ELECTRIC, MAGNETIC AND THERMAL CONDITIONS INSIDE AN INDUCTION HEATED WORK-PIECE

A Dissertation
submitted in partial fulfilment
of the requirements for the Degree
of
MASTER OF ENGINEERING
in
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by
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C E R T I F I C A T E.

Certified that the dissertation entitled "Electric, Magnetic and Thermal Conditions Inside An Induction Heated Work-piece" which is being submitted by Sri Suresh Chandra Gupta in partial fulfilment for the award of the Degree of Master of Engineering in Power System Engineering of the University of Roorkee is a record of student's own work carried out by him under my supervision and guidance. The matter embodied in this dissertation has not been submitted for the award of any other Degree or Diploma.

This is further to certify that he has worked for a period of 8 months from January to August '66 for preparing dissertation for Master of Engineering Degree at the University.

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SYNOPSIS.

In the following pages an effort has been made to present a comprehensive picture of the effect of Induction heating. The analysis has been made considering cylindrical and rectangular section for the work-piece. The approach is made through a familiarity with what happens electrically and magnetically inside a work-piece being heated. Both the electrical and thermal aspects of the problem have been analysed thoroughly. The mathematical expressions have been derived for the penetration of magnetic field, distribution of eddy currents, generated heat and the distribution of temperature, in the two types of work-pieces starting from the fundamental electromagnetic and heat flow theory. The solution of heat flow differential equation in the case of rectangular work-piece has been effected by showing a direct analogy between the heat-flow problem and a particular transmission-line problem. Further an attempt has been made to give a brief review of the development of the subject in the past.

CHAPTER - ONE

HISTORICAL DEVELOPMENTS

1.1. INTRODUCTION:

When an electric current passes through a conductor, heat is generated according to the equation.

$$H = k I^2 R t. \quad \ldots \quad .. (1.1)$$

where,

H = calories of heat

k = 0.239

I = Current through conductor in amperes

R = Resistance of conductor in ohms.

t = time in seconds.

This equation is the basic low governing such heating by an electric current known as resistance heating. Electrical heating methods fall in to two separate classes. In one, the heat is transferred from a source at a high temperature to the article to be heated. In the second, the heat is generated in the article itself and the source of energy is not at a high temperature.

The heat is transferred from a high temperature source by conduction, convection and radiation or by a combination of these methods, as it frequently happens in practice.

Heat may be generated directly in the article to be treated either by conducting a current through it or by inducing a current in it. In the first method, which is used to a limited extent, an alternating current, at a frequency of 50 c/s is passed through the article, or through some part of it. In the second method, alternating currents, are used, and the contact with the article being heated is unnecessary. This method is known as Induction

1.2. GENERAL CLASSIFICATION:

High frequency heating comprises two distinct methods; induction heating for electrical conductors, and dielectric heating for materials which are normally regarded as electrical insulators. Thus in induction heating the temperature of any material is raised by electro-magnetic generation of heat within the material itself, and not by any other method such as conduction, convection or radiation. The material being heated should not be part of any electrical circuit directly supplied by electrical conductors from a source of electrical energy. Hence the principles upon which high-frequency induction and dielectric heating are based, are-

(i) The heat produced in a current carrying conductor.
(ii) The heat produced in magnetic materials when they are placed in an alternating magnetic field and
(iii) Heat produced in electrical insulators when subjected to the forces of an electric field.

It is not surprising that early scientists soon recognised these phenomena and considered them in their laws and theorems. The earliest designs of electrical equipment included allowances for copper losses, which were due to the heat produced by the currents flowing in the electrical conductors of the apparatus. Compensation was made for the core losses in electromagnetic machines such as motors, generators and transformers. These core losses are the sum of the heat losses produced by the circulating electric currents in the magnetic circuit and the hysteresis losses due to the subjection ofthe magnetic circuit to an alternating magnetic field. Lastly, in the design of high voltage condensers and cables, the insulating material was alway known to undergo a temperature rise as a result of the losses produced by the voltage stresses to which it was alternately exposed. But considerable time had passed before any one considered the useful application of the principles which caused these heat losses and developed ways and means of utilizing them

1.3. CHARACTERISTICS OF INDUCTION HEATING:

When an electric conducting material, is placed within the turns of a coil carrying a current, a current is induced in the material and heat is generated in it. The current is greatest at the surface of the body and decreases as the depth below the surface increases. Consequently the heat is generated in a thin surface layer of the conducting material and reaches the inside by thermal conduction. The depth of this layer decreases as the frequency increases.

The heating effect increases with the magnitude and frequency of the current in the coil, and also with the resistivity and permeability of the material being heated. Hence electric resistivity and relative permeability are the two most important properties of the material. Resistivity depends upon temperature, which is turn depends upon the internal power distribution (always greatest at the surface), specific heat, density thermal conductivity, rate of heating, and surface thermal losses. Permeability depends upon degree of magnetic saturation and also upon whether the temperature is below or above the curie-value.

Hysteresis loss is also present when heating magnetic materials but it is insignificant relative to eddy current loss since the eddy current loss increases at a much greater rate with frequency.

1.4. DEVELOPMENT IN THE PAST:

Induction heating is a very spectacular process, it dates back many years, It was in the earlier part of the nineteenth century that first mention is made in the literature of the use of induced currents for heating metal. By this time, Michael Faraday had completed his experiments with coils carrying rapidly reversing currents and he recorded that when one such current carrying coil is inductively coupled with another, a voltage is induced in the second coil. This was the birth of thetransformer, and the addition of a magnetic circuit to its workability The literature discloses, that shortly after the middle of the nineteenth century, a concerted effort was being made to use high-frequency induced currents for heating metals, and numerous patents were filed in both the United States and foreign countries describing such equipment. The objective of most of these applications was the melting of metal, utilising a graphite or metallic crucible heated by induction to a temperature above that of the charge, which in turn was melted bythermal conduction from the crucible.

Ferranti and Colby(14) both presented data on an induction melting furnace, which induced currents directly in to the charge. Kjellin(14) did further work along these lines and also presented an adaptation of the Colby design, which eliminated certain difficulties in getting the maximum amount of energy in to the melt. These melting furnaces were all operated on relatively low frequencies, ranging from 5 to 60 cycles, largely because no means was available at that time for producing electrical energy at higher frequencies.

In the early 1900's, Dr. E.F. Northrup(14) invented the high-frequency melting furnace in which the material to be melted was not in the shape of ring, but was placed in a retaining crucible of non-conducting material. During all this development work, however, very little interest was given to any phase of induction heating except melting, and it was not until about 1925 that any mention is made of induction heating for metallurgical or metal-joining applications.

Accurate methods for designing induction heating coils(12) and of calculating their electrical performance in advance have been available since the 1930's and are still being improved. The papers which have made substantial contribution in this field are those by Dwight and Bagai (1935)(41), Baker (1944)(2), Vaughan and Williamson (1945)(5), Vaughan and Williamson (1946)(7), Baker (1957)(9), Baker has been working on this problem for years and his contribution is very noteworthy. He has conducted long term studies of this problem and its applications and has obtained very valuable information.

After 1935, Induction heating has put to some very unusual and worthwhile industrial uses. It is felt that this unique form of electric heating has played a very important role in the program of industralization in Mexico(8), particularly in view of the additional electric power that is anticipated in that country. Induction & Dielectric heating have had a spectacular growth in the United-States from 1938 onwards, upto 1948, there had been 500,000 KW of installed power for this type of heating in the United States, the

$ 50,000,000.

1.5. TECHNICAL ADVANTAGES OF INDUCTION HEATING:

The advantages of this method of heating may be summed up under the following headings:

1. A better product
2. More convenient operation
3. Increased speed of production.

1. A better product:

Induction heating has the outstanding advantage over the other methods due to the fact that in this method the zone to be heated can be localized and accurately controlled. In other methods of heating such as flame or furnace heating the whole of the article has to be raised to temperature or even if attempts are made to localize the heating, appreciable heating of adjacent parts inevitably takes place, partly due to the heat they receive directly from the source and partly because of conduction of heat to them. Induction heating overcomes this trouble in two ways. In the first place the coil can be designed to focus the heat only on the required area, secondly the heating time is very short, so there is very little opportunity for heat to be conducted away from the heated area to the adjacent parts. Since Induction heating is characterised by high concentration of heat per unit volume and close control of transmitted heat, it is particularly suitable for surface hardening because the heat producing eddy currents have a tendency to flow next to the surface of the charge. If a shaft is heated to hardening temperature at its surface zone only and is subsequently quenched, then only the surface zone hardens while its core remains soft. This distribution of hardness is often desirable, because it gives the shaft a hard wear-resistant surface, whereas the core remains soft and thus avoids brittleness attendant to hardness. The application of highly concentrated heat for surface hardening and forging shortens the heating time. This is important not only because of the saving of time in itself, but also because a short heat interval prevents or minimises scale formation.

2. <u>More convenient operation:</u>

The only high temperature produced in heating by induction is at the area within the coil itself, and there is consequently very little heating of other parts of the work, so that it can be handled easily with bare hands without risk of burning. The absence of radiant heat and fumes from the apparatus itself enables it to be installed in ordinary workshops without the necessity for a separate heat treatment department.

3. <u>Greater speed of production:</u>

As the heating itself is very rapid, most jobs take only a few seconds. Since un-necessarily large masses of metal are not heated up, the subsequent time required for cooling in cases where quenching is not used is reduced to a minimum. By suitably arranging the heating equipment it is frequently possible to carry out two or three operations at once. Induction heating is more easily adopted to continuous operation than many heat treatment methods.

4. <u>Saving of money:</u>

Most of the advantages which have been enumerated above involve a saving of money. Economy is also affected in the following ways. The concentration of heat on a highly localized area means that the heating is far more efficient. In many cases as much as 95% of the heat generated is actually used for the purpose for which it is intended low-grade labour can be employed since the process is entirely automatic.

CHAPTER - TWO

THEORY & ANALYSIS OF INDUCTION HEATING

2.1. INTRODUCTION:

In this Chapter an analysis is made of the effect of Induction heating. The analysis is carried out and mathematical equations are derived for two cases.

1. Cylindrical workpiece
2. Rectangular work-piece

In order to heat a charge, itis placed coaxially inside a coil, which isknown as inductor. The coil is energised from a a.c. generator. The generator energises the coil and thus an alternating magnetic flux is produced. This flux induces electromotive forces within the charge and naturally as a result current flows circumferentially through the cylindrical charge. Hence the heating effect is the result of I^2R losses due to the eddy currents in combination with the electrical resistance of the charge. So the theory of Induction heating is based on the fundamental transformer principles. The heating coil theoretically becomes the primary and the work-piece acts as a secondary which is short circuited, and similar to the case of transformer, the closer is the coil to the work's surface the more intense is the transfer of magnetic flux.

2.2. HEATING OF MAGNETIC MATERIALS:

Heat producing losses are those internal energy losses which cause a temperature rise in the material linked by the magnetic lines of force. In magnetic material these losses are derived in to two classes - hysteresis losses and eddy current losses. Hysteresis loss is the characteristic property of ferromagnetic materials. The probable cause of of hysteresis loss is that the molecules of magnetic materials are in themselves small magnets, which vibrate at the same frequency as the alternating magnetic field linking the material. The hysteresis loss is the heat generated by the friction between the rapidly oscillating molecules of the material as they attempt to align themselves with the rapidly alternating magnetic field. Hysteresis loss is present only in the magnetic materials, and it ceases when the magnetic change point or curie point has passed.

The heat expressed by hysteresis can be expressed as follows:

$$W_h = 0.83\ h\ f\ B^{1.6} \times 10^{-7} \text{ watts per cu. in.} \qquad \ldots (2.1)$$

where,

W_h = Heat generated due to hysteresis in watts.

h = Hysteresis constant for the material.

B = Flux density in lines per sq. inch.

It is noted from the above equation that the hysteresis loss is directly proportional to the frequency and to the flux density to the power 1.6. The equation applies strictly to flux ranges between about 4,000 and 12,000 gausses. The value 1.6 is called the steinmetz exponent and is the average value for silicon steel. For other materials, this exponent will have some different value.

Eddy current losses are resistance losses due to current circulation in the material, resulting from electromotive forces induced by varying induction. If the magnetic material is replaced by some non-magnetic metal the hysteresis loss will be eliminated but the eddy currents will still be induced and hence heat generated. Therefore magnetic as well as non-magnetic materials respond to Induction heating, eddy current loss is proportional to the square of the frequency.

When the frequency of the alternating magnetic flux is increased, hysteresis & eddy current losses increase. The eddy current loss, however increases at a much greater rate than the hysteresis loss. At frequencies of the order of 10,000 cycles and more the eddy current loss is predominant and the hysteresis loss is negligible. Hence in most of the induction heating applications the hysteresis loss is insignificant relative to eddy current loss and therefore for all practical purposes, the hysteresis loss drops completely from the picture.

2.3. ELECTROMAGNETIC EQUATION:

Following Maxwell, equations can be written, when an alternating current is passed through the winding of the coil, which sets up an alternating flux through the section of the specimen;

$$\nabla \times \bar{H} = \bar{i} \quad \ldots \quad \ldots (2.2.)$$

$$\nabla \times \bar{E} = - \frac{\partial \bar{B}}{\partial t} \quad \ldots \quad \ldots (2.3\ )$$

$$\nabla \cdot \bar{B} = 0 \quad \ldots \quad \ldots (2.4\ )$$

Also, by ohms law-

$$\bar{E} = \rho \bar{i} \quad \ldots \quad \ldots (2.4\ )$$

where,

$\bar{H}$ = Magnetizing force vector.

$\overline{i}$ = Current density vector.

$\overline{B}$ = Flux density vector.

$\overline{E}$ = Electric field intensity vector.

ρ = Resistivity of the material

∇ = $(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z})$ Differential operator.

Now from equation (2.2) and (2.5),

$$\nabla \times \overline{H} = \frac{1}{\rho}\text{[illegible]}.$$

or,

$$\nabla \times \nabla \times \overline{H} = \nabla \times \frac{1}{\rho}\overline{E} \qquad \ldots \qquad \ldots (2.6)$$

Combining equation (2.3) and (2.6),

$$\nabla \times \nabla \times \overline{H} = \frac{1}{\rho} \cdot \nabla \times \overline{E}$$

$$\pm - \frac{1}{\rho}\frac{\partial \overline{B}}{\partial t}$$

or,

$$-\nabla^2 \overline{H} + \nabla . \overline{H} = -\frac{1}{\rho}\frac{\partial \overline{B}}{\partial t} \qquad \ldots \qquad \ldots (2.7)$$

But from equation (2.4) we have,

$$\nabla . \overline{B} = 0$$

$$\therefore \mu \nabla . \overline{H} = 0$$

$$\therefore \nabla . \overline{H} = 0.$$

where μ = Permeability of the material.

Therefore, from equation (2.7),

$$-\nabla^2 \overline{H} = -\frac{1}{\rho}\frac{\partial \overline{B}}{\partial t}$$

or

$$\nabla^2 \overline{H} = \frac{1}{\rho}\frac{\partial \overline{B}}{\partial t}$$

or

$$\nabla^2 \overline{H} = \frac{\mu}{\rho}\frac{\partial \overline{H}}{\partial t} \qquad \ldots \qquad \ldots (2.8)$$

Equation (2.8) gives the space time variation of flux density in the workpiece and can be used to find the flux density distribution at any instant for any particular mode of time variation of B.

Two cases have been analysed by the author and they are dealt in the following pages.

2.4. ELECTRIC & MAGNETIC CONDITIONS INSIDE INDUCTION HEATED WORK-PIECE:

In the following pages an effort is made to improve the general understanding of induction heating through an approach with what happens electrically & magnetically inside a work-piece being heated. A.I.E.E. std. no.54 defines Induction heating as the heating of a nominally conducting material in a varying magnetic field due to its internal losses. This definition is sufficiently general to include not only intentional induction heating as practical industrially for different applications but it also includes the eddy current and hysteresis heating which appear as core losses in motors and transformers.

2.4.1. Conducting Cylinder surrounded by a Solenoidal Coil:

A long, round, solid bar of homogeneous conducting material is located coaxially inside a solenoidal coil, as shown in fig. (2.1). The coil carries a sinusoidal alternating current. The configuration is used frequently in industrial induction heating installations. The same principles apply to more complicated cases also.

2.4.2. Qualitative Explanation-

The bar is considered to be composed of many thin coaxial sleeves. The bar and coil are assumed very long so the end effects are neglected. The magnetic field is parallel to the center line of bar. The primary current in the coil and the induced current in the bar follow coaxial paths around the same center line. Resistivity and permeability are assumed to be uniform and hysteresis loss is neglected.

The magnetic intensity at the surface of the bar (i.e. $r = a$) is identical with the airgap intensity and the magnetic field intensity outside of the solenoid is zero since the return flux ouside of the solenoid spreads over an infinitely large area, thus reducing the flux density to zero. The field distribution is the same for any cross-section perpendicular to the axis of the cylinder and it varies only with the distance from the center line and is independent of the coordinate θ, leading to the unidirectional flow of flux.

2.4.3. Thin Tubular charge inside the Coil:

The magnetic field intensity in the outer most sleeve is equal to the airgap intensity. The total flux surrounded by this sleeve induces a voltage in the sleeve. Therefore, a current flows circumferentially in the sleeve. The magnitude of this current is determined by the induced voltage and the resistance of the sleeve,for the time being if the solid charge is considered to be made up of only one sleeve, it acts as if a thin walled tubular metallic charge is inserted coaxially inside the solenoidal coil. The variable magnetic flux surrounded by the charge induces voltages and currents in the charge in a similar manner as in the secondary of a transformer, consisting of one turn which is short circuited. Since the coil is energised by a constant alternating current, an alternating field intensity H_o results in the interspace between the coil and the tubular charge as shown in fig. (2.2). Similar to the current I_c in the coil, a current I_A circulates circumferentially in the charge and produces a magnetic intensity ΔH_o which is constant through-out the space surrounded by charge and which is zero for any point outside. The resulting field intensity at any instant can be regarded as a result of superposing the magnetic intensities caused by the currents in the coil & charge. The magnetic intensity in the space surrounded by the charge is

$$\overline{H}_1 = \overline{H}_o + \Delta \overline{H}_o \qquad \ldots \qquad \ldots (2.9)$$

where,

$\overline{H}_o$ = Intensity produced by the coil current alone.

$\Delta \overline{H}_o$ = In tensity produced by the charge current alone.

The induction or flux density in the space surrounded by the charge is-

$$B_1 = \mu H_1 \qquad \ldots \qquad \ldots (2.10)$$

Magnetic flux surrounded by the charge is-

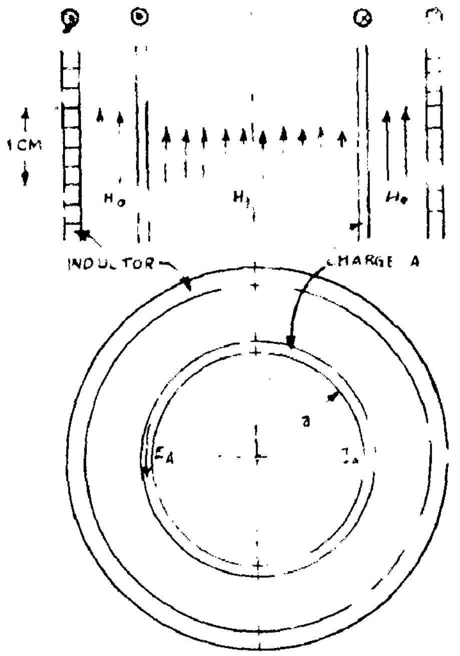
$$\phi_1 = B_1 a^2 \pi = \mu H_1 a^2 \pi \qquad \ldots \qquad \ldots (2.11)$$

where,

a = Radius of the charge

The vector diagram for this case is shown in fig.(2.3) and can be explained as follows-

The e.m.f. E_A generated in the charge A by flux ϕ_1 is lagging 90° behind it. Overcoming the resistance of the path E_A causes a current I_A to flow through the charge which obviously is in phase with E_A. ϕ_1 and H_1 are parallel vectors ΔH_o is



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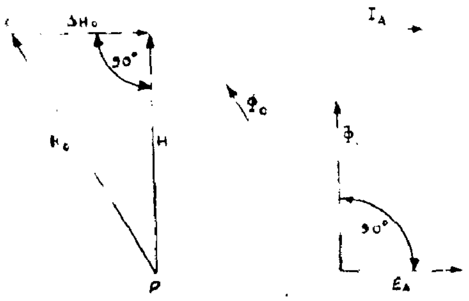

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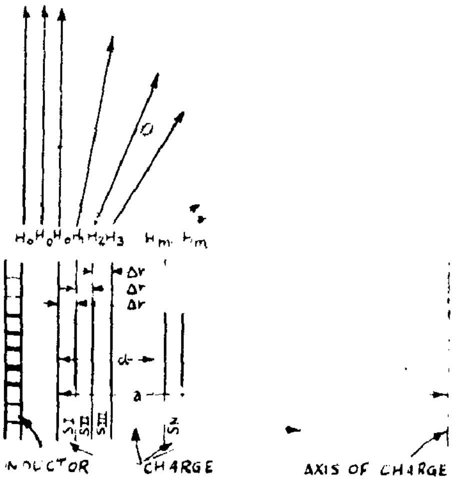

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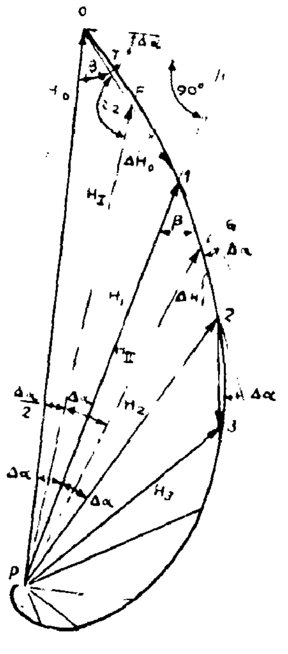
Charge



[illegible]
[illegible] magnetic induction.

parallel to l_A. Hence H_0 being the vector difference of H_1 & ΔH_0 from equation (2.9) follows as OP. The flux ϕ_0 in the interspace between the coil and charge is parallel to H_0.

An inspection of the vector diagram shown in fig.(2.3) reveals the following interesting points:

(1) The magnetic intensity H_0 outside the charge is reduced to intensity H_1 as it penetrates the charge. This effect is due to the circulation of eddy currents in the charge and is known as screening effect.

(2) The flow of current causes a time lag between the two intensities i.e. they are not in phase, H_1 lagging behind H_0 and therefore the flux ϕ_1 surrounded by the charge lags behind the flux ϕ_0 outside of the charge.

2.4.4. Solid Cylindrical Charge inside the Coil:

After the analysis of a thin tubular charge, the case of a solid cylindrical charge can be analysed by considering the charge to be made up by a large number of thin walled tubular sleeves which are telescoped in to each other. The eddy current heating of a solid charge is approached by considering the current distribution in the individual thin walled sleeves and then reducing the thickness of the sleeve towards zero.

The charge is shown diagrammitically in fig.(2.4), in which S_1, S_2, S_3.......S_N ... etc. refer to the different sleeves which constitute the charge and the thickness of each sleeve is Δr.

H_0 is the intensity outside of the charge and H_1, H_2, H_3.........H_m.....etc. , are the intensities which emerge from the sleeves S_1, S_2, S_3......... S_m..... etc. respectively.

From the case of a single thin walled sleeve, it is expected that the magnetic intensity H_0 will be reduced by a certain percentage after penetrating the sleeve S_1 (fig.2.4) and that it will emerge from sleeve S_1 as the intensity H_1, it is also expected that the emerging field intensity H_1 will lag behind the entering intensity H_0. The vectors of the magnetic intensities are shown on a large scale in fig.(2.5).

The angle OIP between ΔH_0 and H_1 was 90° in the case of a single sleeve (fig.2.3), but now in this case it can not remain 90° as shown in fig. (2.5). The reason for this dissimilarity is that in the case of a single sleeve the current I_1 flowing in sleeve S_1 is determined by the electromagnetic action of the sleeve S_1 only, but in this case it is no longer determined by the electromagnetic action of sleeve S_1 only, but also by the electromagnetic action of all the other sleeves i.e. S_2, S_3......S_N.....etc. which are inside of sleeve S_1.

The current I_1 in sleeve S_1 is a function of the total flux surrounded by sleeve S_1. Now this flux is a function of the different currents in all the sleeves. Therefore the magnetic intensity H_0, by penetrating sleeve S_1, of thickness Δr, emerges as intensity H_1, lagging by an angle $\Delta\alpha$ behind H_0 and is reduced in magnitude by a definite percentage of its entering value. The magnetic intensity H_1 now enters the sleeve S_2. Since the wall thickness of the sleeve S_1 and S_2 is the same, it is obvious that ~~t~~the emerging intensity H_2 from sleeve S_2 will lag behind H_1 by the same angle $\Delta\alpha$, as H_1 was lagging behind H_0 in sleeve S_1. The ratio of reduction of H_1 to H_2 is therefore equal to H_0 to H_1.

These assumptions are justified by leading to interdependent systems of magnetic intensities & fluxes, electromagnetic forces and currents, which fulfill all physical requirements.

2.4.5. Vector Diagram of Magnetic Intensities:

The vector diagram is shown in fig. (2.5). Let-

$$\frac{H_1}{H_0} = \frac{H_2}{H_1} = \ldots\ldots\ldots \quad \frac{H_m}{H_{m-1}} = \ldots = q = \text{Constant} \quad .(2.12)$$

$$OPI = IP2 = 2P3 = \Delta\alpha = \text{Constant} \qquad (2.13)$$

The triangles P01, P12, P23 .... etc. are similar (two homologous sides and one included angle $\Delta\alpha$) These geometric relationships are mathematically expressed, by-

$$H_m = H_o \quad e^{-(m \cdot \Delta\alpha)\,\mathrm{Cot}\,\beta} \quad \ldots \qquad (2.14)$$

as shown in Appendix I.

where,

β = Angle between H_o and Δ H_o. fig. (2.5).

α = Phase angle of H_m w.r.t. H_o.

or,

$$\alpha = m \cdot \Delta\alpha \quad \ldots \quad \ldots \qquad (2.15)$$

$$\therefore \quad H_m = H_o\, e^{-\alpha\,\mathrm{Cot}\cdot\beta} \ldots \quad \ldots \qquad (2.16)$$

Equation (2.16) shows that the locus of magnetic intensity is a logarithmic spiral.

2.4.6. Mathematical Analysis-

The external energy source in induction heating is the magnetizing force of field intensity, for this reason it is of great importance to determine how this field intensity is distributed throughout the work-piece. The field intensity created by the alternating current, in turn induces the heating current in the work-piece, which bears a direct relation to it. Therefore, the distribution of magnetic field intensity gives a direct indication of the distribution of heat and temperature gradient in the work-piece.

From equation (2.8) $\nabla^2 \bar{H} = \frac{\mu}{\rho} \frac{\partial \bar{H}}{\partial t}$.

Reducing the above equation to cylindrical coordinates and assuming unidirectional flow of flux (i.e. in the axial direction), the equation left to be solved is-

$$\frac{1}{r} \cdot \frac{d}{dr} \left(r \frac{dH}{dr}\right) = \frac{\mu}{\rho} \frac{\partial H}{\partial t} \quad \ldots \qquad (2.17)$$

where, the field is independent of coordinate θ and it varies with the distance r from the center line of the bar.

And for a radius a of the section, the boundary conditions are established as below:

$$\text{at} \quad r = a, \quad H = H_0 \quad \text{i.e. at the surface.}$$

$$= U_0 \operatorname{Cos} \omega t.$$

$$* \operatorname{Re} (U_0 e^{j \omega t}).$$

and at r = 0, i.e. at the center 'H' will have some finite value however smaller it may be.

Now assuming a solution for (2.17) of the form-

$$H = U(r) \cos \omega t.$$

$$= \mathrm{Re}\, [U(r)\, e^{j\omega t}].$$

where, U(r) is a function of r only.

Therefore from equation (2.17),

$$\frac{1}{r} \frac{d}{dr} \left[r \frac{d}{dr} \left[\mathrm{Re}\, [U(r)\, e^{j\omega t}] \right] \right] =$$

$$\frac{\mu}{\rho} \cdot \frac{\partial}{\partial t} \left[\mathrm{Re}\, [U(r)\, e^{j\omega t}] \right]$$

Suppressing the 'Re' symbol, and dividing throughout by the common factor $e^{j\omega t}$.

$$\frac{1}{r} \frac{d}{dr} \left(r \frac{dU}{dr}\right) = \frac{\mu}{\rho} U \cdot j\omega$$

or

$$\frac{d_2U}{dr^2} + \frac{1}{r} \frac{dU}{dr} = j \frac{\mu}{\rho} \cdot \omega \cdot U$$

or,

$$\frac{d_2U}{dr^2} + \frac{1}{r} \frac{dU}{dr} - j\omega \cdot \frac{\mu}{\rho} U = 0$$

or,

$$\frac{d_2U}{dr^2} + \frac{1}{r} \frac{dU}{dr} - \frac{j}{c^2} U = 0 \qquad \ldots \qquad (2.18)$$

where, $c^2 = \dfrac{\rho}{\mu \cdot \omega}$ ... (2.19)

now, $\mu = \mu_r \mu_o = 4\pi \times 10^{-9} \mu_r.$

and $\omega = 2\pi f$-

where μ_r = Relative permeability.

and f = frequency in c/s.

