TIME OPTIMAL PROBLEM OF MULTIDIMENSIONAL DISTRIBUTED PARAMETER SYSTEM

A DISSERTATION
submitted in partial fulfilment of the
requirements for the award of the Degree
of
MASTER OF ENGINEERING
in
ELECTRICAL ENGINEERING
(System Engineering and Operation Research)

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C E R T I F I C A T E

Certified that the dissertation entitled "TIME OPTIMAL PROBLEM OF MULTIDIMENSIONAL DISTRIBUTED PARAMETER SYSTEM" which is being submitted by Mr. Rajiv Goyal in partial fulfilment of the requirement for the award of Master of Engineering in Electrical Engineering (System Engineering and Operation Research) of the University of Roorkee is a record of the student's own work carried out by him under my supervision and guidance. The matter embodied in this dissertation has not been submitted for the award of any other degree.

This is to further certify that Mr. Goyal has worked for a period of $5\frac{1}{2}$ months from 1st April 1976 to 18th September 1976 for preparing this dissertation.

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A C K N O W L E D G E M E N T

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(RAJIV GOYAL)

CONTENTS

SCOPE OF THE WORK

Distributed Parameter System (DPS) is a system in which mass or energy is distributed over all the spatial dimensions. Such systems arise in various application areas, such as : Bending of beams, Heat transfer, chemical process systems, communication systems. Dynamics of such systems is described by partial differential equations. Compared to ordinary differential equations, very small amount of work has been done towards the solution of partial differential equations. Owing to these difficulties study of optimal control of DPS is formidable compared to its counterpart in Lumped Parameter Systems (LPS).

There are many physical systems which are described by diffusion equations, for example, heating of solids, flow of viscous fluids, diffusion of gases, flux distribution in a solid rotor. Very often it is ~~described~~ desired to achieve a particular type of distribution in such systems by applying manipulative control on its boundaries. It is also desirable to achieve such distributions by suitably designing the controller, in shortest possible time. From physical point of view, the control may sometimes have physical constraints.

This dissertation proposes to investigate time optimal control of multidimensional DPS considering Linear Diffusion Equations as illustrative examples. The dimensions of the systems considered are one, two and three. The results are extended to N-dimensions. The boundary controlled Diffusion Systems are transformed to integral equations using successively Seperation Variable and Laplace Transformation techniques, for Time Optimal Control studies. Structure of controller has been studied following optimal control theory of LPS, which demands the solution of a set of simultaneous non-linear equations involving exponential functions in time. The difficulties of solving these equations have been illustrated in the design of Time Optimal Control of fourth order - LPS. This leads to the belief that solution of these equations in the case of diffusion systems is much more difficult when convergent results are desired. A transformation technique has been applied which overcomes these difficulties. The controllers thus obtained are tested for various systems and results then compared. The study ends with a conclusion emphasising extension of the present work for future studies.

CHAPTER - I

INTRODUCTION

A distributed parameter system is a system whose mass or energy is distributed over all its spatial dimensions. The behaviour of a such a system, in contrast to lumped parameter systems which depend on a single factor and whose parameters are concentrated at a point, depend on various factors, say, time, distance. Behaviour of a multitude of physical systems can be represented by distributed parameter systems, flux distribution in a solid rotor (Fig. 1.1), oscillations of fluid in a tank, (fig. 1.2), transfer of power in transmission lines (Fig. 1.3), vibration of beams (Fig. 1.4) and circling line when an aircraft flies with a long flexible cable attached to it [9] (Fig. 1.5) are a few examples. Many other control and design problems faced in electronics, hydraulics, metallurgical and chemical plants are also distributed parameter problems. The usual phenomenon in a continuous industrial process is the flow of material through a number of processing zones which are distributed consecutively in space. The control action for them, e.g. the control of temperature are also distributed in space and act over the entire length of the processing zones. In metallurgy industry, the operation of a rolling mill, or any other system, for hot processing of metal by applying pressure depends entirely on heating rate. Here then, it is required to control the heating to obtain a specified temperature distribution throughout the entire volume of metal, in minimum time. Another very popular example, which has been the topic of considerable importance in research works, is heat flow or diffusion equation. In general, all dynamical systems are basically distributed systems whose behaviour is described by partial differential equations, integral equations, integro-differential equations and sometimes more general and complex functionals and relations.

The analysis of the distributed parameter system is a very complex and laborious task. Sometimes on neglection of some parameters of a distributed parameter system rest of the parameters can be lumped together, reducing the labour needed in analysis and computation. Such a system is lumped parameter system and its behaviour is described by ordinary differential equations, e.g. If there has to be transmission of power in a short line (upto 60 or 80 Kms) then the line capacitance can be neglected. The resistance and inductance of the line are then lumped and the analysis of this line is much easier to that when the capacitance was considered. Thus in a nutshell, lumped parameter system is a particular case of distributed parameter system.

1.1 OPTIMAL CONTROL OF DISTRIBUTED PARAMETER SYSTEMS

The theory of optimal systems, based on the recent advances in mathematics and engineering, has been rapidly developing during recent years. As the technology is progressing, demands from automatic control system are growing more diversive. The range of objects operated by automatic control are rapidly increasing. In various processes automatic control systems are required to ensure highest productivity for a given expenditure of raw materials, fuel or energy. High accuracy of operation of a system or a unit and high speed of operation are often required, some specified state is required to be approached with least expenditure of available means. Many industrial systems operate in a manner such that, the potentiality of the plant is not fully utilized. It is, therefore, necessary to create methods of control that enable to utilize the potentialities of the plant to the fullest extent and create systems that are optimal in any specified sense [10, 32]

Taking into account the importance of optimality in modern technology, optimal control of distributed parameter system is a topic of keen interest as behaviour of a multitude of dynamical systems is described in distributed parameters. But due to complications involved in treatment of partial differential equations and co mputation work, less attention has been paid towards it. Looking into the literature available, it is seen that, in general, the approach for numerical solution of optimal control problems in distributed parameter system is to prepare an approximate model of the physical system which can be solved by conventional techniques[10]. Reviewing the various methods used for approximating the system, they can be put into two categories.

(1) In the first category, the optimal control problem is formulated for the equations describing the distributed parameter systems. Conditions for optimality are obtained and an approximate solution to the optimal control policy is found out by utilizing the conditions for optimality.

(2) In the second category, the distributed parameter system is approximated to a lumped parameter system by some technique. The specified performance index is then approximated to one specified for the lumped parameter system model and an optimal control policy is determined for lumped system. In this approach it is necessary to choose a lumped parameter system model which will yield optimal control law for the lumped system, which is sufficiantly close to the optimal control law for distributed parameter system.

The first category makes use of either of the four methods [9] whichever is suitable for the purpose of approximating distributed parameter system to a model in the problem -

(a) Eigen function expansion or Harmonic Truncation

(b) Space quantisation

(c) Time and space quantisation

(d) Transfer function approximation

Approximated model is then subjected to either of the four approaches [9,10] to develop the necessary conditions for optimality

(i) Variational method

(ii) Moment method

(iii) Dynamic programming method

(iv) Function space method

Frequently a combination of these methods is used to find the optimal control.

Vinter [17] gave a generalised form of Pontryagins Maximum Principle and derived canonical equations yielding minimum time contfol for linear systems w ith quadratic control effort constraints. He observed that by taking a restricted class of system solution could be obtained in terms of system parameters and the same procedure could be employed for minimum energy problems. Lu and Shen [30] presented a practical approach to the problem of optimal boundary control synthesis in one dimensional linear stationary distributed parameter s ystem and obtained optimal control function directly from a generalised quadratic performance index by using gradient methods extended to the functional space. Kin and Erzberger [20] considered a N-dimensional wave equation and desired Riccati equation for optimum boundary control with unconstrained control function and quadratic error measure. He solved these equations by using certain type of weighting factors in the quadratic error index and employing the seperable variable method. Greenberg [21] took parabolic differential equation and formulated them as ordinary differential equations in Hilbert space. By formulating the quadratic cost criterion as inner products on this space he proved the existence of optimum control both when the system operator was coercive and the infinitesimal generator of a semigroup of operators. Wang & Tung [25] discussed general method of optimum control of distributed parameter dynamical system ~~giving~~ with main features as mathematical description of distributed system, formulation of optimum control problem and derivation of a maximum principle for a particular class of system. Brogan [27] treated the distributed parameter problem involving non-homogeneous boundary conditions by dynamic programming technique and showed that it can be as powerful as the variational approach. Brogan 28 also took the wildland fire suppression from control point of view and taking the fire fighting as removal of heat he made a distributed parameter model of it. He then suggested many optimization problems but was not able to present a final solution due to scarcity of qualitative experimental data.

In this dissertation time optimal control of multidimensional distributed parameter system has been investigated by considering linear diffusion equation as an example.

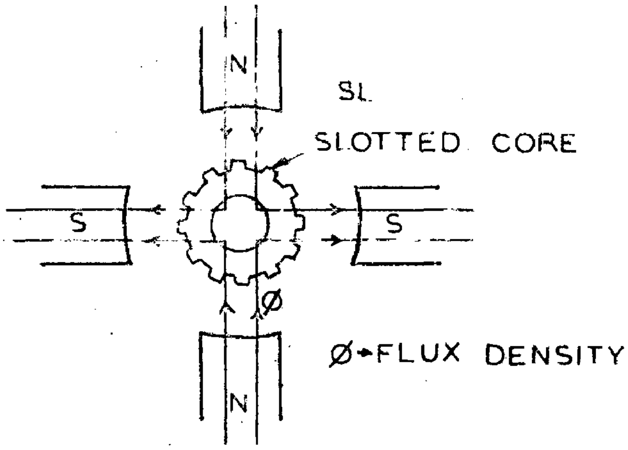
1.3 TIME OPTIMAL CONTROL OF LINEAR DIFFUSION SYSTEMS

Leibowitz and Surendran [29] developed a procedure of time optimal control of Linear Diffusion Systems consisting of two iteration s chemes and based on singular perturbation techniques and calculas of variation s. Khatri and Goodson [18] synthesised one and two dimensional heat transfer system by reducing partial differential equations and boundary conditions to an integral equation and applying variation procedure. Boody [24] applied a noniterative finite difference method for solution of Poisson and Laplace equations with linear boundary conditions. He observed the method to be simpler and more accurate than iterative procedure and the computational work was vastly introduced. McCausland [8] applied three methods : subdivision method, fourier series method and the parabolic method for finding time optimal control in a single dimension heat conduction equation and observed that subdivision and harmonic method gives the correct control input taking sufficiently high order approximation to the actual distributed system but the parabolic method gave an inherent inaccuracy which could not be removed. McCausland [16] obtained a lumped model using truncated eigen function expansion for the distributed system but this approach was limited to the problems where the time and space variables could be seperated. Prabhu and McCausland [11] applied Galerkin's method for modelling the diffusion system to lumped system and observed the obtained switching instants of the optimal control to be in a very good agreement with available literature. Mahapatra [13] converted the diffusion systems to integral equation and had proposed some transformation for time optimal control studies using Knudson's method applicable to lumped parameter systems [14]. These transformations have been used in finding solution of non-linear equations (5.1) in this dissertation, which reduces the computation work tremendously. Sakawa[12] gave two methods : variational method and reduction of problem to linear or non-linear programming, for time optimal study of heat conduction equation in single dimension. Upon use of variational method he derived Freedholm's integral equations as necessary conditions for time optimal control.

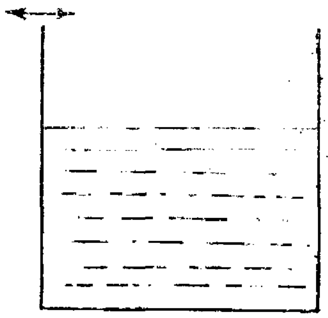
In this dissertation diffusion problems giving heat conduction is one, two and three spatial dimensions have been stated in Chapter II. They are then subjected to Laplace Transform [5] and Seperable Variable [5] techniques in Chapter IV. Using the property of orthogonality, the system is then converted into lumped parameter integral equations. The technique has been extended to n-dimensions. Thus the problem of time optimal control in distributed parameter system is converted into time optimal control of lumped parameter system. The Pontryagin's Maximum Principle[2] then gives the structure of control to be of Bang-Bang type, having (n-1) switchings for nth order system.

This leaves the problem as the solution of non-linear simultaneous equations involving exponential of time. It is this part which has been given the maximum stress in this dissertation. Taking three illustrative examples of fourth order lumped parameter system in Chapter III and converting them into non-linear equations (of the nature obtained in distributed parameter system), it has been shown that tremendous difficulty is faced in their solution if they are kept in time domain. Owing to these difficulties only a few persons have touched this portion of computation. A transformation[13] has been used in Chapter V which reduces the labour needed in computation of the solution of these non-linear equations.

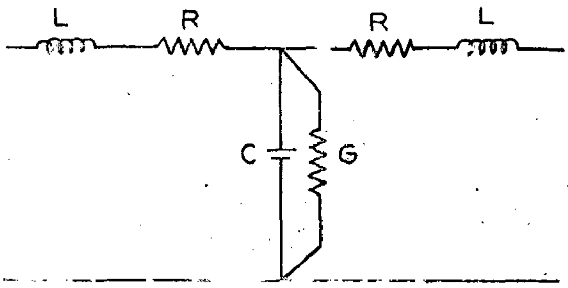
Lastly, using harmonic truncation method [10] , the syst response is obtained for increasing number of switchings. The computational work has been carried out on one, two and three dimensional linear diffusion systems.



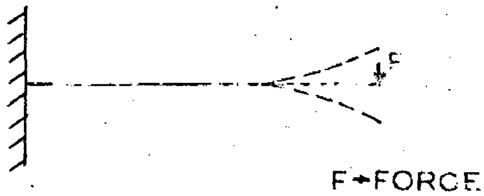
SOLID ROTOR
FIG.1.1



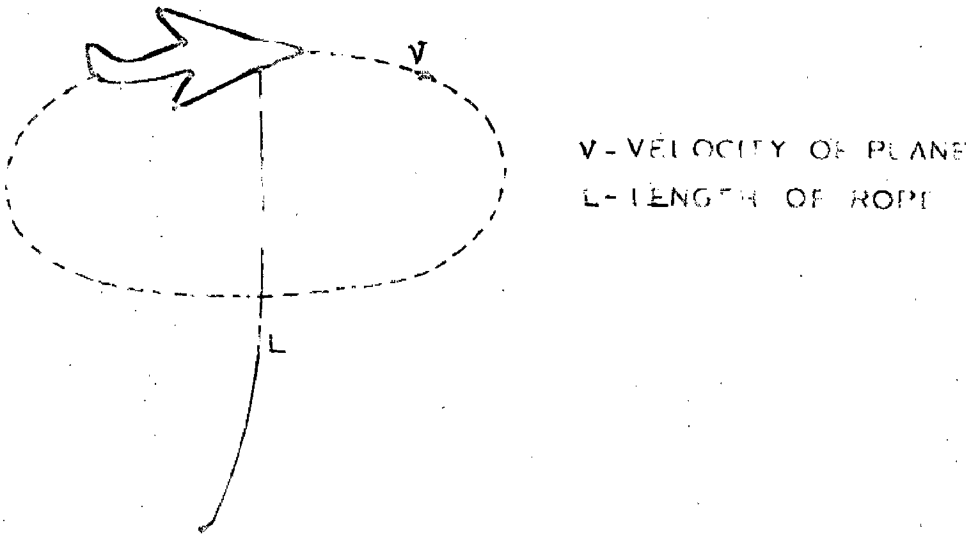
FLUID IN TANK
FIG. 1.2



R - RESISTANCE/km, L-INDUCTANCE/km
G - CONDUCTANCE/km, C-LINE CAPACITANCE/km
LONG TRANSMISSION LINE
FIG. 1.3



BEAM
FIG. 1.4



CIRCLING LINE
FIG. 1.5

CHAPTER - II

PROBLEM STATEMENT

The chapter describes various systems under investigation for Time Optimal Control studies. These systems include (i) One dimensional Diffusion System, (ii) Two dimensional Diffusion System, and (iii) Three dimensional Diffusion System.

2.1. PROBLEM-A: ONE DIMENSIONAL DIFFUSION SYSTEM

Consider a rod of length L and of negligible width and thickness. The length is taken along the spatial coordinate - x, as shown in Fig.2.1.1. Heat is added at one end of the rod at x = 0 and the other end (x = L) is maintained at zero temperature. It is desired to attain a steady state temperature distribution $\theta_s(x)$ in time T. Let u(t) be the source of heat (hence called controlled function) applied at x = 0. Let us assume the initial temperature distribution of the rod to be zero. Neglecting the heat flow along y and z direction, the behaviour of temperature θ at distance x and time t is characterised by partial differential equation:

$$\frac{\partial \theta (x,t)}{\partial t} = \alpha^2 \frac{\partial \theta (x,t)}{\partial x^2} \quad \forall \; x \in (0,L) \qquad ..(2.1)$$

With boundary conditions:

$$\theta(o,t) = u(t); \quad \theta(L,t) = 0 \tag{2.2}$$

and with initial and final conditions:

$$\theta(x,0) = 0; \quad \theta(x,T) = \theta_s(x) \tag{2.3}$$

Where α^2 is the conductivity of the material of the rod.

When (2.1 - 2.3) are normalised with respect to spatial coordinate

x, then the system is tranformed to:

$$\frac{\partial\theta(x,t)}{\partial t} = \frac{\partial \theta(x,t)}{\partial x^2} \quad ; \quad x \in (0,1) \tag{2.4}$$

With boundary conditions;

$$\theta(o,t) = u(t); \quad \theta(1,t) = 0 \tag{2.5}$$

and initial and final conditions as:

$$\theta(x,0)=0; \quad \theta(x,T) = \theta^*(x) = \sin \pi x/\pi, \text{ (say)} \tag{2.6}$$

2.2. PROBLEM-B: TWO DIMENSIONAL DIFFUSION SYSTEM

Consider a square plate of length L in spatial coordinates x and y as shown in fig.2.2.1.Heat is added uniformly at the surface x = 0 and all the other surfaces are maintained at zero temperature. The temperature distribution θ (x,y,t), for this system, when normalised with respect to x and y coordinates, is governed by the partial differential equation:

$$\frac{\partial\theta(x,y,t)}{\partial t} = \frac{\partial^2\theta(x,y,t)}{\partial x^2} + \frac{\partial^2\theta(x,y,t)}{\partial y^2} ;$$

$$x,y \in (0,1) \qquad (2.7)$$

Let u(t) be the heat source, hence called control function, applied at x = 0. Therefore, the boundary conditions are:

$$\theta(0,y,t) = u(t) ; \quad \theta(1,y,t) = 0 \qquad (2.8)$$

$$\theta(x,0,t) = 0 ; \quad \theta(x,1,t) = 0$$

Let us assume that the plate is initially released and at t = T, the steady state temperature distribution is $\theta^*(x,y)$. The initial and final conditions are then given as:

$$\theta(x,y,0) = 0; \; \theta(x,y,T) = \theta^*(x,y)$$

$$= \frac{\sin \pi x \sin \pi y}{\pi^2}, \text{(say)} \qquad (2.9)$$

2.3 PROBLEM-C: THREE DIMENSIONAL DIFFUSION SYSTEMS

Consider a cube of arm L as shown in Fig.2.3.1. Heat is added uniformly at the surface x = 0 and all the other faces are maintained at zero temperature. The temperature distribution $\theta(x,y,z,t)$, for this system, when normalized with respect to x, y and z-spatial coordinates, is governed by the partial differential equation:

$$\frac{\partial\theta(x,y,z,t)}{\partial t} = \frac{\partial^2\theta(x,y,z,t)}{\partial x^2} + \frac{\partial^2\theta(x,y,z,t)}{\partial y^2} + \frac{\partial^2\theta(x,t,z,t)}{\partial z^2} \; ; \quad x,y,z, \in (0,1) \qquad (2.10)$$

Let u(t) be the heat source, hence called control function, applied at x = 0. Therefore, the boundary condition are:

$$\begin{aligned} \theta\,(0,y,z,t) &= u\,(t) \; ; & \theta\,(1,y,z,t) &= 0 \\ \theta\,(x,0,z,t) &= 0 \; ; & \theta\,(x,1,z,t) &= 0 \\ \theta\,(x,y,0,t) &= 0 \; ; & \theta\,(x,y,1,t) &= 0 \end{aligned} \qquad (2.11)$$

Like previous problems, let us assume the initial temperature distribution of the plate to be zero. If $\theta^*(x,y,z)$ be the steady state temperature distribution at t = T, then the initial and final conditions are given as:

$$\theta(x,y,z,0) = 0 \; ; \quad \theta(x,y,z,T) = \theta^*(x,y,z)$$

$$= \frac{\sin \pi x \sin \pi \sin \pi z}{\pi^3} \text{, (say)} \qquad (2.12)$$

2.4 TIME OPTIMAL CONTROL

From physical point of view, the heat source u(t) can have magnitude constraints of the type :

$$|u(t)| \leq 1.0 \qquad (2.13)$$

The time optimal control problem is to design the structure of u(t) for all the three systems mentioned earlier, such that, the steady state distributions $\theta^*(x)$ for one dimensional, $\theta^*(x,y)$ for two dimensional, $\theta^*(x,y,z)$ for three dimensional Diffusion systems stated in equation (2.4 - 2.6),(2.7 - 2.9) and (2.10 -2.12) respectively, can be obtained in minimum time T.

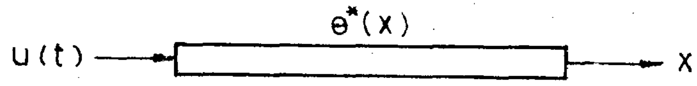


FIG. 2.1.1

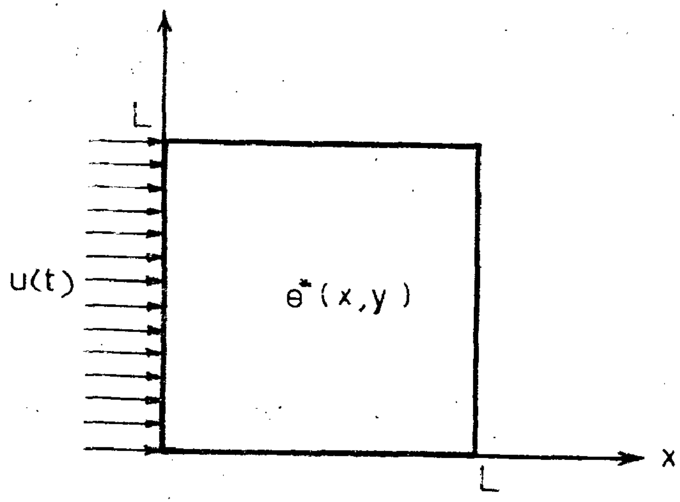


FIG. 2.2.1

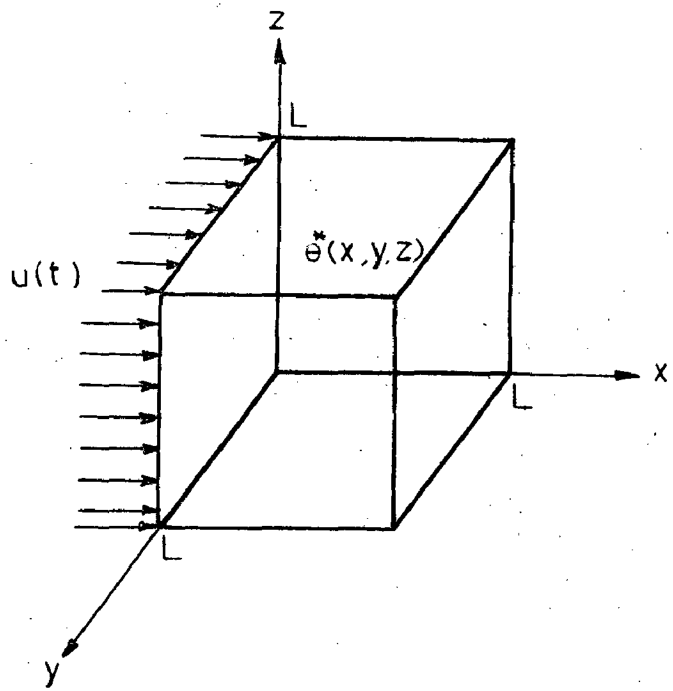


FIG. 2.3.1

CHAPTER - III

TIME OPTIMAL CONTROL OF LUMPED PARAMETER SYSTEMS

This chapter examines the computational problem of time optimal control of systems described by ordinary linear differential equations. The control function is a scalar one. Results of this chapter will be useful to study the time optimal control of linear diffusion equations described in later chapters.

