COMPENSATION
IN
THE PARAMETER PLANE

A Dissertation
submitted in partial fulfilment
of the requirements for the Degree
of
MASTER OF ENGINEERING
in
ADVANCED ELECTRICAL MACHINES ENGINEERING
(ELECTRICAL ENGINEERING)

By

MUKTA NANDA SARMA

DEPARTMENT OF ELECTRICAL ENGINEERING
UNIVERSITY OF ROORKEE
ROORKEE
(INDIA)

1967

CERTIFICATE

Certified that the dissertation entitled "Compensation in the Parameter Plane" which is being submitted by Sri Mukta Nanda Sarma in partial fulfilment for the award of the Degree of Master of Engineering in Advanced Electrical Machines Engineering of the University of Roorkee is a record of student's own work carried out by him under my supervision and guidance. The matter embodied in this dissertation has not been submitted for the award of any other Degree or Diploma.

This is further to certify that he has worked for a period of SIX months from DECEMBER '66 to JULY '67 for preparing dissertation for Master of Engineering Degree of the University.

Roorkee
Dated: July 12, 1967

(P. Das)
Lecturer,
Deptt. of Elect.Engg.,
University of Roorkee,
Roorkee, U.P., India.

ACKNOWLEDGEMENTS

The author wishes to express his deep sense of gratitude to Sri P. Das, Lecturer, Electrical Engineering, University of Roorkee, Roorkee for initiating this topic and extending very valuable guidance and suggestions throughout the compilation of this work. He is also indebted to Prof. C.S. Ghosh, Prof. and Head of the Electrical Engineering Department for the various facilities provided in connection with this work.

The author also takes the opportunity to thank D.D. Siljak, Visiting Professor, Department of Electrical Engineering, School of Engineering, University of Santa Clara, Santa Clara, California for the valuable help he has rendered during the course of this work.

Grateful acknowledgement is paid by the author to Mr. R.S. Khurana of Electrical Engineering, University of Roorkee and to the computer centre, S.E.R.C., Roorkee for their active cooperation and sincere help.

Mukta Nanda Sarma.

(Mukta Nanda Sarma)

CONTENTS

SYNOPSIS

The analysis and synthesis of control systems have very advantageously been done by the parameter plane method. This has opened up a big scope for its application to the design of Linear continuous systems, sampled data systems and non-linear systems. The sensitivity of some performance index due to some parameter fluctuation can be obtained by this method. Some of the frequency domain specifications can also be simultaneously satisfied along with the time domain specifications. The Chebyshev functions are used to facilitate the complicated calculations that arise in the analysis. The methods are illustrated by two examples.

CHAPTER - 1

I_N_T_R_O_D_U_C_T_I_O_N.

Various methods for the study of stability of control systems are available but they do not fulfil all the requirements of a design problem like the essential features of the system response. Better techniques for the analysis and synthesis of control systems in terms of the system response have therefore become essential. The parameter plane method is a good tool used for this purpose. Analysis and synthesis of linear continuous, sampled data, nonlinear systems and sensitivity problems have been very efficiently handled by means of this method.

1.1. Review of Earlier Works:

When the design information required is of a frequency response nature, the Bode diagram, Nyquist plot, Nichols chart techniques are usually used. When frequency response methods are to be used with the transfer functions, the stability test most commonly used is the Nyquist criterion. This is a test for the existence of roots in the right hand half of the s-plane. It is based on Cauchy's principle of argument, and the manipulation involved is a conformal mapping of the imaginary axis of the s-plane on a polar plane defined by the loop transfer function. Absolute stability and steady state accuracy are determined rather easily from the plots, particularly from the Bode diagram which permits evaluation of the transfer function gain. It is also relatively simple to check the effect of gain adjustments and gain variations on stability and accuracy. The frequency response approach has certain disadvantages in that (i) the time-domain characteristics such as overshoot and setting time can hardly be recognized from the frequency characteristic and (ii) frequency response techniques are not suitable for the design of multiloop control systems and particularly useless in the cases having more than one adjustable parameter.

A correlation between the frequency and transient responses is absolutely essential in the majority of control systems and Laplace transform with the concept of complex frequency s becoming an important mathematical tool in the analysis and synthesis of feedback control systems. This inspires one to develop synthesis techniques in the algebric domain where the characteristics of both transient and frequency responses are evident.

The idea of investigating the transient response of feedback control systems in the algebric domain was first mooted by Vishnegradsky. The approach developed by him designates that the two middle coefficients of the 3rd degree characteristic equation be considered variables. But this work does not provide scope for studying the higher degree characteristic equations.

Vishnegradsky's work was extended by Neimark in his D-partition method for the stability analysis of control systems. D-partition method enables a designer to assume two system parameters which appear linearly in coefficients of the nth degree characteristic equation to be variables. With the help of this method it is possible to determine the number of left half plane roots of the characteristic equation in various areas of the parameter plane. Though the work may entail time-consuming inconvenience, the method may be extended to investigate the degree of stability. The main drawback of this method lies in its inability to apply to design problems in terms of transient response.

The root locus method presented by Evans provides information about all the roots of the characteristic equation, and it permits a simple numerical evaluation of these roots for different values of the open loop gain. Applying full potentials of the Laplace transform, this procedure admits control over both time and frequency domain characteristics. But this method has two main limitations - (1) it is basically a single parameter method and (2) it makes the synthesis of multiloop systems inconvenient. Thus root locus method suffers from the same difficulties experienced in applying frequency response techniques to the design of multiloop structures with more than one adjustable parameter. Moreover, accurate plotting of the locus requires a considerable amount of labour. Although the root locus technique has been well developed and widely used for the analysis and design of linear feedback control systems, it was originally defined with only the open loop gain K as the varying parameter; that is the root locus diagram is defined as a plot of the poles of the closed loop function C(s)/R(s) as K varies from 0 to ∞. In most design problems, the effects on the closed loop system poles must be studied when parameters other than K are varied and frequently there are more than one variable parameter. In feedback control systems with multiple feedback loops, the forward path gain K does not appear only as a multiplying factor in the open-loop transfer function; therefore, the conventional root locus technique, again can not be applied directly. The root contour technique is utilized to construct the root loci when parameters other than K are varied in a feedback control system. In this procedure K is held constant but the open loop function poles and zeros are varied (due to the variation of parameters other than K). In the design of compensating networks for single loop or multiloop system, the root contours give a clear indication of the proper choice of the parameters of the compensating functions.

Mitrovic's method brings out solution to the algebric problem of control system synthesis in an extremely effective manner. The original method designates that the last two coefficients of the characteristic equation be considered variables. This is a limitation in that the free system parameters are offen found to appear in any arbitrary pair of coefficients in the characteristic equation. Generalization of Mitrovic's method [5] removes this difficulty. Mitrovic's method can be extended to take into account the frequency domain characteristics [6].

After the application of the Laplace transform, the coefficients of the obtained characteristic equation are given as known functions of system parameters. Some of these parameters are adjusted and should be determined by the synthesis procedure to meet the system specifications of performance which are given in advance. In the algebric domain, these specifications are met by adjusting the system parameters so that the roots of the characteristic equation are set at some desired locations. Therefore, a useful method to obtain a simple correlation between adjustable system parameters and the root locations has been long recognized and desired.

The parameter plane method has very creditably been put to wide use by D.D. Siljak. His works have opened up a big scope in the field of analysis and synthesis of control system problems. Results of some of the illustrative examples presented by him are given here in brief.

The first example has been taken from the paper [1] which corresponds to a linear continuous system. Let us consider a control system with the block diagram given in Fig.1.3 and transfer functions.

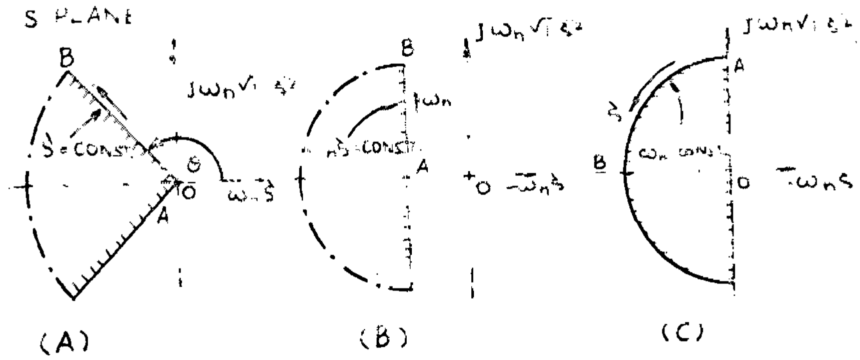


FIG. 11 THE S-PLANE CONTOURS.

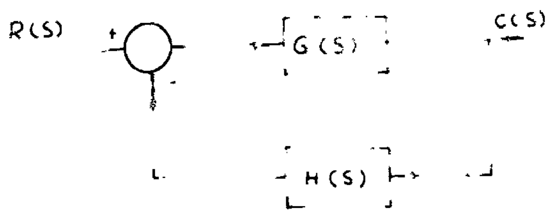
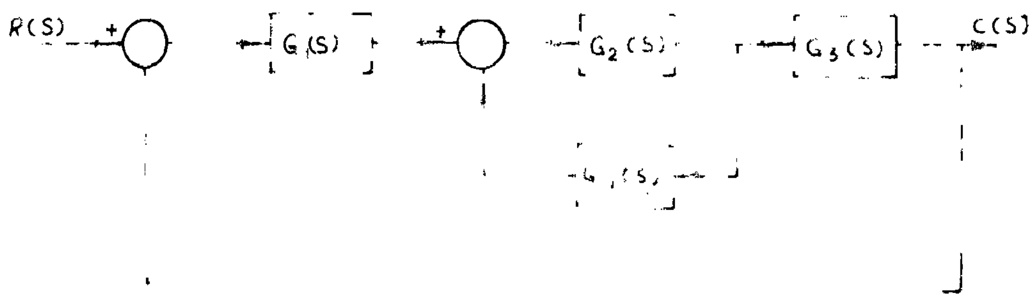


FIG. 12 - SINGLE LOOP FEEDBACK CONTROL SYSTEM



F . 13 SYSTEM BLO. DIAGRAM

$$\left.\begin{aligned} G_1(s) &= 2 \\ G_2(s) &= \frac{K_2}{0.2s^2 + 0.8s + 1} \\ G_3(s) &= \frac{1}{0.4s + 1} \\ G_{-1}(s) &= \frac{0.5K_{-1}s}{0.5s + 1} \end{aligned}\right\} \quad \ldots \quad \ldots \quad (1.1)$$

It is necessary to determine all the possible values of the variable parameters K_2 and K_{-1} so that the corresponding characteristic equation-

$$f(s) = 0.04s^4+0.34s^3+(0.2K_{-1}K_2+1.12)s^2+(0.5K_{-1}K_2+K_2+1.7)s+2K_2+1=0 \quad \ldots \quad \ldots \quad (1.2)$$

has all the roots within the s plane contour given in the upper half corner of Fig.1.4.

Before the specified s-plane contour is mapped onto the $\alpha\beta$ plane, it is to be noted that the variable parameters do not appear linearly in the coefficients of the characteristic equation (1.2).

Let us take-

$$\left.\begin{aligned} K_{-1}K_2 &= \alpha \\ K_2 &= \beta \end{aligned}\right\} \quad \ldots \quad \ldots \quad (1.3)$$

From the Fig.1.4 it is clear that the specified contour is a combination of all the three contours shown in Fig.1.1. Thus each part of the contour should be mapped onto the $\alpha\beta$ plane separatel[y]. The values of the constants are found to be

$$\left.\begin{aligned} B_1 &= 0.2\, w_n^2, \qquad\qquad C_1 = -2 \\ B_2 &= 0.2\, w_n^2\, U_2 - 0.5w_n, \qquad C_2 = -w_n \\ D_1 &= 0.04\, w_n^4 U_3 - 0.34\, w_n^3\, U_2 + 1.12\, w_n^2 - 1 \\ D_2 &= 0.04\, w_n^4 U_4 - 0.34\, w_n^3\, U_3 + 1.12\, w_n^2\, U_2 - 1.7\, w_n \end{aligned}\right\} \quad \ldots \quad (1.4)$$

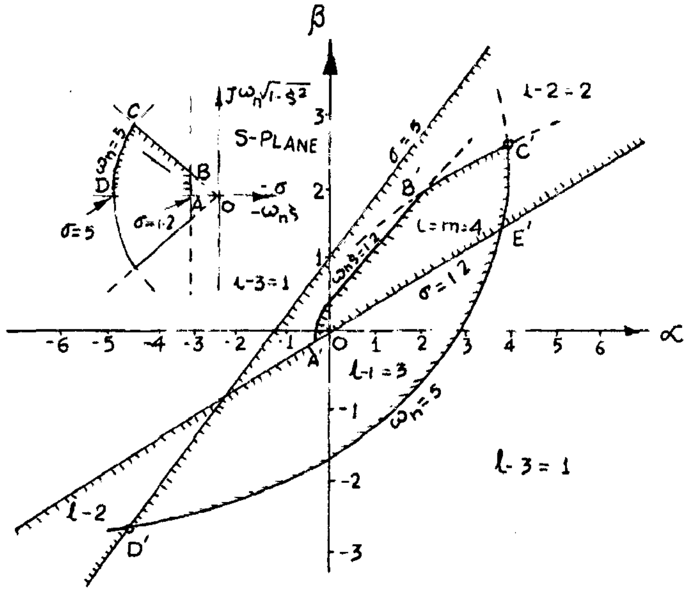


FIG. 1.4- THE S-PLANE CONTOUR AND PARAMETER PLANE DIAGRAM.

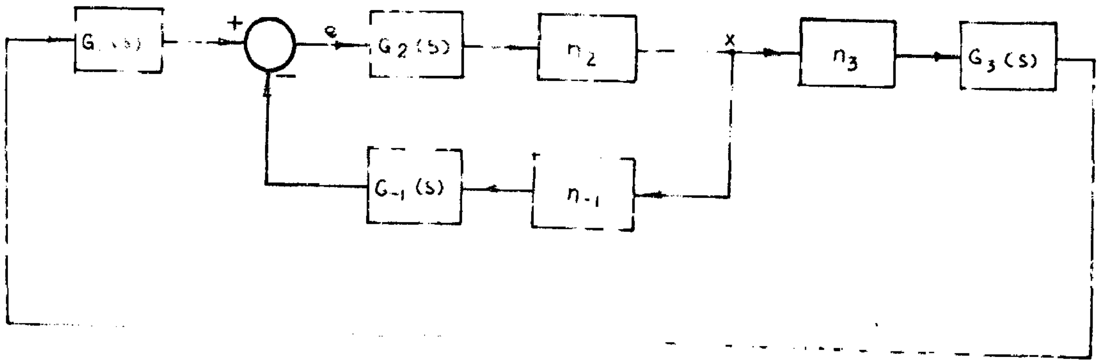


FIG 1.5 SYSTEM BLOCK DIAGRAM.

The real root boundaries which correspond to $\sigma = 1.2$ and $\sigma = 5$ are-

$$\left.\begin{array}{ll} -0.312\alpha + 0.8\beta + 0.07 = 0 & \text{for } \sigma = 1.2 \\ 2.5\alpha - 3.0\beta + 3.0 = 0 & \text{for } \sigma = 5 \end{array}\right\} \quad \ldots \quad (1.5)$$

From the mapping contour in the $\alpha\beta$ plane it is possible to earmark the region which corresponds to all the roots within the s-plane contour i.e. 1 = m = 4. The required values of the parameters K_2 and K_{-1} can be obtained very easily from the region A'B'C'E'.

The second example is taken from the paper [2] which corresponds to a real describing function in the nonlinear system. The system block diagram of Fig.1.5 has the following transfer functions.

$$\left.\begin{array}{l} G_1(s) = \dfrac{K_1}{0.2s + 1} \\ G_2(s) = \dfrac{K_2 s}{s + 1} \\ G_3(s) = \dfrac{K_3}{s(2s + 1)} \\ G_{-1}(s) = \dfrac{K_{-1}}{0.5s + 1} \end{array}\right\} \quad \ldots \quad (1.6)$$

The nonlinearity shown in Fig.1.5 which relates to the purely and only amplitude dependant describing function $N_1(A)$ is located in the feedback path of the minor control loop. The nonlinearities n_2 and n_3 are not considered here.

The characteristic equation of the system under investigation is-

$$f(s) = N_1K_{-1}K_2s^2(0.2s+1)(2s+1)+K_1K_2K_3(0.5s+1)+s(0.2s+1)(0.5s+1)\times(s+1)(2s+1) = 0 \quad \ldots \quad \ldots \quad (1.7)$$

If K_1 is considered an adjustable system parameter with

$$\left.\begin{aligned} K_1 &= \alpha \\ K_1 &= \beta \end{aligned}\right\} \quad \ldots \quad \ldots \quad (1.8)$$

The curve $\xi = 0$ is plotted as a complex root absolute stability boundary in Fig.1.6 with normalized coordinated axes. The real root boundary is simply the α-axis. After these boundaries are purely shaded, the stable region is found to be the area between the boundaries.

The third example is taken from the paper [2]. This is a case of relative stability in the nonlinear system. The system block diagram is same as Fig.1.5 and transfer functions are-

$$\left.\begin{aligned} G_1(s) &= 1 \\ G_2(s) &= \frac{2}{0.2s^2+0.8s+1} \\ G_3(s) &= \frac{0.5\,(s+1)}{0.2s+1} \\ G_{-1}(s) &= \frac{2K_{-1}}{T_{-1}s+1} \end{aligned}\right\} \quad \ldots \quad \ldots \quad (1.9)$$

The nonlinearity n_{-1} of Fig.1.5 is considered. It is necessary to study the relative stability of the system which is specified by $\xi = 0.5$ for various values of system parameters K_{-1}, T_{-1}, S/D and the initial signal level.

The characteristic equation of the above system is-

$$f(s) = 0.04\alpha s^4+(0.36\alpha+0.04)s^3+(2\alpha+0.36)s^3+(2\alpha+0.8K_{-1}\beta+2)s+$$

$$4K_{-1}\beta + 2 = 0. \quad \ldots \quad \ldots \quad (1.10)$$

where,

$\alpha = T_{-1}$ and $\beta = K_1$

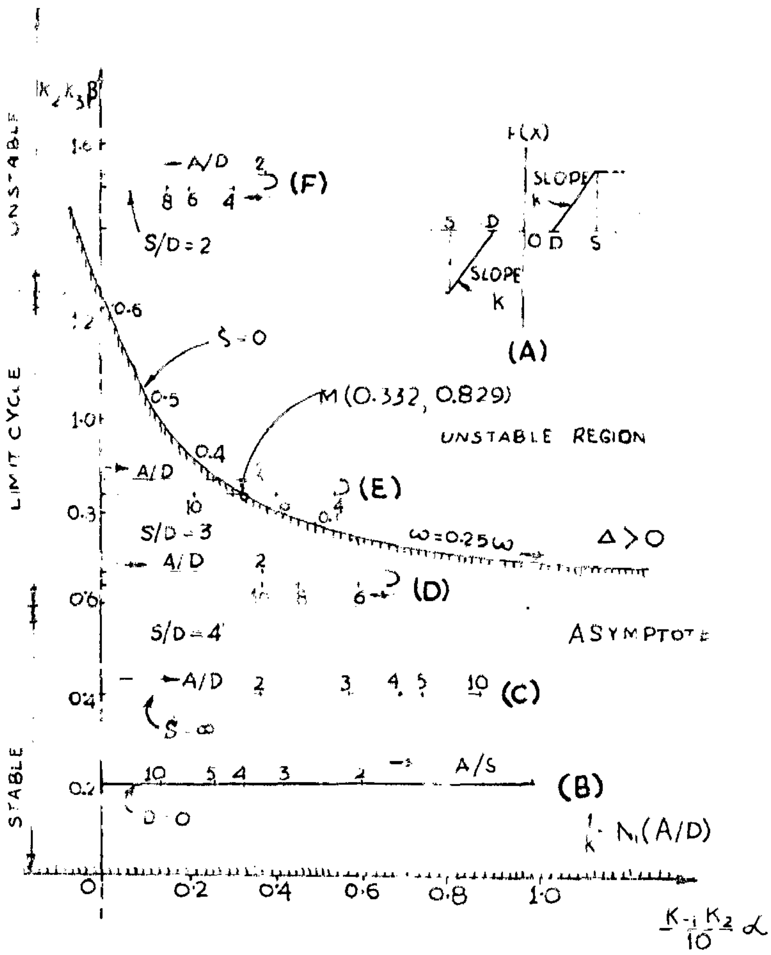


Fig. 1·6 - *PARAMETER PLANE DIAGRAM FOR A PURELY DESCRIBING FUNCTION.*

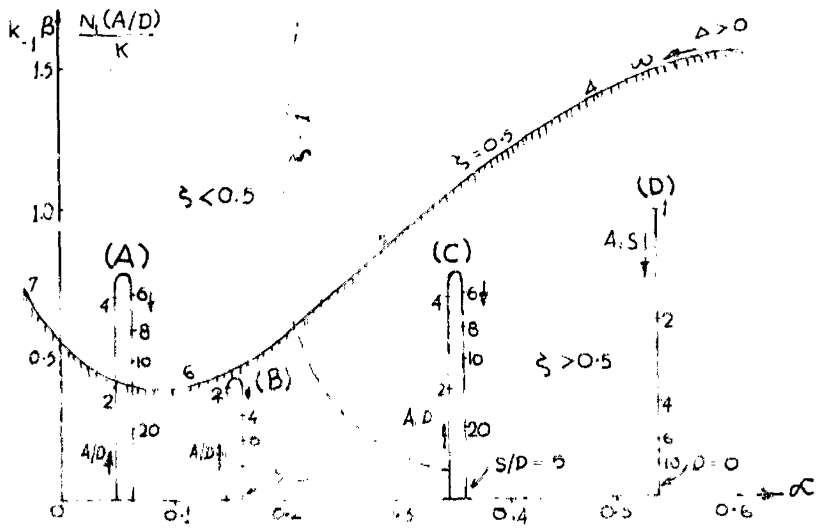


Fig. 1·7 - *PARAMETER PLANE DIAGRAM FOR RELATIVE ANALYSIS.*

The variation of N_1 resulting from changes in the amplitude A would be drawn along the β axis by using standard diagrams. This obtains the Fig.1.7, in which several M loci are plotted for different values of the linear system parameters $T_{-1} = \alpha$ and the ratio S/D of the nonlinear system parameters. Fig.1.7 indicates that the locus (A) is unsatisfactory, since a portion of the locus is outside the region $\xi > 0.5$. However, if the initial value of the amplitude A is chosen so that it lies on the left side of the M locus and under the curve $\xi = 0.5$, the relative stability requirement is fulfilled. The relative stability requirements is also satisfied by the locus (C), which is the same locus (A), it is shifted the α- axis. The loci (B) and (D) which are plotted for different values of S/D and $T_{-1} = \alpha$, also satisfy the prescribed degree of relative stability.

