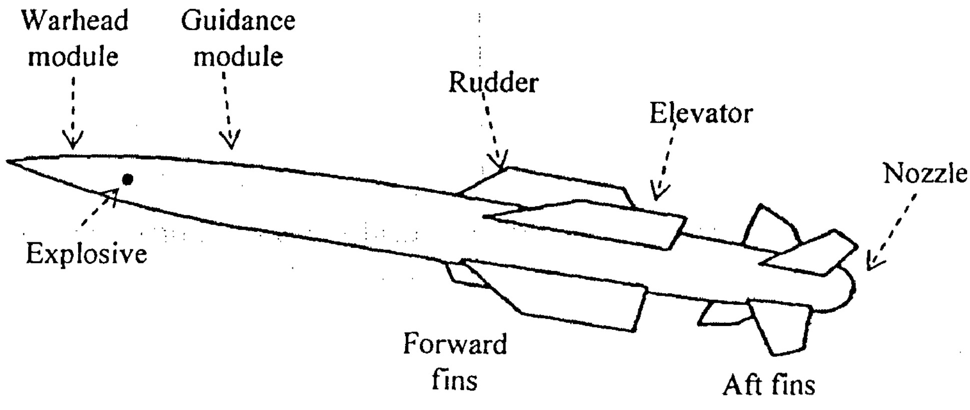

Fig 2.8 Functional modules of a typical missile

2.3 Guidance Laws

Proportional Navigation[3]

Nearly all missiles using homing guidance, at their terminal phase, adopt the proportional navigation (PN) trajectory. The principle of this system can be understood by referring to figure 2.9. This depicts a motorcar and lorry approaching a crossroad. The driver of the car observes the lorry as he approaches the intersection. If the lorry appears to fall back in the line of sight of the car driver, the car will arrive at the crossroad in front of the lorry. If, on the other hand, the sight line to the lorry moves more into the direct ahead region of the car driver's vision then the lorry will cross in front of the car. The dangerous situation will be if the line of sight from the car to lorry appears as a set of parallel lines. In such a case, the car and lorry will impact at the crossroads.

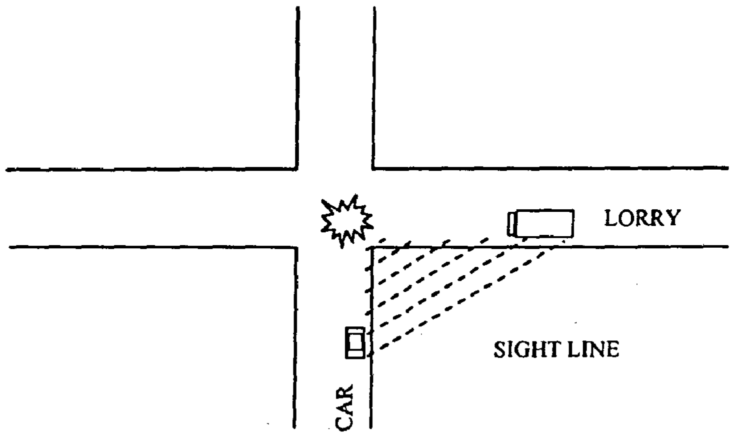

Fig 2.9 PN Trajectory

Homing missiles [11] makes use of the concept of proportional navigation to achieve impact with the target as shown in figure 2.10. A target tracker in the nose of the missile is locked onto the target and establishes the sight line from missile to target at all times. Instruments in the missile measure the rate at which the sight line is swinging in space; usually rate gyroscopes are used to do this. The rate is passed to the missile control system which causes the missile flight path to change at a rate which reflects the rate of turn of the sight line. Thus the rate of change of flight path is made proportional to the rate of turn of sight line. If this is continued, it will be found that the missile quickly steers on to a constant direction such that the rate of turn of sight line between missile and target is zero.

The consequence will be the impact between missile and target. This simple form of guidance, requires little computing effort in the missile although the instruments needed to track the target from the missile are costly.

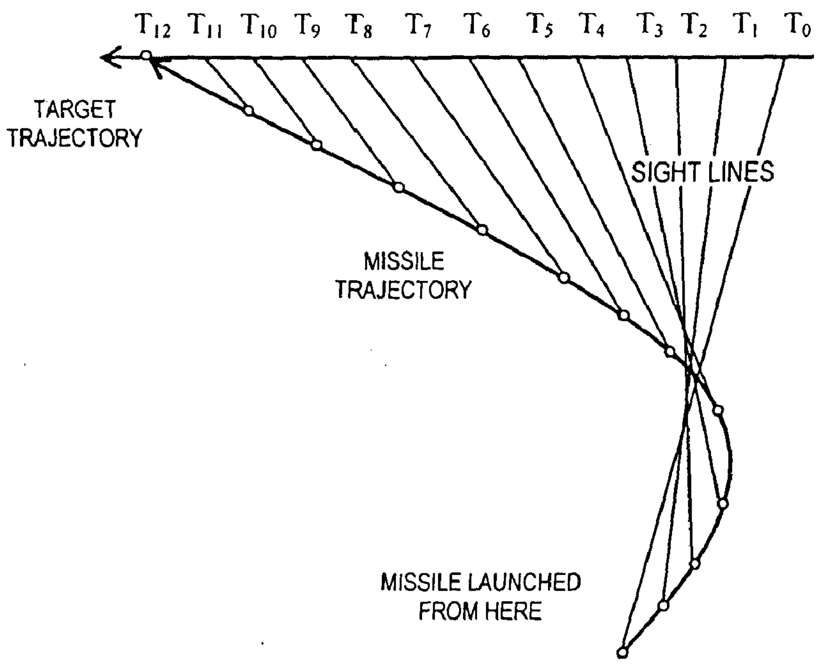

Fig 2.10 Homing kinematics

Straight-line Trajectory

The straight-line trajectories can only be used at short ranges against stationary or slow moving targets and it is not very common in guided weapon systems. The straight line between launch point and impact point is established before launch. If the target has a crossing rate, a slight aim off is allowed. The missile is then launched usually to supersonic speed along the trajectory, and it is maintained parallel to the original straight line by internal gyroscopic instruments acting through its control system. In some case, this technique is used for the initial flight path of a missile, which then follows a ballistic trajectory.

Cruise Trajectory

A cruise trajectory is one, which involves the missile flying at constant speed and constant height for most of its journey from launch to target. In many cases the missile may go through a number of phases involving a change of height and perhaps a change of speed, but then each phase will be a segment of cruise trajectory. Weapon systems using this type of path are strategic cruise missiles, and anti-shipping sea skimming missiles.

Ballistic trajectory

The ballistic trajectory, also called a free fall trajectory, is the path followed by an object or projectile when it is moving without any force acting upon it except the force of gravity. No missile passing through the atmosphere can adopt this path unaided because it will be subject to air pressure forces brought into play by its aerodynamic shape. However a missile moving outside the atmosphere with no motors running will execute a free fall trajectory. The application of simple ballistic mathematics to such missiles show that for short range missiles, when the distances are such that the surface of the earth can be regarded as flat, the ballistic trajectory is a segment of a parabolic curve. Long range weapons moving over a significant percentage of the earth's surface behave as if they are artificial satellites, and they have an elliptical trajectory with the centre of the earth as one focus.

Guided missiles paths passing through the atmosphere is adjusted so that they approximate to a ballistic trajectory, by measuring accurately the force acting on them at any instant, other than gravitation, and providing a carefully metered thrust which exactly cancels the force [1].

Line of Sight (LOS) Trajectory

The missile following LOS trajectory will normally follow a large curved trajectory, towards the end of their flight path, requiring considerable lateral acceleration (latax) capability [2]. The objective of the guidance system is to constrain the missile to lie as nearly as possible on the line joining the tracker and the target called the Line of Sight. There are many types of LOS systems and the main categories are Beam riding systems, Manual systems, Semi automatic systems, LOS using differential trackers and Command off the LOS systems.

Optimal Guidance Law

In optimal missile guidance problem, two important mission parameters i.e. missile target engagement time and the energy needed can be reduced by utilising optimal control. Optimal guidance law for short-range homing missiles to intercept highly manoeuvrable targets is available in literature [13]. The guidance problem that needs to be solved for the interception is to find the optimal missile trajectory such that the total time for the interception is minimised.

The performance index is given by, $J = t_f + k\int_0^{t_f} u^2 dt$

Where k is the weighting factor, u is the lateral acceleration command of the missile, and t_f is the time of flight.

The guidance law achieves the best performance in terms of the miss distance and interception time in comparison to the proportional navigation guidance. However, a major disadvantage of this law is that the target's future trajectory must be known in advance which is impossible to evaluate in a realistic environment [5].

Chapter - 3
MODELLING OF GUIDED MISSILE SYSTEM

3.1 Introduction

Modelling of complex dynamic system is one of the important subject in engineering. A model is often complicated to be used in real problems and hence approximation procedures based on physical considerations or using mathematical approaches must be used to achieve simpler models than the original one [8]. This chapter describes the approximate modelling of a missile guidance system to obtain linear time invariant system model. Chapter-4 describes four methods of model order reduction, for further simplification of the system model by approximation procedures using mathematical approaches.

The signals to the missile are usually in the form of up-down and left-right commands which are transmitted to two separate servos called rudder servos and elevator servos. Guided missiles usually have one or two axes of symmetry and hence the two control loops used for elevation and azimuth control can have identical characteristics with the exception of a constant bias of gravitational pull in the control loop of elevation [2].

There are essentially two control loops, which constitute the guidance system. They are the target tracking loop and the missile guidance loop. Block diagram of a beam riding system is shown below in the figure 3.1.

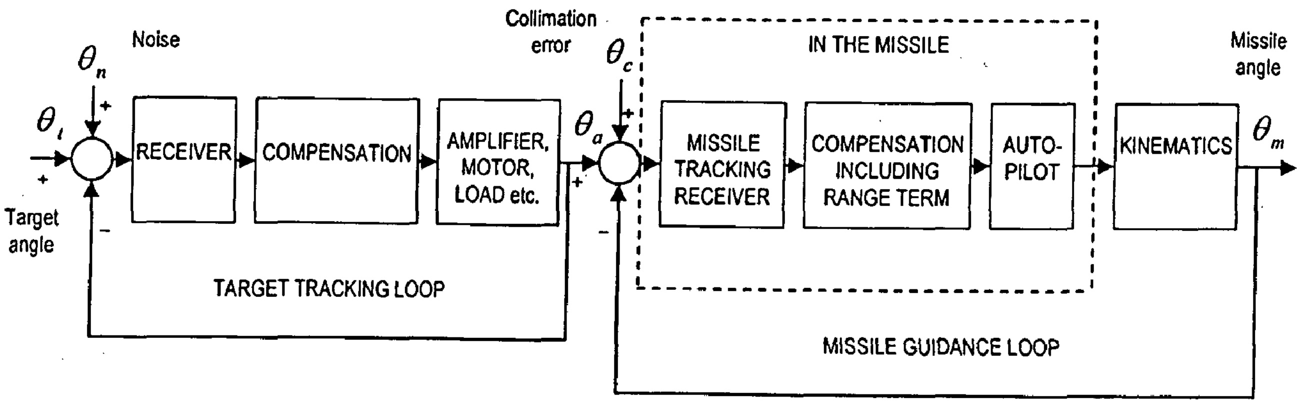

Fig. 3.1 Beam riding system

3.2 Target Tracker Servo Loop

A target tracker attempts to align its electrical null axis or bore-sight in elevation and azimuth with the line of sight (LOS). There are two identical servo systems to do this and only one need to be considered. The requirement of the servo is to produce two signals, one up-down and the other left-right proportional to the misalignment between the LOS and the bore-sight. The angular error detecting mechanism associated with the tracker is either a radar receiver or an optical signal processing system. Most of these error detectors are very non-linear for large misalignments but are essentially linear for small misalignments of about 1^0 or less and tracking errors are rarely larger than this. Figure 3.2 shows a typical target tracking system.

The angle channel receiver produces a signal proportional to the misalignment between target and its own bore-sight $(\theta_t - \theta_a)$. Since it is a linear device, and no reckonable time lag is associated its transfer function is simple gain k_l volts/radian. This error signal is fed to a proportional plus integral amplifier whose transfer function is $k_2 / (1+1/T_1\, s)$ and the servo is now of type 2 which can give a constant steady state error to an acceleration input.

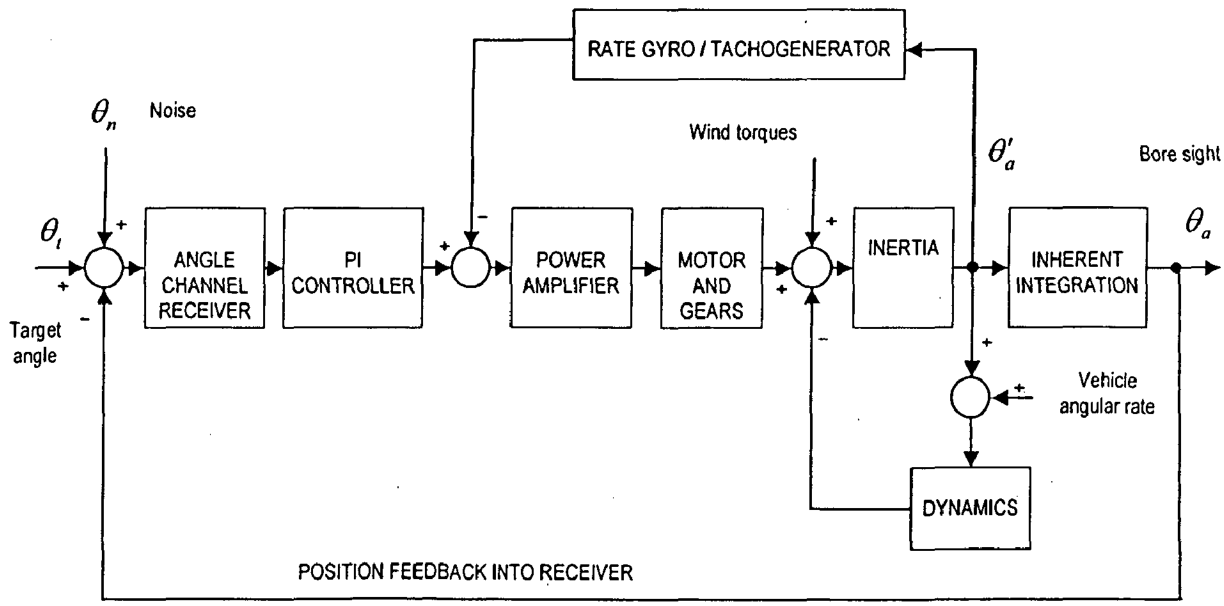

Fig 3.2 Target tracker

The other servo components, which follow, are a power amplifier, motor (electric or hydraulic) and a gear speed reducer together with the lumped inertia and viscous friction. Some angular rate feed back is usually provided by a tacho-generator or a rate gyro and the output from it is subtracted from the proportional plus integral amplifier. The purpose of the rate feedback is two fold. Firstly it improves stability margin and secondly to assist in rejecting outside disturbances.

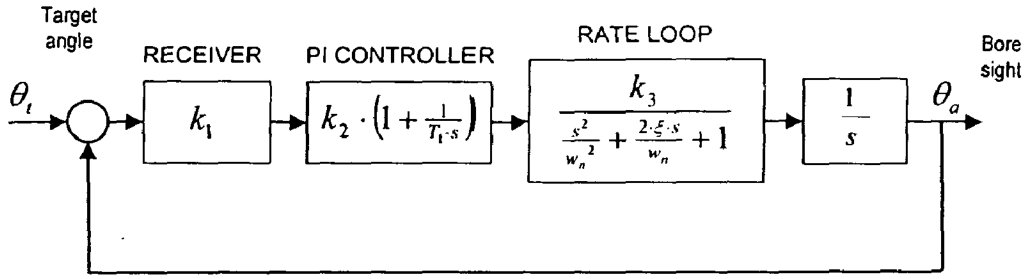

Fig 3.3 Block diagram of a target tracker

The integral term produces velocity lag in the tracking servo. If we wish to eliminate the velocity lag by removing the integral term, then the system become type-1, which cannot follow the velocity input. The transfer function is given below.

$$\frac{\theta_a}{\theta_t} = \frac{1}{\frac{s^3}{w_n^2 k_1 k_2 k_3} + \frac{2\xi \cdot s^2}{w_n k_1 k_2 k_3} + \frac{s}{k_1 k_2 k_3} + 1}$$

3.3 Missile Guidance Loop

The guidance loop consists of a guidance receiver which produces signals proportional to the misalignment of the missile from the line of sight. The guidance signals are passed through compensation network which consists of phase advance networks and PI controller. Phase advance network ensure the closed loop stability and the PI controller reduces the steady state error and hence the miss distance. In order to maintain constant sensitivity to missile linear displacement from LOS, the signals are also multiplied by the measured or assumed missile range R_m and then passed to the missile servos. The deflection of the missile fins alters the direction of the missile according to the aerodynamics, which is represented by the aerodynamic transfer function block and kinematics block. The missile guidance loop is represented in the figure 3.4 shown below.

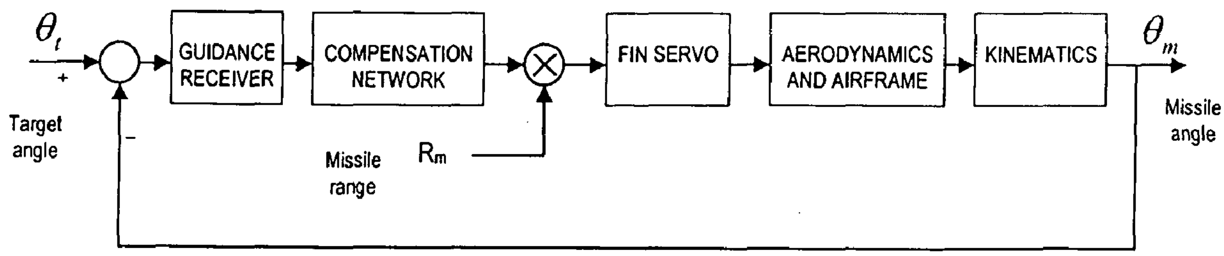

Fig 3.4 Missile guidance loop

3.3.1 Aerodynamic Transfer Function

The reference axis system standardised in the guided weapon industry is centred on the centre of gravity and fixed in the body as shown in the figure 2.4 shown below[2]. x-axis is called the roll axis, y-axis is called the pitch axis and z-axis is called the yaw axis.