3.1 PROBLEM STATEMENT

Consider a linear, time-invariant lumped parameter system (LPS) described by a first order, ordinary vector differential equation:

$$\dot{X}(t) = A\,X(t) + Bu(t) \qquad \ldots (3.1)$$

with initial conditions: $X(0) = X_o \qquad \ldots (3.2)$

where A is the system matrix of order n x n.

B is the control matrix of order n x 1.

X(t) is state vector of order n x 1.

u(t) is a scalar quantity, called the control function for the system (3.1 - 3.2).

It is desired to design the structure of the control u(t), such that the system described by (3.1) is driven from the initial state X_o to a final X* in minimum time T. The control being limited in magnitude by the constraint

$$|u(t)| \leq 1 \quad \ldots \quad (3.4)$$

3.2 SOLUTION TECHNIQUE

If the lumped parameter system is of order two or maximum three, $u^*(t)$ can be designed using methods given in [2,3]. The principle adopted here to design the optimal control $u^*(t)$ is in the line of Smith [1].

Consider the diagonalisation of matrix A in (3.1), assuming it to have (for the sake of simplicity) distinct eigen values λ_i's, $i = 1, 2, \ldots, n$.

Let

$$X(t) = PZ(t) \quad \ldots \quad (3.5)$$

where P is a non singular matrix

then (3.1) can be written as

$$\dot{Z} = P^{-1}APZ + P^{-1}B\,u \quad \ldots \quad (3.6)$$

making the substitution

$$Q = P^{-1}B; \; D = P^{-1}AP \quad \ldots \quad (3.7)$$

The system (3.1) gets transformed in the canonical form

$$\dot{Z} = DZ + Qu \quad \ldots \quad (3.8)$$

where

$$D = \begin{bmatrix} \lambda_1 & & & & \\ & \lambda_2 & & & \\ & & \cdot & 0 & \\ & 0 & & \cdot & \\ & & & & \lambda_n \end{bmatrix} \quad \ldots (3.9)$$

From (3.8) and (3.9),

$$\dot{z}_i = \lambda_i z_i + q_i u \; ; \quad i = 1,2, \ldots, n \quad \ldots (3.10)$$

Applying Laplace Transform technique to (3.10), the solution is given as

$$z_i(t) = z_i(0) e^{\lambda_i t} + q_i \int_0^t e^{\lambda_i(t-\tau)} u(\tau) \, d\tau \; ; \quad i = 1,2,\ldots,n \quad .. (3.11)$$

At t = T

$$z_i(T) = z_i(0) e^{\lambda_i T} + q_i \int_0^T e^{\lambda_i(T-t)} u(t) dt \; ; \quad i = 1,2,\ldots,n \quad .. (3.12)$$

or

$$\int_0^T e^{\lambda_i(T-t)} u(t) dt = \frac{z_i(T) - z_i(0) e^{\lambda_i T}}{q_i} \; ; \; i = 1,2,\ldots,n \quad .. (3.13)$$

Assuming $X_o = 0$; hence giving $z_i(0) = 0$

Thus from (3.13)

$$\int_0^T e^{\lambda_i(T-t)} u(t) \, dt = c_i \; ; \; i=1,2,\ldots,n \quad \ldots (3.14)$$

where $c_i = \dfrac{z_i(T)}{q_i}$ ... (3.15)

From Pontryagin's Maximum Principle [2] , the structure of u(t) will be Bang-Bang type, having (n-1) switchings for nth order system (3.1). Since in our problem the initial state vector is zero, Bang-Bang control will be + 1 at time t = 0 (see fig-3.1). Breaking the integral into n-sections with control u(t) changing in each section (3.14) can be written as

$$\int_0^{t_1} e^{\lambda_i (T-t)} u(t)\,dt - \int_{t_1}^{t_2} e^{\lambda_i (T-t)} u(t)\,dt + \dots + (-1)^{n-1} \int_{t_{n-1}}^{T} e^{\lambda_i (T-t)} u(t)\,dt = c_i;$$

$$i = 1,2,\dots,n \quad \dots(3.16)$$

where $t_1, t_2, \dots, t_{n-1}$ are the time of switchings and T the optimal time in which the control transforms system (3.1) from initial state X_o to steady state X*

Solving (3.16)

$$2e^{\lambda_i (T-t_1)} - 2e^{\lambda_i (T-t_2)} + \dots + (-1)^{n-2}\, 2e^{\lambda_i (T-t_{n-1})} - e^{\lambda_i T} + (-1)^{n-1} + \lambda_i c_i = 0;\ i=1,2,\dots,n. \quad \dots (3.17)$$

Substituting

$$F_i(t_1,t_2,\dots,t_{n-1},T) = 2e^{\lambda_i(-t_1+T)} - 2e^{\lambda_i (T-t_2)} + \dots + (-1)^{n-2}\, 2e^{\lambda_i (T-t_{n-1})} - e^{\lambda_i T} + (-1)^{n-1} + \lambda_i c_i;\ i=1,2,\dots,n \quad \dots (3.18)$$

we get

$$F_i(t_1, t_2, \ldots, t_{n-1}, T) = 0 \;;\; i = 1,2,\ldots, n \quad \ldots \quad (3.19)$$

Thus problem of time optimal control of system (3.1-3.3) with constraint (3.4) is finally reduced to the solution of non-linear equations (3.19).

3.3 SOLUTION OF NON LINEAR EQUATIONS

Newton-Raphson method for solution of non-linear simultaneous equations have been adopted here for the solution of (3.19). Let $\bar{t}_1$, $\bar{t}_2$, ..., $\bar{t}_n$ be the approximate roots of (3.19) ($\bar{t}_n$ being the approximate value of optimal value of optimal time T). If these approximate values are located, respectively, at intervals h_1, h_2, ..., h_n from the exact values of the roots, then

$$t_j = \bar{t}_j + h_j \;;\; j = 1,2, \ldots, n \quad \ldots (3.20)$$

and

$$F_i(\bar{t}_1 + h_1, \bar{t}_2 + h_i, \ldots, \bar{t}_n + h_n) = 0 \;;\; i=1,2,\ldots,n. \quad \ldots (3.21)$$

Expanding F_i's by Taylor's series and neglecting higher powers of h_j's in (3.21).

$$F_i(\bar{t}_1, \bar{t}_2, \ldots, \bar{t}_n) + h_1 \left(\frac{\partial F_i}{\partial t_1}\right)_{t_1=\bar{t}_1} + \ldots + h_n \left(\frac{\partial F_i}{\partial t_n}\right)_{t_n=\bar{t}_n} = 0;$$

$$i=1,2, \ldots, n \quad \ldots (3.22)$$

From (3.18),

$$\frac{\partial F_i}{\partial t_j} = (-1)^j 2\lambda_i \, e^{\lambda_i (T-t_j)} \;;\; j = 1,2,\ldots,n-1 \ldots$$

and

$$\frac{\partial F_i}{\partial T} = - \lambda_i \, e^{\lambda_i T} \quad \ldots (3.23)$$

Substituting these in (3.22), n-simultaneous linear equations in unknown $h_1, h_2, \ldots, h_n$ are obtained. Solutions of these equations are obtained by Gauss -Siedal method.

Computer Program for solving these equations (3.19) by the above method has been provided in Appendix - 3.4.2 .

3.4 ILLUSTRATIVE EXAMPLE

Let

$$\dot{X} = \begin{bmatrix} -6.8190 & 4.2700 & -0.8120 & 0.2368 \\ 4.27 & -7.6300 & 4.5070 & -0.8120 \\ -0.812 & 4.5070 & -7.6300 & 4.2700 \\ 0.2368 & -0.8120 & 4.2700 & -6.8170 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix} + \begin{bmatrix} -0.4872 \\ 1.0150 \\ -1.9530 \\ 4.6620 \end{bmatrix} u(t) \quad \ldots (3.24)$$

given that

$$X(T) = \begin{bmatrix} 0.1870 \\ 0.3010 \\ 0.3010 \\ 0.1870 \end{bmatrix} \text{ and } X(0) = 0 \quad \ldots (3.25)$$

While proceeding towards the final solution, it was found that

$$D=\begin{bmatrix} -0.9861 & 0 & 0 & 0 \\ 0 & -3.9147 & 0 & 0 \\ 0 & 0 & -8.7190 & 0 \\ 0 & 0 & 0 & -15.2780 \end{bmatrix}$$

$$P=\begin{bmatrix} 0.371 & -0.601 & -0.601 & -0.371 \\ 0.601 & -0.371 & 0.371 & 0.601 \\ 0.601 & 0.371 & 0.371 & -0.601 \\ 0.371 & 0.601 & -.601 & 0.371 \end{bmatrix}$$

$$P^{-1}=\begin{bmatrix} 0.371 & 0.601 & 0.601 & 0.371 \\ -0.601 & -0.371 & 0.371 & 0.601 \\ -0.601 & 0.371 & 0.371 & -0.601 \\ -0.371 & 0.601 & -0.601 & -0.371 \end{bmatrix}$$

$$Z(T)=\begin{bmatrix} 0.50114 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$Q = \begin{bmatrix} 0.9876 \\ 0.9936 \\ -2.8599 \\ 3.6995 \end{bmatrix} \quad ; \quad C = \begin{bmatrix} 0.5074 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Computer programme for obtaining these values is shown in Appendix - 34.1

The integral equations, which are obtained on using all the above values, are then

$$\int_0^T e^{-0.9861\,(T-t)}\, u(t)\,dt = 0.5074$$

$$\int_0^T e^{-3.9147\,(T-t)}\, u(t)\,dt = 0$$

$$\int_0^T e^{-8.7190(T-t)}\, u(t)\,dt = 0$$

$$\int_0^T e^{-15.2780\,(T-t)}\, u(t)\,dt = 0$$

The initial values given as the approximate solution are seen in the first row of table - 3.1. The rest of the table gives an idea of the intermediate stages reached, while proceeding towards final solution. The final solution obtained (giving the structure of the control u*(t)) is as shown in table - 3.2 . Appendix 3.3 summarizes results of two more examples of fourth order system.

3.5 CONCLUDING REMARKS

Finding time optimal control switchings necessitates the solution of non linear equations (3.19) involving exponental functions in time. The Newton-Ralphson method adopted here for the solution of these equations needs some estimated value of the variables before proceeding towards the final solution. It has been seen practically that a great deal of difficulty is experienced in assuming these initial values of the variables t_1, t_2, t_3 and t_4. These finite timings can lie any where on the positive real line. This makes the choice of initial values, a matter of guess and intution. It has been further found that these equations are very sansitive to the selection of initial values. Many a time, on giving a set of initial values, overflow problems were expe rienced when working on IBM 1620. The programme had to be brought to an end and refed with a different set of initial values. The initial value chosen, thus, may or may not lead to the solution. The difficulty experienced in solution of fourth order system leads to the belief that still more difficulties will be expe rienced in finding time optimal Control switchings for a higher order system.

APPENDIX 3.4.1

```
C   C  EIGEN VALUE AND EIGEN VECTOR CALCULATION RAJIV GOYAL
       DIMENSION A(10,10),B(10,10),BT(10),YT(10),LEMDA(10),PTRAN(10,10),
      1D(10),Z(10),C(10)
       N=4
       READ 2,((A(I,J),J=1,N),I=1,N)
     2 FORMAT(4F20.8)
       READ 2,(BT(I),I=1,N)
       READ 2,(YT(I),I=1,N)
       CALL EIGEN (A,B,N,10)
       PUNCH 4
     4 FORMAT(20X,14HEIGEN   VALUES)
       PUNCH 2,(A(I,I),I=1,N)
       PUNCH 5
     5 FORMAT(20X,15HEIGEN   VECTORS)
       PUNCH 2,((B(I,J),J=1,N),I=1,N)
       DO 10 I=1,N
       JP=I
       DO 10 J=JP,N
       PTRAN(I,J)=B(J,I)
    10 PTRAN(J,I)=B(I,J)
       DO 11 I=1,N
       D(I)=PTRAN(I,1)*BT(1)
       Z(I)=PTRAN(I,1)*YT(1)
       DO 11 J=2,N
       D(I)=D(I)+PTRAN(I,J)*BT(J)
    11 Z(I)=Z(I)+PTRAN(I,J)*YT(J)
       DO 12 I=1,N
    12 C(I)=Z(I)/D(I)
       PUNCH 2,(Z(I),I=1,N)
       PUNCH 2,(D(I),I=1,N)
       STOP
       END
```

Eigen

APPENDIX 3.4.2

```
C  RAJIV GOYAL        TIME OPTIMAL CONTROL OF LUMPED SYSTEMS
   DIMENSION A(10,10),D(10,10),DY(10),R(10),F(10),T(10),X(10),C(10)
200 FORMAT(8E10.4)
   EPS=0.0001
   N=3
   N1=N-1
   M=N+1
   READ 200,(R(I),I=1,N),(C(I),I=1,N)
   READ 200,(T(I),I=1,N)
90  DO 25 I=1,N1
25 X(I)=T(N)-T(I)
   X(N)=T(N)
   DO 30 I=1,N
30 F(I)=2.*EXP(R(I)*X(1))-2.EXP(R(I)*X(2))+EXP(R(I)*X(3))-1.0+
  1R(I)*C(I)
   DO 35 I=1,N
   DO 35 J=1,N1
   JA=J+1
35 A(I,J)=-2.0*(-1.0)**JA*R(I)*EXP(R(I)*X(J))
   DO 36 I=1,N
36 A(I,N)=R(I)*(F(I)+1.0-R(I)*C(I))
   DO 38 I=1,N
38 A(I,M)=-F(I)
   SOLUTION BY GAUSS METHOD STARTS
   DO 5 K=1,N1
   DO 6 J=K,M
 6 D(K,J)=A(K,J)/A(K,K)
   KK=K+1
   DO 3 I=KK,N
   DO 4 J=K,M
 4 D(I,J)=A(I,J)-A(I,K)*D(K,J)
 3 CONTINUE
   DO 11 MM=1,N
   DO 11 NN=1,M
   A(MM,NN)=0.0
11 A(MM,NN)=D(MM,NN)
 5 CONTINUE
   DY(N)=A(N,M)/A(N,N)
   DO 20 I=1,N1
   K=N-I
   SUM=0.0

   J=N-K
   DO 7 L=1,J
   LL=M-L
 7 SUM= SUM+A(K,LL)*DY(LL)
   DY(K)=(A(K,M)-SUM)/A(K,K)
20 CONTINUE
   PUNCH 200,(T(I),I=1,N),(F(I),I=1,N)
   PUNCH 200,(DY(I),I=1,N)
   IF(ABSF(F(1))-EPS)15,15,60
15 IF(ABSF(F(2))-EPS)16,16,60
16 IF(ABSF(F(3))-EPS)17,17,60
```

```
17 IF(ABSF(F(4))-EPS)18,18,60
60 DO 40 I=1,N
40 T(I)=T(I)+DY(I)
   GO TO 90
18 STOP
   END
```

APPENDIX - 3.4.3

EXAMPLE

$$\dot{X} = \begin{bmatrix} -6.409 & 3.860 & -0.6115 & 0.1736 \\ 3.860 & -7.021 & 4.034 & -0.6115 \\ -0.6115 & 4.034 & -7.021 & 3.860 \\ 0.1736 & -0.6115 & 3.860 & -6.409 \end{bmatrix}$$

$$\begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix} + \begin{bmatrix} -0.3339 \\ 0.7548 \\ -1.4810 \\ 4.1380 \end{bmatrix} u(t)$$

given that $X(T) = \begin{bmatrix} 0.1870 \\ 0.3010 \\ 0.3010 \\ 0.1870 \end{bmatrix}$

It was found that

$$D = \begin{bmatrix} -0.9793 & 0 & 0 & 0 \\ 0 & -3.8193 & 0 & 0 \\ 0 & 0 & -8.2431 & 0 \\ 0 & 0 & 0 & -13.8183 \end{bmatrix}$$

$$P = \begin{bmatrix} 0.371 & -0.601 & -0.601 & -0.371 \\ 0.601 & -0.371 & 0.371 & 0.601 \\ 0.601 & 0.371 & 0.371 & -0.601 \\ 0.371 & 0.601 & -0.601 & 0.371 \end{bmatrix}$$

$$P^{-1} = \begin{bmatrix} 0.371 & 0.601 & 0.601 & 0.371 \\ -0.601 & -0.371 & 0.371 & 0.601 \\ -0.601 & 0.371 & 0.371 & -0.601 \\ -0.371 & 0.601 & -0.601 & 0.371 \end{bmatrix}$$

$$z(T) = \begin{bmatrix} 0.50114 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$Q = \begin{bmatrix} 0.9774 \\ 1.8588 \\ -2.5581 \\ 3.0071 \end{bmatrix} \quad ; \quad C = \begin{bmatrix} 0.5129 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The solution, i.e. the time of switchings and final times T, were found as shown in table - 3.3

TABLE - 3.3

t_1	t_2	t_3	T
5.322	5.619	5.742	5.773

EXAMPLE

$$\dot{X} = \begin{bmatrix} -5.0 & 2.5 & 0 & 0 \\ 2.5 & -5.0 & 2.5 & 0 \\ 0 & 2.5 & -5.0 & 2.5 \\ 0 & 0 & 2.5 & -5.0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 2.5 \end{bmatrix} u(t)$$

given that

$$X(T)= \begin{bmatrix} 0.1870 \\ 0.3010 \\ 0.3010 \\ 0.1870 \end{bmatrix}$$

During the solution, it was found that

$$D = \begin{bmatrix} -0.954 & 0 & 0 & 0 \\ 0 & -3.454 & 0 & 0 \\ 0 & 0 & -6.545 & 0 \\ 0 & 0 & 0 & -9.045 \end{bmatrix}$$

$$P = \begin{bmatrix} 1.0 & 1.0 & 1.0 & 1.0 \\ 1.62 & 0.618 & -0.618 & -1.62 \\ 1.62 & -0.618 & -0.618 & 1.62 \\ 1.0 & -1.0 & 1.0 & -1.0 \end{bmatrix}$$

$$P^{-1} = \begin{bmatrix} 1.0 & 1.62 & 1.62 & 1.0 \\ 1.0 & 0.618 & -0.618 & -1.0 \\ 1.0 & -0.618 & -0.618 & 1.0 \\ 1.0 & -1.62 & 1.62 & -1.0 \end{bmatrix} \quad Z(T) = \begin{bmatrix} 1.346 \\ 0.0 \\ 0.0 \\ 0.0 \end{bmatrix}$$

$$Q = \begin{bmatrix} 2.5 \\ -2.5 \\ 2.5 \\ -2.5 \end{bmatrix} \quad ; \quad C = \begin{bmatrix} 0.538 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The time optimal control switchings are shown in table 3.4.

TABLE - 3.4

t_1	t_2	t_3	$T(=t_4)$
5.864	6.178	6.282	6.331

TABLE - 3.1

Iteration No.	t_1	t_2	t_3	$t_4(=T)$	F_1	F_2	F_3	F_4
1	4.398	4.682	4.787	4.826	-.007198	-.04694	-0.9846	-.1164
2	4.918	5.210	5.328	5.361	-.0008749	-.000608	-.0008376	-.002534
3	5.243	5.536	5.654	5.688	$-.2367\times10^{-3}$	$.1100\times10^{-5}$	$-.6\times10^{-5}$	$-.12\times10^{-5}$
4	5.309	5.609	5.720	5.754	$-.736\times10^{-5}$	$.3\times10^{-5}$	$.5\times10^{-6}$	$.4\times10^{-6}$

TABLE 3.2

t_1	t_2	t_3	T
5.309	5.609	5.720	5.754

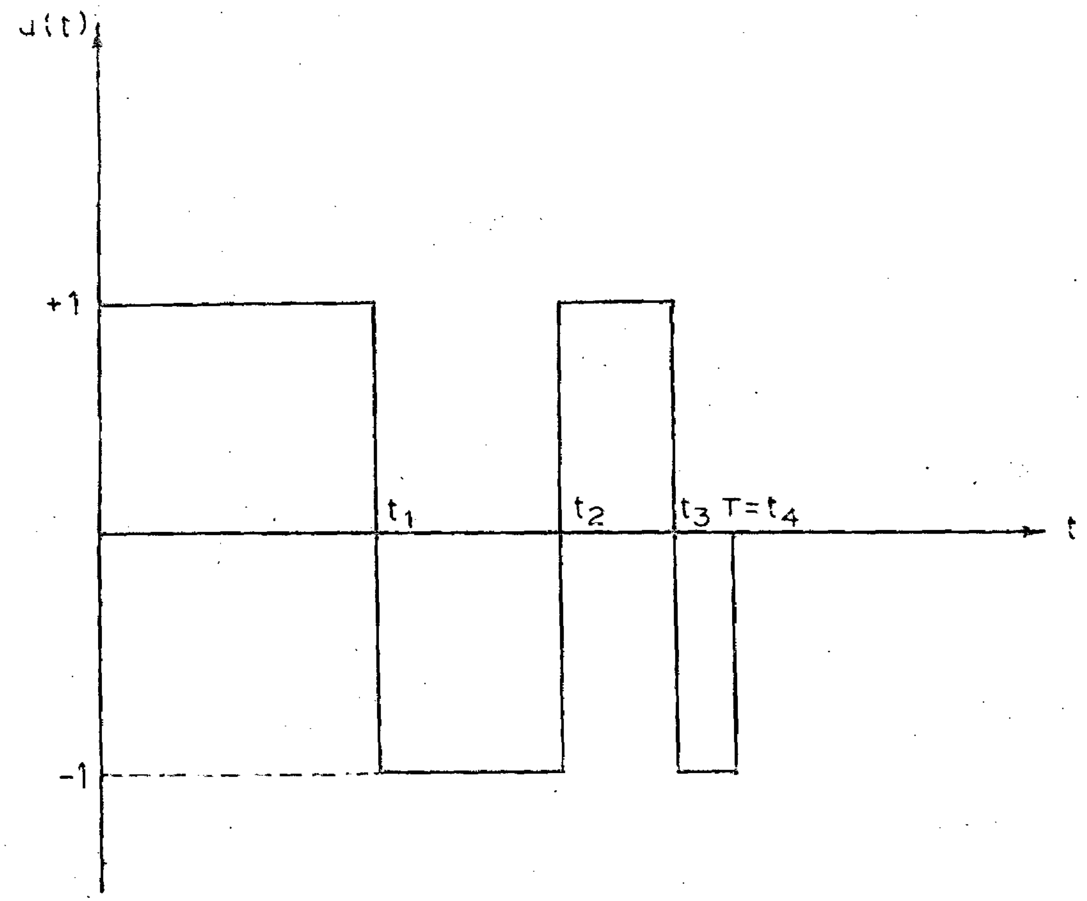


FIG. 3.2.1 BANG BANG CONTROL $u(t)$

CHAPTER - IV

CONVERSION OF LINEAR DIFFUSION EQUATIONS TO LUMPED PARAMETER SYSTEMS FOR TIME OPTIMAL CONTROL STUDIES

This chapter presents a technique to convert linear diffusion systems into lumped equations using seperation variable and Laplace transform techniques[59]. These lumped equations are utilized to study the time optimal control problems, stated in Chapter-II. Results are then extended to n-dimensions systems.

4.1. ONE DIMENSIONAL LINEAR DIFFUSION EQUATION

Consider Problem-A, stated in Chapter-II (equations 2.4 - 2.6).

Taking Laplace transform on both the sides of equations (2.4 - 2.5) with respect to t,

$$s\,\theta(x,s) - \theta(x,0) = \frac{d^2\,\theta(x,s)}{dx^2} \qquad (4.1)$$

and

$$\theta(0,s) = u(s)\ , \quad \theta(1,s) = 0 \qquad (4.2)$$

Also

$$s\theta(x,s) = \frac{d^2\theta(x,s)}{dx^2}\ ; \ \text{since}\ \ \theta(x,0) = 0 \qquad (4.3)$$

Where r_k are the poles of F(s) and A_k (x) is function of x only.

