# RELIABILITY ANALYSIS OF COMPLEX SYSTEMS

A DISSERTATION
submitted in partial fulfilment
of
the requirements for the award of the Degree
of
MASTER OF ENGINEERING
in
POWER SYSTEM ENGINEERING

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# CERTIFICATE

Certified that the dissertation entitled, 'RELIABILITY ANALYSIS OF COMPLEX SYSTEMS' which is being submitted by Shri Rakesh Khetarpal in partial fulfilment for the award of the Degree of Master of Engineering in Power System Engineering of the University of Roorkee, Roorkee is a record of the student's own work carried out by him under my supervision and guidance. The matter embodied in this dissertation has not been submitted for the award of any degree or Diploma.

This is further to certify that he has worked for a period of Six Months from January'75 to March'75 and October'76 to December 76 for preparing this dissertation at this University.

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# S Y N O P S I S

For analysing the reliability of a system, the first step is to evaluate the system reliability. As the size and complexity of the system increases, the system reliability evaluation is quite time consuming and complicated. In this dessertation some techniques are presented for evaluation of system reliability.

In Chapter I, a method is presented to evaluate the reliability of a hierarchical system. The expected number of subsystem pairs communicating through the first level subsystem is considered as a reliability measure.

In Chapter II maintained systems are considered. Two method i.e. Successive displacement and Graph theoretic approaches are presented for finding the steady state availability of a system. Spectral decomposition method, Canonical transformation method and state transition matrix method are presented to find inherent availability of the system.

# CHAPTER I

## 1.1 Introduction

The electrical power system is hierarchical with a central control, center at the top of hierarchy, a number of generating plants and substations in the bottom, and several regional and local control centers inbetween. The control centers, interfaced with human operators, are coupled together to a greater or lesser extent by communication nets ranging from loose coupling in the case of teléphone line message transfer to direct computer-to-computer data transmission through cyclic digital data transmission and other communication means. Thus in the power control system a variety of controllers and control computers are located at diversed points in the system and multiple control functions are being executed simultaneously. The concept of communication with and through the root is particularly important here when dealing with such centralized computer networks where all communication must take place through some central computer. In terms of the new reliability measure to be detailed in the following chapter the expected number of node pairs communicating through the root is taken as a reliability measure which is different from conventional reliability measures such as MTBF and availability which essentially are based on two valued logics, i.e., the overal system state is either up or down.

## 1.2 Tree Network Reliability

We define a 'rooted tree' as a finite set T of one or more nodes such that:

(a) There is one specially designated node called the root of the tree, root(T); and

(b) The remaining nodes (excluding the root) are partitioned into $M \geq 0$ disjoint sets $T_1$, $T_2$, $T_3, \ldots, T_m$ and each of these sets in turn is a rooted tree. The trees $T_1, \ldots, T_m$ are called subtrees of the root.

Here for significant computational advantage the ancestory relation in the basis tree is exploited.

In a family tree immediate successors of a given parent node are 'brothers'. We extend the seniority to a set of brothers also. For example, the leftmost to be the eldest and right most to be the youngest. Characterizing the extended seniority relation fully, a link from each node to its 'eldest son' is called the successor index and to its next younger brother is called the brother index. If a node has no sons or no younger brother, then the successor (or brother) index is set equal to zero. Similarly the common ancestor of all nodes, which has no father, has a 0 predecessor index.

Now we proceed to calculate the reliability of a tree network assuming the reliability of its elements, nodes and links are known. A state vector with the root of each of the subtrees is associated. This state vector contains the information such as the expected number of nodes which can communicate with that node and the expected number of node pairs communicating through the root. The state vector of a given tree is obtained by a set of recursion relations, provided the state of its subtrees are know. We join the rooted subtrees into larger and larger rooted subtrees using the recursion relation until the state of the entire network is obtained.

Suppose there are two subtrees with roots I and J. Let J = P(I) i.e. J is a predecessor of I. If T(I) is a state vector associated with the root I of a subtree and say it gives the expected number of nodes in the subtree which communicate with the root I, including I. Similarly let T(J) be the state vector associated with the root J of a subtree. We assume the state of I and J i.e. T(I) and T(J) are known. Now joining I and J by the link (I,J) leads to a new state of J i.e. T(J)' which we wish to compute. If the link (I,J) and the node J are operational T(J)' = T(I) + T(J); if not, than T(J)' = T(J). Putting the two together we have the recurrence relation

$$T(J)' = T(J) + T(I)\ QN(J)\ QL(I)$$

Where QN(J) is the probability that node J is operative and QL(I) is the probability that the link (I,J) is operative.

Now taking node pairs communicating through the root as our criterion we consider a state component R(I) where R(I) is the expected number of node pairs (pairs including I are allowed) both of which are connected to the root node I. The recurrence relation for R(I) is

$$R(J)' = R(J) + \left[T(I)T(J) + R(I)QN(J)\right] QL(I)$$

An algorithm for the calculation of the reliability of a tree network is developed now. To facilitate the computation of algorithms we associate level with each node. Levels are defined in ascending order from left to night. The resulting algorithm is:

Step 0: Set T(K) = QN(K) and R(K) = 0, for K = 1,N Set I=1

Step 1: If S(I) = L(I) = 0, go to step 3, otherwise go to next step.

Step 2: Set I = I+1 and go to step 1

Step 3: Set J = P(I), If J=0 go to step 8, otherwise calculate R(J) and T(J) with the help of following relations.

$$R(J) = R(J) + (R(I)QN(J) + T(I)T(J))QL(I)$$

$$T(J) = T(J) + T(I)QN(J)\ QL(I)$$

Step 4 : If B(I) > 0, go to next step. Otherwise go to step 8.

Step 5- Set I = B(I)

Step 6 : If S(I) > 0, go to next step. Otherwise go to step 3.

Step 7 : Set I = S(I) and go to step 6.

Step 8 : If J =0, stop. Otherwise set I = J and go to step 3.

The calculation can be carriedout in two ways. In the first way link and node probabilities, PL(I),QL(I), PN(I),QN(I) can be considered as numbers and reliability criterion can be evaluated as a number. The evaluation can also be functional, that is the reliability of the subtrees can be represented as poly nomial functions of the link and node probabilities. We assume all links are operative with probability QL = p and all arcs operative with probability QN = q.

As an example using the algorithm the reliabity is evaluated for the tree network of Fig 1.2. The various indexes for the same are define as follows

| Node.I | Predecessor.P(I) | Sucessor.S(I) | Brother.B(I) | Level.L(I) |
|---|---|---|---|---|
| 1 | 7 | 0 | 0 | 1 |
| 2 | 9 | 7 | 0 | 3 |
| 3 | 10 | 9 | 0 | 3 |
| 4 | 0 | 10 | 0 | 3 |
| 5 | 10 | 11 | 3 | 0 |
| 6 | 12 | 0 | 0 | 4 |
| 7 | 2 | 1 | 8 | 1 |

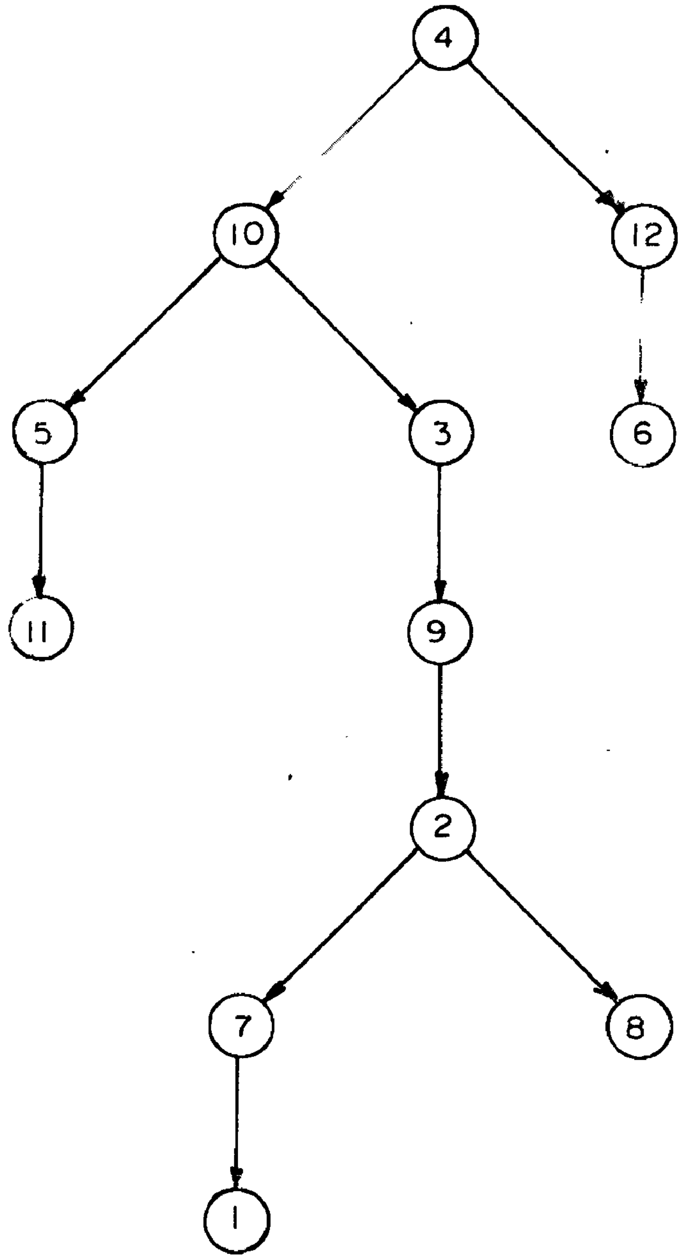


FIG.1.2 A TREE NETWORK.

| | | | | |
|---|---|---|---|---|
| 8 | 2 | 0 | 0 | 4 |
| 9 | 3 | 2 | 0 | 3 |
| 10 | 4 | 5 | 12 | 1 |
| 11 | 5 | 0 | 0 | 0 |
| 12 | 4 | 6 | 0 | 4 |

Let the link and node probabilities be given as follows

| I | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| QN(I)= | .95 | .92 | .98 | .91 | .91 | .92 | .97 | .93 | .93 | .92 | .93 | .91 |
| QL(I)= | .89 | .88 | .89 | 0 | .89 | .91 | .92 | .96 | .92 | .91 | .92 | .91 |

Using a suitable computer program on the basis of the algorithm developed, the tree network reliability was evaluated to be equal to 7.618519E-1

# CHAPTER-II

## INTRODUCTION

When allowing repair of failed components, we observe it is not meaningful to speak of component reliability, but it is meaningful to speak of the total system reliability to which a contribution is made by successful component performance. The difficulty lies in the fact that component reliability does not allow consideration of component repairs. Similarly system reliability does not consider the effects of system repairs either. Consequently, since it should be to our advantage to repair failed systems and components as rapidly as possible, especially if their operation is critical to some desired objective, we need some additional measure of system performance that considers the effects of repair. Such a measure is provided by the concept of system dependability, that is, whether it is operating or operable when we want it to be, given that the system passes through up and down cycles during its life time.

One measure of system dependability is provided by its reliability, which is the probability that the system will operate without failure for a specified period of time.

Three additional measures of system dependability that are desired to be considered are defined as follows:

(1) Point availability, defined to be the probability that the system is in an up state (i.e. either operating or operable) at a specified time.

(2) Interval availability, defined to be the expected fractional amount of an interval of specified length that the system is in an up state.

(3) Inherent availability defined to be the expected fractional amount of time in a continuum of operating time that the system is as an up state.

