ANALYSIS OF THE HEMODYNAMICS OF THE ARTERIAL TREES AND THEIR ELECTRICAL MODELLING

A DISSERTATION
submitted in partial fulfilment of the
requirements for the Degree
of
MASTER OF ENGINEERING
in
ELECTRICAL ENGINEERING
(Measurement and Instrumentation)

By

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C E R T I F I C A T E

Certified that the dissertation entitled 'Analysis of the Hemodynamics of the Arterial Trees and their Electrical Modelling' which is being submitted by SRI RAJKUMAR, KUMAR in partial fulfilment for the award of the Degree of MASTER OF ENGINEERING in ELECTRICAL ENGINEERING (Measurements and Instrumentation) of University of Roorkee is a record of bonafide work carried out by him under my supervision and guidance. The matter embodied in this dissertation has not been submitted for the award of any other Degree or Diploma.

This is further certified that he has worked for Five months from July 1974 to December 1974 for preparing this dissertation.

Dated 30/12/74

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C O N T E N T S

CHAPTER-ONE

INTRODUCTION

It has been recognised for a long time that the vascular system consists largely of a complex configuration of branched elastic tubes. According to Poiseuille's law, the flux of a viscous incompressible fluid through a rigid tube is a linear function of the pressure difference between the ends of the tube. However, in the vascular beds of mammals, the pressure flow relation is always non-linear. This non-linearity has been ascribed to the elastic nature of blood vessels and their consequent rather large distensibility.

A great variety of mathematical and physical models of the human arterial system has been introduced, since the start of investigation in this field. They can basically be divided in two groups: the 'Windkessel-models' and the 'Transmission models'. It is generally felt now-a-days that a model should be more sophisticated and should have counterparts of the essential hemo-dynamic quantities in the actual system such as pressure-flow relationships, reflection coefficients, pulse wave velocity etc. The 'Windkessel' models can not be expected to do this because they lack in their original concept, a representation of pulse wave velocity. The Windkessel is equivalent to a single chamber, that is, a lumped system, in which the wave velocity is infinite and pressure and flow pulses change simultaneously.

Transmission models require division of the arteries into segments. Inserting the parameters of each segment into the equation of motion and the equation of continuity results in a relationship between pressure-gradient and flow on the one hand, and between flow gradient and pressure on the other. The form of an electrical delay line is determined where voltage stands for pressure and current for flow. Therefore, the construction of a passive electrical equivalent of the arterial system seems possible.

Second, third and fourth chapters deal with the fundamental aspect of the arterial circulation. The fifth chapter deals with the mathematical analysis of the arterial circulation in quite details first taking the artery to be of circular cross-section and later considering it to be of elliptical cross-section. The sixth chapter deals with the electrical models of the arterial trees taking into consideration of sleeve effect and non-newtonian properties of blood. Last chapter gives an account of arterial vs. venous hemodynamics.

CHAPTER-TWO

PHYSIOLOGY OF THE HEMODYNAMICS

2.1 INTRODUCTION

Fluid dynamics is largely a consideration of the relationship between pressure and flow. Hemodynamics, being the fluid dynamics of blood, is concerned with the specific case of an inhomogeneous, viscous fluid contained within a series of flexible, branched tubes whose properties may be time varying. The majority of work in the field of hemodynamics has been done on the arterial system. In analyzing the flow of blood in the arterial-system we must first understand the blood dynamics from the physiological point of view.

2.2 THE BLOOD

The blood is a red fluid of alkaline reaction and is salty in taste, its specific gravity is 1.050-1.060. The blood consists of cells and plasma. The cells make up 40-45 percent of total amount of blood and the plasma makes up 55-60 percent. The body of an adult contains about five litres of blood which weigh one thirteenth of the total body weight.[1]

2.3 FUNCTIONS OF THE BLOOD

The blood performs an important function in metabolism; it delivers nutrients to the tissues of all the organs and carries the waste products away.

The blood performs a most important function in respiration, it delivers oxygen to the tissues of the organs and carries carbon dioxide away. Oxygen enters the blood through the lungs. Carbon dioxide is eliminated from the blood mainly through the lungs.

The blood effects 'humoral regulation of the activities' of various organs; it transports various substances (harmones etc.) round the organism.

The blood also has a 'protective function,' it contains cells which possess properties of phagocytosis, and special products called antibodies, which play a protective role.

The blood takes part in distributing heat within the organism and in maintaining a constant body temperature. Because of the movement of blood through the blood vessels heat is transported from warmer parts of the body to cooled parts. The blood gives off the excess of heat into the external environment, and the organism therefore does not become overheated.

It should be noted that part of the blood does not circulate through the blood vessels, but is stored in so called blood depots (in the capillaries of the spleen, liver and subcutaneous tissue). Under different conditions the volume of blood circulating in the organism may increase or decrease through a change in the volume of blood depot. For example, during muscular work and in cases of blood loss the blood from the depots is released in to general circulation.[2,3]

2.4 CARDIOVASCULAR SYSTEM

The blood continuously moves through the organism; this movement is called blood circulation. All organs of the human body communicate with each other through the circulation of the blood. The blood flows through blood vessels, which are elastic tubes with varying diameters. A closed network of blood vessels radiates throughout the entire body. The heart, which is a hollow, muscular organ, contracts rhythmically and pumps the blood throughout the the organism.

2.5 BLOOD-VESSELS

There are three types of blood vessels- (1) arteries, (2) capillaries and (3) veins. They differ from each other in structure and in function.

ARTERIES- Arteries are vessels through which the blood flows from the heart to organs. They have comparatively thick walls made up of three coats; an outer coat, a middle coat and an inner coat fig.(1). The outer coat, or tunica adventitia, consists of connective tissue. The middle coat, or tunica media, consists of smooth muscle tissue and contains elastic connective tissue fibres. Contractions of this coat decrease the lumen of the blood vessel. The inner coating, or tunica intima, is made up of connective tissue and is lined with a layer

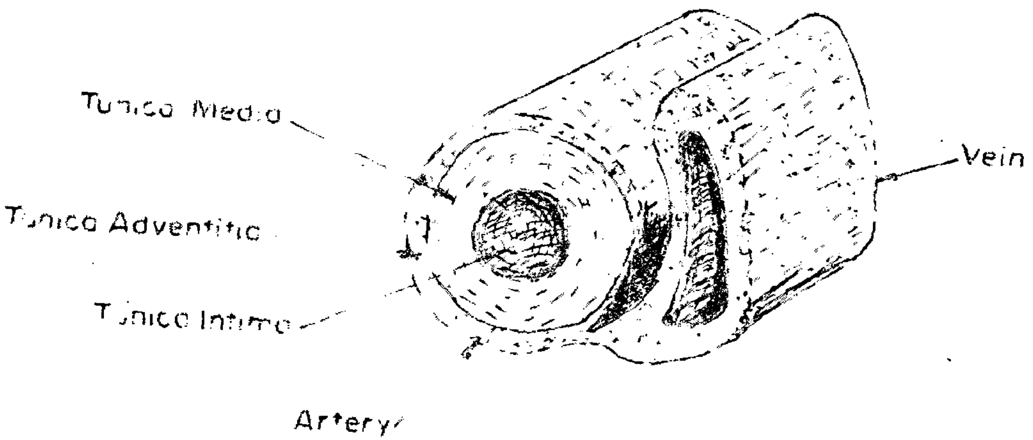


FIG. I STRUCTURE OF ARTERY AND ADJACENT VEIN.

of flat cells, the endothelium. The arteries differ in diameter; the farther from the heart, the smaller the diameter. Inside each organ the artery divides into smaller branches. The smallest arterial vessels are called arterioles. The arterioles divide into capillaries.

Capillaries

Capillaries are minute blood vessels which are visible only under the microscope. The lumens of the capillaries vary and average 7.5 μ; the length of a capillary does not exceed 0.3 mm. There are several hundred capillaries per square millimetre of tissue of any organ. The total lumen of the capillaries of the entire body is 500 times that of the aorta. When an organ is in a state of rest most of its capillaries are contracted and no blood flows through them. In an active organ the number of functioning capillaries increases. The wall of a capillary consists of one layer of endothelial cells. The interchange of substances between the blood and the tissues takes place only through the capillary walls. Various nutrients and oxygen, and part of the blood plasma of which lymph is formed, pass from the blood into the tissues. Carbon dioxide and other waste products pass from the tissues into the blood. The endothelium of the capillaries plays an active role in allowing the substances to pass from the blood into the tissues and vice-versa. The interchange of substances depends not only on the state of the capillary walls, but also on the main substance of the connective tissue surrounding the capillaries. As it flows through a capillary arterial blood changes to venous blood, which drains into the veins.

VEINS- Veins are vessels through which blood flows from the organs to the heart. Like arteries, they have walls composed of three coats fig.(1), but they contain fewer elastic and muscle fibres and so are less resilient and collapse easily. Unlike arteries, veins have valves which open in the direction of the blood flow. This helps the blood in the veins to flow in the direction of the heart. The smallest veins are called venules. Closer to the heart the venous vessels increase in diameter. The total lumen of the veins is larger than that of the arteries, but smaller than that of the capillaries.

Each region or organ of the body is usually supplied with blood by several vessels. One of them, the largest in diameter, is called the main vessel, while the smaller ones are called the accessory or collateral vessels. Some arteries communicate with each other through connecting vessels, called anastomoses. There are also anastomoses between veins.

If the blood ceases to flow in one vessel (if the vessel is cut or compressed by a tumor, etc) the circulation through the collateral vessels and anastomoses will increase. New collateral vessels and anastomoses may gradually develop in addition to the existing ones. The blood circulation is thus restored.

2.6 SYSTEMIC (GREATER) AND PULMONARY (LESSER) CIRCULATION

All the blood vessels in the human body compose two circuits of blood circulation; the systemic (or greater) circuit and the pulmonary (or lesser) circuit.

The systemic circulation begins with the aorta which leads from the left ventricle and carries arterial blood to all the organs. The aorta divides into numerous branches, the arteries. The arteries enter the organs where they divide into smaller branches which then form network of capillaries. From the capillaries the blood, now venous, passes into small veins which form larger veins. From all the veins of the systemic circulation the blood is collected into the superior and inferior venae cavae which empty into the right atrium. Thus, the systemic circulation is a system of vessels through which the blood travels from the left ventricle to the organs and from the organs to the right atrium.

The pulmonary circulation begins with the pulmonary trunk which arises from the right ventricle and conveys venous blood to the lungs. The arterial blood flows from the lungs through the pulmonary veins into the left atrium. In other words, the pulmonary circulation is a system of

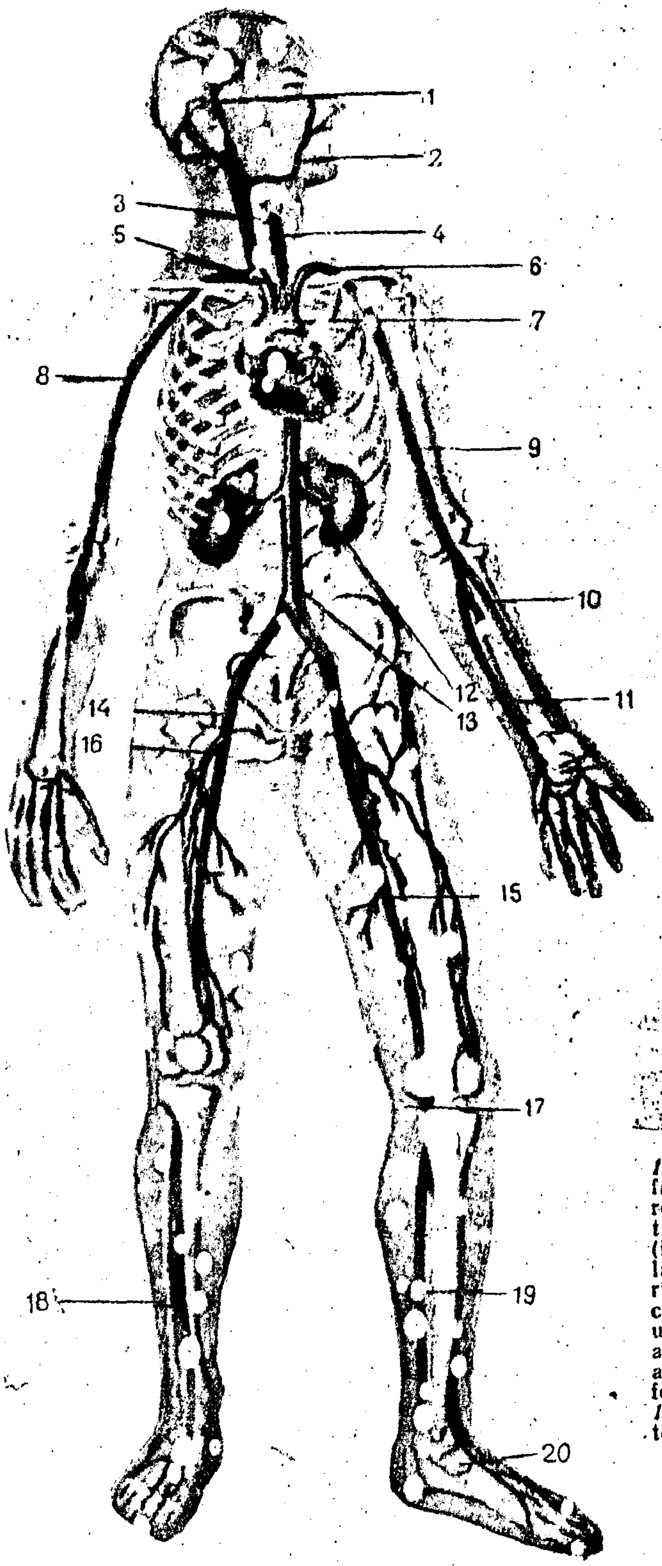


Fig 2. **Arterial system (diagram)**

1—superficial temporal artery; *2*—facial artery; *3*—right common carotid artery; *4*—left common carotid artery; *5*—brachiocephalic trunk (innominate artery); *6*—left subclavian artery; *7*—arch of aorta; *8*—right axillary artery; *9*—left brachial artery; *10*—radial artery; *11*—ulnar artery; *12*—renal artery; *13*—abdominal aorta; *14*—external iliac artery; *15*—femoral artery; *16*—deep femoral artery; *17*—popliteal artery; *18*—anterior tibial artery; *19*—posterior tibial artery; *20*—dorsalis pedis artery of vessels through which the blood moves from the right ventricle to the lungs and from the lungs to the left atrium[1].

2.7 ARTERIAL-SYSTEM

It consists of the following main arteries. Fig (2)

1. Superficial temporal artery,
2. facial artery,
3. right common carotid artery,
4. Left common carotid artery,
5. brachiocephalic trunk (innonimate artery)
6. left subclavian artery,
7. arch of aorta,
8. right axillary artery,
9. left brachial artery,
10. radial artery,
11. ulnar artery,
12. renal artery,
13. abdominal aorta,
14. external iliac artery,
15. femoral artery,
16. deep femoral artery,
17. popliteal artery,
18. anterior tibial artery,
19. posterior tibial artery,
20. dorsalis pedis artery,

CHAPTER-THREE

PROPERTIES OF THE ARTERIAL WALL

3.1 THE STATIC ELASTIC PROPERTIES OF THE ARTERIAL WALL

The Elastic Modulus

The Young's modulus of an isotropic tube, which does not change in length on inflation, is given by Love,

$$E = \frac{\Delta P}{\Delta R_o} \times \frac{2(1-\sigma^2)R_i^2 R_o}{(R_o^2 - R_i^2)} \qquad \ldots (1)$$

where,

ΔR_o = change in external radius,

ΔP = change in pressure,

R_i = Internal radius

σ = Poisson's ratio.

Poisson's ratio is the ratio of transverse to longitudinal strain, all materials becoming narrower when they are stretched in length. If $\sigma = 0.5$ no change in the volume of the material occurs for a very small strain. The equation assumes the isotropy of the material, that is the mechanical properties are identical in all directions. The arterial wall is more extensible longitudinally than circumferentially, but when no change in length occurs the effective circumferential modulus is a function only of the true radial and circumferential moduli.[5]

With these assumptions the incremental modulus is

$$E_{inc}\, p_2 = \frac{p_3 - p_1}{R_{o3} - R_{o1}} \; \frac{2(1-\sigma^2)R_{i2}^2 R_{o2}}{(R_o^2 - R_i^2)} \qquad \ldots (2)$$

where the subscripts 1,2,3 represent successive measurements of pressure and radius. If no volume change occurs in the wall, then $R_o^2-R_i^2$ is constant. The units of stress are force per unit area. Strain is a ratio of length and is dimensionless, thus elastic modulus has the same units as stress. It is, therefore, necessary to measure the internal-pressure, radius and wall thickness of the arteries.

The increase in modulus with increasing pressure depends both on the elastic properties of the collagen, elastin and muscle within the arterial wall, and ont their arrangement and linkages. Table 1 shows the elastic moduli(E) of arterial wall constituents.

TABLE 1.The elastic moduli(E) of arterial wall constituents.

Tissue	E(dynes/ cm^2x10^6)
Collagen(tendon)	100
Elastin(ligamentum nuchae)	6
Smooth muscle (resting)	2.5

It is generally assumed that the smooth muscle, collagen and elastin in the arterial wall function in parallel, and each bears some load at all internal pressures. The elastic modulus of resting vascular smooth muscle is probably in the region of 1x10^6 dynes/cm^2.

The arterial wall becomes stiffer as it is extended. This increase is less marked in the thoracic arota upto a pressure of 100 mm Hg. Mean values for the static elastic modulus (dynes/cm^2 x 10^6) at 100 mm Hg. pressure were thoracic aorta 4.3, abdominal aorta 8.7, femoral artery 6.9, carotid artery 6.4.[4]

3.2 The dynamic Elastic Properties of the Arterial Wall.

Although the response of the arterial tree to relatively slow changes in blood pressure is determined by its static elastic properties, the rapid pressure changes occuring at each heart beat will result in rather different behaviour. This is due to the visco-elastic properties of the arterial wall. The mechanical response of a visco-elastic material depends both on the force applied (elastic response) and on the time it acts (viscous response). These substances display 'creep' (continuing extension at constant load) and stress relaxation (tension decay at constant length)[6].

The dynamic elastic modulus (E') is given by

$$E' = \frac{\Delta P}{\Delta R_o} \frac{2(1-\sigma^2)R_o R_i^2}{R_o^2 - R_i^2} \qquad \ldots (3)$$

E' may be resolved into two components, elastic and viscous. These are termed E_{dyn} and $\eta\omega$ respectively, and are defined as follows:

$$E_{dyn.} = E'\cos\phi \qquad \ldots (4)$$

$$\eta\omega = E'\sin\phi \qquad \ldots (5)$$

where $\emptyset$ is the phase angle between pressure changes (ΔP leading) and radius changes (ΔR_o).