The poles of F(s) are given by

$$\sinh \sqrt{s} = 0$$

This gives

$$s = -k^2 \pi^2 ; \quad k = 0,1,2, \ldots\ldots \qquad (4.8)$$
$$= r_k$$

$A_o(x)$ is given by

$$A_o(x) = \lim_{s \to r_o} (s - r_o) \frac{\sinh \sqrt{s}\,(1-x)}{\sinh \sqrt{s}}$$

$$= \lim_{s \to 0} s \frac{e^{\sqrt{s}(1-x)} - e^{-\sqrt{s}(1-x)}}{e^{\sqrt{s}} - e^{-\sqrt{s}}}$$

$$= 0 \qquad (4.9)$$

A_k (x) is given by

$$A_k(x) = \frac{p\,(r_k)}{\left(\frac{dq}{ds}\right)_{s = r_k}} ; \quad k > 0 \qquad (4.10)$$

Where p is the numerator and q denominator of F(s)

Let $\theta(x,s)$ be denoted by θ for the sake of simplicity. Solution of (4.3) is thus given by

$$\theta = \bar{A} \sinh \sqrt{s}\, x + \bar{B} \cosh \sqrt{s}\, x \tag{4.4}$$

at $x = 1$ $\quad \bar{A} \sinh \sqrt{s} + \bar{B} \cosh \sqrt{s} = 0$

or,

$$\bar{A} = - \frac{\bar{B} \cosh \sqrt{s}}{\sinh \sqrt{s}}$$

at $x = 0$, From (4.2) and (4.4)

$$\bar{B} = u(s)$$

Thus solution of (4.3) is given as

$$\theta = \left(- \frac{\cosh \sqrt{s}}{\sinh \sqrt{s}} \sinh \sqrt{s}\, x + \sinh \sqrt{s}\, x \right) u(s)$$

or,

$$\theta = \frac{\sinh \sqrt{s}\,(1-x)}{\sinh \sqrt{s}}\, u(s) \tag{4.5}$$

Here let,

$$F(s) = \frac{\sinh \sqrt{s}\,(1-x)}{\sinh \sqrt{s}} \tag{4.6}$$

Writing F(s) in the form [5]

$$F(s) = \sum_k A_k(x) \frac{1}{s - r_k} \tag{4.7}$$

Now

$$p\,(r_k) = \pm j \sin k\pi\,(1-x) \qquad (4.11)A$$

and

$$\left(\frac{dq}{ds}\right)_{s=r_k} = \left(\frac{\cosh\sqrt{s}}{2\sqrt{s}}\right)_{s=-k^2\pi^2}$$

$$= \pm \frac{(-1)^k}{2\,jk\,\pi} \qquad (4.11B)$$

From (4.10), (4.11A) and (4.11B)

$$A_k(x) = 2\,k\pi \sin k\pi\,x \qquad (4.12)$$

Thus from (4.7), (4.9) and (4.12)

$$F(s) = 2\,\pi \sum_{k=1}^{\infty} k \frac{\sin k\pi\,x}{s + k^2\,\pi^2}$$

Thus from (4.5),

$$\theta\,(x,t) = 2\,\pi \sum_{k=1}^{\infty} k \sin k\pi x \int_0^t e^{-k^2\,\pi^2(t-\tau)}\,u(\tau)\,d\tau$$

$$(4.13)$$

assuming expansion of $\theta\,(x,t)$ in coordinate x and time t, is a convergent series in the sense of Wein berger [5]. This gives the complete solution of system (2.4 - 2.6). While obtaining the time optimal control, we see that at $t = T$, from (2.6) and (4.13)

$$2\pi \sum_{k=1}^{\infty} k \sin k\pi x \int_{0}^{T} e^{-k^2 \pi^2 (T-t)} u(t)\,dt = \frac{\sin \pi x}{\pi} \qquad (4.14)$$

Multiplying both the sides by $\sin m\pi x$; $m = 1, 2, \ldots$ and integrating within limits 0 and 1 (using the property of orthogonality) [5,10]

$$2\pi m \int_{0}^{T} e^{-m^2 \pi^2 (T-t)} u(t)\,dt \int_{0}^{1} \sin^2 m\pi x\, dx = \int_{0}^{1} \frac{\sin m\pi x \cdot \sin \pi x}{\pi}\, dx$$

$$m = 1, 2, 3, \ldots$$

or

$$\int_{0}^{T} e^{-m^2 \pi^2 (T-t)} u(t) = \frac{1}{2\pi^2 m}\,; \quad m = 1$$

$$= 0\,; \quad m = 2, 3, \ldots \qquad (4.15)$$

The one dimensional linear dimension equation for time optimal control studies, given by (2.4-2.6), is converted into a set of non-linear equation in infinite domain. These type of equations have been discussed in Chapter-III for time optimal control studies of lumped parameter systems described by state equations. Therefore, these infinite number of integeral equation (4.15) can be approximated in a finite domain and the non-linear equations governing switching instants can be obtained following the procedure adopted in Chapter III. Further, the diffusion system under consideration is a heating phenomenon ($\theta(x,o) = 0$ and $\theta^* = \sin \pi x/\pi$). Therefore, the initial sign value of the control should be +1. Hence, the non-linear equations to be solved are given by

$$-e^{-m^2 \pi^2 T} + 2e^{-m^2 \pi^2 (T-t_1)} - 2e^{-m^2 \pi^2 (T-t_2)} + \ldots\ldots + (-1)^{n-2} e^{-m^2 \pi^2 (T-t_{n-1})} + (-1)^{n-1} + m^2 \pi^2 C_{m2} = 0 \; ; \; m = 1,2,3,\ldots\ldots \qquad (4.16A)$$

Where

$$C_m 2 = \frac{1}{2 \pi^2 m} \; ; \quad m = 1$$

$$= 0 \; ; \quad m = 2,3, \ldots\ldots \qquad (4.16B)$$

4.2. TWO DIMENSION LINEAR DIFFUSION EQUATION

Consider Problem - B, stated in Chapter- II: (equations 2.7 - 2.9).

Taking Laplace transform on both the sides of(2.7) and (2.8) with respect to t and imposing zero initial condition (2.9).

Taking Laplace transform on both the sides of (2.7) and (2.8) with respect to t and imposing zero initial conditon (2.9).

$$\frac{\partial^2 \theta(x,y,s)}{\partial x^2} + \frac{\partial^2 \theta(x,y,s)}{\partial y^2} - s\theta(x,y,s) = 0 \tag{4.17}$$

and

$$\theta(0,y,s) = u(s)\ ; \quad \theta(1,y,s) = 0$$

$$\theta(x,0,s) = 0\ ; \quad \theta(x,1,s) = 0 \tag{4.18}$$

Now assuming

$$\theta(x,y,s) = X(x,s)\ Y(y) \tag{4.19}$$

(4.17) gives

$$s\,X(x,s)\ Y(y) = Y(y)\,\frac{d^2 X(x,s)}{d\,x^2} + X(x,s)\,\frac{d^2 Y(y)}{d\,y^2} \tag{4.20}$$

With boundary conditions

$$X(1,s) = 0$$

$$Y(0) = 0\ ; \quad Y(1) = 0 \tag{4.21}$$

Writing $X = X(x,s)$, $Y = Y(y)$ and dividing (4.20) by XY

$$\frac{1}{X}\ \frac{d^2 X}{dx^2} - s = -\frac{1}{Y}\ \frac{d^2 Y}{dy^2} = \lambda^2, \text{(say)} \tag{4.22}$$

This gives

$$\frac{d^2 X}{dx^2} - (s + \lambda^2) X = 0 \tag{4.23}$$

and

$$\frac{d^2 Y}{dy^2} + \lambda^2 Y = 0 \tag{4.24}$$

Let the solution of (4.24), be given by

$$Y = A' \sin \lambda y + B' \sin \lambda y \tag{4.25}$$

Applying boundary conditions at y = 0 and y = 1 (4.21), the solution is given by

$$Y_n = A'_n \sin n \pi y \;;\; \lambda^2 = n^2 \pi^2 \;;\; n=1,2,3,---- \tag{4.26}$$

Which in turn gives the solution of (4.19)as

$$\theta = \sum_{n=1}^{\infty} A'_n \sin n\pi y \, X_n (x,s) \tag{4.27}$$

at x = 0, (4.21) and (4.27) give

$$\sum_{n=1}^{\infty} A'_n \sin n \pi y \, X_n (0,s) = u(s) \tag{4.28}$$

Multiply (4.28) by sin k π y ; k = 1,2,3,.....on both sides and integerating within limits 0 and 1 (using property of orthogonality)

$$X_k(0,s) = \frac{4}{\pi k A'_k} u(s) \;;\; k \text{ is odd}$$
$$= 0 \;;\; k \text{ is even} \tag{4.29}$$

Let the solution of (4.23) by given by

$$X_n = C_n \sinh \sqrt{s+\lambda^2}\, x + D_n \cosh \sqrt{s+\lambda^2}\, x \quad (4.30)$$

at x = 0

$$D_n = X_n(0,s)$$

$$= \frac{4}{\pi n A'_n} u(s) \; ; \quad n \text{ is odd}$$

at x = 1, from (4.21),

$$C_n \sinh \sqrt{s+\lambda^2} + D_n \cosh \sqrt{s+\lambda^2} = 0$$

Or

$$C_n = - \frac{4}{\pi n A'_n} \; \frac{\cosh \sqrt{s+n^2\pi^2}}{\sinh \sqrt{s+n^2\pi^2}} u(s) \; ;$$

since $\lambda^2 = n^2 \pi^2$

Hence (4.30) can be written as

$$X_n = \frac{4}{\pi n A'_n} \; \frac{\sinh \sqrt{s+n^2\pi^2}\,(1-x)}{\sinh \sqrt{s+n^2\pi^2}} u(s) \; ;$$

$$n = 1,3,5, \ldots \quad (4.31)$$

Thus solution of (4.17) is given as

$$\theta(x,y,s) = \frac{4}{\pi} \sum_{n=1,3}^{\infty} \frac{1}{n} \sin n\pi y.$$

$$\frac{\sinh \sqrt{s+n^2\pi^2}\,(1-x)}{\sinh \sqrt{s+n^2\pi^2}} u(s) \; ;$$

(4.32)

Following the procedure adopted in 4.1 to find the laplace inverse of θ (x,s), we get

$$\theta(x,y,t) = 8 \sum_{n=1,3}^{\infty} \sum_{k=1,2}^{\infty} \frac{k}{n} \sin n\pi y \sin k\pi x$$

$$\int_0^t e^{-(n^2+ k^2 \pi^2(t-\tau)} \, u(\tau)\, d\tau \quad \ldots(4.3.3)$$

Assume that the double series expansion of θ (x,y,t) in x and y coordinates and time t, is a convergent series in the sense of Weinberger [5].

To convert this into lumped integeral equations, we see that at t = T (2.9) gives

$$8 \sum_{n=1,3}^{\infty} \sum_{k=1,2}^{\infty} \frac{k}{n} \sin n\pi y \sin k\pi x \int_0^T e^{-(n^2+k^2)\pi^2(T-t)}$$

$$u(t)\,dt = \frac{\sin \pi x \sin \pi y}{\pi^2}$$

Multiplying both the sides by sin p π x, p=1,2,... and integerating within limits 0 and 1 (using property of orthogonality)

$$\sum_{n=1,3}^{\infty} \frac{\sin n\pi y}{n} \int_0^T e^{-(n^2+p^2)\ \pi^2(T-t)} u(t)\,dt = \frac{\sin \pi y}{8\, p\pi^2} \ ; \ p=1$$

$$= 0 \ ; \ p=2,3,\ldots$$

Operating similarly with sin q π y, q = 1,3,5, ... we obtain

$$\int_0^T e^{-(p^2+q^2)\pi^2(T-t)} u(t)\, dt = \frac{q}{8p\pi^2} \; ; \; p, q = 1 \qquad \ldots (4.34)$$

$$= 0 \; ; \; p = 2,3,4,\ldots\ldots \; ; \; q=3,5,7\ldots$$

or $$\int_0^T e^{-m^2\pi^2(T-t)} u(t)\, dt = C_{m^2}; \quad C_{m^2} = \frac{q}{8p\pi^2} \; ; \; m^2=2$$

$$= 0 \; ; \; \text{otherwise} \ldots (4.35)$$

where $m^2 = (p^2 + q^2)$ ...(4.36)

The infinite number of integeral equations (4.35-4.36) represent the equivalent lumped equations of two dimensional diffusion systems stated in chapter-III by equations (3.14-3.15) for time optional control studies. These equations resemble those of one dimensional diffusion equations (4.15). Only The constants are these equations (4.15) and (4.35-4.36) are ~~only~~ different. Further, the time optional control involves heating phenomenon in both the situations, therefore the non-linear equations to be solved for determining the switching instants in the case of two dimension diffusion systems can be referred to equation (4.16A), m^2 being given by (4.36)

4.3 THREE DIMENSIONAL LINEAR DIFFUSION EQUATION

Consider Problem-C, stated in chapter-II, (equations 2.10-2.12).

Taking Laplace transforms on both the sides of (2.10) and (2.11), with respect to t and imposing zero initial condition (2.12)

$$\frac{\partial^2 \theta}{\partial x^2}(x,y,z,s) + \frac{\partial^2 \theta(x,y,z,s)}{\partial y^2} + \frac{\partial^2 \theta(x,y,z,s)}{\partial z^2} - s\,\theta(x,y,z,s)=0 \qquad \ldots (4.37)$$

and

$$\theta(0,y,z,s) = u(s)\ ;\ \theta(1,y,z,s) = 0$$
$$\theta(x,0,z,s) = 0\ ;\quad \theta(x,1,z,s) = 0 \qquad \ldots (4.38)$$
$$\theta(x,y,0,s) = 0\ ;\quad \theta(x,y,1,s) = 0$$

Now assuming

$$\theta(x,y,z,s) = Z(z)\ \theta^1(x,y,s) \qquad \ldots (4.39)$$

(4.37) gives

$$Z(z)\left(\frac{\partial^2\theta^1 x,y,s)}{\partial x^2} + \frac{\partial^2\theta^1(x,y,s)}{\partial y^2}\right) + \theta^1(x,y,s)\frac{d^2Z(z)}{dz^2}$$
$$-sZ(z)\theta^1(x,y,s) = 0 \qquad \ldots (4.40)$$

with boundary conditions

$$Z(0) = 0\ ;\ Z(1)=0 \qquad \ldots (4.41)$$
$$\theta^1(1,y,s)=0\ ;\ \theta^1(x,0,s) = 0\ ;\ \theta^1(x,1,s) = 0$$

Writing $Z = Z(z)$; $\theta^1 = \theta^1(x,y,s)$ and dividing (4.44) by $Z\theta^1$

$$\frac{1}{\theta^1}\left(\frac{\partial^2\theta^1}{\partial x^2} + \frac{\partial^2\theta^1}{\partial y^2}\right) - s = -\frac{1}{Z}\ \frac{d^2Z}{dz^2} = K',\ (\text{say})$$
$$\ldots (4.42)$$

This gives

$$\frac{\partial^2\theta^1}{\partial x^2} + \frac{\partial^2\theta^1}{\partial y^2} - (s + K')\ \theta^1 = 0 \qquad \ldots (4.43)$$

and

$$\frac{d^2Z}{dz^2} + K'Z = 0 \qquad \ldots (4.44)$$

Similar to (4.24), applying boundary conditions (4.41), the solution of (4.44) is given as

$$Z_n = G_n \sin n\pi z^*,\ K' = n^2\pi^2,\ n = 1,2,3,\ \ldots\ldots \qquad (4.45)$$

which in turn gives the solution of (4.37) as

$$\theta = \sum_{n=1}^{\infty} G_n \sin n\pi z \, \theta_n^{1*} \qquad \ldots \quad (4.46)$$

Assuming

$$\theta_n^1 = Y(y)\, X(x,s) \qquad \ldots \quad (4.47)$$

(4.43) gives

$$Y_{(y)} \frac{d^2X(x,s)}{dx^2} + X(x,s) \frac{d^2Y(y)}{dy^2} - (s+\kappa_n')=0 \qquad \ldots \quad (4.48)$$

with boundary conditions

$$Y(0) = 0; \quad Y(1) = 0$$

$$X(1,s) = 0 \qquad \ldots \quad (4.49)$$

Writing $Y(y) = Y$, $X(x,s) = X$ and dividing by XY

$$\frac{1}{X} \frac{d^2X}{dx^2} - (s+\kappa_n') = \frac{1}{Y} \frac{d^2Y}{dy^2} = \kappa'' \text{ (say)} \qquad \ldots \quad (4.50)$$

This gives

$$\frac{d^2X}{dx^2} - (s + \kappa_n' + \kappa'') \; X = 0 \qquad \ldots \quad (4.51)$$

and

$$\frac{d^2Y}{dy^2} + \kappa'' \, Y = 0 \qquad \ldots \quad (4.52)$$

Applying boundary conditions (4.49), the solutions of (4.52) is given by

$$Y_d = H_d \sin d\pi y \; ; \; \kappa'' = d^2\pi^2, d = 1,2,\ldots \quad (4.53)$$

which in turn gives the solution of (4.47) as

$$\theta_n^1 = \sum_{d=1}^{\infty} H_\alpha \sin d\pi y \, X_d \quad \ldots (4.54)$$

Following the procedure adopted in solution of (4.23), solution of (4.51) is obtained as:

$$x_d = x_d(0,s) \; \frac{\sinh \sqrt{s+(n^2+d^2)\pi^2}\,(1-x)}{\sinh \sqrt{s+(n^2+d^2)\pi^2}}$$

which

$$\theta(x,y,z,s) = \sum_{n=1}^{\infty} \sum_{d=1}^{\infty} G_n H_d \sin n\pi z \sin d\pi y \, x_d(x,s) \quad \ldots (4.55)$$

at x=0, (4.38) gives

$$\sum_{n=1}^{\infty} \sum_{d=1}^{\infty} G_n H_d \sin n\pi z \sin d\pi y \, x_d(0,s) = u(s)$$

using the property of orthogonality in the case of double series, we get

$$X_j(0,s) = \frac{16}{ij\, G_i H_j \pi^2} u(s) \; ; \; i \text{ and } j \text{ are odd} \quad \ldots (4.56)$$

$$= 0 \; ; \; i \text{ or } j \text{ is even}$$

Thus (4.55) can be wtitten as

$$\theta\,(x, y, z, s) = \frac{16}{\pi^2} \sum_{n=1,3}^{\infty} \sum_{d=1,3}^{\infty} \frac{1}{nd} \sin n\pi z\, \gamma$$

$$\sin d\pi y\, \frac{\sin h\sqrt{s+(n^2+d^2)\pi^2}\,(1-x)}{\sin h\sqrt{s+(n^2+td^2)\pi^2}}\, u(s) \qquad \ldots(4.57)$$

Following the proc edure, a dopted in 4.1 and 4.2, to find Laplace inverse of $\theta\,(x,s)$ and $\theta\,(x,y,s)$

$$\theta\,(x, y, z, t) = \frac{32}{\pi} \sum_{n=1,3}^{\infty} \sum_{d=1,3}^{\infty} \sum_{k=1,2}^{\infty} \frac{k}{nd} \sin n\pi z$$

$$\sin d\,\pi\, y \sin k\,\pi\, x \int_0^t$$

$$e^{-(x^2+k^2+d^2)\,\pi^2(t-\tau)}\, u(\tau)\, d\tau \qquad \ldots (4.58)$$

Assume that the triple series expansion of $\theta(x,y,z,t)$ in x, y, z coordinates and time t, is a convergent series in the sense of Weingberger [5] . To convert this into lumped parameter system, at t = T (2.12) gives

$$\frac{32}{\pi} \sum_{n=1,3}^{\infty} \sum_{d=1,3}^{\infty} \sum_{k=1,2}^{\infty} \frac{k}{nd} \sin n\pi\, z \sin d\pi y \sin k\pi x$$

$$\int_0^T e^{-(n^2+k^2+d^2)\,\pi^2\,(T-t)}\, u(t)\, dt$$

$$= \frac{\sin\pi x \sin\pi y \sin\pi z}{\pi^3}$$

Using the property of orthogonality for a triple series, we obtain

$$\int_0^T e^{-(p^2+q^2+r^2)\pi^2(T-t)}\, u(t)\, dt$$

$$= \frac{qr}{32\pi^2 p} \; ; \quad p, q, r = 1$$

$$= 0 \; ; \quad p = 2,3,4, \ldots\ldots\ldots ;$$

$$q = 3,5,7, \ldots\ldots ; \quad r = 3,5,7,$$

(4.59)

or

$$\int_0^T e^{-m^2\pi^2(T-t)}\, u(t)\, dt$$

$$= C_m 2 \; ; \quad C_m 2 = \frac{qr}{32\pi^2 p} \; ;$$

$$m^2 = 3 \qquad (4.60)$$

$$= 0, \text{ otherwise}$$

Where

$$m^2 = (p^2 + q^2 + r^2)$$

The infinite number of integeral equations (4.60-4-61) represent the equivalent lumped equations of three dimensional diffusion system stated in Chapter III by equations (3.14-3.15) for time optimal control studies. Similar sort of representation was also seen in (4.15), (4.35-4.36) for one dimensions and two dimension systems respectively. The difference being in the eigen values and constants involved in them. As here also the time optimal control problem involves heating from ~~oriential~~ an initial state $\theta(x,y,z,0) = 0$ to a steady state θ^*, hence the initial value of the control will be +1. These integeral equations (4.60 - 4.61) on simplification give rise to a set of non-linear equations with the switching instants of the control u(t) as their unknowns.

4.4. n-DIMENSIONAL LINEAR DIFFUSION EQUATION

It is hard to attach any physical significance to n-dimensional linear diffusion equations. But only for the sake of academic interest the method, adopted for time optimal control studies of one, two and these dimension linear diffusion equations, is extended to n-dimensional spatial coordinates.

On the lines of problems-A, B and C in Chapter II, n-dimensional linear diffusion system (in normalized form) is taken to be described by

$$\frac{\partial \theta}{\partial t}(x_1, x_2, \ldots\ldots, x_n, t) = \sum_{i=1}^{n} \frac{\partial^2 \theta(x_1, x_2, \ldots\ldots, x_n, t)}{\partial x_i^2} \quad ;$$

$$x_i \in (0,1) \; ; \quad i = 1,2, \ldots\ldots n \qquad (4.62)$$

With boundary conditions

$$\theta\,(0,\, x_2\,,\, \ldots\ldots,\, x_n,\, t) = u(t)\;;$$

$$\theta(1, x_2,\, \ldots\ldots,\, x_n,\, t) = 0$$

$$\theta(x_1, 0,\, \ldots\ldots x_n, t) = 0\;;\; \theta\,(x_1,\, 1,\, \ldots,\, x_n, t) = 0$$

$$\vdots \qquad\qquad (4.63)$$

$$\theta\,(x_1,\, x_2,\, \ldots\ldots,\, x_{n-1},\, 0, t) = 0\;;$$

$$\theta\,(x_1,\, x_2,\, \ldots\ldots,\, x_{n-1},\, 1, t) = 0$$

and initial and final conditions

$$\theta\,(x_1,\, \ldots\ldots,\, x_n,\, 0) = 0\;;$$

$$\theta\,(x_1,\, \ldots\ldots,\, x_n,\, T) = \theta^* = \prod_{i=1}^{n} \frac{\sin \pi x_i}{\pi^n}\,, \text{(say)}$$

$$(4.64)$$

The constraint on the magnitude of so called control function, is taken to be same as (2.13) for the time optimal control problem described in 2.4.

