OPTIMUM MODEL REDUCTION IN FREQUENCY DOMAIN

A DISSERTATION

Submitted in partial fulfilment of the
requirements for the award of the degree
of
MASTER OF ENGINEERING
in
ELECTRICAL ENGINEERING
(System Engineering & Operational Research)

By
SANJAY KUMAR CHATURVEDI

DEPARTMENT OF ELECTRICAL ENGINEERING
UNIVERSITY OF ROORKEE
ROORKEE-247 667 (INDIA)
APRIL, 1990

CANDIDATE'S DECLARATION

I hereby certify that the work which is being presented in the thesis entitled 'OPTIMUM MODEL REDUCTION IN FREQUENCY DOMAIN' in partial fulfilment of the requirement for the award of the Degree of MASTER OF ENGINEERING, submitted in the Department of 'ELECTRICAL ENGINEERING' of the University is an authentic record of my own work carried out during a period from 17th Oct. 1989 to 23rd March 1990 under the supervision of Shri Rajendra Prasad, Lecturer, Department of Electrical Engineering, University of Roorkee, Roorkee.

The matter embodied in this thesis has not been submitted by me for the award of any other degree.

21/4/90

(SANJAY KUMAR CHATURVEDI)

This is to certify that the above statement made by the candidate is correct to the best of my knowledge.

R. Prasad 21/4/90

(RAJENDRA PRASAD)
Lecturer,
Electrical Engineering Deptt.,
University of Roorkee,
Roorkee - 247 667

ACKNOWLEDGEMENT

I wish to express my deep and sincere gratitude to Shri Rajendra Prasad, Lecturer, Electrical Engineering Department, University of Roorkee for his valuable guidance during each and every phase of this work. Inspite of his busy schedule, he rendered his generous help in form of going through manuscript and giving useful suggestions.

I wish to acknowledge to Prof. R.B.Saxena, Head of the Department, Electrical Engineering Department, University of Roorkee for providing necessary facilities which have made this work possible.

It is beyond expression in words my gratitude to Prof. J.D.Sharma, Prof. M.K.Vasantha and Ms. Indira Gupta (Lecturer), Electrical Engineering Department, University of Roorkee, without whose indispensible help and continued encouragement this work would had not been possible.

I am also much thankful to Mr. Harsh Kumar, M.E I Yr. (SEOR) for help during this work. Thanks are to Mr. Dharampal Tyagi for expert typing of the thesis.

ROORKEE

(SANJAY KUMAR CHATURVEDI)

Dated : 21, April, 1990.

**

ABSTRACT

The work included in this thesis deals with model reduction techniques in frequency domain i.e. based on a transfer function description of the original system.

The first chapter introduces model reduction problem, its necessity and a broad classification of various model reduction techniques. This is followed by a detailed procedure for minimization of a performance index, the Integral square error (ISE), in Chapter-2 Stability based reduction methods are described in Chapter-3. The mixed methods to obtain reduced order model (ROM) are presented in Chapter-4, by combining the stability based reduction methods of Chapter-3 and error minimization technique. The denominator of the ROM is obtained by the stability based reduction methods and optimal coefficients of numerator polynomial are obtained by minimizing, the performance index, ISE. The respective step-response of the illustrative examples are shown for comparison purposes. The mixed methods described in Chapter-4, are extended to reduce the order of discrete-time systems in Chapter-5. A scheme to design a controller, using ROMs obtained from mixed methods, is given in Chapter-6.

The computer programs, in FORTRAN, for both continuous and discrete time cases have also been developed and implemented successfully on a PC.

**

C-O-N-T-E-N-T-S

****

chapter :1

introduction

1.1 NECESSITY OF MODEL REDUCTION :

There are many existing large scale systems, presenting a great challange to both system analyst and control system designers. Such systems may be traced to almost every facet of human activity e.g. networks, structures, power system, control system, socio-economic system, transportation, process industries etc.

The mathematical procedure of system modelling often leads to comprehensive description of a process in the form of high order differential equations (For continuous time systems)and difference equations (for discrete case) which are difficult to use either for analysis or controller synthesis. It is hence useful, and sometimes necessary, to find equations of the same type but of lower order that may be considered to adequately reflect the dominant characterstics of system under consideration. Some of the main reasons for using reduced order models of higher order linear system could be :

(a) To have a better understanding of system :

A system of uncomfortably high order poses difficulties in its analysis, synthesis or identification, An obvious method to deal with such type of system is to approximate it by a low order system which reflect the characteristics of original system such as time constant, damping ratio, natural frequency etc.

(b) To reduce complexity :

The development of state-space methods and optimal control techniques have made the design of control system for high order multivariable system quite feasible. When the order of systems become high, special numerical techniques are required to permit the calculation to be done at a reasonable cost on fast digital computers. In such cases an adequate low order model, if available may substantially reduce the computational burden, hence saving in both the memory and time requirement of computer.

(c) To reduce hardware complexity :

A control system design for a high order system is likely to be very complicated and of a high order itself. This is particularly true for controllers based on optimal control theory. Controllers designed based on a low order model will be less costly and easy to implement and maintain.

(d) To make feasible design :

Reduced order models, may effectively be used in special situations like

(i) On line interactive system modelling.

(ii) Sub optimal control derived by simplified model.

(iii) Adaptive control.

(iv) Prediction of transient response sensitivity, dynamic error of high order systems.

1.2 STATEMENT OF MODEL REDUCTION :

The reduction of a high order system into its lower order approximants can be done in frequency-domain and time-domain as well. In frequency domain the problem can be stated as -

Given a transfer function description of a higher order single input - single output (SISO) system :

$$G_o(s) = \frac{a_o + a_1 s + a_2 s^2 + \ldots + a_n \; s^{n-1}}{b_o + b_1 s + b_2 s^2 + \ldots + b_{n+1} s^n} = \frac{N(s)}{D(s)}$$

where n is the order of the system.

A reduced order model is desired, which can adequately describe the significant dynamic behaviour of the original system and can be expressed as

$$G_r(s) = \frac{c_o + c_1 s + c_2 s^2 + \ldots + c_r \; s^{r-1}}{d_o + d_1 s + d_2 s^2 + \ldots + d_{r+1} s^r} \;, \; \forall \; r < n$$

where r is the order of the reduced order system.

In time domain, the systems can be described by the following state space equations

Original System	Reduced Order System (r < n)
$\dot{\underline{X}}(t) = A\,\underline{X}(t) + B\underline{u}(t)$	$\dot{\underline{X}}_r(t) = A_r\underline{X}_r(t) + B_r\underline{u}(t)$
$y(t) = C\underline{X}(t) + D\underline{u}(t)$	$y(t) = C_r\underline{X}_r(t) + D\underline{u}(t)$
where,	where,
$\underline{X}(t)$ = nx1 state vector	$\underline{X}_r(t)$ = rx1 state vector
$\underline{u}(t)$ = mx1 input vector	A_r = rxr system matrix
A = nxn system matrix	B_r = rxm input matrix
B = nxm input matrix	C_r = ℓxr output matrix
y(t) = ℓx1 output vector	D_r = ℓxm transmission matrix
C = ℓxn output matrix	
D = ℓxm transmission matrix	

(For SISO ℓ = m = 1) and in physical systems, transmission matrix, in general, is zero.

1.3 CLASSIFICATION OF MODEL REDUCTION TECHNIQUES :

The order reduction techniques can broadly be classified as -

1.3.1 Time domain simplification techniques :

In time domain reduction techniques the original and reduced systems are expressed in state space form. The order of matrices A_r, B_r, C_r are less than A, B, C and the output y_r will be a close approximation to y for specified inputs. The time domain techniques belong to either of the following categories :

(1) Modal Analysis : This category attempts to attain the dominant eigen values of the original system and then obtains the remaining parameters of the low order model in such a way that its response, to a certain specified input should approximate closely to that of high order system. The method proposed by DAVISON [2], AOKI [12] belong to this category. Davison's method consists of diagonalising of the system matrix and neglecting the large eigen values. In this case, the input is taken as step function and all the eigen values are assumed to be distinct. This restriction, however, was removed by CHIDAMBARA [4] and DAVISON [3]. AOKI [12] took a more general approach based on aggregation. A method to improve the quality of simplified aggregated models

of systems without increasing order of the state differential equations has been given by GRUCA et.al. [5]. It consisted of introduction of delay in the output vector of aggregated model to minimize a quality index function of the output error vector. However, the numerical difficulties and the absence of guide lines for selecting the weighting matrices in performance index of this method were well observed by the researchers. INOOKA et.al.[6] proposed a method based on combining the method of aggregation and integral square error (ISE) criterian. An important variation of dominant eigen value concept was proposed by KUPPURAJULU and ELANGOVAN [7] wherein the high order system is replaced by three models, successively representing the initial, intermediate and final stages of the transient response.

The above out-lined approaches, though useful in many applications, suffer from the following disadvantages :

(i) The computation of eigen values, eigen vectors and the aggregation matrix may be quite formidable for a very high order system.

(ii) In cases, where the eigen values of a system are close together or where the eigen values are not easily identified or for system without dominant mode, these methods obviously fail.

(iii) There may be considerable difference between the steady state responses of the high order system and its low order model to certain inputs [2]. However this shortcoming was removed by CHIDAMBARA [4] at the cost of poor matching during transient period. The above mentioned points led to the optimum model order reduction approach.

(2) Optimum model reduction : This second group is based on obtaining a low order model of a given high order system so that its impulse or step response will match to that of the original system in optimum manner with no restriction on the location of eigen values. Such techniques aim at minimizing a selected performance criterion. Which in general, is a function of error between the responses of the original high order system and its reduced order approximant. The parameters of reduced order model (ROM) are then obtained either from the necessary conditions of optimality or by means of a search algorithm. The approximations have been studied for step and impulse responses.

Chidambara (1969) gave two techniques for model order reduction where the integral of the squared error between the step responses of the exact and simplified model is minimized. SINHA and BERZNAI [8] solved the problem by using pattern-search algorithms. BANDLR et.al.[9] used three different gradient techniques for the minimization of performance index in the simplification problem. YAHAGI [10] obtained optimal low order model by using the technique of least square fit, linear, programming and parameter optimization. For state space representation the most important results were obtained by WILSON et. al.[11]. But this also requires the solution of Lyapunov type equations.

But whatever be the approach to the problem, the main objective is that the reduced order approximant should reproduce the significant characteristic of the parent system as closely as possible.

1.3.2 Frequency domain simplification techniques :

Most frequency domain simplification methods start with the transfer function (T.F.) description of the original system. The objective in this case is that the frequency domain properties of the parent system and its reduced order approximant should match as closely as possible. They can mainly be classified as :

(1) Continued fraction expansion and truncation (CFE) : This method was first proposed by CHEN and SHIEH [13]. Since then various improvements and extension of this approach have been presented such as by CHEN and SHIEH [17], CHEN and HAN [19] etc. In this approach the transfer function of the original system is expanded in continued fraction form and then some of its terms are truncated to get the desired order reduced model. CHEN [16] extended the CFE technique to multivariable system's reduction

and their design. In this technique, the continued fraction expansion and subsequent inversion operations to find the simplified model are extremely time consuming and laborious . Though the computer oriented algorithms for expansion into CFE and their inversion [14] have been divised for various cauer forms [17,18], but the serious disadvantage associated with the method is that the stability of the ROM is not guaranteed, even though the original system is stable. This problem may be avoided by CHEN, CHANG and HAN [19]. CHUANG [15] suggested an alternative CFE technique to have expansions about $s = 0$ and $s = \infty$ alternatively showing good agreement in both transient and steady state regions.

(2) Pade approximation technique : In pade approximation, the power series expansions of high order transfer function and that of approximants are equated to obtain the prarameters of ROM. Though the method renders various advantages such as computational simplicity, fitting of steady state value of output of the system and ROM for inputs of the form $\alpha_i t^i$, However it suffers from serious disadvantage that ROM may be unstable (stable) even though the original system is stable (unstable). CHUANG [21] proposed a partial solution to the stability problem through the homographic transformation $s = \omega/(\alpha + \beta\omega)$, that

gives a family of reduced - models of same order. CHEN et.al.[19] have obtained stable reduced order pade approximants by using stability equation method.

(3) Moment matching technique : This technique aims at equating a few lower order moments of the reduced order model to those of the original system, and no consideration is given to other moments. This would preserve the low frequency response of the system while transient response will be error prone BOSLEY et.al.[22] discussed the similarity between CFE and moment matching method. However some researchers have shown that the methods based on moment matching, pade approximation or CFE are essentially equivalent [36]. The disadvantage of this method is that the transient performance of the reduced order model (ROM) may not always be satisfactory and also, the stability of ROM may not be guaranteed.

(4) Reduction based on stability criteria : HUTTON and FRIED LAND [23] based their method on $\alpha - \beta$ expansion that uses the Routh table of the original transfer function. The advantage of this method is that the ROM will be stable if the original model is stable. Another Routh-based model reduction scheme has been suggested by KRISHANAMURTHY et.al.[24]. This has the advantage that once the Routh-Hurwitz arrays for

the numerator and denominator polynomial of the original model are constructed, the various ROM are produced by mere inspection. SHAMASH [25] combined this method with Pade approximation technique to get stable ROMS. CHEN et.al.[26] have given a technique which uses stability equations for finding stable reduced order models.

This chapter has introduced the model reduction problem along with a broad classification of reduction techniques. Only appropriate references have been cited for the methods which will be used in this thesis.

1.4 MOTIVATION AND SCOPE OF THE THESIS :

The objective of this thesis is to study some mixed methods of model order reduction using error minimization techniques for continuous and discrete time systems and then check the suitability of reduction methods for controller designes. The author has developed the computer programs for reduction of continuous and discrete time SISO systems. This work is based on error minimization technique which has been described in detail in the next chapter.

**

chapter :2

adjustment of parameters
to
minimize integral square error

In previous chapter it has been stated that in some reduction techniques we chose certain performance criterion which is a function of unknown system parameters. The maximum or minimum value of this index then corresponds to optimum set of parameter values. A number of such performance criterion are used in practice . The most common being the integral square error (ISE) given by

$$ISE = \int_0^\infty e^2(t)\, dt$$

which is nothing but the square of the error between input and output when a step input is given to it. This chosen performance index (P.I.) is then minimized as under.

2.1 PROCEDURE TO CALCULATE I.S.E. [27] :

With reference to the Fig. 2.1 let

$$E(s) = \frac{\sum_{i=o}^{n-1} c_i s^i}{\sum_{i=o}^{n} d_i s^i} = \frac{C(s)}{D(s)} \qquad \ldots(2.1)$$

the error transfer function

$$E(s) = \frac{R(s)}{1+G(s)} \qquad \ldots(2.2a)$$

The I.S.E. is given by

$$I_n = \int_0^\infty e^2(t)\, dt \qquad \ldots(2.2b)$$

$$= \frac{1}{2\pi j} \int_{-j\infty}^{+j\infty} E(s)\, E(-s)\, ds \text{ (by using parseval theorem)} \qquad \ldots(2.2c)$$

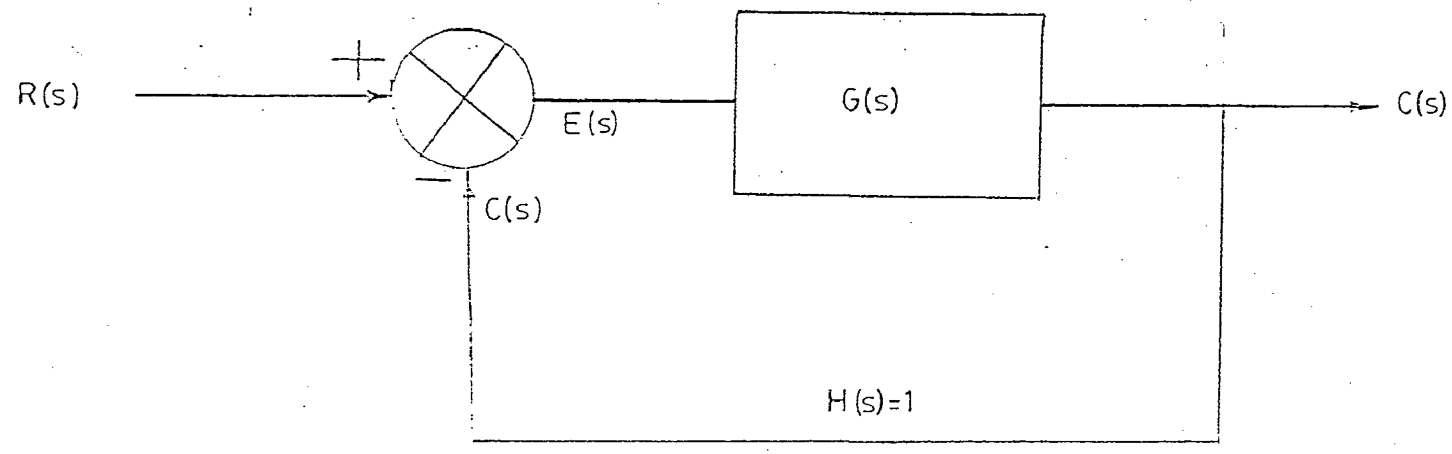


Fig 2.1 UNITY FEED BACK SYSTEM

The steps involved are

(i) Start with Fourier transform of error as a function of the complex frequency s. This function will involve the free parameters of the system as unknown coefficients.

(ii) Use Parseval's theorem, to get I.S.E. in terms of the error transform, E(s) [equation (2.1)].

(iii) Solve the integral encountered, solution of which will be interms of coefficients appearing in E(s), i. e. $I_n = f\,(c_0, c_1 \ldots\ldots, c_{n-1}; d_0, d_1 \ldots\ldots d_n)$

(iv) Use the standard minimization procedures, to obtain the values of unknowns. In other words, to minimize ISE, c_0, c_1 $c_2 \ldots\ldots c_{n-1}$ and d_0, $d_1 \ldots\ldots d_n$ are adjusted by equating partial derivatives of I_n to zero with respect to these parameters and solving the resultant set of equations for the values of parameters.

2.2 METHODS TO SOLVE DEFINITE INTEGRAL :

There are a number of methods available to solve the integral as in equation (2.2c). Here, however we give two methods to solve the integral.

2.2.1 Method 1 : Using initial value theorem :

Taking equation (2.1) as a reference point, the ISE for a n^{th} order system will be

$$I_n = \frac{1}{2\pi j} \int_{-j\infty}^{+j\infty} \frac{C(s)}{D(s)} \times \frac{C(-s)}{D(-s)} \, ds \qquad \ldots(2.3)$$

where,

C(s) is a polynomial in s, and can have roots in right half of complex frequency plane and D(s) is a polynomial in s, and can have roots only in left half of complex frequency plane.

For $s = j\omega$

$\overline{C(s)} = C(\bar{s}) = C(-s)$, where bar denotes the complex conjugate

and

$\overline{D(s)} = D(-s)$

Let

$$\frac{C(s)}{D(s)} \times \frac{C(-s)}{D(-s)} = \frac{A(s)}{D(s)} + \frac{B(s)}{D(-s)} \qquad \ldots(2.4)$$

where,

$$A(s) = \sum_{i=o}^{n-1} a_i s^i \qquad \ldots(2.4a)$$

$$B(s) = \sum_{i=o}^{n-1} b_i s^i \qquad \ldots(2.4b)$$

Since, C(s) x C(-s) will be an even function of 's' (complex conjugate property)

$\Longrightarrow B(s) = A(-s)$

So, integral in equation (2.3) can be rewritten as

$$I_n = \frac{1}{2\pi j} \int_{-j\infty}^{+j\infty} \left[\frac{A(s)}{D(s)} + \frac{A(-s)}{D(-s)} \right] ds$$

$$= \frac{1}{2\pi j} \left[\int_{-j\infty}^{+j\infty} \frac{A(s)}{D(s)} \, ds + \int_{-j\infty}^{+j\infty} \frac{A(-s)}{D(-s)} \, ds \right]$$

$$= \frac{2}{2\pi j} \left[\int_{-j\infty}^{+j\infty} \frac{A(s)}{D(s)} \, ds \right] \quad \text{(by change of variables)} \qquad \ldots(2.5)$$

Equation (2.5) shows that the I_n will be equal to the one half of Laplace inverse of $A(s)/D(s)$ evaluated at $t = 0$.

Thus, using initial value theorem,

$$I_n = 2 \times \mathcal{L}^{-1} \left. \frac{A(s)}{D(s)} \right|_{t=0} = 2 \times \mathcal{L}^{-1}\left[\frac{1}{2} \lim_{t \to 0} \frac{A(s)}{D(s)} \right]$$

$$= \lim_{t \to 0} \mathcal{L}^{-1}\left[\frac{A(s)}{D(s)}\right] = \lim_{s \to \infty} s \times \frac{A(s)}{D(s)} = \frac{a_{n-1}}{d_n} \qquad ...(2.6)$$

Thus only one coefficient a_{n-1} is needed to compute the integral's value.