Then equation (2.22) reduces to the simple form,

$$U = A\ J_o(mr) \quad \ldots \qquad \ldots \qquad (2.23)$$

now at $r = a$, $U = U_o$ (at the surface)

$$\therefore \quad U_o = A\ J_o\ (ma)$$

$$\therefore \quad A = \frac{U_o}{J_o\ (ma)}$$

Substituting the value of A in equation (2.23),

$$U = U_o \frac{J_o(mr)}{J_o(ma)}$$

now,

$$H = \text{Re}\ (U.\ e^{j\omega t}).$$

$$= \text{Re} \left[U_o.\ \frac{J_o\ (mr)}{J_o\ (ma)}\ .\ e^{j\omega t} \right]$$

Since $H_o = \text{Re}\ (\ U_o\ e^{j\omega t})$

$$\therefore \quad H = H_o.\ \frac{J_o(mr)}{J_o(ma)} \qquad \ldots \qquad (2.24)$$

The Bessel function $J_o\ (mr)$ has the form of an infinite series and is given by(11)

$$J_o(mr) = 1 - \frac{(mr)^2}{2^2} + \frac{(mr)^4}{2^2.4^2} - \frac{(mr)^6}{2^2.4^2.6^2} + \frac{(mr)^8}{2^2.4^2.6^2.8^2} \ldots$$

$$\ldots \qquad (2.25)$$

$$\text{now } m = \frac{\sqrt{-j}}{c}$$

$$\therefore J_o\ (mr) = J_o\ (\sqrt{-j}\ .\frac{r}{c}) = 1 + j\frac{(r/c)^2}{2^2} - \frac{(r/c)^4}{2^2.4^2} +$$

$$j\ \frac{(r/c)^6}{2^2.4^2.6^2} + \frac{(r/c)^8}{2^2.4^2.6^2.8^2} + \ldots$$

$$\ldots \qquad (2.26)$$

Putting $r/c = x$ we have,

Substituting these values in equation (2.19),

$$c^2 = \frac{\rho}{4\pi x\ 10^{-9}.\ \mu_r\ x\ 2\pi f} = \frac{\rho\ x\ 10^9}{8\pi^2\ \mu_r . f}$$

$$\therefore\ c = \frac{1}{2\pi\sqrt{2}} \left(\frac{\rho\ x10^9}{\mu_r \cdot f} \right)^{1/2} \text{cm.}$$

let $s = c\sqrt{2} = \frac{1}{2\pi} \left(\frac{\rho\ x10^9}{\mu_r \cdot f} \right)^{\frac{1}{2}}$ cm. ... (2.20)

The quantity 's' is known as reference depth and is of great importance. 's' has the dimension of length from equation (2.18) we get,

$$\frac{d^2U}{dr^2} + \frac{1}{r}\frac{dU}{dr} + m^2 U = 0 \qquad \ldots \qquad (2.21)$$

where, $m^2 = -\ j/c^2$

Equation (2.21) is Bessel's differential equation and its general solution is given by$^{(11)}$.

$$U = A\ J_o\ (mr) + B\ K_o\ (mr) \qquad \ldots \qquad (2.22)$$

In the above equation,

A & B are arbitrary constants,

$J_o(mr)$ is a Bessel function of the first kind and zeroth order.

$K_o(mr)$ is the Bessel's function of second kind and zeroth order.

Since $K_o(mr)$ becomes infinite as r approaches zero and since the magnetic intensity must have a finite value at r = 0, we may drop this function from the solution.

Therefore $B = 0$

Then equation (2.22) reduces to the simple form,

$$U = A\ J_o(mr) \quad \ldots \qquad \ldots \qquad (2.23)$$

now at $r = a$, $U = U_o$ (at the surface)

$$\therefore \quad U_o = A\ J_o\ (ma)$$

$$\therefore \quad A = \frac{U_o}{J_o\ (ma)}$$

Substituting the value of A in equation (2.23),

$$U = U_o \frac{J_o(mr)}{J_o(ma)}$$

now,

$$H = \text{Re}\ (U.\ e^{j\omega t}).$$

$$= \text{Re} \left\{ \left[U_o.\ \frac{J_o\ (mr)}{J_o\ (ma)}\ .\ e^{j\omega t} \right] \right\}$$

Since $H_o = \text{Re}\ (\ U_o\ e^{j\omega t})$

$$\therefore \quad H = H_o.\ \frac{J_o(mr)}{J_o(ma)} \qquad \ldots \qquad (2.24)$$

The Bessel function $J_o\ (mr)$ has the form of an infinite series and is given by[(11)]

$$J_o(mr) = 1 - \frac{(mr)^2}{2^2} + \frac{(mr)^4}{2^2.4^2} - \frac{(mr)^6}{2^2.4^2.6^2} + \frac{(mr)^8}{2^2.4^2.6^2.8^2} \ldots$$

$$\ldots \qquad (2.25)$$

$$\text{now } m = \frac{\sqrt{-j}}{c}$$

$$\therefore J_o\ (mr) = J_o\ (\sqrt{-j}\ .\frac{r}{c}) = 1 + j\frac{(r/c)^2}{2^2} - \frac{(r/c)^4}{2^2.4^2} + j\ \frac{(r/c)^6}{8^2.4^2.6^2} + \frac{(r/c)^8}{2^2.4^2.6^2.8^2} + \ldots$$

$$\ldots \qquad (2.26)$$

Putting $r/c = x$ we have,

$$J_o(\sqrt{-j}.\ r/c) = J_o(\sqrt{-j}\ .\ x)$$

$$= 1+j\frac{x^2}{2^2} - \frac{x^4}{2^2.\ 4^2} - j\frac{x^6}{2^2.\ 4^2.6^2} + \frac{x^8}{2^2.4^2.6^2.8^2} + \ldots. \qquad \ldots \qquad (2.27)$$

When the real and imaginary terms are separated, two separate series are obtained. The real terms form a series which Kelvin named ber (x), the real part of the Bessel function, while the other series is called bei (x), the imaginary part of the Bessel function, where,

$$\mathrm{ber}(x) = 1 - \frac{x^4}{2^2.4^2} + \frac{x^8}{2^2.4^2.6^2.8^2} - \frac{x^{12}}{2^2.4^2.6^2.8^2.10^2.12^2} + \ldots. \qquad \ldots \qquad (2.28)$$

and,

$$\mathrm{bei}(x) = \frac{x^2}{2^2} - \frac{x^6}{2^2.4^2.6^2.} + \frac{x^{10}}{2^2.4^2.6^2.8^2.10^2} - \ldots. \qquad \ldots \qquad (2.29)$$

Hence, $J_o(mr) = J_o(\sqrt{-j}\ .r/c) = \mathrm{ber}\ (r/c)+j\ \mathrm{bei}\ (r/c)$ ... (2.30)

Therefore from equation (2.24) we have,

$$H = H_o\frac{\mathrm{ber}\ (r/c) + j\ \mathrm{bei}\ (r/c)}{\mathrm{ber}\ (a/c) + j\ \mathrm{bei}\ (a/c)} \qquad \ldots \qquad (2.31)$$

$$\therefore \frac{H}{H_o} = \frac{\mathrm{ber}\ (r/c) + j\ \mathrm{bei}(r/c)}{\mathrm{ber}\ (a/c) + j\ \mathrm{bei}\ (a/c)} \qquad \ldots \qquad (2.32)$$

$$\therefore H = A\left[\mathrm{ber}\ (r/c) + j\ \mathrm{bei}\ (r/c)\right] \qquad (2.33)$$

where,

$$A = \frac{H_o}{\mathrm{ber}\ (a/c) + j\ \mathrm{bei}(a/c)} \qquad \ldots \qquad (2.34)$$

Current density

$$i_r = -\frac{\partial H}{\partial r} = -\frac{A}{c}\left[\text{ber}'(r/c) + j\,\text{bei}'(r/c)\right] \qquad (2.35)$$

At the surface r = a,

$$i_a = -\frac{A}{c}\left[\text{ber}'\,(a/c) + j\,\text{bei}'\,(a/c)\right] \qquad (2.36)$$

or,

$$\frac{i_r}{i_a} = \frac{\text{ber}'\,(r/c)+j\,\text{bei}'\,(r/c)}{\text{ber}'\,(a/c) + j\,\text{bei}'(a/c)} \quad \ldots \qquad (2.37)$$

$$\left|\frac{i_r}{i_a}\right| = \left[\frac{\text{ber}'^2\,(r/c) + \text{bei}'^2(r/c)}{\text{ber}'^2\,(a/c) + \text{bei}'^2(a/c)}\right]^{\frac{1}{2}} \ldots \qquad (2.38)$$

This ratio i_r/i_a is plotted for different values of (a/c) in fig.(2.6), which gives the current distribution in the cylinder.

From the above curves given in fig.(2.6), it is investigated that the skin effect is not so pronounced at the lower values of a/c, but is very much pronounced at the higher values of a/c, for example studying the curve for a/c = 4, we find that i_r/i_a = 0.5, at r/c = 2.8, or we can say that the current density has dropped to 50% of its surface value at a radius of 2.8/4 or 70% of the work radius i.e. the current density has dropped to half the value at the surface at 30% of the work radius in from the surface, therefore, the skin effect, is not so pronounced.

Now studying the curve for a/c = 12, it is noticed that the current density drops to 50% of its surface value at r/c = 11, i.e. at a radius of 11/12 or 92% of its work radius or in other words the current density becomes half at a radius only 8% in from the surface, so obviously it results in a

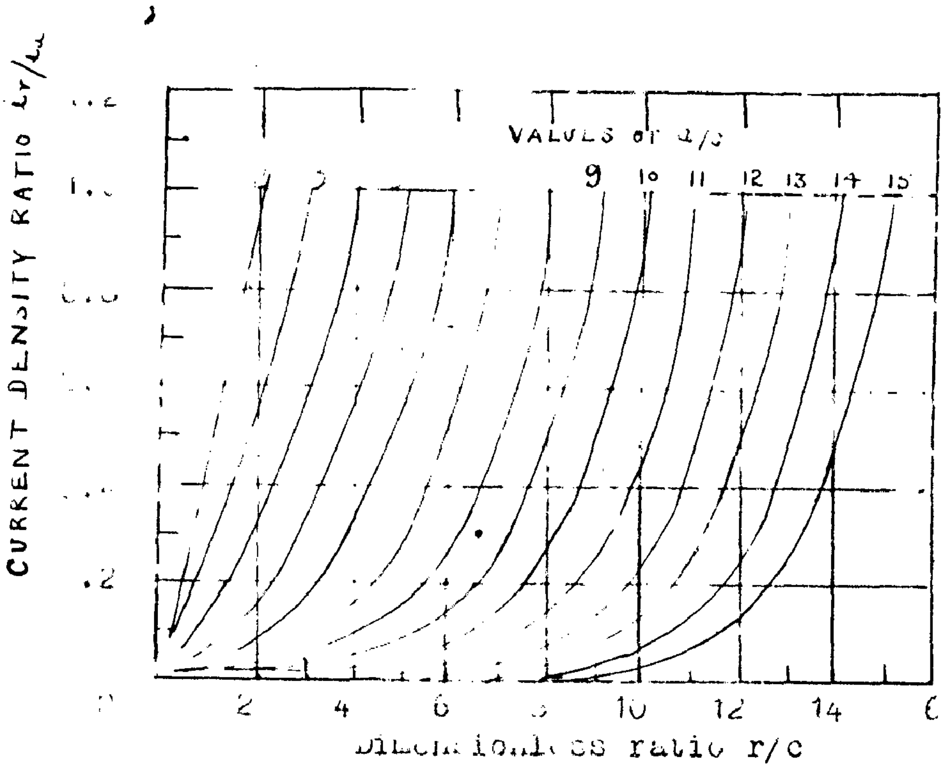


Fig.2.6 - Current Density Distribution

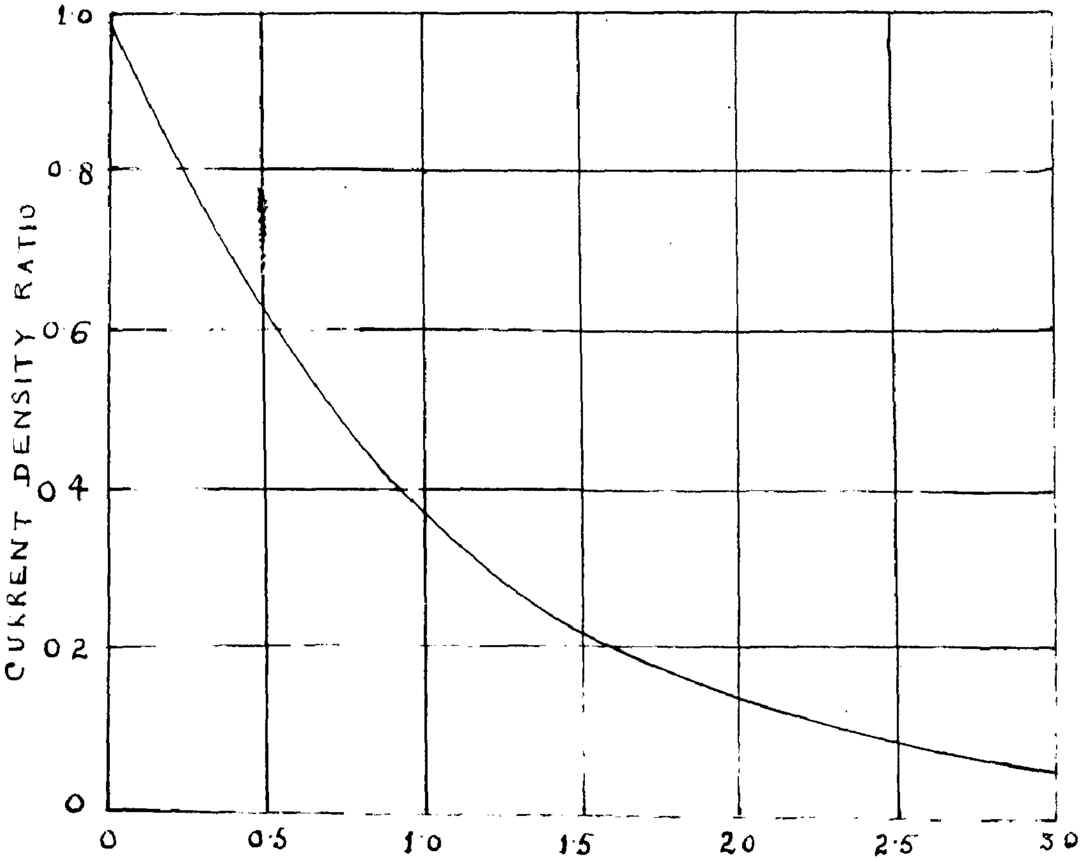


FIG 2·7 - DIMENSIONLESS RATIO $\frac{x}{S}$

pronounced skin effect.

For a fixed value of radius the severity of skin-effect is determined by the quantity c which is equal to $s/\sqrt{2}$, where s is the reference depth. The reference depth which first appeared in equation (2.20) is inversely proportional to the square root of frequency. Hence higher the frequency, smaller is the value of reference depth and more pronounced is the skin-effect, which has been justified from the curves also.

Many examples can be simply derived by approximating equation (2.37) for the large values of r/c i.e. when either the frequency of the induced current or the work radius at the point considered is large, The Bessel functions can be written under these conditions as(11)

$$\mathrm{ber}(r/c) = \frac{e^{r/c\sqrt{2}}}{\sqrt{2\pi r/c}} \cdot \mathrm{Cos}\left(\frac{r}{c\sqrt{2}} - \frac{\pi}{8}\right)$$

$$\mathrm{bei}\,(r/c) = \frac{e^{r/c\sqrt{2}}}{\sqrt{2\pi r/c}} \cdot \mathrm{Sin}\left(\frac{r}{c\sqrt{2}} - \frac{\pi}{8}\right)$$

$$\mathrm{ber}'(r/c) = \frac{e^{r/c\sqrt{2}}}{\sqrt{2\pi r/c}} \cdot \mathrm{Cos}\left(\frac{r}{c\sqrt{2}} + \frac{\pi}{8}\right)$$

$$\mathrm{bei}'(r/c) = \frac{e^{r/c\sqrt{2}}}{\sqrt{2\pi r/c}} \cdot \mathrm{Sin}\left(\frac{r}{c\sqrt{2}} + \frac{\pi}{8}\right)$$

From equation (2.37) we have,

$$\frac{i_r}{i_a} = \frac{\mathrm{ber}'\ r/c + j\ \mathrm{bei}'\ r/c}{\mathrm{ber}'\ a/c + j\ \mathrm{bei}'\ a/c}$$

$$= \left\{\left(\frac{2\pi.a/c}{2\pi.r/c}\right)^{\frac{1}{2}} \cdot e^{(1/c\sqrt{2})(r-a)}\right\} x \left\{\frac{\mathrm{Cos}(\frac{r}{c\sqrt{2}}+\frac{\pi}{8})+j\mathrm{Sin}(\frac{r}{c\sqrt{2}}+\frac{\pi}{8})}{\mathrm{Cos}(\frac{a}{c\sqrt{2}}+\frac{\pi}{8})+j\mathrm{Sin}(\frac{a}{c\sqrt{2}}+\frac{\pi}{8})}\right\} \tag{2.39}$$

Since $c\sqrt{2} = s$ = Reference depth.

and let $a - r = x$ = Distance from the surface

The equation (2.39) reduces to-

$$\frac{i_r}{i_a} = (a/r)^{\frac{1}{2}}.e^{-x/s} \;, \quad e^{-j.x/s} \qquad \ldots \qquad (2.40)$$

The above equation indicates that at a depth x where x = a [illegible] , the phase of the current d[illegible]ity has been retarded by [illegible] or $\frac{(a-r)}{c\sqrt{2}}$, The phase will b[illegible]versed completely, when $x = s\pi = \sqrt{2} . c . \pi$

Then in that case as a and r both become very large, the equation (2.40) further reduces to:

$$\frac{i_r}{i_a} = e^{-x/s} \qquad \ldots \qquad (2.41)$$

This equation is plotted in fig. 2.7.

It is seen that when $x = s$

$$\frac{i_r}{i_a} = \frac{1}{e}$$

or,

$$i_r = (1/e)\, i_a \qquad \ldots \qquad (2.42)$$

i.e. i_r [illegible] reduced to 36.7% of its surface valu[illegible] x = s. This i[illegible]e "Equivalent current depth".

By a similar analysis we have-

$$H_r = H_o\, e^{-x/s} \qquad \ldots \qquad (2.43)$$

The curve plotted in fig.(2.7) gives only approximate results but the fig.(2.6) gives the accurate results. The difference lies in the fact that fig.(2.6) is based on the accurate formula in equation (2.38), which takes in to account the effects of the current flow from the opposite side of the cylinder, whereas fig. (2.7) is based on the approximation in equation (2.41), which assumes no effect from the current on the opposite side.

However, fig. (2.7) can be used with reasonable accuracy when $s \angle 1/5\ a$, i.e. the current depth is less than one fifth of the radius.

2.4.7. Total Flux and Power in a Solid Cylinder:

Considering the elementary ring of radius r and width r as shown in fig. (2.8), if ϕ_r is the total flux in the ring then,

$$\phi_r = \mu.\ H.\ 2\ \pi\ r.\ \ \delta r. \qquad \ldots \qquad (2.44)$$

The total flux ϕ_{tr} inside the area enclosed by the ring at r is given by integrating equation (2.44)-

$$\phi_{tr} = \int_0^r \mu\, H_r\ 2\ \pi\ r.\ \ dr$$

Putting the value of H from equation (2.33),

$$\phi_{tr} = 2\pi\mu \int_0^r A\ (\mathrm{ber}\ \frac{r}{c} + j\ \mathrm{bei}\ r/c).r.dr. \ldots \qquad (2.45)$$

From the properties of Bessel function, we can write the following useful derivatives$^{(11)}$.

$$\mathrm{bei}'\ (r/c) = \frac{1}{rc} \int r\ \mathrm{ber}\ (r/c)\ dr. \qquad \ldots \qquad (2.46)$$

and

$$\mathrm{ber}'\ (r/c) = -\frac{1}{rc} \int r\ \mathrm{bei}\ (r/c)\ dr \qquad \ldots \qquad (2.47)$$

∴ from (2.46)

$$\int r\ \mathrm{ber}\ (r/c)\ dr = rc\ \ \mathrm{bei}'\ (r/c) \qquad \ldots \qquad (2.48)$$

and,

$$\int r\ \mathrm{bei}\ (r/c)\ dr = -\ rc\ \mathrm{ber}'\ (r/c) \qquad \ldots \qquad (2.49)$$

Substituting equation (2.48) and (2.49) in equation (2.45)

$$\therefore \phi_{tr} = 2\pi\mu A \left[\Big[rc \text{ bei}'(r/c) \Big]_0^r + j\Big[-rc \text{ ber}'(r/c) \Big]_0^r \right]$$

$$= 2\pi\mu .A.\, rc \left[\text{bei}' (r/c) - j \text{ ber}' (r/c) \right]$$

Substituting the value of constant A from equation (2.34),

$$\phi_{tr} = (2\pi\mu .rc).H_o \left[\frac{\text{bei}'(r/c) - j \text{ ber}'(r/c)}{\text{ber } (a/c) + j \text{ bei } (a/c)} \right] \qquad ..(2.50)$$

At $r = a$, $\phi_{tr} = \phi_a$

where ϕ_a = Total flux inside the work-piece.

$$\therefore \phi_a = (2\pi\mu .a.c.)\, H_o \left[\frac{\text{bei}'(a/c) - j \text{ ber}'(a/c)}{\text{ber } (a/c) + j \text{ bei},(a/c)} \right] \qquad ...\ (2.51)$$

After rationalising Equation (2.51) we get,

$$\phi_a = (2\pi\mu \cdot a.c.\, H_o) \frac{(\text{bei}'a/c - j \text{ ber}'a/c)(\text{ber } a/c + j \text{ bei } a/c)}{\text{ber}^2(a/c) + \text{bei}^2 (a/c)} \qquad ...\ (2.52)$$

Now area of the work-piece is given by $A_w = \pi a^2$

Hence from (2.52)

$$\phi_a = \frac{2\ \mu\ H_o .c.A_w}{a} \left[\frac{(\text{bei}'a/c \text{ ber } a/c - \text{ber}'a/c \text{ bei } a/c) - j(\text{bei}'a/c \text{ bei } a/c + \text{ber}'\ a/c \text{ ber } a/c)}{\text{ber}^2\ a/c + \text{bei}^2\ a/c} \right] \qquad ...\ (2.53)$$

$$\text{or, } \phi_a = \mu H_o\ A_w\ (P - jQ) = \phi_p - j\ \phi_q \qquad ...\ (2.54)$$

$$\text{where, } P = \frac{2c}{a} \left| \frac{\text{bei}'a/c \text{ ber } a/c - \text{ber}'a/c \text{ bei } a/c}{\text{ber}^2\ a/c + \text{bei}^2\ a/c} \right| \qquad (2.55)$$

and,

$$Q = \frac{2c}{a}\left[\frac{\text{bei}'\ a/c.\ \text{bei}\ a/c+\text{ber}'\ a/c\ \text{ber}\ a/c}{\text{ber}^2\ a/c\ +\ \text{bei}^2\ a/c}\right] \quad \text{.. (2.56)}$$

These are called the P and Q functions of the cylinder.

Total power in the cylinder:

The heat generated in the solid cylinder is derived by considering the total current flowing in the shell of width δr in 1 cm. of work length.

The current flowing through the shell is given by-

$$I_{tr} = i_r \cdot \delta r \qquad \text{...} \quad (2.57)$$

The resistance of the shell is given by-

$$R_r = \frac{\rho\ 2\pi r}{\delta r} \qquad \text{...} \quad (2.58)$$

Therefore, the power or heat dissipation loss is-

$$P_{tr} = I_{tr}^2 \cdot R_r = i_r^2 \cdot 2\pi\rho\ r.\ \delta r.$$

Integrating over the complete radius, the power loss per cm. length is-

$$P_t = \int_0^a i_r^2 \cdot 2\pi \cdot \rho \cdot r \cdot dr \qquad \text{...} \quad (2.59)$$

Putting the value of i_r from (2.35),

$$P_t = \int_0^a \frac{(A)^2}{c^2} \cdot 2\pi\rho \cdot r\ (\text{ber}'^2\ r/c+\text{bei}'^2\ r/c)dr.$$

$$P_t = \frac{A^2}{c^2} . 2\pi . \rho .\ \ a.c.\ (\text{ber}\ a/c\ \text{ber}'a/c+\text{bei}a/c.\text{bei}'a/c)$$

$$\text{...} \quad (2.60)$$

$$\therefore P_t = \frac{|A|^2}{c} \cdot 2\pi \cdot \rho . a\ (\text{ber}\ a/c\ \text{ber}'a/c+\text{bei}\ a/c+\text{bei}'a/c)$$

$$\text{...} \quad (2.61)$$

$$P_t = 2\pi\rho H_o^2 . a/c \left[\frac{\text{ber } a/c . \text{ber}' a/c + \text{bei } a/c \text{ bei}' a/c}{\text{ber}^2 \; a/c + \text{bei}^2 \; a/c}\right] \quad \ldots \quad (2.62)$$

or,

$$P_t = 2\pi \; H_o^2 . \frac{\sqrt{2}.a}{s}\left[\frac{\text{ber}(\frac{\sqrt{2}a}{s}).\text{ber}'(\frac{\sqrt{2}a}{s}) + \text{bei}(\frac{\sqrt{2}a}{s}) \text{ bei}'(\sqrt{2}a/s)}{[\text{ber}(\sqrt{2}a/s)]^2 + [\text{bei} \; (\sqrt{2}a/s)]^2}\right] \quad \ldots \quad (2.63)$$

or,

$$P_t \text{ (watts per unit length)} = 2\pi H_o^2 . \rho . a/s. F. \quad \ldots \quad (2.64)$$

where,

$$F = \sqrt{2}\, \frac{\text{ber}(\sqrt{2}a/s) \; \text{ber}' \; (\sqrt{2}a/s) + \text{bei}(\sqrt{2}a/s)\text{bei}'(\sqrt{2}a/s)}{[\text{ber} \; (\sqrt{2}a/s)]^2 + [\text{bei} \; (\sqrt{2}a/s)]^2} \quad \ldots \quad (2.65)$$

If the length of the work-piece is l_w cms.

Then,

$$\text{Total power } P_t = 2\pi H_o^2 \rho . \frac{a}{s} F. l_w \text{ watts} \quad (2.66)$$

In the equation (2.66), H_o is in Ampere turns/cm.

The functions F and a/s.F can be plotted as a function of a/s, the radius measured in terms of the skin-thickness, they are shown in fig.(2.9) and fig. (2.10) respectively.

The values of the functions F and a/s. F corresponding to different values of a/s are tabulated in the table No.2.1.

TABLE 2.1

$a/c = \sqrt{2}a/s$	$a/s = a/c\sqrt{2}$	F $\sqrt{2}\,\frac{\mathrm{ber}(a/c)\mathrm{ber}'(a/c)+\mathrm{bei}(a/c)\mathrm{bei}'(a/c)}{\mathrm{ber}'^2 a/c + \mathrm{bei}^2 a/c.}$	a/s.F.
0.5	.354	0.011	0.0039
1.0	.707	0.087	0.0615
1.5	1.06	0.260	0.276
2.0	1.414	0.488	0.690
2.5	1.77	0.660	1.17
3.0	2.12	0.764	1.62
3.5	2.48	0.805	2.00
4.0	2.83	0.826	2.34
4.5	3.18	0.840	2.67
5.0	3.54	0.855	3.03
10.0	7.07	0.930	6.57
20.0	14.14	0.965	13.65
30.0	21.2	0.972	20.60
50.0	35.4	0.980	34.70
70.0	49.5	0.985	48.70
100.00	70.70	0.990	70.00
140.00	99.00	0.999	99.00

It is seen that when a/s is less than unity the quantity a/s.F is approximated quite well by the expression $\frac{1}{4}(a/s)^4$. Under this condition the power per unit length is

$$P_t = \frac{1}{2}\,\pi\,H_o^2\,\rho\,(a/s)^4 \qquad .. \quad (2.67)$$

When a/s is greater than 5 the quantity a/s.F. becomes approximately a/s, so that the power per unit length is-