Taking Laplace transform on both the sides of equations (4.62) and (4.63) with respect to time t and applying zero initial condition (4.64).

$$\sum_{i=1}^{n} \frac{\partial^2 \theta (x_1, x_2, \ldots\ldots\ldots, x_n, s)}{\partial x_i^2}$$

$$- s\theta(x_1, x_2, \ldots\ldots, x_n, s) = 0 \qquad (4.65)$$

and

$$\theta (0, x_2, \ldots\ldots x_n, s) = u(s) \; ;$$

$$\theta (1, x_2, \ldots\ldots, x_n, s) = 0$$

$$\theta (x_1, 0, \ldots\ldots, x_n, s) = 0 \; ;$$

$$\theta (x_1, 1 \ldots\ldots, x_n, s) = 0 \qquad (4.66)$$

$$\vdots$$

$$\theta (x_1, x_2, \ldots\ldots, 0, s) = 0 \; ;$$

$$\theta(x_1, x_2, \ldots\ldots\ldots, 1, s) = 0$$

Now assuming

$$\theta (x_1, x_2, \ldots\ldots, x_n, s) = X^n(x_n)\, \theta^{n-1} (x_1, x_2, \ldots\ldots x_{n-1}, s) \qquad (4.67)$$

(4.65) gives

$$X^n(x_n) \sum_{i=1}^{n-1} \frac{\partial^2 \theta^{n-1}(x_1, x_2, \ldots\ldots, x_{n-1}, s)}{\partial x_{n-1}^2}$$

$$+ \theta^{n-1} (x_1, \ldots\ldots, x_{n-1}, s) \frac{d^2 X^n(x_n)}{d x_n^2}$$

$$- s X^n(x_n)\, \theta^{n-1}(x_1, \ldots\ldots, x_{n-1}, s) = 0$$

$$(4.68)$$

With boundary conditions

$$X^n(0) = 0 \; ; \quad X^n(1) = 0$$

$$\theta^{n-1}(1, x_2, \ldots, x_{n-1}, s) = 0$$

$$\theta^{n-1}(x_1, 0, \ldots, x_{n-1}, s) = 0 \; ;$$

$$\theta^{n-1}(x_1, x_2, \ldots, x_{n-1}, s) = 0 \quad (4.69)$$

$$\vdots$$

$$\theta^{n-1}(x_1, x_2, \ldots, 0, s) = 0 \; ;$$

$$\theta^{n-1}(x_1, x_2, \ldots, 1, s) = 0$$

Writing $X^n = X^n(x_n)$; θ^{n-1} $\theta^{n-1}(x_1, x_2, \ldots x_{n-1}, s)$

and dividing (4.68) by $\theta^{n-1} X^n$

$$\frac{1}{\theta^{n-1}} \sum_{i=1}^{n-1} \frac{\partial^2 \theta^{n-1}}{\partial x_i^2} - s$$

$$= - \frac{1}{X^n} \frac{d^2 X^n}{d x_n^2} = A^n \; ; \quad \text{(say)}$$

This gives $$\sum_{i=1}^{n-1} \frac{\partial^2 \theta^{n-1}}{\partial x_i^2} - (s + A^n)\, \theta^{n-1} = 0 \qquad (4.71)$$

and $$\frac{d^2 X^n}{d x_n^2} + A^n X^n = 0 \qquad (4.72)$$

Applying boundary conditions (4,69), solution of (4.72) is obtained as

$$X^n_{m_n} = B^n_{m_n} \sin m_n \pi x_n ;$$

$$A^n_{m_n} = m_n^2 \pi^2 ; \quad m_n = 1, 2, \ldots\ldots\ldots \tag{4.73}$$

which in~~tension~~ turn gives the solution of (4.65) as

$$\theta\, (x_1, x_2, \ldots\ldots, x_n, s)$$

$$= \sum_{m_n=1}^{\infty} B^n_{m_n} \sin m_n \pi\, x_n\, \theta^{n-1}_{m_n} \tag{4.74}$$

Assuming

$$\theta^{n-1}_{m_n} = X^{n-1}(x_{n-1})\, \theta^{n-2}(x_1, x_2, \ldots\ldots, x_{n-2}, s) \tag{4.75}$$

(4.71) gives

$$X^{n-1} \sum_{i=1}^{n-2} \frac{\partial^2 \theta^{n-2}}{\partial x_i^2} + \theta^{n-2} \frac{d^2 X^{n-1}}{d x_{n-1}^2}$$

$$- (s + A^n_{m_n})\, X^{n-1} \theta^{n-2} = 0 \tag{4.76}$$

With boundary conditions

$$X^{n-1}(0) = 0 ; \quad X^{n-1}(1) = 0$$

$$\theta^{n-2}(1, x_2, \ldots\ldots, x_{n-2}, s) = 0$$

$$\theta^{n-2}(x_1, 0, \ldots\ldots x_{n-2}, s) = 0 \quad ;$$

$$\theta^{n-2}(x_1, 1, \ldots\ldots, x_{n-2}, s) = 0 \qquad (4.72)$$

$$\vdots$$

~~$\theta^{n-1}(x_1, 0, \ldots\ldots, x_{n-2}, s) = 0$;~~

$$\theta^{n-2}(x_1, 1, \ldots\ldots, 0, s) = 0 \quad ;$$

$$\theta^{n-2}(x_1, x_2, \ldots\ldots, 1, s) = 0$$

Where $X^{n-1} = X^{n-1}(x_n)$ and $\theta^{n-2} = \theta^{n-2}(x_1, \ldots\ldots, x_{n-2}, s)$

Dividing (4.70) by $X^{n-1}\,\theta^{n-2}$

$$\frac{1}{\theta^{n-2}} \sum_{i=1}^{n-2} \frac{\partial^2 \theta^{n-2}}{\partial x_i^2} - (s + A^n_{m_n})$$

$$= \frac{1}{X^{n-1}} \frac{d^2 X^{n-1}}{d x_{n-2}^2} = A^{n-2}, \text{ say} \quad (4.78)$$

which gives

$$\sum_{i=1}^{n-2} \frac{\partial^2 \theta^{n-2}}{\partial x_i^2} - (s + m_n^2 \pi^2 + A^{n-1})\,\theta^{n-2} = 0$$

(4.79)

and

$$\frac{d^2 X^{n-1}}{d x_{n-1}^2} + A^{n-1} X^{n-1} = 0 \qquad (4.80)$$

Applying boundary conditions (4.77), the solution of (4.80) is obtained as

Which in turn gives solution of (4.65) as

$$\theta = \sum_{m_n=1}^{\infty} \sum_{m_{n-1}=1}^{\infty} B^n_{m_n} \quad B^{n-1}_{m_{n-1}}$$

$$\sin m_n \pi \; x_n \; \sin m_{n-1} \; \theta^{n-2}_{m_{n-1}} \; (x_1, \ldots\ldots, x_{n-2}, s) \qquad (4.82)$$

Similarly successively applying the seperable variable techniques as in (4.67) and (4.75), we reach

$$\theta = \sum_{m_n=1}^{\infty} \quad \sum_{m_2=1}^{\infty} \quad \ldots\ldots\ldots$$

$$\sum_{m_2=1}^{\infty} \quad B^n_{m_n} \quad B^{n-1}_{m_{n-1}} \quad \ldots\ldots\ldots\ldots\ldots$$

$$B_{m_2} \quad \sin m_n \; x_n \; \sin m_{n-1} \; x_{n-1} \text{-------}$$

$$\sin m_2 \; \pi \; x_2 \quad \theta^1_{m_2} \; (x_1, s) \quad \ldots \qquad (4.83)$$

and

$$\frac{d^2\theta^1 \; (x_1, s)}{d \; x_1^2} - (s + (m_n^2 + m_{n-1}^2 + \; \ldots\ldots + m_2^2)\pi^2) = 0 \qquad (4.84)$$

On the lines of procedure of solution of (4.23), solution of (4.84) is obtained as

$$\theta^{1}_{m_2}(x_1, s) = \theta^{1}_{m_2}(0,s) \frac{\sinh\sqrt{s + M\pi^2}(1-x)}{\sinh\sqrt{s + M\pi^2}} ;$$

$$M = (m_n^2 + m_{n-1}^2 \cdots\cdots m_2^2) \tag{4.85}$$

Thus

$$\theta = \sum_{m_n=1}^{\infty} \sum_{m_{n-1}=1}^{\infty} \cdots\cdots\cdots$$

$$\sum_{m_2=1}^{\infty} B^{n}_{m_n} B^{n-1}_{m_{n-1}} \cdots\cdots\cdots\cdots\cdots$$

$$B^{2}_{m_2} \sin m_n\pi x_n \sin m_{n-1}\pi x_{n-1} \cdots\cdots \sin m_2\pi x_2$$

$$\theta^{1}_{m_2}(x_1, s) \tag{4.86}$$

at

$x_1 = 0$, (4.60) gives

$$\sum_{m_n=1}^{\infty} \sum_{m_{n-1}=1}^{\infty} \cdots\cdots\cdots \sum_{m_2=1}^{\infty} B^{n}_{m_n} B^{n-1}_{m_{n-1}} \cdots\cdots\cdots$$

$$B^{2}_{m_2} \sin m_n \pi x_n \sin m_{n-1} \pi x_{n-1} \cdots\cdots\cdots$$

$$\sin m_2 x_2 \, \theta^{2}_{m_2}(0,s)$$

$$= u(s)$$

Using property of orthoganality in the case of n-order series, we get

$$\theta^{1}_{m_2}(0,s) = 2^{n-1}\,\frac{1-(-1)^{i_n}}{i_n\;B^{n}_{i_n}}\;\frac{1-(-1)^{i_{n-1}}}{i_{n-1}\;B^{n-1}_{i_{n-1}}}\cdots\cdots\;\frac{1-(-1)^{i_2}}{i_2\;B^{2}_{i_2}}\cdot\frac{1}{\pi^{n-1}}\;u(s)$$

$$= \frac{2^{2n-2}}{\pi^{n-1}\prod_{j=2}^{n} i_j\;B^{j}_{i_j}}\;u(s)\;;\quad \text{all } i_j \text{ are odd}$$

$$= 0 \quad ; \quad \text{oterhwise} \tag{4.87}$$

Thus(4.86) can be writeen as

$$\theta(x_1, x_2, \ldots\ldots\ldots, x_n, s) =$$

$$= \frac{\partial^{2n-2}}{\pi^{n-1}} \sum_{m_n=1,3}^{\infty}\;\sum_{m_{n-1}=1,2}^{\infty}\cdots\cdots\cdots\cdots \sum_{m_2=1,3}^{\infty} \frac{1}{m_n m_{n-1}\cdots m_2}\;\sin m_n\pi x_n\;\sin m_{n-1}\pi x_{n-1}\cdots\cdots\cdots$$

$$\sin m_2\pi x_2\;\frac{\sinh\sqrt{s+M\pi^2}\,(1-x_1)}{\sinh\sqrt{s+M\pi^2}}\;u(s) \tag{4.88}$$

Following the procedure adopted in 4.1 and 4.2 to find Lplace increase of θ (x, s) and θ(x,y,s).

$$\theta(x_1, x_2, \ldots\ldots, x_n\ t) = \frac{2^{2n-1}}{\pi^{n-2}} \sum_{m_n=1,3}^{\infty} \sum_{m_{n-1}=1,3}^{\infty} \cdots\cdots$$

$$\sum_{m_2=1,3}^{\infty} \sum_{m_1=1,2}^{\infty} \frac{m_1}{m_n\, m_{n-1} \ldots m_2} \sin m_n \pi x_n \ \sin m_{n-1} x_{n-1} \ldots$$

$$\sin m_2 \pi x_2 \ \sin m_1 x_1 \int_0^t e^{-M\pi^2(t-\tau)} u(\tau)\, d\tau$$

(4.89)

Assume that the n-order series expansion of θ in $x_1, x_2, \ldots$ x_n coordinates and time t, is a convergent series in the sense of Weinberger (5).

To convert this into lumped parameter system, at t = T (4.64) gives

$$\frac{2^{n-1}}{\pi^{n-2}} \sum_{m_n=1,3}^{\infty} \cdots\cdots\cdots\cdots \sum_{m_2=1,3}^{\infty} \sum_{m_1=1,2}^{\infty} \frac{m_1}{m_n \ldots m_2}$$

$$\sin m_n \pi x_n \ldots\ldots \sin m_1 \pi x_1 \int_0^T e^{-M\pi^2(T-t)} u(t)\, dt$$

$$= \prod_{i=1}^{n} \sin \pi x_i$$

Following the methods used in 4.1,4.2 and 4.3 (using the property of or thegonality), we get

$$\int_0^T e^{-(p_1^2 + p_2^2 + \cdots\cdots + p_n^2)\pi^2(T-t)} u(t)\, dt$$

$$= \frac{p_2 p_3 \cdots p_n}{2^{2n-1}\pi^2 p_1} \quad ; \quad p_j = 1 \ ; \ j = 1,2,3,,,,,$$

$$= 0 \quad ; \quad p_1 = \not{1},2,3,4,\ldots\ldots ;$$

$$p_j = 3,5,7, \ldots, \not{1}_j = 2,3,4,\ldots \qquad (4.90)$$

or $$\int_0^T e^{-m^2\pi^2(T-t)} u(t)\, dt = C_{m^2} = \frac{p_2 p_3 \cdots p_n}{2^{n-1}\pi^2 p_1} \ ;$$

$$m^2 = n \qquad (4.91)$$

$$= 0 \ ; \text{ otherwise}$$

where $m^2 = (p_1^2 + p_2^2 + \cdots + p_n^2)$ (4.92)

Thus the integeral equations of the form (4.15), (4.35-4.36) and (4.60-4.61) are reached.

CHAPTER - V

A TRANSFORMATION TECHNIQUE FOR SOLVING NON-LINEAR EQUATIONS

In this chapter, ther non-linear equations obtained from sections 4.1 , 4.2 and 4.3 are transformed from time domain to S-domain with the help of some transformations [13] These transfromed equations are then solved by Newton-Ralphson method. The geometry of the transformations indicate that the variables , thus involved in new equations, lie on the real line within a region 0 and 1 (excluding zero and one). The switching instants for one dimensional, two dimensional and three dimensional systems are then solved by this method. The performance of the systems are compared with the help of graphs given at the end of this chapter.

5.1 TRANSFORMATIONS

The non linear equations to be solved for obtaining the switching timings in one, two and three dimensional linear diffusion systems are of the type

$$e^{-m^2\pi^2 T} + 2e^{-m^2\pi^2(T-t_1)} - 2e^{-m^2\pi^2(T-t_2)} + \ldots\ldots$$

$$+ (-1)^{n-2}\, 2e^{-m^2\pi^2(T-t_{n-1})} + (-1)^{n-1} - m^2\pi^2 C_{m2} = 0 \qquad (5.1)$$

The value of m^2 being given by (4.16A), (4.35) and (4.60) and value of C_{m^2} by (4.16B) , (4.36), (4.61) for one dimensional two dimensional and three dimensional systems, respectively.

Let,

$$e^{-\pi^2(T-t)} = S \ ; \quad S \in (0,1) \tag{5.2}$$

Therefore,

$$e^{-\pi^2 T} = S_o \quad \text{and} \quad e^{-\pi^2(T-t_j)} = S_j ; \ j=1,2,..n-1 \tag{5.3}$$

Let

$$\mu = m^2, \tag{5.4}$$

Thus the equation (5.1) is transformed to

$$F_{\mu}(S_{\pi}, S_o) = -S_o^{\mu} + 2S_1^{\mu} - 2S_2^{\mu} + \ldots\ldots + (-1)^{n-1} S_{n-1}^{\mu}$$

$$+ (-1)^{n-1} - M\,\pi^2 (C_{\mu}) = 0 \tag{5.5}$$

Observing the behaviour of S with respect to time in fig. 5.1.1

$$S_o < S_1 < S_2 < \ldots\ldots < S_{n-1} \ ; \quad S_{\kappa} \in (0,1) ; \quad \kappa = 0,1,2,\ldots,(n-1) \tag{5.6}$$

From (5.3),

$$T = -\frac{1}{\pi^2} \log S_o ; \quad t_j = T + \frac{1}{\pi^2} \log S_j , \ j=1,2,\ldots,(n-1) \tag{5.7}$$

Solution of equations (5.5) are obtained by Newton-Raphson method described in Chapter III and switching instants are obtained from (5.7)

5.2 COMPUTATIONAL RESULTS FOR ONE DIMENSIONAL LINEAR DIFFUSION SYSTEM

The time optimal control switching instants can be obtained from equations (4.15), following the method adopted in 5.1. The equations (4.15) are converted to non-linear equations

$$F_n(a_0, a_r) = - a_0^{n^2} + 2a_1^{n^2} - 2\, a_2^{n^2} + \dots\dots + (-1)^{r-2}\, 2\, a_{r-1}^{n^2} + (-1)^{r-1}$$

$$- n^2 \pi^2 C_{n^2} = 0 \; ; \quad n = 1,2,3 \qquad (5.8)$$

with $C_{n^2} = 0.5$; n = 1

$= 0$; n = 2,3,4

The solution of these equations for r = 2,3,4 have been obtained and summarised in Table 5.1. The computer programme for this, based upon N-R method is given in Appendix 5.1.1. The complete iteration for r = 4 has been shown in Appendix 5.2.2. All these equations have been solved with accuracy of 10^{-7}.

5.2.1 COMPARISON OF PERFORMANCE

Solution for $\theta(x,t)$ is obtained in equation (4.13) and substituting the Bang-Bang control, having (r-1) switchings, the steady state distribution is given by

$$\theta^{*}(x) = 2\pi \sum_{k=1}^{\infty} K \sin K\pi x \Bigg[\frac{1}{K^2\pi^2} \Big(-e^{-K^2\pi^2 T} + 2e^{-K^2\pi^2(T-t_1)}$$

$$- 2e^{-K^2\pi^2(T-t_2)} + \ldots + (-1)^{n-2} 2e^{-K^2\pi^2(T-t_{n-1})}$$

$$+ (-1)^{n-1} \Big) \Bigg] \qquad (5.9)$$

Taking first five harmonic terms in (5.9) , $\theta^{*}(x)$ has been computed on IBM 1620 (See appendix 5.2.3 for computer program) for various final time T (see Table 5.1). These are shown in Fig. 5.2.1. It is seen from these graphs that as the number of integral equations (4.16A) increase, the steady state distribution under bang-bang control tends to the given distribution of function $\theta^{*}(x) = \frac{\sin \pi x}{\pi}$.

5.3 COMPUTATIONAL RESULTS FOR TWO DIMENSIONAL LINEAR DIFFUSION SYSTEM

The time optimal control switching instants can be obtained from equations (4.34), following the method adopted in 5.1. The equations (4.34) can be converted to non-linear equations:

$$F_m(s_0, s_\pi) = -s_0^{m^2} + 2s_1^{m^2} - 2s_2^{m^2} + \ldots + (-1)^{n-2} s_{n-1}^{m^2} + (-1)^{n-1} -$$

$$m^2 \pi^2 C_{m^2} = 0 \;;\; m^2 = p^2 + q^2 \;;\; p = 1,2,3,\ldots,\; q = 1,3,5.. \qquad (5.10)$$

with $C_{m^2} = 0.125$, $m^2 = 2$,

$= 0$; otherwise

The solution of these equations for $m^2 = 2, 5, 10, 13$ ($p = 1,2,3,;\ q = 1$) have been obtained and summarised in Table 5.2. The computer program for this, based upon N-R method is given in appendix 5.3.1. The complete iteration for $m^2 = 13$, has been shown in appendix 5.3.2. All the equations have been solved with an accuracy of 10^{-7}.

5.3.1 COMPARISON OF PERFORMANCE

Solution for $\theta(x,y,t)$ is obtained in equation(4.32) and substituting for the bang=bang control with (n-1) switchings, the steady state distribution $\theta^*()x,y)$ is given by

$$\theta^*(x,y) = 8 \sum_{n=1,3}^{\infty} \sum_{k=1,2}^{\infty} \frac{k}{n} \sin m\pi y \sin k\pi x \left[\frac{1}{m^2\pi^2} \left(e^{-m^2\pi^2 T} + 2e^{-m^2\pi^2(T-t_1)} - 2e^{-m^2\pi^2(T-t_2)} + \dots + (-1)^{n-1} 2e^{-m^2\pi^2(T-t_{n-}} + (-1)^{n-1} \right) \right] \tag{5.11}$$

Taking first five harmonics, $\theta^*(x)$ for y=0,2,0.4, 0.6,0.8 and $\theta^*(y)$ for x = 0.2, 0.4, 0.6, 0.8 have been computed on IBM 360 (see appendix 5.3.3 for computer program) for various final time T (see table 5.2). These are shown in fig. (5.3.1 - 5.3.8). It is seen from the graphs that, as the number of integral equations (4.35) increase, the steady-state distribution under bang-bang control, tends to the given distribution function (see 2.9).

5.4 COMPUTATIONAL RESULTS FOR THREE DIMENSIONAL LINEAR DIFFUSION SYSTEM

The time optimal control switching instants for three dimensional diffusion system can be obtained from equations (4.60), following the method adopted in 5.1. These equations are converted to non-linear algebraic equations

$$F_m(S_o, S_K) = - S_o^{m^2} + 2S_1^{m^2} - 2S_2^{m^2} + \ldots + (-1)^{n-2} S_{n-1}^{m^2} + m^2\pi^2 C_{m^2} = 0;$$

$$m^2 = p^2 + q^2 + r^2;$$

$$p = 1, 2, 3, \ldots$$

$$q = 1, 3, 5, \ldots$$

$$r = 1, 3, 5 \ldots \qquad (5.13)$$

with

$$C_{m^2} = 0.03125 \;; \quad m^2 = 3$$

$$= 0 \quad ; \quad \text{otherwise.}$$

The solution of these equations for $m^2 = 3, 6, 11, 14, 19$ ($p = 1, 2, 3$; $q = 1, 3$; $r = 1, 3$) have been obtained and summarised in Table 5.3. The computer program for this, based on N-R method, is given in appendix 5.4.1. The complete iteration for $m^2 = 19$ has been shown in appendix 5.4.2. All the equations have been solved with an accuracy of 10^{-7}.

5.4.1 COMPARISON OF PERFORMANCE

Solution for $\theta(x,y,z,t)$ is obtained in equation (4.58) and substituting the bang-bang control with (n-1) swtichings, the steady state distribution $\theta^*(x,y,z)$ is given by

$$\theta^{*}(x,y,z) = \frac{32}{\pi} \sum_{n=1,3}^{\infty} \sum_{d=1,3}^{\infty} \sum_{k=1,2}^{\infty} \frac{k}{nd} \sin k\pi x \sin d\pi y$$

$$\sin n\pi z \left[\frac{1}{m^2\pi^2} \left(-e^{-m^2\pi^2 T} + 2e^{-m^2\pi^2(T-t_1)} - 2e^{-m\pi^2(T-t_2)} + \ldots + (-1)^{n-2} 2e^{-m^2\pi^2(T-t_{n-1})} + (-1)^{n-1}\right]\right. \qquad (5.14)$$

Taking first five harmonics, $\theta^{*}(x)$ for y,z =0.2, 0.4, 0.6,0.8 $\theta^{*}(y)$ for x,z = 0.2, 0.4, 0.6, 0.8; and $\theta^{*}(z)$ for x,y=0.2,0.4, 0.6, 0.8 have been computed on IBM 360 (see appendix 5.4.3) for computer pnogram) for various final time T (See Table 5.3). These are shown in Fig. (5.4.1 - 5.4.8). It is seen from graphs that as the number of integral equation (4.58) increase, the steady state distribution under bang-bang control, tends to the given distribution (see 2.12).

```
C  RAJIV GOYAL TIME OPTIMAL CONTROL IN DISTRIBUTED PARAMETER SYSTEMS 1D
   DIMENSION S(20),PHI(20),PHID(20,20),C(20),A(20,21),X(20),
  1H(20),B(20),T(20)
   READ 1,N
 1 FORMAT(I2)
   READ 2,(S(I),I=1,N)
 2 FORMAT(4E20.8)
   READ 2,ACC
   N1=N-1
   N2=N+1
   K=0
   C(1)=(-1.)**N1+0.5
   DO 3 I=2,N
 3 C(I)=(-1.)**N1
 8 DO 9 I=1,N
   Y=I**2
   IY=Y
   II=IY-1
   PHI(I)=-1.*S(1)**IY+C(I)
   PHID(I,1)=-1.*Y*S(1)**II
   DO 9 J=2,N
   PHI(I)=PHI(I)+((-1.)**J)*2.*S(J)**IY
 9 PHID(I,J)=2.*((-1.)**J)*Y*S(J)**II
   PUNCH 2,(PHI(I),I=1,N)
   DO 13 I=1,N
   H(I)=ABSF(PHI(I))-ACC
   IF(H(I))13,13,11
13 CONTINUE
   GO TO 14
11 K=K+1
   PUNCH 12,K
12 FORMAT(I6)
   DO 5 I=1,N
   B(I)=-PHI(I)
   DO 5 J=1,N
   A(I,J)=PHID(I,J)
 5 A(I,N2)=B(I)
15 CALL SOLEQN(A,N,20)
   DO 22 I=1,N
22 X(I)=A(I,N2)
   DO 7 I=1,N
 7 S(I)=S(I)+X(I)
   PUNCH 2,(S(I),I=1,N)
   GO TO 8
14 PUNCH 20
20 FORMAT(20X,21HTIME   OF   SWITCHING)
   PI2=(22./7.)**2
   T(1)=-LOGF(S(1))/PI2
   DO 16 I=2,N
16 T(I)=T(1)+LOGF(S(I))/PI2
   PUNCH 2,(T(I),I=1,N)
   STOP
   END
```

APPENDIX 5.2.3

```
C  PLOTTING OF ACTUAL                 VALUE   RAJIV GOYAL      1D
   DIMENSION P(20),X(20),EXP CT(20),Z(20),T(20),Q(20),ACTU L(20),
  1B(20)
   READ 1,M
 1 FORMAT(I2)
 2 FORMAT(8F10.6)
   EXPECTED  VALUE
   L=19
   X(1)=0.05
   DO 5 J=1,L
   J1=J+1
   X(J1)=X(J)+0.05
   P(J)=X(J)*(22./7.)
 5 EXP CT(J)=SINF(p(J))/(22./7.)
   PUNCH 2,(EXP CT(K),K=1,L)
   ACTUAL  VALUE
   NT=4
   DO 100 JP=2,M
   READ 1,N
   READ 3,(T(I),I=1,N)
 3 FORMAT(4E20.8)
   N1=N-1
   DO 6 J=1,N1
 6 Z(J)=T(N)-T(J)
   DO 8 K=1,L
   ACTU L(K)=0.
   DO 8 I=1,NT
   G=-I**2
   Y=(22./7.)**2
   Q(I)=G*Y
   DO 7 J=1,N1
   J1=J-1
   F=2.*EXPF(G(I)*Z(J))*((-1.)**J1)
   IF(J-1)15,15,20
15 B(1)=F
   GO TO 7
20 B(J)=B(J1)+F
 7 CONTINUE
   F=B(N1)
   F=(F-EXPF(Q(I)*T(N))+((-1.)**N1))/(-Q(I))
   XT=I
   PQ=SINF((22./7.)*X(K)*XT)
 8 ACTU L(K)=(2.*(22./7.)*XT*PQ*F)+ACTUL(K)
   PUNCH 1,JP
   PUNCH 2 ,(ACTU L(K),K=1,L)
00 CONTINUE
   STOP
   END
```