From equation (2.4)

$$A(s) \times D(-s) + A(-s)\, D(s) = C(s) \times C(-s) \qquad ...(2.7)$$

in expanded form ,

$$A(s) \times D(-s) = \sum_{i=0}^{n-1} a_i s^i \times \sum_{j=0}^{n} d_j (-s)^j$$

$$= \sum_{i=0}^{n-1} \sum_{j=0}^{n} (-1)^j \times a_i \times d_j \times s^{i+j}$$

Likewise,

$$A(-s) \times D(+s) = \sum_{i=0}^{n-1} \sum_{j=0}^{n} (-1)^i \times a_i \times d_j \times s^{i+j}$$

collecting equal powers of s in eqn. (2.7)

$$A(s) \times D(-s) + A(-s) \times D(s) = \sum_{m=0}^{m=2n-1} E_m s^m + \sum_{m=0}^{2n-1} (-1)^m E_m s^m$$

$$= \sum_{m=0}^{m=2n-1} (1+(-1)^m) \times E_m s^m$$

where,

$$E_m = \begin{vmatrix} \sum_{i=0}^{m} (-1)^{m-i} a_i d_{m-i}; \text{ for } 0 \leq m \leq n-1 \\ \sum_{i=m-n}^{n-1} (-1)^{m-i} a_i d_{m-i}; \text{ for } n \leq m \leq 2n-1 \end{vmatrix} \quad \ldots(2.8)$$

$$C(s) \times C(-s) = \sum_{i=0}^{n-1} c_i s^i \times \sum_{j=0}^{n-1} c_j(-s)^j$$

$$= \sum_{i,j=0}^{n-1} (-1)^j c_i c_j s^{i+j}$$

Again, collecting equal powers of 's'

$$C(s) \times C(-s) = \sum_{m=0}^{2n-2} 2C_m x s^m$$

where,

$$2 \times C_m = \begin{vmatrix} \sum_{k=0}^{m} (-1)^k c_k c_{m-k} : \text{ for } 0 \leq m \leq n-1 \\ \sum_{k=m-n+1}^{n-1} (-1)^k c_k c_{m-k}; \text{ for } n \leq m \leq 2n-2 \end{vmatrix} \quad \ldots(2.9)$$

Putting all the deduced values in equation (2.4) yields

$$\sum_{m=0}^{2n-1} \left(\frac{1+(-1)^m}{2}\right) E_m s^m = \sum_{m=0}^{2n-2} C_m s^m \quad \ldots(2.10)$$

Due to the cancellation of coefficients, only even powers of s will appear. Equating even powers of 's' will give rise to n non-linear algebraic equations i.e. $E_m = C_m (m=0,2,\ldots,2n-2)$

$$\sum_{i=0}^{m} (-1)^i \, a_i d_{m-i} = C_m \; ; \text{ for } 0 \leq m \leq n-1$$

$$\sum_{j=m-n}^{n-1} (-1)^j \, a_j \, d_{m-j} = C_m \; ; \text{ for } n \leq m \leq 2n-1$$

The coefficient matrix will slightly differ according to 'n' whether even or odd. That is to say

(i) for n even, the equations will be

$$\begin{bmatrix} d_0 & 0 \; 0 \ldots\ldots 0 \\ d_2 & d_1 d_0 \ldots\ldots 0 \\ \vdots & \vdots \\ d_n & d_{n-1} \ldots\ldots d_1 \\ 0 & 0 \; d_n \ldots d_3 \\ 0 & 0 \; 0 \ldots d_{n-1} \end{bmatrix} \begin{bmatrix} a_0 \\ -a_1 \\ \vdots \\ (-1)^{n-1} a_{n-1} \end{bmatrix} = \begin{bmatrix} C_0 \\ C_2 \\ \vdots \\ C_{2n-2} \end{bmatrix} \qquad \ldots(2.11)$$

(ii) for n odd,

$$\begin{bmatrix} d_0 & 0 \; 0 \ldots 0 \\ d_2 & d_1 \; d_0 \ldots 0 \\ d_{n-1} & d_{n-2} \ldots d_0 \\ 0 & d_n \; d_{n-1} \ldots d_2 \\ 0 & 0 \ldots\ldots d_{n-1} \end{bmatrix} \begin{bmatrix} a_0 \\ -a_1 \\ \vdots \\ (-1)^{n-1} a_{n-1} \end{bmatrix} = \begin{bmatrix} C_0 \\ C_2 \\ \vdots \\ C_{2n-2} \end{bmatrix} \qquad \ldots(2.12)$$

In general $DA = C$ $\qquad \ldots(2.13)$

Hence, the coefficient a_{n-1} can be evaluated.

2.2.2 Using residue theorem :

(a) Considering the first order transfer function :

$$G(s) = \frac{c_0}{d_0 + d_1 s} = \frac{c_0}{d_1(s+\alpha)}$$

where,

$$\alpha = \frac{d_0}{d_1}$$

$$\text{ISE}\ ,\ I_1 = \frac{1}{2\pi j}\int_{-j\infty}^{+j\infty} \frac{c_o}{d_o + d_1 s} \times \frac{c_o}{d_o - d_1 s}\ ds$$

The roots of the G(-s) will be complex conjugate of G(s) and will lie outside the contour, and thus will have no effect. So,

$$I_1 = \frac{1}{2\pi j}[2\pi j\ \Sigma \text{ residues at all L.H.S. poles}]$$

$$= \frac{c_o^2}{d_1^2\ [\ \frac{d_o}{d_1} + \alpha]}$$

$$= \frac{c_o^2}{d_1^2(\frac{d_o}{d_1} + \frac{d_o}{d_1})}$$

So,

$$I_1 = \frac{c_o^2}{2d_o d_1}$$

(b) Considering a second order transfer function G(s) i.e.

$$G(s) = \frac{c_o + c_1 s}{d_o + d_1 s + d_2 s^2} = \frac{c_o + c_1 s}{(s + \alpha_1)(s+\alpha_2)}$$

where,

α_1 and α_2 are poles of G(s) and ,

$\alpha_1 + \alpha_2 = - \; d_1/d_2$

$\alpha_1 . \alpha_2 = d_o/d_2$

Then,

$$I_2 = \frac{1}{2\pi j}\int_{-j\infty}^{+j\infty} \frac{c_o + c_1 s}{d_o + d_1 s + d_2 s^2} \times \frac{c_o - c_1 s}{d_o - d_1 s + d_2 s^2}\, ds$$

Again the roots of the G(-s) will be complex conjugate of G(s) and will lie outside the contour, and so,

$$I_2 = \frac{1}{2\pi j}[\; 2\pi j \;\Sigma \text{ residues at all L.H.S. poles }]$$

$$= \frac{c_o^2 - c_1^2\,\alpha_1^2}{d_2^2(\alpha_2-\alpha_1)(2\alpha_1)(\alpha_1+\alpha_2)} + \frac{c_o^2 - c_1^2\,\alpha_2^2}{d_2^2\,(-\alpha_2+\alpha_1)(-\alpha_2-\alpha_1)(-2\alpha_2)}$$

$$= \frac{c_o^2\,(\alpha_2-\alpha_1) + c_1^2\alpha_1\alpha_2(\alpha_2 - \alpha_1)}{d_2^2(\alpha_2-\alpha_1)(\alpha_1+\alpha_2)(2\alpha_1\alpha_2)} = \frac{c_o^2 + c_1^2\alpha_1\alpha_2}{d_2^2(\alpha_1+\alpha_2)(2\alpha_1\alpha_2)}$$

(by putting values of $\alpha_1\alpha_2$ and $(\alpha_1 + \alpha_2)$)

$$I_2 = \frac{c_o^2 d_2 + c_1^2 d_1}{2d_o d_1 d_2}$$

Likewise,

$$I_3 = \frac{c_2^2 d_o d_1 + (c_1^2 - 2c_o c_2) d_o d_3 + c_o^2 d_2 d_3}{2d_o d_3\,(-d_o d_3 + d_1 d_2)}$$

(for other values of I.S.E. refer Newton et.al.[27]).

The error calculated from above procedures comes in terms of the unknown system parameters whose optimum values can be found by usual gradient method.

The error minimization technique is employed to obtain reduced order models in a different way - i.e. instead of taking the error between output and input, the performance criterion is taken as the error between the step responses of original and reduced systems. This error can always as the ratio of two polynomials as in equation (2.1).

The next chapter introduces the stability criterion based reduction methods which will be utilized in subsequent chapter.

****

chapter : 3

stable reduction methods

In this chapter we will introduce the four different methods used to find the stable denominator of the ROM. The methods are -

3.1 DOMINANT POLE RETENTION METHOD : [2]

In this method, the reduced denominator (Dr) is formed by selecting the dominant poles, which are generally the poles nearer to imaginary axis in s-plane. The magnitude of residues at respective poles can be considered as the guiding factor to decide about dominant poles. The poles with large residues are said to be dominant.

Considering the Dr of the original system of order n

$$D_n(s) = (s + \alpha_1)(s + \alpha_2) \ldots\ldots\ldots (s + \alpha_n)$$

A reduced denominator is formed by selecting r dominant poles, i.e.

$$D_r(s) = (s + \alpha_1)(s + \alpha_2) \ldots\ldots (s + \alpha_r)$$

However, this method suffers from the following disadvantages:

(i) There is no enough justification (apart from the stability) as to why a reduced order model must have only those poles that are present in the large order system and none else.

(ii) The computation of roots of polynomial may be quite formidable for high order system.

(iii) In case, the poles are close together, the method will obviously fail.

3.2 METHODS BASED ON STABILITY CRITERIA :

3.2.1 Routh approximation method : [23]

The basic idea underlined in this method is to construct Routh- Hurwitz array for denominator of the original system and then to construct the denominator for ROM in a manner such that the coefficients of its Routh- Hurwitz array agree upto a given order with those of the original model. The approximation ensures the stability of ROM. The procedure to construct denominator of ROM is as follows :

Any transfer function of a system can always as written as :

$$G(s) = \beta_1 F_1(s) + \beta_2 F_2(s) F_1(s) + \beta_3 F_3(s) F_2(s) F_1(s) + \ldots\ldots + \beta_n F_n(s) \ldots\ldots F_2(s) F_1(s).$$

$$= \sum_{i=1}^{n} \beta_i \prod_{j=1}^{i} F_j(s) \qquad \ldots(3.1)$$

where, β_i are constants and F_i are defined by CFE,

$$F_i(s) = \cfrac{1}{\alpha_i s + \cfrac{1}{\alpha_{i+1} s + \cfrac{1}{\alpha_{i+2} s + \ddots \, \alpha_{n-1} s + \cfrac{1}{\alpha_n s}}}} \qquad \ldots(3.2)$$

For i = 1, the definition is modified slightly , i.e. the first term in CFE is taken as $(1 + \alpha_1 s)$ instead of $\alpha_1 s$. The canonical form of eqn. (3.1) is refered as the α-β expansion of G(s) and plays a fundamental role in theory of Routh-approximation. The table constructed with denominator polynomial of

G(s) is called α-expansion and with that of numerator is called as β-expansion . Here, we are showing as how the α-table is constructed, as shown in Table - 3.1

Table - 3.1 : Alpha (Routh) Table.

	$d_o^o = d_o$	$d_2^o = d_2$	$d_4^o = d_4$	...
	$d_o^1 = d_1$	$d_2^1 = d_3$	$d_4^1 = d_5$	
$\alpha_1 = \frac{d_o^o}{d_o^1}$	$d_o^2=d_2^o-\alpha_1 d_2^1$	$d_2^2=d_4^o-\alpha_1 d_4^1$	....	....
$\alpha_2 = \frac{d_o^1}{d_o^2}$	$d_o^3=d_2^1-\alpha_2 d_2^2$	$d_2^3=d_4^1-\alpha_2 d_4^2$	....	....
⋮	⋮	⋮	⋮	⋮
$\alpha_n = \frac{d_o^{n-1}}{d_o^n}$	$d_o^{n+1}=d_n^{n-1}-\alpha_n d_2^n$		....	....

The first two rows of the table are constructed from the denominator of the original system under the assumption,

$$d_j^o = 0 = d_{j-1}^1 \; \forall \; j > n$$

The reamaining entries are formed by cross-multiplication rule;

$$\left.\begin{array}{l} d_o^{i+1} = d_2^{i-1} - \alpha_i d_2^i \\ d_2^{i+1} = d_4^{i-1} - \alpha_i d_4^i \quad ; \\ \dots \\ \dots \\ d_{n-i-2}^{i+1} = d_{n-i}^{i-1} - \alpha_i \, d_{n-i}^i \end{array}\right| \; ; \text{ for } i = 1,2,3,\dots, n-1 \qquad \dots(3.3)$$

(where denominator $D_n(s) = \sum_{i=o}^{n+1} d_i s^i$)

The α_i are marginal entries given by

$$\alpha_i = \frac{d_o^{i-1}}{d_o^i} ;$$

for (n - i) odd, the last equation of eqn. (3.3) is modified -

$$d_{n-i-1}^{i+1} = d_{n-i+1}^{i-1} \qquad \ldots(3.4)$$

Assuming the G(s) to be asymtotically stable i.e. all the poles have negative real parts, the Routh -stability test guarantees that all d_o^i, in first coloumn of Table - 3.1 are non-zero and will have the same sign. The denominator for ROM is now computed from the α-expansion Table - 3.1 by introducing Routh-convergents.

3.2.1.1 <u>Routh convergents</u> : The r^{th} order Routh-convergent for G(s) is obtained by truncating the α-expansion and arranging the results as a rational polynomial in s. The truncation eliminates those terms in α-expansion containing α_{r+1}, α_{r+1},... α_n and thus depends on only first α_r coefficients i.e.

$$\left.\begin{aligned} D_{r1}(s) &= 1 + \alpha_1 s \\ D_{r2}(s) &= \alpha_1\alpha_2 s^2 + \alpha_2 s + 1 \\ &\ldots\ldots \\ &\ldots\ldots \\ D_{rr}(s) &= \alpha_r s \times D_{r(r-2)}(s) + D_{r(r-2)}(s) \end{aligned}\right| \qquad \ldots(3.5)$$

For control application, it is preferable to obtain a low frequency response approximation by applying the reciprocal transformation, which is merely the operation of reversing the order of polynomial coefficients.

This means,

$$\bar{D}(s) = \frac{1}{s} D_n(1/s) = d_n\ s^n + d_{n-1}s^{n-1} + \ldots. + d_o$$

It is obvious that a second transformation will give the original denominator. Thus for r^{th} order model,

$$D_r(s) = \frac{1}{s} \bar{D}_r(s) \qquad \ldots(3.6)$$

Let us ellaborate a step-wise procedure by taking an example.

<u>Ex. 3.2.1</u> - Let $D_4(s) = 2s^4 + 36s^3 + 204s^2 + 360s + 240$

<u>Step (i)</u> : $\bar{D}(s) = \frac{1}{s} D_4(1/s) = 240s^4 + 360s^3 + 204s^2 + 36s + 2$

<u>Step (ii)</u>: Construct the table for $\bar{\alpha}$ - expansion

	240	204	2
	360	36	0
$\alpha_1 = 2/3$	180	2	0
$\alpha_2 = 2$	32	0	0
$\alpha_3 = 45/8$	2	0	0
$\alpha_4 = 16$			

For second order (say)

$$\bar{D}_2(s) = 4/3\ s^2 + 2s + 1$$

$$= 1/3\ (4s^2 + 6s + 3)$$

$$D_2(s) = 3s^2 + 6s + 4$$

3.2.2 Routh-Hurwitz array method : [24]

This method does not require reciprocal transformation, reduces the mathematical manipulation encountered in the calculation of α-expansion coefficients. This is an alternative approach of obtaining Routh type models. It uses the Routh-Hurwitz array formulation of the denominator of the original system. Let the denominator of the original system be -

$$D_n(s) = c_{11}s^n + c_{21}s^{n-1} + c_{12}s^{n-2} + c_{22}s^{n-3} + \ldots\ldots$$

The Routh -stability is depicted in Table - (3.2).

Table - 3.2 : Routh - Hurwitz array.

	c_{11}	c_{12}	c_{13}	....
	c_{21}	c_{22}	c_{23}	....
	c_{31}	c_{32}	...	....
	c_{41}	c_{42}	...	....
	⋮	⋮	⋮	⋮
	$c_{n-1,1}$	$c_{n-1,2}$	...	...
	$c_{n,1}$	....	...	...
	$c_{n+1,1}$	....	...	...

The first two rows of the table are constructed from the coefficients of denominator, $D_n(s)$ of $G(s)$. The first row consists of the odd numbered coefficients and second row is formed by the even numbered coefficients. While the numbering is done from higher power of s to lower one. The rest part of the table is completed in conventional way by computing the coefficients of succeeding rows by -

$$c_{ij} = c_{i-2,j+1} - [c_{i-2,1} \times c_{i-1,j+1}] / c_{i-1,1} \qquad \ldots(3.7)$$

for $\quad i \geq 3$ and $1 \leq j \leq [(n-i+3)/2]$

Where [.] stands for integral part of the quantity.

A reduced order polynomial can always be constructed with this table. It should be noted that the effect of all coefficients of the previous two rows have already been taken into account while computing the subsequent rows. For a r^{th} order model $(n-r+1)^{th}$ and $(n-r+2)^{th}$ rows are chosen to form the denominator polynomial of the reduced order model.

Ex. 3.3.2 : $D_4(s) = 240\, s^4 + 360\, s^3 + 204\, s^2 + 36\, s + 2$

s^4	240	204	2
s^3	360	36	
s^2	180	2	
s^1	32		
s^0	2		

For a second order reduced order model, the denominator will be

$$D_2(s) = 180\, s^2 + 32\, s + 2$$

3.2.3 <u>Stability equation method</u> : [26]

In this method only two equations (called stability equations) with one-half of the order of original model need to be factored. The method proceeds in the following way

Let,

$$H(s) = \frac{a_{21} + a_{22}s + a_{23}s^2 + \ldots + a_{2,n}\, s^{n-1}}{a_{11} + a_{12}s + \ldots\ldots + a_{1,n+1}s^n} = \frac{F_n(s)}{F_d(s)} \qquad \ldots(3.8)$$

For stable systems, the even and odd parts i.e. stability equation can be factored as

$$F_{de}(s) = a_{11} \prod_{i=1}^{\ell_1} (1+s^2/x_i^2)$$

and $$F_{do}(s) = a_{12}s \prod_{i=1}^{\ell_2} (1 + s^2/y_i^2) \qquad \ldots(3.9)$$

Where, ℓ_1 and ℓ_2 are integer parts of n/2 and (n-1)/2 respectively and

$$x_1^2 < y_1^2 < x_2^2 < y_2^2 < \ldots\ldots$$

The y_i^2 and x_i^2 are in sequence. Discarding the factor with larger magnitude of y_i^2 or x_i^2 alternatly is the process of reducing stability equations.

The reduced stability equation with desired order r are written as :

$$F_{de}^{r}(s) = a_{11} \prod_{i=1}^{m_1} (1 + s^2/x_i^2)$$
$$F_{do}^{r}(s) = a_{12} s \prod_{i=1}^{m_2} (1 + s^2/y_i^2) \quad \ldots(3.10)$$

where, m_1 and m_2 are integer part of $r/2$ and $(r-1)/2$, respectively. Then the reduced denominator can be constructed as

$$F_{dr}(s) = F_{de}^{r}(s) + F_{do}^{r}(s)$$
$$= \sum_{j=o}^{r} a_{1,j+1}^{r} s^j \quad \ldots(3.11]$$

Again, the method is well-illustrated by an example.

<u>Ex. 3.2.3</u> : Let a system with a transfer function

$$H(s) = \frac{1}{(s^3 + 3s^2 + 2.99\, s + 0.99)}$$

$$F_{de}(s) = 0.99\, (1 + \frac{s^2}{.33})$$

$$F_{do}(s) = 2.99\, s\, (1 + \frac{s^2}{2.99})$$

discarding the factor with $y_1^2 = 2.99$, the reduced denominator will be

$$F_{d2}(s) = 3s^2 + 2.99\, s + 0.99.$$

Various stability based reduction methods have been described in this chapter for reducing the denominator which will be used for finding the ROMs. The mixed methods using advantages of error minimization technique and stability based methods forms the content of Chapter - 4.

****

chapter : 4

mixed methods using error minimization

4.1 INTRODUCTION :

The importance and necessity of the subject of reduced order modelling of high order complex systems have been detailed in first chapter. The topic has aroused widespread interest which is evident from the large number of research publications. The work included in this chapter deals with frequency domain model reduction techniques based on transfer function discription of the original high order system.

4.2 METHOD NO. 1 :

Mixed Method Using Dominant Pole Retention :

In this method the denominator of ROM is found using dominant pole retention method, while the numerator coefficients of the ROM are found by error minimization technique described in Chapter - 2.

