STABILITY OF PLANE COUETTE FLOW OF POWERLAW FLUID PAST A NEO-HOOKEAN DEFORMABLE SOLID

A DISSERTATION

Submitted in partial fulfillment of the requirements for the award of the degree of

MASTER OF TECHNOLOGY

in

CHEMICAL ENGINEERING

(With Specialization in Industrial Safety & Hazards Management)

By

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JUNE - 2013

DECLARATION

I hereby declare that the work being presented by me in this thesis entitled **"Stability of plane Couette flow of power-law fluid past a Neo-Hookean deformable solid"** in partial fulfillment of the requirements for the award of the degree of Master of Technology in Chemical Engineering with specialization in "Industrial Safety and Hazards Management" submitted to Department of Chemical Engineering, Indian Institute of Technology, Roorkee; is an authentic record of my original work carried out under the guidance of Dr. Gaurav, Assistant Professor, Department of Chemical Engineering, IIT Roorkee. The matter embodied in this thesis has not been submitted for the award of any other degree.

Date: 14.06.13 P Harish

Place: IIT Roorkee

CERTIFICATE

This is to certify that P Harish has completed the thesis entitled **"Stability of plane Couette flow of power-law fluid past a Neo-Hookean deformable solid"** under my supervision. This is to certify that the above statement made by the candidate is correct to the best of my knowledge.

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ACKNOWLEDGEMENT

It is a pleasure to thank the many people who made this thesis possible. Firstly, I thank the **Almighty** for everything he has given me.

It is difficult to overstate my gratitude to my supervisor, **Dr. Gaurav**. With his enthusiasm, his inspiration, and his great efforts to explain things clearly and simply, he helped to make the work interesting. Throughout my thesis-writing period, he provided encouragement, sound advice, good teaching, and lots of good ideas. I would have been lost without him. I am and shall be ever grateful to him for all the kindness and affection he had bestowed on me.

I am very thankful to my various course instructors for providing me good knowledge of various subjects. I am indebted to my many student colleagues, especially **Atul, Shubhro** and **Amit** for providing a stimulating and fun environment in which to learn and grow. A special note of thanks goes to **Mahender** for always being there to lend me a helping hand whenever I needed and inciting me to do my very best. This work would have been incomplete, if not for the inputs of these four people. I gratefully acknowledge IIT Roorkee for providing me an excellent academic environment and facilities to carry out research.

Lastly, and most importantly, I wish to thank my parents and my family, for supporting, teaching and loving me. To them I dedicate this thesis.

P Harish

ABSTRACT

The stability of Couette flow of power-law fluid of thickness R past Neo-Hookean deformable solid of thickness HR subjected to shear flow is considered in this work. Power-law fluid is chosen as it is the simplest model of fluid which can show the effects of shear thickening and shear thinning behavior. Whereas Neo Hookean solid, which is a nonlinear constitutive model accurately captures the behavior of flow as it leads to values of critical shear rate which are smaller than those obtained by using the linear viscoelastic solid model. Four key dimensionless parameters, i.e. γ (Imposed shear rate), n (power-law index), T (interfacial tension) & H (thickness ratio) characterizes the problem. Linear stability analysis is performed to find the stability of the system. Shear flow of the fluid due to Couette flow tends to destabilize the surface fluctuations. Various diagrams have been plotted between growth rates as a function of wavenumber showing the study of parameters how they affect the flow. For large values of H, i.e. solid to fluid thickness ratio, critical shear rate goes on decreasing and shear thickening fluids has more stabilizing effect in comparison with shear thinning fluids keeping all other governing parameters constant. The results obtained are potentially of great interest for enhancing mixing in microfluidic devices.

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CHAPTER 1

INTRODUCTION

1.1 Motivation

Many recent experimental works (XII) and studies show that fluid flow past deformable solid is qualitatively different from fluid flow past rigid surfaces. Flow of fluids past soft or deformable solids induces oscillations in the surface and these oscillations in turn changes the dynamic characteristics of flow. They deform under the action of tangential and normal stresses at the interface. If these disturbances grow with time they can change the pattern of fluid flow, creating complicated time dependent flows. Flow past flexible surfaces is encountered in a wide variety of applications such as in biological systems in which flows of fluids such as blood occurs in arteries and veins and are generally made of soft tissues. If the fluid is power-law i.e. non-Newtonian, then it exhibits properties such as shear thickening and shear thinning. The stresses at the fluid solid interface depend on this additional factor which changes the growth or decay of interfacial disturbances.

There are several applications for which fluid–solid interfacial instabilities are relevant. Fluids in microfluidic devices may be shear-thinning , such as blood, or viscoelastic, such as DNA solutions, and are often difficult to mix due to the associated length scales and flow rates. However, one may be able to design a microfluidic device with flow channels made of, or coated with, a deformable solid and induce the instability between the flowing fluid and the deformable solid to create complicated flow patterns, thereby enhancing mixing.

Instabilities can be either desirable or undesirable. For example, mass transfer and heat transfer operations are effective when the interface between the phases is wavy as it leads to and consequently better mixing. On other hand, in case of photographic film formation, surfacing of the one layer on other should be done steadily since the optical properties of the resulting film would be degraded if instability of interface prevails. This shows that flow instabilities can be desirable or undesirable depending upon the objective of a particular process and a strategy is frequently required for inducing or suppressing the flow instabilities.

Previous studies in this field provide many details about the stability of fluid flows past soft polymer gels or deformable solids. Newtonian fluids have been largely considered to this point. Many potential applications of these flows, however, may use shear-thinning or shear thickening fluids. In the creeping flow limits stability of power-law fluid past deformable solid was studied by Roberts and Kumar (2006). To address the issue of how shear-thickening and shear thinning affects the elastohydrodynamic instability of the interface between a flowing fluid and a deformable solid, we focus our study on the case of plane Couette flow of a power-law fluid past an incompressible and impermeable neo-Hookean solid. Fig 1.1 shows the flow configuration considered in this study.

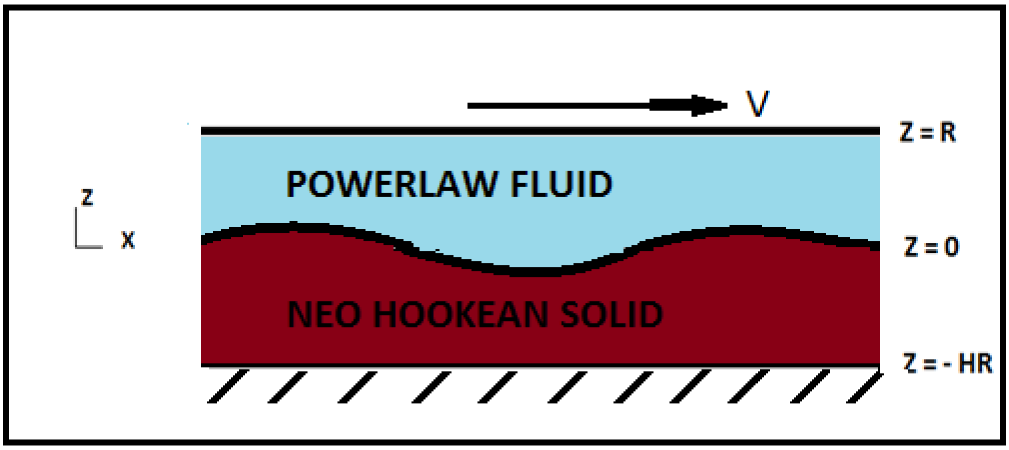


Fig 1.1: Problem Geometry showing the deformation of solid due to Couette flow

1.2 Couette Flow

It is a type of flow in which the displacement in the fluid is caused due to the movement of the plate or wall with which fluid is in contact. Due to no slip condition, the fluid which is in contact with the moving plate also moves with the same velocity. As seen in fig 1.1 the bottom plate with which the neo-Hookean solid is attached remains fixed and the top plate with which the power-law fluid is in contact is moving with velocity V in x direction with respect to the fixed plate. Hence shear stress sets up the flow field. This type of flow is known as Couette flow.

1.3 Power-law fluid

It is clear from the previous discussions that the objective of this thesis is to determine the affects of shear thickening and shear thinning for the stability of flow past deformable solid. Since power-law fluid model is the simplest constitutive model equation that captures a shear rate dependent viscosity so we use power-law model to study the effects of shear dependent viscosity.

A power-law fluid is a type of generalized Newtonian fluid for which the shear stress is given by

$$\tau = k \left(\frac{\partial u}{\partial y}\right)^{n}$$

Where τ is the shear stress of the fluid, k is the flow consistency constant, $\frac{\partial u}{\partial y}$ is the velocity gradient or the shear rate which is perpendicular to the plane of shear and n is the power-law index.

The quantity

$$\mu_{eff} = k \left(\frac{\partial u}{\partial y}\right)^{n-1}$$

is the effective viscosity which is a function of the shear rate or the velocity gradient. The above model is also known as the Ostwald de Waele power law model. This relationship is used because of its simplicity, but it approximately describes the behaviour of a real non-Newtonian fluid. This point can be justified by taking an example, if n is kept less than one, then the power law model predicts that the effective viscosity would decrease with increasing shear rate indefinitely, requiring a fluid with zero viscosity as the shear rate approaches infinity, and infinite viscosity when the fluid is at rest. This is in contradiction because a real fluid always possesses a minimum and a maximum effective viscosity whose value can’t be zero or infinity. Therefore, the power law model is only used to describe the fluid behaviour depending on the range of shear rates at which the power-law coefficient n is properly fitted. Other than power-law model there are certain other models which also describe the flow characteristics and behaviour of shear thickening and shear thinning, but the problem with other models is they do so at the expense of simplicity, so we are using power law model to describe fluid behaviour.

Power-law fluids are generally subdivided into three different types of fluids based on the value n i.e. power-law index.

n	Type of fluid
< 1	Pseudo plastic, shear thinning fluids
= 1	Newtonian fluids
> 1	Dilatants, shear thickening fluids

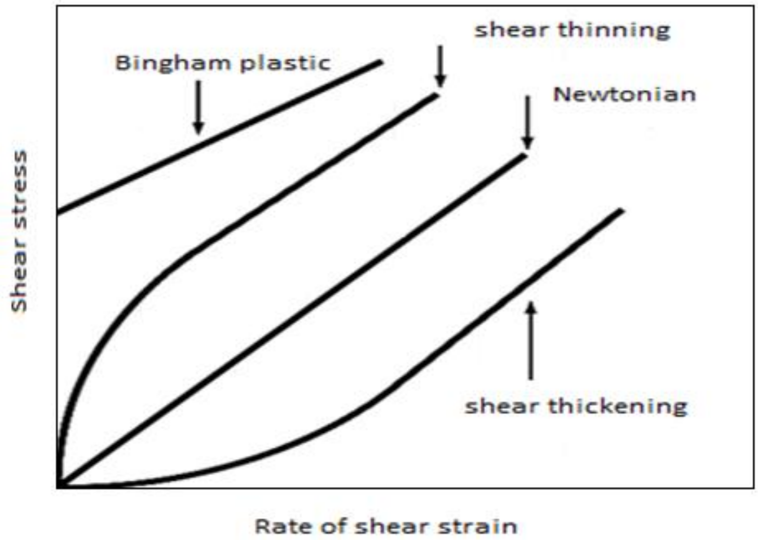


Fig 1.2 Types of fluid

1.3.1 Pseudoplastic

Pseudoplastic or shear-thinning fluids are the one which have a lower apparent viscosity at higher shear rates, and these are generally the solutions of polymeric substance. It is usually assumed that the large molecular chains tumble at random and affects huge volume of fluid under low shear, but they gradually align themselves in the direction of increasing shear and produce less resistance to the flow.

Shear thinning property is found in many complex solutions such as blood, ketchup, paint, lava, nail polish, and whipped cream. This property is a common property of many polymeric solutions and molten polymers.

1.3.2 Dilatant Fluid

Dilatant or shear-thickening fluids are the one for which apparent viscosity increases with increasing shear rates. These are rarely seen, uncooked paste of cornstarch and water is the example of shear thickening or dilatant fluid. With high shear stress in the paste water is squeezed out from the starch molecules, these are able to interact more strongly. Another example of dilatant fluid is sand completely soaked with water. It is easily seen that when we walk on wet sand a dry area appears below our foot.

1.4 Neo-Hookean Solid

The neo-Hookean solid model is the generalisation of Hooke's law and it is also valid for small and finite deformation-gradients. The stress-strain relationship for a neo-Hookean deformable solid is nonlinear.

As we are dealing in this thesis with the issue of changes in stability characteristics of fluid flow past deformable solid, so it's very important to appropriately choose a model which can accurately captures the characteristics and behaviour of deformable solids. Many of the research papers in the field of flow past deformable solid have employed a linear constitutive relationship to describe the characteristics of deformable solid layer. But the problem of using the linearised elasticity is that it is only applicable when the deformation gradients in the solid layer are small as compared to unity. If the deformation gradients are large, then this linear elastic model can't be used to predict the dynamic behaviour of deformable solid and it is quite necessary to use a frame invariant model which takes into account of the nonlinearities between stress and strain in the soft solid. Since the stress strain relationship for neo-Hookean deformable solid is nonlinear, as a consequence of this neo-Hookean model exhibits a first normal stress difference. On the other hand, the first normal stress difference is zero for linear viscoelastic model.

Neo-Hookean solid model is one of the simplest model which gives nonlinear constitute relation for elastic solids and accurately captures the behaviour of deformable solids in the real system like rubber reasonably well. In the present thesis, we use neo-Hookean solid model to represent the dynamics of deformable solid layer.

CHAPTER - 2

LITERATURE REVIEW

Yih (1967) and Hickox (1971) first studied the stability of Couette flow of two Newtonian fluids with fluids having different viscosities. They performed a longwave asymptotic analysis and concluded that viscosity stratification is sufficient to cause the interfacial instability, at any non zero Reynolds number.

Silberberberg and co-workers (1987) have studied the flow of Newtonian fluid through tubes with gel coated walls and concluded that the pressure drop required for maintaining the fluid flow is higher than what required for rigid walled tubes. This result is valid if Reynolds number is much below the transition Reynolds number. This increased pressure drop is a result of increase in dissipation of energy due to the oscillations at the interface.

Fredrickson and Kumaran (1994) have studied the Couette flow of Newtonian fluid past linear viscoelastic solid and shown that as the work is done at the interface by the mean flow, instabilities is caused due to this work done or the energy transfer from mean flow to fluctuations. They concluded that the interfacial waves become unstable beyond a critical dimensionless strain.

Gkanis and kumar (2003) investigated for Couette flow of a Newtonian fluid past neo-Hookean deformable solid. They carved out the linear stability analysis to show how Newtonian fluid flow over a Neo-Hookean solid can become unstable due to the fact that waves may propagate along the solid–fluid interface. They have assumed inertial effects to be negligible as they were working creeping flow limit. In the base state solution of the solid they found that Neo-Hookean solid exihibits a first normal stress difference, and this leads to instability behaviour that is significantly different from what is observed in using a linear constitutive equation. This highlighted the importance of using a nonlinear constitutive equation for solid layer. While neglecting the interfacial tension, the first normal stress difference which the Neo-Hookean solid is exhibiting gives rise to a new short-wave instability. They concluded that for thin solids, high-wavenumber modes is getting unstable first for a wider range of wavenumber with the increasing strain imposed on the system, while for thick solids, it was shown that a small range of first order wavenumbers becomes unstable first.

They had compared the results by using the linear elastic model and neo-Hookean model and found that Neo-Hookean model leads to larger values of the critical wavenumber and smaller values of the critical imposed strain, but this difference rapidly minimizes as the solid thickness goes on increasing. The result of this study highlights the importance of using nonlinear constitutive model when modelling for elastohydrodynamic instabilities accounting for large displacement gradients.

Gkanis & Kumar (2005) have studied the "effect of pressure gradients on the stability of creeping flows of Newtonian fluids in channels lined with an incompressible and impermeable neo-Hookean material". They concluded similarly as Gkanis and kumar (2003) that it is necessary to account for non linear rheological behaviour in the solid layer. Further, they pointed out that the stability characteristics of pressure driven flow past a neo-Hookean solid is significantly different from that of Couette flow past a neo-Hookean solid.

Gaurav and Shankar (2007) have studied for stability of Newtonian liquid flow down an inclined plane lined with a deformable solid layer. They carried the analysis for both linear viscoelastic and neo-Hookean solid at zero and finite Reynolds number. At finite Reynolds number, they concluded that for both the solid models, free-surface instability in flow down a rigid plane can be suppressed at all wavelengths by the deformability of the solid layer. They had shown that the neutral curves which were associated with instability suppression were found to be identical for both linear viscoelastic and neo-Hookean solid models. It was concluded that a soft elastomeric coatings offers a passive route to control and suppress the interfacial instabilities.

All the work mentioned above used Newtonian fluid. There is limited work done for the case of non Newtonian fluid. If the fluid is non Newtonian, it will show the properties of shear thickening and shear thinning. These additional factors may also amplify or suppress the surface instabilities. So in thesis we are interested in knowing the effects of these additional parameters on the stability of the system. Below are the some works which have been done using the non Newtonian fluid.

Khomami (1990) studied the "interfacial stability and deformation of two stratified power-law fluids in plane Poisseuille flow". It was observed that dependence of viscosity on shear rate has a huge affect on the interfacial stability regime in comparison with the effective viscosity change. They also concluded that effect of shear thinning viscosity mainly shows the effects in the less viscous fluid if the viscosity ratio is less than one, while if viscosity ratio is greater than one then shear thinning in both the fluid layers affects the stability regime, this effect is generally shown at small depth ratios.

Waters (1983) and Waters and Keeley (1987) studied the effects of shear thinning only and combined effect of shear thinning and elasticity. They had done the analysis using longwave asymptotics method. They concluded that the presence of shear thinning in the fluids has significant effect in the stability of the system and while considering elasticity they found that it can stabilize or destabilize the system in the presence of viscosity stratification.

Our objective in this thesis is to find the effects of shear thickening and shear thinning on the stability of plane Couette flow past neo-Hookean deformable solid and to find all the other parameters which can affect the stability of the system. We start our work with the problem formulation in which we write the governing equations for both fluid and solid, and then we carried out further to find out the base state solution. In chapter 4 we will be linearising our equations using the linear stability analysis. In chapter 5 we will be using spectral collocation method so as to solve our linearised equation in Matlab.

