APPLICATIONS OF CONTINUED FRACTI EXPANSIONS AND INVERSION TO LINEAR SYSTEM REDUCTION

A DISSERTATION

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requirements for the award of the degree
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MASTER OF ENGINEERING
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ELECTRICAL ENGINEERING
(SYSTEMS ENGINEERING AND OPERATION RESEARCH)

By

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Dedicated to
my
Beloved Parents

CANDIDATES DECLARATION

I hereby certify that the work which being presented in the dissertation entitled 'APPLICATIONS OF CONTINUED FRACTION EXPANSIONS AND INVERSION TO LINEAR SYSTEM REDUCTION' in partial fulfilment of the requirements for the degree of Master of Engineering in Electrical Engineering (System Engineering and Operation Research) submitted in the department of Electrical Engineering, University of Roorkee, Roorkee is an authentic record of my own work, carried out under the supervision of Mr. Rajendra Prasad, Lecturer, Electrical Engineering Department, University of Roorkee, Roorkee, during a period from 1.8.86 to 31.1.87.

The matter embodied in this dissertation has not been submitted for the award of any other degree or diploma.

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This is to certify that the above statement made by the candidate is correct to the best of my knowledge.

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CHAPTER - 1

INTRODUCTION

1.1 NECESSITY OF MODEL REDUCTION

Every physical system can be translated into mathematical model. The mathematical model of large scale systems are very complex and they can not be reduced by hand canclulations. Fast digital computers can only be used to reduce these complex models.

The mathematical procedure of system modelling often leads to comprehensive description of a process in the form of high order differential equations, which are difficult to use either for analysis or controller synthesis . It is hence useful and some - times necessary to find the possibility of finding some equation of the same type but of lower order that may be considered to adequately reflect the dominant characteristics of the system under consideration. Some of the reasons for using reduced order models of high order linear systems could be

(1) A system of uncomfortably high order poses difficulties in its analysis, synthesis or identification. So in its analysis, synthesis or identification an obvious method of dealing with such systems is to approximate it by a low order system for which characteristics such as time constant, damping ratio, natural frequency and their inter relationships are well known.

1.3 MODEL REDUCTION USING CFE

The principal philosophy underlying the derivation of simplified models by CFE stems from the fact that the continued fraction expansion resembles multiple feedback loops and feed forward paths with blocks corresponding to the quotients. As quotients descend lower and lower in position, or equivalently the blocks develop to more or more inner loops, they have less and less significance as far as the overall system performance is concerned. Therefore, truncating the continued fraction often some terms is equivalent to ignoring the inner, less important loops.

Suppose an nth order model is expressed as

$$G(s) = \frac{b_1 s^{n-1} + b_2 s^{n-2} + \dots + b_n}{s^n + a_1 s^{n-1} + a_2 s^{n-2} + \dots + a_n} \qquad \dots(1)$$

Because a general control system is a low pass filter in nature, in the simplification, we should take care of the steady state first and then the transient part. This means that we have to start the continued fraction expansion from the constant terms, or arrange the polynomials in the ascending order.

So first rewrite the polynomials in ascending order

$$G(s) = \frac{b_n + b_{n-1} s + \dots + b_2 s^{n-2} + b_1 s^{n-1}}{a_n + a_{n-1} s + \dots + a_2 s^{n-2} + a_1 s^{n-1} + s^n} \qquad \dots(2)$$

(b) The development of state space methods and optimal control techniques has made the design of a control system for high order multivariable systems quite feasible. When the order of the system becomes very high special numerical techniques are required to permit the calculations to be done at a reasonable cost on a typical digital computer. So reduced order model reduces computational complexity and computational burden as well hence a saving in both the memory and time requirement of computer.

1.2 APPLICATIONS OF REDUCED ORDER MODELS

The reduced order models and reduction techniques have been widely used for the analysis and synthesis of high order systems. Some of the uses to which these have been put are

(1) Prediction of the transient response sensitivity of high order systems using low order models.

(2) Predicting dynamic errors of high order systems using low order equivalents.

(3) Suboptimal controls derived by simplified models.

(4) Control system design.

$$= \cfrac{1}{H_1 + \cfrac{1}{\cfrac{H_2}{s} + \cfrac{1}{H_3 + \cfrac{1}{\cfrac{H_4}{s} + \ddots}}}} \qquad (3)$$

Based on (3) we can draw a general feedback block diagram as shown in Fig. 1 (a). If an m order simplified model is desired we only keep 2m quotients in (3) and omit the remaining ones, and consequently the general feedback block diagram reduces as given in Fig. 1(b).

The most important properties of the continued fraction expansion are

(1) It converges faster than other series expansions.

(2) It contains most of the essential characteristics of the original model in the first few terms.

(3) It does not require any knowledge of the model eigen spectrum.

(4) Since the denominator coefficients of the simplified model depend on both the numerator and denominator coefficients of the original model, stability of the simplified model can not be guaranteed even if the original model is stable.

Fig.1(a) Block diagram corresponds to continued Fraction Expansion

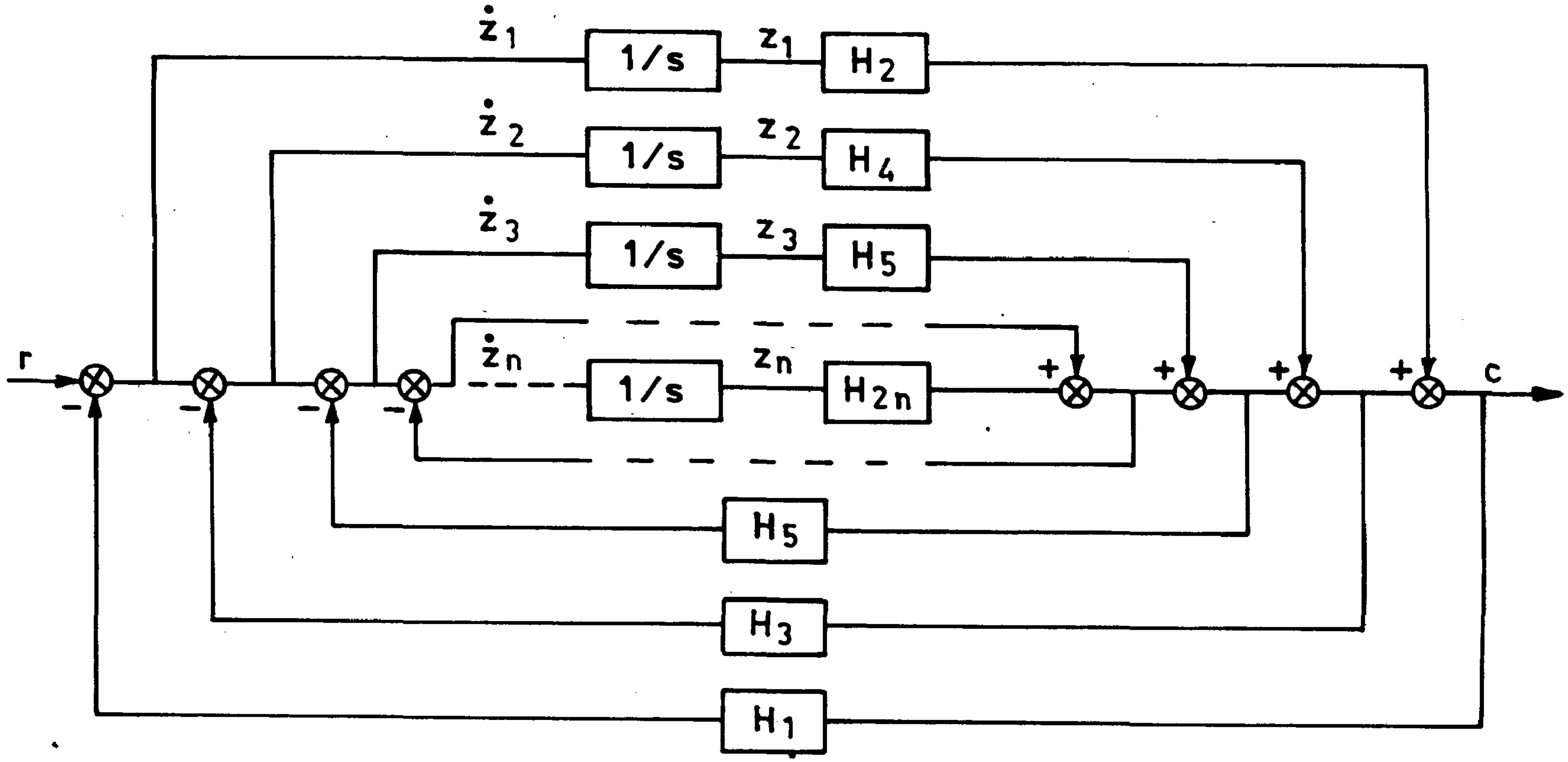
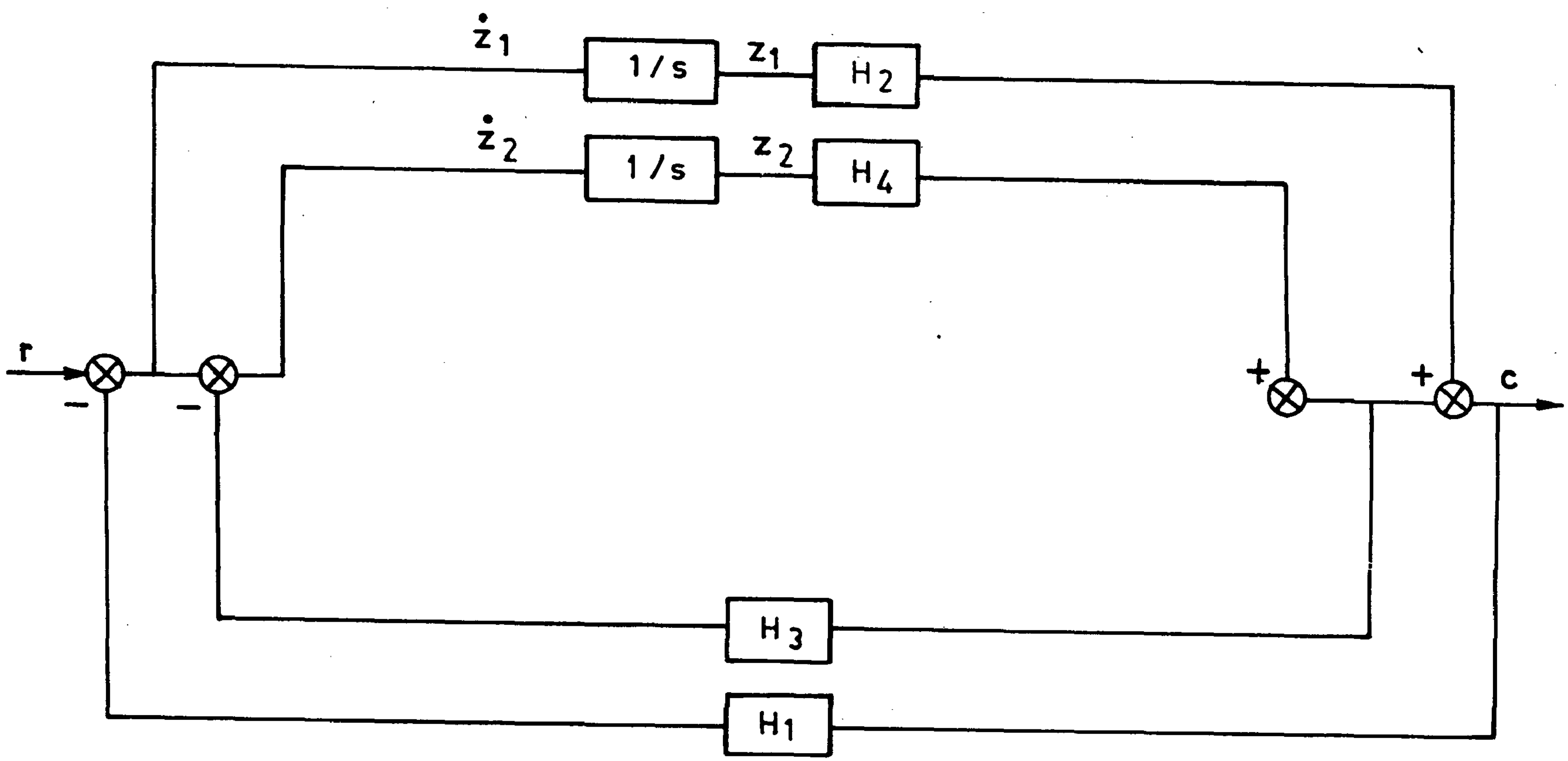


Fig.1(b) Simplified model corresponds to Continued Fraction Expansion



1.4 ORGANIZATION OF THE THESIS

In the present work a few selected model order reduction techniques namely Routh approximation. Routh Hurwitz array, stability equation, Modal and polynomial differentiation and each mixed with Cauer 2nd form and Cauer 3rd form are applied to

(1) SISO systems

(2) MIMO system

(3) Power system simplification

The thesis deals with frequency domain model order reduction techniques.

In chapter 2 various form of continued fraction expansions and their use in model reduction is dealt with.

Third chapter is devoted on various stability based reduction methods and a few other important reduction techniques.

Fourth chapter describes the model reduction using mixed method. Mixed reduction methods are obtained by mixing the methods described in chapter - 3, with Cauer second and third forms.

Chapter-5 deals with comparison of various reduction methods. The transfer functions of various SISO and MIMO models are reduced by pre-described reduction techniques, and unit step and frequency responses of reduced models are plotted with the responses of original models. The final responses show the validity of each method and their relative drawbacks.

Chapter - 6 describes the development of a power system model. The problem is taken from [18] and the development of the model is also from the same reference. In the same chapter various reduction tech - niques are applied to power system model. The transfer functions obtained from different methods are given in this chapter and all the reduced models are summarised and their responses are compared in the last showing the validity and drawbacks of reduction techniques used there in.

CHAPTER - 2

REDUCTION USING CONTINUED FRACTION EXPANSIONS AND INVERSION

The continued fraction expansion (CFE) method for obtaining reduced order models was first proposed by Chen and Shieh [1]. Various ramification and extensions of the GFE have since been presented by Chen and Shieh [2,3], and Chen and Huang [4]. As pointed out by Wilson, 'Chen's results are probably the best that have been obtained, although the method is only applicable to single input single output systems'.

Bosley and Leus [5] have compared the step responses of Chen and Shieh's reduced model and original system and have found very little error. Chen [6] has extended the CFE techniques [1,3] to model reduction and design of multivariable control systems. Shieh et.al [7] have demonstrated that the First, Second [6] and Third Cauer form formulations for order reduction give good approximations in the transient, steady state and overall region of the response curve respectively. Shieh and Goldman [8] have shown that a mixture of the first and second Cauer forms give good approximations for both the transient and the steady state responses. Davidson and Lucas [9] have formulated CFE method about a general point to allow good approximation to both transient, and steady state behaviour. One difficulty with the CFE approach is that the stability of the model is not guaranteed, even though the original system is stabled Chuang [10] has modified the

the original CFE techniques to have expansions about $s=o$ and $s=\infty$ alternatively thereby showing good agreement in both the transient and steady state regions.

CAUER FORMS OF CFE

Consider the following rational transfer function.

$$T(s) = \frac{A_{2,n}\, s^{n-1} + A_{2,n-1}\, s^{n-2} + \ldots A_{2,4}\, s^3 + A_{23}\, s^2 + A_{22} s + A_{21}}{A_{1,n+1}\, s^n + s_{1,n}\, s^{n-1} + \ldots A_{1,4}\, s^3 + A_{1,3}\, s^2 + A_{1,2} s + A_{1,1}} \qquad (4)$$

where $A_{i,j}$ are constants.

Equation (4) can be expanded into the following four different Cauer form representations.

2.1 THE CAUER FIRST FORM

$$T_1(s) = \cfrac{1}{H_1' s + \cfrac{1}{H_2' + \cfrac{1}{H_3' s + \cfrac{1}{H_4' + \cfrac{1}{\ddots}}}}} \qquad (5)$$

2.2 THE CAUER SECOND FORM

$$T_2(s) = \cfrac{1}{h_1' + \cfrac{1}{h_2'/s + \cfrac{1}{h_3' + \cfrac{1}{h_4/s + \cfrac{1}{\ddots}}}}} \tag{6}$$

2.3 THE CAUER THIRD FORM

$$T_3(s) = \cfrac{1}{h_1 + H_1 s + \cfrac{1}{\frac{h_2}{s} + H_2 + \cfrac{1}{h_3 + H_3 s + \cfrac{1}{\frac{h_4}{s} + H_4 + \cfrac{1}{\ddots}}}}} \tag{7}$$

2.4 THE CAUER MODIFIED FORM

$$T_4(s) = \cfrac{1}{h_1'' + \cfrac{s}{H_1'' + \cfrac{1}{h_2'' + \cfrac{s}{H_2'' + \cfrac{1}{\ddots}}}}} \tag{8}$$

Equation (7) is a combination of the Cauer first and second forms in such a way that if we let the h or H in (7) approach zero, then (7) will be identical with (5) or (6) respectively.

2.5 EXPANSION BY GENERALIZED ROUTH'S ALGORITHM

$$T(s) = \frac{A_{21}+A_{22}s+A_{23}s^2+A_{24}s^3+\dots+A_{2,n-1}s^{n-2}+A_{2,n}s^{n-1}}{A_{11}+A_{12}s+A_{13}s^2+A_{14}s^3+\dots+A_{1n}s^{n-1}+A_{1,n+1}s^n} \qquad (9(a))$$

$$=1/\Big[\frac{A_{11}}{A_{21}}+\frac{A_{1,n+1}}{A_{2n}}+\frac{(A_{12}-\frac{A_{11}A_{22}}{A_{21}}-\frac{A_{1,n+1}A_{21}}{A_{2n}})s+(A_{13}-\frac{A_{11}A_{23}}{A_{21}}-\frac{A_{1,n+1}A_{22}}{A_{2n}}\dots}{A_{21}+A_{22}s+A_{23}s^2+\dots}$$

$$+\frac{(A_{1n}-\frac{A_{11}A_{2n}}{A_{21}}-\frac{A_{1,n+1}A_{2,n-1}}{A_{2n}})s^{n-1}}{+A_{2n}s^{n-1}}\Big] \qquad (9(b))$$

Define

$$h_p = \frac{A_{p,1}}{A_{p+1,1}}, \qquad p +1,2,3,\dots,n \qquad (10)$$

$$H_p = \frac{A_{p,n+2-p}}{A_{p+1,n+1-p}} \qquad p=1,2,3,\dots,n \qquad (11)$$

where $h_p \neq 0, H_p \neq 0$, and substitute (10) and (11) into (9(b)) and we have

$$T(s) = 1/[h_1+H_1 s$$

$$+\frac{(A_{12}-h_1A_{22}-H_1A_{21})s+(A_{13}-h_1A_{23}-H_1A_{22})s^2+\dots+(A_{1n}-h_1A_{2n}-H_1A_{2,n-1})s^{n-1}}{A_{21}+A_{22}s+A_{23}s^2+\dots+A_{2n}s^{n-1}}] \qquad (12)$$

in which $(A_{12}-h_1A_{22}-H_1A_{21})$, $(A_{13}-h_1A_{23}-H_1A_{22}),\dots,(A_{1n}-h_1A_{2n}-H_1A_{2,n-1})$ can be written as $A_{31},A_{32},\dots,A_{3,n-1}$, respectively. Therefore we have

$$T(s) = 1/[h_1+H_1 s+\frac{A_{31}s+A_{32}s^2+\dots+A_{3,n-1}s^{n-1}}{A_{21}+A_{22}s+\dots+A_{2n}s^{n-1}}] \qquad (13)$$

Dividing again, we have

$$T(s) = 1/[h_3+H_1 s+1/\{ \frac{A_{21}}{A_{31}s} + \frac{A_{2n}}{A_{3,n-1}} + \frac{(A_{22}-\frac{A_{21}A_{32}}{A_{31}}-\frac{A_{2n}A_{31}}{A_{3,n-1}})s}{A_{31}s + A_{32}s^2+\ldots}$$

$$\frac{(A_{23}-\frac{A_{21}A_{33}}{A_{31}}-\frac{A_{2n}A_{32}}{A_{3,n-1}})s^2+\ldots+(A_{2,n-1}-\frac{A_{21}A_{3,n-1}}{A_{31}}-\frac{A_{2n}A_{3n-2}}{A_{3,n-1}})}{+A_{3,n-1}s^{n-1}} \qquad (14)$$

Finally, we have the expression

$$T(s) = \cfrac{1}{h_1+H_1 s + \cfrac{1}{h_2/s+H_2+ \cfrac{1}{h_3+H_3 s + \cfrac{1}{h_4/s+H_4+ \cfrac{1}{\ddots}}}}} \qquad (15)$$

The quotients in the expansion of (15) can be obtained by the following generalized Routh algorithm and the modified Routh array.

The coefficients in (9(a)) can be expressed by the following double-subscript notation.

$$\begin{array}{cccccc} A_{11} & A_{12} & A_{13} \cdots & A_{1n} & A_{1,n+1} \\ A_{21} & A_{22} & A_{23} \cdots & A_{2n} & \end{array} \qquad (16)$$

and the elements of the third, fourth, and subsequent rows can be evaluated from the following algorithm

$$A_{j,k}=A_{j-2,k+1}-h_{j-2}A_{j-1,k+1}-H_{j-2}A_{j-1,k} \quad J=3,4,\ldots,n+1, k=1,2\ldots \tag{17}$$

and

$$h_p=\frac{A_{p,1}}{A_{p+1,1}}, \quad H_p=\frac{A_{p,n+2-p}}{A_{p+1,n+1-p}}, \quad p=1,2,3,\ldots,n \tag{18}$$

The complete array is

$$h_1=\frac{A_{11}}{A_{21}} < \begin{matrix} A_{11} & A_{12} & A_{13}\cdots A_{1n} & A_{1,n+1} \\ A_{21} & A_{22} & A_{23}\cdots A_{2n} & \end{matrix} > H_1=\frac{A_{1,n+1}}{A_{2n}}$$

$$h_2=\frac{A_{21}}{A_{31}} < \qquad\qquad > H_2=\frac{A_{2,n}}{A_{3,n-1}}$$

$$h_3=\frac{A_{31}}{A_{41}} < \begin{matrix} A_{31} & A_{32} & \cdots & A_{3,n-1} \\ A_{41} & \cdots & A_{4,n-2} & \end{matrix} > H_3=\frac{A_{3,n-1}}{A_{4,n-2}}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \tag{19}$$

$$h_{n-1}=\frac{A_{n-1,1}}{A_{n,1}} < \begin{matrix} A_{n-1,1} & A_{n-1,2} & A_{n-1,3} \\ A_{n,1} & A_{n,2} & \end{matrix} > H_{n-1}=\frac{A_{n-1,3}}{A_{n,2}}$$

$$h_n=\frac{A_{n,1}}{A_{n+1,1}} < A_{n+1,1} \qquad H_n=\frac{A_{n,2}}{A_{n+1,1}}$$

The triangular pattern in the formulation of (19) is called the modified Routh array.

Equation (17) is a generalized Routh algorithm. If all H are zero, (17) is simplified as

$$A_{j,k} = A_{j-2,k+1} - h_{j-2}A_{j-1,k+1},$$
$$j = 3,4,\ldots,k = 1,2,\ldots \tag{20}$$

Equation (20) is a regular Routh algorithm which is commonly used to obtain the quotients of the Cauer second form. On the other hand, if all h are zero (17) reads

$$B_{j,k} = B_{j-2k+1} - H_{j-2}B_{j-1,k+1}$$
$$j=3,4,\ldots,k=1,2,\ldots \tag{21}$$

where

$$B_{1,i} = A_{1,n+2-i},\ i=1,2,\ldots n+1$$

and

$$B_{2j} = A_{2,n+1-j},\ j=1,2,\ldots,n$$

Equation (21) is a regular Routh Algorithm which is used to evaluate the quotients of the Cauer first form. Either formulated pattern by the algorithms shown in (20) or (21) will be a zig-zag pattern. It is noted that the elements $A_{j,k}$ $j=3,4,\ldots$ and $k=1,2,\ldots$, in (20) or (21) do not have the same values as those elements of (19).

Cauer Modified Form

The Cauer modified form (Chuang 1970) is obtained, by carrying out a Taylor series expansion to both S=0 and S = ∞ alternately. This would in effect mean that the expansion begins from the constant term and then from the highest order term. The approximation is good both in the steady state and transient period. The Cauer modified form is

$$T(s) = \cfrac{1}{h_1'' + \cfrac{S}{H_1'' + \cfrac{S}{h_2'' + \cfrac{1}{\quad}}}}$$

The transfer function.

T(s) is expanded into a Cauer type CFE about S = 0 and S = ∞. $h_1''.h_2''\ldots,H_1'', H_2''\ldots,$ are evaluated by modified routh array.

$$h_1'' = \frac{a_{11}}{b_{11}} < \begin{matrix} a_{11} & a_{12}\ldots\ldots a_{1,n-1} & a_{1n} & 1 \\ b_{11} & b_{12}\ldots\ldots b_{1,n-1} & b_{1n} & \end{matrix} > H_1'' = b_{1,n}$$

$$h_2'' = \frac{a_{21}}{b_{21}} < \begin{matrix} a_{21} & a_{22}\ldots\ldots a_{2,n-1} & 1 \\ b_{21} & \end{matrix}$$

$$\vdots \qquad b_{22}\ldots\ldots b_{2,n-1} > H_2'' = b_{2,n-1}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$h_n'' = \frac{a_{n1}}{b_{n1}} < \begin{matrix} a_{n1} \\ b_{n1} \end{matrix} > H_n'' = b_{n,1}$$

where

$$a_{j+1,k} = a_{j,k+1} - h_j''\, b_{j,k+1} \quad j=1,2\ldots n-1$$

$$b_{j+1,k} = b_{j,k} - H_j''\, a_{j+1,k} \quad k=1\ldots\ldots n-j$$

where

$$h_j'' = \frac{a_{j,1}}{b_{j,1}} \qquad H_j'' = \frac{b_{j,n+1-j}}{a_{j+1,n+1-j}} \quad j=1\ldots n$$

2.6 CONTINUED FRACTION INVERSION

If the quotients of a continued fraction of the Cauer third form are given, or all h and H are known, what is the corresponding transfer function. This is the problem of continued fraction inversion.

From (9(b)) is noted that

$$A_{n,1} = h_n A_{n+1,1}$$

$$A_{n-1,1} = h_{n-1} A_{n,1} = h_{n-1} h_{n+1,1}$$

$$A_{n-2,1} = h_{n-2} A_{n-1,1} = h_{n-2} h_{n-1} A_{n+1,1}$$

$$A_{31} = h_3 A_{41} = h_3 h_4 \ldots h_n A_{n+1,1}$$

$$A_{21} = h_2 A_{31} = h_2 h_3 \ldots h_n A_{n+1,1}$$

$$A_{11} = h_1 A_{21} = h_1 h_2 \ldots h_n A_{n+1,1} \qquad (22)$$

and

$$A_{n,2} = H_n A_{n+1,1}$$

$$A_{n-1,3} = H_{n-1} A_{n,1} = H_{n-1} H_n A_{n+1,1}$$

$$A_{n-1,3} = H_{n-1} A_{n-2} = H_{n-1} H_n A_{n+1,1}$$

$$\vdots$$

$$A_{n-1,3} = H_2 A_{3,n-1} = H_2 H_3 \ldots H_n A_{n-1,1}$$

$$A_{1,n} = H_1 A_{2n} = H_1 H_2 \ldots H_n A_{n+1,1} \qquad (23)$$

Equation (22) and (23) can be written as the following general equation. Let $A_{n+1,1} = 1$, then

$$A_{j,1} = \prod_{p=j}^{n} h_p, \qquad p=j,j+1,\ldots,n \tag{24}$$

and

$$A_{j,n+2-j} = \prod_{p=j}^{n} H_p, \quad p=j,j+1,\ldots,n \tag{25}$$

where j is the row number in the modified Routh array. The intermediate terms can be evaluated from (17), starting from the element in the last row of the modified Routh array and ending up at the elements in the first row. Or if we substitute j = n+1 and k=1, yields

$$A_{n+1,1} = A_{n-1,2} - h_{n-1} A_{n,2} - H_{n-1}A_{n,1} \tag{26}$$

Likewise, if we rearrange the order of (26), we have

$$A_{n-1,2} = A_{n+1,1} + h_{n-1} A_{n,2} + H_{n-1} A_{n,1}$$

If we perform the same procedures on other elements, we have

$$\begin{aligned} A_{n-2,2} &= A_{n,1} + h_{n-2}A_{n-1,2} + H_{n-2}A_{n-1,1} \\ A_{n-2,3} &= A_{n,2} + h_{n-2}A_{n-1,3} + H_{n-2}A_{n-1,2} \\ &\vdots \\ A_{1n} &= A_{3,n-1} + h_1A_{2n} + H_1A_{2,n-1} \end{aligned} \tag{27}$$

The general form for (27) is

$$A_{j,k} = A_{j+2,k-1} + h_jA_{j+1,k} + H_jA_{j+1,k-1},$$

$$j = n-1, n-2, \ldots, 1, \qquad k=2,3,\ldots,n+1-j \tag{28}$$

Equation (24), (25) and (28) are used to obtain the continued fraction inversion.

CHAPTER - 3

STABILITY CRITERIA BASED REDUCTION METHODS

Hutton and Friedland [11] based their reduction method on an $\alpha - \beta$ expansion that uses the Routh table of the original transfer function. This has a number of useful properties. If the original Transfer Function is stable, then all approximants are stable. This method was modified by Krishnamurthi et.al [12] to reduce computations by avoiding reciprocal transformation.

Chen. et.al [13] have given a technique which uses the stability equation method for getting the reduced polynomials of the numerator and denominator of the model.

Hutton and Friedland's method has been modified for simplification of unstable systems. Singh [14] has pointed out that Routh approach may lead to the same reduced model for different high order systems, while Shamash [15] has provided examples where such techniques fail to give acceptable models.

Now the detailed description of above three methods is as under

3.1 REDUCTION BY ROUTH APPROXIMATION [BY HUTTON AND FRIEDLAND - 1975]

Consider a linear time invariant (SISO) system having the transfer function (TF)

$$G(s) = \frac{b_1 s^{n-1} + \dots + b_n}{a_0 s^n + a_1 s^{n-1} + \dots + a_n} \qquad (29)$$

A T.F. of the form (29) that is asymptotically stable can always be expanded in the following canonical form.

$$G(s) = \beta_1 F_1(s) + \beta_2 F_1(s) F_2(s) + \cdots\cdots$$
$$\beta_n F_1(s) \quad F_2(s) \ldots F_n(s)$$
$$= \sum_{i=1}^{n} \beta_i \prod_{j=2}^{n} F_j(s) \qquad (30)$$

where $\beta_i (i=1,2\ldots n)$ are constants and $F_j(J=2\ldots.n)$ are defined by continued fraction expansions

$$F_j(s) = \cfrac{1}{\alpha_j s + \cfrac{1}{\alpha_{j+1} s + \cfrac{1}{\alpha_{j+2} s + \ddots \, \alpha_{n-1} s + \cfrac{1}{\alpha_n s}}}} \qquad (31)$$

For $F_1(s)$ definition (31) is modified slightly, the first term in the CFE is $1+\alpha_1 s$ instead of $\alpha_1 s$. The canonical (30) is referred to as the alpha-beta expansion of G(s) and plays a fundamental role in the theory of Routh approximations.