Let $G_o(s)$ be the n^{th} order transfer function description of a system -

$$G_o(s) = \frac{\bar{a}_o + \bar{a}_1 s + \dots + \bar{a}_{n-1} s^{n-1}}{\bar{b}_o + \bar{b}_1 s + \dots + \bar{b}_n s^n} \qquad \dots(4.1)$$

Let $s_1, s_2, \dots, s_r$ be the dominant roots of denominator of $G_o(s)$ then r^{th} order ROM can be chosen $(r<n)$ which will have unknown coefficients in its numerator polynomial, while the denominator polynomial is formed by dominant poles i.e.

$$G_r(s) = \frac{\bar{c}_o + \bar{c}_1 s + \dots + \bar{c}_{r-1} s^{r-1}}{\bar{d}_o + \bar{d}_1 s + \dots + \bar{d}_r s^r} \qquad \dots(4.2)$$

where,

$$(s-s_1)(s-s_2)\ \ldots\ (s-s_r) = \sum_{i=o}^{r} \bar{d}_i s^i$$

4.2.1 Computation of Numerator coefficients :

The numerator of the $G_r(s)$ is determined by minimizing step response error between the original system and its ROM while also satisfying the steady state value matching constraint.

Let $x(t)$= Step response of original system.

$x_r(t)$= Step response of reduced system.

Then, the step response error is given by

$$e = ||x(t) - x_r(t)||^2 = \int_o^\infty [x(t) - x_r(t)]^2\, dt$$

$$= \int_o^\infty [y(t) - y_r(t)]^2\, dt \qquad \ldots(4.3)$$

where,

$$\left. \begin{aligned} y(t) &= x(\infty) - x(t) \\ y_r(t) &= x(\infty) - x_r(t) \end{aligned} \right| \qquad \ldots(4.3a)$$

Matching the steady state values of original and reduced order model,

$$x_r(\infty) = \frac{\bar{c}_o}{\bar{d}_o} = \frac{\bar{a}_o}{\bar{b}_o} (= x(\infty))$$

also

$$Y(s) = \mathcal{L}\{y(t)\}$$

$$Y_r(s) = \mathcal{L}\{y_r(t)\}$$

So,

$$Y(s) = \frac{x(\infty)}{s} - \frac{G_o(s)}{s} = \frac{\bar{a}_o}{\bar{b}_o s} - \sum_{i=1}^{n-1} \bar{a}_i s^i / \sum_{i=o}^{n} \bar{b}_i s^i \times s$$

$$= \frac{a_o + a_1 s + a_2 s^2 + \ldots\ldots + a_{n-1} s^{n-1}}{b_o + b_1 s + b_2 s^2 + \ldots\ldots + b_n s^n} \qquad \ldots (4.4)$$

Likewise,

$$Y_r(s) = \frac{x(\infty)}{s} - \frac{G_r(s)}{s}$$

$$Y_r(s) = \sum_{i=o}^{r-1} c_i s^i \Big/ \sum_{j=o}^{r} d_j s^j \qquad \ldots (4.5)$$

where the following equation can be identified,

$$a_i = \frac{\bar{a}_o b_{i+1}}{b_o} - \bar{a}_{i+1} \; ; \; i = o,\ 1,\ \ldots,\ n-2 \qquad \ldots (4.6a)$$

$$a_{n-1} = \bar{a}_o / b_o x b_n \qquad \ldots (4.6b)$$

$$c_i = \bar{c}_o / d_o \times d_{i+1} - \bar{c}_{i+1} \; ; \; i = o,1,2,\ldots\ r-2 \qquad \ldots (4.6c)$$

$$c_{r-1} = \bar{c}_o / d_o \times d_r \qquad \ldots (4.6d)$$

Expanding equation (4.2)

$$e = \int_o^\infty y^2(t)\ dt - 2\int_o^\infty y(t) y_r(t)\ dt + \int_o^\infty y_r^2(t)\ dt$$

By Parsval's theorm the integrals are transformed into frequency domain,

$$e = \frac{1}{2\pi j} \left[\int_{-j\infty}^{+j\infty} Y(s)\ Y(-s) ds + \int_{-j\infty}^{+j\infty} Y_r(s)\ Y_r(-s)\ ds - 2\int_{-j\infty}^{+j\infty} Y(s) Y_r(-s) ds\right] \qquad \ldots (4.7)$$

The integral

$$\frac{1}{2\pi j}\int_{-j\infty}^{+j\infty} Y(s)\, Y(-s)\, ds \text{ and } \frac{1}{2\pi j}\int_{-j\infty}^{+j\infty} Y_r(s)\, Y_r(-s) ds, \text{ can}$$

be evaluated in terms of a_i, b_i of $Y(s)$ and c_i, d_i of $Y_r(s)$ by using approach given in Chapter - 2. The coefficients a_i, b_i, d_i are already known at this stage.

The integral

$$\frac{1}{2\pi j}\int_{-j\infty}^{+j\infty} Y(s)\, Y_r(-s)\, ds$$

Can also be expressed in terms of c_1, c_2, ... c_{r-1} by extending the approach of Newton et al [27] and is discussed in Appendix - A. Thus

$$e = K + A_{r-1} - [E_{n-1} + V_{r-1}] \qquad \ldots(4.8)$$

where,

$K = \frac{1}{2\pi j}\int_{-j\infty}^{+j\infty} Y(s)\, Y(-s)\, ds$, is completly known and can be calculated from the knowledge of a_i and b_i.

For the second terms in eqn. (4.7) using same approach, we get

$$\begin{bmatrix} 2d_o & 0 & 0 & \cdots & 0 & \cdots & 0 \\ 2d_2 & 2d_1 & 0 & \cdots & 0 & \cdots & 0 \\ \cdot\cdot & \cdot\cdot & \cdot\cdot\ \cdot\cdot & & \cdot\cdot\ \cdot\cdot & & \cdot\cdot \\ \vdots & \vdots & \vdots\ \vdots & & \vdots\ \vdots & & \vdots \\ 2d_r & 2d_{r-1} & 2d_{r-2}\cdot\cdot & & 2d_2 & & 2d_1 \\ \vdots & \vdots & \vdots\ \vdots & & \vdots\ \vdots & & \vdots \\ 0 & 0 & \cdot\cdot\ 0 & & \cdot\cdot\ 2d_{r-3} & & 2d_{r-1} \end{bmatrix} \begin{bmatrix} A_o \\ -A_1 \\ \cdot\cdot \\ \vdots \\ \cdot\cdot \\ \vdots \\ A_{r-1} \end{bmatrix} = \begin{bmatrix} c_o^2 \\ 2c_oc_2-c_1^2 \\ 2c_oc_4-2c_1c_3+c_2^2 \\ \vdots \\ \cdot\cdot \\ \vdots \\ -2c_{r-3}c_{r-1}+c_{r-2}^2 \\ -c_{r-1}^2 \end{bmatrix}$$

Symbolically,

$$DA = C$$

Thus inverting D, an expression for A_{r-1} in terms of c_o, c_1,.... can be found out. Likewise for the third term in equation (4.7) we get,

$$\begin{bmatrix} d_o & 0 & 0 & \cdots & 0 & b_o & 0 & \cdots & 0 \\ -d_1 & d_o & 0 & \cdots & 0 & b_1 & -b_o & \cdots & 0 \\ d_2 & -d_1 & d_o & \cdots & 0 & b_2 & -b_1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & d_{r-1} \cdots & -d_{r-1} & 0 & 0 & 0 & -b_n \end{bmatrix} \begin{bmatrix} E_o \\ E_1 \\ \vdots \\ E_{n-1} \\ V_o \\ V_1 \\ \vdots \\ V_{r-1} \end{bmatrix} = \begin{bmatrix} a_o c_o \\ a_1 c_o - a_o c_1 \\ a_2 c_o - a_1 c_1 + a_o c_2 \\ \vdots \\ a_{n-1} c_{r-2} - a_{n-2} c_{r-1} \\ -a_{n-1} c_{r-1} \\ 0 \end{bmatrix} \quad ..(4.10)$$

Or symbolically

$$NF = P$$

Again by inverting N, expression for E_{n-1} and V_{r-1} can be obtained. Thus, eqn. (4.7) will be

$$e = K + [m_1 m_2 \ldots m_r]\, C - [(p_1 + q_1)(p_2+q_2) \ldots (p_{n+r}+q_{n+r})]P \qquad \ldots(4.11)$$

Where, m_i are the elements of the last row of D^{-1} and p_i and q_i are elements of n^{th} and $(n+r)^{th}$ rows of N^{-1}. In the case of step response error minimization it is obvious from equation(4.7) that c_o, $c_1 \ldots$, c_{r-2} are unknown parameters while c_{r-1} is known one. Minimization of e with respect to c_o, $c_1 \ldots c_{r-2}$ i.e.

$$\frac{\partial e}{\partial c_i} = 0 \quad , i = 0, 1, 2 \ldots r - 2$$

will yield (r - 1) linear simultaneous equations

$$\left.\begin{array}{l} 2m_1 c_o + 2m_2 c_2 + 2m_3 c_4 \ldots - K_1 = 0 \\ \quad - 2m_2 c_1 - 2m_3 c_3 \ldots - K_2 = 0 \\ \quad \ldots \quad \ldots \quad \ldots \\ \quad 2m_{r-1}\, c_{r-2} + \ldots \quad - K_{r-1} = 0 \end{array}\right| \qquad \ldots(4.12)$$

The equations (4.12) can uniquely be solved to obtain c_o, $c_1 \ldots c_{r-2}$, which when substituted in eqn. (4.6) will give $\bar{c}_1$, $\bar{c}_2 \ldots \bar{c}_{r-1}$ This completes r^{th} order reduced model computation.

Ex. 4.1 : Let a system transfer function discription be [23]

$$G_o(s) = \frac{14s^3 + 248s^2 + 900s + 1200}{s^4 + 18s^3 + 102\, s^2 + 120}$$

$$D(s) = (s^4 + 18s^3 + 102s^2 + 120)$$

The roots are -

REAL PART	IMAGINARY PART
-1.196684	-0.693370
-1.196684	+0.693370
-7.803316	-1.357582
-7.803316	+1.357582

To reduce it to a second order approximant, we assumed

$$G_2(s) = \frac{\bar{c}_o + \bar{c}_1 s}{s^2+2.3934s + 1.9128}$$

Where, denominator of $G_2(s)$ is formed by taking dominant poles of the original systems, i.e.

$$D_2(s) = (s+1.196684+j\ 0.69337)(s+1.196684- j\ 0.69337)$$

Matching steady state values of $G_o(s)$ and $G_2(s)$ yields $\bar{c}_o = 19.128$

$$\text{Step response of ROM } Y_r(s) = \frac{s + (2.3934 - \bar{c}_1)}{D_2(s)}$$

$$= \frac{s + c_o}{D_2(s)}$$

Following the procedure described in section 4.2.1; the value of c_o is obtained as - 7.5133 which will give

$$\bar{c}_o = 9.9062 \ (\therefore c_o = 2.3934 - \bar{c}_o)$$

Hence the second order model is

$$G_2(s) = \frac{9.9067\ s + 19.1281}{s^2 + 2.3934\ s + 1.91281}$$

4.3 METHOD No. 2 :

Mixed Method Using Routh Approximation :

In this method, the denominator is obtained by usual procedure discribed in section 3.2.1 while the numerator polynomial is obtained by error minimization technique as detailed in method No. 1. The ROM of ex. 4.1 is obtained as :

$$G_2(s) = \frac{10.2964\ s + 13.3333}{s^2 + 2s + 1.3333}$$

4.4 METHOD No. 3 :

Mixed Method Using Routh Stability :

This method takes the advantage of Routh stability array in combination with the error minization procedure. The denominator is obtained by procedure of section 3.2.2. Hence a reduced second order model is deduced

$$G_2(s) = \frac{1158.148\ s + 1200}{92\ s^2 + 156.5217\ s + 120}$$

4.6 METHOD No. 4 :

Mixed Method Using Stability Equation :

Combining the method of article 3.2.3 and article 4.2.1 the second order model is obtained as

$$G_2\ (s) = \frac{0.1480\ s + 100}{s^2 + 6.6667\ s + 10}$$

Table 4.1 shows the cummulative errors which occur in the output for the step input after every 20 seconds, taken for comparison purposes i.e.

cummulative error, $j = \int_{o}^{N} [y(t_i) - y_r(t_i)]^2 \, dt$ where $y(t_i)$ and $y_r(t_i)$ are the responses of original and reduced order systems at t_i and N is the number of sample period, i =(0, 1, 2, 3...N)

Table - 4.1 : Cummulative error.

Model from method no.	Steady state value	Output (after 20 secs.)	CMERR
1.	10	10.0005	2.13406
2.	10	10.0000	1.89589
3.	10	10.0003	52.93622
4.	10	10.0000	13.09305
Original	10	9.9997	-

The step responses of the system and its approximants are depicted in Fig. 4.1.

<u>Ex. 4.2</u> : Considering an 8th order model [24]

$$G_o(s) = \frac{35\,s^2 + 1086s^6 + 13285s^5 + 84203s^4 + 278376s^3 + 511812s^2 + 482964s + 194480}{s^8 + 33s^7 + 437s^6 + 3017s^5 + 11870s^4 + 27470s^3 + 37492s^2 + 28880s + 9600}$$

The third and second order models (obtained by the methods outlined in this Chapter) for this problem are,

<u>Third Order Models</u> :

$$G_3(s) = \frac{31.1575s^2 + 75.8557s + 40.5167}{s^3 + 3\,s^2 + 4s + 2}$$

(by method No. 1)

$$G_3(s) = \frac{23.5207\ s^2 + 25.4971\ s + 13.0988}{s^3 + 1.3554\ s^2 + 1.6972\ s + 0.6466}$$

(by method no. 2)

$$G_3(s) = \frac{-\ 22212.94s^2 + 5887.145\ s + 11.8557}{s^3 + 2.0613\ s^2 + 1.7606\ s + 0.5852}$$

(by method no. 3)

$$G_3(s) = \frac{18.3163\ s^2 - 21.5941\ s + 17.1101}{s^3 + 0.6980\ s^2 + 0.2808\ s + 0.8446}$$

(by method no. 4)

Fig. 4.2 shows the step responces of the original and the ROMs. Table 4.2 gives the errors which occur in output.

<u>IInd Order Models</u> :

$$G_2(s) = \frac{31.3518\ s + 20.2583}{s^2 + 2s + 1}$$

$$G_2(s) = \frac{16.8766\ s + 9.6642}{s^2 + 0.9002\ s + 0.4770}$$

$$G_2(s) = \frac{18.5134\ s + 6.8574}{s^2 + 1.0183\ s + 0.3385}$$

$$G_2(s) = \frac{4.9888s + 24.5126}{s^2 + 0.4022\ s + 1.21}$$

The table - 4.3 and Fig. 4.3 show the cummulative error and responses respectively.

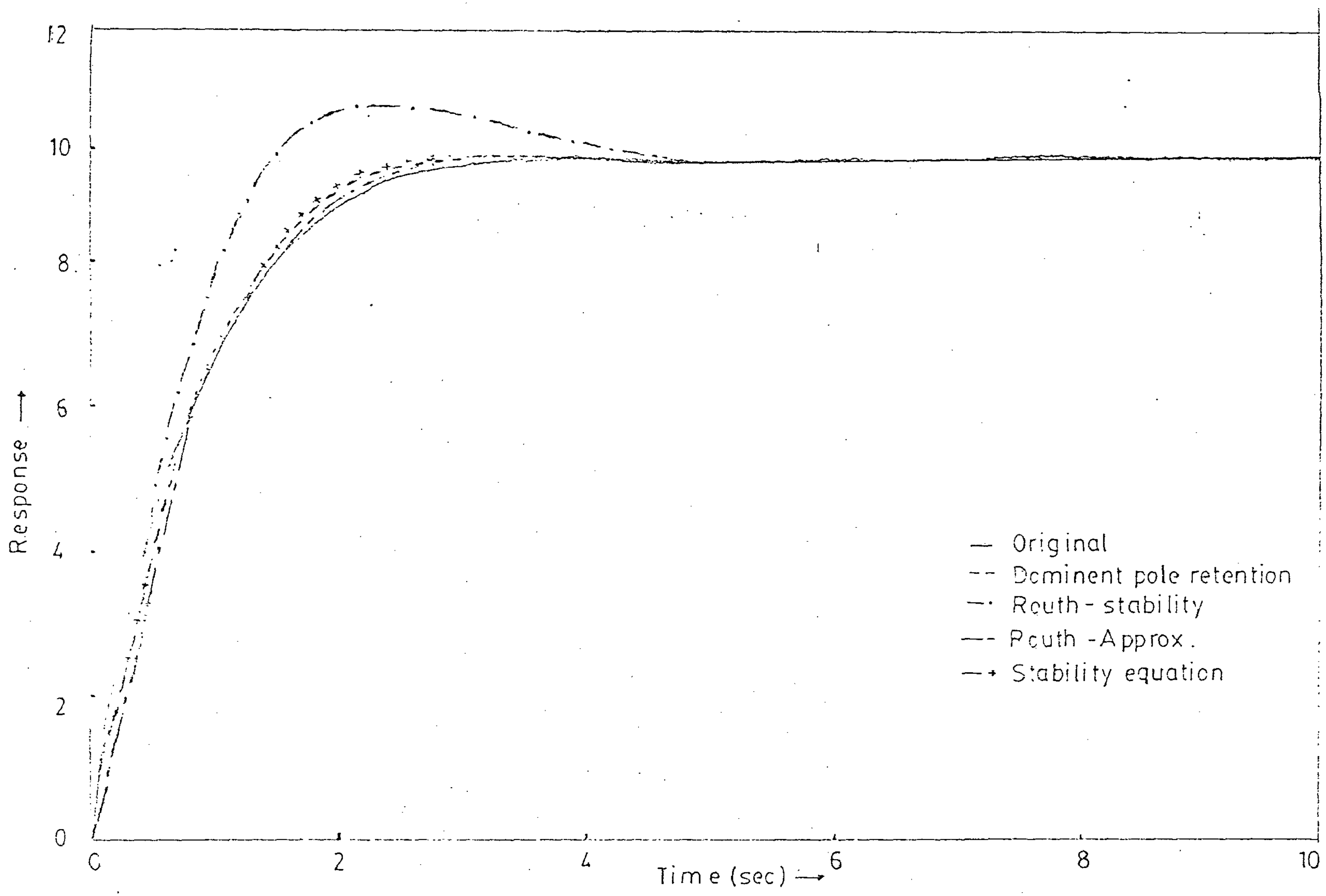


Fig 4.1 STEP-RESPONSES (4th ORDER REDUCED TO SECOND ORDER)

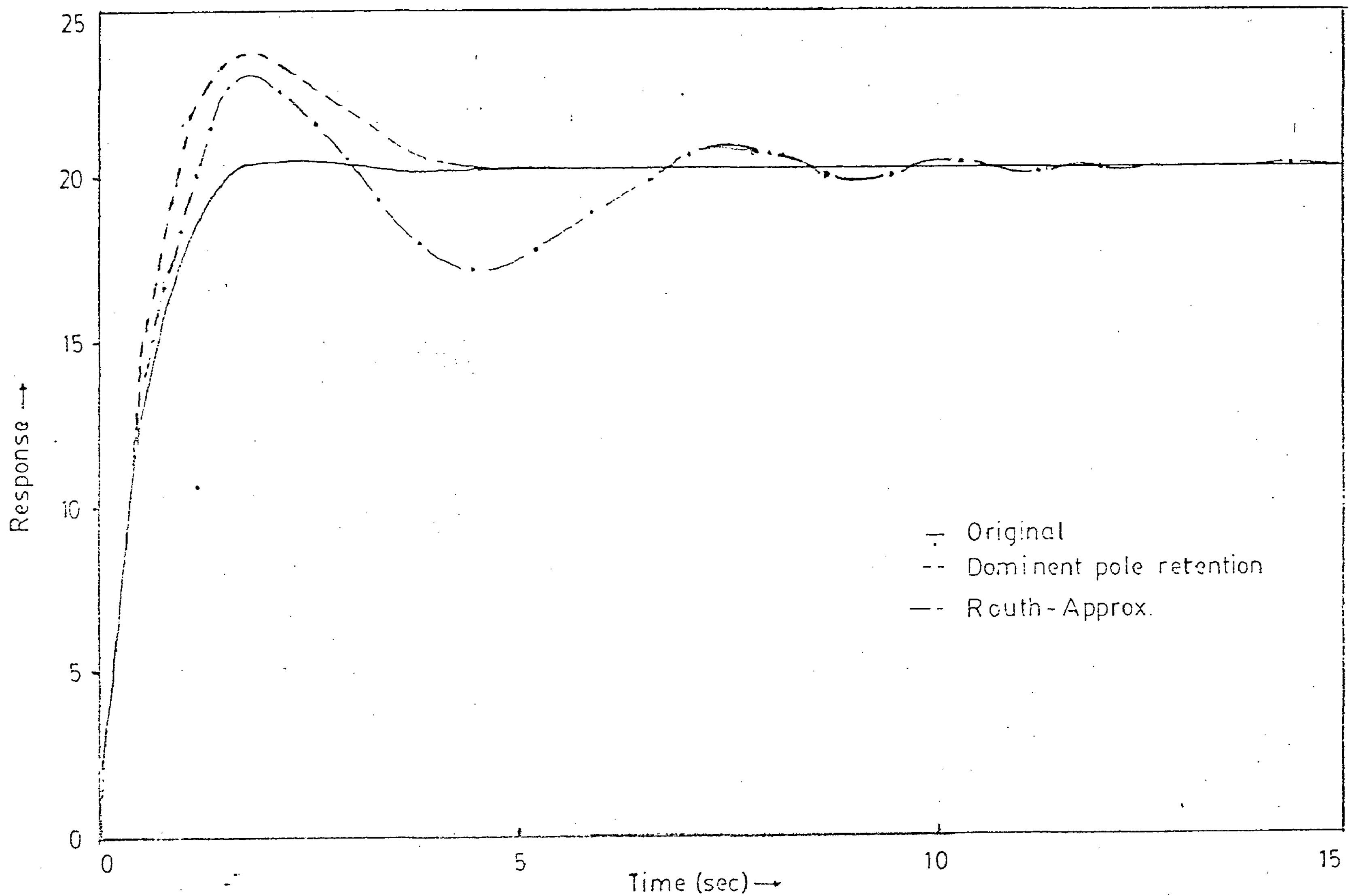


Fig 4.2 STEP RESPONSES (8th ORDER REDUCED TO THIRD ORDER)