Refering to Fig.1.7, as the dotted curve $\xi = 1$ is drawn, it is easy to conclude that the mentioned root configuration is obtained to the left of the curve since two tangents may be drawn to the curve $\xi = 1$. As already known, these tangents determine the real roots of the characteristic equation (1.10). The remaining two roots are the complex control pair related to the ξ curve. Therefore, the M locus shown in (B) of Fig.1.7, represents an optimum combination of system parameters with respect to the prescribed relative stability requirement.

CHAPTER - 2

PROBLEM FORMULATION AND ITS SOLUTION

2.1. Introduction-

The factorization of the characteristic polynomials posed a big problem in the control theory for a long time. Parameter plane method being a useful and graphical procedure eased this difficulty to a minimum. Two parameters which appear linearly in the coefficient of the characteristic equation are considered as coordinates in the parameter plane. The procedure consists of plotting some of specific contours from the s-plane onto the parameter plane, thereby the characteristic root locations with different parameter values can be obtained without further calculations. The introduction of Chebyshev polynomials made the work simple and convenient. Moreover, all analytical and graphical operations are done in the real domain.

The method enables a designer of linear continuous systems to get information about absolute as well as relative stabilities of the systems on adjusting some of the system parameters. It is easier to have control over both frequency and transient responses of the system, besides it is very useful for multiloop systems having more than one adjustable system parameter.

By the describing function techniques this method can very well be applied to the study of oscillations in nonlinear systems. Control systems having nonlinearities with real describing function, complex describing function, with two nonlinearities and nonlinearities with frequency and amplitude dependent describing function can be considered.

A limitation of parameter plane method may come up when adjustable system parameters do not appear linearly in the coefficients of the characteristic equation. The generalized method is based on a specific case when the coefficients are the functions of the linear combination of two parameters and their product. Then the method applied for the stability analysis to the design of a linear multivariable and also to a non-linear multivariable control system with two nonlinearities.

In the design of linear control systems it is desirable that a certain degree of stability and the minimum sensitivity of the system response to the parameter variations be achieved. Though both are contradictory to each other, a satisfactory compromise is arrived at on the basis of parameter plane method. In the procedure, a time-domain sensitivity index is minimized within the required stability constraints.

Extensive works on parameter plane method have been done by D.D. Siljak in a number of published papers. The frequency domain specifications are not dealt in these papers.

In this work we have proposed a method by which the frequency domain specifications can be obtained in the parameter plane. The works of the previous authors along with the work presented here will enable a designer to design any component which may simultaneously satisfy the frequency domain and the time domain specifications. For the compensation in the parameter plane the type of the compensator is first assumed i.e. whether the compensator should be in the forward or in the feedback path or in both. The nature of the transfer functions are also assumed, and only two parameters are taken as variables which can be selected by the designer. The problem is now to specify the limits of the two variable parameters such that all the system specifications are met with.

2.2. Linear Continuous Systems-

The paper [1] presents a method for analysis and synthesis of linear continuous control systems. The method provides the scope for studying both absolute and relative stabilities in the parameter plane. The graphical procedure maps the specific contour (for absolute stability $\xi = 0$; relative stability $0 \leqslant \xi \leqslant 1$) from the s-plane onto the parameter plane. Compared to relative stability, the absolute stability study is an easier one because of ease of manipulations.

Basic equations derived for extensive use in the paper are given below.

Let us consider the characteristic equation-

$$f(s) = \sum_{K=0}^{m} a_K s^K = 0 \qquad \ldots \quad \ldots \quad (2.1)$$

in which the coefficients a_K $(K = 0, 1, 2, \ldots\ldots\ldots, n)$ are real and s is the complex variable.

If s is expressed by

$$s = -w_n \xi + j w_n \sqrt{1 - \xi^2} \qquad \ldots \quad \ldots \quad (2.2)$$

where,

w_n = undamped natural frequency

ξ = relative damping coefficient.

It has been shown that s^K may be given in the form which is demonstrated as follows-

$$s^K = w_n^K \left[T_K(-\xi) + j\sqrt{1 - \xi^2}\; U_K(-\xi) \right] \ldots \quad \ldots \quad (2.3)$$

where,

$$\left.\begin{aligned} T(-\xi) &= (-1)^{K}\, T_K(\xi) \\ U(-\xi) &= (-1)^{K+1}\, U_K(\xi) \end{aligned}\right\} \quad \ldots \quad \ldots \quad (2.4)$$

Functions $T_K(\xi)$ and $U_K(\xi)$ are Chebyshev polynomials of the first and the second kinds respectively. The argument ξ of Chebyshev

funct... $\leqslant |\xi| \leqslant 1$; for the stable system it is $0 \leqslant \xi \leqslant 1$.

... $T_K(\xi)$ and $U_K(\xi)$ may be obtained by applying the

...as.

$$\left.\begin{aligned} 2\xi T_K(\xi) + T_{K-1}(\xi) &= 0 \\ 2\xi U_K(\xi) + U_{K-1}(\xi) &= 0 \end{aligned}\right\} \ldots \quad (2.5)$$

...$_1(\xi) = \xi$, $U_0(\xi) = 0$ and $U_1(\xi) = 1$ Since

... and $U_K(\xi)$ play an important role in the

...ts, their numerical values for pertinent values

...pendix 5.1. For an evaluation of these

... trigonometric functions may also be used

$$\left.\begin{aligned} T_K(\xi) &= \cos(K \arccos \xi) \\ U_K(\xi) &= \frac{\sin(K \arccos \xi)}{\sin(\arccos \xi)} \end{aligned}\right\} \ldots \quad (2.6)$$

Substituting equation (2.3) into equation (2.1) and applying the condition that the summations of the reals and the imaginaries must go to zero independently, enables (2.1) to be rewritten as two simultaneous equations which are as follows.

$$\left.\begin{aligned} &\sum_{K=0}^{m} a_K v_n^K T_K(-\xi) = 0 \\ &\sum_{K=0}^{m} a_K v_n^K U_K(-\xi) = 0 \end{aligned}\right\} \quad \ldots \quad \ldots (2.7)$$

Since the functions $T_K(\xi)$ may be expressed in terms of the functions $U_K(\xi)$ as

$$T_K(\xi) = \xi U_K(\xi) - U_{K-1}(\xi) \quad \ldots \quad \ldots (2.8)$$

Then from equations (2.7) and ((2.4)

$$\left.\begin{aligned} &\sum_{K=0}^{m} (-1)^K a_K v_n^K U_{K-1}(\xi) = 0 \\ &\sum_{K=0}^{m} (-1)^K a_K v_n^K U_K(\xi) = 0 \end{aligned}\right\} \quad \ldots \quad \ldots (2.9)$$

Now the coefficients a_K of the characteristic equation (2.1) are considered as linear functions of variable system parameters α and β as follows.

$$a_K = b_K \alpha + c_K \beta + d_K \quad \ldots \quad \ldots \quad \ldots (2.10)$$

Owing to equation (2.10), equations (2.9) may be rewritten as-

$$\left.\begin{aligned} &\alpha B_1(v_n, \xi) + \beta C_1(v_n, \xi) + D_1(v_n, \xi) = 0 \\ &\alpha B_2(v_n, \xi) + \beta C_2(v_n, \xi) + D_2(v_n, \xi) = 0 \end{aligned}\right\} \ldots (2.11)$$

where,

$$\left.\begin{aligned}
B_1 &= \sum_{K=0}^{p \leqslant m} (-1)^K b_K v_n^K U_{K-1} \\
B_2 &= \sum_{K=0}^{p \leqslant m} (-1)^K b_K v_n^K U_K \\
C_1 &= \sum_{K=0}^{q \leqslant m} (-1)^K c_K v_n^K U_{K-1} \\
C_2 &= \sum_{K=0}^{q \leqslant m} (-1)^K c_K v_n^K U_K \\
D_1 &= \sum_{K=0}^{r \leqslant m} (-1)^K d_K v_n^K U_{K-1} \\
D_2 &= \sum_{K=0}^{r \leqslant n} (-1) d_K v_n^K U_K
\end{aligned}\right\} \cdots \quad \cdots (2.12)$$

Equations (2.11) may be solved for unknowns α and β

$$\left.\begin{aligned}
\alpha &= \frac{C_1D_2 - C_2D_1}{B_1C_2 - B_2C_1} \\
\beta &= \frac{B_2D_1 - B_1D_2}{B_1C_2 - B_2C_1}
\end{aligned}\right\} \cdots \quad \cdots (2.13)$$

In equations (2.12) and (2.13) the arguments w_n and ξ are omitted for simplicity.

If the equations (2.11) are rewritten so that the first variable is α and the second variable is β, then rectangular coordinate axes α and β make a right coordinate system in the parameter plane. In this $\alpha\beta$ plane, equation (2.13) may represent the loci of points corresponding to the roots with relative damping coefficients, undamped natural frequency, or setting-time being constant, depending on which variable among w_n, ζ or $w_n\zeta$ is considered constant. In further developments, these loci will be called the ζ curve, w_n curve and $w_n\zeta$ curve respectively. According to equations (2.12) and (2.13) which indicate that the parameters α and β are expressed as rational functions of w_n and ζ, the curves represent plane algebric curves in the parameter plane.

In some of the applications of the parameter plane method, when attention is focussed on the setting-time curve, equation (2.3) should be rewritten as-

$$s^K = P_K(w_n\zeta\ ,\ w_n^2) + jw_n\sqrt{1-\zeta^2}\ Q_K(w_n\zeta\ ,\ w_n^2) \quad \ldots \quad (2.14)$$

where functions P_K and Q_K are related to the Chebyshev functions as

$$\left.\begin{aligned} P_K(w_n\zeta, w_n^2) &= w_n^K T_K(-\zeta) = (-1)^K w_n^K\ T_K(\zeta) \\ Q_K(w_n\zeta, w_n^2) &= w_n^{K-1} U_K(-\zeta) = (-1)^{K+1} w_n^{K-1}\ U_K(\zeta) \end{aligned}\right\} \ldots \quad (2.15)$$

The functions P_K and Q_K may also be obtained from the general formulas-

$$\left.\begin{aligned} P_{K+1} + 2\, w_n\zeta\, P_K + w_n^2\, P_{K-1} &= 0 \\ Q_{K+1} + 2\, w_n\zeta\, Q_K + w_n^2\, Q_{K-1} &= 0 \\ \text{with } P_o = 1,\quad P_1 = -w_n\zeta,\quad Q_o = 0,\quad &\text{and } Q_1 = 1 \end{aligned}\right\} \ldots \quad (2.16)$$

Since functions P_K may be expressed in terms of Q_K functions

$$P_K = -w_n\zeta\, Q_K - w_n^2\ Q_{K-1} \quad \ldots \quad (2.17)$$

By using equations (2.14) and (2.17) the following relationship may be derived from equation (2.1)-

$$\left.\begin{aligned} &\sum_{K=0}^{m} a_K \, Q_{K-1} = 0 \\ &\sum_{K=0}^{m} a_K \, Q_K = 0 \end{aligned}\right\} \quad \ldots \quad \ldots \quad (2.18)$$

which is similar to equation (2.9)

With equations (2.18), equations (2.13) are the same but the expressions for B_1, B_2, C_1, C_2, D_1, D_2 are changed into the following-

$$\left.\begin{aligned} B_1 &= \sum_{K=0}^{p \leqslant m} b_K \, Q_{K-1} \\ B_2 &= \sum_{K=0}^{p \leqslant m} b_K \, Q_K \\ C_1 &= \sum_{K=0}^{q \leqslant m} c_K \, Q_{K-1} \\ C_2 &= \sum_{K=0}^{q \leqslant m} c_K \, Q_K \\ D_1 &= \sum_{K=0}^{r \leqslant m} d_K \, Q_{K-1} \\ D_2 &= \sum_{K=0}^{r \leqslant m} d_K \, Q_K \end{aligned}\right\} \quad \ldots \quad \ldots \quad (2.19)$$

In equations (2.16) - (2.19), the arguments $w_n \xi$ and w_n^2

are omitted for simplicity.

Since functions $Q_K(w_n \xi, \; w_n^2)$ are polynomials in $w_n \xi$ and w_n^2, as a result of equations (2.19) the variables α and β are expressed by equations (2.13) as rationallfunctions of $w_n \xi$ and w_n^2. Thus settling time curves determined by equations (2.19) and (2.13) represent plane algebric curves.

By the definition of ξ and w_n curves given in equations (2.12) and (2.13) it may be concluded that if certain values of α and β (say α_1, β_1) are evaluated from equations (2.13) for certain values of w_n and ξ say $(w_n)_1$ and ξ_1 then equation (2.1) must be satisfied by the set of values of α_1, β_1, $(w_n)_1$ and ξ_1 according to the substitutions of equations (2.2) and (2.10). This means that for $\alpha = \alpha_1$ and $\beta = \beta_1$, characteristic equation (2.1) has a complex conjugate pair of roots which corresponds to the values of $(w_n)_1$ and ξ_1 as stated in equation (2.2).

If the complex variable s is replaced in equation (1) by

$$s = -\sigma \qquad \ldots \qquad \ldots \quad \ldots \quad (2.20)$$

where σ is the normal symbol for the real coordinate of any point on the s-plane, in accordance with equations (2.10) the characteristic equation (2.1) becomes-

$$\alpha \sum_{K=0}^{p \leqslant m} (-1)^K b_K \, \sigma^K + \beta \sum_{K=0}^{q \leqslant m} (-1)^K e_K \, \sigma^K + \sum_{K=0}^{r \leqslant m} (-1)^K d_K \, \sigma^K = 0 \quad .. (2.21)$$

For a given value of σ, equation (2.21) represents a straight line in the $\alpha\beta$ plane, which is the locus of points corresponding to real roots with value $-\sigma$. If equation (2.21) is satisfied for certain values of α, β and σ, the characteristic equation

(2.1) must have a real root $-\sigma$ according to equation (2.20).

Now it is important to conclude that by applying equations (2.13) and (2.21) the problem of determining the variable parameters α and β may be solved so that the corresponding characteristic equation (2.1) has a prescribed root value. In order to solve that problem in a general manner for all root locations, it is necessary to plot the appropriate curves and straight lines in the parameter plane by using equations (2.13) and (2.21). Then by investigating the position of the point $M(\alpha, \beta)$ in relation to the obtained diagram, the designer is able to adjust variable parameters α and β so that all the roots of equation (2.1) are set at some desired locations. After the mentioned diagram is plotted in the plane, the parameters α and β are determined without any calculations.

Absolute stability for linear control systems may be interpreted by observing the existence of roots in the right half of s-plane. This is done by mapping the imaginary axis and an infinite radius closing semicircle from the s-plane onto the parameter plane. To analyse the relative stability of the linear control systems we shall take any contour of Fig.2.1 or their possible combinations for mapping onto the $\alpha\beta$ plane. The interpretation of relative stability enables a designer to determine adjustable system parameters so that the roots lie in certain areas of the s-plane.

The graphical procedure adopted here utilises the equations (2.13) and (2.21) in mapping a specified contour onto the $\alpha\beta$ plane. The parameter plane is divided into various regions by the complex and real root boundaries. To determine the number of roots lying in each region of $\alpha\beta$ plane corresponding to specified s-plane contour, certain rules and graphical techniques are developed. But we shall first take up some preliminary considerations.

The complex root boundaries in the $\alpha\beta$ plane are curves determined by equations (2.13) while real root boundaries are always straight lines defined by equations (2.21) for a given value of σ. The latter are known as σ lines. If the working point $M(\alpha, \beta)$ crosses a complex root boundary, two complex roots simultaneously cross the specified contour in the s-plane. If the point $M(\alpha, \beta)$ crosses a real root boundary, one of the real roots simultaneously crosses the specified s-plane contour along the real axis. To determine the number of roots in various regions in the parameter plane, it is necessary to know whether a root is leaving or entering the s-plane contour at the moment M goes over a boundary in the parameter plane. Certain rules will be followed to solve this problem. If the s-plane contour is shaded on the left, facing the direction in which W_n increases as shown in Fig.2.1(A) then the crossing of the boundary from shaded side will indicate a pair of roots leaving the contour and the crossing of the boundary from unshaded side will indicate a pair of roots entering the conteur. This is also valid in $\alpha\beta$ plane. But the side to be shaded is ascertained by the sign of the denominator $\Delta = B_1C_2 - B_2C_1$ which appears on the right side of equations (2.13). Facing the direction in which w_n increases, the boundary curves in the plane should be shaded on the left if $\Delta > 0$ and on the right if $\Delta < 0$.

To investigate the number of roots at a point of the plane, the characteristic polynomial is denoted by $f(w_n, \xi)$ and given in the vector form.

$$f(w_n, \xi) = R(w_n, \xi) + jI(w_n, \xi) \qquad \ldots \qquad (2.22)$$

where,

$$\left.\begin{aligned} R(v_n, \xi) &= \sum_{K=0}^{n} C_K v_n^K (-1)^K T_K(\xi) \qquad \ldots \\ I(v_n, \xi) &= \sqrt{1-\xi^2} \sum_{K=0}^{n} C_K v_n^K (-1)^{K+1} U_K(\xi) \end{aligned}\right\} (2.23)$$

When the characteristic equation is given and a point $M(\alpha,\beta)$ is specified, the coefficients C_K are evaluated numerically from equation (2.10). For a particular value of ξ, necessary values of Chebyshev polynomials are read from tables. Then we can obtain $R(v_n, \xi)$ and $I(v_n, \xi)$ in equation (2.23) as polynomials in v_n. Putting different values to v_n and using equations (2.22) and (2.23), a locus of the vector $f(v_n, \xi)$ which maps the radial line $\overline{AB}$ from the s-plane onto the f(s) plane may easily be plotted. Now it is simple to prove that the locus will describe an argument $n\theta$, where n is the number of roots of $f(s) = 0$ which corresponds to the specified point $M(\alpha, \beta)$ and $\theta = \pi - \cos^{-1}\xi$ as indicated in Fig. 2.1(A).