Inherent availability is commonly referred to as the uptime ratio or limiting availability. We shall refer to measures (1)-(3) as the system availability measures.

STEADY STATE AVAILABILITY

2.2 Availability Evaluation of a HT Pump Configuration:

The overall availability of a generating unit is dependent upon the availability of the subsystems which make up that unit. In many cases, a simple combination method can be used to find the system availability as the subsystems can be assumed to operate independently. This approach can also be used in certain cases where the subsystems are dependent and have relatively low availabilities. Detailed analysis of subsystems may require the development of more sophisticated models to include component dependency, spare component policies and corrective and preventive maintenance considerations. The technique used should be as unsophisticated as the problem will permit, but in many cases a complete model of the subsystem must be constructed and solved to obtain the required availability indices. These values can be obtained by a simulation method or direct analytical techniques such as the Markov approach. The latter aspect is discussed here with respect to a practical subsystem application.

The generating station proposed for a Hydrosystem contains four generating units. The primary heat transport system for each generating unit contains four pumps. The loss of more than one of these pumps due to outage results in the total shut down of the reactor. The loss of one pump produces a unit derating of 25 percent of the rated unit output. The pumps have been assumed to be identical in design and function in this analysis. The heat transport pump configuration is not an independent subsystem. The first pump failure results in a 25 percent derating while a second failure results in a complete shut down. The failure rate of the remaining two pumps after shut down is extremely low and they can be assumed to be failure free during this

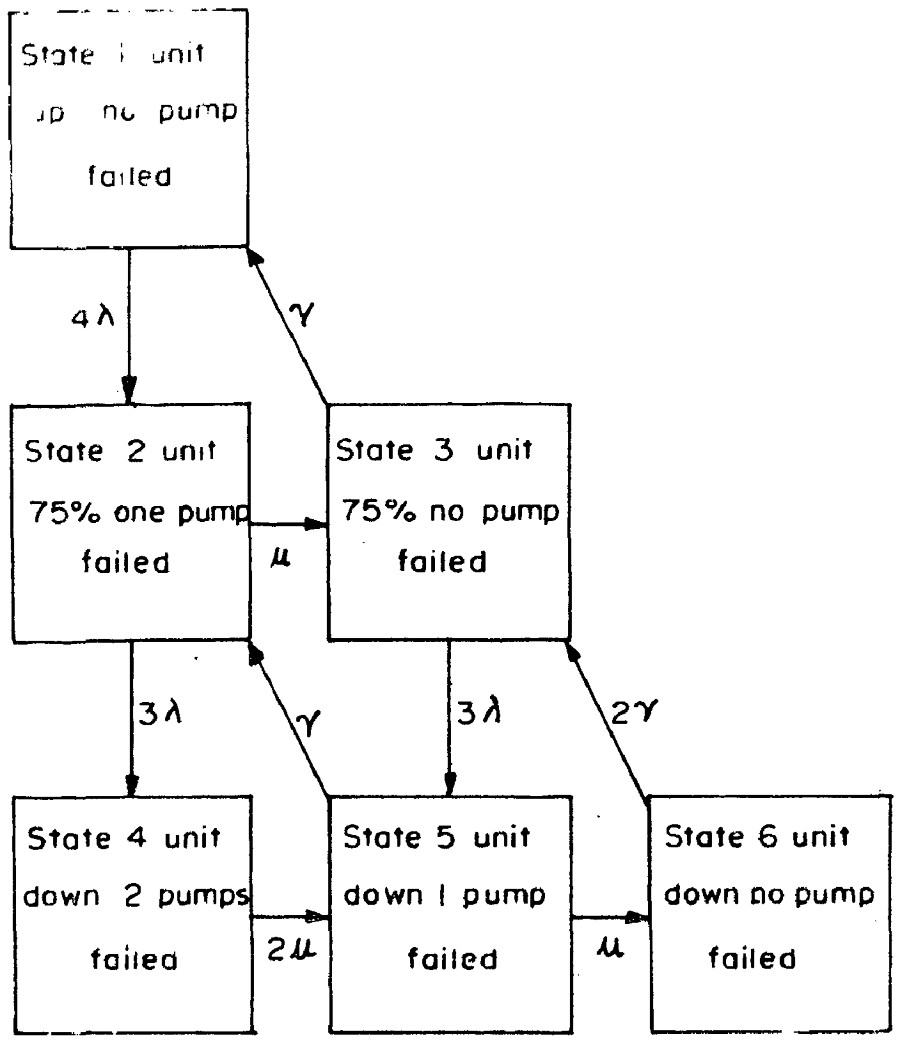


FIG.2.2 STATE SPACE DIAGRAM OF A SINGLE UNIT WITH NO SPARE PUMP.

condition. The pumps can, however be assumed to operate independently with each pump responsible for 25 percent of the output as the probability of having more than one pump on simultaneous outage is very small. No spare pumps are provided to improve the system reliability. Each generating unit within the station is independent with respect to HT pump operation in another unit. All pump failures are grouped into one permanent failure mode. A permanently failed pump must be removed from the operating site for repair. A pump is assumed to be restored to its original quality after repair. Other minor failures are ignored. The following designations are used:

$\lambda$ = failure rate of a pump in operation

$\mu$ = repair rate of a permanently failed pump.

$\gamma$ = installation rate of a repaired pump.

Single unit model:

Figure 2.2 shows the state space diagram for this case. The following stochastic transitional matrix can be obtained from this model.

| | | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| | 1 | $-4\lambda$ | 0 | $\gamma$ | 0 | 0 | 0 |
| | 2 | $4\lambda$ | $-(3\lambda+\mu)$ | 0 | 0 | $\gamma$ | 0 |
| A= | 3 | 0 | $\mu$ | $-(3\lambda+\gamma)$ | 0 | 0 | $2\gamma$ |
| | 4 | 0 | $3\lambda$ | 0 | $-2\mu$ | 0 | 0 |
| | 5 | 0 | 0 | $3\lambda$ | $2\mu$ | $-(\mu+\gamma)$ | 0 |
| | 6 | 0 | 0 | 0 | 0 | $\mu$ | $-2\gamma$ |

A four generating unit model:

Four generating unit models are more complicated than the single unit case. An extremely large number of states are required for the complete model (126 for no spare case). A complete four unit model is not presented here due to its extremely large size. Such models can be constructed by computer program.

2.21 Steady State Availability by Successive Displacement

The use of method of successive displacements or alternately the Gauss-Seidel method is presented here for obtaining the values of the state probabilities. Knowing the values of the state probabilities the values of the system's operating characteristics are easily obtained.

If the system configuration and the transition rates are known a set of stat. equations for probability of being in each state can directly be written, which can be modified to obtain a set of differential equation.

In matrix notation

$$\begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_n \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & & a_{nn} \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_n \end{bmatrix}$$

Where $p_1, p_2 \ldots p_n$ are state probabilities and $a_{11}$ to $a_{nn}$ are elements of the transition rate matrix.

For steady state availability $[\dot{P}] = 0$ we have thus

$$\begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_n \end{bmatrix} = \begin{bmatrix} 0 & -\frac{a_{12}}{a_{11}} & \cdots & -\frac{a_{1n}}{a_{11}} \\ -\frac{a_{21}}{a_{22}} & 0 & \cdots & -\frac{a_{2n}}{a_{22}} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ -\frac{a_{n1}}{a_{nn}} & -\frac{a_{n2}}{a_{nn}} & & 0 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_n \end{bmatrix}$$

We define

$$[A'] = \begin{bmatrix} 0 & -\frac{a_{12}}{a_{11}} & \cdots & -\frac{a_{1n}}{a_{11}} \\ -\frac{a_{21}}{a_{22}} & 0 & \cdots & -\frac{a_{2n}}{a_{22}} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ -\frac{a_{n1}}{a_{nn}} & -\frac{a_{n2}}{a_{nn}} & \cdots & 0 \end{bmatrix}$$

A' can be expressed as the sum of two matrices C and D. Thus P = CP + DP, which suggest the use of an iterative solution procedure called the method of successive displacements, $p^{(k+1)} = Cp^{(k)} + Dp^{(k+1)}$. The vector $p^{(k)}$ contains the estimates of the probabilities obtained on the k th iteration. Thus, the method of successive displacements consists of the following steps:

1. Choose an arbitrary initial approximation vector

$$p^{(o)} = (p_i^{(0)}) \quad , \quad i = 1,2,\ldots n.$$

2. Generate successive approximations $p_i^{(k)}$ by the iteration

$$p_i^{(k+1)} = \sum_{j=1}^{i-1} c_{ij} p_j^{(k)} + \sum_{j=i}^{n} d_{ij} p_j^{(k+1)}$$

for $i = 1,2,\ldots,n$ and $k = 0,1,\ldots$

3. Apply the normalising equation, $\sum_{i=1}^{n} p_i = 1$ after each complete round of iterations by dividing each 'probability' value by the sum of the 'probability' values obtained during the iteration round.

4. Continue until an appropriate convergence criterion is satisfied. Typical convergence criterion include

(a) $\max_{1 \le i \le n} \dfrac{|p_i^{(k+1)} - p_i^{(k)}|}{|p_i^{(k+1)}|} < \epsilon$ for some prescribed $\epsilon$

(b) $\sum_{i=1}^{n} |p_i^{(k+1)} - p_i^{(k)}| < \sigma$ for some prescribed $\sigma$

(c) $k = M$ for a prescribed integer $M$

Example:

Taking the single unit HT pump model as discussed earlier for solution. Assuming the value of $\lambda = 0.2$, $\mu = 2$, and $\gamma = 1$ per month substituting in the transitional rate matrix $\Lambda$ obtained earlier we get

$$\Lambda = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array}\begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} -0.8 & 0 & 1 & 0 & 0 & 0 \\ 0.8 & -2.6 & 0 & 0 & 1 & 0 \\ 0 & 2 & -1.6 & 0 & 0 & 2 \\ 0 & 0.6 & 0 & -4 & 0 & 0 \\ 0 & 0 & 0.6 & 4 & -3 & 0 \\ 0 & 0 & 0 & 0 & 2 & -2 \end{bmatrix} \end{array}$$

We have $\dot{P} = |\Lambda| P$

For steady state solution $\dot{P} = 0$

Thus in matrix notation the balance equation can be expressed as $P = |\Lambda'| P$ where $P = (p_i)$, $i = 1,2,\ldots 6$ . Note that $\Lambda'$ can be expressed as the sum of two matrices C and D, where

$$\begin{bmatrix} 0 & 0 & 1.25 & 0 & 0 & 0 \\ .3076 & 0 & 0 & 0 & 0.3846 & 0 \\ 0 & 1.25 & 0 & 0 & 0 & 1.25 \\ 0 & 0.15 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.2 & 1.333 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1.0 & 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0.307 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1.25 & 0 & 0 & 0 & 0 \\ 0 & 0.15 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.2 & 1.33 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1.0 & 0 \end{bmatrix}$$

$$D = \begin{bmatrix} 0 & 0 & 1.25 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.38 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1.25 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