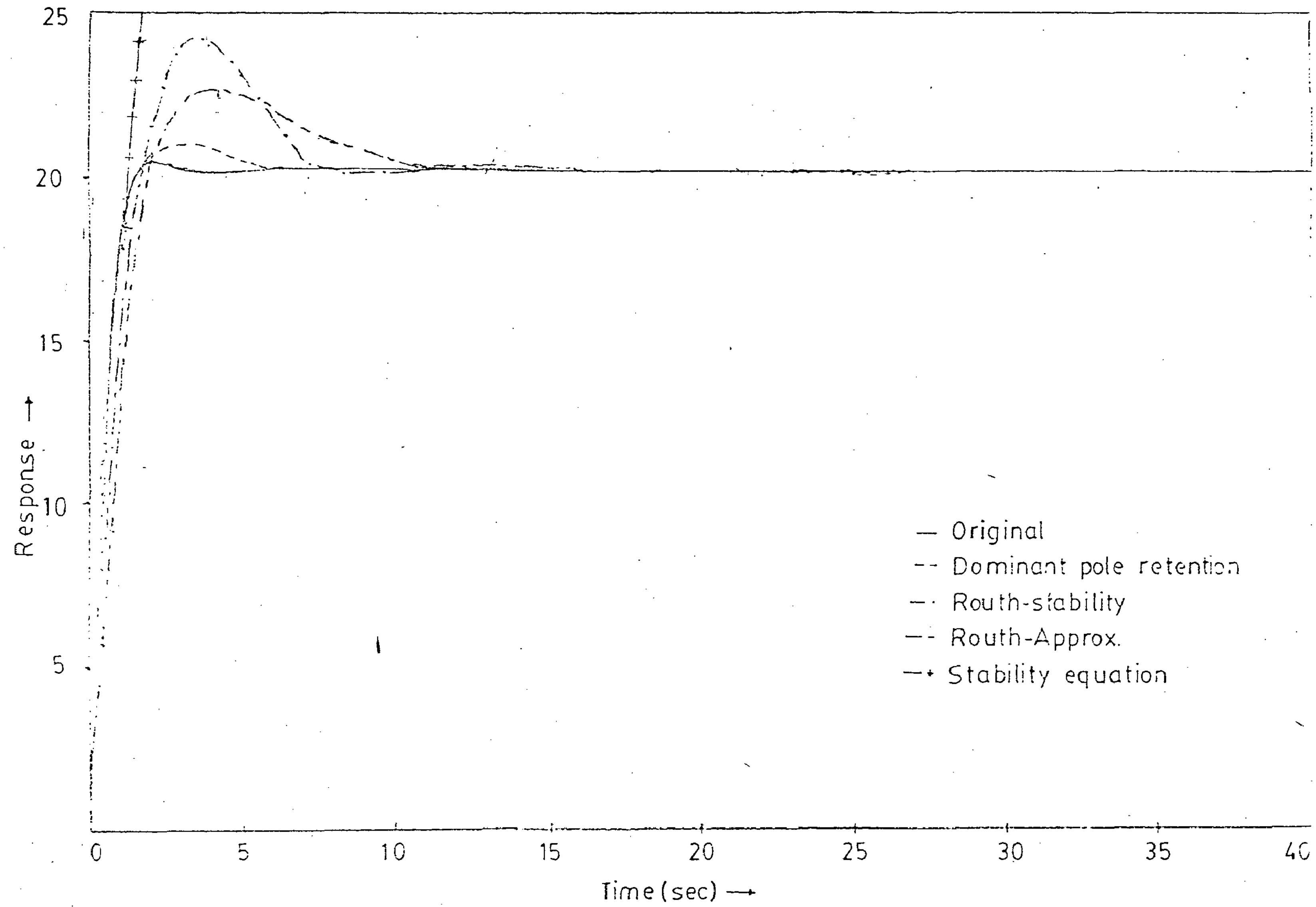


Fig 4.3 STEP-RESPONSES (8th ORDER REDUCED TO SECOND ORDER)

Table - 4.2 : Cummulative error.

Model from method	Steady state value	Output after (20 sec.)	CMERR
1.	20.2584	20.25835	214.709
2.	20.2584	20.218227	Large error
3.	20.2584	20.260999	263.82054
4.	20.2584	20.29067	Large error
Original	20.2584	20.25834	-

Table - 4.3 : Cummulative error.

Model from method	Steady state value	Output after (20 sec.)	CMBRR
1.	20.2583	20.2583	13.742
2.	20.2583	20.2569	369.52
3.	20.2583	20.2600	587.589
4.	20.2583	20.2575	-
Original	20.2584	20.2583	-

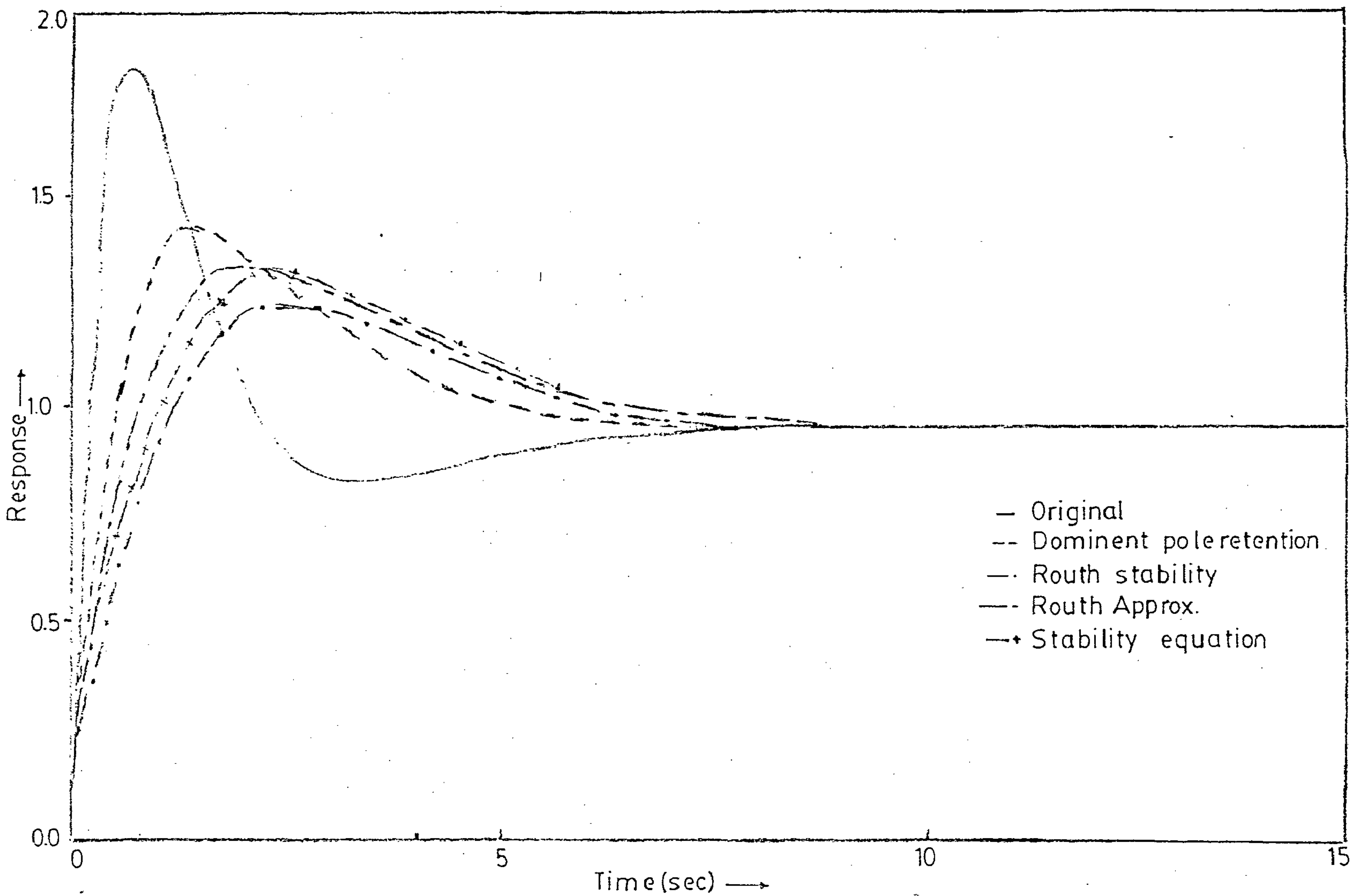


Fig 4.3a CHAUNG MODEL

4.6 OTHER EXAMPLES :

The program output and step responses of the following systems and their respective ROMs are given in (.)

Ex 4.3: $G_o(s) = \dfrac{5 + 15\, s}{1 + 1.8\, s + 7.8\, s^2 + 0.7}$

(WHITEFIELD [35], Fig. 4.4, Second order ROMs)

Ex. 4.4 $G_o(s) = \dfrac{5 + 99.8432\, s + 506.6497\, s^2 + 81.6913\, s^3}{5 + 101.05\, s + 521.01\, s^2 + 105.2\, s^3 + s^4}$

(SHAMASH [25], Fig. 4.5, Third order ROMs)

Ex. 4.5 $G_o(s) = \dfrac{2 + 6\, s + 8s^2}{2 + 5s + 4s^2 + s^3}$

(CHUANG [15], Fig. 4.3a, Second order ROMs)

Ex. 4.6 Multivariable System :

Considering the transfer matrix of a linearized model for a gas turbine developed by MULLER (1971) for which

$$H(s) = \frac{\begin{bmatrix} h_{11}(s) & h_{12}(s) \\ h_{21}(s) & h_{22}(s) \end{bmatrix}}{D(s)}$$

where,

$$h_{11}(s) = 14.90\, s^2 + 1506.473s + 2543.2$$

$$h_{12}(s) = 95150\, s^2 + 1132094.7\, s + 1805947$$

$$h_{21}(s) = 85.2\, s^2 + 8642.888\, s + 12268.8$$

$$h_{22}(s) = 12400\, s^2 + 1492588\, s + 252680$$

$$D(s) = s^4 + 113.225\, s^3 + 1357.275\, s^2 + 3499.75\, s + 2525$$

$$= (s + 1.3471)(s+1.8735)(s+10.0047)(s+99.999)$$

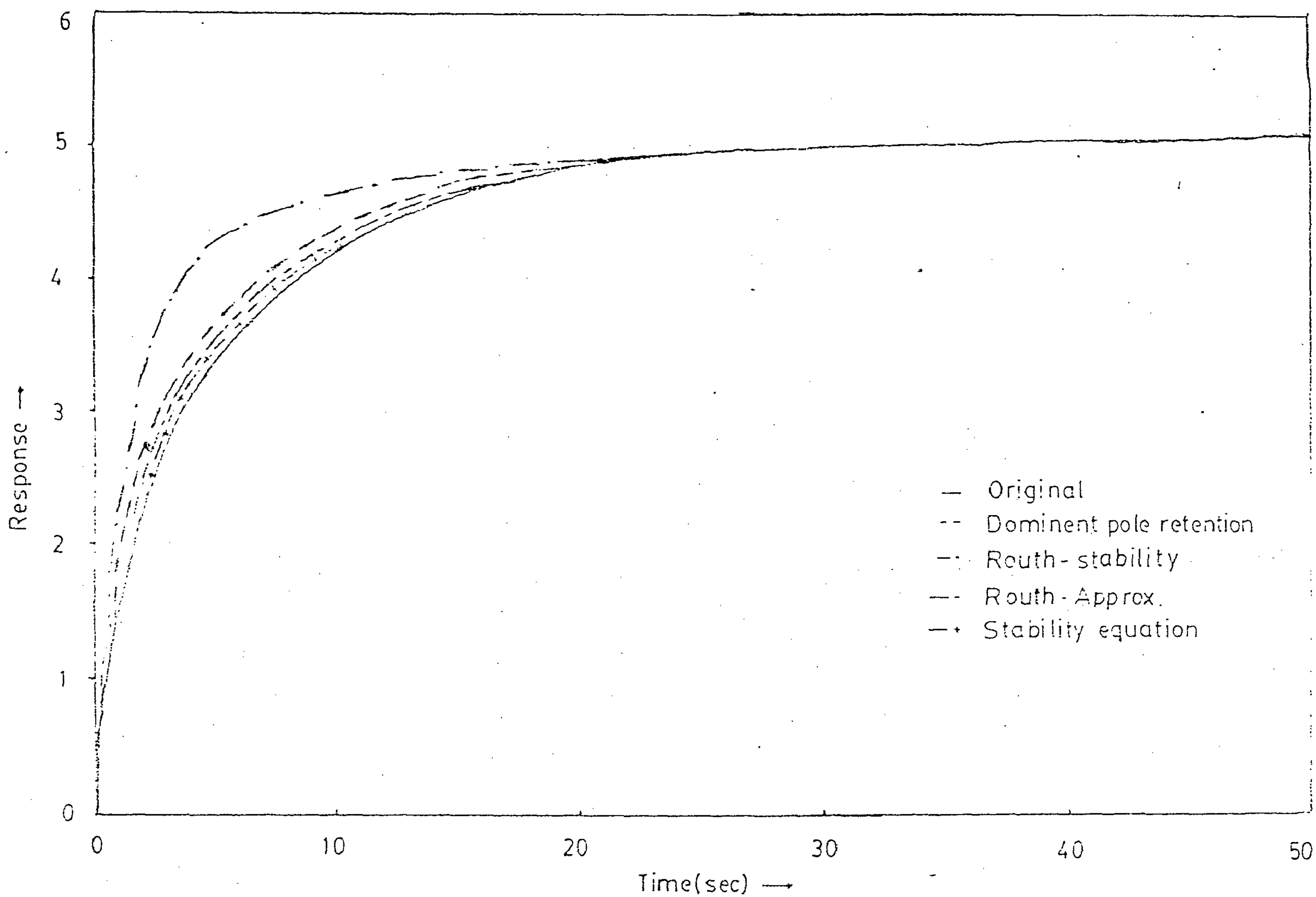


Fig 44 STEP-RESPONSES (3rd ORDER REDUCED TO SECOND ORDER)

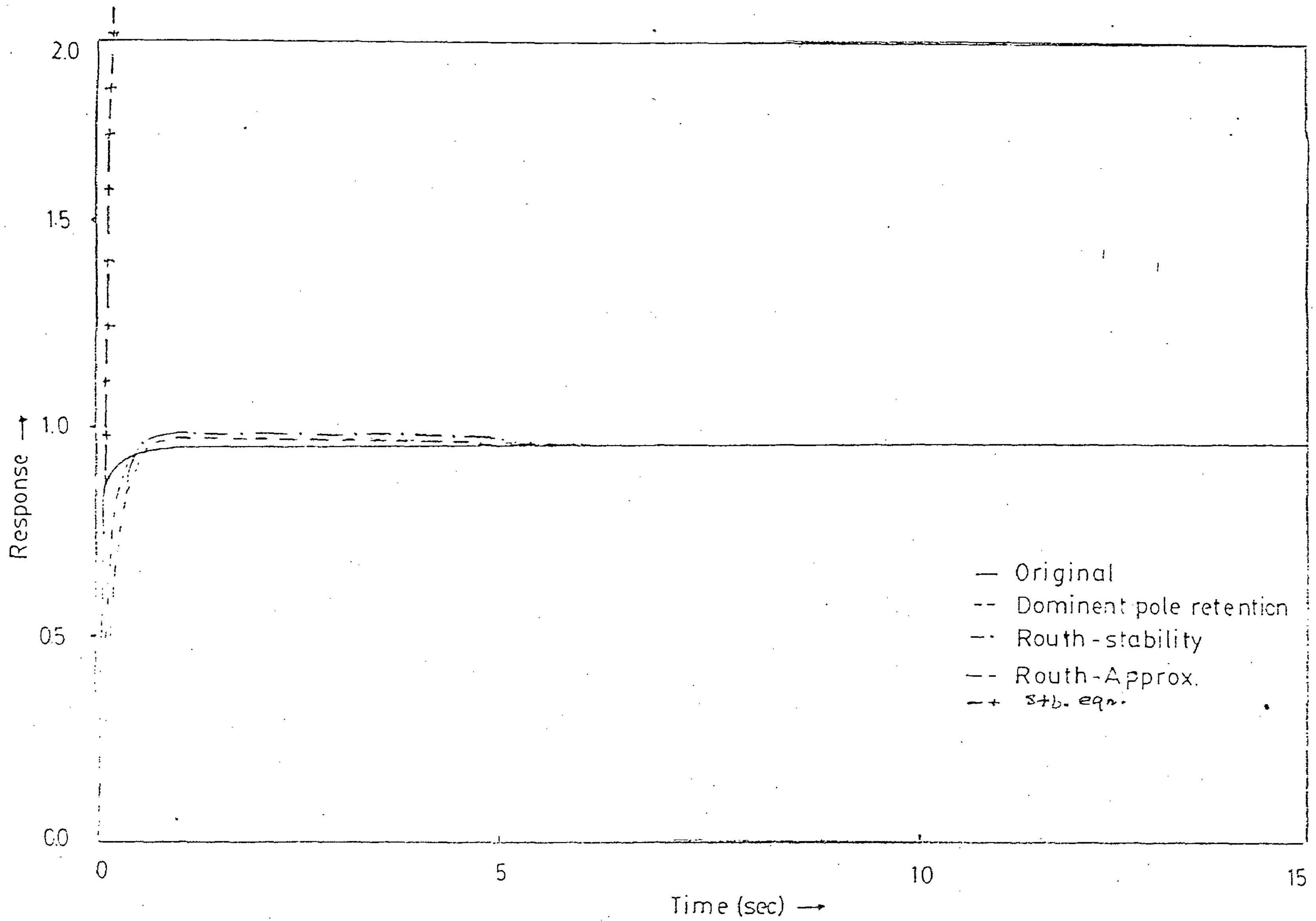


Fig 4.5 STEP-RESPONSES (4th ORDER REDUCED TO THIRD ORDER)

```
NP=  3
NZ=  1

Nr(S) COEFS          5.0000        15.0000

Dr(S) COEFS          1.0000         8.1000         7.8000         .7000

** INTEGRAL SQUARE ERROR** =   52.4664

        **  FOR ORGINAL MODEL  **

  ROOTS ARE :->
           REAL PART              IMAGINARY PART
            -.142857               .000000
            -1.000000              .000000
           -10.000000              .000000
 NP=  2
 NZ=  1
        **  FOR REDUCED ORDER  **

  ROOTS ARE :->
 0         REAL PART              IMAGINARY PART
          -.142857               .000000
         -1.000000               .000000
     ------------------------------
     METHOD NO.(FOR DENOMINATOR)=    1
     ------------------------------
        **DOMINANT POLE**
     ------------------------------

  ROMs D(S) POLYn COEFFS. OF S(0).S(1)..
     .1429        1.1429        1.0000

  ROMs N(S) POLYn COEFFS. S(0).S(1)..
     .7143        2.6156
     METHOD NO.(FOR DENOMINATOR)=    2
     ------------------------------
        **ROUTH STABLITY**
     ------------------------------

  ROMs D(S) POLYn COEFFS. OF S(0).S(1)..
    1.0000        8.0103        7.8000

  ROMs N(S) POLYn COEFFS. S(0).S(1)..
    5.0000        26.2774
     METHOD NO.(FOR DENOMINATOR)=    [illegible]
     ------------------------------
        **ROUTH APPROX.**
     ------------------------------

  ROMs D(S) POLYn COEFFS. OF S(0).S(1)..
     .1296        1.0501        1.0000

  ROMs N(S) POLYn COEFFS. S(0).S(1)..
     .6482        2.4462
     METHOD NO.(FOR DENOMINATOR)=    4
     ------------------------------
        **STBLTY. EQ.**
     ------------------------------

  ROMs D(S) POLYn COEFFS. OF S(0).S(1)..
     [illegible]
```

ROMs N(S) POLYn COEFFS. S(0).S(1)..

