THE STATISTICAL MECHANICS OF FINANCIAL MARKETS

A THESIS

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CANDIDATE'S DECLARATION

I hereby certify that the work which is being presented in the thesis entitled **THE STATISTICAL MECHANICS OF FINANCIAL MARKETS** in the partial fulfilment of the requirements for the award of the Degree of Doctor of Philosophy and submitted in the Department of Management Studies of Indian Institute of Technology Roorkee, Roorkee is an authentic record of my own work carried out during a period of January 2003 to November 2007 under the supervision of **Dr.J.P.Singh**, Professor, Department of Management Studies, Indian Institute of Technology Roorkee, Roorkee.

The matter presented in this thesis has not been submitted by me for the award of any other degree of this or any other Institute.

S.P.—

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This is to certify that the above statement made by the candidate is correct to the best of my knowledge.

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ABSTRACT

Chapter 1 contains a brief introduction to the subject to put the problems and investigations in proper perspective. It also provides a brief introduction to the study, motivation for the research, objectives of the research and an outline of organization of this research work with a Chapter wise summary.

Chapter 2 reviews the literature relevant to this research. Literature review is focused on the contemporary work being done towards convergence of the disciplines of physics and finance. Relevant literature on quantum mechanics has also been reviewed, in particular, those areas that are relevant to the research being envisaged. Literature on the financial applications of various types of stochastic processes including Gaussian processes and levy processes has also be studied to identify gaps in existing knowledge in the field.

The Black Scholes model of option pricing constitutes the cornerstone of contemporary valuation theory. However, the model presupposes the existence of several unrealistic assumptions including the lognormal distribution of stock market price processes. There, now, subsists abundant empirical evidence that this is not the case. Consequently, several generalizations of the basic model have been attempted with relaxation of some of the underlying assumptions. In Chapter 3 we postulate a generalization that contemplates a statistical feedback process for the stochastic term in the Black Scholes partial differential equation. Several interesting implications of this modification emanate from the analysis and are explored.

The Black Scholes model also assumes constancy of the return on the "hedge portfolio". In Chapter 4, we attempt one such generalisation based on the assumption that the return process on the "hedge portfolio" follows a stochastic process similar to the Vasicek model of short-term interest rates.

The return process of stock markets has also been modeled as a Levy process in several studies relating to valuation of contingent claims. In Chapter 5, we attempt a generalization of such results through a deformation of the underlying Levy process.

The cardinal contribution of physicists to the world of finance came from Fischer Black & Myron Scholes through the option pricing formula which bears their epitaph and which won them the Nobel Prize for economics in 1997 together with Robert Merton. They obtained closed form expressions for the pricing of financial derivatives by converting the problem firstly, to a partial differential equation and then to a heat equation and solving it for specific boundary conditions. In Chapter 6, we apply the well-entrenched group theoretic methods to obtain various solutions of the Black Scholes equation for the pricing of contingent claims. We also examine the infinitesimal symmetries of the said equation and explore group transformation properties. The structure of the Lie algebra of the Black Scholes equation is also studied.

In Chapter 7, we apply the well entrenched methods of quantum mechanics and quantum field theory to the modeling of the financial markets and the behaviour of stock prices. After defining the various constituents of the model including creation & annihilation operators and buying & selling operators for securities, we examine the time evolution of the financial markets and obtain the Hamiltonian for the trading activities of the market. We finally obtain the probability distribution of stock prices in terms of the propagators of the evolution equations.

Chapter 8 is devoted to an empirical study of the Indian capital markets with data over the last ten years and it is shown that stock return processes deviate significant from normality. Performance of R/S analysis on the data also showed that memory effects are prevalent in the price time series with a possibility of nonlinearities and chaos.

Chapter 9 contains major findings and significant contributions of the research duly summarized followed by the set of recommendations. The thesis finally ends with the limitations of the study and suggestions for further research.

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CHAPTER 1
INTRODUCTION

1. 1 INTRODUCTION

The specialty of "physics" is the study of interactions between the various manifestations of matter and its constituents. The development of this subject over the last several centuries has led to a gradual refining of our understanding of natural phenomena. Accompanying this has been a spectacular evolution of sophisticated mathematical tools for the modeling of complex systems. These analytical tools are versatile enough to find application not only in point processes involving particles but also aggregates thereof leading to field theoretic generalizations and condensed matter physics.

The standard route to pricing of derivatives and similar financial assets is through the stochastic calculus and Ito's Lemma that leads to the celebrated Black Scholes formula [1-2] for option pricing. A comprehensive theory of quantum mechanics has also been developed as a theory of 'random walks' [3]. The contemporary candidate for a unified theory of the fundamental forces of Nature (i.e. string theory) also makes extensive use of random surfaces [4]. Modeling of non relativistic quantum mechanics as energy conserving diffusion processes is, by now, well known [5]. Unification of the general theory of relativity and quantum mechanics to enable a consistent theory of quantum gravity has also been attempted on "stochastic spaces" [6].

Time evolution of stock prices has been, by suitable algebraic manipulations, shown to be equivalent to a diffusion process [2].

Contemporary empirical research into the behavior of stock market price /return patterns has found significant evidence that financial markets exhibit the phenomenon of anomalous diffusion, primarily super diffusion, wherein the variance evolves with time according to a power law t^{α} with $\alpha > 1.0$. The standard technique for the study of super diffusive processes is through a stochastic process that evolves according to a Langevin equation and whose probability distribution function satisfies a nonlinear Fokker Planck equation [7].

There is an intricate yet natural relationship between the power law tails observed in stock market data and probability distributions that emanate as the solution of the Fokker Planck equation. The Fokker Planck equation is known to describe anomalous diffusion under time evolution. Empirical results [8-11] establish that temporal changes of several financial market indices have variances that that are shown to undergo anomalous super diffusion under time evolution.

The theory of stochastic processes thus constitutes the 'golden thread' that provides the connection between the (hitherto) diverse disciplines of physics and finance.

Though at a nascent stage, the winds of convergence of physics and finance are unmistakably perceptible with several concepts of fundamental physics like quantum mechanics, field theory and related tools of non-commutative probability, gauge theory, path integral etc. being applied for pricing of contemporary financial products and for explaining various phenomena of financial markets like stock price patterns, critical crashes etc. [12-22]. The origin of the association between physics and finance, though, can be traced way back to the seminal works of Pareto [23] and Batchlier [24], the former being instrumental in establishing empirically that the distribution of wealth in several nations follows a power law with an exponent of 1.5, while the latter pioneered the modeling of speculative prices by the random walk and Brownian motion. The cardinal contribution of physicists to the world of finance came from Fischer Black & Myron Scholes through the option pricing formula [1-2] which bears their epitaph and which won them the Nobel Prize for economics in 1997 together with Robert Merton [25]. They obtained closed form expressions for the pricing of financial derivatives by converting the problem to a heat equation and then solving it for specific boundary conditions.

1. 2 PRESENT STATE OF KNOWLEDGE, LITERATURE REVIEW & ISSUES & PROBLEMS RELEVANT TO THE STUDY

While a significant majority of contemporary research in the physical sciences is targeted at evolving a unification of the four fundamental forces of Nature viz electromagnetic, gravitational, electroweak and strong interactions, a perusal of recent literature shows that work is also focused on the convergence of the physical and mathematical sciences with finance and economics. Reported literature in this regard facilitates the inference of the *possible* existence of an underlying symmetry between financial markets and the fundamental theories of physics. This opens the way to using some of the well developed physical and geometrical methods in the analysis of financial markets. Attempts have also been made to develop the dynamics of financial markets in the Lagrangian and Hamiltonian formalism. Sparse work has also been done in applying the maxims of quantization, to the economics of financial markets.

One of the most exhaustive set of studies on stock market data in varying dimensions has been reported in [26-30]. In [30], a phenomenological study was conducted of stock price fluctuations of individual companies using data from two different databases covering three major US stock markets. The probability distributions of returns over varying timescales ranging from 5 min. to 4 years were examined. It was observed that for timescales from 5 minutes upto 16 days the tails of the distributions were well described by a power law decay. For larger timescales results consistent with a gradual convergence to Gaussian behaviour was observed. In another study [26] the probability distributions of the returns on the S & P 500 were computed over varying timescales. It was, again, seen that the distributions were consistent with an asymptotic power law behaviour with a slow convergence to Gaussian behaviour. Similar findings were obtained on the analysis of the NIKKEI and the Hang –Sang indices [26].

As mentioned above, research into the behavior of stock market price /return patterns has found significant evidence that financial markets exhibit the phenomenon of anomalous diffusion, primarily super diffusion, wherein the variance evolves with time according to a power law t^{α} with $\alpha > 1.0$.

Anomalous diffusion is a hallmark of several intensively studied physical systems. It is observed, for example, in the chaotic dynamics of fluid in rapidly rotating annulus [31], conservative motion in a periodic potential [32], transport of fluid in a porous media [33], percolation of gases in porous media [34], crystal growth spreading of thin films under gravity [35], radiative heat transfer [36], systems exhibiting surface to surface growth [37] and so on.

Several analogies between physical systems and financial processes have been explored in the last decade, some of which have already been mentioned above. Perhaps, the most striking one is that between financial crashes witnessed in stock markets and critical phenomena like phase transitions.

Furthermore, with the rapid advancements in the evolution and study of disordered systems and the associated phenomena of nonlinearity, chaos, self organized criticality etc., the importance of generalizations of the extant mathematical apparatus to enhance its domain of applicability to such disordered systems is cardinal to the further development of science and has been attempted in various directions.

A considerable amount of work has already been done and success achieved in the broad areas of q-deformed harmonic oscillators [38], representations of q-deformed rotation and Lorentz groups [39-40]. q-deformed quantum stochastic processes have also been studied with realization of q-white noise on bialgebras [41]. Deformations of the Fokker Planck's equation [42] and Levy processes [43-44] have also been analyzed and results reported.

1. 3 RESEARCH OBJECTIVES

The main objectives of this research study are:-

The overriding objective of this research project is to carry this convergence/unification of physics and finance program further through a study of the symmetry groups of the dynamical equations relevant to financial processes and, as mentioned above, intertwining the physics & finance through stochastic processes in order to facilitate (i) the evolution of a model of financial markets amenable to the quantum mechanical framework and (ii) the generalizations of extant results to enhance their domain of applicability.

Specifically, efforts have been made to attempt

(a) Generalization of the Black Scholes equation by introducing a stochastic process with statistical feedback as a model for stock market returns;

(b) Generalization of financial dynamics of the stock price process as a deformed Levy process;

(c) Generalization of the Black Scholes equation by introducing a stochastic return process in lieu of the risk free returns in the Black Scholes partial differential equation;

(d) Modeling of the financial markets within the framework of quantum mechanics;

(e) Construction & Study the properties of the Lie algebra being the underlying symmetry of the Black Scholes partial differential equation that represents the dynamics of a financial derivative and to explore and interpret new solutions of the said equation;

(f) Empirical study of the Indian capital markets with reference to the normality of return process, existence of significant memory effects and possibility of nonlinear and chaotic behavior.

1. 4 SIGNIFICANT CONTRIBUTIONS OF THE RESEARCH

The following are some of the significant contributions emanating from the research work:-

(a) The Black Scholes approach for the pricing of financial contingent claims works under several restricted, rigid and unrealistic assumptions. Generalization of the Black Scholes equation has been attempted in this study by:-

(i) Introducing a stochastic process with statistical feedback as a model for stock market returns. This model can embrace in its ambit possible nonlinearities and chaotic behaviour in stock price patterns through the deformation parameter;

(ii) Introducing a stochastic return process in lieu of the risk free returns in the Black Scholes partial differential equation and thus considering the Black Scholes equation as a partial differential equation in two stochastic processes;

(b) A toy model of the financial markets has been evolved using the conventional machinery of quantum mechanics and operators pertaining to various trading activities obtained. Quantum dynamical equations are solved and the lognormal distribution of stock prices has been obtained as a fallout vindicating the compatibility of the model with the more well known stochastic models;

(c) The symmetry group of the Black Scholes equation has been obtained and the properties of the Lie algebra of the Black Scholes partial differential equation that represents the dynamics of a financial derivative are studied and new solutions of the said equation are obtained and interpreted;

(d) An empirical study of the Indian capital markets was also conducted with data over the last ten years and it was shown that stock return processes deviate significant from normality. Performance of R/S analysis also showed that memory effects are prevalent in the price time series with a possibility of nonlinearities and chaos.

(e) A generalization of financial dynamics of the stock price process as a deformed Levy process has also been attempted.

1. 5 ORGANIZATION OF THIS THESIS

Brief outlines of different chapters are given below:-

Chapter 1 contains a brief introduction to the subject to put the problems and investigations in proper perspective. It also provides a brief introduction to the study, motivation for the research, objectives of the research and an outline of organization of this research work with a Chapter wise summary.

Chapter 2 reviews the literature relevant to this research. Literature review is focused on the contemporary work being done towards convergence of the disciplines of physics and finance. Relevant literature on quantum mechanics has also been reviewed, in particular, those areas that are relevant to the research being envisaged. Literature on the financial applications of various types of stochastic processes including Gaussian processes and levy processes has also be studied to identify gaps in existing knowledge in the field.

The Black Scholes model of option pricing constitutes the cornerstone of contemporary valuation theory. However, the model presupposes the existence of several unrealistic assumptions including the lognormal distribution of stock market price processes. There, now, subsists abundant empirical evidence that this is not the case. Consequently, several generalizations of the basic model have been attempted with relaxation of some of the underlying assumptions. In Chapter 3 we postulate a generalization that contemplates a statistical feedback process for the stochastic term in the Black Scholes partial differential equation. Several interesting implications of this modification emanate from the analysis and are explored.

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In Chapter 7, we apply the well entrenched methods of quantum mechanics and quantum field theory to the modelling of the financial markets and the behaviour of stock prices. After defining the various constituents of the model including creation & annihilation operators and buying & selling operators for securities, we examine the time evolution of the financial markets and obtain the Hamiltonian for the trading activities of the market. We finally obtain the probability distribution of stock prices in terms of the propagators of the evolution equations.

Chapter 8 is devoted to an empirical study of the Indian capital markets with data over the last ten years and it is shown that stock return processes deviate significant from normality. Performance of R/S analysis on the data also showed that memory effects are prevalent in the price time series with a possibility of nonlinearities and chaos.

Chapter 9 contains major findings and significant contributions of the research duly summarized followed by the set of recommendations. The thesis finally ends with the limitations of the study and suggestions for further research.

1. 6 CONCLUSIONS

The following are some of the important outcomes of this research work:-

(a) Closed form expressions have been obtained for the price of a European call option by modifying the Black Scholes formulation

(i) By generalizing the stock return process to a probability dependent deformed Brownian motion that could accommodate "statistical feedback" processes and, thereby, account for the fat tails usually observed in stock market price distributions. It is seen that that in the standard case the exponential is linear in W and the stock price, therefore, is a monotonically increasing function of W. Hence, the condition $S_t - E > 0$ is satisfied for all values of W that exceed a threshold value. However, in this model, consequent to the noise induced drift; the exponential in the stock price process is now a quadratic function of the deformed Brownian motion U. We, therefore, have two roots of U that meet the condition $S_t - E = 0$. Accordingly, there will exist an interval (U_1, U_2) within which the inequality $S_t - E > 0$ will hold. Furthermore, as $q \to 0$, $U_2 \to \infty$ thereby recovering the standard case.

(ii) By generalizing the Black Scholes option pricing partial differential equation to two stochastic variables by including therein a stochastic return process for the "hedge portfolio" returns.

(b) Pricing of financial contingent claims has also been explored when the distribution of the underlying asset is a deformed Levy process.

(c) A "toy model" of the financial markets has been constructed within the quantum mechanical framework, various operators signifying the market processes have been constructed and the market dynamics explored. We derive the probability distribution of stock prices in market equilibrium and show that the prices follow a lognormal distribution, thereby vindicating the efficacy of this model under suitable assumptions as to the quantum mechanical states and amplitudes..

(d) Solutions of the Black Scholes equation have been obtained from symmetry considerations and their properties studied and with the relevant structure Lie groups.

(e) The various features of the logarithmic return spectrum of the Indian stock markets are examined, performing thereon the various statistical tests for the normality of data like chi-square, ANOVA. The possible existence of dependencies and memory effects in the return processes is also examined. In particular, the rescaled range analysis is carried out to compute the Hurst's exponent. It is seen that there is unambiguous evidence to the effect that the returns deviate significantly from normal behaviour. There is also evidence of the existence of memory effects and consequential nonlinearity.