For the example problem, the k th iteration can be expressed as follows:

$$p_1^{(k)} = 0\ p_1^{(k-1)} + 0\ p_2^{(k-1)} + 1.25 p_3^{(k-1)} + 0\ p_4^{(k-1)} + 0\ p_5^{(k-1)}$$
$$0\ p_6^{(k-1)} \qquad - (1)$$

$$p_2^{(k)} = 0.3076 p_1^{(k)} + 0\ p_2^{(k-1)} + 0\ p_3^{(k-1)} + 0\ p_4^{(k-1)}$$
$$0.3846\ p_5^{(k-1)} + 0\ p_6^{(k-1)} \qquad - (2)$$

$$p_3^{(k)} = 0\ p_1^{(k)} + 1.25 p_2^{(k)} + 0\ p_3^{(k-1)} + 0\ p_4^{(k-1)}$$
$$0\ p_5^{(k)} + 1.25 p_6(k) \qquad - (3)$$

$$p_4^{(k)} = 0\ p_1^{(k)} + 0.15\ p_2^{(k)} + 0\ p_3^{(k)} + 0\ p_4^{(k-1)} + 0\ p_5^{(k-1)} + 0\ p_6^{(k-1)} \quad - (4)$$

$$p_5^{(k)} = 0\ p_1^{(k)} + 0\ p_2^{(k)} + 0.2 p_3^{(k)} + 1.333 p_4^{(k)} + 0\ p_5^{(k-1)} + 0\ p_6^{(k-1)} \quad - (5)$$

$$p_6^{(k)} = 0\ p_1^{(k)} + 0\ p_2^{(k)} + 0 p_3^{(k)} + 0\ p_4^{(k)} + 01.0 p_5^{(k)} + 0.p_6^{(k-1)} \quad - (6)$$

Let the following initial probability estimates be employed:

$$p_1^{(o)} = p_2^{(o)} = p_3^{(o)} = p_4^{(o)} = p_5^{(o)} = p_6^{(o)} = 0.1666$$

From equation (1) the value of $p_1^{(1)}$ is obtained as follows

$$p_1^{(1)} = 1.250 \times 0.1.666 = 0.20825$$

Thus from (2) - (6) respectively

$$p_2^{(1)} = 0.1281,\ p_3^{(1)} = 0.3683,\quad p_4^{(1)} = 0.0192$$
$$p_5^{(1)} = 0.09925, p_6^{(1)} = 0.09925$$

Applying the normalizing equation yields the following revised estimates of the probabilities:

$$p_1^{(1)} = \frac{0.2257}{0.9223} = 0.2257$$

$$p_2^{(1)} = 0.1388\ ,\ p_3^{(1)} = 0.3993,\ p_4^{(1)} = 0.0208,\ p_5^{(1)} = 0.1076$$

$$p_6^{(1)} = 0.1076.$$

This way we further continue the iterations till there is no change in the values of the last two iterations. The following probability values have been obtained, namely

$p_1 = 0.376$ $p_2 = 0.125$ $p_3 = 0.219$

$p_4 = 0.022$ $p_5 = 0.089$ $p_6 = 0.089$

Steady state availability of the system

$$= p_1 + p_2 + p_3 = 0.820$$

2.22 Graph Theoretic Formula Approach for the Steady State Availability Evaluation:

We restrict attention to irreducible process with a finite number of states. For the discrete case, let $\pi = (\pi_1, \pi_2 \ldots \pi_N)$ be the row vector of steady state probabilities, and let P be the (N x N) matrix of one step transition probabilities ($p_{ij} = Pr\left[X_n = j | X_{n-1} = i\right]$ ). Then the equations determining the steady state distribution

$$\pi(P - I) = 0 \qquad (1)$$

$$\sum_{i=1}^{N} \pi_i = 1$$

Here eqn(2) is the normalising equation

For the continuous case, let $\pi = (\pi_1, \pi_2 \ldots \pi_N)$ be the row vector of steady state probabilities and let $\wedge$ be the matrix of instantaneous transition rates

$$\lambda_{ij} = \frac{d}{dt} Pr\left[X(t) = j \mid X(o) = i\right] \Big|_{t=0}$$

The determining equations are

$$\pi \Lambda = 0 \qquad (3)$$

$$\sum_{i=1}^{N} \pi_i = 1 \qquad (4)$$

Here (4) is the normalizing equation.

The matrices (P-I) and $\Lambda$ are not arbitrary, but rather possess certain properties which follow from the probability based definition of the elements. The properties are

(1) The off diagonal elements of each are non negative

(2) The diagonal elements are strictly negative

(3) Each row sums to zero.

From these properties and the irreducibility assumption, it follows that the rank of both P-I and $\Lambda$ will be N-1, so in each case the normalizing equation is required to ensure uniqueness as well as to force the solution to fulfil the requirements of a probability distribution. An unnormalized steady state solution means any non zero vector $C = (c_1, c_2 \ldots c_N)$ satisfying (1) and (3). The unique normalized equation is given by

$$\pi_i = C_i \Big| \sum^{N} C_j, \quad i = 1,2,\ldots,N$$

where $C_i$ are the elements of any unnormalised solution.

The graphs used are simply the usual transition diagrams of the Markov process, with points representing the States, and arcs the transitions. The arc 'weights' correspond to the $p_{ij}$ or $\lambda_{ij}$. The presence or absence of loops will not matter, since they are not used in the formula.

An intree to a specific point i in a directed graph G, denoted $T_i$, is a spanning subgraph of G in which every point but i has exactly one arc emanating from it and i has none. If the orientations of arcs are ignored, an intree to a point is just an ordinary tree in the usual graph theoretic sense. Thus an intree to point i is just a tree with the arcs all oriented towards point i. The weight of an intree, $(T_i)$, is defined to be the product of the weights of all arcs appearing in $T_i$.

An unnormalised solution to the steady state equations of finite, irreducible, discrete or continuous parameter Markov process is given by

$$C_i = \Sigma\ w(T_i),\ i = 1,2,\ldots k.$$

where the sum is the overall intrees to the point i in the transition diagram of the process.

With the state probabilities obtained from the normalised solution the reliability of the system can easily be computed when the transition diagram is large or complicated, it is difficult to be certain that one has found all of the intrees to a point merely by looking at the diagram. One can, ofcourse verify a hypothesized solution by substituting in the steady state equations. Alternatively, by programming the suitable algorithm all the intrees can be found using a digital computer.

For a single unit model discussed earlier the reliability can be evaluated by using this graph theoretic formula.

The transition diagram obtained for the system is shown in the Figure 2.22. The intrees for various states are shown in Figure 2.22(a)-(f). Thus we get

$$C_1 = \gamma.\mu.\gamma.2\mu.2\gamma + \gamma.2\gamma.3\lambda.2\mu.\mu + \mu.\gamma.2\mu.\mu.2\gamma$$

$$= 4\gamma^3\mu^2 + 12\gamma^2\mu^2\lambda + 4\gamma^2\mu^3$$

$$= 57.6$$

$$C_2 = 16\gamma^2\mu^2\lambda + 16\gamma^3\mu\lambda$$

$$= 19.2$$

$$C_3 = 16\gamma\mu^3\lambda + 48\gamma\mu^2\lambda^2 + 16\gamma^2\mu^2\lambda$$

$$= 46.8$$

$$C_4 = 24\gamma^2\mu\lambda^2 + 24\gamma^3\lambda^2 + 72\gamma^2\lambda^3$$

$$= 3.856$$

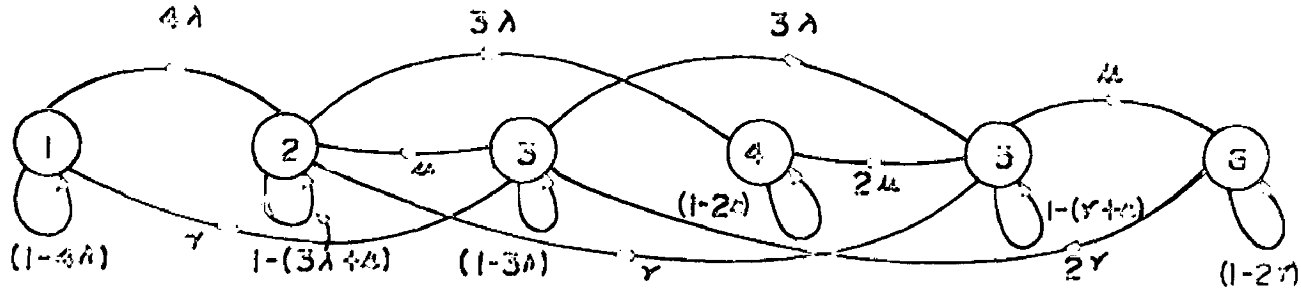


FIG. 2.22 STATE TRANSITION DIAGRAM.

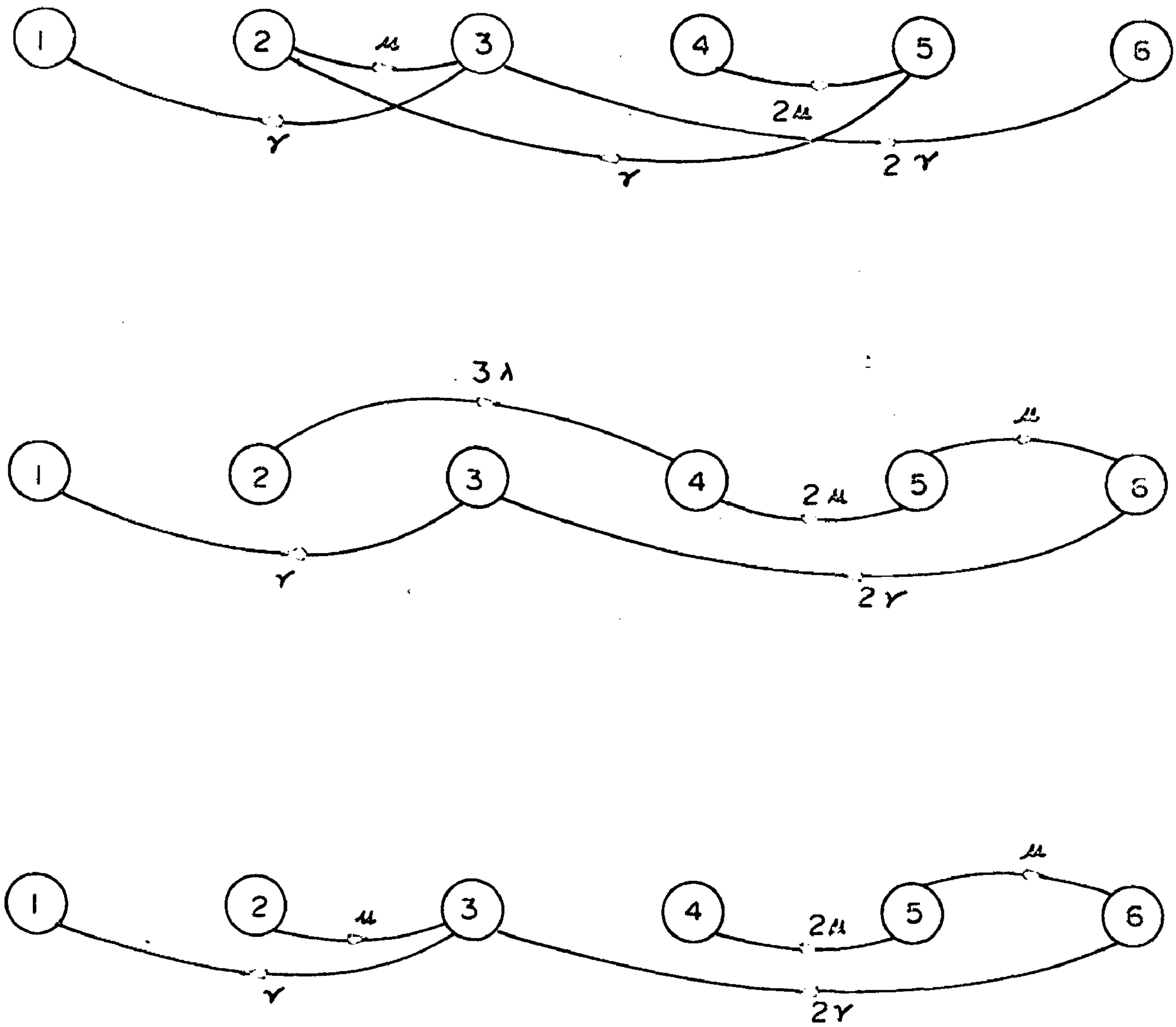


FIG. 2.22 (a) INTREES TO POINT (1)