$\eta\omega$ is the product of the coefficient of viscosity (η) and the angular velocity (ω).[5]

Thus the amplitude of E',

$$|E'| = \left[(E_{dyn})^2 + (\eta\omega)^2\right]^{1/2} \qquad \ldots (6)$$

where $\eta\omega$ is small say $< 10\%\ E_{dyn}$

$$\therefore E_{dyn} \simeq E' \qquad \ldots (7)$$

Bergel suggested that the muscle content of an artery is primarily responsible for its viscosity. 'Wiederhielm' has suggested a model for small arterial vessels and is shown in Fig.(3). The elastin is represented by a number of springs which engage at different degrees of extension and thus simulate the recruitment of fibers that occur at larger deformations.

The elastic properties due to collagen are shown in a similar manner and demonstrate the recruitment of collagen with increasing strain. The collagen fiber jackets surrounding the smooth muscle are simulated by two sets of elastic components, one in parallel and one in series with the muscle fibers Fig.3(a).

In the relaxed state the muscle is quite extensible and the contribution of elastic forces by the muscle and the elastin is trivial since the elastic behaviour is dominated by the

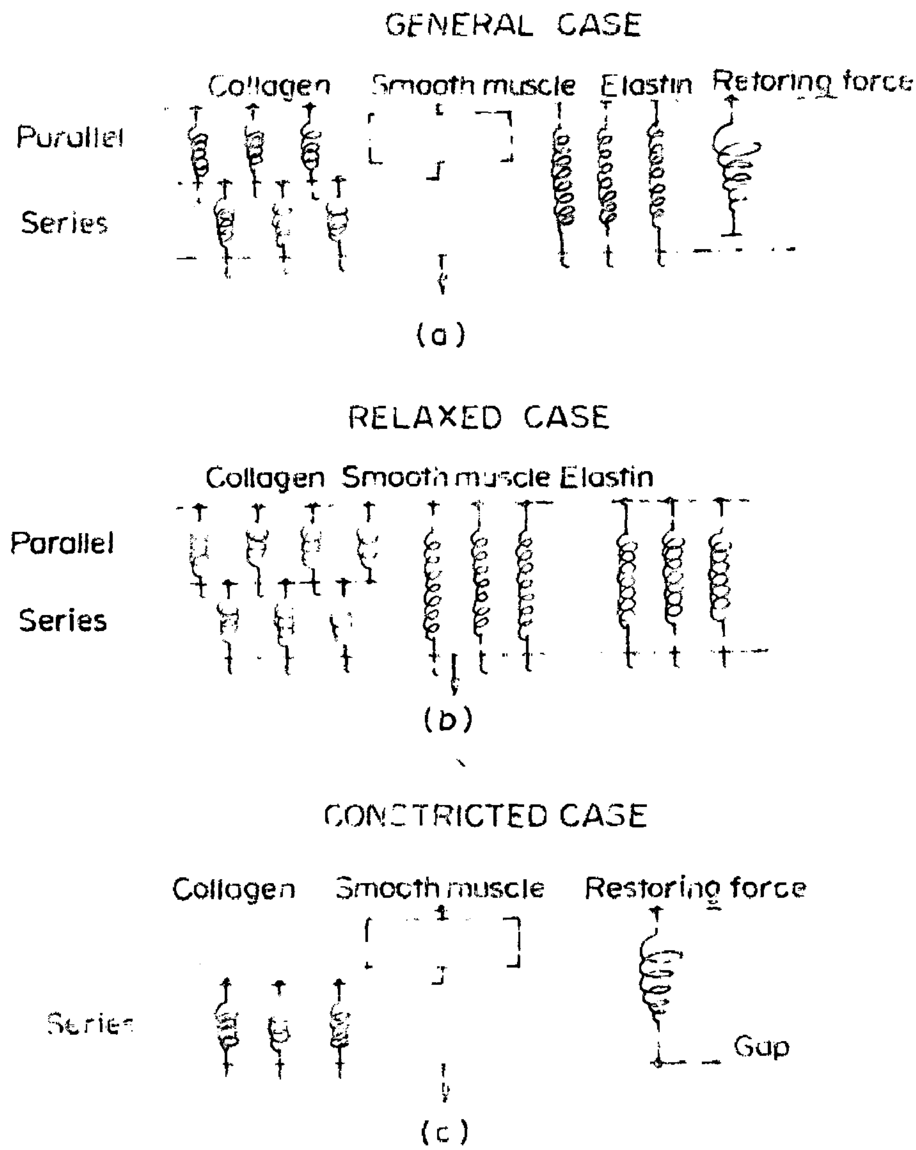


FIG. 3 MODEL OF ARTERIAL WALL.

collagen, Fig.3(b).

In the constricted state, the elastic elements connected in parallel with the vascular smooth muscle will be slack and thus not support any significant amount of tension. The portion of the collagen fibres between the smooth muscle cells will transmit the contractile force from one muscle cell to the next and appear as a series elastic element, fig.3(c).

3.3 MECHANICAL PROPERTIES OF ARTERIES

The functions by which the cardiovascular system serves the biologic organism are mechanical. It is, therefore, necessary to understand the mechanical properties of the cardio-vascular system. One of the major components of cardio-vascular system is its arterial network, the mechanical properties of which determine the propagation of energy from the heart to the periphery. The relationships of blood flow and blood pressure, of intravascular pressure and vessel volume, of pulse wave velocity and blood pressure are but a few of the variables often measured which depend quantitatively on the mechanical properties of the blood vessel walls.[7]

It may be assumed that the relationship between stress and strain in the blood vessel can be expressed by some equation representing the sum of a series of terms of increasing order and their coefficients, for example:

$$P(t) = A\xi + B\frac{d\xi}{dt} + C\frac{d^2\xi}{dt^2} + D\frac{d^3\xi}{dt^3} + \ldots\ldots$$

where A,B,C etc. are coefficients which are analogous to vessel parameters.

It has been found that a first order linear differential equation will match both the contour and the amplitudes when recorded and simulated stress and strain are compared. This suggests that the mass or inertial (second order) and higher terms are negligible in the relationship of stress and strain in the artery wall since the elastic and viscous moduli are the predominant parameters.

The strain which the arteries undergo as a result of arterial pulse pressure variations is normally between 0.01 and 0.04 i.e. between 1 and 4 percent change in circumference. The total strain associated with marked constriction and dilation does not usually exceed 10 percent. Therefore, the circumferential motion of arteries may be characterised as small strain. The mass of the artery wall does not play a significant role in determining the mechanical behaviour of the arteries and can therefore be neglected.

CHAPTER-FOUR

THE PHYSICAL LAWS GOVERNING THE FLOW OF BLOOD

4.1 INTRODUCTION

The biological function of the heart is to pump blood to the tissues of the body. The heart creates a regular, intermittent ejection of blood and so flow in arteries is pulsatile. McDonald[8] described this flow and the accompanying pressure waves by Fourier-series. Such a series is represented by

$$F(t) = A_0 + \sum_{m=1}^{m=\infty} A_m \cos m(\frac{2\pi}{T})t + B_m \sin m(\frac{2\pi}{T})t + \quad \ldots (8)$$

where T = cycle length.

Thus pulsatile flow consists of a set of terms which oscillate around a mean value A_0. This mean value is referred as the steady flow and the remainder as the oscillatory flow. The fact that the arterial system is essentially a set of pipes for distributing blood remindes that the steady or mean flow is the most important component.

4.2 THE STEADY-FLOW

If the shape of a typical flow curve fig.(4) is considered, it will be seen that it is an oscillation of asymmetrical shape. There is a large peak of forward flow, due to systole. Following this peak there is a reversal of flow leading to a backflow. In terms of values about the mean we have

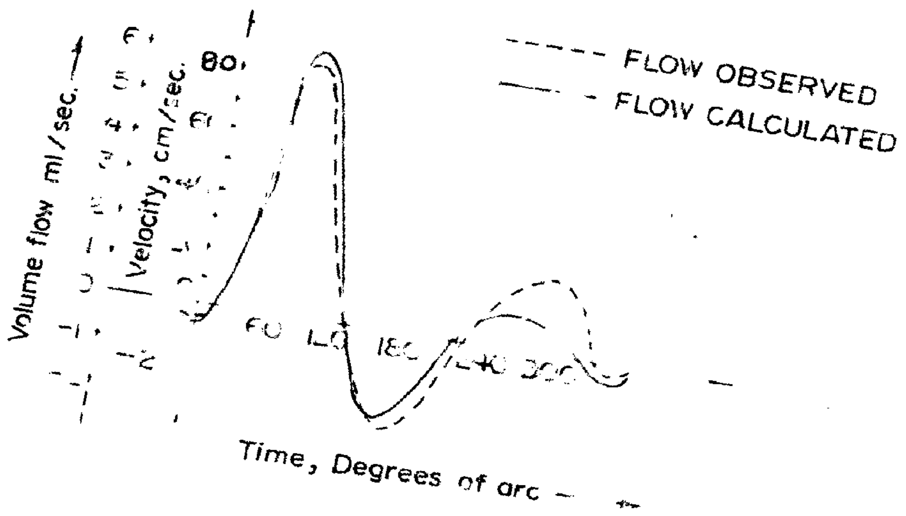


FIG. 4 TYPICAL FLOW IN ARTERY.

a peak value of 5.65 ml/sec. in the forward direction, and 2.85 ml/sec. in the backward direction i.e. towards the heart.

Backflow-

Blood flows towards the heart during part of a cardiac cycle as a normal phenomenon in many arteries. Backflow is expected in all arteries and will only fail to appear when the steady flow is greater than the negative component of the compound oscillatory wave. Physiologically the appearance, or absence, of a backflow will be determined largely by changes in mean flow. It will also depend on changes in the shape of the compound oscillatory flow curve.

The Steady Pressure-gradient

The measurement of steady flow is thus an important factor in calculating arterial flow curves. In the arterial pressure gradient there is a steady term, which is related to the steady flow by POISEUILLE'S formula. The mean pressure drop along arteries is extremely small eg. the gradient corresponding to the mean flow in fig.(4) (12 cm/sec) is only 0.13 mm Hg/cm.

4.3 THE FLOW RELATED TO AN OSCILLATING PRESSURE GRADIENT

If the pulsatile flow and pressure in arteries are expressed as a fourier-series then the elementary case to consider is that of simple harmonic motion of liquid in a cylindrical tube. The mathematical treatment is initially similar to the standard derivation of POISEUILLE's law. The basic assumption made are the same, with the single difference that whereas in steady flow the pressure difference, $P_1 - P_2$ between the two ends of the pipe of length L is constant with time in the present case the pressure gradient $(P_1 - P_2)/L$ oscillates in harmonic motion. The other assumptions on which the analysis is based are[8]:

1. The flow is laminar,
2. The tube is long,
3. The viscosity of the liquid is independent of the rate of shear i.e. it is a Newtonian liquid.
4. There is no 'slip' at the wall.
5. The radius of the tube does not vary e.g. with changes of pressure.

The equation of motion is a unidimensional form of the Navier-Stokes equation and can be written,

$$\frac{\partial^2 \omega}{\partial r^2} + \frac{1}{r}\frac{\partial \omega}{\partial r} + \frac{P_1 - P_2}{\mu\, L} = \frac{1}{\nu}\frac{\partial \omega}{\partial t} \qquad \ldots (9)$$

where, ω is the velocity of the liquid parallel to the axis of the pipe (the z-axis) at a distance 'r' from the axis.

μ = viscosity of the liquid,

ν = kinematic viscosity (μ/ρ)

ρ = density,

R = radius of the pipe.

Taking a pressure gradient which is periodic with time with a circular frequency ω so that

$$(P_1-P_2)/L = -A\,e^{i\omega t} \qquad \ldots (10)$$

and the equation of motion is rewritten,

$$\frac{\partial^2\omega}{\partial r^2} + \frac{1}{r}\cdot\frac{\partial\omega}{\partial r} - \frac{1}{\nu}\,\frac{\partial\omega}{\partial t} = \frac{A}{\mu}\,e^{i\omega t} \qquad \ldots (11)$$

Let $\omega = u\,e^{i\omega t}$ and let the non-dimensional-quantity $R\sqrt{(\omega/\nu)}$ be denoted by α. Then the equation for u is,

$$\frac{\partial^2 u}{\partial y^2} + \frac{1}{y}\,\frac{\partial u}{\partial y} + i^3\alpha^2 u = \frac{AR^2}{\mu} \qquad \ldots (12)$$

(where $y = r/R$)

which is a form of Bessel's equation and the required solution after replacing ω is

$$\omega = \frac{AR^2}{\mu}\,\frac{1}{i^3\alpha^2}\left\{1-\frac{J_o(\alpha y i^{3/2})}{J_o(\alpha i^{3/2})}\right\} e^{i\omega t} \qquad \ldots (13)$$

Womersley integrated equation (13) to give the solution for the volume flow,

$$Q = \frac{\pi R^4}{\mu}\,\frac{A}{i^3\alpha^2}\left\{1-\frac{2J_1(\alpha i^{3/2})}{\alpha i^{3/2}J_o(\alpha i^{3/2})}\right\} e^{i\omega t} \qquad \ldots (14)$$

If the real part of the pressure gradient $Ae^{i\omega t}$ is $M\cos(\omega t-\emptyset)$ then the equation for volume flow is

$$Q = \frac{\pi R^4}{\mu}\,\frac{M.M'_{10}}{\alpha^2}\,\sin(\omega t-\emptyset+\varepsilon_{10}) \qquad \ldots (15)$$

M'_{10}/α^2 and ε_{10} are functions of α.

The Significance of the parameter α in determining Oscillatory Flow

If the equation (15) is compared with that of POISEUILLE'S formula which is,

$$Q = \frac{\pi R^4(P_1-P_2)}{8\mu L} \qquad \ldots (16)$$

the similarity can be seen at once, remembering that the pressure gradient $(P_1-P_2)/L$ is written as $M \cos(\omega t-\phi)$. The factor M'_{10}/α^2 modifies the modulus, or amplitude, of the flow and, in addition, a phase shift ε_{10} is introduced, so that flow lags $(90-\varepsilon_{10})$ behind the pressure-gradient.

Thus for a given pipe, if the pressure gradient oscillates at a very low frequency so that α is small, then the flow oscillate with it with a negligible phase lag and its amplitude will be nearly that given by Poiseuille's formula for steady flow. As the frequency inwreases the amplitude of the oscillation of the fluid, with a constant-pressure oscillation, will diminish progressively and the phase lag will approach 90°. The valme of α increases linearly with the radius of the pipe and with the square root of the frequency, that is

$$\alpha = R\sqrt{(\omega/\nu)} \qquad \ldots (18)$$

4.4 ARTERIAL-IMPEDANCE

For flow in a rigid walled tube, under a pressure gradient $-Ae^{i\omega t}$, we have the average velocity[11],

$$\bar{\omega} = \frac{Q}{\pi R^2} = \frac{AR^2}{i\mu} \frac{M'_{10}}{\alpha^2} e^{i\varepsilon_{10}} e^{i\omega t}$$

$$= A\, e^{i\omega t} \frac{M'_{10}}{i\omega\rho} e^{i\varepsilon_{10}} \qquad \ldots (19)$$

then the longitudinal impedance of the tube per unit length is

$$Z = \frac{A\, e^{i\omega t}}{\bar{\omega}}$$

$$= \frac{\omega\rho}{M'_{10}} \sin\varepsilon_{10} + \frac{i\omega\rho}{M'_{10}} \cos\varepsilon_{10} \qquad \ldots (20)$$

It can be seen that for high frequencies the impedance approaches $i\omega\rho$, that is, the motion of the fluid is governed by its density only, and the impedance is a pure inductance.

The input impedance (Z_o) of a conducting system depends both on the longitudinal impedance (Z) and the wave velocity (C) and is defined as,

$$Z_o = Z.\ C/i\omega \qquad \ldots (21)$$

For the 'tethered and loaded tube' Womersley found

$$Z_o = \frac{C_o}{\sqrt{1-\sigma^2}} \cdot \frac{1}{\sqrt{M'_{10}}} \exp(-i^{1/2}\varepsilon_{10}) \qquad \ldots (22)$$

It can be seen that with increasing α, and M'_{10} tends to 1 from below, and ε_{10} tends to 0 from above, the input impedance approaches that for the inviscid system. In a perfectly elastic system C_o denotes the value given by the Moens-Korteweg formula. If the wall material is elasto-viscous, then C_o will not be constant, but will rise with frequency.

4.5 THE BEHAVIOUR OF THE ARTERIAL WALL

The first model of an artery considered was that of a thin-walled elastic tube, containing a viscous fluid in oscillatory motion.

The movements of the wall are also of physiological interest. It is well known that arteries dilate with each cardiac ejection. Womersley pointed out that in the body the arterial expansion is more directly related to the flow.

$$\frac{2\xi}{R} = \frac{\bar{\omega}}{C} \qquad \ldots (23)$$

where,

$\bar{\omega}$ is the average velocity of flow,

C the wave velocity,

ξ the radial displacement.

R the mean radius of the artery.

In addition to the radial movement, the viscous drag of the fluid will also cause a longitudinal movement of the wall of a free elastic tube. This effect should in fact be larger in magnitude than the radial movement. Further observations have shown that the arteries are largely tethered by the connective tissue around them, and thus it is not realistic to use the free elastic tube as a model.

WOMERSLEY considered the model of an elastic tube subjected to an external longitudinal restraint, and loaded by the mass of the tissues around it. Mathematically,

$$K = \frac{h}{R} \text{ (for free-elastic tube)}$$

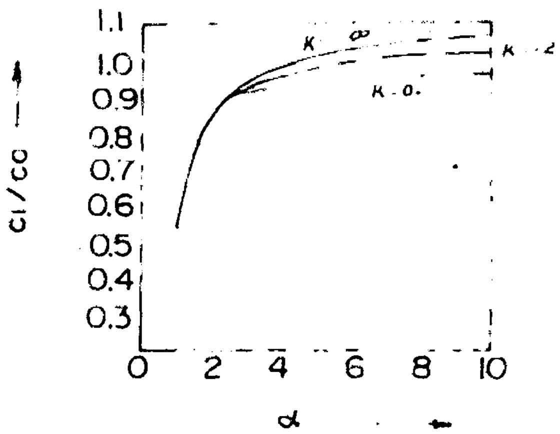


FIG.5(a) VARIATION OF PHASE VELOCITY (c_1) WITH α

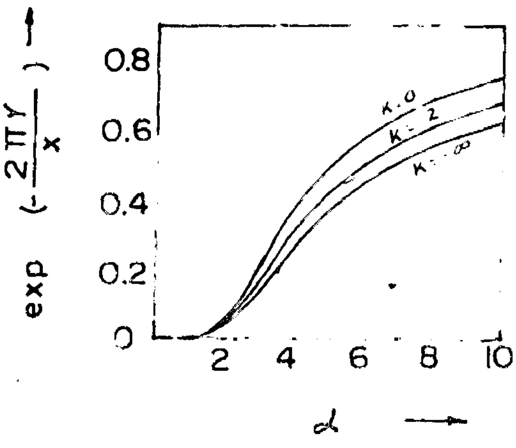


FIG.5(b) VARIATION OF DAMPING WITH α

where h is the wall thickness.