APPENDIX - 5.2.2 (ONE DIMENSION PROBLEM)

Iteration	So	S_1	S_2	S_3	F_1	F_2	F_3	F_4
1	0.3	0.6	0.82	0.93				
2								
3	0.11350056	0.74777952	0.90643955	0.9654103	0	0.0123367	-0.2230488	-0.257539
4	0.02949726	0.65326279	0.84622532	0.95771115	0	0.0211944	-0.0460829	-0.1342911
5	0.010051433	0.65268298	0.86657305	0.96891578	$-0.1x10^{-6}$	-0.0022222	-0.0029355	-0.0066275
6	0.008026571	0.64955391	0.86356917	0.96802859	$-0.1x10^{-6}$	$-0.224x10^{-4}$	$-0.150x10^{-3}$	$-0.1602x10^{-3}$
7	0.0079629804	0.64949367	0.86354465	0.96803246	0	$-0.1x10^{-6}$	$-0.1x10^{-6}$	$-0.2x10^{-6}$
8	0.0079630672	0.64949372	0.86354467	0.96803247	0	0	$-0.1x10^{-6}$	$-0.1x10^{-6}$
9	0.0079630092	0.64949368	0.86354466	0.96803247	$-0.1x10^{-6}$	0	0	0
10	0.0079629092	0.64949368	0.85354466	0.96803247	0	0	0	0

APPENDIX 5.3.1

```
C  RAJIV GOYAL TIME OPTIMAL CONTROL IN DISTRIBUTED PARAMETER SYSTEMS 28
      DIMENSION S(40),PHI(40),PHID(40,40),C(40),A(40,41),X(40),
     1H(40),B(40),T(40),N(40)
      READ 1,L
    1 FORMAT(I2)
      READ 1,LP
      READ 1,KPC
      READ 2,(S(I),I=1,KPC)
    2 FORMAT(4E20.8)
      READ 2,ACC
      KP=0
      DO 50 I=1,L,2
      DO 50 J=1,LP
      KP=KP+1
      N(KP)=(I**2)+(J**2)
      IF(KP-1)45,50,45
   45 KPT=KP-1
      DO 49 M=1,KPT
      IF(N(KP)-N(M))49,51,49
   51 KP=KP-1
      GO TO 50
   49 CONTINUE
   50 CONTINUE
   12 FORMAT(I6)
      KP1=KPC-1
      KP2=KPC+1
      K=0
      C(1)=(-1.)**KP1-1./8.
      DO 3 I=2,KPC
    3 C(I)=(-1.)**KP1
    8 DO 9 I=1,KPC
      Z=N(I)
      II=N(I)
      IZ=N(I)-1
      PHI(I)=-S(1)**II   +C (I)
      PHID(I,1)=-Z*S(1)**IZ
      DO 9 J=2,KPC
      PHI(I)=PHI(I)+2.*((-1.)**J)*S(J)**II
    9 PHID(I,J)=2.*((-1.)**J)*Z*S(J)**IZ
      PUNCH 2,(PHI(I),I=1,KPC)
      DO 13 I=1,KPC
      H(I)=ABSF(PHI(I))-ACC
      IF(H(I))13,13,11
   13 CONTINUE
      GO TO 14
```

```
11 K=K+1
   PUNCH 12,K
   DO 5 I=1,KPC
   B(I)=-PHI(I)
   DO 5 J=1,KPC
   A(I,J)=PHID(I,J)
 5 A(I,KP2)=B(I)
15 CALL SOLEQN (A,KPC,40)
   DO 22 I=1,KPC
22 X(I)=A(I,KP2)
   DO 7 I=1,KPC
 7 S(I)=S(I)+X(I)
   PUNCH 2,(S(I),I=1,KPC)
   GO TO 8
14 PUNCH 20
20 FORMAT(20X,21HTIME   OF   SWITCHING)
   PI2=(22./7.)**2
   T(1)=-LOGF(S(1))/PI2
   DO 16 I=2,KPC
16 T(I)=T(1)+LOGF(S(I))/PI2
   PUNCH 2, (T(I),I=1,KPC)
   STOP
   END
```

APPENDIX 5.3.3

Iteration		F_3	F_4
1	0119	0.070905	-0.0687666
2	4876	-0.0282406	-0.0827806
3	5071	0.0290372	- .0099415
4	3609	-0.0054201	-0.0064914
5	$x10^{-4}$	$0.73x10^{-5}$	$-0.99x10^{-5}$
6	0^{-6}	0	$-0.1x10^{-6}$

APPENDIX - 5.3.2 (TWO DIMENSION PROBLEM)

Iteration	SO	S_1	S_2	S_3	F_1	F_2	F_3	F_4
1	0.48	0.62	0.82	0.96	-0.0882	0.0470119	0.070905	-0.0687666
2	0.568936	0.77112566	0.89255452	0.96957441	0.0274233	0.0664876	-0.0282406	-0.0827806
3	0.45028327	0.68751566	0.86430523	0.96818314	-0.0016894	0.0255071	0.0290372	- .0099415
4	0.45012430	0.70366314	0.88004482	0.97116678	0.0000439	-0.0013609	-0.0054201	-0.0064914
5	0.44730594	0.70161699	0.87932796	0.97133529	$-0.7x10^{-6}$	$0.154x10^{-4}$	$0.73x10^{-5}$	$-0.99x10^{-5}$
6	0.44728567	0.7016139	0.87933185	0.97133656	$0.1x10^{-6}$	$0.1x10^{-6}$	0	$-0.1x10^{-6}$

APPENDIX 5.3.3

```
C  PLOTTING OF ACTUAL                  VALUE RAJIV GOYAL 2 DIMENSION    2A
   DIMENSION X(20),P(20),Q(20),EXPCT(20),T(20),Z(20),ACTUL(20,20),
  1PACT(20),GS(20),B(20),Y(20)
 1 FORMAT(I2)
 2 FORMAT(F10.6)
 3 FORMAT(4E20.8)
 4 FORMAT(20X,6F10.6)
   X(1)=0.2
   Y(1)=0.05
   FX=0.2
   FM=0.05
   ML=4
   L=19
   PRINT 81
   PUNCH 81
81 FORMAT(1X,1HN,2X,1HX,25X,31HINTERVALS  OF  0.05  AT  Y-AXES)
   DO 5 I=1,ML
   I1=I+1
 5 X)I1*=X)I*.FX
   DO 6 J=1,L
   J1=J+1
 6 Y(J1)=Y(J)+FM
   ACTUAL VALUE
   PRINT 70
   PUNCH 70
70 FORMAT(15HACTUAL   VALUES)
   NT=5
   M=4
   DO 100 JP=2,M
   READ 1,N
   READ 3,(T(I),I=1,N)
   N1=N-1
   PRINT 1,N
   PUNCH 1,N
   DO 7 J=1,N1
 7 Z(J)=T(N)-T(J)
   DO 25 K=1,ML
   DO 30 IK=1,L
   ACTUL(IK,K)=0.
   DO 20 JC=1,NT,2
   PACT(K)=0.
   DO 21 I=1,NT
   G      =-((I**2)+(JC**2))
   YM=(22./7.)**2
   Q(I)=G*YM
   DO 22 J=1,N1
   J1=J-1
   F=2.*EXP (Q(I)*Z(J))*((-1.)**J1)
   IF(J-1)15,15,18
15 B(1)=F
   GO TO 22
18 B(J)=B(J1)+F
22 CONTINUE
```

```
      F=B(N1)
      F=(F-EXP (Q(I)*T(N))+((-1.)**N1))/(-Q(I))
      XT=I
      PQ=SIN ((22./7.)*X(K)*XT)
      PACT(K)=(XT*PQ*F)+PACT(K)
   21 CONTINUE
      ZM=JC
      FQ=SIN ((22./7.)*Y(IK)*ZM)/ZM
   20 ACTUL(IK,K)=FQ*PACT(K)+ACTUL(IK,K)
   30 CONTINUE
      PRINT 2,X(K)
      PRINT 4,(ACTUL(IK,K),IK=1,L)
      PUNCH 2,X(K)
      PUNCH 4,(ACTUL(IK,K),IK=1,L)
   25 CONTINUE
  100 CONTINUE
      STOP
      END

C   C PLOTTING OF ACTUAL                    VALUE RAJIV GOYAL 2 DIMENSION   2B
      DIMENSION X(20),P(20),Q(20),EXPCT(20),T(20),Z(20),ACTUL(20,20),
     1PACT(20),GS(20),B(20),Y(20)
    1 FORMAT(I2)
    2 FORMAT(F10.6)
    3 FORMAT(4E20.8)
    4 FORMAT(20X,6F10.6)
      X(1)=0.05
      Y(1)=0.2
      FX=0.05
      FM=0.2
      ML=19
      L=4
      PRINT 81
      PUNCH 81
   81 FORMAT(1X,1HN,2X,1HY,25X,31HINTERVALS  OF  0.05  AT  X-AXES)
      DO 5 I=1,ML
      I1=I+1
    5 X(I1)=X(I)+FX
      DO 6 J=1,L
      J1=J+1
    6 Y(J1)=Y(J)+FM
C     ACTUAL VALUE
      PRINT 70
      PUNCH 70
   70 FORMAT(15HACTUAL   VALUES)
      NT=5
      M=4
      DO 100 JP=2,M
      READ 1,N
      READ 3,(T(I),I=1,N)
      N1=N-1
      PRINT 1,N
      PUNCH 1,N
      DO 7 J=1,N1
    7 Z(J)=T(N)-T(J)
      DO 30 IK=1,L
      DO 25 K=1,ML
      ACTUL(IK,K)=0.
      DO 20 JC=1,NT,2
```

```
      PACT(K)=0.
      DO 21 I=1,NT
      G     =-((I**2)+(JC**2))
      YM=(22./7.)**2
      Q(I)=G*YM
      DO 22 J=1,N1
      J1=J-1
      F=2.*EXP (Q(I)*Z(J))*((-1.)**J1)
      IF(J-1)15,15,18
   15 B(1)=F
      GO TO 22
   18 B(J)=B(J1)+F
   22 CONTINUE
      F=B(N1)
      F=(F-EXP (Q(I)*T(N))+((-1.)**N1))/(-Q(I))
      XT=I
      PQ=SIN ((22./7.)*X(K)*XT)
      PACT(K)=(XT*PQ*F)+PACT(K)
   21 CONTINUE
      ZM=JC
      FQ=SIN ((22./7.)*Y(IK)*ZM)/ZM
   20 ACTUL(IK,K)=FQ*PACT(K)+ACTUL(IK,K)
   25 CONTINUE
      PRINT 2,Y(IK)
      PRINT 4,(ACTUL(IK,K),K=1,ML*
      PUNCH 2,Y(IK)
      PUNCH 4,(ACTUL(IK,K),K=1,ML)
   30 CONTINUE
  100 CONTINUE
      STOP
      END
```

APPENDIX 5.4.1

```
C   C  RAJIV GOYAL TIME OPTIMAL CONTROL IN DISTRIBUTED PARAMETER SYSTEMS 3D
       DIMENSION S(40),PHI(40),PHID(40,40),C(40),A(40,41),X(40),
      1H(40),B(40),T(40),N(40)
       READ 1,L
     1 FORMAT(I2)
       READ 1,LP
       READ 1,IIP
       READ 1,KPC
       READ 2,(S(I),I=1,KPC)
     2 FORMAT(4E20.8)
       READ 2,ACC
       KP=0
       DO 50 I=1,L,2
       DO 50 JQ=1,IIP,2
       DO 50 J=1,LP
       KP=KP+1
       N(KP)=(I**2)+(J**2)+(JQ**2)
       IF(KP-1)45,50,45
    45 KPT=KP-1
       DO 49 M=1,KPT
       IF (N(KP)-N(M))49,51,49
    51 KP=KP-1
       GO TO 50
    49 CONTINUE
    50 CONTINUE
    12 FORMAT(I6)
       KP1=KPC-1
       KP2=KPC+1
       K=0
       C(1)=(-1.)**KP1-1./32.
       DO 3 I=2,KPC
     3 C(I)=(-1.)**KP1
     8 DO 9 I=1,KPC
       Z=N(I)
       II=N(I)
       IZ=N(I)-1
       PHI(I)=-S(1)**II  +C (I)
       PHID(I,1)=-Z*S(1)**IZ
       DO 9 J=2,KPC
       PHI(I)=PHI(I)+2.*((-1.)**J)*S(J)**II
     9 PHID(I,J)=2.*((-1.)**J)*Z*S(J)**IZ
       PUNCH 2,(PHI(I),I=1,KPC)
       DO 13 I=1,KPC
       H(I)=ABSF(PHI(I))-ACC
       IF(H(I))13,13,11
    13 CONTINUE
       GO TO 14
    11 K=K+1
       PUNCH 12,K
       DO 5 I=1,KPC
       B(I)=-PHI(I)
       DO 5 J=1,KPC
       A(I,J)=PHID(I,J)
     5 A(I,KP2)=B(I)
    15 CALL SOLEQN (A,KPC,40)
       DO 22 I=1,KPC
```

```
22 X(I)=A(I,KP2)
   DO 7 I=1,KPC
 7 S(I)=S(I)+X(I)
   PUNCH 2,(S(I),I=1,KPC)
   GO TO 8
14 PUNCH 20
20 FORMAT(20X,21HTIME    OF    SWITCHING)
   PI2=(22./7.)**2
   T(1)=-LOGF(S(1))/PI2
   DO 16 I=2,KPC
16 T(I)=T(1)+LOGF(S(I))/PI2
   PUNCH 2, (T(I),I=1,KPC)
   STOP
   END
```

Iterati		F_2	F_3	F_4	F_5
1	24017	-0.0244839	-0.1358442	-0.2572719	-0.1942589
2	037161	-0.0106673	-0.017395	0.022724	-0.019605
3	000282	-0.0000336	-0.0002504	-0.0003633	-0.0003199
4	2×10^{-6}	-0.2×10^{-6}	-0.2×10^{-6}	-0.1×10^{-6}	-0.2×10^{-6}
5		0	0.1×10^{-6}	0	0.1×10^{-6}

```
22 CONTINUE
   F=B(N1)
```

APPENDIX - 5.4.2 (THREE DIMENSION PROBLEM)

Iteration	So	S_1	S_2	S_3	S_4	F_1	F_2	F_3	F_4	F_5
1	0.55	0.68	0.8	0.92	0.99	0.024017	-0.0244839	-0.1358442	-0.2572719	-0.1942589
2	0.55032293	0.68451231	0.82678458	0.92371659	0.98188255	0.0037161	-0.0106673	-0.017395	0.022724	-0.019605
3	0.54468927	0.67791014	0.82214708	0.92193007	0.98062382	0.0000282	-0.0000336	-0.0002504	-0.0003633	-0.0003199
4	0.54434116	0.67773752	0.8221461	0.92194670	0.98061531	-0.2×10^{-6}	-0.2×10^{-6}	-0.2×10^{-6}	-0.1×10^{-6}	-0.2×10^{-6}
5	0.54434171	0.677738	0.82214645	0.9219465	0.98061536	0	0	0.1×10^{-6}	0	0.1×10^{-6}

APPENDIX 5.4.3

```
C  C  PLOTTING OF ACTUAL                  VALUE RAJIV GOYAL 3 DIMENSION    3A
      DIMENSION X(20),Y(20),W(20),Z(20),P(20),Q(20),R(20),EXP T(20),
     1T(20),ACTUL(20),PACT(20),GS(20),QPACT(20),B(20)
    1 FORMAT(I2)
    2 FORMAT(2F10.6)
    3 FORMAT(4E20.8)
    4 FORMAT(30X,5F10.6)
      X(1)=0.05
      Y(1)=0.2
      W(1)=0.2
      FL=0.05
      FM=0.2
      ML=19
      L=4
      PIE=22./7.
      DO 5 I=1,L
      I1=I+1
      Y(I1)=Y(I)+FM
    5 W(I1)=W(I)+FM
      DO 6 J=1,ML
      J1=J+1
    6 X(J1)=X(J)+FL
      PRINT 70
      PUNCH 70
   70 FORMAT(15HACTUAL    VALUES)
      NT=5
      M=4
      DO 100 JP=2,M
      READ 1,N
      READ 3,(T(I),I=1,N)
      N1=N-1
      PRINT 1,N
      PUNCH 1,N
      DO 7 J=1,N1
    7 Z(J)=T(N)-T(J)
      DO 30 IK=1,L
      DO 25 K=1,ML
      ACTUL(    K)=0.
      DO 72 JQ=1,NT,2
      QPACT(K)=0.
      DO 20 JC=1,NT,2
      PACT(K)=0.
      DO 21 I=1,NT
      G      =-((I**2)+(JC**2)+(JQ**2))
      YM=PIE**2
      Q(I)=G*YM
      DO 22 J=1,N1
      J1=J-1
      F=2.*EXP (Q(I)*Z(J))*((-1.)**J1)
      IF(J-1)15,15,18
   15 B(1)=F
      GO TO 22
   18 B(J)=B(J1)+F
   22 CONTINUE
      F=B(N1)
```

```
      F=(F-EXP (Q(I)*T(N))+((-1.)**N1))/(-Q(I))
      XT=I
      PQ=SIN (PIE*X(K)*XT)
      PACT(K)=(XT*PQ*F)+PACT(K)
   21 CONTINUE
      ZM=JC
      FQ=SIN (PIE*Y(IK)*ZM)/ZM
   20 QPACT(K)=FQ* PACT(K)+QPACT(K)
      ZQ=JQ
      FT=SIN (PIE*W(IK)*ZQ)/ZQ
   72 ACTUL(    K)=FT*QPACT(K)+ACTUL(    K)
   25 CONTINUE
      PRINT 2,Y(IK),W(IK)
      PRINT 4,(ACTUL(    K),K=1,ML)
      PUNCH 2,Y(IK),W(IK)
      PUNCH 4,(ACTUL(    K),K=1,ML)
   30 CONTINUE
  100 CONTINUE
      STOP
      END
```

```
C   C PLOTTING OF ACTUAL                  VALUE RAJIV GOYAL 3 DIMENSION    3B
      DIMENSION X(20),Y(20),W(20),Z(20),P(20),Q(20),R(20),EXP T(20),
     1T(20),ACTUL(20),PACT(20),GS(20),QPACT(20),B(20)
    1 FORMAT(I2)
    2 FORMAT(2F10.6)
    3 FORMAT(4E20.8)
    4 FORMAT(30X,5F10.6)
      X(1)=0.2
      Y(1)=0.05
      W(1)=0.2
      FL=0.05
      FM=0.2
      ML=19
      L=4
      PIE=22./7.
      DO 5 I=1,L
      I1=I+1
      X(I1)=X(I)+FM
    5 W(I1)=W(I)+FM
      DO 6 J=1,ML
      J1=J+1
    6 Y(J1)=Y(J)+FL
      PRINT 70
      PUNCH 70
   70 FORMAT(15HACTUAL    VALUES)
      NT=5
      M=4
      DO 100 JP=2,M
      READ 1,N
      READ 3,(T(I),I=1,N)
      N1=N-1
      PRINT 1,N
      PUNCH 1,N
      DO 7 J=1,N1
    7 Z(J)=T(N)-T(J)
      DO 30 IK=1,L
      DO 25 K=1,ML
      ACTUL(K)=0.
```

```
      DO 72 JQ=1,NT,2
      QPACT(K)=0.
      DO 20 JC=1,NT,2
      PACT(K)=0.
      DO 21 I=1,NT
      G     =-((I**2)+(JC**2)+(JQ**2))
      YM=PIE**2
      Q(I)=G*YM
      DO 22 J=1,N1
      J1=J-1
      F=2.*EXP (Q(I)*Z(J))*((-1.)**J1)
      IF(J-1)15,15,18
   15 B(1)=F
      GO TO 22
   18 B(J)=B(J1)+F
   22 CONTINUE
      F=B(N1)
      F=(F-EXP (Q(I)*T(N))+((-1.)**N1))/(-Q(I))
      XT=I
      PQ=SIN (PIE*X(IK)*XT)
      PACT(K)=(XT*PQ*F)+PACT(K)
   21 CONTINUE
      ZM=JC
      FQ=SIN (PIE*Y(K)*ZM)/ZM
   20 QPACT(K)=FQ* PACT(K)+QPACT(K)
      ZQ=JQ
      FT=SIN (PIE*W(IK)*ZQ)/ZQ
   72 ACTUL(K)=FT*QPACT(K)+ACTUL(K)
   25 CONTINUE
      PRINT 2,X(IK),W(IK)
      PRINT 4,(ACTUL(K),K=1,ML)
      PUNCH 2,X(IK),W(IK)
      PUNCH 4,(ACTUL(K),K=1,ML)
   30 CONTINUE
  100 CONTINUE
      STOP
      END


C  C  PLOTTING OF ACTUAL                    VALUE RAJIV GOYAL 3 DIMENSION   3C
      DIMENSION X(20),Y(20),W(20),Z(20),P(20),Q(20),R(20),EXP T(20),
     1T(20),ACTUL(20),PACT(20),GS(20),QPACT(20),B(20)
    1 FORMAT(I2)
    2 FORMAT(2F10.6)
    3 FORMAT(4E20.8)
    4 FORMAT(30X,5F10.6)
      X(1)=0.2
      Y(1)=0.2
      W(1)=0.05
      FL=0.05
      FM=0.2
      ML=19
      L=4
      PIE=22./7.
      DO 5 I=1,L
      I1=I+1
      X(I1)=X(I)+FM
    5 Y(I1)=Y(I)+FM
```

```
      DO 6 J=1,ML
      J1=J+1
    6 W(J1)=W(J)+FL
      PRINT 70
      PUNCH 70
   70 FORMAT(15HACTUAL   VALUES)
      NT=5
      M=4
      DO 100 JP=2,M
      READ 1,N
      READ 3,(T(I),I=1,N)
      N1=N-1
      PRINT 1,N
      PUNCH 1,N
      DO 7 J=1,N1
    7 Z(J)=T(N)-T(J)
      DO 30 IK=1,L
      DO 25 K=1,ML
      ACTUL(K)=0.
      DO 72 JQ=1,NT,2
      QPACT(K)=0.
      DO 20 JC=1,NT,2
      PACT(K)=0.
      DO 21 I=1,NT
      G     =-((I**2)+(JC**2)+(JQ**2))
      YM=PIE**2
      Q(I)=G*YM
      DO 22 J=1,N1
      J1=J-1
      F=2.*EXP (Q(I)*Z(J))*((-1.)**J1)
      IF(J-1)15,15,18
   15 B(1)=F
      GO TO 22
   18 B(J)=B(J1)+F
   22 CONTINUE
      F=B(N1)
      F=(F-EXP (Q(I)*T(N))+((-1.)**N1))/(-Q(I))
      XT=I
      PQ=SIN (PIE*X(IK)*XT)
      PACT(K)=(XT*PQ*F)+PACT(K)
   21 CONTINUE
      ZM=JC
      FQ=SIN (PIE*Y(IK)*ZM)/ZM
   20 QPACT(K)=FQ* PACT(K)+QPACT(K)
      ZQ=JQ
      FT=SIN (PIE*W(K)*ZQ)/ZQ
   72 ACTUL(K)=FT*QPACT(K)+ACTUL(K)
   25 CONTINUE
      PRINT 2,X(IK),Y(IK)
      PRINT 4,(ACTUL(K),K=1,ML)
      PUNCH 2,X(IK),Y(IK)
      PUNCH 4,(ACTUL(K),K=1,ML)
   30 CONTINUE
  100 CONTINUE
      STOP
      END
```

2

	t_2	t_3	t_4	T
83	-	-	-	0.17234
13	0.26779	-	-	0.27449
59	0.47443	0.48599	-	0.48928

ets.

TABLE - 5.1 (ONE DIMENSIONAL PROBLEM)

No. of Integral Equations=N	No. of Iterations	So	S_1	S_2	S_3	S_4	t_1	t_2	t_3	t_4	T
2	4	0.18233 (0.3)	0.84113 (0.9)	-	-	-	0.15483	-	-	-	0.17234
3	6	0.06645 (0.0001)	0.719241 (0.7187)	0.93602 (0.9382)	- -	-	0.24113	0.26779	-	-	0.27449
4	10	0.00796 (0.03)	0.64949 (0.6)	0.86354 (0.82)	0.96803 (0.93)	-	0.44559	0.47443	0.48599	-	0.48928

Note : The initial values of S's, fed to the computer, are given within brackets.