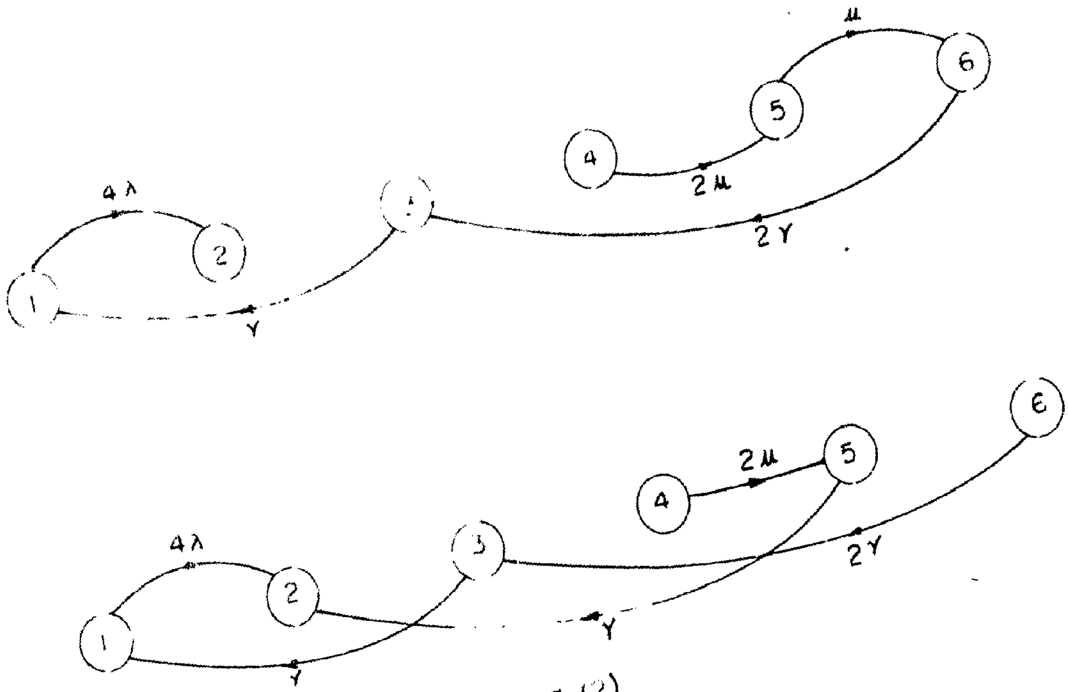


FIG. 2.22(b) INTREES TO POINT (2)

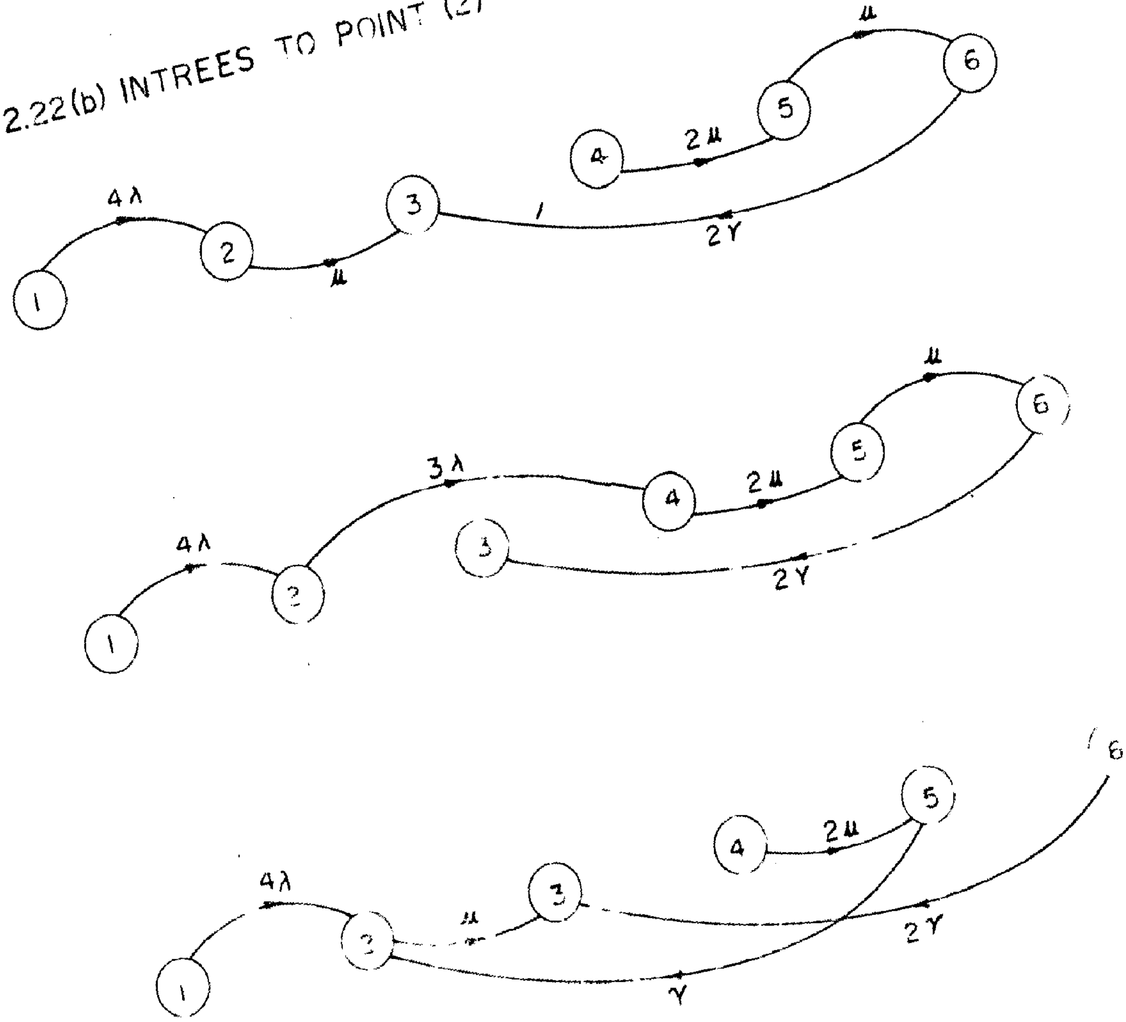


FIG.2.22(c) INTREES TO POINT (3)

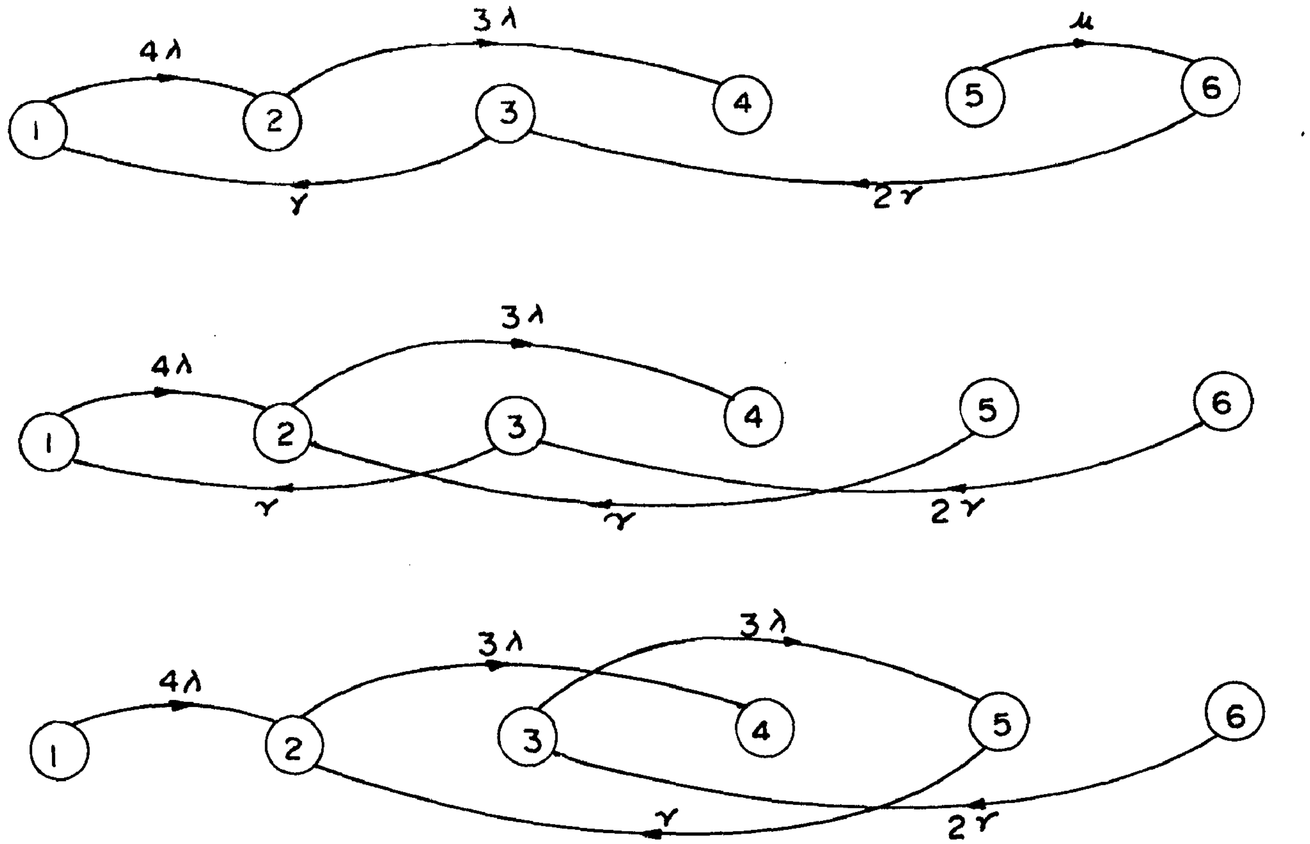


FIG.2.22(d) INTREES TO POINT (4)