With the 'tethered and loaded tube' the term K is defined as,[8]

$$K = (1 - \frac{h_1 \rho_1 R_1}{h \rho R})(1 - \frac{m^2}{\omega^2}) \qquad \ldots (24)$$

where, h_1 = thickness of the loading mass

ρ, ρ_1 = densities,

R, R_1 = radius.

m = natural frequency of the longitudinal restraint.

and ω = circular frequency of the oscillation.

With a fairly stiff constraint (as in the body) $m \gg \omega$ and K is negative, tending to $-\infty$ as m is increased indefinitely. The effect of this modification of wall behaviour on wave transmission is shown in fig.(5). It can be seen that the asymptotic value of the phase-velocity is now greater than that for a non-viscous fluid in the same tube, and that the damping has increased. Both these effects are due to an increase in stiffness of the tube. Thus the 'tethered and loaded tube' is the satisfactory model of the artery.

4.6 Wave-Transmission

The study of dynamic local distensibility of arteries shows that the arterial tree behaves as a wave transmission system.

The distensibility of a segment of artery is defined as the increase in volume per unit increase in transmural pressure. Local vessel distensibility per unit length may be defined as dS/dp where dS is the increment in cross-sectional area S and dP is the increment in pressure p.

In the description of the arterial circulation as a wave transmission system, the heart is considered to be a wave generator, linked to the periphery via an intensive set of short vessel segments undergoing multiple branching. In analogy with transmission line theory, each vessel segment is fully characterised by a set of three quantities[14]-

(a) Longitudinal fluid impedance per unit length (Z_ℓ) defined per harmonic as the ratio of pressure gradient, -dP/dZ, over pulsatile flow, dQ.

(b) Transverse wall admittance per unit length, y_t, defined per harmonic as the ratio of flow gradient, -dQ/dZ, over pulsatile pressure, dP. Transverse impedance $Z_t = 1/y_t$

(c) Segment length, L.

The pressure-flow relationship in such a segment is characterized by one more quantity.

(d) Input impedance, Z_i defined per harmonic as the ratio of pressure,P, overflow Q, at the entrance of the segment.

The two impedances Z_ℓ and Z_t have equal importance in defining the transmission of pressure and flow waves in the arterial tree. They can be described by such quantities as characteristic impedance $Z_o = \sqrt{Z_\ell Z_t}$, propagation constant $\gamma = \sqrt{Z_\ell / Z_t}$. All of the above quantities are complex and frequency dependent and thus we have for the propagation constant

$$\gamma(\omega) = \alpha(\omega) + j\beta(\omega) \qquad \ldots (25)$$

in which α determines the wave attenuation and β the phase velocity 'C' (ω = angular frequency)

$$C(\omega) = \omega/\beta(\omega) \qquad \ldots (26)$$

For a vessel segment of cross-sectional area S and infinitesimal length dZ, through which the flow of incompressible fluid is Q, the net increase in volume per unit time (t) and per unit length can be expressed as $-dQ/dZ$ on one hand and as dS/dt on the other hand. Thus we have,

$$-\frac{dQ}{dZ} = \frac{dS}{dt} \qquad \ldots (27)$$

For the harmonic of S with angular frequency ω we have

$$dS/dt = j\omega dS \qquad \ldots (28)$$

and so we can write,

$$\begin{aligned} y_t &= (-dQ/dZ)dp \\ &= j\omega(dS/dp) \qquad \ldots (29) \end{aligned}$$

$$\therefore Z_t = 1/j\omega\ (dP/dS) \qquad \ldots (30)$$

Thus the transverse wall impedance is inversely proportional to local vessel distensibility per unit length.

CHAPTER - FIVE

THE MATHEMATICAL ANALYSIS OF THE ARTERIAL CIRCULATION

5.1 INTRODUCTION

For the analysis of pulsatile blood flow through a distensible vessel it is necessary to develop a system of simultaneous equations which express the balance of forces and the conservation of mass at every point in the fluid and its boundary. The equations of motion and of continuity for the blood and the vessel wall represent two sets of four independent partial differential equations which contain the three coordinates and time as independent variables. The equation of state, which relates pressure, density and temperature is usually not required under physiological conditions, since blood may be treated as an incompressible fluid of density ρ at constant temperature.

5.2 NAVIER-STOKES EQUATIONS

Assuming no leakage flow through the walls the basic flow equations are given by the continuity equation[15]

$$\frac{\partial V_r}{\partial r} + \frac{V_r}{r} + \frac{\partial V_z}{\partial Z} = 0 \qquad \ldots (31)$$

and the Navier-Stokes equation for an incompressible fluid

$$\frac{\partial V_z}{\partial t} + V_r \frac{\partial V_z}{\partial r} + V_z \frac{\partial V_z}{\partial Z} = F_z - \frac{1}{\rho} \frac{\partial p}{\partial Z} + \nu \left(\frac{\partial^2 V_z}{\partial r^2} + \frac{1}{r} \frac{\partial V_z}{\partial r} + \frac{\partial^2 V_z}{\partial Z^2} \right) \qquad \ldots (32)$$

$$\frac{\partial V_r}{\partial t} + V_r \frac{\partial V_r}{\partial r} + V_z \frac{\partial V_r}{\partial Z} = F_r - \frac{1}{\rho} \frac{\partial p}{\partial r} + \nu \left(\frac{\partial^2 V_r}{\partial r^2} + \frac{1}{r} \frac{\partial V_r}{\partial r} - \frac{V_r}{r^2} + \frac{\partial^2 V_r}{\partial Z^2} \right) \qquad \ldots (33)$$

where, r and Z are the radial and longitudinal coordinates respectively, t = time, p = the pressure, r_i = the internal vessel radius, ρ = the density of the blood, $\nu = \frac{\mu}{\rho}$ the kinematic viscosity of the blood, μ = the viscosity of the blood, V_z = the instantaneous velocity parallel to the vessel axis, V_r = the instantaneous fluid velocity along a radial coordinate, and F = the sum of external body forces such as gravity etc.

The forces associated with the pressure gradient $[(1/\rho)(\partial p/\partial Z)]$ and $[1/\rho(\partial p/\partial r)]$ are balanced by the inertial forces which are left hand sides of the equations (32) and (33) the frictional forces (the bracketed terms on the right hand side of equation (32) and (33) and the body forces F.

5.3 ASSUMPTIONS FOR PULSATILE BLOOD FLOW

The validity of these equations rests primarily on the assumptions that,

1) The blood behaves like a Newtonian fluid. In a Newtonian fluid stress and rate of strain are linearly related (the viscosity is independent of the shear rate).

2) Only laminar flows, without secondary flows or turbulence, are present.

For pulsatile flow the total acceleration consists of two terms, (a) the local acceleration due to the variation of driving pressure with time ($\partial p/\partial t$) and(b) the acceleration due to changes in geometry of the flow channels, such as an increase or decrease in cross-section.

Certain assumptions are made to simplify the Navier-Stokes equations.

1. The artery or vessel is an elastic one.
2. The heart pressure variation is a periodic one.
3. The velocity has both radial and axial components.
4. The nonlinear terms of the equations are neglected.
5. The artery is of circular cross-section.
6. External forces to the body i.e. F_z and F_r are negligibly small.

On imposing the above assumptions, equation (32) simplifies to

$$\left[\frac{\partial^2 V_z}{\partial Z^2} + \frac{\partial^2 V_z}{\partial r^2} + \frac{1}{r}\frac{\partial V_z}{\partial r}\right] = \frac{1}{\nu}\left[\frac{\partial V_z}{\partial t}\right] \qquad \ldots (34)$$

For a sinusoidal pressure wave

$$p = A_1 \exp\left[j\omega\left(t - \frac{Z}{C}\right)\right] \qquad \ldots (35)$$

where,

C = wave velocity,

ω = angular frequency

A_1 = Modulus of pressure wave.

5.4 LONGITUDINAL VELOCITY (V_z)

Let V_z = RTZ

$$= R\left[e^{j\omega(t-Z/C)}\right]$$

where, R and Z are space variables,

T = time variable

$$\therefore RTZ'' = \frac{\partial^2 V_z}{\partial Z^2}$$

$$\left.\begin{aligned} R''TZ &= \frac{\partial^2 V_z}{\partial r^2} \\ R'ZT &= \frac{\partial V_z}{dr} \\ RZT' &= \frac{\partial V_z}{\partial t} \end{aligned}\right\} \quad \ldots (36)$$

Putting in (34) the above values, we get,

$$RTZ'' + R''TZ + \frac{1}{r}R'ZT = 1/_v \left[RZT' \right] \quad \ldots (37)$$

Dividing throughout by RTZ we get

$$\frac{Z''}{Z} + \frac{R''}{R} + \frac{1}{r}\frac{R'}{R} = \frac{1}{v}\left[\frac{T'}{T}\right] \quad \ldots (38)$$

$$\frac{Z''}{Z} = \frac{(j\omega)^2}{C^2} = -\frac{\omega^2}{C^2} \text{ and } \frac{T'}{T} = j\omega$$

Putting in (38) we get ,

$$-\frac{\omega^2}{C^2} + \frac{R''}{R} + \frac{1}{r}\frac{R'}{R} = \frac{1}{v}(j\omega)$$

or $$\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} - \frac{j\omega}{v} = \frac{\omega^2}{C^2} = 0 \quad \ldots (39)$$

(since ω is very small as compared to C, hence $(\omega/C)^2$ is very very small)

Multiplying (39) by r^2R

$$\therefore r^2R'' + rR' - r^2\left(\frac{j\omega}{v}\right)R = 0 \quad \ldots (40)$$

Let $\frac{j^3\omega}{v} = \alpha^2 = -\frac{j\omega}{v}$ therefore (40) becomes

$$\therefore r^2R'' + rR' + (r^2\alpha^2)R = 0 \quad \ldots (41)$$

Let $r^2\alpha^2 = x^2$ (this is done to bring equation in the Bessel form)

$$\therefore r\alpha = x$$

or $$\alpha . \partial r = \partial x$$

Thus , we get

$$x^2 \frac{\partial^2 R}{\partial x^2} + x \frac{\partial R}{\partial x} + x^2 R = 0 \qquad \ldots (42)$$

The equation (42) is the Bessel form of the differential equation.

Solution- $R = A_o J_o(x) + B_o Y_o(x)$

$$= A_o J_o(\alpha r) + B_o Y_o(\alpha r) \qquad \ldots (43)$$

$Y_o(o) = \infty$, when $r = 0$

To get the FINITE VELOCITY at $r = 0$, $B_o = 0$

$$\therefore R = A_o J_o(\alpha r) \qquad \ldots (44)$$

We have, $RZT = R\, e^{j\omega(t-Z/C)} = V_z$

or $V_z = A_o J_o(\alpha r)\left[e^{j\omega(t-Z/C)}\right]$

when $r = a$ (outer radius of artery), velocity $= C_1$

$\therefore A_o = \frac{C_1}{J_o(\alpha a)}$ and the complementary solution is

$$V_z = C_1 \frac{J_o(\alpha r)}{J_o(\alpha a)} e^{j\omega(t-Z/C)} \qquad \ldots (45)$$

To find the particular solution

Gradient of the velocity = 0

$$\frac{\partial V_z}{\partial t} = -\frac{1}{\rho}\frac{\partial}{\partial Z}\left[A_1 e^{j\omega(t-Z/C)}\right]$$

$$= -\frac{1}{\rho} A_1 \left[-\frac{j\omega}{C}\right] e^{j\omega(t-Z/C)}$$

$$\therefore \qquad \partial V_z = -\frac{1}{\rho} A_1 \left[-\frac{j\omega}{C}\right] e^{j\omega(t-Z/C)}\, \partial t$$

Integrating

$$\int \partial V_z = V_z = \frac{j\omega A_1}{\rho C} \int e^{j\omega(t-Z/C)} \partial t$$

$$= \frac{A_1}{\rho C} \left\{ e^{j\omega(t-Z/C)} \right\}$$

$$V_z = \frac{p}{\rho C} \qquad \ldots (46)$$

∴ Complete solution is, = C.F. + P.I.

$$V_z = \left[\frac{C_1 J_o(\alpha r)}{J_o(\alpha a)} + \frac{A_1}{\rho C} \right] e^{j\omega(t-Z/C)} \qquad \ldots (47)$$

The above is the expression for longitudinal velocity.

5.5 RADIAL VELOCITY (V_r)

Continuity Expression,

$$\frac{\partial V_z}{\partial Z} + \frac{V_r}{r} + \frac{\partial V_r}{\partial r} = 0 \qquad \ldots (48)$$

$$\frac{\partial V_r}{\partial r} = 0 \qquad \therefore \frac{\partial V_z}{\partial Z} + \frac{V_r}{r} = 0$$

$$\frac{V_r}{r} = -\frac{\partial V_z}{\partial Z}$$

$$\frac{1}{r} \frac{\partial}{\partial r}(r.V_r) = -\frac{\partial V_z}{\partial Z}$$

$$= -\frac{\partial}{\partial Z} \left[C_1 \frac{J_o(\alpha r)}{J_o(\alpha a)} + \frac{A_1}{\rho C} \right] e^{j\omega(t-Z/C)}$$

$$\int x J_o(x) dx = x\, J_1(x)$$

$$\partial/\partial r\ (r\ V_r) = -r \left[\frac{C_1 J_o(\alpha r)}{J_o(\alpha a)} + \frac{A_1}{\rho C} \right] \left(-\frac{j\omega}{C}\right) e^{j\omega(t-Z/C)}$$

$$\partial(r.V_r) = -r \left[\frac{C_1 J_o(\alpha r)}{J_o(\alpha a)} + \frac{A_1}{\rho C} \right] - \frac{j\omega}{C} e^{j\omega(t-Z/C)} \partial r$$

Integrating the above within the limits 0 to r, finally,

$$V_r = \frac{j\omega}{C} \left[\frac{C_1 J_1(\alpha r)}{\alpha J_o(\alpha a)} + \frac{A_1 r}{2\rho C} \right] e^{j\omega(t-Z/C)} \qquad \ldots (49)$$

Thus the radial velocity (V_r) can be calculated.

5.6 FLOW RATE OF THE FLUID(I)

I = Area x longitudinal velocity

$$I = \int_0^{r_i} V_z (2\pi r) dr \qquad \ldots (50)$$

$$= \int_0^{r_i} \left[C_1 \frac{J_o(\alpha r)}{J_o(\alpha a)} + \frac{A_1}{\rho C} \right] e^{j\omega(t-Z/C)} (2\pi r)\, dr$$

$$= \pi r_i^2 \frac{A_1}{C} \left[1 + 2C_1 \frac{\rho C}{A_1} \frac{J_1(\alpha r_i)}{\alpha J_o(\alpha r_i)} \right] e^{j\omega(t-Z/C)}$$

Let $F_{10} = \frac{2J_o(\alpha r_i)}{\alpha J_o(\alpha r_i)}$ = Womersley's constant.

$$\therefore \quad I = \pi r_i^2 \frac{A_1}{\rho C} \left[1 + \frac{C_1 \rho C}{A_1} F_{10} \right] e^{j\omega(t-Z/C)} \qquad \ldots (51)$$

This gives the fluid flow rate.

5.7 LONGITUDINAL IMPEDANCE (Z_ℓ)

It is given by,

$$Z_\ell' = -\frac{\partial P}{\partial Z}/I \qquad \ldots (52)$$

where $p = A_1 e^{j\omega(t-Z/C)}$

$$\therefore \quad \frac{\partial p}{\partial Z} = -j\omega/C.\ p$$

Thus $Z_\ell' = \frac{j\omega/C.p}{I}$

$$= \frac{\frac{j\omega}{C} A_1 e^{j\omega(t-Z/C)}}{\pi r_i^2 \frac{A_1}{\rho C} \left[1 + \frac{C_1 \rho C}{A_1} F_{10} \right] e^{j\omega(t-Z/C)}}$$

$$= \frac{j\omega\rho}{\pi r_1^2} \left[1 + \frac{C_1 \rho C}{A_1} F_{10} \right]^{-1}$$

$$= \frac{j\omega\rho}{\pi r_1^2.} \left[1 + \eta F_{10} \right]^{-1} \qquad \ldots (53)$$

where $\eta = \frac{C_1 \rho C}{A_1} = -1 \text{(approx)}$

5.8 TRANSVERSE-IMPEDANCE (Z_t')

It is given by $$Z_t' = -\frac{p}{\partial I/\partial Z} \qquad \ldots (54)$$

$$= -A_1 \frac{e^{j\omega(t-Z/C)}}{\partial I/\partial Z}$$

since $$I = \frac{\pi A_1 r_i^2}{\rho C}\left[1+\frac{C_1 \rho C}{A_1} F_{10}\right] e^{j\omega(t-Z/C)}$$

$$\therefore \frac{\partial I}{\partial Z} = \left(-\frac{j\omega}{C}\right)\left[\frac{\pi A_1 r_i^2}{\rho C}\left\{1+\frac{C_1 \rho C}{A_1} F_{10}\right\} e^{j\omega(t-Z/C)}\right]$$

Hence $$Z_t' = \frac{-A_1 \cdot e^{j\omega(t-Z/C)}}{-\frac{j\omega}{C}\frac{\pi r_i^2}{\rho C} A_1 e^{j\omega(t-Z/C)}\left[1+\frac{C_1 \rho C}{A_1} F_{10}\right]}$$

$$= \frac{\rho C^2}{j\omega\pi r_i^2}\left[1 + C_1 \frac{\rho C}{A_1} F_{10}\right]^{-1}$$

$$= \frac{\rho C^2}{j\omega\pi r_i^2}\left[1+\eta F_{10}\right]^{-1} \qquad \ldots (55)$$

Note- For a rigid walled-tube $\eta = -1$, but for an elastic thin walled tube, η is a function of α, σ and K.

$\therefore$ Characteristic Impedance $(Z_o) = \sqrt{Z_\ell' \, Z_t'}$

and Propogation Constant (γ), $\gamma = \sqrt{Z_\ell'/Z_t'}$

On putting the values of Z_ℓ' and Z_t' we get finally

$$Z_o = \frac{C}{\pi r_i^2}\left[1 + \eta F_{10}\right]^{-1} \qquad \ldots (56)$$

and

$$\gamma = \frac{j\omega}{C} \qquad \ldots (57)$$

Thus characteristic impedance and propagation constant can be determined.

5.9 EQUATIONS OF MOTION

For a thin-walled, uniform , isotropic cylindrical tube having internal viscous, elastic and inertial properties and external coupling to the surrounding tissues the equations of motion become[15]

$$\rho_\omega H \frac{\partial^2 u_r}{\partial t^2} - p + \frac{E^* h}{1-\sigma^{*2}} \cdot \frac{\sigma^* \partial u^2}{r\, \partial Z} + \frac{u_r}{r^2} = 0 \qquad \ldots (58)$$

$$\rho_\omega H \frac{\partial^2 u_z}{\partial t^2} - \rho_\omega H \Omega^2 u_z + \mu\left(\frac{\partial V_z}{\partial r} + \frac{\partial v_r}{\partial Z}\right)_{r=r_i} - \frac{E^* h}{1-\sigma^{*2}}\left(\frac{\partial^2 u_z}{\partial Z^2} + \frac{\sigma^* \partial u_r}{r dZ}\right) = 0 \qquad \ldots (59)$$

where,

ρ_ω = density of the wall material,

H = weighted volume of wall substance taking into account external loading,

E^* = complex modulus of elasticity, real part of which is young's modulus.