The n coefficients α_i appearing in the alpha - beta expansion can be computed using the algorithm for constructing the Rough table as shown in Table No.1. The first two rows of the table are formed from the coefficients of the denominator of G(s) where by assumption the entries $a_j^o = a_{j-1}^1 = 0$ for $j > n$. The remaining entries are formed by cross multiplication rule

$$a_o^{i+1} = a_2^{i-1} - \alpha_i \; a_2^i$$

$$a_2^{i+1} = a_4^{i-1} - \alpha_i \; a_4^i$$

$$\vdots$$

$$a_{n-i-2}^{i+1} = a_{n-1}^{i-1} - \alpha_i \; a_{n-i}^i$$

$$i = 1 \ldots, n-1 \qquad (32)$$

For $n-i$ odd, the last equation in (32) is replaced by

$$a_{n-i-1}^{i+1} = a_{n-1+1}^{i-1} \qquad (33)$$

The α_i are marginal entries given by

$$\alpha_i = \frac{a_o^{i-1}}{a_o^i} \quad i = 1,2 \ldots . \; n \qquad (34)$$

The coefficients β_i appearing in the canonical form can also be obtained by use of tabular algorithm as shown in Table 2. The first two rows of the β - table are obtained from the coefficients of the numerator of G(s). The remaining entries are computed from entries in the Routh table computed as shown in Table-2 and the earlier rows of the beta table, using the following recursion.

$$\beta_i = b_o^i \; / \; a_o^i \qquad i = 1,2 \; \ldots \; n \qquad (35)$$

$$b_{j-2}^{i+2} = b_j^i - \beta_i \; a_j^i \qquad j = \begin{cases} 2,4, \ldots n-i \text{ for } n-i, \text{even} \\ 2,4, \ldots n-i-1 \text{ for } n-i \text{ odd} \end{cases}$$

$$i = 1,2, \ldots \; n-2 \qquad (36)$$

TABLE - 1

ALPHA (ROUTH) TABLE

	$a_o^o = a_o$ $a_o^1 = a_1$	$a_2^o = a_2$ $a_2^1 = a_3$	$a_4^o = a_4$ $a_4^1 = a_5$	$a_6^o = a_6$
$\alpha_1 = \frac{a_o^o}{a_o^1}$	$a_o^2 = a_2^o - \alpha_1 a_2^1$	$a_2^2 = a_4^o - \alpha_1 a_4^1$	$a_4^2 = a_6^o - \alpha_1 a_6^1$	...
$\alpha_2 = \frac{a_o^1}{a_o^2}$	$a_o^3 = a_2^1 - \alpha_2 a_2^2$	$a_2^3 = a_4^1 - \alpha_2 a_4^2$	....	
$\alpha_3 = \frac{a_o^2}{a_o^3}$	$a_o^4 = a_2^2 - \alpha_3 a_2^3$	$a_2^4 = a_4^2 - \alpha_3 a_4^3$	....	
$\alpha_4 = \frac{a_o^3}{a_o^4}$	$a_o^5 = a_2^3 - \alpha_4 a_2^4$	....		
$\alpha_5 = \frac{a_o^4}{a_o^5}$	$a_o^6 = a_2^4 - \alpha_5 a_2^5$	....		
	....			

TABLE - 2

BETA TABLE

	$b_o^1 = b_1$ $b_o^2 = b_2$	$b_2^1 = b_3$ $b_2^2 = b_4$	$b_4^1 = b_5$ $b_4^2 = b_6$
$\beta_1 = \frac{b_o^1}{a_o^1}$	$b_o^3 = b_2^1 - \beta_1 a_2^1$	$b_2^3 = b_4^1 - \beta_1 a_4^1$	....
$\beta_2 = \frac{b_o^2}{a_o^2}$	$b_o^4 = b_2^2 - \beta_2 a_2^2$	$b_2^4 = b_4^2 - \beta_2 a_4^2$	....
$\beta_3 = \frac{b_o^3}{a_o^3}$	$b_o^5 = b_2^3 - \beta_3 a_2^3$	....	....
$\beta_4 = \frac{b_o^4}{a_o^4}$	$b_o^6 = b_2^4 - \beta_4 a_2^4$	....	....
	....		
	....		

ROUTH CONVERGENTS

The K^{th} Routh convergent $R_k(s)$ for the transfer function H(s) is obtained by truncating the alpha-beta expansion (30) and arranging the results as a rational function of s. The truncation eliminates those terms in the alpha-beta expansion containing $\alpha_{k+1},\ldots,\alpha_n$, $\beta_{k+1}\ldots$ $\ldots\beta_n$ and hence depends only on the first k alpha and beta coefficients. Let $A_k(s)$ and $B_k(s)$ denote the denominator and numerator respectively of the kth Routh convergents i.e.

$$A_1(s) = \alpha_1 s+1$$

$$B_1(s) = \beta_1$$

$$A_2(s) = \alpha_1 \alpha_2 s^2 + \alpha_2 s + 1$$

$$B_2(s) = \alpha_2 \beta_1 s+ \beta_2$$

$$A_3(s) = \alpha_1 \alpha_2 \alpha_3 s^3 + \alpha_2 \alpha_3 s^2+(\alpha_1+\alpha_3) s+1$$

$$B_3(s) = \alpha_2 \alpha_3 \beta_1 s^2 + \alpha_3 \beta_2 s+ (\beta_1+\beta_3)$$

More generally,

$$A_k(s) = \alpha_k s A_{k-1}(s) + A_{k-2}(s) \qquad (37(a))$$

and

$$B_k(s) = \alpha_k s B_{k-1}(s) + B_{k-2}(s) +\beta_k$$

$$k = 1,2,\ldots\ldots.37(b) \qquad (37(b))$$

with

$$A_{-1}(s) = 0 \qquad B_{-1}(s) = 0$$

$$A_o (s) = 1 \qquad B_o (s) = 0$$

Illustration -1

Consider an 8th order example.

$$G(s)=\frac{35s^7+1086s^6+13285s^5+82402s^4+238376s^3+511812s^2+482964s+194480}{s^8+33s^7+437s^6+3017s^5+11870s^4+27470s^3+37492s^2+28880s+9600}$$

Step -1

Performing reciprocal transformation to get $\bar{G}(s)$,

$$\bar{G}(s)=\frac{194480s^7+482964s^6+511812s^5+238376s^4+82402s^3+13285s^2+1086s+35}{9600s^8+28880s^7+37492s^6+27470s^5+11870s^4+3017s^3+437s^2+33s+1}$$

Step -2

Constructing α Table

	9600	37492	11870	437
	28880	27470	3017	1
α_1= 0.33240	28360.698	10867.119	436.66759	0
α_2= 1.01831	16403.897	2572.3368	1	
α_3= 1.7289	6419.8058	434.93869	0	
α_4= 2.5552	1460.9808	1		
α_5= 4.394175	430.5445	0		
α_6= 3.39331	1			

Constructing β table

	194480	511812	82402	1086
	482964	278376	13285	35
β_1= 6.7340	326827.04	62085.305	1079.2659	0
β_2=17.0293	93316.124	5848.8387	35	
β_3=19.9237	10834.726			

Step 3 Constructing Routh convergents

$$A_2(s) = \alpha_1 \alpha_2 s^2 + \alpha_2 s + 1$$

$$= 0.338486s^2 + 1.01831s + 1$$

$$B_2(s) = \alpha_2 \beta_1 s + \beta_2$$

$$= 6.8573s + 17.02934$$

$$\overline{R}(s) = \frac{6.8573s + 17.02934}{0.338486s^2+1.01831s+1}$$

Applying reciprocal transformation again to obtain record order reduced model as

$$R(s) = \frac{17.02934s + 6.8573}{s^2 + 1.01831s + 0.338486}$$

3.2 REDUCTION USING ROUTH HURWITZ ARRAY

[BY V.KRISHNAMURTHY et.al.-1978]

This method uses the Routh stability array directly to reduce the order of the system. No algorithm is required to reconstruct the reduced order transfer function.

Let the transfer function of the high order system be

$$G(s) = \frac{b_{11} s^m + b_{21} s^{m-1} + b_{12} s^{m-2} + b_{22} s^{m-3} + \ldots}{a_{11} s^n + a_{21} s^{n-1} + a_{12} s^{n-2} + a_{22} s^{n-3} \ldots}$$

where $m \leq n$ (38)

The Routh stability array for the numerator and denominator polynomials of (38) are shown below in table 3 and 4 respectively.

TABLE -3

NUMERATOR STABILITY ARRAY

b_{11}	b_{12}	b_{13}	b_{14}	•	•	•
b_{21}	b_{21}	b_{23}	b_{24}	•	•	•
b_{31}	b_{32}	b_{33}	•	•	•	•
b_{41}	b_{42}	b_{43}	•	•	•	•
•	•	•	•	•	•	•
•	•	•	•	•	•	•
•	•	•	•	•	•	•
$b_{m,1}$						
$b_{m+1,1}$						

TABLE - 4

DEMONINATOR STABILITY ARRAY

a_{11}	a_{12}	a_{13}	a_{14}	.	.	.	.
a_{21}	a_{22}	a_{23}	a_{24}	.	.	.	.
a_{31}	a_{32}	a_{33}	.	.	.	.	.
a_{41}	a_{42}	a_{43}	.	.	.	.	.
.	.	.					
.	.	.					
$a_{n-2,1}$	$a_{n-2,2}$	.	.	.	.	.	.
$a_{n-1,1}$	$a_{n-1,1}$						
$a_{n,1}$							
$a_{n+1,1}$							

The tables are completed by the algorithm

$$C_{ij} = C_{i-2,j+1} - (C_{i-2,1} \cdot C_{i-1,j+1}) / (C_{i-1,1})$$

$$\text{for } i \geq 3 \text{ and } 1 \leq j \leq [(n-i+3)/2] \qquad (39)$$

The transfer function of a system with reduced order $K(\leq n)$ can easily be constructed with (m+2-k)th and (m+3-k)th rows of table -3 and (n+1-k)th and n+2-k)th rows of table 4 as in (40).

$$G(s) = \frac{b_{(m+2-k),1}s^{k-1} + b_{(m+3-k),1}s^{k-2} + b_{(m+2-k),2}s^{k-3} + \ldots}{a_{(n+1-k),1}s^{k} + a_{(n+2-k),1}s^{k-1} + a_{(n+1-k),2}s^{k-2} + \ldots} \qquad (40)$$

for k > (m+1), the first two rows of Table 3 should be used for the numerator polynomial, while for k = 1, only the last row should be used.

Illustration Consider the same 8th order example art -3.1.

Numerator and denominator stability arrays are

35	13285	278376	482964
1086	82402	511812	194480
10629.3186	261881.1381	476696.2283	
55645.54206	463107.8334	194480	
173419.0523	439546.9831		
322068.9463	194480		
334828.6062			
194480			
--			

Denominator stability array

1	437	11870	37492	9600
33	3017	27470	28880	
345.5757	11037.5757	36616.8485	9600	
1962.9907	23973.3537	27963.26887		
6817.1744	31694.0405	9600		
14847.1229	25198.96951			
20123.7335	9600			
18116.1695				

Using last two rows of the stability arrays, second order approximant is obtained as

$$R_2(s) = \frac{334828.6062s + 194480}{20123.7335s^2+18116.1695s+9600}$$

$$= \frac{16.638516s + 9.664226}{s^2+0.900242s + 0.477049}$$

3.3 REDUCTION USING STABILITY EQUATION METHOD [T.C. CHEN AND C.Y.CHANG - 1979]

The approach in this method is to reduce the order of the stability equations of a transfer function and then the order of the original transfer function can be reduced.

Let the T.F. of a high order system be

$$G(s) = \frac{b_m s^m + b_{m-1} s^{m-1} + \dots + b_1 s + b_o}{a_n s^n + a_{n-1} s^{n-1} + \dots + a_1 s + a_o}$$

$$= \frac{F_N(s)}{F_D(s)} \qquad (41)$$

Where $n \geq m$ and $F_N(s)$ and $F_D(s)$ are the numerator and the denominator of G(s), respectively. Separating $F_N(s)$ and $F_D(s)$ into their even parts and odd parts one has

$$G(s) = \frac{F_{Ne}(s) + F_{No}(s)}{F_{De}(s) + F_{Do}(s)} \qquad (41(a))$$

where,

$$F_{Ne}(s) = \sum_{i=0,2,4}^{m} b_i \; s^i$$

$$F_{No}(s) = \sum_{i=1,3,5}^{m} b_i \; s^i \qquad (41(b))$$

and

$$F_{De}(s) = \sum_{i=0,2,4}^{n} a_i \; s_i$$

$$F_{Do}(s) = \sum_{i=1,3,5}^{n} a_i \; s_i \qquad (41(c))$$

Equation (41(b)) and (41(c)) are called stability equations of numerator and denominator respectively.

For a stable system equation (41(b)) and (41(c)) can be factored as

$$F_{Ne}(s) = \prod_{i=1}^{m} (s^2 + Z_{Ni}^2)$$

$$F_{No}(s) = s \prod_{i=1}^{m} (s^2 + p_{Ni}^2) \qquad (41(d))$$

$$F_{De}(s) = \prod_{i=1}^{n'} (s^2 + z_{Di}^2)$$

$$F_{Do}(s) = s \prod_{i=1}^{n'} (s^2 + p_{Di}^2) \qquad (41(e))$$

$m' = m/2$ if m is even

$= (m-1)/2$ if m is odd

$n' = n/2$ if n is even

$= (n-1)/2$ if n is odd (41(f))

$$p_1^2 < p_2^2 < p_3^2 < \cdots$$

$$z_1^2 < z_2^2 < z_3^2 < \cdots \qquad (41(g))$$

Since poles or zeros with smaller magnitudes are more dominant than those poles or zeroes with larger magnitudes, descarding the poles or zeroes with larger magnitudes is a method of reducing the order of the stability equations. Then, the reduced models of the polynomials $F_N(s)$ and $F_D(s)$ can be constructed and the reduced model of G(s) can be obtained.

In order to make the steady state response of the reduced model the same as that of original system and the coefficients of the reduced model the same as those of original system, the coefficients of reduced stability equations are multiplied by the magnitudes of the poles or zeroes which have been discarded. For example, the reduced stability equations of $F_N(s)$ can be written as

$$F_{Ne}(s) = Z_n^2 \prod_{i=1}^{m-1} (s^2 + Z_i^2) \qquad (42(a))$$

$$F'_{No}(s) = s\, p_n^2 \prod_{i=1}^{m'-1} (s^2 + p_i^2) \qquad (42(b))$$

Then the reduced T.F. is

$$G_{n-1}(s) = \frac{F'_N(s)}{F'_D(s)} \qquad (43)$$

where,

$$G'_N(s) = F_{No} + F'_{Ne}$$

if m is even

$$= F'_{No} + F_{Ne}$$

if m is odd (44)

$$F'_D(s) = F_{Do} + F'_{De} \quad \text{if n is even}$$

$$= F'_{Do} + F_{De} \quad \text{if n is odd} \qquad (45)$$

Following the same procedure the reduced models with lower order can be obtained.

Illustration - Consider the same 8th order example as in(3.1).

The numerator stability equations are

$$F_{No}(s) = 35s^2 + 13285s^5 + 278376s^3 + 482964 = 0$$

$$F_{Ne}(s) = 1086s^6 + 82402s^4 + 511812s^2 + 194480 = 0$$

The denominator stability equations are

$$F_{Do}(s) = 33s^7 + 3017s^5 + 27470s^3 + 28880s = 0$$

$$F_{De}(s) = s^8 + 437s^6 + 11870s^4 + 37492s^2 + 9600 = 0$$

The pole-zero patterns are shown in Fig. 1(c). The reduced order models are obtained as

$$R_2(s) = \frac{482964s + 194480}{34194s^2 + 28880s + 9600}$$

$$= \frac{14.1242s + 5.6875475}{s^2 + 0.84459s + 0.28075}$$

3.4 OTHER METHODS

Two methods, which are quite important have also been included in this chapter. They are not based upon stability criteria but still they are very powerful.

3.4.1 Reduction using polynomial differentiation

Per Olof Gutman proposed this method [16] for model reduction using polynomial differentiation. In this method the reciprocals of numerator and denominator polynomials of the high order transfer function are differentiated, suitably many times to yield the coefficients of reduced order model.

The method is computationally very simple and is equally applicable to unstable and nonminimum phase systems. The question naturally arises how well a polynomial is approximated by its derivative. A partial answer is given by following lemma.

Lemma - Given the polynomial

$$P_n(s) = a_n \prod_{i=1}^{n} (S- z_i) \tag{46}$$

Then the zeros to P'_n (s) do not lie outside the convex hull of zeros of $P_n(s)$ (This result is originally due to F. Gauss and F.Lucas).

A drawback of straight forward differentiation is that zeroes with large modulus tend to be better approximated than those with a small modulus. This problem is remedied e.g. by differentiating the reciprocal polynomial, reciprocating back and normalizing. Given the polynomial (46) the reduced order polynomial then becomes.

$$P_{n-1}(s) = P_n(s) - \frac{S}{n} \cdot P'_n(s) \tag{47}$$

Algorithm

Let the transfer function be

$$G(s) = \frac{q(s)}{p(s)} \tag{48}$$

Factorize $q(s) = \bar{q}(s).\hat{q}(s)$

and $p(s) = \bar{p}(s).\hat{p}(s)$

where $\hat{q}(s)$ and $\hat{p}(s)$ include those zeros and poles of $G(s)$ which we want to retain in the reduced order transfer function $G_{red}(s)$. Reduced the order of $\bar{q}(s)$, k_q times and the order of $\bar{p}(s)$, k_p times, each time according to (47). Let the resulting polynomials be $\bar{q}_{red}(s)$ and $\bar{p}_{red}(s)$, respectively. Construct the reduced-order transfer function.

$$G_{red}(s) = C\frac{\bar{q}_{red}(s) \cdot \hat{q}(s)}{\bar{p}_{red}(s) \cdot \hat{p}(s)}$$

where C is a real constant.
$\hat{q}(s)$ might, e.g. include the zeros whose real parts are non negative, $\hat{p}(s)$ might, e.g. include the unstable poles, the purely imaginary poles, the badly damped high frequency poles and the control poles. k_p and k_q are non-negative, not necessarily equal integers chosen, e.g. such that the pole zero excess of $G_{red}(s)$ is equal to pole zero excess of $G(s)$. C is adjusted to give the best approximation in the relevent frequency range.

<u>Error Analysis</u> It can be shown that if $k_p = k_q = 1$ and if the mean value of the zeros and poles are approximately equal, then C can be chosen such that the relative error

between $G(s)$ and $G_{red}(s)$ is approximately zero in both the high frequency and low frequency ranges.

Illustration

Consider an 8th order model

$$G(s) = \frac{35s^7 + 1086s^6 + 13285s^5 + 82402s^4 + 278376s^3 + 511812s^2 + 482964s + 194480}{s^8 + 33s^7 + 437s^6 + 3017s^5 + 11870s^4 + 27470s^3 + 37492s^2 + 28880s + 9600}$$

Step 1

As the system is stable and minimum phase, we let

$\hat{p}(s) = 1$ and $\hat{q}(s) = 1$

Step 2

We let $k_p = k_q = k$ in order to get approximants $G_r(s)$ that are comparable to those of $G(s)$. This means that the pole zero excess is kept equal to one and that the high frequency slope of the Bode plot is retained.

Step 3

C is chosen such that the low frequency gain of $G_r(s)$ is equal to low frequency gain of

$G(s)$ i.e. $C = 1$

Step 4

Using equation (47) successively six times, separately for numerator and denominator polynomials we get second order reduced model.

$$q_{red}(s) = 347734080s + 980179200$$

$$p_{red}(s) = 26994240s^2 + 145555200s + 193536000$$

$$G_2(s) = 4 \times \frac{347734080s + 980179200}{26994240s^2 + 145555200s + 193536000}$$

$$= \frac{5.1527s + 145.24272}{s^2 + 5.39208s + 7.169529}$$

The pole zero locations are tabulated in following table -5. It is apparent how well the poles and zeros of the reduced order systems approximates those of the original system.

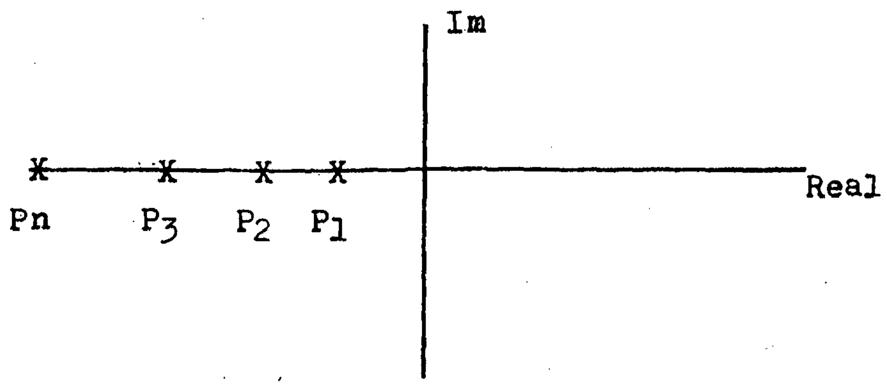
TABLE -5

Order of approximant	Poles	Zeros
r = 8 (Exact)	-1 ± i, -1, -3, -4, -5, -8, -10	-1.03 ± 0.631i, -2.64 -3.83, -4.90, -7.80, -9.78
r=6	-1.27,-1.45±1.102, -3.65,-5.18,-7.72	-1.42±0.696i, -3.32, -4.97, -7.49
r=4	-1.76, -2.29 ± 0.948i, -5.23	-2.15 ± 0.619i,-4.90
r=2	-2.38, -3.01	-2.82
r=1	-2.66	

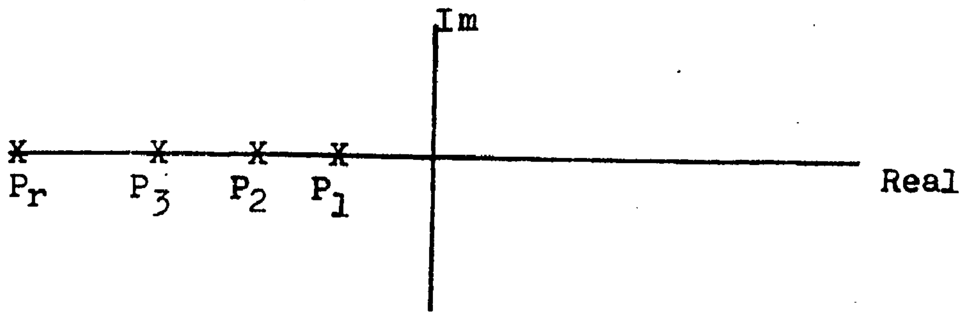
3.4.2 Modal method or model reduction using dominant pole retention

Originally this method was proposed by E.J.Davison [17]. The principle of the method is to neglect eigen-values of the original system which are farthest from the origin, and retain only dominant eigenvalues and hence the dominant time constants of the original system in the reduced model. This implies that the overall behaviour of the approximate system will be very similar to the original system. Since the contribution of the unretained eigenvalues to the system response are impor-tant only at the beginning of the response, whereas the eigenvalues retained are important throughout the whole of the response and, in fact, determine the type of the response which system will have.

Let the poles of an nth order system are shown as



If an rth in order reduced model is needed. We will retain the r poles nearest to the origin and will neglect the rest. So poles of reduced order model can be shown as



This is the gist of the method. [For detailed mathematical description refer [17].

Illustration

Consider eighth order Krishnamurthy Seshadri's model

$$G(s) = \frac{35s^7 + 1086s^6 + 13285s^5 + 82402s^4 + 278376s^3 + 511812s^2 + 482964s + 194480}{s^8 + 33s^7 + 437s^6 + 3017s^5 + 11870s^4 + 27470s^3 + 37492s^2 + 28880s + 9600}$$

System has poles at

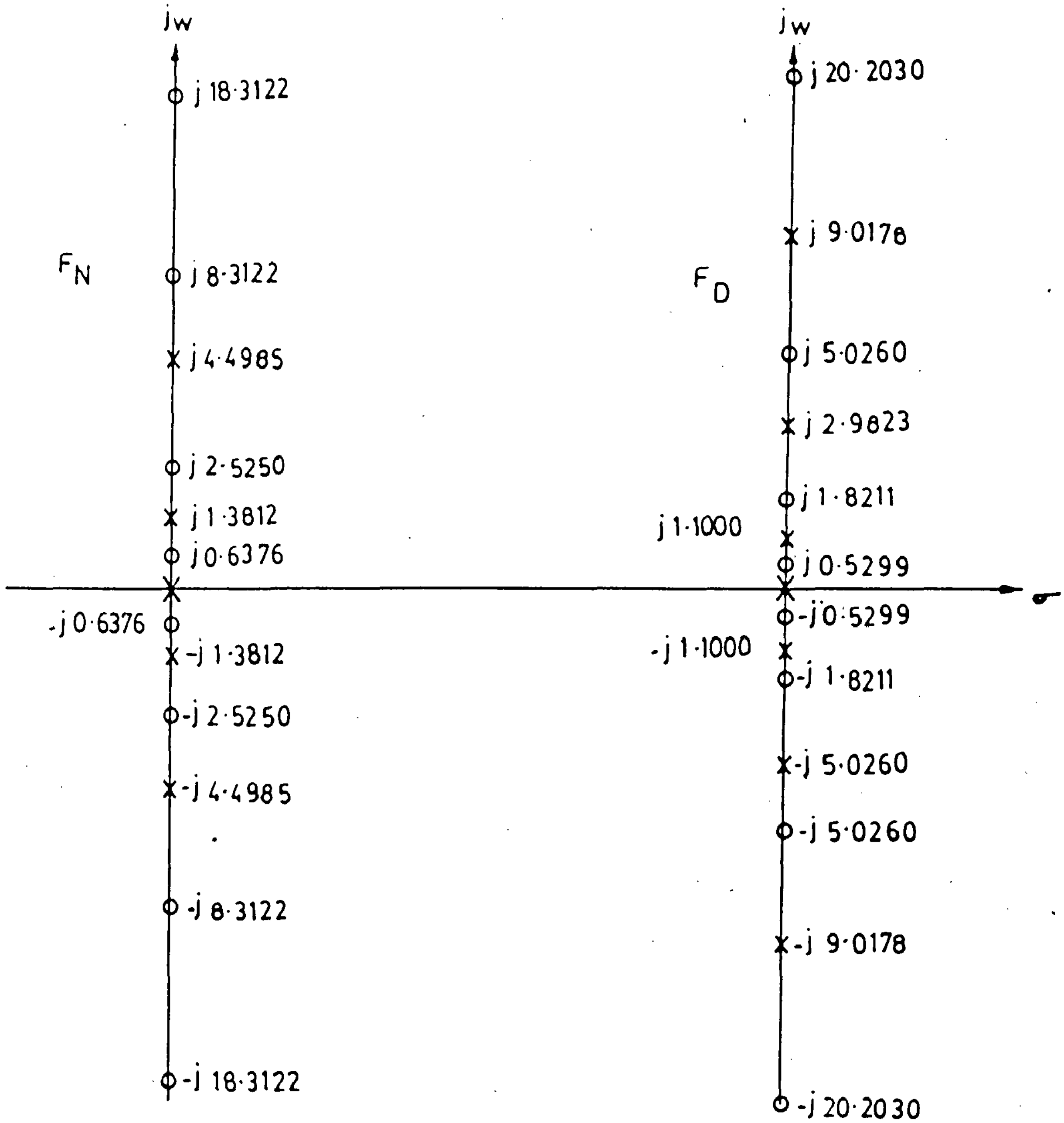
$s = -10, -8, -5, -4, -1 \pm j, -3, -1$

Discarding poles at $s = -10, -8, -5, -4, -1 \pm j$

We get the second order reduced denominator as

$3 + 4s + s^2$

Pole zero pattern of the system is given in Fig. (1(c)).



FIG(1c) PATTERNS OF POLES AND ZEROS OF AN EIGHT ORDER SYSTEM

CHAPTER - 4

MODEL REDUCTION USING MIXED METHODS

Mixed methods are used only to incorporate the advantages of two different methods. CFE method is a very powerful frequency domain method of model reduction. The only drawback of this method is that the reduced model may be unstable even though the original system is stable. To overcome this difficulty, this method has been mixed with various other frequency domain methods by so many researcher and results came out to be very attractive.

In this chapter five methods namely (1) Routh approximation (2) Routh Hurwitz array (3) stability equation (4) Polynomial differentiation (5) Dominant pole retention are mixed with

(1) Cauer second form

(2) Cauer third form

separately.

Each method is followed by an illustrative example in order to reveal under lying technique. In all the mixed methods, denominator is reduced by respective method and coefficients of reduced numerator are found out by matching the cauer coefficients of original and reduced model.

Consider a transfer function is given as

$$G(s) = \frac{A_{2,n}s^{n-1}+A_{2,n-1}s^{n-2}+\ldots+A_{24}s^3+A_{23}s^2+A_{22}s+A_{21}}{A_{1,n+1}s^n+A_{1,n}s^{n-1}+\ldots+A_{14}s^3+A_{13}s^2+A_{12}s+A_{11}}$$

(A) To find out numerator coefficients by matching Cauer IInd forms coefficients.

Step -1

Evaluate Cauer 2nd form coefficients h'_p (p=1,2,3...n) by forming Routh array [18].

$$
\begin{array}{lllllll}
h'_1 = \dfrac{A_{11}}{A_{21}} < & A_{11} & A_{12} & A_{13} & \cdots\cdots & A_{1n} & A_{1,n+1} \\
 & A_{21} & A_{22} & A_{23} & \cdots\cdots & A_{2n} & \\
h'_2 = \dfrac{A_{21}}{A_{31}} < & A_{31} & A_{32} & \cdots & & & \\
h'_3 = \dfrac{A_{31}}{A_{41}} < & A_{41} & \cdots & & & & \\
\vdots \quad \cdots\cdots & & & & & & \\
\vdots \quad \cdots\cdots & & & & & &
\end{array}
$$

where first two rows of this array are copied from the Denominator and numerator coefficients respectively of G(s), and rest elements are computed by well known Routh's algorithm.

$$A_{j,k} = A_{j-2,k+1} - h'_{j-2} A_{j-1,k+1}$$

and $\quad J = 3,4 \ldots n+1$ $\quad k = 1,2 \ldots$ (49(a))

$$h'_p = \frac{A_{p,1}}{A_{p+1,1}} \qquad p = 1,2,3 \ldots n \qquad (49(b))$$

Step -2

Reduce the denominator polynomial by any one of the five methods enumerated in previous chapter. Let the reduced kth polynomial is

$$\Delta_r(s) = B_{11} + B_{12}\, s + B_{13}\, s^2 + \ldots \ldots B_{1r}\, s^r$$

Step -3

Match the Cauer quotients h_i' (i=1,2...r) for finding the numerator terms of the reduced order model (ROM).For this construct inverse Routh array as under

$$B_{i+1,1} = B_{i,1} / h_i' \qquad i=1,2,\ldots r$$

$$B_{j-1,k+1} = (B_{j-2,k+1} - B_{j,k}) / h_{j-2}'$$

$$k = 1,2\ldots.(r-2)$$

$$J = 3,4,5\ldots.(r+1-k) \text{ where } r \leq n$$

for $r = 2$,

$i = 1$

$$B_{2,1} = B_{11} / h_1' \qquad (49(c))$$

$$B_{22} = (B_{1,2} - B_{31}) / h_1' = \left(B_{1,2} - \frac{B_{2,1}}{h_2'}\right) / h_1'$$

$$B_{22} = \frac{h_2' B_{1,2} - B_{2,1}}{h_1' h_2'} \qquad (49(d))$$

Step - 4

Reduced transfer function is

$$R(s) = \frac{B_{22}\ s + B_{21}}{B_{13}\ s^2 + B_{12} s + B_{11}} \qquad (49(e))$$

(B) To find out numerator coefficients by matching cauer 3rd form coefficients .

Step - 1

Evaluate Cauer 3rd form coefficients h_p and H_p $(p=1,2...n)$ by forming Routh array [18].

$$\begin{array}{lllllll}
h_1 = \frac{A_{11}}{A_{21}} & \langle & A_{11} & A_{12} & A_{13} \cdots\cdots & A_{1n}, & A_{1,n+1} \\
 & & A_{21} & A_{22} & A_{23} \cdots\cdots & A_{2n} & \rangle\ H_1 = \frac{A_{1,n+1}}{A_{2n}} \\
h_2 = \frac{A_{21}}{A_{31}} & \langle & A_{31} & A_{32} \cdots\cdots\cdots & A_{3,n-1} & & \rangle\ H_2 = \frac{A_{2n}}{A_{3,n-1}} \\
h_3 = \frac{A_{31}}{A_{41}} & \langle & A_{41} & \cdots\cdots & & & \rangle\ \cdots\cdots \\
 & & \cdots\cdots & & & &
\end{array}$$

Where first two rows of this array are copied from the Denominator and numerator coefficients respectively of G(s) and rest elements are computed by well known Routh's algo - rithm.

$$A_{J,K} = A_{J-2,K+1} - h_{J-2}\ A_{J-1,K+1} - H_{J-2}\ A_{J-1,K}$$

$$J = 3,4 \ldots n+1$$

$$K = 1,2 \ldots \qquad (50(a))$$

and

$$h_p = \frac{A_{p,1}}{A_{p+1,1}}$$

$$H_p = \frac{A_{p,n+2-p}}{A_{p+1,n+1-p}}$$

$$p = 1,2,3 \ldots n \qquad (50(b)$$

Step - 2

Reduce the denominator polynomial by any one of the five methods enumerated in previous chapter. Let the reduced polynomial be

$$\Delta_r(s) = B_{11} + B_{12}\, s + B_{13}\, s^2 + \ldots . B_{1r}\, s^r$$

Step -3

Match the coefficients $B_{1,j}$ (step-2) and h_p, H_p of step 1 by applying the following reverse Routh algorithm.

$$B_{i+1,1} = B_{i,1} / h_i \qquad i=1,2 \ldots r$$

$$B_{j+1,r+1-j} = B_{j,r+2-j}/H_j$$

$$j = 1,2 \ldots (r-1)$$

$$B_{j-1,k+1} = \frac{B_{j-2,k+1} - H_{j-2}B_{j-1,k} - B_{j,k}}{h_{j-2}}$$

$k = 1,2, \ldots .(r-2)$

$j = 3,4,5 \ldots .(r+1-k)$ and $r \leq n$

So for $r = 2$ $\quad i=1$

$$B_{21} = \frac{B_{11}}{H_1} \qquad (50(c))$$

and

$$B_{22} = \frac{B_{13}}{H_1} \qquad (50(d))$$

Step-4

Reduced T.F. is

$$R(s) = \frac{B_{22}\, s + B_{21}}{B_{13}\, s^2 + B_{12} s + B_{11}} \qquad (50(e))$$

An 8th order model is used in all the mixed methods below to illustrate the methods.

$$G(s) = \frac{35s^7 + 1086s^6 + 13285s^5 + 82402s^4 + 278376s^3 + 511812s^2 + 482964s + 194480}{s^8 + 33s^7 + 437s^6 + 3017s^5 + 11870s^4 + 27470s^3 + 37492s^2 + 28880s + 9600}$$

4.1 Mixed method using Routh approximation and CFE

The combination of CFE and Routh approximations for order reduction simply needs to constructing an α table for the denominator and find stable reduced order polynomial for it. However, a short coming of the original Routh approximation as suggested by Hutton and Fried land (1975) is having to go through two reciprocal processes in addition to constructing the α table. This method mixed with CFE ensures the stability of the reduced system.