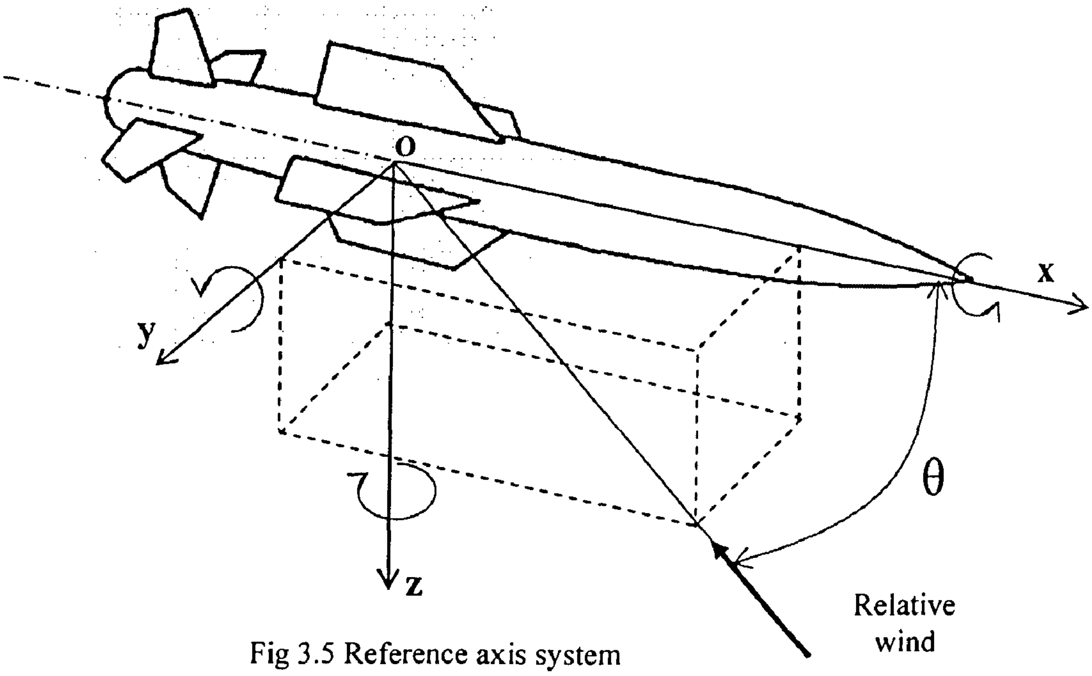

Fig 3.5 Reference axis system

	Roll axis (x)	Pitch axis (y)	Yaw axis (z)
Angular rates	p	q	r
Component of missile velocity, along each axis.	U	v	w
Component of force acting on missile along each axis	X	Y	Z
Moments acting on missile about each axis	L	M	N
Moments of inertia about each axis	A	B	C
Products of inertia	D	E	F

Table 3.1 Symbols used in the air frame

There are six equations of motion for a body with six degrees of freedom; three force equation and three moment equations called Euler's equations[2][7].

$$m(\dot{U} + qw - rv) = X$$
$$m(\dot{v} + rU - pw) = Y$$
$$m(\dot{w} + qU - pv) = Z$$
$$A\dot{p} - (B - C)qr + D(r^2 - q^2) - E(pq + \dot{r}) + F(rp - \dot{q}) = L$$
$$B\dot{q} - (C - A)rp + E(p^2 - r^2) - F(qr + \dot{p}) + D(pq - \dot{r}) = M$$
$$C\dot{r} - (A - B)pq + F(q^2 - p^2) - D(rp + \dot{q}) + E(qr - \dot{p}) = N$$

where, m is the mass of the missile.

The terms q, r, v and w are not large terms and if roll is controlled then p is very small. Then the terms pq, pr, pv, pw can be neglected. Further if the missile is having two axes of symmetry, the terms containing D, E, F can be neglected. Now the set of equations reduces to:

$$m(\dot{U} + qw - rv) = X$$
$$m(\dot{v} + rU) = Y$$
$$m(\dot{w} + qU) = Z$$
$$A\dot{p} = L$$

$$B\dot{q} = M$$

$$C\dot{r} = N$$

Now if lateral acceleration f_y, and angular rate r is considered, we can write

$$m(\dot{v} + rU) = Y = m f_y$$

$$= Y_v \cdot v + Y_r \cdot r + Y_\zeta \cdot \zeta$$

ie, $$f_y = \dot{v} + rU = y_v \cdot v + y_r \cdot r + y_\zeta \cdot \zeta \qquad \text{\_\_\_\_\_\_\_(3.1)}$$

and $$C\dot{r} = N = N_v \cdot v + N_r \cdot r + N_\zeta \cdot \zeta$$

ie, $$\dot{r} = n_v \cdot v + n_r \cdot r + n_\zeta \cdot \zeta \qquad \text{\_\_\_\_\_\_\_(3.2)}$$

where, Y_v is the force derivatives due to velocity along pitch-axis,

Y_r is the force derivatives due to angular rate along yaw-axis,

Y_ζ is the force derivatives due to rudder deflection ζ.

N_v is the moment derivative due to velocity along pitch-axis

N_r is the moment derivative due to angular rate of yaw

N_ζ is the moment derivative due to rudder deflection.

and $y_v = Y_v / m$, $y_r = Y_r / m$, $y_\zeta = Y_\zeta/m$

$n_v = N_v / C$, $n_r = N_r / C$, $n_\zeta = N_\zeta/C$

By eliminating 'v' and 'r' from the equations (3.1) and (3.2) we can have the transfer function between lateral acceleration f_y and rudder deflection ζ as,

$$\frac{f_y}{\zeta} = \frac{y_\zeta \cdot s^2 - y_\zeta \cdot n_r \cdot s - U \cdot (n_\zeta \cdot y_v - n_v \cdot y_\zeta)}{s^2 - (y_v + n_r) \cdot s + y_v \cdot n_r + U \cdot n_v} \qquad \text{\_\_\_\_(3.3)}$$

Similarly by eliminating 'f_y' and 'v' from the equations (3.1) and (3.2) we can have the transfer function between angular rate 'r' and rudder deflection 'ζ' as,

$$\frac{r}{\zeta} = \frac{n_{\zeta} \cdot s - n_{\zeta} \cdot y_{v} - n_{v} \cdot y_{\zeta}}{s^{2} - (y_{v} + n_{r}) \cdot s + y_{v} \cdot n_{r} + U \cdot n_{v}} \qquad \text{\_\_\_(3.4)}$$

The transfer functions described above are known as aerodynamic transfer function in the yaw plane and are useful in determining the effect of rudder deflection on the missile. Transfer function in the pitch plane can also be derived similarly.

3.3.2 Kinematics of the missile

The relationship between the lateral acceleration and the change in the missile angle seen by the tracker is described by the kinematics of the missile. Figure 3.6 shows the fly-plane geometry of the missile guidance system, where O is the tracker, M is the missile, T is the target and OT is the beam or LOS. U_m is the velocity vector of missile at an angle σ_m to the LOS. R_m is the range equal to OM.

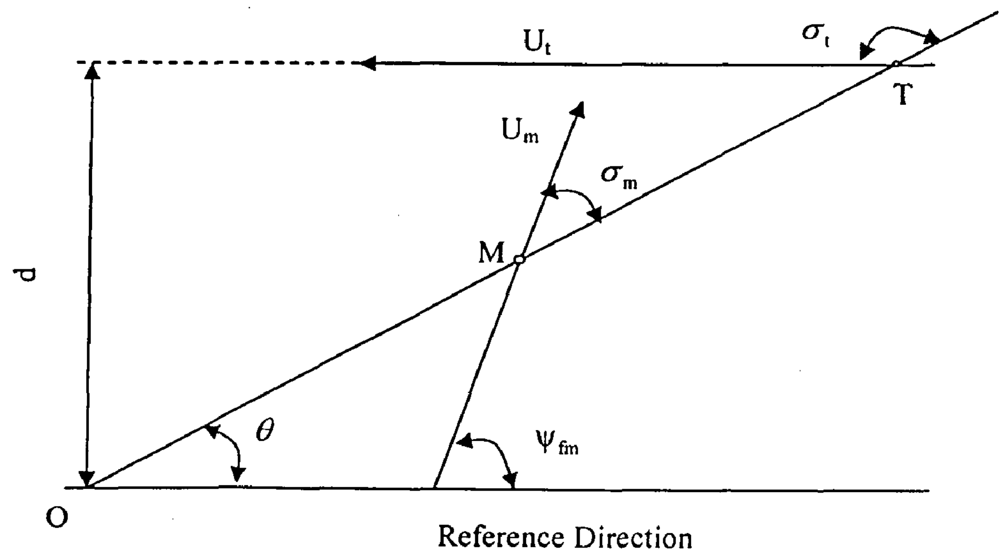

Fig. 3.6 Fly-plane geometry

Referring to figure 3.6 we can write

$$R_m \dot{\theta} = U_m \sin(\sigma_m) \qquad \text{\_\_\_\_\_\_(3.5)}$$

$\dot{R}_m = U_m \cos(\sigma_m)$ ________(3.6)

$f_y \quad = U_m \dot{\psi}_{fm}$ ________(3.7)

Differentiating equation (2.5) we get

$$\dot{R}_m \dot{\theta} + R_m \ddot{\theta} = U_m \cos(\sigma_m) \cdot \dot{\sigma}_m \quad (U_m \text{ is considered constant})$$

$$= U_m \cos(\sigma_m) \cdot (\dot{\psi}_{fm} - \dot{\theta}) \qquad \text{\_\_\_\_\_\_\_\_(3.8)}$$

From equation (3.6) and (3.8) we can get

$$R_m \ddot{\theta} + 2\, U_m \cos \sigma_m \; \dot{\theta} = U_m \cos(\sigma_m) \cdot \dot{\psi}_{fm}$$

$$= f_y \cos(\sigma_m) \quad \text{(from equation 3.7)}$$

ie, $\quad (R_m s^2 + 2\, U_m \cos(\sigma_m)\, s)\; \theta = f_y \cos(\sigma_m)$

therefore,

$$\frac{\theta}{f_y} = \frac{\cos(\sigma_m)}{s\, R_m (s + 2\,(U_m / R_m) \cos \sigma_m)}$$

which gives the kinematic transfer function.

By approximating $\cos(\sigma_m) = 1$ and considering $R_m >> U_m$ the above transfer function can be reduced to

$$\frac{\theta}{f_y} = \frac{1}{s^2 R_m}$$

3.3.3 Autopilot

Accelerometers and/or gyros provide additional feedback into the missile servos to modify the missile motion. The missile control system which consists of servos, control surfaces or thrust vector elements, the airframe, and feedback instruments plus control electronics is called an autopilot. In the case of missiles, autopilot that control the motion in either pitch or yaw plane are called lateral autopilots and that which control roll are called roll autopilots. For a symmetrical missile pitch and yaw autopilots are often identical and a bias of acceleration due to gravity is to be added in the pitch control loop to compensate for the gravitational pull.

An autopilot arrangement where an accelerometer provides the main feedback and the rate gyro is used as a damper is shown in the figure 3.7. The dynamic lags of the rate gyro and the accelerometers are omitted as they have high bandwidth and the lags they introduce in the frequencies of interest are negligible and the dynamics of fin servo is described by a second order transfer function.

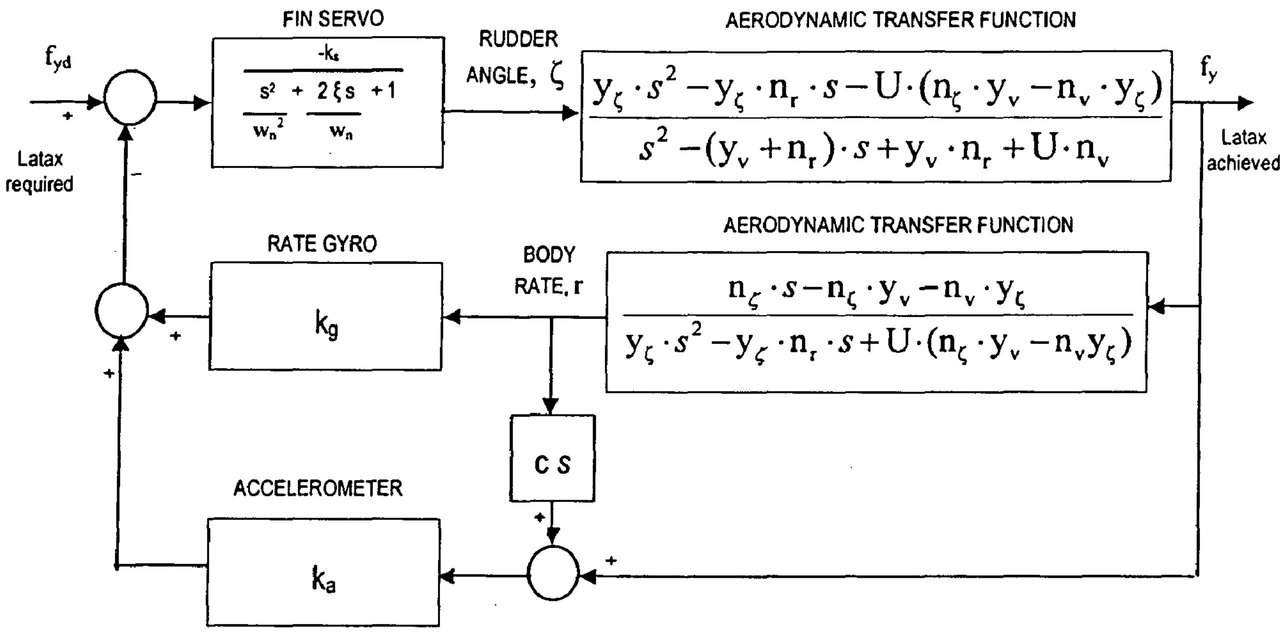

Fig 3.7 Lateral autopilot

The accelerometer is placed forward, about half to two third of the distance from the centre of gravity. Since it is placed at a distance 'c' ahead of the c.g, the total acceleration output it produces will be the acceleration of the c.g. plus the angular acceleration times the distance 'c'.

3.3.4 Compensation network

As described earlier, the purpose of the compensation network is to provide stability and to reduce the error in the guidance loop. Proportional plus integral (PI) controller is used commonly. The PI controller adds additional lag into the system, there by reducing the phase margin of the system. A double phase advance network can be used increase the phase margin of the system as depicted in figure 3.8.

Where K_P is proportional gain,

T_I is integral time constant,

T_C is time constant of double phase advance network,

α is a constant less than 1.

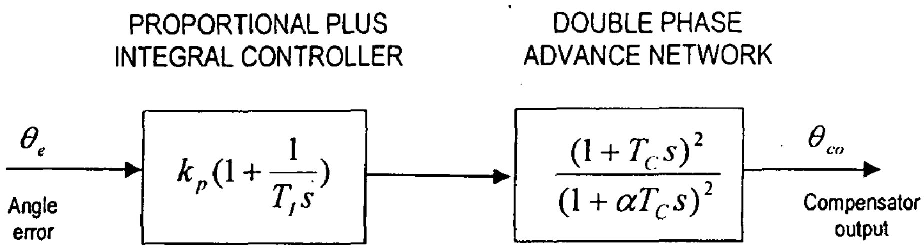

Fig 3.8 Guidance controller

The open loop transfer function can be found by reducing the individual blocks of the missile control loop single transfer function. The open loop transfer function is used to find stability margin and accuracy of the loop.

Chapter - 4
MODEL ORDER REDUCTION

4.1 Introduction

The subject of model order reduction was considered as an open problem in the system theory but now it is considered as a fundamental topic [8]. The main reasons for obtaining low order models are: (i) To simplify the understanding of the system; that is the lower order systems are more comprehensible in terms of their parameters such as time constants, natural frequency, damping ratio etc. This also helps in the heuristic understanding. (ii) To reduce computational efforts in simulation problems; which is evident from the state space model of higher order system, where all the states need to be computed to obtain the output. This also consumes a larger memory. (iii) To make the design of the controller numerically more efficient; which is particularly significant where large amount of numerical calculations are involved, such as optimal control, adaptive control etc. Some adaptive control schemes even involve a real time model reduction to obtain a low-order model so as to design a low-order regulator in real time. (iv) To obtain simpler control laws; which implies that the controller designed will be simpler, when the system is of low order.

But when going for reduced order, the question which a designer has to face is, whether accuracy or simplicity is important, as the reduction in the order simplifies the system, while reducing accuracy of the system.

4.2 Model Reduction using Ruth Table Criteria

This method is direct and can be easily implemented which preserves the stability of the reduced-order transfer function. The reduction algorithm is based on the construction of the Ruth stability array for both numerator and denominator of the transfer function [8]. The Ruth array is constructed with the first row consisting of the odd polynomial coefficients and the second row consisting of the even coefficients. The array is completed by using further coefficients computed by the general formula:

$$r_{i,j} = r_{i-2,j+1} - (r_{i-2,1} \cdot r_{i-1,j+1})/r_{i-1,1} \qquad \text{for } i \geq 3 \text{ and } 1 \leq j \leq (n-i+3)/2$$

If the numerator is a polynomial of order m and denominator of order n, the transfer function is given as the ratio of two polynomial expressions. They are constructed with the $(m+2-r)$ th and $(m+3-r)$ th rows of the array associated with the numerator and taking $(n+1-r)$th and the $(n+2$-k) th rows of the ruth array associated with the denominator of the original transfer function. This method is versatile and is computationally simple, but it is not often accurate compared with other methods.

4.3 Optimal Order Reduction [15]

In the Optimal order reduction approach, the problem of model reduction is regarded as a constrained optimization problem. Considering an LTI system represented by the set of state variable equations,

$$\dot{x} = Ax + Bu$$
$$y = Cx$$

where, $x \in R^n$, is the state vector of the higher order system,

$u \in R^m$, is the input vector,

$y \in R^q$, is the out put vector of the higher order system.

and the matrices A, B, C are constant with appropriate dimensions.

Then the reduced order model of the system is given by,

$$\dot{x}_r = A_r x_r + B_r u$$
$$y_r = C_r x_r$$

where, $x_r \in R^r$, is the state vector of the reduced order system,

$y_r \in R^q$, is the out put vector of the reduced order system.

and the matrices A_r, B_r, C_r are constant with appropriate dimensions and $q \leq r < n$.

The reduction error is defined as,

$$e(t) = y(t) - y_r(t).$$

The optimal reduction technique aims at computing matrices A_r, B_r and C_r by minimizing a function of the reduction error.

The cost function may be written as,

$$J = \sum \int_0^\infty e^T(t).Q.e(t)\,dt, \text{ where Q is a weighing matrix.}$$

The necessary condition for J to have a minimum with respect to matrices A_r, B_r and C_r leads to the matrix relations,

$$F.R + R.F^T + S = 0 \qquad \text{\_\_\_\_\_\_\_\_\_\_(4.1)}$$

$$F^TP + PF + M = 0 \qquad \text{\_\_\_\_\_\_\_\_\_\_(4.2)}$$

Where $F = \begin{bmatrix} A & 0 \\ 0 & A_r \end{bmatrix}$

$$S = \begin{bmatrix} BNB^T & BNB_r^T \\ B_rNB^T & B_rNB_r^T \end{bmatrix}$$

$$M = \begin{bmatrix} C^TQC & -C^TQC_r \\ -C_r^TQC & C_r^TQC_r \end{bmatrix}$$

where, $N = \text{diag}(n_1, n_2, \ldots n_m)$, R and S are solutions of Lyapunov equations (4.1) and (4.2).

with $R = \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix}$

and $P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}$

$$J = \text{trace}[RM)] = \text{trace}[PS]$$

The algorithm for the optimal order reduction is written as [8],

1. Make initial choice of A_r and B_r.
2. Find the R matrix.
3. Compute matrix $R_{11}R_{22}^{-1}$
4. Let $C_r = CR_{11}R_{12}^{-1}$
5. Find the P matrix.
6. Compute the matrix $-P_{22}^{-1}P_{12}^{T}$
7. Compute the matrix B_r
8. If the B_r computed in the previous step is not the same as that previously chosen, then go to step(2) with new B_r. Otherwise the B_r and the C_r obtained are taken as optimal values for the assigned Ar matrix.
9. Compute J for the given A_r and the B_r and C_r obtained. Assign this value of J as J_0.
10. Assign a new value of A_r as,

    $A_r = c. -P_{22}^{-1}P_{12}^{T} A R_{11}R_{22}^{-1} + (1 - c) A_r$, c is chosen between 0 and 1.
11. For this new value of A_r repeat steps (2) to (10). Compute J for the new value of A_r and the previously obtained optimum B_r and C_r. Assign this value of J as J_1.
12. If $J_1 < J_0$, assign to J_0 the value of J_1 and go to step(10)

    If $J_1 > J_0$, go to step(10) decreasing the value of c.