If we consider Fig.2.1(B), the procedure for determining the number of roots at an arbitrary point of the parameter plane should be slightly modified. The Cauchy's theorem is to be applied to the function $f(v_n\xi, v_n^2)$ given as

$$f(v_n\xi, v_n^2) = \sum_{K=0}^{n} C_K P_K + j\sqrt{(v_n)^2 - (v_n\xi)^2} \sum_{K=0}^{n} C_K Q_K \quad \ldots (2.24)$$

The functions P_K and Q_K are defined in equations (2.10) when a characteristic equation is given and the value of $v_n\xi$ specified, by varying v_n and using equation (2.24), a locus of the vector $f(v_n\xi, v_n^2)$ may be plotted onto the f(s) plane which maps the constant settling time straight line $\overline{AB}$ from the s-plane of

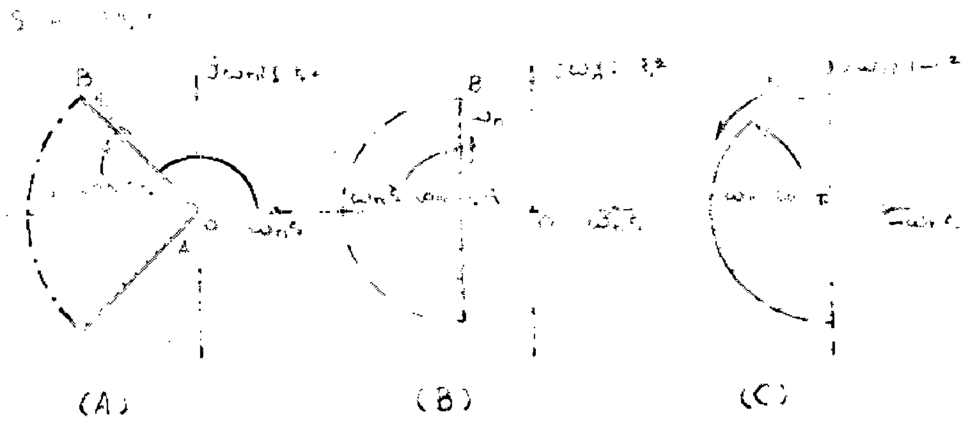


Fig. 2.1 THE S PLANE ...

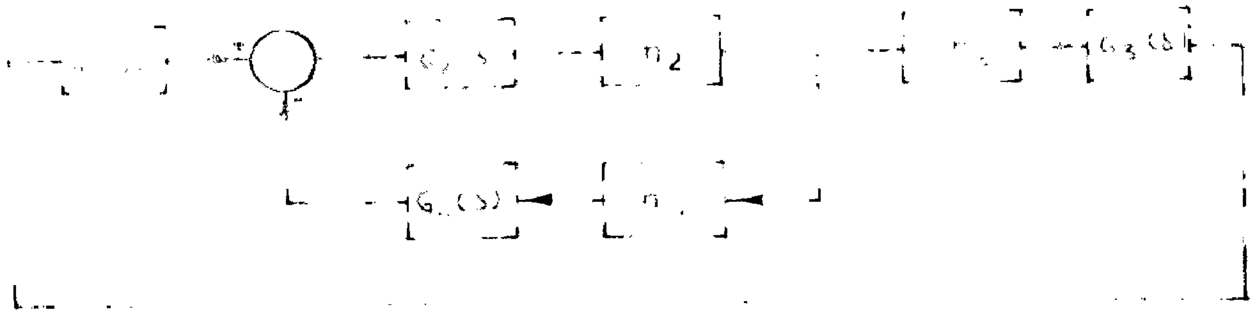


Fig. 2.2 SYSTEM BLOCK DIAGRAM

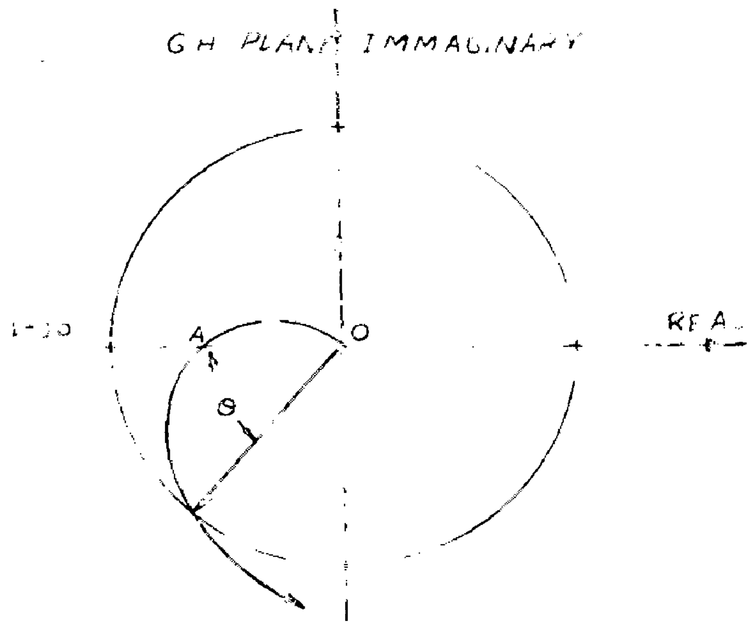


Fig. 2.3 PHASE MARGIN AND GAIN MARGIN DIAGRAM

Fig.2.1(B). The locus will describe an argument $\frac{n\pi}{2}$, where n is the number of roots corresponding to the given characteristic equation $f(s) = 0$, and the chosen point $M(\alpha,\beta)$ in the $\alpha\beta$ plane.

If we consider Fig.2.1(C), the vector defined by equations (2.22) and (2.23) should be considered. When the tracing point moves along the upper half of the contour ABC, the vector of equation (2.22) rotates in a counterclockwise direction describing an argument $n\pi$ where n is the corresponding number of roots.

A feedback control system will meet the performance specifications in both the time and frequency domains if the synthesis is interpreted as an adjustment of the poles and zeros of the closed-loop transfer function which permits the maintenance of control over both transient and frequency responses. We know that the transfer functions of linear continuous control systems are rational functions of the complex variable s. The central problem lies in the factoring of the polynomial equations. The coefficients of polynomials are functions of adjustable system parameters which should be determined so that the zeros of the polynomials are located at some desired locations. To realize the full potentials and advantages of the proposed method, the designer should make rapid transitions from root locations to both frequency and transient responses. In the numerical evaluation of roots, an essential role is played by ζ and w_n curves and σ straight lines. By investigating the position of the working point $M(\alpha,\beta)$ in relation to these curves and straight lines both complex and real roots can simultaneously by evaluated for different values of system parameters. Complex roots are determined by the ζ curves on which the values of w_n are interpolated. Only a few points of these curves should be computed for evaluation of complex roots since a graphical interpolation gives the designed results. Real roots are graphically evaluated from σ straight lines.

Limitations of the parameter plane method may arise when the adjustable system parameters enter nonlinearly into the coefficients of the characteristic equation. The paper [3] considers a general case in which coefficients depend nonlinearly on two system parameters and applies the obtained results to the specific case when the coefficients are functions of the linear combination of two parameters and their product. The presented procedure is then applied to the design of a linear multivariable control system.

In addition certain rules for the mapping of the conteours from the complex variable plane onto the parameter plane which have given intuitively in the reference papers [1] - [2] are now proved. Some additional theorems are introduced to facilitate the interpretation of the parameter plane diagrams. The mapping procedure has been laid down below.

Let us take up the characteristic equation given as in equation (2.1).

$$f(s) = \sum_{K=0}^{m} a_K s^K = 0$$

in which the coefficients $a_K (K = 0, 1, 2, \ldots\ldots\ldots, m)$ are real and nonlinear functions of two system parametes α and β.

$$a_K = a_K(\alpha, \beta) \qquad \ldots \qquad (2.25)$$

and s is the complex variable given as in equation (2.2).

Equations (2.23) can be considered as two equations in the two unknowns α and β which may be solved for α and β as .

$$\alpha = \alpha(v_n, \xi), \quad \beta = \beta(v_n, \xi) \quad \ldots \qquad \ldots \quad (2.26)$$

assuming that the Jacobian $J = J(R, X/\alpha\beta)$ which is

$$J = \frac{\partial R}{\partial \alpha} \cdot \frac{\partial X}{\partial \beta} - \frac{\partial R}{\partial \beta} \cdot \frac{\partial X}{\partial \alpha} \qquad \ldots \quad (2.27)$$

exists and is different from zero. The possibility of solving equations (2.23) in terms of α and β as given in (2.26) depends on the form of functions $c_K(\alpha, \beta)$ involved. The mapping procedure in general is similar as in the previous case. But the side of the boundary to be shaded is determined by the sign of the Jacobian J as proved in Appendix 6.3.

If the root boundaries are plotted and appropriately shaded, the number of roots in each of the bounded regions of the parameter plane can be determined in a straight forward manner as illustrated in the following developments.

Let us consider the case when the coefficients c_K are expressed in terms of α and β as-

$$c_K = b_K \alpha + c_K \beta + h_K \alpha\beta + d_K \qquad \ldots \quad (2.28)$$

where b_K, c_K, h_K and d_K are given numerically. Substituting (2.2) and (2.28) into (2.1) one obtains (2.23) in the form-

$$\left.\begin{aligned} \alpha B_1(v_n, \xi) + \beta C_1(v_n, \xi) + \alpha\beta H_1(v_n, \xi) + D_1(v_n, \xi) = 0 \\ \alpha B_2(v_n, \xi) + \beta C_2(v_n, \xi) + \alpha\beta H_2(v_n, \xi) + D_2(v_n, \xi) = 0 \end{aligned}\right\} (2.29)$$

where,

$$B_1 = \sum_{K=0}^{n} (-1)^K b_K v_n^K T_K(\xi)$$

$$B_2 = \sum_{K=0}^{n} (-1)^{K+1} b_K v_n \sqrt{1-\xi^2}\, U_K(\xi)$$

$$
\left.
\begin{aligned}
C_1 &= \sum_{K=0}^{m} (-1)^K c_K\, v_n^K \; T_K(\xi) \\
C_2 &= \sum_{K=0}^{m} (-1)^{K+1} c_K\, v_n^K \sqrt{1-\xi^2} \; U_K(\xi) \\
H_1 &= \sum_{K=0}^{m} (-1)^K h_K v_n^K \; T_K(\xi) \\
H_2 &= \sum_{K=0}^{m} (-1)^{K+1} h_K\, v_n^K \sqrt{1-\xi^2} \; U_K(\xi) \\
D_1 &= \sum_{K=0}^{m} (-1)^K d_K\, v_n^K \; T_K(\xi) \\
D_2 &= \sum_{K=0}^{m} (-1)^{K+1} d_K\, v_n^K \sqrt{1-\xi^2} \; U_K(\xi)
\end{aligned}
\right\} \quad \dots (2.30)
$$

By using the following notation

$$
\left.
\begin{aligned}
a &= B_2H_1 - B_1H_2 \\
b &= C_2H_1 - C_1H_2 \\
c &= C_1D_2 - C_2D_1 \\
d &= B_1D_2 - B_2D_1 \\
e &= B_2C_1 - B_1C_2 + H_1D_2 - H_2D_1 \\
f &= B_1C_2 - B_2C_1 + H_1D_2 - H_2D_1 \\
j &= -a\alpha + b\beta + B_1C_2 - B_2C_1
\end{aligned}
\right\} \quad \dots (2.31)
$$

and eliminating β from (2.29) one obtains.

$$\alpha_{1,2} = \frac{-e \pm \sqrt{e^2 - 4ac}}{2a}, \quad \beta_{1,2} = -\frac{B_1\alpha_{1,2} + D_1}{H_1\alpha_{1,2} + C_1} \quad ..(2.32)$$

Eliminating from (2.29) yeilds

$$\beta_{1,2} = \frac{-f \pm \sqrt{f^2 - 4bd}}{2b}, \quad \alpha_{1,2} = -\frac{C_1\beta_{1,2} + D_1}{H_1\beta_{1,2} + B_1} \quad ..(2.33)$$

The solutions of (2.29) for α and β in terms of w_n and are given by (2.32) provided $a \neq 0$, if $a = 0$ the solutions are given by (2.33) provided $b \neq 0$, when $a = b = 0$ the solutions for α and β are

$$\alpha = -\frac{c}{e}, \quad \beta = -\frac{d}{f} \quad ... \quad (2.34)$$

In the original parameter plane method where $h_K = 0$ $(K=0,1,2,....,m)$ and $H_1 = H_2 = 0$, the solutions for α and β may be obtained from (2.34).

It is important to note that, in general, for given values of w_n and ξ, there are two pairs of values (α_1, β_1) and (α_2, β_2) which specify (2.29). In addition, since in a linear system analysis parameters are represented by real numbers the following conditions should be satisfied.

$$e^2 - 4ac \geqslant 0, \quad f^2 - 4bd \geqslant 0 \quad ... \quad (2.35)$$

Equations (2.32) or (2.33) represent (2.26) and are used to plot complex root boundaries when the coefficients a_K have the form (2.28). The shading of the boundaries is determined by the sign of the Jacobian given in (2.31).

The real root boundaries are obtained by substituting for a_K from (2.28) into

$$f(-\sigma) = \sum_{K=0}^{m} (-1)^K a_K \sigma^K$$

to obtain

$$\alpha B(-\sigma) + \beta C(-\sigma) + \alpha\beta H(-\sigma) + D(-\sigma) = 0 \qquad \ldots \quad (2.36)$$

where,

$$\left.\begin{aligned} B(-\sigma) &= \sum_{K=0}^{m} (-1)^K b_K \sigma^K \\ C(-\sigma) &= \sum_{K=0}^{m} (-1)^K c_K \sigma^K \\ H(-\sigma) &= \sum_{K=0}^{m} (-1)^K h_K \sigma^K \\ D(-\sigma) &= \sum_{K=0}^{m} (-1)^K d_K \sigma^K \end{aligned}\right\} \qquad \ldots \quad (2.37)$$

Equations (2.36) is solved for β to give

$$\beta = -\frac{B\alpha + D}{H\alpha + C} \qquad \ldots \quad (2.38)$$

Equation (2.38) represents the σ curves in the $\alpha\beta$ plane. For a given value of σ, there is only one pair of the values (α, β) which satisfies (2.36). When the parameters α and β appear linearly $(H = 0)$ in the coefficient a_K of (2.25), curves are straight lines tangent to the curve $\xi = 1$.

2.3. Nonlinear Systems:

Several methods for stability analysis and investigation of self excited oscillations in nonlinear control systems are available. The describing function technique stands out because of its usefulness in engineering problems. Often it is found to be the only mathematical tool for successful solution of these problems. Describing function techniques enable the conventional design methods used for linear control systems with comparative ease in performing the computations involved. The application of the describing function technique or any of the linear methods depends on the nature and specifications of the control problems.

The paper [2] presents the application of the parameter plane analysis to the investigation of stability and self-excited oscillations in nonlinear control systems. The utilisation of the describing function technique makes the same sort of analysis as used for linear continuous systems for the study of nonlinear systems. Variations of the describing functions are drawn along one or both parameter axes and some characteristic curves are plotted. This method enables the designer to maintain control over the amplitude and frequency of the sustained oscillations when adjusting parameters of both linear and nonlinear parts of the systems.

This method is particularly convenient for the design of multiloop nonlinear control systems. The method is applicable to nonlinearity with complex describing functions, two nonlinearities with purely real describing functions and nonlinearities with both frequency and amplitude dependent describing functions. The analysis procedure is suitable for application to both analog and digital computers. All the analytical and graphical operations are performed in the real domain.

In most of the nonlinear control systems, the stability and self excited oscillations are determined by the nonlinear differential equation,

$$C(s) + B(s)\ F(x, sx) = 0 \quad \ldots \quad (2.39)$$

where $s = \frac{d}{dt}$, C(s) and B(s) are polynomials in s with the degree of polynomial C(s) higher than the degree of the polynomial B(s) and function F(x, sx) represents the nonlinearity.

In the describing function method, it is supposed that the solution x = x(t) of the differential equation (2.39) is sufficiently close to the solution.

$$x = A \sin\phi \quad \ldots \quad (2.40)$$

where $\phi = wt$ of the corresponding linear differential equation which has the characteristic equation.

$$C(s) + B(s)\ (N_1 + \frac{N_2 s}{w})x = 0 \quad \ldots \quad (2.41)$$

Equation (2.40) is obtained from equation (2.39) by replacing the function F(x, sx) by

$$F(x, sx) = N_1 x + \frac{N_2}{w} sx \quad \ldots \quad (2.42)$$

where,

$$N_1 = \frac{1}{\pi A} \int_0^{2\pi} F\ (A \sin \phi,\ Aw \cos \phi) \sin \phi\ d\phi \quad \ldots \quad (2.43)$$

In particular when the function F(x, sx) is F(x) only the expressions for N_1 and N_2 reduce to

$$\left.\begin{aligned} N_1 &= \frac{1}{\pi A} \int_0^{2\pi} F(A \sin \phi) \sin \phi\ d\phi \\ N_2 &= \frac{1}{\pi A} \int_0^{2\pi} F(A \sin \phi) \cos \phi\ d\phi \end{aligned}\right\} \quad \ldots \quad (2.44)$$

For the nonlinearity usually encountered in control systems, the integrals of equations (2.43) and (2.44) have been computed once and are collected in standard diagrams.

In cases where the periodic steadystate oscillations are considered the amplitude A and frequency w of equation (2.40) and $s = jw$, the characteristic equation (2.41) corresponding to a linear differential equation with constant coefficients may be written as

$$f(s) = \sum_{K=0}^{m} a_K s^K = 0$$

where the coefficients a_K ($K = 0,1,2,\ldots\ldots,m$) are known function of the describing function $N = N_1 + jN_2$ and system parameters. The coefficients a_K have one of the following forms -

$$a_K = b_K N_1 + c_K \beta + d_K \quad \ldots \quad \ldots \quad (2.45a)$$

$$a_K = b_K N_{11} + c_K N_{12} + d_K \quad \ldots \quad \ldots \quad (2.45b)$$

$$a_K = b_K N_1 + jb_K N_2 + d_K \quad \ldots \quad \ldots \quad (2.45c)$$

depending on the kind of nonlinearity under investigation. These three forms will be discussed below on the basisof the relationships derived for the linear continuous systems refering to equation (2.10) - (2.13).

As already known equation (2.13) represent the ξ curves which are the absolute ($\xi = 0$) or relative ($0 \leqslant \xi \leqslant 1$) stability boundaries in the $\alpha\beta$ plane. After the ξ curves are plotted and the corresponding stable region is determined by studying the position of the point $M(\alpha,\beta)$ in relation to these curves, the designer can readily obtain information about the effects on stability of varying the values of α and β.

For the analysis of stability and sustained oscillations in nonlinear control systems, the describing function $N = N(A, w)$ is substituted with variables α and β as indicated in equations (2.45). Then the curve $\xi = 0$ is plotted in the $\alpha\beta$ plane, with points marked for different values of $w_n = w$, and the stable region is determined. It will be shown that the absolute stability analysis proceeds in the usual fashion with the exception of point $M(\alpha, \beta)$ which moves with changes in amplitude A and frequency w of the sustained signal $x = A \sin wt$ for which the describing function $N(A, w)$ is defined. The relative location of the curve $\xi = 0$ and the M loci described by the point $M(\alpha, \beta)$ determines the stability of the nonlinear system.

The relative stability of nonlinear control systems may be studied advantageously in the parameter plane by using the ξ curves and the approach of N. Kryloff and N. Bogoluiboff.

Now if we assume that the input to the nonlinearity is

$$x = a_o e^{\sigma t} \sin (wt + \phi) \qquad \ldots \quad (2.46)$$

where and w are varying slowly with time.

$$\text{if } a = a_o\, e^{\int^t \sigma dt} \qquad \ldots \quad (2.47)$$

$$\frac{da}{dt} = a\sigma \text{ or, } \quad \frac{da}{a} = \sigma dt \qquad \ldots \quad (2.48)$$

$$\text{when } \sigma = \text{constant}, \quad a = a_o\, e^{\sigma t}$$

$$\text{and } \quad w = \frac{d\phi}{dt}, \quad \phi = \int_o^t w\, dt + \phi_o \qquad \ldots \quad (2.49)$$

From (2.46), (2.48), (2.49) one obtains-

$$x = a_0 e^{\int^t \sigma dt} \sin \left(\int_0^t w dt + \phi_0 \right) = a(t) \sin \phi(t) \qquad \ldots \quad (2.50)$$

$$\therefore \sin \phi = \frac{x}{a}$$

$$\therefore sx = \frac{dx}{dt} = aw \cos \phi + a \sigma \sin \phi = aw \cos \phi + \sigma x$$

from which, $$\cos \phi = \frac{(s - \sigma)}{aw} x \qquad \ldots \quad (2.51)$$

F(x, sx) can be expressed new as

$$F(x, sx) = N_1 x + N_2 \frac{(s - \sigma)}{w} x$$

$$= (N_1 - \frac{\sigma}{w} N_2) + \frac{N_2}{w} sx$$

$$= (N_1 + \frac{N_2}{\sqrt{1 - \xi^2}})x + \frac{1}{w_n \sqrt{1 - \xi^2}} N_2 sx \qquad \ldots . \quad (2.52)$$

where,

$$N_1 = \frac{1}{\pi A} \int_0^{2\pi} F(A \sin \phi, Aw \cos \phi + A\sigma \sin \phi) \sin \phi \, d\phi$$

$$N_2 = \frac{1}{\pi A} \int_0^{2\pi} F(A \sin \phi, Aw \cos \phi + A\sigma \sin \phi) \cos \phi \, d\phi \qquad (2.53)$$

Also we note-

$$s = \sigma + jw = - w_n \xi + j w_n \sqrt{1 - \xi^2}$$

$$\sigma = - w_n \xi$$

$$w = w_n \sqrt{1 - \xi^2}$$

N_1 and N_2 are the functions of A, ξ and w_n. For $\xi = 0$, $w_n = w$ equation (2.53) reduces to that of equations (2.43). With equations (2.52), the characteristic equation of the linearized differential equation which corresponds to equation (2.41) may be written as-

$$C(s) + B(s)\left(N_1 + \frac{N_2}{\sqrt{1-\xi^2}} + \frac{N_2}{w_n\sqrt{1-\xi^2}}\, s\right)x = 0 \quad \ldots \quad (2.54)$$

Substituting $s = -w_n\xi + jw_n\sqrt{1-\xi^2}$ in equation (2.54), the latter may be rewritten as two simultaneous equations in three unknowns A, ξ and w_n which may then be graphically solved in the parameter plane for functions $\xi(A)$ and $w_n(A)$. After the family of and w_n curves is plotted in the usual manner and the M locus is constructed in the same parameter plane diagram according to the function $N(A, \xi, w_n)$ given by equations (2.53), the values of the functions $\xi(A)$ and $w_n(A)$ are read at the corresponding intersections of the net curves and the M locus.