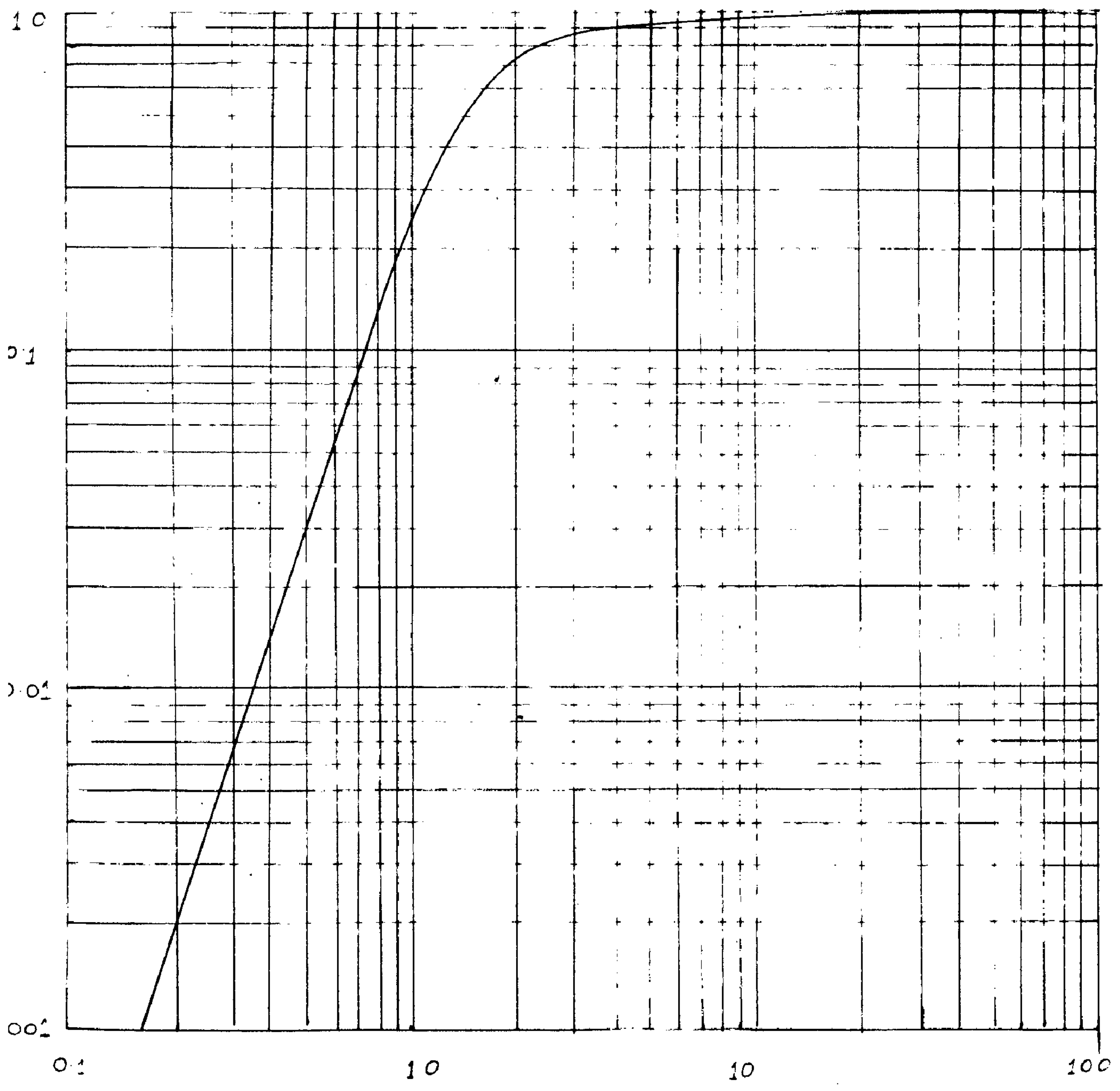


FIG 24 - DIMENSIONLESS RATIO a/s ⟶

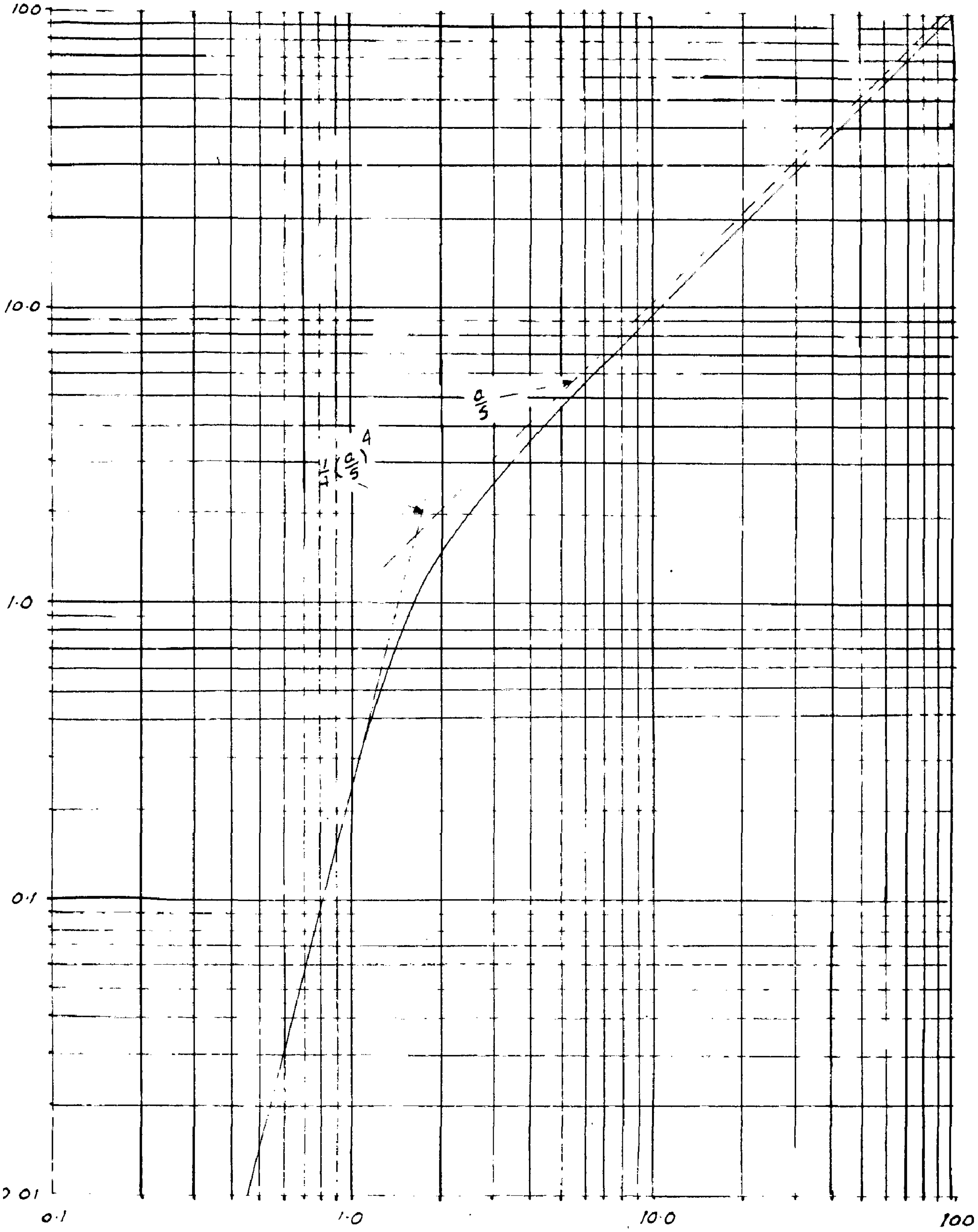


FIG. 2.10 - DIMENSION LESS RATIO a/s

TABLE 2.1

$a/c = \sqrt{2}a/s$	$a/s = a/c/\sqrt{2}$	F $\sqrt{2}\,\dfrac{\mathrm{ber}(a/c)\mathrm{ber}'(a/c)+\mathrm{bei}(a/c)\mathrm{bei}'(a/c)}{\mathrm{ber}'^2 a/c + \mathrm{bei}^2\, a/c.}$	a/s.F.
0.5	.354	0.011	0.0039
1.0	.707	0.087	0.0615
1.5	1.06	0.260	0.276
2.0	1.414	0.488	0.690
2.5	1.77	0.660	1.17
3.0	2.12	0.764	1.62
3.5	2.48	0.805	2.00
4.0	2.83	0.826	2.34
4.5	3.18	0.840	2.67
5.0	3.54	0.855	3.03
10.0	7.07	0.930	6.57
20.0	14.14	0.965	13.65
30.0	21.2	0.972	20.60
50.0	35.4	0.980	34.70
70.0	49.5	0.985	48.70
100.00	70.70	0.990	70.00
140.00	99.00	0.999	99.00

It is seen that when a/s is less than unity the quantity a/s.F is approximated quite well by the expression $\frac{1}{8}\,(a/s)^4$. Under this condition the power per unit length is

$$P_t = \tfrac{1}{2}\,\pi\,H_o^2\,\rho\,(a/s)^4 \qquad \text{.. (2.67)}$$

When a/s is greater than 5 the quantity a/s.F. becomes approximately a/s, so that the power per unit length is–

$$P_t = 2\,\pi\,H_0^2 \cdot \rho \cdot \;a/s \qquad \ldots \qquad (2.68)$$

The curve drawn in fig.(2.10) can be used to estimate the power absorbed per unit length of the cylinder when the impressed field H_0 is known.

A plot of function F as seen in fig. (2.9) helps to establish the upper limit on the frequency necessary for efficient coupling.

In equation (2.63) H_0 has been regarded as a constant from this we draw the implication that the current in the exciting solenoid has been held constant. Therefore the power lost in the solenoid is proportional to the radio frequency resistance of the solenoid. For the copper conductor used in the solenoid, the ratio of radius to skin thickness is usually large i.e. the skin thickness is very small. Now since the skin thickness varies inversely as the square root of frequency so that the resistance varies directly as the square root of frequency. Thus when the function F is plotted, we have effectively divided out the by $(1/s)$ i.e. by the square root of frequency in (2.64). Hence the curve of fig.(2.9 may be regarded as being proportional to the ratio of the power absorbed in the cylinder to the power lost in the exciting solenoid. It is seen from fig. (2.9) that, when a/s is greater than 2.25, the knee of the curve has been passed, and the value of F remains near about 1.0, irrespective of the value of a/s, that is to say that no great benefit is derived by increasing the ratio a/s beyond 2.25. Increasing the ratio a/s means reducing the skin thickness which is achieved by increasing the frequency. Hence it is not very useful to increase the frequency further beyond the point corresponding to a/s = 2.25. The frequency corresponding to a/s = 2.25 is defined as the critical frequency for effective coupling to a cylinder.

We have from equation (2.20)-

$$s = \frac{1}{2\,\pi}\left(\frac{\rho\,x10^9}{\mu_r . f}\right)^{\frac{1}{2}}$$

Putting a/s = 2.25 or s = a/2.25 and denoting the frequency as this point as f_c-

$$\frac{a}{2.25} = \frac{1}{2\,\pi}\left(\frac{\rho\,x10^9}{\mu_r . f_c}\right)^{\frac{1}{2}}$$

or,

$$f_c = \frac{1}{4\,\pi^2}\,(2.25)^2 . \frac{\rho . 10^9}{\mu_r . a^2}$$

or,

$$f_c = \frac{128.5\ x\ 10^6 . \rho}{\mu_r\ .\ a^2} \qquad \ldots \qquad (2.69)$$

where,

f_c = Critical frequency in cycles.

ρ = Resistivity of cylinder in ohm-cm.

$_r$ = Relative permeability of the cylinder.

a = Radius of the cylinder in cm.

2.5. POWER AND FLUX INDUCED IN FLAT METAL SHEETS OF RECTANGULAR CROSS SECTION:

By using an analysis similar to the preceding section, the field distribution in a rectangular slab can be derived. The author has derived the distribution for two cases.:

(i) Current flow in metal sheets of great thickness.

(ii) Current flow in metal sheets of limited finite thickness.

2.6. CURRENT FLOW IN METAL SHEETS OF GREAT THICKNESS:

The sheet is shown in fig.(2.11), in which, the dimension x, measures the distance from the boundary to the point of examination. Considering the one dimensional system, the magnetic intensity varies with the distance x, assuming the unidirectional flow of flux, mathematically equation (2.8) reduces to-

$$\frac{\partial^2 H}{\partial x^2} = \frac{\mu}{\rho} \cdot \frac{\partial H}{\partial t} \quad \ldots \quad (2.70)$$

To solve the equation (2.70) let us assume a solution of the form-

$$H\,(x, t) = \text{Re}\left[U(x).e^{j\omega t}\right] \quad \ldots \quad (2.71)$$

where U(x) is a function of x only and 'Re' indicates 'Real part of it'.

Discarding 'Re' symbol and substituting the above assumed solution in equation (2.70)-

$$e^{j\omega t} \cdot \frac{d_2U}{dx^2} = \frac{\mu}{\rho} \cdot j\omega \cdot e^{j\omega t} .U.$$

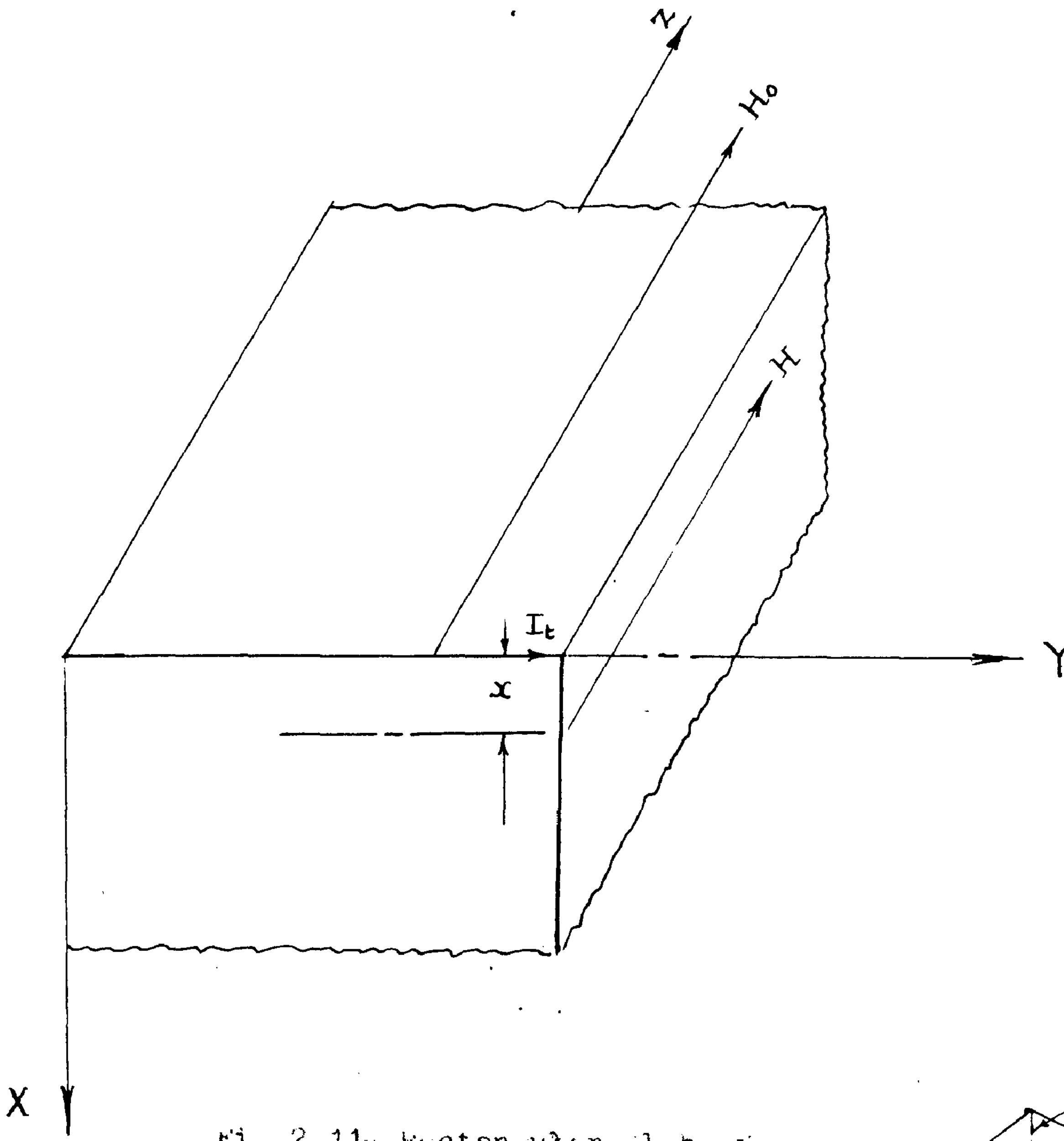


Fig.2.11- Rectangular Slab of Infinite thickness.

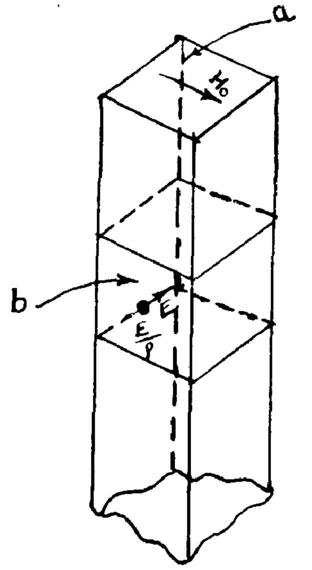


Fig.2.12 - Current flow in a long column of metal.

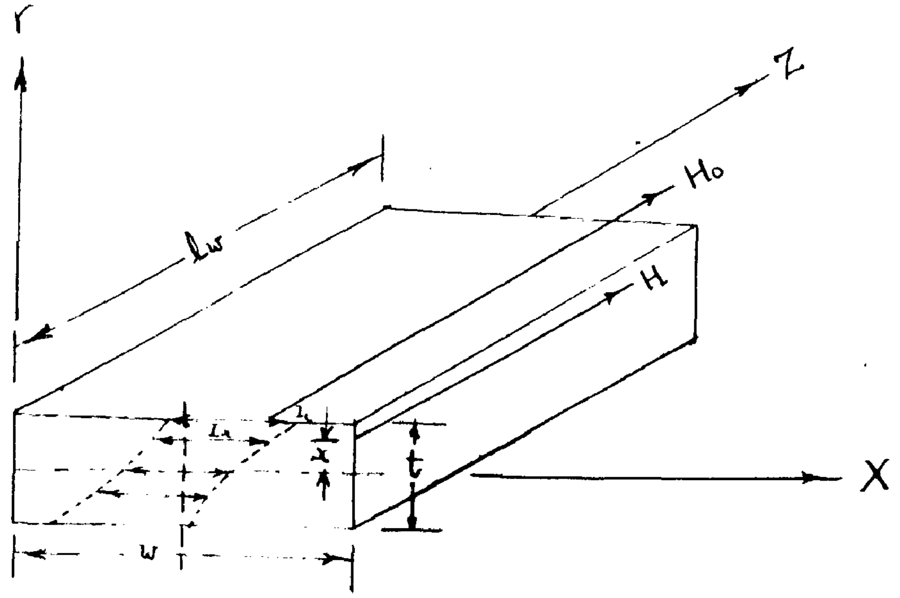


Fig.2.13 - Rectangular [illegible] of Finite [illegible]

or,

$$\frac{d_2U}{dx^2} = j \frac{\mu \cdot \omega}{\rho} U \quad \ldots \quad (2.72)$$

$$= \frac{j}{c^2} U \quad \ldots \quad (2.73)$$

or,

$$\frac{d_2U}{dx^2} - \alpha^2 U = 0 \quad \ldots \quad (2.74)$$

where,

$$\alpha^2 = \frac{j}{c^2} = \frac{1}{c^2} \cdot e^{j(\pi/2)}$$

$$\therefore \alpha = \frac{\sqrt{j}}{c} = \frac{1}{c} \cdot e^{j(\pi/4)}$$

$$= \frac{1}{c} (\mathrm{Cos}\ \pi/4 + j\ \mathrm{Sin}\ \pi/4)$$

$$= \frac{1}{c} \cdot \left(\frac{1+j}{\sqrt{2}} \right)$$

Since $c\sqrt{2} = s$ (from equation 2.20)

$$\therefore \alpha = \frac{1+j}{s} \quad \ldots \quad (2.75)$$

Solution of the differential equation (2.74) is given by

$$U = A\ e^{-\alpha x} + B\ e^{\alpha x} \quad \ldots \quad (2.76)$$

where A and B are the arbitrary constants.

It is seen from the equation (2.76), that the second term becomes infinite as the distance x becomes very large. Since $U \neq \infty$, at $x = \infty$

The constant $B = 0$

Therefore, the equation (2.76) reduces to-

$$U = A \cdot e^{-\alpha x} \quad \ldots \quad (2.77)$$

now at the surface i.e. $x = 0$.

$$H = H_0 = U_0\ \mathrm{Cos}\ \omega\ t = \mathrm{Re}\ (U_0 \cdot e^{j\omega t})$$

i.e. at $x = 0$, $U = U_0$

from (2.77) $A = U_o$

$\therefore \quad U = U_o \, . \, e^{-\alpha x}$

Putting the value of from equation (2.75)-

$$U = U_o \, e^{-(1+j) . \frac{x}{s}} \quad \ldots \quad (2.78)$$

$\therefore \quad H = U \, \mathrm{Cos} \, \omega t = \mathrm{Re} \, (U . e^{j \omega t})$

$$\therefore \quad H = \mathrm{Re} \left[U_o \, e^{-(1+j) . x/s} . e^{j \omega t} \right\} \quad \ldots \quad (2.79)$$

Since $\mathrm{Re} \, (U_o \, e^{j \omega t}) = H_o$

We have from (2.79)-

$$H = H_o \, . \, e^{-(1+j) x/s} \quad \ldots \quad (2.80)$$

$$= H_o . \, e^{-x/s} . \, e^{-j \, x/s}$$

$$= H_o . \, e^{-x/s} \left[\mathrm{Cos} \, \frac{x}{s} - j \, \mathrm{Sin} \, \frac{x}{s} \right] \quad (2.81)$$

From equation (2.35) we have-

$$i_x = - \frac{\partial H}{\partial x}$$

where, i_x = Current density.

now, $\quad i_x = \frac{E}{\rho}$

where, E = Electric Intensity.

and ρ = Resistivity

$$\therefore \quad \frac{E}{\rho} = - \frac{\partial H}{\partial x}$$

or,

$$\frac{\partial H}{\partial x} = - \frac{E}{\rho} \quad \ldots \quad (2.82)$$

From equation (2.80) we have,

$$\frac{\partial H}{\partial x} = - \frac{1}{s} (1+j) . \, H_o . e^{-(1+j) . x/s} \quad \ldots \quad (2.83)$$

From (2.82) and (2.83)-

$$-\frac{E}{\rho} = -\frac{1+j}{s} \cdot H_o e^{-(1+j)x/s}.$$

or,

$$E = (\frac{1+j}{s})\rho \cdot H_o \cdot e^{-(1+j).x/s} \qquad \ldots \qquad (2.84)$$

$$= (\frac{1+j}{s})\rho \cdot H_o \cdot e^{-x/s} \cdot e^{-j\,x/s} \qquad \ldots \qquad (2.85)$$

or,

$$E = (\frac{1+j}{s})\rho \cdot H_o \cdot e^{-x/s} \{\mathrm{Cos}\ x/s - j\ \mathrm{Sin}\ x/s\} \qquad \ldots \qquad (2.86)$$

Consider the fig.(2.12), in which a long column of metal with a cross-section of 1 square cm. is shown.

The current density flowing across the area, b, is from (2.84).

$$i\ (\text{Amperes per sq. cm.}) = \frac{E}{\rho} = (\frac{1+j}{s})H_o.e^{-(1+j)x/s} \qquad \ldots \qquad (2.87)$$

The total current flowing through the side of the column under the patch a, which is 1 cm. on a side, is

$$I_t = \int_{x=0}^{x=\infty} i.dx \qquad \ldots \qquad (2.88)$$

Substituting (2.87) in equation (2.88)-

$$I_t = (\frac{1+j}{s})\ H_o \int_{x=0}^{x=\infty} e^{-(1+j)\ x/s.}\ dx\ .$$

$$= -H_o \left[e^{-(1+j)x/s} \right]_{x=0}^{x=\infty}$$

$$= H_o \qquad \ldots \qquad (2.89)$$

From equation (2.89) we see that the total integrated current I_t is in phase with the magnetic intensity at the surface and is equal to the surface magnetic intensity.

The power dissipated as heat in the little cube of which the area b comprises one face is $\frac{E^2}{\rho}$, where the absolute value of E is used. Then the power density is from equation (2.85)-

$$P(\text{watts for cubic cm.}) = \frac{2}{s^2} \cdot \frac{H_o^2 \cdot e^{-2x/s} \cdot \rho^2}{\rho}$$

$$P(\text{watts for cubic cm.}) = \frac{2}{s^2} \cdot \rho \cdot H_o^2 \cdot e^{-2x/s} \qquad (2.90)$$

when expression (2.90) is integrated throughout the length of the column, the total power in the column beneath the patch, a, which has unit area is found to be-

P_t(watts for square cm. of surface)-

$$P_t = \int_{x=0}^{x=\infty} P dx = \int_{x=0}^{x=\infty} \frac{2}{s^2} \cdot \rho \cdot H_o^2 \cdot e^{-2x/s}$$

$$= \frac{H_o^2}{s} \cdot \rho \qquad \ldots \qquad (2.91)$$

On substituting (2.89) in (2.91)-

$$P_t = \frac{I_t^2}{s} \cdot \rho \qquad \ldots \qquad \ldots \qquad (2.92)$$

Substituting (2.91) in (2.90)-

$$P \text{ (watts per cubic centimeter)} = \frac{2}{s} P_t \cdot e^{-2x/s} \qquad (2.93)$$

From equation (2.87), the current density is given by-

$$i_x = \left(\frac{1+j}{s}\right) H_o \cdot e^{-x/s} \cdot e^{-j\, x/s}$$

At the surface i.e. when x = 0

$$i_o = \left(\frac{1+j}{s}\right) H_o .$$

Hence,

$$i_x/i_o = e^{-x/s} \cdot e^{-j\, x/s} \qquad \ldots \qquad (2.94)$$

or,

$$\frac{i_x}{i_o} = \left| e^{-x/s} \right| \angle - (x/s.180/\pi)^{\circ} \qquad \ldots \qquad (2.95)$$

Similarly from equation (2.93)-

$$\frac{P_x}{P_o} = e^{-2x/s} \qquad \ldots \qquad \ldots \qquad (2.96)$$

where, P_o = Power density at the surface

The values of i_x/i_o and P_x/P_o are tabulated in table no.(2.2) and are plotted in fig.(2.15) against the values of x/s.

TABLE No.2.2

x/s	$i_x/i_o = e^{-x/s}$	Phase Angle of i_x/i_o $= -(x/s.180/\pi)^{\circ}$	$P_x/P_o = e^{-2\,x/s}$
0	1	0°	1
0.25	.778	-14.3°	.606
0.5	.606	-28.6°	.368
1.0	.368	-57.2°	0.136
1.5	.223	-85.8°	0.05
2.0	0.136	-114.4°	.018
2.5	0.082	-143.0°	0.0067
3.0	0.05	171.6°	0.0025

Fig.(2.14), shows the vector current density relations in the metal, and fig.(2.15) shows the Relative current-density and power density in the metal for different values of x/s.

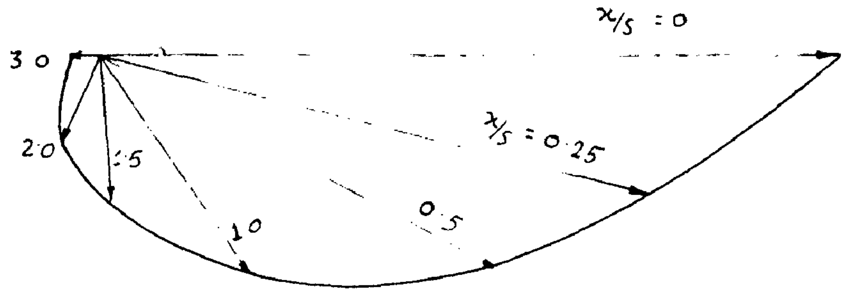


Fig.2.14 - Vector Current Density Variations in the Metal

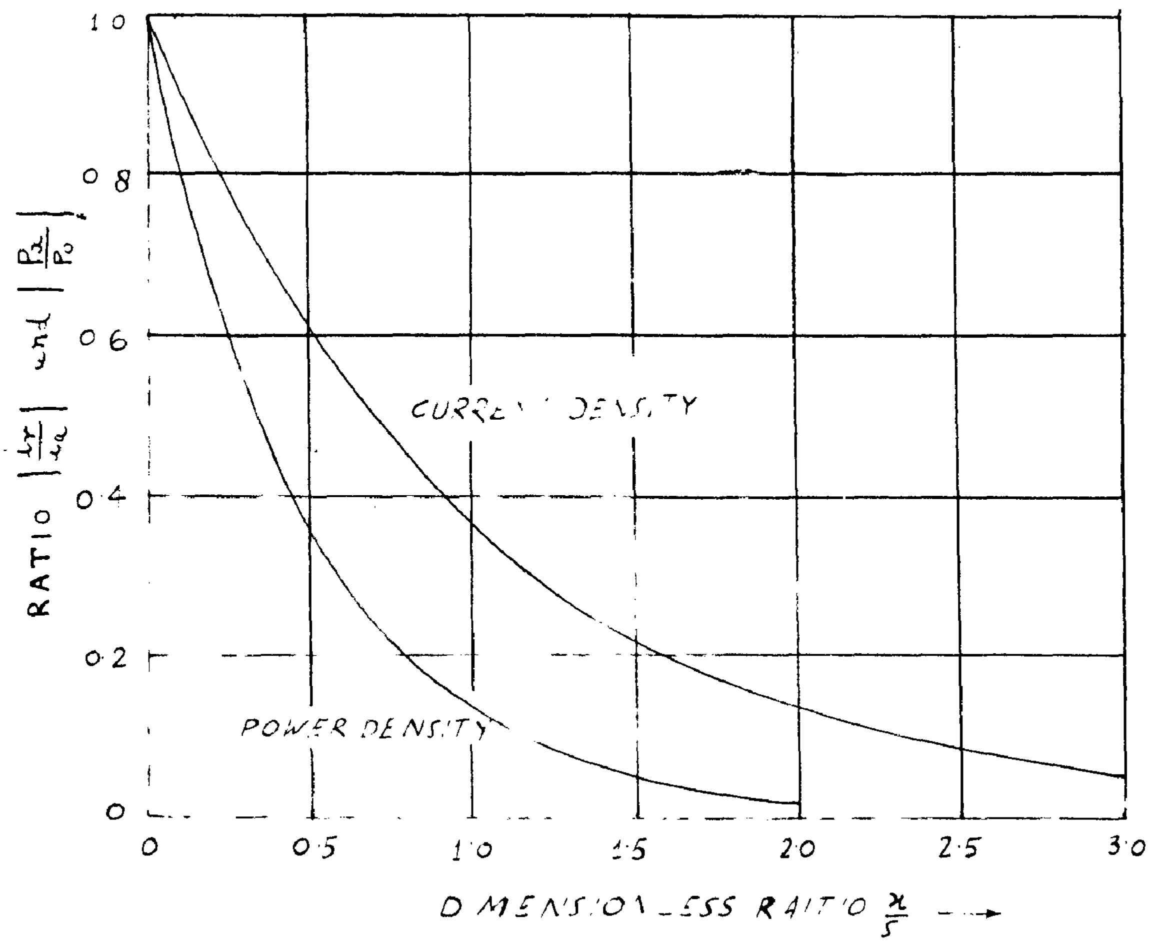


Fig.2.15 - Relative Current and Power Densities in the Metal.

The current density for a number of distances below the surface is shown in fig.(2.14). It is noted from this figure that the current density lags in phase as we go deeper in to the material, and the current density decreases rapidly.

Fig.(2.15), which shows the relative current density as a function of the distance from the surface, reveals that at a depth equal to the skin thickness, the current density is 36.8% of the density at the surface. On the same diagram the relative power density is 13.6% of the density at the surface. Therefore, it is interesting to note that over 96% of the total power is lost in a layer equal to the skin thickness.

2.7. POWER AND FLUX INDUCED IN A RECTANGULAR SLAB OF LIMITED THICKNESS:

The slab is shown in fig.(2.13), referring to it let-

t = thickness of the slab.

w = width of the slab.

and l_w = Axial length of the slab.

The distance x is measured from the centre line as shown in the figure, and it measures the distance from the centre to the point of examination.

Mathematically the equation to be solved is given by (2.70)-

$$\frac{\partial_2 H}{\partial x^2} = \frac{\mu}{\rho} \cdot \frac{\partial H}{\partial t}$$

Assuming a solution $H(x, t) = \text{Re}\,[U(x).e^{j\omega t}]$ as in the section (2.6), the solution of the differential equation is given by from equation (2.76),

$$U = A \,.\, e^{-\alpha x} + B.\, e^{\alpha x} \qquad \ldots \quad \text{from (2.76)}$$

where A & B are arbitrary constants.

$$\alpha = \frac{1+j}{s} \qquad \ldots \quad \text{from (2.75)}$$

The constants A and B determinate from the boundary conditions given below.