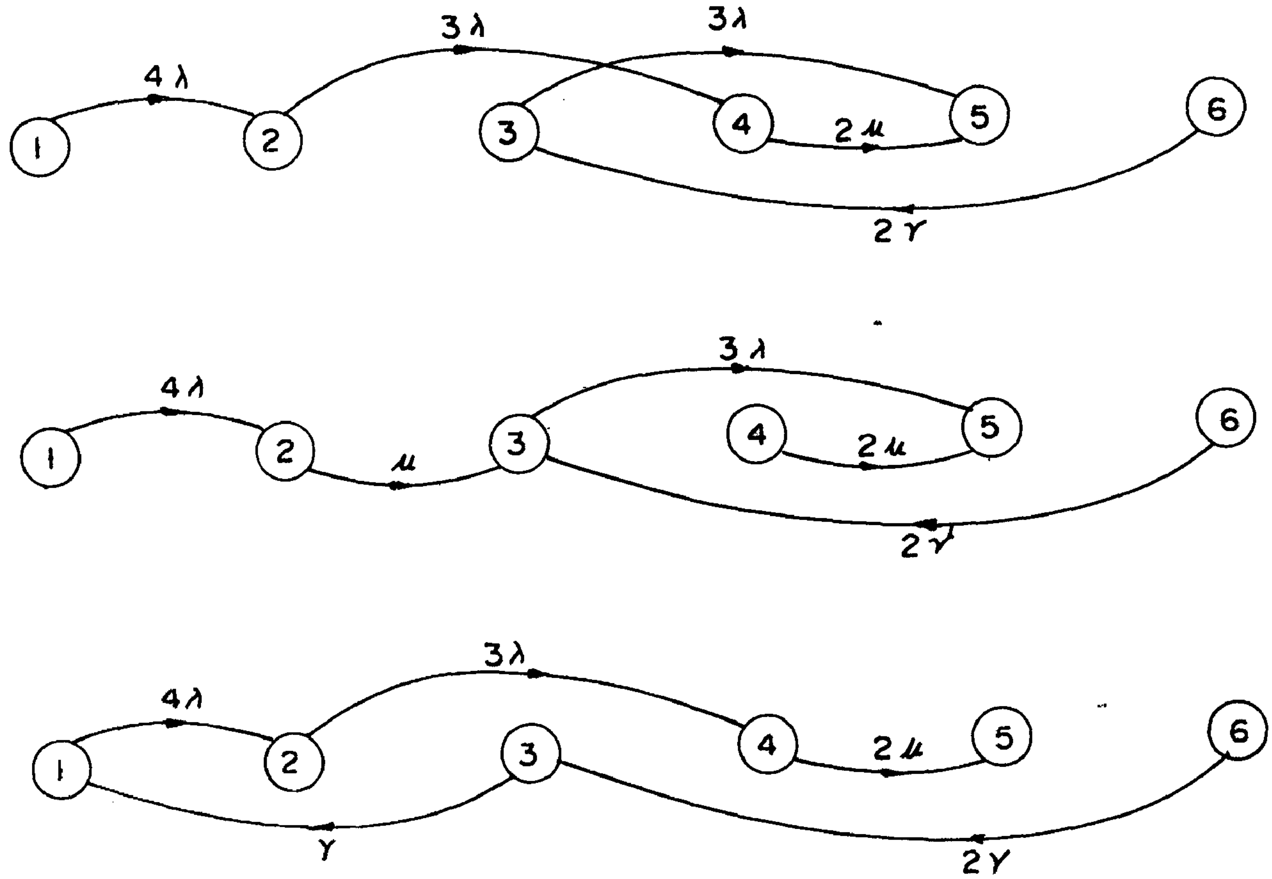


FIG.2.22(e) INTREES TO POINT (5)

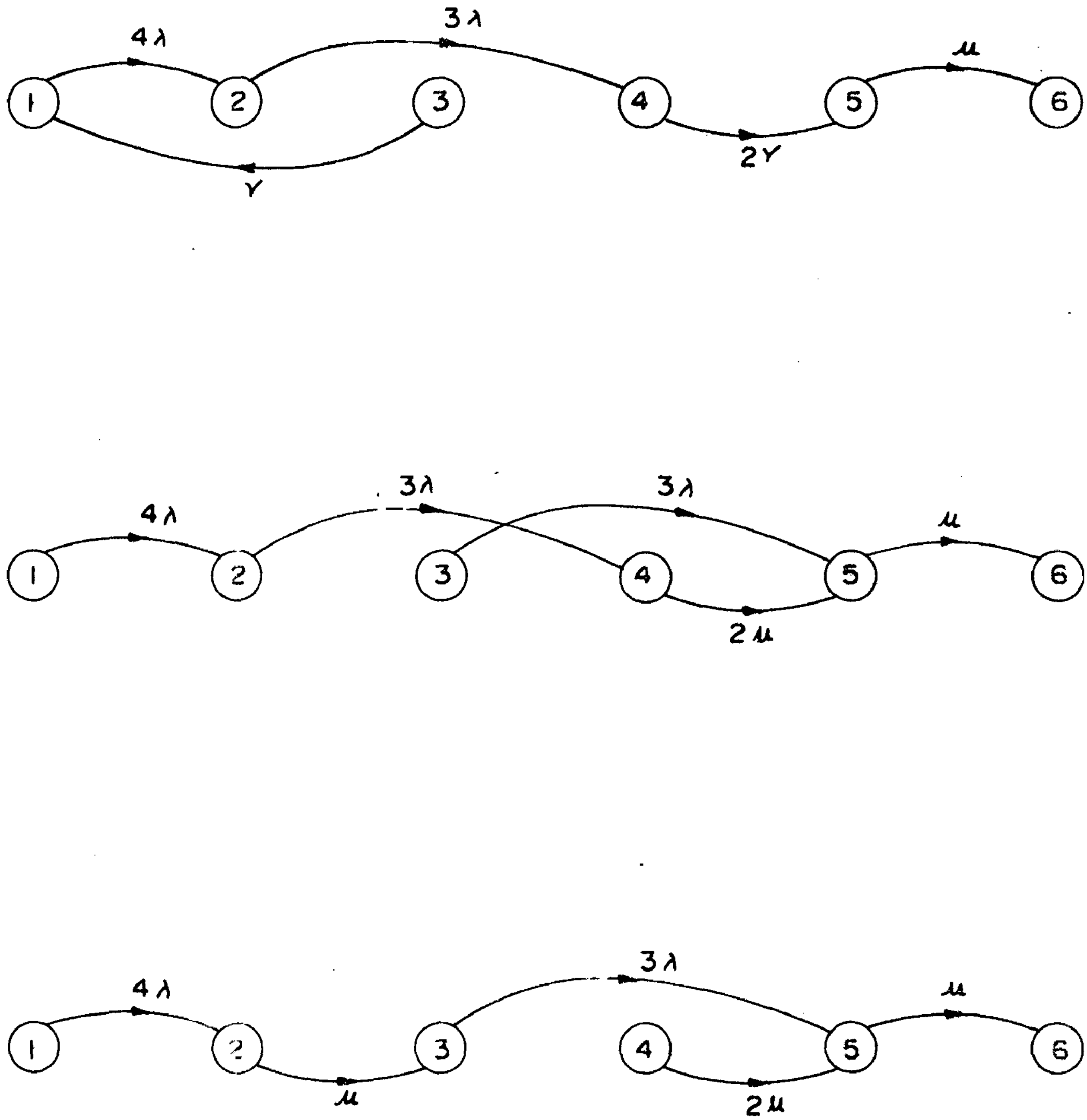


FIG.2.22(f) INTREES TO POINT - (6)

$$C_5 = 144\ \gamma\mu\lambda^3 + 48\ \gamma\mu^2\lambda^2 + 48\ \gamma^2\mu\lambda^2$$

$$= 13.824$$

$$C_6 = 24\ \gamma\mu^2\lambda^2 + 72\ \mu^2\lambda^3 + 24\ \mu^3\lambda^2$$

$$C_1 + C_2 + C_3 + C_4 + C_5 + C_6 = 154.704$$

Thus, $\pi_1 = \dfrac{57.6}{154.704}$

$$= 0.376$$

$\pi_2 = 0.125$, $\pi_3 = 0.219$, $\pi_4 = 0.022$

$\pi_5 = 0.089$, $\pi_6 = 0.089$

Steady state availability of the system = $\pi_1 + \pi_2 + \pi_3 = 0.820$

TIME DEPENDENT AVAILABILITY

2.83 Time dopendant System State Probability Evaluation with Spectral Decomposition

The application of the Markov Process in the solution of multi-derated system models has been discouraged because of the tedious task of using the Laplace transformation to find the general time dependent solution. The application of the Spectral Theory in solving the stochastic differential matrix for general time dependent solution has been overlooked by most of the authors.

The main objective here is to introduce the concepts of the Spectral Theory to solve a large Markovian system, if the system transition rates are known. The system transition probability matrix is derived by using the Spectral Theory. Next, the system reliability indices are computed by using the Spectral Theory. Theory for the solution of large system state probabilities is discussed in detail with example. The approach is analytic. All results are expressed in terms of the system parameters failure and repair rates and can be easily evaluated on a digital computer.

Definitions:

The following definitions are used throughout. All other functions and variables are described as they appear

| | |
|---|---|
| $n$ | No. of system states |
| $i, j$ | $1, 2, 3, \ldots, n$ |
| $A$ | The system transition rate matrix of nxn dimention |
| $a_{i,j}$ | The transition rate from the i th state to the j the state |
| $P$ | The system transition probability matrix |
| $p_{i,j}$ | The transition probability from the i th state to the j th state |
| $M_i$ | $M_1 M_2 \ldots M_n$<br>The projection matrices derived from A to M have the following properties: |
| $M_i M_j = 0$ | Projection matrices are mutually orthogonal |
| $\sum_{i=1}^{n} M_i = I$ | A complete set of projection matrices sums to the unit matrix |

| | |
|---|---|
| $M_i \cdot M_i = M_i$ | A projection matrix is an idempotent matrix |
| I | Identity Matrix |
| $\alpha_i$ | $\alpha_1, \alpha_2, \ldots, \alpha_n$, eigen values of a matrix |
| $\Psi$ | The set of all possible system operating states $1, 2, \ldots, n$. |
| S(T) | State of the system of time T |
| $P_i(T)$ | The probability of the system's i th state at time T: Prob $\left[S(T)=i\right]$ $i \in \Psi$ |
| $\overset{o}{P}(T)$ | The system operating state vector $\lvert p_1(T), p_2(T) \ldots p_n(T) \rvert$ |
| $P_j\left[T \mid \overset{o}{P}(t)\right]$ | The conditional probability that the generator is in the state j at time T are given that state vector at t was $\overset{o}{P}(t)$ Probability $\left[S(T) = j \mid \overset{o}{P}(t)\right]$ |

SYSTEM STATE EQUATIONS

In this section, a simple derivation of the vector partial differential equation which characterizes the conditional probability vector for the system operating state as a function of time is presented.

The first order difference equation associated with the conditional probability function is given by:

$$P_j\left[T+\Delta T \mid \overset{o}{P}(t)\right] = \sum_{i \in \Psi} \text{Prob}\left[S(T+\Delta) = j \mid S(T) = i, \overset{o}{P}(t)\right] \cdot \text{Prob}\left[S(T)=i \mid \overset{o}{P}(t)\right] \qquad (1)$$

Applying the Markov property to equation (1):

$$P_j\left[T+\Delta T \mid \overset{o}{P}(t)\right] = \sum_{i \in \mathbb{U}} P_j\left[T+\Delta T \mid S_j(T)\right] * P_i\left[T \mid \overset{o}{P}(t)\right] \qquad (2)$$

where $P_j\left[T+\Delta T \mid S_i(T)\right]$ is the probability of transition to state j in time interval $\Delta T$ given that the system of time T is in state i. The assumption that the system has a time-homogeneous transition rate between any two states mean that as $\Delta T \to 0$:

$$P_j\left[T+\Delta T \mid S_i(T)\right] = a_{ij} \, \Delta T \quad \text{for } i \neq j \qquad (3)$$

The transition rate from state i to state j, $a_{ij}$, is time homogeneous and non negative. The forward Chapman-Kolmogorv partial differential equation can be derived from above equation (3) and has the following form:

$$\frac{\partial}{\partial T} P_j\left[T \mid \overset{o}{P}(t)\right] = \sum_{i \in \mathbb{U}} a_{ij} \, P_i\left[T \mid \overset{o}{P}(t)\right]$$

or the set of n partial differential equation can be written as

$$\frac{\partial}{\partial T} \overset{o}{P}\left[T \mid \overset{o}{P}(t)\right] = \overset{o}{P}\left[T \mid \overset{o}{P}(t)\right] A \qquad (4)$$

A denotes the nxn matrix of transition rates $a_{ij}$, $\overset{o}{P}\left[T \mid \overset{o}{P}(t)\right]$ is the conditional state vector of the system.

