APPROXIMATION OF ENTIRE FUNCTIONS OF ONE OR SEVERAL COMPLEX VARIABLES

A THESIS

Submitted in partial fulfilment of the requirements for the award of the degree

of

DOCTOR OF PHILOSOPHY

in

MATHEMATICS

By

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AUGUST, 2006

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CANDIDATE'S DECLARATION

I hereby certify that the work which is being presented in the thesis entitled **APPROXIMATION OF ENTIRE FUNCTIONS OF ONE OR SEVERAL COMPLEX VARIABLES** in partial fulfilment of the requirements for the award of the Degree of Doctor of Philosophy and submitted in the Department of Mathematics of the Indian Institute of Technology Roorkee, Roorkee is an authentic record of my own work carried out during a period from **January, 2003** to **August, 2006** under the supervision of **Prof. G. S. SRIVASTAVA.**

The matter presented in this thesis has not been submitted by me for the award of any other degree of this or any other institute.

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This is to certify that the above statement made by the candidate is correct to the best of my knowledge.

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Acknowledgements

I take this opportunity to thank and express profound gratitude to my supervisor, Dr.G.S.Srivastava, Professor, Department of Mathematics, Indian Institute of Technology Roorkee, for his expert guidance, cool temperament and valuable suggestions, support, advice and continuous encouragement during the entire period of my research work. His foresightedness helped me in a big way in completion of this work and I really feel privileged to have worked under his supervision.

I am highly thankful to Prof. S.P.Sharma, Head, Department of Mathematics, all the members of Student Research Committee, Department Research Committee and faculty members for providing me with the necessary facilities and all needed help during my research work.

I owe my sincere thanks to all my friends for their inspirational and moral support of my research work and making my stay in the I.I.T.Roorkee a memorable one.

I would like to thank my family members for providing encouragement and support during my Ph.D. Finally, I would like to thank my parents and brother for inspiring and encouraging me throughout my life. Without their support, this would not have been possible.

G. Ramesh

August 24 , 2006.

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Table of Contents

Introduction

This section contains a brief review of the basic concepts and results that are related to the work presented in the thesis.

0.1 Entire Function

An entire function is a function $f : C \rightarrow C$, which is regular in every finite region of the complex plane. The general theory of these functions originated in the works of Weierstrass [66]; in the beginning it was developed by Picard, Borel, Poincaré, Hadamard and others. In the beginning of twentieth century some new concepts were introduced by eminent mathematicians such as Valiron [64], Lindelöf, Levin, Wiman, Nevanlinna and Hardy etc. Since then, Whittaker [67], Hayman, Boas [5], Holland [21], Clunie, Titchmarch [58] and others have contributed richly to the theory of entire functions.

An entire function $f(z)$ has the representation by a power series of the form

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \quad \lim_{n \to \infty} \sqrt[n]{|a_n|} = 0.$$

This is the simplest class of analytic functions containing all polynomials. Polynomials are classified according to their degree, i.e. according to their growth as $|z| \rightarrow \infty$. An entire function can grow in various ways along different directions. For a generalization of the growth, the function

$$M(r) = M(r; f) = \max_{|z|=r} |f(z)| \tag{0.1.1}$$

is introduced. Then $M(r)$ is said to be the maximum modulus of $f(z)$ for $|z| = r$. It has been established that maximum absolute value of an entire function over a closed disc coincides with the maximum absolute value of that function over its boundary. Blumenthal [4] showed that $M(r)$ is a steadily increasing continuous function of r and is different in adjacent intervals. Further, $\ln M(r)$ is a convex function of $\ln r$ and has the representation [64]

$$\ln M(r) = \ln M(r_0) + \int_{r_0}^{r} \frac{W(x)}{x} dx, \; r > r_0, \tag{0.1.2}$$

where $W(x)$ is a positive indefinitely increasing function of x which is continuous in adjacent intervals. $M(r)$ plays a key role in the study of the growth of entire functions. A.P.Singh and Baloria [49] have studied on maximum modulus and maximum term of composition of entire functions.

In order to estimate the growth of $f(z)$ precisely, the concept of order was introduced. An entire function $f(z)$ is called a function of finite order if $M(r) < \exp(r^k)$ for some $k > 0$. The order of an entire function f is the greatest lower bound of those values of k for which this asymptotic inequality is fulfilled. We shall usually denote the order of an entire function f by ρ. It follows from the definition of the order that

$$e^{r^{\rho-\epsilon}} < M(r) < e^{r^{\rho+\epsilon}}.$$

By taking the logarithm twice we obtain

$$\rho - \epsilon < \frac{\ln \ln M(r)}{\ln r} < \rho + \epsilon,$$

Thus the order ρ of $f(z)$ is given by

$$\rho = \limsup_{r \to \infty} \frac{\ln \ln M(r)}{\ln r}, \quad 0 \leq \rho \leq \infty. \tag{0.1.3}$$

By convention, a constant function is taken to be of order zero.

The concept of type has been introduced to determine the relative growth of two entire functions of same non-zero finite order. Let ρ be the order of an entire function

$f(z)$. The function is said to have a finite type if for some $A > 0$ the inequality

$$M(r) < e^{Ar^{\rho}}$$

is fulfilled. The greatest lower bound for those values of A for which the later asymptotic inequality is fulfilled is called the type τ of the function $f(z)$. It follows from the definition of the type that

$$e^{(\tau-\epsilon)r^{\rho}} < M(r) < e^{(\tau+\epsilon)r^{\rho}}.$$

By taking logarithm and dividing by r^{ρ}, we obtain

$$(\tau - \epsilon) < \frac{\ln M(r)}{r^{\rho}} < (\tau + \epsilon).$$

Thus, an entire function $f(z)$ of order ρ $(0 < \rho < \infty)$ is said to be of type τ if

(0.1.4) $$\tau = \limsup_{r \to \infty} \frac{\ln M(r)}{r^{\rho}}, \quad 0 \leq \tau \leq \infty.$$

The function $f(z)$ is said to be minimal, maximal or normal type according as $\tau = 0$, $\tau = \infty$ or $0 < \tau < \infty$ respectively. An entire function $f(z)$is said to be of growth (ρ, τ) if its order does not exceed ρ, and its type does not exceed τ if it is of order ρ. The function $f(z)$ is of exponential type τ if it is of order less than one,and if of order one, of the type less than or equal to τ, $\tau < \infty$. L.R.Sons [52, 53] have studied on regularity of growth and gaps.

If an entire function $f(z)$ is of zero or infinite order then the usual definition of type has no meaning. Hence the comparison of growth of such functions can not be made by confining to the above concepts. To overcome this difficulty, V.G.Iyer [24] introduced the concept of logarithmic order. Thus for an entire function of order zero, ρ^* is said to be logarithmic order of $f(z)$ if

(0.1.5) $$\rho^* = \limsup_{r \to \infty} \frac{\ln \ln M(r)}{\ln \ln r}, \quad 0 \leq \rho^* \leq \infty.$$

For $1 < \rho^* < \infty$, the logarithmic type τ^* is defined as

$$\tau^* = \limsup_{r \to \infty} \frac{\ln M(r)}{(\ln r)^{\rho}}, \quad 0 \le \tau^* \le \infty. \tag{0.1.6}$$

The entire function $f(z)$ can be expanded in Taylor series around any point $z = z_0$ because it is regular everywhere in the whole plane. However without loss of generality, we may assume $z_0 = 0$. Then $f(z)$ has the representation

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \tag{0.1.7}$$

where the coefficients a_n's are given by [38]

$$a_n = \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(z)}{z^{n+1}} dz = \frac{f^{(n)}(0)}{n!}. \tag{0.1.8}$$

$f^{(n)}(0)$ being the the value of n th derivative of $f(z)$ at $z = 0$.

Various mathematicians such as S.M.Shah [46], Q.I.Rahman [10], A.R.Reddy [40, 41], Juneja and Srivastava [27], Shah and Ishaq [47], Juneja [25], Awasthi [1], G.S.Srivastava [54] and others have found the formulae relating the coefficients of Taylor series with order, lower order, type, lower type,logarithmic order, logarithmic type etc. Juneja [26] has obtained the results, which give formulae for order etc. in terms of the ratio of consecutive coefficients. Recently, J.K.Langely [33] have studied on integer points of entire functions.

For generalizations of the classical characteristics of growth of entire functions M.N.Seremeta [44] defined the generalized order and generalized type with the help of general functions as follows.

Let L^0 denote the class of functions h satisfying the following conditions

(i) $h(x)$ is defined on $[a, \infty)$ and is positive, strictly increasing, differentiable and tends to ∞ as $x \to \infty$,

(ii)

$$\lim_{x \to \infty} \frac{h\{(1 + 1/\psi(x))x\}}{h(x)} = 1,$$

for every function $\psi(x)$ such that $\psi(x) \to \infty$ as $x \to \infty$.

Let Λ denote the class of functions h satisfying condition (i) and

$$\lim_{x\to\infty}\frac{h(cx)}{h(x)}=1$$

for every $c>0$, that is, $h(x)$ is slowly increasing

The generalized order $\rho(\alpha,\beta)$ of an entire function $f(z)$, is defined as [44]

$$\rho(\alpha,\ \beta,\ f)\ =\ \limsup_{r\to\infty}\frac{\alpha[\ln M(r,f)]}{\beta(\ln r)}, \tag{0.1.9}$$

where $\alpha(x)\in\Lambda,\beta(x)\in L^0$.

For $0<\rho<\infty$, the generalized type $\tau(\alpha,\beta)$ is defined as

$$\tau(\alpha,\ \beta,\ f)\ =\ \limsup_{r\to\infty}\frac{\alpha[\ln M(r,f)]}{\beta[(\gamma(r))^{\rho}]} \tag{0.1.10}$$

where $\alpha(x),\ \beta^{-1}(x),\ \gamma(x)\ \in L^0$.

Seremeta obtained the coefficient characterizations of generalized order and generalized type as follows.

Theorem A' [44, Th.1] Let $\alpha(x)\in\Lambda,\ \beta(x)\in L^0$. Set $F(x,c)\ =\ \beta^{-1}[c\alpha(x)]$. If $dF(x,c)/d\ln x=O(1)$ as $x\to\infty$ for all $c,\ 0<c<\infty$, then

$$\rho(\alpha,\ \beta,\ f)\ =\ \limsup_{n\to\infty}\frac{\alpha(n)}{\beta\left(-\frac{1}{n}\ln|a_n|\right)}. \tag{0.1.11}$$

Theorem B' [44, Th.2] Let $\alpha(x)\beta^{-1}(x),\gamma(x)\ \in L^0$. Let ρ be a fixed number, $0<\rho<\infty$. Set $F(x;\sigma,\rho)=\gamma^{-1}\{[\beta^{-1}(\sigma\alpha(x))]^{1/\rho}\}$. Suppose that for all σ, $0<\sigma<\infty$, $F(x;\sigma,\rho)$ satisfies

(i) if $\gamma(x)\ \in\Lambda$ and $\alpha(x)\ \in\Lambda$ then

$$\frac{d\ln F(x;\sigma,\rho)}{d\ln x}\ =\ O(1)\quad \text{as}\quad x\to\infty$$

(ii) if $\gamma(x)\ \in(L^o-\Lambda)$ or $\alpha(x)\ \in(L^o-\Lambda)$ then

$$\lim_{x\to\infty}\ \frac{d\ln F(x;\sigma,\rho)}{d\ln x}\ =\ \frac{1}{\rho}.$$

Then we have

$$(0.1.12) \qquad \tau(\alpha,\ \beta,\ f) \ = \ \limsup_{n\to\infty} \frac{\alpha(\frac{n}{\rho})}{\beta\{[\gamma(e^{1/\rho}|a_n|^{-1/n})]^{\rho}\}}.$$

In the above Theorem A' the relation (0.1.11) was obtained under the condition

$$(0.1.13) \qquad \frac{d[\beta^{-1}(c\alpha(x))]}{d(\ln x)} = O(1) \quad \text{as } x \to \infty.$$

Clearly (0.1.13), is not satisfied for $\alpha = \beta$. To overcome this difficulty, G.P.Kapoor and Nautiyal [29] defined generalized order $\rho(\alpha; f)$ of slow growth with the help of general functions as follows

Let Ω be the class of functions $h(x)$ satisfying (i) and

(iv) there exists a $\delta(x) \in \Lambda$ and x_0, K_1 and K_2 such that

$$0 < K_1 \leq \frac{d(h(x))}{d(\delta(\ln x))} \leq K_2 < \infty \quad \text{for all } x > x_0.$$

Let $\overline{\Omega}$ be the class of functions $h(x)$ satisfying (i) and

(v)

$$\lim_{x\to\infty} \frac{d(h(x))}{d(\ln x)} = K, \quad 0 < K < \infty.$$

Kapoor and Nautiyal [29] showed that class Ω and $\overline{\Omega}$ are contained in Λ. Further, $\Omega \cap \overline{\Omega} = \phi$ and they defined the generalized order $\rho(\alpha; f)$ for entire functions $f(z)$ of slow growth as

$$\rho(\alpha; f) = \limsup_{r\to\infty} \frac{\alpha(\ln M(r, f))}{\alpha(\ln r)},$$

where $\alpha(x)$ either belongs to Ω or to $\overline{\Omega}$.

0.2 Entire Functions of Two Complex Variables

Let $f(z_1, z_2) = \sum a_{m_1,m_2} z_1^{m_1} z_2^{m_2}$ be a function of the complex variables z_1 and z_2, regular for $|\ z_t\ | \leq r_t, t = 1, 2$. If r_1 and r_2 can be taken arbitrarily large, then $f(z_1, z_2)$ represents an entire function of the complex variables z_1 and z_2. Many researchers [13, 37, 28, 31, 30, 32] have studied the growth of entire functions of two complex variables in different ways. Following Bose and Sharma [7], we define the maximum modulus of $f(z_1, z_2)$ as

$$M(r_1, r_2) = \max_{|z_t| \le r_t} | f(z_1, z_2) |, \quad t = 1, 2.$$

The order ρ of the entire function $f(z_1, z_2)$ is defined as [7, p. 219]

$$\rho = \limsup_{r_1, r_2 \to \infty} \frac{\ln \ln M(r_1, r_2)}{\ln(r_1 r_2)}.$$

For $0 < \rho < \infty$, the type τ of an entire function $f(z_1, z_2)$ is defined as [7, p. 223]

$$\limsup_{r_1, r_2 \to \infty} \frac{\ln M(r_1, r_2)}{r_1^\rho + r_2^\rho} = \tau.$$

Bose and Sharma [7], obtained the following characterizations for order and type of entire functions of two complex variables.

Theorem A: The entire function $f(z_1, z_2) = \sum_{m_1, m_2 = 0}^{\infty} a_{m_1, m_2} z_1^{m_1} z_2^{m_2}$ is of finite order if and only if

$$\mu = \limsup_{m_1, m_2 \to \infty} \frac{\ln(m_1^{m_1} m_2^{m_2})}{-\ln | a_{m_1, m_2} |} \tag{0.2.1}$$

is finite and then the order ρ of $f(z_1, z_2)$ is equal to μ.

Define

$$\alpha = \limsup_{m_1, m_2 \to \infty} (m_1^{m_1} m_2^{m_2} | a_{m_1, m_2} |^{\rho})^{1/(m_1 + m_2)}. \tag{0.2.2}$$

Theorem B: If $0 < \alpha < \infty$, the function $f(z_1, z_2) = \sum_{m_1, m_2 = 0}^{\infty} a_{m_1, m_2} z_1^{m_1} z_2^{m_2}$ is an entire function of order ρ and type τ if and only if $\alpha = e\tau\rho$.

0.3 Entire Functions of Several Complex Variables

The first work in general theory of entire functions of several variables appeared as early as the beginning of the last century Borel [6], Sire [50]. However, an intensive investigation of entire functions in C^n, stemming from the general upsurge of interest in the theory of holomorphic functions of several variables, began only 25 to 30 years later. After that this topic became more interest and many researchers [16, 14, 17, 18, 22, 34, 35, 55, 56] and others have worked on analytic and entire functions of several complex variables.

We denote complex n-space by C^n. Thus, $z \in C^n$ means that $z = (z_1, ..., z_n)$, where $z_1, ...z_n$ are complex numbers. A function $f(z)$, $z \in C^n$ is said to be holomorphic or analytic at a point $z^0 \in C^n$ if it can be expanded in some neighborhood of z^0 as an absolutely convergent power series

$$f(z) = \sum_{\|k\|=0}^{\infty} a_k \, (z - z^0)^k.$$

A function $f(z)$ is said to be holomorphic or analytic in a domain G if it is holomorphic at each point of the domain.

A domain $D \subset C^n$ is said to be poly cylindrical, or simply a poly cylinder, if it has the form $D = \{z : z_j \in D_i, i = 1, 2.., n\}$. A poly cylindrical domain D is said to be a circular poly cylinder, or simply a poly disk, if all the domains D_i are disks. A domain $G \subset C^n$ is said to be multi circular with center at z^0 if, together with every point z', G contains any point z whose coordinates z_j satisfy the conditions

$$|z_i - z_i^0| = |z_i' - z_i^0|, \quad i = 1, ..., n.$$

A multi circular domain $G \subset C^n$ with center at z^0 is said to be complete if, together with each point $z' \in G$, it contains the entire poly disk

$$\left\{z : |z_i - z_i^0| = |z_i' - z_i^0| \, , i = 1, ..., n\right\}.$$

Let $G \subset C^n$ be a multi circular domain. we denote by $|G|$ the image in R_+^n of the domain G under the mapping $r_i = |z_i - z_i^0|$, $i = 1, ..., n$.

0.4 Jordan Domain

A domain D is simply-connected if and only if D is homeomorphic to C. There is an alternative definition which states that a domain D is simply-connected if and only if every loop γ in D can be shrunk to a point in D. or more formally is homotopic to a constant loop in $\dot{D}$;

An important class of simply-connected domains is the class of Jordan domains. A loop γ is a Jordan curve if γ is a homeomorphic image of a circle; equivalently, if $z(t)(a \leq t \leq b)$ is a parametrization of γ, then γ is a Jordan curve if $z(t)$ is 1-1 (injective) on $[a, b)$ (z(a) = z(b)). The fundamental Jordan Curve Theorem states that a Jordan curve γ divides the plane into exactly two regions; to be precise γ^c has exactly two components, the unbounded component (exterior of γ), and the bounded component (interior of γ). Sheil - Small [48] defined a Jordan domain is the interior of a Jordan curve. Thus a Jordan curve is the boundary of a Jordan domain. An extended version of the Riemann mapping theorem shows that the union of a Jordan domain and its boundary curve is homeomorphic to a closed disc with the open disc corresponding to the Jordan domain and the circle bounding the disc corresponding to the boundary Jordan curve.

0.5 Faber Polynomials

Most of the mathematicians were attracted by a problem in complex analysis was that of finding a set of polynomials $p_1(z), p_2(z), \ldots$, which belong to a given region, in the sense that any function f analytic in the region can be expanded by a convergent series $a_0 + \sum_{j=1}^{\infty} a_j p_j(z)$, in which the coefficients a_j, but not the polynomials p_j, depend on f. In 1903, Georg Faber [11] published a solution to the problem which was notable both for the basic simplicity of the convergence proof and also for the rich and interesting structure of the polynomials.

After Faber's invention the polynomials became more popular and various mathematicians [9, 51, 45, 57] have worked on these polynomials and applied in different areas. In 1980 Andre Giroux [15] applied these polynomials to study the growth characterizations of order and type of entire functions of one complex variable over Jordan domains in terms of approximation errors.

0.6 Approximation of Entire Functions

At first Bernstein [3] have studied the polynomial approximation of entire transcendental functions. Later Varga [65] have obtained the growth characterizations of order and type of an entire function in the uniform metric space $C[-1,1]$. By these investigations Reddy [41, 42] have studied the behavior of approximation characteristics for entire transcendental functions of slow growth.

Batyrev [2] first extended Bernstein's ideas to the case of the complex plane C. Further, Ibragimov and Shikhaliev [23], Giroux [15], and Vakarchuk [61, 62, 60] continued the investigations in the space C.

In 1989 Vakarchuk [59] carried the investigations to the case of Banach spaces. Ponnusamy [39], Friedrich Haslinger [20] and others have studied on Hardy and Bergman spaces.
The present thesis deals with the study of growth characteristics and polynomial approximations of entire functions of one, two and several complex variables. The organization of the thesis is as follows:

CHAPTER 1: In this chapter we have studied the polynomial approximation of entire functions in Banach spaces (Hardy space, Bergman space and $B(p,q,\kappa)$ space) and then we have obtained characterizations of generalized type of entire functions in terms of approximation errors in the Banach spaces. In the second section of this chapter we have studied the characterizations of entire functions of slow growth in certain Banach spaces as mentioned above, then we have obtained coefficient characterizations of generalized order and generalized type of entire function of slow growth, and then the characterizations of growth characteristics (generalized order and generalized type) have been obtained in terms of approximation errors in the Banach spaces.

In the second section of this chapter, we have defined the generalized type τ of entire functions of slow growth having finite generalized orderρ and their characterizations have been obtained in terms of approximation errors.

CHAPTER 2: In this chapter we have studied the polynomial approximation of entire functions over Jordan domains. We have obtained coefficient characterizations of generalized order and generalized type of entire functions over Jordan domains. Next we have obtained necessary and sufficient conditions of generalized order and generalized type of entire functions in terms of approximation errors by using L^p norm.

CHAPTER 3: In the first section of this chapter, we have studied the approximation of continuous functions on the domain by homogeneous polynomials. First we have obtained necessary and sufficient conditions for a continuous function have an analytic extension in terms of the growth parameters and then we have obtained the coefficient characterizations of order and type of entire functions of two complex variables in terms of approximation errors. In the second section, we have studied the polynomial approximation of entire functions of two complex variables in Banach spaces (Hardy space, Bergman space and $B(p, q, \kappa)$ space) and then we have obtained characterizations of order and type of entire functions of two complex variables in terms of approximation errors in Banach spaces.

CHAPTER 4: In this chapter we have studied the polynomial approximation of entire functions of two complex variables over Jordan domain and obtained coefficient characterizations of order and type of entire functions. Necessary and sufficient conditions for an entire function to have prescribed growth have been obtained in terms of approximation errors by using L^p norm. We have obtained the characterizations of order ρ and type τ of entire functions of two complex variables when f is restriction to the domain D for $2 \leq p \leq \infty$.

CHAPTER 5: The last, i.e. the fifth chapter of this thesis deals with the study of the entire functions of several complex variables. In this chapter, we have studied the polynomial approximation of entire functions of several complex variables in a full region G in R^n_+, and obtained the characterizations of order, type, generalized order, and generalized type in terms of approximation errors.

A list of papers and books referred to in carrying out the present research has been included at the end of the thesis in "Bibliography".

A list of research papers published / communicated from this thesis has been appended.

Chapter 1

Approximation of Entire Functions of One Complex Variable in Certain Banach Spaces

In this chapter we study the polynomial approximation of entire functions in Banach spaces ($B(p, q, \kappa)$ space, Hardy space and Bergman space). The coefficient characterizations of generalized Order and generalized type of entire functions having finite generalized Order of slow growth have been obtained in terms of the approximation errors.

1.1 Introduction

Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be an entire function and $M(r, f) = \max_{|z|=r} |f(z)|$ be its maximum modulus. Recently Vakarchuk and Zhir [63] considered the approximation of entire functions in Banach spaces. Thus, let $f(z)$ be analytic in the unit disc $U = \{z \in \mathbf{C} : |z| < 1\}$ and we set

$$M_q(r, f) = \{\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(re^{i\theta})|^q d\theta\}^{1/q}, \; q > 0.$$

Let H_q denote the Hardy space of functions $f(z)$ satisfying the condition

$$\|f\|_{H_q} = \lim_{r \to 1-0} M_q(r, f) < \infty$$

and let H_q' denote the Bergman space of functions $f(z)$ satisfying the condition

$$\|f\|_{H_q'} = \{\frac{1}{\pi}\int\int_U |f(z)|^q dxdy\}^{1/q} < \infty.$$

For $q = \infty$, let $\|f\|_{H_\infty'} = \|f\|_{H_\infty} = \sup\{|f(z)|, z \in U\}$. Then H_q and H_q' are Banach spaces for $q \geq 1$. Following [63, p.1394], we say that a function $f(z)$ which is analytic in U belongs to the space $\mathbf{B}(p, q, \kappa)$ if

$$\|f\|_{p,q,\kappa} = \{\int_0^1 (1-r)^{\kappa(1/p-1/q)-1} M_q^\kappa(r, f)dr\}^{1/\kappa} < \infty,$$

$0 < p < q \leq \infty$, $0 < \kappa < \infty$ and

$$\|f\|_{p,q,\infty} = \sup\{(1-r)^{1/p-1/q} M_q(r, f)\ 0 < r < 1\} < \infty.$$

It is known [19] that $\mathbf{B}(p, q, \kappa)$ is a Banach space for $p > 0$ and $q, \kappa \geq 1$, otherwise it is a Frechet space. Further [60],

$$(1.1.1) \qquad H_q \subseteq H_q' = \mathbf{B}(q/2, q, q),\ 1 \leq q < \infty.$$

Let X denote one of the Banach spaces defined above and let

$$E_n(X, f) = \inf\{\|f - p\|_X : p \in P_n\}$$

where P_n consists of algebraic polynomials of degree at most n in complex variable z. Vakarchuk and Zhir [63] obtained characterizations of Generalized order in certain Banach spaces ($\mathbf{B}(p, q, \kappa)$ space, Hardy space and Bergman space) as follows :

Theorem C: Let $\alpha \in \Lambda, \beta \in L^0$ and $F(x, c) = \beta^{-1}\{c[\alpha(x)]\}$ Let for all $c \in (0, \infty)$,

$$\frac{df(x, c)}{d\ln x} = O(1) \text{as } x \to \infty,$$

and $\gamma(\alpha, \beta)$ is a finite positive number. Then for a function $f(z) \in \mathbf{B}(p, q, \kappa)$ to be an entire function of generalized order $\gamma(\alpha, \beta)$, it is necessary and sufficient that the following relation to be true:

$$\limsup_{r\to\infty} \frac{\alpha(n)}{\beta[-n^{-1}\ln E_n(\mathbf{B}(p,q,\kappa), f)]} = \gamma(\alpha, \beta). \tag{1.1.2}$$

Theorem D: Suppose the conditions of Theorem C are satisfied and $\delta(\alpha, \beta)$ is a finite positive number. Then for a function $f(z) \in H_q$ to be an entire function of generalized order $\delta(\alpha, \beta)$, it is necessary and sufficient that the following relation to be true:

$$\limsup_{r\to\infty} \frac{\alpha(n)}{\beta[-n^{-1}\ln E_n(H_q, f)]} = \delta(\alpha, \beta). \tag{1.1.3}$$

and also obtained an analog of this Theorem for the Banach spaces H_q' follows from (1.1.1), for $1 \le q < \infty$ and from Theorem D for $q = \infty$.
In the next section we obtained the characterizations of generalized type $\tau(\alpha, \beta)$ of an entire function having finite generalized order $\rho(\alpha, \beta)$ in certain Banach spaces ($\mathbf{B}(p, q, \kappa)$ space, Hardy space and Bergman space) in terms of approximation errors.

1.2 Generalized Type

First we prove

Theorem 1.2.1. *Let $\alpha(x) \in L^o, \beta^{-1}(x) \in L^o$ and $\gamma(x) \in L^o$; let ρ, $0 < \rho < \infty$ be a fixed number. Set $F(x;\sigma,\rho) = \gamma^{-1}\{[\beta^{-1}(\sigma\alpha(x))]^{1/\rho}\}$. Suppose that for all σ, $0 < \sigma < \infty$, we have*
(i) if $\gamma(x) \in \Lambda$ and $\alpha(x) \in \Lambda$ then

$$\frac{d\ln F(x;\sigma,\rho)}{d\ln x} = O(1) \quad as \quad x \to \infty$$

(ii) if $\gamma(x) \in (L^o - \Lambda)$ or $\alpha(x) \in (L^o - \Lambda)$ then

$$\lim_{x\to\infty} \frac{d\ln F(x;\sigma,\rho)}{d\ln x} = \frac{1}{\rho}.$$

Then for any entire function $f(z) \in B(p,q,\kappa)$, we have

$$\text{(1.2.1)} \qquad \limsup_{r\to\infty} \frac{\alpha(\ln M(r,f))}{\beta\{[\gamma(r)]^\rho\}} = \limsup_{n\to\infty} \frac{\alpha(\frac{n}{\rho})}{\beta\{\gamma[e^{1/\rho}E_n(B(p,q,\kappa);f)^{-1/n}]^\rho\}}.$$

provided for $f(z) = \sum a_n z^n$, $|\frac{a_n}{a_{n+1}}|^{1/n} \to d$, $0 < d < \infty$.

Proof. We prove the above result in two steps. First we consider the space $\mathbf{B}(p,q,\kappa)$, $q = 2$, $0 < p < 2$ and $\kappa \geq 1$. Let $f(z) \in \mathbf{B}(p,q,\kappa)$ be of generalized type τ with generalized order ρ. Then from [44, Th 2], we have

$$\text{(1.2.2)} \qquad \limsup_{n\to\infty} \frac{\alpha(\frac{n}{\rho})}{\beta\{[\gamma(e^{1/\rho}|a_n|^{-1/n})]^\rho\}} = \tau.$$

For a given $\epsilon > 0$, and all $n > m = m(\epsilon)$, we have

$$\text{(1.2.3)} \qquad |a_n| \leq \frac{\exp(\frac{n}{\rho})}{[\gamma^{-1}\{[\beta^{-1}(\frac{\alpha(\frac{n}{\rho})}{\tau+\epsilon})]^{1/\rho}\}]^n}.$$

Let $g_n(f,z) = \sum_{j=0}^{n} a_j z^j$ be the n^{th} partial sum of the Taylor series of the function $f(z)$. Following [63, p.1396], we get

$$\text{(1.2.4)} \qquad E_n(\mathbf{B}(p,2,\kappa);f) \leq B^{1/\kappa}[(n+1)\kappa+1;\kappa(1/p-1/2)]\{\sum_{j=n+1}^{\infty} |a_j|^2\}^{1/2}$$

where $B(a,b)$ $(a,b>0)$ denotes the beta function. By using (1.2.1), we have

(1.2.5)

$$E_n(\mathbf{B}(p,2,\kappa);f) \leq \frac{e^{(n+1)/\rho}B^{1/\kappa}[(n+1)\kappa+1;\kappa(1/p-1/2)]}{[\gamma^{-1}\{[\beta^{-1}(\frac{\alpha(\frac{(n+1)}{\rho})}{\tau+\epsilon})]^{1/\rho}\}]^{n+1}}\{\sum_{j=n+1}^{\infty} \psi_j^2(\alpha,\beta)^{1/2}\},$$

where

$$\psi_j(\alpha,\beta) \;\cong\; \frac{\exp\{(j-(n+1))/\rho\}}{[\gamma^{-1}\{[\beta^{-1}(\frac{\alpha(\frac{(j)}{\rho})}{\tau+\epsilon})]^{1/\rho}\}]^{j}\;[\gamma^{-1}\{[\beta^{-1}(\frac{\alpha(\frac{(n+1)}{\rho})}{\tau+\epsilon})]^{1/\rho}\}]^{-(n+1)}}.$$

Set

$$\psi(\alpha,\beta) \;\cong\; \frac{e^{(1/\rho)}}{[\gamma^{-1}\{[\beta^{-1}(\frac{\alpha(\frac{(1)}{\rho})}{\tau+\epsilon})]^{1/\rho}\}]}.$$

Since $\alpha(x)$ is increasing and $j \geq n+1$, we get

$$(1.2.6)\qquad \psi_j(\alpha,\beta) \;\leq\; \frac{\exp\{(j-(n+1))/\rho\}}{[\gamma^{-1}\{[\beta^{-1}(\frac{\alpha(\frac{(n+1)}{\rho})}{\tau+\epsilon})]^{1/\rho}\}]^{j-(n+1)}} \leq \psi^{j-(n+1)}(\alpha,\beta).$$

By above relation (1.2.6) and since $\psi(\alpha,\beta) < 1$, we get from (1.2.5),

$$(1.2.7)\qquad E_n(\mathbf{B}(p,2,\kappa);f) \;\leq\; \frac{e^{(n+1)/\rho}B^{1/\kappa}[(n+1)\kappa+1;\kappa(1/p-1/2)]}{(1-\psi^2(\alpha,\beta))^{1/2}[\gamma^{-1}\{[\beta^{-1}(\frac{\alpha(\frac{(n+1)}{\rho})}{\tau+\epsilon})]^{1/\rho}\}]^{(n+1)}}.$$

For $n > m$, (1.2.7) yields

$$\tau+\epsilon \;\geq\; \frac{\alpha(\frac{n+1}{\rho})}{\beta\{\gamma[e^{1/\rho}E_n^{-1/n+1}(\mathbf{B}(p,2,\kappa);f)[\frac{B^{1/\kappa}[(n+1)\kappa+1;\kappa(1/p-1/2)]}{(1-\psi^2(\alpha,\beta))^{1/2}}]^{1/n+1}]^{\rho}\}}.$$

Now

$$B[(n+1)\kappa+1;\kappa(1/p-1/2)] \;=\; \frac{\Gamma((n+1)\kappa+1)\Gamma(\kappa(1/p-1/2))}{\Gamma((n+1/2+1/p)\kappa+1)}.$$

Hence

$$B[(n+1)\kappa+1;\kappa(1/p-1/2)] \;\simeq\; \frac{e^{-[(n+1)\kappa+1]}[(n+1)\kappa+1]^{(n+1)\kappa+3/2}\Gamma(1/p-1/2)}{e^{[(n+1/2+1/p)\kappa+1]}[(n+1/2+1/p)\kappa+1]^{(n+1/2+1/p)\kappa+3/2}}.$$

Thus

$$(1.2.8) \qquad \{B[(n+1)\kappa+1;\kappa(1/p-1/2)]\}^{1/(n+1)} \cong 1.$$

Proceeding to limits, we obtain

$$(1.2.9) \qquad \tau \geq \limsup_{n\to\infty} \frac{\alpha(\frac{n}{\rho})}{\beta\{\gamma[e^{1/\rho}E_n(\mathbf{B}(p,q,\kappa);f)^{-1/n}]^{\rho}\}}.$$

For reverse inequality, by [63, p.1398], we have

$$(1.2.10) \qquad |a_{n+1}|B^{1/\kappa}[(n+1)\kappa+1;\kappa(1/p-1/2)] \leq E_n(\mathbf{B}(p,2,\kappa);f).$$

Then for sufficiently large n, we have

$$\frac{\alpha(\frac{n}{\rho})}{\beta\{\gamma[e^{1/\rho}E_n^{-1/n}(\mathbf{B}(p,2,\kappa);f)]^{\rho}\}}$$

$$\geq \frac{\alpha(\frac{n}{\rho})}{\beta\{\gamma[e^{1/\rho}(|a_{n+1}|^{-1/n}B^{-1/n\kappa}[(n+1)\kappa+1,;\kappa(1/p-1/2)])^{\rho}\}}$$

$$\geq \frac{\alpha(\frac{n}{\rho})}{\beta\{\gamma[e^{1/\rho}(|a_n|^{-1/n}B^{-1/n\kappa}[(n+1)\kappa+1,;\kappa(1/p-1/2)])^{\rho}\}}.$$

By applying limits and from (1.2.2), we obtain

$$(1.2.11) \qquad \limsup_{n\to\infty} \frac{\alpha(\frac{n}{\rho})}{\beta\{\gamma[e^{1/\rho}E_n(\mathbf{B}(p,q,\kappa);f)^{-1/n}]^{\rho}\}} \geq \tau,$$

From (1.2.9), and (1.2.11), we obtain the required equality

$$(1.2.12) \qquad \limsup_{n\to\infty} \frac{\alpha(\frac{n}{\rho})}{\beta\{\gamma[e^{1/\rho}E_n(\mathbf{B}(p,q,\kappa);f)^{-1/n}]^{\rho}\}} = \tau.$$

In the second step, we consider the spaces $\mathbf{B}(p,q,\kappa)$ for $0 < p < q, q \neq 2$, and $q, \kappa \geq 1$.