σ^* = complex poisson-ratio

h = wall thickness,

Ω = natural frequency of the longitudinal elastic constraint,

u_r and u_2= displacement of a point on the inner surface of the wall in the radial (u_r) or axial (u_2) direction

r_i = Internal radius.

The three terms in equation (58) represent, respectively, radial inertial force, radial stress of the fluid pressure (transmural pressure), and radial stress related to visco-elastic deformation of the wall substance. ~~All forces are viscoelastic deformation of the wall substance~~. All forces are expressed in terms of per unit area of inner wall.

The four terms in equation (59) express the inertial force, the spring force of the external constraint, the drag force from the underlying flow and the force of visco-elastic deformation of the wall substance all acting in Z direction.

Assumptions to Analyse the Equations of Motion

1. The vessel wall is thin i.e. $h/r_i < 0.1$. Thus we can neglect radial velocity gradients within the wall substance.
2. The displacements u_r and u_z and their derivatives are small.
3. The physical properties of the wall material are linear.
4. The wall material is isotropic and homogeneous.
5. The magnitudes of the real parts of E^* and σ^* are much greater than those of the imaginary parts.

The equations for motion of the fluid and of the wall are coupled by the condition that the fluid does not slip along the surface of the wall, i.e.

$$V_r = \left.\frac{\partial u_r}{\partial t}\right|_{r=r_i}$$

$$\text{and} \quad V_z = \left.\frac{\partial u_z}{\partial t}\right|_{r=r_i} \qquad \ldots (60)$$

We have from the previous derivations,

$$V_z = \left[C_1 \frac{J_o(\alpha r)}{J_o(\alpha r_i)} + \frac{A_1}{\rho C} \right] e^{j\omega(t-Z/C)} \qquad \ldots (61)$$

and $$V_r = \frac{j\omega}{2C}\left[\frac{2C_1 J_o(\alpha r)}{J_o(\alpha r_i)} + \frac{A_1 r}{\rho C}\right] e^{j\omega(t-Z/C)} \qquad \ldots (62)$$

The wall equations (58) and (59) are already linear and may be solved by substituting

$$u_z = B_1 \exp\left[j\omega(t- Z/C)\right] \qquad \ldots (63)$$

and $$u_r = D_1 \exp\left[j\omega(t-Z/C)\right] \qquad \ldots (64)$$

Substituting the derivatives of (63) and (64) in (58) and (59) we get finally,

$$\rho_\omega H\omega^2 D_1 + A_1 - \frac{E^* h}{1-\sigma^{*2}}\left[\frac{D_1}{r_i^2} - j\omega\,\frac{\sigma^* B_1}{r_i C}\right] = 0 \qquad \ldots (64)$$

and

$$\rho_\omega H B_1(\omega^2-\Omega^2) - \frac{\mu}{r_i}\left(\frac{\omega^2 r_i^2 A_1}{2C^3{}_o} - \frac{j^3\alpha^2 F_{10} C_1}{2}\right) - \frac{E^* h}{1-\sigma^{*2}}\left(\frac{\omega^2 B_1}{C^2} + \frac{j\omega\sigma^{*2} D_1}{r_i C}\right)$$
$$= 0 \qquad \ldots (65)$$

where B_1 and D_1 are additional constants.

From the boundary conditions

$$V_r = \frac{\partial u_r}{\partial t}\Big|_{r=r_i}$$

$$\frac{j\omega}{2C}\left[\frac{2C_1 J_o(\alpha r)}{J_o(\alpha r_i)} + \frac{A_1 r_i}{\rho C}\right] e^{j\omega(t-Z/C)} = \frac{\partial}{\partial t} D_1 e^{j\omega(t-Z/C)}$$
$$= j\omega D_1 e^{j\omega(t-Z/C)}$$

$$D_1 = \frac{r_i}{2C}\left[\frac{2C_1 J_o(\alpha r)}{r_i J_o(\alpha r_i)} + \frac{A_1}{\rho C}\right]$$

$$= \frac{r_i}{2C}\left[F_{10} C_1 + \frac{A_1}{\rho C}\right] \qquad \ldots (66)$$

From second boundary condition

$$V_z = \frac{\partial u_z}{\partial t}\bigg|_{r=r_i}$$

or $$\left[\frac{C_1 J_o(\alpha r_i)}{J_o(\alpha r_i)} + \frac{A_1}{\rho C}\right] e^{j\omega(t-Z/C)} = \frac{\partial}{\partial t}\left[B_1 e^{j\omega(t-Z/C)}\right] = j\omega B_1 e^{j\omega(t-Z/C)}$$

$$\therefore\ j\omega B_1 = \left[C_1 + \frac{A_1}{\rho C}\right] \qquad \ldots (67)$$

Equations (64) to (67) represent a system of linear homogeneous equations for the complex constants, A_1, B_1, C_1 and D_1 and the condition for the existence of a non-trivial solution requires that the determinant of the coefficients of these quantities vanish.[16]

Assuming, external coupling of the tissues to the wall is not there i.e. $\Omega = 0$ and $H = h$. Finally we get,

$$\left[\frac{h\,E}{(1-\sigma^2)\,r_i \rho_\omega C^2}\right]^2 (1-\sigma^2)(1-F_{10}) - \frac{hE}{(1-\sigma^2)\,r_i \rho C^2}\left[2+K(1-F_{10}) + F_{10}\left(\frac{1}{2} - 2\sigma\right)\right] + F_{10} + 2K = 0 \qquad \ldots (68)$$

where $K = h/r_i$

This equation has roots,

$$\frac{hE}{r_i \rho_\omega C^2} = G \pm \sqrt{G^2 - (1-\sigma^2)H} \qquad \ldots (69)$$

with $$G = \frac{5/4 - \sigma}{1-F_{10}} + \frac{K}{2} + \sigma - \frac{1}{4} \qquad \ldots (70)$$

and $$H = \left[\frac{1 + 2K}{1 - F_{10}} - 1\right] \qquad \ldots (71)$$

Denoting,

$$\left(\frac{hE}{2r_o \rho C^2} \right)^{1/2} = X - jY \qquad \ldots (72)$$

and
$$C_o = \sqrt{2\frac{hE}{\rho r_o}} \qquad \ldots (73)$$

the phase velocity C_1 equals

$$C_1 = \frac{C_o}{X} \qquad \ldots (74)$$

and the amplitude reduction in terms of the wavelength λ,

$$e^{-2\pi YZ/X\lambda} \qquad \ldots (75)$$

Womersley provided detailed tables for C_o/C, X^{-1}, $2\pi Y/X$ and $e^{-2\pi Y/X}$ as a function of α for $\alpha = 1(0.05)10$ and for a number of values for K and σ . Figure (   ) shows C_1/C_o as a function of α for $K = 0, -2, -\infty$.

$K = 0$, represents a very thin unconstrained tube

$K = -\infty$, a tube with complete longitudinal constraint and

$K = -2$, one with a small degree of constraint.

5.10 ELLIPTICAL CROSS-SECTION OF ARTERY

We have considered the artery to be of circular cross-section. Now considering it to be of elliptical cross-section and applying a suitable transformation we can transform the cross-section to a circular one. Thus we can apply the well known equations for the analysis,

$$\frac{u^2}{a^2} + \frac{v^2}{b^2} = 1 \qquad \ldots (76)$$

is the equation of the ellipse.

A transformation $W = Z + \frac{\alpha^2}{Z}$ is tried, ... (77)

$$\frac{dW}{dZ} = 1 - \frac{\alpha^2}{Z^2}$$

The derivative is analytic every where except at $Z = 0$ putting $W = u + jv$

and $Z = X + jY$ in equation (77) and equating real and imaginary parts, we get,

$$u = x + \frac{\alpha^2 x}{x^2+y^2}$$

$$v = y - \frac{\alpha^2 y}{x^2+y^2}$$

Now $$\frac{\partial u}{\partial x} = 1 + \frac{\alpha^2\left[(x^2+y^2)-2x^2\right]}{(x^2+y^2)^2}$$

$$= 1 + \frac{\alpha^2(y^2-x^2)}{(x^2+y^2)^2}$$

$$\frac{\partial v}{\partial y} = 1 - \frac{\alpha^2\left[(y^2+x^2)-2y^2\right]}{(x^2+y^2)^2}$$

$$= 1 + \frac{\alpha^2(y^2-x^2)}{(x^2+y^2)^2}$$

and $\frac{\partial u}{\partial y} = -\alpha^2 \frac{2\,xy}{(x^2+y^2)^2}$

$$\frac{\partial v}{\partial x} = \alpha^2 \frac{2xy}{(x^2+y^2)^2}$$

As $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$, the function is analytic.

If a circle of radius R is inscribed in z-plane then $Re^{j\theta} = Z$

$$W = u+jv = Z + \frac{\alpha^2}{Z}$$

$$= Re^{j\theta} + \frac{\alpha^2}{Re^{j\theta}}$$

$$\therefore \quad u = (R + \frac{\alpha^2}{R})\cos\theta = a\cos\theta$$

$$v = (R - \frac{\alpha^2}{R})\sin\theta = b\sin\theta$$

Thus $\frac{u^2}{a^2} + \frac{v^2}{b^2} = 1$

In other words, the circle of radius R is transferred to ellipse of semiaxes a, and b.

Since, $R + \frac{\alpha^2}{R} = a$ and $R - \frac{\alpha^2}{R} = b$

$$\therefore R = \frac{a+b}{2} \text{ and } \alpha^2 = (\frac{a-b}{2})R = \frac{a^2-b^2}{4}$$

Now c = distance of focii from the centre and is given by $c^2 = a^2-b^2$. Thus $\alpha^2 = \frac{a^2-b^2}{4} = \frac{c^2}{4}$

$$\therefore \alpha = \frac{c}{2}$$

Thus the ellipse is transferred to the circle.

Writing the Navier-Stokes equation,

$$\frac{\partial V_z}{\partial t} + V_r \frac{\partial V_z}{\partial r} + V_z \frac{\partial V_z}{\partial Z} = F_z - \frac{1}{\rho} \frac{\partial p}{\partial z} + \nu \left(\frac{\partial^2 V_z}{\partial r^2} + \frac{1}{r} \frac{\partial V_z}{\partial r} + \frac{\partial^2 V_z}{\partial Z^2} \right)$$

Neglecting nonlinear terms and body forces,

$$\frac{\partial p}{\partial Z} = -\rho \frac{\partial V_z}{\partial t} + \mu \left[\frac{1}{r} \frac{\partial V_z}{\partial r} + \frac{\partial^2 V_z}{\partial Z^2} \right] \qquad \ldots (78)$$

Assuming that the tube being considered is broken up axially into short segments of length ΔZ, and radially into N concentric shells.

The radii to the midpoint of the nth annulus are,

$$r_n = \frac{2n-1}{2N-1} R, \qquad n = 1, 2 \ldots . N \qquad \ldots (79)$$

and the separation between the midpoints

$$\Delta r = \frac{2R}{2N-1} \qquad \ldots (80)$$

Let $I_{i,j}$ be the longitudinal flow rate for an annulus.

$$\therefore I_{i,j} = A_{i,j} V_{ij} \qquad \ldots (81)$$

where A_{ij} is the area of the jth annular cross-section of the ith longitudinal section

$$A_{i,j} = \pi \left[(r_n + \frac{\Delta r}{2})^2 - (r_n - \frac{\Delta r}{2})^2 \right]$$

$$I_{ij} = 2\pi r_n \Delta r V_{ij}$$

$$= \frac{4\pi R^2 (2n-1)}{(2N-1)^2} V_{ij}$$

$$\therefore V_{ij} = \frac{(2N-1)^2}{4\pi R^2\ (2n-1)}\, I_{ij} \qquad \ldots (82)$$

Writing equation (78) in finite difference notation,

$$\frac{\Delta P}{\Delta Z} = -\rho\,\frac{dV_{ij}}{dt} + \mu\left[\frac{V_{i,j+1}-V_{i,j-1}}{2\,r_n\Delta r} + \frac{V_{i,j+1}+V_{i,j-1}-2V_{i,j}}{(\Delta r)^2}\right]$$

$$= -\rho\,\frac{dV_{ij}}{dt} - \frac{2\,\mu V_{ij}}{\Delta r^2} + \mu\ V_{i,j+1}\left(\frac{1}{\Delta r^2} + \frac{1}{2r_n\Delta r}\right)$$

$$+ \mu\ V_{i,j-1}\left(\frac{1}{\Delta r^2} - \frac{1}{2r_n\Delta r}\right) \qquad \ldots (83)$$

On substituting from equations (79) and (80) we get;

$$\left(\frac{1}{\Delta r^2} + \frac{1}{2r_n\Delta r}\right) = \frac{(2N-1)^2}{2R^2}\left[\frac{n}{2N-1}\right]$$

Similarly $\left(\frac{1}{\Delta r^2} - \frac{1}{2r_n\Delta r}\right) = \frac{(2N-1)^2}{2R^2}\left(\frac{n-1}{2n-1}\right)$

Putting above values in equation (83), we get

$$\frac{\Delta P}{\Delta Z} = -\rho\,\frac{dV_{ij}}{dt} + \frac{(2N-1)^2}{2R^2}\,\mu\left[-V_{i,j} + \frac{n}{2n-1}\,V_{i,j+1} + \frac{n-1}{2n-1}\,V_{i,j-1}\right] \qquad \ldots (84)$$

But $\Delta P = p_o - p_i$ = difference between output and input pressures.

$$\therefore \frac{p_o-p_i}{\Delta Z} = -\frac{\rho(2N-1)^2}{4\pi R^2(2n-1)}\,\frac{dI_{ij}}{dt} + \frac{(2N-1)^2}{2R^2}\,\mu\left[-\frac{(2N-1)^2}{4\pi R^2(2n-1)}\,I_{i,j}\right]$$

$$= -\frac{\rho(2N-1)^2}{4\pi R^2(2n-1)}\,\frac{dI_{ij}}{dt} - \frac{\mu(2N-1)^4}{8\pi R^4(2n-1)}\,I_{i,j} \qquad \ldots (85)$$

For the simplest case, when N = 2, n = 1

$$\therefore p_o - p_i = -\frac{9\ \rho\Delta Z}{4\pi R^2}\,\frac{dI_{i,j}}{dt} - \frac{81\ \mu\ \Delta Z}{8\pi R^4}\,I_{i,j} \qquad \ldots (86)$$

The coefficient of the last term in equation (86) gives the fluid resistance,

$$Rm = \frac{81\ \mu\Delta Z}{8\pi R^4} \qquad \ldots (87)$$

Note:- The finite difference approximation gives a fluid-resistance only 81/64 larger than that given by Poiseuille's law.

The coefficient of the other term on the right side of equation (86) gives the fluid inductance

$$Lm = \frac{9\ \rho\Delta Z}{4\pi R^2} \qquad \ldots (88)$$

Note: This is 9/4 greater than the inductance obtained by assuming a flat velocity profile, since the velocity always drops off at the edges of the tube. This value of Lm may be regarded as a better approximation.

CHAPTER-SIX

THE ELECTRICAL MODELS OF ARTERIAL TREES

6.1 INTRODUCTION

The application of network theory to the study of circulatory phenomena presents a new and different method of approach which has both advantages and disadvantages. The striking similarities between hemodynamic and electrical quantities such as pressure and voltage, flow and current led to the construction of electrical equivalents of hemodynamic phenomena.

The study of wave transmission problems in an arterial tree seems particularly suited for the electrical analog approach.

Transmission models require division of the arteries into segments. Inserting the parameters of each segment into the equation of motion and the equation of continuity results in a relationship between pressure gradient and flow on the one hand, and between flow gradient and pressure on the other. The form of an electrical delay line is determined by the same relationships, where voltage stands for pressure and current for flow.Therefore, the construction of a passive electrical equivalent of the arterial system seems possible.

6.2 RATIONAL SYSTEM OF UNITS IN HEMODYNAMICS

For many years, the concept of resistance has been used in hemodynamics by analogy with electric circuits. Recently, the analogy has been extended to include the concept of impedance, and inertance and compliance have been defined by analogy with inductance and capacitance, respectively since electrical analogs have been increasingly used as models of circulatory phenomena.

By rational is meant consistent with the fundamental units of physics. Hemodynamics units can rationally be defined in terms of either the C.G.S. or M.K.S. system.[18]

TABLE II

HEMODYNAMIC UNITS AND CGS AND MKS EQUIVALENT

Quantity	MKS Hemodynamic unit	CGS Equivalent	MKS Equivalent
Resistance	Hemodynamic ohm (hohm)	$10^7 g\ cm^{-4} sec^{-1}$	$kg\ m^{-4} sec^{-1}$
Inertance	Wome	$10^7 g\ cm^{-4}$	$kg\ m^{-4}$
Compliance	Frank	$10^{-7} g^{-1} cm^4 sec^2$	$kg^{-1} m^4 sec^2$
Reactance, Impedance	Hohm	$10^7 g\ cm^{-4} sec^{-1}$	$kg\ m^{-4} sec^{-1}$
Power	Watt	$10^7 g\ cm^2 sec^{-3}$	$kg\ m^2 sec^{-3}$

In an ideal hemodynamic resistance (negligible inertance and compliance), the power dissipation is given by

$$W = Q\Delta P \qquad \ldots (1)$$

where,

Q = flow in $cm^3\ sec^{-1}$

ΔP = pressure drop in joules cm^{-3}

W = power dissipation in watts

Thus the hemodynamic ohm or hohm,

$$R = \Delta P/Q \qquad \ldots (2)$$

where R = hemodynamic resistance in hohms.

One hohm is equal to one joule-second-cm^{-6} or 10^7 erg-second-cm^{-6}.

Hemodynamic impedance has been differently defined by different authors. Some define it in terms of pressure and linear velocity (cm/sec). Others define it in terms of pressure and volume flow. For a sinusoidal pressure and flow, the impedance in hohms is

$$Z = \frac{\Delta P_m}{Q_m}, \qquad \ldots (3)$$

where, Z = modulus of hemodynamic impedance in hohms,

ΔP_m = amplitude of pressure sinusoid in joules cm^{-3}

Q_m = amplitude of flow sinusoid in cm^3 sec.$^{-1}$

The inertance is given by

$$L = \Delta P_L/\dot{Q} \qquad \ldots (4)$$

where, L = inertance in joule-second^2-cm^{-6}

ΔP_L = pressure due to inertance in joules cm^{-3}

$\dot{Q}$ = rate of change of flow in cm^3 $second^{-2}$

It is proposed that the unit of inertance be named in honor of late J.R. Womersley and hence it is 'Wome'. One wome is equal to one-joule-sec^2-cm^{-6} or 10^7 erg-sec^2-cm^{-6}.