Accuracy = 10^{-7}

TABLE - 5.2 (TWO DIMENSION PROBLEM)

No. of integral Equations = N	No. of Iterations	So	S_1	S_2	S_3	S_4	t_1	t_2	t_3	t_4	T
2	4	0.69715 (0.59)	0.89750 (0.86)	-	-	-	0.02557	-	-	-	0.03652
3	6	0.53919 (0.59)	0.77753 (0.75)	0.94694 (0.92)	-	-	0.03706	0.05701	-	-	0.06253
4	6	0.44729 (0.48)	0.70161 (0.62)	0.87933 (0.82)	0.97134 (0.96)	-	0.04558	0.06843	0.07851	-	0.08145

Note : The initial values of S's, fed to the computer, are given within brackets.

Accuracy = 10^{-7}

TABLE - 5.3 (THREE DIMENSION PROBLEM)

No. of Integral Equations = N	No. of Iterations	So	S_1	S_2	S_3	S_4	t_1	t_2	t_3	t_4	T
2	5	0.87718	0.94842	-	-	-	0.00790	-	-	-	0.01327
		(0.74)	(0.905)								
3	5	0.74843	0.84929	0.96094	-	-	0.01280	0.02530	-	-	0.02934
		(0.62)	(0.78)	(0.93)							
4	6	0.65734	0.77130	0.90046	0.97571	-	0.01619	0.03186	0.03999	-	0.04248
		(0.65)	(0.77)	(0.88)	(0.98)						
5	5	0.54434	0.67774	0.82215	0.92195	0.98062	0.02219	0.04175	0.05334	0.05959	0.06157
		(0.55)	(0.68)	(0.80)	(0.92)	(0.99)					

Note : The initial values of S's, fed to the computer, are given within brackets.

Accuracy = 10^{-7}

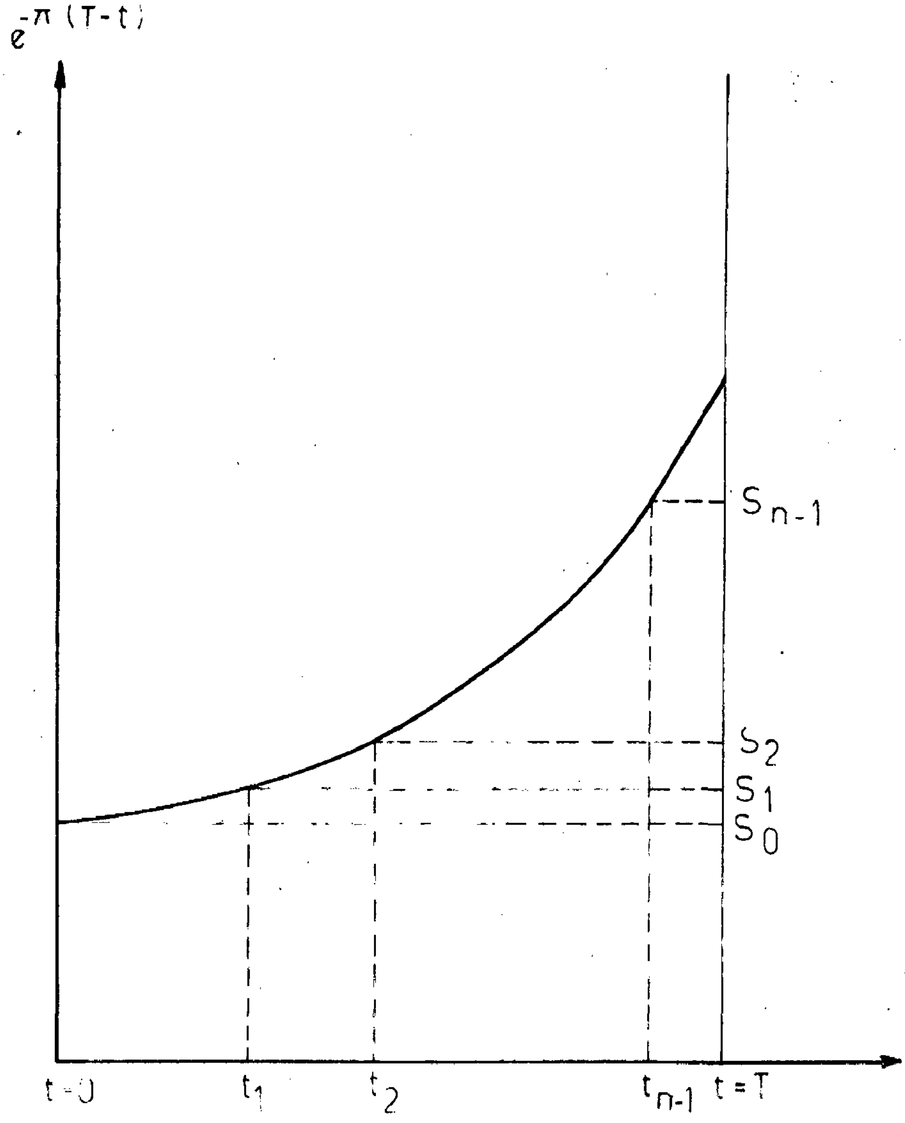


FIG. 5.1.1

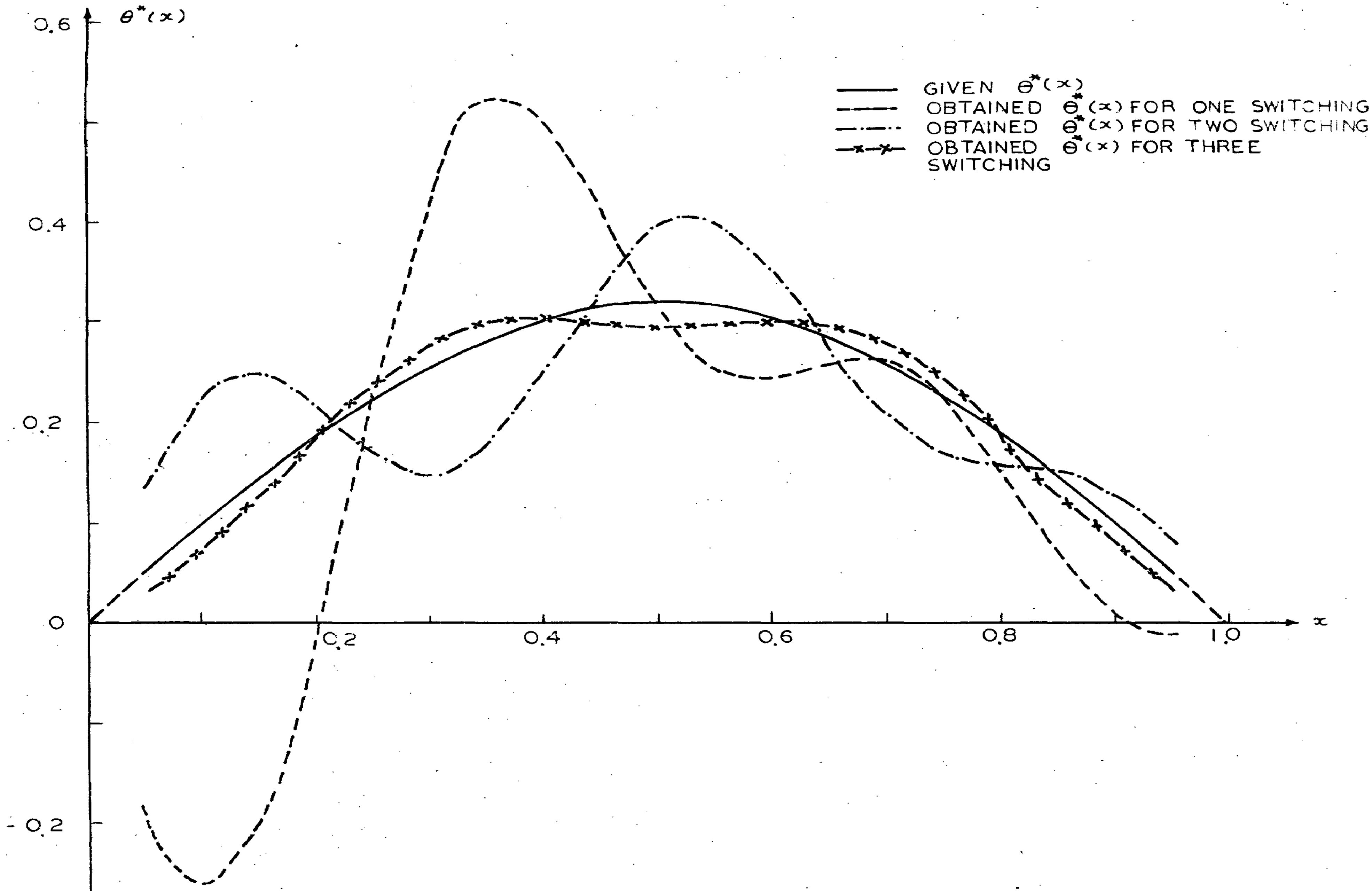


FIG.5.21 COMPARISON OF OBTAINED $\theta^*(x,y)$ AND THE GIVEN $\theta^*(x)$

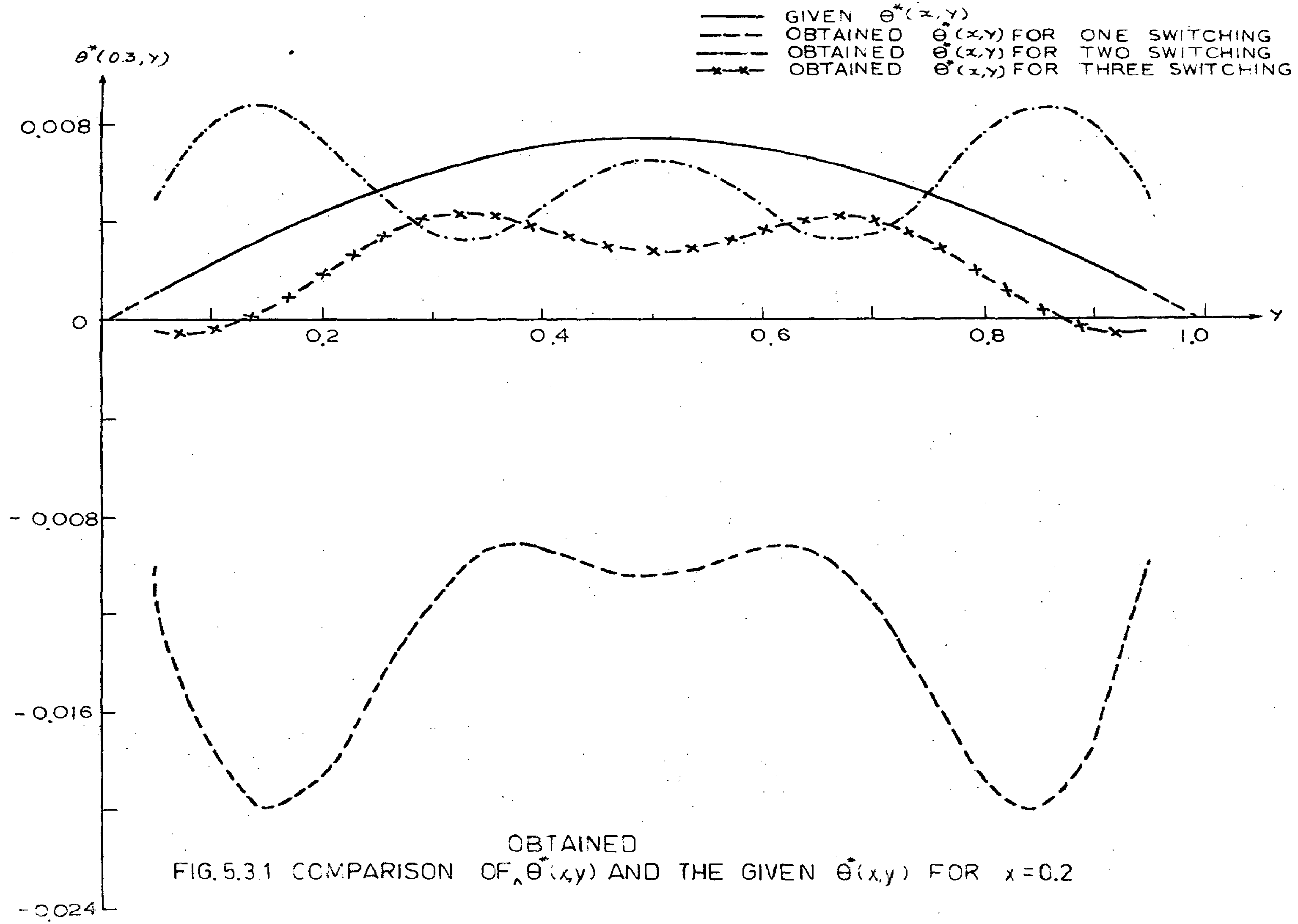


FIG. 5.3.1 COMPARISON OF OBTAINED $\hat{\theta}^*(x,y)$ AND THE GIVEN $\theta^*(x,y)$ FOR $x=0.2$

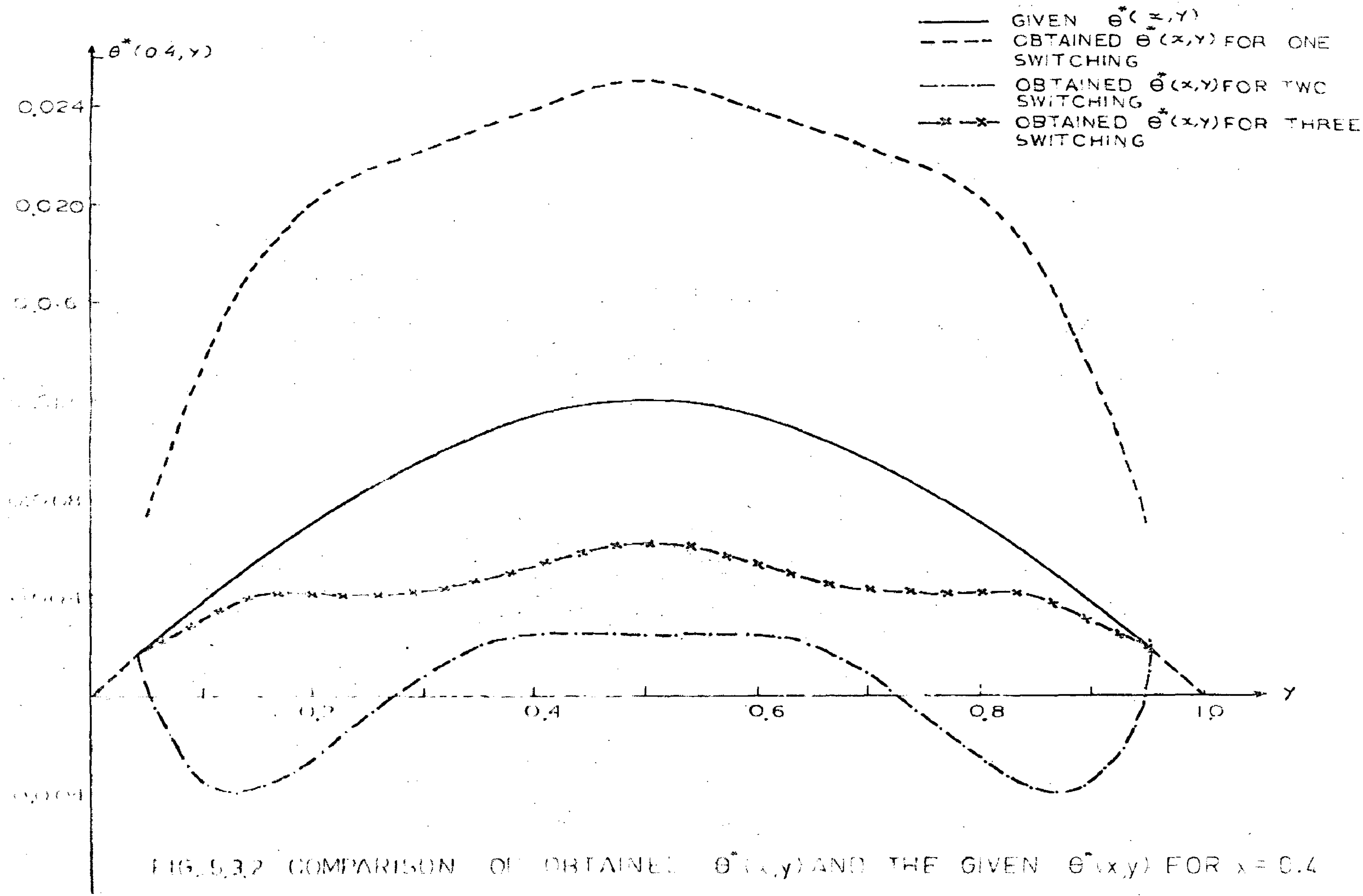


FIG. 5.3.2 COMPARISON OF OBTAINED $\theta^*(x,y)$ AND THE GIVEN $\theta^*(x,y)$ FOR $x = 0.4$

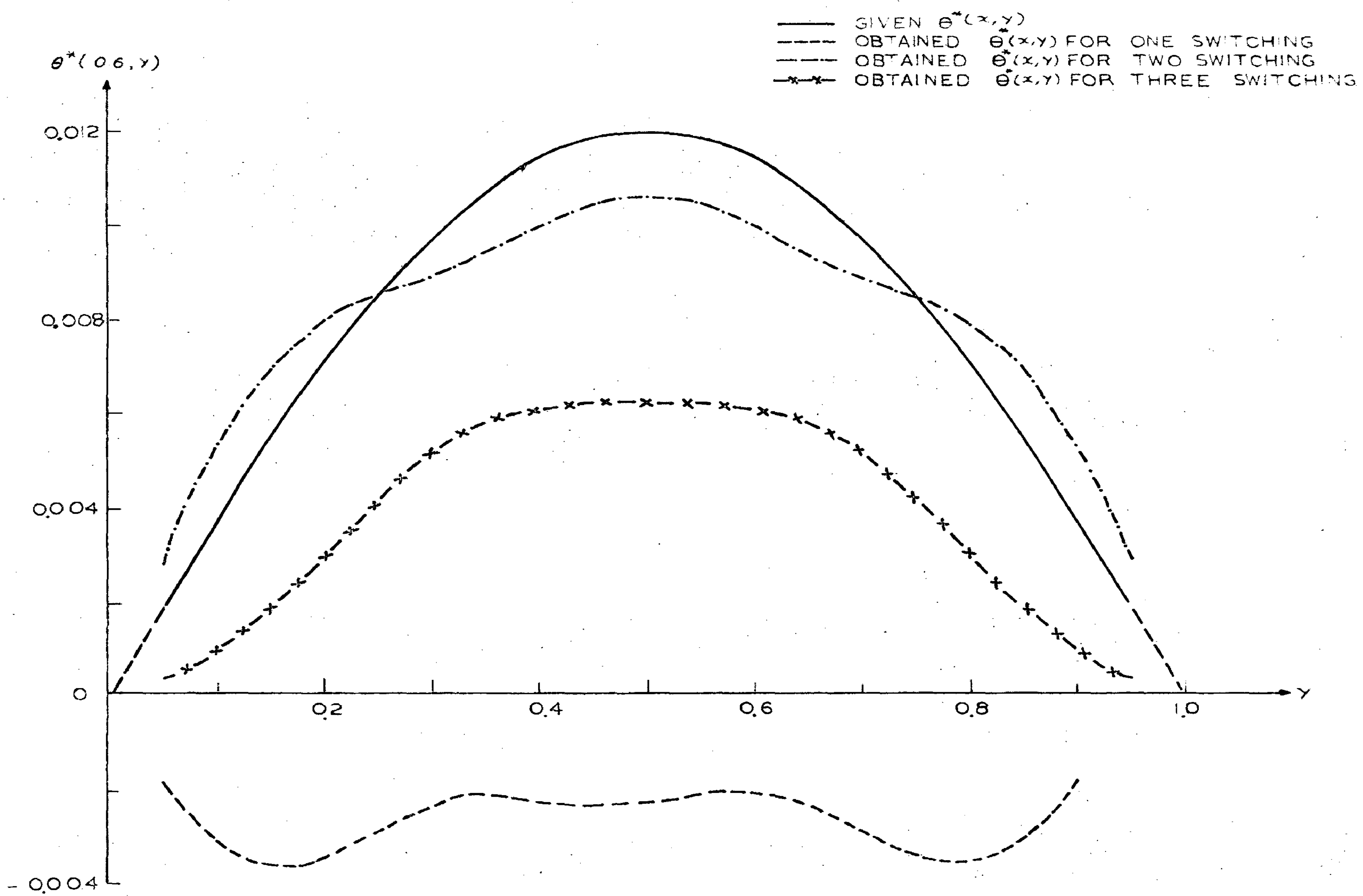


FIG. 5.3.3 COMPARISON OF OBTAINED $\theta^*(x,y)$ AND THE GIVEN $\theta^*(x,y)$ FOR $x = 0.6$

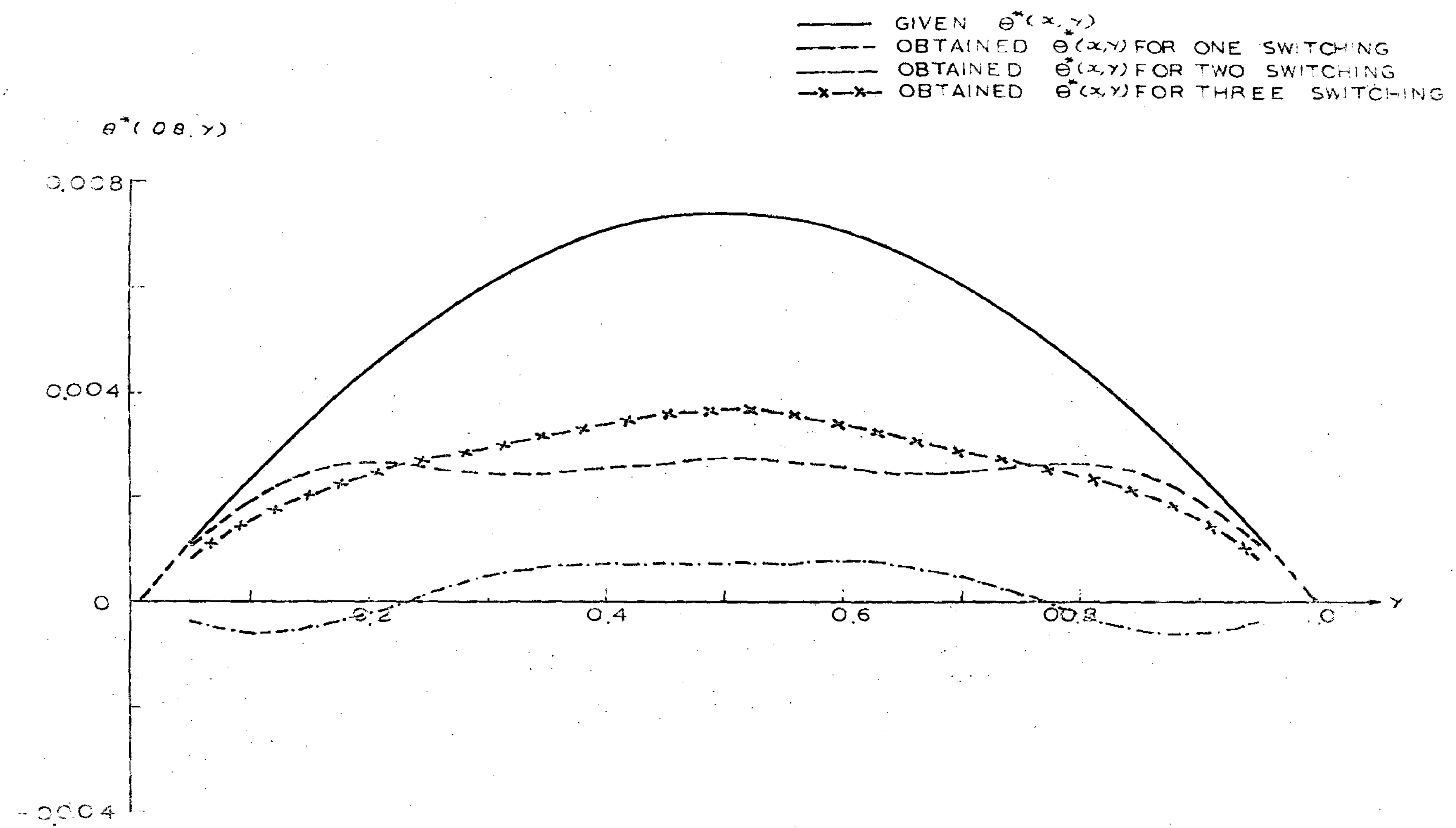


FIG.5.3.4 COMPARISON OF OBTAINED $\theta^*(x,y)$ AND THE GIVEN $\theta^*(x,y)$ FOR x = 0.8