Gvaradze [19] showed that, for $p \geq p_1$, $q \leq q_1$, and $\kappa \leq \kappa_1$, if at least one of the inequalities is strict, then the strict inclusion $\mathbf{B}(p,q,\kappa) \subset \mathbf{B}(p_1,q_1,\kappa_1)$ holds and the following relation is true:

$$\|f\|_{p_1,q_1,\kappa_1} \leq 2^{1/q-1/q_1}[\kappa(1/p-1/q)]^{1/\kappa-1/\kappa_1}\|f\|_{p,q,\kappa}.$$

For any function $f(z) \in \mathbf{B}(p,q,\kappa)$, the last relation yields

$$(1.2.13) \qquad E_n(\mathbf{B}(p_1,q_1,\kappa_1);f) \leq 2^{1/q-1/q_1}[\kappa(1/p-1/q)]^{1/\kappa-1/\kappa_1}E_n(\mathbf{B}(p,q,\kappa);f).$$

For the general case $\mathbf{B}(p,q,\kappa)$, $q \neq 2$, we prove the necessity of condition (1.2.1).

Let $f(z) \in \mathbf{B}(p,q,\kappa)$ be an entire transcendental function having finite generalized order $\rho(\alpha,\beta;f)$ whose generalized type is defined by (1.2.2). Using the relation (1.2.3), for $n > m$ we estimate the value of the best polynomial approximation as follows

$$E_n(\mathbf{B}(p,q,\kappa);f) = \|f-g_n(f)\|_{p,q,\kappa} \leq \left(\int_0^1 (1-r)^{(\kappa(1/p-1/q)-1)}M_q^\kappa dr\right)^{1/\kappa}.$$

Now

$$|f|^q = |\sum a_n z^n|^q \leq (\sum |a_n r^n|)^q \leq (r^{n+1}\sum_{k=n+1}^{\infty}|a_k|)^q, \quad \text{hence}$$

$$M_q^q \leq (r^{n+1}\sum_{k=n+1}^{\infty}|a_k|)^q,$$

or

$$M_q^\kappa \leq r^{(n+1)\kappa}(\sum_{k=n+1}^{\infty}|a_k|)^\kappa.$$

Hence

$$(1.2.14) \quad E_n(\mathbf{B}(p,q,\kappa);f) \leq B^{1/\kappa}[(n+1)\kappa+1;\kappa(1/p-1/q)]\sum_{k=n+1}^{\infty}|a_k|$$

$$\leq \frac{e^{(n+1)/\rho}B^{1/\kappa}[(n+1)\kappa+1;\kappa(1/p-1/q)]}{(1-\psi(\alpha,\beta))[\gamma^{-1}\{[\beta^{-1}(\frac{\alpha(\frac{(n+1)}{\rho})}{\tau+\epsilon})]^{1/\rho}\}]^{(n+1)}}.$$

For $n > m$, (1.2.14) yields

$$\tau+\epsilon \geq \frac{\alpha(\frac{n+1}{\rho})}{\beta\{\gamma[e^{1/\rho}E_n^{-1/n+1}(\mathbf{B}(p,2,\kappa);f)[\frac{B^{1/\kappa}[(n+1)\kappa+1;\kappa(1/p-1/q)]}{(1-\psi(\alpha,\beta))}]^{1/n+1}]^{\rho}\}}.$$

Proceeding to limits, we obtain

$$\tau \geq \limsup_{n\to\infty} \frac{\alpha(\frac{n}{\rho})}{\beta\{\gamma[e^{1/\rho}E_n(\mathbf{B}(p,q,\kappa);f)^{-1/n}]^{\rho}\}}.$$

For the reverse inequality, let $0 < p < q < 2$ and $\kappa, q \geq 1$. By (1.2.13), where $p_1 = p, q_1 = 2$, and $\kappa_1 = \kappa$, and the condition (1.2.1) is already proved for the space $\mathbf{B}(p,2,\kappa)$, we get

$$\begin{aligned}\limsup_{n\to\infty} \frac{\alpha(\frac{n}{\rho})}{\beta\{\gamma[e^{1/\rho}E_n{}^{-1/n}(\mathbf{B}(p,q,\kappa);f)]^{\rho}\}} &\geq \limsup_{n\to\infty} \frac{\alpha(\frac{n}{\rho})}{\beta\{\gamma[e^{1/\rho}E_n{}^{-1/n}(B(p,2,\kappa);f)]^{\rho}\}} \\ &= \tau.\end{aligned}$$

Now let $0 < p \leq 2 < q$. Since we have

$$M_2(r,f) \leq M_q(r,f), \quad 0 < r < 1.$$

Therefore

$$E_n(\mathbf{B}(p,q,\kappa);f) \geq |a_{n+1}|\, B^{1/\kappa}[(n+1)\kappa+1;\kappa(1/p-1/q)]. \tag{1.2.15}$$

From relations (1.2.2) and (1.2.1), (1.2.4) yields

$$\limsup_{n\to\infty} \frac{\alpha(\frac{n}{\rho})}{\beta\{\gamma[e^{1/\rho}E_n(\mathbf{B}(p,q,\kappa);f)^{-1/n}]^{\rho}\}} \geq \limsup_{n\to\infty} \frac{\alpha(\frac{n}{\rho})}{\beta\{\gamma[e^{1/\rho}|a_n|^{-1/n}]^{\rho}\}} = \tau.$$

Now we assume that $2 \le p < q$. Set $q_1 = q, \kappa_1 = \kappa$, and $0 < p_1 < 2$ in the inequality (1.2.13), where p_1 is an arbitrary fixed number. Substituting p_1 for p in (1.4.24), we get

$$E_n(\mathbf{B}(p,q,\kappa);f) \ \ge\ |a_{n+1}|B^{1/\kappa}[(n+1)\kappa+1;\kappa(1/p_1-1/q)]. \tag{1.2.16}$$

Using (1.2.16) and applying the same analogy as in the previous case $0 < p \le 2 < q$, we obtain

$$\limsup_{n\to\infty} \frac{\alpha(\frac{n}{\rho})}{\beta\{\gamma[e^{1/\rho}E_n(\mathbf{B}(p,q,\kappa);f)^{-1/n}]^{\rho}\}} \ \ge\ \tau.$$

From relations (1.2.9) and (1.2.11), we obtain the required relation (1.2.12). This completes the proof of Theorem 1.2.1. □

Theorem 1.2.2. *Assuming that the conditions of the Theorem 1.2.1 are satisfied and $\xi(\alpha,\beta)$ is a positive number, a necessary and sufficient condition for a function $f(z) \in H_q$ to be an entire function of generalized type $\xi(\alpha,\beta)$ having finite generalized order ρ is that*

$$\limsup_{n\to\infty} \frac{\alpha(\frac{n}{\rho})}{\beta\{\gamma[e^{1/\rho}E_n(H_q;f)^{-1/n}]^{\rho}\}} \ =\ \xi(\alpha,\beta). \tag{1.2.17}$$

An analog of this theorem for the Bergman Spaces follows from (1.1.1) *for* $1 \le q < \infty$ *and from Theorem 1.2.2 for* $q = \infty$.

Proof. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be an entire transcendental function whose generalized type τ having finite generalized order ρ. Since

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} \ =\ 0 \tag{1.2.18}$$

and $f(z) \in \mathbf{B}(p,q,\kappa)$, where $0 < p < q \le \infty$ and $q, \kappa \ \ge 1$, from relation (1.1.1), we get

$$E_n(\mathbf{B}(q/2,q,q);f) \le \varsigma_q E_n(H_q;f), \quad 1 \le q < \infty. \tag{1.2.19}$$

where ς_q is a constant independent of n and f. In the case of Hardy space H_∞,

$$E_n(\mathbf{B}(p,\infty,\infty);f) \le E_n(H_\infty;f), \quad 1 < p < \infty. \tag{1.2.20}$$

Since

$$\begin{aligned} \xi(\alpha,\beta;f) &= \limsup_{n\to\infty} \frac{\alpha(\frac{n}{\rho})}{\beta\{\gamma[e^{1/\rho}E_n^{-1/n}(H_q;f)]^\rho\}} \\ &\ge \limsup_{n\to\infty} \frac{\alpha(\frac{n}{\rho})}{\beta\{\gamma[e^{1/\rho}E_n^{-1/n}(\mathbf{B}(q/2,q,q);f)]^\rho\}} \\ &\ge \tau, \quad 1 \le q < \infty. \end{aligned} \tag{1.2.21}$$

Using estimate (1.2.20) we prove inequality (1.2.21) in the case $q = \infty$.
For the reverse inequality

$$\xi(\alpha,\beta;f) \le \tau, \tag{1.2.22}$$

use the relation (1.2.3), which is valid for $n > m$, and estimate from above the generalized type τ of an entire transcendental function $f(z)$ having finite generalized order ρ as follows. We have

$$\begin{aligned} E_n(H_q;f) &\le \|f-g_n\|_{H_q} \\ &\le \sum_{j=n+1}^{\infty} |a_j| \\ &\le \gamma^{-1}\{[\beta^{-1}(\frac{\alpha(\frac{n+1}{\rho})}{\tau+\epsilon})]^{1/\rho}\}^{-1/n+1} e^{(n+1)/\rho} \sum_{j=n+1}^{\infty} \psi_j(\alpha,\beta). \end{aligned}$$

Using (1.2.6),

$$
\begin{aligned}
E_n(H_q;f) &\leq \|f-g_n\|_{H_q} \\
&\leq (1-\psi(\alpha,\beta))^{-1}\gamma^{-1}\{[\beta^{-1}(\frac{\alpha(\frac{n+1}{\rho})}{\tau+\epsilon})]^{1/\rho}\}^{-1/n+1}e^{(n+1)/\rho}.
\end{aligned}
$$

This yields

$$
(1.2.23) \qquad \tau+\epsilon \geq \frac{\alpha(\frac{n+1}{\rho})}{\beta\{\gamma[e^{1/\rho}[E_n^{-1/n+1}(H_q;f)][1-\psi(\alpha,\beta)]^{1/n+1}]^{\rho}\}}.
$$

Since $\epsilon > 0$ is arbitrary, passing to the limit as $n \to \infty$ in (1.2.23), we obtain inequality (1.2.22). Thus we have finally

$$
(1.2.24) \qquad \xi(\alpha,\beta;f) = \tau.
$$

This completes the proof of Theorem 1.2.2. □

1.3 Generalized Order of Entire Functions of Slow Growth

In this section we consider the generalized order of functions of slow growth in terms of the approximation errors $E_n(\mathbf{B}(p,q,\kappa);f)$. We now prove

Theorem 1.3.1. *Let $\alpha(x) \in \overline{\Omega}$, then necessary and sufficient condition for an entire function $f(z) \in B(p,q,\kappa)$ of finite generalized order ρ is*

$$
(1.3.1) \qquad \limsup_{n\to\infty} \frac{\alpha(n)}{\alpha[-n^{-1}\ln E_n(B(p,q,\kappa);f)]} = \Upsilon(\alpha) = \rho - 1.
$$

Proof. From [29, Th 4], we have

$$(1.3.2) \qquad \rho(\alpha; f) = 1 + L, \qquad \text{where } L = \limsup_{n\to\infty} \frac{\alpha(n)}{\alpha\{(1/n)\ln|a_n^{-1}|\}}.$$

For a given $\epsilon > 0$ and all $n > m = m(\epsilon)$, we have

$$(1.3.3) \qquad |a_n| < \frac{1}{\exp\{n\alpha^{-1}[\alpha(n)/(\rho+\epsilon)]\}}$$

From (1.2.4),

$$(1.3.4) \quad E_n(\mathbf{B}(p,2,\kappa); f) \leq \frac{B^{1/\kappa}[(n+1)\kappa+1; \kappa(1/p-1/2)]\{\sum_{j=n+1}^{\infty}\psi_j{}^2(\alpha)\}^{1/2}}{\exp\{(n+1)\alpha^{-1}[\alpha(n+1)/((\rho-1)+\epsilon)]\}}$$

where

$$\psi_j(\alpha) = \exp\{(n+1)\alpha^{-1}[\alpha(n+1)/((\rho-1)+\epsilon)] - j\alpha^{-1}[\alpha(j)/((\rho-1)+\epsilon)]\}.$$

Set

$$\psi(\alpha) \cong \frac{1}{\exp\{\alpha^{-1}[\alpha(1)/((\rho-1)+\epsilon)]\}}.$$

Then we have

$$(1.3.5) \quad \psi_j(\alpha) \leq \exp\{(n+1-j)\alpha^{-1}[\alpha(n+1)/((\rho-1)+\epsilon)]\} \leq \psi^{j-(n+1)}(\alpha).$$

By relation (1.3.5) and since $\psi(\alpha) < 1$, (1.3.4) takes the form

(1.3.6)

$$E_n(\mathbf{B}(p,2,\kappa); f) \leq \frac{B^{1/\kappa}[(n+1)\kappa+1; \kappa(1/p-1/2)]}{(1-\psi^2(\alpha))^{1/2}\exp\{(n+1)\alpha^{-1}[\alpha(n+1)/((\rho-1)+\epsilon)]\}}$$

for $n > m$, estimate (1.3.6) yields

$$(1.3.7)\quad (\rho-1)+\epsilon \;\geq\; \frac{\alpha(n+1)}{\alpha\{(n+1)^{-1}[-\ln E_n(\mathbf{B}(p,2,\kappa);f)+\ln \frac{B^{1/\kappa}[(n+1)\kappa+1;\kappa(1/p-1/2)]}{(1-\psi^2(\alpha))^{1/2}}]\}}$$

and passing to limits as $n \to \infty$ in (1.3.7), we get

$$(1.3.8)\qquad \rho - 1 \;\geq\; \Upsilon(\alpha).$$

For the reverse inequality, from relation (1.4.24), for sufficiently large n, we have

$$\frac{\alpha(n)}{\alpha[-n^{-1}\ln E_n(\mathbf{B}(p,2,\kappa);f)]}$$

$$\geq \frac{\alpha[(1-\frac{1}{n+1})(n+1)]}{\alpha\{(1+\frac{1}{n})[\ln(1/\sqrt[n+1]{|a_{n+1}|})+\ln(1/\sqrt[(n+1)\kappa]{B((n+1)\kappa+1;\kappa(1/p-1/2))})]\}}.$$

Proceeding to limits, we obtain

$$(1.3.9)\qquad \Upsilon(\alpha) \;\geq\; \rho - 1.$$

From relations (1.3.8) and (1.3.9), we will obtain the required equality

$$(1.3.10)\qquad \Upsilon(\alpha) \;=\; \rho - 1.$$

This completes the proof of Theorem 1.3.1. □

Theorem 1.3.2. *Let $\alpha(x) \in \overline{\Omega}$, then a necessary and sufficient condition for an entire function $f(z) \in H_q$ to be of finite generalized order ρ is*

$$(1.3.11)\qquad \limsup_{n\to\infty} \frac{\alpha(n)}{\alpha[-n^{-1}\ln E_n(H_q;f)]} \;=\; \rho - 1.$$

Proof. From (1.2.19)

$$E_n[\mathbf{B}(q/2,q,q);f] \leq \varsigma_q\, E_n(H_q;f) \quad 1 \leq q < \infty$$

where ς_q is a constant independent of n and f. Hence

$$\begin{aligned} \limsup_{n\to\infty} \frac{\alpha(n)}{\alpha[-n^{-1}\ln E_n(H_q;f)]} &\geq \limsup_{n\to\infty} \frac{\alpha(n)}{\alpha[-n^{-1}\ln E_n(\mathbf{B}(q/2,q,q);f)]} \\ &\geq \rho - 1,\ 1 \leq q < \infty. \end{aligned} \tag{1.3.12}$$

Using estimate (1.2.20), the above inequality is true for the case when $q = \infty$.
For reverse inequality, we use the relation (1.3.3), which is valid for $n > m$, and estimate from above the generalized order ρ of an entire function $f(z)$ as follows. We have

$$\begin{aligned} E_n(H_q;f) &\leq \|f - g_n\|_{H_q} \\ &\leq \sum_{j=n+1}^{\infty} |a_j| \\ &\leq \exp\left\{-(n+1)\alpha^{-1}[\frac{\alpha(n+1)}{(\rho-1)+\epsilon}]\right\} \sum_{j=n+1}^{\infty} \psi_j(\alpha). \end{aligned}$$

Using (1.3.5), we get

$$E_n(H_q;f) \leq (1-\psi(\alpha))^{-1} \exp\left\{-(n+1)\alpha^{-1}[\frac{\alpha(n+1)}{(\rho-1)+\epsilon}]\right\}.$$

This yields

$$(\rho - 1) + \epsilon \geq \frac{\alpha(n+1)}{\frac{1}{n+1}\left[\ln \frac{1}{E_{n+1}(H_q;f)} + \ln \frac{1}{1-\psi(\alpha)}\right]}.$$

Now proceeding to limits, we obtain

$$(1.3.13) \qquad \rho - 1 \geq \limsup_{n\to\infty} \frac{\alpha(n)}{\alpha\{-n^{-1}\ln E_n(H_q; f)\}}.$$

From (1.3.12), and (1.3.13), we get the required relation (1.3.11). This completes the proof of Theorem 1.3.2. □

1.4 Generalized Type of Entire Functions of Slow Growth

We define the generalized type $\tau(\alpha; f)$ of an entire function $f(z)$ having finite generalized order $\rho(\alpha; f)$ as

$$\tau(\alpha; f) = \limsup_{r\to\infty} \frac{\alpha(\ln M(r, f))}{[\alpha(\ln r)]^{\rho}}$$

where $\alpha(x)$ either belongs to Ω or to $\overline{\Omega}$.

Now we prove

Theorem 1.4.1. *Let $\alpha(x) \in \overline{\Omega}$, then the entire function f(z) of generalized order ρ, $1 < \rho < \infty$, is of generalized type τ if and only if*

$$(1.4.1) \qquad \tau = \limsup_{n\to\infty} \frac{\alpha(\frac{n}{\rho})}{\{\alpha[\frac{\rho}{\rho-1}\ln|a_n|^{-1/n}]\}^{\rho-1}},$$

provided $dF(x\,;\,\tau\,,\,\rho)/d\ln x = O(1)$ as $x \to \infty$ for all τ, $0 < \tau < \infty$.

Proof. Let

$$\limsup_{R\to\infty} \frac{\alpha(\ln M(R, f))}{[\alpha(\ln R)]^{\rho}} = \tau.$$

We suppose $\tau < \infty$. Then for every $\epsilon > 0$, $\exists\, R(\epsilon)$ such that for all $R \geq R(\epsilon)$, we have

$$\frac{\alpha(\ln M(R, f))}{[\alpha(\ln R)]^{\rho}} \leq \tau + \epsilon = \overline{\tau}$$

or

$$\ln M(R,f) \;\leq\; (\alpha^{-1}\{\overline{\tau}[\alpha(\ln R)]^{\rho}\}).$$

Choose $R = R(n)$ to be the unique root of the equation

$$n = \frac{\rho}{\ln R}\, F[\ln R\,;\, \overline{\tau}\,,\, \frac{1}{\rho}]. \tag{1.4.2}$$

Then

$$\ln R = \alpha^{-1}[(\frac{1}{\overline{\tau}}\, \alpha(\frac{n}{\rho}))^{1/(\rho-1)}] = F[\frac{n}{\rho}\,;\, \frac{1}{\overline{\tau}}\,, \rho - 1]. \tag{1.4.3}$$

By Cauchy's inequality,

$$\begin{aligned} |a_n| \;&\leq\; R^{-n}M(R;f) \\ &\leq\; \exp\{-n\ln R \;+\; (\alpha^{-1}\{\overline{\tau}\;[\alpha(\ln R)]^{\rho}\})\} \end{aligned}$$

By using (1.4.2) and (1.4.3), we get

$$|a_n| \leq \exp\{-nF \;+ \frac{n}{\rho}F\}$$

or

$$\frac{\rho}{\rho-1}\ln|a_n|^{-1/n} \geq \alpha^{-1}\{[(\frac{1}{\overline{\tau}}\,\alpha(\frac{n}{\rho}))^{1/(\rho-1)}]\}$$

or

$$\overline{\tau} \;=\; \tau + \epsilon \geq \frac{\alpha(\frac{n}{\rho})}{\{\alpha[\frac{\rho}{\rho-1}\ln|a_n|^{-1/n}]\}^{\rho-1}}.$$

Proceeding to limits, we obtain

$$\tau \geq \limsup_{n\to\infty} \frac{\alpha(\frac{n}{\rho})}{\{\alpha[\frac{\rho}{\rho-1}\;\ln|a_n|^{-1/n}]\}^{(\rho-1)}}. \tag{1.4.4}$$

Inequality (1.4.4) obviously holds when $\tau = \infty$.

Conversely, let

$$\limsup_{n\to\infty} \frac{\alpha(\frac{n}{\rho})}{\{\alpha[\frac{\rho}{\rho-1}\ \ln|a_n|^{-1/n}]\}^{(\rho-1)}} = \sigma.$$

Suppose $\sigma < \infty$. Then for every $\epsilon > 0$, $\exists N(\epsilon)$ such that for all $n \geq N$, we have

$$\frac{\alpha(\frac{n}{\rho})}{\{\alpha[\frac{\rho}{\rho-1}\ \ln|a_n|^{-1/n}]\}^{(\rho-1)}} \leq \sigma + \epsilon = \overline{\sigma}$$

(1.4.5) i.e. $|a_n| \leq \dfrac{1}{\exp\{(\rho-1)\ \frac{n}{\rho}\ F[\frac{n}{\rho}\ ;\ \frac{1}{\overline{\sigma}}\ ,\ \rho-1]\}}.$

The inequality

(1.4.6) $\sqrt[n]{|a_n|R^n} \leq Re^{-(\frac{\rho-1}{\rho})\ F[\frac{n}{\rho}\ ;\ \frac{1}{\overline{\sigma}}\ ,\ \rho-1]} \leq \dfrac{1}{2}$

is fulfilled beginning with some $n = n(R)$. Then

(1.4.7) $\displaystyle\sum_{n=n(R)+1}^{\infty} |a_n|R^n \leq \sum_{n=n(R)+1}^{\infty} \frac{1}{2^n} \leq 1.$

We now express $n(R)$ in terms of R. From inequality (1.4.6),

$$2R \leq \exp\{(\frac{\rho-1}{\rho})\ F[\frac{n}{\rho}\ ;\ \frac{1}{\overline{\sigma}}\ ,\ \rho-1]\},$$

we can take $n(R) = E[\rho\ \alpha^{-1}\{\overline{\tau}\ (\alpha(\ln R + \ln 2))^{\rho-1}\}]$. Consider the function $\psi(x) = R^x \exp\{-(\frac{\rho-1}{\rho})\ x\ F[\frac{x}{\rho}\ ;\ \frac{1}{\overline{\sigma}}\ ,\ \rho-1]\}$ and take its logarithmic derivative and set it equal to zero.

(1.4.8) $\dfrac{\psi'(x)}{\psi(x)} = \ln R - (\dfrac{\rho-1}{\rho})F[\dfrac{x}{\rho}\ ;\ \dfrac{1}{\overline{\sigma}}\ ,\ \rho-1] - \dfrac{dF[\frac{x}{\rho}\ ;\ \frac{1}{\overline{\sigma}}\ ,\ \rho-1]}{d\ln x} = 0.$

As $x \to \infty$, by the assumption of the theorem, for finite σ $(0 < \sigma < \infty)$, $dF[x\ ;\ \overline{\sigma}\ ,\ \rho - 1]\ /\ d\ln x$ is bounded. So there is an $A > 0$ such that for $x \geq x_1$ we have

$$(1.4.9) \qquad \left|\frac{dF[\frac{x}{\rho}\ ;\ \frac{1}{\overline{\sigma}}\ ,\ \rho - 1]}{d\ln x}\right| \leq A.$$

We can take $A > \ln 2$. It is then obvious that inequalities (1.4.6) and (1.4.7) hold for $n \geq n_1(R) = E[\rho\ \alpha^{-1}\{\overline{\sigma}\ (\alpha(\ln R + A))^{(\rho-1)}\}] + 1$. We let n_0 designate the number $\max(N(\epsilon),\ E[x_1] + 1)$. For $R > R_1(n_0)$ we have $\psi'(n_0)/\psi(n_0) > 0$. From (1.4.9)and (1.4.8) it follows that $\psi'(n_1(R))/\psi(n_1(R)) < 0$. We hence obtain that if for $R > R_1(n_0)$ we let $x^*(R)$ designate the point where $\psi(x^*(R)) = \max_{n_0 \leq x \leq n_1(R)} \psi(x)$, then

$$n_0 < x^*(R) < n_1(R) \quad \text{and } x^*(R) \ = \ \rho\alpha^{-1}\{\overline{\sigma}\ (\alpha(\ln R - a(R)))^{\rho-1}\}.$$

where

$$-A < a(R) = \frac{dF[\frac{x}{\rho}\ ;\ \frac{1}{\overline{\sigma}}\ ,\ \rho - 1]}{d\ln x}\Big|_{x=x^*(R)} < A.$$

Further

$$\max_{n_0<n<n_1(R)} (|a_n|R^n) \leq \max_{n_0<x<n_1(R)} \psi(x) \ = \frac{R^{\rho\alpha^{-1}\{\overline{\sigma}\ (\alpha(\ln R - a(R)))^{\rho-1}\}}}{e^{\rho\alpha^{-1}\{\overline{\sigma}(\alpha(\ln R-a(R)))^{\rho-1}\}\ (\ln R - a(R))}}$$

$$= \ \exp\{a(R)\rho\alpha^{-1}\{\overline{\sigma}\ (\alpha(\ln R - a(R)))^{\rho-1}\}\} \ \leq \ \exp\{A\rho\alpha^{-1}\{\overline{\sigma}(\alpha(\ln R + A))^{\rho-1}\}\}.$$

It is obvious that (for $R > R_1(n_0)$)

$$M(R, f) \leq \Sigma_{n=0}^{\infty}|a_n|R^n \ = \ \Sigma_{n=0}^{n_0}|a_n|R^n \ + \ \Sigma_{n=n_0+1}^{n_1(R)}|a_n|R^n \ + \ \Sigma_{n=n_1(R)+1}^{\infty}|a_n|R^n$$

$$\leq \ O(R^{n_0}) \ + \ n_1(R) \max_{n_0<n<n_1(R)} (|a_n|R^n) \ + \ 1$$

$$M(R,f)(1+o(1)) \leq \exp\{(A\rho+o(1))\ \alpha^{-1}[\overline{\sigma}\ (\alpha(\ln R+A))^{\rho-1}]\}$$

$$\alpha(\ln M(R,f)) \leq \overline{\sigma}[\alpha(\ln R+A)]^{\rho-1} \ \leq\ \overline{\sigma}\ [\alpha(\ln R+A)]^{\rho}.$$

We then have

$$\frac{\alpha[(A\rho+o(1))^{-1}\ln M(R,f)]}{[\alpha(\ln R+A)]^{\rho}} \leq \overline{\sigma}\ =\ \sigma+\epsilon.$$

Since $\alpha(x)\ \in \overline{\Omega} \subseteq \Lambda$, now proceeding to limits we obtain

$$\limsup_{R\to\infty} \frac{\alpha(\ln M(R;f))}{[\alpha(\ln R)]^{\rho}} \ \leq\ \sigma. \tag{1.4.10}$$

From inequalities (1.4.4) and (1.4.10), we get the required the result. This completes the proof of Theorem 1.4.1. □

Theorem 1.4.2. *Let $\alpha(x)\ \in \overline{\Omega}$, then a necessary and sufficient condition for an entire function $f(z)\ \in B(p,q,\kappa)$ to be of generalized type τ having finite generalized order ρ, $1<\rho<\infty$ is*

$$\tau\ =\ \limsup_{n\to\infty} \frac{\alpha(\frac{n}{\rho})}{[\alpha\{\frac{\rho}{\rho-1}\ \ln(|E_n(B(p,q,\kappa);f)|^{-1/n})\}]^{(\rho-1)}}. \tag{1.4.11}$$

Proof. We prove the above result in two steps. First we consider the space $\mathrm{B}(p,q,\kappa)$, $q=2,\ 0<p<2$ and $\kappa\geq 1$. Let $f(z)\ \in\ \mathrm{B}(p,q,\kappa)$ be of generalized type τ with generalized order ρ. Then from the Theorem 1.4.1, we have

$$\limsup_{n\to\infty} \frac{\alpha(\frac{n}{\rho})}{\{\alpha[\frac{\rho}{\rho-1}\ln|a_n|^{-1/n}]\}^{\rho-1}} \ =\ \tau. \tag{1.4.12}$$

For a given $\epsilon>0$, and all $n>m=m(\epsilon)$, we have

$$|a_n|\ \leq\ \frac{1}{\exp\{(\rho-1)\ \frac{n}{\rho}\ F[\frac{n}{\rho}\ ;\ \frac{1}{\overline{\tau}}\ ,\ \rho-1]\}}. \tag{1.4.13}$$

Let $g_n(f,z)\ =\ \sum_{j=0}^{n} a_j z^j$ be the n^{th} partial sum of the Taylor series of the function

$f(z)$. Following [63, p.1396], we get

$$(1.4.14) \qquad E_n(\mathbf{B}(p,2,\kappa);f) \ \le\ B^{1/\kappa}[(n+1)\kappa+1;\kappa(1/p-1/2)]\{\sum_{j=n+1}^{\infty}|a_j|^2\}^{1/2}$$

where $B(a,b)$ $(a,b>0)$ denotes the beta function. By using (1.2.3), we have

(1.4.15)

$$E_n(\mathbf{B}(p,2,\kappa);f) \ \le\ \frac{B^{1/\kappa}[(n+1)\kappa+1;\kappa(1/p-1/2)]}{\exp\{(\rho-1)\ \frac{n+1}{\rho}\ F[\frac{n+1}{\rho}\ ;\ \frac{1}{\tau}\ ,\ \rho-1]\}}\left\{\sum_{j=n+1}^{\infty}\psi_j^2(\alpha)\right\}^{1/2},$$

where

$$\psi_j(\alpha) \ \cong\ \frac{\exp\{\frac{n+1}{\rho}(\rho-1)[\alpha^{-1}\{(\frac{\alpha(\frac{n+1}{\rho})}{\tau+\epsilon})^{1/(\rho-1)}\}]\}}{\exp\{\frac{j}{\rho}(\rho-1)[\alpha^{-1}\{(\frac{\alpha(\frac{j}{\rho})}{\tau+\epsilon})^{1/(\rho-1)}\}]\}}.$$

Set

$$\psi(\alpha) \ \cong\ \exp\{-\frac{1}{\rho}(\rho-1)[\alpha^{-1}\{(\frac{\alpha(\frac{1}{\rho})}{\tau+\epsilon})^{1/(\rho-1)}\}]\}.$$

Since $\alpha(x)$ is increasing and $j\ge n+1$, we get

(1.4.16)

$$\psi_j(\alpha) \ \le\ \exp\{\frac{((n+1)-j)}{\rho}\ (\rho-1)[\alpha^{-1}\{(\frac{\alpha(\frac{n+1}{\rho})}{\tau+\epsilon})^{1/(\rho-1)}\}]\} \ \le\ \psi^{j-(n+1)}(\alpha).$$

Since $\psi(\alpha)\ <\ 1$, we get from (1.2.5) and (1.2.6),

(1.4.17)

$$E_n(\mathbf{B}(p,2,\kappa);f) \ \le\ \frac{B^{1/\kappa}[(n+1)\kappa+1;\kappa(1/p-1/2)]}{(1-\psi^2(\alpha))^{1/2}[\exp\{\frac{n+1}{\rho}\ (\rho-1)[\alpha^{-1}\{(\frac{\alpha(\frac{n+1}{\rho})}{\tau+\epsilon})^{1/(\rho-1)}\}]\}]}.$$

For $n>m$, (1.4.17) yields

$$\tau+\epsilon \ge \frac{\alpha(\frac{n+1}{\rho})}{\{\alpha(\frac{\rho}{(1+\frac{1}{n})(\rho-1)}\{\ln(|E_n|^{-1/n})+\ln(\frac{B^{1/\kappa}[(n+1)\kappa+1;\kappa(1/p-1/2)]}{(1-\psi^2(\alpha))^{1/2}})^{1/n}\})\}^{(\rho-1)}}.$$

Now

$$B[(n+1)\kappa+1;\kappa(1/p-1/2)] = \frac{\Gamma((n+1)\kappa+1)\Gamma(\kappa(1/p-1/2))}{\Gamma((n+1/2+1/p)\kappa+1)}.$$

Hence

$$B[(n+1)\kappa+1;\kappa(1/p-1/2)] \simeq \frac{e^{-[(n+1)\kappa+1]}[(n+1)\kappa+1]^{(n+1)\kappa+3/2}\Gamma(1/p-1/2)}{e^{[(n+1/2+1/p)\kappa+1]}[(n+1/2+1/p)\kappa+1]^{(n+1/2+1/p)\kappa+3/2}}.$$

Thus

$$(1.4.18) \qquad \{B[(n+1)\kappa+1;\kappa(1/p-1/2)]\}^{1/(n+1)} \cong 1.$$

Proceeding to limits, we obtain

$$(1.4.19) \qquad \tau \geq \limsup_{n\to\infty} \frac{\alpha(\frac{n}{\rho})}{[\alpha\{\frac{\rho}{\rho-1}\ln(|E_n|^{-1/n})\}]^{(\rho-1)}}.$$