The compliance is given by

$$C = \Delta V/\Delta P_c \qquad \ldots (5)$$

where;

C = compliance in cm^6-$joule^{-1}$

ΔV = change in volume in cm^3

ΔP_c = change in compliant pressure in joules cm^{-3}.

The unit of compliance is named 'Frank' in the honor of Otto-Frank.

6.3 PHYSICAL MODEL OF THE HUMAN VASCULAR SYSTEM

The model of the circulatory system is shown in Fig.(6a). It consists of two elastic reservoirs connected by a long column of blood.[19]

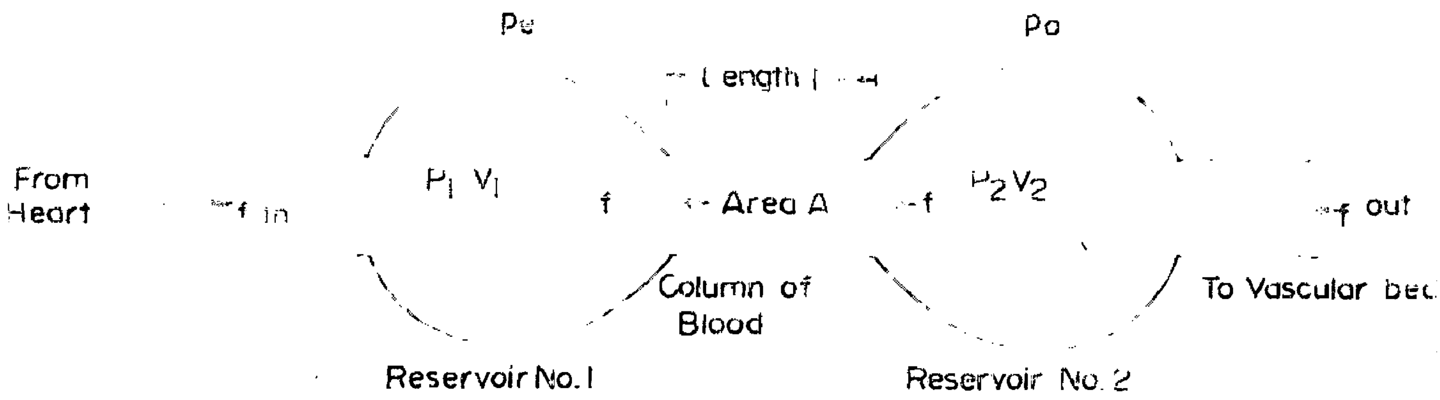
Writing the equation of conservation of mass for the first reservoir, let

f_{in} = volume of blood flow in.

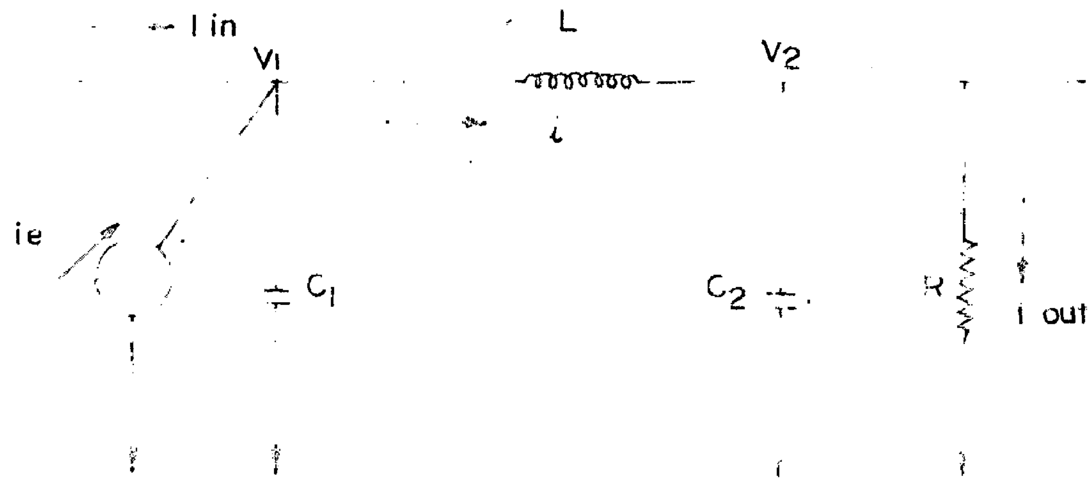
V_1 = volume of reservoir, and

f = flow out of first reservoir.

Then,



IG.6 (a) PHYSICAL MODEL OF THE CIRCULATORY SYSTEM.



IG.6 (b) ELECTRICAL ANALOG OF THE CIRCULATORY SYSTEM.

PHYSICAL MODEL OF THE HUMAN VASCULAR SYSTEM.

$$f_{in} - f = \frac{dV_1}{dt} \qquad \ldots (6)$$

Similarly for the second reservoir,

$$f - f_{out} = \frac{dV_2}{dt} \qquad \ldots (7)$$

Assuming that there is a constant linear relationship between the volume of the reservoir and the net pressure in it, i.e.,

$$p_1 - p_e$$

where, p_1 is the internal pressure and

p_e is some prescribed pressure surrounding the reservoir.

$$V_1 = V|_{unstretched} + K_1(p_1 - p_e) \qquad \ldots (8)$$

and

$$\frac{dV_1}{dt} = K_1 \frac{dp_1}{dt} - K_1 \frac{dp_e}{dt} \qquad \ldots (9)$$

Similarly for the second reservoir, taking p_o as a constant (e.g. tissue pressure)

$$V_2 = V_2|_{unstretched} + K_2(p_2 - p_o) \qquad \ldots (10)$$

and

$$\frac{dV_2}{dt} = K_2 \frac{dp_2}{dt} \qquad \ldots (11)$$

Assuming for simplicity that the blood flowing between the first and second reservoirs is flowing in a rigid tube of uniform cross-sectional area A and of length ℓ. Neglecting dissipation in the tube, conservation of momentum states that,

$$\text{time rate of change of momentum} = \text{Applied force} \qquad \ldots (12)$$

or $\frac{d}{dt}\left(\rho A \ell \frac{f}{A}\right) = p_1 A - p_2 A$ ... (13)

where, ρ is the mass density of the blood,

f is the volume flow rate,

$\therefore$ f/A is the velocity of the flow, Thus,

$$M \frac{df}{dt} = p_1 - p_2 \qquad \text{... (14)}$$

where $M = \rho \ell / A$

$M.A$ = effective inertance per unit cross-sectional area of the blood,

From the second reservoir the discharge is to the distal vascular bed (capillaries etc.). Assuming that there is a linear relationship between pressure in the reservoir and flow into this bed,

$$f_{out} = \frac{p_2}{r} \qquad \text{... (15)}$$

where, r is the total effective peripheral resistance

Defining $$f_e = K_1 \frac{dp_e}{dt} \qquad \text{... (16)}$$

the total system of equations can be written as

$$\left.\begin{aligned} f_{in} - f &= K_1 \frac{dp_1}{dt} - f_e \\ f - f_{out} &= K_2 \frac{dp_2}{dt} \\ M \frac{df}{dt} &= p_1 - p_2 \\ \text{and} \quad f_{out} &= \frac{p_2}{r} \end{aligned}\right\} \qquad \text{... (17)}$$

It is convenient to consider the electrical analog of the system and equations. The following analogies are drawn.

Physical Model	Electrical Analog
flow rate (f_{in}, f, f_{out}, f_e)	Current (i_{in}, i, i_{out}, i_e)
Pressure (p_1, p_2)	Voltage (v_1, v_2)
Compliance (K_1,K_2)	Capacitance (C_1, C_2)
Mass equivalent (M)	Inductance (L)
Peripheral resistance (r)	Resistance (R)

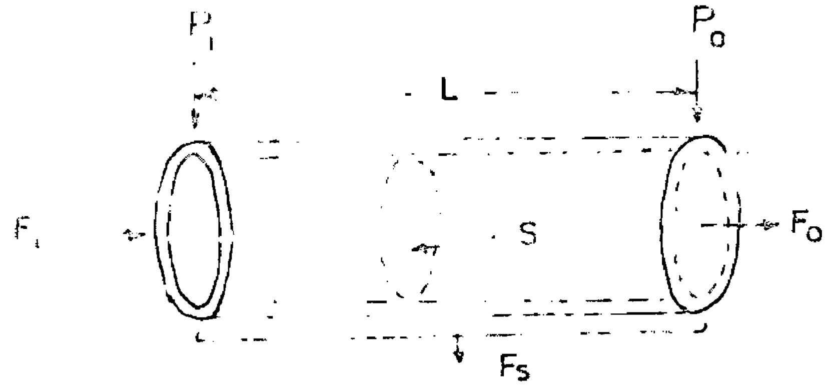
In terms of these variables, the equation become

$$\left.\begin{aligned} i_{in} - i &= C_1 \frac{dv_1}{dt} - i_e \\ i - i_{out} &= C_2 \frac{dv_2}{dt} \\ L \frac{di}{dt} &= v_1 - v_2 \text{ and} \\ i_{out} &= v_2/R \end{aligned}\right\} \qquad \dots (18)$$

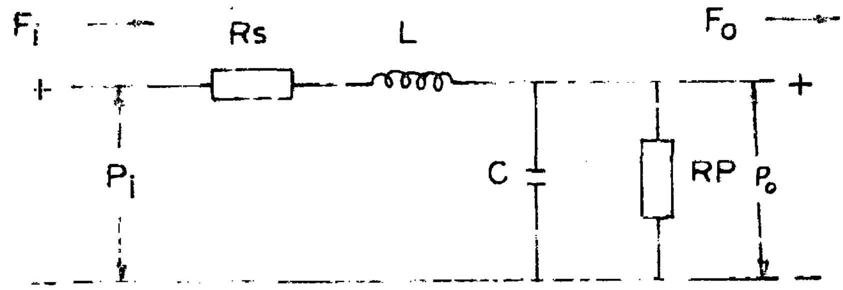
The equivalent circuit is shown in fig.(6b).

During diastole it is assumed that i_e and i_{in} are zero. Physically i_e zero implies that there is no active vascular compression and i_{in} zero implies that during diastole there is neither flow into or out of the heart. With these assumptions the system during diastole becomes,

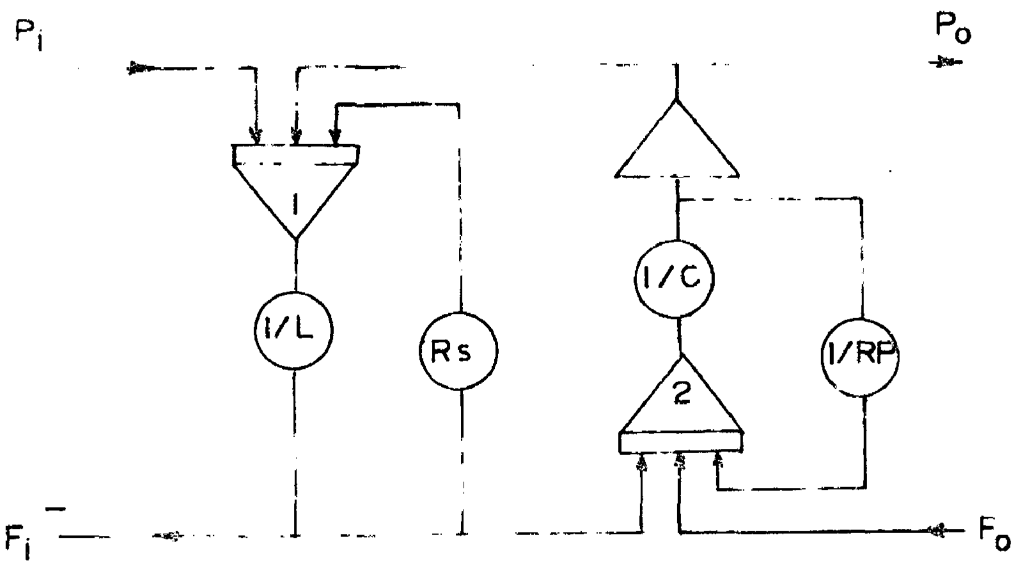
$$\left.\begin{aligned} \frac{di}{dt} &= \frac{1}{L} v_1 - \frac{1}{L} v_2 \\ \frac{dv_1}{dt} &= -\frac{1}{C_1} i \\ \text{and} \quad \frac{dv_2}{dt} &= \frac{1}{C_2} i - \frac{1}{RC_2} v_2 \end{aligned}\right\} \qquad \dots (19)$$



(a) SEGMENT OF A UNIFORM ARTERY.



(b) AN ELECTRICAL ANALOG CIRCUIT.



(c) ANALOG COMPUTER CIRCUITARY

.7 REPERSENTATION OF A SEGMENT ARTERY.

These equations may be written in the vector form

$$\dot{X} = FX \qquad \ldots (20)$$

where

$$X = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} i \\ v_1 \\ v_2 \end{bmatrix} \qquad \ldots (21)$$

and

$$F = \begin{bmatrix} 0 & \frac{1}{L} & -\frac{1}{L} \\ -\frac{1}{C_1} & 0 & 0 \\ \frac{1}{C_2} & 0 & -\frac{1}{RC_2} \end{bmatrix} \qquad \ldots (22)$$

6.4 HEMODYNAMIC PARAMETER ESTIMATION

Hemodynamic parameters such as the internal diameter of an artery or the arterial compliance are important quantities in the evaluation of the condition of the systemic-arterial system in patients. The computation is in effect a parameter-estimation technique, where the response of a system to an input signal is compared with that of a model of the system.[20]

The lumped linear, time-invariant description of a segment of artery, illustrated in fig.(7ab) contained the four components RS, L, C and RP, where

RS --→ represents the laminar poiseuille resistance to blood flow through the arterial segment.

L --→ represents the inertia of the mass of blood contained within the segment.

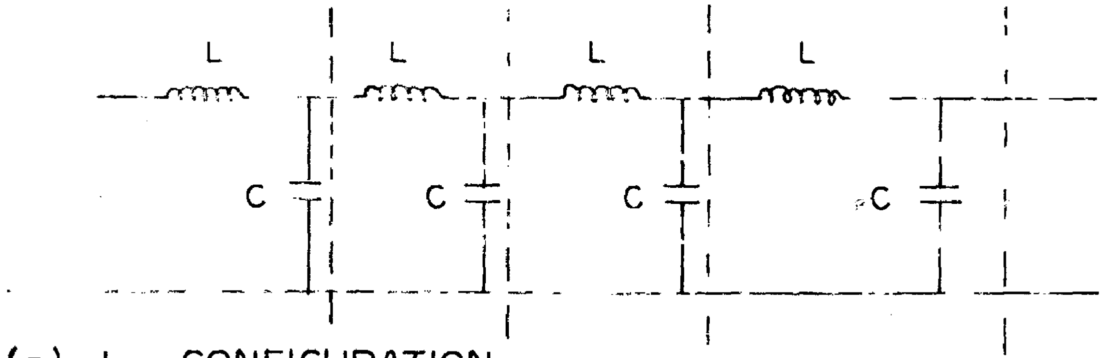
L and RS - together are called the longitudinal impedance.

C —→ represents the compliance of the arterial-segment or the volume increase for a small increment in pressure, and is called the transverse impedance.

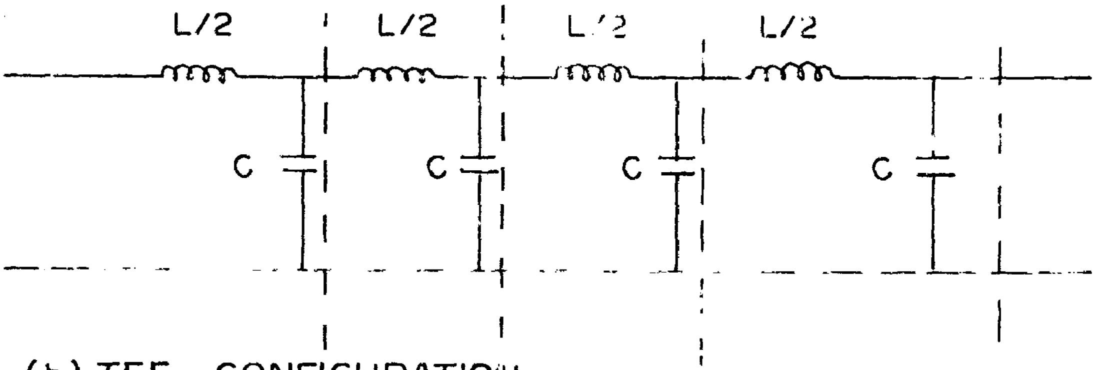
Most of the blood flow directed into the arterial segment will leave at the other end, but some will leak out of the segment via a number of smaller side branches. This secondary flow path is represented by a simple lumped resistance RP.

Contrary to the properties of this model an actual arterial segment is of a distributed nature, has side branches. Fast variations are caused by pulsatile changes in the diameter of the artery during each heart beat. These are relatively small and thus ignored. Slow variations with time are caused by a change in contraction of the smooth muscle contained within the arterial wall material, but the time scale of this effect is such that changes over a small number of heart beats are unappreciable.

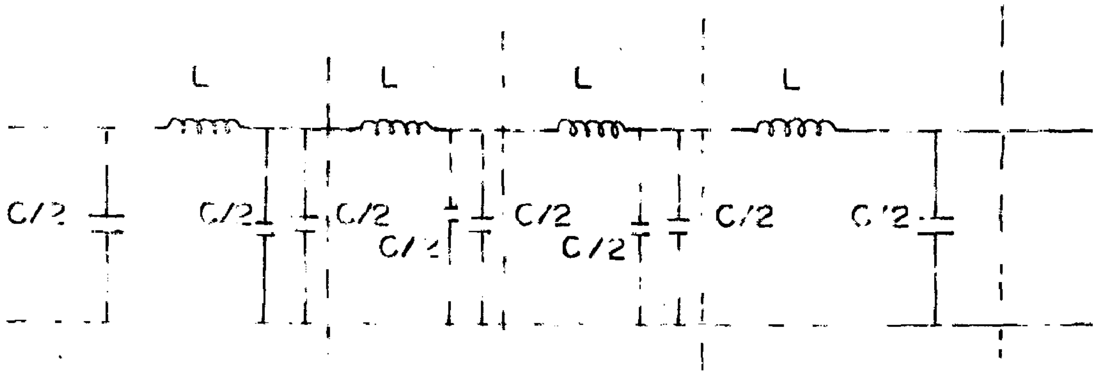
In Fig.(8) are presented three arm models, using L, tee and pi(π) configurations. Although tee and π segments are a more accurate-representation of a continuous transmission line, the L segment was chosen for implementation on the computer to simplify the programming. It can be seen from fig.(8) that the two middle sections are identical in all cases, whatever representation is chosen, The differences occur at the boundaries, giving rise to an 'end-effect' i.e. the parameter values of first and last segment are less accurate.