    If $J_1 = J_0$, compute $RI = P_{22}^{-1}P_{11}^{T} P_{12} R_{22}^{-1}$.

    If it is equal to I_r, the optimum will be reached. If it is not Identity matrix, go to step (10) decreasing the value of c.

4.4 Hunkel based method of order reduction

Here the problem is to find a transfer matrix $G_k(s)$ of degree k<n such that the Hunkel norm of error matrix $E(s) = G(s) - G_k(s)$ is minimised. The step by algorithm for Hunkel based method is given in [8]. For the present work a built in function of MATLAB-7 is linked with the GUI. It is available in the Robust Control Tool box with function name as 'reduce'.

4.5 Genetic algorithm based method of order reduction

Genetic Algorithms (GA) are efficient search methods which is useful for finding the global optimum point in the search space [12]. They are also parallel in nature and are not prone to noisy environments and discontinuity. The steps involved in GA are:

1. Initialise a population with random or some initial value.
2. Find the merits of each individual of the population, known as fitness value.
3. Select the individuals with best performance for the next generation, known as reproduction.
4. Apply genetic operators such as mutation and cross-over to the new population.
5. Continue from (2) to (4) until some condition is met, such as number of generations, time limit, error limits etc.

There can be different areas of implementing GA for the problem of model order reduction. In the present work GA is used to find optimum values of the numerator and denominator coefficients, when the difference between step response of original system and reduced order system are minimum. For a faster convergence initial population is taken as the coefficient obtained using the Hunkel method of order reduction. A limit in the number of generation is given as stopping criteria and the model can be improved by increasing the number of iterations. The disadvantage is that, time taken for computation is large and since it is a random process reduced order model obtained once may not be equal to that obtained at the next time.

4.6 Applications of Model Order Reduction

There are many applications for the technique of model order reduction such as model reduction in control system, reduction of RLC circuit complexity in

VLSI design etc. Two applications related to control systems are discussed below.

4.6.1 Design of PID Controller

The classical control system design involves methods such as state variable feedback technique, PID controller tuning, using phase compensation networks etc. It becomes computationally difficult when a controller or compensator need to designed for higher order systems. This problem can be solved by reducing the model of the system to suitable reduced order.

The design of a PID controller using pole-zero cancellation technique is illustrated in [6]. In this method the original system is reduced to a second order system and the root-locus plot of the reduced system is obtained. Now the zeros of PID controller are placed so as to cancel the effect of poles. This results in improved dynamic performance and stability.

4.6.2 Intelligent Controller

Recent development in control theory is towards intelligent control, to cope with the new challenges in the need for better quality and performance of the military systems. In many of the intelligent control applications, a mathematical model of the original system is required for generating the controller outputs or tuning the controller parameters, such as model reference control, adaptive control, predictive control, optimal control etc [21]. As the order of the system increases, the computational time also increases substantially. Since the controller has to operate in real time, reduced order model of the system can be used instead of the original model of the system.

Chapter - 5
SIMULATION OF MISSILE TRAJECTORY

5.1 Introduction

Simulation of the system is useful for finding the potential faults in the design of a control system, as well as to evaluate the performance, before testing on the physical system. In the present work simulation is done on three dimensional plane, with the LTI model of the missile guidance system, using MATLAB-7.

5.2 Three Dimensional Coordinate System

The coordinate systems used here for modelling on a 3-D space are, Cartesian coordinate system and Spherical coordinate system. For the present application, system model is obtained in a spherical coordinate system, described two angles (azimuth and elevation), and the range. For plotting, Cartesian coordinate system is used. A point in space represented in Cartesian coordinate system (x, y, z) can be converted to spherical coordinate system (θ, ϕ , R) and vice versa by the equation given below.

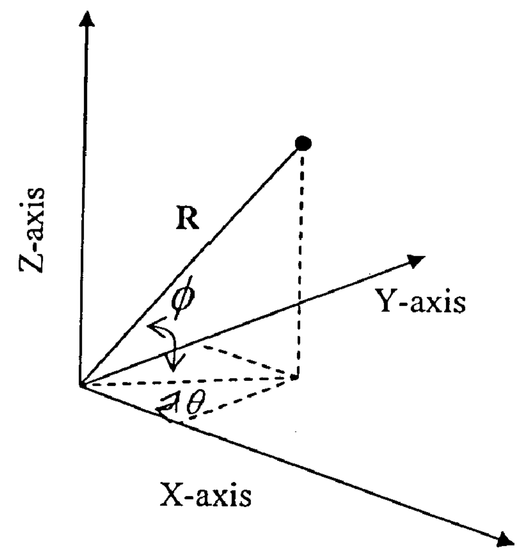

Fig. 5.1 Three-Dimensional coordinate system

θ	$= \tan^{-1}(y/x)$
ϕ	$= \tan^{-1}(z / \mathrm{sqrt}(x^2 + y^2))$
R	$= \mathrm{sqrt}(x^2 + y^2 + z^2)$

x	$= R \,.\, \cos(\phi) \,.\, \cos(\theta)$
y	$= R \,.\, \cos(\phi) \,.\, \sin(\theta)$
z	$= R \,.\, \sin(\phi)$

This can be easily calculated with MATLAB functions, 'cart2sph' and 'sph2cart' for converting Cartesian coordinate system to Spherical and Spherical coordinate system to Cartesian coordinate system respectively.

5.3 Effect of Acceleration due to Gravity and Air-friction

Since the missile experiences the gravitational pull all the time, a thrust opposite in direction and equal in magnitude to the force of gravity is to be generated in order to nullify the effect of gravity. It is achieved by giving a bias signal at the input of the controller proportional to the gravitational effect. In the spherical coordinate system, the correction is to be done to the elevation, magnitude of which vary depending upon the distance of the missile from the place where the missile is launched.

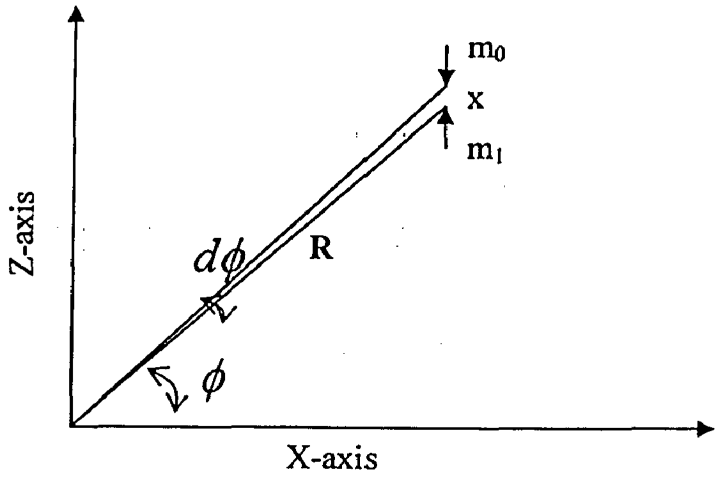

Fig 5.2 Displacement due to gravity

Considering that the missile produces a displacement 'x' by the effect of acceleration due to gravity, and the angle produced is $d\phi$.

Therefore, $d\phi = \sin^{-1}(x/R)$.

But, $x = u + \frac{1}{2}.g.t^2$

$= \frac{1}{2}.g.\Delta t^2$ (considering u=0)

ie, $d\phi = \sin^{-1}(g.\Delta t^2/2.R)$ is the bias needed to be applied at the input of the elevation control loop.

The acceleration due to gravity changes from place to place depending upon the latitude and altitude. David Y. Hsu [16][17] gives an approximate method to calculate the gravity at various location, which is as shown below:

$C_{22} = 1.271727412728199 \times 10^{-18}$;

$C_{21} = 3.034117526395185 \times 10^{-12}$;

$C_{20} = 1.188523953283804 \times 10^{-4}$;

$C_{12} = 1.878969973008548 \times 10^{-16}$;

$C_{11} = -1.347079301177616 \times 10^{-9}$;

$C_{10} = -5.197841463945455 \times 10^{-14}$;

$C_{02} = -6.685260859851881 \times 10^{-14}$;

$C_{01} = 9.411353888873278 \times 10^{-7}$;

$C_{00} = -9.780326582929618$;

$$gu = (C_{22} \times \sin^4(T) + C_{12} \times \sin^2(T) + C_{02}) \times h^2 + (C_{21} \times \sin(T) + C_{11} \times \sin^2(T) + C_{01}) \times h + (C_{20} \times \sin^4(T) + C_{10} \times \sin^2(T) + C_{00});$$

Where, 'T' is the latitude, 'h' is the altitude, and 'gu' is the vertical component of the acceleration due to gravity.

If the velocity of the missile is not controlled after the fuel is burned out, it will decelerate due to air friction. The magnitude of deceleration will be proportional to the velocity of the missile. Thus we can have,

$$\dot{Um} = Um / \tau$$

Where, $\dot{Um}$ is the deceleration of missile, Um is the velocity of missile, and τ is a constant, typically between 8 to 10.

Thus, change in missile velocity, $\Delta Um = -(Um / \tau) \times \Delta t$.

5.4 Simulation of Missile Trajectory

For the present work, simulation of the missile trajectory is carried out which follows the Line of Sight guidance law. The state space model is obtained from the transfer function of the system, by using the command 'ss'. Two identical systems are used to simulate the elevation control and the azimuth control. The initial state of the need to be calculated before starting the simulation. The initial state depends on the angle of fire of the missile.

State equation of the system :

$$\dot{X}_A = A \cdot X_A + B \cdot U_A \qquad \text{(Azimuth control)}$$

$$\dot{X}_E = A \cdot X_E + B \cdot U_E \qquad \text{(Elevation control)}$$

Output equation of the system :

$$Y_A = C \cdot X_A + D \cdot U_A \qquad \text{(Azimuth control)}$$

$$Y_E = C \cdot X_E + D \cdot U_E \qquad \text{(Elevation control)}$$

Where, A, B, C, D are the state-space matrices of the two identical system; X_A and X_E are the state vectors of the azimuth control and elevation control respectively, U_A and U_E are the respective inputs and Y_A and Y_E are the respective outputs.

The algorithm for Simulation of missile trajectory.

1.Initializing the states:

At steady state, $\dot{X}_A = 0$ and $\dot{X}_E = 0$.

Therefore initial states, $X_{A0} = A^{-1} . (-B .U_A)$ and

$X_{E0} = A^{-1} . (-B.U_E)$

Where, U_A and U_E are the firing angle at azimuth and elevation.

2. Simulation of Missile trajectory:
   a. Find target angle in azimuth and elevation, by converting Cartesian coordinate to spherical coordinate.
   b. Apply target angle as the input command to the missile in azimuth and elevation. Also, add the correction for gravity to the elevation input.
   c. Compute the state vector and outputs for the next time interval; subtract the angle due to gravitational effect.
   d. Convert the spherical coordinate to Cartesian coordinate.
   e. Store the data and update plot.
   f. Compute the distance between missile and target.
   g. Check for the stopping criteria; if true display miss-distance and exit. Else go to step (a).

Calculation of successive position of the missile

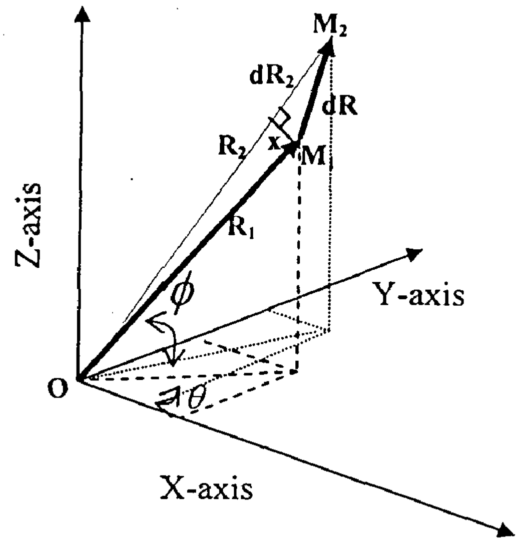

Fig. 5.3 Calculation of missile position

Considering that the missile has advanced from position M_1 to M_2 in time dt. Assuming constant velocity missile the displacement of the missile,

$dR = Um.dt$, where, Um is the missile velocity.

If $d\phi$ and $d\theta$ are the deviation in elevation and azimuth of the missile, the distance 'x' can be calculated as

$$x = .\ sqrt(R_1^2 . \sin^2(d\phi) + R_1^2 .\sin^2(d\theta))$$

where $R_1 = OM_1$ and $R_2 = OM_2$ are the ranges of the missile when it is at position M_1 and M_2 respectively.

Therefore dR_2 which is the increment in the range of the missile can be calculated by considering the right angled triangle as shown in the figure.

ie, $dR_2^2 = dR^2 - x^2$

Therefore the new range,

$$R_2 = R_1 + dR_2$$

$$= R_1 + sqrt(dR^2 - R_1^2 . \sin^2(d\phi) + R_1^2 .\sin^2(d\theta))$$

Now we have the new position of the missile in spherical coordinate, and the Cartesian coordinate can be found out as described earlier.

Chapter - 6
RESULTS AND DISCUSSION

6.1 Mathematical Modelling of the Missile Guidance System

Mathematical modelling of the complete system is carried out in frequency domain using Laplace transform. Control system tool box of the MATLAB-7 is used extensively for this purpose. The software is written as m-files that can run under MATLAB. All the m-files are linked to a Graphical User Interface (GUI), so that the user can alter the parameters and view the results quickly without spending much time in editing the m-file and running it again.

Although the system is a nonlinear multi-input, multi-output system, it is linearized at some specific operating point and only single input and output is considered at a time as discussed earlier in chapter-3. Considering that the missile is symmetrical along its cross-section, the two control loops which are acting on the azimuth and elevation in a spherical coordinate system. The target-tracking loop and the missile guidance loop are considered separately.

6.1.1 Missile Guidance-Loop

The transfer functions of the different functional blocks in the missile control loops are obtained separately, and cascaded to form a single transfer function. The various functional blocks are: autopilot, missile kinematics and controller.

Auto pilot

The block diagram of autopilot is reproduced here for convenience (figure 6.1) The aero dynamic transfer unction shown in the literature [2] is verified using symbolic constants in the MATLAB-7.

Derivation of aerodynamic transfer function using symbolic constants in MATLAB-7

Simultaneous equations involving higher degree and large number of terms can be solved using symbolic constants in MATLAB-7. For this purpose the variable used equation are declared as symbols, using 'sym' command. Then the sets of equations are formed with matrix notation. Now the solution is obtained by various matrix manipulation available in MATLAB.

As discussed in chapter-3, from equations 3.1 & 3.2 the aerodynamic equations are given as,

$$f_y = \dot{v} + rU \qquad \text{\_\_\_(6.1)}$$

$$f_y = y_v \cdot v + y_r \cdot r + y_\zeta \cdot \zeta \qquad \text{\_\_\_(6.2)}$$

$$\dot{r} = n_v \cdot v + n_r \cdot r + n_\zeta \cdot \zeta \qquad \text{\_\_\_(6.3)}$$

This is rewritten as

$$f_y - y_\zeta \cdot \zeta = y_v \cdot v + y_r \cdot r \qquad \text{\_\_\_(6.4)}$$

$$-n_\zeta \cdot \zeta = n_v \cdot v + n_r \cdot r - \dot{r} \qquad \text{\_\_\_(6.5)}$$

Two expressions are obtained for 'v' and 'r' by solving the above set of equations using the symbolic constants of MATLAB. An expression 'exp1' is formed from the equation 6.1 as,

$$exp1 = v' + rU - f_y = 0$$

The result obtained for exp1 by running a MATLAB program (listed in the Appendix-A) is shown below. Please note that the MATLAB variable 'C' should be treated as ζ and derivative terms are replaced with laplace coefficient 's'.

exp1 =

```
-(Nv*Yr*fy +Nv*U*Yc*C -Nv*U*fy -Nr*Yv*fy-Nr*s*Yc*C
+Nr*s*fy +Nc*C*s*Yr -Nc*C*U*Yv +s*Yv*fy +s^2*Yc*C -
s^2*fy)/(-Nr*Yv +s*Yv +Nv*Yr)
      = 0.
```

Neglecting Yr and rearranging, we get transfer function between lateral acceleration and fin deflection.

```
fy        Yc.s^2 - Yc.Nr.s - U(Nr.Yv - Nv.Yc)
---   =   ------------------------------------
C           s^2 - (Yv+Nr).s + Yv.Nr + U.nv
```
__(6.6)

Similarly

$$v \quad = (y_\zeta \cdot \zeta - rU) / (s - y_v)$$

Expression 'exp2' is formed from 6.3 as

$$\exp2 \quad = n_v \cdot v + n_r \cdot r + n_\zeta \cdot \zeta - \dot{r} = 0$$

The result obtained for exp1 by running a MATLAB program is shown below.

exp2 =

```
-(-Nv*Yc*C +Nv*r*U -Nr*r*s +Nr*r*Yv -s*Nc*C +Yv*Nc*C
+s^2*r -s*r*Yv)/(s -Yv)
```
= 0

Therefore,

```
 r          Nc.s - Nc.Yv + Nv.Yc
--- = --------------------------------
 C     s^2 - (Yv+Nr).s + Yv.Nr + U.nv
```
__(6.7)

From equations 6.6 and 6.7 we get the transfer function between body rate and lateral acceleration.

```
 r          Nc.s - Nc.Yv + Nv.Yc
--- = ------------------------------------
fy     Yc.s^2 - Yc.Nr.s - U(Nr.Yv - Nv.Yc)
```
__(6.8)

The results obtained are conforming to the transfer function described in [2].

Block-diagram Reduction

The block diagram of the autopilot shown above is reduced as shown in figure 6.1a through 6.1c.

I.