For reverse inequality, by [63, p.1398], we have

$$(1.4.20) \qquad |a_{n+1}|B^{1/\kappa}[(n+1)\kappa+1;\kappa(1/p-1/2)] \leq E_n(\mathbf{B}(p,2,\kappa);f).$$

Then for sufficiently large n, we have

$$\frac{\alpha(n/\rho)}{[\alpha\{\frac{\rho}{\rho-1}\ln(|E_n|^{-1/n})\}]^{(\rho-1)}}$$

$$\geq \frac{\alpha(\frac{n}{\rho})}{[\alpha\{\frac{\rho}{\rho-1}\{\ln(|a_{n+1}|^{-1/n}) + \ln(B^{-\rho/n\kappa}[(n+1)\kappa+1;\kappa(1/p-1/2)])\}\}]^{(\rho-1)}}$$

$$\geq \frac{\alpha(\frac{n}{\rho})}{[\alpha\{\frac{\rho}{\rho-1}\{\ln(|a_n|^{-1/n}) + \ln(B^{-\rho/n\kappa}[(n+1)\kappa+1;\kappa(1/p-1/2)])\}\}]^{(\rho-1)}}.$$

By applying limits and from (1.4.12), we obtain

$$(1.4.21) \qquad \limsup_{n\to\infty} \frac{\alpha(\frac{n}{\rho})}{[\alpha\{\frac{\rho}{\rho-1}\ln(|E_n|^{-1/n})\}]^{(\rho-1)}} \geq \tau.$$

From (1.4.19), and (1.4.21), we obtain the required relation

$$(1.4.22) \qquad \limsup_{n\to\infty} \frac{\alpha(\frac{n}{\rho})}{[\alpha\{\frac{\rho}{\rho-1}\ln(|E_n|^{-1/n})\}]^{(\rho-1)}} = \tau.$$

In the second step, we consider the spaces $\mathbf{B}(p,q,\kappa)$ for $0 < p < q, q \neq 2$, and $q, \kappa \geq 1$. Gvaradze [19] showed that, for $p \geq p_1$, $q \leq q_1$, and $\kappa \leq \kappa_1$, if at least one of the inequalities is strict, then the strict inclusion $\mathbf{B}(p,q,\kappa) \subset \mathbf{B}(p_1,q_1,\kappa_1)$ holds and the following relation is true:

$$\|f\|_{p_1,q_1,\kappa_1} \leq 2^{1/q-1/q_1}[\kappa(1/p-1/q)]^{1/\kappa-1/\kappa_1}\|f\|_{p,q,\kappa}.$$

For any function $f(z) \in \mathbf{B}(p,q,\kappa)$, the last relation yields

$$(1.4.23) \qquad E_n(\mathbf{B}(p_1,q_1,\kappa_1);f) \leq 2^{1/q-1/q_1}[\kappa(1/p-1/q)]^{1/\kappa-1/\kappa_1}E_n(\mathbf{B}(p,q,\kappa);f).$$

For the general case $\mathbf{B}(p,q,\kappa)$, $q \neq 2$, we prove the necessity of condition (1.4.11). Let $f(z) \in \mathbf{B}(p,q,\kappa)$ be an entire transcendental function having finite generalized order $\rho(\alpha;f)$ whose generalized type is defined by (1.4.12). Using the relation (1.4.13), for $n > m$ we estimate the value of the best polynomial approximation as follows

$$E_n(\mathbf{B}(p,q,\kappa);f) = \|f - g_n(f)\|_{p,q,\kappa} \leq (\int_0^1 (1-r)^{(\kappa(1/p-1/q)-1)}M_q^\kappa dr)^{1/\kappa}.$$

Now

$$|f|^q = |\sum a_n z^n|^q \leq (\sum |a_n r^n|)^q \leq (r^{n+1}\sum_{k=n+1}^{\infty}|a_k|)^q.$$

Hence

$$E_n(\mathbf{B}(p,q,\kappa);f) \leq B^{1/\kappa}[(n+1)\kappa+1;\kappa(1/p-1/q)]\sum_{k=n+1}^{\infty}|a_k|$$

$$\leq \frac{B^{1/\kappa}[(n+1)\kappa+1;\kappa(1/p-1/q)]}{(1-\psi(\alpha))[\exp\{\frac{n+1}{\rho}(\rho-1)[\alpha^{-1}\{(\frac{\alpha(\frac{n+1}{\rho})}{\tau+\epsilon})^{1/(\rho-1)}\}]\}]}.$$

For $n > m$, (1.4.24) yields

$$\tau+\epsilon \geq \frac{\alpha(\frac{n+1}{\rho})}{\{\alpha(\frac{\rho}{(1+\frac{1}{n})(\rho-1)}\{\ln(|E_n|^{-1/n}) + \ln(\frac{B^{1/\kappa}[(n+1)\kappa+1;\kappa(1/p-1/2)]}{(1-\psi(\alpha))})^{1/n}\})\}^{(\rho-1)}}.$$

Since $\psi(\alpha) < 1$, and $\alpha \in \overline{\Omega}$, proceeding to limits and using (1.2.8), we obtain

$$\tau \geq \limsup_{n\to\infty} \frac{\alpha(\frac{n}{\rho})}{[\alpha\{\frac{\rho}{\rho-1}\ln(|E_n|^{-1/n})\}]^{(\rho-1)}}.$$

For the reverse inequality, let $0 < p < q < 2$ and $\kappa, q \geq 1$. By (1.4.23), where $p_1 = p, q_1 = 2$, and $\kappa_1 = \kappa$, and the condition (1.4.11) is already proved for the space $\mathbf{B}(p, 2, \kappa)$, we get

$$\limsup_{n\to\infty} \frac{\alpha(n/\rho)}{[\alpha\{\frac{\rho}{\rho-1}\ln(|E_n(\mathbf{B}(p,q,\kappa);f)|^{-1/n})\}]^{(\rho-1)}}$$

$$\geq \limsup_{n\to\infty} \frac{\alpha(n/\rho)}{[\alpha\{\frac{\rho}{\rho-1}\ln(|E_n(\mathbf{B}(p,2,\kappa);f)|^{-1/n})\}]^{(\rho-1)}} = \tau.$$

Now let $0 < p \leq 2 < q$. Since we have

$$M_2(r,f) \leq M_q(r,f),\ 0 < r < 1,$$

therefore

(1.4.24) $$E_n(\mathbf{B}(p,q,\kappa);f) \geq |a_{n+1}|\ B^{1/\kappa}[(n+1)\kappa+1;\kappa(1/p-1/q)].$$

Then for sufficiently large n, we have

$$\frac{\alpha(n/\rho)}{[\alpha\{\frac{\rho}{\rho-1}\ln(|E_n|^{-1/n})\}]^{(\rho-1)}}$$

$$\geq \frac{\alpha(\frac{n}{\rho})}{[\alpha\{\frac{\rho}{\rho-1}\{\ln(|a_{n+1}|^{-1/n}) + \ln(B^{-\rho/n\kappa}[(n+1)\kappa+1;\kappa(1/p-1/q)])\}\}]^{(\rho-1)}}$$

$$\geq \frac{\alpha(\frac{n}{\rho})}{[\alpha\{\frac{\rho}{\rho-1}\{\ln(|a_n|^{-1/n}) + \ln(B^{-\rho/n\kappa}[(n+1)\kappa+1;\kappa(1/p-1/q)])\}\}]^{(\rho-1)}}.$$

By applying limits and from (1.4.12), we obtain

$$\limsup_{n\to\infty} \frac{\alpha(\frac{n}{\rho})}{[\alpha\{\frac{\rho}{\rho-1}\ln(|E_n|^{-1/n})\}]^{(\rho-1)}} \geq \limsup_{n\to\infty} \frac{\alpha(\frac{n}{\rho})}{[\alpha\{\frac{\rho}{\rho-1}\ln(|a_n|^{-1/n})\}]^{(\rho-1)}} = \tau.$$

Now we assume that $2 \leq p < q$. Set $q_1 = q, \kappa_1 = \kappa$, and $0 < p_1 < 2$ in the inequality (1.4.23), where p_1 is an arbitrary fixed number. Substituting p_1 for p in (1.4.24), we get

$$E_n(\mathbf{B}(p,q,\kappa);f) \geq |a_{n+1}|B^{1/\kappa}[(n+1)\kappa+1;\kappa(1/p_1-1/q)]. \tag{1.4.25}$$

Using (1.4.25) and applying the same analogy as in the previous case $0 < p \leq 2 < q$, for sufficiently large n, we have

$$\frac{\alpha(n/\rho)}{[\alpha\{\frac{\rho}{\rho-1}\ln(|E_n|^{-1/n})\}]^{(\rho-1)}}$$

$$\geq \frac{\alpha(\frac{n}{\rho})}{[\alpha\{\frac{\rho}{\rho-1}\{\ln(|a_{n+1}|^{-1/n}) + \ln(B^{-\rho/n\kappa}[(n+1)\kappa+1;\kappa(1/p_1-1/q)])\}\}]^{(\rho-1)}}$$

$$\geq \frac{\alpha(\frac{n}{\rho})}{[\alpha\{\frac{\rho}{\rho-1}\{\ln(|a_n|^{-1/n}) + \ln(B^{-\rho/n\kappa}[(n+1)\kappa+1;\kappa(1/p_1-1/q)])\}\}]^{(\rho-1)}}.$$

By applying limits and using (1.4.12), we obtain

$$\limsup_{n\to\infty} \frac{\alpha(\frac{n}{\rho})}{[\alpha\{\frac{\rho}{\rho-1}\ln(|E_n|^{-1/n})\}]^{(\rho-1)}} \geq \tau.$$

From relations (1.4.19) and (1.4.21), and the above inequality, we obtain the required relation (1.4.22). This completes the proof of Theorem 1.4.2. □

Theorem 1.4.3. *Assuming that the conditions of theorem 1.4.2 are satisfied and $\xi(\alpha)$ is a positive number, a necessary and sufficient condition for a function $f(z) \in H_q$ to be an entire function of generalized type $\xi(\alpha)$ having finite generalized order ρ is that*

$$\limsup_{n\to\infty} \frac{\alpha(\frac{n}{\rho})}{[\alpha\{\frac{\rho}{\rho-1}\ln(|E_n(H_q;f)|^{-1/n})\}]^{\rho-1}} = \xi(\alpha). \tag{1.4.26}$$

Proof. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be an entire transcendental function having finite generalized order ρ and generalized type τ . Since

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = 0 \tag{1.4.27}$$

$f(z) \in \mathbf{B}(p, q, \kappa)$, where $0 < p < q \leq \infty$ and $q, \kappa \geq 1$. From relation (1.1.1), we get

$$E_n(\mathbf{B}(q/2, q, q); f) \leq \varsigma_q E_n(H_q; f), \quad 1 \leq q < \infty. \tag{1.4.28}$$

where ς_q is a constant independent of n and f. In the case of Hardy space H_∞,

$$E_n(\mathbf{B}(p, \infty, \infty); f) \leq E_n(H_\infty; f), \quad 1 < p < \infty. \tag{1.4.29}$$

Since

$$\xi(\alpha; f) = \limsup_{n\to\infty} \frac{\alpha(n/\rho)}{[\alpha\{\frac{\rho}{\rho-1}\ln(|E_n(H_q;f)|^{-1/n})\}]^{(\rho-1)}}$$

$$\geq \limsup_{n\to\infty} \frac{\alpha(n/\rho)}{[\alpha\{\frac{\rho}{\rho-1}\ln(|E_n((\mathbf{B}(q/2,q,q);f)|^{-1/n})\}]^{(\rho-1)}}$$

$$\geq \tau, \quad 1 \leq q < \infty. \tag{1.4.30}$$

Using estimate (1.4.29) we prove inequality (1.4.30) in the case $q = \infty$.

For the reverse inequality

$$\xi(\alpha; f) \leq \tau, \tag{1.4.31}$$

we use the relation (1.4.13), which is valid for $n > m$, and estimate from above, the generalized type τ of an entire transcendental function $f(z)$ having finite generalized order ρ, as follows. We have

$$\begin{aligned} E_n(H_q; f) &\leq \|f - g_n\|_{H_q} \\ &\leq \sum_{j=n+1}^{\infty} |a_j| \\ &\leq \frac{1}{[\exp\{(\rho-1)\,\frac{n+1}{\rho}\,[\alpha^{-1}\{(\frac{\alpha(\frac{n+1}{\rho})}{\tau+\epsilon})^{1/(\rho-1)}\}]\}]} \sum_{j=n+1}^{\infty} \psi_j(\alpha). \end{aligned}$$

Using (1.4.16),

$$\begin{aligned} E_n(H_q; f) &\leq \|f - g_n\|_{H_q} \\ &\leq \frac{1}{(1-\psi(\alpha))\,[\exp\{(\rho-1)\,\frac{n+1}{\rho}\,[\alpha^{-1}\{(\frac{\alpha(\frac{n+1}{\rho})}{\tau+\epsilon})^{1/(\rho-1)}\}]\}]}. \end{aligned}$$

$$\frac{1}{E_n(H_q; f)} \geq (1-\psi(\alpha))\,\exp\left\{(\rho-1)\frac{(n+1)}{\rho}\left[\alpha^{-1}\left\{\left(\frac{\alpha(\frac{(n+1)}{\rho})}{\overline{\tau}}\right)^{\frac{1}{(\rho-1)}}\right\}\right]\right\}.$$

This yields

$$\tau + \epsilon \geq \frac{\alpha(\frac{n+1}{\rho})}{[\alpha\{\frac{\rho}{\rho-1}[\ln(|E_n(H_q;f)|^{-1/n+1}) + \ln((1-\psi(\alpha))^{-1/n+1})]\}]^{(\rho-1)}}. \tag{1.4.32}$$

Since $\psi(\alpha) < 1$ and by applying the properties of the function α, passing to the limit as $n \to \infty$ in (1.4.32), we obtain inequality (1.4.31). Thus we have finally

$$\xi(\alpha) = \tau. \tag{1.4.33}$$

This completes the proof of Theorem 1.4.3. □

Remark 1.4.1. An analog of this theorem for the Bergman Spaces follows from (1.1.1) for $1 \leq q < \infty$ and from Theorem 1.4.3 for $q = \infty$.

Chapter 2

Approximation of Entire Functions of One Complex Variable Over Jordan Domain

In this present Chapter, we study the polynomial approximation of entire functions over Jordan domains by using Faber polynomials. The coefficient characterizations of generalized order and generalized type of entire functions have been obtained in terms of the approximation errors.

2.1 Introduction

Let C be an analytic Jordan curve, D its interior and E be its exterior. Let φ map E conformally onto $\{w : |w| > 1\}$ such that $\varphi(\infty) = \infty$ and $\varphi'(\infty) > 0$. Then for sufficiently large $|z|$, $\varphi(z)$ can be expressed as

(2.1.1) $$w = \varphi(z) = \frac{z}{d} + c_0 + \frac{c_1}{z} + \frac{c_2}{z^2} + \ldots.$$

An arbitrary Jordan curve can be approximated from the inside as well as from the outside by analytic Jordan curves. Since C is analytic, φ is holomorphic on C as well. The n th Faber polynomial $F_n(z)$ of C is the principal part of $(\varphi(z))^n$ at ∞, so that

$$F_n(z) = \frac{z^n}{d^n} + \ldots.$$

Faber [12] proved that as $n \to \infty$,

$$(2.1.2) \qquad F_n(z) \sim (\varphi(z))^n$$

uniformly for $z \in E$ and

$$(2.1.3) \qquad \lim_{n\to\infty} \left(\max_{z\in C} |F_n(z)| \right)^{1/n} = 1.$$

A function f holomorphic in D can be represented by its Faber series

$$(2.1.4) \qquad f(z) = \sum_{n=0}^{\infty} a_n F_n(z)$$

where

$$a_n = \frac{1}{2\pi i} \int_{|w|=r} f(\varphi^{-1}(w)) w^{-(n+1)} dw$$

and $r < 1$ is sufficiently close to 1 so that φ^{-1} is holomorphic and univalent in $|w| \geq r$, the series converging uniformly on compact subsets of D. Let $M(r, f) = \max_{|z|=r} |f(z)|$ be the maximum modulus of $f(z)$. The growth of $f(z)$ is measured in terms of its order ρ and type τ defined as under

$$(2.1.5) \qquad \limsup_{r\to\infty} \frac{\ln \ln M(r, f)}{\ln r} = \rho,$$

$$(2.1.6) \qquad \limsup_{r\to\infty} \frac{\ln M(r, f)}{r^{\rho}} = \tau,$$

for $0 < \rho < \infty$.

Let $L^p(D)$ denote the set of functions f holomorphic in D and such that

$$\|f\|_{L^p(D)} = \left(\frac{1}{A} \int \int_D |f(z)|^p dx dy \right)^{1/p} < \infty$$

where A is the area of D. For $f \in L^p(D)$, set

$$E_n^p = E_n^p(f; D) = \min_{\pi_n} \|f - \pi_n\|_{L^p(D)}$$

where π_n is an arbitrary polynomial of degree at most n.

2.2 Generalized Order and Generalized Type

In this section we obtain the growth characterizations in terms of the coefficients $\{a_n\}$ of the Faber series (2.1.4). We first prove

Theorem 2.2.1. *Let $\alpha(x) \in \Lambda$, $\beta(x) \in L^0$. Set $H(x;c) = \beta^{-1}[c\ \alpha(x)]$, then f is restriction to the domain D of an entire function of finite generalized order ρ iff*

(2.2.1)
$$\limsup_{n\to\infty} \frac{\alpha(n)}{\beta\left(-\frac{1}{n}\ln|a_n|\right)} = \rho,$$

provided $dH(x;c)/d\ln x = O(1)$ as $x \to \infty$ for all c, $0 < c < \infty$.

Proof. Let $f(z) = \sum_{n=0}^{\infty} a_n F_n(z)$ be an entire function of finite generalized order ρ, where

$$a_n = \frac{1}{2\pi i}\int_{|w|=R} f(\varphi^{-1}(w))\ w^{-(n+1)} dw$$

with arbitrarily large R. From (2.1.1), we have

$$\lim_{|w|\to\infty} \frac{\varphi^{-1}(w)}{w} = d.$$

Hence for sufficiently large $|w|$,

$$(d-\epsilon)|w| \ \leq\ |\varphi^{-1}(w)| \leq (d+\epsilon)|w|.$$

Therefore

$$|f(\varphi^{-1}(w))| \leq \exp\left\{\alpha^{-1}\left[\overline{\rho}\,\beta\left(\ln\left(d+\epsilon\right)|w|\right)\right]\right\},\ \overline{\rho}=\rho+\epsilon,$$

and from Cauchy's inequality, we have

$$|a_n| \leq R^{-n}\exp\left\{\alpha^{-1}\left[\overline{\rho}\,\beta\left(\ln\left(d+\epsilon\right)|w|R\right)\right]\right\}.$$

for all R sufficiently large. To minimize the right member of this inequality, choose $R = R(n) = \frac{1}{d+\epsilon}\exp\left\{H(n;\frac{1}{\overline{\rho}})\right\}$. Substituting this value of R in the above inequality, we have

$$-\ln|a_n| \geq nH(n;\frac{1}{\overline{\rho}}) - n\ln\left(d+\epsilon\right) - \alpha^{-1}\left[\overline{\rho}\,\beta\left(H(n;\frac{1}{\overline{\rho}})\right)\right]$$

$$\Rightarrow \quad -\frac{1}{n}\ln|a_n| \;\geq\; H\left(n;\frac{1}{\overline{\rho}}\right) \;=\; \beta^{-1}\left[\frac{1}{\overline{\rho}}\,\alpha(n)\right]$$

$$\Rightarrow \quad \beta\left(-\frac{1}{n}\ln|a_n|\right) \;\geq\; \frac{1}{\overline{\rho}}\alpha(n)$$

$$\Rightarrow \quad \rho+\epsilon \;\geq\; \frac{\alpha(n)}{\beta\left(-\frac{1}{n}\ln|a_n|\right)}.$$

Now proceeding to limits and since ϵ is arbitrary, we have

$$\rho \;\geq\; \limsup_{n\to\infty}\frac{\alpha(n)}{\beta\left(-\frac{1}{n}\ln|a_n|\right)}. \tag{2.2.2}$$

Conversely, let

$$\limsup_{n\to\infty}\frac{\alpha(n)}{\beta\left(-\frac{1}{n}\ln|a_n|\right)} = \sigma.$$

Suppose $\sigma < \infty$. Then for every $\epsilon > 0$, $\exists\, N(\epsilon)$ such that for all $n \geq N$, we have

$$\frac{\alpha(n)}{\beta\left(-\frac{1}{n}\ln|a_n|\right)} \;\leq\; \sigma+\epsilon \;=\; \overline{\sigma},$$

$$\Rightarrow \quad |a_n| \;\leq\; \exp\left\{-n\,H\left(n;\frac{1}{\overline{\sigma}}\right)\right\}.$$

Since $f(z) = \sum_{n=0}^{\infty} a_n F_n(z)$, therefore

$$|f(z)| \leq \sum_{n=0}^{\infty} \exp\left\{-nH\left(n; \frac{1}{\overline{\sigma}}\right)\right\} |F_n(z)|.$$

But from (2.1.2), we have for some $K > 0$, $|F_n(z)| \leq K|\varphi(z)|^n \ \forall\ z \in E$ and from (2.1.1), for all sufficiently large $|z|$, we have

(2.2.3) $$|\varphi(z)| \leq \frac{|z|}{d-\epsilon}.$$

Therefore the above inequality reduces to

(2.2.4) $$|f(z)| \ \leq\ K \sum_{n=0}^{\infty} \exp\left\{-nH\left(n; \frac{1}{\overline{\sigma}}\right)\right\} \left(\frac{|z|}{d-\epsilon}\right)^n.$$

By considering the function $\psi(x) \ = \ (\frac{R}{d-\epsilon})^x \ \exp\left\{-xH\left(x; \frac{1}{\overline{\sigma}}\right)\right\}$ and proceeding on the lines of proof of Theorem 1 of Seremeta [44, p 294], we obtain

$$M(R; f)(1 + o(1)) \ \leq\ \exp\{(G + o(1))\alpha^{-1}[\overline{\sigma}\ \beta(\ln R + G)]\}$$

$$\Rightarrow \quad \frac{\alpha[(G + o(1))^{-1} \ln M(R; f)]}{\beta(\ln R + G)} \leq \ \overline{\sigma} = \sigma + \epsilon.$$

Since $\alpha(x) \in \Lambda$ and $\beta(x) \in L^0$, on letting $R \to \infty$ and since ϵ is arbitrary, we get

(2.2.5) $$\limsup_{R\to\infty} \frac{\alpha(\ln M(R; f))}{\beta(\ln R)} \ \leq \sigma \ = \ \limsup_{n\to\infty} \frac{\alpha(n)}{\beta\left(-\frac{1}{n} \ln |a_n|\right)}.$$

The above inequality holds obviously of $\sigma = \infty$. From (2.2.2) and (2.2.5), we obtain the required result (2.2.1). This completes the proof of Theorem 2.2.1. □

Theorem 2.2.2. *Let $\alpha(x)$, $\beta^{-1}(x)$, $\gamma(x) \in L^0$; let ρ be a fixed number, $0 < \rho < \infty$. Set $H(x; \sigma, \rho) = \gamma^{-1}\left\{[\beta^{-1}(\sigma\ \alpha(x))]^{1/\rho}\right\}$, then f is the restriction to the domain D of* *an entire function of generalized order ρ and finite generalized type τ if and only if*

$$(2.2.6)\qquad \limsup_{n\to\infty} \frac{\alpha\left(\frac{n}{\rho}\right)}{\beta\left\{\left[\gamma\left(de^{1/\rho}|a_n|^{-1/n}\right)\right]^{\rho}\right\}} = \tau$$

provided if $\gamma(x) \in \Lambda$ and $\alpha(x) \in \Lambda$, $dH(x;\sigma,\rho)/d\ln x = O(1)$ as $x \to \infty$.

Proof. Proceeding as in the proof of Theorem 2.2.1, we have

$$(d-\epsilon)|w| \le |\varphi^{-1}(w)| \le (d+\epsilon)|w|.$$

Let f be an entire function of generalized type τ having finite generalized order ρ. Then we have

$$|f(\varphi^{-1}(w))| \le \exp\left\{\alpha^{-1}\left\{\bar{\tau}\,\beta\left[(\gamma((d+\epsilon)|w|))^{\rho}\right]\right\}\right\},$$

and from Cauchy's inequality, we have

$$|a_n| \le R^{-n}\ \exp\left\{\alpha^{-1}\left\{\bar{\tau}\,\beta\left[(\gamma((d+\epsilon)|w|))^{\rho}\right]\right\}\right\},$$

for all R sufficiently large. To minimize the right hand side of this inequality, choose $R = R(n) = \frac{1}{(d+\epsilon)}H\left(\frac{n}{\rho};\frac{1}{\bar{\tau}},\rho\right)$. Substituting value of R in the above inequality, we have

$$|a_n| \le \frac{(d+\epsilon)^n \exp\left(\frac{n}{\rho}\right)}{\left[(d+\epsilon)\, H\left(\frac{n}{\rho};\frac{1}{\bar{\tau}},\rho\right)\right]^n}$$

$$\Rightarrow\quad (d+\epsilon)e^{1/\rho}|a_n|^{-1/n} \ \ge\ H\left(\frac{n}{\rho};\frac{1}{\bar{\tau}},\rho\right).$$

Proceeding to limits, we have

$$(2.2.7)\qquad \limsup_{n\to\infty} \frac{\alpha\left(\frac{n}{\rho}\right)}{\beta\left\{\left[\gamma\left(de^{1/\rho}|a_n|^{-1/n}\right)\right]^{\rho}\right\}} \ \le\ \tau.$$

Conversely let

$$\limsup_{n\to\infty} \frac{\alpha\left(\frac{n}{\rho}\right)}{\beta\left\{\left[\gamma\left(de^{1/\rho}|a_n|^{-1/n}\right)\right]^{\rho}\right\}} = \eta.$$

Suppose $\eta < \infty$. Then for every $\epsilon > 0$, $\exists\, Y(\epsilon)$ such that for all $n \geq Y$, we have

$$\frac{\alpha\left(\frac{n}{\rho}\right)}{\beta\left\{\left[\gamma\left(de^{1/\rho}|a_n|^{-1/n}\right)\right]^{\rho}\right\}} \leq \eta + \epsilon = \overline{\eta}.$$

$$\Longrightarrow \quad |a_n| \leq \frac{d^n \exp\left(\frac{n}{\rho}\right)}{\left[H\left(\frac{n}{\rho};\frac{1}{\overline{\eta}},\rho\right)\right]^n}.$$

Since $f(z) = \sum_{n=0}^{\infty} a_n\, F_n(z)$, therefore

$$|f(z)| \leq \sum_{n=0}^{\infty} \frac{d^n \exp\left(\frac{n}{\rho}\right)}{\left[H\left(\frac{n}{\rho};\frac{1}{\overline{\eta}},\rho\right)\right]^n}\, |F_n(z)|.$$

As in (2.2.4), we have on using the estimate of $F_n(z)$,

$$\begin{aligned} |f(z)| &\leq \sum_{n=0}^{\infty} \frac{d^n \exp\left(\frac{n}{\rho}\right)}{\left[H\left(\frac{n}{\rho};\frac{1}{\overline{\eta}},\rho\right)\right]^n} \left(\frac{d\,|z|}{d-\epsilon}\right)^n \\ &\leq \sum_{n=0}^{\infty} \frac{d^n \exp\left(\frac{n}{\rho}\right)}{\left[H\left(\frac{n}{\rho};\frac{1}{\overline{\eta}},\rho\right)\right]^n}\, R^n. \end{aligned}$$

To estimate the summation of the right hand side of above inequality, we consider the function $\psi(x) = \left(R\, e^{1/\rho}\right)^x \left[H\left(\frac{n}{\rho};\frac{1}{\overline{\eta}},\rho\right)\right]^{-x}$. Then following the proof of Seremeta [44, Th .2, Page 296], we obtain

$$M(R;f) \leq \exp\left\{(A\,\rho + o(1))\ \alpha^{-1}\left\{\overline{\eta}\,\beta\left[\left(\gamma\ \left(Re^{\frac{1}{\rho}}\right)\right)^{\rho}\right]\right\}\right\}.$$

By using the definition of the class L^0, Λ, and proceeding to limits, we obtain

(2.2.8) $$\tau = \limsup_{R\to\infty} \frac{\alpha\,[\ln M(R;f)]}{\beta[(\gamma(R))^\rho]} \le \eta.$$

From (2.2.7) and (2.2.8), we get the required result. This completes the proof of Theorem 2.2.2. □

2.3 L^p - Approximation

In this section we consider the approximations of an entire function over the domain D. Consider the polynomials

$$p_n(z) = \lambda_n z^n + \ldots (\lambda_n > 0)$$

defined through

$$\frac{1}{A}\int\int_D p_n(z)\,\overline{p_m(z)}dx\,dy = \delta_{n,m}.$$

These polynomials were first considered by T. Carleman [8] who proved that

(2.3.1) $$p_n(z) \sim \left(\frac{(n+1)A}{\pi}\right)^{1/2} \varphi'(z)\,(\varphi(z))^n \quad \text{as} \quad n\to\infty$$

uniformly for $z \in E$ where A and $\varphi(z)$ are as defined earlier. Any function $f \in L^2(D)$ can be expanded in terms of these polynomials in a series

(2.3.2) $$f(z) = \sum_{n=0}^{\infty} b_n\, p_n(z)$$

where

$$b_n = \frac{1}{A}\int\int_D f(z)\,\overline{p_n(z)}dx\,dy$$

and the series converges uniformly on compact subsets of D.

Parseval's relation yields

$$E_n^2 = \left(\sum_{k=n+1}^{\infty} |b_k|^2\right)^{1/2}. \tag{2.3.3}$$

We now prove

Lemma 2.3.1. *Let $\alpha(x) \in \Lambda$, $\beta(x) \in L^0$, then*

$$\limsup_{n\to\infty} \frac{\alpha(n)}{\beta\left(-\frac{1}{n}\ln|b_n|\right)} = \limsup_{n\to\infty} \frac{\alpha(n)}{\beta\left(-\frac{1}{n}\ln E_n^2\right)}. \tag{2.3.4}$$

Proof. From (2.3.3), we have

$$|b_{n+1}| \le E_n^2,$$

$$\Longrightarrow \frac{1}{n} \ln \frac{1}{|b_{n+1}|} \ge \frac{1}{n} \ln \frac{1}{E_n^2}$$

Since $\beta \in L^0$, we have

$$\beta\left(-\frac{1}{n} \ln |b_{n+1}|\right) \ge \beta\left(-\frac{1}{n} \ln E_n^2\right).$$

Since $\alpha \in \Lambda$, proceeding to limits, we have

$$\limsup_{n\to\infty} \frac{\alpha(n)}{\beta\left(-\frac{1}{n}\ln|b_n|\right)} \le \limsup_{n\to\infty} \frac{\alpha(n)}{\beta\left(-\frac{1}{n}\ln E_n^2\right)}. \tag{2.3.5}$$

Conversely, let

$$\limsup_{n\to\infty} \frac{\alpha(n)}{\beta\left(-\frac{1}{n}\ln|b_n|\right)} = \rho.$$

Suppose $\rho < \infty$. Then for every $\epsilon > 0$, $\exists\, G(\epsilon)$ such that for all $n \ge G$, we have

$$\frac{\alpha(n)}{\beta\left(-\frac{1}{n}\ln|b_n|\right)} \le \rho + \epsilon = \overline{\rho}$$

$$\Longrightarrow |b_n| \leq G \exp\left\{-n\,\beta^{-1}\left[\frac{1}{\rho}\,\alpha(n)\right]\right\}$$
$$\leq G \exp\left\{-n\,H(n;\frac{1}{\rho})\right\}.$$

Therefore

$$\begin{aligned}
\left(E_n^2\right)^2 &\leq G \sum_{k=n+1}^{\infty} \exp\left\{-2kH(k;\frac{1}{\rho})\right\} \\
&\leq G \exp\left\{-2(n+1)H(n+1;\frac{1}{\rho})\right\} \left(1-\frac{1}{e^{2H(n+1;\frac{1}{\rho})}}\right)^{-1} \\
&\leq G \exp\left\{-2(n+1)H(n+1;\frac{1}{\rho})\right\}
\end{aligned}$$

$$\Longrightarrow \quad \ln\frac{1}{E_n^2} \geq (n+1)\,H(n+1;\frac{1}{\rho})$$

or

$$\beta\left(-\frac{1}{n+1}\,\ln E_n^2\right) \geq \frac{1}{\rho}\,\alpha(n+1).$$

Proceeding to limits, we obtain

$$\text{(2.3.6)} \qquad \limsup_{n\to\infty} \frac{\alpha(n)}{\beta\left(-\frac{1}{n}\ln E_n^2\right)} \leq \rho = \limsup_{n\to\infty} \frac{\alpha(n)}{\beta\left(-\frac{1}{n}\ln|b_n|\right)}$$

from (2.3.5) and (2.3.6), we get the required result. This completes the proof of Lemma 2.3.1. □

Lemma 2.3.2. *Let $\alpha(x)$, $\beta^{-1}(x)$, $\gamma(x) \in L^0$; let ρ be a fixed number, $0 < \rho < \infty$. Set $H(x;\sigma,\rho) = \gamma^{-1}\left\{\left[\beta^{-1}\left(\sigma\,\alpha(x)\right)\right]^{1/\rho}\right\}$, then*

$$\text{(2.3.7)} \qquad \limsup_{n\to\infty} \frac{\alpha\left(\frac{n}{\rho}\right)}{\beta\left\{\left[\gamma\left(de^{1/\rho}|b_n|^{-1/n}\right)\right]^{\rho}\right\}} = \limsup_{n\to\infty} \frac{\alpha\left(\frac{n}{\rho}\right)}{\beta\left\{\left[\gamma\left(de^{1/\rho}(E_n^2)^{-1/n}\right)\right]^{\rho}\right\}}.$$

Proof. From (2.3.3), we have

$$|b_{n+1}| \leq E_n^2,$$

$$e^{1/\rho}\, |b_{n+1}|^{-1/n} \geq e^{1/\rho}\, (E_n^2)^{-1/n}$$

since $\gamma \in L^0$, we have

$$\gamma\big[d\, e^{1/\rho}\, |b_{n+1}|^{-1/n}\big] \;\geq\; \gamma\Big[d\, e^{1/\rho}\, (E_n^2)^{-1/n}\Big]$$

$$\Longrightarrow\; \beta\Big\{\big(\gamma\big[d\, e^{1/\rho}\, |b_{n+1}|^{-1/n}\big]\big)^{\rho}\Big\} \;\geq\; \beta\Big\{\Big(\gamma\Big[d\, e^{1/\rho}\, (E_n^2)^{-1/n}\Big]\Big)^{\rho}\Big\}.$$

Hence

$$\frac{\alpha\left(\frac{n+1}{\rho}\right)}{\beta\{(\gamma[d\, e^{1/\rho}\, |b_{n+1}|^{-1/n}])^{\rho}\}} \leq \frac{\alpha\left(\frac{n+1}{\rho}\right)}{\beta\Big\{\Big(\gamma\Big[d\, e^{1/\rho}\, (E_n^2)^{-1/n}\Big]\Big)^{\rho}\Big\}}.$$

By applying limits, since $\alpha \in L^0$, we obtain

$$\text{(2.3.8)} \qquad \limsup_{n\to\infty} \frac{\alpha\left(\frac{n}{\rho}\right)}{\beta\left\{[\gamma\,(de^{1/\rho}|b_n|^{-1/n})]^{\rho}\right\}} \leq \limsup_{n\to\infty} \frac{\alpha\left(\frac{n}{\rho}\right)}{\beta\left\{[\gamma\,(de^{1/\rho}(E_n^2)^{-1/n})]^{\rho}\right\}}.$$

Conversely, let

$$\limsup_{n\to\infty} \frac{\alpha\left(\frac{n}{\rho}\right)}{\beta\left\{[\gamma\,(de^{1/\rho}|b_n|^{-1/n})]^{\rho}\right\}} = \tau.$$

Suppose $\tau < \infty$. Then for every $\epsilon > 0$, $\exists\, V(\epsilon)$ such that for all $n \geq V$, we have

$$\gamma^{-1}\left[\left\{\beta^{-1}\left[\frac{1}{\tau+\epsilon}\,\alpha\left(\frac{n}{\rho}\right)\right]\right\}^{1/\rho}\right] \leq d\, e^{1/\rho}\, |b_n|^{-1/n}$$

$$\Longrightarrow\; |b_n| \leq d^n\, e^{n/\rho}\left[H\left(\frac{n}{\rho}\,;\,\frac{1}{\overline{\tau}}\,,\,\rho\right)\right]^{-n}.$$

Therefore

$$
\begin{aligned}
\left(E_n^2\right)^2 &\le \sum_{k=n+1}^{\infty} d^{2k}\, e^{2k/\rho} \left[H\left(\frac{k}{\rho}\,;\,\frac{1}{\overline{\tau}}\,,\,\rho\right)\right]^{-2k} \\
&\le \left[\frac{d\,e^{1/\rho}}{H\left(\frac{n+1}{\rho}\,;\,\frac{1}{\overline{\tau}}\,,\,\rho\right)}\right]^{2(n+1)} \left[1-\left(\frac{d\,e^{1/\rho}}{H\left(\frac{n+1}{\rho}\,;\,\frac{1}{\overline{\tau}}\,,\,\rho\right)}\right)^2\right]^{-1} \\
&\le O(1)\left[\frac{d\,e^{1/\rho}}{H\left(\frac{n+1}{\rho}\,;\,\frac{1}{\overline{\tau}}\,,\,\rho\right)}\right]^{2(n+1)}
\end{aligned}
$$

for $n > 2de^{1/\rho}$.

$$
\begin{aligned}
\Longrightarrow d\,e^{1/\rho}\,\left(E_n^2\right)^{-1/(n+1)} &\ge H\left(\frac{n+1}{\rho}\,;\,\frac{1}{\overline{\tau}}\,,\,\rho\right) \\
&\ge \gamma^{-1}\left\{\left[\beta^{-1}\left(\frac{1}{\overline{\tau}}\,\alpha\left(\frac{n+1}{\rho}\right)\right)\right]^{1/\rho}\right\}
\end{aligned}
$$

$$
\beta\left[\left\{\gamma\left[d\,e^{1/\rho}\,\left(E_n^2\right)^{-1/(n+1)}\right]\right\}^{1/\rho}\right] \ge \frac{1}{\overline{\tau}}\alpha\left(\frac{n+1}{\rho}\right)
$$

$$
\Longrightarrow \tau+\epsilon \ge \frac{\alpha\left(\frac{n+1}{\rho}\right)}{\beta\left[\left\{\gamma\left[d\,e^{1/\rho}\,\left(E_n^2\right)^{-1/(n+1)}\right]\right\}^{1/\rho}\right]}
$$

Since $\alpha(x)$, $\beta^{-1}(x)$ and $\gamma(x) \in L^0$, proceeding to limits, we have

$$
(2.3.9) \qquad \limsup_{n\to\infty} \frac{\alpha\left(\frac{n}{\rho}\right)}{\beta\left\{\left[\gamma\left(de^{1/\rho}|b_n|^{-1/n}\right)\right]^{\rho}\right\}} \ge \limsup_{n\to\infty} \frac{\alpha\left(\frac{n}{\rho}\right)}{\beta\left\{\left[\gamma\left(de^{1/\rho}(E_n^2)^{-1/n}\right)\right]^{\rho}\right\}}.
$$

From (2.3.8) and (2.3.9), we get the required result (2.3.7). This completes the proof of Lemma 2.3.2. □

2.4 Main Results

In this section we obtain the growth characterizations (Generalized order and Generalized type) of an entire function in terms of approximation errors E_n^p.