In a plate of thickness 't', the distribution must be symmetrical about the center line of the plate. Thus any quantity at $x = + t/2$ and $x = - t/2$ must be identical. This implies that $A = B$ in the equation (2.76).

Therefore the equation (2.76) reduces to-

$$U = A\left[e^{-\alpha x} + e^{\alpha x}\right]$$

$$= 2A \cosh \alpha x \qquad \ldots \qquad (2.97)$$

In order to determinate the constant A in (2.97) we take the surface boundary condition in to account.

i.e. At $x = \pm t/2$, $H = H_o = \text{Re}\,\{U_o e^{j\ \ t}\}$

or $x = \pm t/2$, $U = U_o$.

Hence from equation (2.97),

$$U_o = 2A \cosh \frac{\alpha t}{2} .$$

$$\therefore \quad A = U_o/2.\ \text{Cosh}.\alpha t/2 \qquad \ldots \qquad (2.98)$$

Substituting this value of A in (2.97)-

$$U = U_o.\ \frac{\text{Cosh}\,\alpha x}{\text{Cosh}\ \alpha t/2} \qquad \ldots \qquad (2.99)$$

$$\therefore H = \text{Re}\ (U.e^{j\omega t}) = \text{Re}\ (U_o.\text{Cosh}\,\alpha x.e^{j\omega t}/\text{Cosh}\,\alpha t/2\ )$$

$$\therefore H = H_o\ .\ \text{Cosh}\,\alpha x/\ \text{Cosh}\,\alpha t/2 \qquad \ldots \qquad (2.100)$$

Putting the value of   from (2.75)-

$$H = H_o \frac{\text{Cosh}\ (\frac{1+j}{s}).x}{\text{Cosh}\ (\frac{1+j}{2s}).t} \qquad \ldots \qquad (2.101)$$

It can be shown mathematically that-

$$\text{Cosh}\ (1+j)\ \theta = \text{Cosh}\ \theta\ \text{Cos}\ \theta - j\ \text{Sinh}\ \theta\ \text{Sin}\ \theta \qquad \ldots \qquad (2.102)$$

Rationalising (2.102) we get,

$$\text{Cosh}\ (1+j)\theta = \left[\tfrac{1}{2}\ (\text{Cosh}\ 2\theta + \text{Cos}\ 2\theta)\right]^{\frac{1}{2}} \angle \phi \qquad \ldots \qquad (2.103)$$

where, $\tan \phi = \tanh \theta \ \ \tan \theta$

Utilising the relation given in (2.103), in (2.101) we get,

$$H_x = H_o \left[\frac{\mathrm{Cosh}\,\frac{2x}{s} + \mathrm{Cos}\,\frac{2x}{s}}{\mathrm{Cosh}\,\frac{t}{s} + \mathrm{Cos}\,\frac{t}{s}} \right]^{\frac{1}{2}} \angle \phi_1 - \phi_2 \qquad (2.104)$$

where,

$$\tan \phi_1 = \tanh \frac{x}{s} \tan \frac{x}{s}$$

and

$$\tan \phi_2 = \tanh \frac{t}{2s} \tan \frac{t}{s}$$

Hence equation (2.104) gives the field distribution through the slab.

2.7.1. Total flux in the slab:

The total flux within the slab is given by-

$$\phi_w = 2 \int_0^{t/2} \mu \cdot H_x.\ w.\ dx \qquad \ldots \qquad (2.105)$$

Putting the value of H_x from (2.101) we get-

$$\phi_w = 2 \int_0^{t/2} \mu \cdot w.\ H_o \frac{\mathrm{Cosh}(1+j)\ x/s}{\mathrm{Cosh}(1+j)\ t/2s} dx$$

$$= \frac{2\mu .\ w.\ H_o}{\mathrm{Cosh}(1+j)t/2s} \int_0^{t/2} \mathrm{Cosh}\ (1+j)\ x/s.dx.$$

$$= \left[\frac{2\mu.\ w.\ H_o}{\mathrm{Cosh}(1+j)t/2s} \right] . \left(\frac{s}{1+j}\right) \Big[\mathrm{Sinh.}\ (1+j)x/s \Big]_0^{t/2}$$

$$= 2\mu.w.\ H_o \left(\frac{s}{1+j}\right) . \frac{\mathrm{Sinh}(1+j)t/2s}{\mathrm{Cosh}\ (1+j)t/2s} \qquad \ldots (2.106)$$

or,

$$\phi_w = \frac{2\mu.w.\ H_o.s}{1+j} . \frac{\mathrm{Sinh}\frac{t}{2s} . \mathrm{Cos}\,\frac{t}{2s} + j\mathrm{Cosh}\frac{t}{2s}\ \mathrm{Sin}\frac{t}{2s}}{\mathrm{Cosh}\,\frac{t}{2s}\ \mathrm{Cos}\frac{t}{2s} - j\mathrm{Sinh}\,\frac{t}{2s}\ \mathrm{Sin}\,\frac{t}{2s}} \qquad \ldots (2.107)$$

After rationalising equation (2.107) and making some mathematical manipulations, shown in Appendix 2, we get,

$$\phi_w = \mu H_o \cdot A_w \cdot \frac{s}{t} \left[\frac{(\mathrm{Sinh}\,\frac{t}{s} + \mathrm{Sin}\frac{t}{s}) - j(\mathrm{Sinh}\,\frac{t}{s} - \mathrm{Sin}\frac{t}{s})}{(\mathrm{Cosh}\,\frac{t}{s} + \mathrm{Cos}\,\frac{t}{s})} \right] \qquad \ldots \quad (2.108)$$

where,

$$A_w = w.t = \text{Area of the slab} \qquad \ldots \quad (2.109)$$

Equation (2.108) can be written as-

$$\phi_w = \mu H_o \cdot A_w \cdot (P - jQ) = \phi_p - j\,\phi_q \quad (2.110)$$

where,

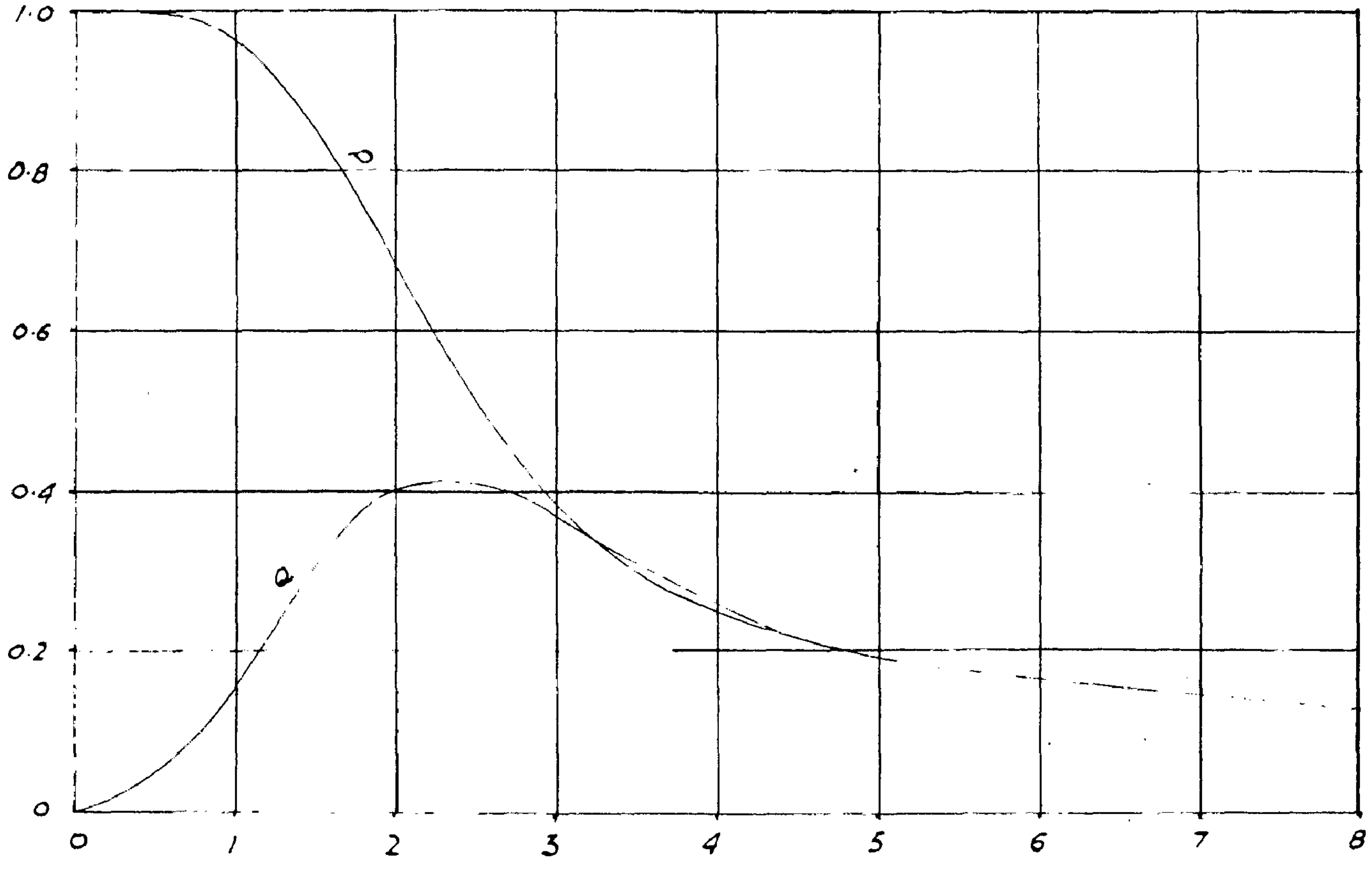
$$P = \frac{s}{t} \left\{ \frac{\mathrm{Sinh}\,\frac{t}{s} + \mathrm{Sin}\,\frac{t}{s}}{\mathrm{Cosh}\,\frac{t}{s} + \mathrm{Cos}\,\frac{t}{s}} \right\} \qquad \ldots \quad (2.111)$$

and,

$$Q = \frac{s}{t} \left[\frac{\mathrm{Sinh}\,\frac{t}{s} - \mathrm{Sin}\,\frac{t}{s}}{\mathrm{Cosh}\,\frac{t}{s} + \mathrm{Cos}\,\frac{t}{s}} \right] \qquad \ldots. \quad (2.112)$$

P and Q functions are extremely valuable and are tabulated in table (2.3).

The P & Q functions are plotted for different values of $\frac{t}{s}$ in fig. (2.16). It is seen from the graph that for $\frac{t}{s} > 5$, the values of P and Q are equal. Also when $\frac{t}{s} > 8$, P = Q = s/t.



[illegible] - P and Q Functions - Solid Rectangular Slab.

TABLE (2.3)

P & Q FUNCTIONS FOR DIFFERENT VALUES OF t/s

$\frac{t}{s}$	P	Q
0	1	0
1	0.967	0.161
2	0.68	0.406
3	0.373	0.364
4	0.248	0.263
5	0.200	0.202
6	0.165	0.166
7	0.143	0.143
8	0.125	0.125

CHAPTER - THREE

CHAPTER 3

THERMAL CONDITIONS

3.1. INTRODUCTION

This chapter deals with the temperature distribution in the work-piece. The measure cause of the conduction of heat encountered in Induction heating is the skin effect. Due to skin-effect the heat is generated inside the surface of the metal, and this amount of heat falls off exponentially toward the center of thepiece, just as the current drops towards the center. The power generated (or the heating effect) falls off from the surface about three to four times as rapidly as the current effect. Therefore while considering the Induction heating applications, the relation between the heat depth and current depth is of considerable importance.

3.2. INFLUENCE OF TEMPERATURE ON THE PHYSICAL PROPERTIES OF METAL:

The major factors in heat and temperature distribution are current depth, time and heat conductivity. Current depth is a function of frequency, resistivity and permeability. The characteristic physical properties of the metal (emmissivity, resistivity, thermal conductivity and permeability) play an important part in induction heating. During the heating process most of these properties change in value, so it is very important to know their relationship to temperature. In general, the resistivity increases with temperature in nearly all metals and hence the current depth increases with temperature. Usually this increase in resistivity is linear with temperature rise, so the average or integrated resistivity between room temperature and the final required temperature can be used to drive a reasonably approximate value for the average current depth.

Fig.(3.1) and (3.2) show the integrated resistivity of pure aluminium and common steel alloys respectively.

The values of thermal conductivity usually rise with temperature, the exception being steel whose thermal conductivity falls with temperature.

Permeability varies considerably with temperature and thereby effect the current depth and power input to the work-piece since they are the functions of permeability. In general, the permeability becomes unity at temperatures between 1275 and 1600°F, depending upon the intensity of the magnetising field and the alloy content of the steel,

Hopkinson(16) showed that for very weak fields, the permeability actually rises, whereas in strong fields it falls off rapidly at 1400 to 1475° F to about one-hundredth of its value. The following table gives the magnetic change points of some common metals(17).

TABLE 3-1

Magnetic Change points

Metal ...	... Temperature, °F.
Iron ...	... 1420.
Cobalt. ...	... 2105
Nickel ...	... 680
Carbon steel (medium)	... 1330

This critical temperature is called the curie point.

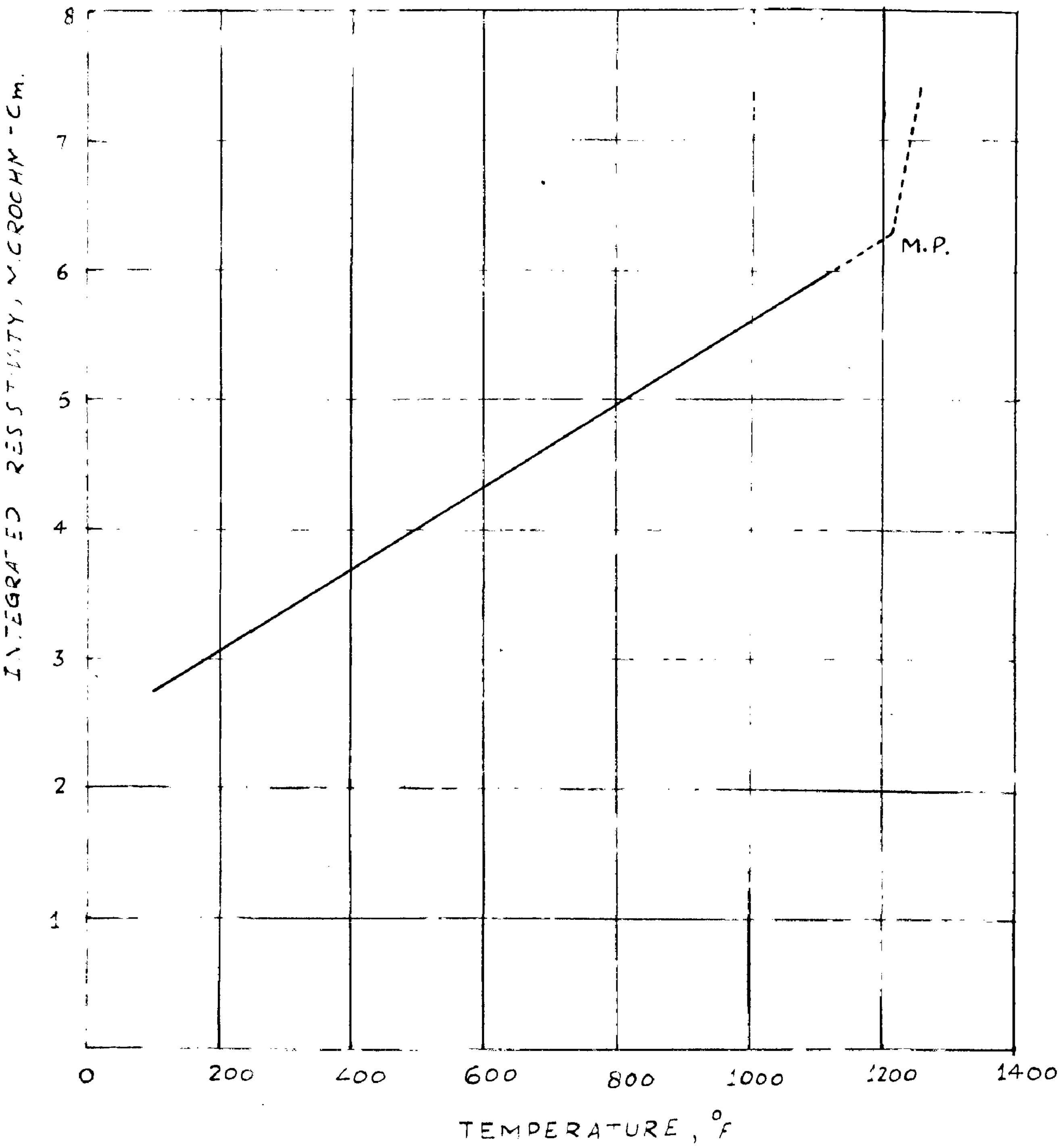
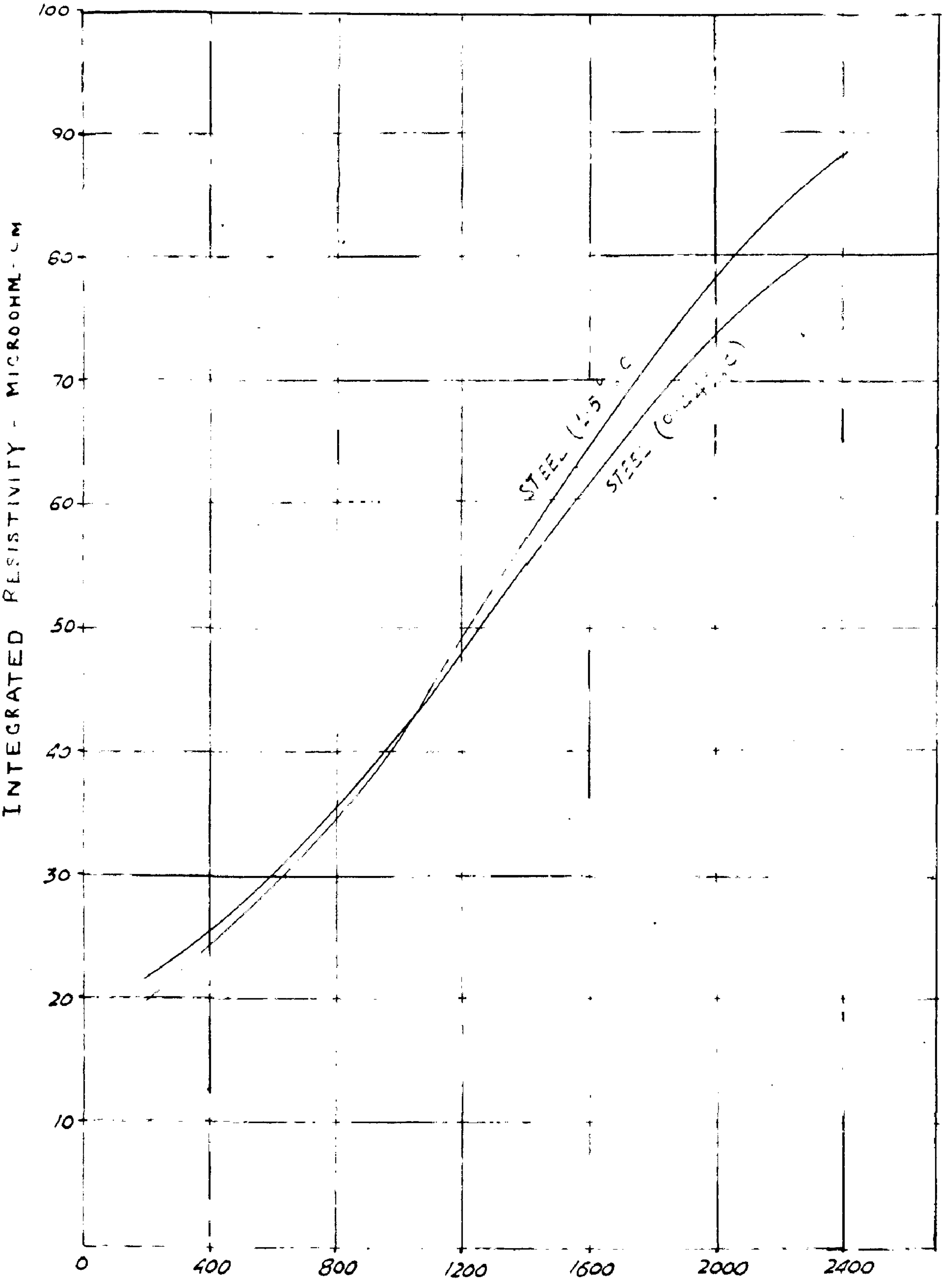


Fig. 3.1 - Integrated Resistivity vs. Temperature-Pure Aluminum.

(Integrated from room temperature 70 °F)


INTEGRATED RESISTIVITY - MICROOHM-CM
100
90
80
70
60
50
40
30
20
10
0
400
800
1200
1600
2000
2400
STEEL (1.5% C)
STEEL (0.24% C)

[illegible]

3.3. SPECIFIC HEAT EFFECTS:

In order to use the average specific heat over a certain range of temperature, it is converted to Pounds per kilowatthour. Fig.(3.3) shows the Pounds per kilowatthour of Aluminium, steel and copper.

The definition of Pounds per kilowatt-hour (lb.per Kwhr) is as follows:

$$\text{lb/Kwhr} = \frac{3413}{k_s \cdot \Delta\theta} \quad \ldots \quad (3.1)$$

where,

$\Delta\theta$ = Temperature rise, °F.

k_s = Average specific heat over $\Delta\theta$.

The thermal or useful power required to raise the temperature by $\Delta\theta$ is obtained from-

$$P_t = \frac{\text{lb/hr.}}{\text{lb/Kwhr}} \quad \ldots \quad (3.2)$$

3.4. HEAT LOST BY RATIATION AND CONVECTION:

The radiation loss is given by-

$$P_R = 37\ e\ (T_s^4 - T_A^4).\ 10^{-12}\ \text{watts/sq.in.} \quad (3.3)$$

where,

e = emissivity coefficient of the surface.

T_s = surface temperature, °K.

T_A = Ambient temperature, °K.

Fig.(3.4) shows a set of curves for typical metals. These figures are based on a normal surface of work-piece.

Convection loss encountered in Induction heating is very small so it can very well be neglected.

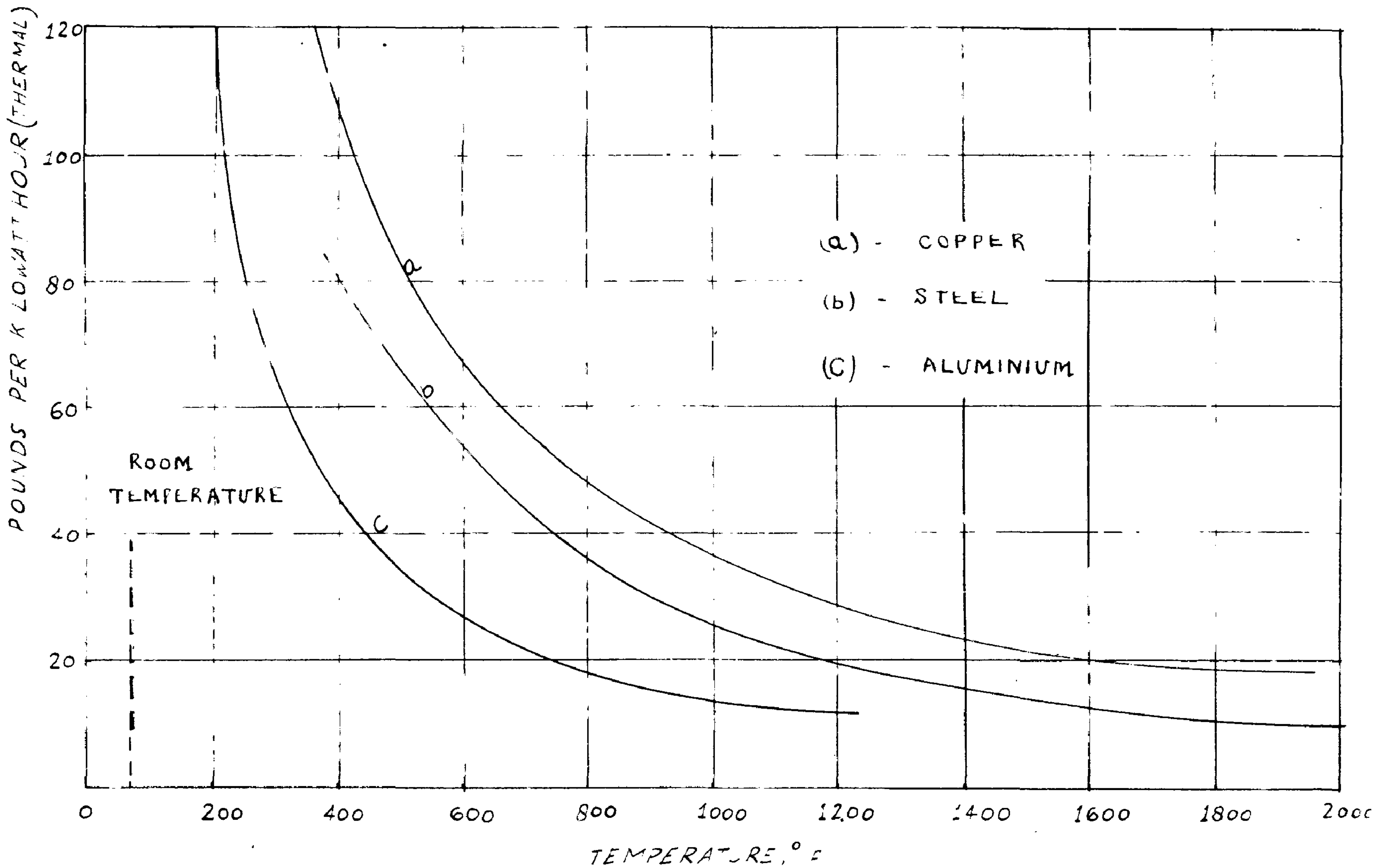


FIG. 3.3 - POUNDS PER KILOWATTHOUR Vs TEMPERATURE

3.5. THE DIFFERENTIAL EQUATION OF HEAT-FLOW IN ONE DIMENSION:

The differential equation of heat flow is established, by restricting the problem to flow in one direction or dimension. Therefore, the limiting condition is that heat flows only in the x-direction and under this condition a small rectangular box within the body of the material is examined. This box shown in Fig.(3.5), has a width of 1 cm, a height of 1 cm., and a length dx. Heat may be generated in each cubic centimeter by electrical means.

Let H = Rate of heat generation in gram-calories per second per cubic centimeter.

H is a function of both time and distance.

Heat generated within the small box, in a short time interval dt is given by-

$$Q_o = H.\ dx.\ dt. \qquad \ldots \qquad \ldots \qquad (3.4).$$

The temperature at the point 0 is U°c. Then the increase in heat stored in the box during the time, dt, is -

$$Q_1 = \gamma \cdot k_s \frac{\partial U}{\partial t} dt \ .\ dx \qquad ,,, \qquad ,,, \qquad (3.5)$$

where,

γ = Density of the material

k_s = Specific heat of the material.

The temperature gradient at point 0 is $(\frac{\partial U}{\partial x})_o$.

The gradient at point B on the right hand face of the box is

$$(\frac{\partial U}{\partial x})_B = (\frac{\partial U}{\partial x})_o + \left[\frac{\partial}{\partial x}(\frac{\partial U}{\partial x}) \right]_o \frac{dx}{2}$$

and the gradient at point A on the left hand face of the box is -

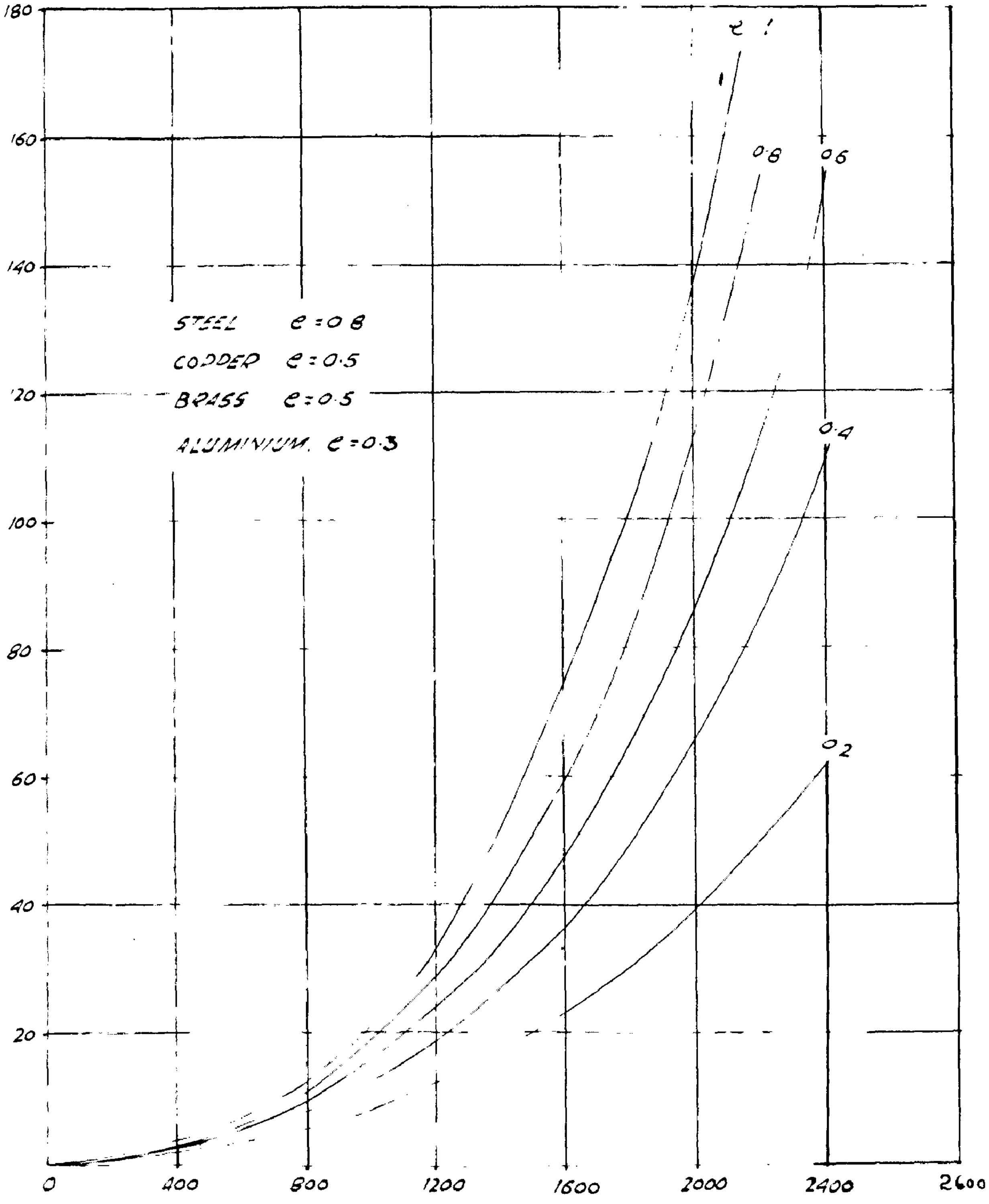


FIG. 3·4 - RADIATION LOSSES.