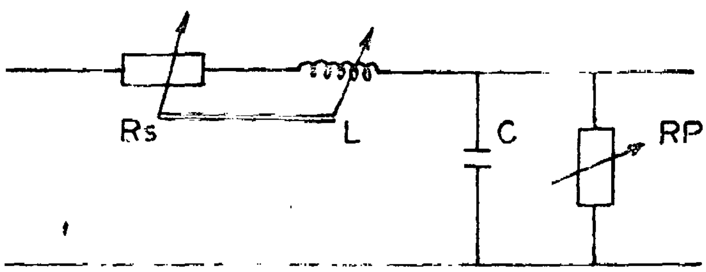
(a) L - CONFIGURATION.



(b) TEE - CONFIGURATION.



(c) π - COFIGURATION.



(d) EQUIVALENT DIAGRAM OF L - SECTION.

FIG.8 DIFFERENT CONFIGURATIONS OF ARTERIES

The following set of equations approximately describe the properties of a uniform arterial-segment.

$$RS = \frac{8\pi\eta}{S^2}$$

$$L = \frac{\rho\ell}{S} \;;\quad C = dV/dp \;;\quad RP = \frac{p_o}{F_s} \qquad \ldots (23)$$

$$F_i = \frac{1}{L}\int (P_i - p_o - F_i RS)\, dt$$

$$p_o = \frac{1}{C}\int (F_i - F_o - \frac{p_o}{RP})\, dt \qquad \ldots (24)$$

Equivalent Diagram of L Section

The actual form of each L section is depicted in fig.(8d). The values of RS and L follow from the arterial cross-sectional area S and the segment length . By maintaining the length of each segment at a known and fixed value, it is possible to express RS in terms of L by eliminating S in the equations for RS and L. It follows that $RS = AL^2$, with A a constant which is inversely proportional to the segment length ℓ. $A = 0.8/\ell$ (approx).

The linking of RS to L is important for two reasons.

1) It reduces the number of adjustable parameters.

2) It improves the uniqueness of the solution i.e. the input and output pressure pulses of an artery may provide the transfer of this 'system' as a function of frequency, but not the impedance level.

6.5 THE ELECTRICAL ANALOG OF A SEGMENT OF ARTERY

The derivation of the electrical equivalent of the arterial tree is based upon the analogy between the linearised Navier-Stokes equation and the equation of continuity on one hand and the telegraph equations on the other.

Writing the Navier-Stokes equation,

$$-\frac{1}{\rho}\frac{\partial p}{\partial Z} = \frac{\partial V_z}{\partial t} + V_r\frac{\partial V_z}{\partial r} + V_z\frac{\partial V_z}{\partial Z} - \nu\left[\frac{\partial^2 V_z}{\partial Z^2} + \frac{1}{r}\frac{\partial V_z}{\partial r} + \frac{\partial^2 V_z}{\partial r^2}\right] \quad \ldots (25)$$

In this expression for the pressure gradient, the first three terms in the right-hand member concern inertial properties, the last three viscous. Noordergraaf[16] neglected the second and third terms in the right-hand member, with respect to the first, and multiplied by the cross-sectional area S, which gives,

$$-\frac{S}{\rho}\frac{\partial p}{\partial Z} = S\frac{\partial V_z}{\partial t} - \nu\, S\left[\frac{\partial^2 V_z}{\partial Z^2} + \frac{1}{r}\frac{\partial V_z}{\partial r} + \frac{\partial^2 V_z}{\partial r^2}\right] \quad \ldots (26)$$

For simplified model, we assume the longitudinal velocity V_z to be independent of 'r' radius of the vessel, i.e. velocity is having a flat profile. The first inertial term in equation (26) on the right hand side represents the flat profile for velocity. The second bracketed viscous term represents the parabolic-profile.

Viscous term is represented by Poission's flow and is equal to $\frac{8\mu}{\pi r^4}$ I, where I denotes the flow.

$$\therefore -\frac{S}{\rho}\frac{\partial p}{\partial Z} = \frac{\partial I}{\partial t} - S\,\nu\left[\frac{8\mu}{\pi r^4} I\right] \quad \ldots (27)$$

or $$-\frac{\partial p}{\partial Z} = \frac{\rho}{S}\frac{\partial I}{\partial t} + \frac{8\mu}{\pi r^4} I$$

or $$-\frac{\partial p}{\partial Z} = \frac{\rho}{S}\ \frac{\partial I}{\partial t} + W\ I \qquad \ldots (28)$$

Considering again the arterial segment to be short, and equating the difference between inflow and outflow with the sum of the uptake of blood an equation of continuity can be written:

$$-\frac{\partial I}{\partial Z} = \frac{dS}{dp}\ \frac{\partial p}{\partial t} + W'p \qquad \ldots (29)$$

where, $-\frac{\partial I}{\partial Z}$ = flow gradient (Difference of the inflow and outflow)

Term $\frac{dS}{dp} \cdot \frac{\partial p}{\partial t}$ represents the distensible property of the tube

$W'p$ = Escape of the blood due to lateral path ways,

W' = Leakage of the blood which is escaped from the segments through the lateral arteries.

$\frac{dS}{dp}$ = distensibility per length = $\frac{2(1-\sigma^2)\pi r^3}{Eh}$

σ = Poisson ratio for the wall material,

E = Young modulus,

h = wall thickness

The analogous telegraph equations are

$$-\frac{\partial V}{\partial Z} = L'\ \frac{\partial i}{\partial t} + R'i \qquad \ldots (30)$$

$$-\frac{\partial i}{\partial Z} = C'\ \frac{\partial V}{\partial t} + G'V \qquad \ldots (31)$$

where,

$V = V(Z,t)$ = voltage,

L' = Inductance per length

C' = capacitance per length

$i = i(Z,t)$ = current,

R' = resistance per length,

$\frac{1}{R'_\ell} = G'$ = conductance per length

On comparing the equations (28), (29) and (30), (31) yields,

pressure p $\sim$ voltage V

flow I $\sim$ current i

Inertance per length ρ/S $\sim$ inductance per length L'

Resistance per length $8\mu/\pi r^4$ $\sim$ resistance per length R'

Compliance per length dS/dp $\sim$ capacitance per length C'

leakage per length W' $\sim$ conductance per length G'

The corresponding passive electrical network fig.(9) representing a finite length of artery ΔZ consists of four passive elements.[16]

$$\left.\begin{aligned} L &= L'\Delta Z = \frac{\rho}{S}\,\Delta Z \\ R &= R'\Delta Z = \frac{8\mu}{\pi r^4}\,\Delta Z \\ C &= C'\Delta Z = \frac{dS}{dp}\cdot\Delta Z \\ G &= G'\Delta Z = W'\Delta Z \end{aligned}\right\} \qquad \ldots (32)$$

The ratio $-\frac{\partial V}{\partial Z} / i$ is called the longitudinal impedance per length, the ratio $-V/\partial i/\partial Z$ is called the transverse impedance per length.

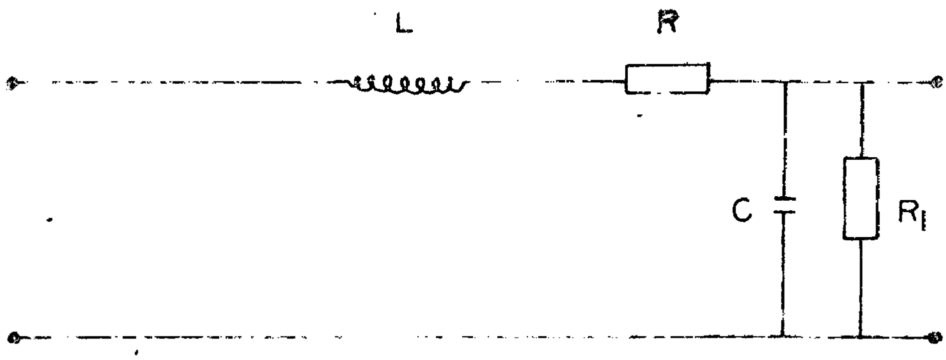


FIG. 9 PASSIVE ELECTRICAL NETWORK FOR A SEGMENT OF ARTERY.

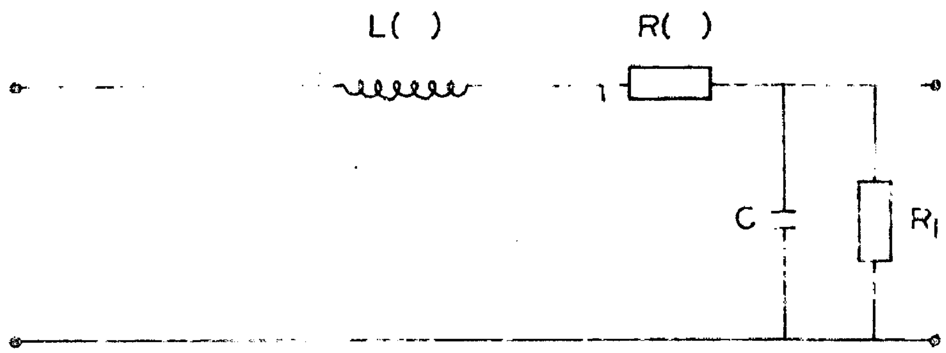


FIG. 10 PASSIVE ELECTRICAL NETWORK CONSIDERING THE SLEEVE EFFECT.

6.6 SLEEVE-EFFECT

In the above derivations we assumed flat velocity profile in the calculation of the inertial term and a parabolic velocity profile in the viscous term. This leads to frequency independent values for L' and R' in the analogous equation (30). In special cases this may be an adequate approximation. For many arteries this is not true because of the pulsatile nature of blood flow. In those arteries the interaction between viscous and inertial terms has to be taken into account, determining a velocity profile different from the two assumed. A modified passive equivalent electrical network for 'sleeve-effect' considerations is to be designed.

Womersley derived a mathematical expression for the relationship between flow Q and pressure gradient $\frac{\partial p}{\partial Z}$ for laminar oscillatory flow.[21]

$$Q = \pi r^2 \frac{A}{j\omega\rho}\left[1 - \frac{2J_1(\alpha j^{3/2})}{\alpha j^{3/2} J_0(\alpha j^{3/2})}\right] e^{j\omega t} \qquad \ldots (33)$$

where, A = amplitude,

ω = circular frequency = $2\pi f$

$A\,e^{j\omega t}$ = pressure gradient = $-\frac{\partial p}{\partial Z}$

$\alpha = r\sqrt{\frac{\omega}{\nu}}$ where ν = kinematic viscosity = μ/ρ,

J_0 and J_1 are the zero and first order Bessel functions of the complex argument

$$\alpha j^{3/2} \; ; \; j^{3/2} = \frac{j-1}{\sqrt{2}} \; ; \; j = \sqrt{-1}$$

The relationship between flow gradient and pressure reduces for this limiting case to equation (29) . From this it follows that the transverse impedance remains unchanged, so that the sleeve effect requires a modification of the longitudinal impedance only. The longitudinal impedance Z_ℓ is derived from equation (33)

$$Z_\ell' = Z_\ell/\Delta Z, \quad \Delta Z = \text{segment length.}$$

$$Z_\ell' = -\frac{\partial p/\partial Z}{Q}$$

$$= \frac{j\mu\alpha^2}{\pi r^4}\left[1 - \frac{2J_1(\alpha j^{3/2})}{\alpha j^{3/2} J_o(\alpha j^{3/2})}\right]^{-1} \qquad \ldots (34)$$

Writing this as,

$$Z_\ell' = j\omega L'(\omega) + R'(\omega) \qquad \ldots (35)$$

then
$$L'(\omega) = \frac{\rho}{\pi r^2 M_{10}'}\cos\varepsilon_{10}'$$

$$R'(\omega) = \frac{\mu\alpha^2}{\pi r^4 M_{10}'}\sin\varepsilon_{10}'$$

where

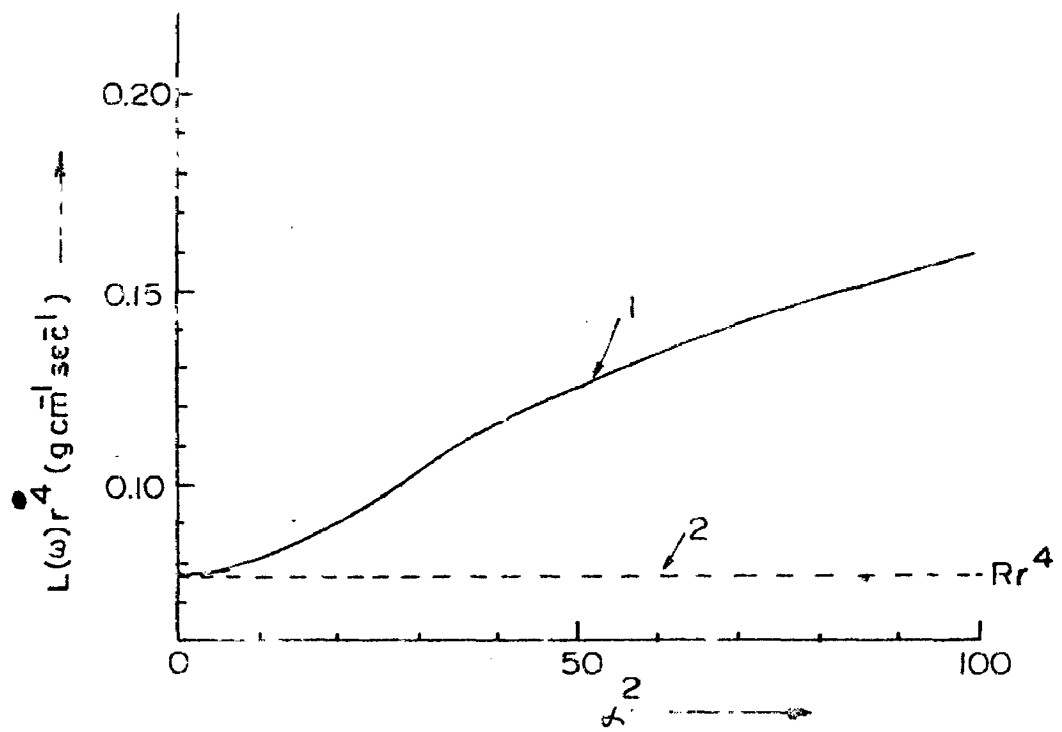
$$M_{10}' = \text{modulus}\left\{1 - \frac{2J_1}{\alpha j^{3/2} J_o}\right\}$$

$$\varepsilon_{10}' = \text{phase}\left\{1 - \frac{2J_1}{\alpha j^{3/2} J_o}\right\}$$

The corresponding circuit is given in fig.(10). $L'(\omega)$ and $R'(\omega)$ are frequency dependent. The difference between L' and $L'(\omega)$ and between R' and $R'(\omega)$ is shown in Fig.(11).

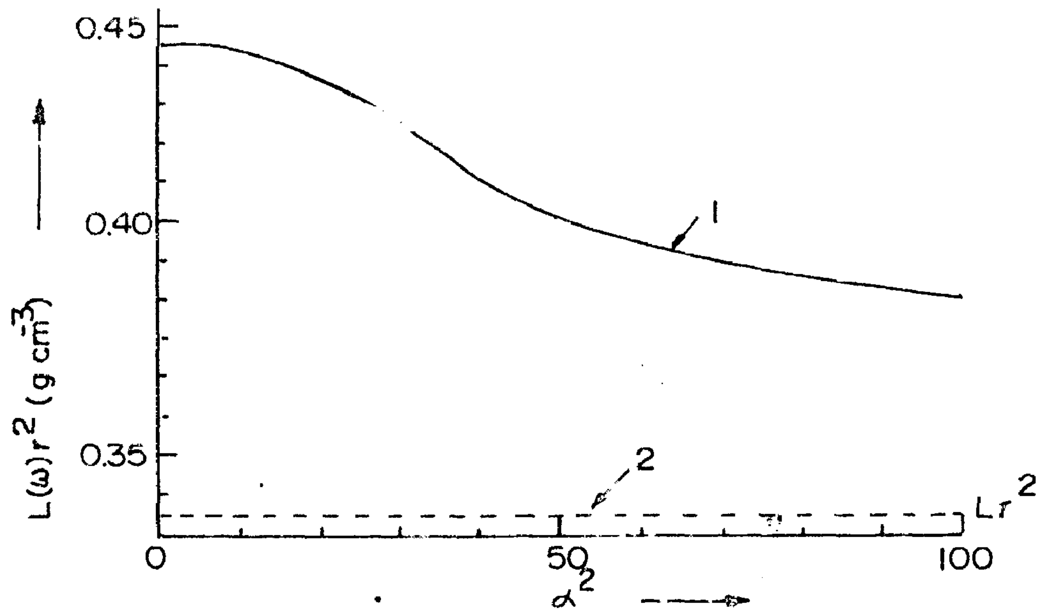
From the recurrence formula for Bessel functions

$$J_n(x) + J_{n+2}(x) = \frac{2(n+1)}{x}J_{n+1}(x) \qquad \ldots (36)$$



(a) REAL PART OF THE LONGITUDINAL IMPEDANCE.

1. WITHOUT SLEEVE EFFECT. 2. WITHOUT SLEEVE EFFECT.



(b) IMAGINARY PART OF THE LONGITUDINAL IMPEDANCE.

G.11 REPRESANTATION OF SLEEVE EFFECT.