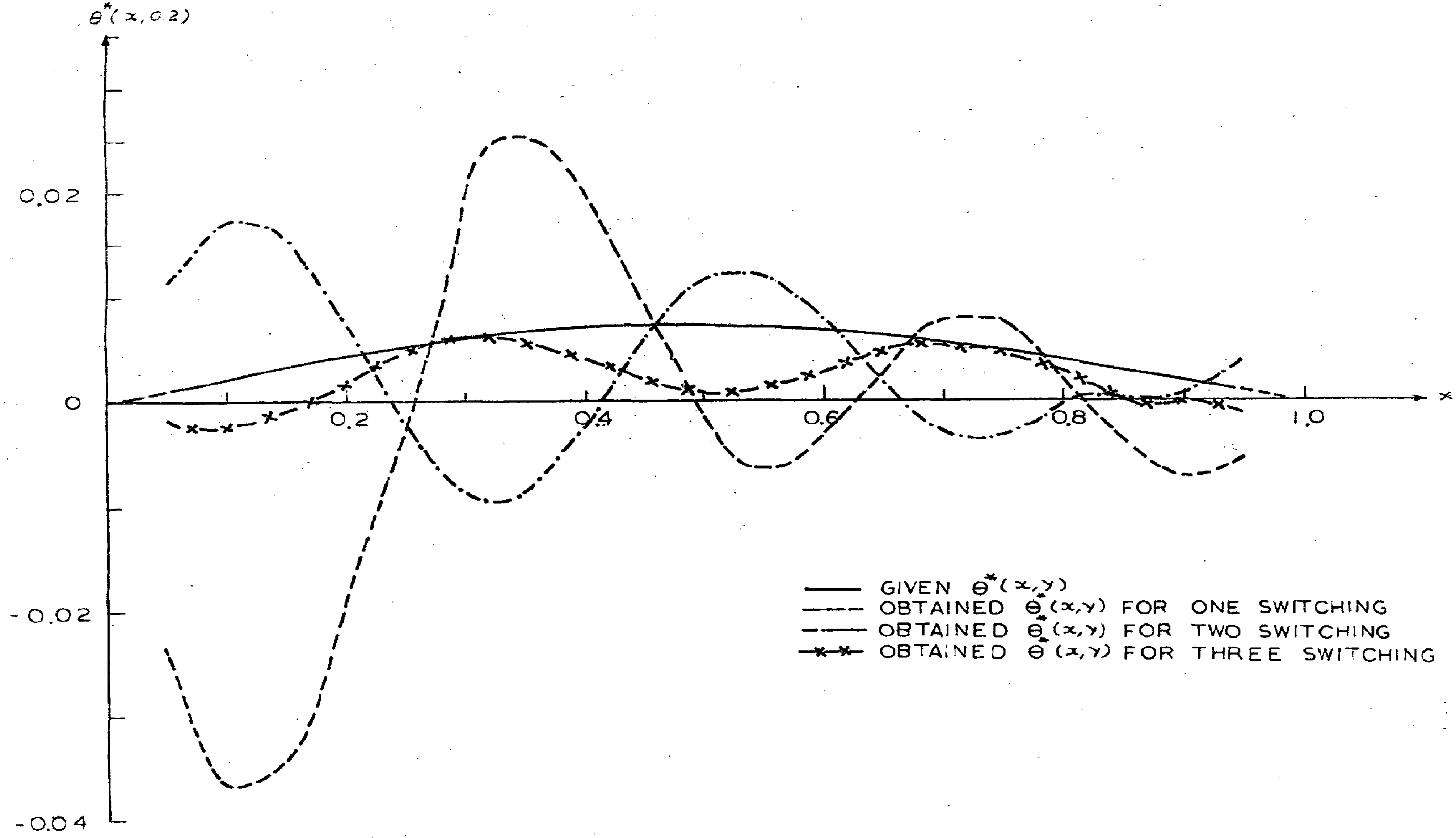


FIG. 5.3.5 COMPARISON OF OBTAINED $\theta^*(x,y)$ AND THE GIVEN $\theta^*(x,y)$ FOR $y=0.2$

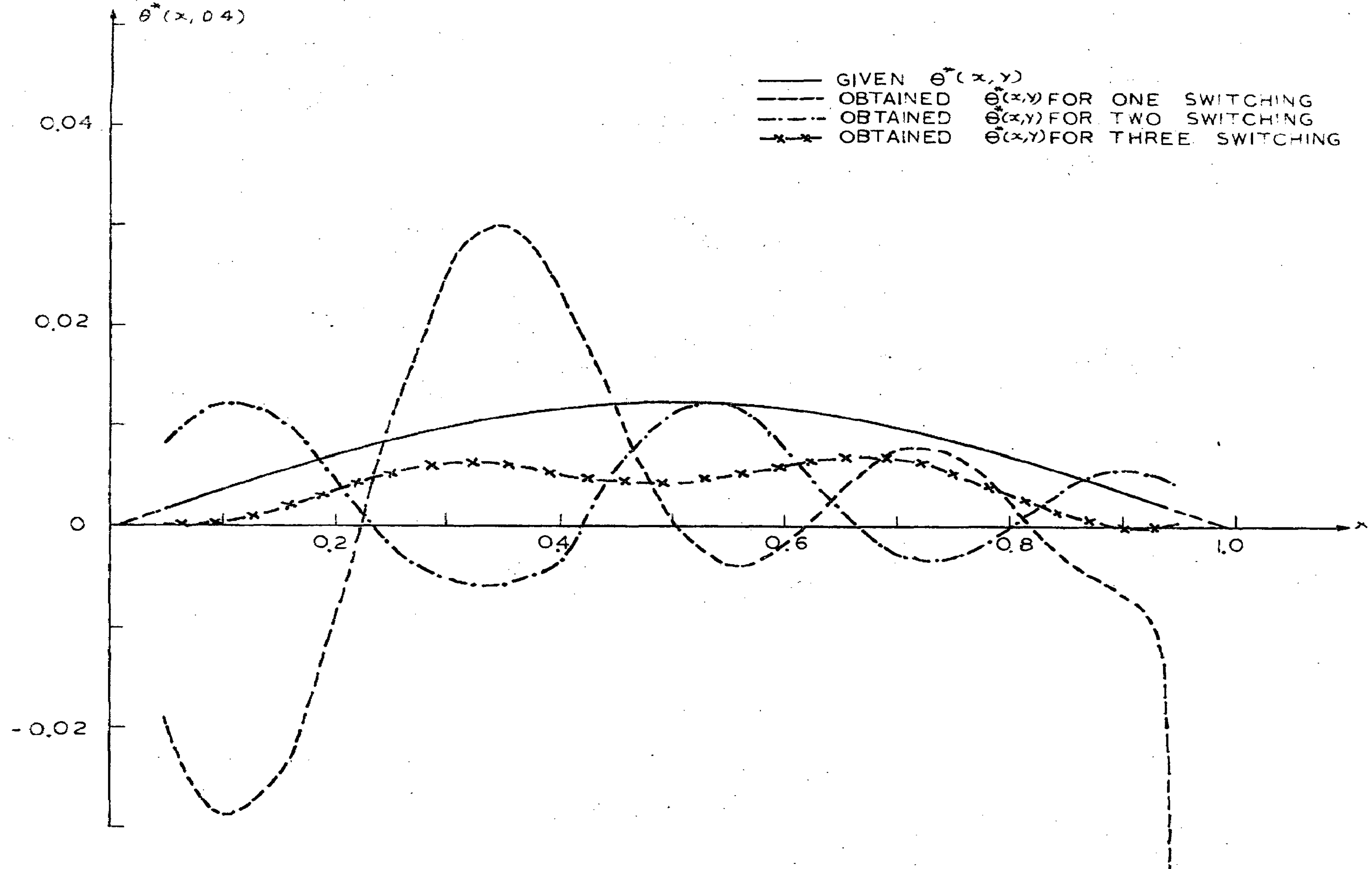


FIG. 5.3.6 COMPARISON OF OBTAINED $\theta^*(x,y)$ AND THE GIVEN $\theta^*(x,y)$ FOR $y = 0.4$

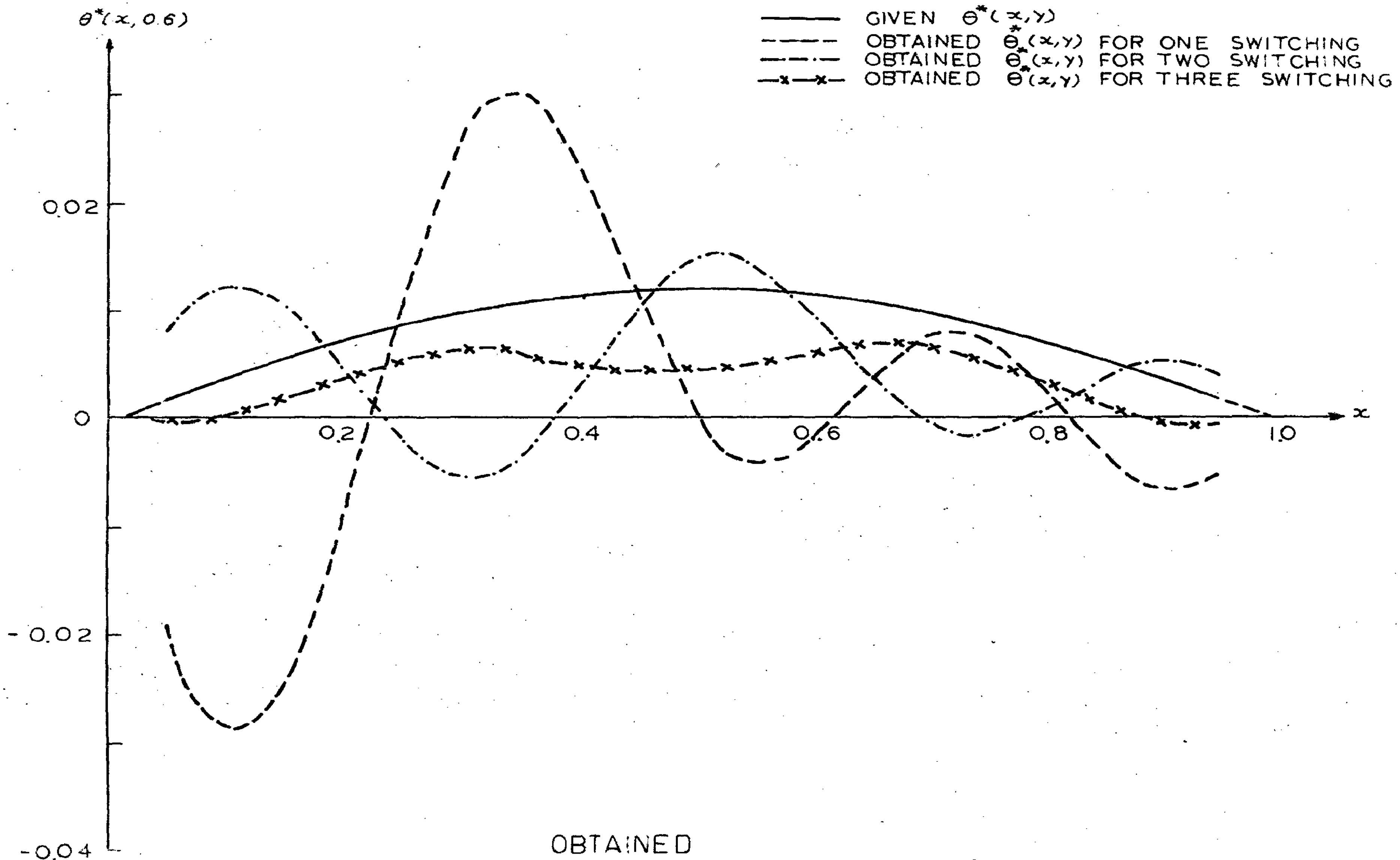


FIG.5.3.7 COMPARISON OF OBTAINED $\theta^*(x,y)$ AND THE GIVEN $\theta^*(x,y)$ FOR $y = 0.6$

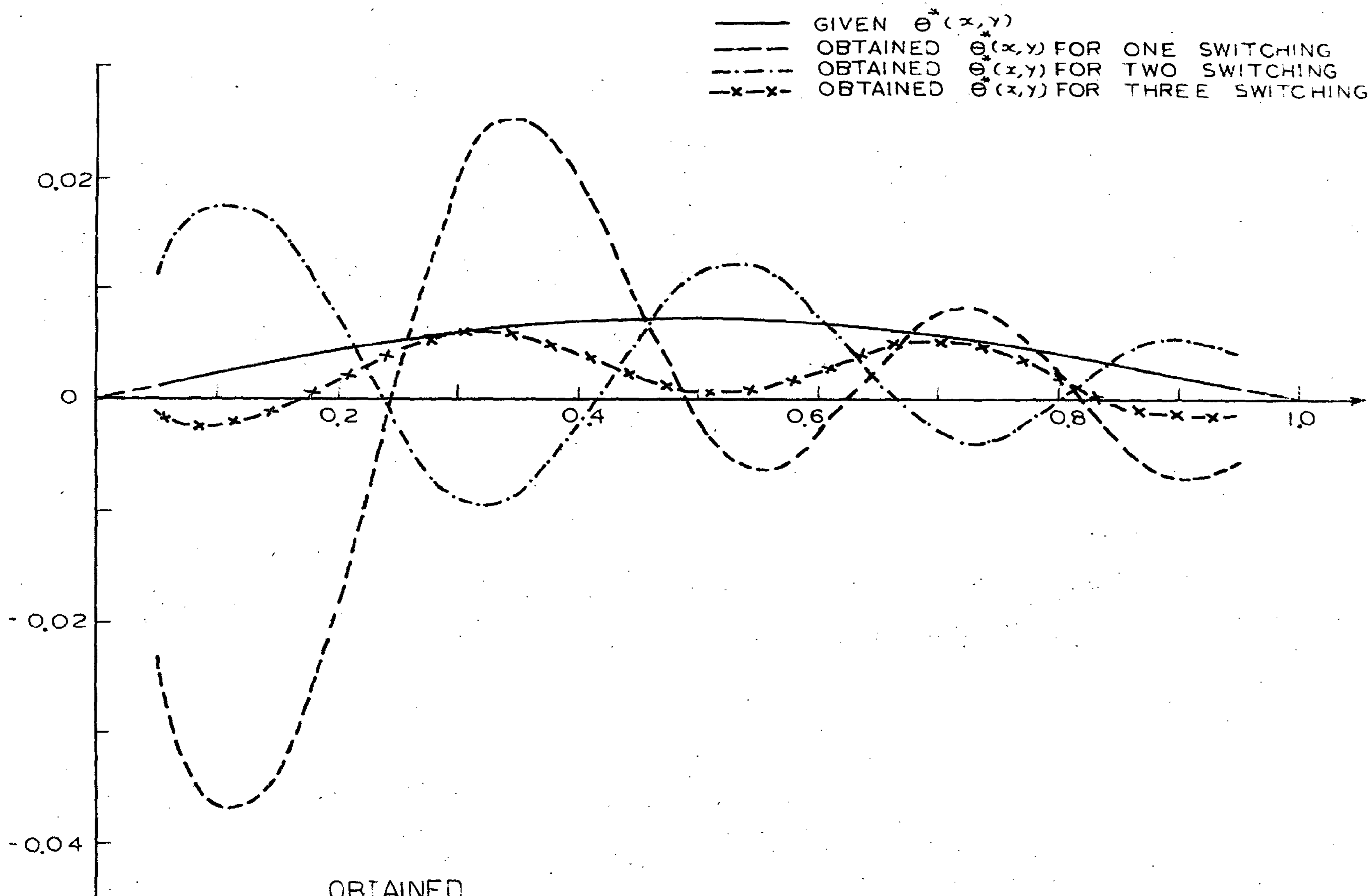


FIG. 5.3.8 COMPARISON OF OBTAINED $\theta^*(x,y)$ AND THE GIVEN $\theta^*(x,y)$ FOR $y = 0.8$

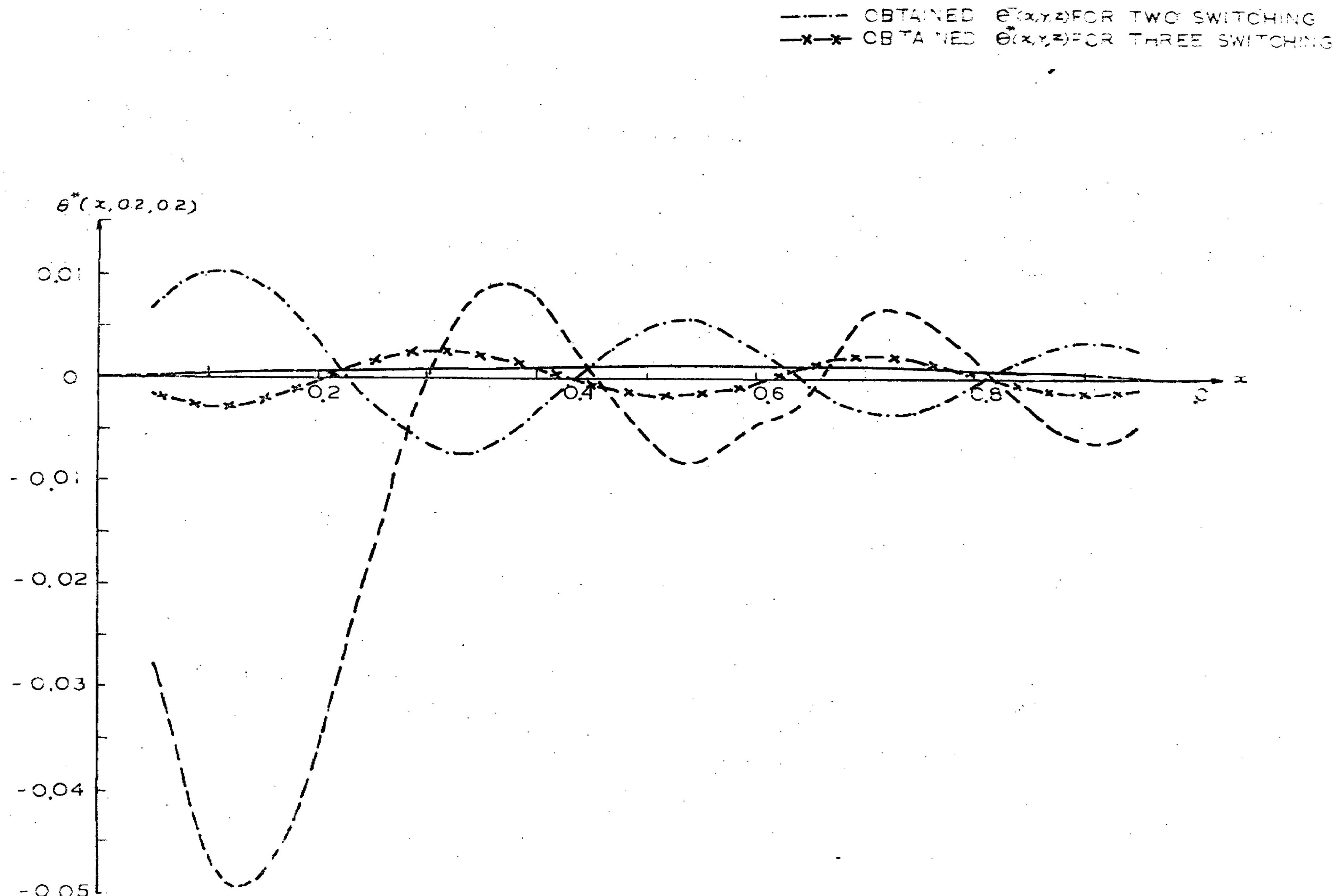
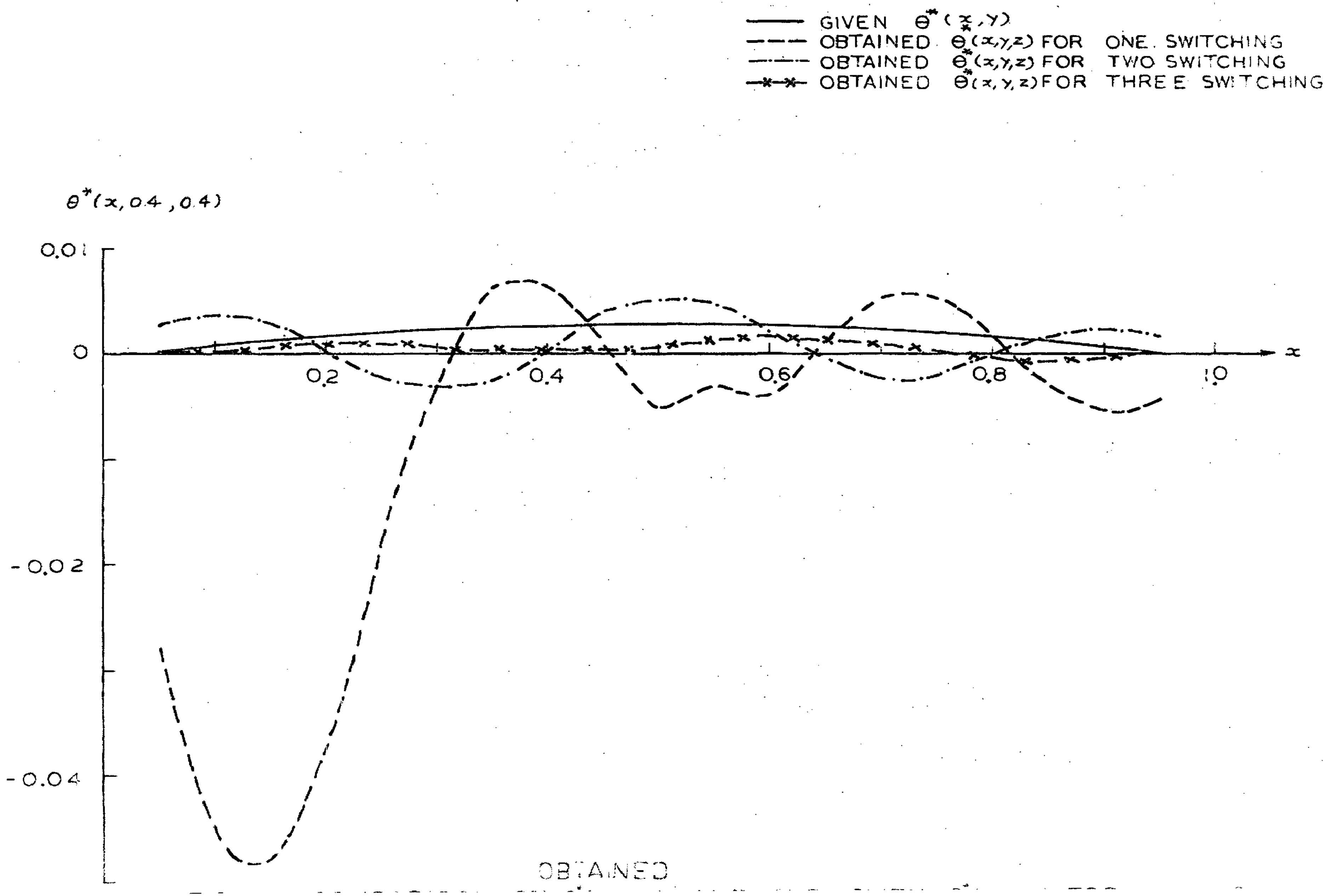


FIG. 5.41 COMPARISON OF OBTAINED $\theta^*(x,y,z)$ AND THE GIVEN θ^* ...