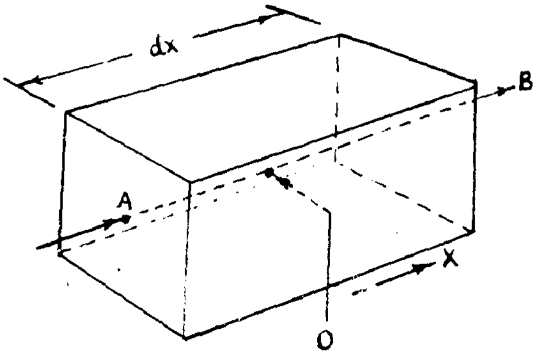


FIG 3.5. ELEMENTARY Box

$$\left(\frac{\partial U}{\partial x}\right)_A = \left(\frac{\partial U}{\partial x}\right)_0 - \left[\frac{\partial}{dx}\left(\frac{\partial U}{\partial x}\right)\right]_0 \cdot \frac{dx}{2}$$

The heat flowing out through the face at A in time dt is-

$$Q_2 = k_c \left(\frac{\partial U}{\partial x}\right)_A . \ dt. \qquad \ldots \qquad \ldots \qquad (3.6)$$

where,

k_c = Thermal conductivity of the material

The heat flowing out through the face at B is-

$$Q_3 = - k_c \left(\frac{\partial U}{\partial x}\right)_B . dt.$$

From the conservation of energy,

$$Q_o = Q_1 + Q_2 + Q_3.$$

Thus,

$$H. \ dx. \ dt = \nu . \ K_s \ \frac{\partial U}{\partial t} \ . \ dx. \ dt - k_c \left[\frac{\partial}{\partial x}\left(\frac{\partial U}{\partial x}\right)\right]_o dx.dt. \qquad \ldots \qquad \ldots \qquad (3.7)$$

or,

$$H = \nu . \ k_s . \ \frac{\partial U}{\partial t} - k_c \ . \ \frac{\partial^2 U}{\partial x^2} \qquad \ldots \qquad \ldots \qquad (3.8)$$

This is the differential equation of heat flow in one dimension.

3.6. SOLUTION OF HEAT CONDUCTION EQUATION IN THE CASE OF CYLINDRICAL WORK-PIECE.

The following assumptions are made:

(i) There is no axial variation of the temperature.

(ii) There is no angular variation of the temperature.

(iii) There is no convection in the air-gap i.e. the air-gap is fairly small.

(iv) Thermal conductivity of both air and solid are not temperature dependent.

(v) At time t = o (initially), every thing is at room temperature.

(vi) Initial temperature is uniform through out.

Let,

θ_a = Ambient temperature.

r = Radial distance from the center, cm.

a = Radius of the cylinder, cm.

Hence in cylindrical co-ordinates, the heat conduction equation to be solved is-

$$\frac{\partial^2\theta}{\partial r^2} + \frac{1}{r} \cdot \frac{\partial \theta}{\partial r} - \frac{1}{k} \cdot \frac{\partial \theta}{\partial t} = 0 \quad \ldots \qquad (3.9)$$

where,

k = Thermal diffusivity ($k = \frac{k_c}{\nu \cdot k_s}$)

now let us call a function-

$$U = \theta - \theta_a$$

Then since θ_a is constant, we can write-

$$\frac{\partial U}{\partial r} = \frac{\partial \theta}{\partial r}, \quad \frac{\partial^2 U}{\partial r^2} = \frac{\partial^2 \theta}{\partial r^2}, \text{ and } \frac{\partial U}{\partial t} = \frac{\partial \theta}{\partial t}$$

Hence with zero initial temperature, the equation (3.9) can be written as:

$$\frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r} - \frac{1}{k}\frac{\partial U}{\partial t} = 0 \qquad (0 \leq r < a,\ t > 0) \qquad \ldots \qquad (3.10)$$

This equation is to be solved with the boundary condition that at the surface-

i.e. At $r = a$, $k_c \frac{\partial U}{\partial r} = P_o$ ... ... (3.11).

where,

P_o = Surface power density, cal/sec./sq.cm.

The subsidiary equation is-

$$\frac{d^2\bar{U}}{dr^2} + \frac{1}{r}\frac{d\bar{U}}{dr} - \frac{p}{k}\,\bar{U} = 0$$

or,

$$\frac{d^2\bar{U}}{dr^2} + \frac{1}{r}\frac{d\bar{U}}{dr} - q^2\,\bar{U} = 0 \qquad \ldots \qquad \ldots \qquad (3.12)$$

where, $q^2 = \frac{p}{k}$, (p being the Laplace operator) (3.12a)

The equation (3.12) is to be solved with the boundary condition that at the surface-

i.e. At $r = a$, $k_c \frac{d\bar{U}}{dr} = \frac{P_o}{p}$ ... ... (3.13)

Hence the solution is given by-

$$\bar{U} = A\, I_o\,(qr) + B\, K_o\,(qr) \qquad \ldots \qquad \ldots \qquad (3.14)$$

where,

A and B are the arbitrary constants-

$I_o(qr)$ and $K_o(qr)$ are modified Bessel's functions of zeroth order and first and second kind respectively. Since $K_o(qr)$ tends to infinity at $r = 0$, it is excluded from the solution and then we have-

$$\overline{U} = A\, I_o\,(qr) \qquad \ldots \qquad \ldots \qquad (3.15)$$

$$\therefore \frac{d\overline{U}}{dr} = A\, \frac{d}{dr}\left\{\left[I_o\,(q.r)\right]\right\} = A.q.I_1(q.r.) \qquad \ldots \qquad (3.16)$$

Hence from (3.13) and (3.16),

$$k_c,\ A.\ q.I_1\,(q.a) = \frac{P_o}{p}$$

or,

$$A = \frac{P_o}{p.k_c.q.\ I_1\,(q.a)}$$

putting the value of q from (3.12a)-

$$A = \frac{P_o k^{\frac{1}{2}}}{p^{3/2}.k_c.I_1(q.a)} \qquad \ldots \qquad \ldots \qquad (3.17)$$

Putting the value of constant A in (3.15), the solution is-

$$\overline{U} = \frac{P_o\, k^{\frac{1}{2}}.\ I_o\,(qr)}{p^{3/2}.k_c.I_1(qa)} \qquad \ldots \qquad \ldots \qquad (3.18)$$

Solving the equation (3.18) with the help of inversion thoerem(22), the temperature U_r at a distance r from the center after a time t is given by-

$$U_r = \frac{P_o a}{k_c}\left\{\left[\frac{2kt}{a^2} + \frac{r^2}{2a^2} - \frac{1}{4} - 2\sum_{n=1}^{\infty} e^{-\frac{k\beta_n^2 t}{a^2}} . \frac{J_o(r.\ \beta_n/a)}{\beta_n^2 J_o(\,\beta_n)}\right]\right\}$$

$$\ldots \qquad \ldots \qquad (3.19)$$

where,

U_r = Temperature in °c, at any distance r from the center.

P_o = Surface Power density, cal./sec./sq.cm.

a = Radius of cylinder, cm.

k_c = Thermal conductivity, cal./sec./cm./°c.

k = Thermal diffusivity = $\frac{k_c}{\nu \cdot k_s}$,

t = time in seconds.

k_s = Specific heat, cal/gm/°C

ν = Density gms/cu.cm.

r = Radius (variable), cm.

β_n = Positive roots of $J_1(\ ) = 0$

$J_o(x)$= Bessel's function of first kind & zero order.

The expression given in (3.19) can be simplified by using the dimensionless function T, where-

$$T = \frac{k.t}{a^2} \qquad \ldots \qquad (3.20)$$

Substituting the value of T from (3.20) in (3.19)-

$$U_r = \frac{P_o.a}{k_c} \left\{ 2\,T + \frac{r^2}{2a^2} - \frac{1}{4} - 2 \sum_{n=1}^{\infty} e^{-T.\beta_n^2} . \frac{J_o(r.\beta_n/a)}{\beta_n^2\, J_o(\beta_n)} \right\} \qquad \ldots \qquad (3.21)$$

The summation series in eqn. (3.21) becomes zero by making T greater than 0.25 and eqn. (3.21) reduces to-

$$U_r = \frac{P_o.a}{k_c} \left(2T + \frac{r^2}{2a^2} - \frac{1}{4} \right) \qquad \ldots \qquad (3.22)$$

The values of the function $U_r/ (P_o.a/k_c)$ are calculated for different values of T, corresponding to various values of r/a, and are tabulated in the table no.(3.2). The calculations for this are made on digital computer and the sample programme is shown in Appendix 3 (programme no.2).

TABLE (3.2)

Function $F = U_r /(P_o . a/k_c)$ for different values of T and r/a.

r/a	Function $F = U_r / (P_o . a/k_c)$						
	T=0.0125	T=0.025	T=0.05	T =0.10	T =0.25	T = 0.5	T= 0.75
0.00	0.00003	0.00003	0.00116	0.02689	0.25000	0.75000	1.25000
0.25	0.00003	0.00005	0.00411	0.04189	0.28125	0.78125	1.28125
0.50	0.00005	0.00288	0.02356	0.09659	0.37500	0.87500	1.37500
0.75	0.00915	0.03612	0.09758	0.21463	0.53125	1.03125	1.53125
0.80	0.01796	0.05377	0.12402	0.24804	0.57000	1.07000	1.57000
0.90	0.05564	0.10799	0.19186	0.32576	0.65500	1.15500	1.65500
1.00	0.13285	0.19218	0.28106	0.41833	0.75000	1.25000	1.75000

In fig.(3.6) the eqn. (8.21) is plotted for $U_r/(P_o.a/k_c)$ as a function of r/a for different values of T. The interesting fact investigated from fig.(3.6) is that up to T = 0.25, the heat flow is in transient state, with the surface rising faster than the center. Above this value a "steady-state" condition is achieved and all points on the radius rise at the same value, assuming that all other values in eqn. (3.19) remain constant.

After the steady-state condition has been reached, eqn. (3.22) can be used to determine the temperature difference between surface U_s and center U_c, by using the values r/a =1 and r/a = 0. Therefore,

$$U_s = \frac{P_o.a}{k_c}\,(2T + \tfrac{1}{4}) \quad \ldots \quad (3.23)$$

and $$U_c = \frac{P_o.a}{k_c}\,(2T - \tfrac{1}{4}) \quad \ldots \quad (3.24)$$

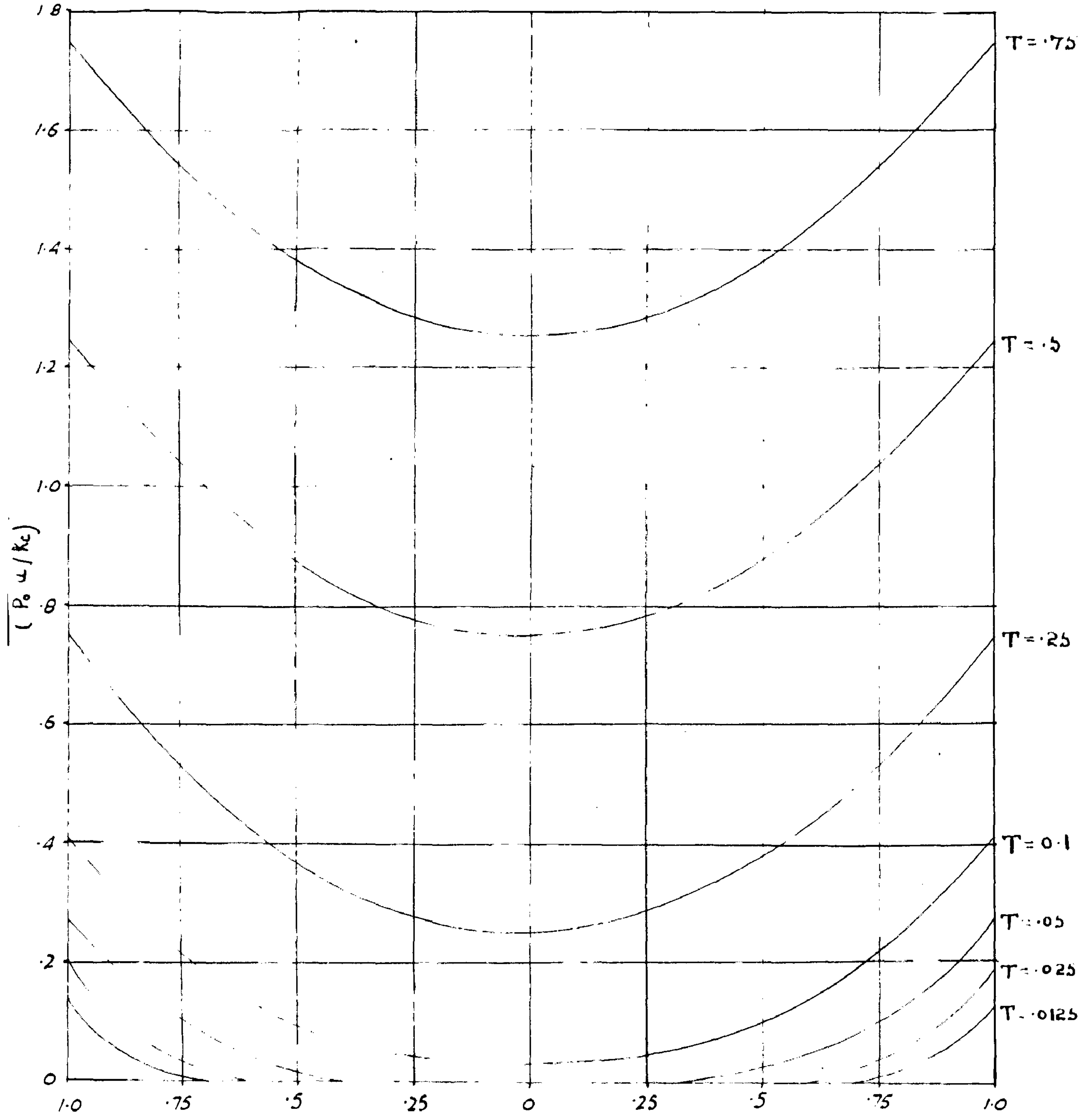
$$\therefore \quad U_s - U_c = \frac{P_o.a}{k_c} \quad \ldots \quad (3.25)$$

$$U_r - U_c = \frac{P_o.a}{2k_c} \cdot \frac{r^2}{a^2} \quad \ldots \quad (3.26)$$

All these equations are based on the assumption that radiation losses are negligible and that the heat is generated at the surface, so corrections are to be made for both factors.

3.7. CORRECTION FOR FINITE CURRENT DEPTH:

The correction for the finite current depth, and therefore for power generation inside the surface, is contained in the following equation, the proof of which is given in Appendix 4.



1.8
1.6
1.4
1.2
1.0
·8
·6
·4
·2
0
(P₀ L / Kc)
1·0
·75
·5
·25
0
·25
·5
·75
1·0
T = ·75
T = ·5
T = ·25
T = 0·1
T = ·05
T = ·025
T = ·0125

[illegible]

[illegible]

$$U_r - U_c = \frac{P_o . a}{2k_c} \left[\frac{r^2}{a^2} - \frac{1}{k_2} \left(\frac{X(k_2\ r/a) -1}{Z(k_2)} \right) \right] \quad (3.27)$$

where,

$$X(k_2\ r/a) = \mathrm{ber}^2\ (k_2 . r/a) + \mathrm{bei}^2\ (k_2\ r/a)$$

$$Z\ (k_2) = \mathrm{ber}\ \ k_2\ \mathrm{ber}'\ k_2 + \mathrm{bei}\ k_2\ \mathrm{bei}'\ k_2 .$$

$$k_2 = k_1 . a$$

Here k_1 is a function of current depth such that

$$k_1 = \frac{\sqrt{2}}{\delta} \quad \ldots \quad (3.28)$$

Equation (3.27) is based on the assumption that sufficient time t_1 has elapsed so that all points on the radius are rising at a uniform rate, i.e. $T \geq 0.25$ in eqn. (3.20).

The time t for this value of T is given by-

$$T = \frac{kt_1}{a^2} = \frac{k_c . t_1}{\nu . k_s \cdot a^2} = 0.25 \quad \ldots \quad (3.29)$$

or $$t_1 = \frac{\nu \cdot k_s . a^2}{4k_c} \quad \ldots \quad (3.30)$$

The surface-to-center temperature differential is given by putting r/a = 1, in eqn. (3.27) -

$$U_s - U_c = \frac{P_o . a}{2k_c} \left[1 - \frac{1}{k_2} . \frac{X(k_2)-1}{Z(k_2)} \right] \quad (3.31)$$

3.8. CORRECTION FOR RADIATION LOSSES:

Radiation losses are the difference between total and net power inputs.

let-

P_a = Total power input, cal $\mathrm{sec.}^{-1}$ $\mathrm{sq.cm}^{-1}$

P_n = Net or effective power input (after radiation).

Then eqn. (3.31) is corrected for radiation losses by the following equation–

$$U_s - U_c = \frac{P_n . a}{2k_c} \left[1 - \left[\left(\frac{1}{P_n/P_a} \right) \left(\frac{1}{k_2} \right) \left(\frac{X.(k_2) - 1}{Z\ (k_2} \right) \right] \right] \quad (3.32)$$

or

$$U_s - U_c = \frac{P_n . a}{2k_c} \,..\, F\ (P_n/P_a,\ k_2) \quad \ldots \quad (3.33)$$

where,

F.(P_n/P_a, k_2) is the correction factor given by

$$F\ (P_n/P_a,\ k_2) = 1 - \left[\left(\frac{1}{P_n\ /P_a} \right) \left(\frac{1}{k_2} \right) \left[\frac{X(k_2)\ -1}{Z(k_2)} \right] \right] \quad \ldots \quad (3.34)$$

when the radiation losses are negligibly small, $P_n/P_a \rightarrow 1$, and $P_n \rightarrow P_o$. Eqn. (3.32) then gives the same results as eqn. (3.31).

Converting eqns. (3.29) and (3.33) in to practical units or fps,

$$U_s - U_c = \frac{0.084\ P_n . a}{k_c} \ .\ F\ (P_n/P_a,\ k_2) \quad \ldots \quad (3.35)$$

where,

$U_s - U_c$ = Surface-to-center differential , °F.

P_n = Net power input, watts per sq.inch.

a = Radius, inch.

k_c = Thermal conductivity, cal sec.$^{-1}$, cm^{-1}, °c^{-1}

γ = Density, lb. per cu.inch.

The correction factor F (P_n/P_a, k_2) is plotted on Fig.(3.7) for different values of the ratio of radius to current depth a/s.

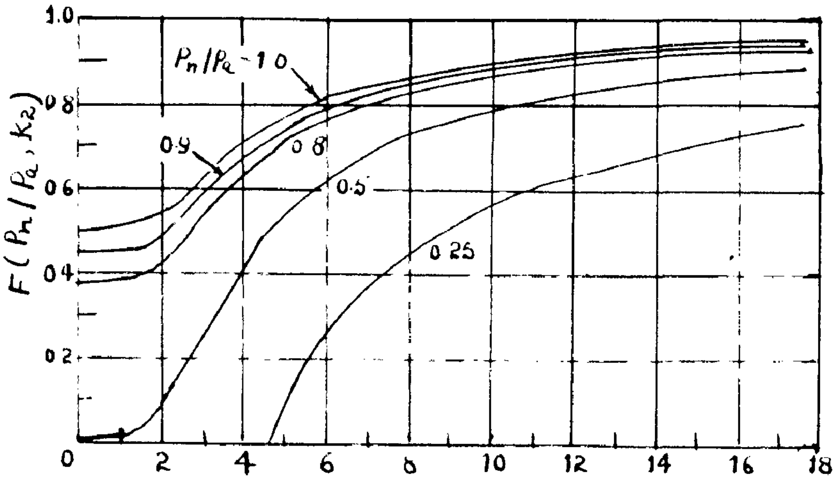


Fig. 5. [illegible] radiation correction factors.

The temperature distribution throughout the radius is determined by taking the generalized version of eqn.(3.32)-

$$U_r - U_c = \frac{P_n . a}{2k_c} \left[\frac{r^2}{a^2} - \left(\frac{1}{P_n/P_a}\right) \left(\frac{1}{k_2}\right) \left\{ \frac{X(k_2\ r/a)-1}{Z(k_2)} \right\} \right] \quad \ldots \quad (3.36)$$

3.8. ILLUSTRATIVE EXAMPLE:

The following data has been choosen for the calculation of heating time and surface-to-center temperature differential.

Metal	Aluminium
Temperature rise, ΔU	875°F(70 to 945°F)
Production rate	4500 Ib per hour
Radius, a	3.5 inch
Length, lw	40 inch
Thermal conductivity over ΔU, k_c	0.40 cal $cm^{-1} sec^{-1} °c^{-}$
Density ν	0.096 lb per cu.in.
Specific heat over ΔU, k_s	0.250
Permeability, μ	1
Frequency, f	60 cps
Resistivity at θ, ρ (from fig. 3.1)	5.45 micro-ohm-cm.
Assumed Room temperature	70°F.

First, the time of heating t is determined from the mass of the billet M and the given production rate-

$$M = \pi a^2\ l_w . \nu$$

$$= (3.5)^2 . (40)\ (0.096) = 148 \text{ lb.}$$

Therefore Heating time, $t = \frac{M \times 3600}{lb/hr} = \frac{148 \times 3600}{4500}$

$$= 118 \text{ sec.}$$

The factor T is given from eqn. (3.20)

$$T = \frac{k_c . t}{\nu \cdot k_s \, a^2}$$

$$= \frac{(0.40)\ (118)}{(0.096 \times 453.5/2.54^3)(0.250)(3.5 \times 2.54)^2}$$

$$= \frac{(0.40)\ (118)\ (2.54)}{(0.096)(453.5)\ (0.250)\ (3.5)^2}$$

$$= 0.90$$

This value of T is greater than 0.25; therefore the steady-state heating time has been reached and equation (3.35) can be used.

The time at which the conditions change from transient to steady-state can be evaluated by putting T = 0.25

$$t = \frac{T . \nu \cdot k_s . a^2}{k_c}$$

$$= \frac{(0.25)\ (0.096)\ (0.250)(3.5)^2(453.5)}{(0.40)\ (2.54)}$$

$$= 32.8 \text{ sec.}$$

Therefore up to 32.8 sec, the heating is in a transient state with the surface rising faster than the center. Metal pounds per kilowatt hour from fig. (3.3) = 16.2 lb/Kw hr. Hence from eqn. (3.2), the useful thermal power density in to the work is calculated as–

$$P_t = \frac{4500}{16.2} = 278 \text{ Kw.}$$

where P_t is the required thermal power.

The net power density is–

$$P_n = \frac{P_t \times 10^3}{\text{Surface area}} = \frac{P_t \times 10^3}{2\pi . a . l_w}$$

$$= \frac{(278)\ (10)^3}{(2)\ (\pi)\ (3.5)\ (40)} = 316 \text{ watts/sq.in.}$$

Now the current depth s is-

$$s = \frac{1}{2\pi} \left(\frac{\rho \times 10^9}{\mu . f}\right)^{\frac{1}{2}} \text{ cm.} \qquad \text{(from eqn.2.20)}$$

$$\therefore \quad s = \frac{1}{2\pi} \left(\frac{5.45 \times 10^{-6} \times 10^9}{1 \times 60}\right)^{\frac{1}{2}}$$

$$= 1.52 \text{ cm.}$$

$$\therefore \text{ Ratio } \frac{a}{s} = \frac{3.5 \times 2.54}{1.52} = 5.85$$

Radiation at the given temperature is found from fig.(3.4). This is-

$$P_r = 6 \text{ watts /sq.in.}$$

Therefore the total power-density input is P_a-

$$P_a = P_n + P_r = 316 + 6 = 322 \text{ watts/sq.in.}$$

The ratio $\frac{P_n}{P_a} = \frac{316}{322} = 0.980$

Using the curve of $P_n/P_a = 1.0$ from fig. (3.7), the value of $F(P_a/P_n, k_2)$ is 0.82, at $\frac{a}{s} = 5.85$.

Finally eqn. (3.35) is used to determine the actual surface to-center temperature differential.

$$U_s - U_c = \frac{0.084 . P_n . a}{k_c} . F\ (P_n/P_a, k_2)$$

$$= \frac{(0.084)\ (316)\ (3.5)\ (0.82)}{(0.40)}$$

$$= \underline{190°F.}$$

CHAPTER - FOUR

SOLUTION OF HEAT FLOW DIFFERENTIAL EQUATION IN THE CASE OF RECTANGULAR WORK_PIECE

4.1. INTRODUCTION:

The differential equation in single dimension was established in sectinn (3.5) and is given by-

$$H = \nu k_s . \frac{\partial U}{\partial t} - k_c \frac{\partial^2 U}{\partial x^2} \qquad \text{... from eqn. (3.8)}$$

In this chapter, the solution of the above differential equation is effected by showing the direct analogy between the heat-flow problem and a particular transmission-line problem.

4.2. SOLUTION OF THE TRANSMISSION_LINE PROBLEM:

The transmission line of length a is shown in fig.(4.1). The uniformly distributed constants of the line are-

R = Series resistance per unit length.

L = Series inductance per unit length,

G = Shunt conductance per unit length.

C = Shunt capacitance per unit length.

At both the ends, the line is terminated in impedances. At one end, x = 0, the line is terminated in an impedance, Z_o, while at x = a, the line is terminated in an impedance Z_a.

A voltage intensity E(x) measured in volts/cm., is induced all along the line by an external force. This voltage may be brought about by an impinging electromagnetic field or by inserting a battery or generator in series with the line. E(x) may be a function of both time and distance.

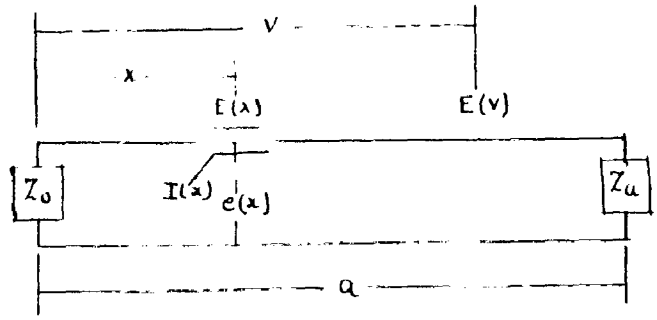


Fig. 4.1 - The transmission line used as an analogue to the heat-flow problem.

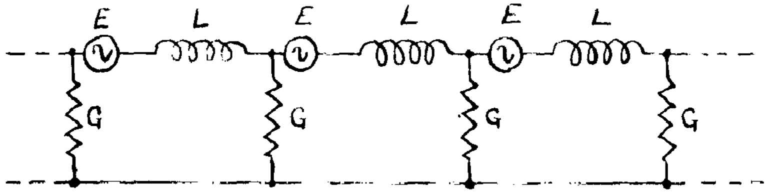


Fig. 4.2 - The transmission line analogous to the heat-flow problem.

Let-

e = the voltage across the line at a distance x from Z_o;

and-

I = the current in the line at a distance x from Z_o.

Then,

$$\frac{\partial e}{\partial x} = - RI - L\frac{\partial I}{\partial t} + E(x) \quad \ldots \quad (4.1)$$

and,

$$\frac{\partial I}{\partial x} = - Ge - C\frac{\partial e}{\partial t} \quad \ldots \quad (4.2)$$

or making use of the Heaviside differential operator p,

$$\frac{\partial e}{\partial x} = - (R+pL)\ I + E(x) \quad \ldots \quad (4.3)$$

and,

$$\frac{\partial I}{\partial x} = - (G + pc)e \quad \ldots \quad (4.4)$$

From (4.4),

$$e = - \frac{1}{(G+pc)} \cdot \frac{\partial I}{\partial x} \quad \ldots \quad (4.5)$$

Differentiating eqn. (4.5) with respect to x, we get-

$$\frac{\partial e}{\partial x} = - \frac{1}{(G+ pC)} \cdot \frac{\partial^2 I}{\partial x^2} \quad \ldots \quad (4.6)$$

Substituting (4.6) in (4.3), we get-

$$E(x) = (R + pL)\ I - \frac{1}{(G + pC)} \cdot \frac{\partial^2 I}{\partial x^2} \quad (4.7)$$

or

$$\frac{\partial^2 I}{\partial x^2} - \gamma^2 I = - (G + pC) \cdot E(x) \quad \ldots \quad (4.8)$$

where,

$$\gamma^2 = (R + pL)\ (G+pC) \quad \ldots \quad (4.9)$$

The complete solution of the differential equation (4.8) is given in Appendix (5) and from there the solution for the current I at point x is-

$$\begin{aligned}
I\ (x) = \Bigg[& (Z_c^2 + Z_a Z_o) \int_{v=x}^{v=a} E(v)\ \mathrm{Cosh}\ \gamma\ (a + x - v)\ dv \\
& + (Z_c^2 - Z_a Z_o) \int_{v=0}^{v=a} E(v)\ \mathrm{Cosh}\ \gamma\ (a - x - v)\ dv \\
& + (Z_c^2 + Z_a Z_o) \int_{v=0}^{v=x} E(v)\ \mathrm{Cosh}\ \gamma\ (a - x + v)\ dv \\
& + (Z_c Z_a + Z_c Z_o) \int_{v=x}^{v=a} E(v)\ \mathrm{Sinh}\ \gamma\ (a + x - v)\ dv \\
& + (Z_c Z_a - Z_o Z_o) \int_{v=0}^{v=a} E(v)\ \mathrm{Sinh}\ \gamma (a - x - v)\ dv \\
& + (Z_c Z_a - Z_c Z_o) \int_{v=0}^{v=x} E(v)\ \mathrm{Sinh}\ \gamma (a - x + v)\ dv \Bigg] \\
& \div 2\ Z_c\ \mathrm{Sinh}\ (\gamma a) \left[Z_c^2 - Z_o Z_a + (Z_o + Z_a) Z_c\ \mathrm{Coth}\ (\gamma a) \right]
\end{aligned} \qquad \ldots \qquad (4.10)$$

where, $Z_c = \left(\frac{R + pL}{G + pC} \right)^{\frac{1}{2}}$ ... (4.11)

4.3. ANALOGY BETWEEN HEAT FLOW AND TRANSMISSION LINE PROBLEM:

By making R = C = 0, the transmission line has only the constants of series inductance L and shunt leakage G, as shown in fig.(4.2). Under this condition the equation (4.7) reduces to-

$$E(x) = pLI - \frac{1}{G} \cdot \frac{\partial^2 I}{\partial x^2} = L \frac{\partial I}{\partial t} - \frac{1}{G} \frac{\partial^2 I}{\partial x^2} \qquad \ldots \qquad (4.12)$$

and equation (4.5) reduces to-

$$e = -\frac{1}{G}\ \frac{\partial I}{\partial x} \qquad \ldots \qquad (4.13)$$

Comparing equation (4.12) with (3.8), we see that if I(x) is analogous to the temperature U, the following equalities must exist:

$$L = \nu \cdot k_s \qquad \ldots \qquad (4.14)$$

$$\frac{1}{G} = k_c \qquad \ldots \qquad (4.15)$$

and,

E(x) = H gram - calories per second per cubic centimeter

$$\ldots \qquad (4.16)$$

From equation (4.13)

$$e = -\frac{1}{G} \cdot \frac{\partial I}{\partial x} = -k_c \cdot \frac{\partial U}{\partial x} \qquad \ldots \qquad (4.17)$$

From equation (4.17) we see that the voltage on the line is analogous to the heat flow across a surface and is proportional to the temperature gradient.