Putting n = 0 in the above expression, we get

$$J_o(x) + J_2(x) = \frac{2}{x} J_1(x)$$

or $$xJ_o(x) - 2J_1(x) - 2J_1(x) = -x\, J_2(x) \qquad \ldots (37)$$

Equation (34) can be rewritten

$$Z_l' = \frac{j\mu\alpha^2}{\pi r^4} \left\{ \frac{\alpha\, j^{3/2} J_o(\alpha j^{3/2}) - 2J_1(\alpha j^{3/2})}{\alpha j^{3/2} J_o(\alpha j^{3/2})} \right\}^{-1} \qquad \ldots (38)$$

On applying (37), (38) reduces to

$$Z_l' = -\frac{j\mu\alpha^2}{\pi r^4} \; \frac{J_o(\alpha j^{3/2})}{J_2(\alpha j^{3/2})} \qquad \ldots (39)$$

Applying the series development of the Bessel functions

$$J_n(x) = \sum_{m=o}^{\infty} \frac{(-1)^m (\frac{x}{2})^{2m+n}}{m!\,(m+n)!}$$

to equation (39) and substituting $\lambda = \frac{j\alpha^2}{4}$ leads to

$$Z_l' = \frac{4\mu}{\pi r^4} \; \frac{\sum\limits_{m=o} \frac{\lambda^m}{m!\; m!}}{\sum\limits_{m=o} \frac{\lambda^m}{m!\,(m+2)!}} \qquad \ldots (40)$$

Application of a continued fraction expansion results in (See Appendix I) .

$$Z' = \frac{4\mu}{\pi r^4}\left(a_1\lambda + b_1 + \frac{1}{\lceil (a_2\lambda)^{-1}} + \frac{1}{\lceil b_2} + \frac{1}{\lceil (a_3\lambda)^{-1}} + \frac{1}{\lceil b_3} + \ldots\right) \qquad \ldots (41)$$

$$\left.\begin{aligned} &\text{with } \frac{1}{\lceil x} + y = \frac{1}{x+y} \\ &a_m = \frac{1}{2m-1} \\ &b_m = 2m \end{aligned}\right\} \qquad \ldots (42)$$

The element values thus become

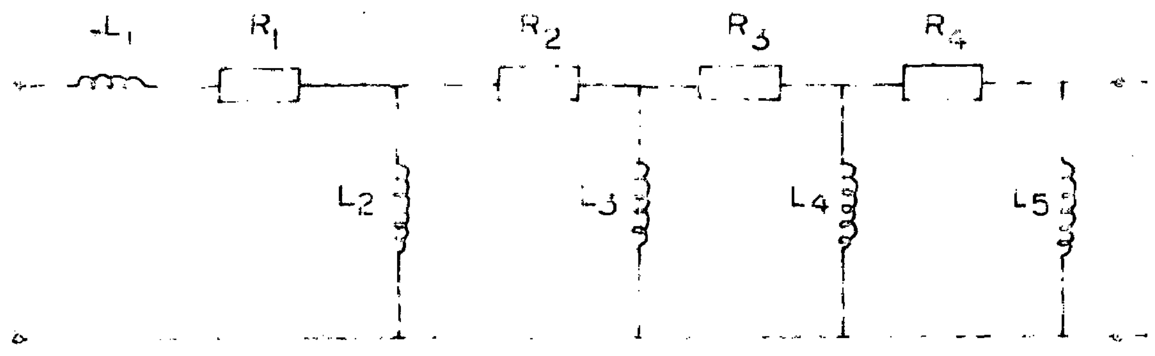
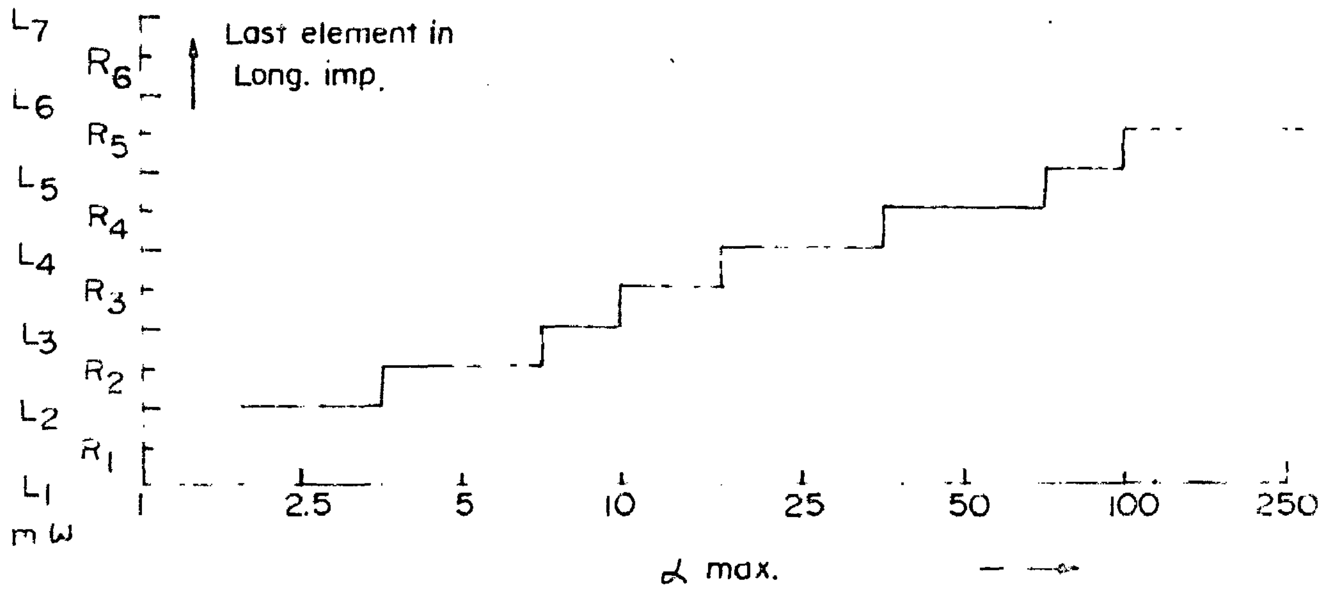


FIG.12 EQUIVALENT NETWORK FOR THE LONGITUDINAL IMPEDANCE.



ELEMENTS IN THE LONGITUDINAL IMPEDANCE.

$$\left.\begin{aligned} L'_m &= \frac{\rho}{S}\;\frac{1}{2m-1} \\ R'_m &= \frac{8\mu}{\pi r^4}\,m \end{aligned}\right\} \qquad m = 1,2,3\;\ldots. \qquad \ldots (43)$$

The corresponding passive network for the longitudinal impedance is drawn in fig.(12).

In the extreme case that α is sufficiently small, so that R_2 + following elements may be omitted, the circuit of fig.(12) reduces to L_1, R_1 and L_2 in series, giving an impedance per length of:

$$Z'_\ell = \frac{4}{3}\frac{\rho}{S}\,j\omega + \frac{8\pi\mu}{S^2} \qquad \ldots (44)$$

For direct current (ω=0) equation (44) reduces to

$$Z'_\ell = \frac{8\pi\mu}{S^2} \qquad \ldots (45)$$

which is equal to the resistance.

For large values of α Eqn.(34) asymptotic expansion of the Bessel functions gives

$$Z'_\ell = \frac{\rho}{S}j\omega \qquad \ldots (46)$$

So for large α the term $R'(\omega)$, although an increasing function of α^2, may be neglected since $j\omega L'(\omega)$ increases faster, fig.(11b).

If α is so large that R_1, $R_2 \ldots .$ R_{N-1} may be neglected, L_1, L_2 .... L_n will virtually act as a single inductance L, with

$$\begin{aligned} L &= \frac{\rho}{S}\left[1 + \left\{\sum_{m=2}^{N}(2m-1)\right\}^{-1}\right] \\ &= \frac{\rho}{S}\left(1 + \frac{1}{N^2-1}\right) \qquad \ldots (47) \end{aligned}$$

If a relative error of $\frac{1}{N^2-1}$ is tolerated, R_N + following elements may be omitted. The error decreases with decreasing α, since the resistance increase faster than the inductances. Hence, an accuracy of at least $1/N^2-1$ for the whole α range is obtained by using R_1, R_2 .... R_{N-1} and L_1, L_2 .... L_n only. This accuracy can be attained with even fewer elements for the α's we actually have to deal within the circulatory system.

Elements required for each α in order to obtain an accuracy of 2% in modulus and 3 degrees error in phase, can be calculated. The results are shown in fig.(13). Given the radius of an artery and having decided how many harmonics should be transmitted with this accuracy, fig.(13) indicates the elements necessary to represent the longitudinal impedance of a segment of this artery.

Since L_1 and R_1 fig.(12) turn out to be equal to L and R in the network of fig.(9), the required network, representing the electrical analogy of laminar oscillatory flow impedance, only adds a corrective network to the circuit as shown in figure (14).

For the human systemic arterial tree an average number of correction elements are 5, the maximum being 8. Figure (11) gives magnitude and phase of the input impedance of the entire system for the cases with and without sleeve effect. The inclusion of the sleeve effect shows some tendency to smooth the input impedance. On the whole, the difference has proved not to be large, and so the contradictory assumption originally

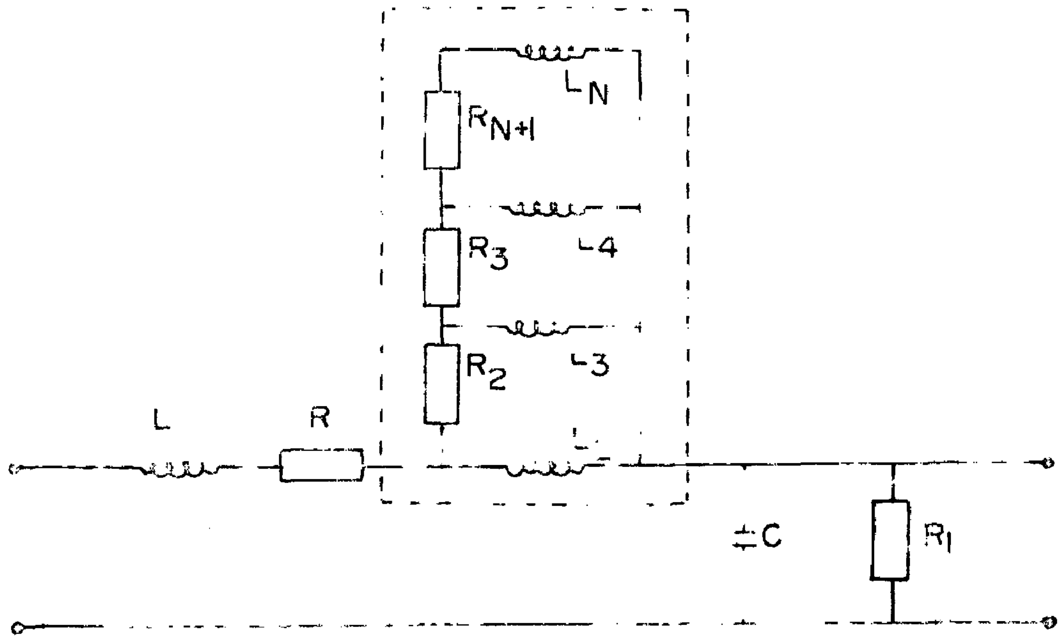


FIG 14 ELECTRICAL ANALOG OF LAMINAR OSCILLATORY FLOW IMPEDANCE WITH SLEEVE EFFECT.

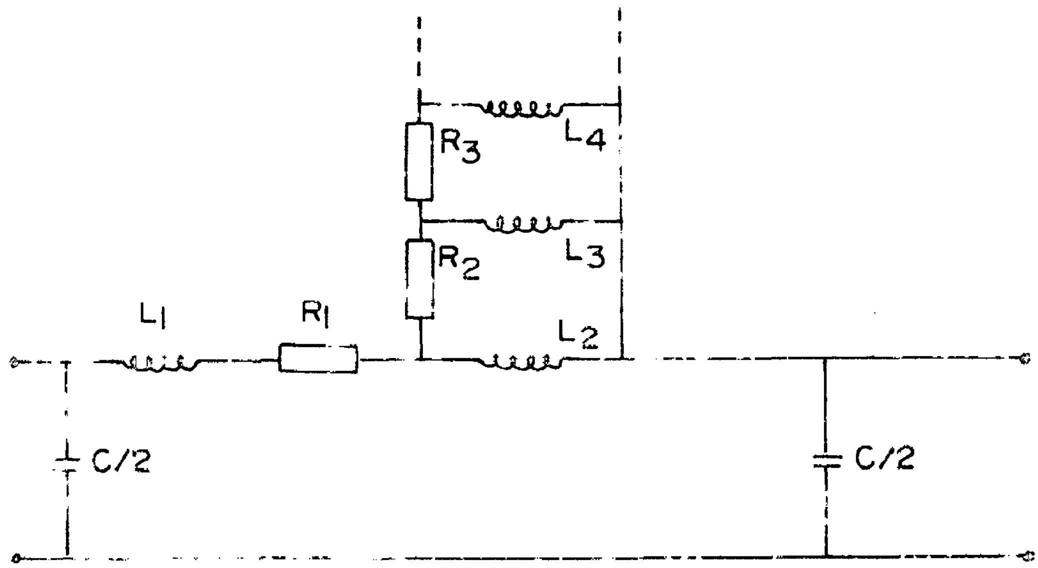


FIG.15 INPUT IMPEDANCE OF THE SYSTEMIC ARTERIAL TREE.

made concerning velocity profiles had no devastating effect.

6.7 ANOMALOUS VISCOSITY OF BLOOD

It has been known for a long time that blood is not a Newtonian fluid, but that the relation between applied hydrostatic pressure gradient and steady flow is not a linear one. This is due to the fact that blood is a suspension. The apparent viscosity depends upon the rate of flow and upon the radius of the tube. Three phenomena may be considered responsible for this:[16]

1. Cohesion of red cells,
2. Inclination of the red cells to move to the axis of the tube.
3. Orientation of the cells.

The rate of shear in blood, depends on the radius of the vessel, and is the determining quantity for the apparent viscosity. For low shear-stresses the apparent viscosity is almost independent of the radius and increases with decreasing stress, while for high shear stresses it is almost independent of the stress and decreases with decreasing radius.

In the construction of an electrical analog of the arterial system the effects of the phenomena, described above, on the equations of motion of blood in the vessels, must be known.

Taylor showed that neglecting the shear dependence of the viscosity in the arterial system as a whole introduces only a small error, because the effects on oscillatory and steady components of the flow tend to extinguish each other.

In small arteries there is a tendency of red cells to move toward the axial region of the artery thus generating a boundary layer of lower viscosity.

Womersley assumed the viscosity of blood to be constant and the longitudinal velocity of blood at the wall to be equal to that of the wall itself. Taylor worked out two modifications of this, namely:

1) the flow equation for the condition that there is is a marginal layer of lower viscosity at the wall of the vessel.

2) the flow equation for the case that the viscosity is constant but that the fluid slips at the wall.

Under the assumption of slippage at the wall, Taylor found,[21]

$$Q = \frac{A\pi r^4}{j\mu\alpha^2}\left[1 - \frac{2J_1}{\alpha j^{3/2}\{J_o - k\alpha j^{3/2}J_1\}}\right] e^{j\omega t} \qquad \ldots (48)$$

where,
$$A\,e^{j\omega t} = -\frac{\partial p}{\partial Z}$$

$$K = \frac{\mu}{r\gamma}$$

and γ = coefficient of friction at the wall.

Taylor, disregarding leakage, also derived an expression for the wave velocity, which can easily be rewritten as:

$$C^2 = \frac{Eh}{2r\rho(1-\sigma^2)} \left[\frac{J_o - \frac{2}{\alpha j^{3/2}} J_1 - K\,\alpha j^{3/2} J_1}{J_o - K\,\alpha\, j^{3/2}\, J_1} \right] \quad \ldots (49)$$

From the relation $Z_t^{-1}\ Z_\ell = - \frac{\omega^2}{C^2}$, he showed that the transverse impedance remains unchanged. The longitudinal impedance is given by

$$Z'_{\ell,s} = \frac{-\partial p/\partial Z}{Q} = j\omega \frac{\rho}{S} \left[\frac{J_o - K\,\alpha\, j^{3/2} J_1}{J_o - \frac{2}{\alpha j^{3/2}} J_1 - K\,\alpha\, j^{3/2} J_1} \right] \quad \ldots (50)$$

In order to obtain the electrical equivalent of the longitudinal impedance a logical approach is to isolate the term containing K.

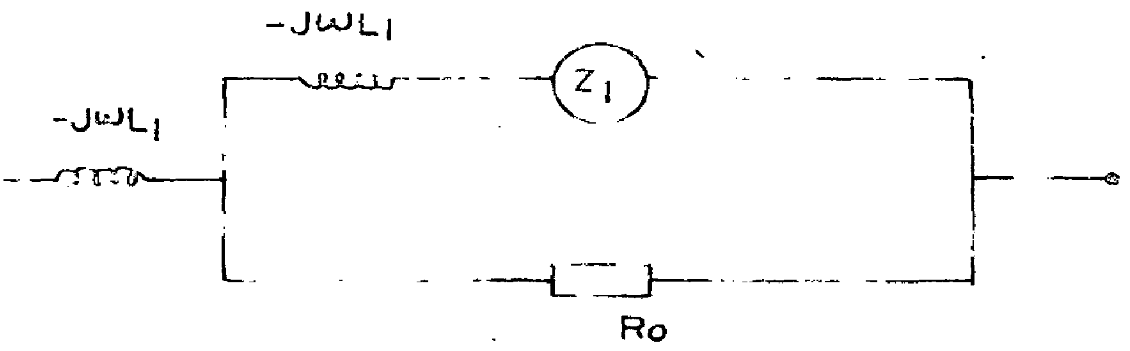
$$\therefore\ Z'_{l,s} = j\omega \frac{\rho}{S} \left[1 + \frac{\frac{2}{\alpha\, j^{3/2}} J_1}{J_o - \frac{2}{\alpha j^{3/2}} J_1 - K\,\alpha\, j^{3/2} J_1} \right]$$

$$= j\omega \frac{\rho}{S} \left[1 + \frac{1}{\frac{\alpha j^{3/2} J_o - 2J_1}{2J_1} - \frac{1}{2} K\,\alpha^2 j^3} \right]$$

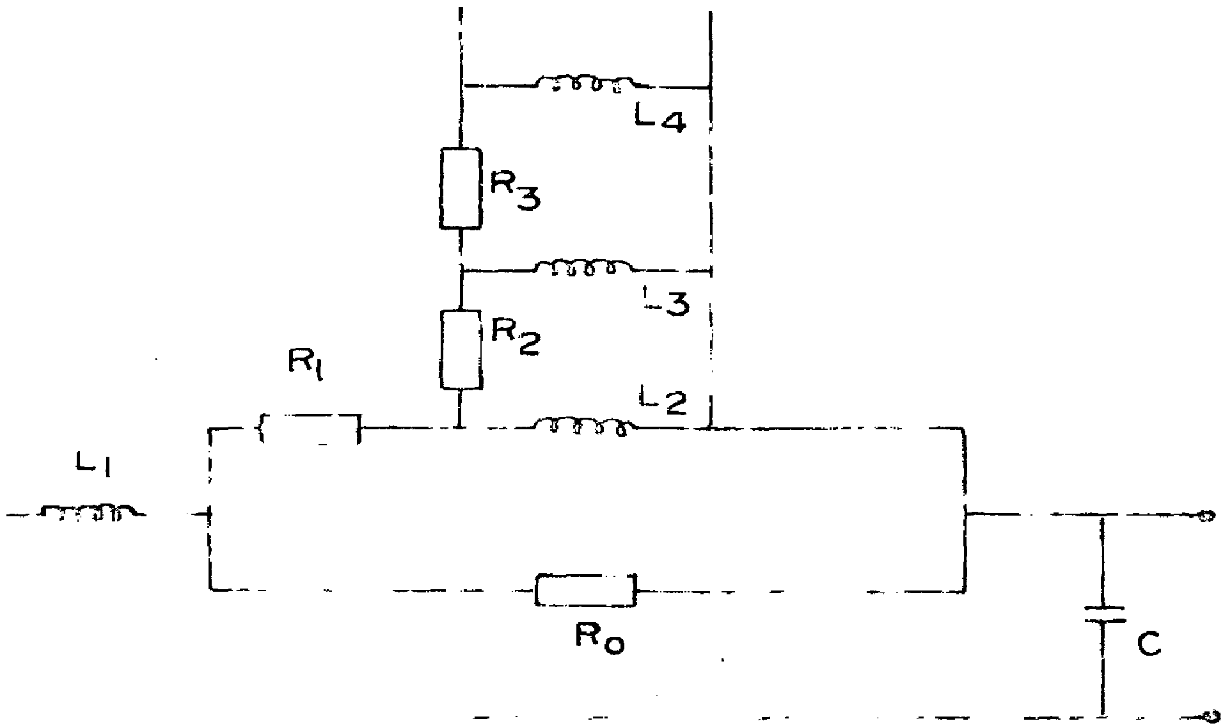
$$= j\omega \frac{\rho}{S} \left[1 + \frac{1}{\left(-1 + \frac{\alpha j^{3/2}\ J_o}{\alpha j^{3/2} J_o - 2J_1}\right)^{-1} + \frac{1}{2} K\ j\ \alpha^2} \right]$$

Denoting $\frac{2J_1}{\alpha j^{3/2} J_o} = F_{10}$

$$\therefore\ Z'_{\ell,s} = j\omega \frac{\rho}{S} \left[1 + \frac{1}{\left\{-1+(1-F_{10})^{-1}\right\}^{-1} + \frac{1}{2} K j \alpha^2} \right]$$



IMPEDANCE SCHEME FOR THE LONGITUDINAL IMPEDANCE WITH ANOMALOUS VISCOSITY.