CHAPTER - 3

PROBLEM FORMULATION

3.1. System configuration:

The system we consider consists of a neo- Hookean solid of thickness HR fixed onto a surface at $z = -HR$ and a layer of power-law fluid of thickness R in the region $0 < z < R$ as shown in the figure 3.1. The fluid is bounded by a solid plate at $z = R$ which moves at a constant velocity V in the x direction. Small perturbations to the interface and other dynamical variables are imposed on the base state variables and we study in this thesis the growth or decay of these perturbations. If the perturbations grow with time, we call it as an unstable configuration while if they decay with time it is referred as stable configuration.

Assumptions :

- Incompressible power-law fluid
- Impermeable and incompressible Neo-Hookean solid
- Two dimensional system
- Top plate is rigid and is moving steadily with velocity V in the x direction
- Densities of power-law fluid and Neo-Hookean solid are assumed to be identical.

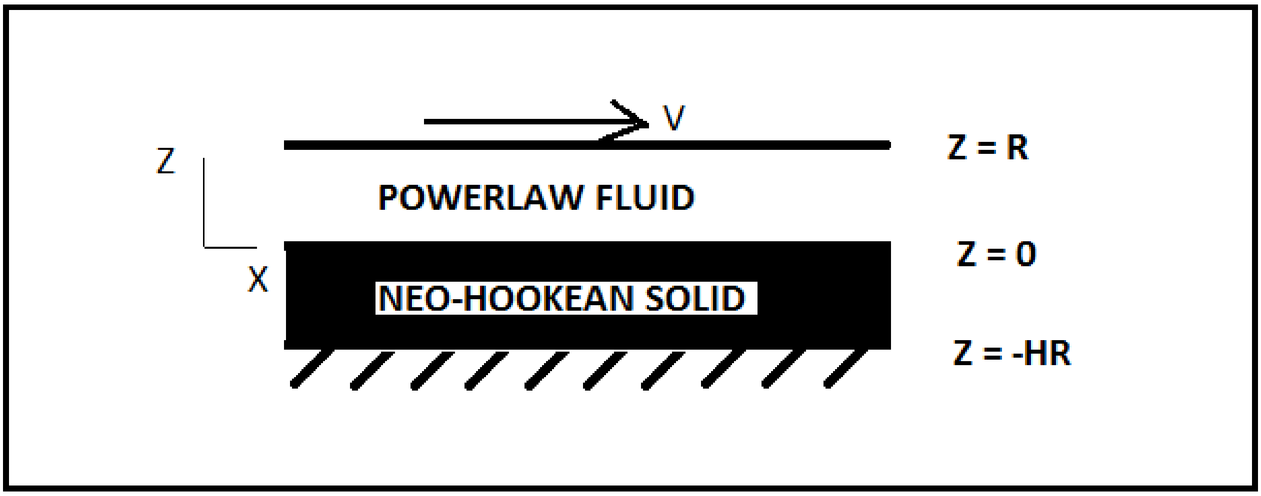


Fig 3.1: System configuration

3.2. Governing equations for power-law fluid

1) Conservation of mass

$$\nabla^* . \; v^* = 0$$

Superscript $[..]^*$ shows the dimensional variables

$$\frac{\partial v_x^*}{\partial x^*} + \frac{\partial v_z^*}{\partial z^*} = 0 \tag{1}$$

2) Conservation of momentum (Navier Stokes equation)

x- momentum

$$\rho\left(\frac{\partial v_x^*}{\partial t^*} + v_x^* \frac{\partial v_x^*}{\partial x^*} + v_z^* \frac{\partial v_x^*}{\partial z^*}\right) = \frac{\partial \tau^*{}_{xx}}{\partial x^*} + \frac{\partial \tau^*{}_{xz}}{\partial z^*} \tag{2}$$

z- momentum

$$\rho\left(\frac{\partial v_z^*}{\partial t^*} + v_x^* \frac{\partial v_z^*}{\partial x^*} + v_z^* \frac{\partial v_z^*}{\partial z^*}\right) = \frac{\partial \tau^*{}_{xz}}{\partial x^*} + \frac{\partial \tau^*{}_{zz}}{\partial z^*} \tag{3}$$

Where dimensional stress τ^* is

$$\tau^* = -p_f^* \, \delta + m^* \left(\frac{1}{2}\, \pi^*\right)^{(n-1)/2} \gamma^*$$

Rate of strain tensor γ^*

$$\gamma^* = \nabla v^* + (\nabla v^*)^T$$

Second invariant of the rate of strain tensor π^*

$$\pi^* = -\frac{1}{2}\,[(tr\gamma^*)^2 - tr(\gamma^*)^2]$$

Where ,

p_f^* = fluid pressure

v^* = fluid velocity

δ = identity tensor

$m^* \left(\frac{1}{2}\pi^*\right)^{(n-1)/2}\gamma^*$ = is apparent viscosity and it is replaced by μ as in Newtonian fluid.

m^* = is the consistency constant its value is dependent on power-law index n , for keeping the scope of this work reasonable we set the value of 'm' is set as unity.

- for shear thinning fluids, n < 1
- for shear thickening fluids, n > 1

$$\boldsymbol{\gamma}^* = \nabla \boldsymbol{v}^* + (\nabla \boldsymbol{v}^*)^{\boldsymbol{T}}$$

$$\gamma^* = \begin{bmatrix} \frac{\partial v_x^*}{\partial x^*} & \frac{\partial v_z^*}{\partial x^*} \\ \frac{\partial v_x^*}{\partial z^*} & \frac{\partial v_z^*}{\partial z^*} \end{bmatrix} + \begin{bmatrix} \frac{\partial v_x^*}{\partial x^*} & \frac{\partial v_z^*}{\partial x^*} \\ \frac{\partial v_x^*}{\partial z^*} & \frac{\partial v_z^*}{\partial z^*} \end{bmatrix}$$

$$\boldsymbol{\gamma}^* = \begin{bmatrix} 2\frac{\partial v_x^*}{\partial x^*} & \frac{\partial v_z^*}{\partial x^*} + \frac{\partial v_x^*}{\partial z^*} \\ \frac{\partial v_x^*}{\partial z^*} + \frac{\partial v_z^*}{\partial x^*} & 2\frac{\partial v_z^*}{\partial z^*} \end{bmatrix} \tag{4}$$

Therefore we get

$$\text{tr}\gamma^* = 2\left(\frac{\partial v_x^*}{\partial x^*} + \frac{\partial v_z^*}{\partial z^*}\right)$$

Where tr is the trace of the matrix, which is sum of the diagonal elements. From conservation of mass, equation no. (1)

We get, $\text{tr}\,\gamma^* = 0$

$$(\textbf{tr}\boldsymbol{\gamma}^*)^2 = \mathbf{0} \tag{5}$$

Now,

$$\gamma^{*2} = \begin{bmatrix} 2\frac{\partial v_x^*}{\partial x^*} & \frac{\partial v_z^*}{\partial x^*} + \frac{\partial v_x^*}{\partial z^*} \\ \frac{\partial v_x^*}{\partial z^*} + \frac{\partial v_z^*}{\partial x^*} & 2\frac{\partial v_z^*}{\partial z^*} \end{bmatrix} \begin{bmatrix} 2\frac{\partial v_x^*}{\partial x^*} & \frac{\partial v_z^*}{\partial x^*} + \frac{\partial v_x^*}{\partial z^*} \\ \frac{\partial v_x^*}{\partial z^*} + \frac{\partial v_z^*}{\partial x^*} & 2\frac{\partial v_z^*}{\partial z^*} \end{bmatrix}$$

$$\gamma^{*2} = \begin{bmatrix} \left(2\frac{\partial v_x^*}{\partial x^*}\right)\left(2\frac{\partial v_x^*}{\partial x^*}\right) + \left(\frac{\partial v_z^*}{\partial x^*} + \frac{\partial v_x^*}{\partial z^*}\right)^2 & 2\frac{\partial v_x^*}{\partial x^*}\left(\frac{\partial v_z^*}{\partial x^*} + \frac{\partial v_x^*}{\partial z^*}\right) + 2\frac{\partial v_z^*}{\partial z^*}\left(\frac{\partial v_z^*}{\partial x^*} + \frac{\partial v_x^*}{\partial z^*}\right) \\ 2\frac{\partial v_x^*}{\partial x^*}\left(\frac{\partial v_z^*}{\partial x^*} + \frac{\partial v_x^*}{\partial z^*}\right) + 2\frac{\partial v_z^*}{\partial z^*}\left(\frac{\partial v_z^*}{\partial x^*} + \frac{\partial v_x^*}{\partial z^*}\right) & \left(2\frac{\partial v_z^*}{\partial z^*}\right)\left(2\frac{\partial v_z^*}{\partial z^*}\right) + \left(\frac{\partial v_z^*}{\partial x^*} + \frac{\partial v_x^*}{\partial z^*}\right)^2 \end{bmatrix}$$

$$\gamma^{*2} = \begin{bmatrix} 4\left(\frac{\partial v_x^*}{\partial x^*}\right)^2 + \left(\frac{\partial v_z^*}{\partial x^*} + \frac{\partial v_x^*}{\partial z^*}\right)^2 & 2\frac{\partial v_x^*}{\partial x^*}\left(\frac{\partial v_z^*}{\partial x^*} + \frac{\partial v_x^*}{\partial z^*}\right) + 2\frac{\partial v_z^*}{\partial z^*}\left(\frac{\partial v_z^*}{\partial x^*} + \frac{\partial v_x^*}{\partial z^*}\right) \\ 2\frac{\partial v_x^*}{\partial x^*}\left(\frac{\partial v_z^*}{\partial x^*} + \frac{\partial v_x^*}{\partial z^*}\right) + 2\frac{\partial v_z^*}{\partial z^*}\left(\frac{\partial v_z^*}{\partial x^*} + \frac{\partial v_x^*}{\partial z^*}\right) & 4\left(\frac{\partial v_z^*}{\partial z^*}\right)^2 + \left(\frac{\partial v_z^*}{\partial x^*} + \frac{\partial v_x^*}{\partial z^*}\right)^2 \end{bmatrix}$$

$$\mathrm{tr}\gamma^{*2} = 4\left(\frac{\partial v_x^*}{\partial x^*}\right)^2 + \left(\frac{\partial v_z^*}{\partial x^*} + \frac{\partial v_x^*}{\partial z^*}\right)^2 + \left(\frac{\partial v_z^*}{\partial x^*} + \frac{\partial v_x^*}{\partial z^*}\right)^2 + 4\left(\frac{\partial v_z^*}{\partial z^*}\right)^2$$

$$\mathrm{tr}\gamma^{*2} = 4\left(\left(\frac{\partial v_x^*}{\partial x^*}\right)^2 + \left(\frac{\partial v_z^*}{\partial z^*}\right)^2\right) + 2\left(\frac{\partial v_z^*}{\partial x^*} + \frac{\partial v_x^*}{\partial z^*}\right)^2$$

we will show in the next chapter, chapter no. (4) when we linearised all equations that we neglect the higher order perturbation terms, which would simply mean we are interested in state of vey small disturbances.

Hence we get $$\mathbf{tr}\boldsymbol{\gamma}^{*2} = \mathbf{2}\left(\frac{\partial \mathbf{v}_z^*}{\partial \mathbf{x}^*} + \frac{\partial \mathbf{v}_x^*}{\partial \mathbf{z}^*}\right)^{\mathbf{2}} \tag{6}$$

$\boldsymbol{\pi}^* = -\frac{\mathbf{1}}{\mathbf{2}}\,[(\boldsymbol{tr\gamma}^*)^{\mathbf{2}} - \boldsymbol{tr}(\boldsymbol{\gamma}^*)^{\mathbf{2}}]$

from equation no (5) and (6) we get

$$\pi^* = -\frac{1}{2}\left[0 - 2\left(\frac{\partial v_z^*}{\partial x^*} + \frac{\partial v_x^*}{\partial z^*}\right)^2\right]$$

$$\pi^* = -\frac{1}{2}\left[-2\left(\frac{\partial v_z^*}{\partial x^*} + \frac{\partial v_x^*}{\partial z^*}\right)^2\right]$$

$$\boldsymbol{\pi}^* = \left(\frac{\partial \mathbf{v}_z^*}{\partial \mathbf{x}^*} + \frac{\partial \mathbf{v}_x^*}{\partial \mathbf{z}^*}\right)^{\mathbf{2}} \tag{7}$$

$\boldsymbol{\tau}^* = -\boldsymbol{p_f}^*\boldsymbol{\delta} + \boldsymbol{m}^*\left(\frac{\mathbf{1}}{\mathbf{2}}\boldsymbol{\pi}^*\right)^{(\boldsymbol{n}-\mathbf{1})/\mathbf{2}}\boldsymbol{\gamma}^*$

from equation no. (4) and (7)

$$\tau^* = -p_f{}^* \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + m^* \left(\frac{1}{2} \left(\frac{\partial v_x^*}{\partial z^*} + \frac{\partial v_z^*}{\partial x^*} \right)^2 \right)^{(n-1)/2} \begin{bmatrix} 2\frac{\partial v_x^*}{\partial x^*} & \frac{\partial v_z^*}{\partial x^*} + \frac{\partial v_x^*}{\partial z^*} \\ \frac{\partial v_x^*}{\partial z^*} + \frac{\partial v_z^*}{\partial x^*} & 2\frac{\partial v_z^*}{\partial z^*} \end{bmatrix}$$

$$\tau^* = -p_f{}^* \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + m^* \frac{1}{2}^{(n-1)/2} \left(\frac{\partial v_x^*}{\partial z^*} + \frac{\partial v_z^*}{\partial x^*} \right)^{(n-1)} \begin{bmatrix} 2\frac{\partial v_x^*}{\partial x^*} & \frac{\partial v_z^*}{\partial x^*} + \frac{\partial v_x^*}{\partial z^*} \\ \frac{\partial v_x^*}{\partial z^*} + \frac{\partial v_z^*}{\partial x^*} & 2\frac{\partial v_z^*}{\partial z^*} \end{bmatrix}$$

$$\boldsymbol{\tau^*{}_{xx} = -p_f{}^* + m^* \frac{1}{2}^{(n-1)/2} \left(\frac{\partial v_x^*}{\partial z^*} + \frac{\partial v_z^*}{\partial x^*} \right)^{(n-1)} 2\frac{\partial v_x^*}{\partial x^*}} \tag{8}$$

$$\tau^*{}_{xz} = \tau^*{}_{zx} = m^* \frac{1}{2}^{(n-1)/2} \left(\frac{\partial v_x^*}{\partial z^*} + \frac{\partial v_z^*}{\partial x^*} \right)^{(n-1)} \left(\frac{\partial v_x^*}{\partial z^*} + \frac{\partial v_z^*}{\partial x^*} \right)$$

$$\boldsymbol{\tau^*{}_{xz} = \tau^*{}_{zx} = m^* \frac{1}{2}^{(n-1)/2} \left(\frac{\partial v_x^*}{\partial z^*} + \frac{\partial v_z^*}{\partial x^*} \right)^{n}} \tag{9}$$

$$\boldsymbol{\tau^*{}_{zz} = -p_f{}^* + m^* \frac{1}{2}^{(n-1)/2} \left(\frac{\partial v_x^*}{\partial z^*} + \frac{\partial v_z^*}{\partial x^*} \right)^{(n-1)} 2\frac{\partial v_z^*}{\partial z^*}} \tag{10}$$

Substituting $\tau^*{}_{xx}$ and $\tau^*{}_{xz}$ in equation no. (2), we get

X- momentum

$$\rho \left(\frac{\partial v_x^*}{\partial t^*} + v_x^* \frac{\partial v_x^*}{\partial x^*} + v_z^* \frac{\partial v_x^*}{\partial z^*} \right) =$$
$$\frac{\partial}{\partial x^*} \left(-p_f{}^* + m^* \frac{1}{2}^{(n-1)/2} \left(\frac{\partial v_x^*}{\partial z^*} + \frac{\partial v_z^*}{\partial x^*} \right)^{(n-1)} 2\frac{\partial v_x^*}{\partial x^*} \right) + \frac{\partial}{\partial z^*} \left(m^* \frac{1}{2}^{(n-1)/2} \left(\frac{\partial v_x^*}{\partial z^*} + \frac{\partial v_z^*}{\partial x^*} \right)^{n} \right)$$

- $$\rho \left(\frac{\partial v_x^*}{\partial t^*} + v_x^* \frac{\partial v_x^*}{\partial x^*} + v_z^* \frac{\partial v_x^*}{\partial z^*} \right) = -\frac{\partial p_f{}^*}{\partial x^*} + m^* \frac{1}{2}^{(n-1)/2} \left(\frac{\partial v_x^*}{\partial z^*} + \frac{\partial v_z^*}{\partial x^*} \right)^{(n-1)} 2\frac{\partial^2 v_x^*}{\partial x^{*2}} + 2m^* \frac{1}{2}^{\frac{n-1}{2}} \frac{\partial v_x^*}{\partial x^*} (n-1) \left(\frac{\partial v_x^*}{\partial z^*} + \frac{\partial v_z^*}{\partial x^*} \right) \left(\frac{\partial v_x^*}{\partial z^*} + \frac{\partial v_z^*}{\partial x^*} \right)^{(n-2)} \left(\frac{\partial^2 v_x^*}{\partial x^* \partial z^*} + \frac{\partial^2 v_z^*}{\partial z^{*2}} \right) + m^* \frac{1}{2}^{(n-1)/2} n \left(\frac{\partial v_x^*}{\partial z^*} + \frac{\partial v_z^*}{\partial x^*} \right)^{n-1} \left(\frac{\partial^2 v_x^*}{\partial z^{*2}} + \frac{\partial^2 v_z^*}{\partial x^* \partial z^*} \right)$$

As we will be dealing with the linear stability analysis , so the higher order perturbation terms are neglected. So the terms which we neglect are

$$\frac{\partial v_x^*}{\partial x^*}\left(\frac{\partial v_x^*}{\partial z^*}+\frac{\partial v_z^*}{\partial x^*}\right)\left(\frac{\partial v_x^*}{\partial z^*}+\frac{\partial v_z^*}{\partial x^*}\right)^{(n-2)}\left(\frac{\partial^2 v_x^*}{\partial x^* \partial z^*}+\frac{\partial^2 v_z^*}{\partial\ z^{*2}}\right)$$

So the above equation reduces to

➢ $$\boldsymbol{\rho\left(\frac{\partial v_x^*}{\partial t^*}+v_x^*\frac{\partial v_x^*}{\partial x^*}+v_z^*\frac{\partial v_x^*}{\partial z^*}\right)=-\frac{\partial p_f{}^*}{\partial x^*}+m^*\frac{1}{2}^{(n-1)/2}\left(\frac{\partial v_x^*}{\partial z^*}+\frac{\partial v_z^*}{\partial x^*}\right)^{(n-1)}2\frac{\partial^2 v_x^*}{\partial\ x^{*2}}+ m^*\frac{1}{2}^{(n-1)/2}\ n\ \left(\frac{\partial v_x^*}{\partial z^*}+\frac{\partial v_z^*}{\partial x^*}\right)^{n-1}\left(\frac{\partial^2 v_x^*}{\partial\ z^{*2}}+\frac{\partial^2 v_z^*}{\partial x^*\partial z^*}\right)} \tag{11}$$

Now, for Z-momentum, substituting $\tau^*{}_{xz}$ and $\tau^*{}_{zz}$ from equation (9) and (10) in equation (3)

$$\rho\left(\frac{\partial v_z^*}{\partial t^*}+v_x^*\frac{\partial v_z^*}{\partial x^*}+v_z^*\frac{\partial v_z^*}{\partial z^*}\right)=$$

$$\frac{\partial}{\partial x^*}\left(m^*\frac{1}{2}^{(n-1)/2}\left(\frac{\partial v_x^*}{\partial z^*}+\frac{\partial v_z^*}{\partial x^*}\right)^n\right)+\frac{\partial}{\partial z^*}\left(-p_f{}^*+m^*\frac{1}{2}^{(n-1)/2}\left(\frac{\partial v_x^*}{\partial z^*}\frac{\partial v_z^*}{\partial x^*}\right)^{(n-1)}2\frac{\partial v_z^*}{\partial z^*}\right)$$

Upon simplification

$$\boldsymbol{\rho\left(\frac{\partial v_z^*}{\partial t^*}+v_x^*\frac{\partial v_z^*}{\partial x^*}+v_z^*\frac{\partial v_z^*}{\partial z^*}\right)=-\frac{\partial p_f{}^*}{\partial z^*}+m^*\frac{1}{2}^{(n-1)/2}n\left(\frac{\partial v_x^*}{\partial z^*}+\frac{\partial v_z^*}{\partial x^*}\right)^{n-1}\left(\frac{\partial^2 v_z^*}{\partial\ x^{*2}}+\frac{\partial^2 v_x^*}{\partial x^*\partial z^*}\right)} \tag{12}$$

3.3. Non-dimensionalisation of fluid equations :

Dimensional variables and quantities are non -dimensionalized using following scales

'R ' for lengths and displacements;

' $\frac{\eta_f}{E}$ ' for time

' E ' for pressure and stresses, E is the shear modulus of the neo-Hookean solid.