If the nonlinear function is F(x), the coefficients N_1 and N_2 may be computed using equations (2.44) derived for $\xi = 0$ and $w_n = w$, and the standard diagrams of describing functions may be utilised.

For the analysis of absolute stability and investigation of sustained oscillations or limit cycle, two loci are to be drawn in the parameter plane, namely the curve $\xi = 0$ and the M - locus. The linear part of the system is represented by the curve $\xi = 0$ and the nonlinear part by the M - locus. Along the curve $\xi = 0$, various values of w are marked and along the locus M values of the amplitude A and frequency w are shown. The absolute stability of the nonlinear system depends on the relative location of curve $\xi = 0$ and the M - locus. Limit cycles are found at the intersections.

In the developments followed the stability is discussed in regard to all the cases in equation (2.45). The system block diagram of Fig.2.2 serves to illustrate the general technique. As seen in the Fig.2.2 the system has two control loops with nonlinearities in both forward and feedback paths.

If the nonlinearity with purely real describing function N_1 is located somewhere in the control loop of a feedback system, the coefficients a_K of the corresponding characteristic equation (2.1) have the form (2.45a). In this equation, β represents an adjustable parameter of the linear portion of the system and is called the linear parameter. For convenience, parameters determining the nonlinearity are named the nonlinear parameters. The stability analysis of a control system with one real nonlinearity can be performed with respect to initial conditions on the one hand and to the linear and nonlinear system parameters on the other.

The problem of two real non-linearities often arises in the design of nonlinear control systems. It involves a rather general graphical solution in the parameter plane.

When the describing function is complex, the stability analysis is similar to that presented in the previous sections; the only difference is the complex coefficients a_K of equation (2.45a) which changes equation (2.11) to the form-

$$\left.\begin{aligned} B_1\alpha - B_2\beta + D_1 &= 0 \\ B_2\alpha + B_1\beta + D_2 &= 0 \end{aligned}\right\} \quad \ldots \quad (2.55)$$

where, $\alpha = N_1$ and $\beta = N_2$.

From equations (2.55) (2.13) obtain the form-

$$\left.\begin{aligned} \alpha &= -\frac{B_1 D_1 + B_2 D_2}{B_1^2 + B_2^2} \\ \beta &= \frac{B_2 D_1 - B_1 D_2}{B_1^2 + B_2^2} \end{aligned}\right\} \quad \ldots \quad (2.56)$$

The plotting of the ξ curves with equations (2.56) is performed in the usual manner, except the determinant $\triangle = B_1^2 + B_2^2$ of equations (2.56) is always greater than or atleast equal to zero and the shading of the curves is on the left side facing the direction of increasing w_n.

We shall now consider a control system with the block diagram of Fig.2.2, the only nonlinearity n_2 is located in the forward path of the inner control loop. This block diagram may represent a positional control system with a tachogenerator feedback. The components in the forward path of the inner control loop are the amplifier, generator and servomotor. If the motor exhibits a quadratic coulomb friction described by the function

$$F(sx) = k_1 (sx)^2 \sin sx \qquad \ldots \qquad (2.57)$$

where, sx represents the angular velocity of the motor shaft. Then,

$$(T_2 s+1)\, sx = \frac{K_2}{T_1 s + 1}\, e - k_2 F(sx) \qquad \ldots \qquad (2.58)$$

where,

x is the shaft position

e is the error signal of the inner control loop

K_2 is the gain constant

T_1 is the time constant of the amplifier

T_2 is the time constant of the servomotor

k_2 is the proportionality factor.

By applying the describing function method, the nonlinear function F(sx) may be expressed as

$$F(sx) = \frac{8k_1}{3} Av\, sx \qquad \ldots \qquad (2.59)$$

and, in accordance with equation (2.58), overall transfer function of amplifier, generator and servomotor becomes-

$$G_2\ (s,\ N_1) = \frac{K_2}{s(T_1 s + 1)(T_2 s + 1 + N_1)} \quad \ldots \quad (2.60)$$

in which $N_1 = (\frac{8k}{3\pi})$ Av is the corresponding describing function and, for convenience, $k = k_1 k_2$. The derivation of equations (2.58)—(2.60) is given only in brief.

In applying conventional design techniques, difficulties arise because the overall transfer function is $G_2(s,\ N_1)$, rather than the simple product $N_1 G_2(s)$. In the parameter plane analysis, this entails no real difficulty, since N_1 can be considered as variable α or β in the usual manner.

The parameter plane enables a relative stability analysis of nonlinear control systems to be performed on the basis of the functions ζ (A) and w_n(A) defined previously. The functions are obtained graphically in the parameter plane by plotting a family of ζ and w_n curves and plotting the corresponding M locus from equations (2.52) and (2.53). At the points of intersections between the family curves and the M-locus, the corresponding values of w_n and ζ are read as functions of the amplitude A which is marked on the M locus. However, for most practical purposes, it is not necessary to determine functions ζ (A) and w_n(A) in entirely, since the evaluation of their extreme values is sufficient to predict the system performance characteristics. Therefore, a relative stability analysis may be directed towards the determination of system parameters which result in a minimum value for the function ζ (A) which is greater than a prescribed value ζ_0 for all possible values of the amplitude A.

In terms of the parameter plane analysis, the formulated relative stability requirement is readily interpreted as determination of system parameters so that the entire M locus is located inside the relative stability region specified for $\xi > \xi_0$.

It is assumed in the relative stability analysis that the corresponding characteristic equation has only one pair of complex control roots, and that other roots are real. In normal control system design, such a root configuration is desired. Thus the parameter plane method which permits the evaluation of all roots in a simple manner, enables the complete relative stability analysis of nonlinear control systems.

2.4. Frequency domain Specifications:

The performance indices of a servo system in the frequency domain are known as the frequency domain specifications. The gain margin and the phase margin are the two of them which measure the degree of stability quantitatively.

Gain margin is the ratio of the gain at which the system becomes unstable to the actual system gain, assuming that the phase of all vectors remains unchanged. The system is just unstable if the polar plot passes through the -1+j0 point. From the plot of Fig.2.3, this condition would exist if the vector along the negative real axis was increased in length until its tip reached the -1+j0 point. The gain margin is then 1/OA and as shown in the Fig. the gain margin is positive.

Phase margin is the amount of negative phase shift which must be introduced (without gain increase)to make the curve pass through -1+j0 point. The vector which must be shifted is obviously that vector which is unit long, and it may be located by drawing a circle with centre at the origin and unit radius. The phase margin is then the angle between this vector and the negative real axis as shown by angle θ in Fig23. The phase margin has the significance of estimating the stability effect of changes of the parameters of the system which affect the phase of G(s)H(s).

It is usually preferable to evaluate the gain margin and the phase margin of a control system from its Bode plot because of ease of construction and the way results obtained from the plot. Here the gain margin and phase margin are obtained from Nyquist plot.

The characteristic equation of a unity feedback closed loop system is given in terms of the parameters α and β by

$$f(s) = P(s)\alpha + Q(s)\beta + R(s) = 0 \qquad \ldots \qquad (2.61)$$

The system can be represented with an equivalent single loop (Refer to Fig.1.2) having the open-loop transfer function-

$$GH(jw) = \frac{P(jw)\alpha + Q(jw)\beta}{R(jw)}$$

$$\therefore \quad 1 + GH = f(s)$$

$$\therefore \quad GH(jw) = \frac{P(jw)\alpha + Q(jw)\beta}{R(jw)} \qquad \ldots \qquad (2.62)$$

We know from the definition that the magnitude of the GH plot when it intersects the unity radius circle is unity and the angle is $-(180^\circ - \theta^\circ)$ where θ° is the phase margin. Therefore (2.62) reduces to-

$$GH = \frac{P(jw)\alpha + Q(jw)\beta}{R(jw)} = 1\underline{/-}(180^\circ - \theta^\circ).$$

$$= \cos(-\theta^\circ) + j\sin(-\theta^\circ)$$

$$= a + jb \ldots$$

Then we have

$$\left[P_1(w) + jP_2(w)\right]\alpha + \left[Q_1(w) + jQ_2(w)\right]\beta = (a+jb)\left[R_1(w) + jR_2(w)\right] \quad \ldots \quad (2.63)$$

Equating real and imaginary terms, one obtains

$$\left.\begin{aligned} P_1(w)\alpha + Q_1(w)\beta &= aR_1(w) - bR_2(w) \\ P_2(w)\alpha + Q_2(w)\beta &= aR_2(w) + bR_1(w) \end{aligned}\right\} \ldots \quad (2.64)$$

From the above equations (2.64), we can obtain two expressions for α and β in terms of w.

$$\alpha = \frac{\begin{vmatrix} aR_1(w) - bR_2(w) & Q_1(w) \\ aR_2(w) + bR_1(w) & Q_2(w) \end{vmatrix}}{\begin{vmatrix} P_1(w) & Q_1(w) \\ P_2(w) & Q_2(w) \end{vmatrix}} \quad \ldots \quad (2.65)$$

$$\beta = \frac{\begin{vmatrix} P_1(w) & aR_1(w) - bR_2(w) \\ P_2(w) & aR_2(w) + bR_1(w) \end{vmatrix}}{\begin{vmatrix} P_1(w) & Q_1(w) \\ P_2(w) & Q_2(w) \end{vmatrix}} \quad \ldots \quad (2.66)$$

Therefore, giving different values to w we can plot the values of α and β in the parameter plane. Hence once the phase margin is given the stability of a particular system can be ascertained in the parameter plane.

From the definition of the gain margin-

20 log $|GH|$ in db = Gain margin in db.

When the phase of GH(jw) is $-180°$

$$GH(jw) = \frac{P(jw)\alpha + Q.(jw)\beta}{R(jw)} = c$$

or $$\left[P_1(w) + jP_2(w)\right]\alpha + \left[Q_1(w) + j\,Q_2(w)\right]\beta = c\left[R_1(w) + jR_2(w)\right] \quad \ldots (2.67)$$

$$\left.\begin{aligned} \text{or, } P_1(w)\alpha + Q_1(w)\beta &= cR_1(w) \\ P_2(w)\alpha + Q_2(w)\beta &= cR_2(w) \end{aligned}\right\} \quad \ldots \quad (2.68)$$

Once the gain margin is known we can proceed exactly the similar way as in the case of phase margin to get two expressions for α and β in terms of w.

$$\alpha = \frac{\begin{vmatrix} cR_1(w) & Q_1(w) \\ cR_2(w) & Q_2(w) \end{vmatrix}}{\begin{vmatrix} P_1(w) & Q_1(w) \\ P_2(w) & Q_2(w) \end{vmatrix}} \quad \ldots \quad (2.69)$$

$$\beta = \frac{\begin{vmatrix} P_1(w) & cR_1(w) \\ P_2(w) & cR_2(w) \end{vmatrix}}{\begin{vmatrix} P_1(w) & Q_1(w) \\ P_2(w) & Q_2(w) \end{vmatrix}} \quad \ldots \quad (2.70)$$

The plotting of the values of α and β in the parameter plane will enable one to check the stability of the system for the given gain margin.

Now a family of curves for constant phase margins and constant gain margins can be plotted in the parameter plane. The values of the parameters α and β can be selected to satisfy the required frequency domain specifications.

2.8. Sensitivity due to some parameter variations:

Frequency domain sensitivity helps to consider the influence of small parameter changes on the frequency response characteristics, and to minimize the mean squared variation of the system response. Side by side with the sensitivity consideration, the standard frequency response technique is used to determine system stability and response characteristics. The frequency domain sensitivity also helps to find out the change of a system parameter which will place a stable system on the verge of instability.

Algebric methods may be applied to the design of a control system and the pole-zero sensitivity can be used to perform the sensitivity analysis.

With the help of time-domain sensitivity an efficient computer method has been devised to investigate control system sensitivity.

These methods enable the designer to satisfy both stability and sensitivity requirements. The design process may involve interchanging the steps for stability and sensitivity considerat-ions until a satisfactory solution is obtained. The methods may help to meet a desired set of system performance characteristics, but this achievement is meagre when compared to the tedious work they involve.

Based on the application of the parameter plane method this paper [4] presents a procedure which enables the stability and sensitivity requirements to be considered simultaneously in control system design. Here, a time-domain sensitivity index is minimized within the required stability limits. Limitations of this technique may come up due to the fact that the form of the system must be defined before hand and also because only two system parameters can be adjusted simultaneously.

In a linear control system subject to an input signal, the error, $e = e\ (t, q_1, q_2, \ldots\ldots, q_r)$ is a function of the time t and a set of system parameters $q_1, q_2, \ldots\ldots, q_r$. To investigate how the small changes in a parameter q_i affect the error signal, parameter influence coefficients u_i have been defined as

$$u_i \triangleq \frac{\partial e}{\partial q_i} , \quad i = (1, 2, \ldots\ldots\ldots\ldots, r) \ldots \qquad (2.71)$$

For a given set of values of the system parameters $q_1, q_2 \ldots \ldots\ldots, q_r$, the function $u_i = u_i(t)$ is a function of time. The shape of the function $u_i(t)$ determines the sensitivity of the error signal to differential changes in the parameter q_i. The function $u_i(t)$ also depends on the input signal which is usually a test signal such as an impulse, step or ramp function. Therefore in order for $u_i(t)$ to be used in the sensitivity analysis of a control system, the input signal must be defined. In this paper, a unit impulse, and step input are chosen as the input test signals.

The main weakness of the coefficient u_i as a measure or index of a system sensitivity is that a function cannot readily serve as a metric system performance. The purpose of the performance index is to represent in a single measure, or in a limited group of such numbers, a quality measure for the performance of the system. The system then can be optimised on the basis of this number. A general index is introduced by

$$I = \int_0^\infty F(u_1, u_2, \ldots\ldots\ldots\ldots, u_r, t)\, dt \ldots\ldots \qquad (2.72)$$

which establishes an integral as a measure of the system sensitivity to small parameter variations. A set of parameter values which minimizes the above integral provides a least-sensitive system according to the defined performance index. In a system design, the choice of the function F should be governed by usefulness in that the particular index which is selected must be a convenient one to use as well as one which yeilds practical results. It should also be noted that, in the design, the parameter influence coefficients should be normalized to become-

$$u_i \triangleq \frac{\partial e}{\partial l_n q_i} \quad , \quad i = (1, 2, \ldots\ldots\ldots, r) \ldots \qquad (2.73)$$

But here for simplicity we shall take the sensitivity index-

$$I_i = \int_0^{\infty} u_i^2 (t)\, dt \qquad \ldots \qquad \ldots \qquad (2.74)$$

The index considers the influence of the small variations in a system parameter q_i. By using Parceval's theorem, the index I_i of (2.74) readily can be expressed as a function of system parameters.

When the test input signal (a unit impulse) is applied to a linear control system, the Laplace transformation of the error signal is a rational function of a given as-

$$E(s) = \frac{h(s)}{f(s)} \qquad \ldots \qquad \ldots \qquad (2.75)$$

where,

$$\left.\begin{aligned} f(s) &= \sum_{K=0}^{m} a_K\, s^K \\ h(s) &= \sum_{K=0}^{n} b_K s^K \quad m \geqslant n \end{aligned}\right\} \ldots \qquad (2.76)$$

The coefficients a_K and b_K are functions of the system parameters-

$$\left.\begin{aligned} a_K &= a_K\, (q_1, q_2, \ldots\ldots\ldots\ldots, q_r) \\ b_K &= b_K\, (q_1, q_2, \ldots\ldots\ldots\ldots, q_r) \end{aligned}\right\} \ldots \qquad (2.77)$$

From (2.71), it follows that the Laplace transformation $U_i(s)$ of the coefficient $u_i(t)$ is-

$$U_i(s) = \frac{\partial E(s)}{\partial q_i} \quad \ldots \quad \ldots \quad \ldots \quad (2.78)$$

using (2.75) - (2.78), one obtains the function $U_i(s)$ in the form

$$U_i(s) = \frac{\sum_{K=0}^{m} a_K s^K \sum_{K=0}^{n} \frac{\partial b_K}{\partial q_i} s^K - \sum_{K=0}^{m} \frac{\partial a_K}{\partial q_i} s^K \sum_{K=0}^{n} b_K s^K}{\left[\sum_{K=0}^{m} a_K s^K\right]^2} \quad (2.79)$$

Thus, $U_i(s)$ is a rational function in s, and Parceval's theorem can be used to express the integral I_i of (2.74) as

$$I_i = \frac{1}{2\pi j} \int_{-j\infty}^{+j\infty} U_i(s)\, U_i(-s)\, ds \quad \ldots \quad (2.80)$$

By substituting (2.79) into (2.80) and using the standard integral tables (Appendix 6.2), the integral I_i of (2.80) can be calculated in terms of the system parameters to obtain

$$I_i = I_i\,(q_1, q_2, \ldots\ldots\ldots\ldots, q_r) \quad \ldots \quad (2.81)$$

Thus the minimization of the sensitivity is reduced to a minimization of the function I_i with respect to the system parameters $q_1, q_2, \ldots\ldots\ldots, q_r$.

It should be noted that the proposed index of sensitivity can be readily applied to any signal in the system other than the error signal. Furthermore, the form of the index can be extended to the case when the sensitivity of several signals with respect to multiparameter variations is to be considered.

After the sensitivity index is expressed as a function of system parameters, the minimization procedure can be done formally in the standard manner of equating the partial derivatives of the obtained function to zero and solving the resultant set of equations.

$$\frac{\partial I_i}{\partial q_j} = 0, \qquad j = (1, 2, \ldots\ldots\ldots\ldots, r) \qquad \ldots \qquad (2.82)$$

for the values of the parameters. The set of parameters which simultaneously satisfies (2.82) corresponds to a system which is relatively insensitive to small variations in the parameter q_i. Before the solution set of parameters is accepted, the higher order derivatives should be calculated to verify the minimum of function I_i.

The formal minimization procedure outlined above, suffers from a serious limitation in that the obtained set of parameters may result in a unstable, or even unrealizable, system. Therefore, in the minimization of the sensitivity index, certain stability limits have to be introduced. The design objective here will be to determine values of system parameters which minimize the sensitivity index under the constraint that all the roots of the system characteristic equation lie at certain desired root locations. These simultaneous stability and sensitivity considerations are performed in the parameter plane.

In the parameter plane method, the characteristic equation of the system is considered in the form given in (2.1) and the complex variable s is given in (2.2), two system parameters α and β are given in (2.26).

Parameters α and β are chosen among the parameters $q_1, q_2, \ldots\ldots\ldots, q_r$ which appear in the coefficients a_k (2.77). They can be chosen as either parameters of the controller or as parameters of the controlled part of the system.

In the $\alpha\beta$ plane, characteristic curves are plotted using

(3.1), and the regions are determined for which all the roots of (3.23) have the damping coefficient ξ greater than a specified value. Furthermore, since the parameter plane diagram places in evidence all the root values, the system response characteristics can be observed easily. Then, if all the parameters $q_1, q_2, \ldots\ldots,$ q_r with the exception of α and β, are specified, the performance index I_d is only a function of these two variable parameters. Therefore, the I_d = constant curves can be plotted in the parameter plane along with the diagrams which determine the system stability and response. The values of the parameters α and β, which minimize the sensitivity index inside a desired relative damping region, correspond to a design solution. After the parameter plane diagram is plotted and the I_d = constant curves are introduced, the solution values of α and β are found by inspection.

CHAPTER - 3

APPLICATION OF THE METHOD TO SOME NUMERICAL EXAMPLES

The parameter plane method is here applied to two numerical examples of compensation. The nature of the transfer function of the compensator is assumed to be known and only two parameters are assumed to be adjustable. The realization of the compensator is not our problem and so it is not discussed here.

In the first example, the compensator is used only in the inner feedback path. Both the gain and time constants are taken to be the adjustable parameters.