CTRICAL EQUIVALENT OF THE OSCILLATORY FLOW EDANCE WITH SLEEVE AND ANOMALOUS VISCOSITY ECTS.

$$= j\omega \frac{\rho}{S} + \frac{1}{\left\{-j\omega\frac{\rho}{S} + j\omega\frac{\rho}{S}(1-F_{10})^{-1}\right\}^{-1} + \frac{1}{j\omega}\cdot\frac{\pi r^2}{\rho}\frac{1}{2}Kj\frac{r^2\omega}{\gamma}}$$

$$= j\omega \frac{\rho}{S} + \frac{1}{(-j\omega\frac{\rho}{S} + Z_\ell')^{-1} + K\frac{\pi r^4}{2\mu}} \qquad \dots (51)$$

The electrical representation of $Z_{\ell,s}$ is thus shown in fig.(16). Thus $Z_{\ell,s}$ differs from Z_ℓ only by the presence of the resistor R_0, which is equal to $R_0 = \frac{2\mu}{K\pi r^4}$ the inverse of the last term in the denominator in equation (51).

As no modification is required in the transverse impedance, the passive electrical network, representing the oscillatory flow impedance of a segment of artery, accounting for both the sleeve-effect and slip at the wall, which is determined by the friction coefficient γ, takes the form illustrated in fig.(17). To the capacitor C a resistor may be placed in parallel, to account for possible leakage as in fig.(14). The values of the elements are

$$\left.\begin{aligned} L_m' &= \frac{\rho}{S}\cdot\frac{1}{2m-1} \\ R_m' &= \frac{8\pi\mu}{S^2}\cdot m \qquad\qquad m = 1,2,3\dots \\ R_0' &= \frac{8\pi\mu}{S^2}\cdot\frac{1}{4K} \end{aligned}\right\} \qquad \dots (52)$$

Taylor also derived the flow equation for the case of a boundary layer of lower viscosity μ' and width Δr assuming perfect adherance of the fluid to the wall. The longitudinal impedance $Z_{\ell,a}'$ is equal to the $Z_{\ell,s}'$. He also showed that the transverse impedance is unaffected, just as in the case of slippage. Therefore, the electrical network representing the flow impedance is identical to that illustrated in fig. (17).

6.8 ELECTRICAL-ANALOG OF PULMONARY ARTERIAL TREE

Pollack, Reddy and Noordergraf[22] applied the network equivalent shown in fig.(12), with some modifications, to the larger branches of the pulmonary arterial tree.

The values of the electrical elements of fig.( ) are as follows-

$$\left.\begin{aligned} L_m &= \frac{\rho}{S}\,\frac{1}{2m-1}\,\ell \\ R_m &= \frac{8\pi}{S^2}\,\eta m\,\ell \\ \text{and}\quad C &= \frac{3S}{E}\,\frac{(a+1)^2}{2a+1}\,\ell \end{aligned}\right\} \qquad \ldots (53)$$

For vessels with sufficiently small cross-section S, the resistive term R_m are much larger than the inductive and capacitive terms, thus reducing the circuit of fig.(14) to a simple series resistor. Therefore, only the larger vessels need be represented by the complex network of fig.(14). Smaller vessels (radius under 2mm) are represented simply by resistors.

As an improvement over the network of fig.(12) with its rotated L configuration Noordergraaf divided the transverse impedance into two sections, forming a 'π' network. This tends to reduce the error caused by lumping by distributing the compliance of each segment. The resulting network is shown in fig.(18). The resistor R_{LPS}(post-segmental leakage resistor) represents leakage through a small vessel originating at or near the distal end of segment. The generator marked

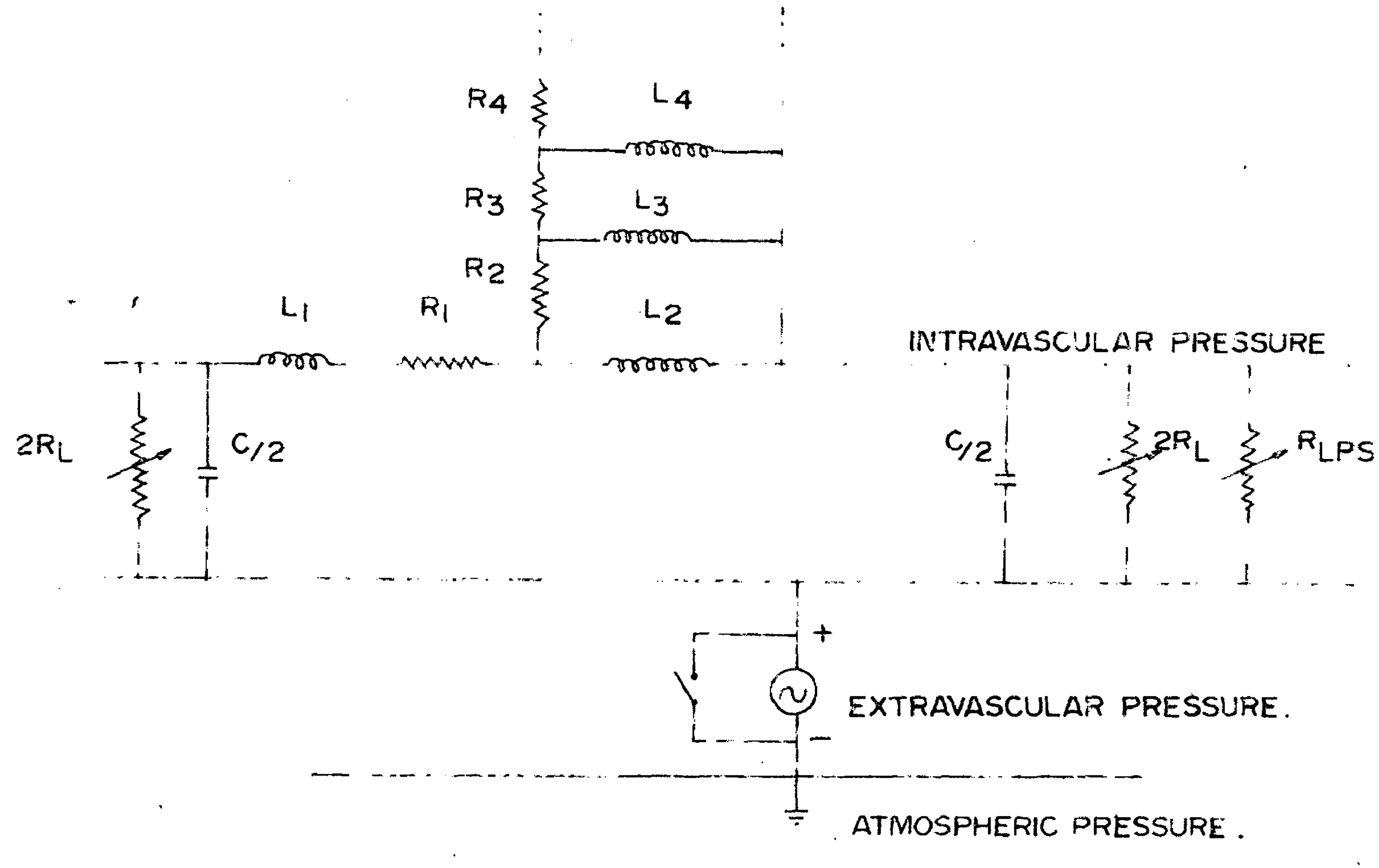


FIG.18 ELECTRICAL ANALOG OF PULMONARY ARTERIAL TREE.

'extra vascular pressure' is used to simulate any desired perivascular-pressure fluctuation resulting from respiration-phenomena.

The dynamics of the pulmonary arterial system are influenced by perivascular pressure continuously, fluctuating with respiratory rhythem. In the theory, such extramural pressure fluctuations were ignored. This constraint of aero extramural pressure has been removed by placing a voltage generator in each segment between the transverse impedance and ground fig.(18). Each generator can supply the voltage corresponding to local extravascular pressure.

<u>Peripheral Resistance</u>- Total peripheral resistance is determined by dividing the mean arterio-post capillary pressure difference by the cardiac output. Mean pulmonary arterial pressure in the recumbent position is taken to be 15 mm Hg. Although, no exact data are available on post-capillary pressure, its value can be estimated to 8.5 mm Hg from the measurements of Agostoni. Using 6.5 mm Hg(15 mm-8.5 mm) arterio-post capillary pressure difference and assuming a cardiac output of 5 litres per minute, the peripheral resistance comes about 100 g S^{-1} cm^{-4} or 100 ohm expressed in electrical language.

In the model each peripheral resistor R_L or R_{LPS} is chosen in inverse proportion to the cross-sectional area of the vessel. Since the parallel combination of these peripheral resistors constitutes the total peripheral resistance, the second condition in determining their value is that the resistance of the parallel combination is 100 ohms.

CHAPTER-SEVEN

ARTERIAL VERSUS VENOUS HEMODYNAMICS

7.1 INTRODUCTION

Following the studies of Hagen and poiseuille, the science of hydrodynamics for flow through rigid circular tubes has been widely developed by engineers and biologists. In contrast, it is only during recent years that studies have been carried out concerning the flow through collapsible tubes and these have been few in comparison to the studies of flow through rigid tubes.

Approximately one sixth of the total blood volume of the vascular system is contained within a system of branching circular tubes (arterial system) in which the blood flow is described by the classical laws of hydrodynamics for circular tubes. In contrast, approximately two thirds of the blood volume is located within a branching system of collapsible tubes (capillaries, venules and veins) in which the flow differs from that in tubes of circular cross-section.

7.2 ANATOMY OF VEINS

The veins are composed of the same structural elements as the arteries, but there are some important quantitative differences.

The venous walls are some five to ten times thinner than the walls of the corresponding arteries. Thus they offer little resistance to collapse if either intravascular pressure decreases (as during periods of transient flow acceleration) or extravascular pressure increases (as during muscular contraction). The absence of a significant number of elastic elements in the media constitutes the primary structural difference between veins and arteries.

The presence of valves is a unique feature of the venous system. The functional contribution of the valves consists in increasing the efficiency with which extramurally applied forces acting as an axiliary pump propel the blood toward the heart.[23]

7.3 TRANSMURAL PRESSURE

The pressure within the veins is low and pulsatile in nature. Because of these conditions the variations in the pressure external to the veins are also of consequence since the cross-sectional shape is affected. Hence, a third parameter, in addition to the pressure and flow, must be introduced, which need not be considered in case of arteries. This is transmural pressure, and is defined as the pressure difference from inside the vein to outside the vein.

The extravascular pressure primarily due to respiration and muscular contraction, are of the same order of nitude as the intravascular pressure. Thus they play an important role in establishing the transmural pressure.

Transmural pressure in the venous system can be zero or negative in regions of the venous system such as near the entrance of the thorax . The low transmural pressure plays an important role in venous modelling. The elastic modulus decreases non-linearily with decreasing transmural pressure especially as transmural-pressure approaches zero. The Young's modulus for a vein, is at least 2 to 3 times smaller than that found for an artery.

The thin wall, the low transmural pressure, and small Young's modulus all contribute to making the veins structurally non-self-supporting and hence collapsible. It is this collapse phenomena that makes a theoretical analysis of the veins more difficult than for the arteries.

In the flow of liquids through collapsible tubes the cross-section of the tube is free to change as the transmural pressure changes. In a collapsible tube there is a liquid-solid interface extending over the entire surface, even though the tube is free to change its cross-section. When a collapsible tube becomes distended to the point that its cross-section is circular, and its wall is stretched, it no longer behaves as a collapsible tube but functions as a distensible, circular tube, and the flow through it it is described by the classical laws of hydro-dynamics for circular tubes.[24,25]

7.4 DIFFICULTIES IN VENOUS HEMODYNAMICS

For the reasons listed below it is not possible to apply the analytical treatment applicable to the arterial case directly to the case of veins.[26]

1. Three rather than two variables must be related. Since two of these are independent, whole families of solutions are obtained, rather than just one solution.

2. The cross-section of the vessels cannot be assumed to be circular. Therefore, this introduces problems in the solution of the fluid flow equations such as to find velocity profiles.

3. The cross-sectional area of the veins changes as the veins flatten out.This implies that the relationship between the pressure and the flow is non-linear.

4. One of the most useful methods for studying the properties of the arterial system has been the electric analog. Since the arterial system is approximately linear, linear techniques of circuit-synthesis have been used successfully to design the circuit, and the circuits could be constructed easily from standard electric circuit elements. Since the pressure flow relationship within veins is nonlinear, linear synthesis techniques can not be applied directly.

CHAPTER-EIGHTH

C O N C L U S I O N S

Equations are known which describe the laminar flow of viscous fluids for all fluid flow systems. Although some turbulence may occur near bifurcations, which dies out rapidly at small distances from the bifurcation, so that the flow through the vessel is laminar. The equations of fluid flow consist of four partial differential equations in four unknown quantities, the three components of velocity and the pressure, of the four equations are expressed conveniently as a single vector equation known as the Navier-Stokes equation.

The Navier-Stokes equation has the unfortunate property of possessing nonlinear terms. In the mathematical-analysis of the arteries, they were usually ignored. The reason for doing this was that these terms were proportional to the square of the velocity, and that, since the velocity itself was small as compared to the phase velocity, its square must be even smaller and hence could be considered negligible with respect to other terms.

In the analog computer for the human systemic circulatory system, the electrical equivalents of a segment of artery as designed up till now are not satisfactory, because they lack representation of sleeve effect, which results from the interaction between viscous and inertial forces during pulsatile blood flow. A network is discussed taking into account the sleeve effect and anomalous viscosity of blood. These effects do not alter the circuit completely but only require adding to the former circuit a corrective network.

The main dissimilarities between the venous system and the arterial system have been discussed. The primary area of concern appears to be due to the thin self supporting vessels whose wall properties vary with transmural pressure. This attributes to a decreased phase velocity which makes accurate modelling difficult.

APPENDIX-I

Introducing the notation,

$$F(2S) = \sum_{o} \frac{\lambda^m}{m!\,(m+2s-1)!} \Big/ \sum_{o} \frac{\lambda^m}{m!\,(m+2s)!} \quad , \; s = 1,2,3,\ldots, \qquad \text{(A)}$$

$$= 2s + \frac{\left(\sum\limits_{o} = \sum\limits_{1}\right) \dfrac{m\,\lambda^m}{m!\,(m+2s)!}}{\sum\limits_{o} \dfrac{\lambda^m}{m!\,(m+2s)!}}$$

$$= 2s + \frac{\sum\limits_{o} \dfrac{\lambda^{m+1}}{m!\,(m+2s+1)!}}{\sum\limits_{o} \dfrac{\lambda^m}{m!\,(m+2s)!}}$$

$$= 2s + \left[\frac{2s+1}{\lambda} + \frac{\left(\sum\limits_{o} = \sum\limits_{1}\right) \dfrac{m\lambda^m}{m!\,(m+2s+1)}}{\sum\limits_{o} \dfrac{\lambda^{m+1}}{m!\,(m+2s+1)!}} \right]^{-1}$$

$$= 2s + \left[\frac{2s+1}{\lambda} + \frac{\sum\limits_{o} \dfrac{\lambda^m}{m!\,(m+2s+2)!}}{\sum\limits_{o} \dfrac{\lambda^m}{m!\,(m+2s+1)!}} \right]^{-1} = 2s + \left[\frac{2s+1}{\lambda} + \frac{1}{F(2s+2)} \right]^{-1}$$

$$= 2s + \frac{1\rfloor}{\lceil \frac{2s+1}{\lambda}} + \frac{1\rfloor}{\lceil 2s+2} + \frac{1\rfloor}{\lceil \frac{2s+3}{\lambda}} + \frac{1\rfloor}{\lceil 2s+4} + \cdots\cdots \qquad \text{... (B)}$$

where, $\frac{1\rfloor}{\lceil x} + y = \frac{1}{x+y}$

Rewriting Eqn.(40),

$$Z_l' = \frac{4\,\mu}{\pi\, r^4}\left[\lambda + \frac{1 + \sum \frac{\lambda^m}{m!\,m!} - \sum \frac{\lambda^{m+1}}{m!\,(m+2)!}}{\sum\limits_0 \frac{\lambda^m}{m!\,(m+2)!}}\right]$$

$$= \frac{4\,\mu}{\pi r^4}\left[\lambda + \frac{1 + \sum\limits_0 \frac{\lambda^{m+1}}{(m+1)!\,(m+1)!} - \sum\limits_0 \frac{\lambda^{m+1}}{m!\,(m+2)!}}{\sum\limits_0 \frac{\lambda^m}{m!\,(m+2)!}}\right]$$

$$= \frac{4\,\mu}{\pi r^4}\left[\lambda + \frac{1 + \sum\limits_0 \frac{\lambda^{m+1}}{(m+1)!\,(m+2)!}}{\sum \frac{\lambda^m}{m!\,(m+2)!}}\right]$$

$$= \frac{4\,\mu}{\pi r^4}\left[\lambda + \frac{1 + \sum\limits_1 \frac{\lambda^m}{m!\,(m+1)!}}{\sum\limits_0 \frac{\lambda^m}{m!\,(m+2)!}}\right]$$

$$= \frac{4\,\mu}{\pi r^4}\quad \lambda + \frac{\sum\limits_0 \frac{\lambda^m}{m!\,(m+1)!}}{\sum\limits_0 \frac{\lambda^m}{m!\,(m+2)!}}$$

$$= \frac{4\,\mu}{\pi r^4}\left[\lambda + F(2)\right]$$

Applying equation (B) with s=1 leads to,

$$Z_l' = \frac{4\,\mu}{\pi r^4}\left[\lambda + 2 + \frac{1|}{|\frac{3}{\lambda}} + \frac{1|}{|4} + \frac{1|}{|5/\lambda} + \frac{1|}{|6} + \cdots\right]$$

in general form we can write,

$$Z_l' = \frac{4\mu}{\pi r^4}\left[a_1\lambda + b_1 + \frac{1|}{|(a_2\lambda)^{-1}} + \frac{1|}{|b_2} + \frac{1|}{|(a_3\lambda)^{-1}} + \frac{1|}{|b_3} + \cdots\right] \qquad \ldots\ldots (41)$$

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