Now we prove

Theorem 2.4.1. *Let* $2 \le p \le \infty$. $\alpha(x) \in \Lambda$, $\beta(x) \in L^0$. *Set* $H(x;c) = \beta^{-1}[c\,\alpha(x)]$, *then f is restriction to the domain D of an entire function of finite generalized order* ρ *if and only if*

$$\limsup_{n\to\infty} \frac{\alpha(n)}{\beta\left(-\frac{1}{n}\ln(E_n^p)\right)} = \rho, \tag{2.4.1}$$

provided $dH(x;c)/d\ln x = O(1)$ *as* $x \to \infty$ *for all* c, $0 < c < \infty$.

Proof. We prove the theorem in two steps. First we consider the case for $p = 2$. Assume f is of finite generalized order ρ. Then from (2.2.1), we have

$$|a_n| \le e^{-n\,H(n;\frac{1}{\rho})}.$$

Now by considering the orthonormality of the polynomials $p_n(z)$, we have

$$b_n = \frac{1}{A}\sum_{k=n+1}^{\infty} a_k \int\int_D F_k(z)\,\overline{p_n(z)}dx\,dy.$$

Hence

$$|b_n| \le \sum_{k=n+1}^{\infty} |a_k| \max_{z\,\in\,C} |F_k(z)|.$$

Since from (2.1.3), we have

$$\max_{z\,\in\,C} |F_k(z)| \;\le\; L\,(1+\epsilon)^k, \tag{2.4.2}$$

therefore, we have

$$\begin{aligned}
|b_n| &\leq L \sum_{k=n+1}^{\infty} e^{-k\,H(k;\frac{1}{\rho})}\,(1+\epsilon)^k \\
&\leq L\,e^{-(n+1)\,H((n+1);\frac{1}{\rho})}\,(1+\epsilon)^{(n+1)}\left[1-\frac{1+\epsilon}{e}\right]^{-1} \\
&\leq O(1)\,L\,e^{-(n+1)\,H((n+1);\frac{1}{\rho})}
\end{aligned}$$

since $H(x;\frac{1}{\rho})$ is an increasing function $\to\infty$ as $x\to\infty$. Hence

$$\ln\frac{1}{|b_n|} \geq (n+1)\,H((n+1);\frac{1}{\rho})$$

$$\Longrightarrow\ -\frac{1}{n+1}\,\ln|b_n| \geq \beta^{-1}\left[\frac{1}{\rho}\,\alpha(n+1)\right]$$

or

$$\beta\Big(-\frac{1}{n+1}\,\ln|b_n|\Big) \geq \frac{1}{\rho}\,\alpha(n+1)$$

$$\Longrightarrow\ \rho+\epsilon \geq \frac{\alpha(n+1)}{\beta\left(-\frac{1}{n+1}\,\ln|b_n|\right)}.$$

Since $\beta \in L^0$, proceeding to limits, we get

$$\limsup_{n\to\infty}\frac{\alpha(n)}{\beta\left(-\frac{1}{n}\,\ln|b_n|\right)} \leq \rho. \tag{2.4.3}$$

Conversely, let

$$\limsup_{n\to\infty}\frac{\alpha(n)}{\beta\left(-\frac{1}{n}\ln|b_n|\right)} = \sigma.$$

Suppose $\sigma<\infty$. Then for each $\epsilon>0$, $\exists\ Z(\epsilon)$ such that for all $n\geq Z$, we have

$$|b_n| \leq e^{-n\,H\left(n;\frac{1}{\sigma}\right)}.$$

By Carleman's result, as $n \to \infty$, we have

$$p_n(z) \sim \left(\frac{(n+1)A}{\pi}\right)^{1/2} \varphi'(z) \, (\varphi(z))^n$$

uniformly for $z \in E$. Therefore for all $z \in E$, we have

$$|p_n(z)| \le L' \, (n+1)^{1/2} \, |\varphi'(z)| \, |\varphi(z)|^n,$$

$$|\varphi'(z)| \le T \quad \forall \, z \in E,$$

$$\text{and} \qquad |\varphi(z)| \le \frac{|z|}{d-\epsilon}$$

for all z with sufficiently large modulus. Therefore

$$|f(z)| \le L \sum_{n=0}^{\infty} e^{-n \, H(n; \frac{1}{\bar\sigma})} \, (n+1)^{1/2} \, \left(\frac{|z|}{d-\epsilon}\right)^n.$$

Now consider $n \, \beta^{-1} \left[\frac{1}{\bar\sigma} \, \alpha(n)\right] - \frac{1}{2} \ln(n+1) = g(n)$.

$$\Longrightarrow \beta^{-1} \left[\frac{1}{\bar\sigma} \, \alpha(n)\right] = \frac{g(n)}{n} \left[1 + \frac{1}{2} \, \frac{\ln(n+1)}{g(n)}\right]$$

or

$$\frac{1}{\bar\sigma} \, \alpha(n) = \beta \left[\frac{1}{n} \, g(n) \left\{1 + \frac{1}{2} \, \frac{\ln(n+1)}{g(n)}\right\}\right].$$

Since $\beta \in L^0$, we have

$$\frac{1}{\bar\sigma} \, \alpha(n) \simeq \beta \left[\frac{1}{n} \, g(n)\right]$$

or

$$\alpha(n) = (\bar\sigma + \epsilon) \, \beta \left[\frac{1}{n} \, g(n)\right] + o(\beta(n))$$

$$\Longrightarrow g(n) = n\beta^{-1} \left[\frac{1}{\sigma + 2\epsilon} \, \alpha(n)\right].$$

Therefore

$$|f(z)| \le L \sum_{n=0}^{\infty} e^{-n\,H\left(n;\frac{1}{\sigma}\right)} \left(\frac{|z|}{d-\epsilon}\right)^n.$$

Consider the function $\chi(x) = \left(\frac{R}{d-\epsilon}\right)^x \; exp\left[-x\,H\left(x;\frac{1}{\sigma+2\epsilon}\right)\right]$. Take its logarithmic derivative and set it equal to zero. Then we have

$$\frac{\chi'(x)}{\chi(x)} = \ln\left(\frac{R}{d-\epsilon}\right) - H\left(x;\frac{1}{\sigma+2\epsilon}\right) - \frac{d\,H\left(x;\frac{1}{\sigma+2\epsilon}\right)}{d\,\ln x} = 0.$$

By assumption of the theorem $\exists\, K' > 0$ such that for $x \ge x_1$

$$\left|\frac{d\,H\left(x;\frac{1}{\sigma+2\epsilon}\right)}{d\,\ln x}\right| \le K'.$$

Let $K_1(R) = E\left[\alpha^{-1}\left\{(\sigma+2\epsilon)\,\beta(\ln R + K')\right\}\right] + 1$ and $k_0 = \max\left(K'(\epsilon), E[x_1]+1\right)$. For $R > R_1(k_0)$, $\psi'(k_0)/\psi(k_0) > 0$, and $\psi'(K_1(R))/\psi(K_1(R)) < 0$. Let $x^*(R)$ be the point where the function ψ attains its maximum such that

$$\psi(x^*(R)) = \max_{k_0 \le x \le K_1(R)} \psi(x),$$

then $k_0 < x^*(R) < K_1(R)$ and $x^*(R) = \alpha^{-1}\left((\sigma+2\epsilon)\,\beta(\ln R - a(R))\right)$, where

$$-K' < a(R) = \frac{d\,H\left(x;\frac{1}{\sigma+2\epsilon}\right)}{d\,\ln x}\Big|_{x=x^*(R)} < K'.$$

Further

$$\begin{aligned}
\max_{k_0 \le k \le K_1(R)} \left(|b_k|\,R^k\right) &\le \max_{k_0 \le x \le K_1(R)} \psi(x) \\
&= \frac{R^{\alpha^{-1}\{(\sigma+2\epsilon)\,\beta(\ln R - \ln(d-\epsilon) - a(R))\}}}{e^{\alpha^{-1}\{(\sigma+2\epsilon)\,\beta(\ln R - \ln(d-\epsilon) - a(R))\}\,[\ln R - \ln(d-\epsilon) - a(R)]}} \\
&= \exp\left\{K'\,\alpha^{-1}\left[(\sigma+2\epsilon)\,\beta(\ln R - \ln(d-\epsilon) - a(R))\right]\right\} \\
&\le \exp\left\{K'\,\alpha^{-1}\left[(\sigma+2\epsilon)\,\beta(Y)\right]\right\}
\end{aligned}$$

where $\beta(Y) = \beta(\ln R - \ln(d-\epsilon) + K')$. Therefore for $R > R_1(k_0)$, we have

$$M(R;f) \leq \sum_{k=0}^{\infty} |b_k| R^k = \sum_{k=0}^{k_0} |b_k| R^k + \sum_{k=k_0+1}^{k_1(R)} |b_k| R^k + \sum_{k=k_1(R)+1}^{\infty} |b_k| R^k$$

$$\leq O(R^{k_0}) + 1 + k_1(R) \max_{k_0 \leq k \leq K_1(R)} (|b_k|\ R^k).$$

Hence

$$\begin{aligned} M(R;f)\ (1+o(1)) &\leq \left(\alpha^{-1}\left[(\sigma+2\epsilon)\ \beta(Y)\right]+1\right)\ \exp\left\{K'\ \alpha^{-1}\left[(\sigma+2\epsilon)\ \beta(Y)\right]\right\} \\ &\leq \exp\left\{(K'+o(1))\ \alpha^{-1}\left[(\sigma+2\epsilon)\ \beta(Y)\right]\right\}. \end{aligned}$$

Then, we have

$$\frac{\alpha\left[(K'+o(1))^{-1}\ \ln M(R;f)\right]}{\beta(Y)} \leq \sigma + 2\epsilon.$$

Since $\alpha(x) \in \Lambda$ and $\beta(x) \in L^0$, proceeding to limits as $R \to \infty$, we obtain

(2.4.4) $$\rho = \limsup_{R\to\infty} \frac{\alpha(\ln M(R;f))}{\beta(\ln R)} \leq \sigma.$$

Combining (2.4.3) and (2.4.4),we obtain

$$\limsup_{n\to\infty} \frac{\alpha(n)}{\beta\left(-\frac{1}{n}\ln|b_n|\right)} = \rho.$$

The result now follows on using Lemma 2.3.1 for the case of $p = 2$.

Now we consider the case $p > 2$. Since, we have

(2.4.5) $$E_n^2 \leq E_n^p \leq E_n^{\infty}$$

for $2 \leq p \leq \infty$, it is sufficient to consider the case $p = \infty$. Suppose f is an entire function of generalized type having finite generalized order ρ. Then

$$\begin{aligned} E_n^\infty &\le \max_{z\in C}\left| f(z) - \sum_{k=0}^{n} a_k\, F_k(z)\right| \\ &\le \sum_{k=n+1}^{\infty} |a_k| \max_{z\in C} |F_k(z)|. \end{aligned}$$

Since by Theorem 2.2.1, we have

$$|a_n| \le e^{-n\, H(n;\frac{1}{\rho})}.$$

and since we have

$$\max_{z\in C} |F_k(z)| \le K(1+\epsilon)^k,$$

therefore the above inequality becomes

$$\begin{aligned} E_n^\infty &\le K \sum_{k=n+1}^{\infty} e^{-n\, H(n;\frac{1}{\rho})}\,(1+\epsilon)^k \\ &\le K\, e^{-(n+1)\, H(n+1;\frac{1}{\rho})}\,(1+\epsilon)^{(n+1)}\left[1-\frac{(1+\epsilon)}{e}\right]^{-1} \\ &\le O(1)\, K\, e^{-(n+1)\, H(n+1;\frac{1}{\rho})}\,(1+\epsilon)^{(n+1)} \end{aligned}$$

$$\Longrightarrow \ln\frac{1}{E_n^\infty} \ge (n+1)\, H((n+1);\frac{1}{\rho})$$

or

$$\beta\left(-\frac{1}{n+1}\ln E_n^\infty\right) \ge \frac{1}{\rho}\,\alpha(n+1).$$

Since $\alpha \in L^0$, proceeding to limits, we get

$$\limsup_{n\to\infty}\frac{\alpha(n)}{\beta\left(-\frac{1}{n}\ln E_n^\infty\right)} \le \rho.$$

In view of inequalities (2.4.5) and the fact that (2.4.1) holds for $p = 2$, this last inequality is an equality. This completes the proof of Theorem 2.4.1. □

Theorem 2.4.2. *Let $2 \leq p \leq \infty$. $\alpha(x)$, $\beta^{-1}(x)$, $\gamma(x) \in L^0$; and ρ be a fixed number, $0 < \rho < \infty$. Set $H(x;\sigma,\rho) = \gamma^{-1}\left\{[\beta^{-1}(\sigma\,\alpha(x))]^{1/\rho}\right\}$, then f is restriction to the domain D of an entire function of generalized order ρ and finite generalized type τ if and only if*

$$\limsup_{n\to\infty} \frac{\alpha\left(\frac{n}{\rho}\right)}{\beta\left\{[\gamma\,(de^{1/\rho}(E_n^p)^{-1/n})]^{\rho}\right\}} = \tau \tag{2.4.6}$$

provided if $\gamma(x) \in \Lambda$ and $\alpha(x) \in \Lambda$, $dH(x;\sigma,\rho)/d\ln x = O(1)$ as $x \to \infty$.

Proof. We prove the result in two steps. First we consider the case when $p = 2$. Assume f is an entire function of generalized type τ having finite generalized order ρ. From equation (2.2.6), we have

$$|a_n| \leq d^n\, e^{n/\rho}\, \left[H\left(\frac{n}{\rho}\,;\,\frac{1}{\overline{\tau}}\,,\,\rho\right)\right]^{-n},$$

where $\overline{\tau} = \tau + \epsilon$. As before we mentioned in the proof of Theorem 2.4.1, we obtain

$$|b_n| \leq \sum_{k=n+1}^{\infty} |a_k|\, \max_{z\,\in\, C} |F_k(z)|.$$

Since from (2.1.3), we have

$$\max_{z\,\in\, C} |F_k(z)| \leq L'\,(1+\epsilon)^k.$$

Therefore, we have

$$|b_n| \leq L' \sum_{k=n+1}^{\infty} d^k\, e^{k/\rho}\, \left[H\left(\frac{k}{\rho}\,;\,\frac{1}{\overline{\tau}}\,,\,\rho\right)\right]^{-k} (1+\epsilon)^k$$

$$\leq \; L'[d\,(1+\epsilon)]^{(n+1)} e^{\frac{n+1}{\rho}} \left[H\left(\frac{n+1}{\rho}\,;\,\frac{1}{\overline{\tau}}\,,\,\rho\right)\right]^{-(n+1)} \left[1-\frac{(1+\epsilon)\,e^{1/\rho}}{d\,\left[H\left(\frac{n+1}{\rho}\,;\,\frac{1}{\overline{\tau}}\,,\,\rho\right)\right]}\right]^{-1}$$

$$\leq \; O(1)\,L'\,d^{(n+1)}\,e^{\frac{n+1}{\rho}} \left[H\left(\frac{n+1}{\rho}\,;\,\frac{1}{\overline{\tau}}\,,\,\rho\right)\right]^{-(n+1)} (1+\epsilon)^{(n+1)}$$

$$\Longrightarrow \; |b_n|^{1/(n+1)} \leq \frac{d\,e^{1/\rho}\,(1+\epsilon)}{\left[H\left(\frac{n+1}{\rho}\,;\,\frac{1}{\overline{\tau}}\,,\,\rho\right)\right]}$$

or

$$\frac{d\,e^{1/\rho}\,(1+\epsilon)}{|b_n|^{1/(n+1)}} \geq \left[H\left(\frac{n+1}{\rho}\,;\,\frac{1}{\overline{\tau}}\,,\,\rho\right)\right]$$

or

$$\beta\Big\{\left[\gamma\left(d\,e^{1/\rho}\,(1+\epsilon)\,|b_n|^{-1/(n+1)}\right)\right]^{\rho}\Big\} \;\geq\; \frac{1}{\overline{\tau}}\,\alpha\left(\frac{n+1}{\rho}\right).$$

Since $\alpha(x)$, $\beta^{-1}(x)$, $\gamma(x) \in L^0$, proceeding to limits, since ϵ is arbitrary, we obtain

$$\tau \geq \limsup_{n\to\infty} \frac{\alpha\left(\frac{n}{\rho}\right)}{\beta\left\{\left[\gamma\left(d\,e^{1/\rho}\,|b_n|^{-1/n}\right)\right]^{\rho}\right\}}. \tag{2.4.7}$$

Conversely, let

$$\limsup_{n\to\infty} \frac{\alpha\left(\frac{n}{\rho}\right)}{\beta\left\{\left[\gamma\left(d\,e^{1/\rho}\,|b_n|^{-1/n}\right)\right]^{\rho}\right\}} = \sigma.$$

Suppose $\sigma < \infty$. Then for each $\epsilon > 0$, $\exists\, L(\epsilon)$ such that for all $n \geq L$, we have

$$|b_n| \leq d^n\,e^{n/\rho} \left[H\left(\frac{n}{\rho}\,;\,\frac{1}{\overline{\sigma}}\,,\,\rho\right)\right]^{-n},$$

where $\overline{\sigma} = \sigma + \epsilon$. As in the proof of Theorem 2.2.1, we obtain

$$|f(z)| \leq L \sum_{n=0}^{\infty} d^n\,e^{n/\rho} \left[H\left(\frac{n}{\rho}\,;\,\frac{1}{\overline{\sigma}}\,,\,\rho\right)\right]^{-n} (n+1)^{1/2} \left(\frac{|z|}{d-\epsilon}\right)^{n}.$$

Consider

$$\frac{(n+1)^{\rho/2n}}{\left[H\left(\frac{n}{\rho}\,;\,\frac{1}{\bar\sigma}\,,\,\rho\right)\right]^{\rho}} = g(n/\rho)$$

$$\Longrightarrow \left(\frac{g(n/\rho)}{e}\right)^{-1/\rho} = \frac{\left[H\left(\frac{n}{\rho}\,;\,\frac{1}{\bar\sigma}\,,\,\rho\right)\right]}{(n+1)^{1/2n}}$$

$$n^{1/2n}\ (1+1/n)^{1/2n}\ \left(\frac{g(n/\rho)}{e}\right)^{-1/\rho} = \gamma^{-1}\left\{\left[\beta^{-1}\left(\frac{1}{\bar\sigma}\,\alpha(\frac{n}{\rho})\right)\right]^{1/\rho}\right\}.$$

As $n \to \infty$, we have

$$\gamma\left[(1+o(1))\ \left(\frac{g(n/\rho)}{e}\right)^{-1/\rho}\right] = \left[\beta^{-1}\left(\frac{1}{\bar\sigma}\,\alpha(\frac{n}{\rho})\right)\right]^{1/\rho}.$$

Since $\gamma(x) \in L^0$, by using the property of L^0 class, we have

$$\simeq \gamma\left[\left(\frac{g(n/\rho)}{e}\right)^{-1/\rho}\right] = \left[\beta^{-1}\left(\frac{1}{\bar\sigma}\,\alpha(\frac{n}{\rho})\right)\right]^{1/\rho}$$

$$\Longrightarrow \alpha(\frac{n}{\rho}) = (\bar\sigma+\epsilon)\ \left\{\left(\beta\left[\gamma\left(\frac{g(n/\rho)}{e}\right)^{-1/\rho}\right]^{\rho}\right)\right\} + o(\gamma(v))$$

$$g(n/\rho) = \left[H\left(\frac{n}{\rho}\,;\,\frac{1}{\bar\sigma+\epsilon}\,,\,\rho\right)\right]^{-\rho}.$$

Therefore, using above approximation of $g(n/\rho)$, we get

$$|f(z)| \le L\sum_{n=0}^{\infty} e^{n/\rho}\ \left[H\left(\frac{n}{\rho}\,;\,\frac{1}{\bar\sigma+\epsilon}\,,\,\rho\right)\right]^{-n}\ \left(\frac{d\,|z|}{d-\epsilon}\right)^{n}.$$

Consider the function $\zeta(x) = (Re^{1/\rho})^{x}\ \left[H\left(\frac{x}{\rho}\,;\,\frac{1}{\bar\sigma+\epsilon}\,,\,\rho\right)\right]^{-x}$. Set its logarithmic derivative equal to zero. Then

$$\frac{\zeta'(x)}{\zeta(x)} = \ln R + \frac{1}{\rho} - \ln\left(H\left(\frac{x}{\rho}\,;\,\frac{1}{\bar\sigma+\epsilon}\,,\,\rho\right)\right) - \frac{d\,\ln\left(H\left(\frac{x}{\rho}\,;\,\frac{1}{\bar\sigma+\epsilon}\,,\,\rho\right)\right)}{d\,\ln x} = 0.$$

If $\alpha(x), \gamma(x) \in \Lambda$, then by hypothesis of theorem, $\exists\ A > 0$, such that for $x > x_1$, we have

$$\left| \frac{d\ \ln\left(H\left(\frac{x}{\rho}\,;\ \frac{1}{\overline{\sigma}+\epsilon}\,,\ \rho\right)\right)}{d\ \ln x} \right| < A.$$

By replacing $\overline{\sigma}$ by $\overline{\sigma}+\epsilon$, the rest of the minimization process follows from the proof of converse part of Seremeta [44, Th .2, Page 296]. Then we get,

$$M(R;f) \leq \exp\left\{(A\rho + o(1))\ \alpha^{-1}\ \left\{(\overline{\sigma}+\epsilon)\,\beta\left[\left(\gamma\left(R\, e^{\frac{1}{\rho}+A}\right)\right)^{\rho}\right]\right\}\right\}.$$

Since $\alpha(x),\ \gamma(x) \in L^0$, proceeding to limits, we obtain

$$\tau = \limsup_{R\to\infty} \frac{\alpha\left(\ln M(R;f)\right)}{\beta[(\gamma(R))^{\rho}]} \leq \sigma. \tag{2.4.8}$$

Combining (2.4.7) and (2.4.8), we obtain

$$\limsup_{n\to\infty} \frac{\alpha\left(\frac{n}{\rho}\right)}{\beta\left\{\left[\gamma\left(de^{1/\rho}|b_n|^{-1/n}\right)\right]^{\rho}\right\}} = \tau. \tag{2.4.9}$$

The result now follows on using Lemma 2.3.2 for the case of $p = 2$.

Now we consider the other case $p > 2$. Since from (2.4.5), it is sufficient to consider the case $p = \infty$. Suppose f is an entire function of generalized type τ having finite generalized order ρ. Then

$$\begin{aligned} E_n^{\infty} &\leq \max_{z\in C}\left| f(z) - \sum_{k=0}^{n} a_k\ F_k(z)\right| \\ &\leq \sum_{k=n+1}^{\infty} |a_k|\ \max_{z\in C}|F_k(z)|. \end{aligned}$$

Since by Theorem 2.2.2, we have

$$|a_n| \le K\, d^n\, e^{n/\rho} \left[H\left(\frac{n}{\rho}\,;\, \frac{1}{\mp}\,,\, \rho\right)\right]^{-n},$$

and since we have

$$\max_{z\in C} |F_k(z)| \le K(1+\epsilon)^k,$$

therefore the above inequality becomes

$$\begin{aligned}
E_n^\infty &\le K \sum_{k=n+1}^{\infty} d^k\, e^{k/\rho} \left[H\left(\frac{k}{\rho}\,;\, \frac{1}{\mp}\,,\, \rho\right)\right]^{-k} (1+\epsilon)^k \\
&\le K[d\,(1+\epsilon)]^{n+1} e^{(n+1)/\rho} \left[H\left(\frac{n+1}{\rho}\,;\, \frac{1}{\mp}\,,\, \rho\right)\right]^{-(n+1)} \left[1 - \frac{d\,(1+\epsilon)\,e^{1/\rho}}{\left[H\left(\frac{n+1}{\rho}\,;\, \frac{1}{\mp}\,,\, \rho\right)\right]}\right]^{-1} \\
&\le O(1)\, K\, d^{n+1}\, e^{(n+1)/\rho} \left[H\left(\frac{n+1}{\rho}\,;\, \frac{1}{\mp}\,,\, \rho\right)\right]^{-(n+1)} (1+\epsilon)^{(n+1)}
\end{aligned}$$

$$\Longrightarrow\ (E_n^\infty)^{1/(n+1)} \le \frac{d\,e^{1/\rho}\,(1+\epsilon)}{\left[H\left(\frac{n+1}{\rho}\,;\, \frac{1}{\mp}\,,\, \rho\right)\right]}$$

$$\frac{d\,e^{1/\rho}\,(1+\epsilon)}{(E_n^\infty)^{1/(n+1)}} \ge \left[H\left(\frac{n+1}{\rho}\,;\, \frac{1}{\mp}\,,\, \rho\right)\right]$$

or

$$\beta\left\{\left[\gamma\left(d\,e^{1/\rho}\,(1+\epsilon)\,(E_n^\infty)^{-1/(n+1)}\right)\right]^{\rho}\right\} \ge \frac{1}{\mp}\,\alpha\left(\frac{n+1}{\rho}\right).$$

Since $\alpha(x)$, $\beta^{-1}(x)$, $\gamma(x) \in L^0$, proceeding to limits we obtain

$$\tau \ge \limsup_{n\to\infty} \frac{\alpha\left(\frac{n}{\rho}\right)}{\beta\left\{\left[\gamma\left(d\,e^{1/\rho}\,(E_n^\infty)^{-1/n}\right)\right]^{\rho}\right\}}.$$

In view of inequalities (2.4.5) and the fact that (2.4.6) holds for $p = 2$, this last inequality is an equality. This completes the proof of Theorem 2.4.2. □

Chapter 3

Approximation of Entire Functions of Two Complex Variables

In the first section of this chapter we study the approximation of continuous function $f(x, y)$ on the domain $[-1,1] \times [-1,1]$ by homogeneous polynomials has been considered. Necessary and sufficient conditions for $f(x, y)$ having analytic function extension $f(z_1, z_2)$ of two complex variables have also been obtained in terms of the growth parameters. In the next section, we study the polynomial approximation of entire functions of two complex variables in Banach spaces. The characterizations of order and type of entire functions of two complex variables have been obtained in terms of the approximation errors.

3.1 Introduction

Let $f(x)$ be a real valued continuous function defined on $[-1,1]$ and Π_n the set of real polynomials of degree at most n.Then

$$E_n(f) = \inf_{p \in \Pi_n} \|(f - p)\|_{L^\infty[-1,1]} \quad n = 0, 1, 2...,$$

denotes the minimum error in the chebyshev approximation of $f(x)$ over the set Π_n. Bernstein [3, p. 118], proved that

$$\lim_{n \to \infty} \sqrt[n]{E_n} = 0$$

if and only if $f(x)$ has an analytic extension $f(z)$ such that $f(z)$ is an entire function. Later, Varga [65, p. 176], proved that $f(x)$ has entire function extension $f(z)$ of order ρ if and only if

$$\limsup_{n\to\infty} \frac{n \ln n}{-\ln E_n(f)} = \rho.$$

Further $f(x)$ has an analytic extension $f(z)$ such that $f(z)$ is an entire function of order ρ and type τ if and only if

$$\limsup_{n\to\infty} E_n^{\frac{\rho}{n}}(f) = e\rho\tau.$$

In this chapter we consider the approximation of real valued continuous functions of two variables in terms of the minimum error defined as follows. Let $f(x, y)$ be a real-valued continuous function defined on the square $-1 \le x, y \le 1$ of R^2. We consider the class $\Pi_{m,n}$ of all real homogeneous polynomials in (x, y) of degree at most m and n in x and y respectively. Define the error

$$(3.1.1) \qquad E_{m,n}(f) = \inf_{p \in \Pi_{m,n}} \|(f-p)\|_{L^\infty[-1,1]^2} \qquad \text{for } m, n = 0, 1, 2...$$

We derive conditions under which the function $f(x, y)$ can be extended to an analytic function $f(z_1, z_2)$ of two variables z_1 and z_2 in the poly disc $|z_1| \le |\rho_1|$, $|z_2| \le |\rho_2|$, $1 < \rho_1 < \rho_2 \le \infty$. To study the approximation of functions of two variables, we introduce the polynomials of least deviation from 0 on $[-1, 1] \times [-1, 1]$. For one variable, these are known to be [36, Th. 11], the polynomials given by :

$$p_n = 2^{-n+1} C_n(x).$$

Where

$$C_n(x) = \frac{(x + \sqrt{x^2 - 1})^n + (x - \sqrt{x^2 - 1})^n}{2}.$$

On substituting $x = \cos t$, we get $C_n(x) = \cos nt$.

Let $f(x, y)$ be a continuous function defined for $-1 \le x \le 1, -1 \le y \le 1$. We consider its expansion on $[-1,1]^2$ into series of Chebyshev polynomials. Thus

$$(3.1.2) \qquad f(x,y) = \frac{a_{0,0}}{2} + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{m,n} C_{m,n}(x,y).$$

Where $C_{m,n}(x,y)$ is a polynomial in x and y of degree m and n respectively. We write $x = \cos t_1$ and $y = \cos t_2$. It is easily seen that $f(x,y)$ is an even, periodic function of period 2π with respect to both variables t_1 and t_2 and can be expanded in a double Fourier series

$$f(\cos t_1, \cos t_2) = \frac{a_{0,0}}{2} + \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} a_{m,n} \cos mt_1 \cos nt_2.$$

Where

$$a_{m,n} = \frac{1}{\pi^2} \int_{t_1=-\pi}^{\pi} \int_{t_2=-\pi}^{\pi} f(\cos t_1, \cos t_2) \cos mt_1 \cos nt_2 dt_1 dt_2.$$

Substituting $cost_1 = x$ and $cost_2 = y$, we get the required expression (3.1.2).