CHAPTER 2
LITERATURE REVIEW

2.1 INTRODUCTION

The pioneering work in adapting statistical methods to the analysis of stock market behaviour and return patterns is credited to the French mathematician Louis Bachelier [45], who applied the formal tenets of probabilistic calculus to study the price movements of stocks, bonds, futures, and options in the relevant trading marketplaces (stock exchanges/commodities exchanges etc.). Bachelier's paper [45], in retrospection, was a work of incredible foresight, many years ahead of its time. It completely revolutionized the study of finance, paving the way for the origin of a distinct branch of study called "quantitative finance". In fact, not only this, another far reaching implication of Batchlier's work was the recognition that the random walk process (later formalized by Weiner) [46] is Brownian motion. Einstein rediscovered this result several years later [47].

Bachelier's thesis was, undoubtedly, revolutionary, but received little attention for several decades. During the decades of the 1920s through the 1940s, market analysis was dominated by fundamental analysts (followers of Graham and Dodd) and technical analysts. It was only in the 1950s that the quantitative analysts (followers of Bachelier) became active and came to the fore.

Little work was done in the application of statistical analysis to study stock market behavior until the late 1940s. However, thereafter, progress was rapid. Cootner compiled his classic volume The "Random Character of Stock Market Prices" [48], published in 1964 which strongly facilitated the progression of quantitative analysis.

Cootner's work [48] lays down the premises for the development of the "Efficient Market Hypothesis" that was formally propounded by Fama [49-51] in the 1960s.

2.2 STOCK PRICES & RANDOM WALKS

Modeling of stock prices as a "Random Walk" is formalized by Osborne in his paper on Brownian motion [52]. Osborne models stock market behaviour as a process in which changes in stock market prices can be equivalent to the movement of a particle in a fluid, commonly called Brownian motion [53]. He does so on the premises of a number of assumptions and drawing conclusions from these results. Briefly stated, his assumptions were that

(a) Minimum price movements are discrete e.g. one-eighth of a dollar;

(b) The number of transactions per day is finite;

(c) "Market Price" and "Investor Value" of traded instruments are related and that this relationship between "price" and "value" is the prime determinant of market returns;

(d) Given two securities with different expected returns, the rational investor would invest in the stock with the higher expected return;

(e) Buyers and sellers are unlikely to trade unless there is equality of opportunity to profit. In other words, the buyer cannot have an advantage over the seller or vice versa, if a transaction is to be accomplished.

These assumptions, taken in consortium lead to the radical conclusion that stock prices would be normally distributed. Assumption (e) would subsist in the marketplace because investors are most concerned with paying the right price for value (Assumption c), and, given two variables with expected values, investors will pick the one with the higher expected return Assumption (d). As a result, a buyer and seller find a particular mutually advantageous. In other words, because investors are able to rationally equate price and value, they will trade at the equilibrium price based on the information available at that time. The sequence of price changes is independent, because price is already equated to available information. This, further, implies that because price changes are independent (i.e., they are a random walk), we would expect the distribution of changes to be normal, with a stable mean and finite variance as mandated by the Central Limit Theorem of probability calculus, or the Law of Large Numbers[54].

Thus, the fallout of Osborne's work was, in essence, that because the stock markets are large systems that have a large number of degrees of freedom (or investors), current prices must reflect the information everyone already has. Changes in price would come only from unexpected new information. Therefore, stock prices should behave as independent identically distributed (IID) variables. This paved the way for application of a vast array of statistical tools for the analysis and modeling of such prices and the consequential returns.

2.3 EFFICIENT MARKET HYPOTHESIS

Fama [49-50] in 1965 formalized the work of Osborne into the Efficient Market Hypothesis (EMH), which states, in technical terms, that the market is a martingale, or "fair game"; that is, information cannot be used to profit in the marketplace.

The theory of "efficient markets." constitutes the bedrock of quantitative capital market theory, and in the last four decades, research in capital market has depended on it. One of the important consequences of the EMH is that it justifies the use of probability calculus in analyzing capital markets.

In this context, it is pertinent to mention that if the stock prices show nonlinear characteristics, then the use of standard statistical analysis can give misleading results, particularly if a random walk model is used.

The cardinal philosophy of EMH is that all assets are priced so that all public information, both fundamental and price history, is already discounted in the prevailing market prices. Prices, therefore, move only when new information is received. One cannot outperform an efficient market by gaming because not only do the prices reflect known information, but the large number of investors will ensure that the prices are fair. Investors are considered rational in efficient markets so that they know, in a collective sense, what information is important and what is not. After digesting the information and assessing the risks involved, the collective consciousness of the market finds an equilibrium price. Essentially, the EMH says that the market is made up of too many people to be wrong.

If the aforesaid assumption is true, then today's change in price is caused only by today's unexpected news. Yesterday's news is no longer important, and today's return is unrelated to yesterday's return; the returns are independent. If returns are independent, then they are random variables and follow a random walk. If enough independent price changes are collected, in the limit (as the number of observations approaches infinity), the probability distribution becomes the normal distribution. This assumption regarding the normality of returns enables a large spectrum of statistical tools, tests and modeling techniques to be adopted for studying price behaviour of assets. This is the random walk version of the EMH.

It must, however, be emphasized here that, technically, market efficiency does not necessarily imply a random walk, but a random walk does imply market efficiency. Therefore, the assumption that returns are normally distributed is not necessarily implied by efficient markets. But all the same, the EMH in any version says that past information does not affect market activity, once the information is generally known.

This independence assumption between market moves naturally lends itself first to a random walk theory, and then to more general martingale and submartingale models.

The concept of efficient markets gradually got rooted into capital market theory and went on contradict fundamental analysis as well as technical analysis. In the initial stages, EMH propounded that past price information was not related to future prices. However, in 1973 Lorie and Hamilton [55] remarked that.

The assertion that a market is efficient is vastly stronger than the assertion that successive changes in stock prices are independent of each other. The latter assertion—"the weak form" of the "efficient market hypothesis"—merely says that current prices of stocks fully reflect all that is implied by the historical sequence of prices so that a knowledge of that sequence is of no value in forming expectations about future prices. The assertion that the market is efficient implies that current prices reflect and impound not only all of the implications of the historical sequence of prices, but also all that is knowable about the companies whose stocks are being traded ... it suggests the fruitlessness of efforts to earn superior rates of return by the analysis of all public information."

This attack on fundamental analysis was not well received by a significant investment community, and the EMH was split into the "weak" and "strong" forms [56]. The strong form suggested that fundamental analysis was a useless activity, because prices already reflected "all that is knowable," or all public and private (insider) information whereas the weak form postulated that prices reflect only past price histories.

As a compromise, the "semistrong" form was articulated [57]. In the semistrong version of the EMH, prices reflect all "public" information. Security analysts, using Graham-and-Dodd techniques [58], formulate value based on information that is available to all investors. A large number of independent estimates results, in. a "fair" value by the aggregate market. Analysts, thus, become the reason for making markets efficient. Fundamental analysts form a fair price by consensus.

The semistrong form of the EMH was much more acceptable to the investment community because it said that markets were efficient because of security analysis, not in spite of it. In addition, the semistrong form implied that changes in stock prices were random because of influences outside the price series itself. That is, price changes were random because of the evaluation of the changing fundamentals of a company, caused by both micro- and macro-economics. By the mid-1970s, the semistrong version of the EMH was the generally accepted theory.

In conclusion, semistrong version of EMH claims that markets are efficient because prices reflect all public information. A weak - form efficient market is one in which the price changes are independent and may be a random walk.

2.4 EMPIRICALSTUDIES ON STOCK MARKET PRICES

Before the EMH was even fully formed, exceptions to the normality assumption were being found. When, in 1964, Osborne [52] plotted the density function of stock market returns, he found them only to be "approximately normal" with extra observations in the tails of the distribution (kurtosis). The tails were fatter than they should be.

The existence of "fat tails" was almost universally acknowledged by 1964but the implications of this departure from normality were widely debated. Mandelbrot's chapter in the Cootner volume [48] suggested that returns may belong to a family of "Stable Paretian" distributions, which are characterized by undefined or infinite variance. Cootner contested the suggestion, which would have seriously weakened the Gaussian hypothesis, and offered an alternative in which sums of normal distributions may result in a distribution that looks fat-tailed but is still Gaussian.

In 1965, in an extensive study, Fama [49, 50] observed that returns were negatively skewed: more observations were in the left-hand (negative) tail than in the right-hand tail, the tails were fatter and the peak around the mean was higher than predicted by the normal distribution. This statistical condition is called "leptokurtosis."

In 1970, Sharpe [59] compared annual returns to the normal distribution, and found that extremal values of stock returns occurred much more often that that predicted by the normal distribution.

Turner and Weigel [60] in 1990 performed an extensive study of volatility, using daily S&P index returns from 1928 through 1990. They also obtained similar results. Table 2.1 summarizes their findings. They found that "daily return distributions for the Dow Jones and S&P 500 are negatively skewed and contain a larger frequency of returns around the mean interspersed with infrequent very large or very small returns as compared to a normal distributions."

A graph of the frequency distribution of returns of the 5-day logarithmic first difference in prices for the S&P 500 from January 1928 to December 1989 is presented in Figure 2.1 (a) as adapted from the above study [60]. The returns have been normalized so that they have a zero mean and a standard deviation of one. A frequency distribution for an equal number of Gaussian random numbers is also shown. The high peak and fat tails can be clearly seen. In addition, the return data have a number of four- and five-sigma events in both tails. Figure 2.1 (b) illustrates the differences between the two curves. The negative skewness can be seen at the count three standard deviations below the mean. The stock market's probability of a three-sigma event is roughly twice that of the Gaussian random numbers.

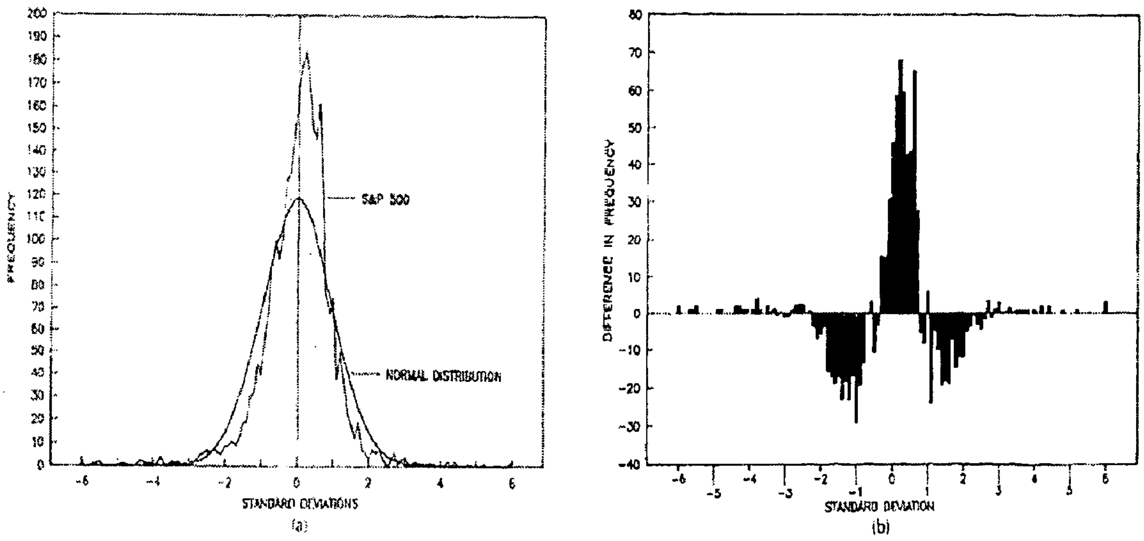


Fig 2.1

A more explicit representation of the above features can be seen in Table 2.1 which is also adapted from the same study [60].

Table 2.1 Volatility Study: Daily S&P 500 Returns, 1/28-12/89

Decade	Mean	Standard Deviation	Skewness	Kurtosis
1920s	0.0322	1.6460	-1.4117	18.9700
1930s	-0.0232	1.9150	0.1783	3.7710
1940s	0.0100	0.8898	-0.9354	10.8001
1950s	0.0490	0.7050	-0.8398	7.8594
1960s	0.0172	0.6251	-0.4751	9.8719
1970s	0.0062	0.8652	0.2565	2.2935
1980s	0.0468	1.0989	-3.7752	79.6573
Overall	0.0170	1.1516	-0.6338	21.3122

Adapted from Turner and Weigel (1990) [60]

In another recent study of quarterly S&P 500 returns, from 1946 through 1988, Friedman and Laibson [61] point out that "in addition to being leptokurtotic, large movements have more often been crashes than rallies" and significant leptokurtosis "appears regardless of the period chosen."

Sterge [62], in a study of financial futures prices of Treasury Bond, Treasury Note, and Eurodollar contracts, found the same leptokurtotic distributions. Sterge notes that "very large (three or more standard deviations from the norm) price changes can be expected to occur two to three times as often as predicted by normality."

These studies offer ample evidence that U.S. stock market returns are not normally distributed. If stock returns are not normally distributed, then much statistical analysis, particularly diagnostics such as correlation coefficients and t-statistics [63], is seriously weakened and may give misleading answers. The case for a random walk in stock prices is also seriously weakened.

It is pertinent to mention that tests of normality have also been conducted in several stock markets of other countries including the United Kingdom [64], Japan [65], Hong Kong [66] & India [67].

Tests on normality of stock prices have also be attempted from a different perspective viz. the scaling of volatility or standard deviation. The variance is stable and finite for the normal distribution alone. In fact, if the capital markets fall into the "Stable Paretian" family of distributions, as postulated by Mandelbrot, they would have infinite variance.

Studies of volatility have tended to focus on stability over time. In the normal distribution, the variance of $n-day$ returns should be n times the daily return i.e. the variance scales in proportion to time. This constitutes a useful test for the normality of the underlying data. This scaling feature of the normal distribution is referred to as the $T^{1/2}$ Rule, where T is the increment of time.

The investment community often "annualizes" risk, using the $T^{1/2}$ Rule. Annual returns are usually reported, but volatility is calculated based on monthly returns. The monthly standard deviation is therefore converted to an annual number by multiplying it by the square root of 12—a perfectly acceptable method, if the distribution is normally distributed.

Studies show that standard deviation does not scale according to the $T^{1/2}$ Rule. Turner and Weigel [60] found that monthly and quarterly volatility were higher than it should be, compared to annual volatility but daily volatility was lower than it should be.

Studies of volatility have also been conducted using the autoregressive conditional heteroskedastic (ARCH) model of Engle [68]. This model sees volatility as conditional upon its previous level. Thus high volatility levels are followed by more high volatility, while low volatility is followed by more low volatility. This is consistent with Mandelbrot's observation [69] that the size of price changes (ignoring the sign) seems to be correlated. Statistical evidence compiled by Engle and LeBaron [70] among others supports the ARCH model. In recent years this has led to increasing recognition that standard deviation is not a standard measure, at least over the short term. ARCH also results in fat tailed probability distributions. Therefore, ARCH has had the most impact upon option pricing and technical trading rules.

In addition, there have been numerous market anomalies in which excess nonmarket returns has be achieved, contrary to the "fair game" of the semi-strong EMH. In the stock market, these include the small firm effect [71], the low P/E effect, and the January effect. Rudd and Clasing [72] document excess returns realized from nonmarket-factor returns generated by the BARRA El six-factor risk model. This CAPM [73] -based model found that four sources of nonmarket risk (market variability, low valuation and unsuccessful, immaturity and smallness, and financial risk) all offered the opportunity for significant nonmarket returns. Rudd and Clasing [72] say that these factor returns are "far from random," suggesting that the semistrong EMH is flawed. These anomalies have long suggested that the current paradigm requires an adjustment that takes these anomalies into account.

2.5 PHYSICS & THE MODELLING OF FINANCIAL PROCESSES

Though at a nascent stage, the winds of convergence of physics and finance are unmistakably perceptible with several concepts of fundamental physics like quantum mechanics , field theory and related tools of non-commutative probability, gauge theory, path integral etc. being applied for pricing of contemporary financial products and for explaining various phenomena of financial markets like stock price patterns, critical crashes [12-22] etc. The origin of the association between physics and finance, though, can be traced way back to the seminal works of Pareto [23] and Batchlier [24], the former being instrumental in establishing empirically that the distribution of wealth in several nations follows a power law with an exponent of1.5, while the latter pioneered the modeling of speculative prices by the random walk and Brownian motion. The cardinal contribution of physicists to the world of finance came from Fischer Black & Myron Scholes through the option pricing formula [1] which bears their epitaph and which won them the Nobel Prize for economics in 1997 together with Robert Merton [25]. They obtained closed form expressions for the pricing of financial derivatives by converting the problem to a heat equation [74] and then solving it for specific boundary conditions.