.6410 2.4220

Ex. 4.4

NP= 4
NZ= 3

Nr(S) COEFS 5.0000 99.6452 506.0497 [illegible]
13

Dr(S) COEFS 5.0000 101.0500 521.0100 105.20
00 1.0000

,INTEGRAL SQUARE ERROR = .0125

** FOR ORGINAL MODEL **

ROOTS ARE :->

REAL PART	IMAGINARY
-.100020	.000000
-.099980	.000000
-5.000000	.000000
-100.000000	.000000

NP= 3
NZ= 2

** FOR REDUCED ORDER **

ROOTS ARE :->

REAL PART	IMAGINARY PART
-.099980	.000000
-.100020	.000000
-5.000000	.000000

METHOD NO.(FOR DENOMINATOR)= 1

DOMINANT POLE

ROMs D(S) POLYn COEFFS. OF S(0).S(1)..
.0500 1.0100 5.2000 1.0000

ROMs N(S) POLYn COEFFS. S(0).S(1)..
.0500 1.0102 5.0638
METHOD NO.(FOR DENOMINATOR)= 2

ROUTH STABLITY

ROMs D(S) POLYn COEFFS. OF S(0).S(1)..
5.0000 101.0500 520.0494 105.2000

ROMs N(S) POLYn COEFFS. S(0).S(1)..
5.0000 100.[illegible] 513.2252
METHOD NO.(FOR DENOMINATOR)= 3

ROMs D(S) POLYn COEFFS. OF S(0).S(1)..

.0476	.9623	4.9617	1.0000

ROMs N(S) POLYn COEFFS. S(0).S(1)..

.0476	.9621	4.8327

METHOD NO.(FOR DENOMINATOR)= 4

SIBLTY. EQ.

ROMs D(S) POLYn COEFFS. OF S(0).S(1)..

.1940	.0096	.2019	1.0000

ROMs N(S) POLYn COEFFS. S(0).S(1)..

.1940	-.9883	.2083

NP= 3
NZ= 2

Nr(S) COEFS 2.0000 6.0000 8.0000

Dr(S) COEFS 2.0000 5.0000 4.0000 1.0000

** INTEGRAL SQUARE ERROR** = .6944

** FOR ORGINAL MODEL **

ROOTS ARE :->

0	REAL PART	IMAGINARY PART
	-.999999	.000000
	-1.000001	.000000
	-2.000000	.000000

NP= 2
NZ= 1

** FOR REDUCED ORDER **

ROOTS ARE :->

0	REAL PART	IMAGINARY PART
	-.999999	.000000
	-1.000001	.000000

METHOD NO.(FOR DENOMINATOR)= 1

DOMINANT POLE

ROMs D(S) POLYn COEFFS. OF S(0).S(1)..
1.0000 2.0000 1.0000

ROMs N(S) POLYn COEFFS. S(0).S(1)..
1.0000 3.1111
METHOD NO.(FOR DENOMINATOR)= 2

ROUTH STABLITY

ROMs D(S) POLYn COEFFS. OF S(0).S(1)..
2.0000 4.5000 4.0000
EX:4.5

ROMs N(S) POLYn COEFFS. S(0).S(1)..
2.0000 5.5289
METHOD NO.(FOR DENOMINATOR)= 3

ROUTH APPROX.

ROMs D(S) POLYn COEFFS. OF S(0).S(1)..
.5556 1.3889 1.0000

ROMs N(S) POLYn COEFFS. S(0).S(1)..
.5556 1.9023
METHOD NO.(FOR DENOMINATOR)= 4

STBLTY. EQ.

ROMs D(S) POLYn COEFFS. OF S(0).S(1)..
.5000 1.2500 1.0000

ROMs N(S) POLYn COEFFS. S(0).S(1)..
.5000 1.6824

Let us select the denominator by domainant pole retention method.

Then

$$G_r(s) = \frac{B_o + B_1\, s}{b_o + b_1 s + b_2 s^2}$$

where,

$$b_o + b_1 s + b_2 s^2 = (s + 1.8235)(s + 1.3471)$$

$$= s^2 + 3.2206\ s + 2.5238$$

To match the steady state value of the system and ROM

$$B_o = \begin{bmatrix} 2.542 & 1805.10 \\ 12.2630 & 252.56 \end{bmatrix}$$

Minimization of error will yield four unknown linear equation, which will give

$$B_1 = \begin{bmatrix} 1.2033 & 932.8647 \\ 7.154 & 1436.2 \end{bmatrix}$$

by other methods, the coefficients matrices are :

Method No.	B_o	B_1
2.	$\begin{bmatrix} 2543.2 & 1805947 \\ 12268.8 & 252680 \end{bmatrix}$	$\begin{bmatrix} 2344.9 & 1699679 \\ 11998.7 & 903236.4 \end{bmatrix}$
3.	$\begin{bmatrix} 1.9938 & 1415.779 \\ 9.6182 & 198.0894 \end{bmatrix}$	$\begin{bmatrix} 1.1884 & 900.9204 \\ 6.8422 & 1201.063 \end{bmatrix}$
4.	$\begin{bmatrix} 31.1325 & 22107.43 \\ 150.188 & 3093.173 \end{bmatrix}$	$\begin{bmatrix} -1.6918 & -62.3381 \\ 0.7476 & 8512.143 \end{bmatrix}$

The step responses of the system are shown in Fig. 4.6, 4.7, 4.8, 4.9 respectively.

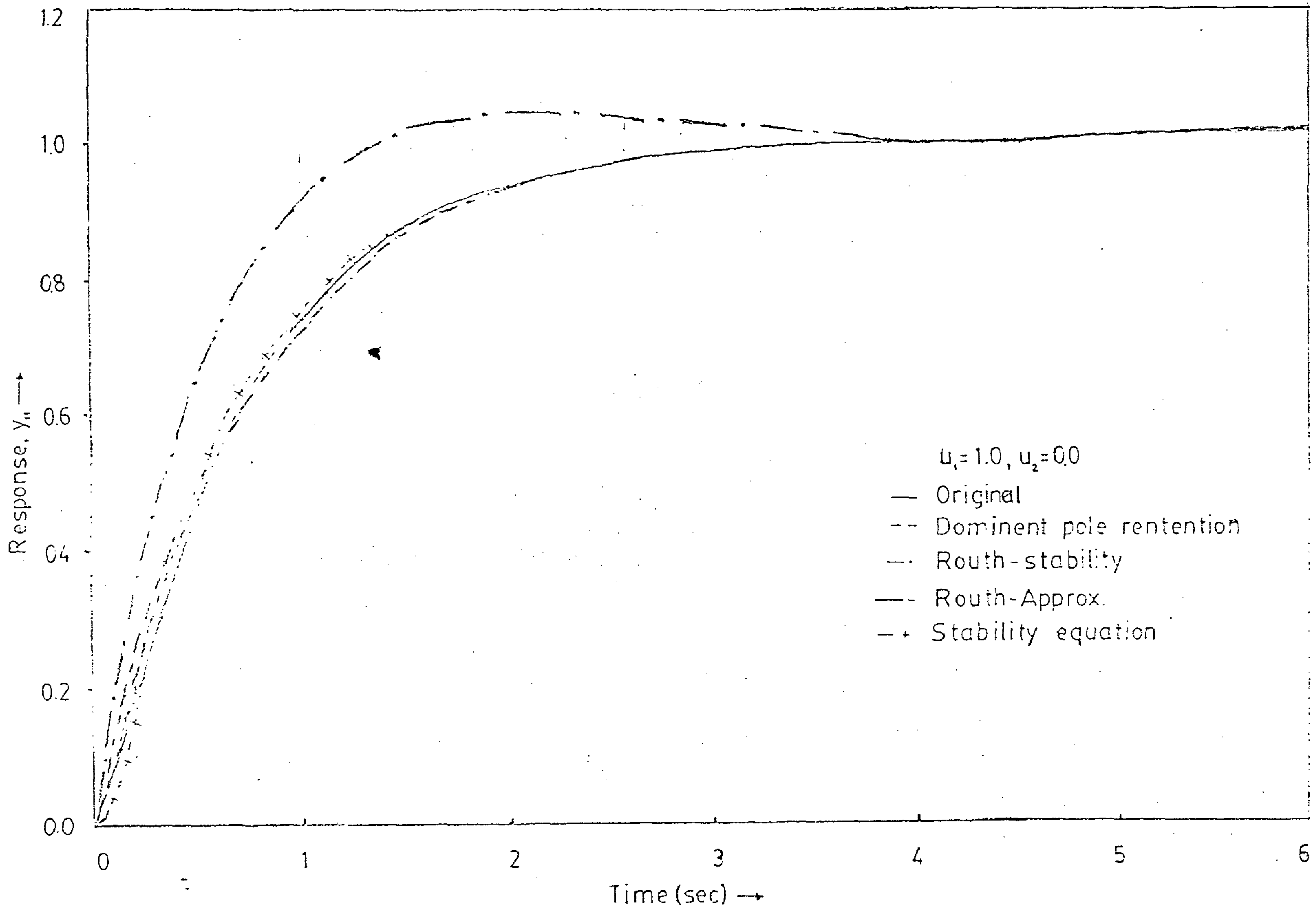


FIG 4.6 STEP RESPONSES - MULTI VARIABLE SYSTEM

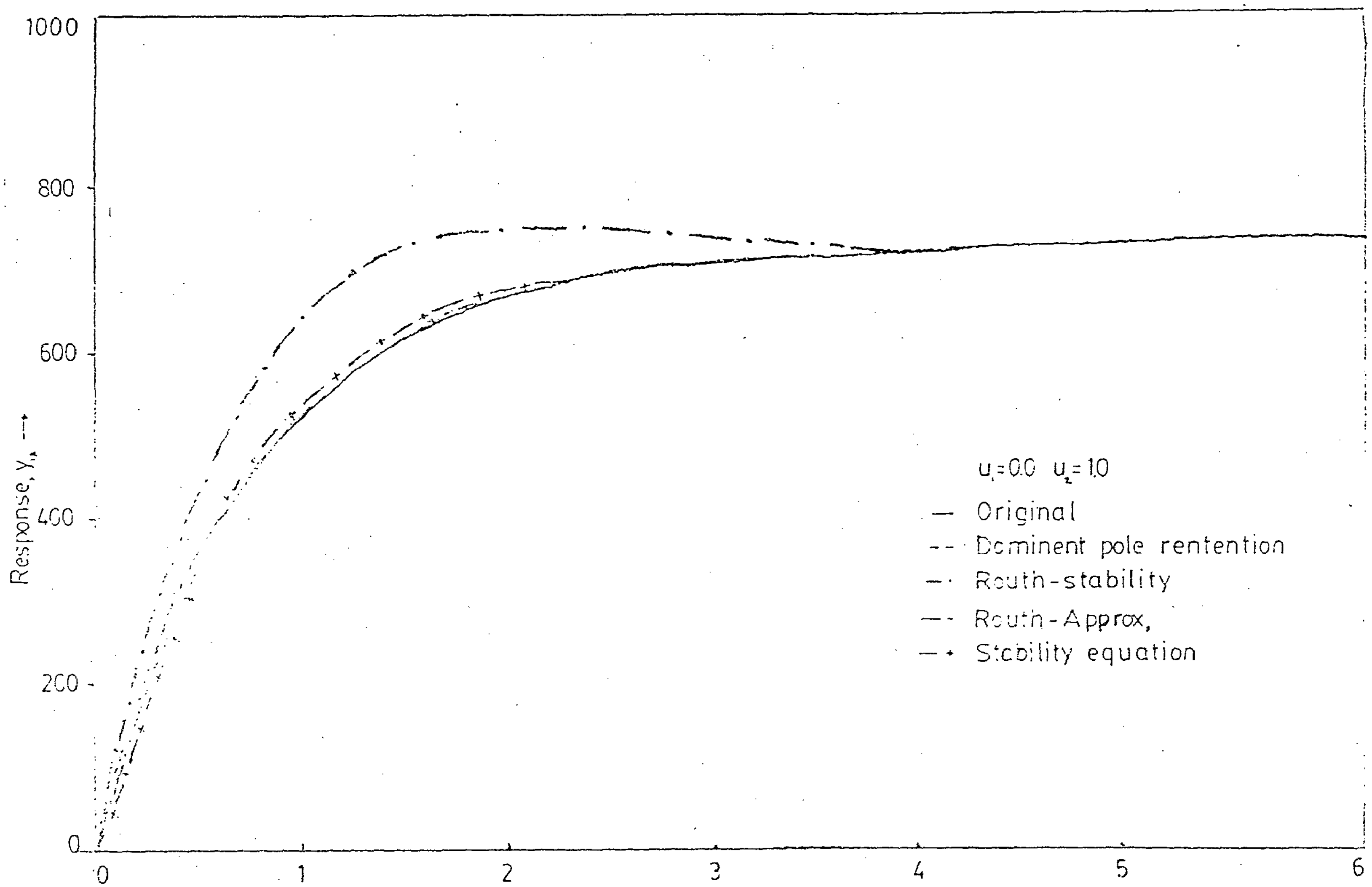


Fig 4.3 STEP RESPONSES MULTI VARIABLE SYSTEM

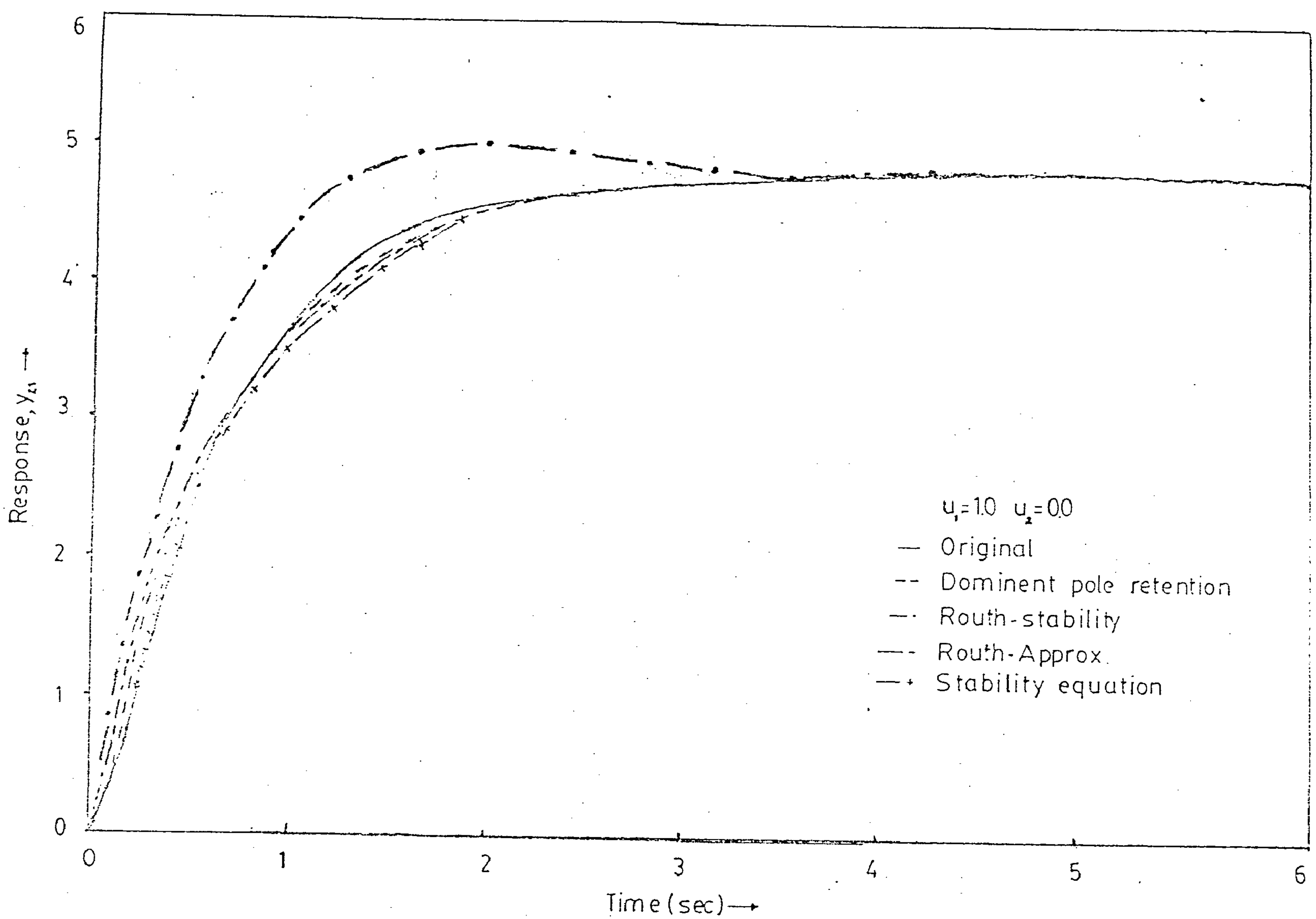


Fig 4.8 STEP RESPONSES - MULTI VARIABLE SYSTEM

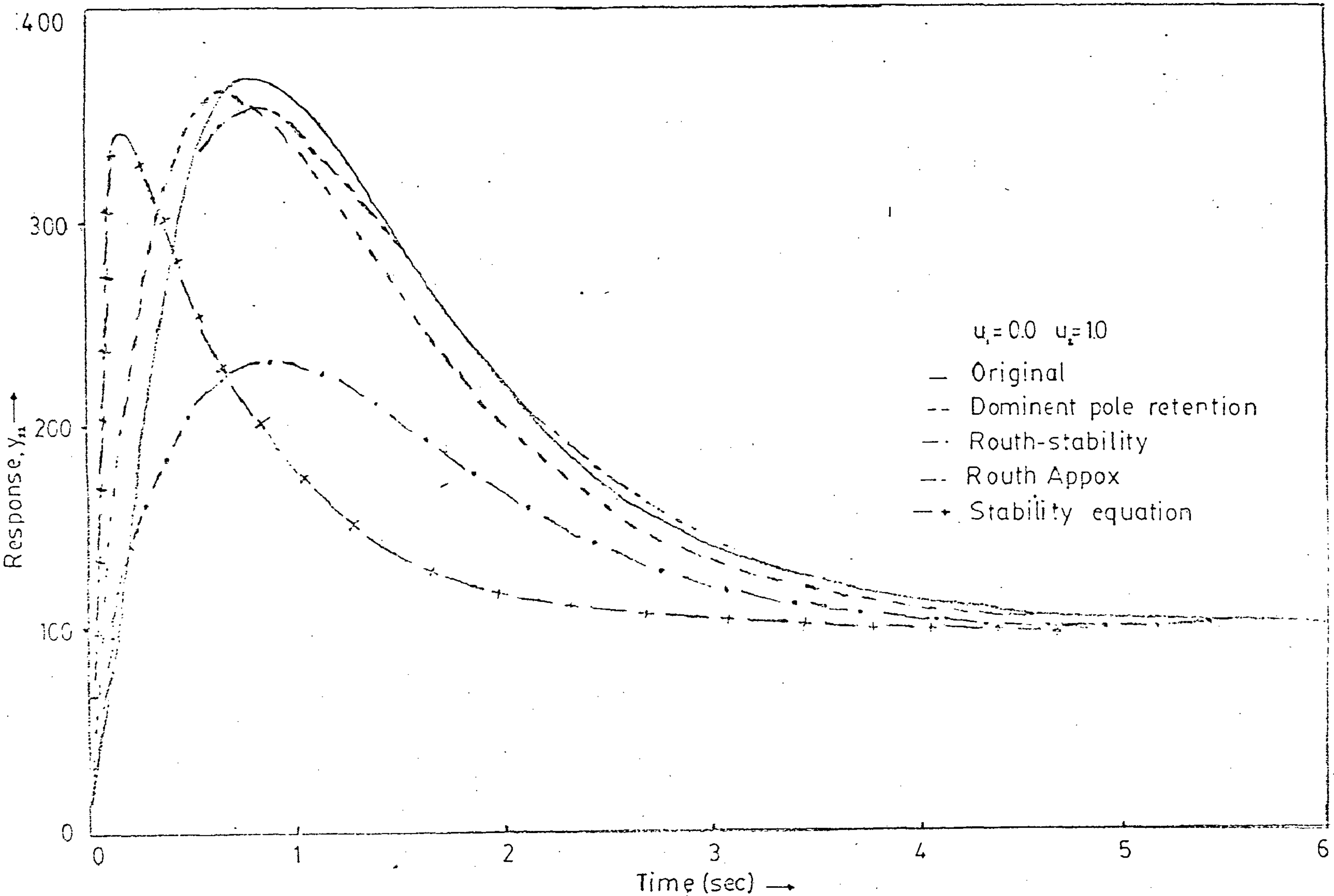


Fig 4.9 STEP RESPONSES MULTI VARIABLE SYSTEM

On the basis of the results of the examples taken and from their respective step responses, the following conclusions may be drawn -

In the example 4.1, responses of the ROMs are well matched for all methods, while method number 3 gives some error in transient zone, but the steady state response is matched properly. For the example 4.2, the system is reduced to its third and second order approximants. The step responses of third order ROMs are shown only for method number 1 and 2. It may be seen that method number 1 gives satisfactory results while method number 2 shows some oscillatory behaviour (Fig. 4.2). This may be due to the existance of complex poles of ROM. When the same system is reduced to its second order ROM, the better response is given by method number 1 as compared to method number 2 and 3 respectively. This has also been confirmed by the step response error in Table 4.3.