Next, we define certain Banach spaces of two complex variables as follows Let H_q, $q > 0$ denote the space of functions $f(z_1, z_2)$ analytic in the unit bi disc $U = \{z_1, z_2 \in C : |z_1| < 1, |z_2| < 1\}$ such that

$$\|f\|_{H_q} = \lim_{r_1, r_2 \to 1-0} M_q(f; r_1, r_2) < \infty,$$

where

$$M_q(f; r_1, r_2) = \left\{ \frac{1}{4\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} |f(r_1 e^{it_1}, r_2 e^{it_2})|^q \, dt_1 dt_2 \right\}^{1/q},$$

and let $H_q^{'}$, $q > 0$ denote the space of functions $f(z_1, z_2)$ analytic in U and satisfying the condition

$$\|f\|_{H_q^{'}} = \left\{ \frac{1}{\pi^2} \int_{|z_1|<1} \int_{|z_2|<1} |f(z_1, z_2)|^q \, dx_1 dy_1 dx_2 dy_2 \right\}^{1/q} < \infty.$$

Set

$$\|f\|_{H'_\infty} = \|f\|_{H_\infty} = \sup\{|f(z_1, z_2)| \ : \ z_1, z_2 \in U\}.$$

H_q and H'_q are Banach spaces for $q \geq 1$. In analogy with spaces of functions of one variable, we call H_q and H'_q the Hardy and Bergman spaces respectively.

The function $f(z_1, z_2)$ is analytic in U and belongs to the space $\mathbf{B}(p, q, \kappa)$, where $0 < p < q \leq \infty$, and $0 < \kappa \leq \infty$, if

$$\|f\|_{p,q,\kappa} = \left\{\int_0^1 \int_0^1 \{(1-r_1)(1-r_2)\}^{\kappa(1/p-1/q)-1} M_q^\kappa(f, r_1, r_2) dr_1 dr_2\right\}^{1/\kappa} < \infty,$$

for $0 < \kappa < \infty$, and for $\kappa = \infty$,

$$\|f\|_{p,q,\infty} = \sup\{[(1-r_1)\ (1-r_2)\}^{(1/p-1/q)-1}\ M_q(f, r_1, r_2) \ : \ 0 < r_1, r_2 < 1\} < \infty.$$

The space $\mathbf{B}(p, q, \kappa)$ is a Banach space for $p > 0$ and $q, \kappa \geq 1$, otherwise it is a Fréchet space. Further, we have

(3.1.3) $$H_q \subset H'_q = \mathbf{B}(q/2,\ q,\ q), \quad 1 \leq q < \infty.$$

Let X be a Banach space and let $E_{m,n}(f, X)$ be the best approximation of a function $f(z_1, z_2) \in X$ by elements of the space P that consists of algebraic polynomials of degree $\leq m+n$ in two complex variables :

(3.1.4) $$E_{m,n}(f, X) = \inf\{\|f-p\|_x \ ; \ p \in P\}.$$

Notation : For reducing the length of expressions we use the following notations in main results.

$B^{1/\kappa}[(n+1)\kappa+1\ ;\ \kappa(1/p-1/2)] = B[n, p, 2, \kappa]$
$B^{1/\kappa}[(m+1)\kappa+1\ ;\ \kappa(1/p-1/2)] = B[m, p, 2, \kappa]$
$B^{1/\kappa}[(n+1)\kappa+1\ ;\ \kappa(1/p-1/q)] = B[n, p, q, \kappa]$
$B^{1/\kappa}[(m+1)\kappa+1\ ;\ \kappa(1/p-1/q)] = B[m, p, q, \kappa].$

3.2 Growth Characterizations in $[-1,1] \times [-1,1]$

In this section first we prove the the necessary and sufficient condition the function $f(z_1, z_2)$ to be entire and then we obtained growth characterizations of order and type on the domain $[-1,1] \times [-1,1]$ in terms of approximation errors. We now prove

Theorem 3.2.1. *Let $f(x,y)$ be a real valued continuous function defined on $[-1,1]^2$. Then $f(x,y)$ has an analytic extension $f(z_1,z_2)$ such that $f(z_1,z_2)$ is an entire function, if and only if*

(3.2.1)
$$\lim_{m,n\to\infty} E_{m,n}^{\frac{1}{(m+n)}}(f) = 0.$$

Proof. First we show that, if f is analytic in $D_{\delta\eta}, 1 < \delta \le \infty, 1 < \eta \le \infty$, where $D_{\delta\eta}$ is the poly ellipse $\{(w_1, w_2), w_1 \in$ closed elliptic region in the z_1 plane bounded by ellipse E_{δ_1} with foci $(\pm 1, 0)$ and sum of the semi axes is δ_1, similarly $w_2 \in$ closed elliptic region in the z_2 plane bounded by ellipse E_{δ_2} with foci $(\pm 1, 0)$ and sum of semi of axes equal to $\delta_2\}$ then

(3.2.2)
$$\limsup_{m,n\to\infty} E_{m,n}^{\frac{1}{(m+n)}}(f) \le \frac{1}{\delta\eta}.$$

We begin by considering the expansion of $f(x,y)$ given by (3.1.2).
Since $|C_{m,n}(x,y)| \le 1$ for $-1 \le x, y \le 1$, we have from (3.1.2),

(3.2.3)
$$E_{m,n}(f) \le \sum_{l=m+1}^{\infty} \sum_{k=n+1}^{\infty} |a_{k,l}|.$$

Now we estimate $|a_{k,l}|$. The substitution $z_1 = \exp it_1$, $z_2 = \exp it_2$ in the integral for $a_{k,l}$ gives the line integral along the circles $|z_1| = 1, |z_2| = 1$:

(3.2.4)
$$a_{k,l} = \frac{1}{(\pi i)^2} \int_{|z_2|=1} \int_{|z_1|=1} f\left(\frac{z_1 + z_1^{-1}}{2}, \frac{z_2 + z_2^{-1}}{2}\right)\left(\frac{z_1^{k} + z_1^{-k}}{2}\right)\left(\frac{z_2^{l} + z_2^{-l}}{2}\right)\frac{dz_1}{z_1}\frac{dz_2}{z_2}.$$

We take δ_1, δ_2 with $1 < \delta_1 < \delta, 1 < \delta_2 < \eta$. Consider the function

$$g(z_1, z_2) = f\left(\frac{z_1 + {z_1}^{-1}}{2}, \frac{z_2 + {z_2}^{-1}}{2}\right)$$

in the closed rings R_1 and R_2 bounded by the circles $C_1 : |z_1| = {\delta_1}^{-1}$, $C_2 : |z_1| = \delta_1, C_3 : |z_2| = {\delta_2}^{-1}, C_4 : |z_1| = \delta_2$. The annular regions ${\delta_1}^{-1} \leq |z_1| \leq \delta_1$ and ${\delta_2}^{-1} \leq |z_2| \leq \delta_2$ are mapped to ellipses E_{δ_1} and E_{δ_2} by the transformations $w_1 = \frac{z_1+{z_1}^{-1}}{2}$ and $w_2 = \frac{z_2+{z_2}^{-1}}{2}$ respectively. Where sum of semi-axes of E_{δ_1} is δ_1 and sum of semi-axes of E_{δ_2} is δ_2. Since $f(w_1, w_2)$ is analytic in $D_{\delta\eta}$ it follows that $g(z_1, z_2)$ is analytic in the poly disc $\{(z_1, z_2), 0 \leq |z_1| \leq \delta_1, 0 \leq |z_2| \leq \delta_2\}$.

In order to obtain an estimate of $a_{k,l}$, we now transform the path of integration. We split the integral (3.2.4), into four parts. The integral containing z_1^{-k}, z_2^{-l} is taken over a circle with a large radius, and for the integral with z_1^k, z_2^l, we take a circle with a small radius. Thus,

$$\begin{aligned}
a_{k,l} &= \frac{1}{2\pi i}\int_{C_3}\int_{C_1} g(z_1, z_2){z_1}^{k-1}{z_2}^{l-1}dz_1dz_2 \\
&+ \frac{1}{2\pi i}\int_{C_3}\int_{C_2} g(z_1, z_2){z_1}^{-(k-1)}{z_2}^{l-1}dz_1dz_2 \\
&+ \frac{1}{2\pi i}\int_{C_4}\int_{C_1} g(z_1, z_2){z_1}^{k-1}{z_2}^{-(l-1)}dz_1dz_2 \\
&+ \frac{1}{2\pi i}\int_{C_4}\int_{C_2} g(z_1, z_2){z_1}^{-(k-1)}{z_2}^{-(l-1)}dz_1dz_2.
\end{aligned}$$

Let M be the maximum of the absolute value of $f(w_1, w_2)$ on $D_{\delta\eta}$, then the absolute value of the first integral is not greater than

$$\frac{1}{2\pi}M\left(\frac{1}{\delta_1}\right)^{(k-1)}\left(\frac{1}{\delta_2}\right)^{(l-1)}2\pi{\delta_1}^{-1}{\delta_2}^{-1} = M{\delta_1}^{-k}{\delta_2}^{-l}.$$

In the same way, the remaining three integrals are majorized by $M{\delta_1}^{-k}{\delta_2}^{-l}$, and we get $|a_{k,l}| \leq 4M\delta_1{}^{-k}\delta_2{}^{-l}$. Then, by (3.2.3),

$$E_{m,n}(f) \leq 4M \sum_{l=n+1}^{\infty} \sum_{k=m+1}^{\infty} \delta_1{}^{-k}\delta_2{}^{-l} = \frac{4M}{(\delta_1-1)(\delta_2-1)} M\delta_1{}^{-m}\delta_2{}^{-n}.$$

Hence

$$\sqrt[m+n]{E_{m,n}(f)} \leq M_1^{1/(m+n)}\delta_1{}^{-(m+1)/(m+n)}\delta_2{}^{-(n+1)/(m+n)}.$$

Proceeding to limits, we obtain

$$\limsup_{m\to\infty, n\to\infty} \sqrt[m+n]{E_{m,n}(f)} \leq \delta_1{}^{-1}\delta_2{}^{-1}.$$

Since δ_1 and δ_2 are arbitrary, we get (3.2.2). The converse follows in the same manner and the result is proved. This completes the proof of Theorem 3.2.1. □

Next we obtain the characterization of order of entire function $f(z_1, z_2)$ in terms of the approximation error $E_{m,n}$.

Theorem 3.2.2. *Let $f(x, y)$ be a real-valued continuous function on $[-1, 1]^2$. Then*

(3.2.5)
$$\limsup_{m,n\to\infty} \frac{\ln(m^m n^n)}{-\ln E_{m,n}(f)} = \sigma.$$

Where σ is a non-negative real number, if and only if $f(x, y)$ has an analytic extension $f(z_1, z_2)$, which is an entire function of order σ.

Proof. First, we assume that $f(x, y)$ has an analytic extension $f(z_1, z_2)$, which is an entire function of order $\sigma(0 \leq \sigma < \infty)$. For each $m, n \geq 0$, we have

(3.2.6)
$$E_{m,n}(f) \leq \frac{4B(\rho,\eta)}{\rho^m\eta^n(\rho-1)(\eta-1)}$$

for any $\rho, \eta > 1$, where $B(\rho, \eta) \equiv \max_{z_1\in E_\rho, z_2\in E_\eta} |f(z_1, z_2)|$, and E_ρ with $\rho > 1$ denotes the closed interior of the ellipse with foci $(\pm 1, 0)$, half-major axis $\frac{(\rho^2+1)}{2\rho}$ and half-minor axis $\frac{(\rho^2-1)}{2\rho}$ .Simillarly E_η with $\eta > 1$ denotes the closed interior of the ellipse with foci $(\pm 1, 0)$, half-major axis $\frac{(\eta^2+1)}{2\eta}$ and half-minor axis $\frac{(\eta^2-1)}{2\eta}$. The ellipse E_ρ is bounded by the closed disks $D_1(\rho)$ and $D_2(\rho)$ in the sense that

$$D_1(\rho) \equiv \left\{ z_1 : |z_1| \leq \frac{\rho^2 - 1}{2\rho} \right\} \subset E_\rho \subset D_2(\rho) \equiv \left\{ z_1 : |z_1| \leq \frac{\rho^2 + 1}{2\rho} \right\}.$$

Similarly, ellipse E_η is bounded by the closed disks $D_3(\eta)$ and $D_4(\eta)$ i.e.

$$D_3(\eta) \equiv \left\{ z_2 : |z_2| \leq \frac{\eta^2 - 1}{2\eta} \right\} \subset E_\eta \subset D_4(\eta) \equiv \left\{ z_2 : |z_2| \leq \frac{\eta^2 + 1}{2\eta} \right\}.$$

From these inclusions, it follows by definition that

$$M_f\left(\frac{\rho^2-1}{2\rho}, \frac{\eta^2-1}{2\eta}\right) \leq B(\rho,\eta) \leq M_f\left(\frac{\rho^2+1}{2\rho}, \frac{\eta^2+1}{2\eta}\right) \quad \forall\, \rho, \eta > 1. \tag{3.2.7}$$

Consequently, from (3.2.7), we have for each $m, n \geq 0$,

$$E_{m,n}(f) \leq \frac{4M_f(\frac{\rho^2+1}{2\rho}, \frac{\eta^2+1}{2\eta})}{\rho^m \eta^n (\rho - 1)(\eta - 1)} \tag{3.2.8}$$

for any $\rho, \eta > 1$. Since $f(z_1, z_2)$ is by assumption of order σ, given any $\epsilon > 0$, there exists an $R(\epsilon) > 0$, such that $M_f(r_1, r_2) < \exp{(r_1 r_2)^{\sigma+\epsilon}}$ for all $r_1, r_2 \geq R(\epsilon)$. Thus

$$E_{m,n}(f) \leq \frac{4\exp\{(\frac{\rho^2+1}{2\rho})^{\sigma+\epsilon}(\frac{\eta^2+1}{2\eta})^{\sigma+\epsilon}\}}{\rho^m \eta^n (\rho - 1)(\eta - 1)} \tag{3.2.9}$$

for all $\rho, \eta > 4R(\epsilon)$, and all $m, n \geq 0$. The right hand side of this inequality, considered as a function of ρ and η for fixed m, n is minimized by choosing $\rho = 4m^{1/(\sigma+\epsilon)}$, $\eta = 4n^{1/(\sigma+\epsilon)}$ and this choice of ρ, η is compatible with the restriction $\rho, \eta \geq 4R(\epsilon)$

for all m,n sufficiently large. For these choices of ρ and η, we have

$$\left[\left(\frac{\rho^2+1}{2\rho}\right)^{\sigma+\epsilon}\left(\frac{\eta^2+1}{2\eta}\right)^{\sigma+\epsilon}\right]$$

$$\begin{aligned} &= \left(2m^{1/(\sigma+\epsilon)} + \frac{m^{-1/(\sigma+\epsilon)}}{8}\right)^{\sigma+\epsilon}\left(2n^{1/(\sigma+\epsilon)} + \frac{n^{-1/(\sigma+\epsilon)}}{8}\right)^{\sigma+\epsilon} \\ &\approx 4^{\sigma+\epsilon} mn. \end{aligned}$$

Hence we have

$$E_{m,n}(f) \leq \frac{4\exp[4^{\sigma+\epsilon}mn]}{4^{\left(\frac{1}{(m+1)}+\frac{1}{(n+1)}\right)}[m^{(m+1)}n^{(n+1)}]^{\frac{1}{(\sigma+\epsilon)}}}.$$

Proceeding to limits, we have

$$\limsup_{m,n\to\infty} \frac{\ln(m^m n^n)}{-\ln E_{m,n}(f)} \leq \sigma+\epsilon.$$

As ϵ is arbitrary, and $f(z_1, z_2)$ is of order σ, it follows that there exists a finite number $\beta \geq 0$ such that

(3.2.10)
$$\limsup_{m,n\to\infty} \frac{\ln(m^m n^n)}{-\ln E_{m,n}(f)} = \beta \leq \sigma.$$

Then it follows that for any given $\epsilon > 0$, there exist positive integers $m_0(\epsilon)$ and $n_0(\epsilon)$ such that

(3.2.11)
$$E_{m,n}(f) \leq \frac{1}{m^{m/(\beta+\epsilon)} n^{n/(\beta+\epsilon)}}$$

for all $m \geq m_0(\epsilon)$, $n \geq n_0(\epsilon)$, and from (3.2.1), this means that $f(x, y)$ can be extended to an entire function $f(z_1, z_2)$. For each $m, n \geq 0$, there exists a unique polynomial $p_{m,n}(x,y) \in \Pi_{mn}$ such that

$$\|f - p_{m,n}\|_{L^\infty[-1,1]^2} \;=\; E_{m,n}(f),\; m,n = 0,1,2,...$$

By triangle inequality,

$\|p_{m+1,n+1} - p_{m,n}\|_{L^\infty[-1,1]^2}$ is bounded by $4E_{m,n}(f)$.

Thus

$$(3.2.12) \qquad |p_{m+1,n+1}(z_1,z_2) - p_{m,n}(z_1,z_2)| \le 4E_{m,n}(f)\rho^{n+1}\eta^{m+1}$$

for all $z_1 \in E_\rho, z_2 \in E_\eta$, for any $\rho, \eta > 1$, we can write

$$f(z_1,z_2) \;=\; P_{0,0}(z_1,z_2) + \sum_{l=0}^{\infty}\sum_{k=0}^{\infty}(p_{k+1,l+1}(z_1,z_2) - p_{k,l}(z_1,z_2)),$$

and from (3.2.12), it follows that this series converges uniformly in any bounded domain of the hyper plane. Thus, from (3.2.12), we have

$$(3.2.13) \qquad |f(z_1,z_2)| \le |P_{0,0}(z_1,z_2)| + 4\sum_{l=0}^{\infty}\sum_{k=0}^{\infty}E_{k,l}(f)\rho^{k+1}\eta^{l+1}$$

for any $z_1 \in E_\rho, z_2 \in E_\eta$ and consequently, from the definition of $B(\rho,\eta)$, it follows that

$$(3.2.14) \qquad B(\rho,\eta) \le |P_{0,0}(z_1,z_2)| + 4\sum_{l=0}^{\infty}\sum_{k=0}^{\infty}E_{k,l}(f)\rho^{k+1}\eta^{l+1}.$$

With the first inequality of (3.2.7), and (3.2.11), we can write this as

$$M_f\left(\frac{\rho^2-1}{2\rho}, \frac{\eta^2-1}{2\eta}\right) \;\le\; |P_{0,0}(z_1,z_2)|$$

$$
\begin{aligned}
&+ \; 4\left[\sum_{l<m_0(\epsilon)}\sum_{k<n_0(\epsilon)} + \sum_{l\geq m_0(\epsilon)}\sum_{k<n_0(\epsilon)}\right] E_{k,l}(f)\rho^{k+1}\eta^{l+1} \\
&+ \; 4\left[\sum_{l<m_0(\epsilon)}\sum_{k\geq n_0(\epsilon)} + \sum_{l>m_0(\epsilon)}\sum_{k>n_0(\epsilon)}\right] E_{k,l}(f)\rho^{k+1}\eta^{l+1} \\
&\leq \; O(1) + 16 \sum_{l>m_0(\epsilon)}\sum_{k>n_0(\epsilon)} \frac{\rho^{k+1}\eta^{l+1}}{[k^k l^l]^{1/(\beta+\epsilon)}}.
\end{aligned}
$$

Applying Theorem A to the series on the right hand side of above equation , we see that the series representation is an entire function of order $\beta+\epsilon$. Thus there exists an $R(\epsilon) \geq 1$ such that

$$
M_f\left(\frac{\rho^2-1}{2\rho}, \frac{\eta^2-1}{2\eta}\right) \leq P_{n_0,m_0}(\rho,\eta) + \exp\left(\rho^{(\beta+2\epsilon)}\eta^{(\beta+2\epsilon)}\right) \tag{3.2.15}
$$

for all $\rho, \eta > R(\epsilon)$, where

$$
P_{n_0,m_0}(\rho,\eta) \equiv |P_{0,0}(z_1,z_2)| + 4\sum_{l<m_0(\epsilon)}\sum_{k<n_0(\epsilon)} E_{k,l}(f)\rho^{k+1}\eta^{l+1}
$$

is a polynomial. From(3.2.15), it then follows that

$$
\limsup_{\rho,\eta\to\infty} \frac{\ln\ln M_f(\rho,\eta)}{\ln(\rho\eta)} \leq \beta, \tag{3.2.16}
$$

which shows that the entire function $f(z_1,z_2)$ is of order at most β. Summarizing, if $f(z_1,z_2)$ is of finite order σ, then (3.2.10), is valid for some β with $\beta \leq \sigma$. If $\beta < \sigma$, the argument above leading to (3.2.16), shows that $f(z_1,z_2)$ would be of order less than σ, a contradiction. Thus $\beta = \sigma$ and the converse follows as well. This completes the proof of Theorem 3.2.2. □

Theorem 3.2.3. *Let $f(x,y)$ be a real-valued continuous function on $[-1,1]^2$. Then* *there exist constants σ (positive) and α, τ (non negative) such that*

$$\limsup_{m,n\to\infty} \left\{(m^m n^n E^\sigma_{m,n}(f))\right\}^{1/(m+n)} = \alpha$$

if and only if $f(x,y)$ has an analytic extension $f(z_1,z_2)$ such that $f(z_1,z_2)$ is an entire function of order σ and type τ. Where $\alpha = (e\sigma\tau)2^{-\sigma}$.

Proof. From (3.2.8), for each $m, n \geq 0$, and for every $\rho, \eta > 1$, we have

$$E_{m,n}(f) \leq \frac{4M_f(\frac{\rho^2+1}{2\rho}, \frac{\eta^2+1}{2\eta})}{\rho^m\eta^n(\rho-1)(\eta-1)}.$$

Since by assumption, $f(z_1, z_2)$ is of order σ and type τ, given any $\epsilon > 0$, there exists an $R(\epsilon) > 0$ such that $M_f(r_1, r_2) \leq \exp\{(\tau+\epsilon)(r_1^\sigma + r_2^\sigma)\}$ for all $r_1, r_2 > R(\epsilon)$. Thus

$$E_{m,n}(f) \leq \frac{4\exp\left\{(\tau+\epsilon)\{(\frac{\rho^2+1}{2\rho})^\sigma + (\frac{\eta^2+1}{2\eta})^\sigma\}\right\}}{\rho^m\eta^n(\rho-1)(\eta-1)} \tag{3.2.17}$$

for all $\rho, \eta \geq 2R(\epsilon)$, and all $m, n \geq 0$. The right hand side of this inequality considered as a function of ρ and η for fixed m,n, is minimized by choosing $\rho \approx 2(\frac{m}{\sigma(\tau+\epsilon)})^{1/\sigma}$, $\eta \approx 2(\frac{n}{\sigma(\tau+\epsilon)})^{1/\sigma}$, and these choices of ρ and η are compatible with the restriction $\rho, \eta \geq 2R(\epsilon)$ for all sufficiently large values of m and n. For these values of ρ and η, we have

$$\left(\frac{\rho^2+1}{2\rho}\right)^\sigma + \left(\frac{\eta^2+1}{2\eta}\right)^\sigma$$

$$\begin{aligned}
&= \frac{m}{\sigma(\tau+\epsilon)}\left(1+\frac{1}{4}\left(\frac{\tau+\epsilon}{m}\right)^{2/\sigma}\right)^\sigma + \frac{n}{\sigma(\tau+\epsilon)}\left(1+\frac{1}{4}\left(\frac{\tau+\epsilon}{n}\right)^{2/\sigma}\right)^\sigma \\
&\approx \frac{m+n}{\sigma(\tau+\epsilon)}(1+o(1)).
\end{aligned}$$

Hence,

$$E_{m,n}(f) \leq 4\frac{\{\sigma(\tau+\epsilon)\}^{(m+n+2)/\sigma}\exp\{(m+n)/\sigma\}}{2^{m+n+2}m^{(m+1)/\sigma}n^{(n+1)/\sigma}}$$

$$E^{\sigma}_{m,n}(f) \leq 4^{\sigma}\left[\frac{\sigma(\tau+\epsilon)}{2^{\sigma}}\right]^{m+n+2}\frac{\exp(m+n)}{m^{m+1}n^{n+1}}.$$

On proceeding to limits, we get

$$\limsup_{m,n\to\infty}\left\{(m^m n^n)E^{\sigma}_{m,n}(f)\right\}^{1/(m+n)} \leq [e\sigma(\tau+\epsilon)]2^{-\sigma}.$$

As ϵ is arbitrary, the assumption that $f(z_1, z_2)$ is of order σ and type τ implies that there exists a finite $\chi \geq 0$ such that

$$\limsup_{m,n\to\infty}\left\{(m^m n^n)E^{\sigma}_{m,n}(f)\right\}^{1/(m+n)} = \chi \leq \alpha. \tag{3.2.18}$$

From (3.2.18), it follows that given any $\epsilon > 0$, there exist positive integers $m(\epsilon)$ and $n(\epsilon)$ such that

$$E_{m,n}(f) \leq \frac{(\chi+\epsilon)^{(m+n)/\sigma}}{m^{m/\sigma}n^{n/\sigma}} \tag{3.2.19}$$

for all $m \geq m(\epsilon), n \geq n(\epsilon)$. With the first inequality of (3.2.7), and the inequality of (3.2.13), we can write this as

$$\begin{aligned} M_f\left(\frac{\rho^2-1}{2\rho}, \frac{\eta^2-1}{2\eta}\right) \;&\leq\; \left\{|P_{0,0}(z_1,z_2)| + 4\sum_{l<n(\epsilon)}\sum_{k<m(\epsilon)} E_{k,l}(f)\rho^{k+1}\eta^{l+1}\right\} \\ &+\; 4\sum_{l>n(\epsilon)}\sum_{k>m(\epsilon)}\left\{(\chi+\epsilon)^{(k+l)/\sigma}\rho^{k+1}\eta^{l+1}\right\}(k^k l^l)^{-1/\sigma}. \end{aligned}$$

Applying Theorem B to the infinite series on the right hand side of above equation, it follows that it represents an entire function of order σ and type $(\chi+\epsilon)/e\sigma$. Thus, there exists an $R(\epsilon) \geq 1$ such that

$$M_f\left(\frac{\rho^2-1}{2\rho}, \frac{\eta^2-1}{2\eta}\right) \leq P_{n,m}(\rho,\eta) + \exp\{(\chi+\epsilon)(e\sigma)^{-1}(\rho^\sigma+\eta^\sigma)\} \tag{3.2.20}$$

for all $\rho, \eta > R(\epsilon)$. where

$$P_{n,m}(\rho,\eta) \equiv |P_{0,0}(z_1,z_2)| + 4\sum_{l<n(\epsilon)}\sum_{k<m(\epsilon)} E_{k,l}(f)\rho^{k+1}\eta^{l+1}$$

is a polynomial of degree at most $m(\epsilon)$ and $n(\epsilon)$ in ρ and η. From (3.2.20), it then follows that

$$\limsup_{\rho,\eta\to\infty} \frac{\ln M_f(\rho,\eta)}{(\rho^\sigma+\eta^\sigma)} \leq \frac{\chi 2^\sigma}{e\sigma} \leq \tau \tag{3.2.21}$$

which shows that the entire function $f(z_1, z_2)$ is of order σ and type at most τ. Summarizing, if $f(z_1, z_2)$ is of order σ and type at most τ, then (3.2.18) is valid for some χ with $\chi \leq \alpha$. If $\chi < \alpha$, the argument above leading to (3.2.21) shows that $f(z_1, z_2)$ would be of order σ and type less than τ, a contradiction. Thus $\alpha = \chi$ and the circle of reasoning is complete for the converse as well. This completes the proof of Theorem 3.2.3. □

3.3 Growth Characterizations in Banach spaces

In this section, we have obtained growth characterizations of order and type of entire functions of two complex variables on certain Banach spaces in terms of approximation errors. Now we prove

Theorem 3.3.1. *Let* $f(z_1, z_2) = \sum_{m,n=0}^{\infty} a_{m,n}\, z_1^m\, z_2^n$, *then the entire function*

$f(z_1, z_2) \in B(p, q, \kappa)$ *is of finite order* ρ*, if and only if*

$$(3.3.1) \qquad \rho = \limsup_{m,n \to \infty} \frac{\ln (m^m n^n)}{-\ln E_{m,n}(f, B(p, q, \kappa))}.$$

Proof. We prove the above result in two steps. First we consider the space $\mathbf{B}(p, q, \kappa)$, $q = 2$, $0 < p < 2$ and $\kappa \geq 1$. Let $f(z_1, z_2) \in \mathbf{B}(p, q, \kappa)$ of order ρ. From Theorem A, for any $\epsilon > 0$, there exists a natural number $n_0 = n_0(\epsilon)$ such that

$$(3.3.2) \qquad |a_{m,n}| \leq m^{-m/\rho+\epsilon}\, n^{-n/\rho+\epsilon} \quad m, n > n_0.$$

We denote the partial sum of the Taylor series of a function $f(z_1, z_2)$ by $T_{m,n}(f, z_1, z_2) = \sum_{j_1=0}^{m} \sum_{j_2=0}^{n} a_{j_1,j_2} z_1^{j_1} z_2^{j_2}$. We write

$$(3.3.3) \qquad E_{m,n}(f, \mathbf{B}(p, 2, \kappa)) = \|f - T_{m,n}(f)\|_{p,2,\kappa}$$

$$= \left\{ \int_0^1 \int_0^1 \{(1 - r_1)\,(1 - r_2)\}^{\kappa(1/p-1/2)-1} \left(\sum_{j_1} \sum_{j_2} r_1^{2j_1}\, r_2^{2j_2} |a_{j_1,j_2}|^2 \right)^{\kappa/2} dr_1 dr_2 \right\}^{1/\kappa}$$

where

$$\sum_{j_1} \sum_{j_2} r_1^{2j_1}\, r_2^{2j_2} |a_{j_1,j_2}|^2 = S_1 + S_2 + \sum_{j_1=m+1}^{\infty} \sum_{j_2=n+1}^{\infty} r_1^{2j_1}\, r_2^{2j_2} |a_{j_1,j_2}|^2,$$

$$S_1 = \sum_{j_1=0}^{m} \sum_{j_2=n+1}^{\infty} r_1^{2j_1}\, r_2^{2j_2} |a_{j_1,j_2}|^2$$

$$S_2 = \sum_{j_1=m+1}^{\infty} \sum_{j_2=0}^{n} r_1^{2j_1}\, r_2^{2j_2} |a_{j_1,j_2}|^2.$$

Since S_1, S_2 are bounded and $r_1, r_2 < 1$, the above expression (3.3.3) becomes

$$E_{m,n}(f, \mathbf{B}(p, 2, \kappa)) \leq C \left\{ \int_0^1 \{(1 - r)^{\kappa(1/p-1/2)-1}\} r^{(s+1)\kappa} dr \right\} \left\{ \sum_{j_1=m+1}^{\infty} \sum_{j_2=n+1}^{\infty} |a_{j_1,j_2}|^2 \right\}^{1/2}$$

where

$$\left\{\int_0^1 \{(1-r)^{\kappa(1/p-1/2)-1}\} r^{(s+1)\kappa} dr\right\} = \left\{\int_0^1 \{(1-r_1)^{\kappa(1/p-1/2)-1}\} r_1^{(m+1)\kappa} dr_1\right\} \times$$
$$\left\{\int_0^1 \{(1-r_2)\}^{\kappa(1/p-1/2)-1}\, r_2^{(n+1)\kappa} dr_2\right\}.$$

Therefore,

$$(3.3.4) \quad E_{m,n}(f, \mathbf{B}(p,2,\kappa)) \leq C\ B[m,p,2,\kappa] B[n,p,2,\kappa] \left\{ \sum_{j_1=m+1}^{\infty} \sum_{j_2=n+1}^{\infty} |a_{j_1,j_2}|^2 \right\}^{1/2}$$

where C is a constant and $B(a,b)$ $(a,b>0)$ denotes the beta function.

By using (3.3.2), we have

$$\begin{aligned}
\sum_{j_1=m+1}^{\infty} \sum_{j_2=n+1}^{\infty} |a_{j_1,j_2}|^2 &\leq \sum_{j_1=m+1}^{\infty} \sum_{j_2=n+1}^{\infty} j_1^{-\frac{2j_1}{\rho+\epsilon}}\, j_2^{-\frac{2j_2}{\rho+\epsilon}} \\
&\leq \sum_{j_1=m+1}^{\infty} j_1^{-\frac{2j_1}{\rho+\epsilon}} \sum_{j_2=n+1}^{\infty} j_2^{-\frac{2j_2}{\rho+\epsilon}} \\
&\leq O(1)\ (m+1)^{-2(m+1)/\rho+\epsilon}\ (n+1)^{-2(n+1)/\rho+\epsilon}.
\end{aligned}$$

Using above inequality in (3.3.4), we have

$$E_{m,n}(f, \mathbf{B}(p,2,\kappa)) \ \leq\ C\ B[m,p,2,\kappa] B[n,p,2,\kappa]\ (m+1)^{-(m+1)/\rho+\epsilon}\ (n+1)^{-(n+1)/\rho+\epsilon}.$$

$$\Rightarrow\ \rho+\epsilon \geq \frac{\ln\left[(m+1)^{(m+1)}\ (n+1)^{(n+1)}\right]}{-\ln\{E_{m,n}(f,\mathbf{B}(p,2,\kappa))\} + \ln\{B[m,p,2,\kappa]\} + \ln\{B[n,p,2,\kappa]\}}.$$

Now

$$B[(n+1)\kappa+1;\kappa(1/p-1/2)] = \frac{\Gamma((n+1)\kappa+1)\Gamma(\kappa(1/p-1/2))}{\Gamma((n+1/2+1/p)\kappa+1)}.$$

Hence

$$B[(n+1)\kappa+1;\kappa(1/p-1/2)] \simeq \frac{e^{-[(n+1)\kappa+1]}[(n+1)\kappa+1]^{(n+1)\kappa+3/2}\Gamma(1/p-1/2)}{e^{[(n+1/2+1/p)\kappa+1]}[(n+1/2+1/p)\kappa+1]^{(n+1/2+1/p)\kappa+3/2}}.$$

Thus

$$(3.3.5) \qquad \{B[(n+1)\kappa+1;\kappa(1/p-1/2)]\}^{1/(n+1)} \cong 1.$$

Proceeding to limits, we obtain

$$(3.3.6) \qquad \rho \geq \limsup_{m,n \to \infty} \frac{\ln(m^m\, n^n)}{-\ln\{E_{m,n}(f, \mathbf{B}(p,2,\kappa))\}}.$$

For the reverse inequality, since from the right hand side of the inequality (3.3.4), we have

$$(3.3.7) \qquad |a_{m+1,n+1}|\; B[m,p,2,\kappa]\; B[n,p,2,\kappa] \leq E_{m,n}(f, \mathbf{B}(p,2,\kappa)),$$

we have

$$\frac{\ln(m^m\, n^n)}{-\ln E_{m,n}(f, \mathbf{B}(p,2,\kappa))} \geq \frac{\ln(m^m\, n^n)}{-\ln|a_{m+1,n+1}| + \ln\{B[m,p,2,\kappa]\} + \ln\{B[n,p,2,\kappa]\}}.$$

Proceeding to limits, we obtain

$$(3.3.8) \qquad \limsup_{m,n \to \infty} \frac{\ln(m^m\, n^n)}{-\ln E_{m,n}(f, \mathbf{B}(p,2,\kappa))} \geq \rho.$$

From (3.3.6) and (3.3.8), we get the required result.