Physics is the study of connections between an assortment of expression of matter and its constituents. The development of this subject over the last several centuries has led to a gradual refining of our understanding of natural phenomena. Accompanying this has been a amazing evolution of sophisticated mathematical tools for the modeling of complex systems [75-76]. These analytical tools are adaptable enough to find application not only in point processes involving particles but also aggregates thereof leading to field theoretic generalizations and condensed matter physics [78].

Stock market phenomena are assumed to result from complicated interactions among many degrees of freedom, and thus they were analyzed as random processes and one could go to the extent of saying that the Efficient Market Hypothesis [78-79] was formulated with one primary objective – to create a scenario which would justify the use of stochastic calculus [80] for the modeling of capital markets.

The theory of stochastic processes [81] constitutes the "golden thread" that unites the disciplines of physics and finance. Modeling of non relativistic quantum mechanics as energy conserving diffusion processes is, by now, well known [5]. Unification of the general theory of relativity and quantum mechanics to enable a consistent theory of quantum gravity has also been attempted on "stochastic spaces" [6]. Time evolution of stock prices has been, by suitable algebraic manipulations, shown to be equivalent to a diffusion process [2].

Anomalous diffusion is a hallmark of several intensively studied physical systems. It is observed, for example, in the chaotic dynamics of fluid in rapidly rotating annulus [31], conservative motion in a periodic potential [32], transport of fluid in a porous media [33], percolation of gases in porous media [34], crystal growth spreading of thin films under gravity [35], radiative heat transfer [36], systems exhibiting surface to surface growth [37] and so on.

A. Ott el al., [31], have observed anomalously enhanced self- (tracer) diffusion in systems of polymerlike breakable micelles. They have argued that it provides the first experimental realization of a random walk for which the second moment of the jump-size distribution fails to exist (''Levy flight''). The basic mechanism is the following: Due to reptation, short micelles diffuse much more rapidly than long ones. As times goes on, shorter and shorter micelles are encountered by the tracer, and hence the effective diffusion constant increases with time.

In a related study [31], a detailed discussion of the fact that the above anomalous régime only exists in a certain range of concentration and temperature. The theoretical dependence of the asymptotic diffusion constant on concentration is found to be in quite good agreement with the experiment.

C.-K. Peng et al., [31], have found that the successive increments in the cardiac beat-to-beat intervals of healthy subjects display scale-invariant, long-range anti correlations (up to 10^4 heart beats). Furthermore, they find that the histogram for the heartbeat intervals increments is well described by a Lévy stable distribution. For a group of subjects with severe heart disease, they found that the distribution is unchanged, but the long-range correlations vanish. Therefore, the different scaling behavior in health and disease must relate to the underlying dynamics of the heartbeat.

T.H.Solomon et al., [31], have studied the chaotic transport in a laminar fluid flow in a rotating annulus, experimentally by tracking large numbers of tracer particles for long times. Sticking and unsticking of particles to remnants of invariant surfaces (Cantori) around vortices results in superdiffusion: The variance of the displacement grows with time as 12 with $\gamma = 1.65 \pm 0.15$. Sticking and flight time probability distribution functions exhibit power-law decays with exponents 1.6±0.3 and 2.3±0.2, respectively. The exponents are consistent with theoretical predictions relating Lévy flights and anomalous diffusion.

J. Klafter and G. Zumofen, [32], have examined the diffusion in a Hamiltonian system, studied in terms of the continuous-time random walk formulation for Lévy walks. The Lévy-walk scheme is extended (i) to include interruptions by periods of temporal localization and (ii) to describe motion in two dimensions. They analyze a case of conservative motion in a two-dimensional periodic potential. Numerical calculations of the mean-squared displacements and the propagators for intermediate energies are consistent with the Lévy-walk description.

H. Spohn, [33], has considered conventional relaxation dynamics for surfaces, both evaporation dynamics and surface diffusion. They pointed out that the cusp singularity of the surface free energy implies that the relaxation dynamics has to be treated as a free boundary value problem. On this basis they predict, that under appropriate conditions, the spontaneous formation of facets and a finite time of healing for the high symmetry surface.

The unsteady creeping motion of a thin sheet of viscous liquid as it advances over a gently slopping dry bed was examined in [35] with a focus on the motion of the leading edge under various influences and four problems were discussed. In the first problem the fluid travels down an open channel formed by two strategies parallel retaining walls placed perpendicular to an inclined plane. When the channel axis was parallel to the fall line there was a progressive – wave solution with a straight leading edge, but inclination of the axis generated distortions. In the second problem a sheet with a straight leading edge traveling over an inclined plane penetrated a region where the bed was uneven, and the subsequent deformation of the leading edge was followed. The third problem considered the flow down an open channel of circular cross – section (a partially filled pipe) and the time dependent shape of the leading edge was calculated. The fourth problem was that of flow down an inclined plane with a single curved edge retaining wall. These problems were all analyzed by assuming that a length characteristics of the geometry was large compared with the fluid depth divided by the bed slope, and the all the solutions displayed extreme sensitivity to the data.

In another study [34], the classic Marshak wave equation (an equilibrium diffusion radioactive transfer description) was obtained as the lowest order approximately in an asymptotic analysis of a system of time dependent nonequilibrium radiative transfer equation. The next approximation led to a more general equilibrium diffusion approximation, which contained the radiative energy in the description. An asymptotic solution of this higher order equilibrium diffusion approximation was derived by including the smallness parameters in both the independent time variable and the dependent variable of the problem. The solution obtained was applicable over a longer time interval than the solution of the Marshak equation. Its main qualitative feature was that the predicted position of the wave front lags behind the Marshak prediction.

Contemporary empirical research into the behavior of stock market price /return patterns has found significant evidence that financial markets exhibit the phenomenon of anomalous diffusion, primarily superdiffusion, wherein the variance evolves with time according to a power law t^{α} with $\alpha > 1.0$. The standard technique for the study of superdiffusive processes is through a stochastic process that evolves according to a Langevin equation [82] and whose probability distribution function satisfies a nonlinear Fokker Planck equation [7].

There is an intricate yet natural relationship between the power law tails observed in stock market data and probability distributions that emanate as the solution of the Fokker Planck equation. The Fokker Planck equation [6] is known to describe anomalous diffusion under time evolution. Empirical results [8-11] establish that temporal changes of several financial market indices have variances that that are shown to undergo anomalous super diffusion under time evolution.

Several analogies between physical systems and financial processes have been explored in the last decade, some of which have already been mentioned above. Perhaps, the most striking one is that between financial crashes witnessed in stock markets and critical phenomena like phase transitions that is discussed here to place the main theme of this research in its proper perspective.

Stock market crashes are believed to exhibit log periodic oscillations which are characteristic of systems exhibiting discrete scale invariance i.e. invariance through rescaling by integral powers of some length scale like the Serpinski triangle and other similar fractal shapes. In the years preceding the infamous crash of October 19, 1987, the S & P market index was seen to fit the following expression exceedingly precisely

$$(S\&P)_t = \Omega + \Gamma(t_c - t)^\gamma \left\{1 + \Xi \cos\left[\theta \ln(t_c - t) + \phi\right]\right\}.$$

1. J. Feigenbaum & P. G. O. Freund [83], have proposed a picture of stock market crashes as critical points in a system with discrete scale invariance. The critical exponent is then complex, leading to log-periodic fluctuations in stock market indexes. They also presented "experimental" evidence in favor of this prediction. This picture is in the spirit

of the known earthquake-stock market analogy and of recent work on log-periodic fluctuations associated with earthquakes.

2. D Sornette, A. Johansen & J.P. Bouchaud, [84], have presented an analysis of the time behavior of the S&P 500 (Standard and Poors) stock exchange index before and after the October 1987 market crash and identified precursory patterns as well as aftershock signatures and characteristic oscillations of relaxation. Combined, they all suggest a picture of a kind of dynamical critical point, with characteristic log-periodic signatures, similar to what has been found recently for earthquakes. These observations are confirmed on other smaller crashes, and strengthen the view of the stock market as an example of a self-organizing cooperative system.

Physicists working in solid state and condensed matter physics would immediately recognize the analogy of the above expression with the one obtained for critical phenomenon in spin model of ferromagnetism.

In this context P. C. Martin et al [85], have demonstrated the statistical dynamics of a classical random variable that satisfies a nonlinear equation of motion which is recast in terms of closed self-consistent equations in which only the observable correlations at pairs of points and the exact response to infinitesimal disturbances appear. The self-consistent equations are developed by introducing a second field that does not commute with the random variable. Techniques used in the study of the interacting quantum fields are then employed, and systematic approximations obtained. It is also possible to carry out a "charge normalization" eliminating the nonlinear coupling in favor of a dimensionless parameter which measures the deviation from Gaussian behavior. No assumptions of spatial or time homogeneity or of small deviation from equilibrium enter and it is shown that previously inferred renormalization schemes for homogeneous systems were incomplete or erroneous. The application of the method to classical microscopic systems, where it leads from first principles to a coupled-mode description is briefly indicated.

We briefly elucidate the salient features of this model. Crystalline solids comprise of atoms arranged in a lattice. Each such atom generates a magnetic field parallel to the direction of the atom's spin. In the case of substances that do not exhibit ferromagnetic character, these spin directions are randomly oriented so that the aggregate magnetic field vanishes. However, in ferromagnetic substances these spins are polarized in a particular direction resulting in a nonzero aggregate field. Ferromagnetic substances usually exhibit two distinct phases. one in which the spins orient themselves in a particular direction resulting in an aggregate magnetic moment at temperatures below a well defined critical temperature t_c and the other where the spins are disoriented with a zero aggregate moment above the critical temperature. At temperatures below t_c, the coupling force between neighboring atoms predominates resulting in an alignment of spins whereas above t_c the additional energy manifests itself in disorienting (randomizing) the spins.

Renormalization group theory enables us to group these atoms in blocks of spins whose composite spins are equal to the algebraic sum of the spins of the atoms constituting the block. It then provides that a model involving interactions between these composite spins of a block can be constructed that replicates the macroscopic properties of the block and yet cannot depend on the size of the block. That is, the system would exhibit a scaling symmetry, which is discrete, if we allow for the finite size of the atom and continuous otherwise. The magnetic susceptibility of such a magnetic substance defined by $\chi(T) = \left.\frac{\partial M}{\partial B}\right|_{B=0}$, where the symbols have their usual meaning, obeys a power law of the form $\chi(T) = \mathrm{Re}\left[(T - T_c)^{\alpha + i\beta}\right]$ or equivalently $\chi(T) = (T - T_c)^{\alpha}\left\{1 + \beta\cos\left[\ln(T - T_c)\right] + O(\beta^2)\right\}$ which is reminiscent of op cited expression for log periodic oscillations in financial crashes.

The works of physicists in financial and economic systems are designated "econophysics". Such systems are treated as complex systems and are usually driven by "fluctuations", and quantifying fluctuations is a topic that many physicists have contributed to in recent years. It is, therefore, possible that methods and concepts developed in the study of strongly fluctuating systems might yield new results in economics. Besides, economic systems are complex interacting systems for which a tremendous amount of quantitative data exists, much of it is never analyzed. For example in [86] where statistical physicists studying fluctuations have uncovered two new empirical "laws". The first empirical law concerns the histogram giving the relative occurrence probability that a stock experiences a given price change; this histogram decreases as the given price change increases, with an apparent power law tail that describes fluctuations differing by as much as 8 orders of magnitude in this relative occurrence probability. The second empirical law concerns a histogram of size changes of business firms, which has a width that decreases as a power law of the firm size for firms that range over roughly 8 orders of magnitude. In addition to such scaling laws, there appears also the analog of "universality" - e.g., the analogous histogram of country size appears to obey the same scaling law, with the same exponent, as the histogram of firm size.

In another study on volatility [87], volatility of the MIB30–stock–index high–frequency data from November 28, 1994 through September 15, 1995 was studied. The volatility random walk was empirically characterized in the framework of continuous–time finance. To this end, the index volatility was computed by means of the log–return standard deviation. A periodic component was found for the hourly time window, for which data was analyzed. Fluctuations were also studied by means of detrended fluctuation analysis, and long–range correlations were detected. Volatility values were found to be log–stable distributed.

In another study [88] conducted by the same group of workers, a study of the statistical properties of volatility was performed, as measured by locally averaging over a time window T, the absolute value of price changes over a short time interval ΔT. An analysis of the S&P 500 stock index for the 13-year period Jan. 1984 to Dec. 1996 was conducted. It was found that the cumulative distribution of the volatility is consistent with a power-law asymptotic behavior, characterized by an exponent $\mu \approx 3$, similar to what is found for the distribution of price changes. The volatility distribution retains the same functional form for a range of values of T. Further, the volatility correlations were also studied by using the power spectrum analysis. Both methods supported a power law decay of the correlation function and gave consistent estimates of the relevant scaling exponents. Also, both methods showed the presence of a crossover at approximately 1.5 days. These results were also extended to the volatility of individual companies by analyzing a data base comprising all trades for the largest 500 U.S. companies over the two-year period Jan. 1994 to Dec. 1995.

Random matrix Theory and spin glasses [89] have also been used to develop models of financial systems. In [90] methods of random matrix theory are adopted to analyze the cross-correlation matrix C of stock price changes of the largest 1000 U.S. companies for the 2-year period 1994–1995. It is found that the statistics of most of the eigenvalues in the spectrum of C agree with the predictions of random matrix theory, but there are deviations for a few of the largest eigenvalues and that C has the universal properties of the Gaussian orthogonal ensemble of random matrices.

From a study of the eigenvalue statistics of the cross-correlation matrix constructed from price fluctuations of the leading 1000 stocks, it was also found that the largest 1% of the eigenvalues and the corresponding eigenvectors show systematic deviations from the predictions for a random matrix, whereas the rest of the eigenvalues conform to random matrix behavior suggesting that these 1% of the eigenvalues contain system-specific information about correlated time evolution of different companies [90] .

Another study based [91] on the Random Matrix Theory analysis of stock price provides for a systematic comparison between the statistics of the cross-correlation matrix C - whose elements C_{ij} are the correlation-coefficients between the returns of stock i and j - and that of a random matrix having the same symmetry properties. The analysis shows that Random Matrix Theory can be used to distinguish random and non-random parts of C; the non-random part of C, which deviates from Random Matrix Theory results, provides information regarding genuine cross-correlations between stocks.

Another interesting study is reported in [92]. Results are discussed of three recent phenomenological studies focused on understanding the distinctive statistical properties of financial time series - (i) The probability distribution of stock price fluctuations: Stock price fluctuations occur in all magnitudes, in analogy to earthquakes - from tiny fluctuations to very drastic events, such as the crash of 19 October 1987, sometimes referred to as "Black Monday". The distribution of price fluctuations decays with a power-law tail well outside the Levy stable regime and describes fluctuations that differ by as much as 8 orders of magnitude. In addition, this distribution preserves its functional form for fluctuations on time scales that differ by 3 orders of magnitude, from 1 min up to approximately 10 days. (ii) Correlations in financial time series: While price fluctuations themselves have rapidly decaying correlations, the magnitude of fluctuations measured by either the absolute value or the square of the price fluctuations has correlations that decay as a power-law, persisting for several months. (iii) Volatility and trading activity: A quantification of the relation between trading activity - measured by the number of transactions $N_{\Delta t}$ - and the price change $G_{\Delta t}$ for a given stock, over a time interval $[t;t+\Delta t]$ is attempted and it is found that $N_{\Delta t}$ displays long-range power-law correlations in time which leads to the interpretation that the long-range correlations previously found for $|G_{\Delta t}|$ are connected to those of $N_{\Delta t}$.

In [16] a statistical physics model for the time evolutions of stock portfolios in the spin glass framework is proposed. In this model the time series of price changes are coded into the sequences of up and down spins. The Hamiltonian of the system is introduced and is expressed by spin-spin interactions as in spin glass models of disordered magnetic systems. The interaction coefficients between two stocks are determined by empirical data coded into up and down spin sequences using fluctuation-response theorem. Monte Carlo simulations are performed and the resultant probability densities of the system energy and magnetization show good agreement with empirical data.