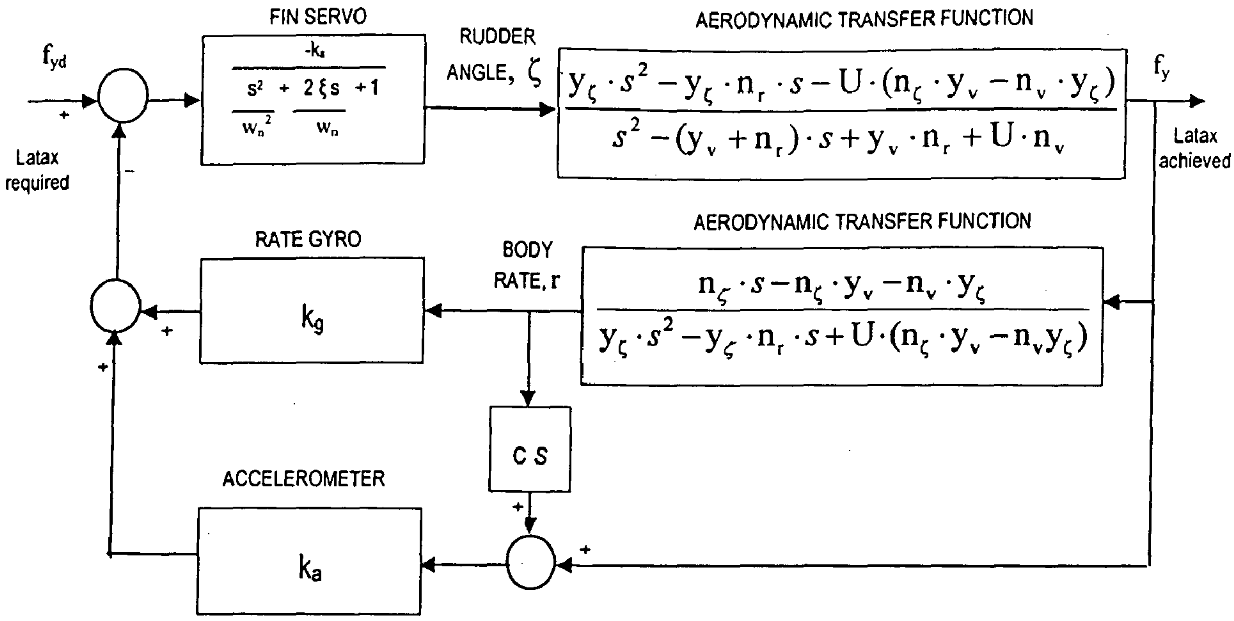

Fig 6.1a Lateral autopilot

II.

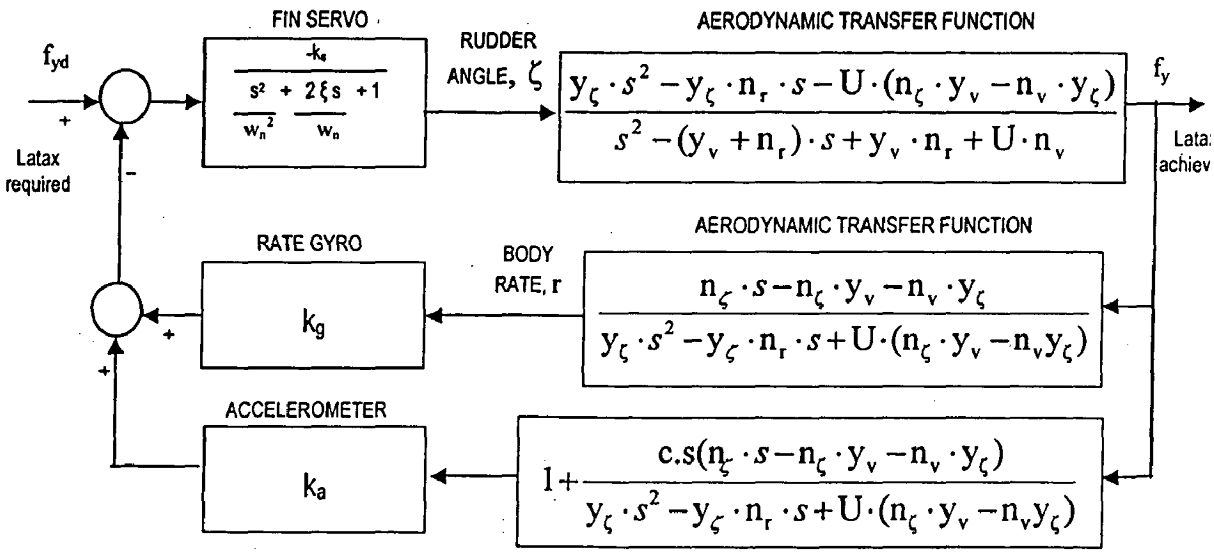

Fig 6.1b Block being reduced

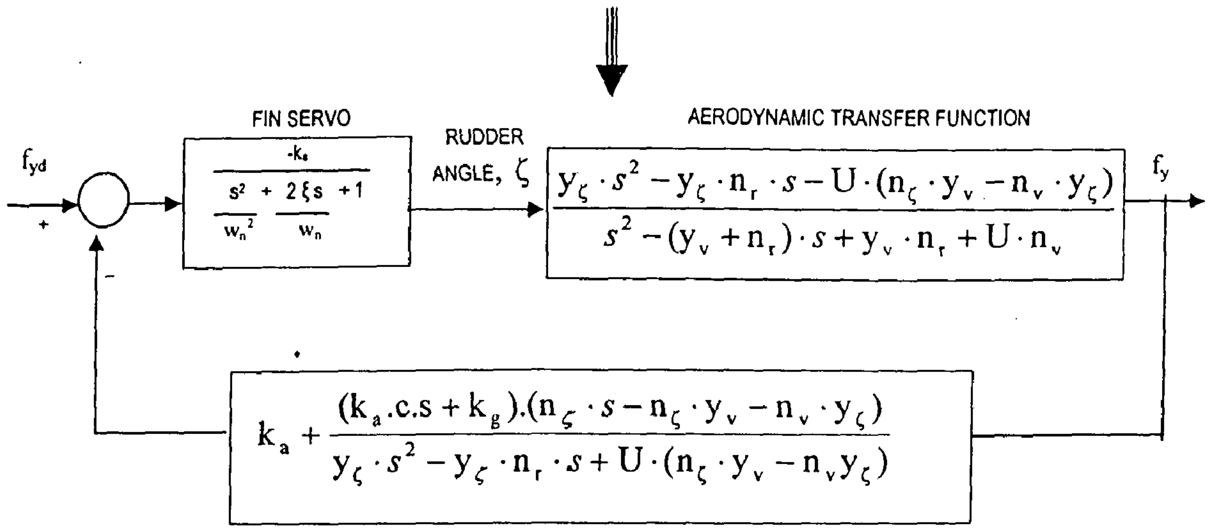

Fig 6.1c Block being reduced

The block diagram can be further simplified to get a single block by using the concept of feedback loop. But as it would become too complicated, for the present work, individual blocks are computed first using the command 'T = tf(N,D)'. Where, 'T' is the transfer function obtained, 'N' is the numerator of the transfer function, 'D' is the denominator of the transfer function and 'tf' is the MATLAB function to form a transfer function object. Then the transfer function objects are combined using operators such as addition (+), subtraction (-), multiplication (*), division (/) etc. The closed loop transfer function is obtained by using the command 'Tcl = Tfw / (1 + Tfw * Tfb)'. Where, 'Tcl' is the closed loop transfer function, 'Tfw' is the forward transfer function and 'Tfb' is the feedback transfer function.

A typical example of missile guidance system given in the literature [2] is used here to obtain a linearized model at the specified operating condition. The missile and other control system parameters are given by :

Missile velocity, U = 500m/s

Lateral acceleration, fy = 250m/s^2

Body incidence, Bta = 0.2rad

Length of missile, lm = 2m

Mass of missile, m = 52kg

Moment of inertia, $C = 14kgm^2$

Static margin, pxx = 4.5% of length (diff. between c.g. and c.p.)

Distance of accelerometer from c.g. of missile, c = 0.5m

Rudder moment arm, Rma = 3/8 of length of missile.

Fin servo gain, ks = 0.007

Natural frequency of fin servo, wns = 180 rad/s

Damping ratio of fin servo, mus = 0.5

Accelerometer gain, Ka = 0.8

Rate gyro gain, Kg = 30

PI controller gain, KP = 1

PI controller integral time, KI = 0.5

Phase advance n/w time constant, Tau = 1

Phase advance n/w coefficient, Alfa = 0.3

The above parameters are entered through the GUI window of the software developed, and aerodynamic derivatives obtained by pressing the 'Calculate' button of the GUI, in the 'Parameter View' panel (figure 6.2). The values obtained for the aerodynamic derivatives are shown below.

Force derivative along y-axis, (due to velocity along y-axis)	$y_v = -3s^{-1}$
Force derivative along y-axis, (due rudder deflection)	$y_\zeta = 180m/s^2$
Moment derivative along z-axis, (due to velocity along y-axis)	$n_v = 1.0029m^{-1}s^{-1}$
Moment derivative along z-axis, (due to angular rate about z-axis)	$n_r = -3s^{-1}$
Moment derivative along z-axis, (due to rudder deflection)	$n_\zeta = -501.4286\text{-}3s^{-1}$

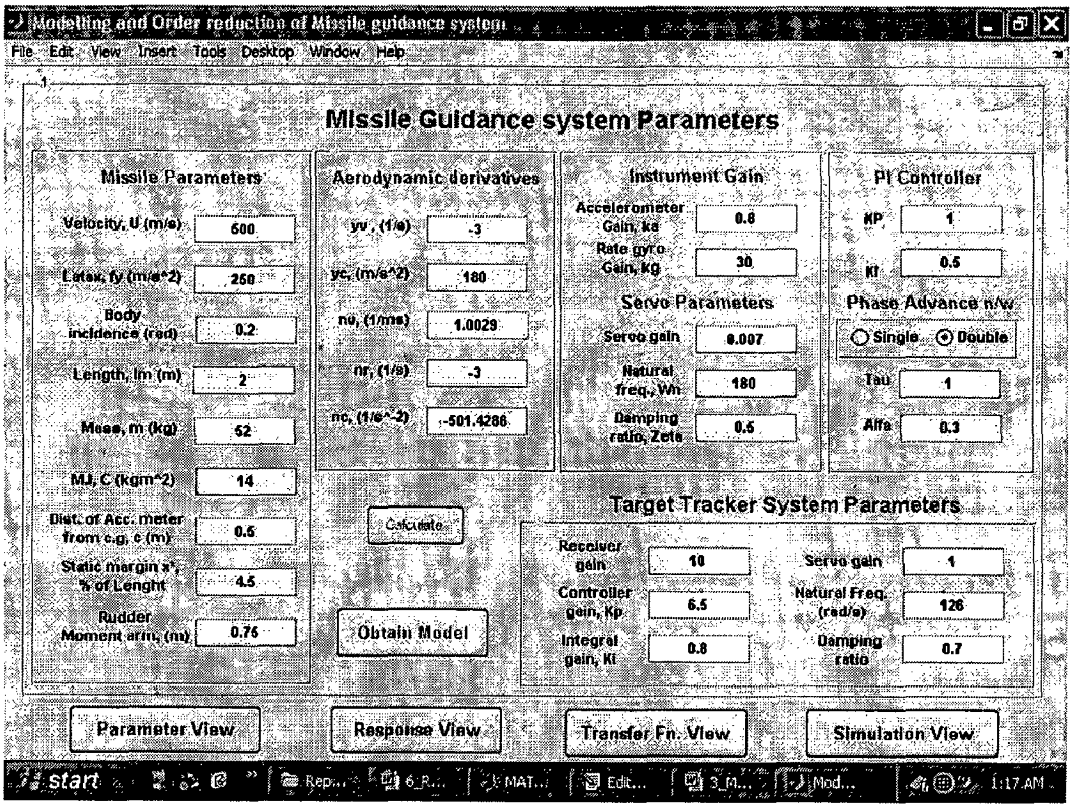

Fig. 6.2 Aerodynamic derivatives obtained in the 'Parameter View' of the GUI

The transfer functions of the different functional blocks obtained are given below :

Fin Servo:

$$\frac{-0.007}{3.086\text{x}10^{-5}\, s^2 + 0.005556\, s + 1} \qquad (6.9)$$

Aerodynamic Derivative (Lateral acceleration/Rudder deflection):

$$\frac{180\, s^2 + 540\, s - 6.619\, \text{x}10^5}{s^2 + 6\, s + 510.4} \qquad (6.10)$$

Aerodynamic Derivative (Body rate/Lateral acceleration) :

$$\frac{-501.4\, s - 1324}{180\, s^2 + 540\, s - 6.619\, \text{x}10^5} \qquad (6.11)$$

Closed loop :

$$\frac{-40824\ s^2 - 1.225\ x10^5\ s + 1.501\ x10^8}{s^4 + 186\ s^3 + 4.682\ x10^4\ s^2 + 3.72\ x10^6\ s + 1.456\ x10^8} \quad (6.12)$$

Figure 6.3 below shows the step responses of the fin-servo, aerodynamic derivatives (Lateral acceleration/Rudder deflection, Body rate/Lateral acceleration) and autopilot.

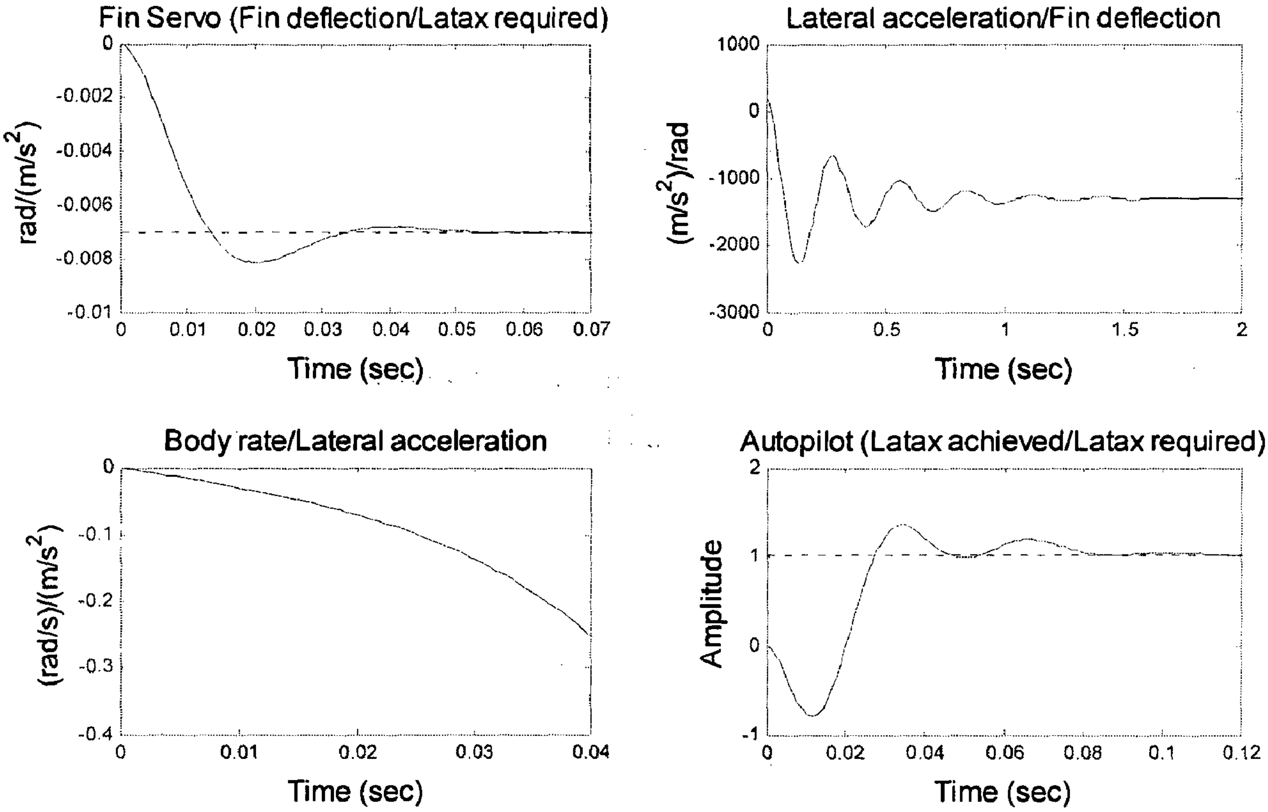

Fig 6.3 Step responses of fin-servo, aerodynamic derivatives, and autopilot

The time response and frequency responses of the autopilot are shown in the figure 6.4 below.

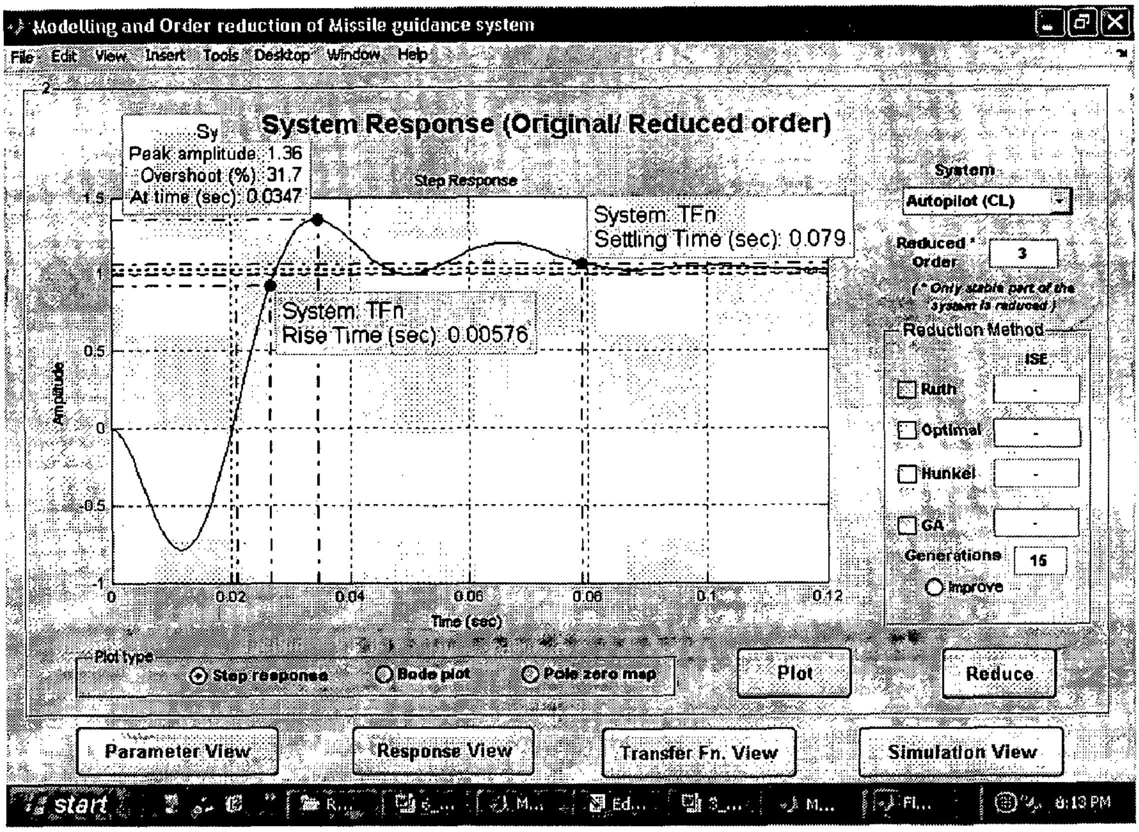

Fig 6.4 Step response of Autopilot

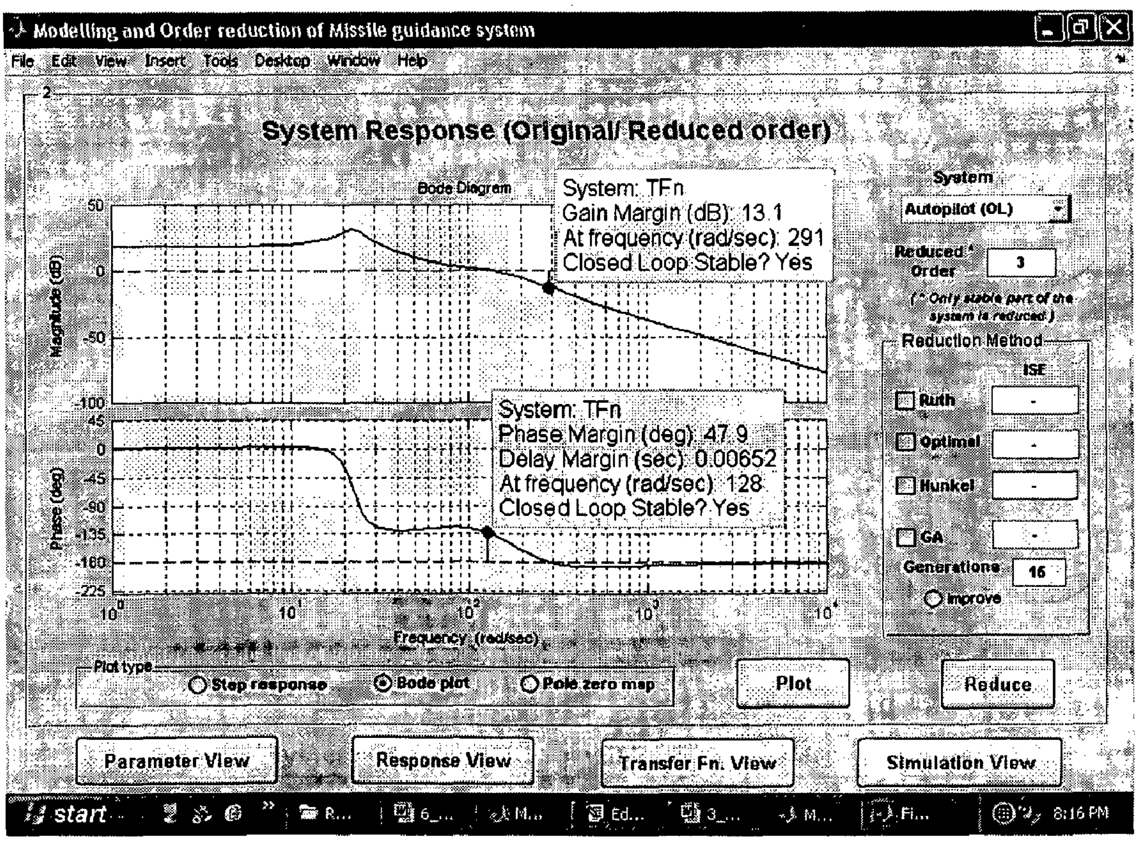

Fig 6.5 Bode plot of Autopilot

Time response

Rise time : 0.00576s.