Routh Table of the Original System

9600,	28880, 37492, 27470, 11870, 3017, 437, 33, 1
194480,	482964, 511812, 278376, 82402, 13285, 1086, 35

$h_1' = 0.0493624$ } 5039.7367

$h_2' = 38.589317$ Cauer second form coefficients

$h_1 = 0.0493624$ } Cauer third form coefficients

$H_1 = 1/35$

(1) Routh approximation mixed with CFE of Cauer second form

Step-1 Cauer coefficients are

$$h_1' = 0.0493624$$

$$h_2' = 38.589317$$

Step -2 2nd order denominator polynomial (refer illustrative example of Article 3.1) reduced by Routh approximation is given as

$$s^2 + 1.01831s + 0.338486$$

Step - 3

As per equation 59(c) and 59(d)

$$B_{21} = \frac{B_{11}}{h_1'} = \frac{0.338486}{0.0493624}$$

$$= 6.857$$

$$B_{22} = \frac{h_2' \; B_{12} - B_{21}}{h_1' \; h_2'}$$

$$= 17.029441$$

Step - 4

As per equation 59(e)

$$R(s) = \frac{17.029441s + 6.8571625}{s^2 + 1.01831s + 0.338486}$$

(II) Routh approximation mixed with CFE of Cauer 3rd form.

Step - 1

Cauer coefficients are (50(b))

$h_1 = 0.0493624$

$H_1 = 1/35$

Step - 2

2nd order reduced polynomial (by Routh approximation) is given as

$s^2 + 1.01831s + 0.338986$

Step - 3

As per equation 50(c) and 50(d).

$$B_{21} = \frac{B_{11}}{H_1} = 6.8571625$$

$$B_{22} = \frac{B_{13}}{H_1} = 35$$

Step - 4

As per equation 50(e)

$$R(s) = \frac{35s + 6.8573}{s^2 + 1.01831s + 0.338486}$$

4.2 Mixed method using Routh Hurwitz array and CFE

A short coming of the original Routh approximation as suggested by Hutton and Friedland (1975) is having to go through two reciprocal processes in addition to constructing the α table. The latter difficulty is not so serious because this table is essentially the standard Routh-Hurwitz array and the former problem can be avoided by a regrouping of the entries of α table as suggested by Krishnamurthy and Sheshadri (1976) which gives the α coefficients of the full model without having to perform a reciprocal transformation. The construction of β table which is more cumbersome than the α table is avoided all together. This method is nothing but Routh Hurwitz array.

Since the numerator in present method is approximated by the CFE method, this mixed CFE-Routh Hurwitz array method makes use of stable reduced polynomials for the denominator and takes advantage of computationally convenient scheme of CFE method for numerator.

(1) Routh Hurwitz array method mixed with CFE of Cauer 2nd form

Step - 1

Cauer coefficients of original system

$h_1' = 0.0493624$

$h_2' = 38.589317$

Step - 2

2nd order reduced (by Routh Hurwitz array method) denominator polynomial (refer illustrative example of Article-3.2) is given as

$$20123.7335s^2 + 18116.1695s + 9600$$

Step - 3

As per equation 49(c) and 49(d).

$$B_{21} = \frac{B_{11}}{h_1'} = \frac{9600}{0.0493624}$$

$$= 194480.01$$

$$B_{22} = \frac{h_2' \; B_{12} - B_{21}}{h_1' \; h_2'}$$

$$= 264906.73$$

Step - 4

Reduced transfer function (eq. 59(e)).

$$R(s) = \frac{264906.73s + 194480.01}{20123.7335\, s^2 + 18116.1695\, s + 9600}$$

$$= \frac{13.163955s + 9.6642177}{s^2 + 0.900242s + 0.477049}$$

(II) Routh Hurwitz array method mixed with CFE of Cauer 3rd form

Step -1

Cauer coefficients of original system

$h_1 = 0.0493624$

$H_1 = 1/35$

Step- 2

2nd order reduced denominator polynomial (refer Art. 3.2).

$$20123.7335s^2 + 18116.1695s + 9600$$

Step -3

As per equation 50(c) and 50(d).

$$B_{21} = \frac{B_{11}}{h_1} = 194480.01$$

$$B_{22} = \frac{B_{31}}{H_1} = 704330.66$$

Step -4

Reduced T.F. 15 (50(e)).

$$\frac{704330.66s + 194480.01}{20123.7335s^2+18116.1695s + 9600}$$

$$= \frac{35s + 9.664226}{s^2+ 0.900242s + 0.477049}$$

4.3 MIXED METHOD USING STABILITY EQUATION AND CFE

The main objective of this method is to make use of the advantages of the stability equation method and continued fraction method.

CFE has a short coming namely the reduced model may be unstable even though the original system is stable. Stability equation method was proposed by chen and Han (1979). The reducing procedure is simple and only two equations with one half of the order of the original system need to be factored. All the reduced models are guaranteed to be stable if the original system is stable. However there is a disadvantage of this method. i.e. it can not be applied directly to reduce the transfer functions of non minimum phase systems. The procedure consists of three steps.

1. to reduce the denominator of a transfer function by stability equation method,
2. to obtain partial quotients by the algorithm of CFE method,
3. to discard the undesired partial quotients and to reconstruct the reduced model of which the denominator is obtained from step-1.

(i) Stability equation method mixed with CFE of Cauer 2nd form

Step -1

Cauer coefficients of original system are

$h_1' = 0.0493624$

$h_2' = 38.589317$

Step -2

2nd order reduced (by stability equation method) denominator polynomial (refer illustrative example of Art 3.3). is given as

$$34194s^2 + 28880s + 9600$$

Step -3

As per equation 49(c) and 49(d)

$$B_{21} = \frac{B_{11}}{h_1'}$$

$$= 194479.3$$

$$B_{22} = \frac{h_2' B_{12} - B_{21}}{h_1' h_2'}$$

$$= 482962.55$$

Step -4

Second order reduced transfer function is given as 49(e).

$$R(s) = \frac{482962.55s + 194479.3}{34194s^2 + 28880s + 9600}$$

$$= \frac{14.12419s + 5.6875271}{s^2 + 0.84459s + 0.28075}$$

(II) Stability equation method mixed with CFE of Cauer 3rd form

Step -1

Cauer coefficients of original system are given as

$h_1 = 0.0493624$

$H_1 = 1/35$

Step -2

Second order reduced denominator polynomial (referf Art 3.3) is given as

$34194s^2 + 28880s + 9600$

Step -3

As per equation 50(c) and 50(d)

$$B_{21} = \frac{B_{11}}{h_1} = 194479.3$$

$$B_{22} = \frac{B_{13}}{H_1} = 1196790$$

Step -4

Second order reduced transfer function is given as 50(e)

$$R(s) = \frac{1196790s + 194479.3}{34194s^2 + 28880s + 9600}$$

$$= \frac{35s + 5.6875475}{s^2 + 0.84459s + 0.28075}$$

4.4 MIXED METHOD USING POLYNOMIAL DIFFERENTIATION AND CFE

Reduction method, polynomial differentiation was proposed by Per Olof Gutman. This is a very simple method and this is equally applicable to unstable and nonminimum phase systems. This method mixed with CFE gives good results. Computationally this method is the simple most one and qualitatively is comparable with other appreciated methods.

(1) Model reduction using polynomial differentiation and CFE of Cauer 2nd form

Step -1

Cauer coefficients of original system are

$h'_1 = 0.0493624$

$h'_2 = 38.589317$

Step -2

2nd order reduced (by polynomial differentiation method) denominator polynomial (refer illustrative example of Article (3.4.1) is given as

$$269942 40s^2 + 145555200s + 193536000$$

Step -3

As per equation 49(c) and 49(d)

$$B_{21} = \frac{B_{11}}{h'_1}$$

$$= 3.92071 \times 10^9$$

$$B_{22} = \frac{h'_2 \; B_{12} - B_{21}}{h'_1 \; h'_2}$$

$$= 1.39093 \times 10^9$$

Step -4

Reduced transfer function eq. 49(e).

$$R(s) = \frac{1.39093 \times 10^9 s + 3.92071 \times 10^9}{26994240s^2 + 145555200s + 193536000}$$

$$= \frac{51.527152s + 145.24272}{s^2 + 5.39208s + 7.169529}$$

(II) Model reduction using polynomial differentiation and CFE of Cauer 3rd form

Step -1

Cauer coefficients of original system are

$h_1 = 0.0493624$

$H_1 = 1/35$

Step -2

2nd order reduced (by polynomial differentiation method) denominator polynomial (refer illustrative example of Art 3.4.1) is given as

$$26994240\ s^2 + 145555200s + 193536000$$

Step -3

As per equation 50(c) and 50(d)

$$B_{21} = \frac{B_{11}}{h_1} = 3.92071 \times 10^9$$

$$B_{22} = \frac{B_{13}}{H_1} = 9.44798 \times 10^8$$

Step -4

Reduced transfer function is (equation 50(e).

$$\frac{9.44798 \times 10^8 s + 3.92071 \times 10^9}{26994240s^2 + 145555200s + 193536000}$$

$$= \frac{35s + 145.24272}{s^2+5.39208s + 7.169529}$$

4.5 MODEL REDUCTION USING DOMINANT POLE RETENTION AND CFE

An alternative method especially for reduction of MIMO systems has been introduced by Shieh and Wei (1975) which retains the dominant poles of the full model and applies the matrix continued fraction method to find a reduced order numerator matrix polynomial. The method eliminates the unpredictable results of the straight matrix continued fraction. Such as providing higher order reduced models.

(1) Model reduction using modal method and CFE of Cauer 2nd form

Cauer coefficients of original system are -

Step -1

$h_1' = 0.0493624$

$h_2' = 38.589317$

Step -2

2nd order reduced (by Modal method) denominator polynomial (refer illustrative example of Art 3.4.2) is given as

$s^2 + 4s + 3$

Step -3

As per equation 50(c) and 50(d)

$$B_{21} = \frac{B_{11}}{h_1'} = 60.775003$$

$$B_{22} = \frac{h_2' \, B_{12} - B_{21}}{h_1' \, h_2'}$$

$$= 49.128125$$

Step -4

Reduced transfer function (eq. 50(e)).

$$R(s) = \frac{49.128125s + 60.775003}{s^2 + 4s + 3}$$

(II) Model reduction using modal method and CFE of Cauer 3rd form

Cauer coefficients of original system are

Step -1

$$h_1 = 0.0493624$$

$$H_1 = 1/35$$

Step -2

Second order reduced (by modal method) denominator polynomial (refer illustrative example of Art 3.4.2) is given as

$$s^2 + 4s + 3$$

Step -3

As per equation 50(c) and 50(d)

$$B_{21} = \frac{B_{11}}{h_1} = 60.775003$$

$$B_{22} = \frac{B_{31}}{H_1} = 35$$

Step -4

Reduced transfer function (eq. 50(e)

is

$$R(s) = \frac{35s + 60.775003}{s^2 + 4s + 3}$$

CHAPTER - 5

COMPARISON OF PROPOSED REDUCTION METHODS

In this chapter the reduction methods proposed in earlier chapters i.e. Chapter 3 and Chapter - 4 are applied to various SISO and MIMO systems. The step responses of original system and of reduced models are compared. Following methods are used for comparative study of step responses.

Method 1 - Routh approximation method

Method 2 - Routh approximation mixed with Cauer 2nd form.

Method 3 - Routh approximation mixed with Cauer 3rd form.

Method 4 - Routh Hurwitz array method.

Method 5 - Routh Hurwitz array mixed with Cauer 2nd form.

Method 6 - Routh Hurwitz array mixed with Cauer 3rd form.

Method 7 - Stability equation method

Method 8 - Stability equation method mixed with Cauer 2nd form.

Method 9 - Stability equation method mixed with Cauer 3rd form.

Method 10 - Polynomial differentiation method.

Method 11 - Polynomial differentiation mixed with Cauer 2nd form.

Method 12 - Polynomial differentiation mixed with Cauer 3rd form.

Method 13 - Modal method mixed with Cauer 2nd form.

Method 14 - Modal method mixed with Cauer 3rd form.

5.1 ILLUSTRATIVE EXAMPLES

Three celebrated SISO models, namely

1. Hutton's model [11]

$$G_{11}(s) = \frac{14s^3 + 248s^2 + 900s + 1200}{s^4 + 18s^3 + 102s^2 + 180s + 120}$$

2. Chuang's model [10]

$$G_{12}(s) = \frac{8s^2 + 6s + 2}{s^3 + 4s^2 + 5s + 2}$$

3. Krishnamurthy and Sheshadri's model [12].

$$G_{13}(s) = \frac{35s^7 + 1086s^6 + 13285s^5 + 82402s^4 + 278376s^3 + 511812s^2 + 482964s + 194480}{s^8 + 33s^7 + 437s^6 + 3017s^5 + 11870s^4 + 27470s^3 + 37492s^2 + 28880s + 9600}$$

have been reduced by aforesaid methods and step responses have also been plotted in order to compare them qualitatively in Art 5.1.1.

One 4th order MIMO model has also been reduced and step responses are plotted in Art 5.1.2.

5.1.1 Illustrative Examples

(SISO system)

1. Example -1 (Hutton's model)

2nd order reduced models by various methods enlisted in the begining of this chapter are -

1. By method 1

$$R_{11}(s) = \frac{10s + 13.32}{s^2 + 2s + 1.332}$$

2. By method 2

$$R_{12}(s) = \frac{10.01s + 13.32}{s^2 + 2s + 1.332}$$

3. By method 3

$$R_{13}(s) = \frac{14s + 13.32}{s^2 + 2s + 1.332}$$

4. By method 4

$$R_{14}(s) = \frac{9.04628s + 13.0434}{s^2 + 1.70132s + 1.304}$$

5. By method 5

$$R_{15}(s) = \frac{7.2332s + 13.04}{s^2 + 1.70132s + 1.304}$$

6. Method 6

$$R_{16}(s) = \frac{14s + 13.0434}{s^2 + 1.70132s + 1.304}$$

7. Method 7

$$R_{17}(s) = \frac{8.9277176s + 11.903624}{s^2 + 1.7855435s + 1.1903624}$$

8. Method 8

$$R_{18}(s) = \frac{8.927717s + 11.903624}{s^2 + 1.7855435s + 1.1903624}$$

9. Method 9

$$R_{19}(s) = \frac{14s + 11.903624}{s^2 + 1.7855435s + 1.1903624}$$

10. Method (10)

$$R_{1,10}(s) = \frac{17.64711s + 70.588235}{s^2 + 5.2941s + 7.0588235}$$

11. Method 11

$$R_{1,11}(s) = \frac{70.588235}{s^2 + 5.2941s + 7.0588235}$$

12. Method 12

$$R_{1,12}(s) = \frac{14s + 70.588235}{s^2 + 5.2941s + 7.0488235}$$

13. Method 13

$$R_{1,13}(s) = \frac{9.587571s + 19.128148}{s^2+2.3933682s + 1.9128148}$$

14. Method 14

$$R_{1,14}(s) = \frac{14s + 19.128148}{s^2+ 2.3933682s + 1.9128148}$$

TIME RESPONSES

TABLE -6

EX-1, METHOD 1,2,3

No.	Time (Sec)	$G_{11}(s)$	$R_{11}(s)$	$R_{12}(s)$	$R_{13}(s)$
1.	0	o	0	0	0
2.	0.5	4.674	4.185	4.1876	5.3810
3.	1	7.0025	6.9182	6.9186	8.3066
4.	1.5	8.48169	8.5565	8.5544	9.7297
5.	2	9.3351	9.4567	9.4522	10.3085
6.	2.5	9.7713	9.9009	9.8947	10.4583
7.	3	9.9645	10.0865	10.079	10.4195
8.	3.5	10.0324	10.1389	10.1307	10.3194
9.	4	10.0442	10.1316	10.1229	10.2171
10.	4.5	10.0359	10.1038	10.0949	10.135
11.	5	10.0239	10.0741	10.0651	10.0772
12.	5.5	10.0137	10.0499	10.0408	10.0401
13.	6	10.007	10.0326	10.0235	10.0182
14.	7	9.9992	10.0147	10.0057	10.0008
15.	8	9.9999	10.0093	10.0003	9.9980
16.	9	9.9999	10.0084	9.9994	9.9986
17.	10	9.9999	10.0086	9.9996	9.9994

TIME RESPONSES

TABLE - 7 EX-1 METHOD - 4,5,6

No.	Time Sec.	$G_{11}(s)$	$R_{14}(s)$	$R_{15}(s)$	$R_{16}(s)$
1.	0	0	0	0	0
2.	0.5	4.674	4.1053	3.5267	5.6849
3.	1	7.0025	7.1191	6.4165	9.0352
4.	1.5	8.48169	9.0226	8.4166	10.6725
5.	2	9.3351	10.0451	9.6093	11.2281
6.	2.5	9.7713	10.4728	10.2023	11.2029
7.	3	9.9645	10.5549	10.4122	10.9351
8.	3.5	10.0324	10.472	10.4138	10.6209
9.	4	10.0442	10.3362	10.3261	10.3540
10.	4.5	10.0359	10.207	10.2189	10.1644
11.	5	10.0239	10.1081	10.1264	10.0482
12.	5.5	10.0137	10.0433	10.0598	9.9883
13.	6	10.007	10.0069	10.0186	9.9652
14.	7	9.9992	9.986	9.9884	9.9697
15.	8	9.9999	9.9915	9.9894	9.9876
16.	9	9.9999	9.9988	9.9955	9.9978
17.	10	9.9999	10 .0022	9.9991	10.000

TIME RESPONSE

TABLE 8, EX-1, METHOD -7,8,9

No.	Time Secs.	$G_{11}(s)$	$R_{17}(s)$	$R_{18}(s)$	$R_{19}(s)$
1.	0	0	0	0	0
2.	0.5	4.674	3.9126	3.9126	5.5091
3.	1	7.0025	6.6841	6.6841	8.6277
4.	1.5	8.48169	8.4558	8.4558	10.1684
5.	2	9.3351	9.4780	9.4780	10.767
6.	2.5	9.7713	9.9968	9.9968	10.8648
7.	3	9.9645	10.2099	10.2099	10.7388
8.	3.5	10.0324	10.2568	10.2568	10.5453
9.	4	10.0442	10.2267	10.2267	10.3614
10.	4.5	10.0359	10.1709	10.1709	10.2166
11.	5	10.0239	10.1152	10.1152	10.1157
12.	5.5	10.0137	10.0702	10.0702	10.0522
13.	6	10.007	10.0383	10.0383	10.0161
14.	7	9.9992	10.0061	10.0061	9.9913
15.	8	9.9999	9.9976	9.9976	9.9915
16.	9	9.9999	9.9973	9.9973	9.9958
17.	10	9.9999	9.9986	9.9986	9.9986

TIME RESPONSE

TABLE - 9 EX-1 METHOD-11,12,13

No.	Time Sec	G_{11}	$R_{1,10}$	$R_{1,11}$	$R_{1,12}$
1.	0	0	0	0	0
2.	0.5	4.674	6.1834	3.8396	5.6990
3.	1	7.0025	8.6899	7.4501	8.4336
4.	1.5	8.48169	9.5775	9.0877	9.4762
5.	2	9.3351	9.8693	9.6982	9.8340
6.	2.5	9.7713	9.9611	9.9051	9.9495
7.	3	9.9645	9.9887	9.9714	9.9851
8.	3.5	10.0324	9.9968	9.9917	9.9957
9.	4	10.0442	9.9991	9.9976	9.9988
10.	4.5	10.0359	9.9997	9.9993	9.9996
11.	5	10.0239	9.9999	9.9998	9.9999
12.	5.5	10.0137	9.9999	9.9999	9.9999
13.	6	10.007	9.9999	9.9999	9.9999
14.	7	9.9992	9.9999	9.9999	9.9999
15.	8	9.9999	9.9999	9.9999	9.999
16.	9	9.9999	10	10	10
17.	10	9.9999	10	10	10

TIME RESPONSE

TABLE -10 EX-1 METHOD 13,14

No.	Time sec	$G_{11}(s)$	$R_{1,13}(s)$	$R_{1,14}(s)$
1.	0	0	0	0
2.	0.5	4.674	4.188	5.377
3.	1	7.0025	7.013	8.242
4.	1.5	8.48169	8.667	9.579
5.	2	9.3351	9.524	10.096
6.	2.5	9.7713	9.911	10.226
7.	3	9.9645	10.051	10.205
8.	3.5	10.0324	10.08	10.143
9.	4	10.0442	10.06	10.086
10.	4.5	10.0359	10.045	10.046
11.	5	10.0239	10.026	10.021
12.	5.5	10.0137	10.013	10.008
13.	6	10.007	10.006	10.002
14.	7	9.9992	10.000	9.9999
15.	8	9.9999	9.9999	9.9999
16.	9	9.9999	9.9999	9.9999
17.	10	9.9999	9.9999	9.9999

Fig.2 Comparison of unit step responses (Example 1)

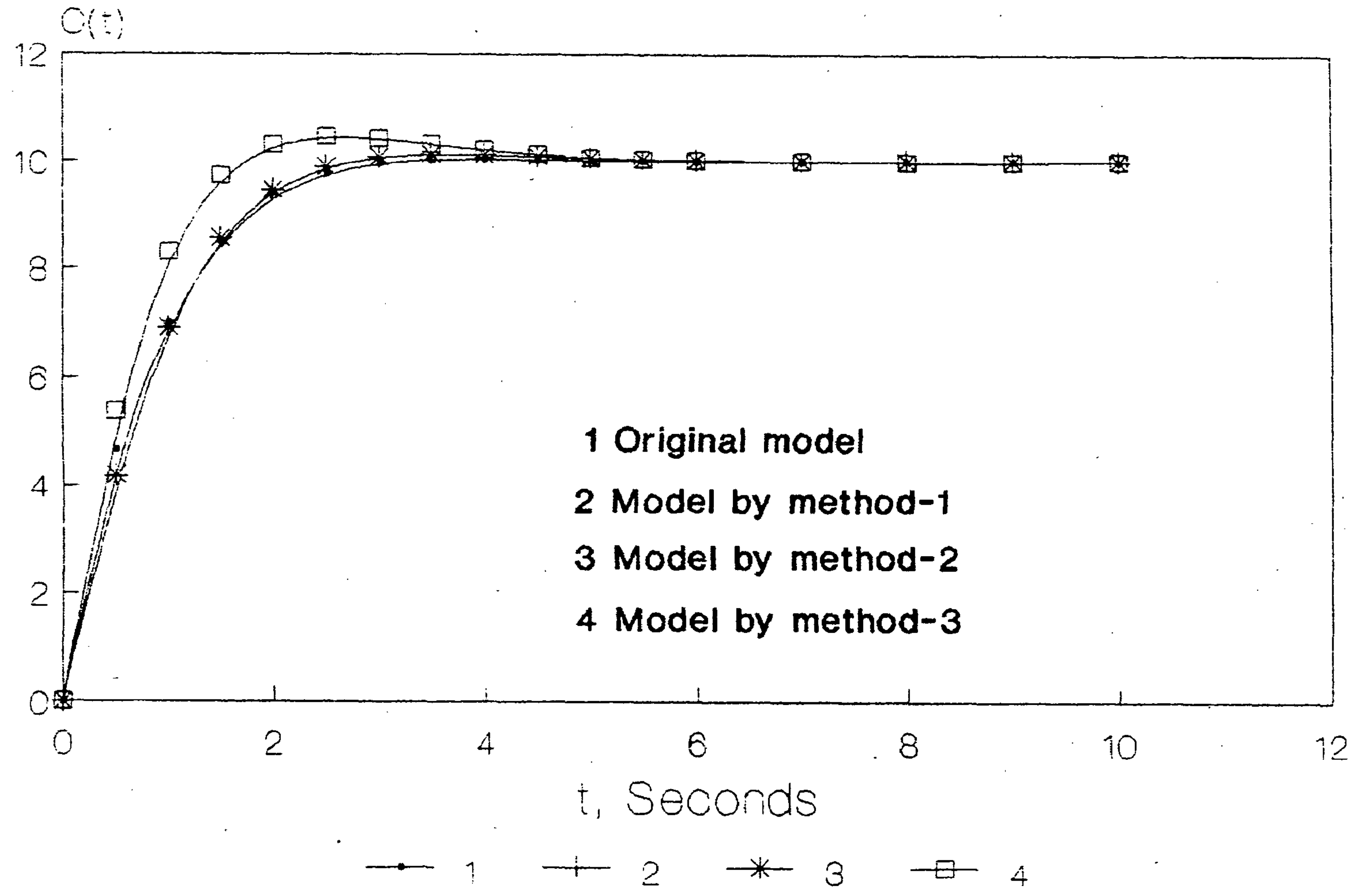


Fig.3 Comparison of unit step responses (Example 1)

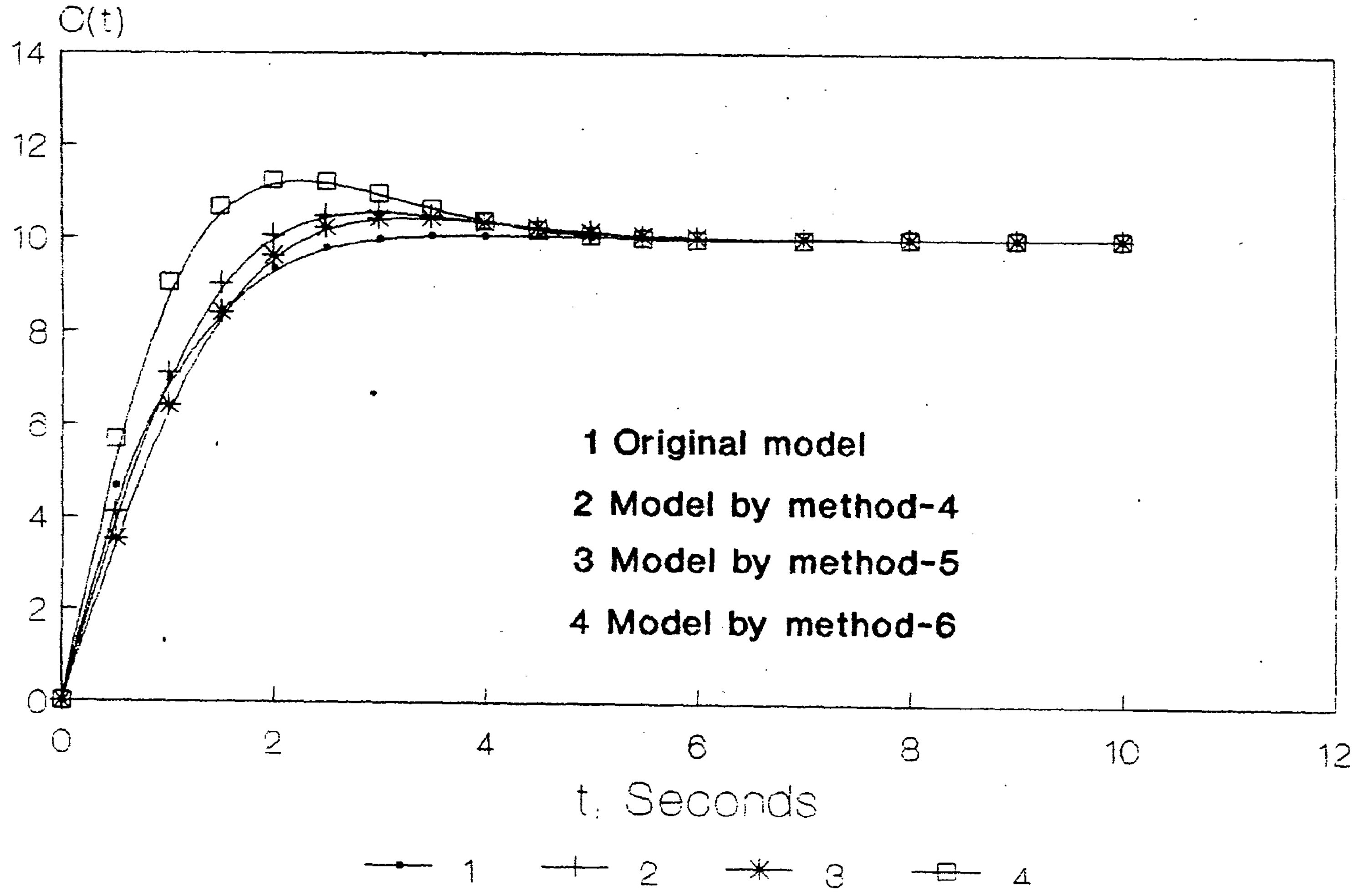


Fig.4 Comparison of unit step responses (Example 1)

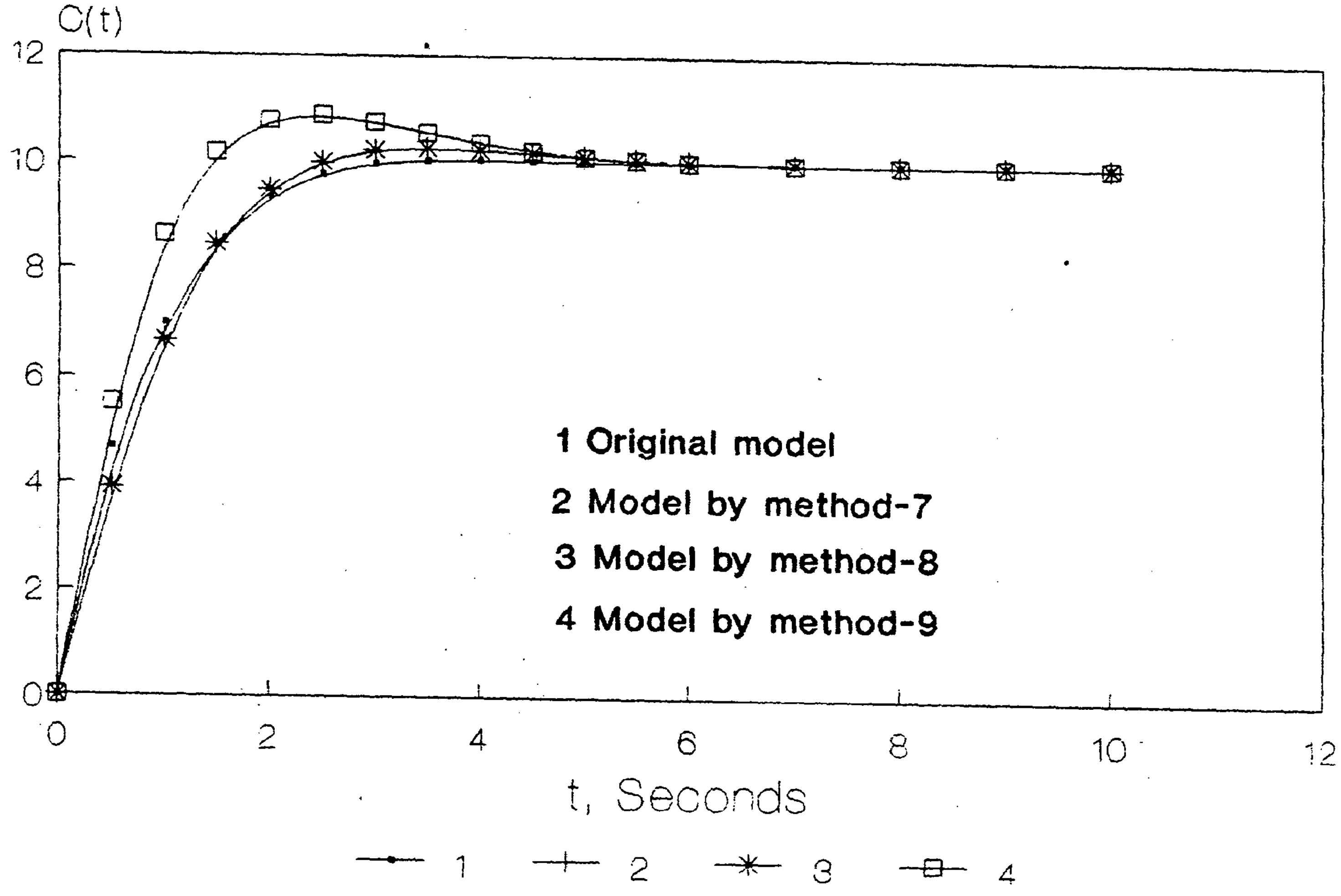


Fig.5 Comparison of unit step responses (Example 1)