$x=\frac{x^*}{R}\Rightarrow x^*=xR$

$z=\frac{z^*}{R}\Rightarrow z^*=zR$

$t=\frac{\eta_f}{E}t^*\Rightarrow\ t^*=\frac{\eta_f}{E}t$

$p=\frac{p^*}{E}\Rightarrow\ p^*=Ep$

$$v_x^* = \frac{x^*}{t^*} = \frac{xR}{\frac{\eta_f}{E}t} = \frac{xER}{\eta_f t} = \frac{v_x RE}{\eta_f} \qquad\qquad v_z^* = \frac{v_z RE}{\eta_f}$$

Now substituting the above dimensional variables in terms of non dimensionalised so as to convert all the governing equations into non dimensionalised equation.

$$\frac{\partial \mathrm{v}_\mathrm{x}^*}{\partial \mathrm{x}^*} + \frac{\partial \mathrm{v}_\mathrm{z}^*}{\partial \mathrm{z}^*} = 0$$

$$\Rightarrow \quad \frac{\partial}{\partial xR}\left(\frac{v_x RE}{\eta_f}\right) + \frac{\partial}{\partial zR}\left(\frac{v_z RE}{\eta_f}\right) = 0$$

$$\Rightarrow \quad \boldsymbol{\frac{\partial v_x}{\partial x} + \frac{\partial v_z}{\partial z} = 0} \tag{13}$$

X momentum

$$\rho\left(\frac{\partial v_x^*}{\partial t^*} + v_x^* \frac{\partial v_x^*}{\partial x^*} + v_z^* \frac{\partial v_x^*}{\partial z^*}\right) = -\frac{\partial \mathrm{p_f}^*}{\partial x^*} + \mathrm{m}^* \frac{1}{2}^{(\mathrm{n}-1)/2}\left(\frac{\partial \mathrm{v}_\mathrm{x}^*}{\partial \mathrm{z}^*} + \frac{\partial \mathrm{v}_\mathrm{z}^*}{\partial \mathrm{x}^*}\right)^{(\mathrm{n}-1)} 2\frac{\partial^2 \mathrm{v}_\mathrm{x}^*}{\partial\ x^{*2}} + \mathrm{m}^* \frac{1}{2}^{(\mathrm{n}-1)/2} \mathrm{n}\left(\frac{\partial \mathrm{v}_\mathrm{x}^*}{\partial \mathrm{z}^*} + \frac{\partial \mathrm{v}_\mathrm{z}^*}{\partial \mathrm{x}^*}\right)^{\mathrm{n}-1}\left(\frac{\partial^2 \mathrm{v}_\mathrm{x}^*}{\partial\ z^{*2}} + \frac{\partial^2 \mathrm{v}_\mathrm{z}^*}{\partial \mathrm{x}^*\partial z^*}\right)$$

$$\Rightarrow \left(\frac{\partial\frac{v_x RE}{\eta_f}}{\partial\frac{\eta_f}{E}t} + \frac{v_x RE}{\eta_f}\ \frac{\partial\frac{v_x RE}{\eta_f}}{\partial xR} + \frac{v_z RE}{\eta_f}\frac{\partial\frac{v_x RE}{\eta_f}}{\partial zR}\right) = -\frac{\partial \mathrm{Ep_f}}{\partial xR} + \mathrm{m}^* \frac{1}{2}^{\frac{\mathrm{n}-1}{2}}\left(\frac{\partial\frac{v_x RE}{\eta_f}}{\partial \mathrm{zR}} + \frac{\partial\frac{v_z RE}{\eta_f}}{\partial \mathrm{xR}}\right)^{(\mathrm{n}-1)} 2\frac{\partial^2\frac{v_x RE}{\eta_f}}{\partial (xR)^2}$$

$$+ \mathrm{m}^* \frac{1}{2}^{(\mathrm{n}-1)/2} \mathrm{n}\left(\frac{\partial\frac{v_x RE}{\eta_f}}{\partial \mathrm{zR}} + \frac{\partial\frac{v_x RE}{\eta_f}}{\partial \mathrm{xR}}\right)^{\mathrm{n}-1}\left(\frac{\partial^2\frac{v_x RE}{\eta_f}}{\partial (zR)^2} + \frac{\partial^2\frac{v_z RE}{\eta_f}}{\partial \mathrm{xR}\partial zR}\right)$$

$$\Rightarrow \quad \frac{\rho R^2 E^2}{\eta_f{}^2}\left(\frac{\partial v_x}{\partial t} + v_x\ \frac{\partial v_x}{\partial x} + v_z \frac{\partial v_x}{\partial z}\right) = -\frac{\partial \mathrm{p_f}}{\partial x} + \mathrm{m}^* \frac{1}{2}^{(\mathrm{n}-1)/2}\left(\frac{E}{\eta_f}\right)^{\mathrm{n}}\left(\frac{\partial v_x}{\partial \mathrm{z}} + \frac{\partial v_z}{\partial \mathrm{x}}\right)^{(\mathrm{n}-1)} 2\frac{\partial^2 v_x}{\partial x^2} + \mathrm{m}^* \frac{1}{2}^{(\mathrm{n}-1)/2} \mathrm{n}\left(\frac{E}{\eta_f}\right)^{\mathrm{n}}\left(\frac{\partial v_x}{\partial \mathrm{z}} + \frac{\partial v_z}{\partial \mathrm{x}}\right)^{\mathrm{n}-1}\left(\frac{\partial^2 v_x}{\partial z^2} + \frac{\partial^2 v_z}{\partial \mathrm{x}\partial z}\right)$$

$\mathrm{m}^*\left(\frac{E}{\eta_f}\right)^{\mathrm{n}} = \mathrm{m}$ is dimensionless Parameter, in this thesis its value is set to unity in order to keep the scope of this work reasonable.

$$\Rightarrow \frac{\rho R^2 E}{\eta_f{}^2}\left(\frac{\partial v_x}{\partial t} + v_x\ \frac{\partial v_x}{\partial x} + v_z \frac{\partial v_x}{\partial z}\right) = -\frac{\partial \mathrm{p_f}}{\partial x} + \mathrm{m}\frac{1}{2}^{(\mathrm{n}-1)/2}\left(\frac{\partial v_x}{\partial \mathrm{z}} + \frac{\partial v_z}{\partial \mathrm{x}}\right)^{(\mathrm{n}-1)} 2\frac{\partial^2 v_x}{\partial x^2} + \mathrm{m}\frac{1}{2}^{(\mathrm{n}-1)/2} \mathrm{n}\left(\frac{\partial v_x}{\partial \mathrm{z}} + \frac{\partial v_z}{\partial \mathrm{x}}\right)^{\mathrm{n}-1}\left(\frac{\partial^2 v_x}{\partial z^2} + \frac{\partial^2 v_z}{\partial \mathrm{x}\partial z}\right)$$

$$\mathrm{m}\frac{1}{2}^{(\mathrm{n}-1)/2} = \mu$$

Multiply the above equation on both L.H.S and R.H.S by $\frac{V\eta_f}{RE}$

We get

$$\frac{\rho VR}{\eta_f}\left(\frac{\partial v_x}{\partial t}+v_x\,\frac{\partial v_x}{\partial x}+v_z\frac{\partial v_x}{\partial z}\right)=\frac{V\eta_f}{RE}\left(-\frac{\partial \mathrm{p_f}}{\partial x}+\mu\left(\frac{\partial v_x}{\partial \mathrm{z}}+\frac{\partial v_z}{\partial \mathrm{x}}\right)^{(\mathrm{n}-1)}2\,\frac{\partial^2 v_x}{\partial x^2}+\mu\,\mathrm{n}\left(\frac{\partial v_x}{\partial \mathrm{z}}+\frac{\partial v_z}{\partial \mathrm{x}}\right)^{\mathrm{n}-1}\left(\frac{\partial^2 v_x}{\partial z^2}+\frac{\partial^2 v_z}{\partial \mathrm{x}\partial z}\right)\right)$$

Where, Reynolds number $Re=\frac{\rho VR}{\eta_f}$ and $\gamma=\frac{V\eta_f}{RE}$

Hence

$$\boldsymbol{\frac{Re}{\gamma}\left(\frac{\partial v_x}{\partial t}+v_x\,\frac{\partial v_x}{\partial x}+v_z\frac{\partial v_x}{\partial z}\right)=-\frac{\partial \mathrm{p_f}}{\partial x}+2\mu\left(\frac{\partial v_x}{\partial \mathrm{z}}+\frac{\partial v_z}{\partial \mathrm{x}}\right)^{(\mathrm{n}-1)}\frac{\partial^2 v_x}{\partial x^2}+\mu\mathrm{n}\left(\frac{\partial v_x}{\partial \mathrm{z}}+\frac{\partial v_z}{\partial \mathrm{x}}\right)^{\mathrm{n}-1}\left(\frac{\partial^2 v_x}{\partial z^2}+\frac{\partial^2 v_z}{\partial \mathrm{x}\partial z}\right)} \tag{14}$$

Z momentum (Non Dimensionalisation)

$$\rho\left(\frac{\partial v_z^*}{\partial t^*}+v_x^*\,\frac{\partial v_z^*}{\partial x^*}+v_z^*\frac{\partial v_z^*}{\partial z^*}\right)=-\,\frac{\partial \mathrm{p_f}^*}{\partial z^*}+\mathrm{m}^*\,\frac{1}{2}^{(\mathrm{n}-1)/2}\,n\left(\frac{\partial \mathrm{v_x^*}}{\partial \mathrm{z}^*}+\frac{\partial \mathrm{v_z^*}}{\partial \mathrm{x}^*}\right)^{\mathrm{n}-1}\left(\frac{\partial^2 \mathrm{v_z^*}}{\partial\ \ x^{*2}}+\frac{\partial^2 \mathrm{v_x^*}}{\partial \mathrm{x}^*\partial z^*}\right)$$

Similarly we get

$$\boldsymbol{\frac{Re}{\gamma}\left(\frac{\partial v_z}{\partial t}+v_x\,\frac{\partial v_z}{\partial x}+v_z\frac{\partial v_z}{\partial z}\right)=-\frac{\partial \mathrm{p_f}}{\partial z}+2\mu\left(\frac{\partial v_x}{\partial \mathrm{z}}+\frac{\partial v_z}{\partial \mathrm{x}}\right)^{(\mathrm{n}-1)}\frac{\partial^2 v_z}{\partial z^2}+\mu\mathrm{n}\left(\frac{\partial v_x}{\partial \mathrm{z}}+\frac{\partial v_z}{\partial \mathrm{x}}\right)^{\mathrm{n}-1}\left(\frac{\partial^2 v_x}{\partial z^2}+\frac{\partial^2 v_z}{\partial \mathrm{x}\partial z}\right)} \tag{15}$$

This completes the governing equations of the power-law fluid. Now, we move forward to write the governing equations for the gel i.e. neo-Hookean solid.

3.4 Governing equations for Neo Hookean solid

Spatial position of a material particle given by vector X = (*X*, *Z*) at time *t* = 0. This initial unstressed state is chosen as the reference configuration in the Lagrangian method for describing the motion of particle. Let the solid body be deformed to a new state at a later time *t* so that each of the material particle moves to a new position with respect to its position in initial unstressed (reference) configuration. In this new deformed state, the current positions of material particles are denoted by (x, z). Note that the independent variables are the spatial position in reference (unstressed) configuration (X = (*X*, *Y*, *Z*)) and time *t*.

(i) Conservation of mass, continuity equation

$$\mathrm{Det}\,\mathrm{F} = 1$$

where F is the deformation gradient tensor given by $\mathrm{F} = \nabla_X . x$

$$F = \begin{bmatrix} \frac{\partial x}{\partial X} & \frac{\partial x}{\partial Z} \\ \frac{\partial z}{\partial X} & \frac{\partial z}{\partial Z} \end{bmatrix}$$

Therefore

$$\mathrm{Det}\,\mathrm{F} = \frac{\partial x}{\partial X}\frac{\partial z}{\partial Z} - \frac{\partial x}{\partial Z}\frac{\partial z}{\partial X} = 1 \qquad (16)$$

(ii) Conservation of momentum, Navier Stokes equation

X momentum

$$\nabla_X . \mathrm{T} = \rho \frac{\partial^2 \mathrm{x}}{\partial \mathrm{t}^2}$$

Z momentum

$$\nabla_X . \mathrm{T} = \rho \frac{\partial^2 \mathrm{z}}{\partial \mathrm{t}^2}$$

where T is the piola kirchoff stress tensor, $T = F^{-1}\sigma$

$$\sigma = -p_s \delta + FF^T$$

$$F = \begin{bmatrix} \frac{\partial x}{\partial X} & \frac{\partial x}{\partial Z} \\ \frac{\partial z}{\partial X} & \frac{\partial z}{\partial Z} \end{bmatrix} \qquad F^T = \begin{bmatrix} \frac{\partial x}{\partial X} & \frac{\partial z}{\partial X} \\ \frac{\partial x}{\partial Z} & \frac{\partial z}{\partial Z} \end{bmatrix} \qquad F^{-1} = \begin{bmatrix} \frac{\partial z}{\partial Z} & -\frac{\partial x}{\partial Z} \\ -\frac{\partial z}{\partial X} & \frac{\partial x}{\partial X} \end{bmatrix}$$

X- momentum

$$\nabla_X . \mathrm{T} = \rho \frac{\partial^2 \mathrm{x}}{\partial \mathrm{t}^2}$$

$$\nabla_X.(F^{-1}\,\sigma) = \rho\frac{\partial^2 \mathrm{x}}{\partial \mathrm{t}^2}$$

$$\nabla_X.\left(F^{-1}\,(-\,p_s\delta + FF^T)\right) = \rho\frac{\partial^2 \mathrm{x}}{\partial \mathrm{t}^2}$$

$$\nabla_X.(-\,p_sF^{-1} + F^T) = \rho\frac{\partial^2 \mathrm{x}}{\partial \mathrm{t}^2}$$

$$\frac{\partial}{\partial X}(-p_sF_{11}^{-1}) + \frac{\partial}{\partial Z}(-p_sF_{21}^{-1}) + \frac{\partial}{\partial X}(F_{11}^{T}) + \frac{\partial}{\partial Z}(F_{21}^{T}) = \rho\frac{\partial^2 \mathrm{x}}{\partial \mathrm{t}^2}$$

$$\frac{\partial}{\partial X}\left(-p_s\frac{\partial z}{\partial Z}\right) + \frac{\partial}{\partial Z}\left(p_s\frac{\partial z}{\partial X}\right) + \frac{\partial}{\partial X}\left(\frac{\partial x}{\partial X}\right) + \frac{\partial}{\partial Z}\left(\frac{\partial x}{\partial Z}\right) = \rho\frac{\partial^2 \mathrm{x}}{\partial \mathrm{t}^2}$$

$$-\boldsymbol{\frac{\partial p_s}{\partial X}\left(\frac{\partial z}{\partial Z}\right) + \frac{\partial p_s}{\partial Z}\left(\frac{\partial z}{\partial X}\right) + \frac{\partial^2 x}{\partial X^2} + \frac{\partial^2 x}{\partial Z^2} = \rho\frac{\partial^2 \mathrm{x}}{\partial \mathrm{t}^2}} \tag{17}$$

Z- momentum

$$\nabla_X.\mathrm{T} = \rho\frac{\partial^2 \mathrm{z}}{\partial \mathrm{t}^2}$$

$$\nabla_X.(F^{-1}\,\sigma) = \rho\frac{\partial^2 \mathrm{z}}{\partial \mathrm{t}^2}$$

$$\nabla_X.\left(F^{-1}\,(-\,p_s\delta + FF^T)\right) = \rho\frac{\partial^2 \mathrm{z}}{\partial \mathrm{t}^2}$$

$$\nabla_X.(-\,p_sF^{-1} + F^T) = \rho\frac{\partial^2 \mathrm{z}}{\partial \mathrm{t}^2}$$

$$\frac{\partial}{\partial X}(-p_sF_{12}^{-1}) + \frac{\partial}{\partial Z}(-p_sF_{22}^{-1}) + \frac{\partial}{\partial X}(F_{12}^{T}) + \frac{\partial}{\partial Z}(F_{22}^{T}) = \rho\frac{\partial^2 \mathrm{z}}{\partial \mathrm{t}^2}$$

$$\frac{\partial}{\partial X}\left(-p_s\frac{\partial x}{\partial Z}\right) + \frac{\partial}{\partial Z}\left(p_s\frac{\partial x}{\partial X}\right) + \frac{\partial}{\partial X}\left(\frac{\partial z}{\partial X}\right) + \frac{\partial}{\partial Z}\left(\frac{\partial z}{\partial Z}\right) = \rho\frac{\partial^2 \mathrm{z}}{\partial \mathrm{t}^2}$$

$$-\boldsymbol{\frac{\partial p_s}{\partial Z}\left(\frac{\partial x}{\partial X}\right) + \frac{\partial p_s}{\partial X}\left(\frac{\partial x}{\partial Z}\right) + \frac{\partial^2 z}{\partial X^2} + \frac{\partial^2 z}{\partial Z^2} = \rho\frac{\partial^2 \mathrm{z}}{\partial \mathrm{t}^2}} \tag{18}$$

3.5 Non dimensionalisation of Neo-Hookean solid equations

Using the same scales we used earlier, we non dimensionalized the governing equations in the same manner.