Settling time : 0.079s.

Overshoot : 31.7% at 0.0347s.

Frequency response

Gain margin : 13.1 dB; Phase crossover frequency : 291 rad/s

Phase margin : 47.9 °; Gain crossover frequency : 128 rad/s

Missile Kinematics

As described in chapter-3, the linearized kinematics of the missile without the range term is given by $1/s^2$. This makes the guidance loop a type-2 system, and hence a phase compensation network is inevitable as evident from the bode diagram shown below.

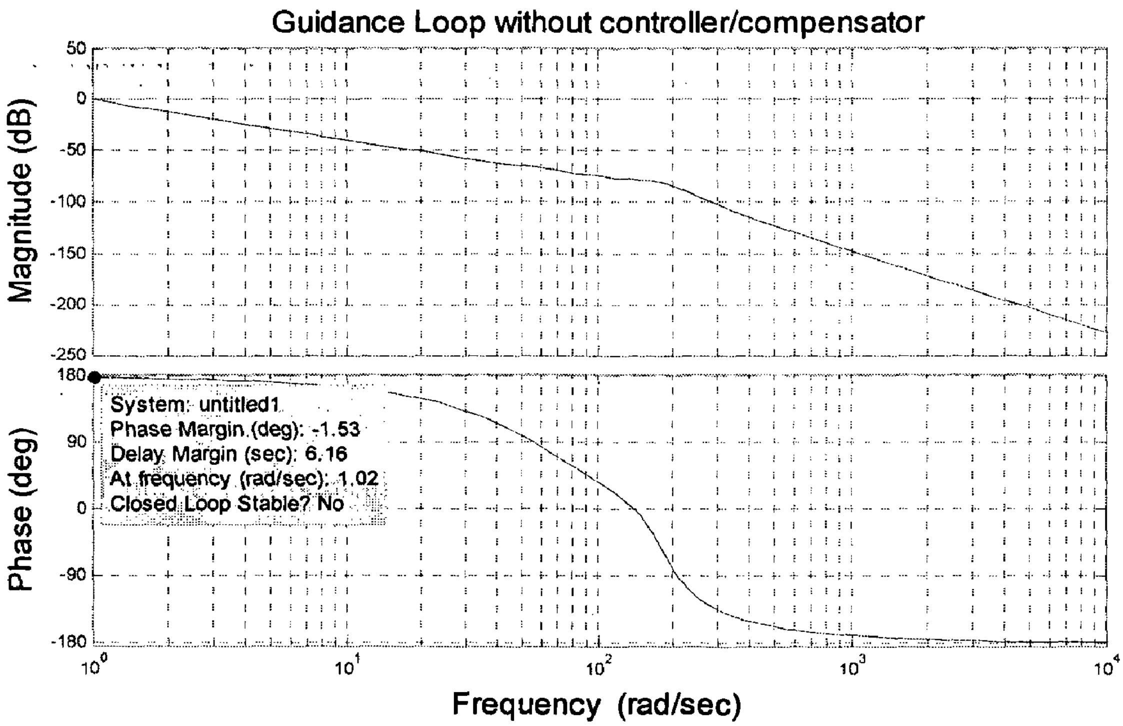

Fig 6.6 Bode plot of Guidance loop without controller

Here the phase margin is negative (-1.53°) and there is no gain margin; hence the system is unstable. The following discussion shows how the PI Controller along with double phase advance network makes the system stable.

Controller and compensation network

To reduce the velocity error a PI Controller is used with Proportional gain, K_P= 1, and Integral gain K_I = 0.5. The transfer function and bode plot are given below.

Transfer function: $\frac{s + 0.5}{s}$

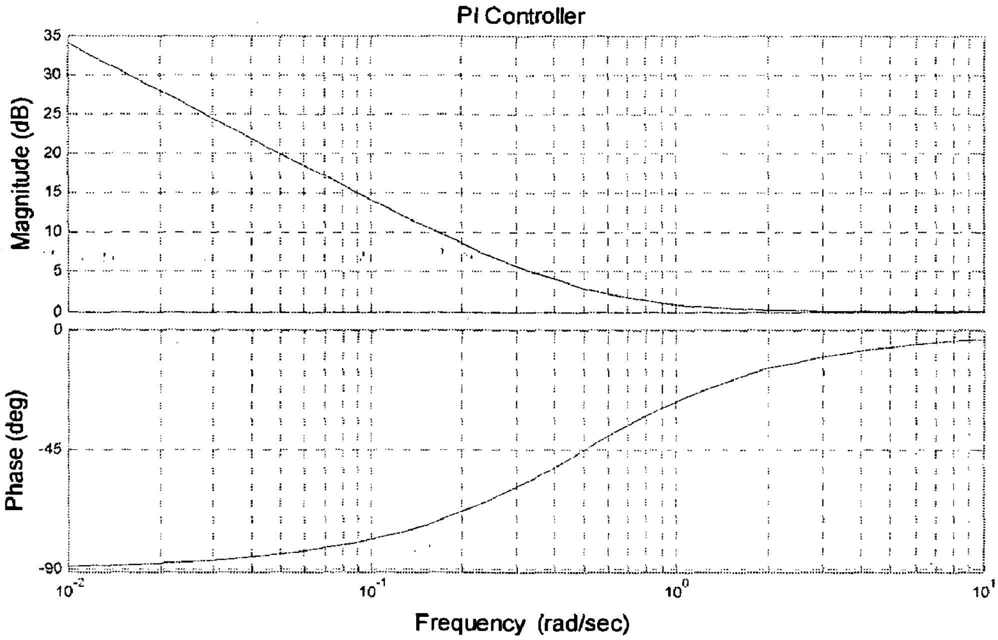

Fig 6.7 Bode plot of PI controller

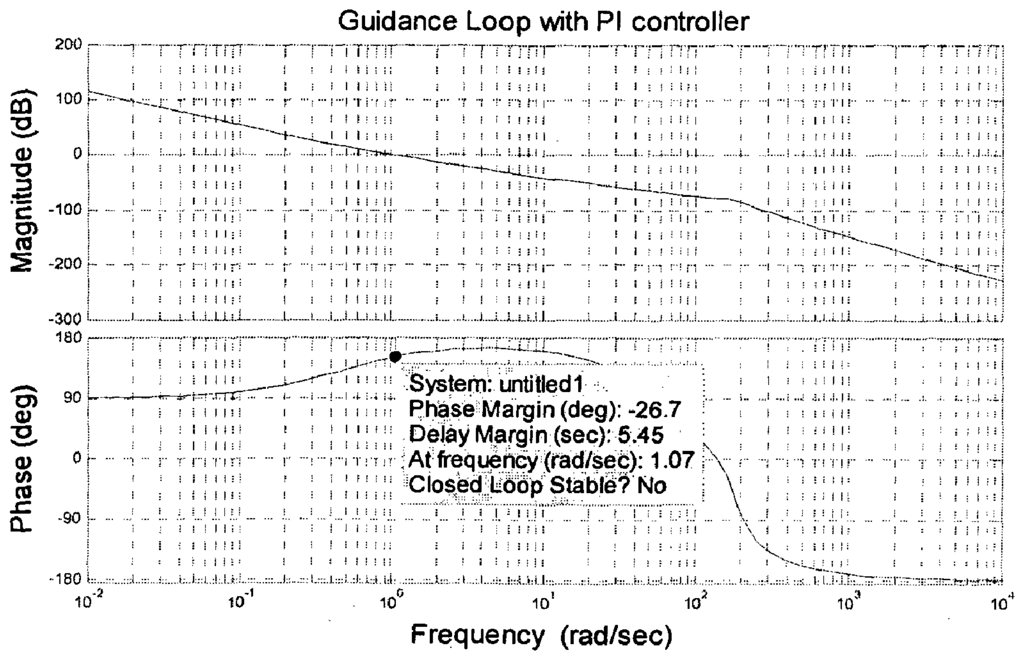

Fig 6.8 Bode plot of Guidance loop with PI controller

The transfer function and bode plot of single phase advance network and double phase advance network with $\alpha = 0.3$ and $\tau = 1$, is shown below.

Single phase-advance n/w	Double phase-advance n/w
$\frac{s+1}{0.3\,s+1}$	$\frac{s^2+2\,s+1}{0.09\,s^2+0.6\,s+1}$

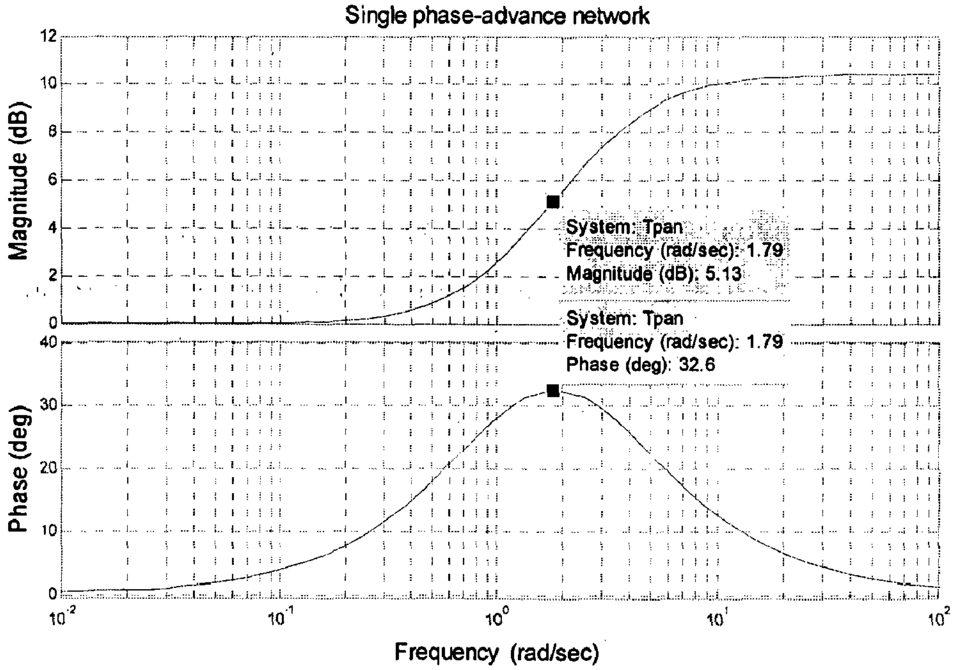

Fig 6.9 Bode plot of single phase-advance network

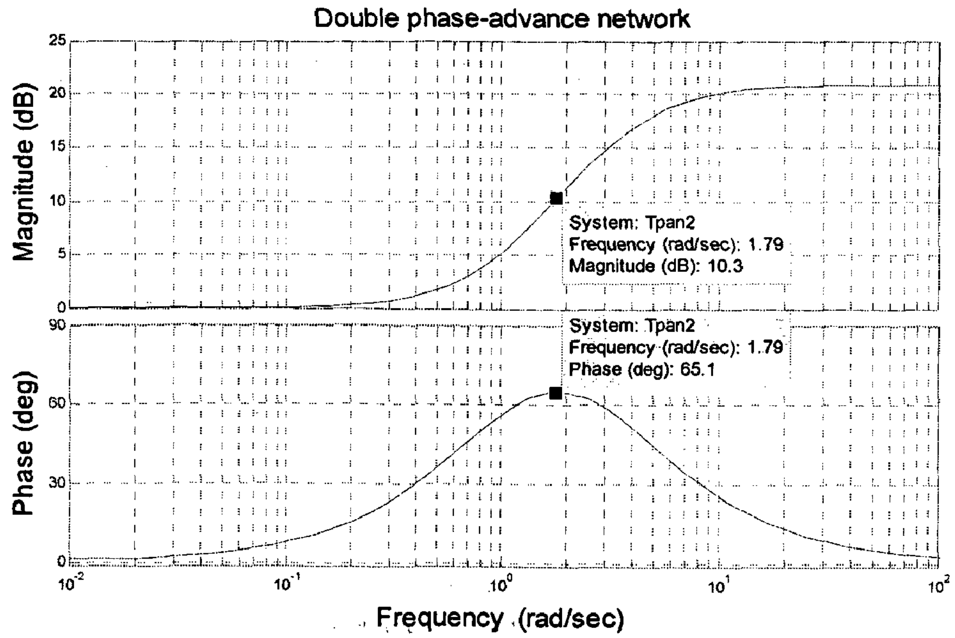

Fig 6.10 Bode plot of double phase advance network

The bode plot shows that the system which was originally unstable becomes stable with a single phase-advance network (low phase margin: 8.11°) and the system becomes much more stable when a double with a double phase-advance network (phase margin: 47.7°).

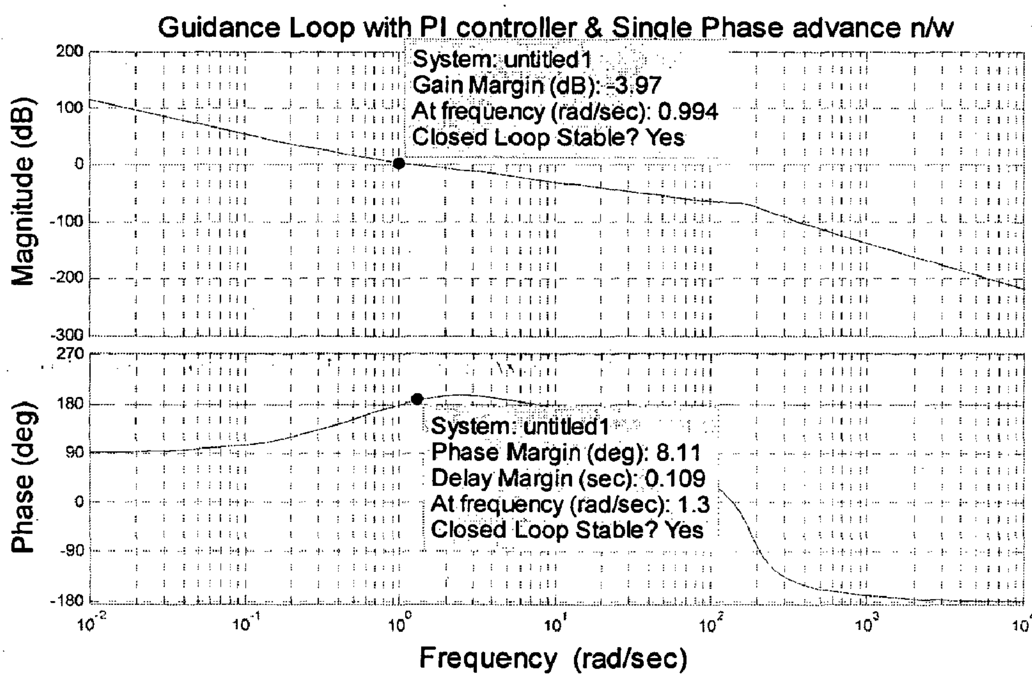

Fig 6.11 Bode plot of Guidance loop with PI controller and single phase advance network

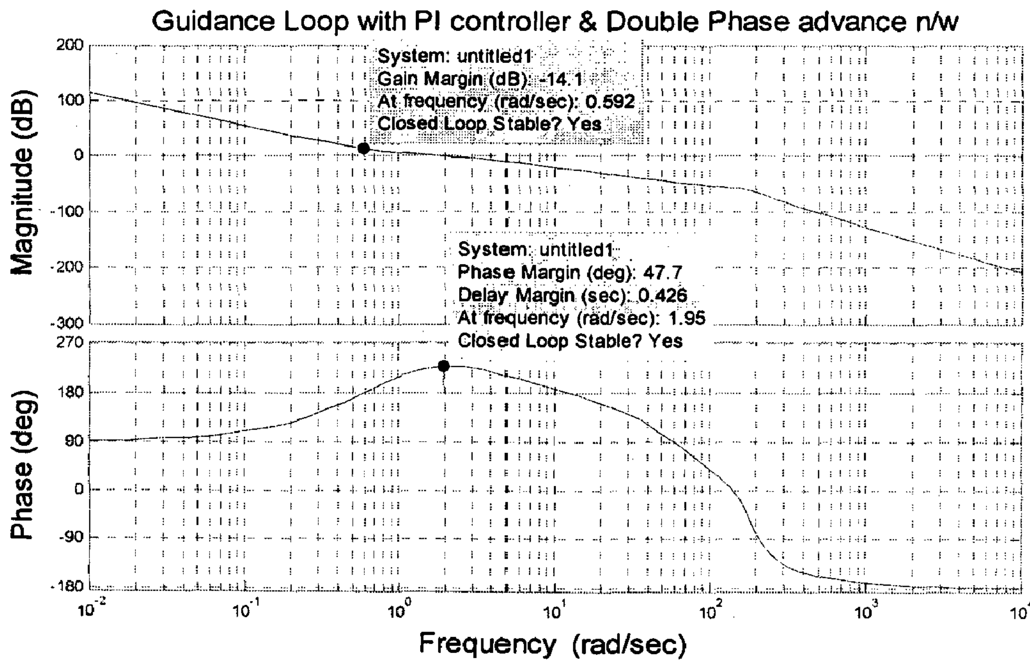

Fig 6.12 Bode plot of Guidance loop with PI controller and double phase advance network