Also,

$$Z_c = \sqrt{\frac{pL}{G}} = \sqrt{p \cdot \nu \cdot k_s\, k_c} \qquad \ldots \qquad (4.18)$$

and

$$\gamma^2 = pLG = p. \frac{\nu - k_s}{k_c} \qquad \ldots \qquad (4.19)$$

4.4. HEAT FLOW PROBLEM IN A PLATE OF FINITE THICKNESS:

Now the specific problem of heat flow in a plate of thickness, a, will be treated, where the power is generated by induced currents.

The expression for the power delivered per cubic cm. in the case of infinite plate thickness was developed in Chapter 2 (section 2.6), where the power density at a depth x was found as-

$$P = \frac{2}{s} P_t \cdot e^{-\frac{2x}{s}} \qquad \text{...from eqn.(2.93)}$$

where,

P = Power density in watts per cubic cm.

P_t= Total power in watts per sq.cm.

However, for a finite thickness of metal sheet, the quantity P_t in (2.93) is no longer exactly the total power delivered per square centimeter of surface. The total power, with the necessary correction term is obtained in the following manner. The power density at a depath x is-

$$P = \frac{2}{s} \cdot P'_t \, e^{-\frac{2x}{s}} \qquad \ldots \qquad (4.20)$$

where,

P'_t = Apparent total power.

The true total power, P_t, is found by integrating eqn. (4.20) from x = 0 to x = a.

Therefore the true total power-

$$P_t = \int_{x=0}^{x=a} \frac{2}{s} P'_t \, e^{-\frac{2x}{s}} \cdot dx = P'_t \left(1 - e^{-\frac{2a}{s}}\right) \qquad (4.21)$$

or

$$P'_t = \frac{P_t}{1-e^{-2a/s}} \qquad \ldots \qquad (4.22)$$

Substituting eqn. (4.22) in (4.20), the power density is given by-

$$P = \frac{2}{s} \cdot \frac{P_t \cdot e^{-\frac{2x}{s}}}{1-e^{-2a/s}} \qquad \ldots \qquad (4.23)$$

Putting $\frac{2}{s} = \beta$, for convenience in (4.23) we get-

$$P = \frac{\beta \cdot P_t \cdot e^{-\beta . x}}{1 - e^{-\beta a}} \qquad \ldots \qquad (4.24)$$

In our analogy to a transmission line, the quantity E(x) is analogous to the heat generated per cubic cm., expressed in gram-calories per second. Since a watt is equal to 1/4.187 gram-calories per second, we have from (4.24)-

$$E(x) = \frac{\beta \cdot P_t \cdot e^{-\beta x}}{4.187\ (1-e^{-\beta a})} \qquad \ldots \qquad (4.25)$$

Assuming that no energy is lost by radiation at either surface of the metal, that is, at x = 0 and x = a, the temperature gradient at these two points must be zero. Consequently in our transmission line analogy, eqn. (4.17) reveals that the voltage at each end of the line must be zero at all times, This implies that the line is short-circuited at each end i.e. $Z_o = Z_a = 0$. Substituting this in eqn. (4.10), we get,

$$I(x) = \left\{ \int_{v=x}^{v=a} E(v) \cosh \gamma (a + x - v)\ dv + \int_{v=0}^{v=a} E(v) \cosh \gamma (a - x - v)\ dv. + \int_{v=0}^{v=x} E(v) \cosh \gamma (a - x + v)\ dv \right\} \div 2\ Z_c \sinh \gamma a \qquad \ldots \qquad (4.26)$$

Substituting (4.25) in (4.26), and after carrying out the necessary integration, we obtain-

$$I(x) = \frac{\beta . P_t}{4.187(1 - e^{-\beta a})} \cdot \left[\frac{-\gamma . e^{-\beta . x}}{Z_c(\beta^2 - \gamma^2)} + \frac{\beta}{Z_c(\beta^2 - \gamma^2)} \times \left[\frac{\cosh \gamma(a-x) - e^{-\beta . a} \cosh \gamma x}{\sinh \gamma . a} \right] \right] \quad \ldots \quad (4.27)$$

Eqn. (4.27) is an operational equation which must still be integrated. To establish the temperature relationship, substituting $Z_c = \sqrt{pL/G}$ and $\gamma^2 = pLG$ from (4.18) and (4.19) in eqn. (4.27) we get–

$$I(x) = \frac{\beta . P_t}{4.187(1-e^{-\beta \cdot a})} \cdot \left[\frac{-G . e^{-\beta x}}{(\beta^2 - pLG)} + \frac{\beta}{\sqrt{(pL/G)}(\beta^2 - pLG)} \times \left[\frac{\cosh\left[(a-x)\sqrt{pLG}\right] - e^{-\beta a} \cosh x\sqrt{pLG}}{\sinh \sqrt{pLG} \, . \, a} \right] \right] \quad \ldots \quad (4.28)$$

$$= \frac{\beta \cdot P_t}{4.187(1-e^{-\beta a})} \cdot \frac{1}{p} \cdot \left[\frac{-Gp \, e^{-\beta x}}{(\beta^2 - pLG)} + \frac{\beta\sqrt{Gp/L} \left[\cosh (a-x)\sqrt{pLG} - e^{-\beta a} \cosh x\sqrt{pLG}\right]}{(\beta^2 - pLG)(\sinh a\sqrt{pLG})} \right] (\quad \ldots \quad (4.29)$$

$$= \frac{\beta \cdot P_t}{4.187(1-e^{-\beta a})} \cdot \frac{1}{p} \cdot \left\{ \frac{-Gp \, e^{-\beta x} \sinh a\sqrt{pLG} + \beta\sqrt{Gp/L} \left[\cosh(a-x)\sqrt{pLG} - e^{-\beta a} \cosh x\sqrt{p}\right.}{(\beta^2 - pLG) \sinh\left(a\sqrt{pLG}\right)} \right. \quad \ldots \quad (4.30)$$

$$\text{or } I(x) = \frac{\beta . P_t}{4.187(1-e^{-\beta a})} . \frac{1}{p} . \frac{1}{Z(p)} \qquad (4.31)$$

where the operational function Z(p) is given by-

$$Z(p) = \frac{(\beta^2 - pLG) \sinh a\sqrt{pLG}}{\left\{-Gpe^{-\beta x}.\sinh a\sqrt{pLG} + \beta\sqrt{Gp/L}\left[\cosh(a-x)\sqrt{pLG} - e^{-\beta a} . \cosh x\sqrt{pLG}\right]\right\}} \quad \ldots \quad (4.32)$$

From eqn. (4.31), we get-

$$I(x) = \frac{P_t . G}{4.187\,\beta} . \frac{1}{p} \frac{\beta^2}{G(1-e^{-\beta a})} . \frac{1}{Z(p)}$$

$$= \frac{P_t . G}{4.187\,\beta} f(\theta, x) \quad \ldots \quad (4.33)$$

where,

$$f(\theta, x) = \frac{1}{p} . \frac{\beta^2}{G(1-e^{-\beta a})} . \frac{1}{Z(p)} \qquad (4.34)$$

The operational relation f(θ, x) has been solved by means of Heaviside Expansion Theorem and is evaluated in the following manner-

From eqn. (4.32), we have-

$$\frac{1}{Z(p)} = \frac{\beta\sqrt{Gp/L}. \left\{\cosh(a-x)\sqrt{pLG} - e^{\beta a}\cosh x\sqrt{pLG}\right\} - Gpe^{-\beta x}\sinh a\sqrt{pLG}}{(\beta^2 - pLG) \sinh a\sqrt{pLG}}$$

$$= \frac{a\sqrt{pLG}}{\sinh a\sqrt{pLG}} \times \left\{ \frac{\frac{\beta}{aL}\left\{\cosh(a-x)\sqrt{pLG} - e^{-\beta a}\cosh x\sqrt{pLG}\right\} - \frac{Gpe^{-\beta x}\sinh a\sqrt{pLG}}{a\sqrt{pLG}}}{(\beta^2 - pLG)} \right.$$

Then from eqn. (4.34)-

$$f(\theta,x) = \frac{1}{(1-e^{-\beta a}).p} \cdot \frac{Y(p).\ a\sqrt{pLG}}{\sinh\ (a\sqrt{pLG}\ )}$$

$$= \frac{1}{(1-e^{-\beta a}).p} \cdot \frac{Y(p)}{Z_1(p)} \qquad (4.35)$$

where,

$$Y(p) = \frac{\frac{\beta}{aLG}\left\{\cosh(a-x)\sqrt{pLG}\ \ -e^{-\beta a}.\cosh\ x\sqrt{pLG}\right\} - \frac{p.\bar{e}^{\beta x}\sinh\ a\sqrt{pLG}}{a\sqrt{pLG}}}{\left(1-\frac{pLG}{\beta^2}\right)} \qquad \ldots \qquad (4.36)$$

and,

$$Z_1(p) = \frac{\sinh\ a\sqrt{pLG}}{a\sqrt{pLG}} \qquad \ldots \qquad (4.37)$$

let-

$$y = \frac{Y(p)}{Z_1(p)}$$

By Heaviside Expansion Theorem$^{(21)}$ we have-

$$y = \left[\frac{Y(p)}{Z_1(p)}\right]_{p=0} + \left[\frac{Y(p)e^{pt}}{p\ \frac{dZ_1}{dp}}\right]_{p=p_1} + \left[\frac{Y(p)e^{pt}}{p\ \frac{dZ_1}{dp}}\right]_{p=p_2} + \ldots\ldots,$$

$$\ldots \qquad (4.38)$$

where p_1, p_2, etc. are roots of $Z_1(p)$, that is, the values of p at which $Z_1(p)$ vanishes.

For finding the roots of $Z_1(p)$, we define an arbitrary number m as follows:

$$-m^2 = pLG \qquad \ldots \qquad (4.39)$$

$$\text{or}\ \sqrt{pLG} = jm \qquad \ldots \qquad (4.39a)$$

Then eqn. (4.37) becomes-

$$Z_1(p) = \frac{\sin\ ma}{jma} \qquad \ldots \qquad (4.40)$$

Roots of eqn. (4.40) are-

$$m = \frac{n\pi}{a} ,$$

where n = 1, 2, 3.. etc. and from eqn. (4.39), the roots are given by-

$$p_n = \frac{-1}{LG} \left(\frac{n\pi}{a}\right)^2 = - \frac{n^2 \pi^2}{a^2\ LG} \quad \ldots \qquad (4.41)$$

where n =1, 2, 3.... etc.

Therefore from eqn. (4.38) we have,

$$y = \left[\frac{Y(p)}{Z_1(p)} \right]_{p=0} + \sum_{n=1}^{n=\infty} \left[\frac{Y(p).e^{pt}}{p \frac{dZ_1}{dp}} \right]_{p= \frac{-n^2 \pi^2}{a^2 \cdot LG}} \qquad (4.42)$$

since $\frac{-n^2\ \pi^2}{a^2\ LG}$, where n = 1, 2, 3..., etc, are the roots of $Z_1(p)$.

Differentiating $Z_1(p)$ and multiplying by p gives-

$$p \frac{dZ_1}{dp} = \frac{(a \sqrt{pLG}\ ) \cosh (a\sqrt{pLG}) - \sinh (a\sqrt{pLG})}{2\ (a\sqrt{pLG})}$$

Then,

$$\left[p\ \frac{dZ_1}{dp} \right]_{p= \frac{-n^2 \pi^2}{a^2\ LG}} = \frac{(-1)^n}{2} \quad \ldots \qquad (4.43)$$

Also from equations (4.37) and (4.36)-

$$Z_1(0) = 1 \quad \ldots \qquad (4.44)$$

$$Y(0) = \frac{\beta}{aLG} (1 - e^{-\beta a}) \quad \ldots \qquad (4.45)$$

and finally-

$$\left[Y(p)\right]_{p=\frac{-n^2\pi^2}{a^2\cdot LG}} = \frac{\left\{\cos\left(n\pi-\frac{n\pi x}{a}\right) - e^{-\beta a}\cos\left(\frac{n\pi x}{a}\right)\right\}}{aLG\left(1+\frac{n^2\pi^2}{\beta^2\ a^2}\right)}$$

$$= \frac{\beta\left[(-1)^n - e^{-\beta a}\right]\cos\left(\frac{n\pi x}{a}\right)}{aLG\left(1+\frac{n^2\pi^2}{\beta^2\ a^2}\right)} \qquad \ldots \qquad (4.46)$$

Substituting the values from equations (4.43), (4.44), (4.45) and (4.46) in eqn.(4.42), we get-

$$y = \frac{\beta}{aLG}\left(1-e^{-\beta a}\right) + \sum_{n=1}^{n=\infty} \frac{2\beta\left[1-(-1)^n e^{-\beta a}\right]}{aLG\left(1+\frac{n^2\pi^2}{\beta^2\cdot a^2}\right)} \cos\left(\frac{n\pi x}{a}\right) e^{\frac{-n^2\pi^2 t}{LGa^2}} \qquad \ldots \qquad (4.47)$$

Therefore from eqn. (4.35) we get-

$$f(\theta,x) = \frac{1}{p}\left[\frac{\beta}{aLG} + \frac{2\beta}{aLG(1-e^{-\beta a})} \sum_{n=1}^{n=\infty}\left[\frac{\left[1-(-1)^n\ e^{-\beta a}\right]}{\left(1+\frac{n^2\pi^2}{\beta^2\cdot a^2}\right)}\cdot e^{\frac{-n^2\pi^2 t}{LGa^2}} \cos\left(\frac{n\pi x}{a}\right)\right]\right]$$

now since $\frac{1}{p}$ is the operation $\int_0^t dt$, we get-

$$f(\theta,x) = \frac{\beta . t}{aLG} + \frac{2\beta}{aLG(1-e^{-\beta a})} \cdot \sum_{n=1}^{n=\infty}\left\{\frac{\left[1-(-1)^n e^{-\beta a}\right]\left[1-e^{\frac{-n^2\pi^2 t}{LGa^2}}\right]\cos\left(\frac{n\pi x}{a}\right)}{\frac{n^2\pi^2}{LGa2}\left(1+\frac{n^2\pi^2}{\beta^2\ a^2}\right)}\right\} \qquad \ldots \qquad (4.48)$$

putting $\theta = \frac{\beta^2 t}{LG}$ ... (4.49)

$$f(\theta,x) = \frac{\theta}{\beta\cdot a} + \frac{2\beta a}{{}^2(1-e^{-\beta a})} \cdot \sum_{n=1}^{n=\infty} \left\{ \frac{\left[1-e^{\frac{-n^2\pi^2}{(\beta a)^2}\theta}\right]\left[1-(-1)^n e^{-\beta a}\right]\cos\left(\frac{n\pi x}{a}\right)}{n^2\left[1+\left(\frac{n\pi}{a}\right)^2\right]} \right\}$$

... (4.50)

So the solution for current is given in eqn. (4.33), where the function f(θ, x) is equal to the value given in eqn. (4.50).

Since the current I(x) is analogous to the temperature U(x), we can write for the temperature -

$$U(x) = \frac{P_t \cdot G}{4.187\beta} \cdot f(\theta, x)$$

Since $\beta = \frac{2}{s}$, in terms of the thermal constants of the material, the temperature is given by-

$$U(x) = \frac{sP_t}{8.374k} \cdot f\,(\theta,x)$$ ... (4.51)

The function f(θ, x) given by eqn. (4.50) has been calculated for different values of θ, the ratio x/a and the ratio a/s. The values of θ are varied from 10 to 10,000, the x/a is varied from 0 to 1, and the ratio a/s is varied from 1 to 20. The values of f(θ, x) are plotted in table no.1 to table no.10. All these calculations have been made on the digital computer and the sample programme is shown in Appendix 3, (programme no.1).

TABLE NO. 4.1

FUNCTION F(THETA,X) AT X/A= 0.00

THETA	A/S= 1.0	A/S= 2.5	A/S= 5.0	A/S= 10.0	A/S= 20.
10.	5.1659	2.8612	2.7356	2.7328	2.728
20.	10.1659	4.8752	4.1737	4.1634	4.158
30.	15.1659	6.8754	5.3346	5.2757	5.271
50.	25.1659	10.8754	7.4168	7.0521	7.047
100.	50.1659	20.8754	12.4300	10.3751	10.329
200.	100.1659	40.8754	22.4301	15.6822	14.987
400.	200.1659	80.8754	42.4301	25.7104	21.664
500.	250.1659	100.8754	52.4301	30.7106	24.478
700.	350.1659	140.8754	72.4301	40.7106	29.740
900.	450.1659	180.8754	92.4301	50.7106	34.816
1000.	500.1659	200.8754	102.4301	55.7106	37.330
2000.	1000.1659	400.8754	202.4301	105.7106	62.347
5000.	2500.1659	1000.8754	502.4301	255.7106	137.347
7000.	3500.1659	1400.8754	702.4301	355.7106	187.347
10000.	5000.1659	2000.8754	1002.4301	505.7106	262.347

TABLE NO. 4.2

FUNCTION F(THETA,X) AT X/A= .25

THETA	A/S= 1.0	A/S= 2.5	A/S= 5.0	A/S= 10.0	A/S= 20.
10.	5.1085	2.5105	1.8181	1.1115	.590
20.	10.1085	4.5204	3.1231	2.0893	1.161
30.	15.1085	6.5206	4.2367	2.9626	1.712
50.	25.1085	10.5206	6.2949	4.4823	2.761
100.	50.1085	20.5206	11.3042	7.5596	5.122
200.	100.1085	40.5206	21.3042	12.7766	9.037
400.	200.1085	80.5206	41.3042	22.7966	15.213
500.	250.1085	100.5206	51.3042	27.7967	17.935
700.	350.1085	140.5206	71.3042	37.7967	23.120
900.	450.1085	180.5206	91.3042	47.7967	28.174
1000.	500.1085	200.5206	101.3042	52.7967	30.684
2000.	1000.1085	400.5206	201.3042	102.7967	55.696
5000.	2500.1085	1000.5206	501.3042	252.7967	130.696
7000.	3500.1085	1400.5206	701.3042	352.7967	180.696
10000.	5000.1085	2000.5206	1001.3042	502.7967	255.696

TABLE NO. 4.3

FUNCTION F(THETA,X) AT X/A= .50

THETA	A/S= 1.0	A/S= 2.5	A/S= 5.0	A/S= 10.0	A/S= 2
10.	5.0000	2.0000	1.0000	.5000	.2
20.	10.0000	4.0000	2.0000	1.0000	.5
30.	15.0000	6.0000	3.0000	1.5000	.7
50.	25.0000	10.0000	5.0000	2.5000	1.2
100.	50.0000	20.0000	10.0000	5.0000	2.5
200.	100.0000	40.0000	20.0000	10.0000	5.0
400.	200.0000	80.0000	40.0000	20.0000	10.0
500.	250.0000	100.0000	50.0000	25.0000	12.5
700.	350.0000	140.0000	70.0000	35.0000	17.5
900.	450.0000	180.0000	90.0000	45.0000	22.5
1000.	500.0000	200.0000	100.0000	50.0000	25.0
2000.	1000.0000	400.0000	200.0000	100.0000	50.0
5000.	2500.0000	1000.0000	500.0000	250.0000	125.0
7000.	3500.0000	1400.0000	700.0000	350.0000	175.0
10000.	5000.0000	2000.0000	1000.0000	500.0000	250.0

TABLE NO. 4.4

FUNCTION F(THETA,X) AT X/A= .75

THETA	A/S= 1.0	A/S= 2.5	A/S= 5.0	A/S= 10.0	A/S= 2
10.	4.8914	1.4894	.1818	.0000	.0
20.	9.8914	3.4795	.8768	.0000	.0
30.	14.8914	5.4793	1.7632	.0373	.0
50.	24.8914	9.4793	3.7050	.5176	.0
100.	49.8914	19.4793	8.6957	2.4403	.0
200.	99.8914	39.4793	18.6957	7.2233	.9
400.	199.8914	79.4793	38.6957	17.2033	4.7
500.	249.8914	99.4793	48.6957	22.2032	7.0
700.	349.8914	139.4793	68.6957	32.2032	11.8
900.	449.8914	179.4793	88.6957	42.2032	16.8
1000.	499.8914	199.4793	98.6957	47.2032	19.3
2000.	999.8915	399.4793	198.6957	97.2032	44.3
5000.	2499.8915	999.4794	498.6957	247.2032	119.3
7000.	3499.8915	1399.4794	698.6957	347.2032	169.3
10000.	4999.8915	1999.4794	998.6958	497.2032	244.3

TABLE NO. 4.5

FUNCTION F(THETA,X) AT X/A= 1.00

THETA	A/S= 1.0	A/S= 2.5	A/S= 5.0	A/S= 10.0	A/S= 20.0
10.	4.8539	1.3573	.1139	.0004	.0000
20.	9.8539	3.3434	.6895	.0079	.0000
30.	14.8539	5.3431	1.5289	.0565	.0000
50.	24.8539	9.3431	3.4467	.3622	.0000
100.	49.8539	19.3431	8.4335	2.0524	.0408
200.	99.8539	39.3431	18.4334	6.7455	.6896
400.	199.8539	79.3431	38.4334	16.7172	4.0409
500.	249.8539	99.3431	48.4334	21.7171	6.2265
700.	349.8539	139.3431	68.4334	31.7170	10.9652
900.	449.8539	179.3431	88.4334	41.7170	15.8891
000.	499.8539	199.3431	98.4334	46.7170	18.3747
000.	999.8540	399.3431	198.4334	96.7170	43.3579
000.	2499.8540	999.3432	498.4334	246.7170	118.3578
000.	3499.8540	1399.3432	698.4334	346.7170	168.3578
000.	4999.8540	1999.3432	998.4335	496.7170	243.3578

TABLE NO. 4.6

FUNCTION F(THETA,X) AT A/S= 1.0

HETA	X/A= 0.00	X/A= .25	X/A= .50	X/A= .75	X/A= 1.00
10.	5.1659	5.1085	5.0000	4.8914	4.8539
20.	10.1659	10.1085	10.0000	9.8914	9.8539
30.	15.1659	15.1085	15.0000	14.8914	14.8539
50.	25.1659	25.1085	25.0000	24.8914	24.8539
00.	50.1659	50.1085	50.0000	49.8914	49.8539
00.	100.1659	100.1085	100.0000	99.8914	99.8539
00.	200.1659	200.1085	200.0000	199.8914	199.8539
00.	250.1659	250.1085	250.0000	249.8914	249.8539
00.	350.1659	350.1085	350.0000	349.8914	349.8539
00.	450.1659	450.1085	450.0000	449.8914	449.8539
00.	500.1659	500.1085	500.0000	499.8914	499.8539
00.	1000.1659	1000.1085	1000.0000	999.8915	999.8540
00.	2500.1659	2500.1085	2500.0000	2499.8915	2499.8540
00.	3500.1659	3500.1085	3500.0000	3499.8915	3499.8540
00.	5000.1659	5000.1085	5000.0000	4999.8915	4999.8540

TABLE NO. 4.7

FUNCTION F(THETA,X) AT A/S= 2.5

THETA	X/A= 0.00	X/A= .25	X/A= .50	X/A= .75	X/A= 1.00
10.	2.8612	2.5105	2.0000	1.4894	1.3573
20.	4.8752	4.5204	4.0000	3.4795	3.3434
30.	6.8754	6.5206	6.0000	5.4793	5.3431
50.	10.8754	10.5206	10.0000	9.4793	9.3431
100.	20.8754	20.5206	20.0000	19.4793	19.3431
200.	40.8754	40.5206	40.0000	39.4793	39.3431
400.	80.8754	80.5206	80.0000	79.4793	79.3431
500.	100.8754	100.5206	100.0000	99.4793	99.3431
700.	140.8754	140.5206	140.0000	139.4793	139.3431
900.	180.8754	180.5206	180.0000	179.4793	179.3431
1000.	200.8754	200.5206	200.0000	199.4793	199.3431
2000.	400.8754	400.5206	400.0000	399.4793	399.3431
5000.	1000.8754	1000.5206	1000.0000	999.4794	999.3432
7000.	1400.8754	1400.5206	1400.0000	1399.4794	1399.3432
10000.	2000.8754	2000.5206	2000.0000	1999.4794	1999.3432

TABLE NO. 4.8

FUNCTION F(THETA,X) AT A/S= 5.0

THETA	X/A= 0.00	X/A= .25	X/A= .50	X/A= .75	X/A= 1.00
10.	2.7356	1.8181	1.0000	.1818	.1139
20.	4.1737	3.1231	2.0000	.8768	.6895
30.	5.3346	4.2367	3.0000	1.7632	1.5289
50.	7.4168	6.2949	5.0000	3.7050	3.4467
100.	12.4300	11.3042	10.0000	8.6957	8.4335
200.	22.4301	21.3042	20.0000	18.6957	18.4334
400.	42.4301	41.3042	40.0000	38.6957	38.4334
500.	52.4301	51.3042	50.0000	48.6957	48.4334
700.	72.4301	71.3042	70.0000	68.6957	68.4334
900.	92.4301	91.3042	90.0000	88.6957	88.4334
1000.	102.4301	101.3042	100.0000	98.6957	98.4334
2000.	202.4301	201.3042	200.0000	198.6957	198.4334
5000.	502.4301	501.3042	500.0000	498.6957	498.4334
7000.	702.4301	701.3042	700.0000	698.6957	698.4334
10000.	1002.4301	1001.3042	1000.0000	998.6958	998.4335

TABLE NO. 4.9

FUNCTION F(THETA,X) AT A/S= 10.0

THETA	X/A= 0.00	X/A= .25	X/A= .50	X/A= .75	X/A= 1.00
10.	2.7328	1.1115	.5000	.0000	.0004
20.	4.1634	2.0893	1.0000	.0000	.0079
30.	5.2757	2.9626	1.5000	.0373	.0565
50.	7.0521	4.4823	2.5000	.5176	.3622
100.	10.3751	7.5596	5.0000	2.4403	2.0524
200.	15.6822	12.7766	10.0000	7.2233	6.7455
400.	25.7104	22.7966	20.0000	17.2033	16.7172
500.	30.7106	27.7967	25.0000	22.2032	21.7171
700.	40.7106	37.7967	35.0000	32.2032	31.7170
900.	50.7106	47.7967	45.0000	42.2032	41.7170
1000.	55.7106	52.7967	50.0000	47.2032	46.7170
2000.	105.7106	102.7967	100.0000	97.2032	96.7170
5000.	255.7106	252.7967	250.0000	247.2032	246.7170
7000.	355.7106	352.7967	350.0000	347.2032	346.7170
10000.	505.7106	502.7967	500.0000	497.2032	496.7170

TABLE NO. 4.10

FUNCTION F(THETA,X) AT A/S= 20.0

THETA	X/A= 0.00	X/A= .25	X/A= .50	X/A= .75	X/A= 1.00
10.	2.7280	.5907	.2500	.0000	.0000
20.	4.1587	1.1611	.5000	.0000	.0000
30.	5.2710	1.7123	.7500	.0000	.0000
50.	7.0471	2.7618	1.2500	.0000	.0000
100.	10.3292	5.1224	2.5000	.0000	.0408
200.	14.9873	9.0375	5.0000	.9624	.6896
400.	21.6643	15.2133	10.0000	4.7866	4.0409
500.	24.4789	17.9357	12.5000	7.0642	6.2265
700.	29.7402	23.1205	17.5000	11.8794	10.9652
900.	34.8163	28.1743	22.5000	16.8256	15.8891
1000.	37.3307	30.6845	25.0000	19.3154	18.3747
2000.	62.3475	55.6964	50.0000	44.3035	43.3579
5000.	137.3475	130.6964	125.0000	119.3035	118.3578
7000.	187.3475	180.6964	175.0000	169.3035	168.3578
10000.	262.3475	255.6964	250.0000	244.3035	243.3578

The function $f(\theta,x)$ is plotted for different values of θ, corresponding to different values of the ratios x/a and a/s. These graphs are plotted in Fig. no.(4.3) to Fig.no.(4.12). With the help of these graphs the value of the function $f(\theta, x)$ can be evaluated throughout the section of the metal sheet and hence the temperature distribution can be evaluated from eqn. (4.51).