The emergence and consequences of large scale regularities, which, in particular, occur in the presence of fat tails in probability distributions in macro-economy and quantitative finance are studied in [17].

In a closed economic system, money should be conserved. Thus, by analogy with energy, the equilibrium probability distribution of money must follow the exponential Gibbs law characterized by an effective temperature equal to the average amount of money per economic agent. In [19], it is shown how the Gibbs distribution emerges in computer simulations of economic models. A thermal machine is then considered, in which the difference of temperatures allows one to extract a monetary profit. The role of debt, and models with broken time-reversal symmetry for which the Gibbs law does not hold are also discussed.

In [18], by analogy with energy, the equilibrium probability distribution of money is postulated to follow the exponential Boltzmann-Gibbs law characterized by an effective temperature equal to the average amount of money per economic agent. A thermal machine which extracts a monetary profit can, then, be constructed between two economic systems with different temperatures. Using data from several sources, it is found that the distribution of income is described for the great majority of population by an exponential distribution, whereas the high-end tail follows a power law. The Lorenz curve and Gini coefficient have been calculated in this study and are shown to be in good agreement with both income and wealth data sets. The Heston model, where stock-price dynamics is governed by a geometrical (multiplicative) Brownian motion with stochastic variance, is also studied. The corresponding Fokker-Planck equation is solved exactly. Integrating out the variance, an analytic formula for the time-dependent probability distribution of stock price changes (returns) is found. The formula is in excellent agreement with the Dow-Jones index for the time lags from 1 to 250 trading days.

Different levels of complexity which are observed in the empirical investigation of financial time series are considered in [20]. Recent empirical and theoretical work is reviewed showing that statistical properties of financial time series are rather complex in several ways. Specifically, they are complex with respect to their (i) temporal and (ii) ensemble properties. Moreover, the ensemble return properties show a behavior which is specific to the nature of the trading day reflecting if it is a normal or an extreme trading day. Important work in regard to classical renewal theorems have been reported in [170-171].

In [21], the authors have used an analogy with statistical physics to describe probability distributions of money, income, and wealth in society. By making a detailed quantitative comparison with the available statistical data, they show that these distributions are described by simple exponential and power-law functions.

[22] is another review article wherein econophysics developments in four areas, including empirical statistical properties of prices, random-process models for price dynamics, agent-based modeling, and practical applications have been elaborated.

Enrique Canessa in [93], addresses the issue of stock market fluctuations within Langevin Dynamics (LD) framework and the thermodynamic definitions of multifractality in order to study second-order characterization given by the analogous specific heat C_q, where q is an analogous temperature relating the moments of the generating partition function for the financial data signals. Due to non-linear and additive noise terms within the Langevin Dynamics, it was found that C_q can display a shoulder to the right of its main peak as also found in the S&P500 historical data which may resemble a classical phase transition at a critical point.

Statistical analyses of the general and sectorial historical M.I.B. indices of the Milan stock exchange were performed in [11]. The analysis showed that the price indices have statistical properties which are compatible with a Lévy random walk. The time evolution of the daily variations of indices was intermittent on a time scale of years and the variance of almost all indices displayed a superdiffusive behavior. By using the theory of enhanced diffusion in Lévy walks as theoretical framework the authors ascribed the superdiffusive behavior to a nonlocal memory coupling price and time.

Statistical properties of the number of shares traded $Q_{\Delta t}$ for a given stock in a fixed time interval Δt were studied in [27].Transaction data for the largest 1000 stocks for the two-year period 1994–95 are analyzed. It is found that for transaction for all securities in three major US stock markets. That the distribution $P(Q_{\Delta t})$displays a power-law decay, and that the time correlations in $Q_{\Delta t}$ display long-range persistence. Further, the authors also investigate the relation between $Q_{\Delta t}$ and the number of transactions $N_{\Delta t}$ in a time interval Δt, and find that the long-range correlations in $Q_{\Delta t}$ are largely due to those of $N_{\Delta t}$. Their results are consistent with the interpretation that the large equal-time correlation previously found between $Q_{\Delta t}$ and the absolute value of price change $|G_{\Delta t}|$ (related to volatility) are largely due to $N_{\Delta t}$.

In [28], another similar empirical study an attempt to quantify the relation between trading activity—measured by the number of transactions N—and the price change $G(t)$ for a given stock, over a time interval $[t, t+\Delta t]$ was made. The time-dependent standard deviation of price changes—volatility—to two microscopic quantities: the number of transactions $N(t)$ in Δt and the variance $W^2(t)$ of the price changes for all transactions in Δt was examined. It was observed that the long-ranged volatility correlations are largely due to those of N. The authors also argue that the tail-exponent of the distribution of N is insufficient to account for the tail-exponent of $P\{G > x\}$. Since N and W display only weak inter-dependency, they claim that their results show that the fat tails of the distribution $P\{G > x\}$ arises from W, which has a distribution with power-law tail exponent consistent with the estimates for G.

A phenomenological study of stock price fluctuations of individual companies is presented in [30]. A systematic analysis of two different databases covering securities from the three major U.S. stock markets: (a) the New York Stock Exchange, (b) the American Stock Exchange, and (c) the National Association of Securities Dealers Automated Quotation stock market is performed. Specifically, the authors consider (i) the trades and quotes database, for which they analyze 40 million records for 1000 U.S. companies for the 2-yr period 1994–95; and (ii) the Center for Research and Security Prices database, for which they analyze 35 million daily records for approximately 16 000 companies in the 35-yr period 1962–96. A study of the probability distribution of returns over varying time scales Δt, where Δt varies by a factor of $\approx 10^5$, from 5 min up to conducted ≈ 4. For time scales from 5 min up to approximately 16 days, it is found that the tails of the distributions can be well described by a power-law decay, characterized by an exponent $2.5 < \alpha 4$, well outside the stable Lévy regime $0 < \alpha 2$. For time scales $\Delta t \gg (\Delta t)_x \approx 16$ days, results consistent with a slow convergence to Gaussian behavior emerge. The examine the role of cross correlations between the returns of different companies and relate these correlations to the distribution of returns for market indices.

Furthermore, with the rapid advancements in the evolution and study of disordered systems and the associated phenomena of nonlinearity, chaos, self organized criticality etc., the importance of generalizations of the extant mathematical apparatus to enhance its domain of applicability to such disordered systems is cardinal to the further development of science.

A considerable amount of work has already been done and success achieved in the broad areas of q-deformed harmonic oscillators [38], representations of q-deformed rotation and Lorentz groups [39-40]. q-deformed quantum stochastic processes have also been studied with realization of q-white noise on bialgebras [41]. Deformations of the Fokker Planck's equation [42], Langevin equation [94] and Levy processes [43-44] have also been analyzed and results reported.

In [94], using the system bath interaction model with the bath consisting of q - deformed harmonic oscillators, a q- deformed version of the quantum Langevin equation is derived, time correlation commutation relations for the deformed noise operator were computed and an expression for the frequency dependent damping coefficient was arrived at. Nonlinearity interpretations of q-deformations were also discussed.

A J Macfarlane [38] , has discussed the quantum group $SU(2)_q$ by a method which is comparable to the method used by Schwinger to develop the quantum theory of angular momentum. q-analogue of the quantum harmonic oscillator is required for this purpose and is developed.

L C Biedenharn [39], has constructed a new realization of quantum group $SU_q(2)$by means of a q-analogue to the Jordan-Schwinger mapping, thus determining both the complete representation structure and q-analogues to the Wigner and Racah operators. To achieve this realisation, a new elementary object is defined, a q-analogue to the harmonic oscillator. The position and momentum of q - harmonic oscillator posses a very uncertain and unusual relation.

S Zakrzewski [40], has selected quantum Lorentz groups H and quantum Minkowski spaces V. The natural structure of a quantum space $G = V \times H$ is introduced, defining a quantum group structure on G only for triangular $H(q=1)$. We show that it defines a braided quantum group structure on G for $|q| = 1$

The q-Poincaré group of M. Schlieker *et al.* [95] is shown to have the structure of a semi direct product and co product $B \times SO_q$ where B is a braided-quantum group structure on the q-Minkowski space of four-momentum with braided-co product $\Delta_{p=p\otimes 1+1\otimes p}$. Here the necessary B is not a usual kind of quantum group, but one with braid statistics. Similar braided vectors and covectors $V(R), V \times R$ exist for a general R- matrix. The abstract structure of the q-Lorentz group is also studied.

Michael Schurmann [41], in his article has established a connection between the Azema martingales and certain quantum stochastic processes with increments satisfying q-commutation relations. This leads to a theory of q-white noise on q-*-bialgebras and to a generalization of the Fock space representation theorem for white noise on *-bialgebras. In particular, quantum Azema noise, q-interpolations between Fermion and Boson quantum Brownian motion and unitary evolutions with q-independent multiplicative increments are studied. It follows from their results that the Azema martingales and the q-interpolations are central limits of sums of g-independent, identically distributed quantum random variables.

C. Blecken and K.A. Muttalib [42], have shown the effect of an external perturbation on the energy spectrum of a mesoscopic quantum conductor which can be described by a Brownian motion model developed by Dyson who wrote a Fokker-Planck equation for the evolution of the joint probability distribution of the energy levels. For weakly disordered conductors, which can be described by a Gaussian random matrix ensemble, the solution of the Fokker-Planck equation has recently been obtained to give the correlation of level densities at different energies and different parameter values. In this paper the author generalize this calculation to the case of a q-random matrix ensemble which should be relevant for conductors at stronger disorder.

U Franz *et al* [43], have indicated that evolution equations like the heat or diffusion equation or the Schrödinger equation can be associated with stochastic processes. In this paper they have studied the relation between equations of the form $\partial_t u = Lu$ and Lévy processes (i.e. quantum stochastic processes with independent and stationary increments) on quantum groups and braided groups. Solutions of such equations are calculated as Appell systems. Wigner distributions of these processes are defined and it is proven that they satisfy a Fokker-Planck equation.

V.I. Man'ko et al [44],have studied a nonlinearity of electromagnetic field vibrations described by q-oscillators which is shown to produce an essential dependence of second order correlation functions on the intensity and deformation of the Planck distribution. Experimental tests of such a nonlinearity are suggested. They have also suggested that q-oscillators are associated to the simplest non-commutative example of Hopf algebra and may be considered to be the basic building blocks for the symmetry algebras of completely integrable theories. They may also be interpreted as a special type of spectral nonlinearity, which may be generalized to a wider class of f-oscillator algebras. In the framework of this nonlinear interpretation, the authers have discussed the structure of the stochastic process associated to q-deformation. The role of the q-oscillator as a spectrum-generating algebra for fast growing point spectrum, the deformation of fermion operators in solid-state models and the charge-dependent mass of excitations in f-deformed relativistic quantum fields are also discussed.

A study of the quantum features of the Universe evolution, proposing the problem of using ``local" physical laws even on cosmic scale was conducted in [97-99] thereby proving a rationale for attempting their applicability in the behavioural sciences.

A new model of Evolutionary Neural Gas (ENG) with any topological constraints, trained by probabilistic laws depending on the local distortion errors and the network dimension has been contributed to the literature in [96-98] and it is shown that the network, considered as a population of nodes that coexist in an ecosystem sharing local and global resources, because of these features is quickly able to adapt to the environment, according to its dimensions.

2.6 NONLINEARITIES, CHAOS & STOCK MARKETS

There exist two traditional approaches to the modeling of a dynamical system. In the first approach, the dynamical deterministic equations of motion are obtained from first principles as differential / difference equations that are integrated forward in time and solved as an initial value problem. This methodology, although strongly preferred due to its exactness, is sometimes impracticable, particularly when we are analyzing the dynamics of many particle systems with complicated interactions among the constituents. In such cases, either the number of degrees of freedom becomes so large as to make the first-principles model intractable or the initial conditions pertaining to each degree of freedom become inaccessible. Attempts are, then, made to model the dynamics as a random process with stochastic, though linear, laws of motion. There was believed to be no region of overlap between these two well-defined approaches.

Chaos, as a physical phenomenon, has attained recognition relatively recently. Its origin, in its modern form, may be traced to the revolutionizing work of the master French mathematician Henri Poincare in the 1890s', on the mathematical aspects of planetary motion, treating it as a three-body problem. Through the use of topological methods, he established that there is no simple solution to the three-body problem. During the course of his analysis, he realized that if one takes two different readings on the position of a planet, then, irrespective of the proximity of the two readings, the orbits of the planets might separate away from each other, after enough time. Hence, accurate prediction of the orbit of any planetary body was impossible. Chaos was, thus, born.

The most apt yet striking manifestation of chaos is summarized in the following statement attributed to Edward Lorenz:-

"The flapping of butterfly wings in Rio de Janeiro could bring on a tornado in Texas several weeks later!"

This is what Edward Lorenz concluded one fine day when he was running a mathematical model of the weather on his computer. It so happened that in order to recheck some results on his weather forecasting model, he decided to re-input his data from the earlier printouts and run the program again. The results were quite inconceivable – although the immediate values of the variable were identical, major divergences surfaced as the run steps was extended – in fact, no significant resemblance was observed between the results of the two runs after a sufficiently long period. Thus, starting from nearly the same initial conditions, weather patterns were produced that grew further and further apart until all association disappeared.

On investigation, the cause of this highly paradoxical scenario was traced to a very trivial matter – while the printer had printed data upto six decimal points which constituted the data fed for the second run, the computer had calculated data upto eight decimals. The data that was fed for the second run was, therefore, minutely different from the data that was used in the first run. Amazing as it may sound, it was these minute differences that manifested themselves as gigantic divergences in the output – this, indeed, is chaos.

This property that manifests itself through sensitivity to initial conditions with a consequential unpredictability is generally termed as Chaos. The discovery of chaos has destroyed the deterministic image of the modern world leading to new directions of research and providing a fillip to the ergodic description of systems.

Chaos provides a link between deterministic systems and random processes. In a deterministic system, chaotic dynamics can amplify small differences, which in the long run produce effectively unpredictable behavior. On the other hand, it is possible that not all random-looking behavior is the product of complicated interactions and hence, may well be tractable in the deterministic framework. The existences of non-linearities in only a few degrees of freedom are sufficient to generate chaotic motion. In such cases, it is possible to model the system behavior deterministically and to make short-term predictions that are far better than those that would be obtained from a linear stochastic model. Chaos is thus a double-edged sword: it implies that even approximate long-term predictions may be impossible, but that very accurate short-term predictions may be possible. Hence, chaos has both good and bad implications for the prediction problem.

Decision making of all kinds [100-101] including investments in the capital markets, rests on our ability to predict the future. However, business and, indeed, life in general, is not predictable. Researchers and practitioners in accounting and finance often investigate or advocate particular disciplined trading strategies, but little work investigates the determinants of individual investors' trading-strategy reliance. Experiments in this regard have been reported in [102-107]. Violations of economic rationality have been observed.

Conventionally, in decision theory, this lack of predictability is explained by factors such as lack of information or the limitations of prediction techniques. Chaos theory, however, provides a radically opposite explanation, in that it accepts unpredictability as an inherent attributes of a wide range of phenomena, so that, forecasting may be an entirely futile and wasteful exercise.

Prediction and forecasting have, hitherto, relied essentially on various linear models like regression, linear programming, capital budgeting and so on [108-111]. It is, however, now established beyond doubt that all fundamental processes of Nature have various degrees of non-linearity. In fact, Chaos is a manifestation of the non-linearities inherent in a system in so far as such unpredictable phenomena are forbidden in linear systems by the very virtue of their linearity.

A corollary to this ubiquitous non-linearity is the high degree of approximation incumbent in all the contemporary decision making processes. Chaos theory emphasizes that because of this sensitivity to initial conditions, many events simply cannot be predicted, because it would be impossible to know and monitor all the variations that might have a significant effect on the outcomes.

A compact, concise and universally acceptable definition of Chaos has, hitherto, eluded the scientific community. However, the following are conventionally accepted as the inherent characteristics of a chaotic system:-

- Exponential divergence of neighboring trajectories in phase space;
- Sensitive dependence on initial conditions;
- Existence of fractal dimensions;
- Critical levels and bifurcations at which the system's behavior radically changes;
- Time dependent feedback.