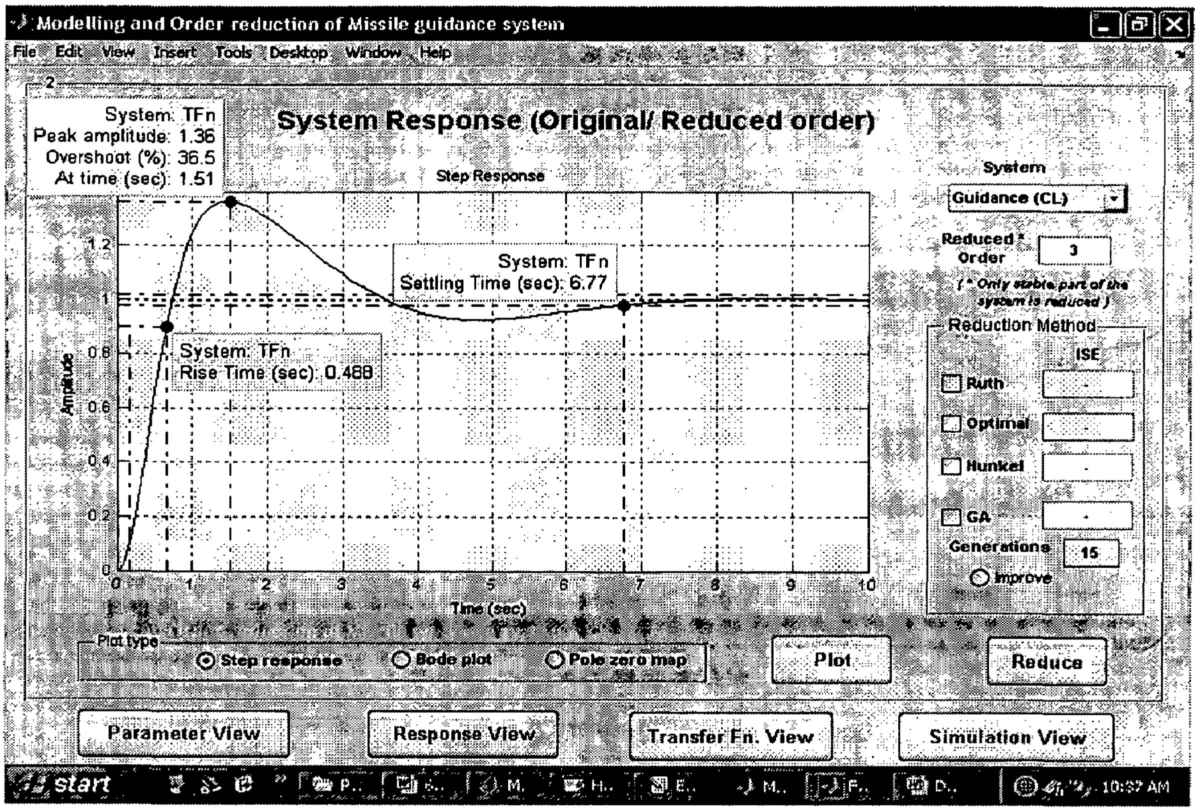

Fig. 6.13 Step response of Missile guidance loop

The transfer function of the Guidance loop obtained by cascading the transfer functions of Autopilot, Missile kinematics, Controller and Double phase advance network, is shown below, which is a 9th order system.

Guidance Loop :

$$\frac{-453600s^5 - 2.495\text{x}10^6 s^4 + 1.664\text{x}10^9 s^3 + 4.167\text{x}10^9 s^2 + 3.335\text{ x}10^9 s + 8.34\text{x}10^8}{s^9 + 192.7 s^8 + 4.807\text{x}10^4 s^7 + 4.034\text{x}10^6 s^6 + 1.705\text{x}10^8 s^7 + 1.01\text{x}10^9 s^4 + 3.282\text{x}10^9 s^3 + 4.167\text{x}10^9 s^2 + 3.335\text{x}10^9 s + 8.34\text{x}10^8} \qquad (6.13)$$

From the step response of the missile guidance loop shown in figure 6.13 the following time response parameters are obtained:

Rise time : 0.488 s.

Settling time : 6.77 s.

Peak overshoot: 36.5 % at 1.51 s.

6.1.2 Target Tracking Loop

The transfer function of the target tracking loop with Receiver gain = 10, Controller gain K_P = 6.5, Integral gain K_I = 0.8, Servo gain = 1, Natural frequency = 126 rad/s, Damping ratio = 0.7, is given below.

Target Tracking Loop:

$$\frac{1.032\times10^{6}\, s + 1.27\times10^{5}}{s^{4} + 176.4\, s^{3} + 1.588e004\, s^{2} + 1.032\times10^{6}\, s + 1.27\times10^{5}} \qquad (6.14)$$

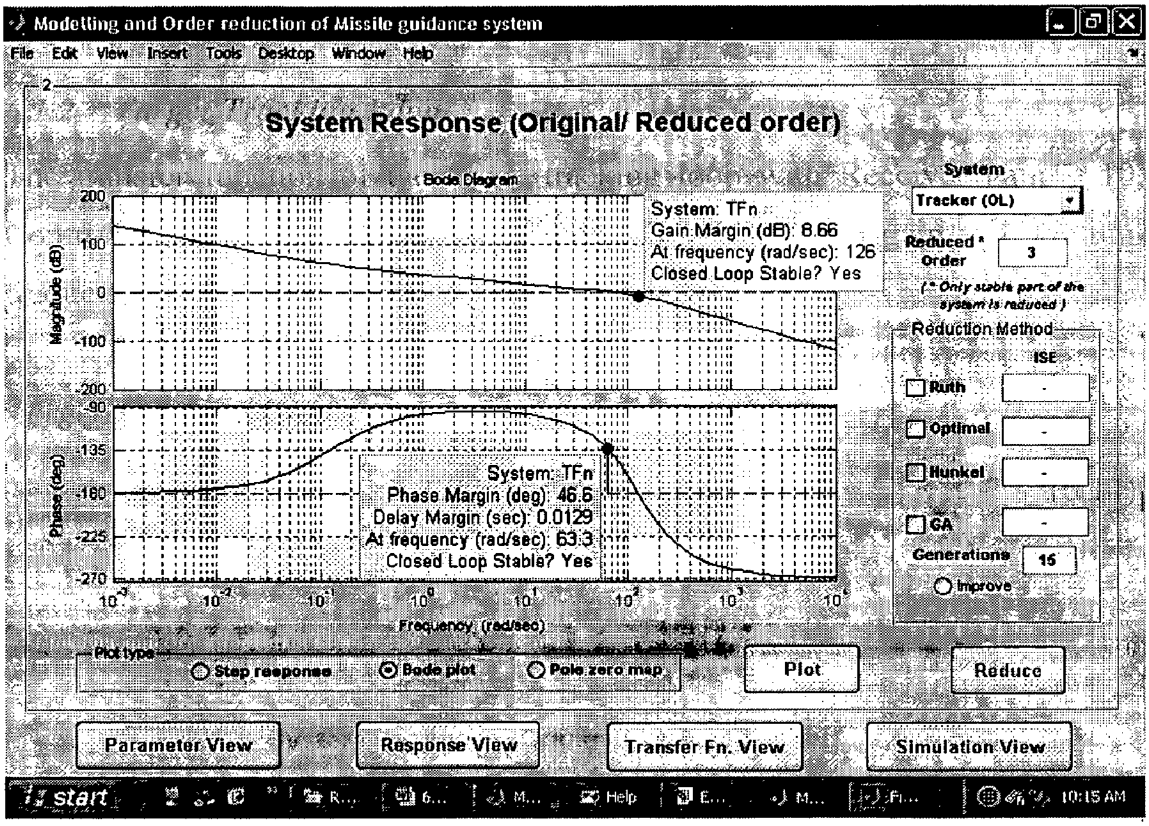

Fig. 6.14 Bode diagram of Target tracking loop (Open loop)

The frequency response obtained in figure 6.14 shows that the system is stable with,

Phase margin: 46.6 (Gain crossover frequency = 63.3 rad/s) and

Gain margin: 8.66 dB (Phase crossover frequency = 126 rad/s).

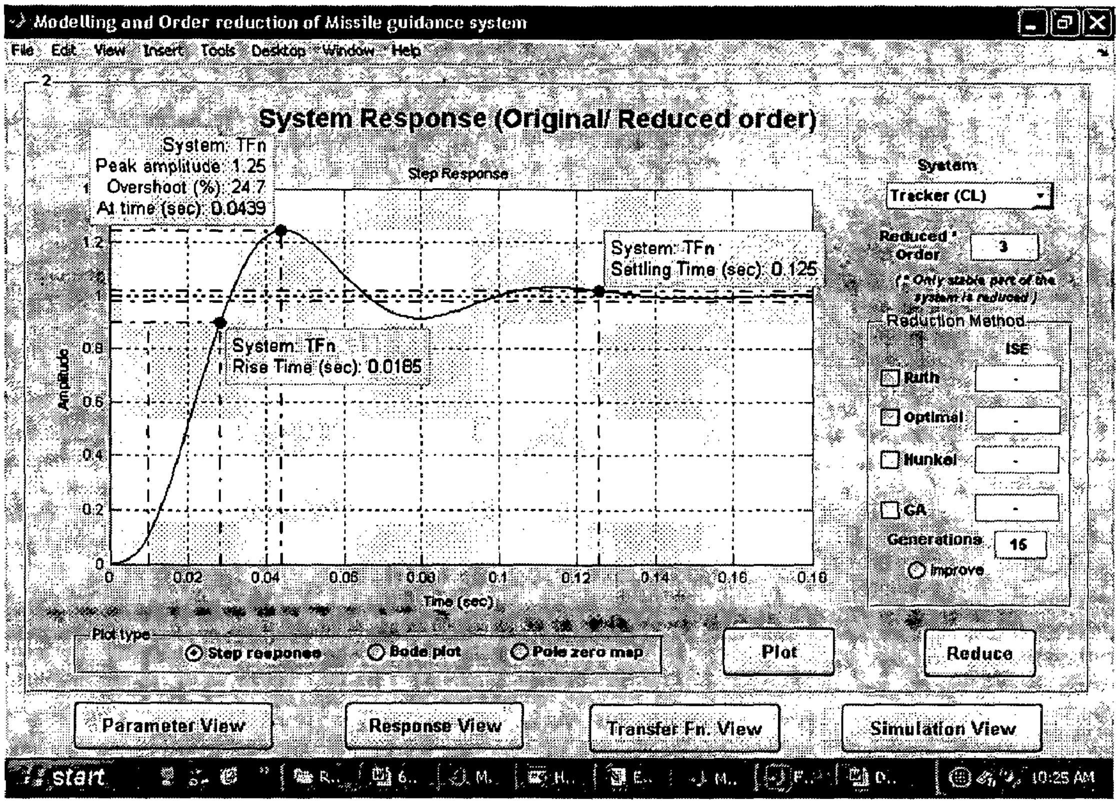

Fig. 6.15 Step response of Target Tracking loop

From the step response shown in figure 6.15, the time response parameters of the Target tracker are:

Rise time : 0.0185s Settling time : 0.125 s

Peak overshoot: 24.7% at 0.0439 s.

6.1.3 Overall System

The overall transfer function is obtained by cascading the Missile guidance loop and Target tracking loop. The transfer function and the step response are shown below, which is a 13th order system.

$$\frac{-4.681\text{x}10^{11}\, s^6 - 2.632\text{x}10^{12}\, s^5 + 1.716\text{x}10^{15}\, s^4 + 4.511\text{x}10^{15}\, s^3 + 3.971\text{x}10^{15}\, s^2 + 1.284\text{x}10^{15}\, s + 1.059\text{x}10^{14}}{s^{13} + 369.1\, s^{12} + 9.793\text{x}10^4\, s^{11} + 1.66\text{x}10^7\, s^{10} + 1.844\text{x}10^9\, s^9 + 1.448\text{x}10^{11}\, s^8 + 7.058\text{x}10^{12}\, s^7 + 1.931\text{x}10^{14}\, s^6 + 1.117\text{x}10^{15}\, s^5 + 3.582\text{x}10^{15}\, s^4 + 4.77\text{x}10^{15}\, s^3 + 3.984\text{x}10^{15}\, s^2 + 1.284\text{x}10^{15}\, s + 1.059\text{x}10^{14}} \quad \text{\_\_\_\_(6.15)}$$

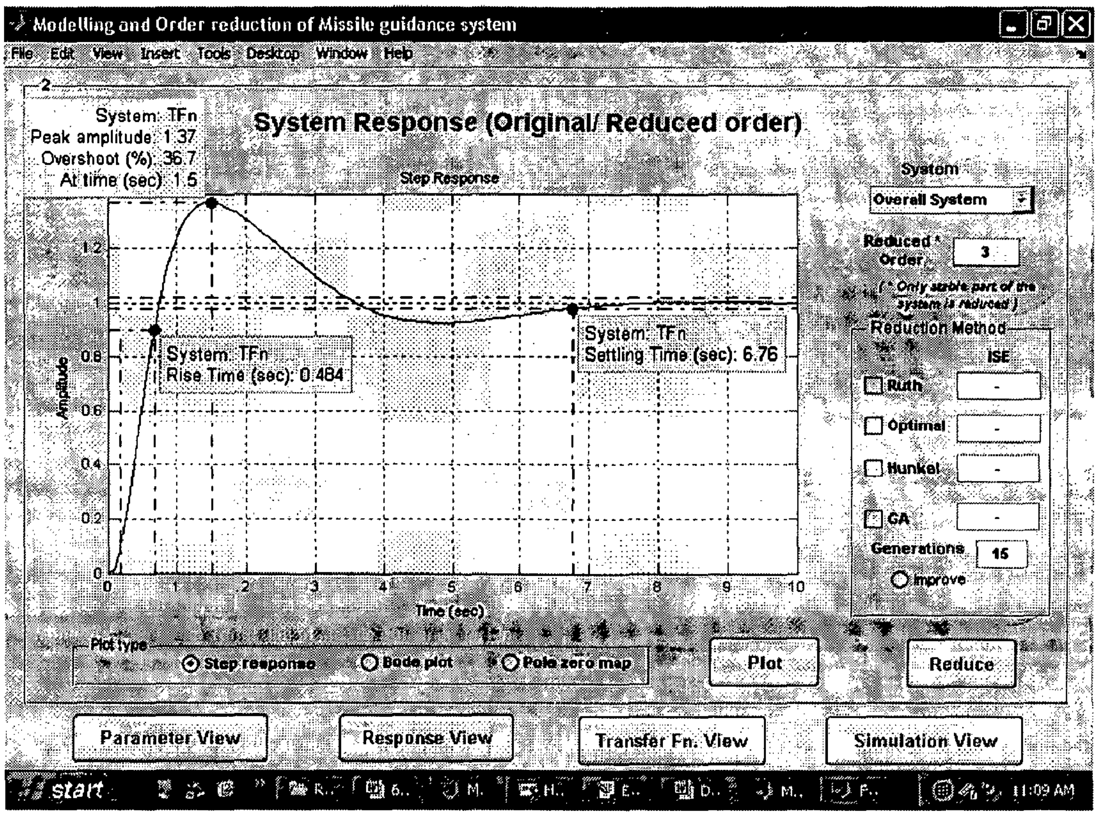

Fig. 6.16 Step response of the overall system

From the step response shown in figure 6.16, the time response parameters of the Overall system are:

Rise time : 0.484s Settling time : 0.6.76 s

Peak overshoot: 36.7% at 1.5 s.

When comparing with the response of the missile guidance loop it is evident that the response of the overall system is dominated by the characteristics of missile guidance loop.

6.2 Model order Reduction.

The order of the three closed loop systems viz. Autopilot, Missile guidance loop and the overall system are reduced by four different methods and the Integral square error (ISE) and the step responses and frequency responses are plotted as given below.