At the exact center of the sheet i.e. $x = a/2$ or $x/a = 0.5$, the function $f(\theta, x)$ given in eqn. (4.50) is greatly simplified. When $x = a/2$, the term $\cos(n\pi x/a)$ is zero for all values of n so at the exact center of the sheet,

$$f(\theta, x) = \frac{\theta}{\beta a} \qquad \ldots \qquad (4.52)$$

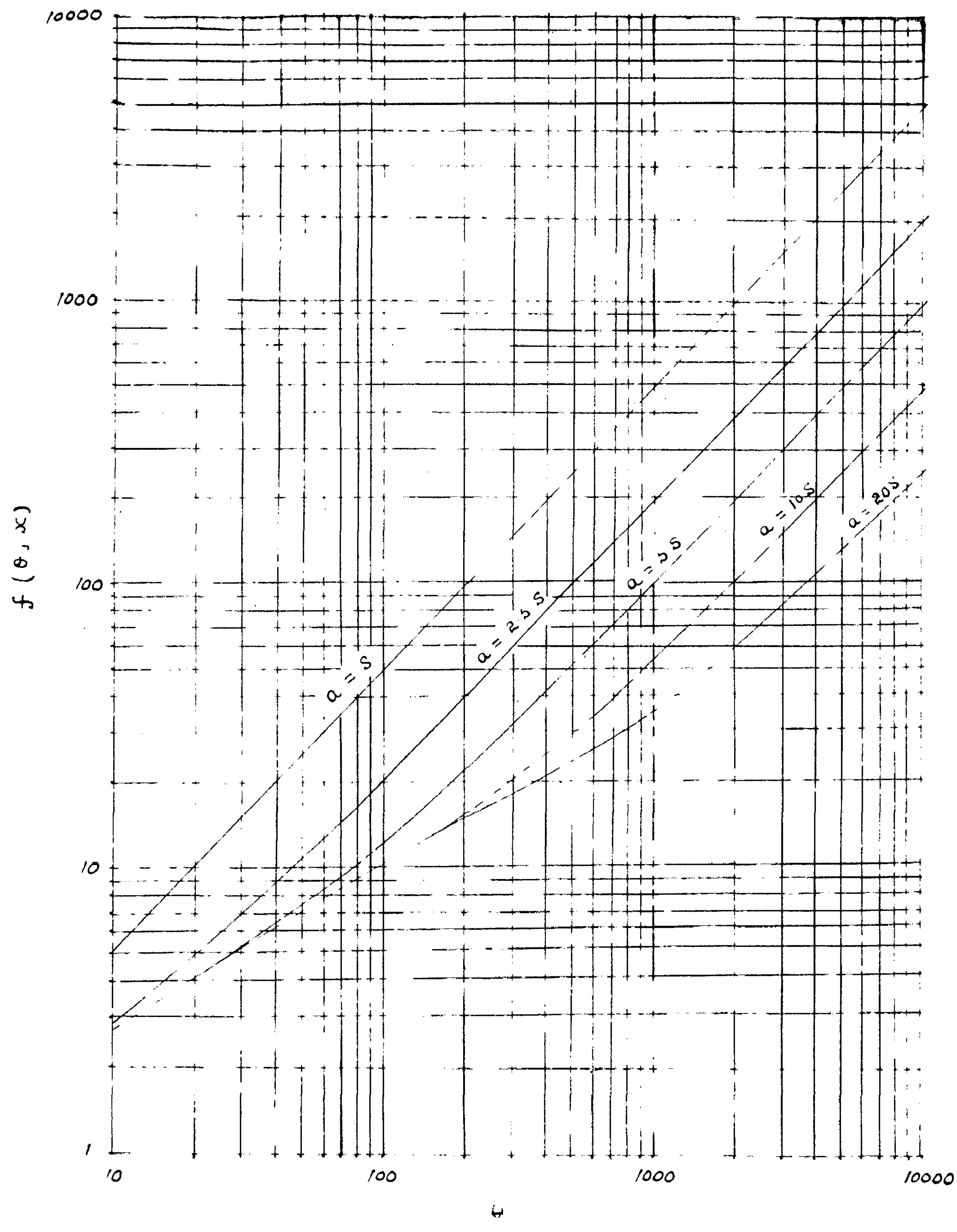
and the temperature is–

$$U(x)_{x=a/2} = \frac{P_t \cdot G \cdot \theta}{4.187\beta^2 a}$$

$$= \frac{P_t \cdot t}{4.187\ L \cdot a}$$

$$= \frac{P_t \cdot t}{4.187 a \cdot \nu \cdot k_s} \qquad \ldots \qquad (4.53)$$

It is very interesting to note from the eqn. (4.53), that the temperature $U(x)$ is a function of time and contains as parameters only the thermal constants of the metal and the power generated per square centimeter. The frequency of the induced currents is not contained in the equation.



[illegible] - $f(\theta, x)$ with $x = 0$, i.e. at the surface of the plate.

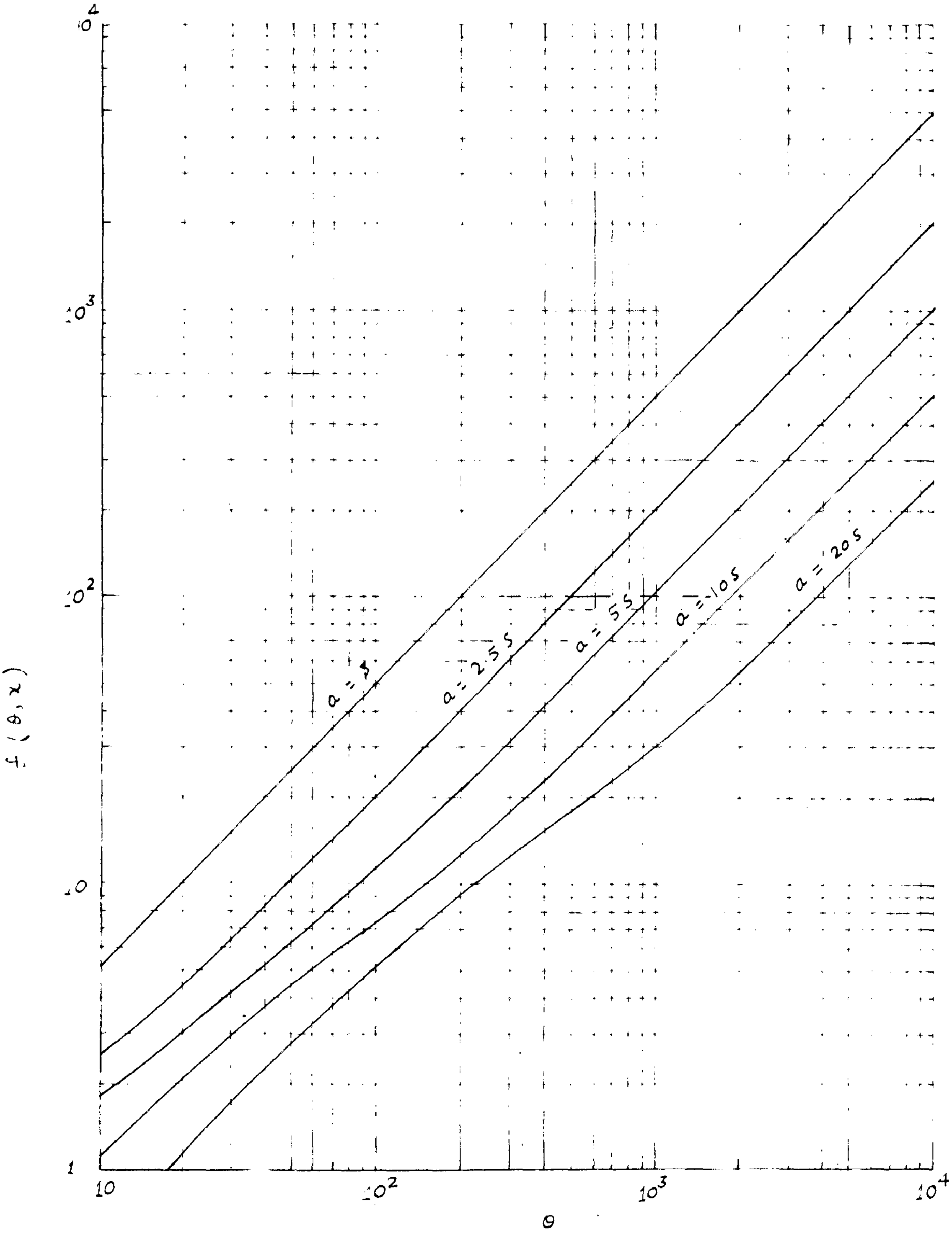


FIG 4 4 - $f(\theta, x)$ AT $x = 0{\cdot}25a$

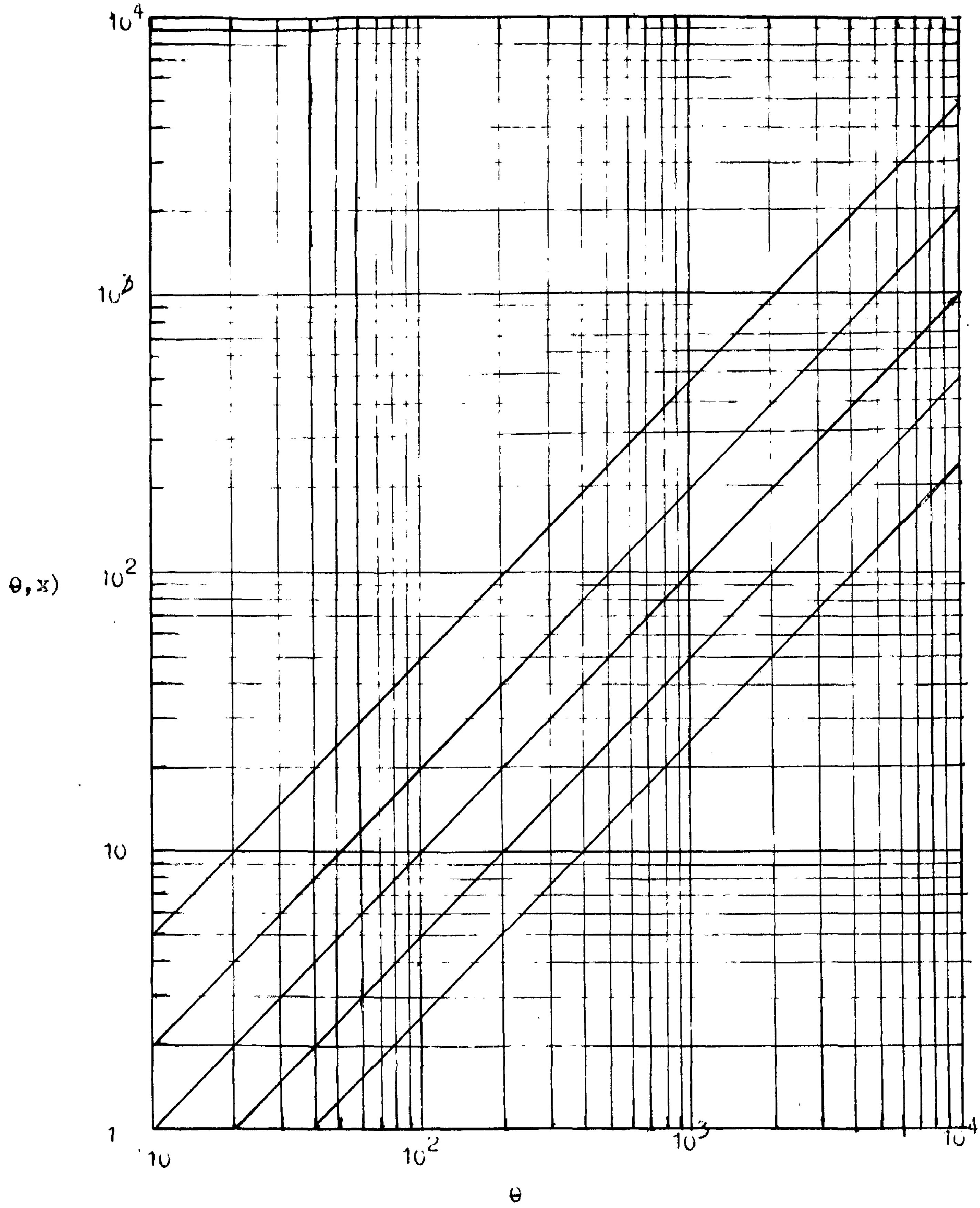


Fig.4.5 - The function $f(\theta,x)$, with $x = 0.5\ a$
ie. at the exact center of the plate.

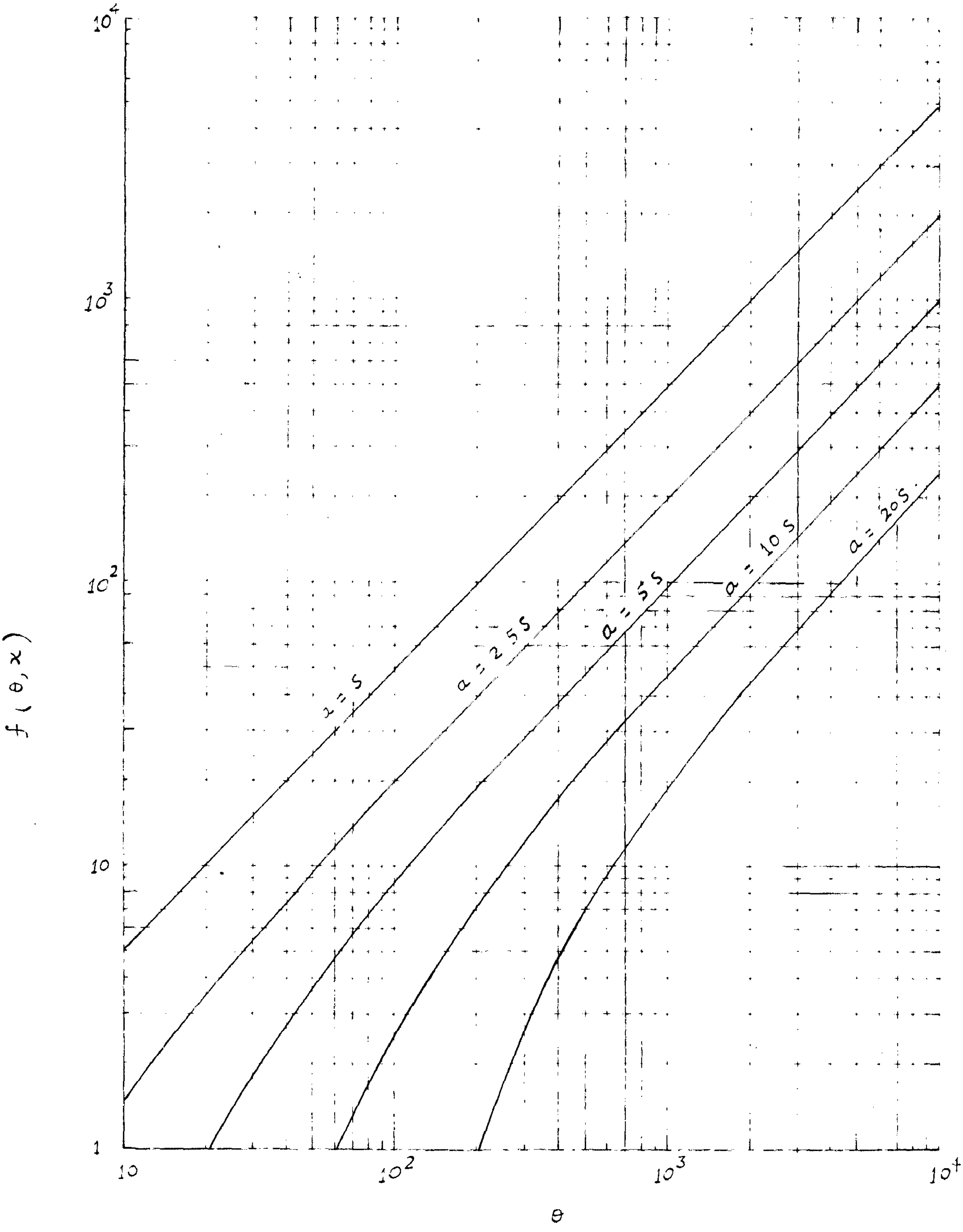


Fig. 4.6 - $f(\theta, x)$ AT $x = 0.75a$

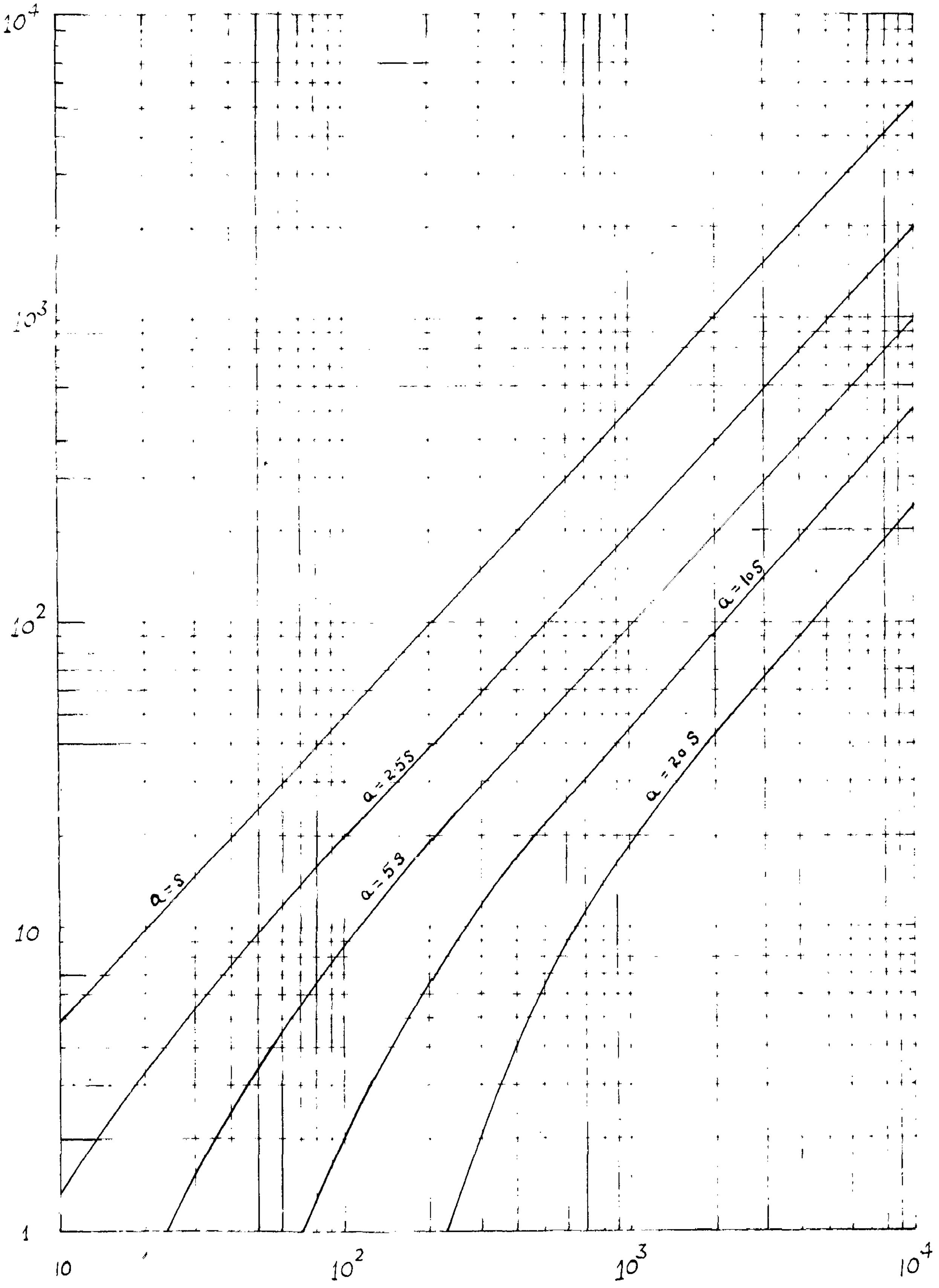


FIG 4.7 - $f(\theta, x)$ WITH $x = a$

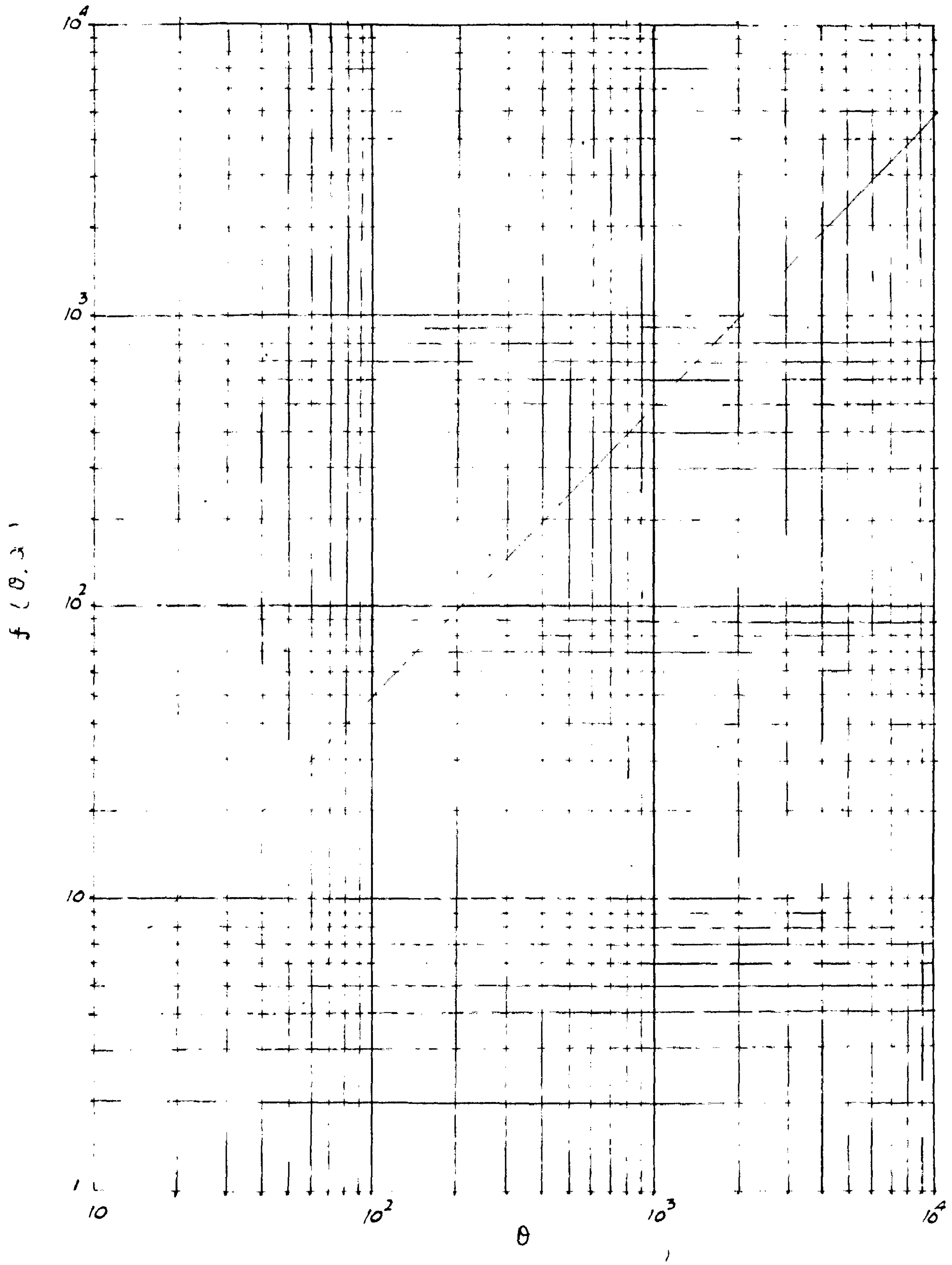
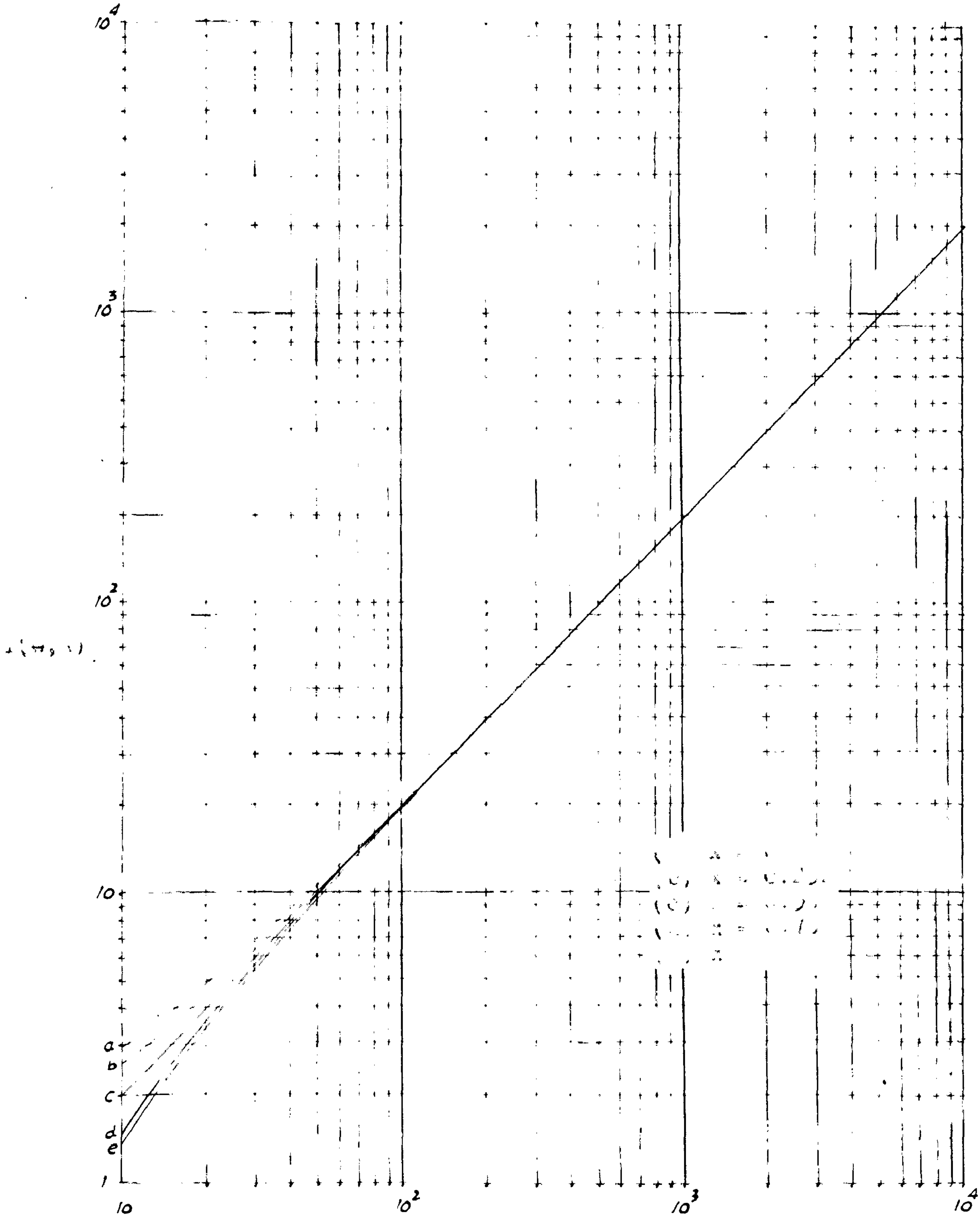
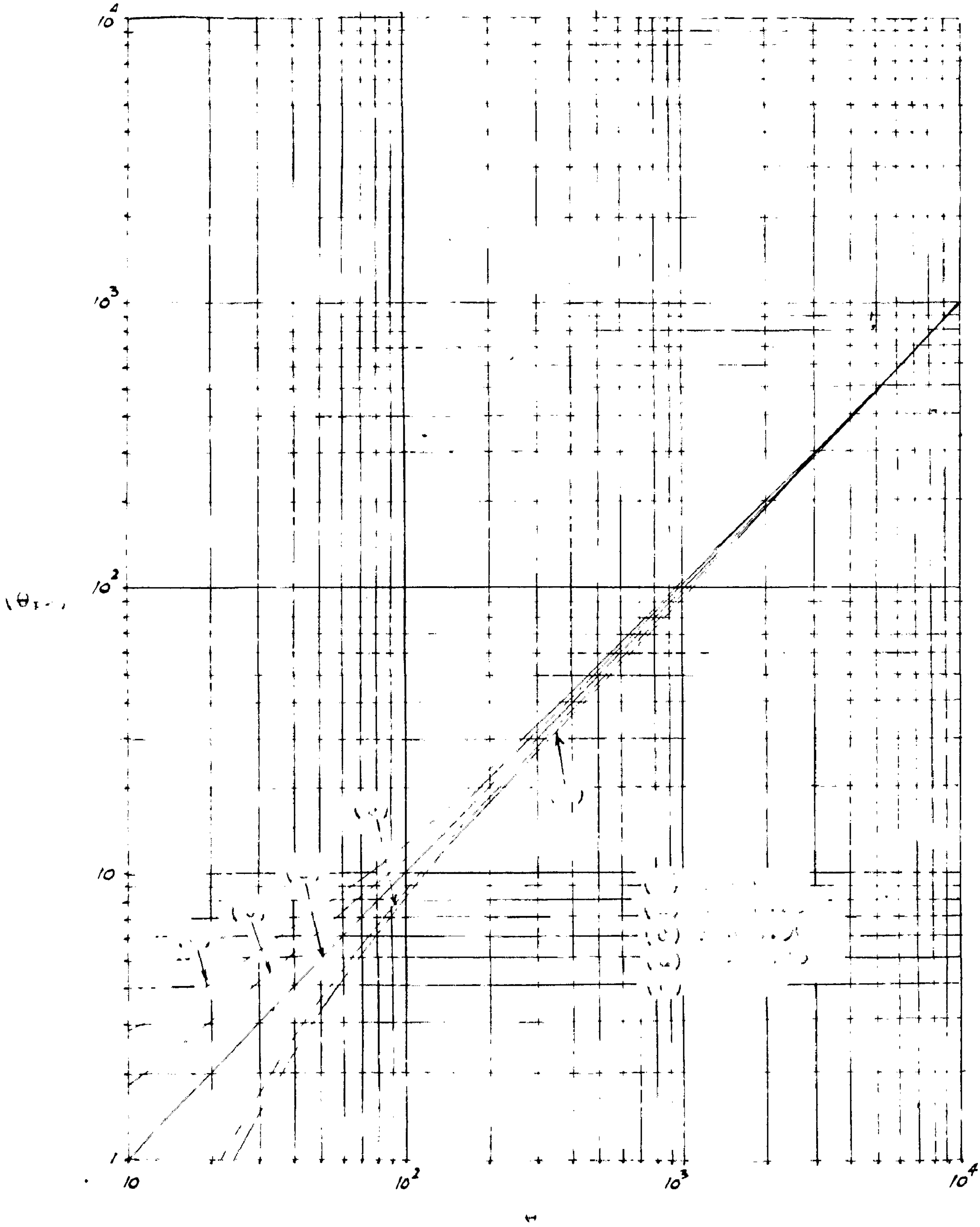