In the second example, the compensator is used both in the forward and in the inner feedback paths. The effective adjustable parameters are the gain K_2 associated with the main plant, and the gain K_{-1} associated with the inner feedback loop.

3.1. Example 1:

Refering to Fig. 1.3, the loop transfer functions are given below-

$$\left.\begin{aligned} G_1(s) &= \frac{s+1}{0.3s+1} \\ G_2(s) &= \frac{10}{s^2+3s+1} \\ G_3(s) &= 1 \\ G_{-1}(s) &= \frac{K_{-1}}{T_{-1}s+1} \end{aligned}\right\} \qquad \ldots \qquad \ldots \qquad (3.1)$$

The characteristic equation corresponding to (3.1) is-

$$f(s) = 0.3T_{-1}s^4 + (1.9T_{-1}+0.3)s^3 + (13.3T_{-1}+1.9)s^2 + (11T_{-1}+3K_{-1}+13.3)s + 11 + 10K_{-1} = 0 \qquad \ldots \qquad \ldots \qquad (3.2)$$

It is considered that-

$T_{-1} = \alpha$ and $K_{-1} = \beta$

We shall check both the absolute ($\xi = 0$) and the relative ($0 \leqslant \xi \leqslant 1$) stabilities of the given system.

For $\xi = 0.0$

$$\left.\begin{aligned} \alpha &= \frac{2.7v_n^2 + 100.0}{0.9v_n^4 - 20.9v_n^2 - 110.0} \quad \ldots \quad \ldots \\ \beta &= \frac{-0.09v_n^6 - 4.37v_n^4 + 135.0v_n^2 + 121.0}{0.9v_n^4 - 20.9v_n^2 - 110.0} \end{aligned}\right\} \quad (3.3)$$

For $\xi = 0.2$

$$\left.\begin{aligned} \alpha &= \frac{-0.36v_n^3 + 3.18v_n^2 - 7.6v_n + 100.0}{0.756v_n^4 - 0.072v_n^3 - 23.94v_n^2 + 53.2v_n - 110.0} \\ \beta &= \frac{0.09v_n^6 - 0.2v_n^5 - 3.7v_n^4 - 9.0v_n^3 + 140.6v_n^2 - 53.72v_n + 121.0}{0.756v_n^4 - 0.072v_n^3 - 23.94v_n^2 + 53.2v_n - 110.0} \end{aligned}\right\} \quad (3.4)$$

For $\xi = 0.3$,

$$\left.\begin{aligned} \alpha &= \frac{-0.54v_n^3 + 3.75v_n^2 - 44.4v_n + 133.0}{0.576v_n^4 + 0.468v_n^3 - 27.74v_n^2 + 79.8v_n - 110.0} \\ \beta &= \frac{0.0899v_n^6 - 0.224v_n^5 + 4.59v_n^4 - 9.92v_n^3 + 142.7v_n^2 + 88.0v_n + 121.0}{0.576v_n^4 + 0.468v_n^3 - 27.74v_n^2 + 79.8v_n - 110.0} \end{aligned}\right\} \quad (3.5)$$

For $\xi = 0.5$,

$$\left.\begin{aligned} \alpha &= \frac{-0.9v_n^3 + 5.7v_n^2 - 19.0v_n + 100.0}{2.7v_n^3 - 39.9v_n^2 + 133.0\, v_n - 110.0} \\ \beta &= \frac{0.09v_n^6 - 0.57v_n^5 - 0.38v_n^4 - 7.37v_n^3 + 187.\,v_n^2 - 146.0v_n + 121.0}{27.v_n^3 - 39.9v_n^2 + 133.0v_n - 110.0} \end{aligned}\right\} \quad (3.6)$$

For a phase margin of 30°, the parameters α and β are given by-

$$\left.\begin{aligned}\alpha &= \frac{-0.45w_n^4+2.38w_n^3+10.45w_n^2+86.71w_n+55.0}{0.9w_n^5-20.9w_n^3-110.0\,w_n}\\ \beta &= \frac{0.078w_n^7+0.001w_n^6-3.78w_n^5-0.935w_n^4+117.1w_n^3+104.83w_n}{0.9w_n^5-20.9w_n^3-110.0\,w_n}\end{aligned}\right\}\quad(3.7)$$

For a phase margin of 45°-

$$\left.\begin{aligned}\alpha &= \frac{-0.636w_n^4+1.9w_n^3-14.8w_n^2+70.66w_n+77.8}{0.9w_n^5-20.9w_n^3-110.0w_n}\\ \beta &= \frac{0.063w_n^7+0.001w_n^6-3.09w_n^5+0.096w_n^4+94.5w_n^3-0.1w_n^2-0.1w_n^2+86.58}{0.9w_n^5-20.9w_n^3-110.0\,w_n}\end{aligned}\right\}\quad(3.8)$$

For a gain margin of 6 db, the parameters are expressed as-

$$\left.\begin{aligned}\alpha &= \frac{1.35w_n^2+50.0}{0.9w_n^4-20.9w_n^2-110.0}\\ \beta &= \frac{0.045w_n^6-2.1w_n^4+67.6w_n^2+60.5}{0.9w_n^4-20.9w_n^2-110.0}\end{aligned}\right\}\quad\ldots\quad(3.9)$$

For a gain margin of 12 db,

$$\left.\begin{aligned}\alpha &= \frac{0.678w_n^2+25.00}{0.9w_n^4-20.9w_n^2-110.0}\\ \beta &= \frac{0.0225\,w_n^6-1.0938w_n^4+33.78w_n^2+30.2}{0.9w_n^4-20.9w_n^2-110.0}\end{aligned}\right\}\quad\ldots\quad(3.10)$$

For an impulse input, the actuating signal-

$$E(s)=\frac{0.3T_{-1}s^4+(1.9T_{-1}+0.3)s^3+(3.3T_{-1}+1.9)s^2+(T_{-1}+3.0K_{-1}+3.3)s+1+10K_{-1}}{0.3T_{-1}s^4+(1.9T_{-1}+0.3)s^3+(13.3T_{-1}+1.9)s^2+(11T_{-1}+3K_{-1}+13.3)s+11+10K_{-1}}\quad\ldots\quad(3.11)$$

The sensitivity index U_β is given by

$$U_\beta = \frac{30.0\alpha s^3 + (130.0\alpha + 30.0)s^2 + (100.0\alpha + 130.0)s + 100.0}{D_1} \qquad (3.12)$$

where $D_1 = \Big[0.09\alpha^2 s^8 + (1.14\alpha^2 + 0.18\alpha)s^7 + (11.59\alpha^2 + 0.09 + 2.28\alpha)s^6$
$(57.1\alpha^2 + 1.8\alpha\beta + 23.18\alpha + 1.14)s^5 + (218.69\alpha^2 + 11.59 +$
$114.24\alpha + 17.4\alpha\beta + 1.8\beta)s^4 + (57.1 + 118.0\alpha\beta + 17.4\beta + 292.2\alpha^2 +$
$438.52\alpha)s^3 + (584.0\alpha + 121.0\alpha^2 + 9.0\beta^2 + 218.69 + 332.0\alpha\beta +$
$117.8\beta)s^2 + (60.0\beta^2 + 332.0\beta + 293.0 + 242.0\alpha + 220.0\alpha\beta)s +$
$220.0\beta + 100\beta^2 + 121.0\Big]$

... (3.13)

∴ We get-

$c_7 = 0,\ c_6 = 0,\ c_5 = 0,\ c_4 = 0,\ c_3 = 30.0\alpha,\ c_2 = 130.0\alpha + 30.0$

$c_1 = 100.0\alpha + 130.0, \quad c_0 = 100.0$

and $d_8 = 0.09\alpha^2, \quad d_7 = 1.14\alpha^2 + 0.18\alpha,$

$d_6 = 11.59\alpha^2 + 0.09 + 2.28\alpha,\ d_5 = 57.1\alpha^2 + 1.8\alpha\beta + 23.18\alpha + 1.14,$

$d_4 = 218.69\alpha^2 + 11.59 + 114.24\alpha + 17.4\alpha\beta + 1.8\beta$

$d_3 = 57.1 + 118.0\alpha\beta + 17.4\beta + 292.2\alpha^2 + 438.52\alpha$

$d_2 = 584.0\alpha + 121.0\alpha^2 + 9.0\beta^2 + 218.69 + 332.0\alpha\beta + 117.8\beta$

$d_1 = 220.0\beta + 100.0\beta^2 + 121.0$

... (3.14)

The sensitivity index for a step input-

$$U_\beta = \frac{30.0\alpha s^3 + (130.0\alpha + 30.0)s^2 + (100.0\alpha + 130.0)s + 100.0}{D_2} \qquad (3.15)$$

where $D_2 = \Big[0.09\,\alpha^2 s^9 + (1.14\,\alpha^2 + 0.18\,\alpha)s^8 + (11.59\,\alpha^2 + 0.09 + 2.28\,\alpha)s^7 +$

$(57.1\,\alpha^2 + 1.8\,\alpha\beta + 23.18\,\alpha + 1.14)s^6 + (218.69\,\alpha^2 + 11.59 +$

$114.24\,\alpha + 17.4\,\alpha\beta + 1.8\,\beta)s^5 + (57.1 + 118.0\,\alpha\beta + 17.4\,\beta +$

$292.2\,\alpha^2 + 438.52\,\alpha)s^4 + (584.0\,\alpha + 121.0\,\alpha^2 + 9.0\,\beta^2 + 218.69 +$

$332.0\,\alpha\beta + 117.8\,\beta)s^3 + (60.0\,\beta^2 + 332.0\,\beta + 293.0 + 242.0\,\alpha + 220.0\,\alpha\beta)$

$(220.0\,\beta + 100.0\,\beta^2 + 121.0)s\Big]$

... ... (3.16)

and

$c_8 = 0,\ c_7 = 0,\ c_6 = 0,\ c_5 = 0,\ c_4 = 0,\ c_3 = 30.0\,\alpha,$

$c_2 = 130.0\,\alpha + 30.0,\quad c_1 = 100.0\,\alpha + 130.0,\quad c_0 = 100.0$

$d_9 = 0.09\,\alpha^2,\ d_8 = 1.14\,\alpha^2 + 0.18\,\alpha,\ d_7 = 11.59\,\alpha^2 + 0.09 + 2.28\,\alpha,$

$d_6 = 57.1\,\alpha^2 + 1.8\,\alpha\beta + 23.18\,\alpha + 1.14,$

$d_5 = 218.69\,\alpha^2 + 11.59 + 114.24\,\alpha + 17.4\,\alpha\beta + 1.8\,\beta,$ (3.17)

$d_4 = 57.1 + 118.0\,\alpha\beta + 292.2\,\alpha^2 + 438.52\,\alpha + 17.4\,\beta,$

$d_3 = 584.0\,\alpha + 121.0\,\alpha^2 + 9.0\,\beta^2 + 218.69 + 332.0\,\alpha\beta + 117.8\,\beta,$

$d_2 = 60.0\,\beta^2 + 332.0\,\beta + 293.0 + 242.0\,\alpha + 220.0\,\alpha\beta,$

$d_1 = 220.0\,\beta + 100.0\,\beta^2 + 121.0,$

$d_0 = 0$

The sensitivity index corresponding to α for an impulse input =

$$U_\alpha = \frac{-(30.0s^3 + 130.0s^2 + 100.0s)\beta}{D_3} \quad \ldots \quad (3.18)$$

where, $D_3 = D_1$

$c_7 = 0,\ c_6 = 0,\ c_5 = 0,\ c_4 = 0,\ c_3 = -30.0\,\beta,$

$c_2 = -130.0\,\beta,\quad c_1 = -100\,\beta,\quad c_0 = 0$

and

$d_8 = 0.09\alpha^2$, $d_7 = 1.14\alpha^2 + 0.18\alpha$, $d_6 = 11.59\alpha^2 + 0.09 + 2.28\alpha$,

$d_5 = 57.1\alpha^2 + 1.8\alpha\beta + 23.18\alpha + 1.14$,

$d_4 = 218.69\alpha^2 + 11.59 + 114.24\alpha + 17.4\alpha\beta + 1.8\beta$,

$d_3 = 57.1 + 118.0\alpha\beta + 17.4\beta + 292.2\alpha^2 + 438.52\alpha$,

$d_2 = 584.0\alpha + 121.0\alpha^2 + 9.0\beta^2 + 218.69 + 332.0\alpha\beta + 117.8\beta$, (3.19)

$d_1 = 60.0\beta^2 + 332.0\beta + 293.0 + 242.0\alpha + 220.0\alpha\beta$,

$d_0 = 220.0\beta + 100.0\beta^2 + 121.0$

Sensitivity index corresponding to α for a step input

$$U_\alpha = \frac{-(30.0s^3 + 130.0s^2 + 100.0s)\beta}{D_4} \quad \ldots \quad \ldots \quad (3.20)$$

where, $D_4 = D_2$;

$c_8 = 0$, $c_7 = 0$, $c_6 = 0$, $c_5 = 0$, $c_4 = 0$, $c_3 = -30.0\beta$, $c_2 = -130.0\beta$

$c_1 = -100.0\beta$, $c_0 = 0$

$d_9 = 0.09\alpha^2$, $d_8 = 1.14\alpha^2 + 0.18\alpha$, $d_7 = 11.59\alpha^2 + 0.09 + 2.28\alpha$,

$d_6 = 57.1\alpha^2 + 1.8\alpha\beta + 23.18\alpha + 1.14$,

$d_5 = 218.69\alpha^2 + 11.59 + 114.24\alpha + 17.4\alpha\beta + 1.8\beta$,

$d_4 = 57.1 + 118.0\alpha\beta + 17.4\beta + 292.2\alpha^2 + 438.52\alpha$, (3.2

$d_3 = 584.0\alpha + 121.0\alpha^2 + 9.0\beta^2 + 218.69 + 332.0\alpha\beta + 117.8\beta$,

$d_2 = 60.0\beta^2 + 332.0\beta + 293.0 + 242.0\alpha + 220.0\alpha\beta$,

$d_1 = 220.0\beta + 100.0\beta^2 + 121.0$,

$d_0 = 0$

3.2. Example 2:

Refering to Fig.1.3, the loop transfer functions are given

$$\left.\begin{array}{l} G_1(s) = \dfrac{s+1}{0.3s+1}, \\[2ex] G_2(s) = \dfrac{K_2}{s^2+3s+1}, \\[2ex] G_3(s) = \dfrac{2}{s+1}, \\[2ex] G_{-1}(s) = \dfrac{K_{-1}}{0.\text{[illegible]}\ 1} \end{array}\right\} \quad \ldots \quad \ldots \quad (3.22)$$

The characteristic equation corresponding to (3.22) is-

$$f(s) = 0.15s^4 + 1.26s^3 + 3.66s^2 + (3.8 + 0.3K_2K_{-1} + K_2)s + K_2K_{-1} + 2K_2 + 1 = 0 \quad \ldots \quad \ldots \quad (3.23)$$

We shall assume-

$$K_2K_{-1} = \alpha, \quad K_2 = \beta$$

For $\xi = 0.0$,

$$\left.\begin{array}{l} \alpha = -0.375\, w_n^4 + 2.625\, w_n^2 + 16.25 \\ \beta = 0.1125\, w_n^4 + 0.4625\, w_n^2 - 8.75 \end{array}\right\} \quad \ldots \quad (3.24)$$

For $\xi = 0.2$,

$$\left.\begin{array}{l} \alpha = -0.315\, w_n^4 - 0.698 w_n^3 + 3.625 w_n^2 - 7.1\, w_n + 16.5 \\ \beta = 0.0945 w_n^4 + 0.099 w_n^3 + 3.65 w_n - 8.75 \end{array}\right\} \quad \ldots \quad (3.25)$$

For $\xi = 0.3$,

$$\left.\begin{array}{l} \alpha = -0.24 w_n^4 - 1.1375\, w_n^3 + 4.875 w_n^2 - 10.65 w_n + 16.25 \\ \beta = 0.072\, w_n^4 + 0.195 w_n^3 - 6.625 w_n^2 + 5.25 w_n - 8.75 \end{array}\right\} \quad \ldots \quad (3.26)$$

For $\xi = 0.5$,

$$\left.\begin{array}{l} \alpha = -2.375 w_n^3 + 8.875 w_n^2 - 17.75\, w_n + 16.25 \\ \beta = 0.5625\, w_n^3 - 2.6375 w_n^2 + 8.875 w_n - 8.75 \end{array}\right\} \quad \ldots \quad (3.27)$$

For a phase margin of 30°, the parameters are-

$$\left.\begin{aligned} \alpha &= \frac{-0.13w_n^5 - 0.525w_n^4 + 0.91w_n^3 - 1.65w_n^2 + 6.7w_n + 1}{0.4w_n} \\ \beta &= \frac{0.04w_n^5 + 0.127w_n^4 + 0.16w_n^3 + 1.2w_n^2 - 3.0w_n - 0.5}{0.4w_n} \end{aligned}\right\} \quad \ldots \quad (3.28)$$

For a phase margin of 45°

$$\left.\begin{aligned} \alpha &= \frac{-0.106w_n^5 - 0.673w_n^4 + 0.74w_n^3 - 2.33w_n^2 + 4.673w_n + 1.414}{0.4w_n} \\ \beta &= \frac{0.0318w_n^5 + 0.1595w_n^4 + 0.132w_n^3 + 1.703w_n^2 - 2.478w_n - 0.707}{0.4w_n} \end{aligned}\right\} \quad (3.29)$$

For a gain margin of 6 db

$$\left.\begin{aligned} \alpha &= -0.19w_n^4 + 1.31w_n^2 + 8.25 \\ \beta &= 0.057w_n^4 + 0.22\,w_n^2 - 4.37 \end{aligned}\right\} \quad \ldots \quad (3.30)$$

For a gain margin of 12 db

$$\left.\begin{aligned} \alpha &= -0.09375\,w_n^4 + 0.66875\,w_n^2 + 4.125 \\ \beta &= 0.028125\,w_n^4 + 0.1165w_n^2 - 2.1875 \end{aligned}\right\} \quad \ldots \quad (3.31)$$

For an impulse input the actuating signal

$$E(s) = \frac{0.15s^4 + 1.25s^3 + (3.8 + 0.3\alpha)s + 1 + \alpha + 3.55\,s^2}{0.15s^4 + 1.25s^3 + 3.55s^2 + (3.8 + 0.3\alpha + \beta)s + \alpha + 2\beta + 1} \quad (3.32)$$

The sensitivity index-

$$U_\beta = \frac{-\left[0.15s^5 + 1.55s^4 + 6.05s^3 + (10.9 + 0.3\alpha)s^2 + (8.6 + 1.6\alpha)s + 2 + 2\alpha\right]}{D_5} \quad (3.33)$$

where
$$D_5 = \Big[0.0225s^8 + 0.375s^7 + 2.6275s^6 + (10.01 + 0.09\alpha + 0.3\beta)s^5 + (22.1 + 1.05\alpha + 3.1\beta)s^4 + (4.63\alpha + 12.1\beta + 30.5)s^3 + (0.09\alpha^2 + 21.5 + 9.38\alpha + 21.8\beta + \beta^2 + 0.6\alpha\beta)s^2 + (8.2\alpha + 17.2\beta + 7.6 + 0.6\alpha^2 + 4.0\beta^2 + 3.2\alpha\beta)s + \alpha^2 + 4.0\beta^2 + 1.0 + 4.0\alpha\beta + 2.0\alpha + 4.0\beta\Big] \quad \ldots\ldots \quad \ldots \quad (3.34)$$

We have,

$$c_7 = 0,\ c_6 = 0,\ c_5 = -0.15,\ c_4 = -1.55,\ c_3 = -6.05,$$
$$c_2 = -(10.9 + 0.3\alpha),\ c_1 = -(8.6 + 1.6\alpha),\ c_0 = -(2 + 2\alpha)$$

and

$$d_8 = 0.0225,\ d_7 = 0.375,\ d_6 = 2.6275,\ d_5 = 10.01 + 0.09\alpha + 0.3\beta$$
$$d_4 = 22.1 + 1.05\alpha + 3.1\beta,\ d_3 = 4.63\alpha + 12.1\beta + 30.5,$$
$$d_2 = 0.09\alpha^2 + 21.5 + 9.38\alpha + 21.8\beta + \beta^2 + 0.6\alpha\beta,$$
$$d_1 = 8.2\alpha + 17.2\beta + 7.6 + 0.6\alpha^2 + 4.0\beta^2 + 3.2\alpha\beta,$$
$$d_0 = \alpha^2 + 4.0\beta^2 + 1.0 + 4.0\alpha\beta + 2.0\alpha + 4.0\beta$$

$$\ldots \quad \ldots \quad (3.35)$$

For a unit step input the sensitivity index-

$$U_\beta = \frac{-\big[0.15s^5 + 1.55s^4 + 6.05s^3 + (10.9 + 0.3\alpha)s^2 + (8.6 + 1.6\alpha)s + 2 + 2\alpha\big]}{D_6} \quad (3.36)$$

where,
$$D_6 = \Big[0.0225s^9 + 0.375s^8 + 2.6275s^7 + (10.01 + 0.09\alpha + 0.3\beta)s^6 + (22.1 + 1.05\alpha + 3.1\beta)s^5 + (4.63\alpha + 12.1\beta + 30.5)s^4 + (0.09\alpha^2 + 21.5 + 9.38\alpha + 21.8\beta + \beta^2 + 0.6\alpha\beta)s^3 + (8.2\alpha + 17.2\beta + 7.6 + 0.6\alpha^2 + 4.0\beta^2 + 3.2\alpha\beta)s^2 + (\alpha^2 + 4.0\beta^2 + 1.0 + 4.0\alpha\beta + 2.0\alpha + 4.0\beta)s\Big] \quad (3.37)$$