As had happened in the case of Edward Lorenz, chaotic systems are highly sensitive to initial conditions insofar as minor differences tend to get magnified manifold with

$$x_{(t+1)} = 1 + y_t - ax_t^2$$
$$y_{(t+1)} = bx_t$$

the evolution of the system. This is illustrated in the Henon Map, defined by the following set of simultaneous difference equations:-

Figure 2.1

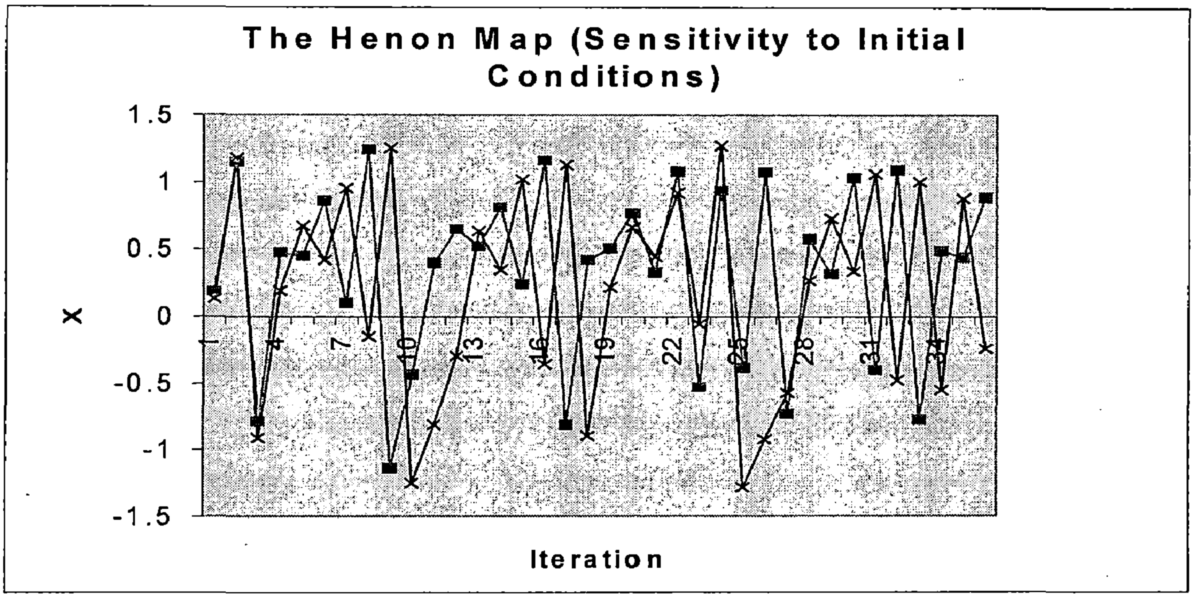
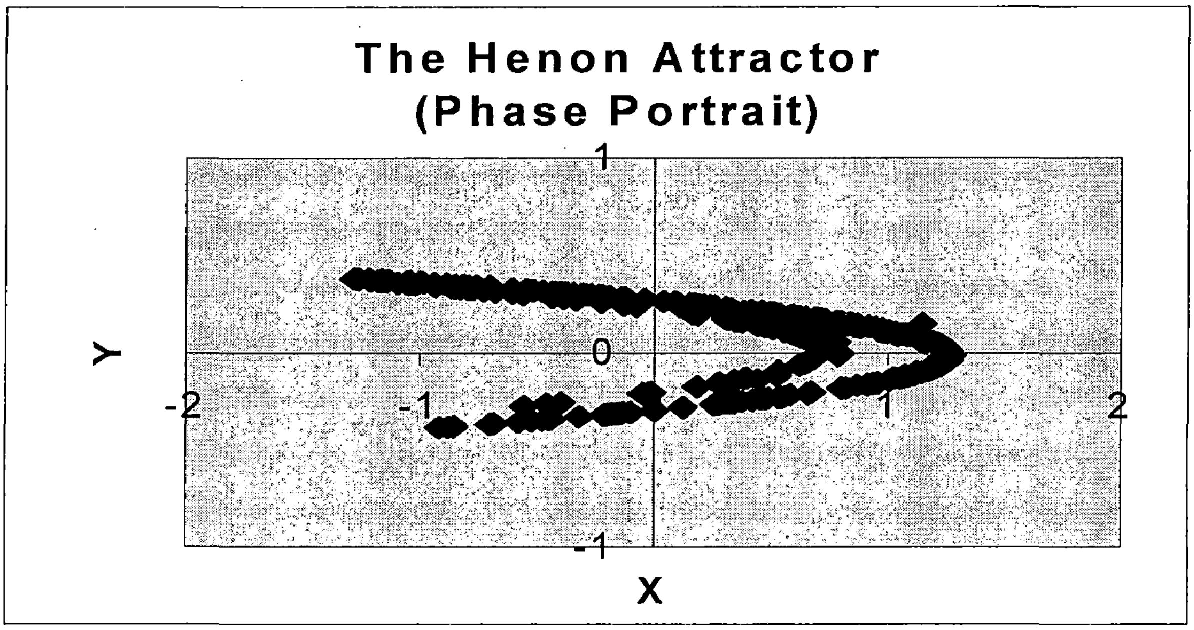


Figure 2.2



While the origin of this sensitive dependence may be attributed to the existence of time dependent feedback mechanisms, the implications are devastating. Unpredictability becomes an inherent attribute and long term forecasting becomes a futile exercise. Marginally small errors in data collection would manifest themselves magnified manifold in forecasted output. For all we know, even with the best available measurement devices, error free measurement is impossible, a fundamental lower limit being imposed by Heisenberg's Uncertainty Principle.

Chaos theory propounds the adoption of a radically new perspective to forecasting. It emphasises the need to acknowledge the true dimensions of uncertainty in its absoluteness and to discard the conventional and traditional so called "rationalistic" models like the Efficient Market Hypothesis. Chaos theory recognises the existence of disorder, discontinuities and randomness as inherent properties or norms rather than as aberrations. Consequent to the acceptance of unpredictability as an inherent property, chaos theory tends to dwell heavily on the necessity of development of adequate "fire fighting" mechanisms as an indispensable part of planning and forecasting.

Several studies [48, 50-51, 112-113] adopting largely diverse and independent approaches have established the existence of the following characteristics in the behavior of stock markets:-

- Long term correlation and memory effects

- Occasional existence of erratic markets
- Existence of fractal dimensions in stock market time series of returns
- Less reliable forecasts with increase in the horizon

thereby establishing the probable existence of chaotic behavior of stock markets.

As mentioned above, the phenomenon of chaos is a manifestation of the non-linearities intrinsic to the dynamic equations of motion that govern the time development of a system. In the next section, we look at some approaches that have been devised to examine the existence or otherwise of nonlinear and chaotic behavior in time series data that represents the evolution of discrete phenomena like the prices of stocks in capital markets.

2.7 RESCALED RANGE ANALYSIS & COMPUTATION OF HURST'S EXPONENT

Variance has been, traditionally, used in one guise or another as the statistical measure of risk. Variance measures the probability that an observation will be a certain distance from the average observation. The larger this number, the wider the dispersion. Wide dispersion would mean that there is a high probability of large swings in returns. The security is risky. However, the use of variance as a measure of risk inherently assumes that the underlying system is random. If the observations are correlated, then the usefulness of variance as a measure of risk is considerably weakened. We illustrate our point by an example. We consider two possible series of stock market returns, say A & B:-

Observation	A	B
1	0.02	0.01
2	-0.01	0.02
3	-0.02	0.03
4	0.02	0.04
5	-0.01	0.05
6	0.02	0.06
Standard Deviation	0.17	0.0171

Table 2.3

A is a trendless series while B has a clear trend. Both have almost the same standard deviation. The two stocks with virtually identical risk (as measured by the standard deviation) have vastly differing return characteristics. The obvious fallacy is that both series are not normally distributed, but then the same is the case with the stock markets. As mentioned above, numerous studies have shown the non-random character of the stock market returns, thereby questioning the usefulness of variance as a comparative measure of risk.

A time series will be truly random when it is influenced by a number of events that are equally likely to occur i.e. .e. the system has a large number of degrees of freedom. In a non-random series the data will clump together to reflect the correlation inherent in its influences and the time series will be a fractal.

The stock markets are modelled as a process that happens in time. As is the case with most systems modelling, this process is treated either as a discrete static process or a continuous random process. However, neither assumption entirely gels with reality and nor extreme is a complete and sophisticated treatment of the subject. The commonality underlying both these assumptions is that they are linear i.e. either they are always static or always random. Time either does not affect the system or does so at a uniform rate.

Most financial returns, including stock returns have shown deviation from Gaussian behaviour at short time scales with the variance not scaling with the sq. root of timescale, an attribute that is symptomatic of the possible existence of power law distributions. A useful measure of quantifying deviations from the Gaussian distribution is the Hurst's exponent. If a population is Gaussian, a Hurst's exponent of 0.5 is mandated. Empirical evidence, however, shows that the Hurst's exponent for typical stock market data is around 0.6 for small timescales of about a day or less and tends to approach 0.5 asymptotically with the lengthening of the timescales. Empirical evidence also demonstrates the existence of memory effects, particularly in stock price volatilities that show long-term memory effects with lag-s autocorrelations. Further, these effects tend to fall off according to a power law rather than exponentially.

CHAPTER 3
OPTION PRICING MODEL WITH DEFORMED BROWNIAN MOTION

Abstract

The Black Scholes model of option pricing constitutes the cornerstone of contemporary valuation theory. However, the model presupposes the existence of several unrealistic assumptions including the lognormal distribution of stock market price processes. There, now, subsists abundant empirical evidence that this is not the case. Consequently, several generalizations of the basic model have been attempted with relaxation of some of the underlying assumptions. In this Chapter, we postulate a generalization that contemplates a statistical feedback process for the stochastic term in the Black Scholes partial differential equation. Several interesting implications of this modification emanate from the analysis and are explored.

3. 1 INTRODUCTION

With the rapid advancements in the evolution and study of disordered systems and the associated phenomena of nonlinearity, chaos, self organized criticality etc., the importance of generalizations of the extant mathematical apparatus to enhance its domain of applicability to such disordered systems is cardinal to the further development of science. A possible mechanism for achieving this objective is through deformation of standard mathematics.

In this Chapter, we attempt a generalization of the Black Scholes formula for the pricing of contingent financial claims based on the deformation of the standard Brownian motion. Section 2, which forms the essence of this Chapter, attempts a deformation of the standard Black Scholes pricing formula. In Section. 3 we illustrate the theory developed in the previous section with a concrete example. Section 4 addresses issues relating to empirical relevance of the model. Section 5 concludes.

3. 2 THE GENERALIZED BLACK SCHOLES MODEL

The standard analysis of the Black Scholes formula for option pricing presupposes that the stock price follows the lognormal distribution. However, significant empirical evidence now subsists of the stock returns deviating from the lognormal distribution with "fat tails" and a "sharp peak" which better fit the truncated Levy flights or other power law distributions [13,114, 115]. To broadbase the Black Scholes model, generalizations by way of "Levy noise" and "jump diffusions" [25] have already been studied. In this Chapter, we propose a model that incorporates a "weighted Brownian motion" as the stochastic (noise) term, where the weights themselves are a function of the "Brownian motion / noise" i.e.,

$$dW_t^P \to dU_t^P = f\left(U_t^P, t\right) dW_t^P \tag{3.1}$$

W_t^P is a regular Brownian motion representing Gaussian white noise with zero mean and δ correlation in time i.e. $E^P\left(dW_t dW_{t'}\right) = dt dt' \delta(t - t')$ and on some filtered probability space $(\Omega, (F_t), P)$. We, further, mandate that the function $f\left(U_t^P, t\right)$ satisfies the Novikov condition and that the process $U_t^P = \int_0^t f\left(U_t^P, s\right) dW_s^P$ is a local P-martingale with a non normal distribution. This requirement is not as restrictive as it may seem at first sight in context of the applications envisaged. We shall address this issue again in the sequel.

This generalization contemplates a statistical feedback process. In this context, several studies on stock market data have shown the existence of nonlinear characteristics and chaotic behavior that lend credence to the existence of a statistical feedback mechanism of market players. Explanations for the existence of "fat tails" in stock market data have been offered through this statistical feedback process e.g. "extremal events" cause "disproportionate reactions" among market players. The deformed noise proposed herein may also capture the "herd behavior" of stock market investors. The model also encompasses time dependent return processes since f is a function of U_t^P and t so that the drift term varies with time.

We define the European call option as a financial contingent claim that entails a right (but not an obligation) to the holder of the option to buy one unit of the underlying asset at a future date (called the exercise date or maturity date) at a price (called the exercise price). The option contract, therefore, has a terminal payoff of $\max(S_T - E, 0) = (S_T - E)^+$ where S_T is the stock price on the exercise date and E is the exercise price.

We consider a non-dividend paying stock, the price process of which follows the geometric Brownian motion with drift $S_t = e^{(\mu t + \sigma U_t^P)}$ under the probability measure P with drift μ and volatility σ. The logarithm of the stock price $Y_t = In\, S_t$ follows the stochastic differential equation

$$dY_t = \mu dt + \sigma dU_t^P = \mu dt + \left[\sigma f\left(U_t^P, t\right)\right] dW_t^P \qquad (3.2)$$

Application of Ito's formula yields the following SDE for the stock price process

$$dS_t = \left(\mu + \frac{1}{2}\left[\sigma f\left(U_t^P, t\right)\right]^2\right) S_t dt + \left[\sigma f\left(U_t^P, t\right)\right] S_t dW_t^P \qquad (3.3)$$

We also introduce a "bond" in our market to factor in the "true value of money", that evolves according to the following price process

$$\frac{dB_t}{B_t} = r dt, B_0 = 1, \qquad (3.4)$$

where r is the relevant risk free interest rate.

Let $C(S_t, t)$ denote the instantaneous price of the call option with exercise price E at any time t before maturity when the price per unit of the underlying is S_t. We assume that $C(S_t, t)$ does not depend on the past price history of the underlying. Applying the Ito formula to $C(S_t, t)$ yields

$$dC_t = \left[\left(\mu + \frac{1}{2}\left[\sigma f\left(U_t^P,t\right)\right]^2\right)S_t\frac{\partial C}{\partial S} + \frac{\partial C}{\partial t} + \frac{1}{2}\left[\sigma f\left(U_t^P,t\right)\right]^2 S_t^2\frac{\partial^2 C}{\partial S^2}\right]dt + \frac{\partial C}{\partial S}\left[\sigma f\left(U_t^P,t\right)\right]S_t dW_t^P, \tag{3.5}$$

Applying Girsanov's theorem to the price process (3.3), we perform a change of measure and define a probability measure Q such that the discounted stock price process $Z_t = S_t e^{-rt}$ or equivalently

$$dZ_t = \left(\mu - r + \frac{1}{2}\left[\sigma f\left(U_t^P,t\right)\right]^2\right)Z_t dt + \left[\sigma f\left(U_t^P,t\right)\right]Z_t dW_t^P \tag{3.6}$$

behaves as a martingale with respect to Q. This is performed by eliminating the drift term through the transformation

$$\frac{\left(\mu - r + \frac{1}{2}\left[\sigma f\left(U_t^P,t\right)\right]^2\right)}{\sigma f\left(U_t^P,t\right)} \rightarrow \gamma_t \tag{3.7}$$

whence $W_t^Q = W_t^P + \gamma_t t$ is a Brownian motion without drift with respect to the measure Q and $dZ_t = \left[\sigma f\left(U_t^Q,t\right)\right]Z_t dW_t^Q$ which is driftless under the measure Q and hence, Z_t is a Q martingale.

The equivalence of $\left[\sigma f\left(U_t^P,t\right)\right]Z_t dW_t^P$ and $\left[\sigma f\left(U_t^Q,t\right)\right]Z_t dW_t^Q$ follows from the fact that both W_t^Q, W_t^P are zero mean Weiner processes and that $f\left(U_t^Q,t\right)$ can be expressed in terms of $f\left(U_t^P,t\right)$ through $dZ_t = \left[\sigma f\left(U_t^Q,t\right)\right]Z_t dW_t^Q$ along with eq. (3.6). The noise terms in $dZ_t = \left[\sigma f\left(U_t^Q,t\right)\right]Z_t dW_t^Q$ and eq. (1.6), will, therefore, be equivalent stochastically.