X-momentum

$$-\frac{\partial p_s}{\partial X}\left(\frac{\partial z}{\partial Z}\right)+\frac{\partial p_s}{\partial Z}\left(\frac{\partial z}{\partial X}\right)+\frac{\partial^2 x}{\partial X^2}+\frac{\partial^2 x}{\partial Z^2}=\frac{Re}{\gamma}\left(\frac{\partial^2 x}{\partial t^2}\right) \quad (19)$$

z- momentum

$$-\frac{\partial p_s}{\partial Z}\left(\frac{\partial x}{\partial X}\right)+\frac{\partial p_s}{\partial X}\left(\frac{\partial x}{\partial Z}\right)+\frac{\partial^2 z}{\partial X^2}+\frac{\partial^2 z}{\partial Z^2}=\frac{Re}{\gamma}\left(\frac{\partial^2 z}{\partial t^2}\right) \quad (20)$$

3.6 Base state solution

It is the steady state solution. The total fluctuation at the interface is the sum of base state solution and the perturbations. Superscript overbar is used to represent the steady state quantity

For base state $\frac{\partial \bar{p}_f}{\partial x}=\frac{\partial \bar{v}_x}{\partial t}=\frac{\partial \bar{v}_x}{\partial x}=\bar{v}_z=\frac{\partial^2 \bar{x}}{\partial t^2}=\frac{\partial \bar{x}}{\partial t}=0$ (∵steady & unidirectional flow)

3.6.1. Base state solution of Power-law fluid :

X momentum

$$\frac{Re}{\gamma}\left(\frac{\partial \bar{v}_x}{\partial t}+\bar{v}_x\frac{\partial \bar{v}_x}{\partial x}+\bar{v}_z\frac{\partial \bar{v}_x}{\partial z}\right)=-\frac{\partial p_f}{\partial x}+2\mu\left(\frac{\partial \bar{v}_x}{\partial z}+\frac{\partial \bar{v}_z}{\partial x}\right)^{(n-1)}\frac{\partial^2 \bar{v}_x}{\partial x^2}+\mu n\left(\frac{\partial \bar{v}_x}{\partial z}+\frac{\partial \bar{v}_z}{\partial x}\right)^{n-1}\left(\frac{\partial^2 \bar{v}_x}{\partial z^2}+\frac{\partial^2 \bar{v}_z}{\partial x\partial z}\right)$$

$$\Rightarrow \quad \mu n\left(\frac{\partial \bar{v}_x}{\partial z}\right)^{n-1}\left(\frac{\partial^2 \bar{v}_x}{\partial z^2}\right)=0$$

For a Couette flow, velocity gradient cannot be equal to zero, so the term $\frac{\partial^2 \bar{v}_x}{\partial z^2}$ should be equals to zero.

$$\Rightarrow \qquad \frac{\partial^2 \bar{v}_x}{\partial z^2} = 0$$

$$\Rightarrow \qquad \frac{\partial \bar{v}_x}{\partial z} = c_1$$

$$\Rightarrow \qquad \bar{v}_x = c_1 z + c_2$$

Boundary conditions

At z = 1 , $v_x^* = V^*$, where V^* is the velocity of the top plate (x direction)

Non dimensionalizing V^*

$$V = \frac{V^*}{\frac{x}{t}} = \frac{v\eta_f}{RE} = \gamma$$

γ is the nondimensional shear rate.

Therefore at z = 1; $\bar{v}_x = \gamma$

at z = 0 ; $\bar{v}_x = 0$

Substituting the above boundary conditions we get

$c_2 = 0$; $c_1 = \gamma$

$$\Rightarrow \qquad \boldsymbol{\bar{v}_x = \gamma z} \tag{21}$$

$$\bar{\tau} = \begin{bmatrix} -\bar{p}_f & 0 \\ 0 & -\bar{p}_f \end{bmatrix} + \mu \left(\frac{\partial \bar{v}_z}{\partial x} + \frac{\partial \bar{v}_x}{\partial z} \right)^{(n-1)} \begin{bmatrix} 2\frac{\partial \bar{v}_x}{\partial x} & \frac{\partial \bar{v}_z}{\partial x} + \frac{\partial \bar{v}_x}{\partial z} \\ \frac{\partial \bar{v}_z}{\partial x} + \frac{\partial \bar{v}_x}{\partial z} & 2\frac{\partial \bar{v}_z}{\partial z} \end{bmatrix}$$

$$\Rightarrow \qquad \bar{\tau} = \begin{bmatrix} -\bar{p}_f & 0 \\ 0 & -\bar{p}_f \end{bmatrix} + \mu \, (\gamma)^{(n-1)} \begin{bmatrix} 0 & \gamma \\ \gamma & 0 \end{bmatrix}$$

$$\Rightarrow \qquad \boldsymbol{\bar{\tau}} = \begin{bmatrix} -\boldsymbol{\bar{p}_f} & \boldsymbol{\mu\gamma^n} \\ \boldsymbol{\mu\gamma^n} & -\boldsymbol{\bar{p}_f} \end{bmatrix} \tag{22}$$

3.6.2. Base state solution of Neo- Hookean Solid :

$$-\frac{\partial \overline{p_s}}{\partial X}\left(\frac{\partial \bar{z}}{\partial Z}\right)+\frac{\partial \overline{p_s}}{\partial Z}\left(\frac{\partial \bar{z}}{\partial X}\right)+\frac{\partial^2 \bar{x}}{\partial X^2}+\frac{\partial^2 \bar{x}}{\partial Z^2}=0$$

$$\Rightarrow \qquad \frac{\partial^2 \bar{x}}{\partial Z^2}=0$$

$$\Rightarrow \qquad \frac{\partial \bar{x}}{\partial Z}=c_1$$

$$\Rightarrow \qquad \bar{x}= c_1 z + c_2$$

Boundary conditions

$$\bar{x}=X \;\; at\; z= -H$$

Tangential stress at z = 0

$$\Rightarrow \qquad (\tau_{xz})_{liquid} = (\sigma_{xz})_{solid}$$

$$\Rightarrow \qquad \mu \;\left(\frac{\partial \bar{v}_z}{\partial x}+\frac{\partial \bar{v}_x}{\partial z}\right)^n = \bar{p}_s\left(\frac{\partial \bar{z}}{\partial X}\right)+\frac{\partial \bar{x}}{\partial Z}$$

$$\Rightarrow \qquad \mu\;\left(\frac{\partial \bar{v}_x}{\partial z}\right)^n = \frac{\partial \bar{x}}{\partial Z}= \mu\,\gamma^n$$

$$\Rightarrow \qquad \mu\;\left(\frac{\partial \bar{v}_x}{\partial z}\right)^n = \frac{\partial \bar{x}}{\partial Z}= \mu\,\gamma^n \qquad \text{at } z=0$$

Applying the boundary conditions we get

$$\Rightarrow \qquad \frac{\partial \bar{x}}{\partial Z}= \mu\,\gamma^n = c_1$$

$$\Rightarrow \qquad X+\mu\,\gamma^n H = c_2$$

$$\Rightarrow \qquad \bar{x}= \mu\,\gamma^n z + X + \mu\,\gamma^n H$$

$$\Rightarrow \qquad \boldsymbol{\bar{x} = X+ \mu\,\gamma^n(H + z)} \qquad (23)$$

$\sigma = -p_s I + FF^T$

$$\Rightarrow \quad \bar{\sigma} = -\bar{p}_s \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} \frac{\partial \bar{x}}{\partial X} & \frac{\partial \bar{x}}{\partial Z} \\ \frac{\partial \bar{z}}{\partial X} & \frac{\partial \bar{z}}{\partial Z} \end{bmatrix} \begin{bmatrix} \frac{\partial \bar{x}}{\partial X} & \frac{\partial \bar{z}}{\partial X} \\ \frac{\partial \bar{x}}{\partial Z} & \frac{\partial \bar{z}}{\partial Z} \end{bmatrix}$$

$$\Rightarrow \quad \bar{\sigma} = -\bar{p}_s \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & \frac{\partial \bar{x}}{\partial Z} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \frac{\partial \bar{x}}{\partial Z} & 1 \end{bmatrix}$$

$$\Rightarrow \quad \bar{\sigma} = -\bar{p}_s \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 + \left(\frac{\partial \bar{x}}{\partial Z}\right)^2 & \frac{\partial \bar{x}}{\partial Z} \\ \frac{\partial \bar{x}}{\partial Z} & 1 \end{bmatrix}$$

$$\Rightarrow \quad \bar{\sigma} = \begin{bmatrix} -\bar{p}_s + 1 + \left(\frac{\partial \bar{x}}{\partial Z}\right)^2 & \frac{\partial \bar{x}}{\partial Z} \\ \frac{\partial \bar{x}}{\partial Z} & 1 - \bar{p}_s \end{bmatrix}$$

$-\bar{p}_s + 1 = -C$

$$\Rightarrow \quad \bar{\sigma} = \begin{bmatrix} -C + \left(\frac{\partial \bar{x}}{\partial Z}\right)^2 & \frac{\partial \bar{x}}{\partial Z} \\ \frac{\partial \bar{x}}{\partial Z} & -C \end{bmatrix} = \begin{bmatrix} -C + (\mu\,\gamma^n)^2 & \mu\,\gamma^n \\ \mu\,\gamma^n & -C \end{bmatrix}$$

$$\Rightarrow \quad \boldsymbol{\bar{\sigma}} = \begin{bmatrix} -\boldsymbol{C} + (\boldsymbol{\mu}\,\boldsymbol{\gamma}^{\boldsymbol{n}})^2 & \boldsymbol{\mu}\,\boldsymbol{\gamma}^{\boldsymbol{n}} \\ \boldsymbol{\mu}\,\boldsymbol{\gamma}^{\boldsymbol{n}} & -\boldsymbol{C} \end{bmatrix} \tag{24}$$

Equation (24) shows the base state solution of the Neo-Hookean solids displacement field. It is clearly seen that as the shear rate is increased the magnitude of displacement gradient of solid also increases. This shows that neo Hookean model gives rise to a first normal stress difference,

$\bar{\sigma}_{11} - \bar{\sigma}_{22} = (\mu\,\gamma^n)^2$, which is not observed when linear viscoelastic solid model is used. Since solid deformation fields and the base state velocity are same for both linear viscoelastic solids as well as neo-Hookean solid, the main difference between both models is in the base state solution of the stress fields; this difference surely influences the stability of the system.

CHAPTER - 4

LINEAR STABILITY ANALYSIS

To study the stability of the base state configuration to small amplitude perturbations, we express each field variable as the sum of the base term and a perturbation term. A normal mode expansion is then applied to the perturbation terms so that each field variable can be written in the form

Small perturbations (denoted by primed quantities) are imposed to the fluid velocity field $v_i = \overline{v}_i + v_i'$ and other dynamical variables in the fluid and the solid displacement field are similarly $u_i = \overline{u}_i + u_i'$ perturbed in order to examine the stability of this fluid-gel system. The evolution of these small perturbations to the base state is determined by this analysis.

The perturbation quantities are expanded in the form of Fourier modes in the *x*-direction, and with an exponential dependence in time

$u_i'(x,z,t) = \tilde{u}_i(z)\exp ik[x - ct]$

$v_i'(x,z,t) = \tilde{v}_i(z)\exp ik[x - ct]$

where,

k is the wave number. (inversely proportional to the wavelength) of perturbations,

c = growth rate

$\tilde{u}_i(z)\ and\ \tilde{v}_i(z)$ are eigen functions determined from the conservation equations

$v_x = \overline{v}_x + v_x' = \overline{v}_x + \tilde{v}_x \exp ik[x - \ ct]$

$v_z = \overline{v}_z + v_z' = \tilde{v}_z \exp ik[x - ct]$

$p = \overline{p} + p' = \overline{p} + \tilde{p}\ \exp ik[x - ct]$

The linearised equation for the fluid displacement field is calculated as follows:

Now governing equations are converted in terms of perturbations:

4.1. Linear Stability analysis of fluid equations

Conservation of mass, from equation no. (13)

$$\frac{\partial v_x}{\partial x}+\frac{\partial v_z}{\partial z}=0$$

➢ $\frac{\partial}{\partial x}\tilde{v}_x \exp ik[x-ct]+\frac{\partial}{\partial z}\tilde{v}_z \exp ik[x-ct]=0$

➢ $ik\tilde{v}_x+D\tilde{v}_z=0$

➢ $\boldsymbol{\tilde{v}_x=\frac{iD\tilde{v}_z}{k}}$ (25)

X-momentum, from equation no. (14)

$$\frac{Re}{\gamma}\left(\frac{\partial v_x}{\partial t}+v_x\frac{\partial v_x}{\partial x}+v_z\frac{\partial v_x}{\partial z}\right)=-\frac{\partial p_f}{\partial x}+2\mu\frac{\partial^2 v_x}{\partial x^2}\left(\frac{\partial v_x}{\partial z}+\frac{\partial v_z}{\partial x}\right)^{n-1}+\mu n\left(\frac{\partial v_x}{\partial z}+\frac{\partial v_z}{\partial x}\right)^{n-1}\left(\frac{\partial^2 v_x}{\partial x^2}+\frac{\partial^2 v_x}{\partial x\partial z}\right)$$

➢ $\boldsymbol{\frac{Re}{\gamma}(-ikc\tilde{v}_x+\gamma zik\tilde{v}_x+\gamma\tilde{v}_z)=}$

$\boldsymbol{-ik\tilde{p}_f-2k^2\mu\gamma^{n-1}\tilde{v}_x+\mu n\gamma^{n-1}D^2\tilde{v}_x+\mu n\gamma^{n-1}ikD\tilde{v}_z}$ (26)

Z –momentum, from equation no. (15)

$$\frac{Re}{\gamma}\left(\frac{\partial v_z}{\partial t}+v_x\frac{\partial v_z}{\partial x}+v_z\frac{\partial v_z}{\partial z}\right)=-\frac{\partial p_f}{\partial z}+2\mu\frac{\partial^2 v_x}{\partial x^2}\left(\frac{\partial v_x}{\partial z}+\frac{\partial v_z}{\partial x}\right)^{n-1}+\mu n\left(\frac{\partial v_x}{\partial z}+\frac{\partial v_z}{\partial x}\right)^{n-1}\left(\frac{\partial^2 v_x}{\partial x^2}+\frac{\partial^2 v_x}{\partial x\partial z}\right)$$

➢ $\boldsymbol{\frac{Re}{\gamma}(-ikc\tilde{v}_z+\gamma zik\tilde{v}_z)=-\frac{\partial\tilde{p}_f}{\partial z}-k^2\mu n\gamma^{n-1}\tilde{v}_z+2\mu\gamma^{n-1}D^2\tilde{v}_z+}$

$\boldsymbol{\mu n\gamma^{n-1}ikD\tilde{v}_x}$ (27)

From equation no. (26)

➢ $\tilde{p}_f=\frac{1}{ik}\left(-2k^2\mu\gamma^{n-1}\tilde{v}_x+\mu n\gamma^{n-1}D^2\tilde{v}_x+\mu n\gamma^{n-1}ikD\tilde{v}_z-\frac{Re}{\gamma}(-ikc\tilde{v}_x+\gamma zik\tilde{v}_x+\gamma\tilde{v}_z)\right)$

➢ $-\frac{\partial\tilde{p}_f}{\partial z}=-\frac{\partial}{\partial z}\frac{1}{ik}\left(-2k^2\mu\gamma^{n-1}\tilde{v}_x+\mu n\gamma^{n-1}D^2\tilde{v}_x+\mu n\gamma^{n-1}ikD\tilde{v}_z-\frac{Re}{\gamma}(-ikc\tilde{v}_x+\gamma zik\tilde{v}_x+\right.$

$\left.\gamma\tilde{v}_z)\right)$

- $-\frac{\partial \tilde{p}_f}{\partial z} = \frac{1}{ik}\left(2k^2\mu\gamma^{n-1}D\tilde{v}_x - \mu n\gamma^{n-1}D^3\tilde{v}_x - \mu n\gamma^{n-1}ikD^2\tilde{v}_z + \frac{Re}{\gamma}(-ikcD\tilde{v}_x + \gamma ik\tilde{v}_x + \gamma zikD\tilde{v}_x + \gamma D\tilde{v}_z)\right)$

From equation no. (25)

- $\tilde{v}_x = \frac{iD\tilde{v}_z}{k}$
- $-\frac{\partial \tilde{p}_f}{\partial z} = \frac{1}{ik}\left(2ik\mu\gamma^{n-1}D^2\tilde{v}_z - \frac{\mathrm{i}}{k}\mu n\gamma^{n-1}D^4\tilde{v}_z - \mu n\gamma^{n-1}ikD^2\tilde{v}_z + \frac{Re}{\gamma}(cD^2\tilde{v}_z - \gamma D\tilde{v}_z - \gamma zD^2\tilde{v}_z + \gamma D\tilde{v}_z)\right)$

Substituting the value of $-\frac{\partial \tilde{p}_f}{\partial z}$ in z momentum equation (27) ,

$$\frac{Re}{\gamma}(-ikc\tilde{v}_z + \gamma zik\tilde{v}_z) = \frac{1}{i}\left(2ik^2\mu\gamma^{n-1}D^2\tilde{v}_z - i\mu n\gamma^{n-1}D^4\tilde{v}_z - \mu n\gamma^{n-1}ik^2D^2\tilde{v}_z + \frac{Rek}{\gamma}(cD^2\tilde{v}_z - \gamma D\tilde{v}_z - \gamma zD^2\tilde{v}_z + \gamma D\tilde{v}_z)\right) - k^4\mu n\gamma^{n-1}\tilde{v}_z + 2\mu\gamma^{n-1}k^2D^2\tilde{v}_z - \mu n\gamma^{n-1}ik^2D^2\tilde{v}_z$$

Rearranging the above equation, we get

- $-ik^3c\frac{Re}{\gamma}\tilde{v}_z + \gamma zik^3\frac{Re}{\gamma}\tilde{v}_z = 2k^2\mu\gamma^{n-1}D^2\tilde{v}_z - \mu n\gamma^{n-1}D^4\tilde{v}_z - \mu n\gamma^{n-1}k^2D^2\tilde{v}_z - ikc\frac{Re}{\gamma}D^2\tilde{v}_z + ikReD\tilde{v}_z + ikRezD^2\tilde{v}_z - ikReD\tilde{v}_z - k^4\mu n\gamma^{n-1}\tilde{v}_z + 2\mu\gamma^{n-1}k^2D^2\tilde{v}_z - \mu n\gamma^{n-1}ik^2D^2\tilde{v}_z$

- $-ikc\frac{Re}{\gamma}D^2\tilde{v}_z + ikRezD^2\tilde{v}_z + ik^3c\frac{Re}{\gamma}\tilde{v}_z - \gamma zik^3\frac{Re}{\gamma}\tilde{v}_z = -2k^2\mu\gamma^{n-1}D^2\tilde{v}_z + \mu n\gamma^{n-1}D^4\tilde{v}_z + \mu\gamma^{n-1}k^2D^2\tilde{v}_z + k^4\mu n\gamma^{n-1}\tilde{v}_z - 2\mu\gamma^{n-1}k^2D^2\tilde{v}_z + \mu\gamma^{n-1}k^2D^2\tilde{v}_z$

- $\boldsymbol{-ikc\frac{Re}{\gamma}D^2\tilde{v}_z + ikRezD^2\tilde{v}_z + ik^3c\frac{Re}{\gamma}\tilde{v}_z - \gamma zik^3\frac{Re}{\gamma}\tilde{v}_z = \mu n\gamma^{n-1}D^4\tilde{v}_z + 2\mu\gamma^{n-1}k^2(n-2)D^2\tilde{v}_z + k^4\mu n\gamma^{n-1}\tilde{v}_z}$ (28)

This is the fourth order Orr Sommerfield type equation for the power-law fluid.