$c_8=0,\ c_7=0,\ c_6=0,\ c_5=-0.15,\ c_4=-1.55,\ c_3=-6.05$

$c_2=-(10.9+0.3\alpha),\ c_1=-(8.6+1.6\alpha),\ c_0=-(2+2\alpha)$

and

$$\begin{aligned}
&d_9=0.0225,\ d_8=0.375,\ d_7=2.6275,\ d_6=10.01+0.09\alpha+0.3\beta,\\
&d_5=22.1+1.05\alpha+3.1\beta,\ d_4=4.63\alpha+12.1\beta+30.5,\\
&d_3=0.09\alpha^2+21.5+9.38\alpha+21.8\beta+\beta^2+0.6\alpha\beta,\\
&d_2=8.2\alpha+17.2\beta+7.6+0.6\alpha^2+4.0\beta^2+3.2\alpha\beta,\\
&d_1=\alpha^2+4.0\beta^2+1.0+4.0\alpha\beta+2.0\alpha+4.0\beta,\\
&d_0=0
\end{aligned} \qquad (3.38)$$

Sensitivity index corresponding to α for an impulse input

$$U_\alpha=\frac{(0.3s^2+1.6s+2.0)\beta}{D_7} \qquad \ldots \qquad \ldots \qquad (3.39)$$

where, $D_7=D_5$,

$c_7=0,\ c_6=0,\ c_5=0,\ c_4=0,\ c_3=0,\ c_2=0.3\beta,$

$c_1=1.6\beta,\ c_0=2.0\beta$

and,

$$\begin{aligned}
&d_8=0.0225,\ d_7=0.375,\ d_6=26275,\\
&d_5=10.01+0.09\alpha+0.3\beta,\ d_4=22.1+1.05\alpha+3.1\beta,\\
&d_3=4.63\alpha+12.1\beta+30.5,\ d_2=0.09\alpha^2+21.5+9.38\alpha+21.8\beta+\beta^2+0.6\alpha\beta\\
&d_1=8.2\alpha+17.2\beta+7.6+0.6\alpha^2+4.0\beta^2+3.2\alpha\beta,\\
&d_0=\alpha^2+4.0\beta^2+1.0+4.0\alpha\beta+2.0\alpha+4.0\beta
\end{aligned}$$

$$\ldots \qquad (3.40)$$

Sensitivity index corresponding to α for a step input signal

$$U_\alpha=\frac{(0.3s^2+1.6s+2)\beta}{D_8} \qquad \ldots \qquad (3.41)$$

where $D_8 = D_6$

$c_8=0,\ c_7=0,\ c_6=0,\ c_5=0,\ c_4=0,\ c_3=0,\ c_2=0.3\beta$,

$c_1=1.6\beta,\ c_0 = 2.0\beta$

$d_9= 0.0225,\ d_8 =0.375,\ d_7 = 2.6275,\ \ d_6=10.01+0.09\alpha+0.3\beta$

$d_5= 22.1 +1.05\alpha+3.1\beta,\ d_4=4.63\alpha +12.1\beta +30.5,$

$$d_3= 0.09\alpha^2 +21.5 + 9.38\alpha+21.8\beta +\beta^2 + 0.6\alpha\beta, \qquad (3.42)$$

$d_2= 8.2\alpha+ 17.2\beta + 7.6 + 0.6\alpha^2 + 4.0\beta^2 + 3.2\alpha\beta,$

$d_1= \alpha^2 + 4.0\beta^2 + 1.0 + 4.0\alpha\beta+ 2.0\alpha +4.0\beta$,

$d_0= 0$

CHAPTER - 4

RESULTS AND DISCUSSION

In both the numerical examples, we have obtained the expressions of α and β in terms of w for the absolute stability, relative stabilities, phase margin, gain margin and sensitivity functions. The corresponding curves are plotted in the parameter plane [shown in Figs. 4.1(a), 4.1(b) and 4.2(a), 4.2(b)] by varying w from o to ∞. Some of the results along with the computer programming have been shown in Appendix 5.4. Taking any point within the stability region and then applying Routh Hurwitz Criterion we can check our calculations and manipulations. The sensitivity of both the systems corresponding to each parameter and within certain stability constraints will be interpreted here.

4.1. Example 1:

Let us take any arbitrary point M(.2,5) within the stable region of Fig.4.1(a).

The characteristic equation corresponding to this point is

$$f(s) = 0.06\,s^4 + 0.68s^3 + 4.56s^2 + 33.5s + 71 = 0 \quad .. \quad (4.1)$$

Routh Hurwitz Criterion is now applied.

We have-

s^4	.06	4.56	0.71
s^3	0.68	33.5	0
s^2	2.2	71	0
s^1	11.0	0	
s^0	71	0	

.. (4.2)

The system has been found stable.

To find a point of minimum sensitivity for a specified damping ratio in the finite $\alpha\beta$ plan is not possible. But

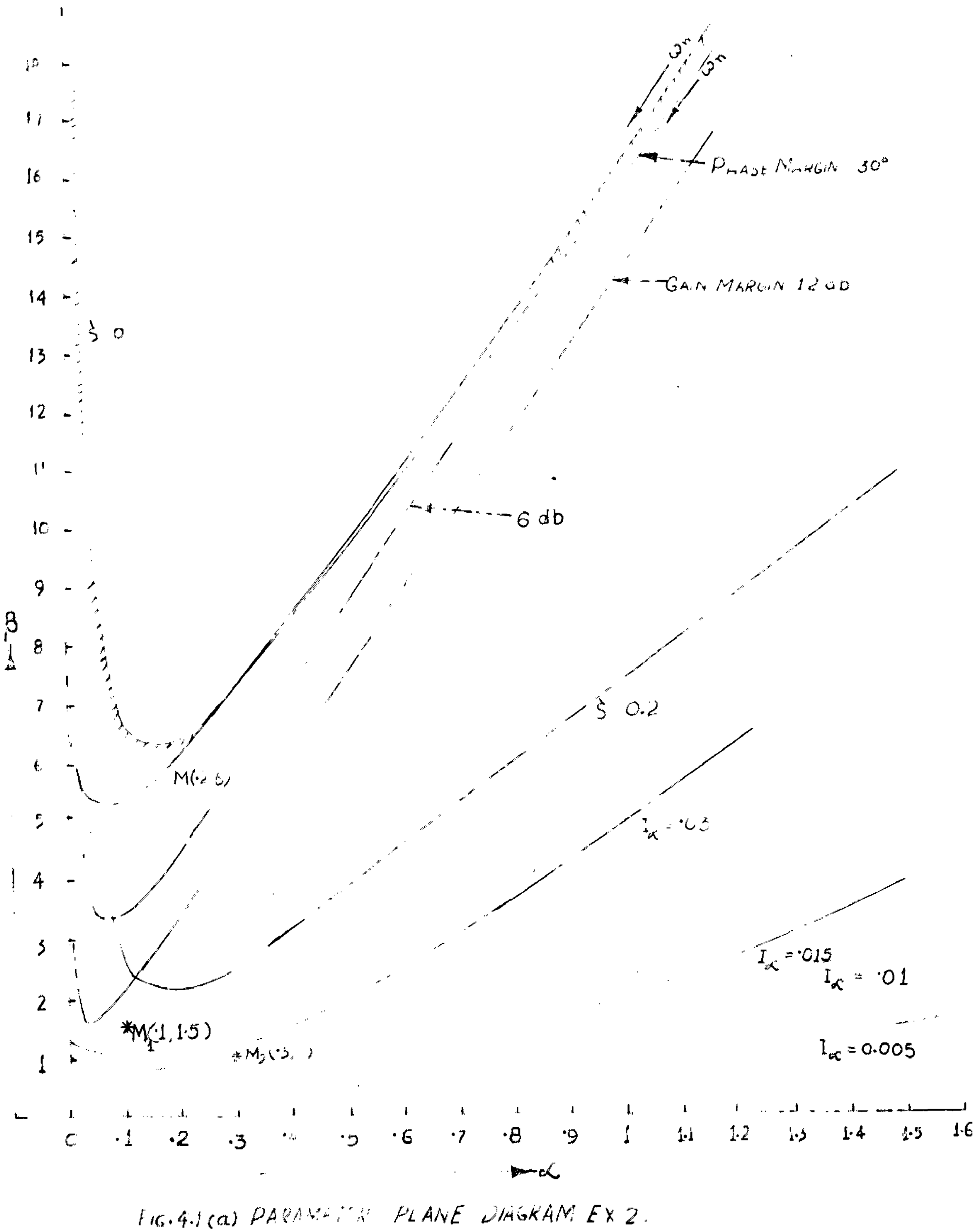


FIG. 4·1 (a) PARAMETER PLANE DIAGRAM EX 2.

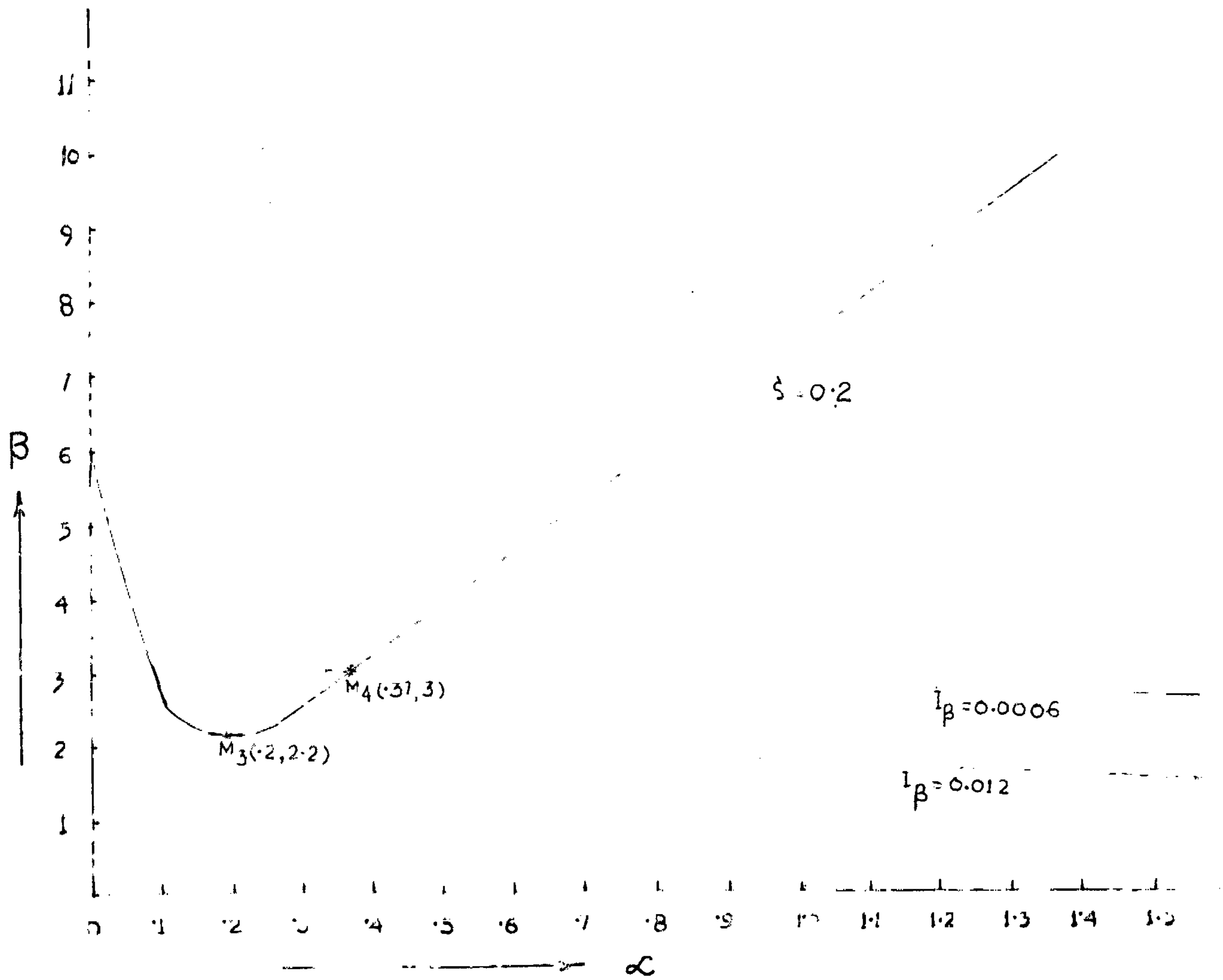


FIG.4.1-(b) PARAMETER PLANE DIAGRAM Ex 1

we can utilise the I = constant curves corresponding to each parameter for choosing a desired system response. In Fig.4.1(a) two operating points $M_1(0.1, 1.5)$ and $M_2(0.3, 1.0)$ may represent two systems which may have satisfactory system response. According to the pole-zero configurations of their closed-loop transfer functions.

$$\left.\begin{array}{lll} M_1 : \text{Poles at} & -1.5, -10, & -1.4 \pm j\,6.86 \\ \ \text{Zeros at} & -10, -25 & \\ M_2 : \text{Poles at} & -1.7, -6 & -1.24 \pm j\,6.09 \\ \ \text{Zeros at} & -3.3, -6.6 & \end{array}\right\} \cdots \quad (4.3)$$

The point M_2 is preferable because M_1 corresponds to higher sensitivity.

Next we shall consider two points $M_3(0.2, 2.2)$ and $M_4(0.37, 3)$ of Fig.4.1(b).

Preceeding as in the previous case we have-

$$\left.\begin{array}{ll} M_3 : \text{Poles at} & -2, -6, -1.3 \pm j6.37 \\ \ \text{Zeros at} & -5, -6 \\ M_4 : \text{Poles at} & -3, -4, \ -1.2 \pm j5.88 \\ \ \text{Zeros at} & -2.7, -11.0 \end{array}\right\} \cdots \quad (4.4)$$

Here we have seen that the value of sensitivity is almost independent of $\mathcal{L}$ and depends only on the value of β.

M_4 is preferable because it corresponds to minimum sensitivity.

4.2. <u>Example 2:</u>

We shall consider the arbitrary point M(3,6) in Fig.4.2(a). The characteristic equation becomes-

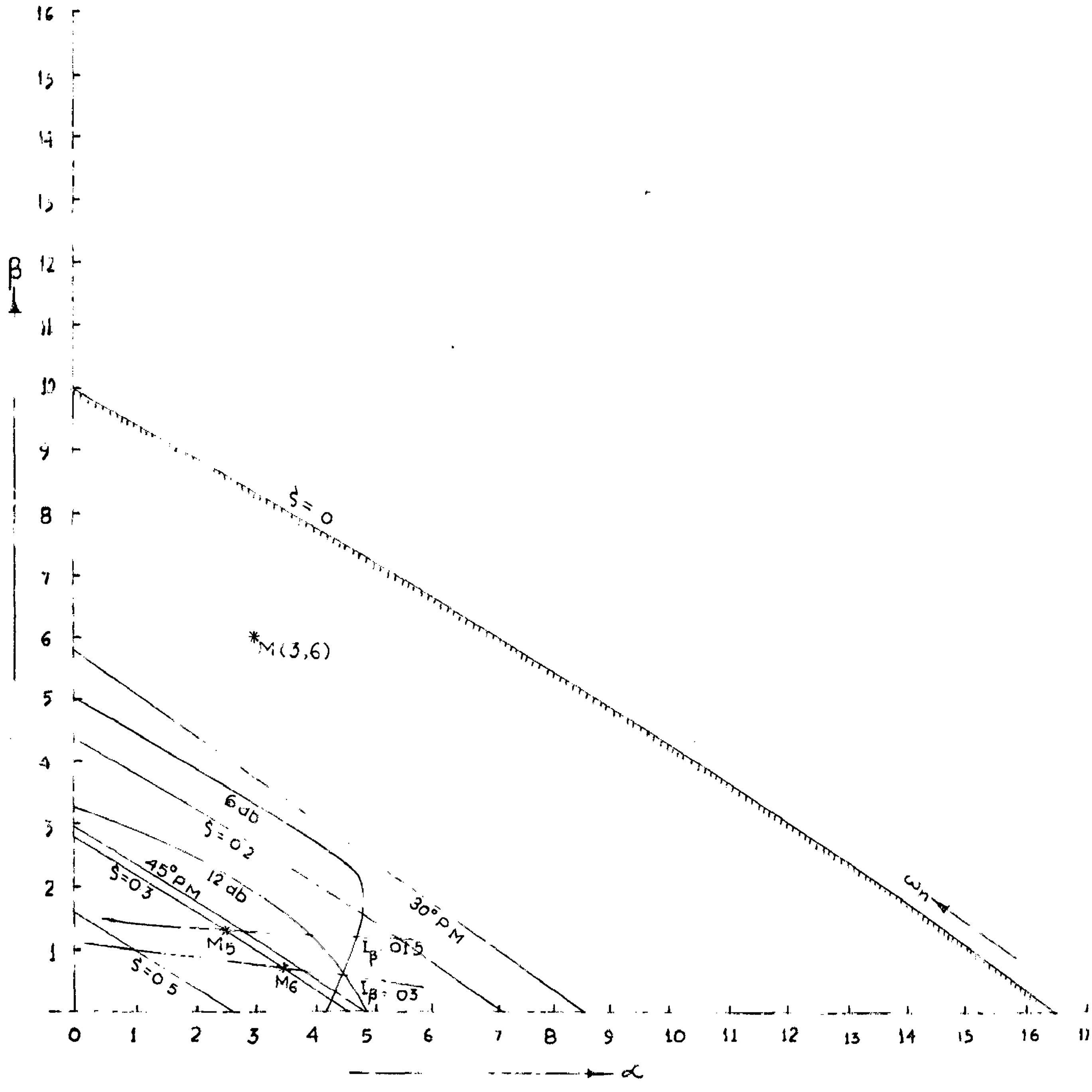


FIG. 4.2 (a) PARAMETER PLANE DIAGRAM EX.2.

$$f(s) = 0.15s^4 + 1.25s^3 + 3.55s^2 + 10.7s + 16 = 0 \quad .. \quad (4.5)$$

As in the previous example we have-

s^4	0.15	3.55	16	
s^3	1.25	10.7	0	
s^2	2.23	16	0	... (4.6)
s^1	1.8	0		
s^0	16	0		

(4.6) shows the system to be stable.

Now if we put the constraint of stability i.e. a damping ratio $\xi = 0.3$ and consider two points M_5 (2.5, 1.3), M_6(3.5, 0.7) proceeding as in the previous example we obtain-

M_5 : Poles at $-2.75,\ -4.5,\ -0.57 \pm j1.8$

Zeros at $-2,\ -6$ ... (4.7)

M_6 : Poles at $-3,\ -4,\ -0.54 \pm j\,1.11$

Zeros at $-2,\ -12$

Here sensitivity curves w.r.t. β are almost independent of the value of α and depends only on the value of β. The point M_5 is preferred because of minimum sensitivity.

Next we shall consider two points M_7(1.85, 1.7), and M_8 (3.35, 0.8) of Fig.4.2(b). The pole zero configuration of closed loop transfer functions of the systems shows that-

M_7 : Poles at $-2.5,\ -5,\ -0.6 \pm j\,1.9$

Zeros at $-2,\ -4.2$

M_8 : Poles at $-3,\ -4,\ -0.54 \pm j\,1.11$ ... (4.8)

Zeros at $-2,\ -10.4$

On the consideration of minimum sensitivity the point M_8 is preferred.