GIVEN θ*(x,y)
OBTAINED θ(x,y,z) FOR ONE SWITCHING
OBTAINED θ*(x,y,z) FOR TWO SWITCHING
OBTAINED θ(x,y,z) FOR THREE SWITCHING
θ*(x,0.4,0.4)
0.01
0
-0.02
-0.04
0.2
0.4
0.6
0.8
1.0
x

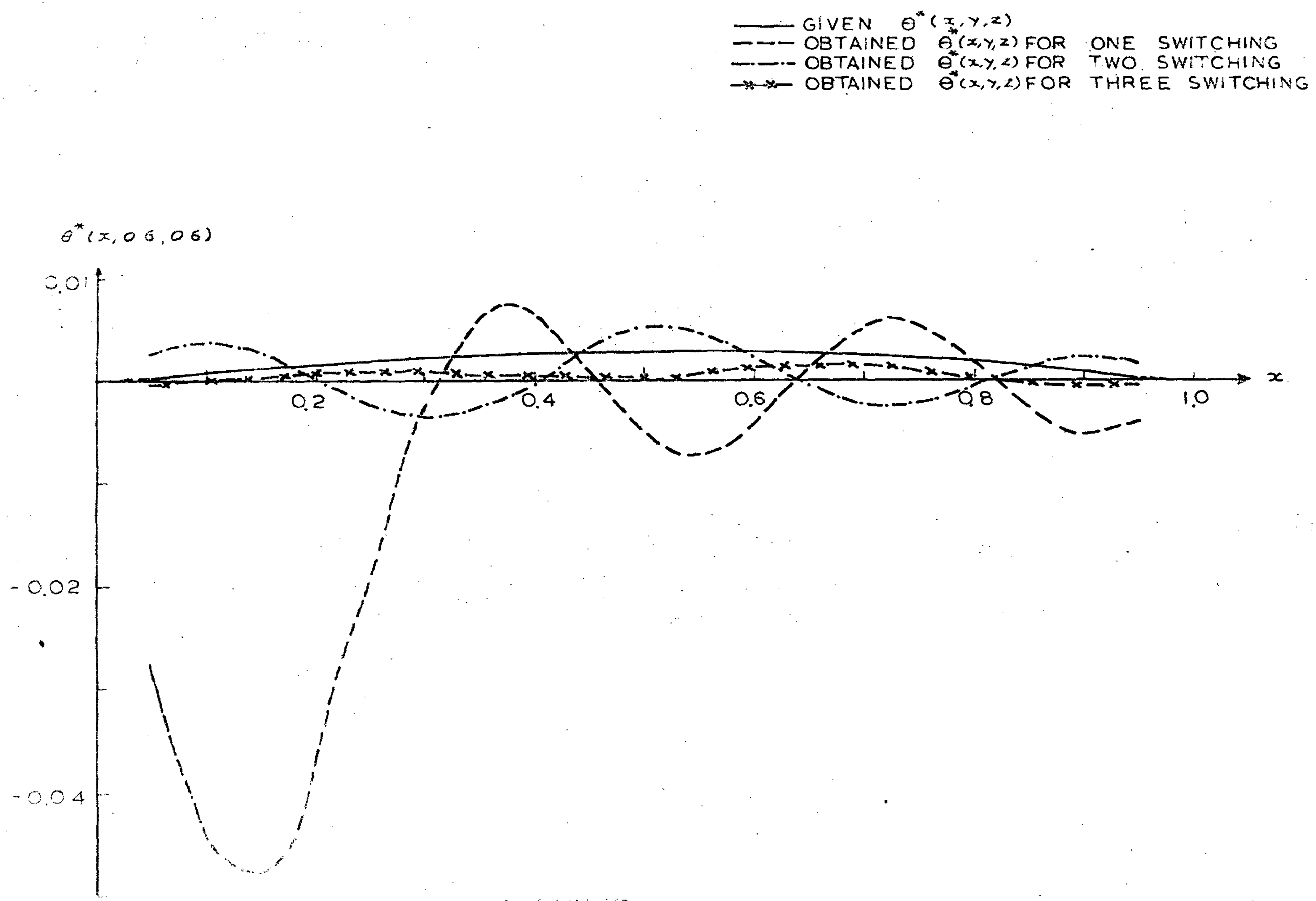


FIG. [illegible] COMPARISON OF OBTAINED $\theta^*(x,y,z)$ AND THE GIVEN $\theta^*(x,y,z)$ AT $y = z = 0.6$

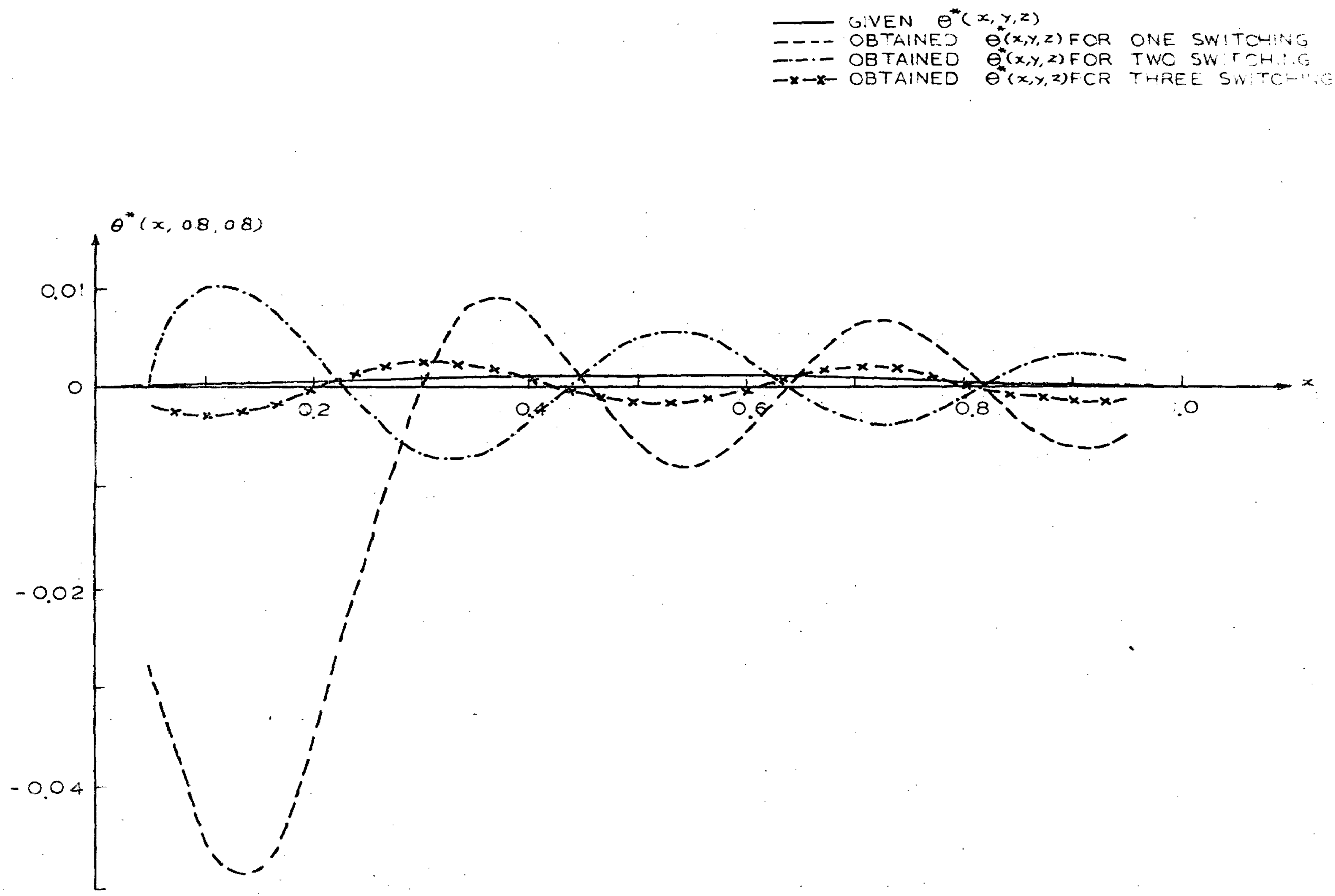


FIG.5.4.4 COMPARISON OF OBTAINED $\theta^*(x,y,z)$ AND THE GIVEN $\theta^*(x,y,z)$ AT $y = z = 0.8$

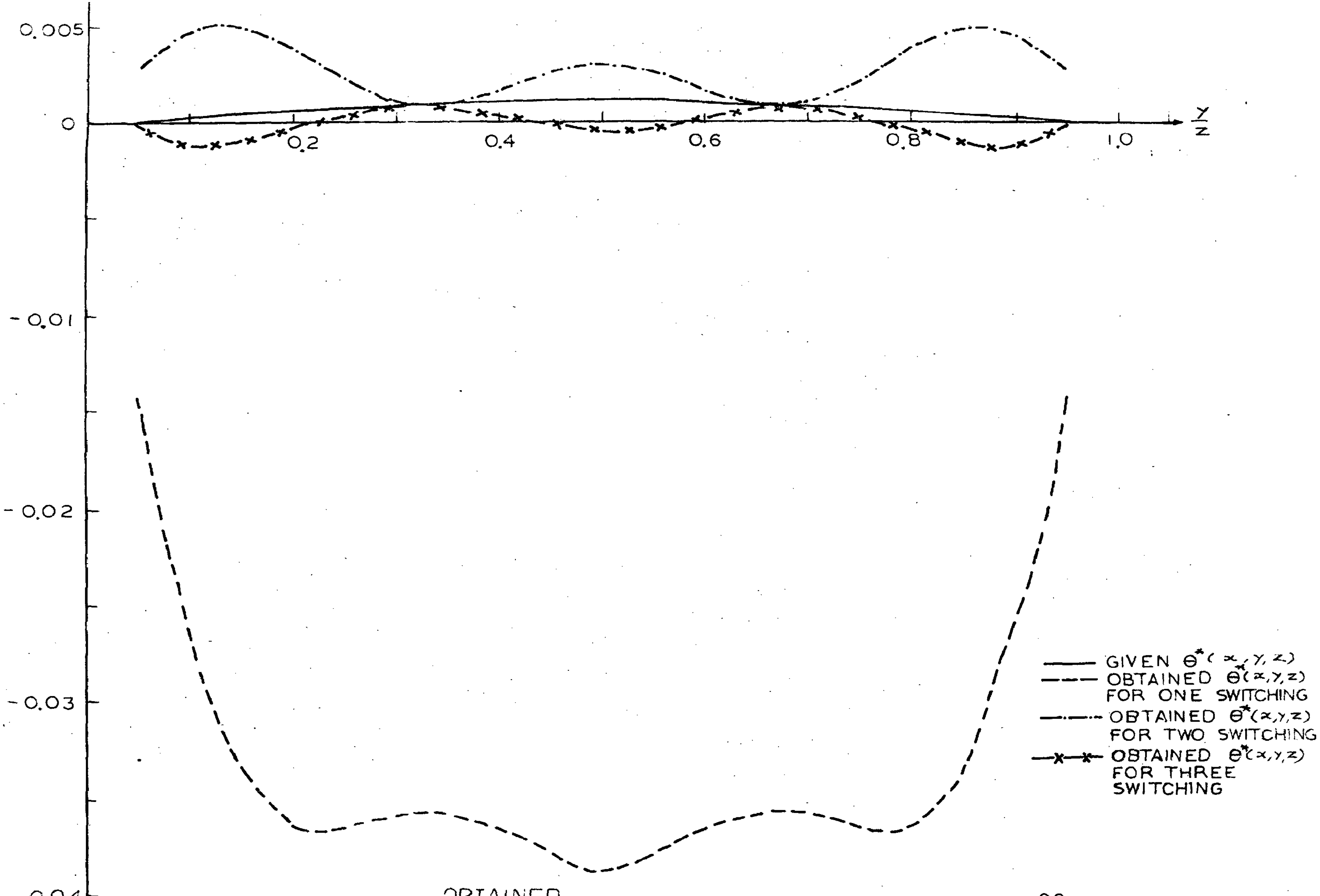


FIG. 5.4.5 COMPARISON OF OBTAINED $\theta^*(x,y,z)$ AND THE GIVEN $\theta^*(x,y,z)$ AT $\frac{x=z=0.2}{x=y=0.2}$

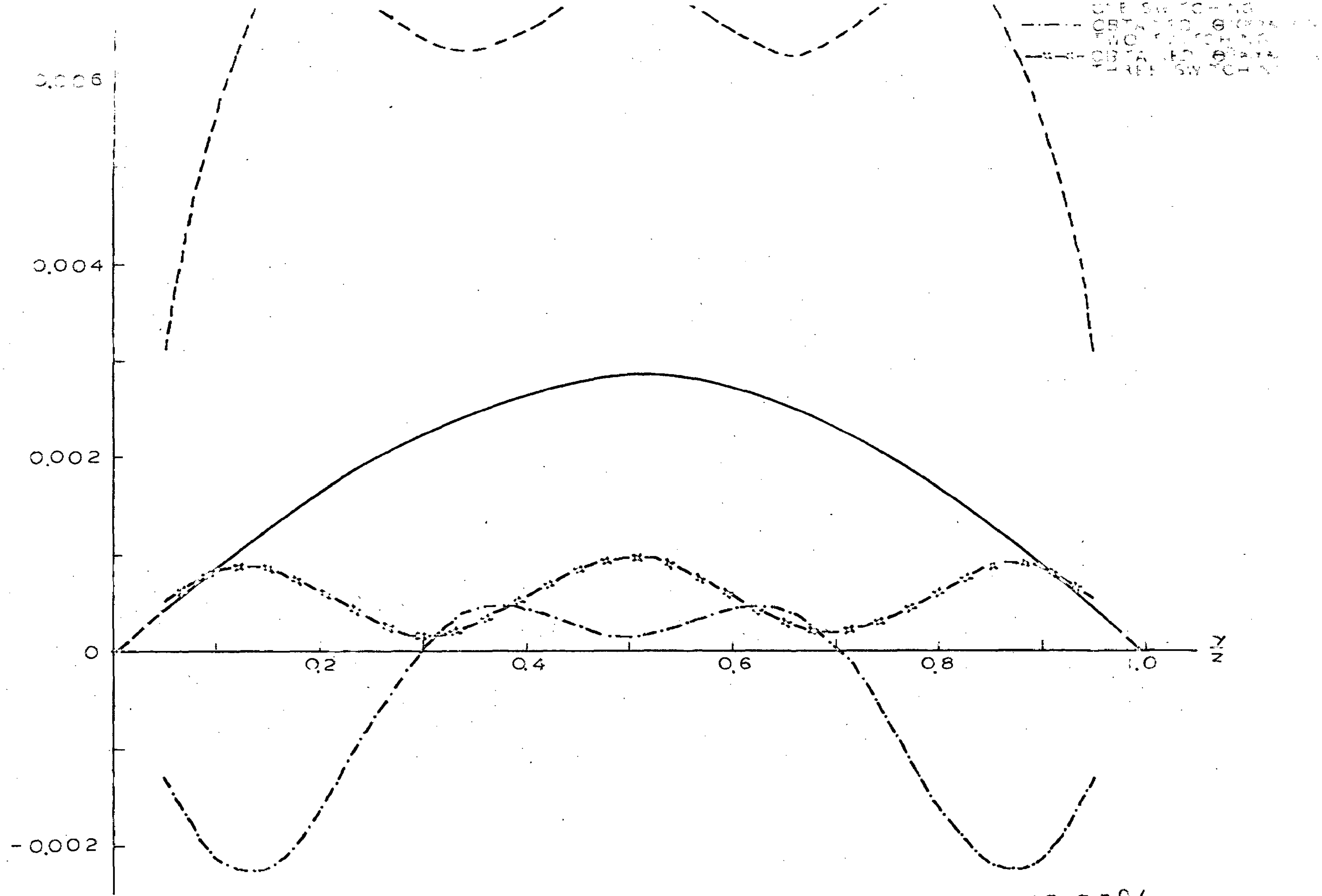


FIG. 5.4.5 COMPARISON OF OBTAINED $\theta(x,y,z)$ AND THE GIVEN $\theta(x,y,z)$ AT $\frac{x = z = 0.4}{x = y = 0.4}$

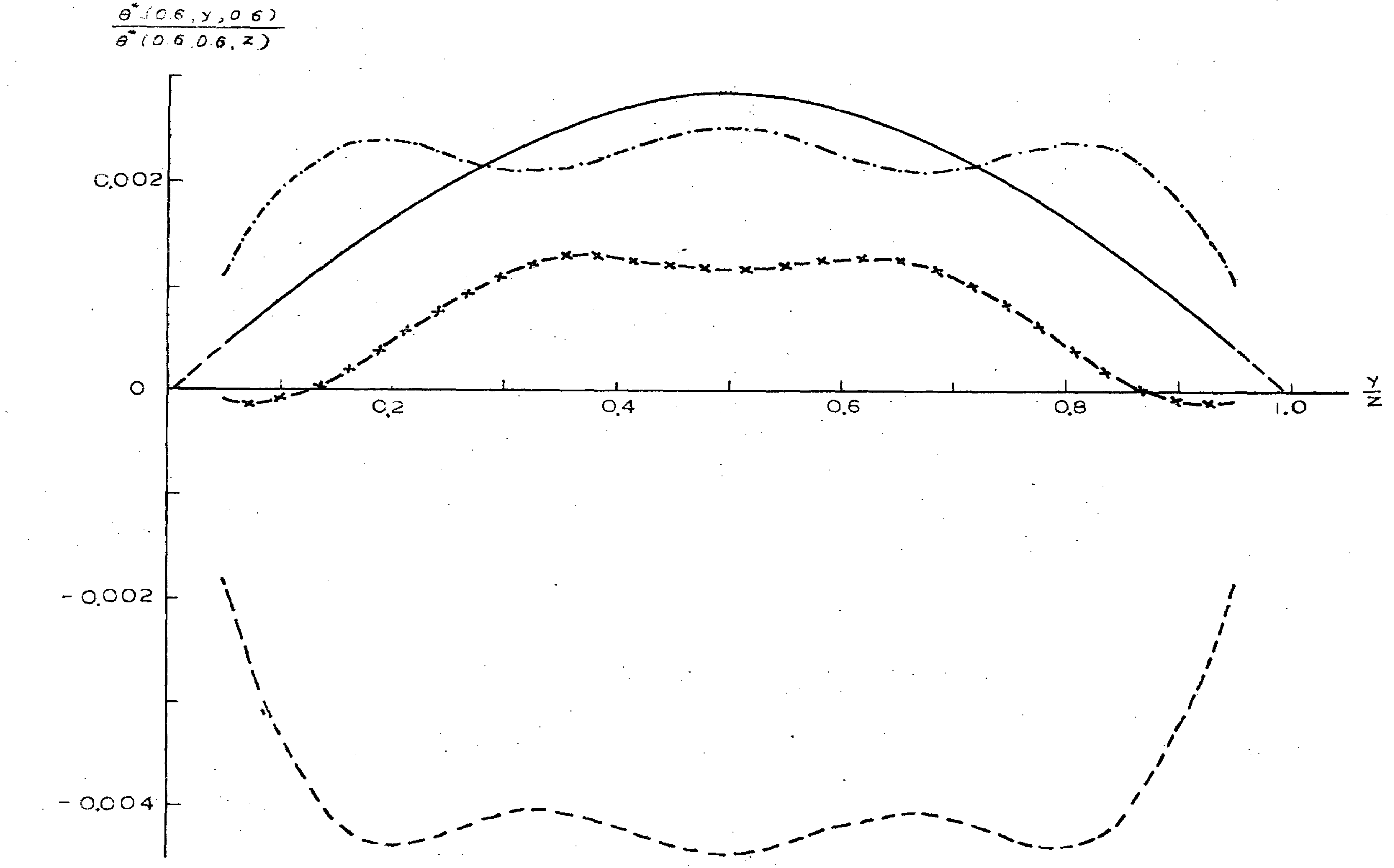


FIG. 5.4.7 COMPARISON OF OBTAINED $\theta^*(x,y,z)$ AND THE GIVEN $\theta^*(x,y,z)$ AT $\frac{x = z = 0.6}{x = y = 0.6}$

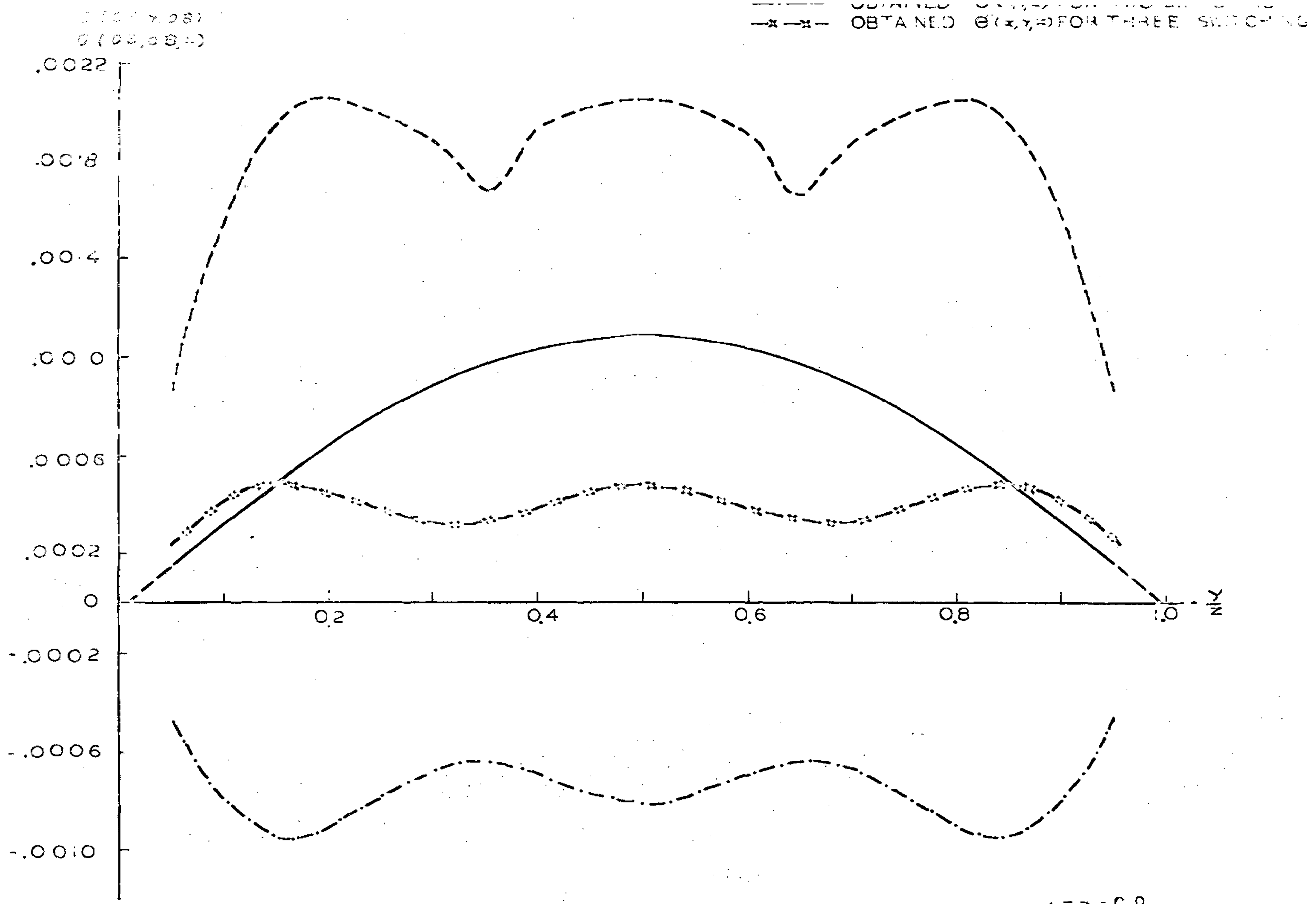


FIG. 5.4.0 COMPARISON OF OBTANED θ″(x,y,z) AND THE GIVEN θ″(x,y,z) AT $\frac{x = z = 0.8}{x = y = 0.8}$

CHAPTER - VI

CONCLUSION

Reviewing the results obtained from computation work and the graphs thus plotted and taking into account first five harmonics only, it is seen that the achieved steady state distribution gradually approaches towards the desired distribution as the number of switchings in the structure of control are increased. Thus if more number of switchings are taken in thestructure of control, steady state temperature response will be still more near to the desired distribution. Also, consideration of more harmonics will give a better access to the distribution.

It is also seen that the transformation given by Mahapatra [13] when utilized for the solution of nonlinear equations(5.1) reduce the computational labour to a large extent. As shown in Chapter III, the computational difficulties came in selecting the initial values of the variables t_1, t_2, t_3,...t_{n-1} while solving non-linear equations (3.17). These timings could be anywhere on the real line, making the range for estimation of initial values very large. But transfer of such equation (in distributed parameter systems) from time domain to S-domain limits the values of the new variables in a range (0, 1). Thus the difficulties experienced during computation are almost eliminated. Above all, these transformations are not only applicable in distributed parameter system, they are also valid for lumped parameter systems. Thus reducing the computation labour there.

5.1 SCOPE OF FURTHER WORK

The geometry of non-linear equations (5.5) show that applications of non-linear programming, can completely eliminate the difficulties left in the computation. If this is achieved it would completely revolutionise the field of distributed parameter systems, which had been drawing less attention due to computational difficulties involved, giving rise to a lot of scope of further work.

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