4.2 linear Stability analysis of neo-Hookean solid governing equations

$\bar{x} = X + \mu\gamma^n(H + z)$

Rewriting the various variables:

$x = \bar{x} + x' = X + \mu\gamma^n(H + z) + \tilde{x}\exp ik[x - ct]$

$z = \bar{z} + z' = \tilde{z}\exp ik[x - ct]$

$p_s = \bar{p}_s + p_s{}' = (1 + c) + \tilde{p}_s \exp ik[x - ct]$

Continuity equation, from equation no. (16)

- $\frac{\partial}{\partial X}(X + \mu\gamma^n(H + z) + \tilde{x}\exp ik[x - ct]) \frac{\partial}{\partial Z}(\tilde{z}\exp ik[x - ct]) - \frac{\partial}{\partial Z}(X + \mu\gamma^n(H + z) + \tilde{x}\exp ik[x - ct]) \frac{\partial}{\partial X}(\tilde{z}\exp ik[x - ct]) = 1$
- $1 + ik\exp[x - ct]\tilde{x} - ik\mu\gamma^n\exp[x - ct]\tilde{z} + \exp ik[x - ct]\, D\tilde{z} = 1$
- $ik\exp[x - ct]\tilde{x} - ik\mu\gamma^n\exp[x - ct]\tilde{z} + \exp ik[x - ct]\, D\tilde{z} = 0$
- $ik\tilde{x} - ik\mu\gamma^n\tilde{z} + D\tilde{z} = 0$

- $$\boldsymbol{\tilde{x} = \frac{ik\mu\gamma^n\tilde{z} - D\tilde{z}}{ik} = \mu\gamma^n\tilde{z} + \frac{iD\tilde{z}}{k}} \qquad (29)$$

X- momentum, from equation no. (17)

$$-\frac{\partial p_s}{\partial X}\left(\frac{\partial z}{\partial Z}\right) + \frac{\partial p_s}{\partial Z}\left(\frac{\partial z}{\partial X}\right) + \frac{\partial^2 x}{\partial X^2} + \frac{\partial^2 x}{\partial Z^2} = \frac{Re}{\gamma}\left(\frac{\partial^2 x}{\partial t^2}\right)$$

- $-\frac{\partial}{\partial X}[(1 + c) + \tilde{p}_s \exp ik[x - ct]\;]\frac{\partial}{\partial Z}[\tilde{z}\exp ik[x - ct]\;] + \frac{\partial}{\partial Z}[(1 + c) + \tilde{p}_s \exp ik[x - ct]\,]\frac{\partial}{\partial X}[\tilde{z}\exp ik[x - ct]\;] + \frac{\partial^2}{\partial X^2}[X + \mu\gamma^n(H + z) + \tilde{x}\exp ik[x - ct]\,] + \frac{\partial^2}{\partial Z^2}[X + \mu\gamma^n(H + z) + \tilde{x}\exp ik[x - ct]\,] = \frac{Re}{\gamma}\left(\frac{\partial^2}{\partial t^2}[X + \mu\gamma^n(H + z) + \tilde{x}\exp ik[x - ct]\,]\right)$

On simplification

$$\Rightarrow \quad \boldsymbol{- i\, k\, \widetilde{p}_s - k^2\, \widetilde{x} + \frac{\partial^2}{\partial Z^2}\, \widetilde{x} = -\frac{Re}{\gamma} k^2\, c^2 \widetilde{x}} \tag{30}$$

$$\Rightarrow \quad \tilde{p}_s = \frac{\mathrm{i}}{k}\left(-\frac{Re}{\gamma} k^2\, c^2 \tilde{x} + k^2\, \tilde{x} - D^2\, \tilde{x}\right)$$

Using equation no. (29)

$$\tilde{p}_s = \frac{\mathrm{i}}{k}\left[-\frac{Re}{\gamma} k^2\, c^2\left(\mu\, \gamma^n \tilde{z} + \frac{iD\tilde{z}}{k}\right) + k^2\left(\mu\, \gamma^n \tilde{z} + \frac{iD\tilde{z}}{k}\right) - D^2\left(\mu\, \gamma^n \tilde{z} + \frac{iD\tilde{z}}{k}\right)\right]$$

$$\tilde{p}_s = \frac{\mathrm{i}}{k}\left[-\frac{Re}{\gamma} k^2\, c^2 \mu\, \gamma^n \tilde{z} - \frac{Re}{\gamma} k\, c^2\, iD\tilde{z} + k^2 \mu\, \gamma^n \tilde{z} + kiD\tilde{z} - \mu\, \gamma^n D^2 \tilde{z} - \frac{i}{k}\, D^3 \tilde{z}\right]$$

$$\boldsymbol{\widetilde{p}_s = \frac{\mathrm{i}}{k^2}\left[-\frac{Re}{\gamma} k^3\, c^2 \mu\, \gamma^n \tilde{z} - \frac{Re}{\gamma} k^2\, c^2\, iD\tilde{z} + k^3 \mu\, \gamma^n \tilde{z} + k^2 iD\tilde{z} - \mu\, \gamma^n k D^2 \tilde{z} - iD^3 \tilde{z}\right]} \tag{31}$$

Z momentum, from equation no. (18)

$$-\frac{\partial p_s}{\partial Z}\left(\frac{\partial x}{\partial X}\right) + \frac{\partial p_s}{\partial X}\left(\frac{\partial x}{\partial Z}\right) + \frac{\partial^2 z}{\partial X^2} + \frac{\partial^2 z}{\partial Z^2} = \frac{\mathrm{Re}}{\gamma}\left(\frac{\partial^2 \mathrm{z}}{\partial \mathrm{t}^2}\right)$$

$$\Rightarrow \quad -\frac{\partial}{\partial Z}[(1 + c) + \tilde{p}_s \exp \tilde{z} \exp ik[x - ct]\ ]\frac{\partial}{\partial X}[X + \mu\, \gamma^n (H + z) + \tilde{x}\tilde{z} \exp ik[x - ct]\ ]\frac{\partial}{\partial X}[(1 + c) + \tilde{p}_s\, \tilde{z} \exp ik[x - ct]\ ]\frac{\partial}{\partial Z}[X + \mu\, \gamma^n (H + z) + \tilde{x}\tilde{z} \exp ik[x - ct]\ ] + \frac{\partial^2}{\partial X^2}\tilde{z}\, \tilde{z} \exp ik[x - ct] + \frac{\partial^2}{\partial Z^2}\tilde{z}\, \tilde{z} \exp ik[x - ct] = \frac{\mathrm{Re}}{\gamma}\frac{\partial^2}{\partial \mathrm{t}^2}(\tilde{z} \exp ik[x - ct]\ )$$

On simplification

$$\boldsymbol{i\, k\, \mu\, \gamma^n\, \widetilde{p}_s - D\widetilde{p}_s + (D^2 - k^2)\, \tilde{z} = -\frac{Re}{\gamma} k^2\, c^2 \tilde{z}} \tag{32}$$

Using equation (31), we get

$$\Rightarrow \quad \frac{Re}{\gamma} k^4\, c^2 (\mu\, \gamma^n)^2 \tilde{z} + \frac{Re}{\gamma} k^3\, c^2\, i\, \mu\, \gamma^n D\tilde{z} - k^4 (\mu\, \gamma^n)^2 \tilde{z} - k^3 i\, \mu\, \gamma^n D\tilde{z} + (\mu\, \gamma^n)^2 k^2 D^2 \tilde{z} + i\, k\mu\, \gamma^n D^3 \tilde{z} + \frac{Re}{\gamma} k^3\, c^2 i\mu\, \gamma^n D\tilde{z} - \frac{Re}{\gamma} k^2\, c^2\, D^2 \tilde{z} - ik^3 \mu\, \gamma^n D\tilde{z} + k^2 D^2 \tilde{z} + i\mu\, \gamma^n k D^3 \tilde{z} - D^4 \tilde{z} + k^2 (D^2 - k^2)\, \tilde{z} = -\frac{Re}{\gamma} k^4\, c^2 \tilde{z}$$

$$\Rightarrow \quad \frac{Re}{\gamma}k^4\, c^2(\mu\,\gamma^n)^2\tilde{z} + \frac{Re}{\gamma}k^3\, c^2\, i\,\mu\,\gamma^n D\tilde{z} \; + \frac{Re}{\gamma}k^3\, c^2 i\mu\,\gamma^n D\tilde{z} - \frac{Re}{\gamma}k^2\, c^2\, D^2\tilde{z} + i\mu\,\gamma^n k D^3\tilde{z} + \frac{Re}{\gamma}k^4\, c^2\tilde{z} = D^4\tilde{z} + ik^3\mu\,\gamma^n D\tilde{z} - k^2D^2\tilde{z} + k^4(\mu\,\gamma^n)^2\tilde{z} + k^3 i\,\mu\,\gamma^n D\tilde{z} - (\mu\,\gamma^n)^2k^2D^2\tilde{z} - i\,k\mu\,\gamma^n D^3\tilde{z} - k^2(D^2 - k^2)\,\tilde{z}$$

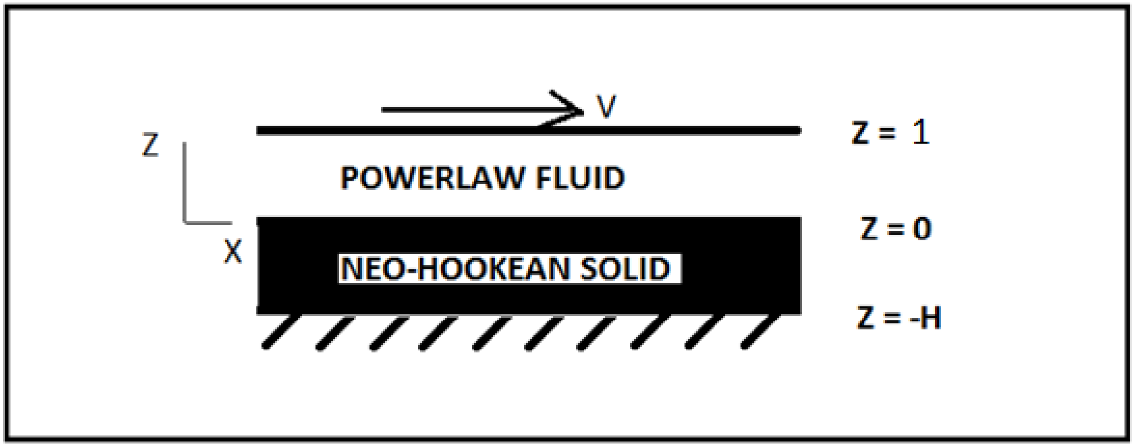
$$\Rightarrow \quad \frac{Re}{\gamma}k^4\, c^2(\mu\,\gamma^n)^2\tilde{z} + \frac{Re}{\gamma}k^3\, c^2\, i\,\mu\,\gamma^n D\tilde{z} \; + \frac{Re}{\gamma}k^3\, c^2 i\mu\,\gamma^n D\tilde{z} - \frac{Re}{\gamma}k^2\, c^2\, D^2\tilde{z} + \frac{Re}{\gamma}k^4\, c^2\tilde{z} = D^4\tilde{z} - k^2D^2\tilde{z} + k^4(\mu\,\gamma^n)^2\tilde{z} - (\mu\,\gamma^n)^2k^2D^2\tilde{z} - 2\,i\,k\mu\,\gamma^n D^3\tilde{z} + 2k^3 i\,\mu\,\gamma^n D\tilde{z} - k^2\, D^2\tilde{z} + k^4\,\tilde{z}$$

$$\Rightarrow \quad -\frac{Re}{\gamma}k^2\, c^2\, D^2\tilde{z} + 2\frac{Re}{\gamma}k^3\, c^2\, i\,\mu\,\gamma^n D\tilde{z} \; + \frac{Re}{\gamma}k^4\, c^2\tilde{z} + \frac{Re}{\gamma}k^4\, c^2(\mu\,\gamma^n)^2\tilde{z} = D^4\tilde{z} - 2i\,k\mu\,\gamma^n D^3\tilde{z} - 2k^2D^2\tilde{z} - (\mu\,\gamma^n)^2k^2D^2\tilde{z} + 2k^3 i\,\mu\,\gamma^n D\tilde{z} + k^4\,\tilde{z} + k^4(\mu\,\gamma^n)^2\tilde{z}$$

$$\Rightarrow \quad -\frac{\boldsymbol{Re}}{\boldsymbol{\gamma}}\boldsymbol{k}^2\, \boldsymbol{c}^2\, \boldsymbol{D}^2\tilde{\mathbf{z}} + 2\frac{\boldsymbol{Re}}{\boldsymbol{\gamma}}\boldsymbol{k}^3\, \boldsymbol{c}^2\, \boldsymbol{i}\,\boldsymbol{\mu}\,\boldsymbol{\gamma}^{\boldsymbol{n}}\boldsymbol{D}\tilde{\mathbf{z}} \; + \frac{\boldsymbol{Re}}{\boldsymbol{\gamma}}\boldsymbol{k}^4\, \boldsymbol{c}^2[1 + (\boldsymbol{\mu}\,\boldsymbol{\gamma}^{\boldsymbol{n}})^2]\tilde{\mathbf{z}} = \boldsymbol{D}^4\tilde{\mathbf{z}} - 2\boldsymbol{i}\,\boldsymbol{k}\boldsymbol{\mu}\,\boldsymbol{\gamma}^{\boldsymbol{n}}\boldsymbol{D}^3\tilde{\mathbf{z}} - \boldsymbol{k}^2[2 + (\boldsymbol{\mu}\,\boldsymbol{\gamma}^{\boldsymbol{n}})^2]\boldsymbol{D}^2\tilde{\mathbf{z}} + 2\boldsymbol{i}\boldsymbol{k}^3\,\boldsymbol{\mu}\,\boldsymbol{\gamma}^{\boldsymbol{n}}\boldsymbol{D}\tilde{\mathbf{z}} + \boldsymbol{k}^4\,[1 + (\boldsymbol{\mu}\,\boldsymbol{\gamma}^{\boldsymbol{n}})^2]\tilde{\mathbf{z}} \tag{33}$$

the above equation is fourth order differential equation for the Neo-Hookean solid.

4.3 Linearized boundary conditions:



Taylor's expansion-

$G|_{z=g(x)} = G|_{z=0} + g\left.\frac{\partial G}{\partial z}\right|_{z=0}$; G is a dynamical quantity.

At interface, i.e. at z $= g(x)$:

The stress continuity conditions at the interface $z = g(x)$ are linearized about the unperturbed interface at $z = 0$.

$z = g(x)$

$\Rightarrow z - g(x) = 0 = f(g, z)$ (say)

Normal to the perturbation is defined as,

$\mathrm{n} = \frac{\nabla f}{|\nabla f|}$

$\nabla f = (\mathrm{e_x}\partial_x + \mathrm{e_z}\partial_z)[\, z - g(x)]$

$\Rightarrow \nabla f = \partial_x g(x)\mathrm{e_x} - \mathrm{e_z}$

$|\nabla f| = \sqrt{1 + [\partial_x g(x)]^2} = 1$ ($\because g$ is a small perturbation)

$\therefore\ \mathrm{n} = -\partial_x g(x)\mathrm{e_x} + \mathrm{e_z}$

Normal in the lower part of perturbation will be

$\mathrm{n} = -\partial_x g(x)\mathrm{e_x} - \mathrm{e_z}$

As $\mathrm{n.t} = 0$; t is the corresponding tangent to the perturbation

$\mathrm{t} = -\mathrm{e_x} + \partial_x g(x)\mathrm{e_z}$

where, $\mathrm{e_x}$ and $\mathrm{e_z}$ are the unit vectors in *x* and *z* directions, respectively.