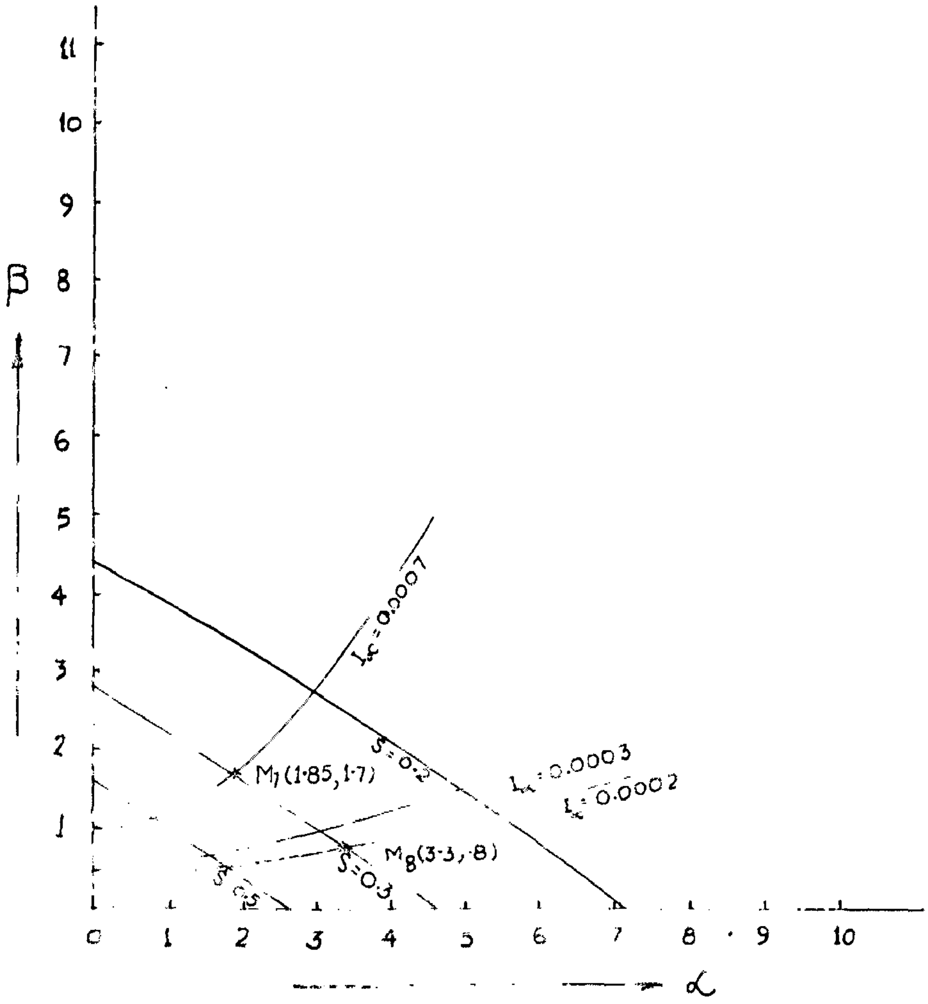


FIG. 4.2(b) – PARAMETER PLANE DIAGRAM Ex 2.

CHAPTER - 5

APPENDICES

5.1. Chebyshev Functions:

The complex variable s of the characteristic equation of a servo system is expressed by-

$$s = -w_n\xi + j\,w_n\sqrt{1-\xi^2} \qquad \ldots \qquad (5.1)$$

where, w_n = undamped natural frequency

ξ = relative damping coefficient.

Given a characteristic equation-

$$f(s) = \sum_{K=0}^{m} a_K s^K = 0, \qquad \ldots \qquad (5.2)$$

s^K can be given by

$$s^K = w_n^K \left[T_K(-\xi) + j\sqrt{1-\xi^2}\,U_K(\xi)\right] \qquad \ldots \qquad (5.3)$$

where,
$$\left.\begin{aligned} T(-\xi) &= (-1)^K\, T_K(\xi) \\ U_K(-\xi) &= (-1)^{K+1} U_K(\xi) \end{aligned}\right\} \ldots \qquad (5.4)$$

Function $T_K(\xi)$ and $U_K(\xi)$ are Chebyshev functions of the first and the second kinds respectively. The argument ξ of the Chebyshev function is $0 \leqslant |\xi| \leqslant 1$; for the stable system it is $0 \leqslant \xi \leqslant 1$

Functions $T_K(\xi)$ and $U_K(\xi)$ may be obtained from the recurrence formulas-

$$\left.\begin{aligned} &T_{K+1}(\xi) - 2\xi T_K(\xi) + T_{K-1}(\xi) = 0 \\ \text{and} \quad &U_{K+1}(\xi) - 2\xi U_K(\xi) + U_{K-1}(\xi) = 0 \end{aligned}\right\} \ldots \qquad (5.5)$$

with $T_o(\xi) = 1$, $T_1(\xi) = \xi$, $U_o(\xi) = 0$ and $U_1(\xi) = 1$.
The numerical values of $T_K(\xi)$ and $U_K(\xi)$ for pertinent values of ξ are given in the tables I and II.

ξ	T_0	T_1	T_8	T_9	T_{10}
0.00		0.00	1.000000000	0.0000000000	-1.000000000
0.05		0.05	0.901036006	0.5321445506	-0.847881549
0.10		0.10	0.695745280	0.7842626560	-0.533892748
0.15		0.15	0.358116803	0.9768306915	-0.065067597
0.20		0.20	-0.040056320	0.9709982720	0.428455628
0.25		0.25	-0.435546875	0.7626953125	0.816894531
0.30		0.30	-0.762225920	0.3988276480	0.995522508
0.35		0.35	-0.960771395	-0.0764725465	0.907241212
0.40		0.40	-0.983689920	-0.5329295360	0.462346291
0.45		0.45	-0.829530395	-0.8720103055	0.044720670
0.50	1	0.50	-0.500000000	-1.0000000000	-0.500000000
0.55		0.55	-0.053449595	-0.8633661045	-0.986255319
0.60		0.60	0.421972480	-0.4721034240	-0.988496583
0.65		0.65	0.812407605	0.0849590365	-0.701960857
0.70		0.70	0.996801280	0.6406865920	-0.098400512
0.75		0.75	0.876953125	0.9755859375	0.586425781
0.80		0.80	0.421972480	0.8815431680	0.988496588
0.85		0.85	-0.270438795	0.2772303985	0.741780472
0.90		0.90	-0.893093120	-0.6076892160	-0.200747483
0.95		0.95	-0.824708795	-0.9600686605	-0.999421659
1.00		1.00	1.000000000	1.0000000000	1.000000000

Table I Functions $T_K(\xi)$

ξ	T_0	T_1	T_2	T_3	T_4	T_5	T_6	T_7	T_8	T_9	T_{10}
0.00		0.00	-1.000	0.0000	1.00000	0.000000	-1.0000000	0.00000000	1.000000000	0.0000000000	-1.000000000
0.05		0.05	-0.995	-0.1495	0.98005	0.347505	-0.9452995	-0.44203495	0.901096005	0.5321445505	-0.847881549
0.10		0.10	-0.980	-0.2960	0.92080	0.480160	-0.8247680	-0.64511360	0.695745280	0.7842626560	-0.538992748
0.15		0.15	-0.955	-0.4365	0.82405	0.683715	-0.6189355	-0.86939565	0.358116803	0.9768306915	-0.065067597
0.20		0.20	-0.920	-0.5680	0.69280	0.845120	-0.3547520	-0.98702080	-0.040056320	0.9709982720	0.428455628
0.25		0.25	-0.875	-0.6875	0.53125	0.953125	-0.0546875	-0.98046875	-0.435546875	0.7626953125	0.816894531
0.30		0.30	-0.820	-0.7920	0.34480	0.998880	0.2545280	-0.84616320	-0.762225920	0.3988276480	0.995522508
0.35		0.35	-0.755	-0.8785	0.14005	0.976535	0.5435245	-0.59606785	-0.960771995	-0.0764725465	0.907241218
0.40		0.40	-0.680	-0.9440	-0.07520	0.883840	0.7822720	-0.25802240	-0.988689920	-0.5329295360	0.462346281
0.45		0.45	-0.595	-0.9855	-0.29195	0.722745	0.9424205	0.12543345	-0.829530395	-0.8720108055	0.044720670
0.50	1	0.50	-0.500	-1.0000	-0.50000	0.500000	1.0000000	0.50000000	-0.500000000	-1.0000000000	-0.500000000
0.55		0.55	-0.395	-0.9845	-0.68795	0.2227755	0.9384905	0.80467355	-0.053449595	-0.8633681045	-0.986255319
0.60		0.60	-0.280	-0.9360	-0.84320	-0.075840	0.7521920	0.97847040	0.421[illegible]480	-0.4721034240	-0.938496589
0.65		0.65	-0.155	-0.8515	-0.95195	-0.386035	0.4501045	0.97117065	0.812407605	0.0849590365	-0.701960857
0.70		0.70	-0.020	-0.7200	-0.99920	-0.670880	0.0599680	0.75483520	0.996801280	0.6406865920	-0.098400512
0.75		0.75	0.125	-0.5625	-0.96875	-0.890625	-0.3671875	0.33984375	0.876953125	0.9755859375	0.586425781
0.80		0.80	0.280	-0.3520	-0.84320	-0.997120	-0.7521920	-0.20638720	0.421972480	0.8815431680	0.988496588
0.85		0.85	0.445	-0.0935	-0.60395	-0.933215	-0.9825185	-0.73706135	-0.270488795	0.2772303985	0.7417804724
0.90		0.90	0.620	0.2160	-0.23120	-0.632160	-0.9066880	-0.99987840	-0.893093120	-0.6076892160	-0.200747[illegible]
0.95		0.95	0.805	0.5795	0.29605	-0.017005	-0.3283595	-0.60687805	-0.824709795	-0.9600686605	-0.999421659
1.00		1.00	1.000	1.000	1.00000	1.000000	1.0000000	1.00000000	1.000000000	1.0000000000	1.0000000000

Table II Functions $U_K(\xi)$

ξ	U_{-1}	U_0	U_1	U_2	U_3	U_4	U_5	U_6	U_7	U_8	U_9	U_{10}
0.00				0.0	-1.00	0.000	1.0000	0.00000	-1.000000	0.0000000	1.00000000	0.000000000
0.05				0.1	-0.99	-0.199	0.9701	0.29601	-0.940499	-0.3900599	0.90149301	0.480809201
0.10				0.2	-0.96	-0.392	0.8816	0.56832	-0.767936	-0.7219072	0.62355456	0.846618112
0.15				0.3	-0.91	-0.573	0.7331	0.79443	-0.499771	-0.9443613	0.21646261	1.009300083
0.20				0.4	-0.84	-0.736	0.5456	0.95424	-0.163904	-1.0198016	-0.24401664	0.922194944
0.25				0.5	-0.75	-0.875	0.3125	1.03125	0.203125	-0.9296875	-0.66796875	0.595703125
0.30				0.6	-0.64	-0.984	0.0496	1.01376	0.553656	-0.6785664	-0.96579584	0.099088896
0.35				0.7	-0.51	-1.057	-0.2299	0.89607	0.857149	-0.2960657	-1.06439499	-0.449010793
0.40				0.8	-0.36	-1.088	-0.5104	0.67968	1.054144	0.1636352	-0.92323584	-0.902223872
0.45				0.9	-0.19	-1.071	-0.7739	0.37449	1.110941	0.6253569	-0.54811979	-1.118664711
0.50	-1	0	1	1.0	0.00	-1.000	-1.0000	0.00000	1.000000	1.0000000	0.00000000	-1.000000000
0.55				1.1	0.21	-0.869	-1.1659	-0.41349	0.711061	1.1956571	0.60416181	-0.531079109
0.60				1.2	0.44	-0.672	-1.2464	-0.82368	0.257984	1.1332608	1.10192896	0.189053952
0.65				1.3	0.69	-0.403	-1.2139	-1.17507	-0.313691	0.7672717	1.31114221	0.837215773
0.70				1.4	0.96	-0.056	-1.0384	-1.39776	-0.918464	0.1119104	1.07513856	1.393283584
0.75				1.5	1.25	0.375	-0.6875	-1.40625	-1.421875	-0.7265625	0.33203125	1.224609375
0.80				1.6	1.56	0.896	-0.1264	-1.09824	-1.630784	-1.5110144	-0.78683904	0.252071936
0.85				1.7	1.89	1.513	0.6821	-0.35343	-1.282931	-1.8275527	-1.82390859	-1.273091903
0.90				1.8	2.24	2.232	1.7776	0.96768	-0.035776	-1.0320768	-1.82196224	-2.247455232
0.95				1.9	2.61	3.059	3.2021	3.02499	2.545381	1.9112339	0.89596341	-0.108903421
1.00				2.0	3.00	4.000	5.0000	6.00000	7.000000	8.0000000	9.00000000	10.000000000

5.2. Evaluation of Definite Integral:

$$I_n = \frac{1}{2\pi j}\int_{-j\infty}^{+j\infty} \frac{c(s)\,c(-s)}{d(s)\,d(-s)}\,ds \qquad \ldots \qquad (5.6)$$

where,

$$\left.\begin{aligned} c(s) &= \sum_{K=0}^{n-1} c_K s^K \\ d(s) &= \sum_{K=0}^{n} d_K s^K \end{aligned}\right\} \ldots \qquad (5.7)$$

Solutions of thefollowing integrals are obtained-

$$I_1 = \frac{c_0^2}{2d_0 d_1} \qquad \ldots \qquad (5.8)$$

$$I_2 = \frac{c_1^2\, d_0 + c_0^2 d_2}{2d_0 d_1\, d_2} \qquad \ldots \qquad (5.9)$$

$$I_3 = \frac{c_2^2 d_0 d_1 + (c_1^2 - 2c_0 c_2) d_0 d_3 + c_0^2 d_2 d_3}{2d_0 d_3(-d_0 d_3 + d_1 d_2)} \qquad (5.10)$$

$$I_8 = \frac{1}{2\Delta_8}\Big[c_7^2 m_0 + (c_6^2 - 2c_5 c_7) m_1 + (c_5^2 - 2c_4 c_6 + 2c_3 c_7) m_2 +$$
$$(c_4^2 - 2c_3 c_5 + 2c_2 c_6 - 2c_1 c_7) m_3 + (c_3^2 - 2c_2 c_4 + 2c_1 c_5 - 2c_0 c_6) m_4 +$$
$$(c_2^2 - 2c_1 c_3 + 2c_0 c_4) m_5 + (c_1^2 - 2c_0 c_2) m_6 + c_0^2 m_7 \Big] \qquad (5.11)$$

where,

$$m_1 = (d_0 d_7 + d_2 d_5)(-d_0 d_1 d_7 + d_0 d_1 d_8 + 2d_1^2 d_6) + (d_3 d_7 - d_5^2)(d_0^2 d_5 +$$
$$d_1 d_2^2) + d_1 d_3 d_8 (d_0 d_3 - d_1 d_2) - d_1^2 d_8 (d_0 d_5 - d_1 d_4) + (-d_2 d_7 + d_3 d_6 -$$
$$d_4 d_5)(d_0 d_3^2 + d_1^2 d_4) - d_1 d_6 (d_1^2 d_6 + 3d_0 d_3 d_5) - d_1 d_2 d_3 (d_3 d_6 - d_4 d_5) +$$
$$2d_0 d_1 d_4 d_5^2 \qquad \ldots \qquad (5.12)$$

$$m_2 = (d_e d_3 - d_1 d_2)(d_e d_7^2 - d_1 d_5 d_8 - d_1 d_6 d_7 + d_2 d_5 d_7) + (d_3 d_8 - d_4 d_7) x$$
$$(-d_e d_1 d_5 + d_e d_3^2 - d_1 d_2 d_3 + d_1^2 d_4) - d_e d_5 d_7 (d_e d_5 - d_1 d_4) +$$
$$d_1^2 d_8 (d_e d_7 - d_1 d_6) \quad \ldots \quad (5.13)$$

$$m_3 = -d_1 (d_1 d_8 - d_2 d_7)^2 + (-d_5 d_8 + d_6 d_7)(d_e d_1 d_5 - d_e d_3^2 + d_1 d_2 d_3 -$$
$$d_1^2 d_4) + d_e d^2{}_7 (-d_e d_5 + d_1 d_4 + d_2 d_3) - 2\, d_e d_1 d_3 d_7 d_8 \quad (5.14)$$

$$m_4 = (-d_5 d_8 + d_6 d_7)\ (2 d_e d_1 d_7 - d_e d_3 d_5 + d_1 d_2 d_5 - d_1^2 d_6) +$$
$$(-d_3 d_8 + d_4 d_7)\ (d_e d_3 d_7 - d_1 d_2 d_7 + d_1^2 d_8) - d_e^2 d_7^3 \quad (5.15)$$

$$m_e = \frac{1}{d_8}(d_6 m_1 - d_4 m_2 + d_2 m_3 - d_e m_4) \quad \ldots \quad (5.16)$$

$$m_5 = \frac{1}{d_e}(d_2 m_4 - d_4 m_3 + d_6 m_2 - d_8 m_1) \quad \ldots \quad (5.17)$$

$$m_6 = \frac{1}{d_e}(d_2 m_5 - d_4 m_4 + d_6 m_3 - d_8 m_2) \quad \ldots. \quad (5.18)$$

$$m_7 = \frac{1}{d_e}\ (d_2 m_6 - d_4 m_5 + d_6 m_4 - d_8 m_3) \quad \ldots \quad (5.19)$$

$$\triangle_8 = d_e\ (d_1 m_7 - d_3 m_6 + d_5 m_5 - d_7 m_4) \quad \ldots \quad (5.20)$$

$$I_9 = \frac{1}{2\triangle_9}\Big[c_8^2\, m_e + (c_7^2 - 2c_6 c_8) m_1 + (c_6^2 - 2c_5 c_7 + 2c_4 c_8) m_2 +$$
$$(c_5^2 - 2c_4 c_6 + 2c_3 c_7 - 2c_2 c_8) m_3 + (c_4^2 - 2c_3 c_5 + 2c_2 c_6 -$$
$$2c_1 c_7 + 2c_e c_8) m_4 + (c_3^2 - 2c_2 c_4 + 2c_1 c_5 + 2c_e c_6) m_5 + (c_2^2 - 2c_1 c_3 +$$
$$2c_e c_4) m_6 + (c_1^2 - 2c_e c_2) m_7 + c_e^2 m_8\Big] \quad \ldots \quad (5.21)$$

where,

$$m_1 = a_1 (a_1 a_{10} - a_2 a_9 + a_3 a_6 + a_3 a_8 + 2a_4 a_6 - a_5^2 - a_5 a_7 - a_7^2) +$$
$$a_2 (-a_2 a_6 - a_3 a_7 + a_4 a_5 + 2a_4 a_7) - a_4^3 \quad \ldots \quad (5.22)$$

$$m_2 = a_1(a_3a_9 + a_4a_9 - a_5a_8 + a_6a_8) + a_2(-a_2a_9 + a_4a_8 + a_7^2) - a_4^2a_7 \qquad (5.23)$$

$$m_3 = a_1(a_3a_{10} + a_4a_{10} + a_7a_9 - a_8^2) + a_2(-a_2a_{10} + a_7a_8) - a_4a_7^2 \qquad (5.24)$$

$$m_4 = a_1(a_5a_{10} + 2a_7a_{10} - a_8a_9) + a_2(a_7a_9 - a_4a_{10}) - a_7^3 \qquad (6.25)$$

$$m_5 = \frac{1}{d_0}(d_2m_4 - d_4m_3 + d_6m_2 - d_8m_1) \quad \ldots \qquad (5.26)$$

$$m_6 = \frac{1}{d_0}(d_2m_5 - d_4m_4 + d_6m_3 - d_8m_2) \quad \ldots \qquad (5.27)$$

$$m_7 = \frac{1}{d_0}(d_2m_6 - d_4m_5 + d_6m_4 - d_8m_3) \quad \ldots \qquad (5.28)$$

$$m_0 = \frac{1}{d_9}(d_7m_1 - d_5m_2 + d_3m_3 - d_1m_4) \quad \ldots\ldots \qquad (5.29)$$

$$m_8 = \frac{1}{d_0}(d_2m_7 - d_4m_6 + d_6m_5 - d_8m_4) \quad \ldots \qquad (5.30)$$

$$\Delta_9 = d_0(d_1m_8 - d_3m_7 + d_5m_6 - d_7m_5 + d_9m_4) \quad \ldots \qquad (5.31)$$

and

$$\left.\begin{array}{ll} a_1 = d_1d_2 - d_0d_3 & a_6 = d_5d_6 - d_4d_7 \\ a_2 = d_1d_4 - d_0d_5 & a_7 = d_1d_8 - d_0d_9 \\ a_3 = d_3d_4 - d_2d_5 & a_8 = d_3d_8 - d_2d_9 \\ a_4 = d_1d_6 - d_0d_7 & a_9 = d_5d_8 - d_4d_9 \\ a_5 = d_3d_6 - d_2d_7 & a_{10} = d_7d_8 - d_6d_9 \end{array}\right\} \qquad (5.32)$$

5.3. Jacobian of Functions:

In mapping a constant ξ contour from the s-plane onto the $\alpha\beta$ plane to obtain the related ξ curve, it is of interest to determine which side of the curve corresponds to a positive increment of $\Delta\xi$. Then that side of the ξ curve should be shaded in order to determine the number of roots in various regions of the $\alpha\beta$ plane. It will be proved here that :-

If the Jacobian $J = J(R, I/\alpha,\beta)$ is positive, then facing direction in which w_n increases the left side of the ξ curve corresponds to a positive increment $\Delta\xi$.