For example 4.3, all the four methods yield good responses. However, model obtained by method number 3 shows poor matching in transient period. In example 4.4, the method number 4 fails to give satisfactory results while others give good matching of responses with that of original system response. In case of CHUNG model (example 4.5), all methods give good results.

All the presented methods have been sucessively applied to each element of transfer matrix of a MIMO system. As shown in Figures (4.6 - 4.9) the results are quite satisfactory.

The methods outlined in this chapter have been extended to discrete time system in chapter- 5.

****

chapter : 5

methods for model reduction

discrete time systems

5.1 INTRODUCTION :

The analysis and design of high order discrete time dynamic systems are often required in many applications. Hence, of late, as evident from the many recent investigations model reduction of discrete - time systems has assumed immense importance.

As for continuous time systems, several investigations have been made for the model order reduction of discrete time systems. Some of these attempts to simplify directly in Z-domain while many other have used the bilinear transformation to extend continuous time reduction methods to discrete time case.

Modal approach of DAVISON [2] for continuous time state-space model has been extended to the discrete time case by WILSON et.al. [28], which is based on obtaining a discrete time ROM on least square fit of ROM to that of original Z-domain system. The CFE method [14] has extended to Z-domain by SHIH and WU [29] and SHAMASH [30]. Pade type approximation and moment matching method have also been used in the same way [32]. A mixed method of PARTHSARTHY et.al. [31] matches a combination of time moments and Markov parameters and is claimed to be computationally superior to that of SHAMASH [32]. CHUANG [21] has given a partial solution to the problem of Pade type ROM (being unstable) by using homographic transformation $z = \omega/(A + B\omega)$ where, A & B are constants, gives a family of ROMs of same order. A mixed method using a combination of Pade-approximation and dominant pole retention have also been proposed. SHIEH et.al. [33] have also given a mixed method utilising the dominant eigen value concept and CFE approach. APPIAH [34] proposed a method using HURWITZ polynomial approximation coupled with Pade-type time moment matching. CHEN et.al. [26] have used the stability equation method to get stable low- order approximants. After SHIH and WU [29], used the bilinear transformation to extend the CFE approach to simplify discrete time systems, it became well known that most of the reduction methods for continuous time domain can also be applied to discrete time systems, such as CFE method [29,33], stability equation method [26] HURWITZ polynomial method [34].

In-spite of the success of the extension of continuous time systems reduction methods to discrete time system using bilinear transformation (or similar transformations), there still remains two distinct disadvantages :

(i) Due to the nature of bilinear transformation, the initial value of the step response of reduced model may not be zero, inspite of zero initial condition of original step-response.

(ii) Most of the methods mentioned in section 5.1 are either in frequency domain or in time domain, and they are designed to secure a good fitting in their respective domain, consequently a reduced order model may be satisfactory in one domain, but unsatisfactory in another domain.

In order to circumvent, the problem associated (with order reduction of discrete systems) as mentioned in the last paragraph, a more enhanced technique i.e. a combination of time and frequency domain approach for order reduction is utilized, with stability guaranteed. The method consists of four steps stated below -

(i) Transformation of denominator polynomial ($D_n(z)$) of Z-transfer function into w-domain by bilinear transformation $z = \omega + 1/ \omega - 1$

(ii) Using one of the method, among the methods available for order reduction of denominator polynomial in frequency domain to reduce $D_n(\omega)$

(iii) Changing the reduced denominator polynomial into Z-domain, by using reverse bilinear transformation $w = z + 1/z - 1$

(iv) To find the optimal coefficients of numerator polynomial in reduced model the sum of the squared errors of unit step response of reduced order model and unit step response of original model is minimized.

For completeness, a brief review of four stable reduction methods for denominator polynomial is first given,

5.2 STABLE REDUCTION METHODS :

5.2.1 Dominant pole retention :

Let the characterstic polynomial of original system in frequency domain be (by using bilinear transformation)

$$D_n(w) = a_n w^n + a_{n-1} w^{n-1} + \dots + a_1 w + a_o \qquad \dots(5.1)$$

The reduced characterstic polynomial $D_r(w)$ of r^{th} order with $r < n$ is obtained by retaining significant poles i.e. retaining poles which are near to imaginary axis. The reverse bilinear transformation will give the $D_r(z)$.

5.2.2 Routh-approximation :

HUTTON and FRIEDLAND [23] first introduced Routh-approximation for reducing the order of continuous-time domain systems. Later a more simpler method was suggested by KRISHNAMURTHY and SHESHADRI [24]. The extension to the ω-domain (Routh approximation) is given as follows

$$\left.\begin{aligned} D_1^+(\omega) &= \omega + \alpha_1 \\ D_2^+(\omega) &= \omega^2 + \alpha_2\omega + \alpha_1\alpha_2 \\ D_3^+(\omega) &= \omega^3 + (\alpha_1+\alpha_3)\ \omega^2 + \alpha_2\alpha_3\omega + \alpha_1\alpha_2\alpha_3 \end{aligned}\right| \qquad \dots(5.2)$$

The reduced order characteristic polynomial will be

$$D_r^+(\omega) = \omega^2 D_{r-2}^+(\omega) + \alpha_r D_{r-1}^+(\omega) \qquad \dots(5.3)$$

with

$$D_{-1}^+(\omega) = 1/\omega \ ; \ D_o^+(\omega) = 1$$

Here the coefficients α_i are obtained from modified Routh table (3.1).

5.2.3 Hurwitz polynomial approximation :

The Hurwitz polynomial approximation proposed by APPIAH [34] is applied in the ω - domain. Equation (5.1) can be rewritten as :

$$D_n(\omega) = g(\omega^2) + \omega h(\omega^2) \qquad \ldots(5.4)$$

where,

$g(\omega^2) = E(\omega) = a_o + a_2\omega^2 + \ldots.$ (Even part of $D_n(\omega)$)

$h(\omega^2) = 1/\omega\ O(\omega) = 1/\omega\ (a_1\omega + a_3\omega^3 + \ldots$ (Odd part of $D_n(\omega)$)

CFE of $h(\omega^2)/g(\omega^2)$ about $\omega^2 = 0$ gives

$$\frac{h(\omega^2)}{g(\omega^2)} = \alpha_o + \cfrac{1}{\beta_o + \cfrac{1}{\alpha_1\omega^{-1} + \cfrac{1}{\beta_1 + \ddots + \cfrac{1}{\beta_{p-1} + \cfrac{1}{\alpha_p \cdot \omega^{-1}}}}}} \qquad \ldots(5.5)$$

where,

$\alpha_o = 0$ and $p = n/2$ if n is even

and, $\alpha_o \neq 0$ and $p = ((n-1)/2)$ if n is odd.

Now, the reduced order characterstic polynomial $D_r(\omega)$ is obtained by truncating the high order terms of CFE i.e.

$$D_r(\omega) = g_r(\omega^2) + \omega h_r(\omega^2) \qquad \ldots(5.6)$$

where,

$$\frac{h_r(\omega^2)}{g_r(\omega^2)} = \alpha_o + \cfrac{1}{\beta_o + \cfrac{1}{\alpha_1\omega^{-1} + \cfrac{1}{\beta_1 + \ddots + \cfrac{1}{\beta_{q-1} + 1/\alpha_q\omega^{-1}}}}} \qquad \ldots(5.7)$$

where,

$$q = (r - 1)/2; \quad \alpha_o \neq 0 \text{ (if r is odd)}$$

$$q = r/2; \quad \alpha_o = 0 \text{ (if r is even)}$$

The above discussion shows that $\alpha_o = 0$ for even, implies that even order characterstic polynomial can only be reduced to even order models and $\alpha_o \neq 0$ for odd case. However, it is learnt that Hurwitz polynomial approximation gives same reduced polynomial as given by Routh approximation technique. Hence it can be said that Hurwitz approximation is only a special case of Routh-approximation by which even and odd characteristic polynomial can be reduced to even or odd order ones.

5.2.4 Stability equation method :

After getting the denominator of Z-domain transfer function in the ω - domain (Eqn. (5.1)), we rduce the stability equations of the $D_n(\omega)$ by the method described in chapter - 3. After getting the reduced order denominator by this method, the reverse bilinear transformation will yield the reduced order denominator for the ROM in Z-domain.

5.3 OPTIMAL COEFFICIENTS OF NUMERATOR POLYNOMIAL :

Once the reduced order characterstic polynomial is obtained by one of the methods of previous section. The numerator coefficients of the reduced order model's are calculated as follows -

Let the reduced polynomial in Z-domain (by inverse bilinear transformation $\omega = z+1/z-1$) be

$$F(z) = a_m^+ z^m + a_{m-1}^+ z^{m-1} + \ldots + a_1^+ z + a_o^+ \qquad \ldots(5.8)$$

and the transfer function of ROM be

$$H^+(z) = \frac{C^+(z)}{U(z)} = \frac{b_{m-1}^+ z^{m-1} + b_{m-2}^+ z^{m-2} + --- + b_o^+}{F(z)} \qquad \ldots(5.9)$$

Here, b_i^+, $i = 0,1,2,\ldots, m-1$ are to be determined by matching the unit step response of the reduced and original system. For unit step input $U(z) = z/z-1$, the response will be

$$C^+(z) = \frac{b_{m-1}^+ z^m + b_{m-2}^+ z^{m-1} + \ldots + b_o^+ z}{a_{m+1}^* z^{m+1} + a_m^* z^m + \ldots + a_1^* z + a_o^*} \qquad \ldots(5.10)$$

$$= c_1^+ z^{-1} + c_2^+ z^{-2} + \ldots$$

where,

$$\left.\begin{aligned} a_i^* &= a_{i-1}^+ - a_i^+ , \quad i = 0,1,2,\ldots, m+1 \\ a_{m+1}^+ &= a_{-1}^+ = 0 \end{aligned}\right| \qquad \ldots(5.11)$$

The relationship between the coefficients a_i^*, b_i^+, c_i^+ is as follow :

$$\left.\begin{aligned}
b_{m-1}^+ &= a_{m+1}^* c_1^+ \\
b_{m-2}^+ &= a_{m+1}^* c_2^+ + a_m^* c_1^+ \\
&\cdots\cdots \\
&\cdots\cdots \\
b_o^+ &= a_{m+1}^* c_m^+ + a_m^* c_{m-1}^+ + \ldots + a_2^* c_1^+ \\
0 &= a_{m+1}^* c_{m+1}^+ + a_m^* c_m^+ + \ldots\ldots + a_2^* c_2^+ + a_1^* c_1^+ \\
0 &= a_{m+1}^* c_{m+2}^+ + a_m^* c_{m+1}^+ + \ldots + a_1^* c_2^+ + a_o^* c_1^+ \\
&\vdots \\
0 &= a_{m+1}^* c_k^+ + a_m^* c_{k-1}^+ + \ldots + a_1^* c_{k-m}^+ + a_o^* c_{k-m-1}^+
\end{aligned}\right| \qquad \ldots(5.12)$$

Equation (5.12) can be rewritten as :

$$
\begin{bmatrix}
a^*_{m+1} & 0 & 0 & \cdots & 0 & 0 & 0 \\
a^*_{m} & a^*_{m+1} & 0 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\
a^*_{1} & a^*_{2} & a^*_{3} & \cdots & 0 & 0 & 0 \\
a^*_{0} & a^*_{1} & a^*_{2} & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & a^*_{m} & a^*_{m+1} & a^*_{m+2} \\
0 & 0 & 0 & \cdots & a^*_{m-1} & a^*_{m} & a^*_{m+1}
\end{bmatrix}_{k\times k}
\begin{bmatrix}
c^+_1 \\ c^+_2 \\ c^+_3 \\ \cdot \\ c^+_m \\ c^+_{m+1} \\ \vdots \\ c^+_{k-1} \\ c^+_k
\end{bmatrix}_{k\times 1}
=
\begin{bmatrix}
1 & 0 & 0 & \cdots & 0 \\
0 & 1 & & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
0 & 0 & \cdot & \cdots & 1 \\
0 & 0 & \vdots & \vdots & 0 \\
\cdot & \cdot & \vdots & \vdots & \cdot \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}_{k\times m}
\begin{bmatrix}
b^+_{m-1} \\ b^+_{m-2} \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ b^+_1 \\ b^+_o
\end{bmatrix}_{m}
$$

or

$$Ac^+ = Vb^+ \qquad \ldots(5.13$$

Where c^+ is response vector at k sampled points, b^+ is unknown m vectors . The solution of c^+ is

$$c^+ = A^{-1} V b^+ \quad \ldots(5.14)$$

In order to preserve steady-state response of the original model, whose steady state response with unit step input is r_∞. Thus

$$\left.\begin{aligned} & b_1^+ + b_2^+ + b_3^+ + \ldots. + b_m^+ = r_o \\ & r_o = r_\infty(a_o^+ + a_1^+ + \ldots. + a_m^+) \\ \text{OR} \quad & S^T b^+ = r_o \end{aligned}\right| \quad \ldots(5.15)$$

where r_o = Steady state value of ROM

and $m \times 1$ vector S is,

$$S = [1 \quad 1 \quad 1 \ldots \quad 1]^T$$

The sum of square errors of the unit step response of the ROM and original models be minimum subject to the constraint eqn. (5.15) i.e.

$$\text{Minimize } J = \sum_{i=1}^{k} (c_i - c_i^+)^2 \quad \ldots(5.16)$$

where c_i is the unit step response of original model at i^{th} sampling point. Letting

$$c = [c_1 \; c_2 \ldots c_k]^T$$

$$\Rightarrow \quad J = (c - c^+)^T (c - c^+)$$

A more general form with different weighting on $(c_i - c_i^+)$ is

$$J = (c - c^+)^T W (c - c^+)$$

Where W = k x k positive definite diagonal matrix. An augmented function $\mathbb{J}$ is formed by introducing Langrange multiplier λ to take care of the constraint (in eqn. (5.15))

$$\mathbb{J} = (c - c^{+})^{T} W(c - c^{+}) + \lambda (S^{T} b^{+} - r_{o}) \qquad \ldots(5.17)$$

Substituting eqn. (5.13) in eqn. (5.17)

$$\mathbb{J} = (c - A^{-1} Vb^{+})^{T} W (c - A^{-1} Vb^{+}) + \lambda (S^{T}.b^{+} - r_{o})$$

The necessary condition to minimize $\mathbb{J}$ is

$$\frac{\partial \mathbb{J}}{\partial b^{+}} = 0$$

or

$$2(A^{-1} V)^{T} W(c - A^{-1} Vb^{+}) - \lambda S = 0 \qquad \ldots(5.18)$$

The equations (5.18) and (5.15) will give (m + 1) linear equations which can be solved to give b^{+} and λ. The above procedure is illustrated by an example.

Ex. 5.1 : Consider an eight order system,

$$H(z) = \frac{1.682 z^{7} + 1.116 z^{6} - 0.21 z^{5} + 0.152 z^{4} - 0.516 z^{3} - 0.262 z^{2} + 0.044 z - 0.006}{8 z^{8} - 5.04 z^{7} - 3.348 z^{6} + 0.63 z^{5} - 0.456 z^{4} + 1.548 z^{3} + 0.786 z^{2} - 0.132 z + 0.018}$$

Using the bilinear transformation the denominator polynomial in ω-domain is obtained as

$$D_{m}(\omega) = 8\omega^{8} + 78.64\omega^{7} + 292.982 \omega^{6} + 526.816 \omega^{5} + 584.144 \omega^{4} + 400.24 \omega^{3} + 139.232 \omega^{2} + 16 \omega + 2$$

By retaining dominant pole the reduced order denominator polynomial is (for a second order ROM)

$$D_2(\omega) = 0.206896 + 0.0926\ \omega + \omega^2$$

$$\text{Thus,}\quad D_2(z) = 0.982808 - 1.958626\ z + 1.113291\ z^2$$

(by inverse bilinear transformation)

The second order model calculated from eqn. (5.15) and eqn. (5.18) with k = 50 and W = I is

$$H^+(z) = \frac{-0.165023 + 0.23935\ z}{0.833638 - 1.75831\ z + z^2}$$

Likewise choosing other methods the models are

$$H^+(z) = \frac{28.7840 - 20.7840\ z}{119.4411 - 212.5067\ z + 101.0656\ z^2}$$

(by Routh approximation + error minimization)

$$H^+(z) = \frac{0.4380 - 0.3483\ z}{1.2018 - 1.9552\ z + 0.8431\ z^2}$$

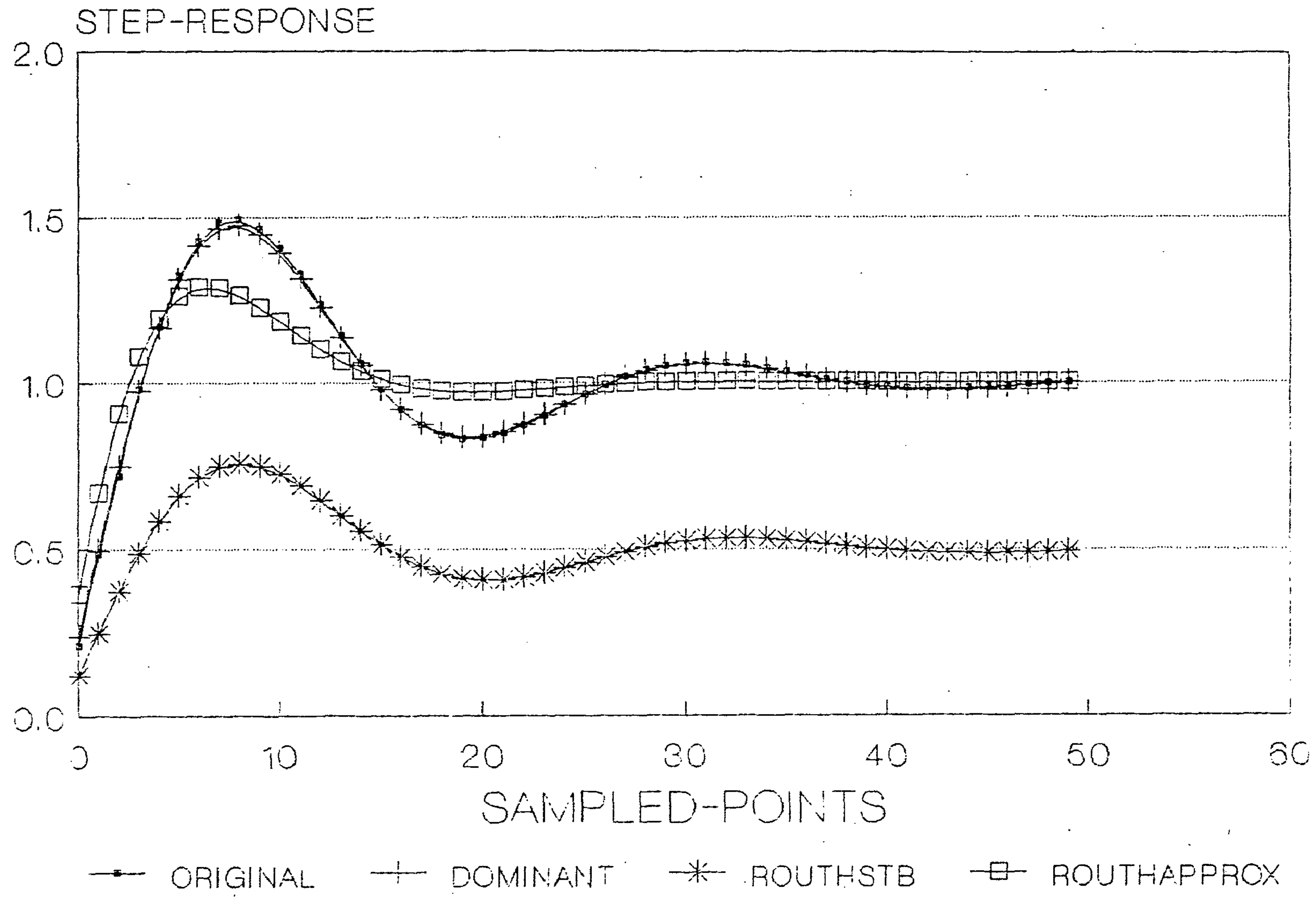
(by Routh stability + error minimization)

$$H^+(z) = \frac{0.0593\ z + 0.1099}{1.0476\ z^2 - 1.9154\ z + 1.0370}$$

(by stability equation + error minimization)

The step responses of the original and ROMs are depicted in Fig. 5.1. The step response of the ROM deduced by method No. 4 is not shown because of the poor matching.