APPLICATION OF SPECTRAL THEORY

A n-state system model is taken to illustrate the procedures of spectral theory for derivation of the time dependent system state probabilities. From equation (4), the matrix differential equations for the n-state generator system can be solved if the initial conditions are known. The different possible initial conditions are:

$P_1(0) = 1.0 \qquad P_2(0) = 0 \qquad P_n(0) = 0$

$P_1(0) = 0 \qquad P_2(0) = 1.0 \qquad P_n(0) = 0$

..............................................

..............................................

$P_1(0) = 0 \qquad P_2(0) = 0 \qquad P_n(0) = 1.0$

The first initial conditions means that at time t = 0, the system is in state 1. For this initial condition, the solution of equation (4) will give the state probabilities of the system at time T. Similarly, other initial conditions can be used to calculate the system's state probabilities at time T.

The transition probability matrix $\overset{o}{P}\left[T|\overset{o}{P}(t)\right]$ and transition rate matrix A can be represented in the spectral form as follows:

$$\overset{o}{P}\left[T|\overset{o}{P}(t)\right] = e^{\sigma_1 T} E_1 + e^{\sigma_2 T} E_2 + \ldots\ldots e^{\sigma_n T} E_n \qquad (6)$$

$$A = \sigma_1 E_1 + \sigma_2 E_2 + \ldots\ldots\ldots \sigma_n E_n \qquad (7)$$

where $\sigma_i$ = The eigenvaluesof A

$E_i$ = The projection matrix of A

$i = 1,2,3,\ldots,n.$

Therefore $\frac{\partial}{\partial T}\overset{o}{P}\left[T|\overset{o}{P}(t)\right]$ can be obtained simply by differentiating equation (6) with respect to T as follows

$$\frac{\partial}{\partial T}\overset{o}{P}\left[T|\overset{o}{P}(t)\right] = \alpha_1 e^{\alpha_1 T} M_1 + \alpha_2 e^{\alpha_2 T} + \ldots \alpha_n e^{\alpha_n T} M_n \qquad (8)$$

and from equation (6) and (7)

$$\overset{o}{P}\left[T|\overset{o}{P}(t)\right] * A = \frac{\partial}{\partial T}\overset{o}{P}\left[T|\overset{o}{P}(t)\right]$$

Since $M_i^2 = M_i$ and $M_i M_j = 0$

It is therefore simply necessary to find the different projection matrices corresponding to different eigenvalues of A for the solution of system state probabilities of time T. For each eigenvalue, the projection matrix $M_i$ can be expressed in terms of its eigen-column vector and eigen row vector as follows;

$$M_i = \frac{V_{ci} * V_{ri}}{V_{ri} * V_{ci}} \qquad (9)$$

Where

$V_{ci}$ is the eigen column vector of $M_i$

$V_{ri}$ is the eigen row vector of $M_i$

The different steps involved in the calculation of system state probabilities by the spectral Theory approach is outlined below.

<u>Step 1</u> : Calculate the eigenvalue of the transition rate

matrix. The characteristic equation of the transition rate matrix A is given by:

$$\alpha^{n} + \alpha^{n-1}C_{n-1} + \alpha^{n-2}C_{n-2} + \ldots\ldots + \qquad = 0$$

Since the determinant of the transition rate matrix is zero, there will be no constant term in the characteristics equation(C.E.). The coefficients of the C.E. are represented by $C_{n-1}$, $C_{n-2}$ etc.

$$C.E. = \alpha(\alpha^{n-1} + \alpha^{n-2}C_{n-1} + \ldots \quad ) \quad = 0$$

One of the eigonvalues of the C.E. of the transition rate matrix will always be zero. In the later part of the discussion it will be seen that the steady state transition probabilities are associated with eigenvalue $\alpha = 0$

Step 2 : Calculate the matrices associated with each eigen value.

$$E(\alpha_1) = A - \alpha_1 I$$

$$E(\alpha_2) = A - \alpha_2 I$$

...............

$$E(\alpha_n) = A - \alpha_n I$$

where I = Identity matrix

Step 3 : Calculate the eigen-column vector and eigen row vector associated with each matrix $E(\alpha_2)$, $E(\alpha_3)$.... $E(\alpha_n)$

Eigen-Column vector

$$= V_c(\alpha_1) = \begin{bmatrix} |D_1(\alpha_1)| \\ (-1)^3|D_2(\alpha_1)| \\ \cdots\cdots \\ \cdots\cdots \\ (-1)^{n+1}|D_n(\alpha_1)| \end{bmatrix}$$

where $D_1, D_2, \ldots D_n$ are the co-factors of the determinant $E(\alpha_1)$ along the first row or $V_c(\alpha_1)$ is the expansion along the first row of the determinant $E(\alpha_1)$.

Eigen-row vector =

$$V_r(\alpha_1) = \left[ |C_1(\alpha_1)|, (-1)^3|C_2(\alpha_2)|, \ldots . (-1)^{n+1}|C_n(\alpha_1)| \right]$$

where $C_1, C_2, \ldots C$ are the cofactors of the determinant $E(\alpha_1)$ along the first column of $V_r(\alpha_1)$ is the expansion along the first column of the determinant $E(\alpha_1)$. Similarly, the eigen-column vectors $V_c(\alpha_2)$, .... $V_c(\alpha_n)$ and eigen-row vectors $V_r(\alpha_2) \ldots V_c(\alpha_n)$ are calculated for the matrices $E(\alpha_2) \ldots E(\alpha_n)$.

Step 4: Calculate the projection matrices $M_1(\alpha_1)$, $M_2(\alpha_2)$ .... $M_n(\alpha_n)$ by using equation(9)

$$M_1(\alpha_1) = \frac{V_c(\alpha_1)V_r(\alpha_1)}{V_r(\alpha_1)V_c(\alpha_1)}$$

$$M_1(\alpha_1) = \frac{\begin{bmatrix} V_{c1}(\alpha_1) \\ V_{c2}(\alpha_2) \\ \cdots\cdots \\ \cdots\cdots \\ V_{cn}(\alpha_1) \end{bmatrix} \left[ V_{r1}(\alpha_1) \; V_{r2}(\alpha_1) \ldots \quad V_{rn}(\alpha_1) \right]}{\left[ V_{c1}(\alpha_1)V_{r1}(\alpha_2) + V_{c2}(\alpha_1)\, V_{r2}(\alpha_1) + \ldots V_{cn}(\alpha_1)V_{rn}(\alpha_1) \right)}$$

$$M_1(\alpha_1) \quad \begin{bmatrix} m_{11}(\alpha_1) & m_{12}(\alpha_1) \;\ldots\ldots & m_{1n}(\alpha_1) \\ m_{21}(\alpha_1) & m_{22}(\alpha_2) \;\ldots\ldots & m_{2n}(\alpha_1) \\ \ldots\ldots & \ldots\ldots & \ldots\ldots \\ \ldots\ldots & \ldots\ldots & \ldots\ldots \\ m_{n1}(\alpha_1) & m_{2n}(\alpha_1) \;\ldots\ldots & m_{nn}(\alpha_n) \end{bmatrix}$$

Similarly $M_2(\alpha_2)$, $M_3(\alpha_3)$ ... $M_n(\alpha_n)$ can be calculated from eigen column vectors and eigen row vectors of matrices $E(\alpha_2)$, $E(\alpha_3)$...$E(\alpha_n)$.

<u>Step 5</u> : Substitute the values of the projection matrices in equation (6) for the solution $\overset{\circ}{P}\left[T|\overset{\circ}{P}(t)\right]$

$$\overset{\circ}{P}\left[T|\overset{\circ}{P}(t)\right] = e^{\alpha_1 T} M_1(\alpha_1) + e^{\alpha_2 T} M_2(\alpha_2) + \ldots\; e^{\alpha_n T} M(\alpha_n)$$

Since $\alpha_1 = 0$ and $e^{\alpha_1 T} = 1$

where P is the transition probability matrix at time T and is given by:

$$\begin{bmatrix} p_{11} & p_{11} & p_{13} \;\cdots & p_{1n} \\ p_{21} & p_{22} & p_{23} \;\cdots & p_{2n} \\ p_{31} & p_{32} & p_{33} \;\cdots & p_{3n} \\ \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ p_{n1} & p_{n2} & p_{n3} & p_{nn} \end{bmatrix}$$

<u>Step 6</u>: Calculate the system state probabilities for any given initial conditions. For example, Consider the initial

Condition:

$$P_1(0) = 1 \qquad P_2(0) = 0 \qquad \dots\dots \qquad P_n(0) = 0$$

The expression for the system state probabilities are

$$P_1(T) = m_{11}(\alpha_1) + m_{11}(\alpha_2)e^{\alpha_2 T} + m_{11}(\alpha_3)e^{\alpha_3 T} \dots\dots$$
$$\dots\ m_{11}(\alpha_n)e^{\alpha_n T}$$

$$P_2(T) = m_{12}(\alpha_1) + m_{12}(\alpha_2)e^{\alpha_2 T} \dots\dots m_{12}(\alpha_n)\ e^{\alpha_n T}$$

$$P_n(T) = m_{1n}(\alpha_1) + m_{1n}(\alpha_2)e^{\alpha_2 T} + \dots\ m_{1n}(\alpha_n)\ e^{\alpha_n T}$$

For steady state time domain solution:

$$T \to \infty$$

$$P_1(\infty) = m_{11}(\alpha_1) \qquad P_2(\infty) = m_{12}(\alpha_1)$$

$$P_3(\infty) = m_{13}(\alpha_1) \qquad P_n(\infty) = m_{1n}(\alpha_1)$$

Similarly the steady state time domain solutions for any other initial conditions can be calculated from equation(11)

Some properties of the projection matrices of equation(11) are

1. The rows of the steady-state projection matrix $M(\alpha = 0)$ always add to 1

$$\sum_{j=1}^{n} m_{ij}(\alpha=0) = 1.0 \qquad i = 1,2,\dots\ n$$

2. The rows of the transient state projection matrices $M(\alpha \neq 0)$ always add to zero.

$$\sum_{j=1}^{n} m_{ij}(\alpha \neq 0) = 0 \qquad i = 1,2,\dots\ n$$

3. The sum of projection matrices given an identifying matrix

$$\sum_{i}^{n} M_i(\alpha) = I$$

4. $M_i(\alpha_i) \circ M_j(\alpha_j) = 0$

5. $M_i(\alpha_j) \circ M_j(\alpha_j) = M_j(\alpha_j)$

In the Appendix A, the proof of the spectral properties of the projection matrices are outlined.