So, we have

$n_1 = -\partial_x g(x)$ $n_2 = -1$

$t_1 = -1$ $t_2 = \partial_x g(x)$

Tangential stress-

$(n.\tau.t) = n_1\tau t_1 + n_1\tau_{12}t_2 + n_2\tau_{21}t_1 + n_2\tau_{22}t_2$

$= \partial_x g(x)\tau_{11} - [\partial_x g(x)]^2\tau_{12} + \tau_{21} - \partial_x g(x)\tau_{22}$

$= \partial_x g(x)\tau_{11} + \tau_{21} - \partial_x g(x)\tau_{22}$ (linearization)

$= \tau_{21} - \partial_x g(x)[\tau_{22} - \tau_{11}]$

Normal stress-

$(\text{n}.\tau.\text{n}) = n_1\tau_{11}n_1 + n_1\tau_{12}n_2 + n_2\tau_{21}n_1 + n_2\tau_{22}n_2$

$= [\partial_x g(x)]^2\tau_{11} + \partial_x g(x)\tau_{12} + \partial_x g(x)\tau_{21} + \tau_{22}$

$= \partial_x g(x)\tau_{12} + \partial_x g(x)\tau_{21} + \tau_{22}$ (linearization)

$= \tau_{22} + \partial_x g(x)[\tau_{12} + \tau_{21}]$

Continuity of velocity at $= \boldsymbol{g(x)}$:

(i) *z*-velocity

$v_z^{\text{fluid}} = v_z^{\text{gel}}$ at $z = g(x)$

$\Rightarrow v_z = \partial_t z$

$\Rightarrow [v_z + g\partial_z v_z]_{z=0} = \partial_t[z + g\partial_z z]_{z=0}$ (∵Taylor's expansion)

$\Rightarrow [v_z' + g\partial_z v_z']_{z=0} = \partial_t[z' + g\partial_z z']_{z=0}$

$\Rightarrow v_z' = \partial_t z'$ (linearization)

$$\boldsymbol{\tilde{v}_z = -ikc\tilde{z}} \quad (34)$$

(ii) *x*-velocity

$v_x^{\text{fluid}} = v_x^{\text{gel}}$ at $z = g(x)$

$\Rightarrow v_x = \partial_t x$

$\Rightarrow [v_x + g\partial_z v_x]_{z=0} = \partial_t [x + g\partial_z x]_{z=0}$ ($\because$ Taylor's expansion)

$\Rightarrow [(\overline{v}_x + v_x') + g\partial_z(\overline{v}_x + v_x')]_{z=0} = \partial_t[(\bar{x} + x') + g\partial_z(\bar{x} + x')]_{z=0}$

$\Rightarrow [(\overline{v}_x + v_x') + g\partial_z\overline{v}_x]_{z=0} = \partial_t[(\bar{x} + x') + g\partial_z\bar{x}]_{z=0}$ (linearization)

$\Rightarrow v_x' + g\partial_z\overline{v}_x|_{z=0} = \partial_t u_x'$

($\because \overline{v}_x|_{z=0} = 0; \partial_t\bar{x} = 0$)

$\Rightarrow \tilde{v}_x + \tilde{g}d_z\overline{v}_x|_{z=0} = -ikc\tilde{x}$

$\Rightarrow \tilde{v}_x + \tilde{z}d_z\overline{v}_x|_{z=0} = -ikc\tilde{x}$

$$\Rightarrow \boldsymbol{\tilde{v}_x + \gamma\tilde{z} = -ikc\tilde{x}} \tag{35}$$

From equation no. (25) and (29)

$\Rightarrow \frac{iD\tilde{v}_z}{k} + \gamma\tilde{z} = -ikc\left(\mu\,\gamma^n\tilde{z} + \frac{iD\tilde{z}}{k}\right)$

$\Rightarrow iD\tilde{v}_z + k\gamma\tilde{z} = -ik^2c\mu\,\gamma^n\tilde{z} + kcD\tilde{z}$

$$\Rightarrow \boldsymbol{iD\tilde{v}_z - kcD\tilde{z} + k\gamma\tilde{z} + ik^2c\mu\,\gamma^n\tilde{z} = 0} \tag{36}$$

Continuity of stresses at $z = g(x)$:

(iii) Linearized tangential stress balance

$(\mathbf{n}.\boldsymbol{\tau}.\mathbf{t})^{\mathbf{fluid}} = (\mathbf{n}.\boldsymbol{\sigma}.\mathbf{t})^{\mathbf{solid}}$ at $z = g(x)$

$(n.\tau.t) = \tau_{21} - \partial_x g(x)[\tau_{22} - \tau_{11}]$ similarly $(n.\sigma.t) = \sigma_{21} - \partial_x g(x)[\sigma_{22} - \sigma_{11}]$

$(\mathrm{n}.\tau.\mathrm{n}) = \tau_{22} + \partial_x g(x)[\tau_{12} + \tau_{21}]$ $(\mathrm{n}.\sigma.\mathrm{n}) = \sigma_{22} + \partial_x g(x)[\sigma_{12} + \sigma_{21}]$

From equation number (8), (9) and (10)

$$\Rightarrow \quad \tau_{11} = -\bar{p}_f - p_f{}' + 2\,\mu \left(\gamma + \frac{\partial v_z{}'}{\partial x} + \frac{\partial v_x{}'}{\partial z}\right)^{(n-1)} \frac{\partial v_X{}'}{\partial x}$$

$$\Rightarrow \quad \tau_{12} = \tau_{21} = \mu \left(\gamma + \frac{\partial v_z{}'}{\partial x} + \frac{\partial v_x{}'}{\partial z}\right)^{n}$$

$$\Rightarrow \quad \tau_{22} = -\bar{p}_f - p_f{}' + 2\,\mu \left(\gamma + \frac{\partial v_z{}'}{\partial x} + \frac{\partial v_x{}'}{\partial z}\right)^{(n-1)} \frac{\partial v_z{}'}{\partial z}$$

$$(n.\tau.t) = \tau_{21} - \partial_x g(x)[\tau_{22} - \tau_{11}]$$

$$\Rightarrow \quad (\mathrm{n}.\tau.\mathrm{t})^{\mathrm{fluid}} = \mu \left(\gamma + \frac{\partial v_z{}'}{\partial x} + \frac{\partial v_x{}'}{\partial z}\right)^{n} - \partial_x g(x)\left[\left(-\bar{p}_f - p_f{}' + 2\,\mu \left(\gamma + \frac{\partial v_z{}'}{\partial x} + \frac{\partial v_x{}'}{\partial z}\right)^{(n-1)} \frac{\partial v_z{}'}{\partial z}\right) - \left(-\bar{p}_f - p_f{}' + 2\,\mu \left(\gamma + \frac{\partial v_z{}'}{\partial x} + \frac{\partial v_x{}'}{\partial z}\right)^{(n-1)} \frac{\partial v_X{}'}{\partial x}\right)\right]$$

$$\Rightarrow \quad (\mathrm{n}.\tau.\mathrm{t})^{\mathrm{fluid}} = \mu\gamma^n \left(1 + \frac{n}{\gamma}\frac{\partial v_z{}'}{\partial x} + \frac{n}{\gamma}\frac{\partial v_x{}'}{\partial z}\right)$$

$$(\mathrm{n}.\tau.\mathrm{t})^{\mathrm{solid}} = \sigma_{21} - \partial_x g(x)[\sigma_{22} - \sigma_{11}]$$

$$\Rightarrow \quad \sigma_{11} = -\bar{p}_s - p_s{}' + 1 + 2\frac{\partial x'}{\partial X} + (\mu\,\gamma^n)^2 + 2\mu\,\gamma^n \frac{\partial x'}{\partial Z}$$

$$\Rightarrow \quad \sigma_{12} = \sigma_{21} = \frac{\partial x'}{\partial Z} + \frac{\partial z'}{\partial X} + \mu\,\gamma^n \frac{\partial z'}{\partial Z} + \mu\,\gamma^n$$

$$\Rightarrow \quad \sigma_{22} = -\bar{p}_s - p_s{}' + 1 + 2\frac{\partial z'}{\partial Z}$$

$$\Rightarrow \quad (\mathrm{n}.\tau.\mathrm{t})^{\mathrm{solid}} = \left(\frac{\partial x'}{\partial Z} + \frac{\partial z'}{\partial X} + \mu\,\gamma^n \frac{\partial z'}{\partial Z} + \mu\,\gamma^n\right) - \partial_x g(x)\left[\left(-\bar{p}_s - p_s{}' + 1 + 2\frac{\partial z'}{\partial Z}\right) - \left(-\bar{p}_s - p_s{}' + 1 + 2\frac{\partial x'}{\partial X} + (\mu\,\gamma^n)^2 + 2\mu\,\gamma^n \frac{\partial x'}{\partial Z}\right)\right]$$

$$\Rightarrow \quad (\mathrm{n}.\tau.\mathrm{t})^{\mathrm{solid}} = -(\mu\,\gamma^n)^2 \frac{\partial z'}{\partial X} + \frac{\partial z'}{\partial X} + \mu\,\gamma^n + \frac{\partial x'}{\partial Z} + \mu\,\gamma^n \frac{\partial z'}{\partial Z}$$

$(\mathbf{n}.\boldsymbol{\tau}.\mathbf{t})^{\mathbf{fluid}} = (\mathbf{n}.\boldsymbol{\sigma}.\mathbf{t})^{\mathbf{solid}}$

$$\Rightarrow \quad \mu\gamma^{n}\left(1+\frac{n}{\gamma}\frac{\partial v_z{}'}{\partial x}+\frac{n}{\gamma}\frac{\partial v_x{}'}{\partial z}\right) = -(\mu\,\gamma^{n})^{2}\frac{\partial z'}{\partial X}+\frac{\partial z'}{\partial X}+\mu\,\gamma^{n}+\frac{\partial x'}{\partial Z}+\mu\,\gamma^{n}\frac{\partial z'}{\partial Z}$$

$$\Rightarrow \quad \mu\gamma^{n}\left(1+\frac{n}{\gamma}\frac{\partial v_z{}'}{\partial x}+\frac{n}{\gamma}\frac{\partial v_x{}'}{\partial z}\right)+(\mu\,\gamma^{n})^{2}\frac{\partial z'}{\partial X}-\frac{\partial z'}{\partial X}-\mu\,\gamma^{n}-\frac{\partial x'}{\partial Z}-\mu\,\gamma^{n}\frac{\partial z'}{\partial Z}=0$$

on simplification

$$\Rightarrow \quad \mu n\gamma^{n-1}D\tilde{v}_x+\mu n\gamma^{n-1}\,i\,k\,\tilde{v}_z-D\tilde{x}-i\,k\,\tilde{z}-\mu\,\gamma^{n}\,D\tilde{z}+(\mu\,\gamma^{n})^{2}\,i\,k\,\tilde{z}=0$$

Using equation no. (25) and (29)

$$\Rightarrow \quad \mu n\gamma^{n-1}D\left(\frac{iD\tilde{v}_z}{k}\right)+\mu n\gamma^{n-1}\,i\,k\,\tilde{v}_z-D\left(\mu\,\gamma^{n}\tilde{z}+\frac{iD\tilde{z}}{k}\right)-i\,k\,\tilde{z}-\mu\,\gamma^{n}\,D\tilde{z}+(\mu\,\gamma^{n})^{2}\,i\,k\,\tilde{z}=0$$

$$\Rightarrow \quad \mathbf{i\mu n\gamma^{n-1}D^{2}\tilde{v}_z+\mu n\gamma^{n-1}\,i\;k^{2}\tilde{v}_z-K\mu\,\gamma^{n}D\tilde{z}-iD^{2}\tilde{z}-\,i\,k^{2}\,\tilde{z}-\mu\,\gamma^{n}\,KD\tilde{z}+(\mu\,\gamma^{n})^{2}\,i\,k^{2}\tilde{z}=0} \qquad (37)$$

(iv) Linearized normal stress balance at $z = g(x)$:

$(\mathbf{n}.\boldsymbol{\tau}.\mathbf{n})^{\mathbf{fluid}} - (\mathbf{n}.\boldsymbol{\sigma}.\mathbf{n})^{\mathbf{solid}} = \boldsymbol{T}\,(\boldsymbol{\nabla}.\mathbf{n})$ at $z = g(x)$

$$\Rightarrow \quad \tau_{22}+\partial_x g(x)[\tau_{12}+\tau_{21}]-\sigma_{22}-\partial_x g(x)[\sigma_{12}+\sigma_{21}]-T\,\frac{\partial^{2}}{\partial x^{2}}\,g(x)=0$$

$$\Rightarrow \quad \left(-\bar{p}_f-p_f{}'+2\,\mu\left(\gamma+\frac{\partial v_z{}'}{\partial x}+\frac{\partial v_x{}'}{\partial z}\right)^{(n-1)}\frac{\partial v_z{}'}{\partial z}\right)+\partial_x g(x)\left[\left(\mu\left(\gamma+\frac{\partial v_z{}'}{\partial x}+\frac{\partial v_x{}'}{\partial z}\right)^{n}\right)+\left(\mu\left(\gamma+\frac{\partial v_z{}'}{\partial x}+\frac{\partial v_x{}'}{\partial z}\right)^{n}\right)\right]-\left(-\bar{p}_s-p_s{}'+1+2\frac{\partial z'}{\partial Z}\right)-\partial_x g(x)\left[\left(\frac{\partial x'}{\partial Z}+\frac{\partial z'}{\partial X}+\mu\,\gamma^{n}\frac{\partial z'}{\partial Z}+\mu\,\gamma^{n}\right)+\left(\frac{\partial x'}{\partial Z}+\frac{\partial z'}{\partial X}+\mu\,\gamma^{n}\frac{\partial z'}{\partial Z}+\mu\,\gamma^{n}\right)\right]-T\,\frac{\partial^{2}z}{\partial x^{2}}=0$$

on simplification

$$\Rightarrow \quad \boldsymbol{-\tilde{p}_f+\tilde{p}_s+2\mu\,\gamma^{n-1}D\tilde{v}_z-2D\,\tilde{z}-k^{2}\,T\,\tilde{z}=0} \qquad (38)$$

Using equation no. (26) and (31) we get

➢ $2k^2\mu\gamma^{n-1}D\tilde{v}_z - \mu n\gamma^{n-1}D^3\tilde{v}_z - \mu n\gamma^{n-1}k^2D\tilde{v}_z - \frac{iRek}{\gamma}(cD\tilde{v}_z - \gamma zD\tilde{v}_z + \gamma\tilde{v}_z) +$
$-\frac{Re}{\gamma}ik^3\ c^2\mu\,\gamma^n\tilde{z} + \frac{Re}{\gamma}k^2\ c^2\ D\tilde{z} + ik^3\mu\,\gamma^n\tilde{z} - k^2D\tilde{z} - i\mu\,\gamma^n kD^2\tilde{z} + D^3\tilde{z} + 2\mu\,\gamma^{n-1}k^2D\tilde{v}_z -$
$2k^2D\,\tilde{z} - k^4\,T\,\tilde{z} = 0$

Arranging the above terms

➢ $$\boldsymbol{-\mu n\gamma^{n-1}D^3\tilde{v}_z + 4k^2\mu\gamma^{n-1}D\tilde{v}_z - \mu n\gamma^{n-1}k^2D\tilde{v}_z - \frac{iRek}{\gamma}cD\tilde{v}_z - iRek\tilde{v}_z + +D^3\tilde{z} - i\mu\,\gamma^n kD^2\tilde{z} + \frac{Re}{\gamma}k^2\ c^2\ D\tilde{z} - k^2D\tilde{z} - 2k^2D\,\tilde{z} + ik^3\mu\,\gamma^n\tilde{z} - \frac{Re}{\gamma}ik^3\ c^2\mu\,\gamma^n\tilde{z} - k^4\,T\,\tilde{z} = 0} \quad (39)$$

(v) **No slip condition at z = 1**

- $\boldsymbol{\tilde{v}_z = 0}$
- $\tilde{v}_x = 0$ using equation no. (25), $\tilde{v}_x = 0 = \mathbf{D}\boldsymbol{\tilde{v}_z = 0}$ (40)

(vi) **No slip condition at z = -h**

- $\boldsymbol{\tilde{z} = 0}$
- $\tilde{x} = 0$ using equation no. (29) $\tilde{x} = 0 = \mathrm{D}\boldsymbol{\tilde{z} = 0}$ (41)

Hence we are left with two fourth order Orr Sommerfield eqn (equation no. 28 and 34) and eight boundary conditions.

- $-ikc\frac{Re}{\gamma}D^2\tilde{v}_z + ikRezD^2\tilde{v}_z + ik^3c\frac{Re}{\gamma}\tilde{v}_z - zik^3Re\tilde{v}_z = \mu n\gamma^{n-1}D^4\tilde{v}_z + 2\mu\gamma^{n-1}k^2(n-2)D^2\tilde{v}_z + k^4\mu n\gamma^{n-1}\tilde{v}_z$

- $-\frac{Re}{\gamma}k^2\ c^2\ D^2\tilde{z} + 2\frac{Re}{\gamma}k^3\ c^2\ i\ \mu\,\gamma^n D\tilde{z} + \frac{Re}{\gamma}k^4\ c^2[1+(\mu\,\gamma^n)^2]\tilde{z} = D^4\tilde{z} - 2i\ k\mu\,\gamma^n D^3\tilde{z} - k^2[2+(\mu\,\gamma^n)^2]D^2\tilde{z} + 2\ ik^3\mu\,\gamma^n D\tilde{z} + k^4\ [1+(\mu\,\gamma^n)^2]\tilde{z}$

No slip condition at z = 1

- $\tilde{v}_z = 0$
- $D\tilde{v}_z = 0$

No slip condition at z = -h

- $\tilde{z} = 0$
- $D\tilde{z} = 0$

At interface i.e. at z = 0

- $-\mu n\gamma^{n-1}D^3\tilde{v}_z + 2k^2\mu\gamma^{n-1}D\tilde{v}_z + 2\mu\,\gamma^{n-1}k^2D\tilde{v}_z - \mu n\gamma^{n-1}k^2D\tilde{v}_z - \frac{iRek}{\gamma}cD\tilde{v}_z - iRek\tilde{v}_z + +D^3\tilde{z} - i\mu\,\gamma^n kD^2\tilde{z} + \frac{Re}{\gamma}k^2\,c^2\,D\tilde{z} - 3k^2D\,\tilde{z} + ik^3\mu\,\gamma^n\tilde{z} - \frac{Re}{\gamma}ik^3\,c^2\mu\,\gamma^n\tilde{z} - k^4\,T\,\tilde{z} = 0$

- $iD\tilde{v}_z - kcD\tilde{z} + k\gamma\tilde{z} + ik^2c\mu\,\gamma^n\tilde{z} = 0$

- $\mu n\gamma^{n-1}D^2\tilde{v}_z + \mu n\gamma^{n-1}\,k^2\tilde{v}_z - D^2\tilde{z} - k^2\,\tilde{z} + 2i\mu\,\gamma^n\,KD\tilde{z} + (\mu\,\gamma^n)^2\,k^2\tilde{z} = 0$

- $\tilde{v}_z = -ikc\tilde{z}$

We will be solving the above equations using pseudospectral also known as collocation method for solving in Matlab and similarly we will also be using Mathematica to solve the fourth order differential equation.