FIG 5.1
DISCRETE SYST.ORDER REDUCTION



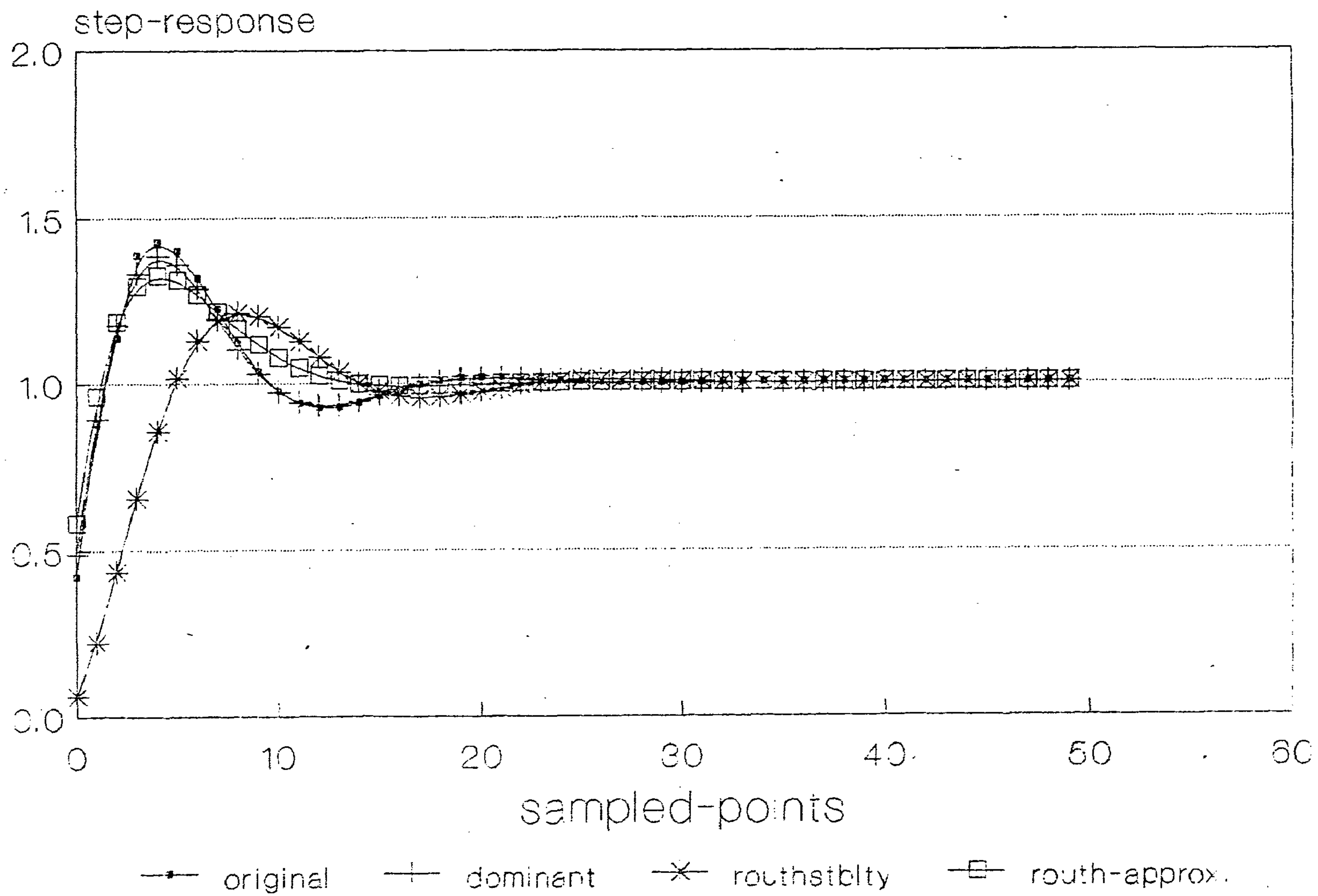
Ex. 5.2 : Considering another system of

$$H(z) = \frac{280.333\ z^7 + 186\ z^6 - 35\ z^5 + 25.333\ z^4 - 86\ z^3 - 43.666\ z^2 + 7.333\ z - 1}{666\ z^8 - 280.333\ z^7 - 186\ z^6 + 35\ z^5 - 25.333\ z^4 + 86\ z^3 + 43.666\ z^2 - 7.333\ z + 1}$$

The program results are given and the step responses are shown in Fig. 5.2.

* * *   * * *

fig 5.2


step-response
2.0
1.5
1.0
0.5
0.0
0
10
20
30
40
50
60
sampled-points
original
dominant
routhstblty
routh-approx.

```
          ** FOR ORGINAL MODEL  **
             -------------------
NPOLE  NZERO  NPROM  SAMPLE-POINTS
----------------------------------------
  8      7      2         50

COEFFS OF N(Z) POLY. Z(N).Z(N-1)....
------------------------------------
  280.3330  186.0000  -35.0000   25.3330  -86.0000

  -43.6660    7.3330   -1.0000

COEFFS OF D(Z) POLY. Z(N).Z(N-1)....
------------------------------------
  666.0000 -280.3330 -186.0000   35.0000  -25.3330

   86.0000   43.6660   -7.3330    1.0000

          **  FOR REDUCED ORDER  **
              ---------------------

        METHOD NO.(FOR DENOMINATOR)=    1
        ---------------------------
            **DOMINANT POLE**
            -----------------

   COEFFS OF D(Z) POLY. Z(N).Z(N-1)....
   ----------------------------------------
    1.2655   -1.9013     .8332
   COEFFS OF N(Z) POLY. Z(N).Z(N-1)....
   ----------------------------------------
     .6201    -.4225
        METHOD NO.(FOR DENOMINATOR)=    2
        ------------------------------
            **ROUTH STABLITY**
             --------------

   COEFFS OF D(Z) POLY. Z(N).Z(N-1)....
   ----------------------------------------
 10691.8500 ********** 7578.8020
   COEFFS OF N(Z) POLY. Z(N).Z(N-1)....
   ----------------------------------------
 1332.0000 1332.0000
        METHOD NO.(FOR DENOMINATOR)=    3
        ------------------------------
            **ROUTH APPROX.**
             --------------

   COEFFS OF D(Z) POLY. Z(N).Z(N-1)....
   ----------------------------------------
    1.3116   -1.9308     .7576
   COEFFS OF N(Z) POLY. Z(N).Z(N-1)....
   ----------------------------------------
     .7615    -.6229
        METHOD NO.(FOR DENOMINATOR)=    4
```

```
--------------------------------
    **SIBLTY EQ.**
    ---------------

COEFFS OF D(Z) POLY. Z(N),Z(N-1)....
-----------------------------------------
 1.0762   -1.8290    1.0748
COEFFS OF N(Z) POLY. Z(N),Z(N-1)....
-----------------------------------------
  .1656     .1768
```

TABLE 5.2: STEP-RESPONSES

a.	original	reduced order models by using			
		dominant-pole	routhstb.	routh-approx.	stbeq.
1	.4209	.4990	.0623	.5806	.1511
2	.8774	.8923	.2234	.9604	.5644
3	1.1345	1.1742	.4347	1.1840	1.1059
4	1.3861	1.3328	.6553	1.2939	1.6041
5	1.4268	1.3854	.8551	1.3265	1.9044
6	1.3981	1.3601	1.0153	1.3109	1.9170
7	1.3184	1.2875	1.1274	1.2692	1.6435
8	1.2242	1.1949	1.1913	1.2168	1.1749
9	1.1204	1.1038	1.2130	1.1637	.6612
10	1.0340	1.0277	1.2021	1.1159	.2635
11	.9739	.9735	1.1693	1.0762	.1037
12	.9386	.9420	1.1250	1.0453	.2269
13	.9246	.9306	1.0782	1.0228	.5893
14	.9274	.9340	1.0352	1.0075	1.0731
15	.9413	.9467	1.0005	.9979	1.5249
16	.9598	.9635	.9758	.9928	1.8045
17	.9785	.9805	.9614	.9907	1.8279
18	.9945	.9948	.9561	.9905	1.5928
19	1.0062	1.0052	.9580	.9916	1.1777
20	1.0133	1.0114	.9646	.9932	.7155
21	1.0161	1.0139	.9739	.9949	.3514
22	1.0158	1.0135	.9838	.9966	.1971
23	1.0135	1.0113	.9930	.9980	.2966
24	1.0101	1.0082	1.0006	.9992	.6140
25	1.0067	1.0051	1.0060	1.0000	1.0459
26	1.0037	1.0024	1.0092	1.0006	1.4554
27	1.0014	1.0004	1.0105	1.0010	1.7151
28	1.0000	.9992	1.0102	1.0013	1.7469
29	.9994	.9986	1.0089	1.0014	1.5454
30	.9994	.9987	1.0069	1.0014	1.1780
31	.9998	.9990	1.0048	1.0014	.7624
32	1.0004	.9996	1.0029	1.0013	.4295
33	1.0011	1.0002	1.0012	1.0013	.2913
34	1.0016	1.0007	1.0000	1.0012	.3606
35	1.0021	1.0011	.9993	1.0012	.6381
36	1.0023	1.0013	.9990	1.0011	1.0235
37	1.0025	1.0014	.9990	1.0011	1.3943
38	1.0025	1.0014	.9993	1.0011	1.6352
39	1.0024	1.0013	.9997	1.0011	1.6735
40	1.0023	1.0012	1.0002	1.0011	1.5012
41	1.0022	1.0011	1.0006	1.0011	1.1752
42	1.0021	1.0010	1.0009	1.0011	.8029
43	1.0020	1.0010	1.0012	1.0011	.4986
44	1.0019	1.0009	1.0014	1.0011	.3571
45	1.0019	1.0009	1.0014	1.0011	.4192
46	1.0019	1.0009	1.0014	1.0011	.6616

47	1.0019	1.0009	1.0014	1.0011	1.0052
48	1.0019	1.0009	1.0013	1.0011	1.3407
49	1.0020	1.0009	1.0012	1.0011	1.5637
50	1.0020	1.0009	1.0011	1.0011	1.6068

chapter : 6

application of reduction methods for controller design

6.1 INTRODUCTION :

The classical techniques of control system synthesis using logarithmic frequency response plots of BODE and NICHOLS, root locus diagrams of EVANS or Nyquist plots are well documented in literature. These above methods are graphical in nature and normally limited to SISO systems. With the advent of state space theory, the optimal control approach has been developed to tackle both deterministic and stochastic signals. This requireshigher order non-linear differential equations' solution and in addition it is difficult to translate industrial specifications into weighting matrices of the performance index which is normally chosen quite arbitrarily. Pole-zero assignment techniques are available but it is not clear how the desired pole-zero location in the case of multivariable systems are to be specified.

With the availability of fast digital computers alongwith interactive graphic display, control system design has entered a new era. In this chapter a procedure for the design of controller is presented. This method is particularly useful for SISO systems. The same technique can also be extended to multi-variable systems with transportation lag.

6.2 PROBLEM STATEMENT :

The problem of model matching can be stated as - ''Given a process whose performance is unsatisfactory and a reference model having the desired performance, derive a controller such that the performance of the augmented process matches with that of the model''.

The results are referred to those cases where the system model is actually implemented and incorporated in the control scheme. In the design of a control system in frequency domain, the specification that are usually considered as design goals may be classified as -

1. The time domain specifications e.g. rise time, over-shoot etc.
2. The frequency domain specifications i.e. bandwidth, phase margin etc.
3. The complex domain specification e.g. damping ratio, undamped natural frequency etc.

6.3 THE DESIGN PROCEDURE : [37]

To improve the efficiency of any design method, it is benificial to have the design goals expressed as mathematical functions or transfer functions (defined as the standard model). The first step in a model matching operation would be to specify such standard model.

Hence, the design procedure proceeds as follows :

STEP 1 : Construction of a specified model whose close-loop system must approximate to that of original closed loop- responses. Let it be specified as

$$G_M(s) = \frac{g_o + g_1 s + g_2 s^2 + \ldots + g_u s^u}{h_o + h_1 s + h_2 s^2 + \ldots + h_v s^v} \quad (u \leq v \text{ and in general } g_o = h_o) \qquad \ldots(6.1)$$

STEP 2 : Specification of structure of controller. Let it be -

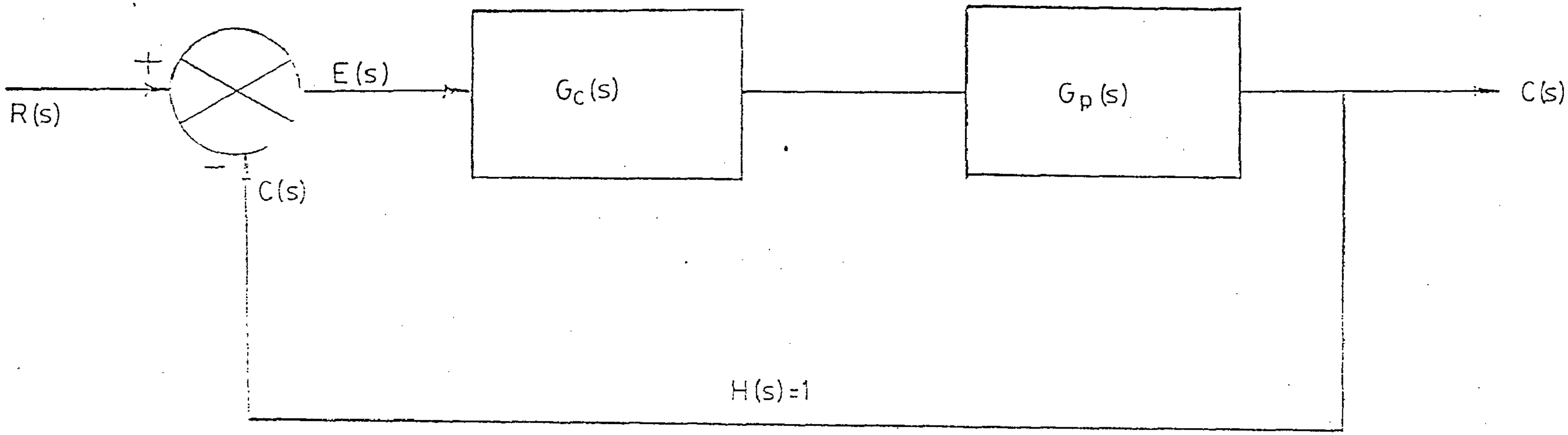


Fig 6.1 FEED BACK SYSTEM WITH PRE-COMPENSATOR

$$G_c(s) = \frac{k_{oo} + k_{o1}s + \dots + k_{oi}s^i}{k_{10} + k_{11}s + \dots + k_{1j}s^j} \qquad \dots(6.2)$$

STEP 3 : Determination of closed-loop transfer function consisting of unknown controller parameters.

If the reduced order plant transfer function be $G_{pr}(s)$ (where r is the order of the model) then

$$G_{pr}(s) = \frac{\alpha_o + \alpha_1 s + \dots + \alpha_m s^m}{\beta_o + \beta_1 s + \dots + \beta_n s^n} \quad (n \geq m \text{ and in general } \alpha_o = \beta_o) \qquad \dots(6.3)$$

and,

$$G_{CL}(s) = \frac{G_C(s)\, G_{pr}(s)}{1 + G_C(s)\, G_{pr}(s)} \quad \text{(Fig. 6.1)}$$

$$= \frac{a_o + a_1 s + a_2 s^2 + \dots + a_q s^q}{b_o + b_1 s + \dots + b_r s^r} \qquad \dots(6.4)$$

$$= c_o + c_1 s + c_2 s^2 + \dots$$

where,

$q = (m + i)$ and $r = (n + j)$

$c_o = 1$ when $a_o = b_o$

The coefficients $a_o, a_1 \dots a_q$; $b_o, b_1 \dots b_r$ and c_o, c_1 $c_2 \dots$ etc. witch contain the unknown controller parameters $k_{oo}, k_{o1} \dots k_{oi}, k_{11}, k_{12} \dots k_{1j}$ and known constant coefficients $\alpha_o, \alpha_1 \dots \alpha_m$ and $\beta_o, \beta_1 \dots \beta_n$. Thus, $G_M(s)$ to be a approximant of $G_{CL}(s)$, we have

$$g_o = h_o c_o \quad (\text{redundant when } a_o = b_o)$$

$$g_1 = h_o c_1 + h_1 c_o$$

$$g_2 = h_o c_2 + h_1 c_1 + h_2 c_o \qquad \dots(6.5)$$

$$\vdots$$

$$g_u = h_o c_u + h_1 c_{u-1} + \dots + h_u c_o$$

$$0 = h\,c + h_1 c + \dots + h\,c$$

(i + j + 3) equations of the above type can sequentially be solved for (i + j +2) unknowns controller parameters of equation (6.2). The particular traingular form of non-linear algebraic equations in eqn. (6.5) make their solutions very easy. The method is well illustrated by an example.

Ex. 6.1 : Consider the high order plant transfer function [37]

$$G_{p4}(s) = \frac{s^3 + 12\,s^2 + 54\,s + 72}{s^4 + 18\,s^3 + 97\,s^2 + 180\,s + 100}$$

The reduced order model for the $G_{p4}(s)$, by the methods described in Chapter - 4 are

$$G_{p2}(s) = \frac{0.6474\,s + 1.44}{s^2 + 3s + 2} \quad \text{(Method No. 1)}$$

or

$$G_{p2}(s) = \frac{79.2586\,s + 72}{87\,s^2 + 159.3103\,s + 100} \quad \text{(Method No. 2)}$$

$$G_{p2}(s) = \frac{0.6311\,s + 0.8276}{s^2 + 2.069\,s + 1.1494} \quad \text{(Method No. 3)}$$

$$G_{p2}(s) = \frac{-0.9520\,s + 7.2}{s^2 + 5.5556\,s + 10} \quad \text{(Method No. 4)}$$

The model transfer function is chosen as [37]

$$G_M(s) = \frac{1 + \alpha(\frac{2\xi}{\omega_n})s}{1 + (\frac{2\xi}{\omega_n})s + \frac{s^2}{\omega_n^2}}$$

where,

ξ - damping ratio

ω_n - undamped natural frequency.

α - design variable which has special significance in so far $\alpha = 0$ in $G_M(s)$ will result in zero displacement error system while $\alpha = 1$ will result in a zero velocity error system.

Choosing $\omega_n = 5.0$, $\xi = 0.707$ and $\alpha = 0.7$,

$$G_M(s) = \frac{25 + 4.242\ s}{25 + 7.07\ s + s^2} = 1 - 0.1131\ s - 0.0080\ s^2 \ldots$$

A close loop system using a P - I (proportional integral) type precompensator

$k_c\ (1 + 1/\tau_1 s)$ and unity feedback, is designed on the basis of $G_{p2}(s)$.

Taking the first, second order model. The close - loop transfer function with this will be (eqn. (6.4))

$$G_{CL}(s) = \frac{0.6474k_c s^2 + (1.44k_c + 0.6474k_c/\tau_1) s + 1.44k_c/\tau_1}{s^3 + (3+0.6474k_c) s + (2+0.6474k_c/\tau_1 + 1.44k_c/\tau_1) s + 1.44k_c/\tau_1}$$

$$= c_o + c_1 s + c_2 s^2 + \ldots.$$

where,

$\Longrightarrow$ $$c_o = 1$$

$$c_1 = \frac{1.44k_c\tau_1 - 2\tau_1 - 1.44k_c}{1.44\ k_c}$$

$$c_2 = \frac{-2.7738\ \tau_1 + 0.6474k_c\tau_1 + 0.2361\ k_c}{1.44\ k_c}$$

From equation (6.5)

$$2\ 5 = 25$$

$$4.242 = 25c_1 + 7.07$$

$$0 = 25c_2 + 7.07c_1 + 1$$

$$0 = 25c_3 + 7.07\ c_2 + c_1$$

In this, the very first equation is redundant and putting the values of c_1, c_2, etc. in the above equations. We get

$k_c = 10.6090$ and $\tau_1 = 0.8636$

The close loop transfer function or original and reduced model are

$$G_{CLO}(s) = \frac{83.376 + 134.53\ s + 67.896\ s^2 + 13.158\ s^2 + s^4}{83.376+143.96s+84.86s^2+22.32s^3+2.6967s^4+0.0943s^5}$$

and

$$G_{CL2}(s) = \frac{15.277 + 20.06\ s + 5.93\ s^2}{15.277 + 21.79\ s+8.522s^2 + 0.8636s^3}$$

Taking the ROM obtained from second method the k_c and τ_1 are found to be i.e.

$k_c = 10.609$ and $\tau_1 = 0.0814$. The close loop transfer function will be

$$G_{CL2}(s) = \frac{8.78 + 14.2778\ s + 5.7821\ s^2}{8.78 + 15.2795\ s + 7.5689\ s^2 + 0.8683\ s^3}$$

The open loop systems' step responses and closed loop step responses are shown in Fig. 6.2 and 6.3 respectively. It is obvious from the responses that they are having a very good match with the original close loop system's response.

We conclude this chapter by giving certain properties of the method as -

(1) Once the desired specifications are translated into a model transfer function this method automatically finds the parameters of a controller of specified structure.