In the second step, for the general case $\mathbf{B}(p,q,\kappa)$, $q \neq 2$, we have

$$(3.3.9) \qquad E_{m,n}(f, \mathbf{B}(p,q,\kappa)) \leq \|f - T_{m,n}(f)\|_{p,q,\kappa}$$

$$= \left\{ \int_0^1 \int_0^1 \{(1-r_1)\,(1-r_2)\}^{\kappa(1/p-1/q)-1} \left(\sum_{j_1} \sum_{j_2} r_1^{qj_1}\, r_2^{qj_2} |a_{j_1,j_2}|^q \right)^{\kappa/q} dr_1 dr_2 \right\}^{1/\kappa},$$

where

$$\sum_{j_1}\sum_{j_2} r_1^{j_1}\, r_2^{2j_2}|a_{j_1,j_2}|^2 = S_1 \;+\; S_2 + \sum_{j_1=m+1}^{\infty}\;\sum_{j_2=n+1}^{\infty} r_1^{2j_1}\, r_2^{2j_2}|a_{j_1,j_2}|^2,$$

$$S_1 = \sum_{j_1=0}^{m}\;\sum_{j_2=n+1}^{\infty} r_1^{2j_1}\, r_2^{2j_2}|a_{j_1,j_2}|^2$$

$$S_2 = \sum_{j_1=m+1}^{\infty}\;\sum_{j_2=0}^{n} r_1^{2j_1}\, r_2^{2j_2}|a_{j_1,j_2}|^2.$$

Since S_1, S_2 are bounded and $r_1, r_2 \;\; < 1$, therefore the above expression (3.3.9) becomes

$$E_{m,n}(f, \mathbf{B}(p,q,\kappa)) \leq C' \left\{\int_0^1 \{(1-r)^{\kappa(1/p-1/q)-1}\} r^{(s+1)\kappa} dr\right\} \left\{\sum_{j_1=m+1}^{\infty}\;\sum_{j_2=n+1}^{\infty} |a_{j_1,j_2}|^q\right\}^{1/q},$$

where

$$\left\{\int_0^1 \{(1-r)^{\kappa(1/p-1/q)-1}\} r^{(s+1)\kappa} dr\right\} = \left\{\int_0^1 \{(1-r_1)^{\kappa(1/p-1/q)-1}\} r_1^{(m+1)\kappa} dr_1\right\} \times$$
$$\left\{\int_0^1 \{(1-r_2)\}^{\kappa(1/p-1/q)-1}\; r_2^{(n+1)\kappa} dr_2\right\}.$$

Therefore

(3.3.10)
$$E_{m,n}(f, \mathbf{B}(p,q,\kappa)) \leq C'\; B[m,p,q,\kappa]\; B[n,p,q,\kappa] \left\{\sum_{j_1=m+1}^{\infty}\;\sum_{j_2=n+1}^{\infty} |a_{j_1,j_2}|^q\right\}^{1/q},$$

where C' is constant and $B[m,p,q,\kappa]$ is Euler's integral of first kind. By using (3.3.2), we have

$$\sum_{j_1=m+1}^{\infty}\;\sum_{j_2=n+1}^{\infty} |a_{j_1,j_2}|^q \;\leq\; \sum_{j_1=m+1}^{\infty} j_1^{\frac{-q\,j_1}{(\rho+\epsilon)}} \sum_{j_2=n+1}^{\infty} j_2^{\frac{-q\,j_2}{(\rho+\epsilon)}}$$
$$\leq\; O(1)\; (m+1)^{\frac{-q\,(m+1)}{(\rho+\epsilon)}}\; (n+1)^{\frac{-q\,(n+1)}{(\rho+\epsilon)}}.$$

Using above inequality in (3.3.10), we have

$$E_{m,n}(f, \mathbf{B}(p,q,\kappa)) \leq C' \; B[m,p,q,\kappa] \; B[n,p,q,\kappa] \; (m+1)^{-(m+1)/\rho+\epsilon} \; (n+1)^{-(n+1)/\rho+\epsilon}.$$

$$\Rightarrow \qquad \rho+\epsilon \geq \frac{\ln\left[(m+1)^{m+1} \; (n+1)^{n+1}\right]}{-\ln E_{m,n}(f,\mathbf{B}(p,q,\kappa)) + \ln\{B[m,p,q,\kappa]\} + \ln\{B[n,p,q,\kappa]\}}.$$

Proceeding to limits, we obtain

$$\text{(3.3.11)} \qquad \rho \geq \limsup_{m,n\to\infty} \frac{\ln(m^m \; n^n)}{-\ln E_{m,n}(f,\mathbf{B}(p,q,\kappa))}.$$

Let $0 < p < q < 2$, and $\kappa, q \geq 1$. Since

$$E_{m,n}(f,\mathbf{B}(p_1,q_1,\kappa_1)) \leq 2^{1/q - 1/q_1} \left[\kappa \; (1/p - 1/q)\right]^{1/\kappa - 1/\kappa_1} E_{m,n}(f,\mathbf{B}(p,q,\kappa)),$$

where $p_1 = p$, $q_1 = 2$ and $\kappa_1 = \kappa$, and the condition (3.3.1) is already proved for the space $\mathbf{B}(p,2,\kappa)$, we have

$$\text{(3.3.12)} \quad \limsup_{m,n\to\infty} \frac{\ln(m^m \; n^n)}{-\ln E_{m,n}(f,\mathbf{B}(p,q,\kappa))} \geq \limsup_{m,n\to\infty} \frac{\ln(m^m \; n^n)}{-\ln E_{m,n}(f,\mathbf{B}(p,2,\kappa))} = \rho.$$

Now let $0 < p \leq 2 < q$. Since

$$M_2(f,r_1,r_2) \leq M_q(f,r_1,r_2), \quad 0 < r_1, r_2 < 1,$$

therefore

$$\begin{aligned}\text{(3.3.13)} \quad E_{m,n}(f,\mathbf{B}(p,q,\kappa)) \; &\geq \; \left\{\int_0^1 \int_0^1 \{(1-r_1)\;(1-r_2)\}^{\kappa(1/p-1/q)-1} \; Q dr_1 dr_2\right\}^{1/\kappa} \\ &\geq \; |a_{m+1,n+1}| \; B[m,p,q,\kappa] \; B[n,p,q,\kappa],\end{aligned}$$

where $Q = \inf\,[M_2^{\kappa}(f-p; r_1, r_2) : p \in P]$. Hence we have

$$\frac{\ln(m^m\, n^n)}{-\ln E_{m,n}(f, \mathbf{B}(p,q,\kappa))} \geq \frac{\ln(m^m\, n^n)}{-\ln|a_{m+1,n+1}| + \ln\{B[m,p,q,\kappa]\} + \ln\{B[n,p,q,\kappa]\}}.$$

Now proceeding to limits, we obtain

$$\limsup_{m,n \to \infty} \frac{\ln(m^m\, n^n)}{-\ln E_{m,n}(f, \mathbf{B}(p,q,\kappa))} \geq \rho. \tag{3.3.14}$$

From (3.3.11) and (3.3.14), we get the required result. This completes the proof of Theorem 3.3.1. □

Theorem 3.3.2. *Let $f(z_1, z_2) = \sum_{m,n=0}^{\infty} a_{m,n} z_1^m z_2^n$, then the entire function $f(z_1, z_2) \in H_q$ is of finite order ρ, if and only if*

$$\rho = \limsup_{m,n \to \infty} \frac{\ln(m^m\, n^n)}{-\ln E_{m,n}(f, H_q)}. \tag{3.3.15}$$

Proof. Let $f(z_1, z_2) = \sum_{m,n=0}^{\infty} a_{m,n} z_1^m z_2^n \in H_q$ be an entire transcendental function. Since f is entire, we have

$$\lim_{m,n \to \infty} \sqrt[(m+n)]{|a_{m,n}|} = 0, \tag{3.3.16}$$

and $f \in H_q$, therefore

$$M_q(f; r_1, r_2) < \infty,$$

and $f(z_1, z_2) \in \mathbf{B}(p,q,\kappa)$, $0 < p < q \leq \infty$; $q, \kappa \geq 1$. By (3.1.3) we obtain

$$E_{m,n}(f, \mathbf{B}(q/2, q, q)) \leq \varsigma_q E_{m,n}(f, H_q), \quad 1 \leq q < \infty. \tag{3.3.17}$$

where ς_q is a constant independent of m , n and f. In the case of space H_∞,

$$(3.3.18) \qquad E_{m,n}(f, \mathrm{B}(p, \infty, \infty)) \leq E_{m,n}(f, H_\infty), \quad 0 < p < \infty.$$

From (3.3.17), we have

$$(3.3.19) \qquad \begin{aligned} \xi(f) &= \limsup_{m,n \to \infty} \frac{\ln(m^m\, n^n)}{-\ln E_{m,n}(f, H_q)} \\ &\geq \limsup_{m,n \to \infty} \frac{\ln(m^m\, n^n)}{-\ln E_{m,n}(f, \mathrm{B}(q/2, q, q))} \\ &\geq \rho, \qquad 1 \leq q < \infty, \end{aligned}$$

and using estimate (3.3.18) we prove inequality (3.3.19) for the case $q = \infty$.

For the reverse inequality

$$(3.3.20) \qquad \xi(f) \leq \rho,$$

since

$$E_{m,n}(f, H_q) \leq O(1) \sum_{j_1=m+1}^{\infty} \sum_{j_2=n+1}^{\infty} |a_{j_1,j_2}(f)|,$$

using (3.3.2), we have

$$\begin{aligned} E_{m,n}(f, H_q) &\leq O(1) \sum_{j_1=m+1}^{\infty} \sum_{j_2=n+1}^{\infty} j_1^{-\frac{j_1}{\rho+\epsilon}}\, j_2^{-\frac{j_2}{\rho+\epsilon}} \\ &\leq O(1) \sum_{j_1=m+1}^{\infty} j_1^{-\frac{j_1}{\rho+\epsilon}} \sum_{j_2=n+1}^{\infty} j_2^{-\frac{j_2}{\rho+\epsilon}} \\ &\leq O(1)\,(m+1)^{-(m+1)/\rho+\epsilon}\,(n+1)^{-(n+1)/\rho+\epsilon}. \end{aligned}$$

$$\Rightarrow \rho + \epsilon \geq \frac{\ln\left[(m+1)^{(m+1)}\,(n+1)^{(n+1)}\right]}{-\ln\left[E_{m,n}(f, H_q)\right]}.$$

Proceeding to limits, then we get (3.3.20). From (3.3.19) and (3.3.20) we obtain the required result.

Now we prove the sufficiency part. Assume that the condition (3.3.15) is satisfied. Then it follows that $\ln [1/E_{m,n}(f, H_q)]^{1/(m+n)} \to \infty$ as $m, n \to \infty$. This yields

$$\lim_{m,n \to \infty} \sqrt[(m+n)]{E_{m,n}(f, H_q)} = 0.$$

This relation and estimate $|a_{m+1,n+1}(f)| \le E_{m,n}(f, H_q)$ yield the relation (3.3.16). This means that $f(z_1, z_2) \in H_q$ is an entire transcendental function. This completes the proof of Theorem 3.3.2. □

Theorem 3.3.3. *Let $f(z_1, z_2) = \sum_{m,n=0}^{\infty} a_{m,n}\, z_1^m\, z_2^n$, then the entire function $f(z_1, z_2) \in B(p, q, \kappa)$ of finite order ρ, is of type τ if and only if*

$$(3.3.21) \qquad \tau = \frac{1}{e\,\rho} \limsup_{m,n \to \infty} \{m^m\, n^n\, E_{m,n}^{\rho}(f, B(p, q, \kappa))\}^{\frac{1}{m+n}}.$$

Proof. We prove the above result in two steps. First we consider the space $\mathbf{B}(p, q, \kappa)$, $q = 2$, $0 < p < 2$ and $\kappa \ge 1$. Let $f(z) \in \mathbf{B}(p, q, \kappa)$ of order ρ. From Theorem B, for any $\epsilon > 0$, there exists a natural number $n_0 = n_0(\epsilon)$ such that

$$(3.3.22) \qquad |a_{m,n}| \le m^{-m/\rho}\, n^{-n/\rho}\, [e\, \rho\, (\tau + \epsilon)]^{\frac{m+n}{\rho}}.$$

By using (3.3.22), we have

$$\begin{aligned} \sum_{j_1=m+1}^{\infty} \sum_{j_2=n+1}^{\infty} |a_{j_1,j_2}|^2 &\le \sum_{j_1=m+1}^{\infty} \sum_{j_2=n+1}^{\infty} j_1^{-\frac{2j_1}{\rho}}\, j_2^{-\frac{2j_2}{\rho}}\, [e\rho(\tau+\epsilon)]^{\frac{2(j_1+j_2)}{\rho}} \\ &\le \sum_{j_1=m+1}^{\infty} j_1^{-\frac{2j_1}{\rho}} [e\rho(\tau+\epsilon)]^{\frac{2j_1}{\rho}} \sum_{j_2=n+1}^{\infty} j_2^{-\frac{2j_2}{\rho+\epsilon}} [e\rho(\tau+\epsilon)]^{\frac{2j_2}{\rho}} \\ &\le O(1)\, (m+1)^{-2(m+1)/\rho}\, (n+1)^{-2(n+1)/\rho} [e\rho(\tau+\epsilon)]^{\frac{2(m+n+2)}{\rho}}. \end{aligned}$$

Using above inequality in (3.3.4), we have

$$E^{\rho}_{m,n}(f, \mathbf{B}(p,2,\kappa)) \leq C^{\rho}\ B^{\rho}[m,p,2,\kappa]\ B^{\rho}[n,p,2,\kappa]\ Y\ [e\ \rho\ (\tau+\epsilon)]^{(m+n+2)},$$

where $Y = (m+1)^{-(m+1)}\ (n+1)^{-(n+1)}$. Proceeding to limits, we have

$$\frac{1}{e\ \rho} \limsup_{m,n\to\infty}\ \{m^m\ n^n\ E^{\rho}_{m,n}(f, \mathbf{B}(p,2,\kappa))\}^{\frac{1}{m+n}} \leq \tau. \tag{3.3.23}$$

For the reverse inequality, since from the right hand side of (??),

$$|a_{m+1,n+1}|\ B[m,p,2,\kappa]\ B[n,p,2,\kappa] \leq E_{m,n}(f, \mathbf{B}(p,2,\kappa))$$

we have

$$m^{m/(m+n)}\ n^{n/(m+n)}\ |a_{m+1,n+1}|^{\rho/(m+n)}\ B^{\frac{\rho}{(m+n)}}[m,p,2,\kappa]\ B^{\frac{\rho}{(m+n)}}[n,p,2,\kappa] \leq \{E^{\rho}_{m,n}\ m^m\ n^n\}^{1/(m+n)}.$$

Proceeding to limits, we obtain

$$\tau \leq \frac{1}{e\ \rho} \limsup_{m,n\to\infty}\ \{m^m\ n^n\ E^{\rho}_{m,n}(f, \mathbf{B}(p,2,\kappa))\}^{\frac{1}{m+n}}. \tag{3.3.24}$$

From (3.3.23) and (3.3.24), we get the required result.

In the second step, for the general case $\mathbf{B}(p,q,\kappa)$, $q \neq 2$. By using (3.3.22), we estimate

$$\begin{aligned}\sum_{j_1=m+1}^{\infty}\sum_{j_2=n+1}^{\infty} |a_{j_1,j_2}|^q &\leq \sum_{j_1=m+1}^{\infty}\sum_{j_2=n+1}^{\infty} j_1^{-\frac{q j_1}{\rho}}\ j_2^{-\frac{q j_2}{\rho}}\ [e\rho(\tau+\epsilon)]^{\frac{q(j_1+j_2)}{\rho}} \\ &\leq \sum_{j_1=m+1}^{\infty} j_1^{-\frac{q j_1}{\rho}}[e\rho(\tau+\epsilon)]^{\frac{q j_1}{\rho}} \sum_{j_2=n+1}^{\infty} j_2^{-\frac{q j_2}{\rho+\epsilon}}[e\rho(\tau+\epsilon)]^{\frac{q j_2}{\rho}}\end{aligned}$$

$$\leq O(1)\ (m+1)^{-q\,(m+1)/\rho}\ (n+1)^{-q\,(n+1)/\rho}[e\rho(\tau+\epsilon)]^{\frac{q\,(m+n+2)}{\rho}}.$$

Using above inequality in (3.3.10), we have

$$E^{\rho}_{m,n}(f, \mathbf{B}(p,q,\kappa)) \leq (C')^{\rho}\ B^{\rho}[m,p,q,\kappa]\ B^{\rho}[n,p,q,\kappa]\ Y\ [e\ \rho\ (\tau+\epsilon)]^{(m+n+2)}.$$

where $Y\ =\ (m+1)^{-(m+1)}\ (n+1)^{-(n+1)}$. Now proceeding to limits, since ϵ is arbitrary, we have

$$\text{(3.3.25)}\qquad \frac{1}{e\ \rho}\ \limsup_{m,n\ \to\ \infty}\ \{m^m\ n^n\ E^{\rho}_{m,n}(f, \mathbf{B}(p,q,\kappa))\}^{\frac{1}{m+n}} \leq \tau.$$

Let $0 < p < q < 2$, and $\kappa, q \geq 1$. Since

$$E_{m,n}(f, \mathbf{B}(p_1, q_1, \kappa_1)) \leq 2^{1/q\,-\,1/q_1}\ [\kappa\ (1/p\ -\ 1/q)]^{1/\kappa\,-\,1/\kappa_1}\ E_{m,n}(f, \mathbf{B}(p,q,\kappa))$$

where $p_1 = p,\ q_1 = 2$ and $\kappa_1 = \kappa$, and the condition (3.3.21) has already been proved for the space $\mathbf{B}(p, 2, \kappa)$, we have

$$\limsup_{m,n\ \to\ \infty}\ \{m^m\ n^n\ E^{\rho}_{m,n}(f, \mathbf{B}(p,q,\kappa))\}^{\frac{1}{m+n}} \geq \limsup_{m,n\ \to\ \infty} \{m^m\ n^n\ E^{\rho}_{m,n}(f, \mathbf{B}(p,2,\kappa))\}^{\frac{1}{m+n}} = \tau.$$

Now let $0 < p\ \leq\ 2 < q$. Since, in this case we have

$$M_2(f, r_1, r_2)\ \leq\ M_q(f, r_1, r_2), \quad 0 < r_1, r_2 < 1,$$

therefore

$$\begin{aligned} \limsup_{m,n\ \to\ \infty} \{m^m\ n^n\ E^{\rho}_{m,n}(f, \mathbf{B}(p,q,\kappa))\}^{\frac{1}{m+n}} &\geq \limsup_{m,n\ \to\ \infty} \{m^m\ n^n\ |a_{m,n}|^{\rho}\}^{\frac{1}{m+n}} \\ \text{(3.3.26)} \qquad &= e\rho\tau. \end{aligned}$$

From (3.3.25) and (3.3.26), we get the required result. This completes the proof of Theorem 3.3.3. □

Theorem 3.3.4. *Let* $f(z_1, z_2) = \sum_{m,n=0}^{\infty} a_{m,n}\, z_1^m\, z_2^n$, *then the entire function* $f(z_1, z_2) \in H_q$ *having finite order* ρ *is of type* τ *if and only if*

$$\tau = \frac{1}{e\,\rho} \limsup_{m,n \to \infty} \{m^m\, n^n\, E^{\rho}_{m,n}(f, H_q)\}^{\frac{1}{m+n}}. \tag{3.3.27}$$

Proof. Since $f(z_1, z_2) = \sum_{m,n=0}^{\infty} a_{m,n}\, z_1^m z_2^n$ is an entire transcendental function, we have

$$\lim_{m,n \to \infty} \sqrt[m+n]{|a_{m,n}|} = 0. \tag{3.3.28}$$

Therefore $f(z_1, z_2) \in \mathbf{B}(p, q, \kappa)$, $0 < p < q \leq \infty$; $q, \kappa \geq 1$. We have

$$\begin{aligned} \xi(f) &= \frac{1}{e\,\rho} \limsup_{m,n \to \infty} \{m^m\, n^n\, E^{\rho}_{m,n}(f, H_q)\}^{\frac{1}{m+n}} \\ &\geq \frac{1}{e\,\rho} \limsup_{m,n \to \infty} \{m^m\, n^n\, E^{\rho}_{m,n}(f, \mathbf{B}(q/2, q, q))\}^{\frac{1}{m+n}} = \tau \end{aligned} \tag{3.3.29}$$

for $1 \leq q < \infty$. Using estimate (3.3.18) we prove inequality (3.3.29) in the case $q = \infty$. For the reverse inequality

$$\xi(f) \leq \tau, \tag{3.3.30}$$

we have

$$E_{m,n}(f, H_q) \leq \sum_{j_1=m+1}^{\infty} \sum_{j_2=n+1}^{\infty} |a_{j_1,j_2}(f)|.$$

Using (3.3.22), we have

$$E^{\rho}_{m,n}(f, H_q) \leq O(1)\,(m+1)^{-(m+1)}\,(n+1)^{-(n+1)}\,[e\,\rho\,(\tau + \epsilon)]^{(m+n+2)}$$

$$\Rightarrow \qquad \tau + \epsilon \geq \frac{1}{e\,\rho} \{(m+1)^{(m+1)} (n+1)^{(n+1)} E^{\rho}_{m,n}(f, H_q)\}^{\frac{1}{(m+n+2)}}.$$

Proceeding to limits, we have

$$\tau \geq \frac{1}{e\,\rho} \limsup_{m,n \to \infty} \{m^m n^n E^{\rho}_{m,n}(f, H_q)\}^{\frac{1}{m+n}}. \tag{3.3.31}$$

From (3.3.29) and (3.3.31), we get the required result.

Now we prove the sufficiency part. Assume that the condition (3.3.27) is satisfied. Then it follows that $\{E^{\rho}_{m,n}(f, H_q)\}^{1/(m+n)} \to 0$ as $m, n \to \infty$. This yields

$$\lim_{m,n \to \infty} \sqrt[(m+n)]{E_{m,n}(f, H_q)} = 0.$$

This relation and estimate $|a_{m+1,n+1}(f)| \leq E_{m,n}(f, H_q)$ yield the inequality (3.3.28). This implies that $f(z_1, z_2) \in H_q$ is an entire transcendental function. This completes the proof of Theorem 3.3.4. □

Chapter 4

Approximation of Entire Functions of Two Complex Variables Over Jordan Domains

In this present Chapter, we study the polynomial approximation of entire functions of two complex variables over Jordan domains by using Faber polynomials. The coefficient characterizations of order and type of entire functions of two complex variables have been obtained in terms of the approximation errors.

4.1 Introduction

Let Γ_1 and Γ_2 be given Jordan curves in the complex plane C and D_j, E_j, j = 1,2., be the interior and exterior respectively, of Γ_j. Let φ_j map E_j conformally onto $\{w_j : |w_j| > 1\}$ such that $\varphi_j(\infty) = \infty$ and $\varphi_j'(\infty) > 0$. Then by [15], for sufficiently large $|z_j|$, $\varphi_j(z_j)$ can be expressed as

(4.1.1) $$w_1 = \varphi_1(z_1) = \frac{z_1}{d_1} + c_0 + \frac{c_1}{z_1} + \frac{c_2}{z_1^2} + ...$$

(4.1.2) $$w_2 = \varphi_2(z_2) = \frac{z_2}{d_2} + c_0' + \frac{c_1'}{z_2} + \frac{c_2'}{z_2^2} + ...$$

where $d_1, d_2 > 0$. Let us put $D = D_1 \times D_2$ and $E = E_1 \times E_2$ in C^2 and let the function φ map E conformally onto the unit bidisc $U = \{|w_1| > 1, |w_2| > 1\}$ such that

$\varphi(z_1, z_2) = \varphi_1(z_1)\ \varphi_2(z_2)$ satisfies the conditions $\varphi(\infty, \infty) = \infty$ and $\varphi'(\infty, \infty) > 0$. Then for sufficiently large $|z_1|, |z_2|$, $\varphi(z_1, z_2)$ can be expressed as

$$(4.1.3) \qquad w_1 w_2 \;=\; \varphi(z_1, z_2) = \frac{z_1}{d_1}\frac{z_2}{d_2} + \sum_{m,n=0}^{\infty} \frac{c_{m,n}}{z_1^m z_2^n}.$$

An arbitrary Jordan curve can be approximated from the inside as well as from the outside by analytic Jordan curves. Since Γ is analytic, φ is holomorphic on Γ as well. The m,n th Faber polynomial $F_{m,n}(z_1, z_2)$ of Γ is the principal part of $(\varphi(z_1, z_2))^{m+n}$ at (∞, ∞), so that

$$F_{m,n}(z_1, z_2) = \left(\frac{z_1}{d_1}\right)^m \left(\frac{z_2}{d_2}\right)^n + \ldots.$$

Following Faber [12] for the one variable case, we can easily see that as $m, n \to \infty$,

$$(4.1.4) \qquad F_{m,n}(z_1, z_2) \sim (\varphi_1(z_1))^m (\varphi_2(z_2))^n$$

uniformly for $z_1 \in E_1, z_2 \in E_2$ and

$$(4.1.5) \qquad \lim_{m,n\to\infty} \left(\max_{z_1, z_2 \in \Gamma} |F_{m,n}(z_1, z_2)| \right)^{1/(m+n)} = 1.$$

A function f holomorphic in D can be represented by its Faber series

$$(4.1.6) \qquad f(z_1, z_2) \;= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} a_{m,n} F_{m,n}(z_1, z_2)$$

where

$$a_{m,n} = \frac{1}{(2\pi i)^2} \int_{|w_1|=r_1} \int_{|w_2|=r_2} f(\varphi_1^{-1}(w_1), \varphi_2^{-1}(w_2)) w_1^{-(m+1)} w_2^{-(n+1)} dw_1 dw_2$$

and $r_1, r_2 < 1$ are sufficiently close to 1 so that for j = 1, 2., φ_j^{-1} are holomorphic and univalent in $|w_j| \geq r_j$ respectively, the series converging uniformly on compact subsets of D. Let $M(r_1, r_2) = \max_{|z_j|=r_j} |f(z_1, z_2)|$, $j = 1, 2$ be the maximum modulus of $f(z_1, z_2)$. The growth of an entire function $f(z_1, z_2)$ is measured in terms of its order ρ and type τ defined as under [7]

$$(4.1.7) \qquad \limsup_{r_1, r_2 \to \infty} \frac{\ln \ln M(r_1, r_2)}{\ln (r_1 r_2)} = \rho,$$

$$(4.1.8) \qquad \limsup_{r_1, r_2 \to \infty} \frac{\ln M(r_1, r_2)}{r_1^\rho + r_2^\rho} = \tau,$$

for $0 < \rho < \infty$.

Let $L^p(D)$ denote the set of functions f holomorphic in D and such that

$$\|f\|_{L^p(D)} = \left(\frac{1}{A} \int \int_D |f(z_1, z_2)|^p dx_1 dy_1 dx_2 dy_2 \right)^{1/p} < \infty$$

where A is the area of D. For $f \in L^p(D)$, set

$$E_{m,n}^p = E_{m,n}^p(f; D) = \min_{\pi_{m,n}} \|f - \pi_{m,n}\|_{L^p(D)}$$

where $\pi_{m,n}$ is an arbitrary polynomial of degree at most $m + n$.

4.2 Order and Type

In this section we obtain the growth characterizations in terms of the coefficients $\{a_{m,n}\}$ of the Faber series (4.1.6). We first prove

Theorem 4.2.1. *The function f is the restriction to domain D of an entire function of finite order ρ if and only if*

$$(4.2.1) \qquad \mu = \limsup_{m,n \to \infty} \frac{\ln m^m n^n}{-\ln |a_{m,n}|}.$$

is finite and then the order ρ of f is equal to μ.

Proof. Let $f(z_1, z_2) = \sum_{m,n=0}^{\infty} a_{m,n} F_{m,n}(z_1, z_2)$ be an entire function of two complex variables in z_1 and z_2, where

$$a_{m,n} = \frac{1}{(2\pi i)^2} \int_{|w_1|=r_1} \int_{|w_2|=r_2} f(\varphi_1^{-1}(w_1), \varphi_2^{-1}(w_2)) w_1^{-(m+1)} w_2^{-(n+1)} dw_1 dw_2$$

with arbitrarily large r_1, r_2. Then

$$|a_{m,n}| = \left| \frac{1}{(2\pi i)^2} \int_{|w_1|=r_1} \int_{|w_2|=r_2} f(\varphi_1^{-1}(w_1), \varphi_2^{-1}(w_2)) w_1^{-(m+1)} w_2^{-(n+1)} dw_1 dw_2 \right|.$$

Since from (4.1.1) and (4.1.2), we have

$$\varphi_1(z_1) = w_1 \implies \varphi_1^{-1}(w_1) = z_1$$

$$\varphi_2(z_2) = w_2 \implies \varphi_2^{-1}(w_2) = z_2.$$

$$\therefore \; |a_{m,n}| \leq M(r_1, r_2)\, r_1^{-m}\, r_2^{-n} \tag{4.2.2}$$

where $M(r_1, r_2) = \max_{|z_t| \leq r_t} |f(z_1, z_2)|$, $t = 1, 2$.

Now we want to show that $\rho \geq \mu$. If $\mu = 0$ then $\rho \geq \mu$, since ρ is not negative. Let $\epsilon > 0$ and $\epsilon < \mu < \infty$. Then from (4.2.1), we have

$$-(\mu - \epsilon) \ln |a_{m,n}| \leq \ln m^m n^n$$

$$\Rightarrow \; \ln |a_{m,n}| \geq -\frac{1}{(\mu - \epsilon)} (m \ln m + n \ln n)$$

for an infinite sequence of values of m and n. From (4.2.2), we have

$$\ln M(r_1, r_2) \;\; \geq \;\; \ln |a_{m,n}| + \ln (r_1^m r_2^n)$$

$$\geq -\frac{1}{(\mu-\epsilon)}(m\ln m + n\ln n) + m\ln r_1 + n\ln r_2$$
$$= m\left(\ln r_1 - \frac{1}{(\mu-\epsilon)}\ln m\right) + n\left(\ln r_2 - \frac{1}{(\mu-\epsilon)}\ln n\right).$$

After minimizing the right hand side of above inequality, we have

$$r_1 = (em)^{\frac{1}{(\mu-\epsilon)}}, \quad r_2 = (en)^{\frac{1}{(\mu-\epsilon)}}.$$

Substitute r_1 and r_2 in the above inequality, then we have

$$\ln M(r_1, r_2) \geq \frac{m}{(\mu-\epsilon)} + \frac{n}{(\mu-\epsilon)} = \frac{r_1^{\mu-\epsilon} + r_2^{\mu-\epsilon}}{e(\mu-\epsilon)}.$$

Since $\mu - \epsilon$ is independent of r_1 and r_2, therefore

$$\rho = \limsup_{r_1, r_2 \to \infty} \frac{\ln\ln M(r_1, r_2)}{\ln(r_1 r_2)} \geq \mu - \epsilon.$$

Since ϵ is arbitrary, therefore we have

(4.2.3) $$\rho \geq \mu.$$

Conversely, let

$$\limsup_{m,n\to\infty} \frac{\ln(m^m n^n)}{-\ln|a_{m,n}|} = \sigma.$$

Suppose $\sigma < \infty$. Then for every $\epsilon > 0$, $\exists\, X(\epsilon), Y(\epsilon)$ such that for all $m \geq X$ and $n \geq Y$, we have

$$|a_{m,n}| \leq K\, m^{-\frac{m}{\sigma+\epsilon}}\, n^{-\frac{n}{\sigma+\epsilon}}.$$

Since $f(z_1, z_2) = \sum_{m,n=0}^{\infty} a_{m,n}\, F_{m,n}(z_1, z_2)$, therefore

$$|f(z_1, z_2)| \leq K \sum_{m,n=0}^{\infty} m^{-\frac{m}{\sigma+\epsilon}}\, n^{-\frac{n}{\sigma+\epsilon}}\, |F_{m,n}(z_1, z_2)|.$$

By using (4.1.5), for all $z_1 \in E_1$ and $z_2 \in E_2$, we have

$$|F_{m,n}(z_1, z_2)| \leq K\ (|\varphi_1(z_1)|)^m(|\varphi_2(z_2)|)^n$$

and by using (4.1.1)and (4.1.2), for all sufficiently large $|z_1|$ and $|z_2|$, we have

$$|\varphi_1(z_1)| \ \leq \ \frac{|z_1|}{d_1 - \epsilon}, \qquad \text{and} \qquad |\varphi_2(z_2)| \ \leq \ \frac{|z_2|}{d_2 - \epsilon}.$$

By applying these inequalities, for all sufficiently large $|z_1|$ and $|z_2|$, we have

$$|f(z_1, z_2)| \leq K \sum_{m,n=0}^{\infty} m^{-\frac{m}{\sigma+\epsilon}}\, n^{-\frac{n}{\sigma+\epsilon}} \left(\frac{|z_1|}{d_1 - \epsilon}\right)^m \left(\frac{|z_2|}{d_2 - \epsilon}\right)^n.$$

Hence

$$\begin{aligned} M(r_1, r_2) \ \leq \ & \sum_{m=0}^{M}\sum_{n=0}^{N} m^{-\frac{m}{\sigma+\epsilon}}\, n^{-\frac{n}{\sigma+\epsilon}} \left(\frac{r_1}{d_1 - \epsilon}\right)^m \left(\frac{r_2}{d_2 - \epsilon}\right)^n \\ & + \sum_{m=M+1}^{\infty}\sum_{n=N+1}^{\infty} m^{-\frac{m}{\sigma+\epsilon}}\, n^{-\frac{n}{\sigma+\epsilon}} \left(\frac{r_1}{d_1 - \epsilon}\right)^m \left(\frac{r_2}{d_2 - \epsilon}\right)^n \\ & + \sum_{m=0}^{M}\sum_{n=N+1}^{\infty} m^{-\frac{m}{\sigma+\epsilon}}\, n^{-\frac{n}{\sigma+\epsilon}} \left(\frac{r_1}{d_1 - \epsilon}\right)^m \left(\frac{r_2}{d_2 - \epsilon}\right)^n \\ & + \sum_{m=M+1}^{\infty}\sum_{n=0}^{N} m^{-\frac{m}{\sigma+\epsilon}}\, n^{-\frac{n}{\sigma+\epsilon}} \left(\frac{r_1}{d_1 - \epsilon}\right)^m \left(\frac{r_2}{d_2 - \epsilon}\right)^n \end{aligned}$$

$$\begin{aligned} M(r_1, r_2) \ \leq \ & A\left(\frac{r_1}{d_1 - \epsilon}\right)^M \left(\frac{r_2}{d_2 - \epsilon}\right)^N \\ & + \sum_{m=M+1}^{\infty}\sum_{n=N+1}^{\infty} m^{-\frac{m}{\sigma+\epsilon}}\, n^{-\frac{n}{\sigma+\epsilon}} \left(\frac{r_1}{d_1 - \epsilon}\right)^m \left(\frac{r_2}{d_2 - \epsilon}\right)^n \\ & + B\left(\frac{r_1}{d_1 - \epsilon}\right)^M \sum_{n=N+1}^{\infty} n^{-\frac{n}{\sigma+\epsilon}} \left(\frac{r_2}{d_2 - \epsilon}\right)^n \\ & + C\left(\frac{r_2}{d_2 - \epsilon}\right)^N \sum_{m=M+1}^{\infty} m^{-\frac{m}{\sigma+\epsilon}} \left(\frac{r_1}{d_1 - \epsilon}\right)^m. \end{aligned} \tag{4.2.4}$$

Let $\sum_1$ be the part of the above double series (4.2.4), for which $m < (\frac{2r_1}{d_1-\epsilon})^{\sigma+\epsilon}$, $n < (\frac{2r_2}{d_2-\epsilon})^{\sigma+\epsilon}$. We estimate $\sum_1$ by taking the largest value of $(\frac{r_1}{d_1-\epsilon})^M\ (\frac{r_2}{d_2-\epsilon})^N$. Then

$$\begin{aligned}\sum_1 &= \sum_{m<(\frac{2r_1}{d_1-\epsilon})^{\sigma+\epsilon}} \sum_{n<(\frac{2r_2}{d_2-\epsilon})^{\sigma+\epsilon}} m^{-\frac{m}{\sigma+\epsilon}} n^{-\frac{n}{\sigma+\epsilon}} \left(\frac{r_1}{d_1-\epsilon}\right)^m \left(\frac{r_2}{d_2-\epsilon}\right)^n \\ &\leq \left(\frac{r_1}{d_1-\epsilon}\right)^{(\frac{2r_1}{d_1-\epsilon})^{\sigma+\epsilon}} \left(\frac{r_2}{d_2-\epsilon}\right)^{(\frac{2r_2}{d_2-\epsilon})^{\sigma+\epsilon}} \sum_{m,n} m^{-\frac{m}{\sigma+\epsilon}} n^{-\frac{n}{\sigma+\epsilon}} \\ &= 0\left(e^{\left(\frac{2r_1}{d_1-\epsilon}\right)^{\sigma+\epsilon}+\left(\frac{2r_2}{d_2-\epsilon}\right)^{\sigma+\epsilon}}\right),\end{aligned}$$

since the above series is convergent and is independent of r_1 and r_2.