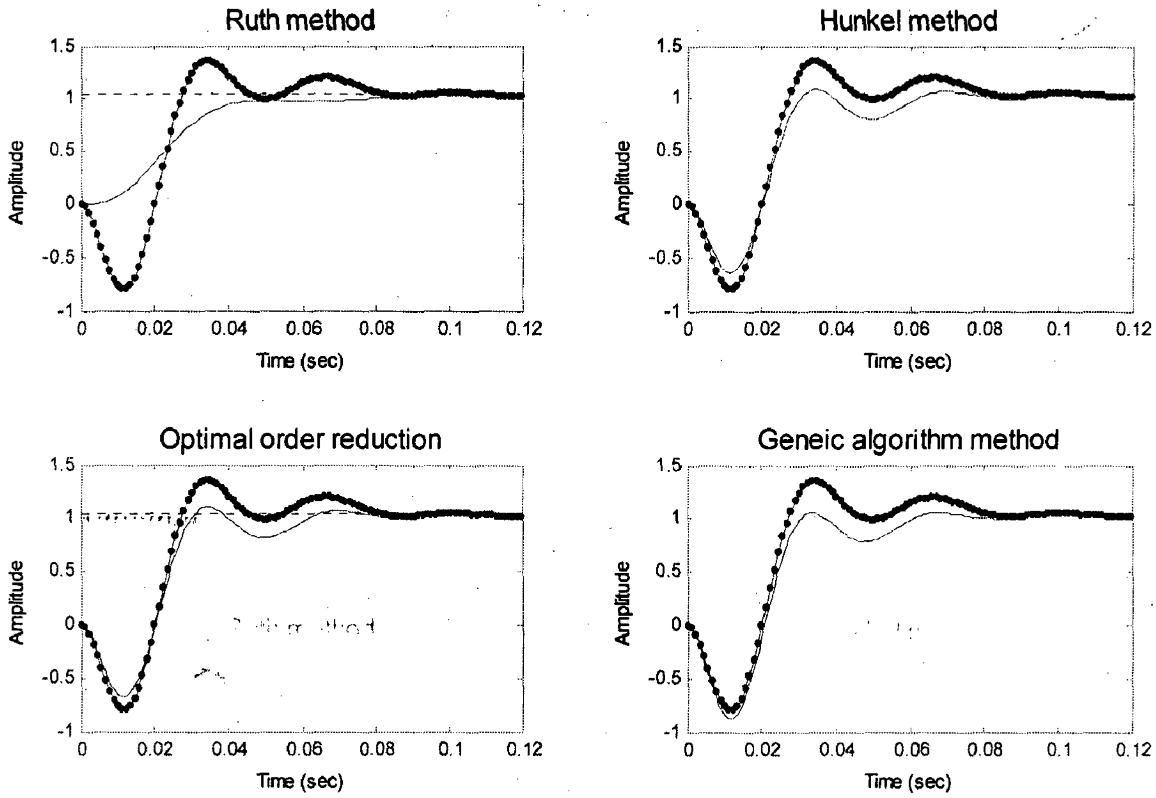

Fig. 6.17 Step response of Original 4^{th} order and reduced 3^{rd} order system

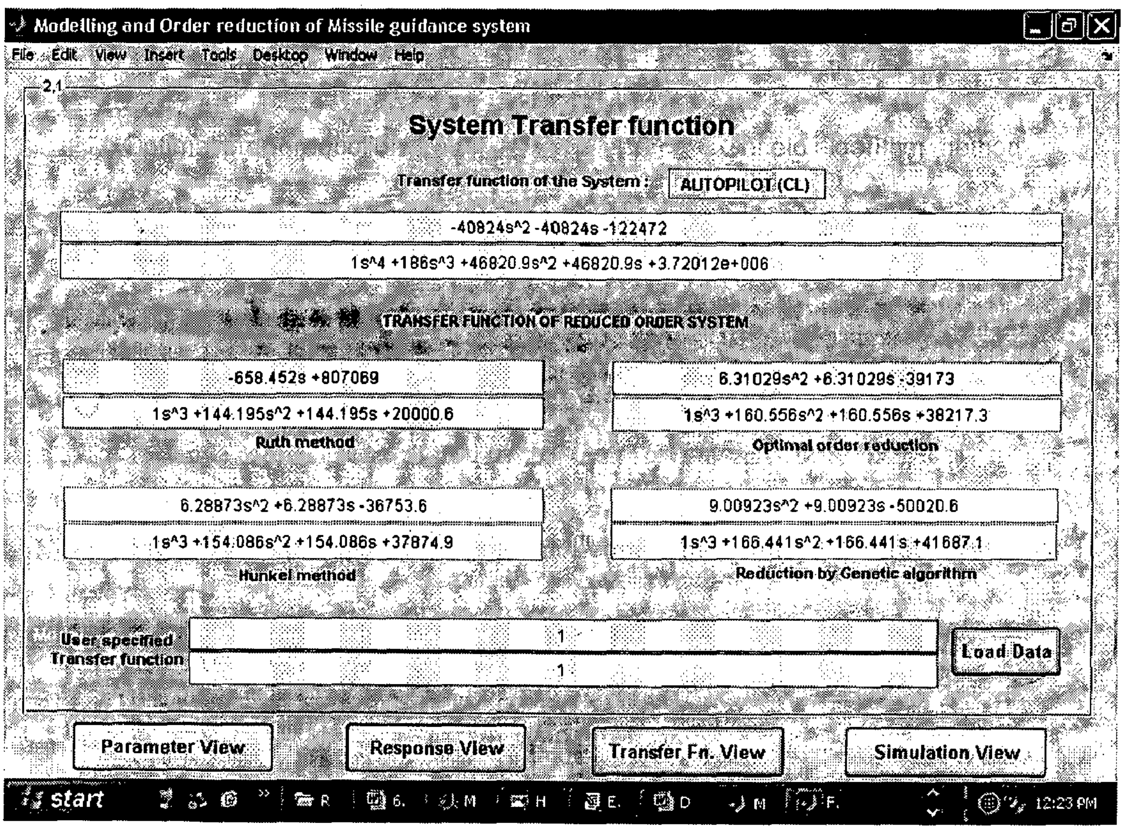

Fig. 6.18 Transfer function of the original and reduced order system

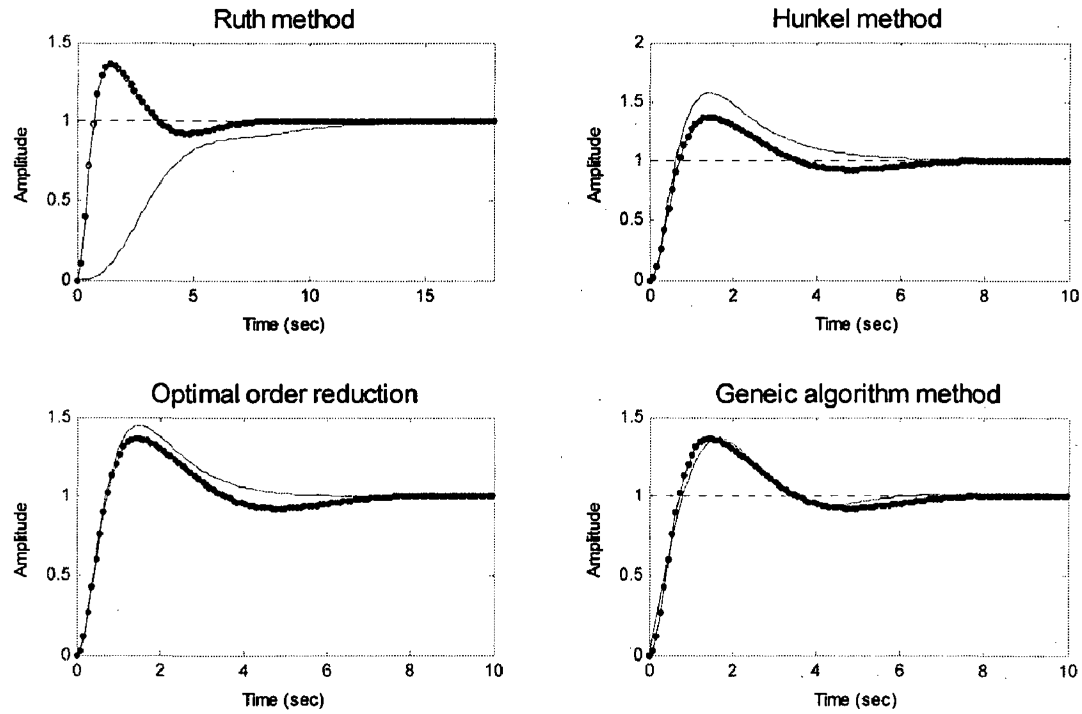

Fig. 6.19 Step response of Original 9th order and reduced 3rd order system

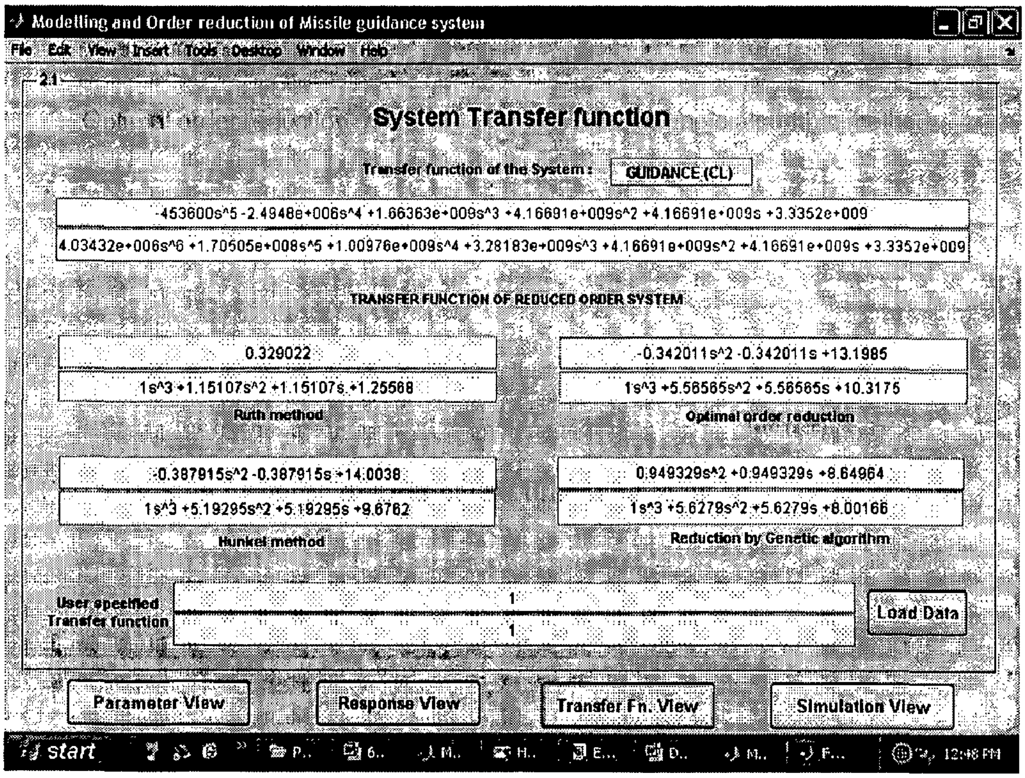

Fig. 6.20 Transfer function of the original and reduced 3rd order system

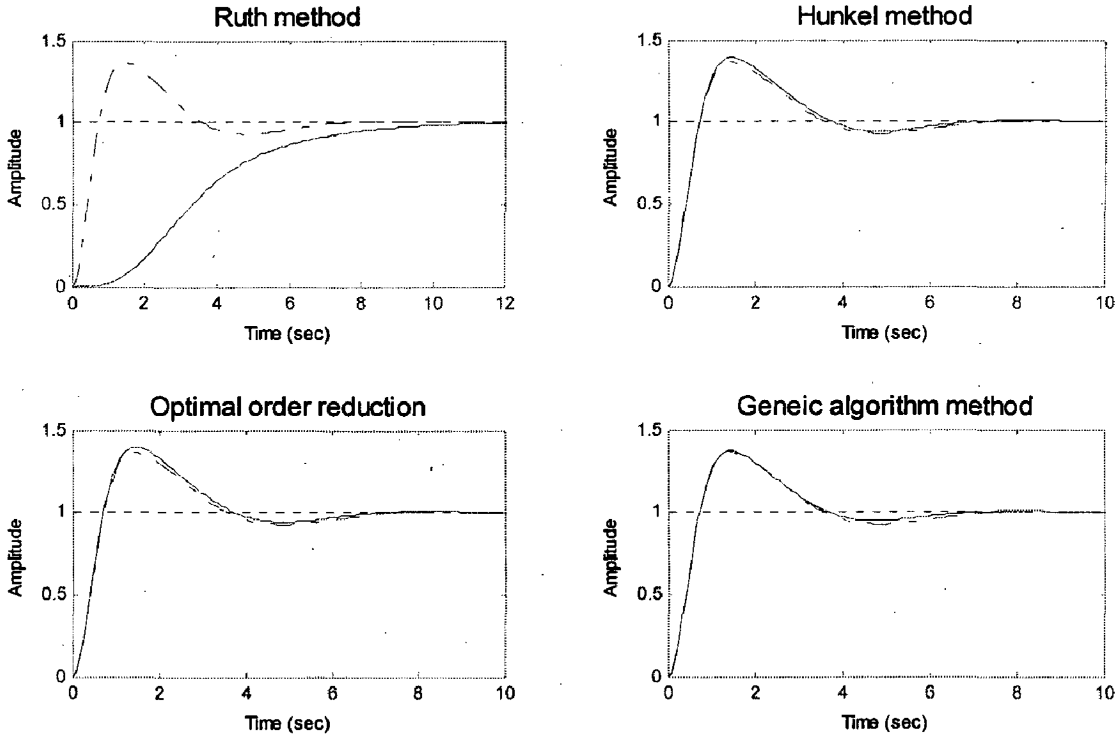

Fig. 6.21 Step response of Original 9th order and reduced 4rd order system

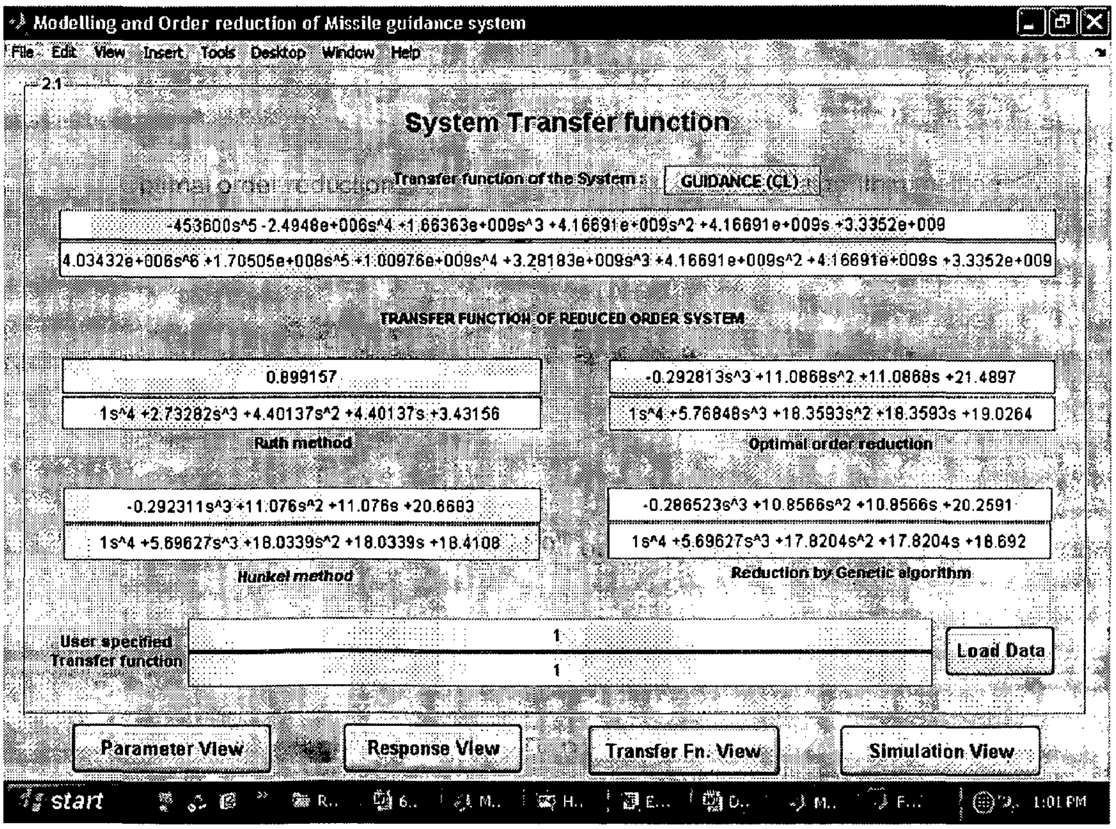

Fig. 6.22 Transfer function of the original and reduced 4th order system

Overall System

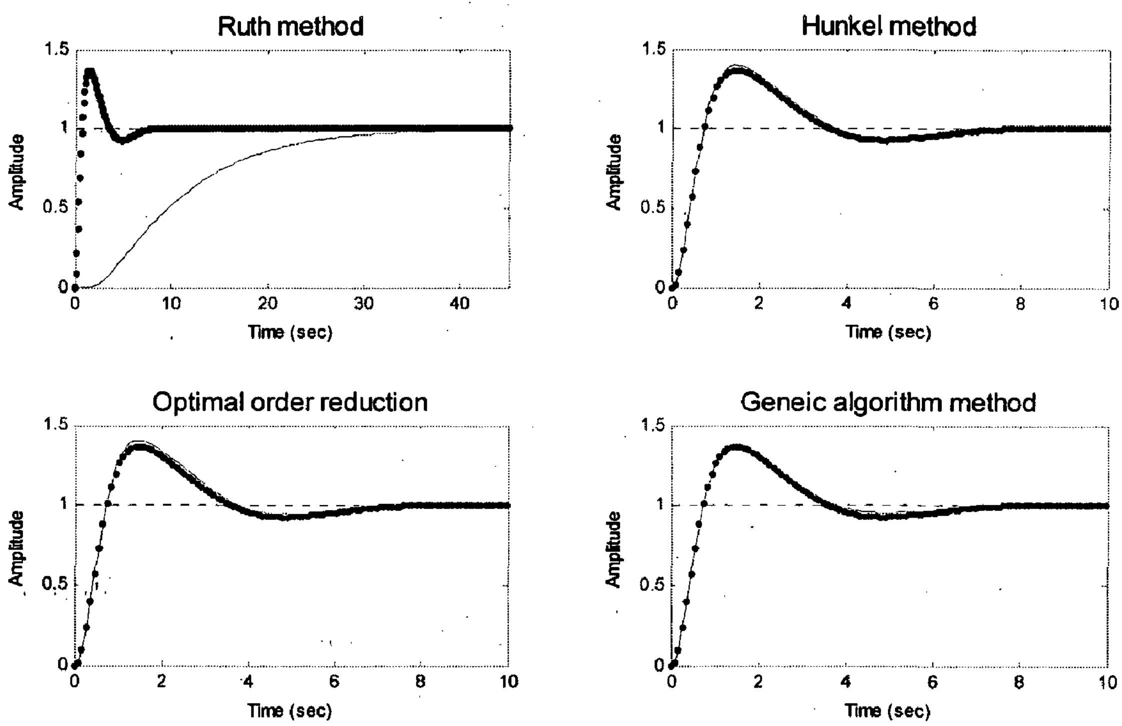

Fig. 6.23 Step response of Original 13th order and reduced 4rd order system

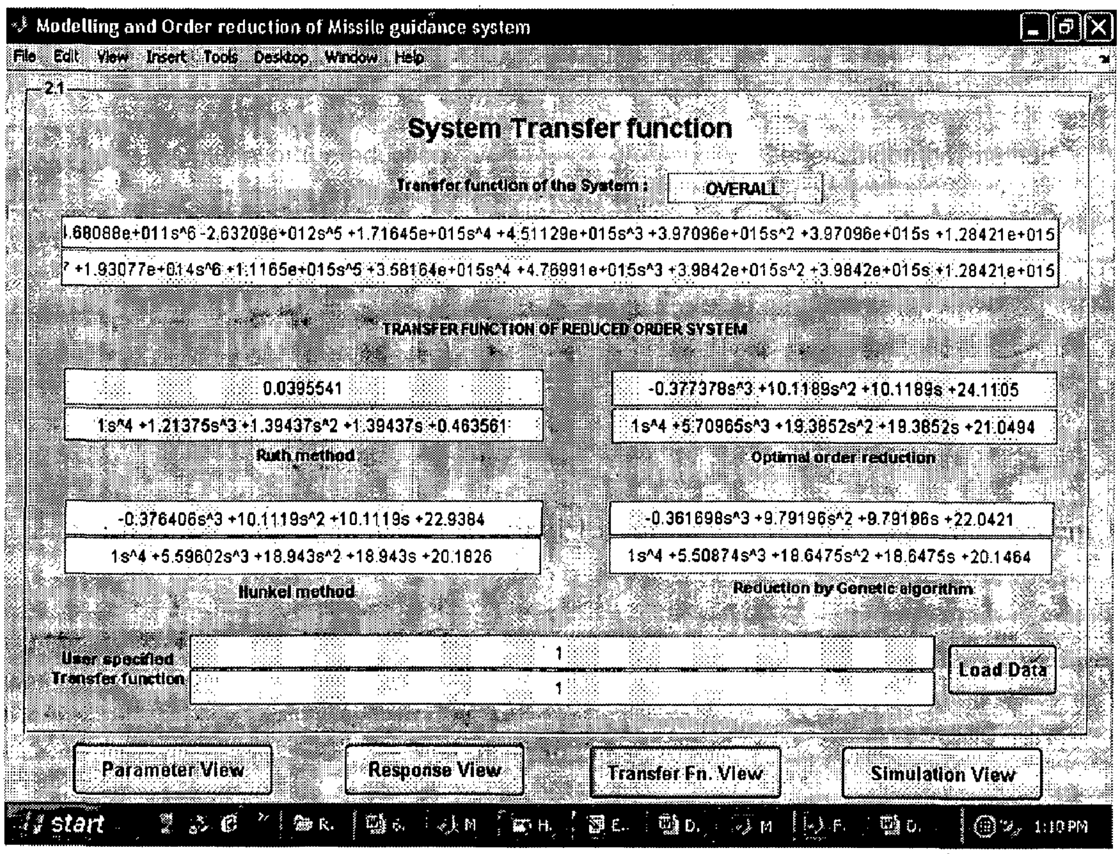

Fig. 6.24 Transfer function of the original and reduced 4th order system

Integral Square Error (ISE)

For the performance evaluation of different algorithms of model order reduction Integral square error was used. The table below shows the ISE obtained for different algorithms and different system.