To prove this statement we shall define a vector-

$\bar{J}_1 = J_1\bar{K}$ as the cross product

$$\bar{J}_1 = \overline{\Delta w_n} \times \overline{\Delta\xi} \qquad \ldots \qquad (5.33)$$

where,

$$\left.\begin{aligned} \overline{\Delta w_n} &= \frac{\partial w_n}{\partial\alpha} i + \frac{\partial w_n}{\partial\beta} j \\ \overline{\Delta\xi} &= \frac{\partial\xi}{\partial\alpha} i + \frac{\partial\xi}{\partial\beta} j \end{aligned}\right\} \qquad \ldots \qquad (5.34)$$

The vectors i and j are unit vectors in the α and β directions, and the vector $\bar{K}$ is a unit vector such that the vectors i, j and $\bar{K}$ form a right-handed system. We must note here that the side of the ξ curve to be shaded is determined by the orientation of the vector $\overline{\Delta\xi}$ since ξ increases in the direction of this vector. The direction of the vector ξ with respect to the vector $\overline{\Delta w_n}$ is determined from (5.33) by the sign of $J_1 = J_1(w_n,\xi/\alpha,\beta)$ given as-

$$J_1 = \frac{\partial w_n}{\partial\alpha} \cdot \frac{\partial\xi}{\partial\beta} - \frac{\partial w_n}{\partial\beta} \cdot \frac{\partial\xi}{\partial\alpha} \qquad \ldots \qquad (5.35)$$

Now it is to be proved that the sign of J_1 is equal to the sign of the Jacobian J defined in-

$$J = \frac{\partial R}{\partial\alpha} \cdot \frac{\partial I}{\partial\beta} - \frac{\partial R}{\partial\beta} \cdot \frac{\partial I}{\partial\alpha} \qquad \ldots \qquad (5.36)$$

It can readily be shown that-

$$J_1\left(\frac{\alpha,\beta}{w_n,\xi}\right) \cdot J_1\left(\frac{w_n,\xi}{\alpha,\beta}\right) = 1 \qquad \ldots \qquad (5.37)$$

$$J\left(\frac{R,I}{\alpha,\beta}\right) \cdot J_1\left(\frac{\alpha,\beta}{w_n,\xi}\right) = J_2\left(\frac{R,I}{w_n,\xi}\right) \qquad \ldots \qquad (5.38)$$

From (5.37), it follows that the two Jacobians $J_1(\alpha,\beta/w_n,\xi)$ and $J_1(w_n,\xi/\alpha,\beta)$ have always the same sign. Then to prove that the Jacobians J and J_1 have also the same sign, it is necessary to show that the Jacobian J_2 on the right side of (5.38) is always positive.

From (5.36), the following can be derived:

$$J_2\left(\frac{R,I}{w_n,\xi}\right) = \frac{\sqrt{1-\xi^2}}{w_n}\left[\left(\frac{\partial R}{\partial w_n}\right)^2 + \left(\frac{\partial I}{\partial \xi}\right)^2\right] \qquad \ldots \qquad (5.39)$$

In deriving (5.37) from (5.34), the relations

$$\frac{dT_K(\xi)}{\partial\xi} = KU_K(\xi), \quad \frac{d\,\sqrt{1-\xi^2}\,U_K(\xi)}{d\xi} = -\frac{KT_K(\xi)}{\sqrt{1-\xi^2}} \qquad (5.40)$$

are used, which exist among Chebyshev functions $T_K(\xi)$ and $U_K(\xi)$.

The Jacobian J_2 of (5.38) is always positive (the case $J_2 = 0$ is not of interest since then (5.34) are not solvable for α and β as function of w_n and ξ), and the signs of Jacobians J and J_1 are the same. Therefore, the statement outlined in the begining of this Appendix is proved.

Before concluding this Appendix it should be noted that the previous theorem extends to the mapping of more general contours from the s or z plane onto the $\alpha\beta$ plane since the proof is valid for other pairs of independent variables such as (w_z, ξ_z), $(w_n^2, w_n\xi)$ etc. In addition, when two parameters α and β appear only linearly in the coefficients of the characteristic equation, the same theorem is valid provided the Jacobian J is reduced to the corresponding determinant $\triangle$ given in the linear continuous or the nonlinear systems.

5.4. Some of the Computer Programmes and Results:

```
C   C  PARAMETER EVALUATION Z M.N.SHARMA. EX 1
       W=0.3
       PUNCH10
10     FORMAT(6X,1HW,10X,2HAP,10X,2HBP,10X,2HAG,10X,2HBG)
1      XP=-0.636*W**4+1.9*W*W*W-14.8*W*W+70.66*W+77.8
       YP=.063*W**7+.001*W**6-3.09*W**5+.096*W**4+94.5*W*W*W-.1*W*W+85.58
       XG=.675*W*W+25.
       YG=.0225*W**6-1.0935*W**4+33.75*W*W+30.2
       Z=.9*W**4-20.9*W*W-110.
       IF(Z-0.)2,3,2
2      AP=XP/(Z*W)
       BP=YP/(Z*W) $ AG=XG/Z $ BG=YG/Z
       PUNCH4,W,AP,BP,AG,BG
4      FORMAT(5F 12.4)
3      W=W+0.3
       IF(W-10.)1,1,5
5      STOP
       END
       W           AP           BP           AG           BG
        .3000     -2.9114      -2.6255       -.2240       -.2970
        .6000     -1.6353      -1.5009       -.2150       -.3595
        .9000     -1.1466      -1.3426       -.2022       -.4498
       1.2000      -.8636      -1.4557       -.1879       -.5541
       1.5000      -.6720      -1.6720       -.1739       -.6615
       1.8000      -.5330      -1.9245       -.1616       -.7657
       2.1000      -.4284      -2.1852       -.1515       -.8648
       2.4000      -.3476      -2.4469       -.1441       -.9606
       2.7000      -.2836      -2.7155       -.1395      -1.0574
       3.0000      -.2314      -3.0071       -.1380      -1.1624
       3.3000      -.1873      -3.3502       -.1401      -1.2869
       3.6000      -.1478      -3.7935       -.1469      -1.4493
       3.9000      -.1095      -4.4261       -.1605      -1.6831
       4.2000      -.0676      -5.4357       -.1858      -2.0581
       4.5000      -.0114      -7.3024       -.2355      -2.7537
       4.8000       .0943     -11.7599       -.3564      -4.4166
       5.1000       .5479     -33.7168       -.9512     -12.6099
       5.4000      -.8639      37.6442        .9750      14.0173
       5.7000      -.3485      12.4685        .2915       4.6218
       6.0000      -.2439       7.8252        .1622       2.8874
       6.3000      -.1962       5.9988        .1083       2.2037
       6.6000      -.1678       5.1109        .0792       1.8698
       6.9000      -.1483       4.6554        .0611       1.6969
       7.2000      -.1338       4.4385        .0490       1.6126
       7.5000      -.1225       4.3701        .0403       1.5834
       7.8000      -.1134       4.4025        .0339       1.5914
       8.1000      -.1057       4.5080        .0290       1.6265
       8.4000      -.0992       4.6695        .0251       1.6821
       8.7000      -.0936       4.8758        .0220       1.7542
       9.0000      -.0886       5.1193        .0194       1.8398
       9.3000      -.0843       5.3946        .0173       1.9371
       9.6000      -.0804       5.6977        .0156       2.0444
       9.9000      -.0769       6.0257        .0141       2.1608
)      STOP  END OF PROGRAM AT  STATEMENT 0005 + 00 LINES
```

```
      PARAMETER EVALUATION FOR GAIN MARGIN 6 DB EX 1 Z M.N.SARMA.
      W=0.3
      PUNCH 10
      FORMAT(8X,1HW,15X,1HA,15X,1HB)
      X=1.35*W*W+50.
      Y=.045*W**6-2.1*W**4+67.6*W*W+60.5
      Z=0.9*W**4-20.9*W*W-110.
      IF(Z-0.)2,3,2
      A=X/Z
      B=Y/Z
      PUNCH 4,W,A,B
      FORMAT(3F16.7)
      W=W+0.3
      IF(W-20.)1,1,5
      STOP
      END
```

W	A	B
.3000000	-.4480186	-.5950194
.6000000	-.4300071	-.7202780
.9000000	-.4064175	-.9015629
1.2000000	-.3757802	-1.1113657
1.5000000	-.3478987	-1.3280186
1.8000000	-.3231390	-1.5392668
2.1000000	-.3029989	-1.7417119
2.4000000	-.2881249	-1.9389318
2.7000000	-.2799406	-2.1401786
3.0000000	-.2759769	-2.3609906
3.3000000	-.2802591	-2.6237346
3.6000000	-.2938460	-2.9683766
3.9000000	-.3210748	-3.4651279
4.2000000	-.3716280	-4.2619071
4.5000000	-.4710954	-5.7376224
4.8000000	-.7128237	-9.2602096
5.1000000	-1.9023641	-26.5957200
5.4000000	1.9490012	29.7144730
5.7000000	.5829040	9.8349624
6.0000000	.3243471	6.1579605
6.3000000	.2165865	4.7025634
6.6000000	.1583042	3.9863249
6.9000000	.1222197	3.6098257
7.2000000	.0979309	3.4202891
7.5000000	.0806242	3.3466052
7.8000000	.0677680	3.3511685
8.1000000	.0579089	3.4122041
8.4000000	.0501551	3.5162231
8.7000000	.0439303	3.6544161

9.0000000	.0388469	3.8707813
9.3000000	.0346350	4.0110869
9.6000000	.0311014	4.2272619
9.9000000	.0281046	4.4520270
10.2000000	.0255387	4.6986327
10.5000000	.0233291	4.9607540
10.8000000	.0213955	5.2373314
11.1000000	.0197069	5.5275265
11.4000000	.0182186	5.8306620
11.7000000	.0168995	6.1461870
12.0000000	.0157243	6.4736457
12.3000000	.0146724	6.8126610
12.6000000	.0137268	7.1629169
12.9000000	.0128733	7.5261459
13.2000000	.0121001	7.8961208
13.5000000	.0113972	8.2786477
13.8000000	.0107567	8.6715608
14.1000000	.0101698	9.0747143
14.4000000	.0096319	9.4879836
14.7000000	.0091377	9.9112586
15.0000000	.0086819	10.3444450
15.3000000	.0082594	10.7874570
15.6000000	.0078689	11.2402190
15.9000000	.0075064	11.7026680
16.2000000	.0071693	12.1747420
16.5000000	.0068552	12.6563910
16.8000000	.0065620	13.1475680
17.1000000	.0062879	13.6482300
17.4000000	.0060311	14.1583350
17.7000000	.0057903	14.6778560
18.0000000	.0055641	15.2067590
18.3000000	.0053514	15.7450150
18.6000000	.0051509	16.2925980
18.9000000	.0049619	16.8494880
19.2000000	.0047835	17.4156620
19.5000000	.0046147	17.9911040
19.8000000	.0044550	18.5757910

STOP END OF PROGRAM AT STATEMENT 0005 + 00 LINES

```
C  C  SENSITIVITY FUNCTION WITH A FOR IMPULSE EX2 7
      A=.5
      PUNCH 10
10    FORMAT(6X,4HALFA,12X,4HBETA,10X,8HINTEGRAL)
11    B=0.
1     DH=.0225
      DG=.375
      DF=2.6275
      DE=10.01+.09*A+.3*B
      DD=22.1+1.05*A+3.1*B
      DC=4.63*A+12.1*B+30.5
      DB=.09*A*A+21.5+9.38*A+21.8*B+B*B+.6*A*B
      DA=8.2*A+17.2*B+7.6+.6*A*A+4.*B*B+9.2*A*B
      DZ=A*A+4.*B*B+1.+4.*A*B+2.*A+4.*B
      CA=1.6*B
      CB=.3*B
      CZ=2.*B
2     FA=(DZ*DG+DB*DF)*(-DZ*DA*DG+DZ*DC*DF+2.*DA*DA*DF)
      FB=(DC*DG-DF*DE)*(DZ*DZ*DE+DA*DD*DB)
      FC=DA*DC*DH*(DZ*DC-DA*DE)-DA*DA*DH*(DZ*DF-DA*DD)
      FD=(-DB*DG+DC*DF-DD*DE)*(DZ*DC*DC+DA*DA*DD)
      FE=-DA*DF*(DA*DA*DF+3.*DZ*DC*DF)
      FF=-DA*DB*DC*(DC*DF-DD*DE)+2.*DZ*DA*DD*DE*DF
      AMA=FA+FB+FC+FD+FE+FF
      FG=(DZ*DC-DA*DB)*(DZ*DG*DG-DA*DE*DH-DA*DF*DG+DB*DF*DG)
      FH=(DC*DH-DD*DG)*(-DZ*DA*DE+DZ*DC*DC-DA*DB*DC+DA*DA*DD)
      FK=-DZ*DC*DG*(DZ*DF-DA*DD)+DA*DA*DH*(DZ*DG-DA*DF)
      AMB=FG+FH+FK
3     FA=(-DE*DH+DF*DG)*(DZ*DA*DE-DA*DC*DC+DA*DB*DC-DA*DA*DD)
      FB=-DA*(DA*DH-DB*DG)*(DA*DH-DB*DG)-2.*DZ*DA*DC*DG*DH
      FC=DZ*DG*DG*(-DZ*DF+DA*DD+DB*DC)
      AMC=FA+FB+FC
      FD=(-DE*DH+DF*DG)*(2.*DZ*DA*DG-DZ*DC*DF+DA*DB*DE-DA*DA*DF)
      FE=(-DC*DH+DD*DG)*(DZ*DC*DG-DA*DB*DG+DA*DA*DH)-DZ*DZ*DG*DG*DG
      AMD=FD+FE
      AME=(DE*AMD-DD*AMC+DF*AMB-DH*AMA)/DZ
      AMF=(DB*AME-DD*AMD+DF*AMC-DH*AMB)/DZ
      AMG=(DB*AMF-DD*AME+DF*AMD-DH*AMC)/DZ
      AMZ=(DF*AMA-DD*AMB+DB*AMC-DZ*AMD)/DZ
      DELTA=DZ*(DA*AMG-DC*AMF+DE*AME-DG*AMD)
4     GA=CB*CB*AMF+(CA*CA-2.*CZ*CB)*AMF
      AI=GA/(2.*DELTA)
      PUNCH 5,A,B,AI
5     FORMAT(4F 16.7)
      B=B+.5
      IF(B-5.)1,1,6
6     A=A+.5
      IF(A-5.)11,11,7
7     STOP
      END
```

```
      ALFA              BETA              INTEGRAL
```

ALFA	BETA	INTEGRAL
.5000000	0.0000000	0.0000000
.5000000	.5000000	.0003367
.5000000	1.0000000	.0007192
.5000000	1.5000000	.0009756
.5000000	2.0000000	.0011289
.5000000	2.5000000	.0012111
.5000000	3.0000000	.0012460
.5000000	3.5000000	.0012498
.5000000	4.0000000	.0012332
.5000000	4.5000000	.0012036
.5000000	5.0000000	.0011659
1.0000000	0.0000000	0.0000000
1.0000000	.5000000	.0002570
1.0000000	1.0000000	.0005853
1.0000000	1.5000000	.0008251
1.0000000	2.0000000	.0009796
1.0000000	2.5000000	.0010705
1.0000000	3.0000000	.0011168
1.0000000	3.5000000	.0011324
1.0000000	4.0000000	.0011272
1.0000000	4.5000000	.0011080
1.0000000	5.0000000	.0010798
1.5000000	0.0000000	0.0000000
1.5000000	.5000000	.0002033
1.5000000	1.0000000	.0004863
1.5000000	1.5000000	.0007073
1.5000000	2.0000000	.0008582
1.5000000	2.5000000	.0009530
1.5000000	3.0000000	.0010065
1.5000000	3.5000000	.0010305
1.5000000	4.0000000	.0010339
1.5000000	4.5000000	.0010230
1.5000000	5.0000000	.0010025
2.0000000	0.0000000	0.0000000
2.0000000	.5000000	.0001652
2.0000000	1.0000000	.0004108
2.0000000	1.5000000	.0006132
2.0000000	2.0000000	.0007580
2.0000000	2.5000000	.0008536
2.0000000	3.0000000	.0009114
2.0000000	3.5000000	.0009414
2.0000000	4.0000000	.0009514
2.0000000	4.5000000	.0009471
2.0000000	5.0000000	.0009328
2.5000000	0.0000000	0.0000000
2.5000000	.5000000	.0001371
2.5000000	1.0000000	.0003518
2.5000000	1.5000000	.0005368
2.5000000	2.0000000	.0006743
2.5000000	2.5000000	.0007688
2.5000000	3.0000000	.0008290
2.5000000	3.5000000	.0008631
2.5000000	4.0000000	.0008780
2.5000000	4.5000000	.0008789
2.5000000	5.0000000	.0008698
3.0000000	0.0000000	0.0000000
3.0000000	.5000000	.0001137

3.0000000	1.0000000	.0003047
3.0000000	1.5000000	.0004737
3.0000000	2.0000000	.000[illegible]36
3.0000000	2.5000000	.0006959
3.0000000	3.0000000	.0007570
3.0000000	3.5000000	.0007939
3.0000000	4.0000000	.0008125
3.0000000	4.5000000	.0008175
3.0000000	5.0000000	.0008126
3.5000000	0.0000000	0.0000000
3.5000000	.5000000	.0000989
3.5000000	1.0000000	.0002665
3.5000000	1.5000000	.0004211
3.5000000	2.0000000	.0005434
3.5000000	2.5000000	.0006326
3.5000000	3.0000000	.0006937
3.5000000	3.5000000	.0007324
3.5000000	4.0000000	.0007538
3.5000000	4.5000000	.0007621
3.5000000	5.0000000	.0007606
4.0000000	0.0000000	0.0000000
4.0000000	.5000000	.0000856
4.0000000	1.0000000	.0002350
4.0000000	1.5000000	.0003768
4.0000000	2.0000000	.0004915
4.0000000	2.5000000	.0005774
4.0000000	3.0000000	.0006378
4.0000000	3.5000000	.0006775
4.0000000	4.0000000	.0007009
4.0000000	4.5000000	.0007117
4.0000000	5.0000000	.0007131
4.5000000	0.0000000	0.0000000
4.5000000	.5000000	.0000748
4.5000000	1.0000000	.0002088
4.5000000	1.5000000	.0003390
4.5000000	2.0000000	.0004466
4.5000000	2.5000000	.0005289
4.5000000	3.0000000	.0005882
4.5000000	3.5000000	.0006283
4.5000000	4.0000000	.0006531
4.5000000	4.5000000	.0006659
4.5000000	5.0000000	.0006696
5.0000000	0.0000000	0.0000000
5.0000000	.5000000	.0000659
5.0000000	1.0000000	.0001867
5.0000000	1.5000000	.0003065
5.0000000	2.0000000	.0004075
5.0000000	2.5000000	.0004861
5.0000000	3.0000000	.0005439
5.0000000	3.5000000	.0005840
5.0000000	4.0000000	.0006097
5.0000000	4.5000000	.0006241
5.0000000	5.0000000	.0006297

0 STOP END OF PROGRAM AT STATEMENT 0007 + 00 LINES

REFERENCES

[1] Siljak, D.D. - "Analysis and Synthesis of Feedback Control Systems in the Parameter Plane I-Linear Continuous Systems". IEEE Transactions on Applications and Industry; November, 1964, pp.449-458.

[2] Siljak, D.D. - "Analysis and Synthesis of Feedback Control Systems in the Parameter Plane - III -Nonlinear Systems". IEEE Transactions on Application and Industry; November 1964, pp.466-473.

[3] Siljak, D.D. - "Generalization of the Parameter Plane Method" IEEE Transaction on Automatic Control; January 1966, pp.63-70.

[4] Burzio, A. and Siljak, D.D. - "Minimization of Sensitivity with Stability Constraints in Linear Continuous Systems" IEEE Transactions on Automatic Control; July 1966, pp.567-569.

[5] Siljak, D.D. - "Generalization of Mitrovic's Method" IEEE Transactions on Applications and Industry, September 1964, pp.314-20.

[6] Choe, H.H. and Thaler, G.J. - "An Extension of Mitrovic's Method: Frequency Response Techniques". IEEE Transactions on Automatic Control, July 1966, pp. 569-573.

[7] George C. Newton Jr., Leonard A. Gould and James F. Kaiser, Analytical Design of Linear Feedback Control, John Wiley & Sons., New York, 1961.

[8] Aizerman, M.A. - Theory of Automatic Control.

[9] Thaler, George J. and Brown, Robert G. - Analysis and Design of Feedback Control Systems, McGraw-Hill Book Company, New York, 1960.

[10] Truxal John G. - Control Engineers' Handbook- McGraw Hill Book Company, New York, 1958.