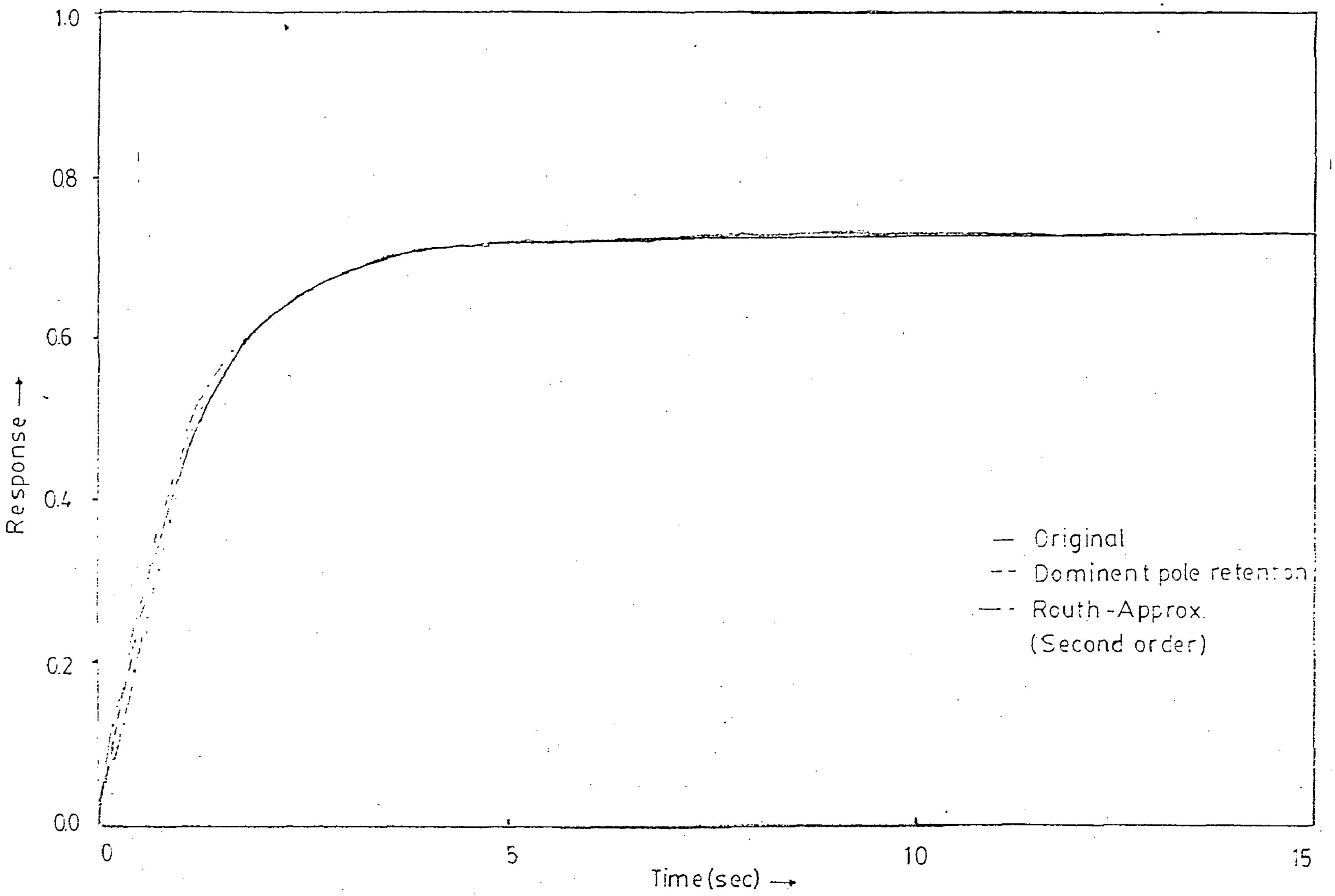


Fig 6.2 OPEN-LOOP STEP-RESPONSES

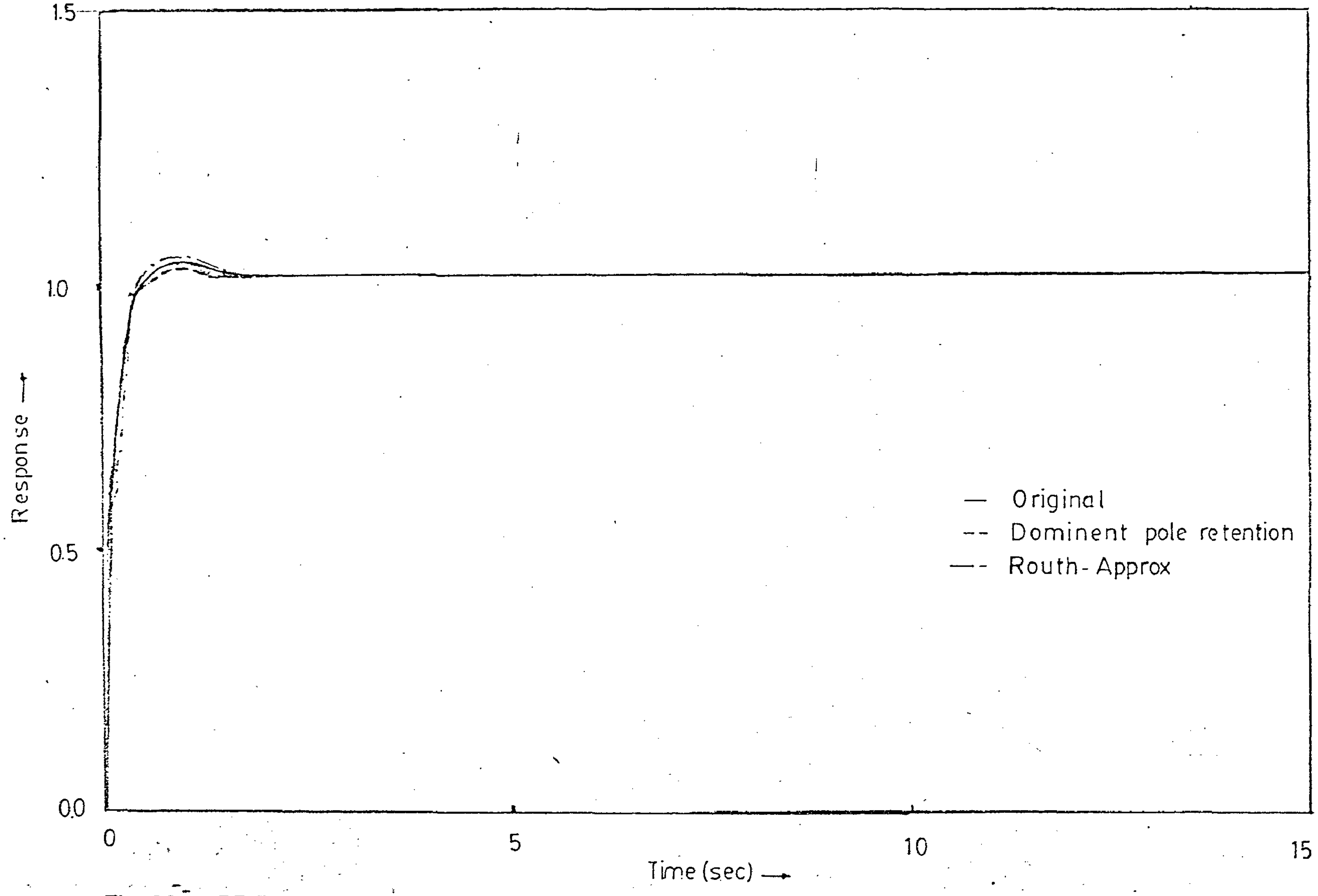


Fig 6.3 PRE-COMPENSATED CLOSE LOOP STEP - RESPONSES

(2) In contrast to classical techniques this is once through design method without resort to any trial and error process.

(3) This is computationally simple.

(4) With a minimum amount of efforts, this method yields practically realisable controllers conforming the desired industrial specifications.

However, this design method should be applied with caution for unstable plants and quite obviously, there is no getting around difficulties of non-minimum phase plants. Because this method is based on approximate model matching and hence, may lead to an unstable overall system due to truncation error. This drawback ~~may be overcome by prespecifying some of the pole zero~~ positions in the compensator to exactly cancel the effect of right hand side poles and zeros. This factor demands further considerations.

****

chapter :7

conclusions

The advantages of system order reduction techniques are well known. The main obvious advantages are saving in computational work in the analysis of large scale systems and economy in the design of associated hardware, for optimal and sub optimal controllers. The CFE or Pade approximation techniques suffer from the inherent difficulties viz. (i) the ROMs may be unstable, (stable), even though the original system is stable (unstable). (ii) The ROMs often show poor matching in transient zone and (iii) It may exhibit non-minimum phase characteristics. The methods presented in this thesis are devoid of these short-comings.

In this thesis several techniques for reducing the order of large scale systems have been tried on typical systems, considered by various researchers, using error minimization techniques for continuous and discrete time systems. The applicability of continuous time reduction method has been tested for controller design. The softwares for the techniques developed in FORTRAN and have been successfully implemented on PC. The error tables and step responses of the original and ROMs have been depicted for the comparison purposes.

The first introductory chapter lists the various possible reasons for going in for ROMs and for the use they have been put to.

In second chapter a detailed procedure for minimizing the error is presented. The third chapter gives the idea about stability based reduction methods which have been combined with error minimization technique, in chapter - 4, to yield the ROMs for continuous time systems. Various mixed methods have been given for obtaining stable ROMs. As shown by different examples in this chapter, all methods work quite well. However it should be pointed out that the efficacy of a model reduction technique depends on the perticular use, the ROM put to. In some cases the stress can be on good matching in low frequency zone while in some other cases the main objective can be to retain the transient zone characteristics.

The extensions of the above model reduction methods to discrete time system are given in chapter - 5. It is well known that by using bilinear transformation, discrete analog of continuous time model reduction can easily be obtained and similar conclusions, as in chapter 4, may be drawn.

A method has been given for controller design in chapter 6 using ROMs. The method is based on Pade approximation and algebraic in nature. The desired performance is converted into a transfer function model which is matched with closed loop system to have identical initial few time moments. The method does not require any trial and error procedure. However, as this method is based on the principle of approximate model matching, it may lead to poor or unstable control for non-minimum phase and unstable systems.

This work is in the direction to provide useful methods for model order reduction and to design controllers for the various systems. The given reduction method can be extended for the reduction of systems with transportation lag and to design sub-optimal controllers.

****

appendix - A

APPENDIX - A

A.1 The integral (Chapter - 4)

$$\frac{1}{2\pi j}\int_{-j\infty}^{+j\infty} Y(s)\, Y_r(-s)\, ds$$

Can be evaluated in the following way

$$J_n = \frac{1}{2\pi j}\int_{-j\infty}^{+j\infty} Y(s) Y_r(-s)\, ds = \frac{1}{2\pi j}\int_{-j\infty}^{+j\infty} \frac{a(s)}{b(s)} \times \frac{c(-s)}{d(-s)} ds$$

$$= \frac{1}{2\pi j}\left[\int_{-j\infty}^{+j\infty} \left(\frac{E(s)}{b(s)} + \frac{D(-s)}{d(-s)}\right)\right] ds$$

Where E(s) and D(s) are two polynomials of the form

$$E(s) = E_o + E_1 s + \ldots\ldots + E_{n-1}\, s^{n-1}$$

$$D(s) = D_o + D_1 s + \ldots\ldots + D_{r-1}\, s^{r-1}$$

and $$J_n = \frac{1}{2\pi j}\int_{-j\infty}^{+j\infty} \frac{E(s)}{b(s)}\, ds + \frac{1}{2\pi j}\int_{-j\infty}^{+j\infty} \frac{D(-s)}{d(-s)}\, ds = \frac{1}{2}\left[E_{n-1} + D_{r-1}\right]$$

E_{n-1} and D_{r-1} can be obtained from

$$a(s)\, c(-s) = E(s)\, d(-s) + D(-s)\, b(s)$$

[refer [27]]

A.2

BILINEAR TRANSFORMATION OF POLYNOMIALS :

In analysing the situation of transformation, the procedure can be subdivided into a sequence of elementry operations on polynomial which involves -

(i) Scaling the magnitude of zeros.

(ii) Replacing the zeros by their reciprocals.

(iii) Shifting the zeros by a real constant.

The first two operations are trivial, and for third, synthetic division is found satisfactory for numerical accuracy. The sub-division of a bilinear transformation into linear transformation, and inversion and then another linear transformation is an established technique.

(i) To scale the magnitude of zeros of polynomial (e.g. given $F(x)$, to replace x by x/K) successive coefficients are multiplied by 1, K, K^2 .... etc.

Thus if

$$F(x) = a_n x^n + a_{n-1}x^{n-1} + \dots + a_1 x + a_0 \qquad \dots(i)$$

then

$$F(x/K) = a_n x^n + a_{n-1}x^{n-1} K + a_{n-2}x^{n-2} K^2 + \cdots + a_1 x K^{n-1} + K^n a_0 \qquad \dots(ii)$$

(ii) To replace the zeros by their reciprocal, $F(x)$ is replaced by its reciprocal polynomial $x^n F(1/x)$ i.e.

$$x^n F(1/x) = a_0 x^n + a_1 x^{n-1} + \dots + a_{n-1}x + a_n$$

or in other words, the coefficient order get reversed.

(iii) Reducing all zeros by real constant, 'd', requires evaluatic of the coefficients of taylor's expansion i.e.

$$F(x+d) = F(d) + F'(d)\,x + \tfrac{1}{2}F''(d)\,x^2 + \dots + \tfrac{1}{n!}F^n(d)\,x^n$$

These coefficients are generated directly by synthetic divisic alogorithm.

The sequence of operations required for transformation $x \rightarrow (x+1)/(x-1)$ is as follows :

$$F(x) \rightarrow F(x+1) \rightarrow F(1/x+1) \rightarrow F(2/x+1) \rightarrow F\left(\frac{2}{x-1}+1\right) \quad \ldots(iii)$$

This completes the transformation.

<u>Ex.</u> : Let $F(x) = 2x^3 + 4x^2 + 6x + 5$

<u>Step 1</u> : <u>decreasing all zeros by 1 i.e. $F(x) \rightarrow F(x+1)$</u>

The synthetic division proceeds as follows :

2	4	6	5
	2	6	12
2	6	12	17
	2	8	
2	8	20	
	2		
2	10		
2			

(i) $F(x+1) = 2x^3 + 10x^2 + 20x + 17$

(ii) <u>Replacing all zeros by their reciprocals</u> :

$F(1/x \pm 1) = 17x^3 + 20x^2 + 10x + 2$

(iii) <u>Scaling all zeros by 2 i.e.</u>

$F(2/x+1) = 17x^3 + 40x^2 + 40x + 16$

(iv) <u>Increase all zeros by 1 i.e. $F(2/x + 1) \rightarrow F(2/x - 1) + 1$:</u>

17	40	40	16
	− 17	− 23	− 17
17	23	17	1
	− 17	− 6	
17	6	11	
	− 17		
17	− 11		
17			

$$\therefore F\left(\frac{x+1}{x-1}\right) = 17x^3 - 11x^2 + 11x + 1$$

* * *

references

REFERENCES :

1. Chidambra, M.R., 'Two simple techniques for the simplification of large dynamic systems, Proc. Joint Automat. control Conf., pp 667 - 676, 1969.

2. Davison, E.J., 'A method for simplifying linear dynamic systems', IEEE Trans. Aut. Contr., Vol. AC - 11, pp 93 - 101, Jan. 1966.

3. Davison, E.J., C A New method for simplifying large linear dynamic systems', IEEE Trans. Automat. Control, Vol. AC - 13, pp 214 - 215, April, 1968.

4. Chidambara, M.R., 'On a method for simplifying linear dynamic systems', IEEE Trans. Automat. Control, Vol. AC -1: pp 119 - 121, Feb. 1967.

5. Gruca, A. and Bertrand, P., 'Approximation of high order systems by low order models with delays', Int. J. Control, Vol. 28, No. 6, pp 953 - 965, 1978.

6. Inooka, H. and Obinata, G., 'Mixed method of aggregation and ISE approaches for system reduction', Electronics Lett. Vol. 13, No. 3, pp 88 - 90, Feb. 1977.

7. Kuppurajulu, A. and Elangovan, S., 'System analysis by simplified models', IEEE Trans. Autom. Control, Vol AC - 1! pp 234 - 237', April 1970.

8. Bereznai, G.T., and Sinha, N.K., (a) 'New minimax objective for automated systems', Electronics Letts. Vol. 6, No. 26, Dec. 1970.
(b) 'Optimum approximation of high order systems by low order models', Int. J. Contr. Vol. 14, No. 5, pp 957 - 959,

9. Bondler, J.W., Markettos, N.D., and Sinha, N.K., 'Optimum systems modelling using recent gradient methods,' Int. J. Systems Sci., Vol. 4, No. I, pp 33 - 44, 1973.

10. Yahagi, T., ' Optimal approximation of transfer functions of linear dynamical systems: IEEE Trans. Circuits and Systems,' Vol. CAS - 24, No. 10, pp 560 - 568, Oct. 1977.

11. Wilson, D.A. and Mishra, R.N., 'Optimal reduction of multivarable systems,', Int. J. Contrd., Vol. 29, No. 2, pp 267 - 278, 1979.

12. Aoki, M., 'Control of large scale dynamic systems by aggregation', IEEE Trans. Automat. Control, Vol AC - 13, pp 246 - 256, 1968.

13. Chen, C.F. and Shieh, L.S., 'A novel approach to linear model simplification,' Int. J. Control, Vol 8, pp. 561 - 570, 1968.

14. Chen, C.F. and Shieh, L.S. 'Continued fraction inversion by Routh algorithm', IEEE Trans. Circuit Theory Vol LT - 16, pp 197 - 202, May 1969.

15. Chuang, S.C., ' Application of continued-fraction method for modelling transfer functions to give more accurate initial transient response', Electron. Lett., Vol. 6, pp 861 - 863, 1970.

16. Chen, C.F., 'Model reduction of multivariable control systems by means of matrix continued fraction', Int. J. Control, Vol 20 No. 2, pp 225 - 238, 1974.

17. Chen C.F. and Shieh, L.S. 'Continued fraction inversion by Routh's algorithm', IEEE Trans. Circuit Theory, Vol. CT - 16, pp 197 - 202, May 1969.

18. Shieh, L.S., Schrneider, W.P., and Willioms, D.R., 'A chain of factored matrices for Routh array inversion and continued fraction inversion,' Int. J. Control, Vol 13, pp 691 - 703, 1971.

19. Chen C.F., Chang, C.Y. and Han, K.W., 'Model reduction using stability equation method and continued fraction expansion', Int. J. Control, Vol. 32, No. 1, pp 87 - 94, 1980.

20. Parthasarthy, R. and John. S., 'System reduction using cauer continued fraction expansion about s = 0 and s = ∞ alternately', Electron. Lett. Vol 14, pp 261 - 262, 1978.

21. Chuang, S.C., 'Homographic transformation for the simplification of continuous time transfer functions by Pade approximation', Int. J. Control, Vol. 23, pp 821 - 826, 1976.

22. Bosley, M.J., and Lees, F.P., 'A survay of simple transfer function derivation from high order state variable models', Automatica, Vol. 8, pp 765 - 775, 1972.

23. Hutton, M.F. and Friedland, B., 'Routh approximations for reducing order of linear, time-invariant system', IEEE Trans. Autom. Control, Vol AC - 20, No. 3, pp 329 - 337, June 1975.

24. Krishnamurthi, V. and Seshadri, V., ' A simple and direct method of reducing the order of linear, time-invariant systems by Routh approximation in the frequency domain', IEEE Trans. Autom. Control, Vol. AC - 20, pp 797 - 799, Oct. 1976.

25. Shamash, Y. 'Model reduction using the Routh stability criterion and the Pade approximation technique', Int. J. Control., Vol. 21, No. 3, pp. 475 - 484, March 1975.

26. Chen, T.C., Chang, C.Y., and Han, K.W., 'Reduction of transfer functions by the stability - equation method', J. Franklin Inst. (USA), Vol. 308, No.4, pp 389 -404, Oct. 1979.

27. Newton, G.C., Gould, L.A. and Kaiser J.F., 'Analytic design of linear feed back control (1964) (London Wiley).

28. Wilson, R.G., Fisher, D.G., and Seborg, D.E., 'Model reduction for discrete - time system', Int. J. Control. Vol. 16, No. 3, pp 549 - 558, 1972.

29. Shih, Y.P. and Wu, W.T., 'Simplification of Z-transfer functions by continued fractions,' Int. J. Control. Vol. 17, No. 5, pp 1089 - 1094, 1973.

30. Shamash, Y., 'Continued fraction method for the reduction of discrete - time dynamic systems', Int. J. Control, Vol. 20, No. 2, pp 267 - 275, 1974.

31. Parthasarathy, R. and Singh, H. 'A mixed method for the simplification of large system dynamics', Proc. IEEE, Vol. 65, No. 11, pp 1604 - 1605, Nov. 1977.