The two measures P & Q are related through the Radon Nikodym derivative which in the deformed case takes the form

$$\xi(t) = \frac{dQ}{dP} = \exp\left(-\int_0^t \gamma_t dW_t^P - \frac{1}{2}\int_0^t \gamma_t^2 dt\right) \tag{3.8}$$

and the expectation operators under the two measures are related as

$$E^Q\left(X_t \middle| F_s\right) = \xi^{-1}(s) E^P\left(\xi(t) X_t \middle| F_s\right) \tag{3.9}$$

Our next step in martingale based pricing is to constitute a Q martingale process that hits the discounted value of the contingent claim i.e., call option. This is formed by taking the conditional expectation of the discounted terminal payoff from the claim under the Q 'measure i.e.

$$E_t = E^Q\left[e^{-rT}\left(S_T - E\right)^+ \middle| F_t\right]. \tag{3.10}$$

We now constitute a self-financing strategy that exactly replicates the claim and whose value is known with certainty.

Making use of ϕ_t units of the underlying asset and ψ_t units of the bond, where $\phi_t = \frac{\partial C(S_t,t)}{\partial S}$, $B_t\psi_t = C(S_t,t) - \phi_t S_t$, we can now construct a trading strategy that has the following properties

(a) it exactly replicates the price process of the call option i.e.

$$\phi_t S_t + \psi_t B_t = C(S_t,t), \forall t \in [0,T]. \tag{3.11}$$

(b) it is self financing i.e. $\phi_t dS_t + \psi_t dB_t = dV_t, \forall t \in [0,T]$. (3.12)

Using eqs. (3.1), (3.3), (3.11) & (3.12) we have

$$dC = \left(\phi_t \mu S_t + \frac{1}{2}\phi_t\left[\sigma f\left(U_t^P,t\right)\right]^2 S_t + \psi_t r B_t\right)dt + \phi_t\left[\sigma f\left(U_t^P,t\right)\right]S_t dW_t^P. \tag{3.13}$$

Matching the diffusion terms of (3.3) & (3.13) and using (3.11), we get the aforesaid expressions for ϕ_t and ψ_t respectively. The value of this portfolio at any time t can be shown to be equal to $V_t = e^{rt}E_t$ with E_t being given by eq.(3.10). It follows that the value of the replicating portfolio and hence of the call option at time t is given by

$$V_t = e^{rt}E_t = e^{-r(T-t)}E^Q\left[(S_T - E)^+ \middle| F_t\right] = e^{-r(T-t)}E^Q\left[(S_T - E)1_{(S_T \geq E)} \middle| F_t\right]$$
$$= e^{-r(T-t)} \int_{\{U_T^Q : S(U_T^Q, T) \geq E\}} \left(S(U_T^Q, T) - E\right)\bar{f}\left(U_T^Q, T \middle| U_t^Q, t\right) dU_T^Q \tag{3.14}$$

The expectation value of the contingent claim $\max(S_T - E, 0) = (S_T - E)^+$ under the measure Q depends only on the marginal distribution of the stock price process S_t under the measure Q which is obtained by writing it in terms of Q Brownian motion W_t^Q. We have, from eq.(3.2), for the deformed stock price process under the measure Q as

$$d(\ln S_t) = \mu dt + \left[\sigma f(U_t^P, t)\right]dW_t^P = \left(r - \frac{1}{2}\left[\sigma f(U_t^Q, t)\right]^2\right)dt + \left[\sigma f(U_t^Q, t)\right]dW_t^Q \tag{3.15}$$

which on integration yields

$$S_t = S_0 \exp\left[\int_0^t \left[\sigma f(U_t^Q, t)\right]dW_t^Q + \int_0^t \left(r - \frac{1}{2}\left[\sigma f(U_t^Q, t)\right]^2\right)ds\right]. \tag{3.16}$$

The value of the call option can now be computed by using eq. (3.14). The existence or otherwise of a closed form solution would depend on the explicit representation of the function $f(U, t)$.

The following observations are cardinal to the above analysis.

(a) We have, implicitly, made the standard assumption of the market satisfying the "No Arbitrage" condition. It is well known that long-term market equilibrium cannot subsist in the presence of arbitrage opportunities. This "No Arbitrage" condition guarantees the existence and measurability of γ_t defined by eq. (3.7) as is proved below:

For this purpose, we assume that there exist values of U_t^P for which $f(U_t^P,t)=0$ and hence, γ_t does not exist. Let $X_t=\{U_t^P : f(U_t^P,t)=0\}$. We construct a portfolio (ϕ,ψ) of the normalized stock process $(\overline{S}_t)$ and the bond process (B_t) where $\phi=\begin{Bmatrix}\theta & for\, U_t^P \in X_t \\ 0 & for\, U_t^P \notin X_t\end{Bmatrix}$ and $\psi_t=\psi_0+\phi_0 S_0+\int_0^t e^{-rs}\phi_s dS_s-\int_0^t re^{-rs}\phi_s ds-e^{-rt}\phi_t S_t$, $B_0=1$ and the normalized stock process i.e. the stock process adapted to a market with zero interest rates being given by $\overline{S}_t=S_t e^{-rt}$ and $d\overline{S}_t=e^{-rt}dS_t-re^{-rt}S_t dt$.

The portfolio is self financing since $V_t=\psi_t+\phi_t\overline{S}_t$.

Further,

$$V_t-V_0=\int_0^t\phi_s d\overline{S}_s=\int_0^t e^{-rs}\left(\mu+\frac{1}{2}\left[\sigma f(U_s^P,s)\right]^2-r\right)\phi_s S_s ds+\int_0^t e^{-rs}\left[\sigma f(U_s^P,s)\right]\phi_s S_s dW_s^P$$

$$=\int_0^t \aleph_{X_s}e^{-rs}\left(\mu+\frac{1}{2}\left[\sigma f(U_s^P,s)\right]^2-r\right)\theta_s S_s ds+\int_0^t\aleph_{X_s}e^{-rs}\left[\sigma f(U_s^P,s)\right]\theta_s S_s dW_s^P=\int_0^t\aleph_{X_s}e^{-rs}(\mu-r)\theta_s S_s ds\geq 0$$

where $\aleph_{X_t}$ is the characteristic function of the set $X_t \forall U,t$ .But under the "No Arbitrage" condition $V_t - V_0 \leq 0$. It, therefore, follows that $\aleph_{X_t} = 0 \, \forall U,t$ and hence, $X_t = \phi$.

(b) In the standard Black Scholes theory, the Novikov condition is automatically satisfied due to the constancy of $\gamma_t \equiv \gamma$. However, in the deformed version, this condition needs to be explicitly imposed to ensure the applicability of the Girsanov's theorem and hence, the existence of the equivalent martingale measure Q . Hence, we require that the function $f(U,t)$ to be such that $E^P\left\{\exp\left[\frac{1}{2}\int_0^T(\gamma_s)^2\, ds\right]\right\} < \infty$. As mentioned above, this condition is not very restrictive insofar as the applications of this model are concerned, since $f(U,t)$ would normally take the form of probability distributions and hence, be non zero bounded functions, thereby, automatically satisfying the square integrability requirements.

(c) Except for the Novikov condition, which needs to be explicitly imposed in the deformed model as mentioned in (b) above, our analysis is equivalent to the standard Black Scholes model since $f(U,t)$ can be expressed as a function of Y, the logarithm of the stock price S through eq. (3.2);

(d) The "No Arbitrage" condition together with the Novikov Condition guarantee the completeness of the market and hence, the availability of replicating portfolios for the valuation of any contingent claim. This is established by showing that there exists a self financing portfolio (ϕ,ψ) defined as in (a) above that exactly replicates the terminal payoff of any lower bounded contingent claim, say $C(S_t,t)$. Mathematically, this implies that there exists a real number ε such that $C(S_T,T)=V_T^{\varepsilon}=\varepsilon+\int_0^T(\phi_t dS_t+\psi_t dB_t)$ or equivalently

$$C(S_T,T)=V_T^{\varepsilon}=\varepsilon+\int_0^T(\phi_t dS_t+\psi_t dB_t)=e^{rT}\left(\varepsilon+\int_0^T\phi_t d\bar{S}_t\right)$$

$$=e^{rT}\left[\varepsilon+\int_0^T e^{-rt}\left(\mu+\frac{1}{2}\left[\sigma\left(U_t^P,t\right)\right]^2-r\right)\phi_t S_t dt+\int_0^T e^{-rt}\left[\sigma\left(U_t^P,t\right)\right]\phi_t S_t dW_t^P\right]=e^{rT}\left[\varepsilon+\int_0^T e^{-rt}\left[\sigma\left(U_t^Q,t\right)\right]\phi_t S_t dW_t^Q\right]$$

By the Martingale Representation Theorem, there exists a function η_t such that $C(S_T,T)=e^{rT}\left\{E^Q\left[e^{-rT}C(S_T,T)\right]+\int_0^T\eta_t S_t dW_t^Q\right\}$. Hence, we can identify $\varepsilon=E^Q\left[e^{-rT}C(S_T,T)\right]$ and $\phi_t=e^{rt}\left[\sigma f\left(U_t^Q,t\right)\right]^{-1}\eta_t$. By selecting the bond component of the portfolio (ψ) according to $\psi_t=\psi_0+\int_0^t e^{-rs}d\lambda_s$ where $\lambda_s=\int_0^s\phi_v dS_v-\phi_s S_s$, we can make our portfolio (ϕ,ψ) self financing. This is shown below. We have,

$$dV_t=d\left(\psi_t e^{rt}+\phi_t S_t\right)=re^{rt}\psi_t dt+e^{rt}d(\psi_t)+d(\phi_t S_t)=re^{rt}\psi_t dt+e^{rt}d(\lambda_t)+d(\phi_t S_t)=re^{rt}\psi_t dt+\phi_t dS_t$$

as required. Furthermore,

$$V_t^{\varepsilon} = e^{rt}\left(\varepsilon + \int_0^t \phi_v d\bar{S}_v\right) = e^{rt}\left(\varepsilon + \int_0^t \eta_v S_v dW_v^Q\right) = e^{rt} E^Q\left(e^{-rT} V_T^{\varepsilon} \middle| F_t\right) = e^{rt} E^Q\left(e^{-rT} C(S_T, T) \middle| F_t\right)$$

showing that V_t^{ε} is lower bounded and hence, establishing the completeness of the market.

3. 3 AN ILLUSTRATION OF THE DEFORMED MODEL

We now present a concrete example as an application of the aforesaid analysis. For the purpose, we consider a Brownian motion of the form

$$dW_t^P \rightarrow dU_t^P = f\left(U_t^P, t\right)^q dW_t^P \tag{3.17}$$

where $f\left(U_t^P, t\right)$ _is a probability density function._

The incorporation of probability dependent term in the stochastic force an enables us to describe nonlinear return processes where the randomness is not uniform across the entire return spectrum. In the standard theory, we envisage a random process that is independent of the level of returns and hence, if a sufficient number of observations are accumulated, the entire spectrum of possible returns will be traversed. However, through this deformed noise function we can model return processes that change with the respective probability of such returns i.e. the degree of randomness changes across the return spectrum – highly frequented regions of the spectrum may have higher/lower returns depending on the nature of the deformation function. Hence, a biased yet random return process can be accommodated. Although, in theory, the entire return spectrum may still be traversed if a sufficient number of observations are made, the dependence on probabilities enable the modeling of systems that require a cleavage of the return spectrum to create an effectively nonergodic space for the system. The model would also be versatile enough to encompass a return spectrum having the character of a multifractal which goes well with contemporary research findings in this area. Furthermore, unlike the standard case where $W_t^P = \int_0^t dW_t^P$ is normally distributed, $U_t^P = \int_0^t f\left(U_t^P, t\right)^q dW_t^P$ is no longer normally distributed but follows a skewed distribution depending on the explicit representation of the function $f\left(U_t^P, t\right)$ and parameter q.

Eq. (3.17) is equivalent to the Langevin equation [82]

$$\frac{dU_t^P}{dt} = f\left(U_t^P, t\right)^q \frac{dW_t^P}{dt} = f\left(U_t^P, t\right)^q \eta(t) \tag{3.18}$$

$\eta(t)$ is a noise function that satisfies

$$\langle \eta(t) \rangle = 0 \tag{3.19}$$

$$\langle \eta(t')dt' \eta(t'')dt'' \rangle = \delta(t'-t'')dt' \tag{3.20}$$

The time evolution of the probability density $f\left(U_t^p, t\right)$ is given by the following equation [93] (The super (sub) scripts are suppressed for the sake of brevity)

$$f(U, t+\Delta t) = \int \bar{f}(U, t+\Delta t | U', t) f(U', t) dU' \tag{3.21}$$

$\bar{f}$ is the transition probability between states. We now set $U' = U - \Delta U$ and expand the integrand as a Taylor's series around $\bar{f}(U+\Delta U, t+\Delta t | U, t) f(U, t)$ to obtain

$$\begin{aligned} \bar{f}(U, t+\Delta t | U', t) f(U', t) &= -\Delta U \frac{d}{dU} \bar{f}(U+\Delta U, t+\Delta t | U, t) f(U, t) + \\ &- \frac{\Delta U^2}{2} \frac{d^2}{dU^2} \bar{f}(U+\Delta U, t+\Delta t | U, t) f(U, t) + \ldots\ldots\ldots \end{aligned} \tag{3.22}$$

Eq. (3.22) on integration gives

$$\begin{aligned} f(U, t+\Delta t) &= -\frac{d}{dU}\left[\left[\int \Delta U \, \bar{f}(U+\Delta U, t+\Delta t | U, t) d\Delta U\right] f(U, t)\right] + \\ &- \frac{1}{2} \frac{d^2}{dU^2}\left[\left[\int \Delta U^2 \, \bar{f}(U+\Delta U, t+\Delta t | U, t) d\Delta U\right] f(U, t)\right] + \ldots\ldots\ldots \end{aligned} \tag{3.23}$$

We can further simplify the above expression, noting that U is a martingale, as follows:-

$$\int \Delta U \, \bar{f}(U+\Delta U, t+\Delta t | U, t) d\Delta U = E_t[\Delta U] = E_t\left[\int_t^{t+\Delta t} f(U_s, s)^q \, dW_s\right] = 0 \tag{3.24}$$

and

$$\int \Delta U^2 \bar{f}(U+\Delta U, t+\Delta t | U, t) d\Delta U = E_t\left[\Delta U^2\right] = E_t\left[\int_t^{t+\Delta t} f(U_s, s)^{2q} ds\right] = f(U_s, t)^{2q} \Delta t + o(\Delta t) \tag{3.25}$$

where the last step follows from Ito isometry. We have ignored terms of second and higher orders in Δt. Using the results in eqs. (3.24) & (3.25) in eq. (3.23) and taking the limit as $\Delta t \rightarrow 0$ we obtain the Fokker Planck equation [92] for the time evolution of the deformed probability density (3.17) as

$$\frac{df}{dt} = \frac{1}{2} \frac{d^2 f^{2q+1}}{dU^2} \tag{3.26}$$

To obtain an explicit solution of eq. (3.26) for the probability density $f(U,t)$, we postulate a normalized scaled solution, which enables the separation of the U and t dependencies through the ansatz

$$f(U,t) = g(t) H(Ug(t)) = g(t) H(z) \tag{3.27}$$

Substitution from eq. (3.27) into eq. (3.26) and simplification yields

$$\frac{g(t)}{g(t)^{2q+3}} \frac{\partial}{\partial z}(zH(z)) = \frac{1}{2} \frac{\partial^2}{\partial z^2} H(z)^{2q+1} \tag{3.28}$$

Writing $\frac{2g(t)}{g(t)^{2q+3}} = -k$, we have

$$g(t) = \left[(q+1)k(t-t_0)\right]^{-\frac{1}{2(q+1)}} \tag{3.29}$$

which gives the solution of eq. (3.26) as

$$f(U,t) = A(t-t_0)^{-\frac{1}{2(q+1)}} \exp_{(1-2q)}\left\{B\left[(U-U_0)(t-t_0)^{-\frac{1}{2(q+1)}}\right]^2\right\} \tag{3.30}$$

where $A = \left[(q+1)k\right]^{-\frac{1}{2(q+1)}}$ $B = -\frac{kA^2}{4(2q+1)}$ and $\exp_q(x) = [1+(1-q)x]^{\frac{1}{1-q}}$ is the q exponential function. k can be determined from the normalization condition $\int_{-\infty}^{\infty} f(U,t)dU = 1$, $f(U,t)$ being a probability density function.