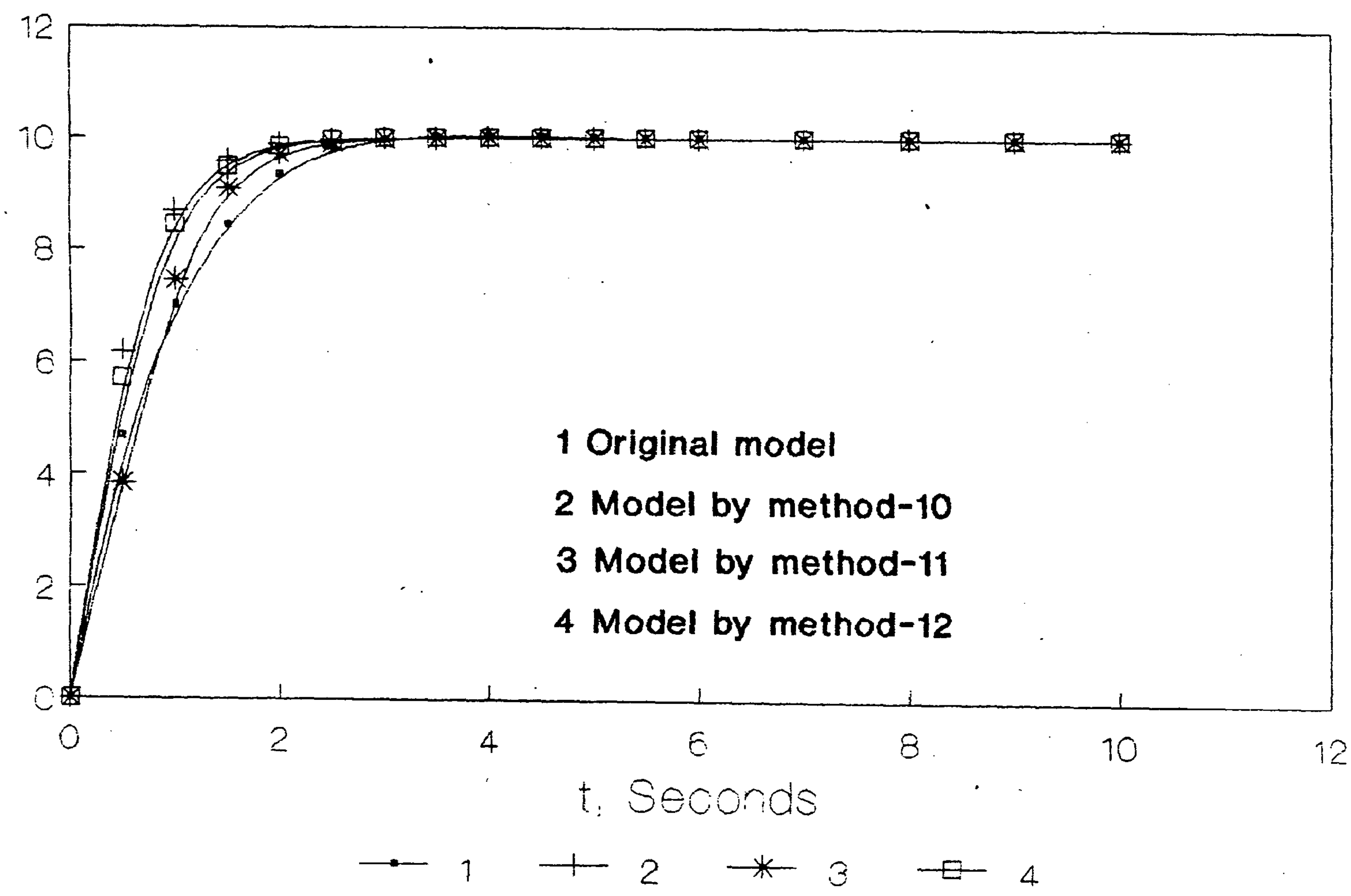
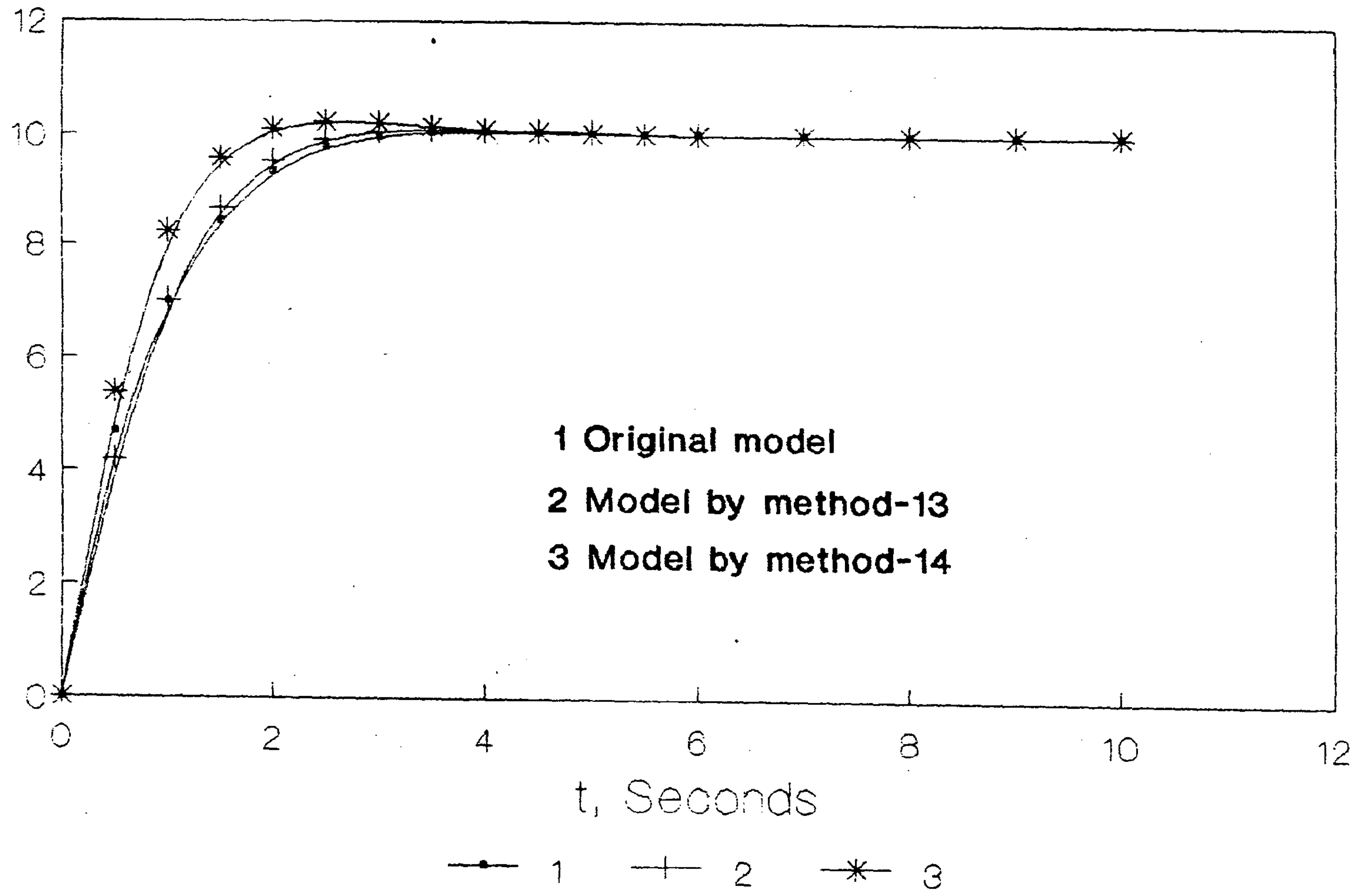


Fig.6 Comparison of unit step responses (Example 1)



EXAMPLE - 2 (CHUAG'S MODEL)

1. Method (1)

$$R_{21}(s) = \frac{1.6667s + 0.55555}{s^2 + 1.3888s + 0.55555}$$

2. Method (2)

$$R_{22}(s) = \frac{1.66655s + 0.5555}{s^2 + 1.3888s + 0.5555}$$

3. Method (3)

$$R_{23}(s) = \frac{8s + 0.5555}{s^2 + 1.3888s + 0.5555}$$

4. Method (4)

$$R_{2,4}(s) = \frac{1.5s + 0.5}{s^2 + 1.125s + 0.5}$$

5. Method (5)

$$R_{2,5}(s) = \frac{1.375s + 0.5}{s^2 + 1.125s + 0.5}$$

6. Method (6)

$$R_{2,6}(s) = \frac{8s + 0.5}{s^2 + 1.125s + 0.5}$$

7. Method (7)

$$R_{2,7}(s) = \frac{1.5s + 0.5}{s^2 + 1.25s + 0.5}$$

Method (8)

$$R_{2,8}(s) = \frac{1.5s + 0.5}{s^2 + 1.25s + 0.5}$$

Method (9)

$$R_{2,9}(s) = \frac{8s + 0.5}{s^2 + 1.25s + 0.5}$$

Method (10)

$$R_{2,10}(s) = \frac{2.25s + 1.5}{s^2 + 2.5s + 1.5}$$

Method (11)

$$R_{2,11}(s) = \frac{3.25s + 1.5}{s^2 + 2.5s + 1.5}$$

Method (12)

$$R_{2,12}(s) \doteq \frac{8s + 1.5}{s^2 + 2.5s + 1.5}$$

Method (13)

$$R_{2,13}(s) = \frac{4s + 1}{s^2 + 2s + 1}$$

Method (14)

$$R_{2,14}(s) = \frac{8s + 1}{s^2 + 2s + 1}$$

TIME RESPONSE

TABLE -11 EX-2 METHOD 1,2,3

No.	Time (Sec)	$G_{2,1}(s)$	$R_{21}(s)$	$R_{22}(s)$	$R_{23}(s)$
1.	0	0	0	0	0
2.	0.5	1.8057	0.64239	0.6422	2.8731
3.	1	1.77187	0.99956	0.9983	4.1224
3.	1.5	1.3449	1.1765	1.176	4.4372
5.	2	1.0692	1.2479	1.2477	4.2541
6.	2.5	0.9259	1.2590	1.2589	3.8406
7.	3	0.8732	1.2389	1.2388	3.3529
8.	3.5	0.8691	1.2052	1.2051	2.8765
9.	4	0.8864	1.168	1.1680	2.4528
10.	4.5	0.9097	1.1328	1.1329	2.0974
11.	5	0.9321	1.1021	1.1021	1.8113
12.	5.5	0.9507	1.0766	1.0767	1.5881
13.	6	0.9652	1.0563	1.0564	1.4185
14.	7	0.9835	1.0287	1.0288	1.2004
15.	8	0.9926	1.013	1.0136	1.0884
16.	9	0.9967	1.0058	1.0059	1.0331
17.	10	0.9986	1.002	1.0023	1.0117
18.	11	0.9999	1.000	1.0007	1.0024

TIME RESPONSE

TABLE -12 EX-2 METHOD 4,5,6

No.	Time sec.	$G_{21}(s)$	$R_{24}(s)$	$R_{25}(s)$	$R_{26}(s)$
1.	0	0	0	0	0
2.	0.5	1.8057	0.6135	0.5667	3.048
3.	1	1.77187	0.9997	0.9306	4.591
4.	1.5	1.3449	1.2197	1.1445	5.1303
5.	2	1.9692	1.3243	1.2527	5.0468
6.	2.5	0.9259	1.3532	1.2904	4.6164
7.	3	0.8732	1.3362	1.2845	4.0289
8.	3.5	0.8691	1.2948	1.2541	3.4076
9.	4	0.8864	1.2432	1.2128	2.8258
10.	4.5	0.9097	1.1909	1.1691	2.3216
11.	5	0.9321	1.143	1.1282	1.9092
12.	5.5	0.9507	1.021	1.0928	1.5882
13.	6	0.9652	1.0692	1.0638	1.3496
14.	7	0.9835	1.0253	1.0245	1.0673
15.	8	0.9926	1.0038	1.0047	0.9562
16.	9	0.9967	0.9957	0.99969	0.9327
17.	10	0.9986	0.9943	0.9956	0.9445
18.	11	0.9994	0.9954	0.9960	0.964

TIME RESPONSE

TABLE 13 EX-2 METHOD 7,8,9

No.	Time (Sec.)	$G_{21}(s)$	$R_{27}(s)$	$R_{28}(s)$	$R_{29}(s)$
1.	0	0	0	0	0
2.	0.5	1.8057	0.5969	0.5969	2.9639
3.	1	1.7187	0.9536	0.9536	4.3697
4.	1.5	1.3449	1.1487	1.1487	4.8122
5.	2	1.0692	1.2395	1.2395	4.6983
6.	2.5	0.9259	1.266	1.266	4.2972
7.	3	0.8732	1.2558	1.2558	3.7792
8.	3.5	0.8691	1.2267	1.2267	3.2463
9.	4	0.8864	1.1903	1.1903	2.7544
10.	4.5	0.9097	1.1534	1.1534	2.3297
11.	5	0.9321	1.1194	1.1194	1.9800
12.	5.5	0.9507	1.0903	1.0903	1.7027
13.	6	0.9652	1.0664	1.0664	1.4897
14.	7	0.9835	1.033	1.033	1.2150
15.	8	0.9926	1.0144	1.0144	1.0774
16.	9	0.9967	1.0051	1.0051	1.0167
17.	10	0.9986	1.0010	1.0010	0.9947
18.	11	0.9994	0.9996	0.9996	0.9899

TIME RESPONSE

TABLE -14 EX-2 METHOD 11,12,13

No.	Time Sec.	$G_{21}(s)$	$R_{2,10}(s)$	$R_{2,11}(s)$	$R_{2,12}(s)$
1.	0	0	0	0	0
2.	0.5	1.8057	0.7288	0.9972	2.2718
3.	1	1.7187	0.9940	1.2835	2.6586
4.	1.5	1.3449	1.0712	1.3066	2.4251
5.	2	1.0692	1.0785	1.2496	2.0623
6.	2.5	0.9259	1.0643	1.1814	1.7378
7.	3	0.8732	1.0469	1.1242	1.4917
8.	3.5	0.8691	1.0321	1.0820	1.3191
9.	4	0.8864	1.0212	1.0529	1.2034
10.	4.5	0.9097	1.0137	1.0336	1.128
11.	5	0.9321	1.0087	1.0210	1.0798
12.	5.5	0.9507	1.0054	1.0131	1.0494
13.	6	0.9652	1.0034	1.0081	1.0304
14.	7	0.9835	1.0012	1.003	1.0114
15.	8	0.9926	1.0004	1.001	1.0042
16.	9	0.9967	1.0001	1.0004	1.0015
17.	10	0.9986	1.000	1.0001	1.0005
18.	11	0.9994	1.000	1.0001	1.0002

TIME RESPONSE

TABLE -15 EX-2 METHOD 13,14

No.	Time (sec)	$G_{21}(s)$	$R_{2,13}(s)$	$R_{2,14}(s)$
1.	0	0	0	0
2.	0.5	1.8057	1.303	2.516
3.	1	1.7187	1.735	3.207
4.	1.5	1.3449	1.781	3.119
5.	2	1.0692	1.676	2.759
6.	2.5	0.9259	1.533	2.354
7.	3	0.8732	1.398	1.995
8.	3.5	0.8691	1.287	1.709
9.	4	0.8864	1.201	1.494
10.	4.5	0.9097	1.138	1.338
11.	5	0.9321	1.094	1.229
12.	5.5	0.9507	1.063	1.153
13.	6	0.9652	1.042	1.101
14.	7	0.9835	1.018	1.043
15.	8	0.9926	1.007	1.018
16.	9	0.9967	1.003	1.007
17.	10	0.9986	1.001	1.003
18.	11	0.9994	1.000	1.001

Fig.7 Comparison of unit step responses (Example 2)

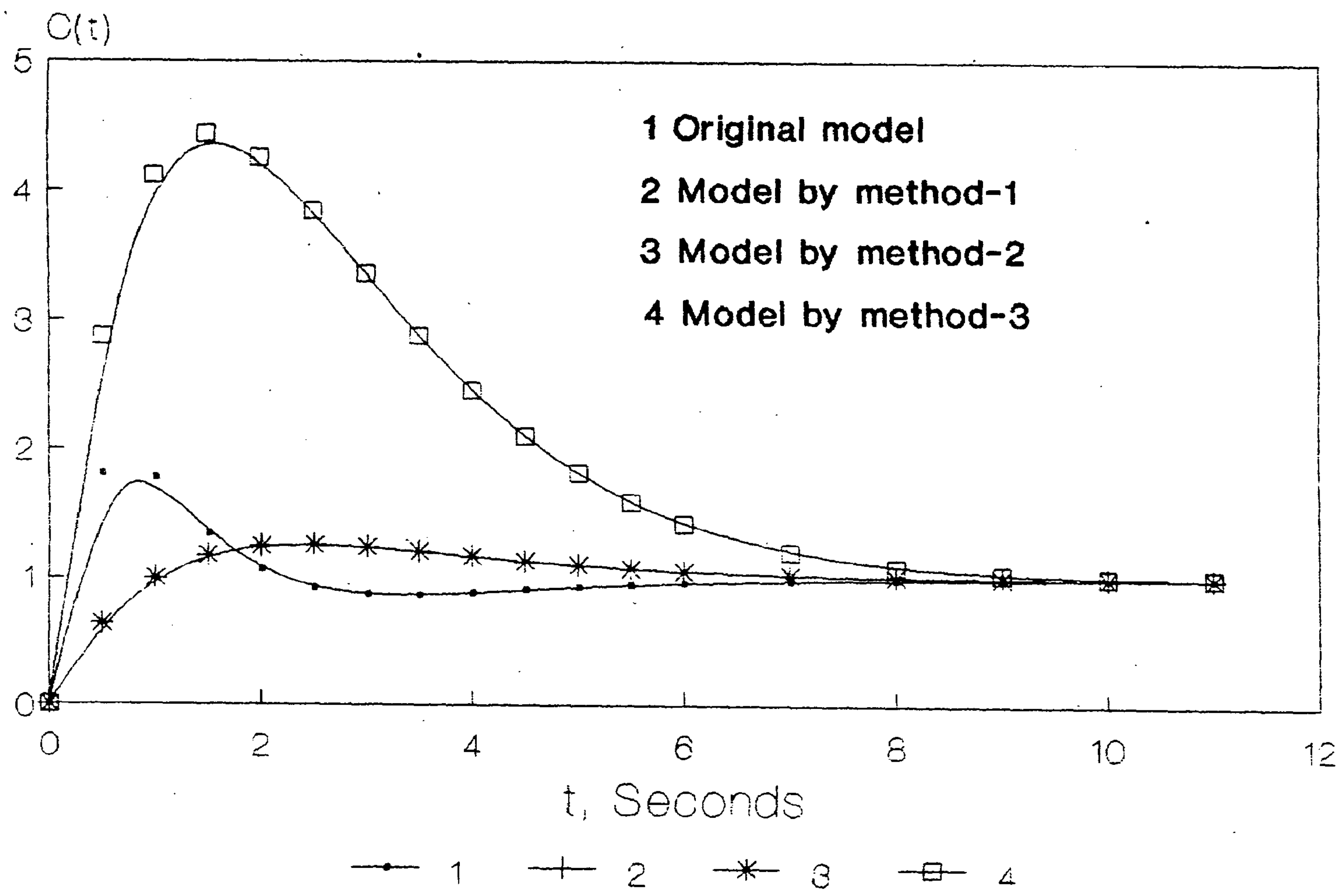


Fig.8 Comparison of unit step responses (Example 2)

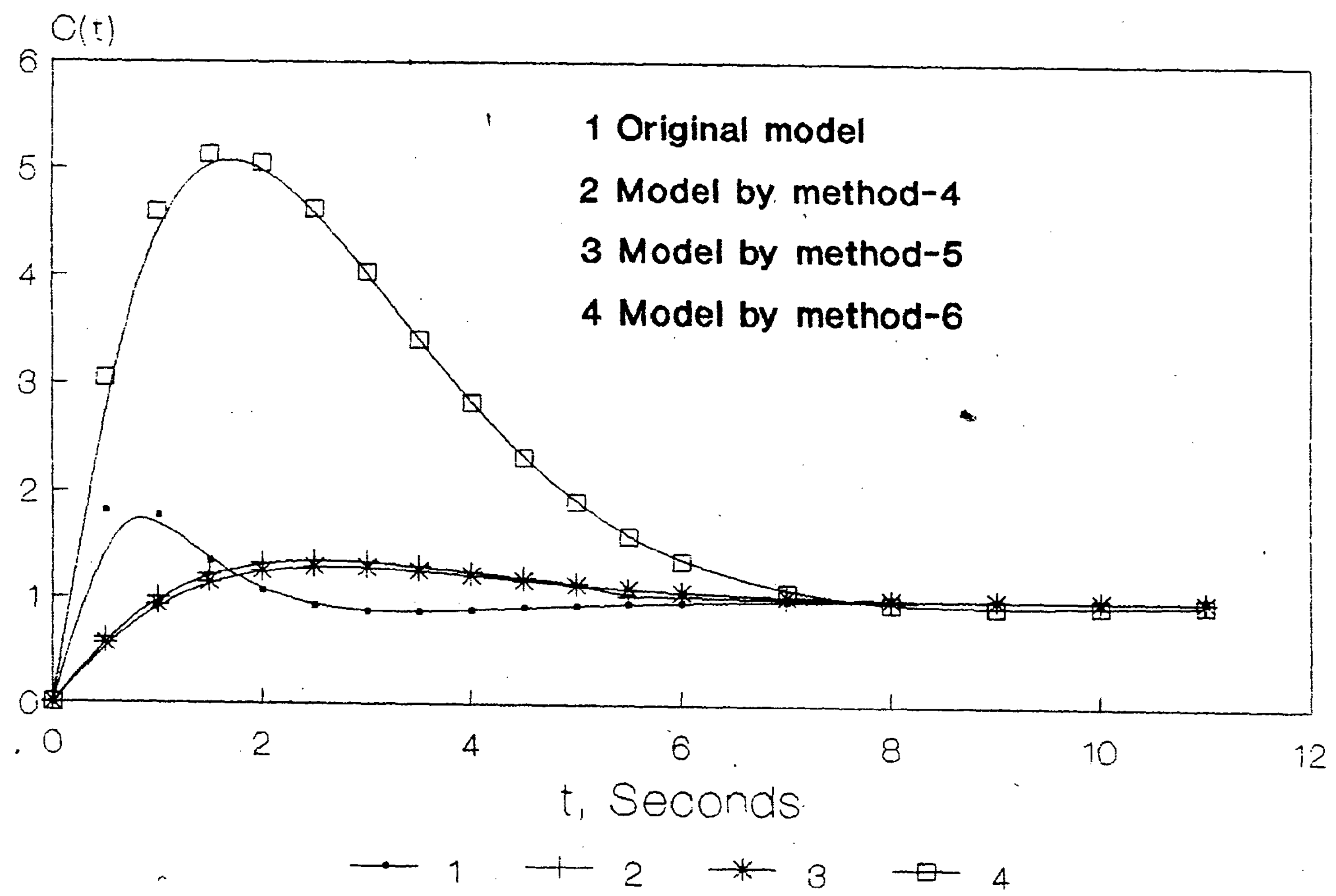


Fig.9 Comparison of unit step responses (Example 2)

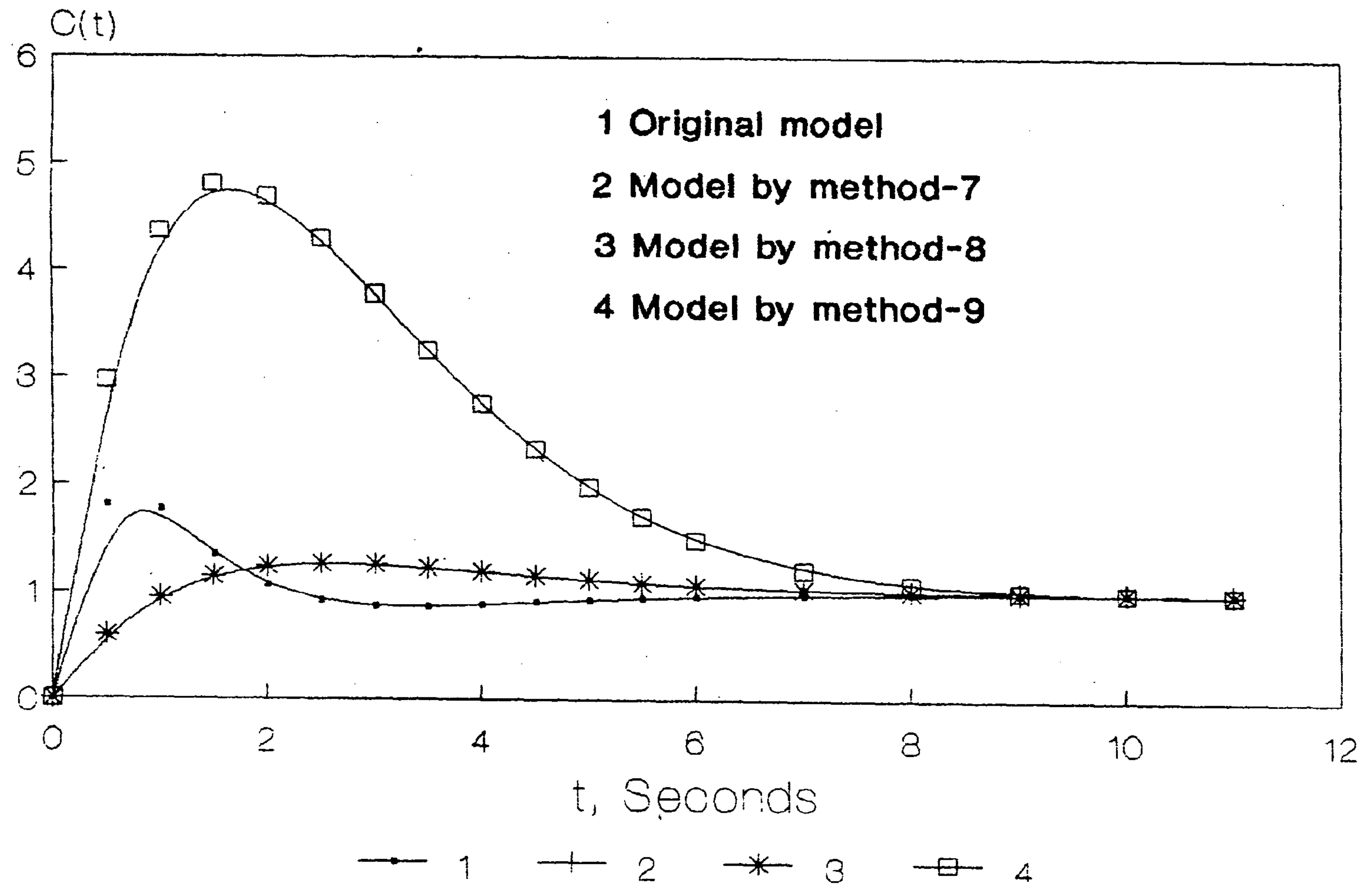


Fig.10 Comparison of unit step responses (Example 2)

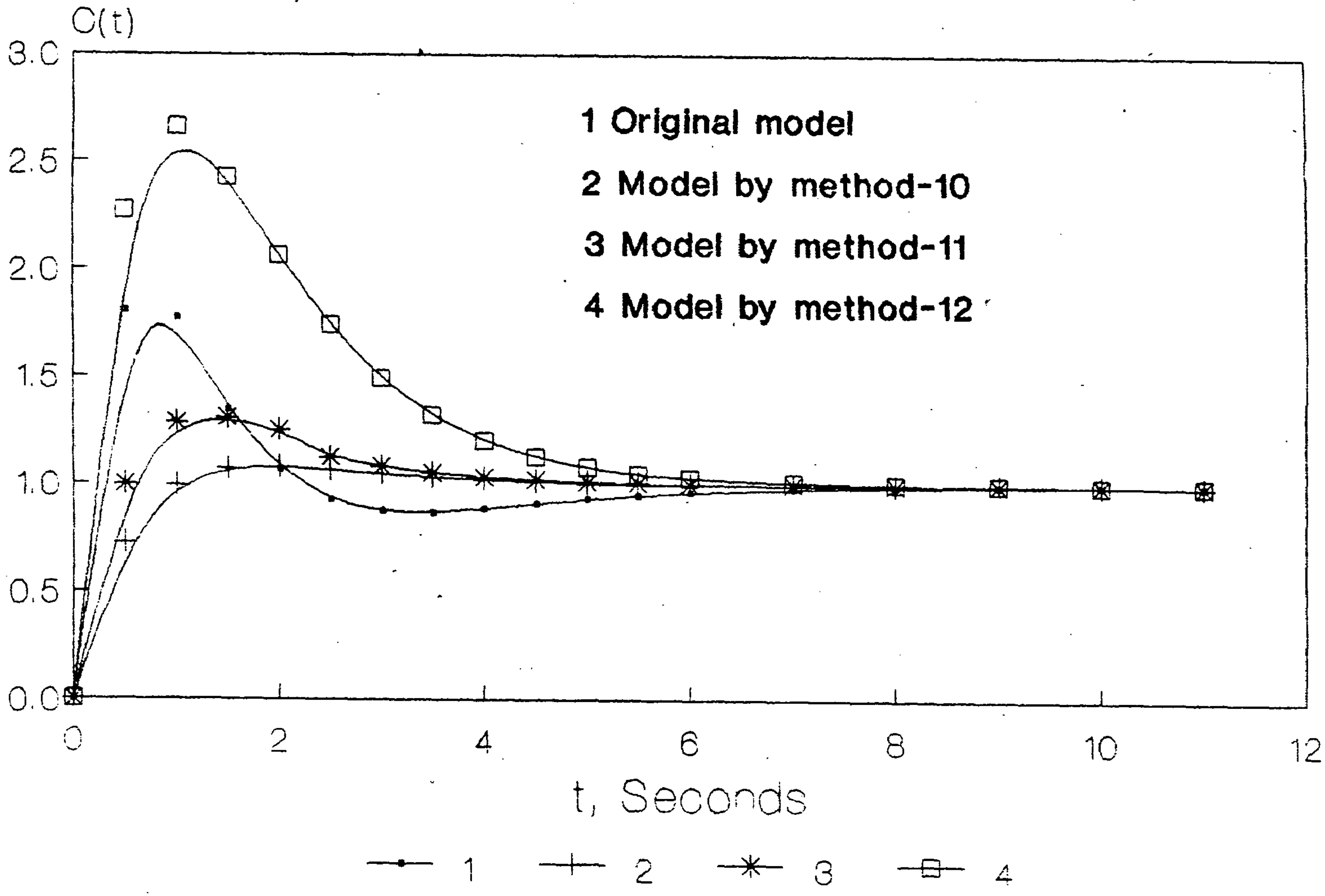
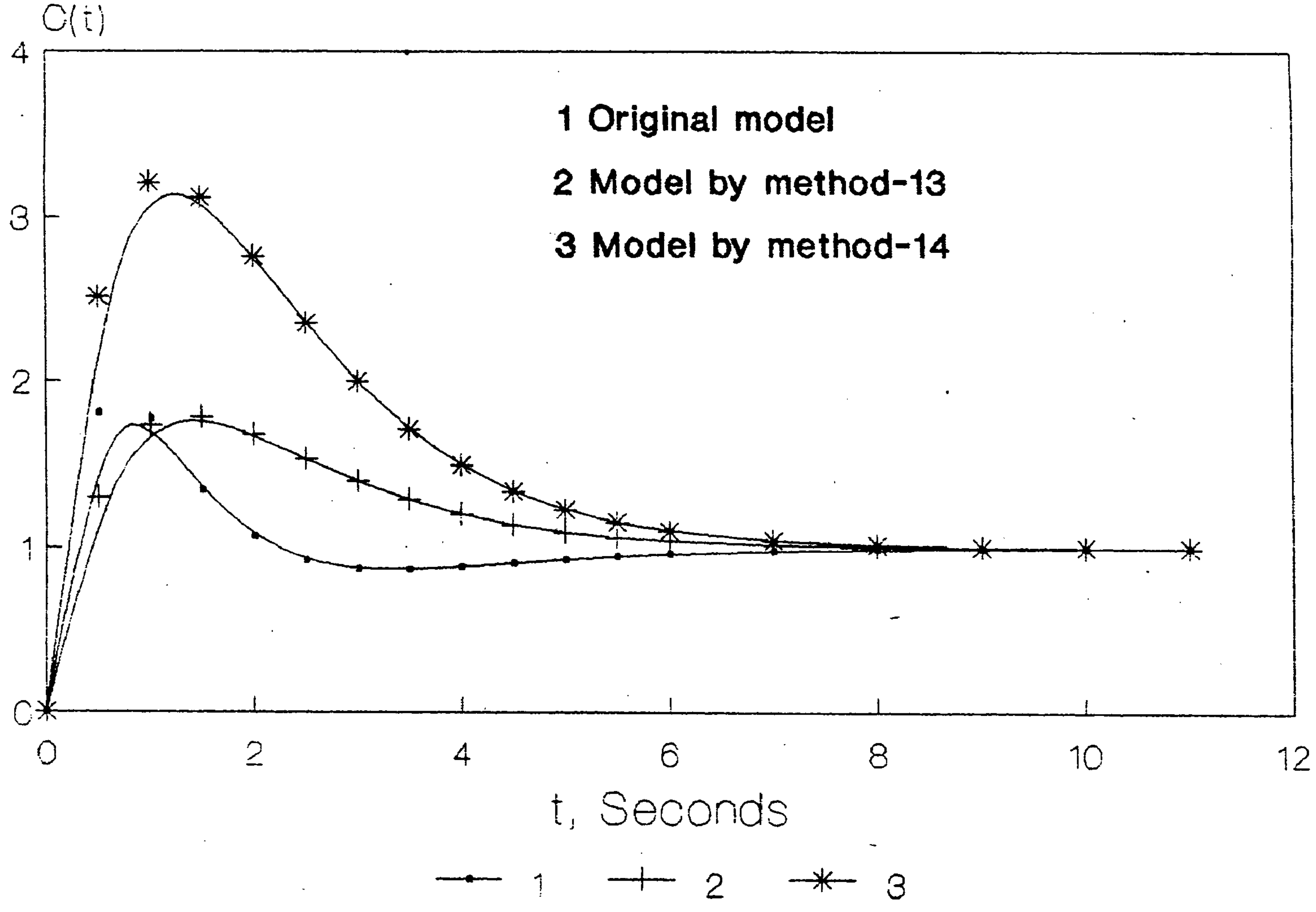