Let $\sum_2$ contain the terms for which $m \geq (\frac{2r_1}{d_1-\epsilon})^{\sigma+\epsilon}$ and $n \geq (\frac{2r_2}{d_2-\epsilon})^{\sigma+\epsilon}$ and so in $\sum_2$, we have $\frac{r_1}{d_1-\epsilon}m^{-1/\sigma+\epsilon} \leq 1/2$ and $\frac{r_2}{d_2-\epsilon}n^{-1/\sigma+\epsilon} \leq 1/2$, and hence

$$\begin{aligned}\sum_2 &= \sum_{m\geq(\frac{2r_1}{d_1-\epsilon})^{\sigma+\epsilon}} \sum_{n\geq(\frac{2r_2}{d_2-\epsilon})^{\sigma+\epsilon}} m^{-\frac{m}{\sigma+\epsilon}} n^{-\frac{n}{\sigma+\epsilon}} \left(\frac{r_1}{d_1-\epsilon}\right)^m \left(\frac{r_2}{d_2-\epsilon}\right)^n \\ &\leq \sum_{m,n}(1/2)^m(1/2)^n \leq 1.\end{aligned}$$

Let $\sum_3$ be the part of the series for which $m < (\frac{2r_1}{d_1-\epsilon})^{\sigma+\epsilon}$ and $n \geq (\frac{2r_2}{d_2-\epsilon})^{\sigma+\epsilon}$ then

$$\sum_3 = \sum_{m<(\frac{2r_1}{d_1-\epsilon})^{\sigma+\epsilon}} m^{-\frac{m}{\sigma+\epsilon}} \left(\frac{r_1}{d_1-\epsilon}\right)^m \sum_{n\geq(\frac{2r_2}{d_2-\epsilon})^{\sigma+\epsilon}} n^{-\frac{n}{\sigma+\epsilon}} \left(\frac{r_2}{d_2-\epsilon}\right)^n.$$

Since

$$\sum_{n\geq(\frac{2r_2}{d_2-\epsilon})^{\sigma+\epsilon}} n^{-\frac{n}{\sigma+\epsilon}} \left(\frac{r_2}{d_2-\epsilon}\right)^n \leq 1$$

and

$$\sum_{m<(\frac{2r_1}{d_1-\epsilon})^{\sigma+\epsilon}} m^{-\frac{m}{\sigma+\epsilon}} \left(\frac{r_1}{d_1-\epsilon}\right)^m \leq 0\{e^{(\frac{2r_1}{d_1-\epsilon})^{\sigma+2\epsilon}}\},$$

therefore

$$\sum_3 \leq 0\{e^{(\frac{2r_1}{d_1-\epsilon})^{\sigma+2\epsilon}}\}.$$

Let the remaining part of the above double series be denoted by $\sum_4$ i.e. for $m \geq (\frac{2r_1}{d_1-\epsilon})^{\sigma+\epsilon}$ and $n < (\frac{2r_2}{d_2-\epsilon})^{\sigma+\epsilon}$, then

$$\sum_4 \leq 0\{e^{(\frac{2r_2}{d_2-\epsilon})^{\sigma+2\epsilon}}\}.$$

Further

$$B\left(\frac{r_1}{d_1-\epsilon}\right)^M \sum_n n^{-\frac{n}{\sigma+\epsilon}} \left(\frac{r_2}{d_2-\epsilon}\right)^n \leq 0\{e^{(\frac{2r_2}{d_2-\epsilon})^{\sigma+2\epsilon}}\}$$

$$C\left(\frac{r_2}{d_2-\epsilon}\right)^N \sum_m m^{-\frac{m}{\sigma+\epsilon}} \left(\frac{r_1}{d_1-\epsilon}\right)^m \leq 0\{e^{(\frac{2r_1}{d_1-\epsilon})^{\sigma+2\epsilon}}\}.$$

Hence, substituting these values in (4.2.4), we have

$$\begin{aligned} M(r_1,r_2) &< \sum_1+\sum_2+\sum_3+\sum_4+0\{e^{(\frac{2r_2}{d_2-\epsilon})^{\sigma+2\epsilon}}\}+0\{e^{(\frac{2r_1}{d_1-\epsilon})^{\sigma+2\epsilon}}\} \\ &\leq 0\left\{e^{\left(\frac{2r_1}{d_1-\epsilon}\right)^{\sigma+2\epsilon}+\left(\frac{2r_2}{d_2-\epsilon}\right)^{\sigma+2\epsilon}}\right\} \\ &\leq 0\left\{e^{\left(\frac{4r_1r_2}{(d_1-\epsilon)(d_1-\epsilon)}\right)^{\sigma+2\epsilon}}\right\}. \end{aligned}$$

Proceeding to limits and since ϵ is arbitrary, we have

$$\limsup_{r_1,r_2\to\infty} \frac{\ln\ln M(r_1,r_2)}{\ln(r_1r_2)} \leq \sigma. \tag{4.2.5}$$

From (4.2.3) and (4.2.5), we obtain the required result (4.2.1). □

Theorem 4.2.2. *Let* $\alpha = \limsup_{m,n\to\infty}\left\{m^m n^n\left(\frac{|a_{m,n}|}{d_1^m d_2^n}\right)^\rho\right\}^{\frac{1}{m+n}}$. *If* $0 < \alpha < \infty$, *the function f is the restriction to domain D of an entire function of finite order ρ and* *type* τ *if and only if*

$$\alpha = e\tau\rho. \tag{4.2.6}$$

Proof. Since f is an entire function of finite order ρ and type τ, therefore

$$|f(\varphi_1^{-1}(w_1), \varphi_2^{-1}(w_2))| \leq e^{(\tau+\epsilon)((d_1+\epsilon)|w_1|)^{\rho}+((d_2+\epsilon)|w_2|)^{\rho}},$$

and from Cauchy's inequality, we have

$$\begin{aligned} |a_{m,n}| &\leq r_1^{-m} r_2^{-n} e^{(\tau+\epsilon)((d_1+\epsilon)|w_1|)^{\rho}+((d_2+\epsilon)|w_2|)^{\rho}} \\ &\leq r_1^{-m} r_2^{-n} e^{(\tau+\epsilon)((d_1+\epsilon)r_1)^{\rho}} e^{(\tau+\epsilon)((d_2+\epsilon)r_2)^{\rho}} \end{aligned}$$

for all r_1, r_2 sufficiently large. To minimize the right hand side of this inequality, we select

$$r_1 = \frac{1}{d_1+\epsilon}\left[\frac{m}{\rho(\tau+\epsilon)}\right]^{1/\rho}, \quad r_2 = \frac{1}{d_2+\epsilon}\left[\frac{n}{\rho(\tau+\epsilon)}\right]^{1/\rho}.$$

Substitute r_1, r_2 in the above inequality, we have

$$|a_{m,n}| \leq \frac{(d_1+\epsilon)^m\ (d_2+\epsilon)^n\ [e\rho(\tau+\epsilon)]^{(m+n)/\rho}}{(m^m n^n)^{1/\rho}}$$

or,

$$\left\{m^m n^n \left(\frac{|a_{m,n}|}{(d_1+\epsilon)^m (d_2+\epsilon)^n}\right)^{\rho}\right\}^{1/(m+n)} \leq e\rho(\tau+\epsilon).$$

Proceeding to limits, since ϵ is arbitrary, we obtain

$$\limsup_{m,n\to\infty} \left\{m^m n^n \left(\frac{|a_{m,n}|}{d_1^m\ d_2^n}\right)^{\rho}\right\}^{1/(m+n)} \leq e\rho\tau. \tag{4.2.7}$$

Conversely, let

$$\limsup_{m,n\to\infty} \frac{1}{e\rho} \left\{ m^m n^n \left(\frac{|a_{m,n}|}{d_1^m\, d_2^n} \right)^{\rho} \right\}^{1/(m+n)} = \sigma.$$

Suppose $\sigma < \infty$. Then for given $\epsilon > 0$, $\exists\, M(\epsilon), N(\epsilon)$ such that for all $m \geq M$ and $n \geq N$, we have

$$|a_{m,n}| \leq K\, m^{-m/\rho}\, n^{-n/\rho}\, d_1^m d_2^n\, [e\rho(\sigma+\epsilon)]^{(m+n)/\rho}.$$

Since $f(z_1, z_2) = \sum_{m,n=0}^{\infty} a_{m,n}\, F_{m,n}(z_1, z_2)$, therefore

$$|f(z_1, z_2)| \leq K \sum_{m,n=0}^{\infty} m^{-m/\rho}\, n^{-n/\rho}\, d_1^m\, d_2^n [e\rho(\sigma+\epsilon)]^{(m+n)/\rho}\, |F_{m,n}(z_1, z_2)|.$$

From (4.1.5), by using the estimate of $F_{m,n}(z_1, z_2)$ in the above inequality, we have

$$\begin{aligned} |f(z_1, z_2)| &\leq K \sum_{m,n=0}^{\infty} m^{-m/\rho}\, n^{-n/\rho}\, d_1^m\, d_2^n\, [e\rho(\sigma+\epsilon)]^{(m+n)/\rho} \left(\frac{|z_1|}{d_1-\epsilon} \right)^m \left(\frac{|z_2|}{d_2-\epsilon} \right)^n \\ &\leq K \sum_{m,n=0}^{\infty} m^{-m/\rho}\, n^{-n/\rho}\, [e\rho(\sigma+\epsilon)]^{(m+n)/\rho} \left(\frac{d_1|z_1|}{d_1-\epsilon} \right)^m \left(\frac{d_2|z_2|}{d_2-\epsilon} \right)^n \\ &\leq K \sum_{m,n=0}^{\infty} m^{-m/\rho}\, n^{-n/\rho}\, [e\rho(\sigma+\epsilon)]^{(m+n)/\rho}\, r_1^m\, r_2^n. \end{aligned}$$

To estimate the right hand side of the above inequality, we proceeded on the similar lines of proof of Theorem V of Bose and Sharma [7, p 224], and we obtain

$$|f(z_1, z_2)| \leq 0\{e^{(\sigma+\epsilon)(r_1^\rho + r_2^\rho)}\}.$$

Hence

$$M(r_1, r_2) \leq 0\{e^{(\sigma+\epsilon)(r_1^\rho + r_2^\rho)}\}.$$

$$\Longrightarrow \frac{\ln M(r_1, r_2)}{r_1^\rho + r_2^\rho} \leq \sigma + \epsilon.$$

On proceeding to limits, we obtain

$$\limsup_{m,n\to\infty} \frac{\ln M(r_1, r_2)}{r_1^{\rho} + r_2^{\rho}} \le \sigma. \tag{4.2.8}$$

From (4.2.7) and (4.2.8), we get the required result. This completes the proof of Theorem 4.2.2. □

4.3 L^p - Approximation

In this section we consider the approximations of an entire function over the domain D. Consider the polynomials

$$p_{m,n}(z_1, z_2) = \lambda_{m,n} z_1^m z_2^n + \ldots (\lambda_{m,n} > 0)$$

defined through the relation

$$\frac{1}{A} \int\int_D p_{m,n}(z_1, z_2)\, \overline{p_{k,l}(z_1, z_2)}\, dx_1 dy_1 dx_2 dy_2 = \delta_{m,n,k,l}.$$

By applying Carleman's result [8] independently on z_1 and z_2, we have

$$p_{m,n}(z_1, z_2) \sim \left(\frac{(m+1)(n+1)A_1 A_2}{\pi^2}\right)^{1/2} \varphi_1'(z_1)(\varphi_1(z_1))^m\, \varphi_2'(z_2)(\varphi_2(z_2))^n \tag{4.3.1}$$

as $m, n \to \infty$, uniformly for $z_1 \in \bar{E}_1$ and $z_2 \in \bar{E}_2$. Any function $f \in L^2(D)$ can be expanded in terms of these polynomials as

$$f(z_1, z_2) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} b_{m,n}\, p_{m,n}(z_1, z_2) \tag{4.3.2}$$

where

$$b_{m,n} = \frac{1}{A} \int\int_D f(z_1, z_2)\, \overline{p_{m,n}(z_1, z_2)}\, dx_1 dy_1 dx_2 dy_2$$

and the series is uniformly convergent on compact subsets of D. Applying Parseval's relation of one variable independently on m and n, we have

$$(4.3.3) \qquad E^2_{m,n} = \left(\sum_{k=m+1}^{\infty} \sum_{l=n+1}^{\infty} |b_{k,l}|^2 \right)^{1/2}.$$

Before going to main results here we state and prove two lemmas which are more useful in the proof of main theorems.

Now we prove

Lemma 4.3.1.

$$(4.3.4) \qquad \limsup_{m,n\to\infty} \frac{\ln(m^m n^n)}{-\ln|b_{m,n}|} = \limsup_{m,n\to\infty} \frac{\ln(m^m n^n)}{-\ln(E^2_{m,n})}.$$

Proof. From (4.3.3), we have

$$|b_{m+1,n+1}| \leq E^2_{m,n},$$

$$\Longrightarrow \; -\ln|b_{m+1,n+1}| \geq -\ln(E^2_{m,n}).$$

Proceeding to limits, we have

$$(4.3.5) \qquad \limsup_{m,n\to\infty} \frac{\ln(m^m n^n)}{-\ln(E^2_{m,n})} \leq \limsup_{m,n\to\infty} \frac{\ln(m^m n^n)}{-\ln|b_{m,n}|}.$$

Conversely, let

$$\limsup_{m,n\to\infty} \frac{\ln(m^m n^n)}{-\ln|b_{m,n}|} = \sigma.$$

Suppose $\sigma < \infty$. Then for each $\epsilon > 0$, $\exists\, M, N$ such that for all $m \geq M$, and $n \geq N$, we have

$$|b_{m,n}| \leq K\, m^{-\frac{m}{\sigma+\epsilon}}\, n^{-\frac{n}{\sigma+\epsilon}}$$

so that

$$
\begin{aligned}
(E^2_{m,n})^2 &\le K \sum_{k=m+1}^{\infty} \sum_{l=n+1}^{\infty} k^{-\frac{2k}{\sigma+\epsilon}}\, l^{-\frac{2l}{\sigma+\epsilon}} \\
&\le K \sum_{k=m+1}^{\infty} \sum_{l=n+1}^{\infty} (m+1)^{-\frac{2k}{\sigma+\epsilon}}\, (n+1)^{-\frac{2l}{\sigma+\epsilon}} \\
&= K(m+1)^{-\frac{2(m+1)}{\sigma+\epsilon}}\, (n+1)^{-\frac{2(n+1)}{\sigma+\epsilon}} \left[1 - \frac{1}{(m+1)^{2/(\sigma+\epsilon)}}\right]^{-1} \left[1 - \frac{1}{(n+1)^{2/(\sigma+\epsilon)}}\right]^{-1} \\
&\le O(1)K(m+1)^{-\frac{2(m+1)}{\sigma+\epsilon}}\, (n+1)^{-\frac{2(n+1)}{\sigma+\epsilon}}.
\end{aligned}
$$

Therefore

$$
E^2_{m,n} \le (m+1)^{-\frac{(m+1)}{\sigma+\epsilon}}\, (n+1)^{-\frac{(n+1)}{\sigma+\epsilon}}
$$

$$
\Longrightarrow\ -\ln(E^2_{m,n}) \ge \frac{1}{\sigma+\epsilon}\, \ln\left((m+1)^{m+1}\right)\left((n+1)^{n+1}\right).
$$

Proceeding to limits and since ϵ is arbitrary, therefore we have

$$
\text{(4.3.6)} \qquad \sigma = \limsup_{m,n\to\infty} \frac{\ln(m^m n^n)}{-\ln|b_{m,n}|} \ \ge\ \limsup_{m,n\to\infty} \frac{\ln(m^m n^n)}{-\ln(E^2_{m,n})}.
$$

From (4.3.5) and (4.3.6), we obtain the required result. This completes the proof of Lemma 4.3.1. □

Lemma 4.3.2. *For any $\rho > 0$,*

(4.3.7)

$$
\limsup_{m,n\to\infty} \frac{1}{e\rho} \left\{ m^m n^n \left(\frac{|b_{m,n}|}{d_1^m d_2^n}\right)^{\rho} \right\}^{1/(m+n)} = \limsup_{m,n\to\infty} \frac{1}{e\rho} \left\{ m^m n^n \left(\frac{E^2_{m,n}}{d_1^m d_2^n}\right)^{\rho} \right\}^{1/(m+n)}.
$$

Proof. From (4.3.3), we have

$$
(|b_{m+1,n+1}|)^{\rho} \le (E^2_{m,n})^{\rho}.
$$

Since $d_1, d_2 > 0$, therefore for all $m, n > 0$, we have

$$\left(\frac{|b_{m+1,n+1}|}{d_1^m d_2^n}\right)^\rho \leq \left(\frac{E_{m,n}^2}{d_1^m d_2^n}\right)^\rho$$

$$\Longrightarrow \left\{m^m n^n \left(\frac{|b_{m,n}|}{d_1^m d_2^n}\right)^\rho\right\}^{1/(m+n)} \leq \left\{m^m n^n \left(\frac{E_{m,n}^2}{d_1^m d_2^n}\right)^\rho\right\}^{1/(m+n)}$$

or

$$\frac{1}{e\rho}\left\{m^m n^n \left(\frac{|b_{m,n}|}{d_1^m d_2^n}\right)^\rho\right\}^{1/(m+n)} \leq \frac{1}{e\rho}\left\{m^m n^n \left(\frac{E_{m,n}^2}{d_1^m d_2^n}\right)^\rho\right\}^{1/(m+n)}.$$

Proceeding to limits, we have

(4.3.8)

$$\limsup_{m,n\to\infty}\frac{1}{e\rho}\left\{m^m n^n \left(\frac{|b_{m,n}|}{d_1^m d_2^n}\right)^\rho\right\}^{1/(m+n)} \leq \limsup_{m,n\to\infty}\frac{1}{e\rho}\left\{m^m n^n \left(\frac{E_{m,n}^2)}{d_1^m d_2^n}\right)^\rho\right\}^{1/(m+n)}.$$

Conversely, let

$$\limsup_{m,n\to\infty}\frac{1}{e\rho}\left\{m^m n^n \left(\frac{|b_{m,n}|}{d_1^m d_2^n}\right)^\rho\right\}^{1/(m+n)} = \sigma.$$

Suppose $\sigma < \infty$. Then for each $\epsilon > 0$, $\exists\ M(\epsilon), N(\epsilon)$ such that for all $m \geq M$ and $n \geq N$, we have

$$|b_{m,n}| \leq \left\{(e\rho(\sigma+\epsilon))^{m+n}\ m^{-m}n^{-n}\right\}^{1/\rho}\ d_1^m d_2^n$$

so that

$$\begin{aligned}
(E_{m,n}^2)^2 &\leq K\sum_{k=m+1}^{\infty}\sum_{l=n+1}^{\infty}\left\{(e\rho(\sigma+\epsilon))^{k+l}\ k^{-k}l^{-l}\right\}^{2/\rho}\ d_1^{2k}d_2^{2l} \\
&\leq K\left\{(e\rho(\sigma+\epsilon))^{(m+1)+(n+1)}\ (s+1)^{-(s+1)}\right\}^{2/\rho}\ d_1^{2(m+1)}\ d_2^{2(n+1)} \\
&\leq O(1)K\left\{(e\rho(\sigma+\epsilon))^{(m+1)+(n+1)}\ (s+1)^{-(s+1)}\right\}^{2/\rho}\ d_1^{2(m+1)}\ d_2^{2(n+1)}
\end{aligned}$$

for $m > 4d_1^\rho\, e\, \rho\, (\sigma + \epsilon)$ and $n > 4d_2^\rho\, e\, \rho\, (\sigma + \epsilon)$,
where $(s+1)^{-(s+1)} = (m+1)^{-(m+1)}(n+1)^{-(n+1)}$, $X_1 = \left[1 - \left(\frac{(e\rho(\sigma+\epsilon))^2 d_1^\rho}{(m+1)^{(m+1)}}\right)^{2/\rho}\right]^{-1}$, and $X_2 = \left[1 - \left(\frac{(e\rho(\sigma+\epsilon))^2 d_2^\rho}{(n+1)^{(n+1)}}\right)^{2/\rho}\right]^{-1}$. Therefore

$$E_{m,n}^2 \leq O(1)K\ \left\{(e\rho(\sigma+\epsilon))^{(m+1)+(n+1)}\ (s+1)^{-(s+1)}\right\}^{1/\rho}\ d_1^{(m+1)}\ d_2^{(n+1)}.$$

Proceeding to limits, we have

(4.3.9)

$$\sigma = \limsup_{m,n\to\infty} \frac{1}{e\rho}\left\{m^m n^n \left(\frac{|b_{m,n}|}{d_1^m d_2^n}\right)^\rho\right\}^{1/(m+n)} \geq \limsup_{m,n\to\infty} \frac{1}{e\rho}\left\{m^m n^n \left(\frac{E_{m,n}^2}{d_1^m d_2^n}\right)^\rho\right\}^{1/(m+n)}.$$

From (4.3.8) and (4.3.9), we get the required result. This completes the proof of Lemma 4.3.2. □

4.4 Main Results

Theorem 4.4.1. *Let $2 \leq p \leq \infty$. Then f is restriction to the domain D of an entire function of finite order ρ if and only if*

(4.4.1)
$$\limsup_{m,n\to\infty} \frac{\ln(m^m n^n)}{-\ln(E_{m,n}^p)} = \rho.$$

Proof. We prove the theorem in two steps. First we consider the case p = 2. Let us assume that f is an entire function having finite order ρ. Then by Theorem 4.2.1, we have

$$|a_{m,n}| \leq K\ m^{-\frac{m}{\rho+\epsilon}}\ n^{-\frac{n}{\rho+\epsilon}}.$$

Now, by considering the property of orthonormality of polynomials $p_{m,n}(z_1, z_2)$, we have

$$b_{m,n} = \sum_{k=m+1}^{\infty} \sum_{l=n+1}^{\infty} a_{k,l} \frac{1}{A} \int \int_D F_{k,l}(z_1, z_2) \overline{p_{m,n}(z_1, z_2)} \, dx_1 dy_1 dx_2 dy_2.$$

Hence

$$|b_{m,n}| \le \sum_{k=m+1}^{\infty} \sum_{l=n+1}^{\infty} |a_{k,l}| \max_{z_1, z_2 \in \Gamma} |F_{k,l}(z_1, z_2)|.$$

Since, by (4.1.5), we have

$$\max_{z_1, z_2 \in \Gamma} |F_{k,l}(z_1, z_2)| \le K(1+\epsilon)^{(k+l)},$$

by substituting all these values the above inequality becomes,

$$\begin{aligned} |b_{m,n}| &\le K \sum_{k=m+1}^{\infty} \sum_{l=n+1}^{\infty} k^{-\frac{k}{\rho+\epsilon}} \, l^{-\frac{l}{\rho+\epsilon}} (1+\epsilon)^{(k+l)} \\ &\le Km^{-\frac{m}{\rho+\epsilon}} \, n^{-\frac{n}{\rho+\epsilon}} (1+\epsilon)^{(m+n)} \end{aligned}$$

for all sufficiently large m and n. Therefore, we have

$$-\ln |b_{m,n}| \ge \frac{1}{(\rho+\epsilon)} \ln (m^m n^n).$$

Proceeding to limits and since ϵ is arbitrary, we obtain

(4.4.2) $$\limsup_{m,n \to \infty} \frac{\ln (m^m n^n)}{-\ln |b_{m,n}|} \le \rho.$$

Conversely, let

$$\limsup_{m,n \to \infty} \frac{\ln (m^m n^n)}{-\ln |b_{m,n}|} = \sigma.$$

Suppose $\sigma < \infty$. Then for each $\epsilon > 0$, $\exists \; L(\epsilon), Z(\epsilon)$ such that for all $m > L$ and

$n > Z$, we have

$$|b_{m,n}| \le K\, m^{-\frac{m}{\sigma+\epsilon}}\, n^{-\frac{n}{\sigma+\epsilon}}.$$

Since by (4.3.1), we have

$$|p_{m,n}(z_1, z_2)| \le K\ (m+1)^{1/2}(n+1)^{1/2}|\varphi_1'(z_1)||\varphi_1(z_1)|^m|\varphi_2'(z_2)||\varphi_2(z_2)|^n$$

for all $z_1 \in E_1$ and $z_2 \in E_2$, we have

$$|\varphi_1'(z_1)| \ \le\ K',\ |\varphi_2'(z_2)|\ \le\ K''$$

where K', K'' are fixed positive constants, and

$$|\varphi_1(z_1)|\ \le\ \frac{|z_1|}{d_1-\epsilon},\ |\varphi_2(z_2)|\ \le\ \frac{|z_2|}{d_2-\epsilon}$$

for all z_1, z_2 with sufficiently large modulus. Hence

$$\begin{aligned} |f(z_1,z_2)| &\le K\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} m^{-\frac{m}{\sigma+\epsilon}} n^{-\frac{n}{\sigma+\epsilon}}(m+1)^{1/2}(n+1)^{1/2}\left(\frac{|z_1|}{d_1-\epsilon}\right)^m\left(\frac{|z_2|}{d_2-\epsilon}\right)^n \\ &\le K\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} m^{-\frac{m}{\sigma+2\epsilon}} n^{-\frac{n}{\sigma+2\epsilon}}\left(\frac{|z_1|}{d_1-\epsilon}\right)^m\left(\frac{|z_2|}{d_2-\epsilon}\right)^n. \end{aligned}$$

To estimate the right hand side of above inequality, following the method used in Theorem 4.2.1, we have

$$\begin{aligned} M(r_1,r_2) &< \sum_1+\sum_2+\sum_3+\sum_4+0\{e^{(\frac{2r_2}{d_2-\epsilon})^{\sigma+2\epsilon}}\}+0\{e^{(\frac{2r_1}{d_1-\epsilon})^{\sigma+2\epsilon}}\} \\ &\le 0\left\{e^{\left(\frac{2r_1}{d_1-\epsilon}\right)^{\sigma+2\epsilon}+\left(\frac{2r_2}{d_2-\epsilon}\right)^{\sigma+2\epsilon}}\right\} \\ &\le 0\left\{e^{\left(\frac{4r_1r_2}{(d_1-\epsilon)(d_1-\epsilon)}\right)^{\sigma+2\epsilon}}\right\}. \end{aligned}$$

Now by applying limits, we obtain

$$(4.4.3) \qquad \rho = \limsup_{r_1, r_2 \to \infty} \frac{\ln \ln M(r_1, r_2)}{\ln (r_1 r_2)} \leq \sigma.$$

From (4.4.2) and (4.4.3), we have

$$\limsup_{m,n \to \infty} \frac{\ln (m^m n^n)}{- \ln |b_{m,n}|} = \rho.$$

By applying Lemma 4.3.1, we have

$$(4.4.4) \qquad \limsup_{m,n \to \infty} \frac{\ln (m^m n^n)}{- \ln (E^2_{m,n})} = \rho.$$

Now we consider the case for $p > 2$. Since

$$(4.4.5) \qquad E^2_{m,n} \leq E^p_{m,n} \leq E^\infty_{m,n} \quad \text{for} \quad 2 \leq p \leq \infty,$$

it is sufficient to consider the case $p = \infty$. Suppose f is an entire function of order ρ. Then

$$E^\infty_{m,n} \leq \max_{z_1, z_2 \in \Gamma} \left| f(z_1, z_2) - \sum_{k=0}^{m} \sum_{l=0}^{n} a_{k,l} \, F_{k,l}(z_1, z_2) \right|$$

$$\leq \sum_{k=0}^{m} \sum_{l=n+1}^{\infty} |a_{k,l}| \max_{z_1, z_2 \in \Gamma} |F_{k,l}(z_1, z_2)| + \sum_{k=m+1}^{\infty} \sum_{l=0}^{n} |a_{k,l}| \max_{z_1, z_2 \in \Gamma} |F_{k,l}(z_1, z_2)|$$

$$(4.4.6) + \sum_{k=m+1}^{\infty} \sum_{l=n+1}^{\infty} |a_{k,l}| \max_{z_1, z_2 \in \Gamma} |F_{k,l}(z_1, z_2)|.$$

The first two summations in the above inequality (4.4.6) are bounded. It is sufficient to estimate the last summation. Since f is an entire function of finite order ρ, therefore by Theorem 4.2.1, we have

$$|a_{m,n}| \leq K\, m^{-\frac{m}{\rho+\epsilon}} n^{-\frac{n}{\rho+\epsilon}}$$

and

$$\max_{z_1,z_2\in\Gamma} |F_{k,l}(z_1,z_2)| \leq (1+\epsilon)^{k+l}.$$

Therefore the above inequality (4.4.6) becomes,

$$\begin{aligned}
E_{m,n}^{\infty} &\leq \sum_{k=m+1}^{\infty}\sum_{l=n+1}^{\infty} k^{-\frac{k}{\rho+\epsilon}} l^{-\frac{l}{\rho+\epsilon}}\,(1+\epsilon)^{k+l} \\
&\leq K \sum_{k=m+1}^{\infty}\sum_{l=n+1}^{\infty} \left(\frac{(1+\epsilon)^{\rho+\epsilon}}{m+1}\right)^{k/(\rho+\epsilon)} \left(\frac{(1+\epsilon)^{\rho+\epsilon}}{n+1}\right)^{l/(\rho+\epsilon)} \\
&\leq K \left(\frac{(1+\epsilon)^{\rho+\epsilon}}{m}\right)^{m/(\rho+\epsilon)} \left(\frac{(1+\epsilon)^{\rho+\epsilon}}{n}\right)^{n/(\rho+\epsilon)}
\end{aligned}$$

$$\Longrightarrow \frac{\ln(m^m n^n)}{-\ln(E_{m,n}^{\infty})} \leq \frac{\ln(m^m n^n)}{[1/(\rho+\epsilon)]\ln(m^m n^n) - \ln K - (m+n)\ln(1+\epsilon)}.$$

Proceeding to limits and since ϵ is arbitrary, we have

$$\limsup_{m,n\to\infty} \frac{\ln(m^m n^n)}{-\ln(E_{m,n}^{\infty})} \leq \rho.$$

In view of inequalities(4.4.5) and the fact that (4.4.1) holds for $p = 2$, this last inequality actually is an equality. Finally assuming (4.4.1) with $p = \infty$, we deduce from (4.4.5), that (4.4.1) will hold for $p = 2$ and hence that f is of order ρ. This completes the proof of Theorem 4.4.1. □

Theorem 4.4.2. *Let $2 \leq p \leq \infty$. Then f is restriction to the domain D of an entire function having finite order ρ of type τ if and only if*

(4.4.7) $$\limsup_{m,n\to\infty} \left\{ m^m n^n \left(\frac{E_{m,n}^{p}}{d_1^m d_2^n}\right)^{\rho} \right\}^{\frac{1}{m+n}} = e\rho\tau.$$

Proof. We prove the theorem in two steps. First we consider the case p = 2. Let us assume that f is an entire function having finite order ρ and finite type τ. Then by Theorem 4.2.2, we have

$$|a_{m,n}| \leq K\, m^{-\frac{m}{\rho}}\, n^{-\frac{n}{\rho}}\, d_1^m\, d_2^n\, (e\rho(\tau+\epsilon))^{(m+n)/\rho}.$$

Now proceeding on the lines of Theorem 4.4.1, we have

$$\begin{aligned} |b_{m,n}| &\leq K \sum_{k=m+1}^{\infty} \sum_{l=n+1}^{\infty} k^{-\frac{k}{\rho}}\, l^{-\frac{l}{\rho}}\, d_1^k\, d_2^l\, (e\rho(\tau+\epsilon))^{(k+l)/\rho}(1+\epsilon)^{(k+l)} \\ &\leq K m^{-\frac{m}{\rho}}\, n^{-\frac{n}{\rho}}\, d_1^m\, d_2^n\, (e\rho(\tau+\epsilon))^{(m+n)/\rho}(1+\epsilon)^{(m+n)} \end{aligned}$$

for all sufficiently large m and n. Therefore, we have

$$m^m n^n |b_{m,n}|^\rho \leq K(d_1^m\, d_2^n)^\rho\, (e\rho(\tau+\epsilon))^{(m+n)}.$$