System	Original Order	Reduced Order	I.S.E			
			Ruth Method	Optimal order reduction	Hunkel Method	By Genetic algorithm
Autopilot	4	3	0.01224	1.91×10^{-3}	2.23×10^{-3}	2.46×10^{-3}
Guidance loop	9	3	3.05	37.47×10^{-3}	127.3×10^{-3}	9.81×10^{-3}
Guidance loop	9	4	3.271	3.355×10^{-3}	2.854×10^{-3}	1.412×10^{-3}
Overall System	13	3	6.745	37.83×10^{-3}	141.2×10^{-3}	14.14×10^{-3}
Overall System	13	4	7.116	4.709×10^{-3}	3.858×10^{-3}	2.101×10^{-3}

Table 6.1 ISE obtained for different reduction methods

6.3 Simulation of Missile Trajectory

Missile trajectory with various target initial conditions are simulated on a 3-D plane and the miss-distance is obtained. Four typical cases are shown in figures 6.25a to 6.25d, with different initial conditions.

Missile velocity : 500m/s, Target velocity : 200m/s.

Target Path

I. From (x=12.0, y=3.0, z=8.0) km towards (x=0.1, y=0.5, z=8.0) km; Miss-distance = 0.88m

II. From (x=12.0, y=3.0, z=8.0) km towards (x=0.1, y=0.5, z=8.0) km; Miss-distance = 0.84m

III. From (x=2.0, y=6.0, z=2.0) km towards (x=1, y=0.5, z=8.0) km; Miss-distance = 1.23m

IV. From (x=2.0, y=8.0, z=5.0) km towards (x=2.0, y=0.7, z=5.0) km; Miss-distance = 2.86m

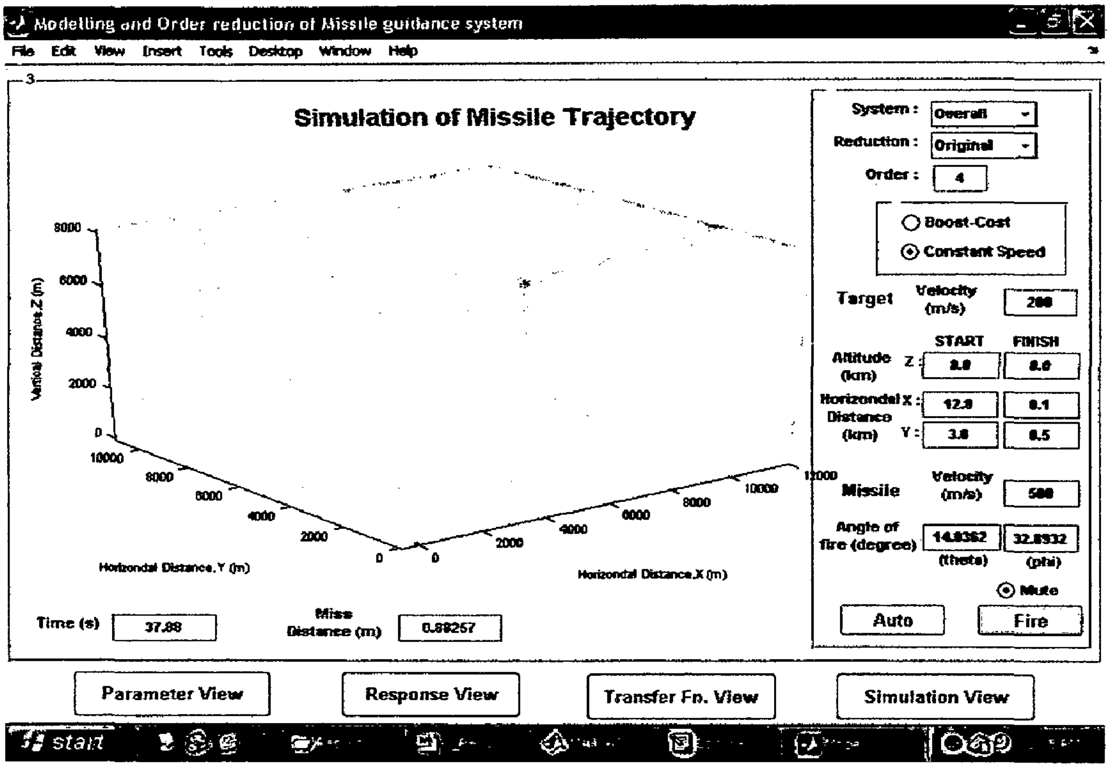

Fig. 6.25a Simulation of Missile trajectory -I

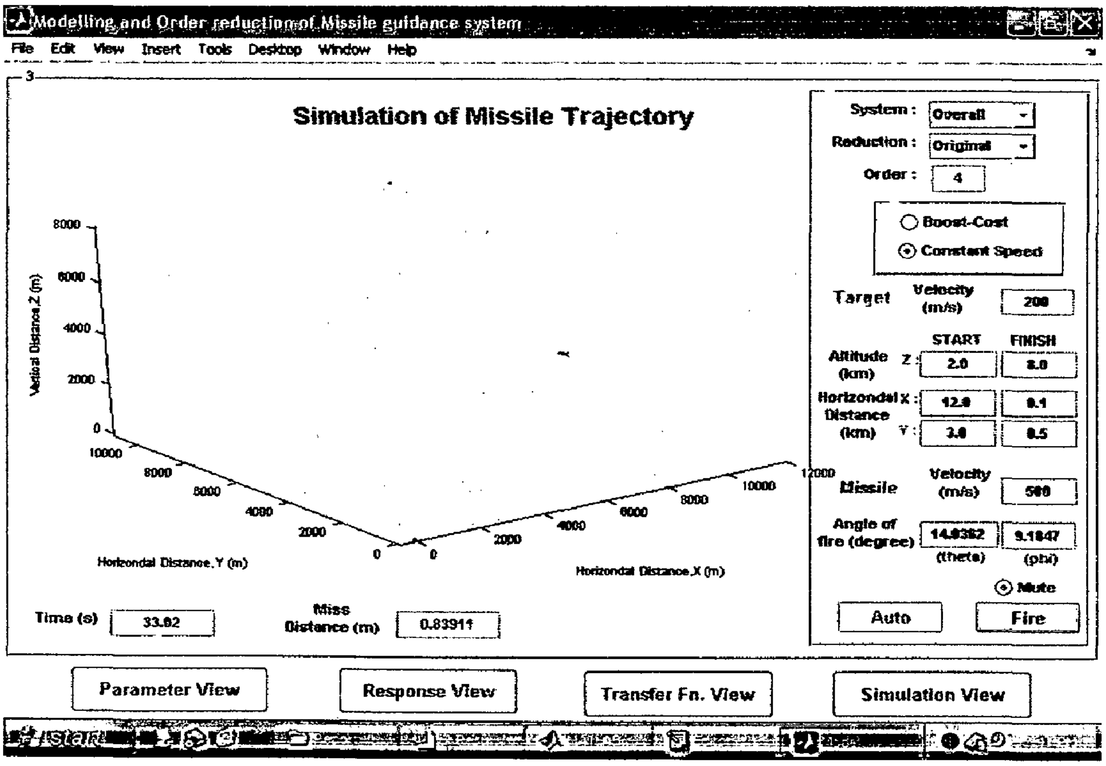

Fig. 6.25b Simulation of Missile trajectory -II

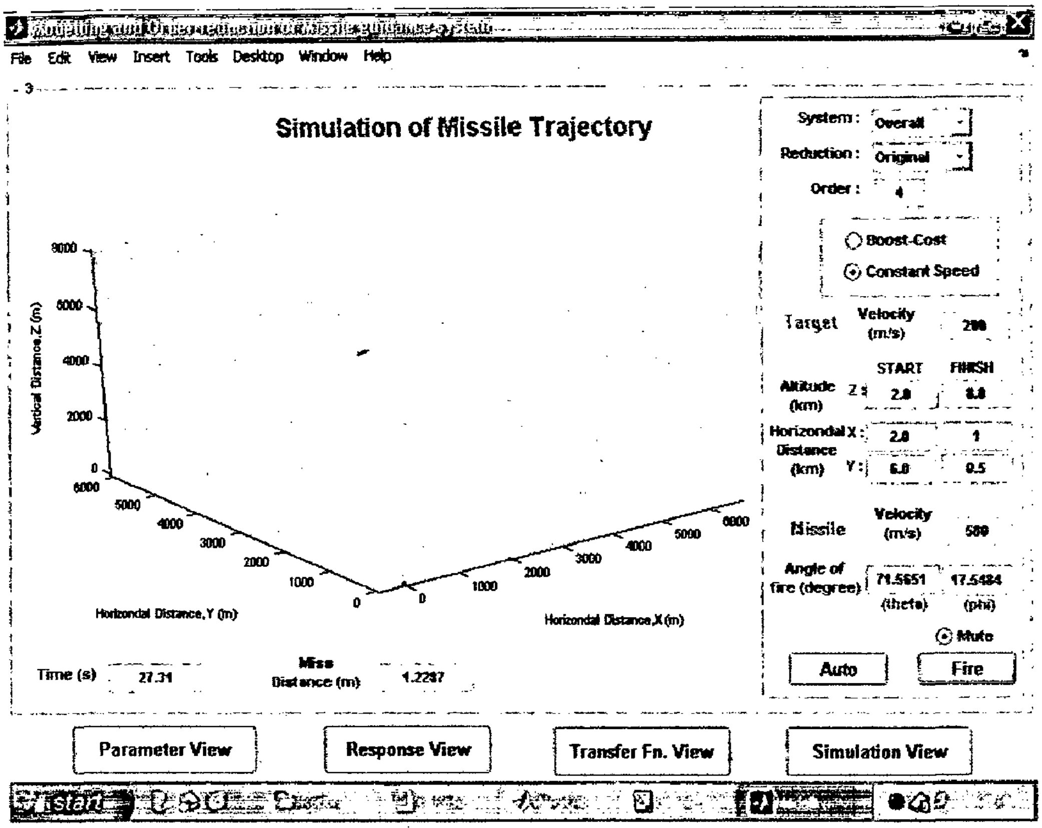

Fig. 6.25c Simulation of Missile trajectory -III

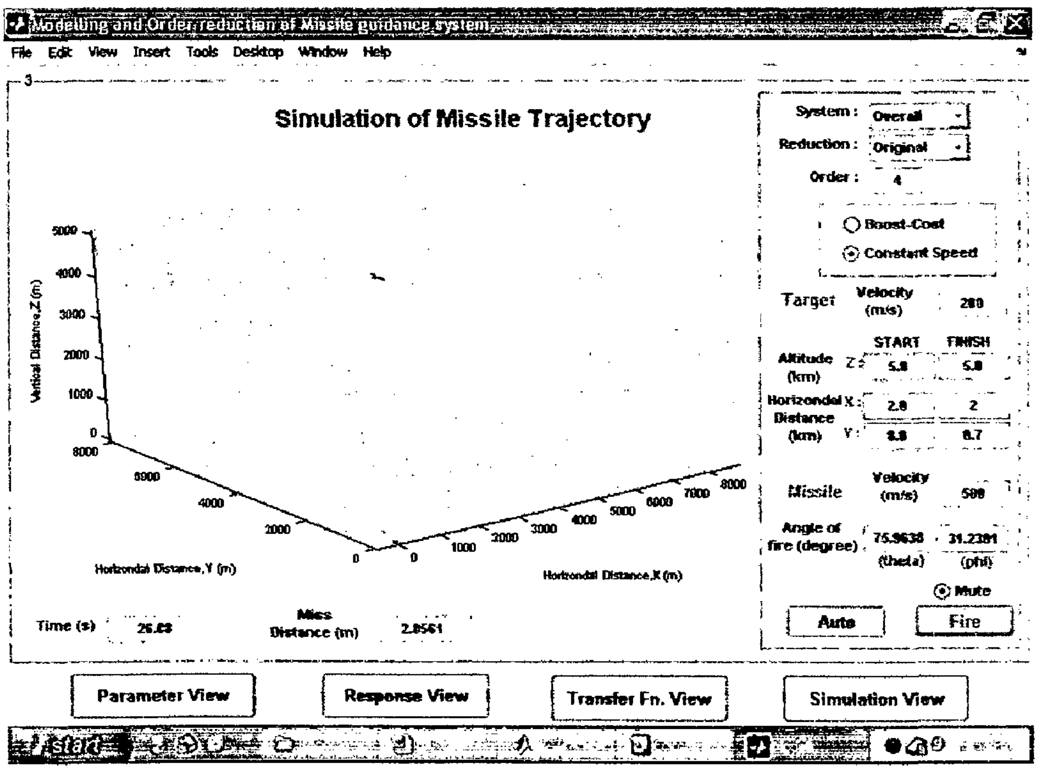

Fig. 6.25d Simulation of Missile trajectory -IV

Target velocity: 200m/s;   Missile velocity: 500m/s

Target flying at constant altitude, towards the point (X=0.5, Y=0.1) km.

Target Altitude (Z) = 5 km

Target Starts from		Miss-distance(m)
X (km)	Y (km)	
12	12	0.77
12	9	0.93
12	6	1.043
12	3	1.153
9	12	0.90
9	9	1.08
9	6	1.20
9	3	1.26
6	12	1.03
6	9	1.23
6	6	1.31
6	3	1.12
3	12	1.15
3	9	1.35
3	6	2.063
3	3	--

Target Altitude (Z) = 3 km

Target Starts from		Miss-distance(m)
X (km)	Y (km)	
12	12	0.57
12	9	0.71
12	6	0.83
12	3	1.02
9	12	0.70
9	9	0.97
9	6	1.18
9	3	1.41
6	12	0.85
6	9	1.20
6	6	1.67
6	3	2.07
3	12	0.10
3	9	1.49
3	6	2.22
3	3	3.57

Table 6.2 Miss-distance calculated with target flying at constant altitude - I

Target velocity: 300m/s; Missile velocity: 700m/s

Target flying at constant altitude, towards the point (X=0.5, Y=0.1) km.

Target Altitude (Z) = 5 km

Target Starts from		Miss-distance(m)
X (km)	Y (km)	
12	12	1.84
12	9	2.17
12	6	2.46
12	3	2.59
9	12	2.19
9	9	2.62
9	6	2.84
9	3	2.83
6	12	2.50
6	9	2.92
6	6	3.02
6	3	2.60
3	12	2.72
3	9	3.23
3	6	6.34
3	3	--

Target Altitude (Z) = 3 km

Target Starts from		Miss-distance(m)
X (km)	Y (km)	
12	12	1.39
12	9	1.77
12	6	2.11
12	3	2.40
9	12	1.86
9	9	2.34
9	6	3.05
9	3	3.65
6	12	2.22
6	9	3.09
6	6	4.28
6	3	5.12
3	12	2.54
3	9	3.85
3	6	5.55
3	3	10.35

Table 6.3 Miss-distance calculated with target flying at constant altitude - II

Target velocity: 300m/s;   Missile velocity: 700m/s

Target flying at varying altitude

Target starts from altitude(Z) = 5 km, flying towards the point (X=0.5, Y=0.1, Z=2.0) km.

Target Starts from		Miss-distance(m)
X (km)	Y (km)	
12	12	0.88
12	9	1.07
12	6	1.34
12	3	1.47
9	12	1.28
9	9	1.45
9	6	1.84
9	3	2.18
6	12	1.47
6	9	1.95
6	6	2.74
6	3	3.34
3	12	1.76
3	9	2.52
3	6	3.88
3	3	5.32

Target starts from altitude(Z) = 5 km, flying towards the point (X=0.5, Y=0.1, Z=2.0) km.

Target Starts from		Miss-distance(m)
X (km)	Y (km)	
12	12	1.86
12	9	1.83
12	6	1.60
12	3	1.36
9	12	1.85
9	9	1.52
9	6	0.76
9	3	0.42
6	12	1.70
6	9	1.02
6	6	1.56
6	3	2.07
3	12	1.54
3	9	1.34
3	6	--
3	3	--

Table 6.4 Miss-distance calculated with target flying at varying altitude

Chapter - 7
CONCLUSION

Modelling of a guided weapon control system with six-degrees of freedom is carried out using MATLAB-7, on a three dimensional space. The model is obtained by linearising the system at user defined operating point, which can be altered through a GUI window.

The overall system is found to be of 13^{th} order, with 4^{th} order Target tracking loop and 9^{th} order Missile guidance loop (Eq. 6.13 – 6.15). The order is reduced to both 4^{th} order and 3^{rd} order by the four methods of order reduction namely order reduction by using Ruth Table Criteria, Optimal order reduction, Hunkel based method of order reduction and by using Genetic algorithm. The Integral square error (ISE) obtained for method using Ruth table criteria is large compared to other three methods. Order reduction by Genetic algorithm takes considerable time, but the ISE obtained is found to be small (Table 6.1).

Algorithm for simulation of the missile trajectory in a 3-D space is developed, considering the effects of acceleration due to gravity and air friction. The three dimensional trajectories of target and missile are obtained by the simulation. Initial states of the missile and target can be changed and the algorithm calculates the miss-distance. Simulation results with target altitude between 3 to 5 km shows that the miss-distance exceeds 5m, only a few times when the target starts at a distance of 3 km or above in x or y coordinate (Table 6.2 – 6.4).

A graphical user interface is also developed with four panels, namely 'Parameter View', 'Response View', 'Transfer function View' and 'Simulation View'. System parameters are changed through the panel 'Parameter View'; step response, bode plot and pole-zero map of the original system and the reduced order system are shown in 'Response View'. Transfer function is obtained on the panel 'Transfer function View' and trajectory simulation is performed on the 'Simulation View'.

Chapter - 8
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Appendix - A

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Derivation of aerodynamic transfer function using symbolic constants   %
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
clc
clear all
% Declaring Aerodynamic symbols
syms s U Yv Yr Yc C r fy Nv Nr Nc;
%
% Aerodynamic equations
%
% fy = v'   + r.U
%    = Yv.v + Yr.r + Yc.C
% r' = Nv.v + Nr.r + Nc.C
%
% X(1) is v & X(2) is r
%
% A.X = B or X = inv(A).B
%
A = [Yv     Yr;
   Nv     Nr-s];
B = [fy-Yc*C;
   -Nc*C   ];
X=inv(A)*B;
% Expression1 : s.v + U.r - fy = 0
exp1 = s*X(1)+U*X(2)-fy;
simplify(exp1)
% exp1 =
% -(Nv*Yr*fy+Nv*U*Yc*C-Nv*U*fy-Nr*Yv*fy-Nr*s*Yc*C+Nr*s*fy+Nc*C*s*Yr
%             -Nc*C*U*Yv+s*Yv*fy+s^2*Yc*C-s^2*fy)/(-Nr*Yv+s*Yv+Nv*Yr)
% Neglecting Yr,
%
% fy       Yc.s^2 - Yc.Nr.s - U(Nr.Yv - Nv.Yc)
% ---  =  ------------------------------------
% C        s^2 - (Yv+Nr).s +.Yv.Nr + U.nv
%
v = (Yc*C-r*U)/(s-Yv);
% Expression2
exp2 = Nv*v+Nr*r+Nc*C-s*r;
simplify(exp2)
% ans =
% -(-Nv*Yc*C+Nv*r*U-Nr*r*s+Nr*r*Yv-s*Nc*C+Yv*Nc*C+s^2*r-s*r*Yv)/(s-Yv)
%
% r      Nc.s - Nc.Yv + Nv.Yc
%--- = -----------------------------
% C   s^2 - (Yv+Nr).s + Yv.Nr + U.nv
%
```