CHAPTER - 5

SPECTRAL COLLOCATION METHOD

For solving the differential equations in Matlab, We need to opt this method so as to find out the eigen values. We need to convert the variables in the y-domain between (-1,1).

5.1 Power-law fluid

Orr-Sommerfeld Equation (from equation no.28)

$$\Rightarrow\ -ikc\frac{Re}{\gamma}D^2\tilde{v}_z + ikRezD^2\tilde{v}_z + ik^3c\frac{Re}{\gamma}\tilde{v}_z - zik^3Re\tilde{v}_z = \mu n\gamma^{n-1}D^4\tilde{v}_z + 2\mu\gamma^{n-1}k^2(n-2)D^2\tilde{v}_z + k^4\mu n\gamma^{n-1}\tilde{v}_z$$

$$\Rightarrow\ -ikc\frac{Re}{\gamma}d_z^2\tilde{v}_z + ik^3c\frac{Re}{\gamma}\tilde{v}_z = \mu n\gamma^{n-1}d_z^4\tilde{v}_z + [2\mu\gamma^{n-1}k^2(n-2) - ikRez]d_z^2\tilde{v}_z + [k^4\mu n\gamma^{n-1} + zik^3Re]\tilde{v}_z$$

Converting the above Orr Sommerfield equation into Y - domain

$$z = \frac{1-Y}{2} \Rightarrow \boldsymbol{y = 1 - 2z} \tag{42}$$

At $z = 0 \Rightarrow y = 1$

And at $z = 1 \Rightarrow y = -1$

Also, $\mathbf{\frac{dy}{dz} = -2} \qquad \mathbf{dz = -\frac{dy}{2}} \quad ; \quad \mathbf{(dz)^2 = \frac{(dy)^2}{4}} \quad ; \quad \mathbf{(dz)^3 = -\frac{(dy)^3}{8}} \quad ;$

$$\mathbf{(dz)^4 = \frac{(dy)^4}{16}} \tag{43}$$

Converting to Y domain, using equation (42,43)

$$\Rightarrow\ -4ikc\frac{Re}{\gamma}d_y^2\tilde{v}_z + ik^3c\frac{Re}{\gamma}\tilde{v}_z = 16\mu n\gamma^{n-1}d_y^4\tilde{v}_z + 4\left[2\mu\gamma^{n-1}k^2(n-2) - ikRe\left(\frac{1-Y}{2}\right)\right]d_y^2\tilde{v}_z + \left[k^4\mu n\gamma^{n-1} + \left(\frac{1-Y}{2}\right)ik^3Re\right]\tilde{v}_z$$

$$\Rightarrow\ -4ikc\frac{Re}{\gamma}d_y^2\tilde{v}_z + ik^3c\frac{Re}{\gamma}\tilde{v}_z = 16\mu n\gamma^{n-1}d_y^4\tilde{v}_z + [8\mu\gamma^{n-1}k^2(n-2) - 2ikRe(1-y)]d_y^2\tilde{v}_z + \left[k^4\mu n\gamma^{n-1} + \left(\frac{1-Y}{2}\right)ik^3Re\right]\tilde{v}_z$$

Let

$$d_y\tilde{v}_z = \varphi_1$$

$$d_y^2\tilde{v}_z = d_y\varphi_1 = \varphi_2$$

$$d_y^3\tilde{v}_z = d_y\varphi_2 = \varphi_3$$

Hence we get,

$$\Rightarrow \ -4\mathbf{ikc}\frac{\mathbf{Re}}{\gamma}\boldsymbol{\varphi}_2 + \mathbf{ik}^3\mathbf{c}\frac{\mathbf{Re}}{\gamma}\tilde{\mathbf{v}}_\mathbf{z} = 16\boldsymbol{\mu}\mathbf{n}\boldsymbol{\gamma}^{\mathbf{n}-1}\mathbf{d_y}\boldsymbol{\varphi}_3{}' + [8\boldsymbol{\mu}\boldsymbol{\gamma}^{\mathbf{n}-1}\mathbf{k}^2(\mathbf{n}-2) - 2\mathbf{ikRe}(1-\mathbf{y})]\boldsymbol{\varphi}_2 + \left[\mathbf{k}^4\boldsymbol{\mu}\mathbf{n}\boldsymbol{\gamma}^{\mathbf{n}-1} + \left(\frac{1-\mathbf{Y}}{2}\right)\mathbf{ik}^3\mathbf{Re}\right]\tilde{\mathbf{v}}_\mathbf{z} \quad (44)$$

$$\Rightarrow \boldsymbol{d_y\tilde{v}_z - \varphi_1 = 0}$$

$$\Rightarrow \boldsymbol{d_y\varphi_1 - \varphi_2 = 0}$$

$$\Rightarrow \boldsymbol{d_y\varphi_2 - \varphi_3 = 0} \quad (45)$$

5.2 Neo-Hookean Solid

Now, converting the solid equations into Y domain

Orr-Sommerfeld Equation (from equation no. (34))

$$-\frac{Re}{\gamma}k^2\, c^2\, D^2\tilde{z} + 2\frac{Re}{\gamma}k^3\, c^2\, i\,\mu\,\gamma^n D\tilde{z} \ \ +\frac{Re}{\gamma}k^4\, c^2[1+(\mu\,\gamma^n)^2]\tilde{z} = D^4\tilde{z} - \ 2i\,k\mu\,\gamma^n D^3\tilde{z} \ - k^2[2+(\mu\,\gamma^n)^2]D^2\tilde{z} + 2k^3 i\,\mu\,\gamma^n D\tilde{z} + k^4\,[1+(\mu\,\gamma^n)^2]\tilde{z}$$

$$-\frac{Re}{\gamma}k^2\, c^2\, d_z^2\tilde{z} + 2\frac{Re}{\gamma}k^3\, c^2\, i\,\mu\,\gamma^n d_z\tilde{z} \ \ +\frac{Re}{\gamma}k^4\, c^2[1+(\mu\,\gamma^n)^2]\tilde{z} = d_z^4\tilde{z} - \ 2ik\mu\,\gamma^n d_z^3\tilde{z} \ - k^2[2+(\mu\,\gamma^n)^2]d_z^2\tilde{z} + 2k^3 i\,\mu\,\gamma^n d_z\tilde{z} + k^4\,[1+(\mu\,\gamma^n)^2]\tilde{z}$$

Converting to Y domain

At z = -H y = -1

And at z = 0 y = 1

Therefore,

$$\mathbf{z} \ = \left(\frac{\mathbf{y}-1}{2}\right)\mathbf{H}$$

$$\boldsymbol{dz} \ = \frac{\mathbf{H}}{2}\mathbf{dy} \quad ; \quad (\boldsymbol{dz})^2 = \frac{\boldsymbol{H^2(dy)^2}}{4} \quad ; \quad (\boldsymbol{dz})^3 = \frac{\boldsymbol{H^3(dy)^3}}{8} \quad ; \quad (\boldsymbol{dz})^4 = \frac{\boldsymbol{H^4(dy)^4}}{16} \quad (46)$$

Using the above equation, converting the Solid Orr Sommerfield equation

$$-\frac{4}{H^2}\frac{Re}{\gamma}k^2\, c^2\, d_y^2\tilde{z} + \frac{4}{H}\frac{Re}{\gamma}k^3\, c^2\, i\,\mu\,\gamma^n d_y\tilde{z} \;\; + \frac{Re}{\gamma}k^4\, c^2[1 + (\mu\,\gamma^n)^2]\tilde{z} = \frac{16}{H^4}d_y^4\tilde{z} - \frac{16}{H^3}ik\mu\,\gamma^n d_y^3\tilde{z} - \frac{4}{H^2}k^2[2 + (\mu\,\gamma^n)^2]d_y^2\tilde{z} + \frac{4}{H}k^3 i\,\mu\,\gamma^n d_Y\tilde{z} + k^4\,[1 + (\mu\,\gamma^n)^2]\tilde{z}$$

Let $d_y\tilde{z} = \psi_1$

$$d_y^2\tilde{z} = d_y\psi_1 = \psi_2$$

$$d_y^3\tilde{z} = d_y\psi_2 = \psi_3$$

Eq. transforms to,

$$\boldsymbol{-4H^2\frac{Re}{\gamma}k^2\, c^2\, \psi_2 + 4\frac{Re}{\gamma}H^3k^3\, c^2\, i\,\mu\,\gamma^n\psi_1 + \frac{Re}{\gamma}H^4k^4\, c^2[1 + (\mu\,\gamma^n)^2]\tilde{z} = 16\psi_3' - 16Hik\mu\,\gamma^n\psi_3 - 4k^2H^2[2 + (\mu\,\gamma^n)^2]\psi_2 + 4H^3k^3 i\,\mu\,\gamma^n\psi_1 + k^4H^4\,[1 + (\mu\,\gamma^n)^2]\tilde{z}} \quad (47)$$

$$\boldsymbol{d_y\tilde{z} - \psi_1 = 0}$$

$$\boldsymbol{d_y^2\tilde{z} - \psi_2 = 0}$$

$$\boldsymbol{d_y^3\tilde{z} - \psi_3 = 0} \quad (48)$$

Now converting the boundary conditions into y-domain

- $\tilde{v}_z + ikc\tilde{z} = 0$

$\Rightarrow$ $\boldsymbol{\tilde{v}_z + ikc\tilde{z} = 0}$

- $\tilde{v}_x + \gamma\tilde{u}_z = -ikc\tilde{x}$

$\Rightarrow$ $id_z\tilde{v}_z - kcd_z\tilde{z} + k\gamma\tilde{z} + ik^2c\mu\,\gamma^n\tilde{z} = 0$

$\Rightarrow$ $-2id_y\tilde{v}_z - \frac{2}{H}kcd_z\tilde{z} + k\gamma\tilde{z} + ik^2c\mu\,\gamma^n\tilde{z} = 0$

$\Rightarrow$ $-2Hid_y\tilde{v}_z - 2kcd_z\tilde{z} + kH\gamma\tilde{z} + ik^2Hc\mu\,\gamma^n\tilde{z} = 0$

$\Rightarrow$ $\boldsymbol{-2Hi\varphi_1 - 2kc\psi_1 + kH\gamma\tilde{z} + ik^2Hc\mu\,\gamma^n\tilde{z} = 0}$

- $\mu n\gamma^{n-1}D^2\tilde{v}_z + \mu n\gamma^{n-1}\, k^2\tilde{v}_z - D^2\tilde{z} - k^2\,\tilde{z} + 2i\mu\,\gamma^n\, KD\tilde{z} + \quad (\mu\,\gamma^n)^2\;\; k^2\tilde{z} = 0$

$\Rightarrow \quad \mu n\gamma^{n-1}d_z^2\tilde{v}_z + \mu n\gamma^{n-1}\, k^2\tilde{v}_z - d_z^2\tilde{v}_z - k^2\,\tilde{z} + 2i\mu\,\gamma^n\, Kd_z\tilde{z} + \quad (\mu\,\gamma^n)^2\;\; k^2\tilde{z} = 0$

$\Rightarrow \quad 4\mu n\gamma^{n-1}d_z^2\tilde{v}_z + \mu n\gamma^{n-1}\, k^2\tilde{v}_z - \frac{4}{H^2}d_y^2\tilde{v}_z - k^2\,\tilde{z} + \frac{4}{H}i\mu\,\gamma^n\, Kd_Y\tilde{z} + \quad (\mu\,\gamma^n)^2\;\; k^2\tilde{z} = 0$

$\Rightarrow \quad \boldsymbol{4\mu n\gamma^{n-1}H^2\varphi_2 + \mu n\gamma^{n-1}H^2k^2\tilde{v}_z - 4\psi_2 - k^2\, H^2\tilde{z} + 4i\mu\,\gamma^n\, HK\psi_1 + H^2\,(\mu\,\gamma^n)^2\;\; k^2\tilde{z} = 0}$

- $\mu n\gamma^{n-1}d_z^3\tilde{v}_z + 4k^2\mu\gamma^{n-1}d_z\tilde{v}_z - \mu n\gamma^{n-1}k^2d_z\tilde{v}_z - \frac{iRek}{\gamma}cd_z\tilde{v}_z - iRek\tilde{v}_z\; + d_z^3\tilde{z} - i\mu\,\gamma^n kd_z^2\tilde{z} + \frac{Re}{\gamma}k^2\; c^2\; d_z\tilde{z} - k^2d_z\tilde{z}\; - \;2k^2d_z\,\tilde{z} + \; ik^3\mu\,\gamma^n\tilde{z} - \frac{Re}{\gamma}ik^3\; c^2\mu\,\gamma^n\tilde{z} - \; k^4\; T\;\tilde{z} = 0$

$\Rightarrow \quad 8\mu n\gamma^{n-1}d_y^3\tilde{v}_z - 8k^2\mu\gamma^{n-1}d_y\tilde{v}_z + 2\mu n\gamma^{n-1}k^2d_y\tilde{v}_z + 2\frac{iRek}{\gamma}cd_y\tilde{v}_z - iRek\tilde{v}_z + \frac{8}{H^3}d_z^3\tilde{z} - \frac{4}{H^2}i\mu\,\gamma^n kd_y^2\tilde{z} + \frac{2}{H}\frac{Re}{\gamma}k^2\; c^2\; d_Y\tilde{z} - \;\frac{6}{H}k^2d_z\,\tilde{z} + \; ik^3\mu\,\gamma^n\tilde{z} - \frac{Re}{\gamma}ik^3\; c^2\mu\,\gamma^n\tilde{z} - \; k^4\; T\;\tilde{z} = 0$

$\Rightarrow \quad 8\mu n\gamma^{n-1}\varphi_3 - 8k^2\mu\gamma^{n-1}\varphi_1 + 2\mu n\gamma^{n-1}k^2\varphi_1 + 2\frac{iRek}{\gamma}c\varphi_1 - iRek\tilde{v}_z + \frac{8}{H^3}\psi_3 - \frac{4}{H^2}i\mu\,\gamma^n k\psi_2 + \frac{2}{H}\frac{Re}{\gamma}k^2\; c^2\;\psi_1 - \;\frac{6}{H}k^2\psi_1 + \; ik^3\mu\,\gamma^n\tilde{z} - \frac{Re}{\gamma}ik^3\; c^2\mu\,\gamma^n\tilde{z} - \; k^4\; T\;\tilde{z} = 0$

$\Rightarrow \quad \boldsymbol{8H^3\mu n\gamma^{n-1}\varphi_3 - 8H^3k^2\mu\gamma^{n-1}\varphi_1 + 2\mu n\gamma^{n-1}k^2H^3\varphi_1 + 2\frac{iRek}{\gamma}H^3c\varphi_1 - iRekH^3\tilde{v}_z + 8\psi_3 - 4Hi\mu\,\gamma^n k\psi_2 + 2H^2\frac{Re}{\gamma}k^2\; c^2\;\psi_1 - \;6H^2k^2\psi_1 + ik^3H^3\mu\,\gamma^n\tilde{z} - \frac{Re}{\gamma}ik^3\; c^2H^3\mu\,\gamma^n\tilde{z} - \; k^4\; H^3T\;\tilde{z} = 0}$

- $\tilde{v}_z\; = 0$ $\qquad\qquad$ $D\tilde{v}_z\; = 0$

$\Rightarrow \quad \boldsymbol{\tilde{v}_z\; = 0}$ $\qquad\qquad$ $\boldsymbol{\varphi_1 = 0}$

- $\tilde{z} = 0$ $\qquad\qquad$ $D\tilde{z} = 0$

$\Rightarrow \quad \boldsymbol{\tilde{z} = 0}$ $\qquad\qquad$ $\boldsymbol{\psi_1 = 0}$

CHAPTER - 6

RESULT AND DISCUSSION

As stated earlier that four dimensionless parameter govern the problem. These parameters are the imposed shear rate γ, solid to fluid thickness ratio, interfacial tension and power-law index. In this section, we will now be showing the parametric study i.e., how the system stability is affected if any of these parameters values are altered and the effects of shear thickening and shear thinning in the stability of the system. With two fourth order differential equations and eight boundary conditions, we solve the equations using symbolic package Mathematica 9.0 and Matlab.

As we had performed linear stability analysis, we study the stability and instability depending upon the value of c. This c is a complex number consists of real and imaginary parts.

$$v_i'(x,z,t) = \tilde{v}_i(z)\exp ik[x - ct]$$

$$c = c_{\mathrm{R}} + \mathrm{i}\, c_{\mathrm{I}}$$

substituting c in the above equation , we get

$$v_i'(x,z,t) = \tilde{v}_i(z)\exp ik[x - c_{\mathrm{R}}\mathrm{t} - \mathrm{i}c_{\mathrm{I}}t]$$

$$v_i'(x,z,t) = \tilde{v}_i(z)\exp[i(kx + t\,\mathrm{k}c_{\mathrm{R}}) + kc_{\mathrm{I}}\mathrm{t}]$$

$$v_i'(x,z,t) = \tilde{v}_i(z)\exp[i(kx + t\,\mathrm{k}c_{\mathrm{R}})\,]\,\exp[kc_{\mathrm{I}}\mathrm{t}]$$

from **De Moivre's theorem**

$$e^{iz} = cosz + isinz$$

Hence we get,

$$v_i'(x,z,t) = \tilde{v}_i(cos(kx + t\,\mathrm{k}c_{\mathrm{R}}) + isin(kx + t\,\mathrm{k}c_{\mathrm{R}}))\exp[kc_{\mathrm{I}}\mathrm{t}]$$

$(cos(kx + t\,\mathrm{k}c_{\mathrm{R}}) + isin(kx + t\,\mathrm{k}c_{\mathrm{R}}))$ has a range [-1,1] So the growth and decay of perturbation depends on the imaginary part of c. Hence flow is stable if c_{I} is negative and its unstable if the value of c_{I} is positive.

6.1Growth rate curves :

We start our discussion showing the effect of power-law index, n on the growth rate curves. It is important because power-law index is the only parameter which is going to decide whether fluid is shear thickening or Newtonian or shear thinning. Depending upon our application, i.e., whether we need stable or unstable mode for the system, we can choose the type of fluid.

Fig 4.1 shows the plot between the growth rate and the wavenumber for thick solid (taking H = 10) at various values of power-law index n. in this plot, shear rate is the critical shear rate for n =1, here the critical condition is meant for the value of c imaginary is either less than or equal to zero for all values of wavenumber.