By applying limits, we have

$$\limsup_{m,n\to\infty} \left\{ m^m n^n \left(\frac{|b_{m,n}|}{d_1^m d_2^n} \right)^\rho \right\}^{\frac{1}{m+n}} \leq e\rho\tau. \tag{4.4.8}$$

Conversely let

$$\limsup_{m,n\to\infty} \frac{1}{e\rho} \left\{ m^m n^n \left(\frac{|b_{m,n}|}{d_1^m d_2^n} \right)^\rho \right\}^{\frac{1}{m+n}} = \sigma.$$

Suppose $\sigma < \infty$. Then for each $\epsilon > 0$, $\exists\, H(\epsilon), G(\epsilon)$ such that for all $m > H$ and $n > G$, we have

$$|b_{m,n}| \leq L\, m^{-\frac{m}{\rho}}\, n^{-\frac{n}{\rho}}\, d_1^m d_2^n\, (e\rho(\sigma+\epsilon))^{(m+n)/\rho}.$$

For sufficiently large r_1, r_2,

$$\begin{aligned} |f(z_1,z_2)| &\leq L\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} g^{-\frac{g}{\rho}}\, d_1^m d_2^n\, (e\rho(\sigma+\epsilon))^{(m+n)/\rho}(s+1)^{1/2}\left(\frac{|z_1|}{d_1-\epsilon}\right)^m\left(\frac{|z_2|}{d_2-\epsilon}\right)^n \\ &\leq L\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} g^{-\frac{g}{\rho}}\, (e\rho(\sigma+2\epsilon))^{(m+n)/\rho}\left(\frac{d_1|z_1|}{d_1-\epsilon}\right)^m\left(\frac{d_2|z_2|}{d_2-\epsilon}\right)^n \\ &\leq L\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} g^{-\frac{g}{\rho}}\, (e\rho(\sigma+2\epsilon))^{(m+n)/\rho}\, r_1^m\, r_2^n \end{aligned}$$

where $g^{-\frac{g}{\rho}} = m^{-\frac{m}{\rho}}\, n^{-\frac{n}{\rho}}$ and $(s+1)^{1/2} = (m+1)^{1/2}(n+1)^{1/2}$. To estimate the right hand side of above inequality we follow the same lines as of Bose and Sharma [7, Theorem V, p 224], and we obtain

$$|f(z_1,z_2)| \leq 0\{e^{(\sigma+\epsilon)(r_1^\rho+r_2^\rho)}\}.$$

Hence

$$M(r_1,r_2) \leq 0\{e^{(\sigma+\epsilon)(r_1^\rho+r_2^\rho)}\}.$$

Now by applying limits, we have

$$\tau = \limsup_{r_1,r_2\to\infty}\frac{\ln M(r_1,r_2)}{r_1^\rho+r_2^\rho} \leq \sigma. \tag{4.4.9}$$

From (4.4.8) and (4.4.9), we have

$$\limsup_{m,n\to\infty}\left\{m^m n^n\left(\frac{|b_{m,n}|}{d_1^m d_2^n}\right)^\rho\right\}^{\frac{1}{m+n}} = e\rho\tau.$$

By applying above Lemma 4.3.2, we have

$$\limsup_{m,n\to\infty}\left\{m^m n^n\left(\frac{E_{m,n}^2}{d_1^m d_2^n}\right)^\rho\right\}^{\frac{1}{m+n}} = e\rho\tau.$$

Now we consider the case for $p > 2$. From (4.4.5), it is sufficient to consider the case

$p = \infty$. Suppose f is an entire function having finite order ρ and of type τ. Then from (4.4.6), the first two summations of the above inequality are bounded. It is sufficient to estimate the last summation. Since f is an entire function of finite type τ, therefore by Theorem 4.2.2, we have

$$|a_{m,n}| \leq K\ m^{-\frac{m}{\rho}} n^{-\frac{n}{\rho}}\ d_1^m\ d_2^n (e\rho(\tau+\epsilon))^{\frac{m+n}{\rho}}.$$

By using above inequality and from (4.4.6), we have

$$\begin{aligned} E_{m,n}^{\infty} &\leq K \sum_{k=m+1}^{\infty} \sum_{l=n+1}^{\infty} k^{-\frac{k}{\rho}} l^{-\frac{l}{\rho}}\ d_1^k\ d_2^l\ (e\rho(\tau+\epsilon))^{\frac{k+l}{\rho}}\ (1+\epsilon)^{k+l} \\ &\leq K \sum_{k=m+1}^{\infty} \sum_{l=n+1}^{\infty} \left(\frac{(1+\epsilon)^{\rho}}{m+1}\right)^{k/\rho} \left(\frac{(1+\epsilon)^{\rho}}{n+1}\right)^{l/\rho} d_1^k\ d_2^l\ (e\rho(\tau+\epsilon))^{k+l} \\ &\leq K \left(\frac{(1+\epsilon)^{\rho}}{m+1}\right)^{m/\rho} \left(\frac{(1+\epsilon)^{\rho}}{n+1}\right)^{n/\rho} d_1^m\ d_2^n\ (e\rho(\tau+\epsilon))^{m+n} \end{aligned}$$

$$\Longrightarrow \left(m^m n^n \left(\frac{E_{m,n}^{\infty}}{d_1^m d_2^n}\right)^{\rho}\right)^{1/(m+n)} \leq (1+\epsilon)\ (e\rho(\tau+\epsilon))$$

$$\limsup_{m,n\to\infty} \left\{ m^m n^n \left(\frac{E_{m,n}^{\infty}}{d_1^m d_2^n}\right)^{\rho} \right\}^{\frac{1}{m+n}} \leq e\rho\tau.$$

In view of inequalities(4.4.5) and the fact that (4.4.7) holds for $p = 2$, this last inequality actually is an equality. Finally assuming (4.4.7)with $p = \infty$, we deduce from (4.4.5),that (4.4.7) will hold for $p = 2$ and hence that f is of type τ. This completes the proof of Theorem 4.4.2. □

Chapter 5

Approximation of Entire Functions of Several Complex Variables

In the present chapter, we study the polynomial approximation of entire functions of several complex variables. The coefficient characterizations of generalized order and generalized type of entire functions of several complex variables have been obtained in terms of the approximation errors.

5.1 Introduction

Let $f(z_1, z_2, ..., z_n)$ be an entire function of n complex variables $z = (z_1, z_2, ..., z_n)$ belongs to C^n. Let G be region in R^n_+ (Positive hyper octant). Let $G_R \subset C^n$ denote the region obtained from G by a similarity transformation about the origin, with ratio of similitude R. Let $d_k(G) = \sup_{z \in G} |z|^k$, where $|z|^k = |z_1|^{k_1}|z_2|^{k_2}...|z_n|^{k_n}$, and let ∂G denotes the boundary of the region G. Let

$$f(z) = f(z_1, z_2, ..., z_n) = \sum_{k_1, k_2, ..., k_n = 0}^{\infty} a_{k_1 ... k_n} z_1^{k_1} ... z_n^{k_n} = \sum_{\|k\|=0}^{\infty} a_k z^k,$$

$\|k\| = k_1 + k_2 + ... + k_n$, be the power series expansion of the function $f(z)$. Let $M_{f,G}(R) = \max_{z \in G_R} |f(z)|$. To characterize the growth of f, order(ρ_G) and type (τ_G) are defined as [14]

$$\rho_G = \limsup_{R \to \infty} \frac{\ln \ln M_{f,G}(R)}{\ln R},$$

$$\tau_G = \limsup_{R \to \infty} \frac{\ln M_{f,G}(R)}{R^{\rho_G}}.$$

For an entire function of several complex variables $f(z) = \sum_{\|k\|=0}^{\infty} a_k\, z^k$, A.A.Gol'dberg [16, Th .1] obtained the order and G-type σ_G in terms of the coefficients of its Taylor expansion by

$$(5.1.1) \qquad \rho = \limsup_{\|k\| \to \infty} \frac{\|k\|\ \ln \|k\|}{-\ln |a_k|}.$$

$$(5.1.2) \qquad (e\, \rho\, \sigma_G)^{1/\rho} = \limsup_{\|k\| \to \infty} \{\|k\|^{1/\rho}\, [|a_k|\, d_k(G)]^{1/\|k\|}\}, \quad (0 < \rho < \infty)$$

where $d_k(G) = \max_{r \in G} r^k; \quad r^k = r_1^{k_1} r_2^{k_2} ... r_n^{k_n}$.
We define error of an entire function f on a region G as

$$E_k(f, G) = \sup\{\|f - p\|_{L^p} : p \in P\}.$$

where

$$\|f\|_{L^p} = \{\int\int_{z_1, z_2, ..., z_k \in G} |f(z_1, z_2, ..., z_k)|^p\, dz_1\, dz_2\, ...dz_k\}^{1/p} < \infty,$$

and $P(z) = \sum_{\|k\|=k} a_k\, z^k$ is a polynomial of degree k.
Before proving main results we state a Lemma.

Lemma 5.1.1. *Let $P'(z) = \sum_{\|k\|=k} E_k\, z^k$ be a polynomial of degree k, where $\|k\| = k_1 + k_2 + ... + k_n$. Let $M_{P',G}(1) = \max_{z \in G} |P'(z)|$. Then*

$$1 \le M_{P',G}(1) \max_{\|k\|=k} \{E_k d_k(G)\} \le (1 + \|k\|)^n.$$

•

5.2 Order and Type

Now we prove

Theorem 5.2.1. *The entire function $f(z) = \Sigma_{\|k\|=0}^{\infty} a_k z^k$ is of order ρ if and only if the following relation holds*

(5.2.1)
$$\rho = \limsup_{\|k\| \to \infty} \frac{\|k\| \ \ln \|k\|}{-\ln E_k \ - \ln |d_k(G)|},$$

where $d_k(G) = \max_{r \in G} \ r^k$.

Proof. From Ronkin [43, Page 131], we have

(5.2.2)
$$\min_{0<t<\infty} e^{\mu t^{\nu}} \ t^{-m} = (\frac{e\mu\nu}{m})^{m/\nu} \quad (\mu > 0, \nu > 0, m \geq 0),$$

(5.2.3)
$$\sup_{0<k<\infty} (\frac{a}{k})^{k/\nu} \ t^k = \exp\{\frac{a}{e\nu} t^{\nu}\} \quad (a > 0, \nu > 0).$$

(5.2.4)
$$E_k(f) \leq \|f - p\|_{L^p} = \| \sum_{\|j\|=\|k\|+1}^{\infty} a_j \ z^j \|_{L^p} \leq \sum_{\|j\|=\|k\|+1}^{\infty} |a_j| \ |z|^j.$$

From (5.1.1), we have

$$|a_k| \leq \frac{1}{d_k(G)} \ \|k\|^{-\|k\|/(\rho+\epsilon)}.$$

By using above inequality and (5.2.4), we get

$$E_k(f) \leq \frac{1}{d_k(G)} \ \frac{1}{(\|k\| + 1)^{(\|k\|+1)/\rho+\epsilon}} \ r^{\|k\|+1} \left[1 - \frac{r}{(\|k\| + 1)^{1/\rho+\epsilon}}\right]^{-1}.$$

By setting $r = 1 + \frac{1}{\|k\|}$ in the above inequality, proceeding to limits and taking into account, the arbitrariness of $\epsilon > 0$, we obtain

$$\limsup_{\|k\| \to \infty} \frac{\|k\| \ \ln \|k\|}{-\ln E_k(f) - \ln d_k(G)} \leq \rho.$$

For reverse inequality, let

$$\limsup_{\|k\| \to \infty} \frac{\|k\| \ \ln \|k\|}{-\ln E_k(f) - \ln d_k(G)} = \sigma.$$

Suppose $\sigma < \infty$. Then for any $\epsilon > 0$ there exists N such that, for all k with $\|k\| > N$,

$$\frac{\|k\| \ \ln \|k\|}{-\ln E_k(f) - \ln d_k(G)} \leq \sigma + \epsilon,$$

and consequently

$$E_k \ d_k(G) \leq \|k\|^{-\|k\|/\sigma+\epsilon}. \tag{5.2.5}$$

Therefore

$$\begin{aligned} M_{f,G}(R) &\leq \max_{r\in G} \sum_{\|k\|=0}^{\infty} E_k r^k R^{\|k\|} \\ &\leq \sum_{\|k\|=0}^{\infty} R^{\|k\|} E_k d_k(G) \\ &\leq \sum_{\|k\|\geq N} R^{\|k\|} \ \|k\|^{-\|k\|/\sigma+\epsilon} + \sum_{\|k\|<N} R^{\|k\|} \ E_k d_k(G) \\ &\leq \sum_{\|k\|=0}^{\infty} R^{\|k\|} \ (1+\|k\|)^n \ \|k\|^{-\|k\|/\sigma+\epsilon} + c_2 \ R^N \ + c_1, \end{aligned}$$

(5.2.6)

where c_1 and c_2 are constants. Set $N(R) = (2R)^{\sigma+\epsilon}$. For $\|k\| \geq N(R)$

$$R^{\|k\|}\left(\frac{1}{\|k\|}\right)^{\|k\|/(\sigma+\epsilon)} \leq 2^{-\|k\|},$$

and by (5.2.3), for any $\|k\|$, in particular, for $\|k\| \leq N(R)$

$$R^{\|k\|}\left(\frac{1}{\|k\|}\right)^{\|k\|/(\sigma+\epsilon)} \leq \exp\left\{\frac{R^{\sigma+\epsilon}}{e(\sigma+\epsilon)}\right\}.$$

Consequently

$$\sum_{\|k\|=0}^{\infty} R^{\|k\|}\,(1+\|k\|)^n\,\|k\|^{-\|k\|/\sigma+\epsilon} =$$

$$\sum_{\|k\|<N(R)} R^{\|k\|}\,(1+\|k\|)^n\,\|k\|^{-\|k\|/\sigma+\epsilon} + \sum_{\|k\|\geq N(R)} R^{\|k\|}\,(1+\|k\|)^n\,\|k\|^{-\|k\|/\sigma+\epsilon}$$

$$\begin{aligned}
&\leq \exp\left\{\frac{R^{\sigma+\epsilon}}{e(\sigma+\epsilon)}\right\}\sum_{\|k\|<N(R)}(1+\|k\|)^n + \sum_{\|k\|\geq N(R)}\frac{(1+\|k\|)^n}{2^{\|k\|}}\\
&\leq (1+N(R))^{n+1}\exp\left\{\frac{R^{\sigma+\epsilon}}{e(\sigma+\epsilon)}\right\} + \sum_{\|k\|\geq N(R)}\frac{(1+\|k\|)^n}{2^{\|k\|}}.
\end{aligned} \tag{5.2.7}$$

Hence by (5.2.6), we conclude that for all $R > 0$ and certain constants c_1, c_2, c_3 and c_4

$$M_{f,G}(R) \leq c_1 + c_2\,R^N + (c_3 + c_4\,R^{(\sigma+\epsilon)\,(n+1)})\,\exp\left\{\frac{R^{(\sigma+\epsilon)}}{e\,(\sigma+\epsilon)}\right\}. \tag{5.2.8}$$

Hence

$$\ln\ln M_{f,G}(R) \leq [1+o(1)]\,(\sigma+\epsilon)\,\ln R.$$

Proceeding to limits, we obtain $\rho \leq \sigma+\epsilon$, and since ϵ is arbitrary, so $\rho \leq \sigma$. We have thus proved (5.2.1). This completes the proof of Theorem 5.2.1. □

Theorem 5.2.2. *The entire function* $f(z) = \Sigma_{\|k\|=0}^{\infty} a_k z^k$ *is of order* ρ *and G- type* σ_G *if and only if the following relation holds*

$$(5.2.9) \qquad (e\rho\sigma_G)^{1/\rho} = \limsup_{\|k\| \to \infty} \{\|k\|^{1/\rho}[E_k d_k(G)]^{1/\|k\|}\}, \quad \rho > 0,$$

where $d_k(G) = \max_{r \in G} r^k$.

Proof. From (5.1.2), we have

$$|a_k| d_k(G) \leq \left(\frac{e\,\rho\,(\sigma_G + \epsilon)}{\|k\|}\right)^{\frac{\|k\|}{\rho}}.$$

By using above inequality and (5.2.4), we get

$$E_k(f) \leq \frac{1}{d_k(G)}\, r^{\|k\|+1} \left\{ e\,(\sigma_G + \epsilon) \left(\frac{\|k\|+1}{\rho}\right)^{-1} \right\}^{\frac{\|k\|+1}{\rho}} \left[1 - r\, \left(\frac{e\rho(\sigma_G + \epsilon)}{\|k\|+1}\right)^{1/\rho}\right]^{-1}.$$

By setting $r = 1 + \frac{1}{\|k\|}$ in the above inequality, proceeding to limits, we obtain

$$(5.2.10) \qquad \limsup_{\|k\| \to \infty} \{\|k\|^{1/\rho}[E_k d_k(G)]^{1/\|k\|}\} \leq \sigma_G.$$

For reverse inequality, let

$$\limsup_{\|k\| \to \infty} \{(E_k d_k(G))^{1/\|k\|}\, \|k\|^{1/\rho}\} = \kappa < \infty.$$

Now we want to show that $\kappa \geq (e\rho\sigma_G)^{1/\rho}$. From the definition of κ, for any $\epsilon > 0$ there exists M such that, for all k with $\|k\| > M$,

$$(E_k d_k(G))^{1/\|k\|}\, \|k\|^{1/\rho} \leq \kappa + \epsilon,$$

and consequently

$$d_k(G)\,\hat{E}_k \le \left(\frac{\kappa+\epsilon}{\|k\|^{1/\rho}}\right)^{\|k\|}. \tag{5.2.11}$$

Therefore

$$\begin{aligned} M_{f,G}(R) &\le \max_{r\in G} \sum_{\|k\|=0}^{\infty} E_k r^k R^{\|k\|} \\ &\le \sum_{\|k\|=0}^{\infty} R^{\|k\|} E_k d_k(G) \\ &\le \sum_{\|k\|\ge M} R^{\|k\|} \left(\frac{\kappa+\epsilon}{\|k\|^{1/\rho}}\right)^{\|k\|} + \sum_{\|k\|\le M} R^{\|k\|}\, E_k d_k(G) \\ &\le \sum_{\|k\|=0}^{\infty} R^{\|k\|}\,(1+\|k\|)^n \left(\frac{(\kappa+\epsilon)^\rho}{\|k\|}\right)^{\|k\|/\rho} + c_1\, R^M + c_2, \end{aligned} \tag{5.2.12}$$

where c_1 and c_2 are constants. Set $N(R) = (2R(\kappa+\epsilon))^\rho$. For $\|k\| \ge N(R)$

$$R^{\|k\|} \left(\frac{(\kappa+\epsilon)^\rho}{\|k\|}\right)^{\|k\|/\rho} \le 2^{-\|k\|},$$

and by (5.2.3), for any $\|k\|$, in particular, for $\|k\| \le N(R)$

$$R^{\|k\|} \left(\frac{(\kappa+\epsilon)^\rho}{\|k\|}\right)^{\|k\|/\rho} \le \exp\left\{\frac{(\kappa+\epsilon)^\rho}{e\rho}\, R^\rho\right\}.$$

Consequently

$$\sum_{\|k\|=0}^{\infty} R^{\|k\|}\,(1+\|k\|)^n \left(\frac{(\kappa+\epsilon)^\rho}{\|k\|}\right)^{\|k\|/\rho} =$$

$$\sum_{\|k\|<N(R)} R^{\|k\|}\,(1+\|k\|)^n \left(\frac{(\kappa+\epsilon)^\rho}{\|k\|}\right)^{\|k\|/\rho} + \sum_{\|k\|\ge N(R)} R^{\|k\|}\,(1+\|k\|)^n \left(\frac{(\kappa+\epsilon)^\rho}{\|k\|}\right)^{\|k\|/\rho}$$

$$\leq \exp\left\{\frac{(\kappa+\epsilon)^\rho}{e\rho} R^\rho\right\} \sum_{\|k\|<N(R)} (1+\|k\|)^n + \sum_{\|k\|\geq N(R)} \frac{(1+\|k\|)^n}{2^{\|k\|}}$$

$$\text{(5.2.13)} \qquad \leq (1+N(R))^{n+1} \exp\left\{\frac{(\kappa+\epsilon)^\rho}{e\rho} R^\rho\right\} + \sum_{\|k\|\geq N(R)} \frac{(1+\|k\|)^n}{2^{\|k\|}}.$$

Hence by (5.2.6), we conclude that for all $R > 0$ and certain constants c_1, c_2, c_3 and c_4

$$\text{(5.2.14)} \qquad M_{f,G}(R) \leq c_1 + c_2 R^M + (c_3 + c_4 R^{\rho(n+1)}) \exp\left\{\frac{(\kappa+\epsilon)^\rho}{e\rho} R^\rho\right\}.$$

$$\sigma_G = \limsup_{R\to\infty} \frac{\ln M_{f,G}(R)}{R^\rho} \leq \frac{(\kappa+\epsilon)^\rho}{e\rho}$$

$$\Rightarrow \qquad (e\rho\sigma_G)^{1/\rho} \leq \kappa+\epsilon.$$

Since ϵ is arbitrary,

$$\kappa \geq (e\rho\sigma_G)^{1/\rho}.$$

From (5.2.10) and (5.2.14), we have

$$\limsup_{\|k\|\to\infty} \{(E_k\, d_k(G))^{1/\|k\|} \|k\|^{1/\rho}\} = (e\rho\sigma_G)^{1/\rho}.$$

We have thus proved (5.2.9). This completes the proof of Theorem 5.2.2. □

5.3 Generalized Order and Generalized Type

Now we prove

Theorem 5.3.1. *Let $\alpha(x) \in L^o$, and $\beta(x) \in \Lambda$. Set $F(x;c) = \beta^{-1}[c\ \alpha(x)]$. If*

$dF(x;c)/d\ln x = O(1)$ *as* $x \to \infty$ *for all* c, $0 < c < \infty$, *then*

$$\limsup_{R\to\infty} \frac{\alpha[\ln M_{f,G}(R)]}{\beta \ln R} = \limsup_{\|k\|\to\infty} \frac{\alpha(\|k\|)}{\beta(-\frac{1}{\|k\|}\ln(E_k(f)d_k(G)))}.$$

Proof. From Seremeta [44, Theorem 1']

$$|a_k| d_k(G) \le e^{-\|k\|\,F(\|k\|;\frac{1}{\rho})}.$$

By using above inequality and (5.2.4), we get

$$E_k(f) \le \frac{1}{d_k(G)}\, e^{-(\|k\|+1)\,F(\|k\|+1;\frac{1}{\rho})}\, r^{\|k\|+1} \left[1 - \frac{r}{e^{F(\|k\|+1;\frac{1}{\rho})}}\right]^{-1}$$

By setting $r = 1 + \frac{1}{\|k\|}$ in the above inequality, proceeding to limits, we obtain

(5.3.1)
$$\limsup_{\|k\|\to\infty} \frac{\alpha(\|k\|)}{\beta(-\frac{1}{\|k\|}\ln(E_k(f)d_k(G)))} \le \rho.$$

Conversely, let

$$\limsup_{\|k\|\to\infty} \frac{\alpha(\|k\|)}{\beta(-\frac{1}{\|k\|}\ln(E_k\, d_k(G)))} = \eta.$$

Suppose $\eta < \infty$. Then for any $\epsilon > 0$ there exists N' such that for all k with $k > N'$, we have

(5.3.2)
$$E_k\, d_k(G) \le \exp\{-\|k\| F(\|k\|\ ;\ 1/\overline{\eta})\}$$

where $\overline{\eta} = \eta + \epsilon$. The inequality

(5.3.3)
$$\sqrt[\|k\|]{R^{\|k\|}\, E_k\, d_k(G)} \;\le\; R\, e^{-F(\|k\|\,;\,1/\overline{\eta})} \;\le\; \frac{1}{2}$$

is fulfilled beginning with some $\|k\| = k(R) = E[\alpha^{-1}[\overline{\eta}\,\beta(\ln R + \ln 2)]]$, where $E[F]$

is an integer part of the function F. Then

$$(5.3.4) \qquad \sum_{\|k\|=k(R)+1}^{\infty} E_k\, d_k(G)\, R^{\|k\|} \leq \sum_{\|k\|\,=\,k(R)+1}^{\infty} \frac{1}{2^{\|k\|}} \leq 1.$$

Now

$$M_{f,G}(R) \leq \sum_{\|k\|\,=\,0}^{\infty} E_k\, d_k(G)\, R^{\|k\|} = \sum_{\|k\|\,=\,0}^{k_0} E_k\, d_k(G)\, R^{\|k\|} +$$

$$(5.3.5) \qquad \sum_{\|k\|=k_0+1}^{k_1(R)} E_k\, d_k(G)\, R^{\|k\|} + \sum_{\|k\|=k_1(R)+1}^{\infty} E_k\, d_k(G)\, R^{\|k\|}$$

by applying above Lemma 5.1.1 and from (5.3.4), the above inequality becomes

$$M_{f,G}(R) \leq (1+\|k\|)^n + k_1(R) \max_{k_0 \leq \|k\| \leq k_1(R)} (E_k\, d_k(G) R^{\|k\|}) + \sum 2^{-\|k\|}.$$

From (5.3.3), we have

$$2R \leq \exp\{F(\|k\|\ ;\ 1/\overline{\eta})\}.$$

Now, we express k in terms of R.

$$\ln 2 + \ln R = F(\|k\|\ ;\ 1/\overline{\eta}) \ = \ \beta^{-1}[\frac{1}{\overline{\eta}}\ \alpha(\|k\|)]$$

where $k_1(R) = k(R) + 1$, and $k_0 = \max\{N', k_1(R)\}$.

$$\max_{k_0 \leq \|k\| \leq k_1(R)} (E_k\, R^{\|k\|}) \leq \max_{k_0 \leq \|k\| \leq k_1(R)} \psi(\|k\|) \leq \exp\{A\, \alpha^{-1}[\overline{\eta}\, \beta(\ln R + A)]\}$$

where $\psi(\|k\|) = R^{\|k\|} \exp\{-\|k\| F(\|k\|\ ;\ 1/\overline{\eta})\}$. From (5.3.5), we have

$$M_{f,G}(R)(1 + o(1)) \leq \exp\{(A + o(1))\, \alpha^{-1}[\overline{\eta}\, \beta(\ln R\ +\ A)]\}.$$

Then we have

$$\frac{\alpha[(A+o(1))^{-1}\ln M_{f,G}(R)]}{\beta(\ln R\ +\ A)} \le \overline{\eta} = \eta + \epsilon.$$

Proceeding to limits, and using the properties of $\alpha(x)$ and $\beta(x)$, we obtain

$$\rho = \limsup_{R\to\infty} \frac{\alpha(\ln M_{f,G}(R))}{\beta(\ln R)} \le \eta. \tag{5.3.6}$$

From (5.3.1) and (5.3.6), we obtain the required result. This completes the proof Theorem 5.3.1. □

Remark 5.3.1. By taking $\alpha(x) = \ln(x)$, $\beta(x) = x$ in the above Theorem 5.3.1, we get (5.2.1).

Theorem 5.3.2. *Let $\alpha(x) \in L^0$, $\beta^{-1}(x) \in L^0$, $\gamma(x) \in L^0$; let ρ be a fixed number, $0 < \rho < \infty$. Set $F(x\ ;\ \sigma\ ,\ \rho) = \gamma^{-1}\ \{[\beta^{-1}\ (\sigma\ \alpha(x))]^{1/\rho}\}$. Suppose that all σ, $0 < \sigma < \infty$, satisfy:*

$$d\ \ln F(x\ ;\ \sigma\ ,\ \rho)\ /d\ \ln x\ = O(1) \quad as\, x\ \to\ \infty;$$

then the following equation holds:

$$\limsup_{R\to\infty} \frac{\alpha(\ln M_{f,G}(R))}{\beta[(\gamma(R))^{\rho}]} = \limsup_{\|k\|\to\infty} \frac{\alpha(\frac{\|k\|}{\rho})}{\beta\{[\gamma(e^{1/\rho}[E_k\ d_k(G)]^{-1/\|k\|})]^{\rho}\}}.$$

Proof. From Seremeta [44, Theorem 2']

$$|a_k|\ d_k(G) \le e^{\|k\|/\rho} \left[F(\frac{\|k\|}{\rho}\ ;\ \frac{1}{\overline{\sigma}}\ ,\ \rho)\right]^{-\|k\|}.$$

By using above inequality and (5.2.4), we get

$$E_k(f) \le \left(\frac{r\ e^{1/\rho}}{[F(\frac{\|k\|+1}{\rho}\ ;\ \frac{1}{\overline{\sigma}}\ ,\ \rho)]}\right)^{(\|k\|+1)} \left[1 - \frac{r\ e^{1/\rho}}{[F(\frac{\|k\|+1}{\rho}\ ;\ \frac{1}{\overline{\sigma}}\ ,\ \rho)]}\right]^{-1}.$$

By setting $r = 1 + \frac{1}{\|k\|}$ in the above inequality, proceeding to limits, we obtain

$$\limsup_{\|k\| \to \infty} \frac{\alpha(\frac{\|k\|}{\rho})}{\beta\{[\gamma(e^{1/\rho}\,\{E_k\, d_k(G)\}^{-1/\|k\|})]^{\rho}\}} \le \sigma. \tag{5.3.7}$$

Conversely, let

$$\limsup_{\|k\| \to \infty} \frac{\alpha(\frac{\|k\|}{\rho})}{\beta\{[\gamma(e^{1/\rho}[E_k\, d_k(G)]^{-1/\|k\|})]^{\rho}\}} = \tau.$$

Suppose $\tau < \infty$. Then for every $\epsilon > 0$ there exists M' such that for all k with $k \ge M'$, we have

$$E_k\, d_k(G) \le \frac{\exp\left(\frac{\|k\|}{\rho}\right)}{[F(\|k\|/\rho\,;1/\overline{\tau}\,,\rho)]^{\|k\|}}$$

where $\overline{\tau} = \tau + \epsilon$. The inequality

$$\sqrt[\|k\|]{E_k\, d_k(G)\, R^{\|k\|}} \le \frac{e^{1/\rho}\, R}{F(\|k\|/\rho\,;1/\overline{\tau}\,,\rho)} \le \frac{1}{2} \tag{5.3.8}$$

is fulfilled for all $\|k\|$ beginning with some $\|k\| = k(R) = E[\rho\,\alpha^{-1}\{\overline{\tau}\,\beta\,[(\gamma\,(2e^{1/\rho}\,R))^{\rho}]\}]$. Then

$$\sum_{\|k\|=k(R)+1}^{\infty} E_k\, d_k(G)\, R^{\|k\|} \le \sum_{\|k\|=k(R)+1}^{\infty} \frac{1}{2^{\|k\|}} \le 1. \tag{5.3.9}$$

Hence

$$M_{f,G}(R) \le \sum_{\|k\|=0}^{\infty} E_k\, d_k(G)\, R^{\|k\|} = \sum_{\|k\|=0}^{k_0} E_k\, d_k(G)\, R^{\|k\|} +$$

$$\sum_{\|k\|=k_0+1}^{k_1(R)} E_k\, d_k(G)\, R^{\|k\|} + \sum_{\|k\|=k_1(R)+1}^{\infty} E_k\, d_k(G)\, R^{\|k\|} \tag{5.3.10}$$

By applying the Lemma 5.1.1 and (5.3.9), the above inequality becomes

$$M_{f,G}(R) \le (1 + \|k\|)^n + k_1(R) \max_{k_0 \le \|k\| \le k_1(R)} (E_k\, d_k(G) R^{\|k\|}) + \sum 2^{-\|k\|},$$

where $k_1(R) = k(R) + 1$, and $k_0 = \max\{M', k_1(R)\}$.

$$\max_{k_0 \le \|k\| \le k_1(R)} (E_k R^{\|k\|}) \le \max_{k_0 \le \|k\| \le k_1(R)} \chi(\|k\|)$$

$$\le \exp\{A \rho \alpha^{-1}\{\overline{\tau} \beta [(\gamma(Re^{\frac{1}{\rho} - A}))^{\rho}]\}\},$$

where $\chi(\|k\|) = (R e^{1/\rho})^{\|k\|} [F(\|k\|/\rho; 1/\overline{\tau}, \rho)]^{-\|k\|}$. From (5.3.10), we have

$$M_{f,G}(R) \le \exp\{(A \rho + o(1)) \alpha^{-1}\{\overline{\tau} \beta [(\gamma (R e^{\frac{1}{\rho} + A}))^{\rho}]\}\}.$$

Since $\alpha(x) \in L^0$, $\beta^{-1}(x) \in L^0$, $\gamma(x) \in L^0$, proceeding to limits, we obtain

$$\text{(5.3.11)} \quad \sigma = \limsup_{R \to \infty} \frac{\alpha(\ln M_{f,G}(R))}{\beta[(\gamma(R))^{\rho}]} \le \tau = \limsup_{\|k\| \to \infty} \frac{\alpha(\frac{\|k\|}{\rho})}{\beta\{[\gamma(e^{1/\rho}[E_k\, d_k(G)]^{-1/\|k\|})]^{\rho}\}}.$$

From (5.3.7) and (5.3.11), we obtain the required result. This completes the proof of Theorem 5.3.2. □

Remark 5.3.2. By taking $\alpha(x) = x$, $\beta(x) = x$ and $\gamma(x) = x$ in the above Theorem (5.3.2), we get (5.2.9).

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List of Research Papers

1. Ramesh Ganti and G.S.Srivastava: Approximation of entire functions of two complex variables in Banach spaces, J. Inequal. Pure Appl. Math., vol.7 (2), art. 51, 2006.

2. Ramesh Ganti, G.S.Srivastava: Approximation of entire functions of slow growth, General Mathematics vol.14 (2), 65-82, 2006.

3. Ramesh Ganti, G.S.Srivastava: Approximation of entire functions of several complex variables in bounded regions, Journal of Mathematical analysis and Approximation Theory, vol 1 (1), 42-48, 2006.

4. Ramesh Ganti, G.S.Srivastava: Generalized order and generalized type of entire functions of several complex variables, Presented at 71st annual conference of Indian Mathematical Society, 2005. Communicated for publication.

5. Ramesh Ganti, G.S.Srivastava: On the approximation of entire functions of two complex variables, Communicated for publication.

6. Ramesh Ganti, G.S.Srivastava: Approximation of entire functions in Banach spaces, Communicated for publication.

7. Ramesh Ganti, G.S.Srivastava: Approximation of entire functions over Jordan domains, Communicated for publication.

8. Ramesh Ganti, G.S.Srivastava: Approximation of entire functions of two complex variables over Jordan domains, Communicated for publication.