The transition probability density $\bar{f}(U,t|U_0,t_0)$, that is the key element in option pricing, is the probability density $f(U,t)$ with a special initial condition $f(U,t_0) = \delta(U-U_0)$ i.e. $\bar{f}(U,t|U_0,t_0)$ also obeys the Fokker Planck equation (3.26). Furthermore, it is seen that the solution for $f(U,t)$ given by eq. (3.30) meets the δ function initial condition in the limit $t \to t_0$, and is, therefore, also a solution for the transition probability density $\bar{f}(U,t|U_0,t_0)$.

As an illustration, the conditional probability density of the logarithm of the stock prices would be

$$\bar{f}\left(Y_{t+\Delta t}|Y_t\right) = A\left(\Delta t\right)^{-\frac{1}{2(q+1)}} \exp_{(1-2q)}\left\{ B\left[\frac{\left(\ln\frac{S_{t+\Delta t}}{S_t} - \mu\Delta t\right)}{\sigma}\left(\Delta t\right)^{-\frac{1}{2(q+1)}}\right]^2\right\}$$

under the probability measure P and $\bar{f}\left(Y_{t+\Delta t}|Y_t\right) = A\left(\Delta t\right)^{-\frac{1}{2(q+1)}} \exp_{(1-2q)}\left\{ B\left[\frac{1}{\sigma}\left(\ln\frac{S_{t+\Delta t}}{S_t}\right)\left(\Delta t\right)^{-\frac{1}{2(q+1)}}\right]^2\right\}$ under Q.

Using the expression (3.30) for $f(U,t)$ with $U_0 = 0, t_0 = 0$ (which does not result in any loss of generality) in eq. (3.16), we derive the expression for the stock price process under the martingale measure Q and, thereby, of the contingent claim using eq. (3.14). To approximate $\int_0^t f(U,s)^{2q}\,ds$ we note that for any arbitrary value of time s, the distribution of the random variable U_s can be mapped onto the distribution of a random variable ω at a fixed time T through the transformation $U_s = \left(\frac{T}{s}\right)^{-\frac{1}{2(1+q)}} U_T$.

Hence, $\int_0^t f(U,s)^{2q} ds = \int_0^t \left[f\left(\left(\frac{T}{s}\right)^{-\frac{1}{2(1+q)}} U_T, s\right)\right]^{2q} ds$

$$= A^{2q} \int_0^t s^{-\frac{q}{(q+1)}} \exp^{2q}_{(1-2q)}\left[B\left(U_T T^{-\frac{1}{2(q+1)}}\right)^2\right] ds = Ct^{\frac{1}{(q+1)}} \exp^{2q}_{(1-2q)}\left[B\left(U_T T^{-\frac{1}{2(q+1)}}\right)^2\right] \quad (3.31)$$

where $C = (q+1)A^{2q}$.

Furthermore, $\int_0^t f(U,t)^q dW = U(t)$, in view of eq. (3.17). Substituting this result and that of eq. (3.31) in eq. (3.16), we get the following expression for the stock price process in the martingale measure Q

$$S_t = S_0 \exp\left\{\sigma U_t + rt - \frac{1}{2}\sigma^2 C t^{\frac{1}{(q+1)}} \exp^{2q}_{(1-2q)}\left[B\left(U_T T^{-\frac{1}{2(q+1)}}\right)^2\right]\right\} \quad (3.32)$$

from which the value of the call option can be recovered using (3.14). It may, however, be noted that in the standard case the exponential is linear in W and the stock price, therefore, is a monotonically increasing function of W. Hence, the condition $S_t - E > 0$ is satisfied for all values of W that exceed a threshold value. However, in this illustration, consequent to the noise induced drift; the exponential in the stock price process is now a quadratic function of the deformed Brownian motion U. We, therefore, have two roots of U that meet the condition $S_t - E = 0$. Accordingly, there will exist an interval (U_1, U_2) within which the inequality $S_t - E > 0$ will hold. Furthermore, as $q \to 0$, $U_2 \to \infty$ thereby recovering the standard case. Hence, we have

$$V_t = e^{-r(T-t)} \int_{U_1}^{U_2} \left(S_0 e^{\left\{ \sigma U_T + rT - \frac{1}{2}\sigma^2 C T^{\frac{1}{(q+1)}} \exp^{2q}_{(1-2q)} \left[B \left(U_T T^{-\frac{1}{2(q+1)}} \right)^2 \right] \right\}} - E \right) f(U_T, T) dU \quad (3.33)$$

As in the standard case, in the martingale measure based risk neutral world, the stock price distribution under Q is dependent on the risk free interest rate r and not on the average return μ. We easily recover the standard results from the generalized model in the limit $q \to 0$.

3. 4 INTERPRETATION OF THE q INDEX

Towards examining the interpretation of the q index in the context of the application being envisaged, we study the impact of the deformation of the standard exponential distribution $g(U, \zeta) = Ce^{BU^2\zeta}$. For this purpose, we note that $f(U,t)$, with $U_0 = 0, t_0 = 0$, can be expanded in the form of a gamma distribution as

$f(U,x)=A\zeta_0^{1/2}\frac{1}{\Gamma\left[(-2q)^{-1}\right]}\int_0^{\infty}x^{-\left(1+\frac{1}{2q}\right)}e^{-x\left(1+2q\zeta_0BU^2\right)}dx$ where $\zeta=t^{-(1+q)^{-1}}$. We assume that there exists a function $h(\zeta)$ that modifies the exponential distribution $g(U,\zeta)$ to $f(U,\zeta)$ i.e. that $f(U,\zeta)=A\int_0^{\infty}h(\zeta)e^{BU^2\zeta}d\zeta$. Identifying $-2q\zeta_0x$ with ζ and comparing the two expressions for f we obtain $h(\zeta)=\zeta_0^{1/2}\frac{1}{\Gamma\left[(-2q)^{-1}\right]}e^{(2q\zeta_0)^{-1}\zeta}(-2q\zeta_0)^{1/2q}\zeta^{-\left(1+\frac{1}{2q}\right)}$. Using this expression for $h(\zeta)$ we obtain the expected values of ζ and ζ^2 as $\langle\zeta\rangle=\zeta_0^{3/2}$ and $\langle\zeta^2\rangle=(1-2q)\zeta_0^{5/2}$ which gives the coefficient of variation as $(1-2q)\zeta_0^{-1/2}-1$. Hence, it follows that if $f(U,t)$ is a probability distribution function that satisfies the nonlinear Fokker Planck eq. (3.26), then its explicit representation is given as in eq. (3.30) where the parameter q is linearly related to the relative variance of $\zeta=t^{-(1+q)^{-1}}$ Furthermore, since the relative variance depends on both q and $\zeta=t^{-(1+q)^{-1}}$, it follows that the function $f(U,t)$ generates an ensemble of returns corresponding to various values of q over a particular time scale and also that, for a given q the distributions of returns evolves anomalously across differing timescales.

3. 5 EMPIRICAL EVIDENCE

The Black Scholes model assumes lognormal distributions of stock prices. However, deviations from such behaviour are, by now, well documented [116]. Empirical evidence testifies that probability distributions of stock returns are negatively skewed, have fat tails and show leptokurtosis [116]. Some of these features of empirical distributions are modelled through Levy distributions [117-120], stochastic volatility [121] or cumulant expansions [119] around the lognormal case. Each of these models, however, attempts to empirically attune the model parameters to fit observed data and hence, is equivalent to interpolating or extrapolating observed data in one form or the other. In contrast, the deformed noise model preserves the analytical framework of the Black Scholes world by retaining only one source of stochasticity and hence remaining within the domain of complete markets. It also provides a complete form solution with enables the prediction of option prices ab initio in lieu of parameter fitting to match observed data.

In this context, the probability distribution function of eq. (3.30) generates power law distributions with consequential fat tails that are characteristic of stock price distributions. This fact is brought out explicitly by writing eq. (3.30), with $U_0 = 0, t_0 = 0$, in the form:-

$$f(U,t) = At^{-\frac{1}{2(q+1)}} \exp_{(1-2q)}\left[B\left(Ut^{-\frac{1}{2(q+1)}}\right)^2\right] = At^{-\frac{1}{2(q+1)}}\left\{1+2q\left[B\left(Ut^{-\frac{1}{2(q+1)}}\right)^2\right]\right\}^{\frac{1}{2q}} \sim \left(2qA^{2q}B\right)^{\frac{1}{2q}} U^{-\frac{1}{q}} t^{-\frac{1}{2q}} \quad (3.34)$$

for sufficiently large values of t.

A plausible explanation of the matching of empirical behaviour referred to in the preceding paragraphs and the probability distribution function (3.30) is based on the observation that if the stock prices show large deviations from the averages, then

$f(U)$ would be small in line with the probabilities of extremal events being small. Since the exponent q is usually negative in the region of interest, the effective volatility would be accentuated. In terms of market behaviour, one could say that the traders would react extremally. On the other hand, mild deviations would cause moderate reactions from market players and hence, the effective volatility gets diminished.

3. 6 CONCLUSIONS

Contemporary empirical research into the behavior of stock market price /return patterns has found significant evidence that financial markets exhibit the phenomenon of anomalous diffusion, primarily super diffusion, wherein the variance evolves with time according to a power law t^{α} with $\alpha > 1.0$. The standard technique for the study of super diffusive processes is through a stochastic process that evolves according to a Langevin equation and whose probability distribution function satisfies a nonlinear Fokker Planck equation of the form (3.26). The very fact that our deformed noise function satisfies the nonlinear Fokker Planck equation is motivation enough for an adoption of this deformed Brownian motion with statistical feedback for the modeling of financial processes.

Ever since the studies of Fama in 1964-65, evidence has been accumulating against the validity of the Efficient Market Hypothesis – the existence of negatively skewed observations and fat tails and distortion around the mean values are but a few [116, [8-9,119-121]. Most financial returns, including stock returns have shown deviation from

Gaussian behaviour at short time scales with the variance not scaling with the sq. root of timescale, an attribute that is symptomatic of the possible existence of power law distributions like the one being envisaged in this study. A useful measure of quantifying deviations from the Gaussian distribution is the Hurst's exponent. If a population is Gaussian, a Hurst's exponent of 0.5 is mandated. Empirical evidence, however, shows that the Hurst's exponent for typical stock market data is around 0.6 for small timescales of about a day or less and tends to approach 0.5 asymptotically with the lengthening of the timescales. Empirical evidence also demonstrates the existence of memory effects, particularly in stock price volatilities that show long term memory effects with lag-s autocorrelations. Further, these effects tend to fall off according to a power law rather than exponentially.

Furthermore, the access to enhanced computing power during the last decade has enabled analysts to try refined methods like the phase space reconstruction methods for determining the Lyapunov Exponents [122] of stock market price data, besides doing Rescaled Analysis [123] etc. A set of several studies has indicated the existence of strong evidence that the stock market shows chaotic behavior with fractal return structures and positive Lyapunov exponents. Results of these studies have unambiguously established the existence of significant nonlinearities and chaotic behavior in these time series [125-128].

As mentioned above, several studies [116, 49-51, 97-98] adopting largely diverse and independent approaches have established the existence of the following characteristics in the behavior of stock markets:-

- Long term correlation and memory effects
- Erratic markets under certain conditions and at certain times
- Fractal time series of returns
- Less reliable forecasts with increase in the horizon

thereby establishing strong evidence for the existence of chaotic behavior. In this context, the following are conventionally accepted as the inherent characteristics of a chaotic system [129-133]:-

- Exponential divergence of trajectories in phase space;
- Sensitive dependence on initial conditions;
- Fractal dimensions;
- Critical levels and bifurcations;
- Time dependent feedback systems;
- Far from equilibrium conditions.

This provides us with a second motivation for the adoption of this deformed Brownian motion structure as a model for the random kicks since our model is based on a statistical time dependent feedback into the system. This feedback may be modeled into the system macroscopically through the explicit representation of the probability distribution function $f(U,t)$ and microscopically through the stochastic process U.

It needs to be emphasized here that the above is purely a phenomenological model for modeling stock behavior. One could, for instance, postulate that the statistical feedback at the microscopic level represents the actions and interaction of the intra trader interactions among traders constituting the market. The statistical dependency in the noise could, further, be representing the aggregate behavior of these traders. Thus, we could model a market with non homogeneous reactions with consequent biased return structures

It is fair to say that the current stage of research in financial processes is dominated by the postulation of phenomenological models that attempt to explain a limited set of market behavior. There is a strong reason for this. A financial market consists of a huge number of market players. Each of them is endowed with his own set of beliefs about rational behavior and it is this set of beliefs that govern his actions. The market, therefore, invariably generates a heterogeneous response to any stimulus. Furthermore, "rationality" mandates that every market player should have knowledge and understanding about the "rationality" of all other players and should take full cognizance in modeling his response to the market. This logic would extend to each and every market player so that we have a situation where every market player should have knowledge about the beliefs of every other player who should have knowledge of beliefs of every other player and so on. We, thus, end up with an infinitely complicated problem that would defy a solution even with the most sophisticated mathematical procedures. Additionally, unlike as there is in physics, financial economics does not possess a basic set of postulates like General Relativity and Quantum Mechanics that find homogeneous applicability to all systems in their domain of validity.

CHAPTER 4
BLACK SCHOLES OPTION PRICING WITH STOCHASTIC RETURNS ON HEDGE PORTFOLIO

Abstract

In this Chapter, we attempt another generalisation of the Black Scholes Model based on the assumption that the return process on the "hedge portfolio" follows a stochastic process similar to the Vasicek model of short-term interest rates.

4.1 INTRODUCTION

In this chapter, we attempt a generalisation of the Black Scholes Model based on the assumption that the return process on the "hedge portfolio" follows a stochastic process similar to the Vasicek model of short-term interest rates. Section 2 lists out the derivation of the Black-Scholes formula through the partial differential equation based on the construction of the complete "hedge portfolio". Section 3, which forms the essence of this chapter, attempts a generalisation of the standard Black Scholes pricing formula on the lines aforesaid. Section 4 concludes.

4.2 THE BLACK SCHOLES MODEL

In order to facilitate continuity, we summarize below the original derivation of the Black Scholes model for the pricing of a European call option [1-2, 25, 114-115] and references therein. The option contract, has a payoff of $\max(S_T - E, 0) = (S_T - E)^+$ on the maturity date where S_T is the stock price on the maturity date and E is the exercise price.

We consider a non-dividend paying stock, the price process of which follows the geometric Brownian motion with drift $S_t = e^{(\mu t + \sigma W_t)}$. The logarithm of the stock price $Y_t = In\ S_t$ follows the stochastic differential equation

$$dY_t = \mu dt + \sigma dW_t \tag{4.1}$$

where W_t is a regular Brownian motion representing Gaussian white noise with zero mean and δ correlation in time i.e. $E(dW_t dW_{t'}) = dtdt'\delta(t - t')$ on some filtered probability space $(\Omega, (F_t), P)$ and μ and σ are constants representing the long term drift and the noisiness (diffusion) respectively in the stock price.

Application of Ito's formula yields the following SDE for the stock price process

$$dS_t = \left(\mu + \frac{1}{2}\sigma^2\right)S_t dt + \sigma S_t dW_t \tag{4.2}$$

Let $C(S,t)$ denote the instantaneous price of a call option with exercise price E at any time t before maturity when the price per unit of the underlying is S. It is assumed that

$C(S,t)$ does not depend on the past price history of the underlying. Applying the Ito formula to $C(S,t)$ yields

$$dC = (\mu S\frac{\partial C}{\partial S}+\frac{1}{2}\sigma^2 S\frac{\partial C}{\partial S}+\frac{\partial C}{\partial t}+\frac{1}{2}\sigma^2 S^2\frac{\partial^2 C}{\partial S^2})dt+\frac{\partial C}{\partial S}\sigma S dW, \tag{4.3}$$

The original option-pricing model propounded by Fischer Black and Myron Scholes envisaged the construction of a "hedge portfolio", Π, consisting of the call option and a short sale of the underlying such that the randomness in one cancels out that in the other. For this purpose, we make use of a call option together with $\partial C/\partial S$ units of the underlying stock.