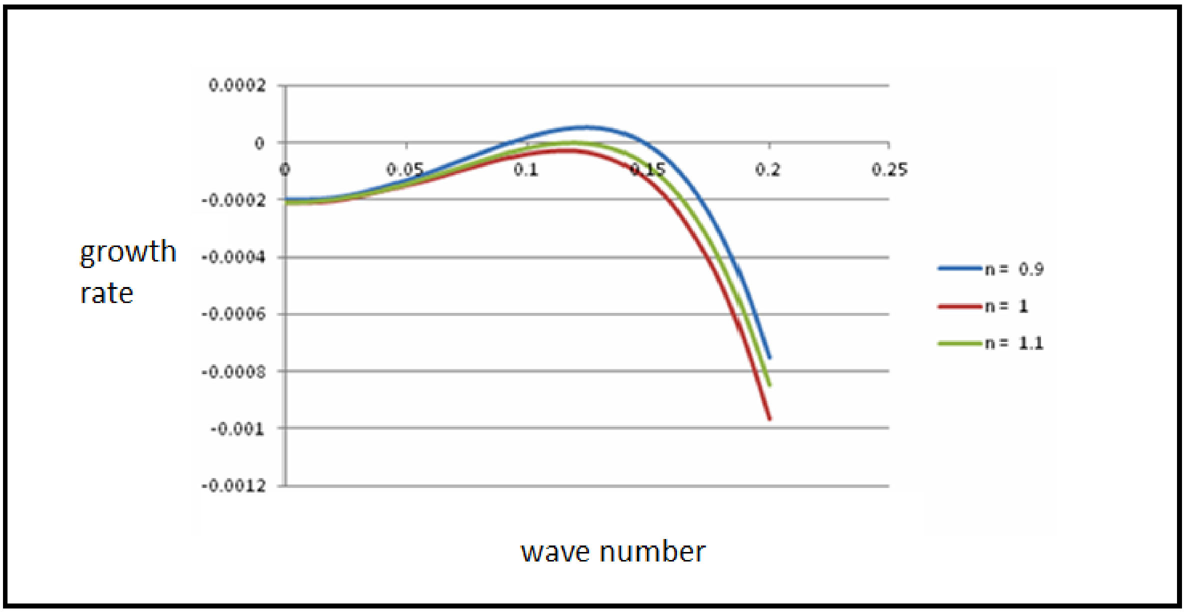


Fig 6.1 Growth rate vs. the wavenumber, when T=10, γ =0.34245, H =10, m=1, for different values of power-law index.

From this plot we can conclude that for thick solids as the value of n is increasing, i.e., fluid is getting shear thickened and the system is tending towards more and more stable.

6.2Variation of shear rate (γ) and thickness ratio ($\boldsymbol{H}$)

We are now interested to know the variation of shear rate in the stability of the system for thin and thick solids. First of all we plot for Newtonian fluid for different values of shear rate, as it would be helpful further for selecting the value of shear rate in the plots where we show the variation of power-law index for thin and thick solids.

First plot is imaginary part of C vs. wavenumber k, n=1, H=2 and neglecting the interfacial tension.

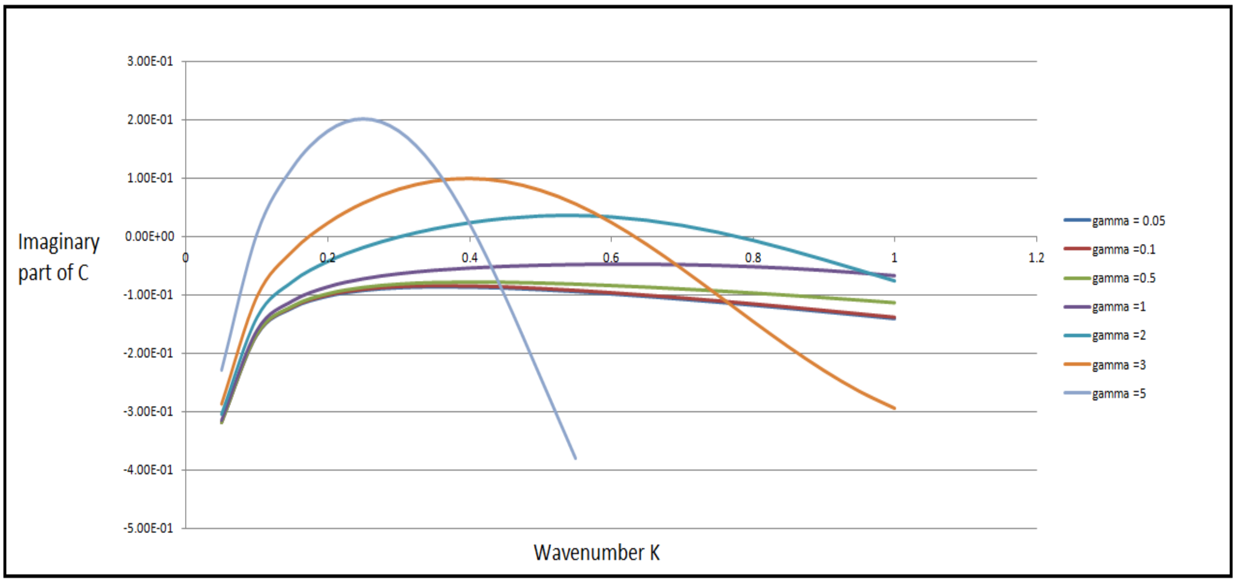


Fig 6.2 Imaginary part of C vs. wavenumber k for n =1, T=0, H=2 and different values of shear rate.

As we are interested in showing the both stable and unstable modes for a particular configuration, So for H=2 we plot taking $\gamma = 3$, as shown in figure the orange curve shows for this value and it is more unstable for a large range of wave number.

From this plot the points which we can note are for constant thickness ratio, we can find unstable modes even with the variation of γ. As the value of γ is increasing the system is becoming more and more unstable although it can also be noted that the range for which the system is becoming unstable goes on decreasing as the shear rate increases. For this particular configuration i.e., H=2, T=0, n=1, the critical shear rate is somewhere around 1.5.

Fig 6.3 shows the plot for more thick solid (H=5), n=1, T=0 and we find that same conclusions as we stated earlier are valid for this case too. The value of γ for which we can find the unstable modes with the variation of power-law index can be selected as 2 from this plot for H=5.

The critical γ for this case is approximately equal to 0.55.

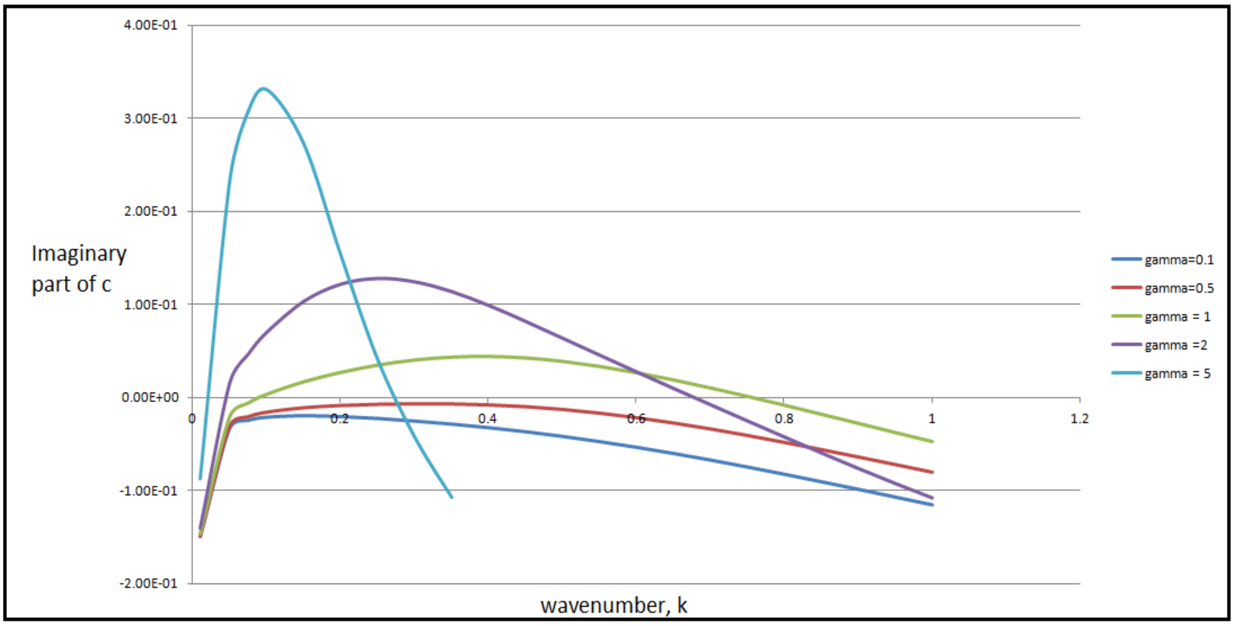


Fig 6.3 Imaginary part of C vs. wavenumber k for n =1, T=0, H=5 and different values of shear rate.

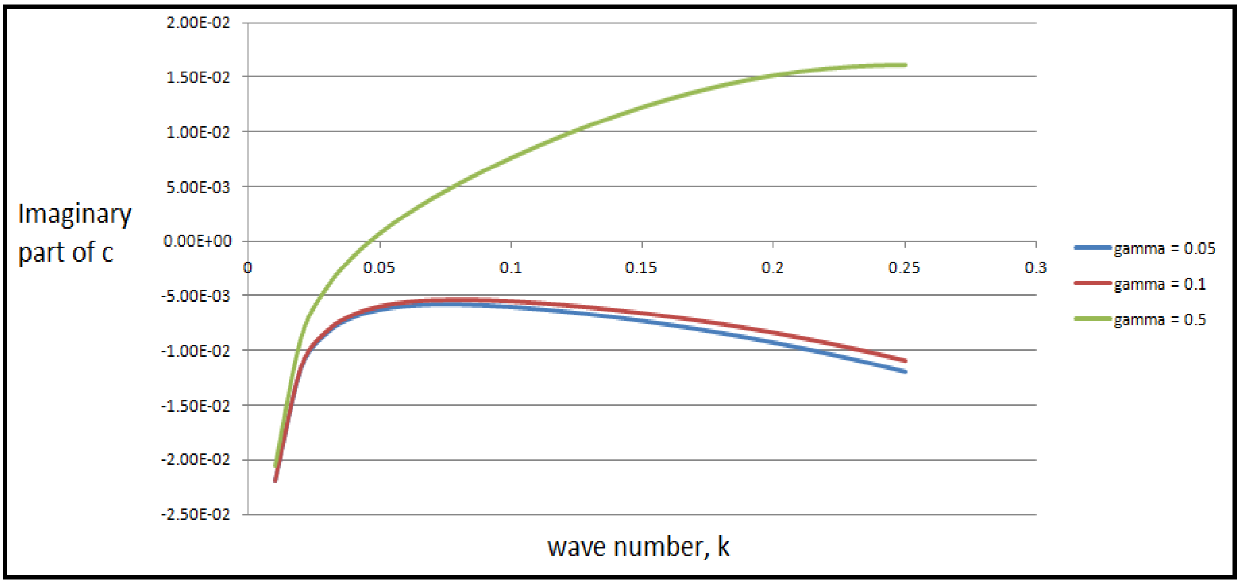


Fig 6.4 Imaginary part of C vs. wavenumber k for n =1, T=0, H=10 and different values of shear rate.

From this plot the shear rate required to show the unstable modes for the variation of power-law index is sufficiently 0.5. The critical γ in this case is calculated and found to be 0.34245.

So from the above three plots we can conclude that with increasing shear rate system becomes more and more unstable and further we can see that for first plot when H = 2 critical shear rate was around 1.5, in second plot when H = 5 critical shear rate was around 0.55, and in third case when H = 10 critical shear rate is found to be 0.34245. This shows that as the thickness of the solid increases, the critical shear rate goes on decreasing.

6.3Variation of power-law index, n

Now we will be showing the effects of shear thickening and shear thinning in the stability of Couette flow. For different thickness ratio we have decided what values of shear rate have to keep for making the system unstable at most values of power-law index.

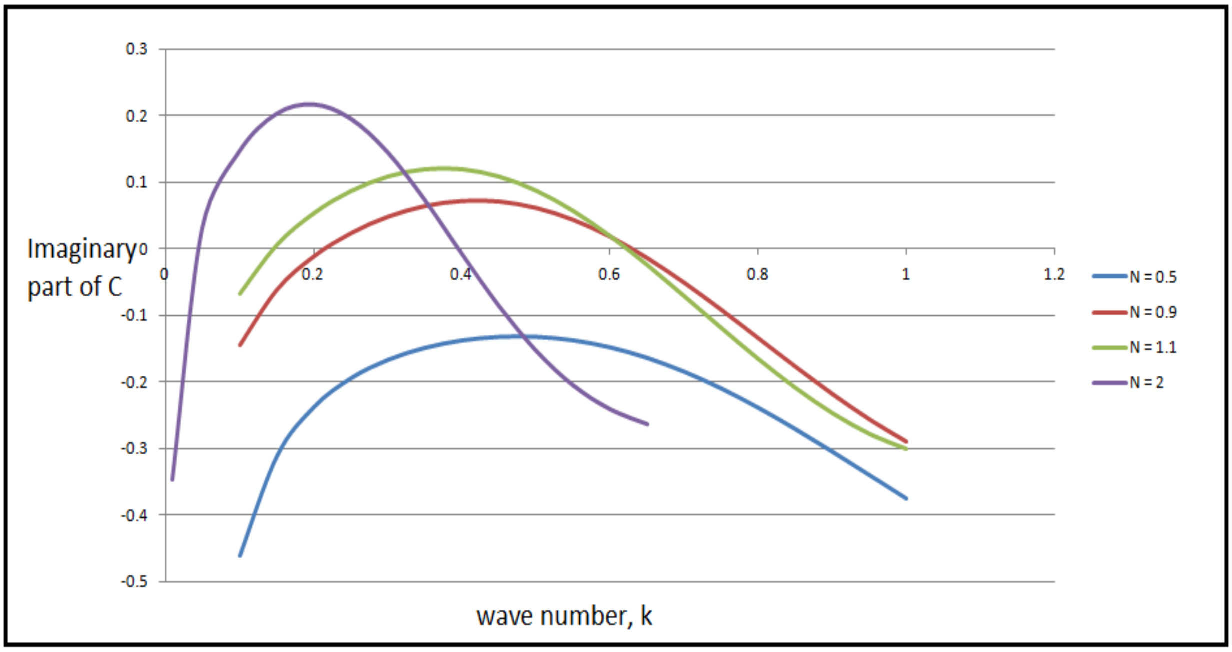


Fig 6.5 Imaginary part of C vs. wavenumber k for γ=3, T=0, H=2 and different values of power-law index, n

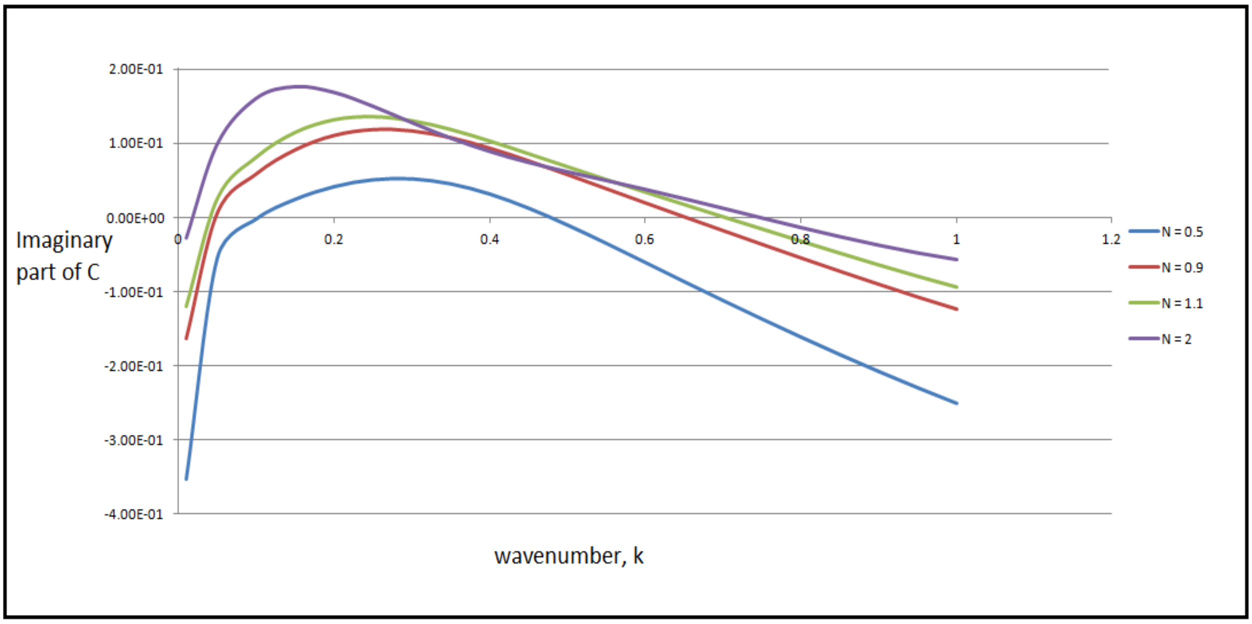


Fig 6.6 Imaginary part of C vs. wavenumber k for γ=3, T=0, H=2 and different values of power-law index, n

From the above two figures the conclusions that can be drawn are, as the value of power-law index goes on increasing i.e. as fluid is getting shear thickened and system is becoming more and more unstable. Moreover, changing n from shear thickening to shear thinning fluid it is seen that the instability mode is moving from a low wavenumber mode to a high wavenumber mode. This is due to the jump in the first normal stress difference across the interface. Hence the use of power-law model alters the instability mode which is selected, but it does not introduce any new modes of instability.

CHAPTER 7

CONCLUSION

We have studied the stability of Couette flow of powerlaw fluid past Neo-Hookean deformable solid and investigated the role of shear thickening and shear thinning of fluid in inducing the surface instabilities. Following are the conclusions which have been drawn from this thesis.

- For thick solids, it was seen that shear thickening fluids has a stabilizing effect.

- Keeping all the parameters constant and varying the shear rate, it was found that with increasing shear rate system is becoming more and more unstable, and it was also found that as the thickness of the solid increases, the critical shear rate goes on decreasing.

- From fig 6.5 and 6.6 it was concluded that as the fluid is getting shear thickened, the system is getting more and more unstable, and the instability mode is changing from a low wavenumber mode to a high wavenumber mode.

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