Fig.11 Comparison of unit step responses (Example 2)



(III) EXAMPLE (3) (SHESHADRI'S MODEL)

1. Method -1

$$R_{31}(s) = \frac{17.0293s + 6.8573}{s^2 + 1.01831s + 0.338486}$$

2. Method -2

$$R_{32}(s) = \frac{17.029441s + 6.8571625}{s^2 + 1.01831s + 0.338486}$$

3. Method -3

$$R_{33}(s) = \frac{35s + 6.8573}{s^2 + 1.01831s + 0.338486}$$

4. Method -4

$$R_{3,4}(s) = \frac{16.638516s + 9.664226}{s^2 + 0.900242s + 0.477049}$$

5. Method -5

$$R_{3,5}(s) = \frac{13.163955s + 9.6642177}{s^2 + 0.9002420s + 0.477049}$$

6. Method -6

$$R_{3,6}(s) = \frac{35s + 9.664226}{s^2 + 0.900242s + 0.477049}$$

7. Method -7

$$R_{37}(s) = \frac{14.1242s + 5.6875475}{s^2 + 0.84459s + 0.28075}$$

8. Method -8

$$R_{38}(s) = \frac{14.12419s + 5.6875271}{s^2 + 0.84459s + 0.28075}$$

9. Method -9

$$R_{3,9}(s) = \frac{35s + 5.6875475}{s^2+0.84459s + 0.28075}$$

10. Method -10

$$R_{3,10}(s) = \frac{51.527152s + 145.24272}{s^2+ 5.39208s + 7.169529}$$

11. Method -11

$$R_{3,11}(s) = \frac{32.986137s + 145.24272}{s^2+ 5.39208s + 17.169529}$$

12. Method -12

$$R_{3,12}(s) = \frac{35s + 145.24272}{s^2+ 5.39208s + 7.169529}$$

13. Method -13

$$R_{3,13}(s) = \frac{49.128127s + 60.775}{s^2+ 4s + 3}$$

14. Method -14

$$R_{3;14}(s) = \frac{35s + 66.775}{s^2+ 4s + 3}$$

TIME RESPONSE

TABLE -16 EX-3 METHOD 1,2,3

No.	Time (Sec)	$G_{31}(s)$	$R_{31}(s)$	$R_{32}(s)$	$R_{33}(s)$
1.	0	0	0	0	0
2.	0.5	11.8765	7.3026	7.302	14.245
3.	1	17.3889	12.5447	12.5447	23.2013
4.	1.5	19.665	16.203	16.203	28.392
5.	2	20.336	18.669	18.668	30.976
6.	2.5	20.355	20.255	20.255	31.823
7.	3	20.210	21.209	21.209	31.570
8.	3.5	20.098	21.720	21.720	30.673
9.	4	20.06	21.93	21.93	29.449
10.	4.5	20.079	21.947	21.947	28.108
11.	5	20.124	21.845	21.845	26.785
12.	5.5	20.208	21.487	21.487	24.475
13.	6	20.248	21.104	21.104	22.709
14.	7	20.258	20.791	20.791	21.635
15	8	20.258	20.568	20.568	20.941
16.	9	20.257	20.423	20.423	20.549
17	10	20.258	20.337	20.337	20.348
18	11	20.258	20.29	20.29	20.256
19	12	20.258	20.266	20.266	20.256
20.	13	20.258	20.256	20.256	20.256

TIME RESPONSE

TABLE -17 EX-3 METHOD 4,5,6

No.	Time sec.	$G_{31}(s)$	$R_{34}(s)$	$R_{35}(s)$	$R_{36}(s)$
1.	0	0	0	0	0
2.	0.5	11.8765	7.602	6.231	14.849
3.	1	17.3889	13.652	11.537	24.83
4.	1.5	19.665	18.138	15.749	30.759
5.	2	20.336	21.199	18.864	33.541
6.	2.5	20.355	23.061	20.982	34.051
7.	3	20.210	23.982	22.263	33.065
8.	3.5	20.098	24.215	22.890	31.218
9.	4	20.06	23.987	23.039	28.999
10.	4.5	20.079	23.487	22.869	26.753
11.	5	20.124	22.862	22.514	24.703
12.	5.5	20.208	21.017	21.617	21.613
13.	6	20.248	20.7	20.842	19.947
14.	7	20.258	20.194	20.351	19.364
15.	8	20.258	20.007	20.123	19.397
16.	9	20.257	20.007	20.071	19.671
17.	10	20.258	20.083	20.106	19.96
18.	11	20.258	20.167	20.167	20.167
19.	12	20.258	20.228	20.218	20.279
20.	13	20.258	20.262	20.252	20.272

TIME RESPONSE

TABLE -18 EX-3 METHOD 7,8,9

No.	Time sec	$G_{31}(s)$	$R_{37}(s)$	$R_{38}(s)$	$R_{39}(s)$
1.	0	0	0	0	0
2.	0.5	11.8765	6.31	6.31	14.725
3.	1	17.3889	11.241	11.241	24.693
4.	1.5	19.665	14.986	14.986	30.975
5.	2	20.336	17.742	17.742	34.484
6.	2.5	20.355	19.691	19.691	35.974
7.	3	20.210	21.001	21.001	36.056
8.	3.5	20.098	21.818	21.818	35.211
9.	4	20.06	22.265	22.265	33.806
10.	4.5	20.079	22.443	22.443	32.113
11.	5	20.124	22.436	22.436	30.329
12.	5.5	20.208	22.106	22.106	26.970
13.	6	20.248	21.622	21.622	24.283
14.	7	20.258	21.161	21.161	22.383
15.	8	20.258	20.796	20.796	21.173
16.	9	20.257	20.539	20.539	20.483
17.	10	20.258	20.377	20.377	20.145
18.	11	20.258	20.285	20.285	20.021
19.	12	20.258	20.241	20.241	20.011
20.	13	20.258	20.251	20.251	20.057

TIME RESPONSE

TABLE-19 EX-3 METHOD-10,11,12

No	Time (sec)	$G_{31}(s)$	$R_{3,10}(s)$	$R_{3,11}(s)$	$R_{3,12}(s)$
1.	0	0	0	0	0
2.	0.5	11.8765	14.527	12.109	12.372
3.	1	17.3889	18.611	17.339	17.477
4.	1.5	19.665	19.778	19.272	19.327
5.	2	20.336	20.117	19.937	19.956
6.	2.5	20.355	20.216	20.156	20.162
7.	3	20.210	20.246	20.226	20.228
8.	3.5	20.098	20.254	20.248	20.249
9.	4	20.06	20.257	20.255	20.255
10.	4.5	20.079	20.258	20.257	20.257
11.	5	20.124	20.258	20.258	20.258
12.	5.5	20.208	20.258	20.258	20.258
13.	6	20.248	20.258	20.258	20.258
14.	7	20.258	20.258	20.258	20.258
15.	8	20.258	20.258	20.258	20.258
16.	9	20.257	20.258	20.258	20.258
17.	10	20.258	20.258	20.258	20.258
18.	11	20.258	20.258	20.258	20.258
19.	12	20.258	20.258	20.258	20.258
20.	13	20.258	20.258	20.258	20.258

TIME RESPONSE

TABLE -20 EX 3 METHOD -13,14

No.	Time (sec)	$G_{31}(s)$	$R_{3,13}(s)$	$R_{3,14}(s)$
1.	0	0	0	0
2.	0.5	11.8765	13.505	10.797
3.	1	17.3889	17.397	15.150
4.	1.5	19.665	18.798	17.3
5.	2	20.336	19.434	18.495
6.	2.5	20.355	19.772	19.196
7.	3	20.210	19.966	19.615
8.	3.5	20.098	20.082	19.869
9.	4	20.060	20.151	20.022
10.	4.5	20.079	20.193	20.115
11.	5	20.124	20.219	20.171
12.	5.5	20.208	20.234	20.205
13.	6.	20.248	20.243	20.226
14.	7	20.258	20.253	20.246
15.	8	20.258	20.256	20.254
16.	9	20.257	20.257	20.256
17.	10	20.258	20.258	20.257
18.	11	20.258	20.258	20.258
19.	12	20.258	20.258	20.258
20.	13	20.258	20.258	20.258

Fig.12 Comparison of unit step responses (Example 3)

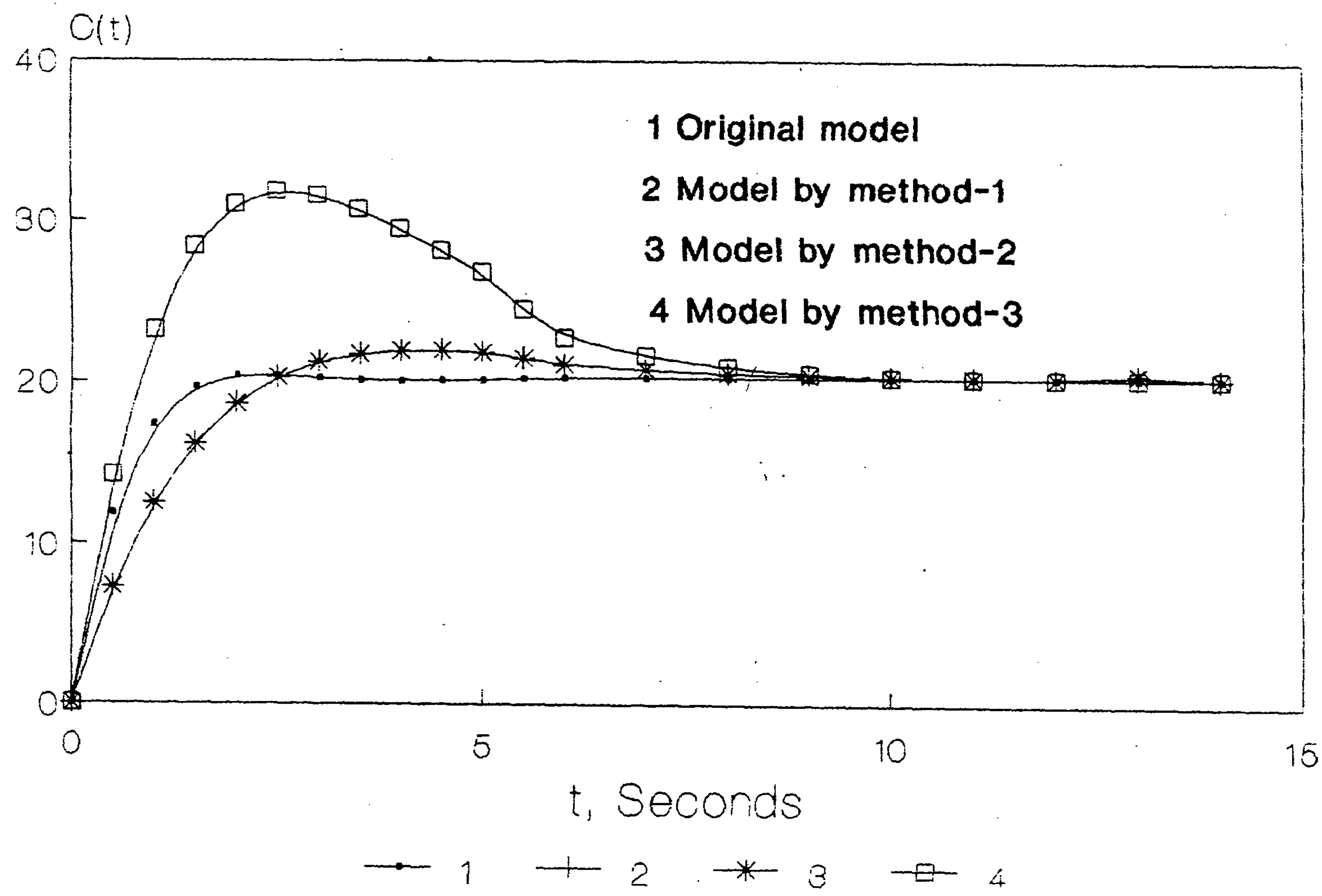


Fig.13 Comparison of unit step responses (Example 3)

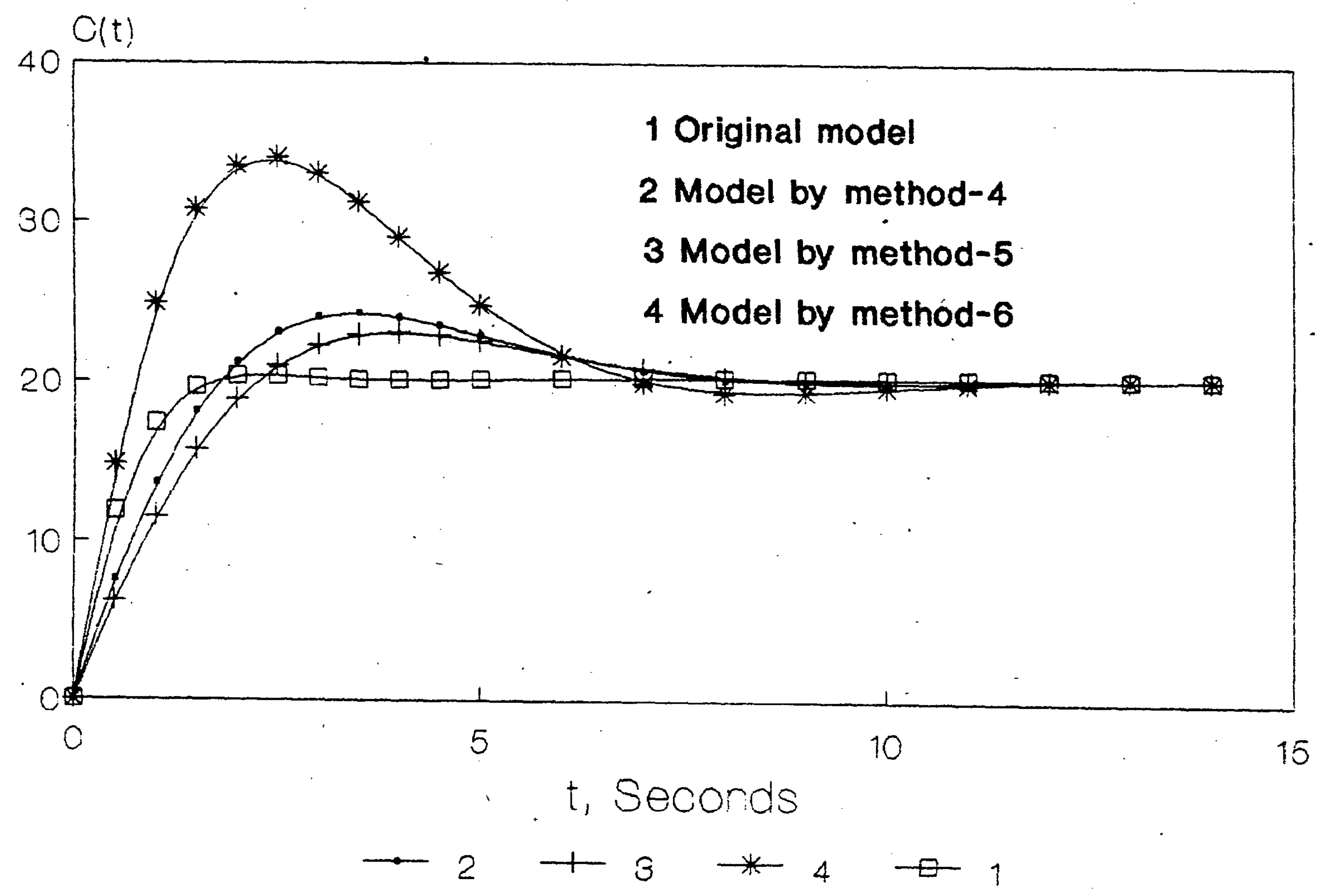


Fig.14 Comparison of unit step responses (Example 3)

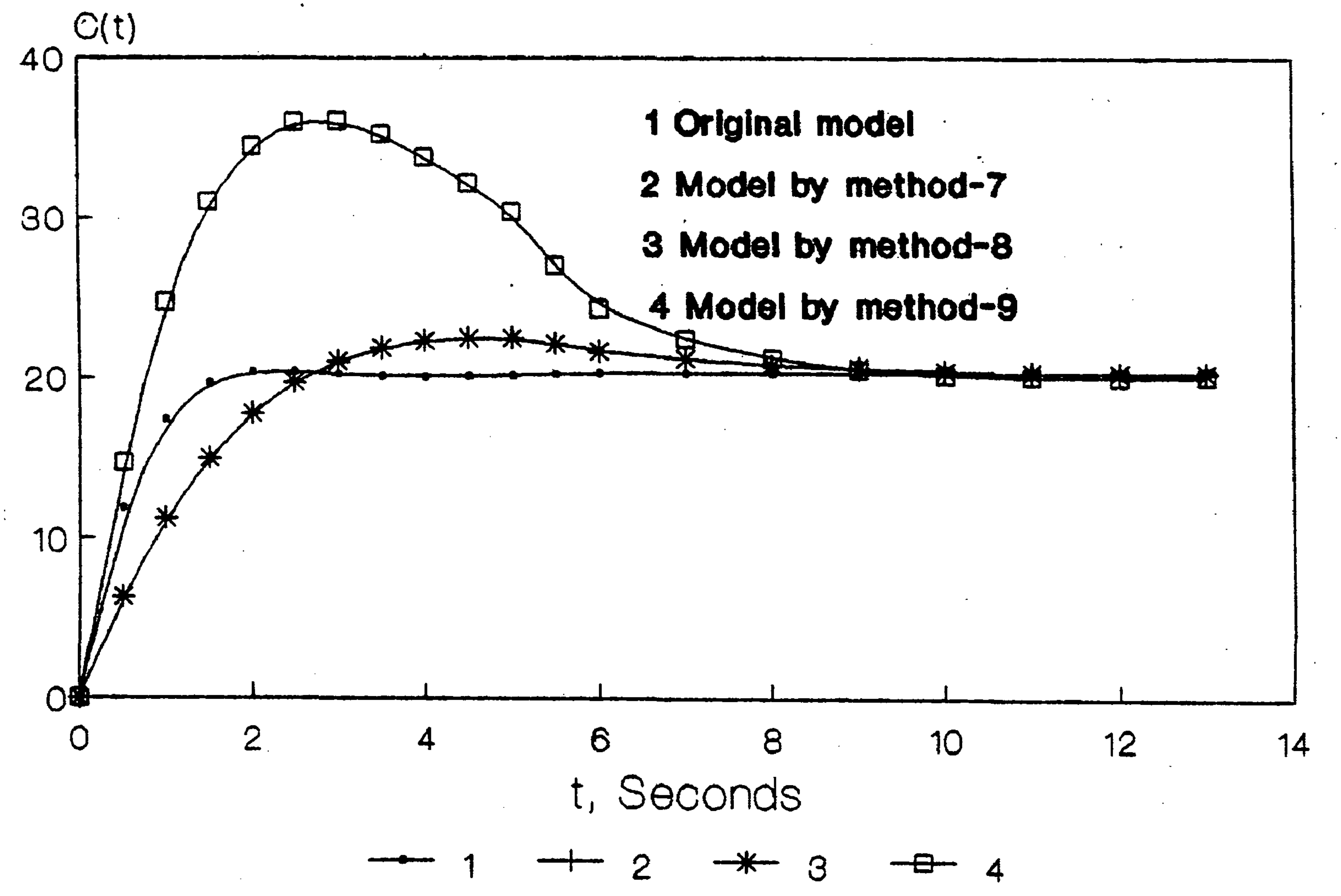


Fig.15 Comparison of unit step responses (Example 3)

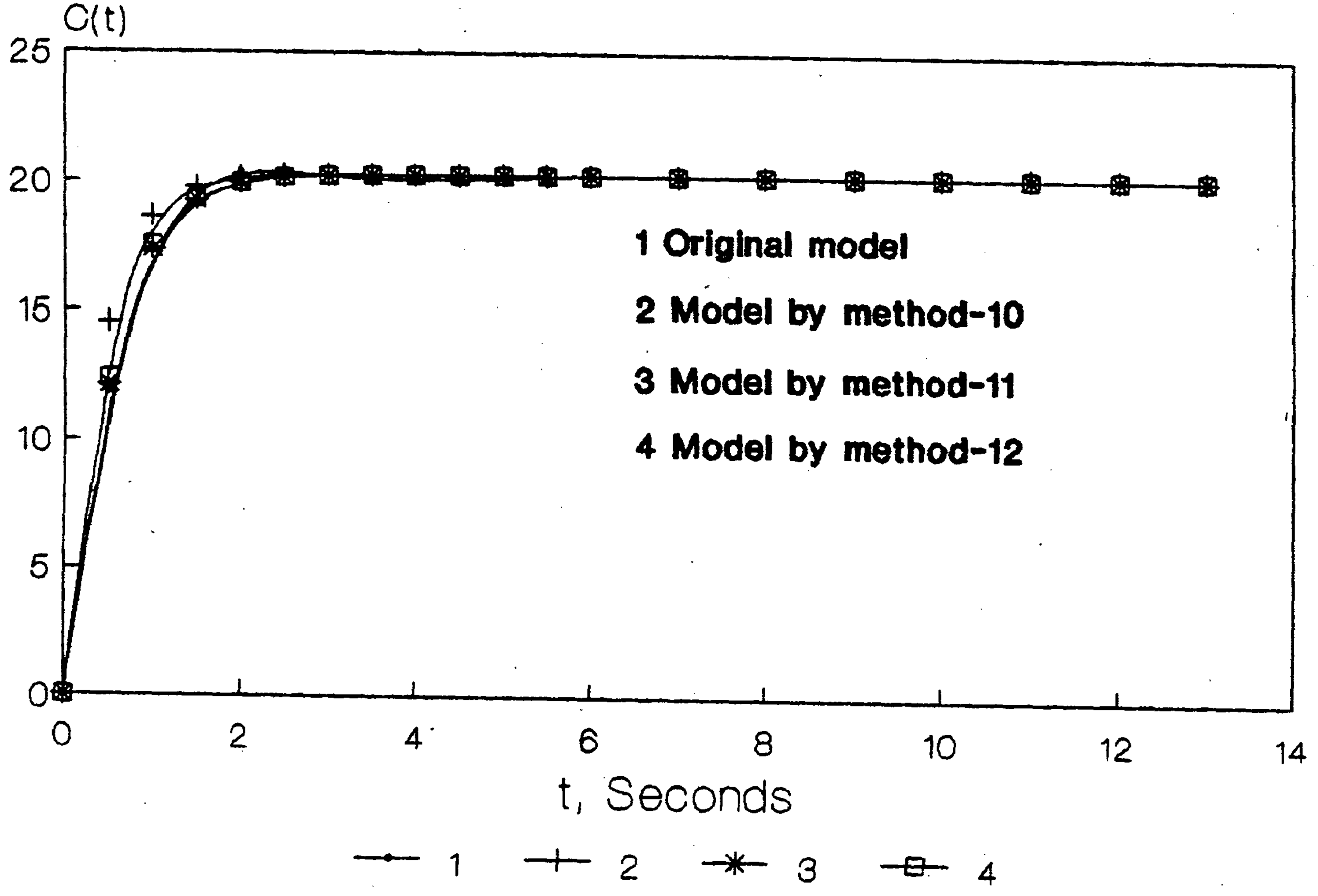
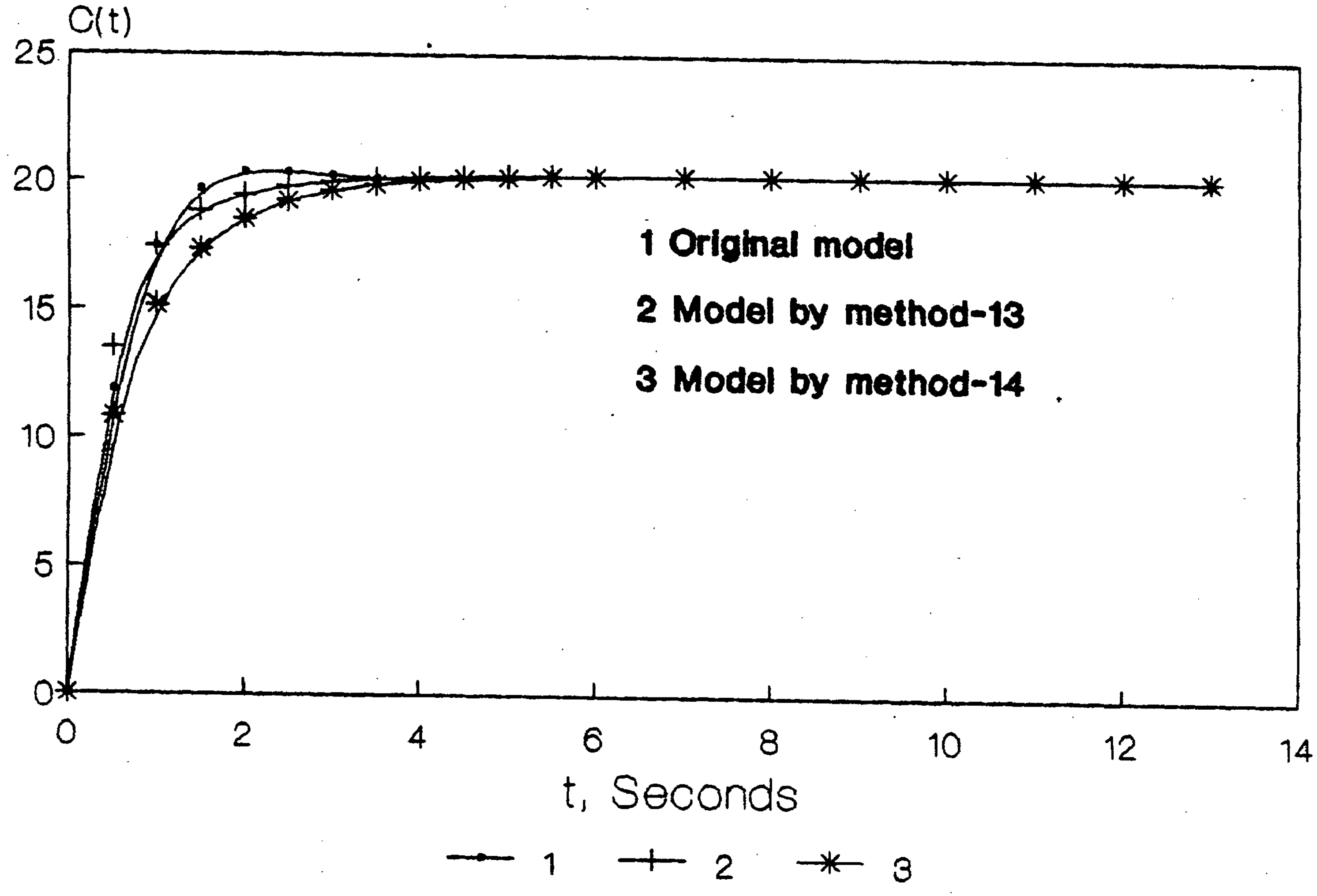


Fig.16 Comparison of unit step responses (Example 3)



Plot of Frequency responses of various reduced models have also been attached herewith.

(5.1.2) Multi Input Multioutput System (MIMO)

One MIMO system has been considered in this section. It is reduced by a few selected reduction techniques mixed with Cauer third form of continued fraction expansion namely.

1. Routh approximation Mixed with CFE of Cauer 3rd form - Method -3.
2. Routh Hurwitz array mixed with CFE of Cauer 3rd form - Method 6.
3. Stability equations mixed with CFE of Cauer 3rd form - Method 9.
4. Polynomial differentiation mixed with CFE of Cauer 3rd form - Method 12.
5. Dominant pole retention mixed with CFE of Cauer 3rd form - Method -14.

MIMO system is given as

$$\dot{X} = A X + B$$

$$Y = C X$$

where,

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -120 & -180 & -102 & -18 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & -2 \\ 11. & 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 1200 & 900 & 248 & 14 \\ 2160 & 720 & 264 & 6 \end{bmatrix}$$

Applying Faddeeva leverrier algorithm as described in section 5.2 of this chapter, we get the Transfer function as

$$G_4(s) = \frac{\begin{bmatrix} 154 & 704 \\ 66 & 1632 \end{bmatrix} s^3 + \begin{bmatrix} 2728 & 12048 \\ 2904 & 28440 \end{bmatrix} s^2 + \begin{bmatrix} 9900 & 62880 \\ 7920 & 160560 \end{bmatrix} s + \begin{bmatrix} 13200 & 68160 \\ 23760 & 226080 \end{bmatrix}}{s^4 + 18s^3 + 102\, s^2 + 180s + 120}$$

Original models and reduced models can be given as

Original model (Example - 4)

$$G_4(s) = \frac{\begin{bmatrix} 154 & 704 \\ 66 & 1632 \end{bmatrix} s^3 + \begin{bmatrix} 2728 & 12048 \\ 2904 & 28440 \end{bmatrix} s^2 + \begin{bmatrix} 9900 & 62880 \\ 7920 & 160560 \end{bmatrix} s + \begin{bmatrix} 13200 & 68160 \\ 23760 & 226080 \end{bmatrix}}{s^4 + 18s^3 + 102s^2 + 180s + 120}$$

Reduced Model by Method 3

$$R_{4,3} = \frac{\begin{bmatrix} 154 & 704 \\ 66 & 1632 \end{bmatrix} s + \begin{bmatrix} 146.52 & 756.576 \\ 263.736 & 2509.488 \end{bmatrix}}{s^2 + 2s + 1.332}$$

Reduced Model by Method - 6

$$R_{4,6}(s) = \frac{\begin{bmatrix} 154 & 704 \\ 66 & 1632 \end{bmatrix} s + \begin{bmatrix} 143.44 & 740.672 \\ 258.192 & 2456.736 \end{bmatrix}}{s^2 + 1.70132s + 1.304}$$

Reduced Model by Method 9

$$R_{4,9}(s) = \frac{\begin{bmatrix} 154 & 704 \\ 66 & 1632 \end{bmatrix} s + \begin{bmatrix} 130.93986 & 676.12584 \\ 235.69176 & 2242.6428 \end{bmatrix}}{s^2 + 1.7855435s + 1.1903624}$$

Reduced model by Method -12

$$R_{4,12}(s) = \frac{\begin{bmatrix} 154 & 704 \\ 66 & 1632 \end{bmatrix} s + \begin{bmatrix} 776.47059 & 4009.4117 \\ 1397.6471 & 13298.823 \end{bmatrix}}{s^2 + 5.2941s + 7.0588235}$$

Reduced Model by Method -14

$$R_{4,14}(s) = \frac{\begin{bmatrix} 154 & 704 \\ 66 & 1632 \end{bmatrix} s + \begin{bmatrix} 210.40963 & 1086.4788 \\ 378.73733 & 3603.7431 \end{bmatrix}}{s^2 + 2.3933682 s + 1.9128148}$$

Suppose original model is given by

$$G_4(s) = \frac{\begin{bmatrix} g_{11}(s) & g_{12}(s) \\ g_{21}(s) & g_{22}(s) \end{bmatrix}}{\Delta(s)}$$

and reduced model is given as

$$R_4(s) = \frac{\begin{matrix} r_{11}(s) & r_{12}(s) \\ r_{21}(s) & r_{22}(s) \end{matrix}}{\Delta r(s)}$$

Then for comparing step responses of original system. We will have to decompose this MIMO system into four SISO systems as given below.

$$G_{4.1}(s) = \frac{g_{11}(s)}{\Delta(s)}$$

$$G_{4,2}(s) = \frac{g_{12}(s)}{\Delta(s)}$$

$$G_{4,3}(s) = \frac{g_{21}(s)}{\Delta(s)}$$

$$G_{4,4}(s) = \frac{g_{22}(s)}{\Delta s}$$

and respective reduced order modes can be given as

$$R_{4,1}(s) = \frac{r_{11}(s)}{\Delta r(s)}$$

$$R_{4,2}(s) = \frac{r_{12}(s)}{\Delta r(s)}$$

$$R_{4,3}(s) = \frac{r_{21}(s)}{\Delta r(s)}$$

$$R_{4,4}(s) = \frac{r_{22}(s)}{\Delta r(s)}$$

Applying this decomposition technique to Example-4, we get following SISO original models and their respective reduced models. The unit step response for every original model and its reduced model has also been shown sequentially.