Example

Let us for example consider an (m,n) system with r parallel repair facilities in which each component has the same constant failure rate λ and constant repair rate μ. Considering (1,2) system with r = 1 we obtain a 3 x 3 transition rate matrix having the form

$$\begin{bmatrix} -2\lambda & \mu & 0 \\ 2\lambda & -(\lambda + \mu) & \mu \\ 0 & \lambda & -\mu \end{bmatrix}$$

such that in matrix notation

$$\frac{\partial}{\partial t} P(T) = \Lambda P(T)$$

Thus, det $(\alpha I - \Lambda) = \alpha(\alpha^2 + \alpha(3\lambda + 2\mu) + (\mu^2 + 2\lambda\mu + 2\lambda^2))$

$$= \alpha(\alpha - \alpha_2)(\alpha - \alpha_3)$$

where

$$\alpha_2 = \frac{-(3\lambda + 2\mu) + \sqrt{\lambda(\lambda + 4\mu)}}{2}$$

$$\alpha_3 = \frac{-(3\lambda + 2\mu) - \sqrt{\lambda(\lambda + 4\mu)}}{2}$$

Let it be given that if per day failure and repair rates for such a system respectively $\lambda = 0.5$ and $\mu = 1$ Then substituting we get

$$\Lambda = \begin{bmatrix} -1 & 1 & 0 \\ 1 & -1.5 & 1 \\ 0 & 0.5 & -1 \end{bmatrix}$$

The eigen values of the matrix $\Lambda$ are

$\alpha_1 = 0$ , $\alpha_2 = -1$, $\alpha_3 = -2.5$

Since in matrix notation, the Chapman Kolmogorov equation can be represented as

$$\frac{\partial}{\partial t} P(T) = P(T)\, \Lambda$$

The matrices associated with eigen values $\alpha_1$, $\alpha_2$, $\alpha_3$ are

$$E(\alpha_1) = \begin{bmatrix} -(1+\alpha_1) & 1 & 0 \\ 1 & -(1.5+\alpha_1) & 0.5 \\ 0 & 1 & -(1+\alpha_1) \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 \\ 1 & -1.5 & 0.5 \\ 0 & 1 & -1 \end{bmatrix}$$

$$E(\alpha_2) = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -0.5 & 0.5 \\ 0 & 1 & 0 \end{bmatrix}, \quad E(\alpha_3) = \begin{bmatrix} 1.5 & 1 & 0 \\ 1 & 1 & 0.5 \\ 0 & 1 & 1.5 \end{bmatrix}$$

The eigen column vector and eigen row vector for the matrices $E(\alpha_1)$, $E(\alpha_2)$ and $E(\alpha_3)$ are

$$V_c(\alpha_1) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad V_c(\alpha_2) = \begin{bmatrix} -0.5 \\ 0 \\ 1 \end{bmatrix} \qquad V_c(\alpha_3) = \begin{bmatrix} 1 \\ -1.5 \\ 1 \end{bmatrix}$$

$$V_r(\alpha_1) = \begin{bmatrix} 1 & 1 & 0.5 \end{bmatrix}$$

$$V_r(\alpha_2) = \begin{bmatrix} -0.5 & 0 & 0.5 \end{bmatrix}$$

$$V_r(\alpha_3) = \begin{bmatrix} 1 & -1.5 & 0.5 \end{bmatrix}$$

The projection matrices $M_1(\alpha_1)$, $M_2(\alpha_2)$ and $M_3(\alpha_3)$ are given by

$$M_1(\alpha_1) = \frac{V_c(\alpha_1)V_r(\alpha_1)}{V_r(\alpha_1)V_c(\alpha_1)} = \frac{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\begin{bmatrix} -0.5 & 0 & 0.5 \end{bmatrix}}{\begin{bmatrix} 1 & 1 & 0.5 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}} = \frac{1}{\sout{0.75}\ 2.5}\begin{bmatrix} 1 & 1 & 0.5 \\ 1 & 1 & 0.5 \\ 1 & 1 & 0.5 \end{bmatrix}$$

$$M_2(\alpha_2) = \frac{V_c(\alpha_2)V_r(\alpha_2)}{V_r(\alpha_2)V_c(\alpha_2)} = \frac{\begin{bmatrix} -0.5 \\ 0 \\ 1 \end{bmatrix}\begin{bmatrix} -0.5 & 0 & 0.5 \end{bmatrix}}{\begin{bmatrix} -0.5 & 0 & 0.5 \end{bmatrix}\begin{bmatrix} 1 \\ -1.5 \\ 1 \end{bmatrix}} = \frac{1}{0.75}\begin{bmatrix} .25 & 0 & -.25 \\ 0 & 0 & 0 \\ -0.5 & 0 & .5 \end{bmatrix}$$

$$M_3(\alpha_3) = \frac{V_c(\alpha_3)V_r(\alpha_3)}{V_r(\alpha_3)V_c(\alpha_3)} = \frac{\begin{bmatrix}1\\1.5\\1\end{bmatrix}\begin{bmatrix}1 & -1.5 & 0.5\end{bmatrix}}{\begin{bmatrix}1 & -1.5 & 0.5\end{bmatrix}\begin{bmatrix}1\\-1.5\\1\end{bmatrix}} = \frac{1}{3.75}\begin{bmatrix}1 & -1.5 & 0.5\\-1.5 & 2.25 & -0.75\\1 & -1.5 & 0.5\end{bmatrix}$$

From Eqn.(1), the spectral representation is therefore

$$\overset{o}{P}\left[T \mid \overset{o}{P}(o)\right] = e^{\alpha_1 T} M_1(\alpha_1) + e^{\alpha_2 T} M_2(\alpha_2) + e^{\alpha_3 T} M_3(\alpha_3)$$

$$= \frac{1}{2.5}\begin{bmatrix}1 & 1 & 0.5\\1 & 1 & 0.5\\1 & 1 & 0.5\end{bmatrix} + \frac{1}{0.75}\begin{bmatrix}0.25 & 0 & -0.25\\0 & 0 & 0\\-0.5 & 0 & 0.5\end{bmatrix}e^{-t} + \frac{1}{3.75}\begin{bmatrix}1 & -1.5 & 0.5\\-1.5 & 2.25 & -.75\\1 & -1.5 & 0.5\end{bmatrix}e^{-2.5t}$$

The expression for the system state probabilities are

$$P_1(T) = 0.4 + 0.333\,e^{-t} + 0.266\,e^{-2.5t}$$

$$P_2(T) = 0.4 - 0.4\,e^{-2.5t}$$

$$A(T) = P_1(T) + P_2(T) = 0.8 + 0.333\,e^{-t} - 0.133\,e^{-2.5t}$$

## 2.4 Canonical Transformation Method:

If a system is described by

$$\dot{X}(t) = A\,X(t)\,, \quad X(o) = X_0 \qquad (1)$$

where $X(t)$ is an n vector which defines the state of the system at time t and A is an nxn constant matrix. The solution of this vector matrix differential equation can be obtained by the following method. Making use of this method system time dependent availability can easily be evaluated.

Any state vector y defined for this system is related to X by

$$X = Py$$

where P = n x n nonsingular matrix

For a 3 x 3 constant matrix A:

$$\text{where} \quad A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

where

$$|A - \lambda I| = -(\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3)$$

and $\lambda_1$, $\lambda_2$ and $\lambda_3$ are different from each other, a diagonalising transformation matrix P that transforms A into a diagonal matrix matrix is given by

$$P = \begin{bmatrix} \begin{vmatrix} e-\lambda_1 & f \\ h & i-\lambda_1 \end{vmatrix} & \begin{vmatrix} e-\lambda_2 & f \\ h & i-\lambda_2 \end{vmatrix} & \begin{vmatrix} e-\lambda_3 & f \\ h & i-\lambda_3 \end{vmatrix} \\ -\begin{vmatrix} d & f \\ g & i-\lambda_1 \end{vmatrix} & -\begin{vmatrix} d & f \\ g & i-\lambda_2 \end{vmatrix} & -\begin{vmatrix} d & f \\ g & i-\lambda_3 \end{vmatrix} \\ \begin{vmatrix} d & e-\lambda_1 \\ g & h \end{vmatrix} & \begin{vmatrix} d & e-\lambda_2 \\ g & h \end{vmatrix} & \begin{vmatrix} d & e-\lambda_3 \\ g & h \end{vmatrix} \end{bmatrix}$$

By means of a suitable transformation $x = Py$ we can transform eqn.(1) into

$$\dot{y} = P^{-1} A Py$$

If $A$ has distinct eigenvectors, then $P^{-1} A P$ can be made equal to $D = diag(\lambda_1, \lambda_2, \ldots, \lambda_n)$ and the solution is found as

$$y(t) = \begin{bmatrix} e^{\lambda_1 t} & & & 0 \\ \cdot & e^{\lambda_2 t} & & \\ \cdot & \cdot & \cdot & \\ 0 & & & e^{\lambda_n t} \end{bmatrix} y(0) = Q(t)y(0)$$

Since

$$Q(t) = \begin{bmatrix} e^{\lambda_1 t} & & & 0 \\ & e^{\lambda_2 t} & & \\ & & \cdot & \\ & & & \cdot \\ 0 & & & e^{\lambda_n t} \end{bmatrix} = e^{Dt}$$

We obtain

$$y(t) = e^{Dt} y(0)$$

or,

Since $y = P^{-1}v$ we obtain

$$y(0) = P^{-1} v(0)$$

Thus if the solution $v(t)$ of eqn.(1) is desired, it is given by

$$X(t) = Py(t) = P\,Q(t)\,P^{-1}\,X(0)$$
$$= P\,e^{Dt}\,P^{-1}\,X(0)$$
$$= P\,e^{(P^{-1}AP)t}\,P^{-1}\,X(0) \qquad (2)$$

In the case where $A$ involves multiple eigen values $\lambda_i$ of multiplicity $m_i (i = 1,2,\ldots,k;\ m_1+ m_2+\ldots+m_k = n)$ and the eigenvectors corresponding to $\lambda_i$ are also multiple with the corresponding multiplicity $m_i$, then $P^{-1}A\,P$ becomes the Jordan canonical form, which we shall denote by J. The solution $X(t)$ in this case is given by

$$X(t) = P\,S(t)\,Q(t)\,P^{-1}\,X(0)$$
$$= P\,e^{Jt}\,P^{-1}\,X(0)$$
$$= P\,e^{(P^{-1}AP)t}\,P^{-1}X(0) \qquad (3)$$

where $S(t)$ is given by

$$S(t) = \begin{bmatrix} S_1(t) & & & & 0 \\ & S_2(t) & & & \\ & & \cdot & & \\ & & & \cdot & \\ 0 & & & & S_k(t) \end{bmatrix}$$

where

$$S_i(t) = \begin{bmatrix} 1 & t & \frac{t^2}{2!} & \cdots & \frac{t^{m_i-1}}{(m_i-1)!} \\ 0 & 1 & t & & \frac{t^{m_i-2}}{(m_i-2)!} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & t \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix} = m_i x m_i \text{ matrix}$$

## 2.5 State Transition Matrix Method:

Another way to find the solution of eqn.(1) is to proceed as follows: By analogy with the scalar case

$\dot{x}(t) = ax(t)$, $x(0) = x_0$

whose solution is $x(t) = e^{at} x_0$, we use the matrix exponential

$$e^{At} = I + \sum_{K=1}^{\infty} \frac{A^K t^K}{K!}$$

for finding the solution of eqn.(1). The series $e^{At}$ converges absolutely and uniformly in any finite interval of the time axis. Since

$$\frac{d}{dt}(e^{At}) = Ae^{At}$$

the solution of eqn(1) satisfying the initial condition $x(t_0)=x_0$ is given by

$$x(t) = e^{AT} x_0 \qquad (4)$$

For $t = 0$, $e^{At}$ reduces to identity matrix. Therefore, eqn(4) clearly satisfies the initial condition.

**Example**

We consider once again the transition matrix A

$$\begin{bmatrix} -1 & 1 & 0 \\ 1 & -1.5 & 1 \\ 0 & 0.5 & -1 \end{bmatrix}$$

whose eigen values were calculated as

$$\lambda_1 = 0 \;, \quad \lambda_2 = -1 \quad , \quad \lambda_3 = -2.5$$

We now proceed to obtain the solution by the above two methods.