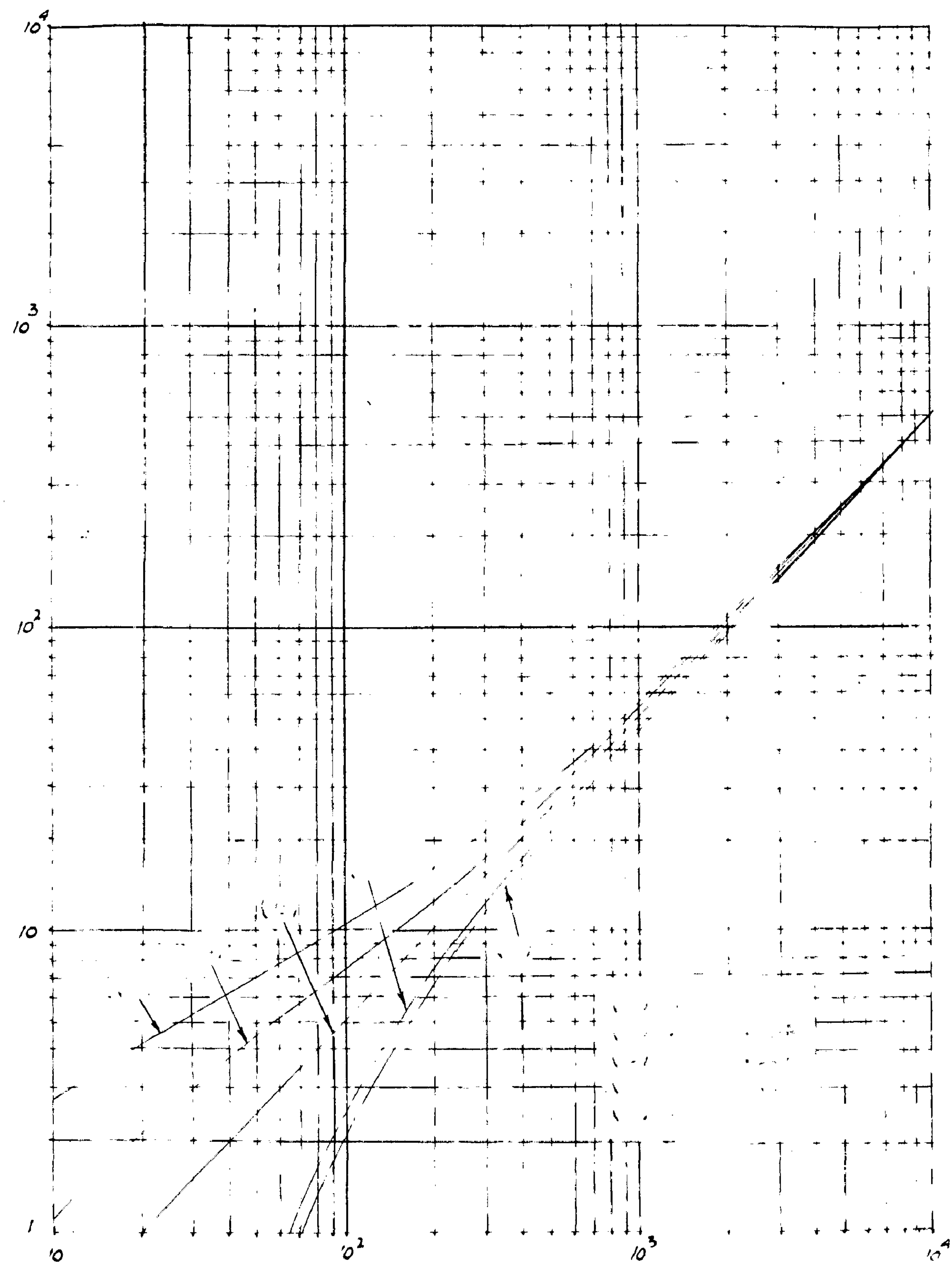
Fig. 4.8 - f(θ, x) with a = 5

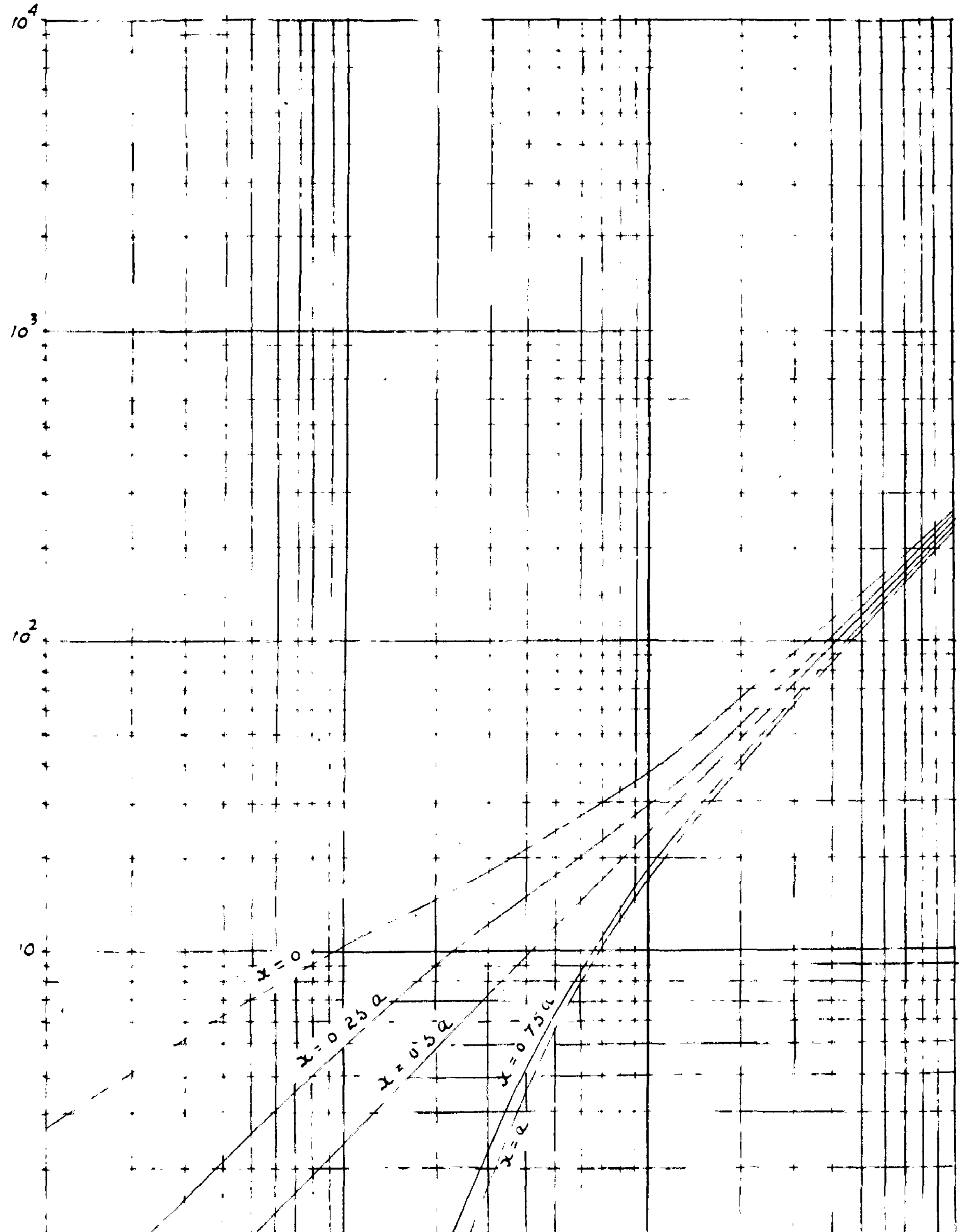


4.9 $f(\theta, x)$, with $a = 2.55$


10^4
10^3
10^2
10
1
10
10^2
10^3
10^4
θ

[illegible]


10^4
10^3
10^2
10
1
10
10^2
10^3
10^4



10^4
10^3
10^2
10
1
10
10^2
10^3
10^4
x = 0
x = 0.25 a
x = 0.5 a
x = 0.75 a
x = a

[illegible]

APPENDICES

APPENDIX - I

Locus of magnetic Intensities

The ratio of two consecutive magnetic intensities from eqn. (2.14) is-

$$H_m = H_o\ e^{-m.\ \Delta\alpha.\cot\beta}$$

$$H_{m-1} = H_o\ e^{-(m-1)\ \Delta\alpha\cot\beta}$$

Therefore,

$$\frac{H_m}{H_{m-1}} = e^{-\Delta\alpha.\cot\beta} = q \quad \ldots \quad (A.1)$$

which is constant for positive, integral m.

Now from fig. (2.5) we have-

$$q = \frac{H_1}{H_o} = \frac{\operatorname{Sin}\beta}{\operatorname{Sin}(\beta+\Delta\alpha)}$$

$$= 1- \frac{\operatorname{Sin}(\beta+\Delta\alpha)-\operatorname{Sin}\beta}{\operatorname{Sin}(\beta+\Delta\alpha).}$$

$$= 1- \frac{2\cos(\beta+\Delta\alpha/2)\ \operatorname{Sin}(\Delta\alpha/2)}{\operatorname{Sin}(\beta+\Delta\alpha)} \quad (A.2)$$

for sleeves of infinitesimal wall thickness.

$$\operatorname{Sin}\frac{\Delta\alpha}{2} = \frac{\Delta\alpha}{2} \quad \ldots \quad (A.3)$$

and $\beta + \frac{\Delta\alpha}{2} = \beta + \Delta\alpha = \beta \quad \ldots \quad (A.4)$

Hence from eqn. (A.2)

$$q = 1- \Delta\alpha \cdot \cot\beta \quad \ldots \quad (A.5)$$

The same result is obtained by using equations (2.14) and (A.1).

i.e. $q = e^{-\Delta\alpha\cot\beta} = 1 - \Delta\alpha.\cot\beta$, for $\Delta\alpha.\cot\beta \ll 1$

APPENDIX II

Total flux in the slab

From eqn. (2.107), we can write,

$$\phi_w = 2\mu.w.\ H_o\ s.X \qquad \ldots \qquad (B.1)$$

where,

$$X = \frac{(\text{Sinh}\,\frac{t}{2s}.\cos\frac{t}{2s} - j\cosh\frac{t}{2s}\sin\frac{t}{2s})}{(1+j)(\cosh\frac{t}{2s}\cos\frac{t}{2s} - j\sinh\frac{t}{2s}\sin\frac{t}{2s})} \qquad (B.2)$$

Putting $\frac{t}{2s} = \theta$ in (B.2) we get,

$$X = \frac{(\sinh\theta\cos\theta - j\cosh\theta\sin\theta)}{(1+j)(\cosh\theta\cos\theta - j\sinh\theta\sin\theta)} \qquad (B.3)$$

Rationalising (B.3) we get,

$$X = \frac{(\sinh\theta\cos\theta - j\cosh\theta\sin\theta)}{(1+j)(\cosh\theta\cos\theta - j\sinh\theta\sin\theta)} \cdot$$

$$\frac{(1-j)(\cosh\theta\cos\theta + j\sinh\theta\sin\theta)}{(1-j)(\cosh\theta\cosh\theta + j\sinh\theta\sin\theta)} \qquad (B.4)$$

Numerator of (B.4) $= (1-j)\ (\sinh\theta\cos\theta - j\cosh\theta\sin\theta).$

$$(\cosh\theta\cos\theta + j\sinh\theta\sin\theta)$$

$$= (1-j)\sinh(1+j)\theta\ .\ \cosh(1-j)\theta$$

$$= (\frac{1-j}{2})\ (\sinh 2\theta + \sinh 2\,j\theta\ )$$

$$= (\frac{1-j}{2})\ (\sinh 2\theta + j\sin 2\,\theta\ )$$

$$= \frac{1}{2}\ (\sinh 2\theta + j\sin 2\theta - j\sinh 2\theta + \sin 2\theta)$$

$$= \frac{1}{2}\ \{\ (\sinh 2\theta + \sin 2\theta) - j(\sinh 2\theta - \sin 2\theta)\} \qquad \ldots \qquad (B.5)$$

Denominator of (B.4)$= (1+j)(1-j).(\cosh\theta\cos\theta - j\sinh\theta\sin\theta).$

$$(\cosh\theta\cos\theta + j\sinh\theta\sin\theta)$$

$$= 2\cosh(1+j)\theta.\cosh(1-j)\theta$$

$$=(\cosh 2\theta + \cosh 2j\theta)$$

$$= (\cosh 2\theta + \cos 2\theta) \qquad \ldots \quad (B.6)$$

Hence from (B.5) and (B.6) we have,

$$X = \frac{(\sinh 2\theta + \sin 2\theta) - j(\sinh 2\theta - \sin 2\theta)}{2(\cosh 2\theta + \cos 2\theta)}$$

$$= \frac{(\sinh\frac{t}{s} + \sin\frac{t}{s}) - j(\sinh\frac{t}{s} - \sin\frac{t}{s})}{2(\cosh\frac{t}{s} + \cos\frac{t}{s})} \qquad \ldots \quad (B.7)$$

Hence ϕ_w from (B.1) is given by-

$$\phi_w = (2\mu.w.H_o.s.)\left[\frac{(\sinh\frac{t}{s} + \sin\frac{t}{s}) - j(\sinh\frac{t}{s} - \sin\frac{t}{s})}{2(\cosh\frac{t}{s} + \cos\frac{t}{s})}\right] \qquad \ldots \quad (B.8)$$

$$= (\mu.t.w.H_o.\frac{s}{t})\left[\frac{(\sinh\frac{t}{s} + \sin\frac{t}{s}) - j(\sinh\frac{t}{s} - \sin\frac{t}{s})}{(\cosh\frac{t}{s} + \cos\frac{t}{s})}\right] \qquad \ldots \quad (B.9)$$

Putting w.t. $= A_w$ = Area of the slab, in eqn. (B.9), we get,

$$\phi_w = \mu H_o.A_w.\frac{s}{t}\left[\frac{(\sinh\frac{t}{s} + \sin\frac{t}{s}) - j(\sinh\frac{t}{s} - \sin\frac{t}{s})}{(\cosh\frac{t}{s} + \cos\frac{t}{s})}\right]+ \qquad \ldots \quad (B.10)$$

which is eqn. (2.108) ...

APPENDIX III

Sample Computer Programmes

PROGRAMME NO. 1

```
C  C  TEMPERATURE DISTRIBUTION IN A METAL SHEET OF FINITE THICKNESS Z
      DIMENSIONTH(25),BA(10),C(10),AK(10),BASQ(10),XBYA(10),F(25,10,10
      READ100,IA,IB,IK,ACC
      READ101,(TH(I),I=1,IA)
      READ101,(BA(I),I=1,IB)
      READ101,(XBYA(I),I=1,IK)
      PY=3.14159265
      PYSQ=PY*PY
      DO1J=1,IB
      BAJ=BA(J)
      C(J)=1./EXPF(BAJ)
      BASQ(J)=BAJ*BAJ
 1    AK(J)=BAJ/2.
      DO2I=1,IA
      DO2J=1,IB
      A=TH(I)/BA(J)
      B=2.*BA(J)/(PYSQ*(1.-C(J)))
      DO2K=1,IK
      SUM=0.
 501  SIGN=-1.
      N=1
 13   AN=N
      SQN=N*N
      G=SQN*PYSQ/BASQ(J)
      H=TH(I)*G
      Z=COSF(AN*PY*XBYA(K))
      IF(H-20.)11,11,12
 11   D=(1.-1./EXPF(H))*(1.-SIGN*C(J))/(SQN*(1.+G))*Z
      GOTO14
 12   D=(1.-SIGN*C(J))/(SQN*(1.+G))*Z
 14   SUM=SUM+D
      E=B*SUM
      F(I,J,K)=A+E
      IF(ABSF(B*D)-ACC)2,2,15
 15   SIGN=-SIGN
      N=N+1
      GOTO13
 2    CONTINUE
      IF(SENSE SWITCH 1)16,17
 16   DO3K=1,IK
      PUNCH200,XBYA(K)
      PUNCH201,(AK(J),J=1,IB)
      DO3I=1,IA
 3    PUNCH202,TH(I),(F(I,J,K),J=1,IB)
      IF(SENSE SWITCH 2)17,18
 17   DO4J=1,IB
      PUNCH203,AK(J)
```

```
      PUNCH204,(XBYA(K),K=1,IK)
      DO4I=1,IA
4     PUNCH202,TH(I),(F(I,J,K),K=1,IK)
18    STOP
100   FORMAT(3I2,E10.2)
101   FORMAT(7F10.0)
200   FORMAT(21X27HFUNCTION F(THETA,X) AT X/A=F5.2/)
201  0FORMAT(7H  THETA,4X4HA/S=F5.1,4X4HA/S=F5.1,4X4HA/S=F5.1,4X4HA/S=F5
     1.1,4X4HA/S=F5.1/)
202   FORMAT(F7.0,(5F13.4))
203   FORMAT(21X27HFUNCTION F(THETA,X) AT A/S=F5.1/)
204  0FORMAT(7H  THETA,4X4HX/A=F5.2,4X4HX/A=F5.2,4X4HX/A=F5.2,4X4HX/A=F5
     1.1,4X4HX/A=F5.2)
      END
```

SAMPLE INPUT

```
13 5 5   .10E-02
10.       20.       30.       50.       100.      500.      700.
900.      1000.     2000.     5000.     7000.     10000.
2.        5.        10.       20.       40.
0.        .25       .5        .75       1.
```

PROGRAMME NO. 2

```
C  C  TEMPERATURE DISTRIBUTION IN A CYLINDRICAL WORKPIECE Z
      DIMENSION T(20),RA(20),B(20),E(20)
      PY=3.14159265
      READ102,ACC
      PUNCH200
      READ100,IT,IRA,IN
      READ101,(T(I),I=1,IT)
      READ101,(RA(I),I=1,IRA)
      READ101,(B(N),N=1,IN)
      DO1I=1,IT
      DO1J=1,IRA
      RAJ=RA(J)
      A=2.*T(I)+RAJ*RAJ/2.-.25
      SS=0.
      DO3N=1,IN
      BN=B(N)
      BNSQ=BN*BN
      C=1./EXPF(T(I)*BNSQ)
      X=BN*RAJ
      L=1
12    IF(X-10.)21,22,22
```

```
21     P=X*.5
       Q=P
       SUM=1.
       FACT=1.
       SIGN=-1.
       M=1
10     AM=M
       FACT=AM*FACT
       W=Q/FACT
       TERM=SIGN*W*W
       SUM=SUM+TERM
       M=M+1
       Q=Q*P
       SIGN=-SIGN
       IF(ABSF(TERM)-ACC)11,11,10
22     AA=SQRTF(2./(PY*X))
       AB=COSF(X-PY*.25)*(1.-4.5/(64.*X*X)+225.*49.0/(24.*(8.*X)**4))
       AC=SINF(X-PY*.25)*(1./(8.*X)-225./(6.*(8.*X)**3))
       SUM=AA*(AB+AC)
11     GOTO(13,14),L
C      R=JO(BN*R/A)
13     R=SUM
       GOTO15
C      S=JO(BN)
14     S=SUM
       GOTO16
15     X=BN
       L=L+1
       GOTO12
16     D=C*R/(S*BNSQ)
3      SS=SS+D
20     E(J)=A-SS-SS
1      PUNCH201,T(I),RA(J),E(J)
       STOP
100    FORMAT(3I2)
101    FORMAT(7F10.0)
102    FORMAT(E11.4)
200    FORMAT(17X1HT,18X3HR/A,14X8HFUNCTION/)
201    FORMAT(F20.4,F20.2,F20.5)
       END
```

SAMPLE INPUT

```
 .1000E-04
 4 7 7
.0125     .025      .05       .1
0.        .25       .5        .75       .8        .9        1.
3.832     7.016     10.173    13.323    16.471    19.616    22.767
```

APPENDIX IV

Correction for finite current-depth

A right circular cylinder of radius a is heated with constant power input by induction. To find the temperature distribution in the cylinder after sufficient time has elapsed for all parts of the cylinder to be increasing in temperature at the same constant rate.

The power flow across any internal cylindrical surface of radius r is equal to the power generated inside this radius minus the power required to raise the temperature of the mass of metal inside this radius at the constant rate speed. This is expressed mathematically by-

$$-2\pi r\, k_c \frac{\partial U}{\partial r} = \int_0^r 2\pi r\, P\, dr - \pi r^2 \nu \cdot k_s \frac{\partial U}{\partial t} \qquad (C.1)$$

where P is the instantaneous volume power density generated at a point in the cylinder. To evaluate the constant rate of temperature rise in terms of average volume power density P_a,

$$\pi a^2 P_a = \pi a^2 \nu \cdot k_s \frac{\partial U}{\partial t}$$

and $$\frac{\partial U}{\partial t} = \frac{P_a}{\nu \cdot k_s}$$

Substituting this value in eqn. (C.1),

$$2\pi r\, k_c \frac{\partial U}{\partial r} = \pi r^2 . P_a - \int_0^r 2\pi r \; P dr.$$

or $$\frac{\partial U}{\partial r} = \frac{P_a . r}{2\, k_c} - \frac{1}{r\, k_c} \int_0^r r\, P dr \qquad (C.2)$$

The power density P in terms of electrical quantities is

$$P = H\,(\mathrm{ber}'^2\, k_1 r + \mathrm{bei}'^2\, k_1 r)$$

where, $k_1 = \frac{(2)^{\frac{1}{2}}}{\delta}$

Substituting this in eqn. (C.2),

$$\frac{\partial U}{\partial r} = \frac{P_a . r}{2k_c} - \frac{M}{r . k_c} \int_0^r r(\text{ber}'^2\, k_1 r + \text{bei}'^2\, k_1 r)\, dr.$$

But,

$$\int_0^r r(\text{ber}'^2\, k_1 r + \text{bei}'^2\, k_1 r)\, dr.$$

$$= \frac{r}{k_1}(\text{ber}\, k_1\, r\, \text{ber}'\, k_1 r + \text{bei}\, k_1 r\, \text{bei}' k_1\, r)$$

$$\therefore\ \frac{\partial U}{\partial r} = \frac{P_a . r}{2k_c} - \frac{M}{k_1 k_c} . (\text{ber}\, k_1 r\, \text{ber}' k_1\, r + \text{bei}\, k_1 r\, \text{bei}' k_1 r) \quad \ldots \quad (C.3)$$

Integrating eqn.(C.3) to find U_r, we get,

$$U_r = \frac{P_a r^2}{4k_c} - \frac{M}{2k_1^2 k_c} . (\text{ber}^2\, k_1 r + \text{bei}^2 k_1 r) + C_1 \quad (C.4)$$

when $r = 0$, $U_r = U_c$, so the value of C_1 is found to be-

$$C_1 = \frac{M}{2k_1^2\, k_c} + U_c \quad \ldots \quad (C.5)$$

Hence from eqns.(C.4) and (C.5),

$$U_r - U_c = \frac{P_a r^2}{4k_c} - \frac{M}{2k_1^2 k_c} . (\text{ber}^2 k_1 r + \text{bei}^2\, k_1 r - 1) \quad \ldots \quad (C.6)$$

now we have,

$$P_a = \frac{1}{\pi a^2} \int_0^a 2\, \pi r .\, P dr. \quad \ldots \quad (C.7)$$

$$\therefore\ P_a = \frac{2M}{a^2} \int_0^a r(\text{ber}'^1\, k_1 r + \text{bei}'^2\, k_1 r)\, dr. \quad ($$

$$\therefore\ P_a = \frac{2M}{k_1 a} (\text{ber}\, k_1 a\, \text{ber}'\, k_1 a + \text{bei}\, k_1 a\, \text{bei}'\, k_1 a)$$

$$\therefore M = \frac{P_a . k_1 a}{2(\text{ber } k_1 a \text{ ber}' k_1 a + \text{bei } k_1 a \text{ bei}' \ k_1 a\ )} \qquad \ldots \qquad (C.8)$$

Putting the value of M from eqn. (C.8) in (C.6)-

$$U_r - U_c = \frac{P_a r^2}{4k_c} - \frac{P_a . k_1 a}{2Z(k_1 a)} . \frac{1}{2k_1^2\ k_c} .(\text{ber}^2 k_1 r + \text{bei}^2 k_1 r - 1)$$

$$= \frac{P_a . a^2}{4k_c} (\frac{r^2}{a^2} - \frac{r^2}{a^2} . \frac{k_1 a}{k_1^2 r^2} . \frac{X\ (k_1 r)\ -1}{Z\ (k_1 a)}) \qquad \ldots \qquad (C.9)$$

$$\text{where } X(k_1 r) = \text{ber}^2\ k_1 r + \text{bei}^2\ k_1 r \qquad \ldots \qquad (C.10)$$

$$\text{and } Z(k_1 a) = \text{ber } k_1 a \text{ ber}' k_1 a + \text{bei } k_1 a \text{ bei}'\ k_1 a \qquad (C.11)$$

Putting $k_2 = k_1 a$ in eqn. (C.9),

$$U_r - U_c = \frac{P_a - a^2}{4k_c} \left[\frac{r^2}{a^2} - \frac{1}{k_2} \quad \frac{X(k_2 . \frac{r}{a}\ ) -1}{Z(k_2)} \right] \qquad \ldots \qquad (C.12)$$

Now the volume power density P_a can be replaced by equivalent surface power density-

$$P_a = \frac{2P_o}{a} \qquad \ldots \qquad (C.13)$$

Then from eqn. (C.12) we get,

$$U_r - U_c = \frac{P_o a}{2k_c} \left[\frac{r^2}{a^2} - \frac{1}{k_2} . \quad \frac{X(k_2\ r/a) - 1}{Z(k_2)} \right] \qquad (C.14)$$

which is the same as eqn. (3.27)

APPENDIX V

A mean of finding the complete solution to the differential equation (4.8)

Rewriting eqn. (4.8) we have-

$$\frac{\partial^2 I}{\partial x^2} - \gamma^2 I = - (G+pC)\ E(x) \qquad \ldots \qquad (D.1)$$

The general solution of the eqn. (D.1) is-

$$I = e^{\gamma x} F_1(x) - e^{-\gamma x} F_2(x) + A\ e^{\gamma x} + B\ e^{-\gamma x} \qquad \ldots \qquad (D.2)$$

where A and B are arbitrary constants and $F_1(x)$ and $F_2(x)$ are indefinite integrals defined below-

$$F_1(x) = - \frac{(G+pC)}{2\gamma} \int e^{-\gamma x} E(x)\ dx \qquad \ldots \qquad (D.3)$$

$$F_2(x) = - \frac{(G+pC)}{2\gamma} \int e^{\gamma x} E(x)\ dx \qquad \ldots \qquad (D.4)$$

The constants A and B may be evaluated in terms of the terminal conditions.

At x = 0, the voltage on the line is-

$$e_{x=0} = - Z_o\ I_{x=0} \qquad \ldots \qquad (D.5)$$

Combining (D.5) with (4.4), we have,

$$\left(\frac{\partial I}{\partial x}\right)_{x=0} = (G+pC)\ Z_o\ I_{x=0} \qquad \ldots \qquad (D.6)$$

Similarly,

$$\left(\frac{\partial I}{\partial x}\right)_{x=a} = - (G+pC)\ Z_a\ I_{x=a} \qquad \ldots \qquad (D.7)$$

now from eqn. (D.2) we have the following relations-

$$\frac{\partial I}{\partial x} = e^{\gamma x}.F_1'(x) + F_1(x)\gamma\ e^{\gamma x} - e^{-\gamma x}.F_2'(x) + \gamma\ e^{-\gamma x} F_2(x) + \gamma A\ e^{\gamma x} - \gamma B\ e^{-\gamma x} \qquad \ldots \qquad (D.8)$$

and
$$I_{x=0} = F_1(0) - F_2(0) + A + B \qquad \ldots \qquad (D.9)$$

Substituting eqn. (D.8) and (D.9) in dqn.(D.6) we have-

$$(G+pC)\ Z_o \left[F_1(0) - F_2(0) + A + B \right]$$

$$= \gamma F_1(0) + \gamma F_2(0) + \gamma A - \gamma B$$

or
$$A \left[\gamma - (G+pC)\ Z_o\right] - B \left[\gamma + (G+pC)\ Z_o\right]$$

$$= - F_1(0) \left[\gamma - (G+pC) Z_o\right] - F_2(0) \left[\gamma + (G+pC)\ Z_o\right] \qquad \ldots \qquad (D.10)$$

Similarly from eqn. (D.7) we have,

$$\left[\gamma + Z_a (G+pC)\right] A e^{\gamma a} - \left[\gamma - Z_a (G+pC)\right] B\ e^{-\gamma a}$$

$$= - \left[\gamma + Z_a (G+pC)\right] F_1(a)\ e^{\gamma a} - \left[\gamma - Z_a (G+pC)\right] F_2(a)\ e^{-\gamma a} \qquad \ldots \qquad (D.11)$$

Also it is noted that-

$$F_2(a) - F_2(0) = - \frac{(G+pC)}{2\gamma} \int_{v=0}^{v=a} E(v)\ e^{\gamma v}\ dv \qquad (D.12)$$

and

$$F_2(x) - F_2(0) = - \frac{(G+pC)}{2\gamma} \int_{v=0}^{v=x} E(v)\ e^{\gamma v}\ dv \qquad (D.13)$$

Similar relations in $F_1(x)$ hold, further more-

$$F_2(a) - F_2(0) + F_1(a) - F_1(0)$$

$$= - \frac{(G+pC)}{\gamma} \int_{v=0}^{v=a} E(v)\ \cosh\ (\gamma v)\ dv \qquad (D.14)$$

After solving the two simultaneous equations (D.10) and (D.11), we get the value of constants A and B. Putting these values in the eqn. (D.2) we get the complete solution of the current equation, and after utilising the relations given in equations (D.12), (D.13) and (D.14) and making some manipulations we see that the equation (D.2) becomes the solution shown by equation (4.10).

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