Reduced order model is denoted by $R_{a,b(c)}(s)$.

where,

a is number of the example

b is number of the method by which it has been reduced

c is number of the decomposed SISO model.

Original model is denoted as

$G_{AB}(s)$

where,

A = number of the example

B = number of the decomposed SISO model.

Original model

$$G_{4,1}(s) = \frac{154s^3 + 2728s^2 + 9900}{s^4 + 18s^3 + 102s^2 + 180s + 120}$$

Reduced models

$$R_{4,3(1)}(s) = \frac{154s + 146.52}{s^2 + 2s + 1.332}$$

$$R_{4,6(1)}(s) = \frac{154s + 143.44}{s^2 + 1.70132s + 1.304}$$

$$R_{4,9(1)}(s) = \frac{154s + 130.93986}{s^2 + 1.7855435s + 1.1903624}$$

$$R_{4,12(1)}(s) = \frac{154s + 776.47059}{s^2 + 5.2941s + 7.0588235}$$

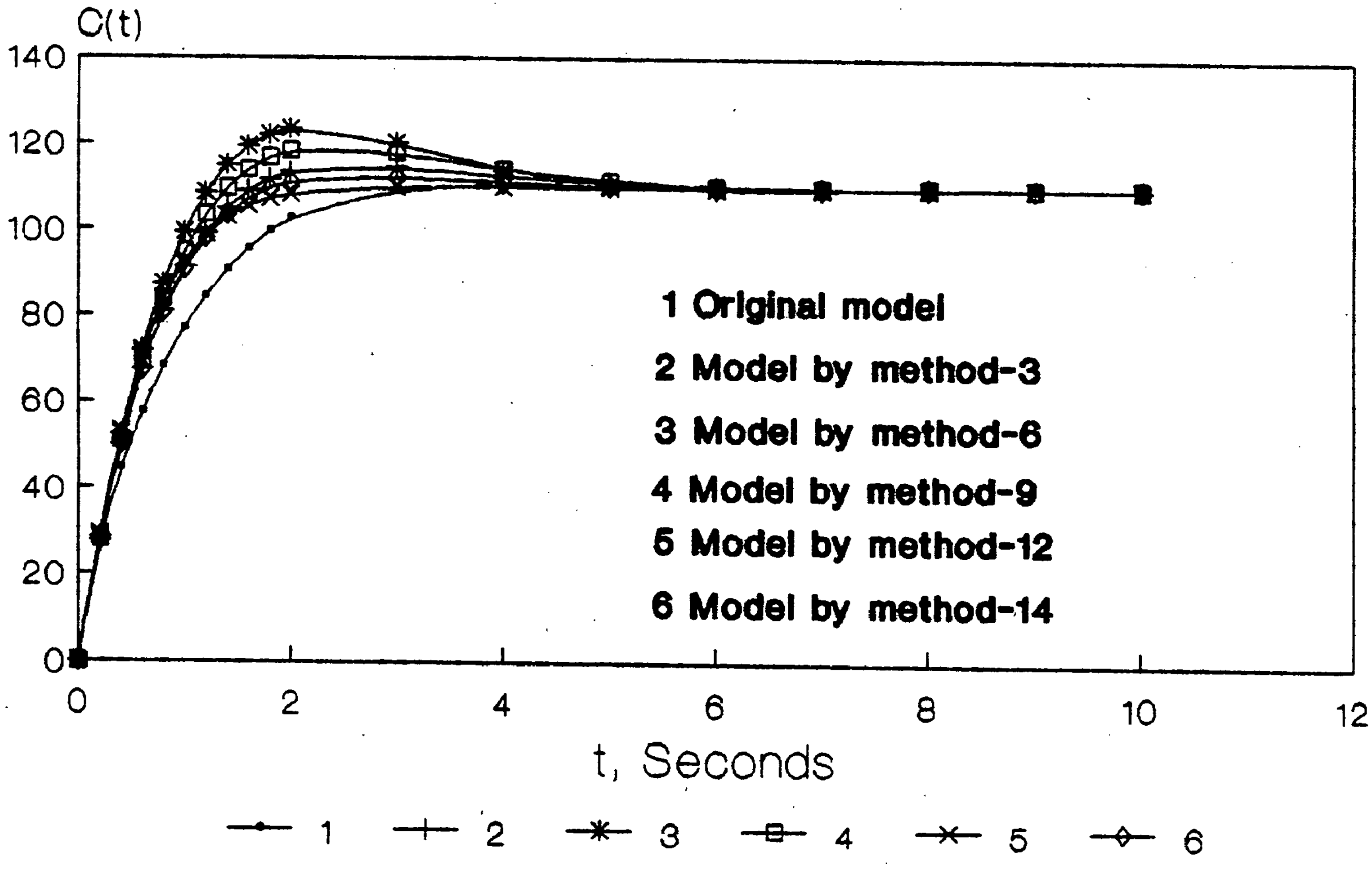
$$R_{4,14(1)}(s) = \frac{154s + 210.40963}{s^2 + 2.3933682s + 1.9128148}$$

TIME RESPONSE

TABLE-21 EX-4 ($G_{4,1}(s)$) METHOD-3,6,9,12,14

No.	Time (Sec)	$G_{4,1}(s)$	$R_{4,3(1)}(s)$	$R_{4,6(1)}(s)$	$R_{4,9(1)}(s)$	$R_{4,12(1)}(s)$	$R_{4,14(1)}(s)$
1.	0	0	0	0	0	0	0
2.	0.2	26.7302	27.725	28.439	28.02	29.13	27.75
3.	0.4	44.527	49.908	52.266	50.89	52.99	49.92
4.	0.6	57.523	67.404	71.763	69.25	70.97	67.28
5.	0.8	68.106	80.994	87.320	83.73	83.85	80.62
6.	1.0	77.028	91.372	99.388	94.90	92.77	90.66
7.	1.2	84.504	99.143	108.442	103.32	98.79	98.05
8.	1.4	90.667	104.827	114.954	109.48	102.79	103.35
9.	1.6	95.652	108.863	119.37	113.81	105.40	107.03
10.	1.8	99.608	111.618	122.09	116.69	107.09	109.50
11.	2	102.686	113.39	123.509	118.43	108.17	111.05
12.	3	109.609	114.61	120.28	118.12	109.83	112.25
13.	4	110.486	112.388	113.89	113.97	109.98	110.95
14.	5	110.263	110.849	110.53	111.27	109.99	110.23
15.	6	110.077	110.20	109.61	110.17	109.99	110.02
16.	7	109.992	110.008	109.66	109.90	110	109.98
17.	8	109.998	109.97	109.86	109.90	110	109.99
18.	9	109.9998	109.98	109.97	109.95	110	109.99
19.	10	110	109.99	110.01	109.98	110	109.999

Fig.17 Comparison of unit step responses
Example 4
G4,1(s)



Original Model

$$G_{4,2}(s) = \frac{704s^3 + 12048s^2 + 62880s + 68160}{s^4 + 18s^3 + 102s^2 + 180s + 120}$$

Reduced Model

$$R_{4,3(2)}(s) = \frac{704s + 756.576}{s^2 + 2s + 1.332}$$

$$R_{4,6(2)}(s) = \frac{704s + 740.672}{s^2 + 1.70132s + 1.304}$$

$$R_{4,9(2)}(s) = \frac{704s + 676.12584}{s^2 + 1.7855435s + 1.1903624}$$

$$R_{4,12(2)}(s) = \frac{704s + 4009.4117}{s^2 + 5.2941s + 7.0588235}$$

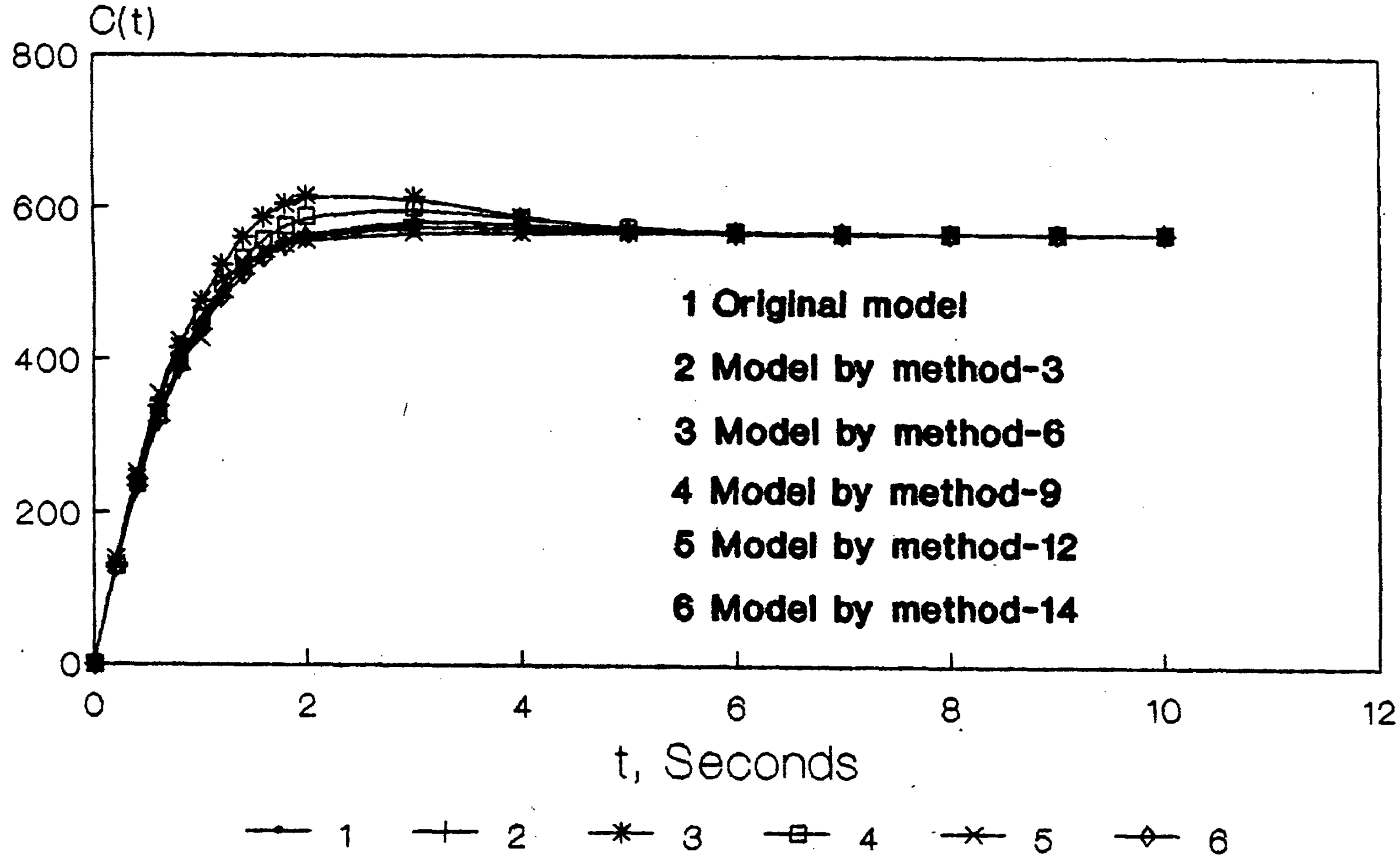
$$R_{4,14(2)}(s) = \frac{704s + 1086.4788}{s^2 + 2.3933682s + 1.9128148}$$

TIME RESPONSE

TABLE-22 EX-4 ($G_{42}(s)$) METHOD-3,6,9,12,14

No	Time (sec)	$G_{4,2}(s)$	$R_{4,3(2)}(s)$	$R_{4,6(2)}(s)$	$R_{4,9(2)}(s)$	$R_{4,12(2)}(s)$	$R_{4,14(2)}(s)$
1.	0	0	0	0	0	0	0
2.	0.2	129.84	128.26	131.52	129.48	139.67	129.001
3.	0.4	238.88	233.46	244.32	237.54	251.01	235.45
4.	0.6	327.36	318.61	338.84	326.24	355.35	321.50
5.	0.8	397.05	386.59	416.19	398.11	424.16	389.71
6.	1	450.64	440.09	477.91	455.11	472.62	442.75
7.	1.2	490.95	481.54	525.79	499.49	505.63	483.17
8.	1.4	520.59	513.10	561.72	533.27	527.67	513.35
9.	1.6	541.82	536.66	587.57	558.30	542.20	535.35
10.	1.8	556.56	553.83	605.10	576.19	551.63	550.95
11.	2	566.38	565.95	615.94	588.39	557.68	561.65
12.	3	577.97	584.05	614.04	600.45	567.06	576.48
13.	4	572.78	578.18	587.73	586.10	567.92	572.52
14.	5	569.31	572.11	571.79	574.56	567.99	569.33
15.	6	568.17	569.15	566.74	569.31	567.99	568.2
16.	7	567.97	568.15	566.53	567.77	567.99	567.97
17.	8	567.96	567.94	567.32	567.62	567.99	567.96
18.	9	567.98	567.94	567.84	567.79	567.99	567.98
19.	10	567.99	567.97	568.03	567.92	567.99	567.99

Fig.18 Comparison of unit step responses
Example 4
G4,2(s)



Original Model

$$G_{4,3}(s) = \frac{60s^3 + 2904s^2 + 7920s + 23760}{s^4 + 18s^3 + 102\, s^2 + 180s + 120}$$

Reduced Model

$$R_{4,3(3)}(s) = \frac{16s + 263.736}{s^2 + 2s + 1.332}$$

$$R_{4,6(3)}(s) = \frac{66s + 258.192}{s^2 + 1.70132s + 1.304}$$

$$R_{4,9(3)}(s) = \frac{66s + 235.69176}{s^2 + 1.7855435s + 1.1903624}$$

$$R_{4,12(3)}(s) = \frac{66s + 1397.6471}{s^2 + 5.2941s + 7.0588235}$$

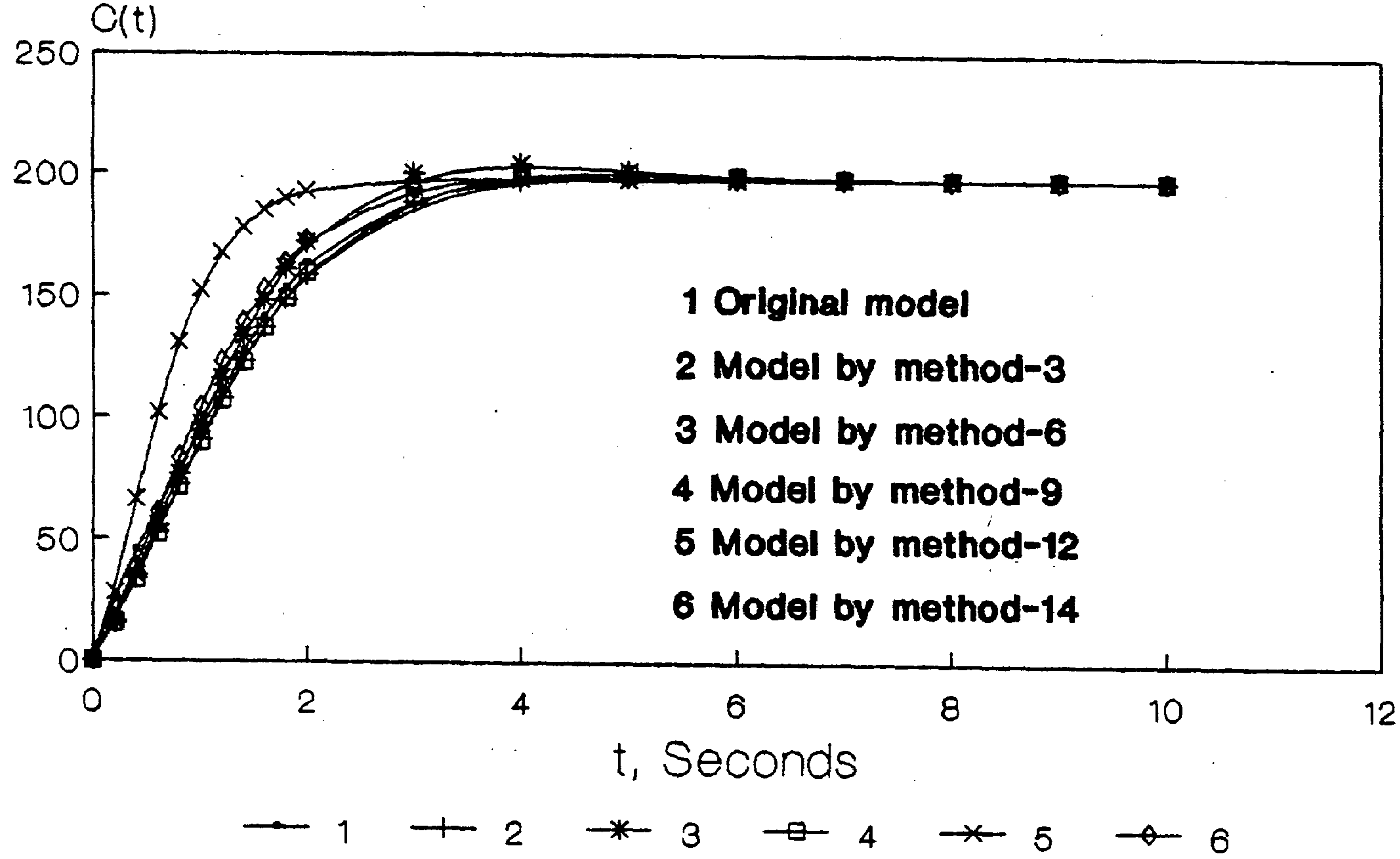
$$R_{4,14(3)}(s) = \frac{66s + 378.73733}{s^2 + 2.3933622s + 1.9128148}$$

TIME RESPONSE

TABLE-23 EX-4 ($G_{4,3}(s)$) METHOD-3,6,9,12,14

No.	Time	$G_{4,3}(s)$	$R_{4,3(3)}(s)$	$R_{4,6(3)}(s)$	$R_{4,9(3)}(s)$	$R_{4,12(3)}(s)$	$R_{4,14(3)}(s)$
1.	0	0	0	0	0	0	0
2.	0.2	24.76	15.39	15.69	15.19	27.56	16.81
3.	0.4	45.37	33.70	34.89	33.11	66.13	38.18
4.	0.6	61.94	53.15	55.73	52.23	101.94	61.07
5.	0.8	78.45	72.53	76.82	71.42	130.72	83.53
6.	1.0	95.35	91.00	97.16	89.89	152.15	104.36
7.	1.2	11.18	108.06	116.07	107.13	167.38	122.92
8.	1.4	127.29	123.42	133.14	122.82	177.87	138.96
9.	1.6	141.11	136.99	148.14	136.79	184.94	152.47
10.	1.8	153.10	148.74	161.00	149.01	189.61	163.58
11.	2	163.25	158.78	171.79	159.51	192.66	172.55
12.	3	191.21	188.28	200.19	190.60	197.49	194.72
13.	4	198.01	197.31	204.13	199.55	197.95	198.80
14.	5	198.63	198.89	201.39	200.26	197.99	198.67
15.	6	198.28	198.64	198.97	199.24	197.99	198.23
16.	7	198	198.27	197.98	198.44	197.99	198.04
17.	8	198	198.10	197.81	198.08	198	198
18.	9	197.99	198.01	197.88	197.98	198	197.99
19.	10	198	197.99	197.96	197.97	198	197.99

Fig.19 Comparison of unit step responses
Example 4
G4,3(s)



Original Model

$$G_{44}(s) = \frac{1632\, s^3 + 28440\, s^2 + 160560s + 226080}{s^4 + 18s^3 + 102\, s^2 + 180s + 120}$$

Reduced model

$$R_{4,3(4)}(s) = \frac{1632s + 2509.488}{s^2 + 2s + 1.332}$$

$$R_{4,6(4)}(s) = \frac{1632s + 2456.736}{s^2 + 1.70132s + 1.304}$$

$$R_{4,9(4)}(s) = \frac{1632s + 2242.6428}{s^2 + 1.7855435s + 1.1903624}$$

$$R_{4,12(4)}(s) = \frac{1632s + 13298.823}{s^2 + 5.2941s + 7.0588235}$$

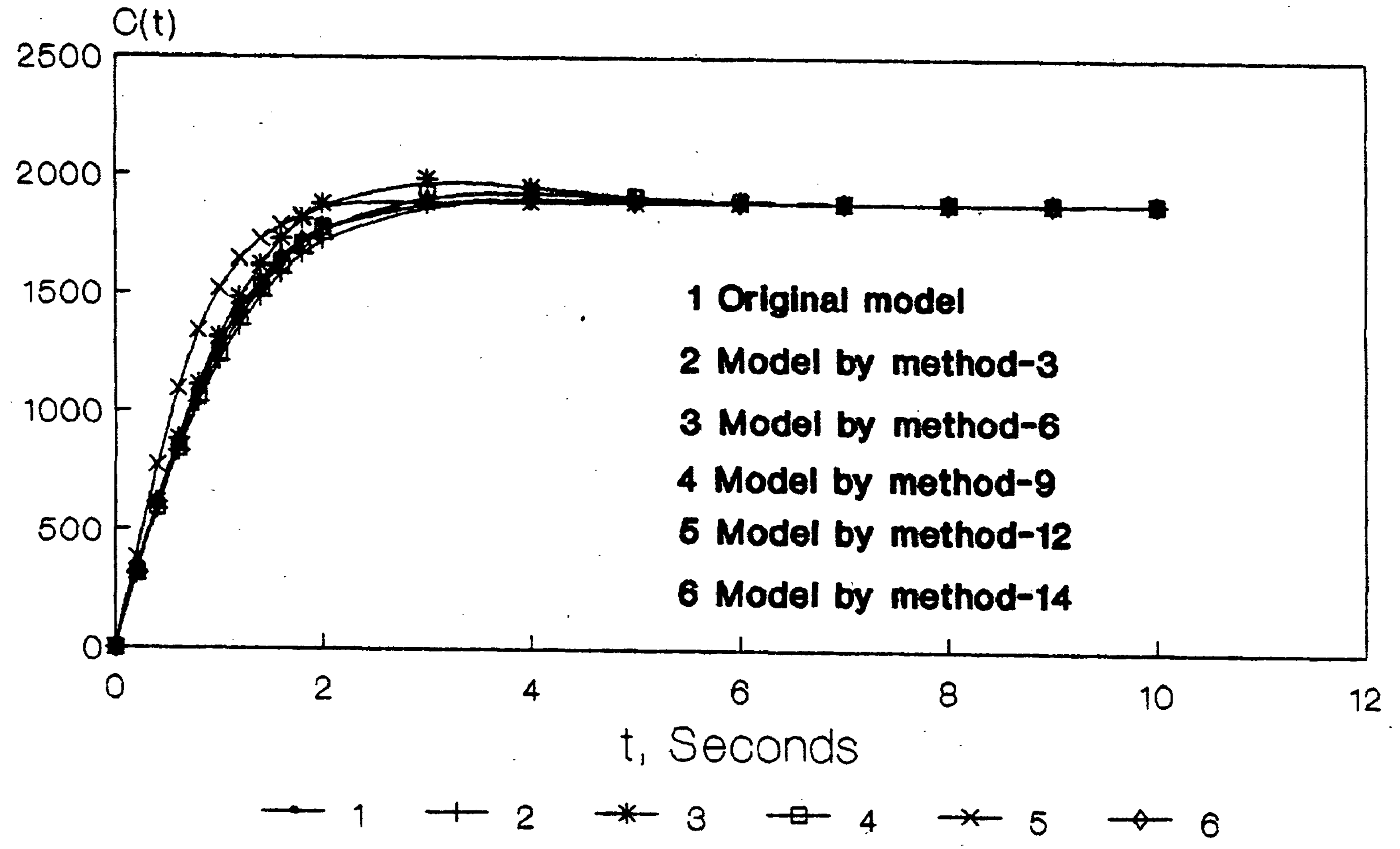
$$R_{4,14(4)}(s) = \frac{1632s + 3603.7431}{s^2 + 2.3933622 + 1.9128148}$$

TIME RESPONSE

TABLE-24 EX-4 ($G_{44}(s)$) METHOD-3,6,9,12,14

No.	Time (sec)	$G_{44}(s)$	$R_{4,3(4)}(s)$	$R_{4,6(4)}(s)$	$R_{4,9(4)}(s)$	$R_{4,12(4)}(s)$	$R_{4,14(4)}(s)$
1.	0	0	0	0	0	0	0
2.	0.2	314.76	310.56	318.09	312.11	380.50	317.55
3.	0.4	608.02	587.53	613.37	593.19	768.35	608.95
4.	0.6	869.68	829.87	879.421	841.33	1092.75	866.53
5.	0.8	1094.10	1038.4	1112.96	1056.55	1339.93	1087.61
6.	1.0	1281.15	1215.15	1313.08	1240.13	1518.26	1272.70
7.	1.2	1433.55	1362.87	1480.61	1394.26	1642.35	1424.32
8.	1.4	1555.2	1484.68	1617.59	1521.64	1726.52	1540.04
9.	1.6	1650.45	1583.78	1726.78	1625.21	1782.52	1641.87
10.	1.8	1723.58	1663.34	1811.40	1708.01	1819.21	1715.88
11.	2	1778.58	1726.31	1874.82	1772.96	1842.97	1771.90
12.	3	1890.05	1877.23	1982.48	1918.34	1880.22	1887.71
13.	4	1896.45	1901.17	1946.55	1925.40	1883.69	1895.97
14.	5	1889.23	1895.51	1904.70	1905.7	1883.97	1889.23
15.	6	1885.29	1888.77	1885.38	1891.44	1883.99	1885.33
16.	7	1884.16	1885.39	1881.08	1885.30	1883.99	1884.14
17.	8	1883.93	1884.21	1881.94	1883.61	1883.99	1883.94
18.	9	1883.96	1883.94	1883.26	1883.52	1883.99	1883.96
19.	10	1883.98	1883.93	1883.91	1883.75	1883.99	1883.98

Fig.20 Comparison of unit step responses
Example 4
G4,4(s)



5.2 FADDEEVA -LEVERRIER AND MODIFIED FADDEEVA LEVERRIER ALGORITHMS FOR DERIVING TRANSFER FUNCTION MIMO SYSTEM

In this section the algorithm due to Leverrier is described with modifications highlighted in [18]. The Leverrier algorithm gives numerical errors when the dimension of matrix A increases. The modified algorithm increases the accuracy.

5.2.1 Faddeeva Leverrier Algorithm

The algorithm widely used to calculate the coefficients of the characteristic polynomial is the algorithm of Leverrier, alternatively called the algorithm of Souriau, Frame or Faddeeva. The algorithm calculates the coefficients a_i of the characteristic polynomial p(s) of matrix A:

$$p(s) = \text{Det } (sI-A) = a_0 s^n + a_1 s^{n-1} + a_2 s^{n-2} + \ldots + a_n \tag{51}$$

and the matrices B_i of the adjoint of (sI-A).

$$\text{adj } (sI-A) = B_0 s^{n-1} + B_1 s^{n-2} + \ldots + B_{n-1} \tag{52}$$

then

$$B_0 = I \qquad a_0 = 1$$
$$a_i = \frac{1}{i} \text{ Trace } (A\, B_{i-1}) \text{ for } i = 1,n \tag{53}$$
$$B_i = A\, B_{i-1} + a_i\, I$$

A nice additional test on the accuracy of this algorithm is given by the equality $B_n = 0$.

Though the method is easy to program but it is a well established fact that nearly all arithmatic operations on a digital computer introduce an error due to the limited accuracy with which the nos. are represented. From equation 53 it can be calculated that these errors will accumulate from a_1 to a_n and from B_1 to B_n, so that a_{i+1} and B_{i+1} will be less accurate than a_i and B_i respectively.

5.2.2 Modified Faddeeva Leverrier Algorithm

Due to the above mentioned deficiency of the ordinary algorithm, the latter coefficients should be obtained in a different manner, Such an approach is possible by using the coefficients b_i of the characteristic polynomial q(s) of the inverse of A and the matrices D_i of the adjoint of $(sI-A^{-1})$,

$$q(s) = \det (sI - A^{-1}) = b_0 s^n + b_1 s^{n-1} + \dots + b_n \quad (54)$$

$$\text{adj } (sI-A^{-1}) = D_0 s^{n-1} + D_1 s^{n-2} + \dots + D_{n-1} \quad (55)$$

Then, the following relations between a_i and b_i and between B_i and D_i can be used.

$$a_n = (-1)^n \det A$$

$$a_{n-i} = a_n b_i, \; B_{n-i} = a_n A^{-1} D_i, \quad \text{for } i=1,n \quad (56)$$

For

$$q(s) = \det (sI-A^{-1}) = \det [(-s\, A^{-1})\, (s^{-1}I-A)]$$
$$= (\det A^{-1})\, (-1)^n\, (a_0+a_1 s+a_2 s^2+\ldots+a_n s^n) \tag{57}$$

Moreover,

$$\mathrm{adj}(sI-A^{-1}) = \det (sI-A^{-1})\, (sI-A^{-1})$$
$$= (\det A^{-1})\, A(-1)^{n-1}\, (I+B_1 s+\ldots+B_{n-1} s^{n-1}) \tag{58}$$

So, from above analysis, it is evident that by using one additional matrix inversion and one determinant evaluation, the same Faddeeva Leverrier algorithm can be used. First to calculate a_i and B_i from A and then b_i and D_i from A^{-1}. Only the first (m-1) elements a_i and B_i of A and the first (n-m) elements b_i and D_i of the A^{-1} need to be calculated. The value of m has to be selected between (n/2) and n. The critical value on average comes as 2n/3 offers good results [18].

The modified algorithm is now

$B_0 = I,\ a_0 = 1.0, \qquad m=2n/3$

$a_i = \frac{1}{i}$ Trace $(A\, B_{i-1})$ for $i=1,\ m-1$

$B_i = A\, B_{i-1} + a_i\, I$

$D_0 = I, \quad a_n = (-1)^n \det A$

$b_i = -\frac{1}{i}$ Trace $(A^{-1}\, D_{i-1})$

$$D_i = A^{-1} D_{i-1} + b_i I \quad \text{for } i = 1, n-m \tag{59}$$

$$a_{n-i} = a_n b_i$$

$$B_{n-i} = -a_n A^{-1} D_i$$

5.2.3 Transfer Function

$$\dot{X} = AX + Bu$$

$$Y = cx$$

$$[G(s)] = \frac{C[\sum_{i=0}^{n-1} s^{n-i-1} B_i] B}{\sum_{i=0}^{n} a_i s^{n-i}} \tag{60}$$

$$= \frac{C [\sum_{i=0}^{n-1} s^{n-i-1} B_i] B}{\Delta(s)}$$

where,

$$\Delta(s) = \sum_{i=0}^{n} a_i s^{n-i}$$

The B_i and a_i are calculated from (53). If the determinant value of A is non zero then the modified algorithm can be applied to calculate a_i and B_i from (59).

5.3 Comparative study of reduction methods

Some of the methods for model order reduction have been described in previous chapter. In this chapter these methods are applied to some typical high order SISO systems, and one MIMO system and a comparison is made with the resultant second order reduced models. The time responses to unit step input of all the reduced models and frequency responses of a few reduced approximants, are calculated and the results are depicted in the form of compact tables and graphs.