**Canonical Transformation Method:**

The matrix P is as given by

$$P = \begin{bmatrix} \begin{vmatrix} -1.5 & 1 \\ 0.5 & -1 \end{vmatrix} & \begin{vmatrix} -15+1 & 1 \\ 0.5 & -1+1 \end{vmatrix} & \begin{vmatrix} -15+2.5 & 1 \\ 0.5 & -1+2.5 \end{vmatrix} \\ \begin{vmatrix} 1 & 1 \\ 0 & -1 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 0 & -1+1 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 0 & -1+2.5 \end{vmatrix} \\ \begin{vmatrix} 1 & -1.5 \\ 0 & 0.5 \end{vmatrix} & \begin{vmatrix} 1 & -15+1 \\ 0 & 0.5 \end{vmatrix} & \begin{vmatrix} 1 & -15+2.5 \\ 0 & 0.5 \end{vmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -0.5 & 1.0 \\ 1 & 0 & -1.5 \\ 0.5 & 0.5 & 0.5 \end{bmatrix}$$

$$|P| = 1.875$$

$$P^{-1} = \frac{1}{1.875}\begin{bmatrix} .75 & .75 & .75 \\ -1.25 & 0 & 2.5 \\ 0.5 & -0.75 & 0.5 \end{bmatrix}$$

$$= \begin{bmatrix} 0.4 & 0.4 & 0.4 \\ -0.666 & 0 & 1.333 \\ .266 & -0.4 & 0.266 \end{bmatrix}$$

$$P^{-1}AP = \begin{bmatrix} 0.4 & 0.4 & 0.4 \\ -0.666 & 0 & 1.333 \\ 0.266 & -0.4 & 0.266 \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 \\ 1 & 1.5 & 1 \\ 0 & 0.5 & -1 \end{bmatrix} \begin{bmatrix} 1 & -0.5 & 1 \\ 1 & 0 & -1.5 \\ 0.5 & 0.5 & 0.5 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2.5 \end{bmatrix}$$

$$X(t) = P\, e^{(P^{-1}AP)t}\, P^{-1}\, X(o)$$

$$\begin{bmatrix} X_1(t) \\ X_2(t) \\ X_3(t) \end{bmatrix} = \begin{bmatrix} 1 & -0.5 & 1 \\ 1 & 0 & -1.5 \\ 0.5 & 0.5 & 0.5 \end{bmatrix} \begin{bmatrix} e^{ot} & 0 & 0 \\ 0 & e^{-t} & 0 \\ 0 & 0 & e^{-2.5t} \end{bmatrix} \begin{bmatrix} 0.4 & 0.4 & 0.4 \\ -0.666 & 0 & 1.333 \\ .266 & -0.4 & .266 \end{bmatrix} \begin{bmatrix} X_1(o) \\ X_2(o) \\ X_3(o) \end{bmatrix}$$

For the initial conditions

$$X_1(o) = 1\,,\; X_2(o) = 0\,,\; X_3(o) = 0$$

$$\begin{bmatrix} X_1(t) \\ X_2(t) \\ X_3(t) \end{bmatrix} \begin{bmatrix} 0.4\, e^{ot} + 0.333\, e^{-t} + 0.266\, e^{-2.5t} \\ 0.4\, e^{ot} - 0.4\, e^{-2.5t} \\ 0.20e^{ot} - 0.333\, e^{-t} + 0.133\, e^{-2.5t} \end{bmatrix}$$

Thus availability is given by

$$A(t) = \pi_1(t) + \pi_2(t)$$

$$= 0.8 + 0.333\, e^{-t} - 0.133\, e^{-2.5t}$$

State Transition Matrix Method

$e^{At}$ may be expanded as series of matrices and then added together into a closed form as follows:

$$e^{At} = I + \sum_{k=1}^{\infty} \frac{A^k t^k}{k!}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} -1 & 1 & 0 \\ 1 & -1.5 & 1 \\ 0 & 0.5 & -1 \end{bmatrix} t + \begin{bmatrix} -1 & 1 & 0 \\ 1 & -1.5 & 1 \\ 0 & 0.5 & -1 \end{bmatrix} \frac{t^2}{2!} + \ldots$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} -1 & 1 & 0 \\ 1 & -1.5 & 1 \\ 0 & 0.5 & -1 \end{bmatrix} t + \begin{bmatrix} 2 & -2.5 & 1 \\ -2.5 & 3.75 & -0.5 \\ 0.5 & 1.25 & 1.5 \end{bmatrix} \frac{t^2}{2!} + \ldots$$

$$= \begin{bmatrix} 1 - t + \frac{2t^2}{2!} + \ldots & t - 2.5\frac{t^2}{2!} + \ldots & \frac{t^2}{2!} + \ldots \\ t - 2.5\frac{t^2}{2!} + \ldots & 1 - 1.5t + 3.75\frac{t^2}{2!} + \ldots & t - 0.5\frac{t^2}{2!} + \ldots \\ 0.5\frac{t^2}{2!} + \ldots & 0.5t + 1.25\frac{t^2}{2!} + \ldots & -t + 1.5\frac{t^2}{2!} + \ldots \end{bmatrix}$$

The solution is

$$X(t) = e^{At} X(o),$$

For the initial condition

$$X_1(o) = 1, \quad X_2(o) = 0, \quad X_3(o)$$

$$\begin{bmatrix} X_1(t) \\ X_2(t) \\ X_3(t) \end{bmatrix} = \begin{bmatrix} 1 - t + \frac{2t^2}{2!} + \ldots \\ t - 2.5 \frac{t^2}{2!} + \ldots \\ 0.5 \frac{t^2}{2!} + \ldots \end{bmatrix}$$

Thus availability is given by

$$A(t) = X_1(t) + X_2(t)$$

$$= 1 - 0.5 \frac{t^2}{2!} + \ldots$$

## 3.0 CONCLUSION

A method is presented to evaluate a measure of reliability of a hierarchical system. The system is coded in a special form to reduce the computation time. A large hierarchical system can be handled by decomposing it into small subsystems.

The successive displacement method for finding the steady state availability of the system requires less storage and computer time rather than other methods as the transition matrix can easily be decomposed in L-U form. During the design phase, various parameters of the system are required to be changed sequentially. To find the steady state availability for each change in system parameter requires the solution of the state equations repeatedly which is quite time consuming. A graph theoretic approach is presented which develops the arithmatic expression for system steady state availability in terms of system parameters. Thus avoids the repeated solution of the state equation.

To find the inherent availability three methods are presented. The state transition matrix method gives an approximate solution. The accuracy of the solution depends on the number of terms considered in the series expension of $e^{AT}$. The spectral decomposition method gives an accurate value of inherent availability. But it requires more memory space.

The canonical transformation method requires simple calculation and less memory requirement than the spectral decomposition method. This also gives the exact value of inherent availability.

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# APPENDIX A

## PROPERTIES OF SPECTRAL MATRICES

To prove the properties of matrix M, mentioned in the preceding sections, let us consider the n matrix R given by:

$$R = V_1, V_2, \ldots, V_n$$

where $V_i$ is the eigen-vector belonging to the eigenvalue $\alpha_i (i = 1,2,\ldots,n)$ from the property of similarity of matrices we have:

$$R^{-1} M R = \Lambda \qquad \text{(A-1)}$$

This implies that

$$M = R \Lambda R^{-1}$$

where

$$= \alpha_1 \begin{bmatrix} 1 & 0 & 0 \ldots & 0 \\ 0 & 0 & 0 \ldots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 & 0 & 0 \ldots & 0 \\ 0 & 1 & 0 \ldots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 \ldots & 0 \end{bmatrix}$$

$$+ \cdots\cdots\cdots + \alpha_n \begin{bmatrix} 0 & 0 & 0 \ldots & 0 \\ 0 & 0 & 0 \ldots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 \ldots & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \alpha_1 & 0 & 0 \ldots & 0 \\ 0 & \alpha_2 & 0 \ldots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 \ldots & \alpha_n \end{bmatrix} \qquad \text{(A-2)}$$

Then, from equation (A-1)

$$M^2 = K\Lambda K^{-1} K\Lambda K^{-1}$$

$$= K\Lambda^2 K^{-1}$$

$$M^3 = K\Lambda^3 K^{-1}$$

$$\cdots\cdots\cdots\cdots$$

$$\cdots\cdots\cdots\cdots$$

$$M^n = K\Lambda^n K^{-1}$$

From equation (A-2)

$$M = K\Lambda K^{-1}$$

$$= \alpha_1 K \begin{bmatrix} 1 & 0 & 0 \dots & 0 \\ 0 & 0 & 0 \dots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 \end{bmatrix} K^{-1} + \alpha_2 K \begin{bmatrix} 0 & 0 & 0 \dots & 0 \\ 0 & 1 & 0 \dots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 \dots & 0 \end{bmatrix} K^{-1}$$

$$+ \alpha_n K \begin{bmatrix} 0 & 0 & 0 \dots & 0 \\ 0 & 0 & 0 \dots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 1 \end{bmatrix} K^{-1}$$

where

$$M = \alpha_1 M_1 + \alpha_2 M_2 + \dots + \alpha_n M_n$$

where

$$M_1 = K \begin{bmatrix} 1 & 0 & 0 \dots & 0 \\ 0 & 0 & 0 \dots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 \end{bmatrix} K^{-1} \qquad \text{(A-3)}$$

$$M_2 = K \begin{bmatrix} 0 & 0 & 0 \dots & 0 \\ 0 & 1 & 0 \dots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 \end{bmatrix} K^{-1}$$

$$M_n = R \begin{bmatrix} 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix} R^{-1}$$

From equation(A-3) therefore

$$M_1^2 = R \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & & 0 \end{bmatrix} R^{-1} * R \begin{bmatrix} 0 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & & 0 \end{bmatrix} R^{-1} =$$

$$= M_1$$

and

$$M_1 * M_2 = R \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & & 0 \end{bmatrix} R^{-1} * R \begin{bmatrix} 0 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & & 0 \end{bmatrix} R^{-1} = 0$$

Therefore, the spectral representation of the transition rate or transition probability matrix is

$$M = \alpha_1 M_1 + \alpha_2 M_2 + \dots \alpha_n M_2$$

## APPENDIX-B

```
C  C   RELIABILITY EVALUATION OF A TREE NETWORK
       DIMENSION IS(20),IB(20),IP(20),QN(20),QL(20),T(20),R(20)
       DIMENSION IL(20)
       READ 1000,N,NN
       DO 1 I = 1,NN
       READ 1000,IP(I),IS(I),IB(I),IL(I)
1      CONTINUE
       READ 2000,(QN(I),I= 1,NN)
       READ 2000,(QL(I),I=1,N)
       DO 5 K = 1, NN
       T(K) = QN(K)
       R(K) = 0.0
5      CONTINUE
       I = 1
6      IF( IS(I) ) 20, 10, 20
10     IF( IL(I) ) 20, 30, 20
20     I = I + 1
       GOTO 6
30     J = IP(I)   IF(J) 31,80,31
31     R(J)  = R(J) + ( R(I) * QN(J) + T(I)T(J)) * QL(I)
       T(J)  = T(J) + T(I) * QN(J) * QL(I)
       IF( IB(I) ) 60, 60, 40
40     I = IP(I)
45     IF( IS(I) )30, 30, 50
50     I = IS(I)
       GOTO 45
60     IF( J) 70, 80, 70
70     I = J
       GOTO 30
80     DO 90 I = 1, NN
       PUNCH 3000, NN, T(I),R(I)
90     CONTINUE
1000   FORMAT(10I5)
2000   FORMAT(8F10.5)
3000   FORMAT(I4,2F10.6)
       STOP
       END
```