For the purpose of comparison of various models an error index is chosen as

$$J = \sum_{i=0}^{N} [y(t_i) - y_r(t_i)]^2$$

where y and y_r are the outputs of a original system G(s) and the reduced model R(s) respectively. N is the number of sampling periods and t_i is the ith sampling instant. Error index J is known as cummulative error, and it has been dipicted in the form of tables for all the four examples as shown below.

COMPARATIVE STUDY

TABLE-25 EX-1

Method No.	Steady state value	Out put Y_r at time t=10	Cummulative error (J)
1.	9.999	.842408E-4	.2112E+01
2.	9.999	.2853E-7	.20424E+01
3.	9.999	.14784E-9	.27515E+02
4.	9.999	.31464E-5	.12138E+02
5.	9.999	.16921E-5	.13049E+02
6.	9.999	.19835E-6	.84901E+02
7.	9.999	.33305E-5	.53186E+01
8.	9.999	.33305E-5	.53186E+01
9.	9.999	.33785E-5	.52135E+02
10.	9.999	.83819E-8	.33086E+02
11.	9.999	.35926E-9	.10998E+02
12.	9.999	.35926E-9	.21691E+02
13.	9.999	.48633E-8	.33659E+00
14.	9.999	.26400E-7	.40940E+01

Y_r = unit step output of reduced model.

COMPARATIVE STUDY

TABLE-26 EX-2

Method No.	Steady state value	Out put Yr at time t=10sec	Cummulative error(J)
1.	.9986	1.0022	.13018E+02
2.	.9986	1.0023	.13021E+02
3.	.9986	1.0117	.25358E+03
4.	.9986	.9943	.15096E+02
5.	.9986	.9952	.15316E+02
6.	.9986	.9445	.77295E+02
7.	.9986	1.0010	.14297E+02
8.	.9986	1.0010	.14297E+02
9.	.9986	.9947	.33331E+03
10.	.9986	1.000	.98865E+01
11.	.9986	1.000	.11090E+01
12.	.9986	1.000	.23409E+02
13.	.9986	1.001	.18327E+01
14.	.9986	1.0003	.13450E+02

Y_r = unit step output of reduced model.

COMPARATIVE STUDY

TABLE-27 EX-3

Method No.	Steady state value	Out put Yr at time t=10sec	Cummulative error (J)
1.	20.257	20.423	.40168E+03
2.	20.257	20.423	.40163E+03
3.	20.257	20.549	.43489E+04
4.	20.257	20.007	.58271E+03
5.	20.257	20.071	.63264E+03
6.	20.257	19.671	.49861E+04
7.	20.257	20.539	.66995E+03
8.	20.257	20.539	.66994E+03
9.	20.257	20.483	.86415E+04
10.	20.257	20.258	.87289E+01
11.	20.257	20.258	.53147E+00
12.	20.257	20.258	.67084E+00
13.	20.257	20.258	.46538E+01
14.	20.257	20.257	.16911

Y_r = unit step output of reduced model.

Fig. 21 Comparison of frequency responses (Example 1)

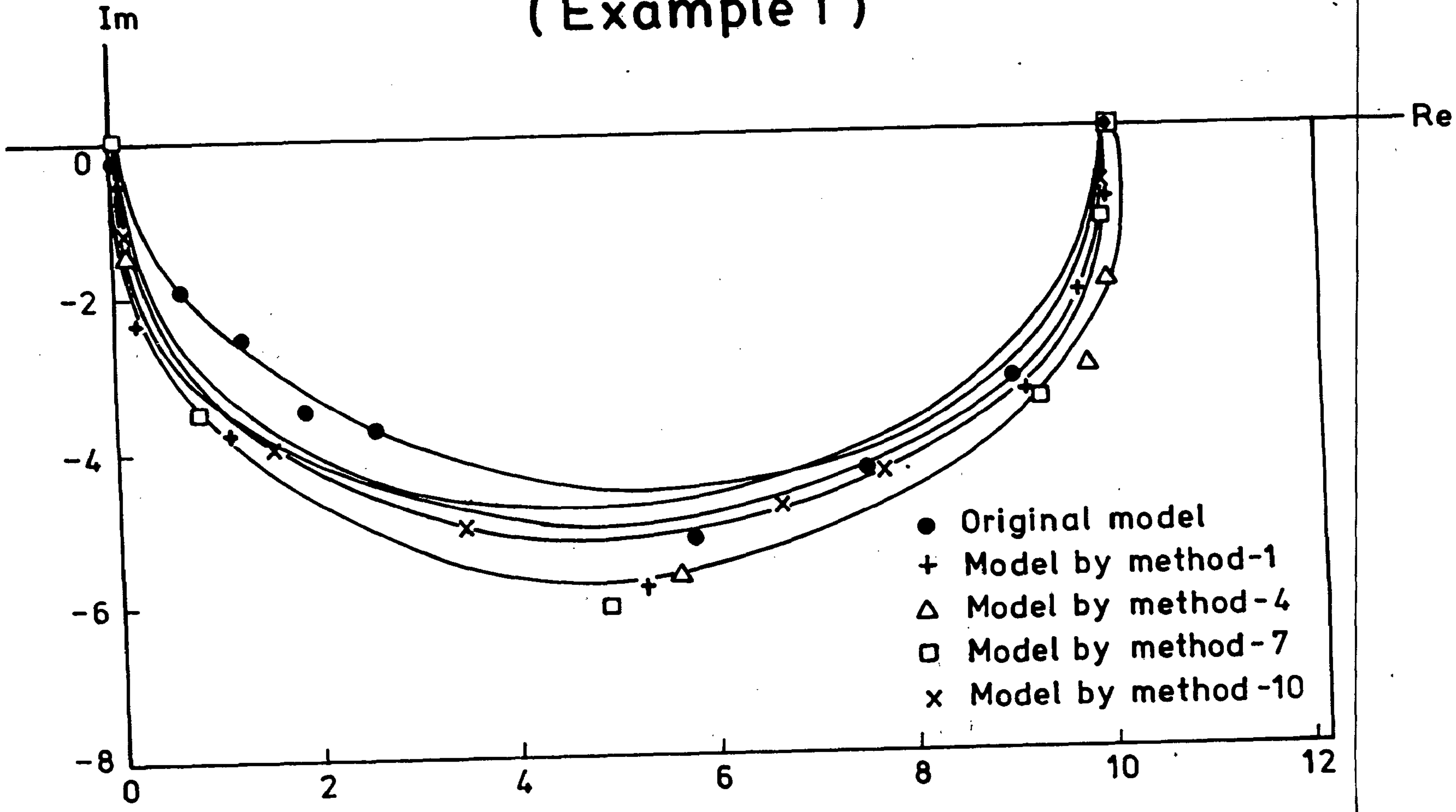


Fig.22 Comparison of frequency responses (Example 2)

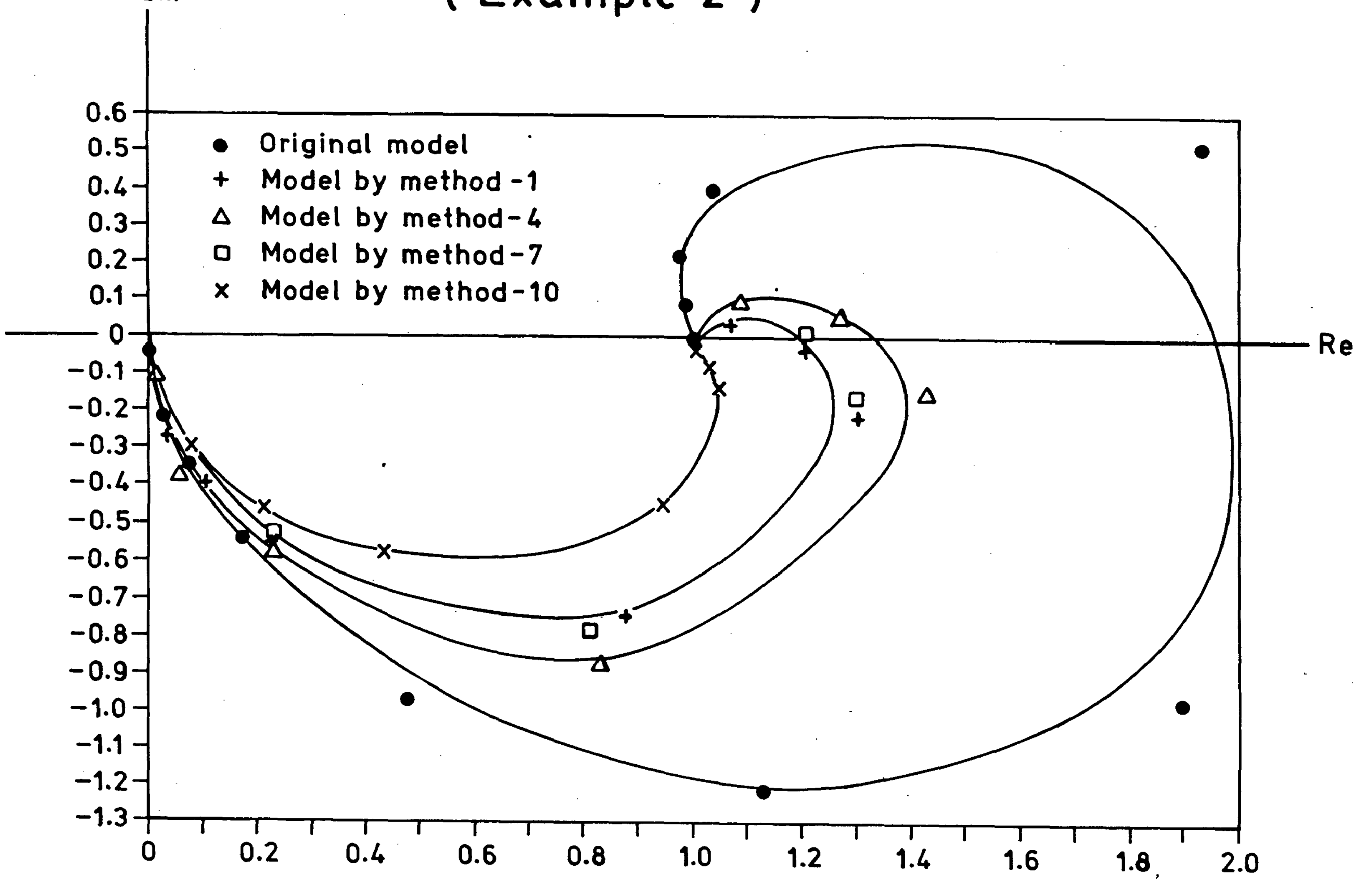
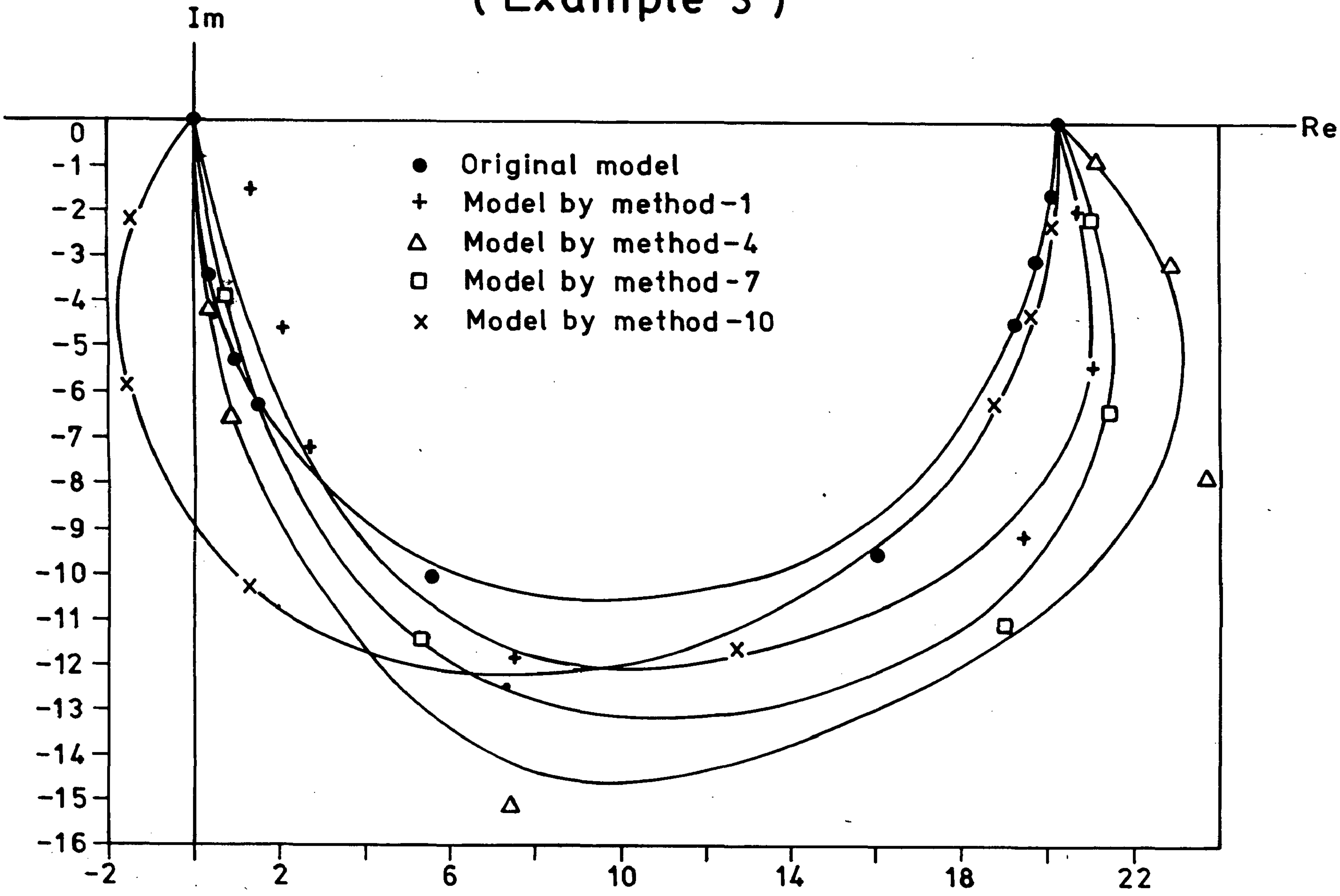


Fig.23 Comparison of frequency responses (Example 3)



COMPARATIVE STUDY

TABLE-28, EX - 4 ($G_{41}(s)$)

No. of Method	Steady state value	Output Yr at time t=10sec	Cummulative error (J)
3.	110.00	109.993	.16646E+04
6.	110.00	11.001	.51365E+04
9.	110.00	10.998	.31451E+04
12.	110.00	11.000	.13122E+04
14.	110.00	10.999	.13361E+04

Y_r = unit step output of reduced model.

COMPARATIVE STUDY

TABLE - 29 EX -4 (G_{42} (s))

No. of method	Steady state value	Output Yr at time t=10sec	Cummulative error (J)
3.	567.997	567.972	.88324E+03
6.	567.997	568.03	.23691E+05
9.	567.997	567.924	.61085E+04
12.	567.997	567.99	.39528E+04
14.	567.997	567.997	.41163E+03

Y_r = unit step output of reduced model.

COMPARATIVE STUDY

TABLE -30 EX-4 ($G_{4,3}(s)$)

No. of method	Steady state value	Output yr at time t=10 sec.	Cummulative error (J)
3	198.00	197.997	.52751E+03
6	198.00	197.964	.12696E+04
9	198.00	197.970	.56014E+03
12	198.00	198.00	.18906E+05
14	198.00	197.998	.99440E+03

Y_r = unit step output of reduced model.

COMPARATIVE STUDY

TABLE 31 EX-4 ($G_{44}(s)$)

No. of method	Steady state value	Output Yr at time t=10 sec.	Cummulative error(J)
3	1883.985	1883.935	.35060E+05
6	1883.985	1883.918	.10819E+06
9	1883.985	1883.750	.17332E+05
12	1883.985	1893.999	.30354E+06
14	1883.985	1883.987	.56773E+03

Y_r = unit step output of reduced model.

From the above we find that the steady state value of all the methods match with original upto third place of decimal except a few ones. The various methods presented here in are algebraic in nature and require simple calculations that can be easily automated. These methods do not require finding the eigen values and eigenvectors of high order system. The solution of high order non-linear equations is not required. Time response of all models require almost same computational time (C.P.U.).

Method 1,2,3 are Routh approximation method, Routh method mixed with Cauer 2nd form and Routh method mixed with Cauer third form respectively. Routh approximation method gives stable reduced order transfer function if the original system is stable. This method can be applied to both SISO and MIMO systems without any modification. Models reduced by method-3 shows an overshoot in transient region which in some cases, is a little bit excessive. Also it stabilizes slower as compared to method 1 and 2 which more or less, give similar response to unit step inputs. Error analysis shows that method 1 and 2 gives nearly same commulative error which is the least also in all the three methods.

Method 4,5,6 are Routh stability array method, Routh stability array mixed with Cauer 2nd form and Routh stability array mixed with Cauer 3rd form. Method 2 is computationally straightforward and simpler than method 1. Model reduced by Method 6 shows overshoot and slow convergence, while model by method 5 converges faster than method 1. However all the three reduced models show good overall time response. Cummulative error is the maximum for model by method 6 and the minimum for the model by method 4.

Method No.7 based on stability equation gives better results. Method-8 and 9 which are actually mixed form of Cauer 2nd and Cauer 3rd form respectively with method 7 also give good results. Models reduced by method 7 and 8 responds exactly to unit step excitation. Cummulative error index shows the least value for models by method 7 and 8 both. Step responses show no undershoot.

Method 10 i.e. polynomial differentiation is computationally simplest of the methods discussed. Method 11 and 12 are its mixed forms with Cauer 2nd and 3rd forms respectively. Model by method 10 and 11 gives faster stabilization even than that of original system. Error analysis shows that the best method among the three is method-11. These methods are equally applicable to unstable and non minimum phase systems also.

Method 13 and 14 employ the mixed form of dominght pole retention and Cauer 2nd and 3rd forms respectively.

Method 13 gives the best unit step response among the two and error analysiswise also method 13 is the best.

In case of example 1 cummulative error is the least for the model reduced by method No.13.

For example 2, model reduced by method No.11 seems to be the best one and for example 3, model reduced by method 14 shows minimum value of cummulative error.

CHAPTER - 6

APPLICATION OF REDUCTION METHODS IN A POWER SYSTEM

This chapter is devoted on

1. Developing a power system model connected to an infinite bus,
2. to derive transfer function from given power system model (81) by [18]
3. to find out reduced order model by applications of reduction techniques enumerated in chapter-4.

6.1 MODEL FOR SINGLE MACHINE POWER SYSTEM CONNECTED TO AN INFINITE BUS

The development of this model is based on [19] and taken from [20]. The single machine power system is connected to an infinite bus and shown in Fig. (24). In this power system, generator is provided with a double time constants speed governor.

MODEL DEVELOPMENT

The electro mechanical oscillation of synchronous generator about a steady state operating point oo can be given by

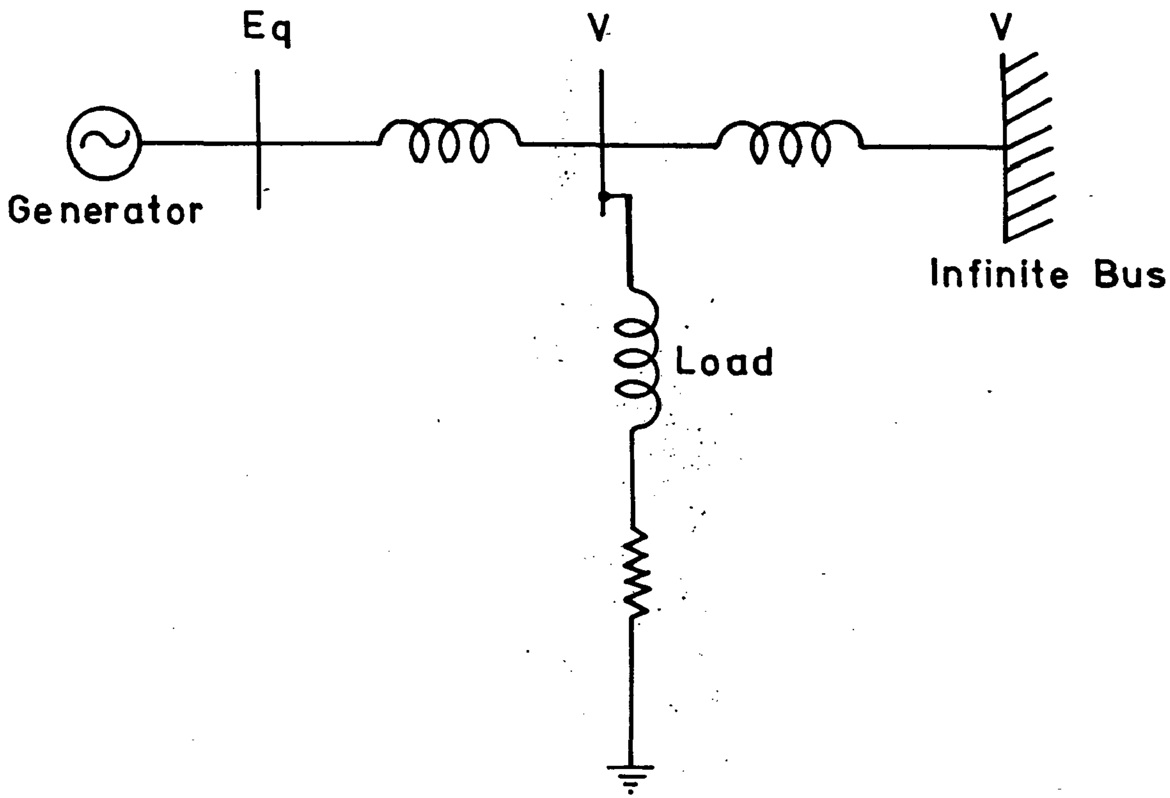
$$M \dot{\Delta} \dot{\delta} + D \Delta \dot{\delta} + \Delta p = p \qquad (61)$$

where,

$$p = c_1 \Delta \delta + b_1 \Delta E_q \qquad (62)$$

$$c_1 = \frac{\partial p}{\partial \delta} = -E_q V Y_{12} \sin (\delta_o = \theta_{12}) \qquad (63)$$

Fig. 24 Single machine power system connected to an infinite bus



$$b_1 = \frac{\partial P}{\partial E_q} = -E_q Y_{11} \cos \theta_{11} + VY_{12} \cos (\delta_0 - \theta_{12}) \tag{64}$$

and

$$p = E_q^2 Y_{11} \cos \theta_{11} + E_q V Y_{12} \cos (\delta_0 - \theta_{12}) \tag{65}$$

The electro magnetic oscillation of the power system can be expressed by

$$\Delta E_q + p T_{d0} \Delta E_q' = \Delta E_{ex} \tag{66}$$

where

$$\Delta E_q' = E_q - (x_d - x_d') I_d \tag{67}$$

and

$$I_q = E_q Y_{11} \cos \theta_{11} + V Y_{12} \cos (\delta_o - \theta_{12}) \tag{68}$$

$$I_d = -E_q Y_{11} \sin \theta_{11} - V Y_{12} \cos (\theta_{12} - \delta_0) \tag{69}$$

From equations (67) and (69) $\Delta E_q'$ can be given by

$$\Delta E_q' = - (x_d - x_d') V Y_{12} \cos (\theta_{12} - \delta_0) \Delta o + 1 + (x_d - x_d') Y_{11} \sin \theta_{11} \Delta E_q$$

$$E_q' = c_2 \Delta \delta + b_2 \Delta E_q \tag{70}$$

where

$$c_2 = -(x_d - x_d') V Y_{12} \cos (\theta_{12} - \delta_0) \tag{71}$$

$$b_2 = 1 + (x_d - x_d') Y_{11} \sin \theta_{11} \tag{72}$$

The terminal voltage V_t is given by

$$V_t^2 = V_d^2 + V_q^2$$

$$V_t = (V_d + V_q)^{/2} \tag{73}$$

where,

$$V_q = E_q - x_d I_d \tag{74}$$

$$V_d = x_q I_q \tag{75}$$

Substituting for I_d and I_q in the above equation we get

$$V_t = \frac{1}{V_t} [V_q \frac{\partial V_q}{\partial \delta} + V_d \frac{\partial V_d}{\partial \delta}] \Delta\delta + \frac{1}{V_t} [V_q \frac{\partial V_q}{\partial E_q} + V_d \frac{\partial V_d}{\partial E_q}] \Delta E_q$$

$$V_t = c_3 \Delta \delta + b_3 \Delta E_q \tag{76}$$

where

$$c_3 = \frac{1}{V_t} [V_q \frac{\partial V_q}{\partial \delta} + V_d \frac{\partial V_d}{\partial \delta}] \tag{77}$$

$$b_3 = \frac{1}{V_t} [V_q \frac{\partial V_q}{\partial E_q} + V_d \frac{\partial V_d}{\partial E_q}] \tag{78}$$

The governor output 'p' in equation (61) can be given by

$$a \frac{d^2 p}{dt^2} + b \frac{dp}{dt} + p = - c\Delta\delta \tag{79}$$

Defining

$$w = \Delta \dot{\delta}$$

$$p_1 = \dot{p}$$

$$U = \frac{\Delta E_{ex}}{T_{d0}} \tag{80}$$

We get the following state equations from (61), (62), (66), (70) and (79).

$$\begin{bmatrix} \Delta\dot{E}_q \\ \Delta\dot{\delta} \\ \dot{w} \\ \dot{p} \\ \dot{p}_1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{T_{do}b_2} & 0 & -\frac{c_2}{b_2} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ -b_1/M & -c_1/M & -D/M & 1/M & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -c/a & -1/a & -b/a \end{bmatrix} \begin{bmatrix} \Delta E_q \\ \Delta\delta \\ w \\ p \\ p_1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} u \qquad (81)$$

6.2 SYSTEM MATRIX A,B,C

The values of described parameters of A are taken from [20] as

$M = 1.000$ $\qquad D = 0.50$

$E_q = 1.482$ $\qquad V_o = 1.00$

$P_o = 2.105$ $\qquad \delta_o = 60^o$

$T_{do} = 5.0$ sec $\qquad y_{11} = 0.266 - j\,1.530$

$x_d' = 0.084$ $\qquad y_{12} = 0.180 + j\,1.080$

$x_d = 0.320$ $\qquad a = T_1 T_2 = 0.05$

$T_1 = 0.100$ sec $\qquad b = T_1 + T_2 = 0.6$ sec

$T_2 = 0.500$ sec $\qquad c = 0.05$

with the help of these values the matrix A is found as

$$A = \begin{bmatrix} -0.183 & 0.0 & 0.227 & 0.0 & 0.0 \\ 0.0 & 0.0 & 1.0 & 0.0 & 0.0 \\ -1.815 & -0.57 & -0.50 & 1.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 1.0 \\ 0.0 & 0.0 & -1.0 & -20.0 & -12.0 \end{bmatrix} \qquad (82)$$

$$B^T = [1.0 \quad 0.0 \quad 0.0 \quad 0.0 \quad 0.0]$$

and

$$C^T = [1.0 \quad 0.0 \quad 0.0 \quad 0.0 \quad 0.0]$$

The state vector

$$X = [\Delta E_q \quad \Delta\delta \quad w \quad p \quad p_1]$$

Using the set of equations from (53) we get the transfer function of the power system with given matrices A,B and C as

$$G(s) = \frac{11.4+17.8s + 26.57s^2 + 12.5s^3 + s^4}{1+12.6881s+29.332s^2+27.7791s^3+22.9928s^4+2.1432s^5} \tag{83}$$

This open loop transfer function can be written in the form given below taking the coefficient of s^5 as unity.

$$G(s) = \frac{5.31915 + 8.3240s + 12.39735s^2+5.83240s^3+ 0.46659s^4}{0.46659+5.92012s+13.68608s^2+12.96146s^3+10.72822s^4+s^5} \tag{84}$$

6.3 APPLICATION OF METHODS

Method -1

2nd order reduced model by this method is given as

$$R_{5,1}(s) = \frac{0.65726860 + 0.4200035}{s^2+0.4674566s + 0.0368422}$$

Method -2

$$R_{5,2}(s) = \frac{0.6572756s + 0.4200035}{s^2 + 0.4674566s + 0.0368422}$$

Method -3

$$R_{5,3}(s) = \frac{0.46659s + 0.4200029}{s^2 + 0.4674566 + 0.0368422}$$

Method -4

$$R_{5,4}(s) = \frac{0.6850248s + 0.6415572}{s^2 + 0.6294779s + 0.0562767}$$

Method -5

$$R_{5,5}(s) = \frac{0.0399463s + 0.6415572}{s^2 + 0.6294779s + 0.0562767}$$

Method -6

$$R_{5,6}(s) = \frac{0.46659s + 0.6415583}{s^2 + 0.6294779s + 0.0562767}$$

Method -7

$$R_{5,7}(s) = \frac{0.6253949s + 0.3996359}{s^2 + 0.4447877s + 0.0350556}$$

Method -8

$$R_{5,8}(s) = \frac{0.6253978s + 0.3996358}{s^2 + 0.4447877s + 0.0350556}$$

Method -9

$$R_{5,9}(s) = \frac{0.46659s + 0.3996359}{s^2 + 0.4447877s + 0.0350556}$$

Method -10

$$R_{5,10}(s) = \frac{1.5205206s + 3.8865341}{s^2 + 1.7302602s + 0.340923}$$

Method -11

$$R_{5,11}(s) = \frac{-23.505486s + 3.8865397}{s^2 + 1.7302602s + 0.340923}$$

Method -12

$$R_{5,12}(s) = \frac{0.46659s + 3.3865341}{s^2 + 1.7302602s + 0.340923}$$

Method-13

$$R_{5,13}(s) = \frac{0.4860036s + 0.5878026}{s^2 + 0.6161566s + 0.0515614}$$

Method-14

$$R_{5,14}(s) = \frac{0.46659s + 0.5878026}{s^2 + 0.6161566s + 0.0515614}$$

TIME RESPONSE

TABLE-32 EX-5 ,(POWER SYSTEM MODEL) METHOD-1,2,3

No.	Time (sec)	$G_5(s)$	$R_{5,1}(s)$	$R_{5,2}(s)$	$R_{5,3}(s)$
1.	0	0	0	0	0
2.	2	1.5536	1.4558	1.455	1.214
3.	4	2.9739	2.9680	2.9680	2.654
4.	6	4.260	4.359	4.359	4.046
5.	8	5.556	5.571	5.571	5.289
6.	10	6.621	6.598	6.598	6.354
7.	12	7.460	7.456	7.456	7.250
8.	14	8.171	8.165	8.165	7.994
9.	16	8.761	8.75	8.75	8.608
10.	18	9.237	9.23	9.23	9.114
11.	20	9.627	9.624	9.624	9.528
12.	25	10.324	10.324	10.324	10.266
13.	30	10.747	10.748	10.748	10.713
14.	35	11.003	11.005	11.005	10.984
15.	40	11.159	11.161	11.161	11.148
16.	45	11.254	11.255	11.255	11.247
17.	50	11.311	11.312	11.312	11.307

TIME RESPONSE

TABLE-33 EX-5 (POWER SYSTEM MODEL) METHOD -4;5,6

No.	Time (sec)	$G_5(s)$	$R_{1,4}(s)$	$R_{1,5}(s)$	$R_{5,6}(s)$
1.	0	0	0	0	0
2.	2	1.5536	1.615	0.9079	1.375
3.	4	2.9739	3.303	2.484	3.026
4.	6	4.260	4.801	4.053	4.547
5.	8	5.556	6.055	5.422	5.841
6.	10	6.621	7.083	6.762	6.907
7.	12	7.460	7.918	7.494	7.774
8.	14	8.171	8.593	8.249	8.476
9.	16	8.761	9.137	8.86	9.043
10.	18	9.237	9.576	9.353	9.5
11.	20	9.637	9.930	9.750	9.869
12.	25	10.324	10.054	10.438	10.407
13.	30	10.747	10.9	10.839	10.879
14.	35	11.003	11.108	11.073	11.096
15.	40	11.159	11.230	11.209	11.223
16.	45	11.254	11.301	11.288	11.296
17.	50	11.311	11.342	11.335	11.339

TIME RESPONSE

TABLE-34 EX-5 (POWER SYSTEM MODEL) METHOD-7,8,9

No.	Time (sec)	$G_5(s)$	$R_{5,7}(s)$	$R_{5,8}(s)$	$R_{5,9}(s)$
1.	0	0	0	0	0
2.	2	1.5536	1.409	1.409	1.204
3.	4	2.9739	2.905	2.905	2.634
4.	6	4.260	4.3	4.3	4.027
5.	8	5.556	5.525	5.525	5.276
6.	10	6.621	6.567	6.567	6.351
7.	12	7.460	7.439	7.439	7.256
8.	14	8.171	8.161	8.161	8.008
9.	16	8.761	8.755	8.755	8.629
10.	18	9.237	9.341	9.241	9.138
11.	20	9.627	9.640	9.640	9.555
12.	25	10.324	10.344	10.344	10.293
13.	30	10.747	10.767	10.767	10.776
14.	35	11.003	11.00	11.00	11.002
15.	40	11.159	11.172	11.172	11.161
16.	45	11.254	11.263	11.263	11.257
17.	50	11.311	11.318	11.318	11.314

TIME RESPONSE

TABLE-35 EX-5 (POWER SYSTEM MODEL) METHOD-10,11,12

No.	Time (sec)	$G_5(s)$	$R_{5,10}(s)$	$R_{5,11}(s)$	$R_{5,12}(s)$
1.	0	0	0	0	0
2.	2	1.5536	3.667	-7.817	3.184
3.	4	2.9739	6.462	-1.402	6.131
4.	6	4.260	8.261	3.235	8.050
5.	8	5.556	9.405	6.211	9.271
6.	10	6.621	10.132	8.102	10.04
7.	12	7.460	10.594	9.005	10.54
8.	14	8.171	10.888	10.068	10.85
9.	16	8.761	11.075	10.554	11.053
10.	18	9.237	11.193	10.862	11.179
11.	20	9.627	11.268	11.058	11.259
12.	25	10.324	11.357	11.290	11.354
13.	30	10.747	11.386	11.364	11.385
14.	35	11.003	11.395	11.388	11.395
15.	40	11.159	11.398	11.396	11.398
16.	45	11.254	11.399	11.398	11.399
17.	50	11.311	11.399	11.399	11.399

TIME RESPONSE

TABLE-36 EX-5 (POWER SYSTEM MODEL) METHOD-13,14

No.	Time (sec)	$G_5(s)$	$R_{5,13}(s)$	$R_{5,14}(s)$
1.	0	0	0	0
2.	2	1.5536	1.338	1.317
3.	4	2.9739	2.902	2.876
4.	6	4.280	4.348	4.325
5.	8	5.556	5.592	5.572
6.	10	6.621	6.632	6.615
7.	12	7.460	7.491	7.477
8.	14	8.171	8.197	8.186
9.	16	8.761	8.776	8.767
10.	18	9.237	9.251	9.244
11.	20	9.627	9.640	9.634
12.	25	9.324	10.332	10.328
13.	30	10.747	10.751	10.749
14.	35	11.003	11.006	11.005
15.	40	11.159	11.161	11.160
16.	45	11.254	11.255	11.254
17.	50	11.311	11.312	11.311

Fig.25 Comparison of unit step responses
Model of power system

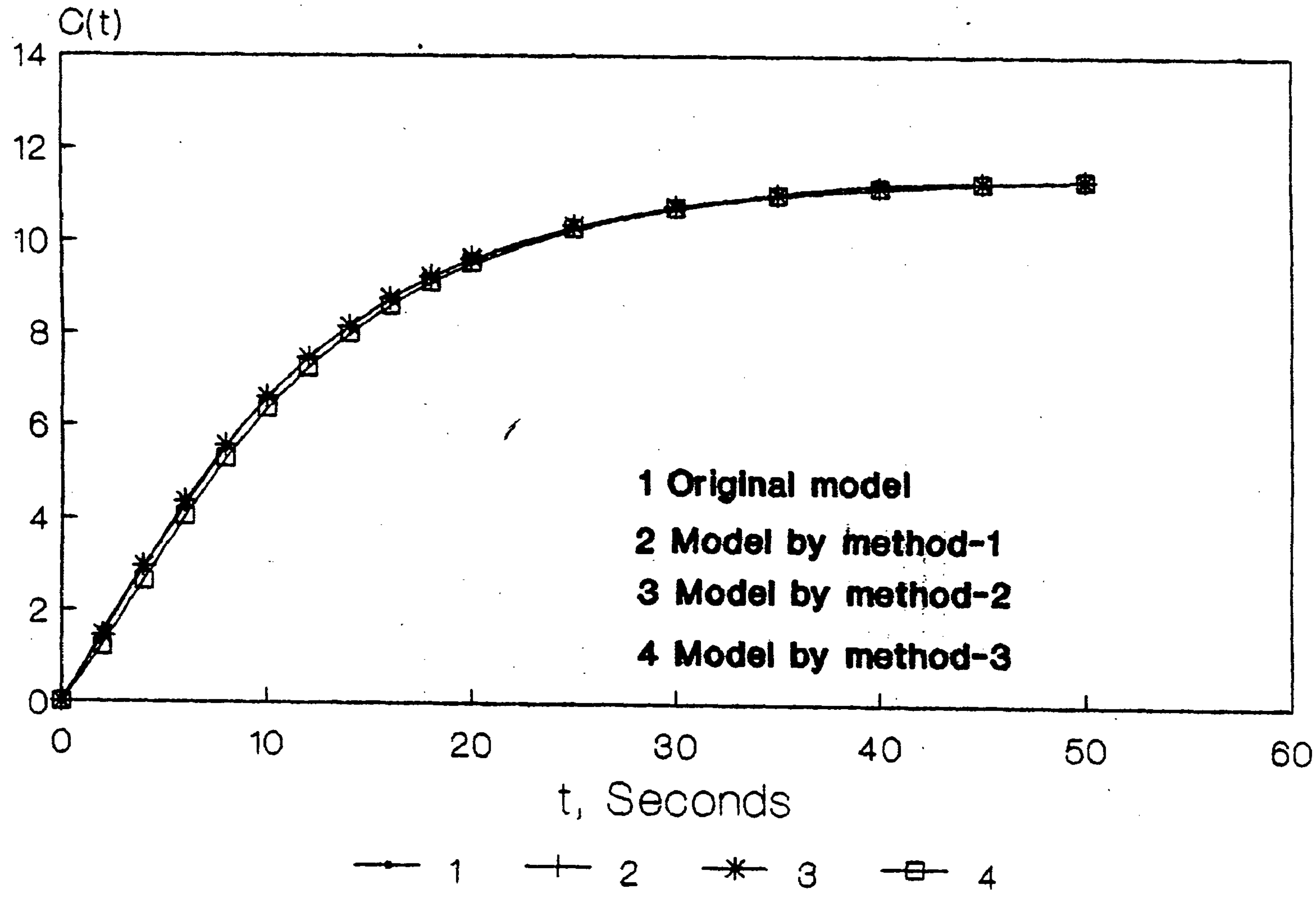


Fig.26 Comparison of unit step responses
Model of power system

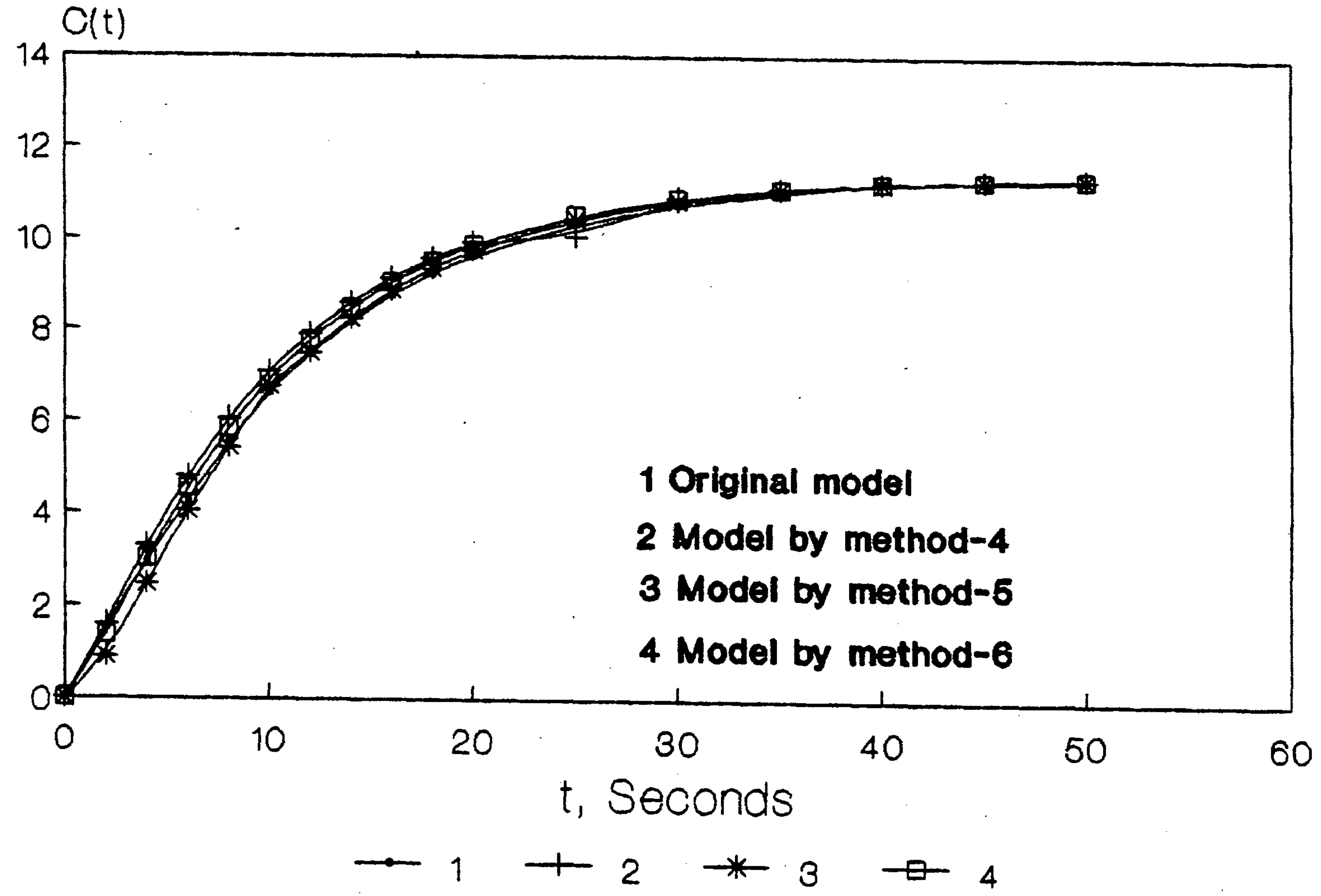


Fig.27 Comparison of unit step responses
Model of power system

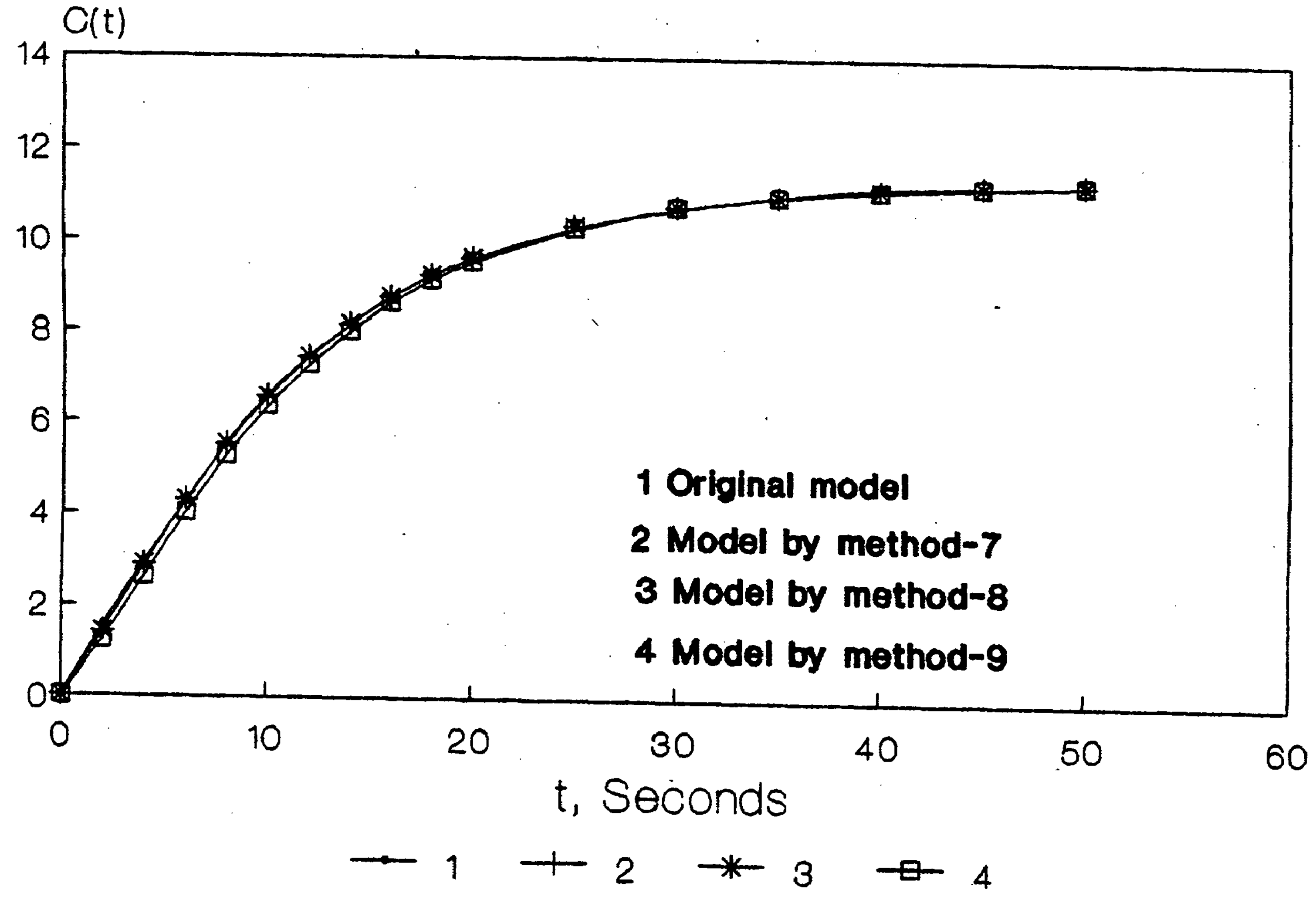


Fig.28 Comparison of unit step responses Model of power system

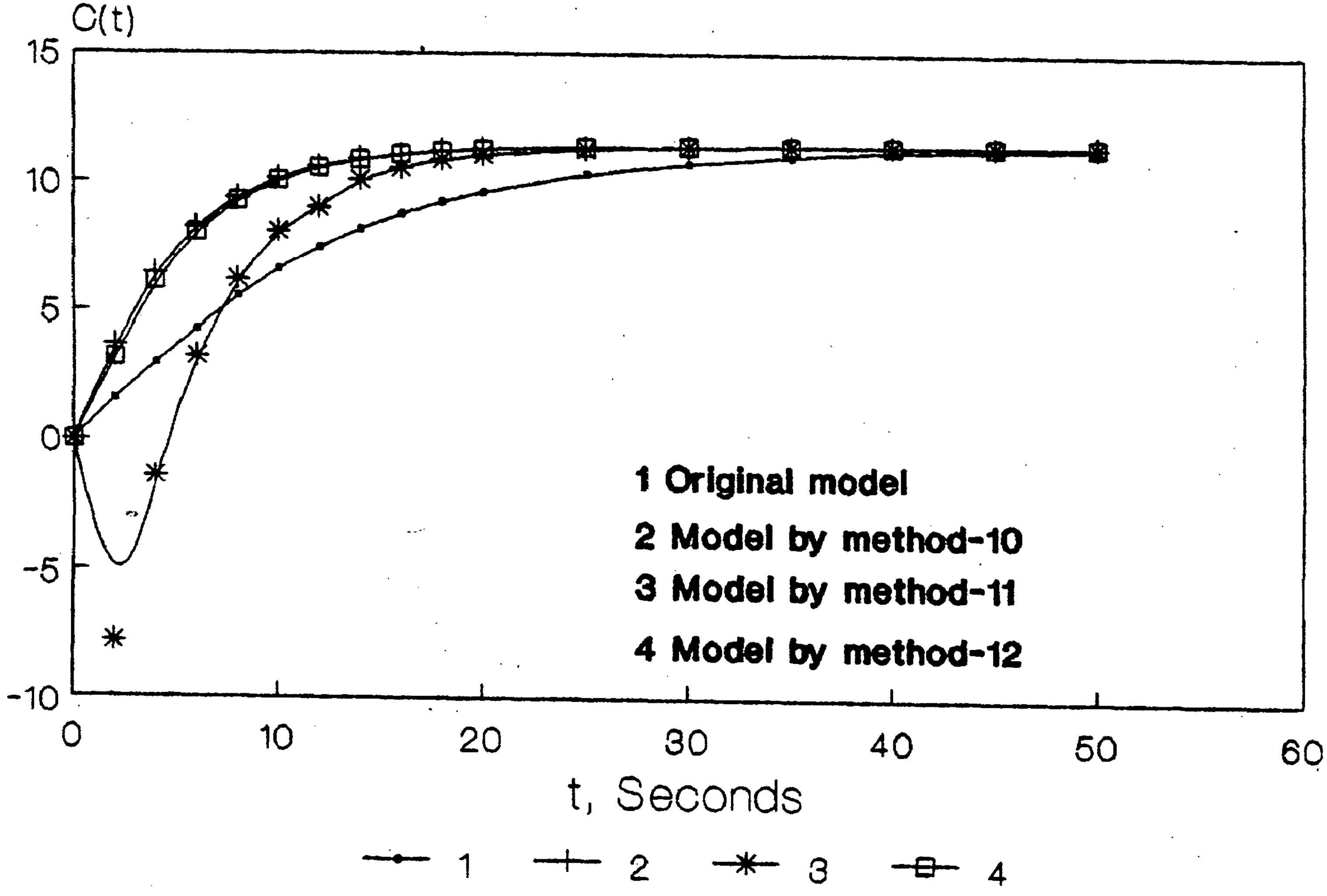
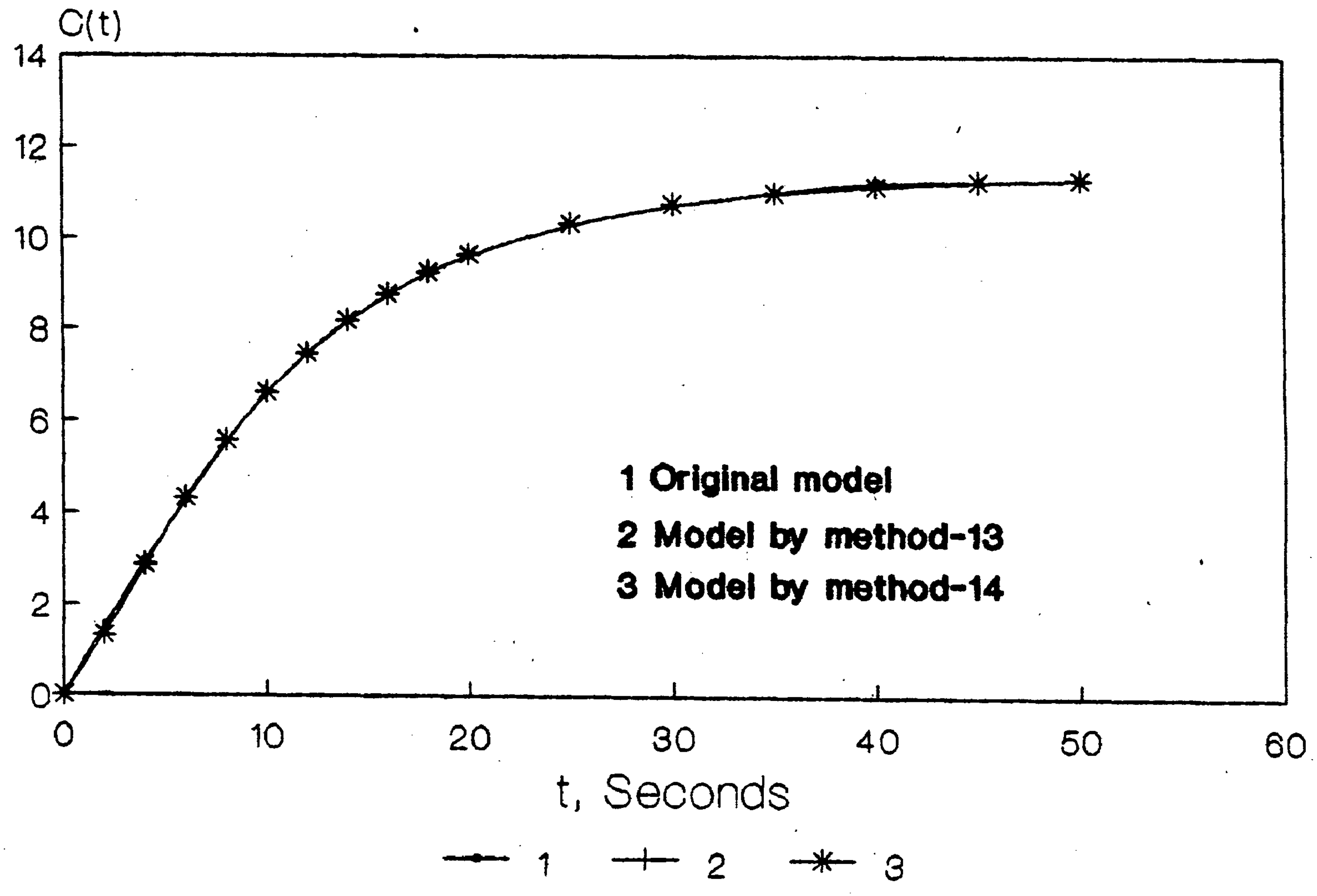


Fig.29 Comparison of unit step responses Model of power system



6.4 COMPARATIVE STUDY

From the above we find that the steady state values of all the methods match with original mostly upto second place of decimal. Method 1 and 2 both gives the least error in the described problem. Time response of each method and model requires almost same computation time (CPU) about 0.9 seconds.

Since the transient response of synchronous machines in electrical power system is equally important as steady state response, therefore the best reduced model will be that which gives good transient and steady state responses.

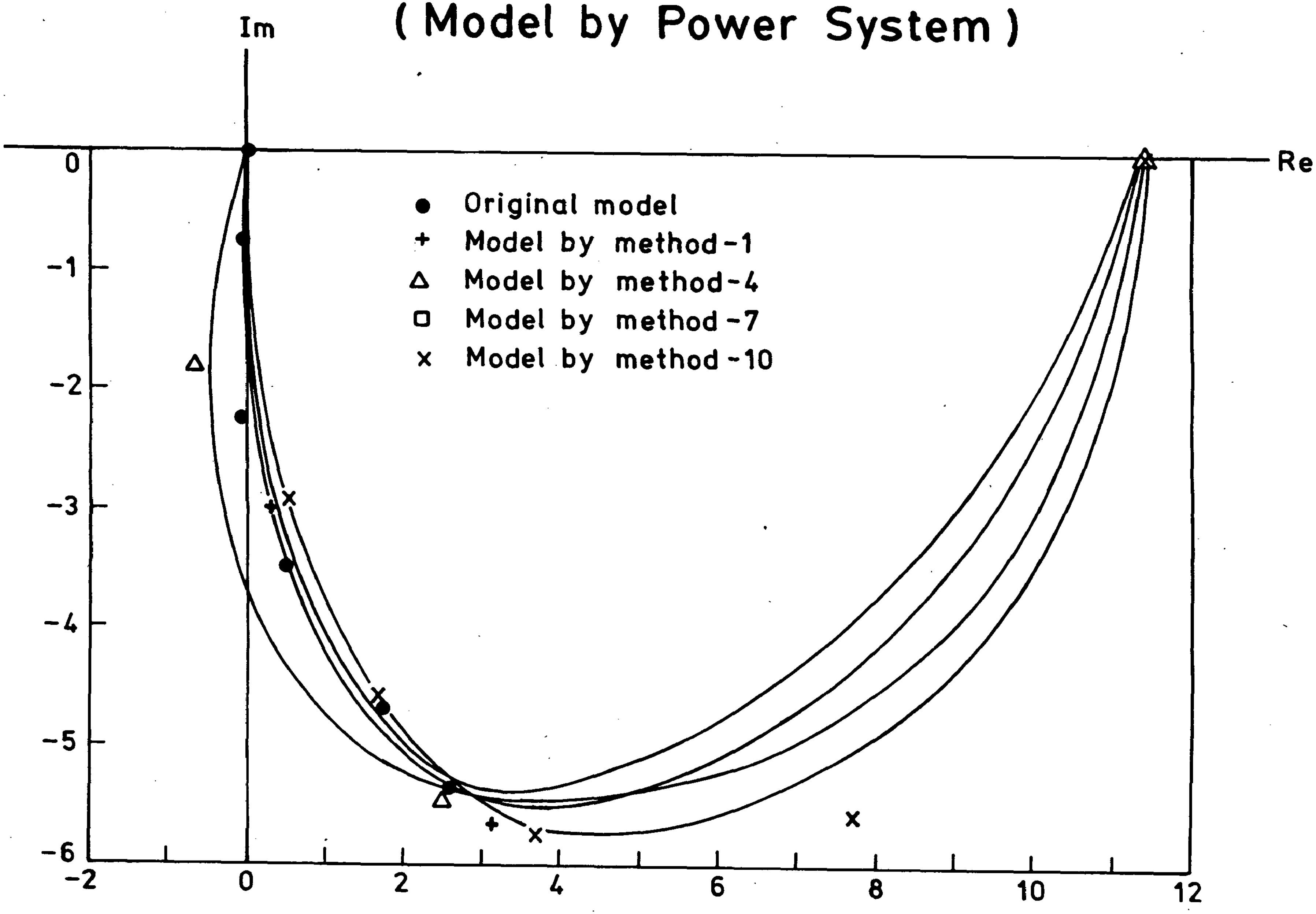
On the basis of unit step responses shown in respective figures, models reduced by method 13 and 14 are the best. However models reduced by method 1, 2, 3, 7, 8 and 9 are also equally good.

COMPARATIVE STUDY

TABLE-37 EX-5 (POWER SYSTEM MODEL)

No of Method	Steady state value	Output Yr at time t=50sec	Cummulative error (J)
1.	11.311	11.312	.40190E-01
2.	11.311	11.312	.40190E-01
3.	11.311	11.307	.10855E+01
4.	11.311	11.342	.37771E+01
5.	11.311	11.335	.17039E+01
6.	11.311	11.339	.17602E+01
7.	11.311	11.318	.67587E-01
8.	11.311	11.318	.67587E-01
9.	11.311	11.314	.10872E+01
10.	11.311	11.399	.18951E+03
11.	11.311	11.399	.30848E+03
12.	11.311	11.399	.17338E+03
13.	11.311	11.312	.12031E+00
14.	11.311	11.311 /	.13462E+00

Fig. 30 Comparison of frequency responses
(Model by Power System)



CHAPTER - 7

CONCLUSION

The development of reduced order models for the analysis and synthesis of high order systems has been an area of active research during the past decade. The present work deals with the applications of methods, with special emphasis on CFE methods for model order reduction to three well known SISO models, namely Hutton's model, Chuang's model and Sheshadri's model and one MIMO model . Moreover these reduction techniques have also been applied to reduce a power system model of a single machine system connected to an infinite bus. The work included here in deals with frequency domain model reduction techniques.

Infact in project work, different models were reduced by pure CFE techniques. This dissertation is an extension of project work, therefore mixed CFE methods have been taken up in this thesis.

The Second Chapter describes in brief CFE based reduction techniques. The CFE approximation technique has an advantage of computational simplicity and it can be used on digital computer for reduction purposes.

The second Cauer CFE originally proposed by Chen and Shieh (1968) is equivalent to a taylor seriesexpansion about s=0. It gives satisfactory approximation in the steady state region. The first Cauer form may also be applied for system reduction. It provides a satisfactory approximation in the transient region with an impulse input but gives error in the steady state region, however as the order of the reduced model increases the error under steady state response becomes negligible. The first Cauer CFE is equivalent to expansion of G(s) about $s = \infty$. The mixed or third Cauer form (Goldman) is a mixture of first and second Cauer CFE forms gives satisfactory approximation for both the transient and steady state responses.

The Cauer modified form (Chuang 1970) is a Taylor series expansion about $s = 0$ and $s = \infty$ both and alternatively. This approximation is good both in the steady state and transient period.

The Third chapter deals with stability criteria based reduction techniques, and a few other ones. Routh approximation method proposed by Hutton and Friedland preserves the stability of the reduced model provided the original model is asymptotically stable. Reduction using the Routh stability array proposed by V.Krishnamurthy and V.Sheshadri is computationally very simple, direct and gives a very similar frequency response as that of original system. Stability equation based reduction method is very convenient if applied by the aid of computer. Since the basic approach of this method is to discard the roots of the stability equations which have large magnitudes, the reduced stability equations will always have their roots in the left side of the s-plane. Therefore all the coefficients of the reduced stability equations as well as reduced transfer function will have positive sign.

Dominant pole retention method was proposed by Davidson in 1966. It preserves stability and it is applicable to MIMO models also. This is a very powerful method. A new method of model reduction was introduced by per-Olof Gutman et.al. based on polynomial differentiation. The reciprocals of the numerator and denominator polynomials of the high order transfer function are differentiated suitably many times to yield coefficients of the reduced order transfer function. The method is computationally very simple and is equally applicable to unstable and norminimum phase systems.

The Fourth chapter describes model reduction using mixed methods namely

CFE and Routh approximation,

CFE and Routh Hurwitz array,

CFE and stability equation,

CFE and polynomial differentiation and CFE and dominant pole retention. In all mixed methods, denominator is reduced by non CFE methods and numerator is reduced by CFE (Cauer 2nd and Cauer 3rd form) by mathing the Cauer coefficients of original system.

In chapter 5, all reduction methods described in previous chapters have been compared by the help of three SISO models and one MIMO model. In the same chapter methods for obtaining transfer function from given state variable equations have also been dealt with. The classical Faddeeva approach is first described. This method is known to give erroneous results if the system matrix A is of high order. A modified algorithm is introduced that removes this problem of inaccuracy computer programmes have been developed for these methods.

The Sixth chapter describes the development of state space model for a power system which consists of synchronous machine connected to an infinite bus. The system model is developed using well known Parks equations.Further various model reduction techniques described in Chapter 3 and 4 are applied to the power system model and a comparative study has been made. The merits and demerits of the various models have also been brought out in tabular form.

Based upon the work carried out in this dissertation, various reduction techniques used here in can be summarized as below.

TABLE -38

GENERALIZED COMPARISON OF VARIOUS REDUCTION TECHNIQUES

Case	Method	Computational efforts	Stability preservation	Applicability to multivariable system	Special feature
1	2	3	4	5	6
1.	Routh α-β expansion	Medium	Yes	Yes	1. can handle unstable full models. 2. Fails to give good approximations (still stable) when there are some pole zero cancellation in the full model.
2.	Routh Hurwitz array (Routh direct)	Low	Yes	Yes	A matching taylor series at s=o can handle unstable full models.
3.	Continued fraction expansion (CFE)	Low	No	Yes (but no. of inputs must be equal to number of outputs.	1.It converges faster than other series expansions. 2.It contains most of the essential characteristics of the original model in the first few terms. 3.It does not require any knowledge of model eigen spectrum.

1.	2	3	4	5	6
4.	Modal	High	Yes	Yes	Computation of eigen values and eigen vector is necessary.
5.	Polynomial differentiation	Low	Yes	Yes	Applicable to non-minimum phase and unstable systems also.
6.	Stability equation	Medium	Yes	Yes	applicable to discrete data system also.
7.	Mixed CFE-Routh(α,β)	Low	Yes	Yes	needs only calculation of α table.
8.	Mixed CFE - Routh Hurwitz array	Lesser than (7)	Yes	Yes	Computationally it is very simple.
9.	Mixed CFE-stability equation.	Medium	Yes	Yes	Can not be applied to reduce non minimum phase systems.
10.	Mixed CFE polynomial differentiation	Lesser than (4)	Yes	Yes	equally applicable to unstable and non minimum phase systems.
11.	Mixed CFE-Modal	Medium	Yes	Yes	-

Note

1. Method 2 and 3 are relatively easily implemented and thus their computer memory and time are minimal. Stability equation method also can be used by computer without any difficulty.

2. Best methods are the mixed methods in which good features of two schemes are combined together. For example in CFE-Routh method, the desired stability feature of Routh and computational convenience of CFE are put together to give a better medhod than any one of those individually.

This work is an effort in the direction of compiling the popular techniques for system order reduction in the frequency domain and to improve them by mixing with CFE techniques especially Cauer 2nd and Cauer 3rd form. A number of systems including SISO and MIMO are chosen and these techniques are applied to determine the reduced order models. Moreover one practical power system model has also been reduced using these techniques. The unit step time response and frequency response for the reduced models alongwith the original system are computed and the graphs are plotted. The accuracy of reduced order models is quantitatively determined by computing the cummulative error.

There is a wide scope for research work in this field especially for finding some innovative generalized technique which could be applied to all type of systems.

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