We then have, on applying Ito's formula to the "hedge portfolio", Π,:-

$$\frac{d\Pi}{dt}=\frac{d}{dt}\left[C(S,t)-S\frac{\partial C(S,t)}{\partial S}\right]=\frac{dC(S,t)}{dt}-\frac{\partial C}{\partial S}.\frac{dS}{dt} \tag{4.4}$$

where the term involving $\frac{d}{dt}\left(\frac{\partial C}{\partial S}\right)$ has been assumed zero since it envisages a change in the portfolio composition. On substituting from eqs. (4.2) & (4.3) in (4.4), we obtain

$$\frac{d\Pi}{dt}=\frac{dC(S,t)}{dt}-\left(\mu+\frac{1}{2}\sigma^2\right)S\frac{\partial C(S,t)}{\partial S}-\sigma S\frac{\partial C}{\partial S}\frac{dW}{dt}=\frac{\partial C(S,t)}{\partial t}+\frac{1}{2}\sigma^2 S^2\frac{\partial^2 C(S,t)}{\partial S^2} \tag{4.5}$$

We note, here, that the randomness in the value of the call price emanating from the stochastic term in the stock price process has been eliminated completely by choosing the portfolio $\Pi = C(S,t) - S\frac{\partial C(S,t)}{\partial S}$. Hence, the portfolio Π is free from any stochastic noise and the consequential risk attributed to the stock price process.

Now $\frac{d\Pi}{dt}$ is nothing but the rate of change of the price of the so-called riskless bond portfolio i.e. the return on the riskless bond portfolio (since the equity related risk is assumed to be eliminated by construction, as explained above) and must, therefore, equal the short-term interest rate r i.e.

$$\frac{d\Pi}{dt} = r\Pi. \tag{4.6}$$

In the original Black Scholes model, this interest rate was assumed as the risk free interest rate r, further, assumed to be constant, leading to the following partial differential equation for the call price:-

$$\frac{d\Pi}{dt} = r\Pi = r\left[C(S,t) - S\frac{\partial C(S,t)}{\partial S}\right] = \frac{\partial C(S,t)}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C(S,t)}{\partial S^2}$$

or equivalently

$$\frac{\partial C(S,t)}{\partial t} + \frac{1}{2}\sigma^2 S\frac{\partial^2 C(S,t)}{\partial S^2} + rS\frac{\partial C(S,t)}{\partial S} - rC(S,t) = 0 \tag{4.7}$$

which is the famous Black Scholes PDE for option pricing with the solution:-

$$C(S,t) = SN(d_1) - Ee^{-r(T-t)}N(d_2) \tag{4.8}$$

where

$$d_1 = \frac{\log\left(\frac{S}{E}\right) + \left(r + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}},\ d_2 = d_1 - \sigma\sqrt{T-t} = \frac{\log\left(\frac{S}{E}\right) + \left(r - \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}} \text{ and}$$

$$N(y) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{y} e^{-\frac{x^2}{2}}dx$$

4.3 THE BLACK SCHOLES MODEL WITH STOCHASTIC RETURNS ON THE HEDGE PORTFOLIO

As mentioned earlier, in the above analysis, the interest rate r, which is essentially a proxy for the return on a portfolio that is devoid of any risk emanating from any variables that cause fluctuations and hence risk in the stock price process, is taken as constant and equal to the risk free rate. However, this return would, nevertheless, be subject to uncertainties that influence returns on the fixed income securities. It is, now, conventional to model these short term interest rates (that are representative of short term returns on fixed income securities) through a stochastic differential equation of the form [134]

$$dr(t) = -\psi\left[r(t),t\right]dt + \eta\left[r(t),t\right]dU(t) \tag{4.9}$$

where $r(t)$ is the short term interest rate at time t, ψ and η are deterministic functions of r,t and $U(t)$ is a Wiener Process.

In our further analysis, we shall assume that this short-term interest rate is represented by the Vasicek model [135] viz.

$$\frac{dr(t)}{dt} + Ar(t) + B - \Sigma\eta(t) = 0 \tag{4.10}$$

where $\eta(t)$ is a white noise stochastic process

$$\langle\eta(t)\rangle = 0, \langle\eta(t)\eta(t')\rangle = \Sigma^2\delta(t-t') \tag{4.11}$$

The call price process now becomes a function of two stochastic variables, the stock price process $S(t)$ and the bond return process (interest rate process) $r(t)$. Hence, application of Ito's formula to $C(S,r,t)$ gives

$$dC = \frac{\partial C}{\partial t}dt + \frac{\partial C}{\partial S}dS + \frac{\partial C}{\partial r}dr + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 C}{\partial S^2}dt + \frac{1}{2}\Sigma^2\frac{\partial^2 C}{\partial r^2}dt \tag{4.12}$$

where dS is given by eq. (4.2) and dr by (4.10) respectively.

As in Section 2, we formulate a "hedge portfolio" Π consisting of a call option $C(S,r,t)$ and a short sale of $\frac{\partial C}{\partial S}$ units of stock $S(t)$ i.e. $\Pi = C - S\frac{\partial C}{\partial S}$. We then have, repeating the same steps as in Section 2 hereof

$$\frac{d\Pi}{dt} = \frac{dC}{dt} - \frac{\partial C}{\partial S}\frac{dS}{dt} = \frac{\partial C}{\partial t} + \frac{\partial C}{\partial r}.\frac{dr}{dt} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + \frac{1}{2}\Sigma^2 \frac{\partial^2 C}{\partial r^2} \tag{4.13}$$

Now, using $\frac{d\Pi}{dt} = r(t)\Pi$, we obtain

$$\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + r(t)S\frac{\partial C}{\partial S} - r(t)C + \frac{1}{2}\Sigma^2 \frac{\partial^2 C}{\partial r^2} + \frac{\partial C}{\partial r}\frac{dr(t)}{dt} = 0 \tag{4.14}$$

This equation defies closed form solution with the extant mathematical apparatus. We can, however, obtain explicit expressions for the call price $\overline{C}(S,t)$ averaged over the stochastic part of the interest rate process, as follows:-

$\overline{C}(S,t)$ would, then, be given by substituting $\frac{\int_t^T r(\tau)d\tau}{\int_t^T d\tau}$ for the constant risk free interest rate r in the Black Scholes formula (4.8).

The averaging process happens to be tedious with extensive computations so we proceed term by term.

We have

$$N(\overline{d}_1) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\overline{d}_1} e^{-\frac{x^2}{2}}dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} H(\overline{d}_1 - x)e^{-\frac{x^2}{2}}dx \qquad (4.15)$$

where $H(x-y)$ is the unit step Heaviside step function defined by [20]

$$H(x,y) = \begin{cases} 0, & x < y \\ 1, & x > y \end{cases}$$

On using the integral representation of $H(x-y)$ as $H(x-y) = Lim_{\varepsilon\to 0}\frac{1}{2\pi i}\int_{-\infty}^{\infty} d\omega \frac{e^{i\omega(x-y)}}{\omega - i\varepsilon}$ [20]

i.e. $$H(\overline{d}_1 - x) = Lim_{\varepsilon\to 0}\frac{1}{2\pi i}\int_{-\infty}^{\infty} d\omega \frac{e^{i\omega(\overline{d}_1 - x)}}{\omega - i\varepsilon} \qquad (4.16)$$

we obtain

$$N\left(\overline{d}_1\right) = Lim_{\varepsilon\to 0}\frac{1}{(2\pi)^{\frac{3}{2}} i}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{e^{-\frac{x^2}{2}+i\omega\left(\overline{d}_1 - x\right)}}{\omega - i\varepsilon}dx d\omega = Lim_{\varepsilon\to 0}\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{e^{i\omega\left(\overline{d}_1 + i\frac{\omega}{2}\right)}}{\omega - i\varepsilon}d\omega \tag{4.17}$$

on performing the Gaussian integration over x in the second step.

Now

$$\overline{d}_1 = \frac{\log\left(\frac{S}{E}\right)+\frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}} + \frac{\int_t^T r(\tau)d\tau}{\sigma\sqrt{T-t}} = d_1^0 + \frac{\int_t^T r(\tau)d\tau}{\sigma\sqrt{T-t}} \tag{4.18}$$

$$\text{where } d_1^0 = \frac{\log\left(\frac{S}{E}\right)+\frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}} \tag{4.19}$$

Since the entire stochastic contribution comes from the expression $\int_t^T r(\tau)d\tau$ in $N\left(\overline{d}_1\right)$, we have

$$N\left(\overline{d}_1\right) = Lim_{\varepsilon\to 0}\frac{1}{2\pi i}\int_{-\infty}^{\infty} d\omega\frac{e^{i\omega d_1^0 - \frac{\omega^2}{2}}}{\omega - i\varepsilon} I_1 \tag{4.20}$$

where $I_1 = \left\langle e^{\frac{i\omega}{\sigma\sqrt{T-t}}\int_t^T r(\tau)d\tau}\right\rangle$ and $\langle\rho\rangle$ denotes the average (expectation) of ρ. (4.21)

Proceeding similarly, we have,

$$N\left(\overline{d_2}\right) = Lim_{\varepsilon \to 0} \frac{1}{2\pi i} \int_{-\infty}^{\infty} d\omega \frac{e^{i\omega d_2^0 - \frac{\omega^2}{2}}}{\omega - i\varepsilon} I_1 \tag{4.22}$$

$$d_2^0 = \frac{\log\left(\frac{S}{E}\right) - \frac{1}{2}\sigma^2 (T-t)}{\sigma\sqrt{T-t}} \tag{4.23}$$

Similarly the discount factor $e^{-r(T-t)}$ will be replaced by $\left\langle e^{-\int_t^T r(\tau)d\tau} \right\rangle = I_2$ (say).

To evaluate the expectation integrals I_1, I_2 we make use of the functional integral formalism [137]. In this formalism, the expectation I_1 would be given by [85]:-

$$I_1 = \frac{\int_{r(t)}^{r(T)} Dr \exp\left[-\frac{1}{2\Sigma^2} \int_t^T d\tau \left(\frac{dr(\tau)}{d\tau} + Ar(\tau) + B \right)^2 + \frac{i\omega}{\sigma\sqrt{T-t}} \int_t^T d\tau r(\tau) \right]}{\int_{r(t)}^{r(T)} Dr \exp\left[-\frac{1}{2\Sigma^2} \int_t^T d\tau \left(\frac{dr(\tau)}{d\tau} + Ar(\tau) + B \right)^2 \right]} = \frac{P}{Q} \tag{4.24}$$

where $Dr = \prod_{\tau=t}^{T} \frac{dr(\tau)}{\sqrt{2\pi}}$ is the functional integration measure.

We first evaluate the functional integral P. Making the substitution $x(\tau) = -\frac{B}{A} - r(\tau)$, we obtain, with a little algebra,

$$P=\int_{x(t)}^{x(T)} Dx\exp\left[-\frac{1}{2\Sigma^2}\int_t^T d\tau\left(\frac{dx(\tau)}{d\tau}+Ax(\tau)\right)^2+\frac{i\omega}{\sigma\sqrt{T-t}}\int_t^T d\tau\left(-\frac{B}{A}-x(\tau)\right)\right]$$

$$=\int_{x(t)}^{x(T)} Dx\exp\left\{\frac{-1}{2\Sigma^2}\int_t^T d\tau\left[\left(\frac{dx(\tau)}{d\tau}\right)^2+A^2x^2(\tau)\right]-\frac{A}{2\Sigma^2}\left[x^2(T)-x^2(t)\right]-\frac{i\omega B(T-t)}{A\sigma\sqrt{T-t}}-\frac{i\omega}{\sigma\sqrt{T-t}}\int_t^T x(\tau)d\tau\right\}$$

$$=\int_{x(t)}^{x(T)} Dx\exp\left\{-\frac{A}{2\Sigma^2}\left[x^2(T)-x^2(t)\right]-\frac{i\omega B(T-t)}{A\sigma\sqrt{T-t}}-I_3\right\} \tag{4.25}$$

where

$$I_3=\frac{1}{2\Sigma^2}\int_t^T d\tau\left[\left(\frac{dx(\tau)}{d\tau}\right)^2+A^2x^2(\tau)\right]+\frac{i\omega}{\sigma\sqrt{T-t}}\int_t^T x(\tau)d\tau$$

$$=\frac{1}{2\Sigma^2}\int_t^T d\tau\left[\left(\frac{dx(\tau)}{d\tau}\right)^2+A^2x^2(\tau)+\frac{2i\omega\Sigma^2}{\sigma\sqrt{T-t}}x(\tau)\right] \tag{4.26}$$

In order to evaluate I_3, we perform a shift of the functional variable $x(\tau)$ by some fixed function $y(\tau)$ i.e. $x(\tau)=y(\tau)+z(\tau)$ where $y(\tau)$ is a fixed functional (whose explicit form shall be defined later) but with boundary conditions $y(t)=x(t), y(T)=x(T)$ so that $z(\tau)$, then, has Drichlet boundary conditions i.e. $z(t)=z(T)=0$.

Substituting $x(\tau) = y(\tau) + z(\tau)$ in (4.26), we obtain

$$I_3 = \frac{1}{2\Sigma^2} \int_t^T d\tau \left[\begin{array}{l} \left(\frac{dy(\tau)}{d\tau}\right)^2 + \left(\frac{dz(\tau)}{d\tau}\right)^2 + 2\left(\frac{dy(\tau)}{d\tau}\right)\left(\frac{dz(\tau)}{d\tau}\right) + A^2 y^2(\tau) + \\ A^2 z^2(\tau) + 2A^2 y(\tau) z(\tau) + \frac{2i\omega\Sigma^2}{\sigma\sqrt{T-t}} y(\tau) + \frac{2i\omega\Sigma^2}{\sigma\sqrt{T-t}} z(\tau) \end{array} \right] \tag{4.27}$$

Integrating the second and third term by parts, we get

$$I_3 = \frac{z(\tau)}{2\Sigma^2}\left(\frac{dz(\tau)}{d\tau} + 2\frac{dy(\tau)}{d\tau}\right)\Bigg|_t^T + \frac{1}{2\Sigma^2} \int_t^T d\tau \left[\begin{array}{l} -z(\tau)\frac{d^2 z(\tau)}{d\tau^2} - 2z(\tau)\frac{d^2 y(\tau)}{d\tau^2} + \left(\frac{dy(\tau)}{d\tau}\right)^2 + A^2 y^2(\tau) + \\ A^2 z^2(\tau) + 2A^2 y(\tau) z(\tau) + \frac{2i\omega\Sigma^2}{\sigma\sqrt{T-t}} y(\tau) + \frac{2iw\Sigma^2}{\sigma\sqrt{T-t}} z(\tau) \end{array} \right] \tag{4.28}$$

Now the boundary terms all vanish since $z(\tau)$ has Drichlet boundary conditions.

Further, if we define the fixed functional $y(\tau)$ in terms of the differential equation

$$-\frac{d^2 y(\tau)}{d\tau^2} + A^2 y(\tau) + \frac{i\omega\Sigma^2}{\sigma\sqrt{T-t}} = 0 \tag{4.29}$$

with boundary condition $y(t) = x(t), y(T) = x(T)$ we obtain

$$I_3 = \frac{1}{2\Sigma^2} \int_t^T d\tau \left\{ \left[\left(\frac{dy(\tau)}{d\tau}\right)^2 + A^2 y^2(\tau) + \frac{2i\omega\Sigma^2}{\sigma\sqrt{T-t}} y(\tau) \right] + \left[-z(\tau)\frac{d^2 z(\tau)}{d\tau^2} + A^2 z^2(\tau) \right] \right\} \tag{4.30}$$

The functional $y(\tau)$ is fixed and is given by the solution of eq (29) as

$$y = \alpha e^{A\tau} + \beta e^{-A\tau} - \gamma \qquad (4.31)$$

where

$$\gamma = \frac{i\omega\Sigma^2}{A^2\sigma\sqrt{T-t}}, \alpha = \frac{x(T)e^{AT} - x(t)e^{At}}{e^{2AT} - e^{2At}} + \gamma\frac{e^{AT} - e^{At}}{e^{2AT} - e^{2At}} \text{ and } \beta = \frac{x(T)e^{-AT} - x(t)e^{-At}}{e^{-2AT} - e^{-2At}} + \gamma\frac{e^{-AT} - e^{-At}}{e^{-2AT} - e^{-2At}}.$$

Integrating out the $y(\tau)$ terms in eq. (4.30) using eq. (4.31), we obtain

$$I_3 = \frac{1}{2\Sigma^2}\left\{A\left[\alpha^2\left(e^{2AT} - e^{2At}\right) - \beta^2\left(e^{-2AT} - e^{-2At}\right) - A\gamma^2(T-t)\right] + \int_t^T d\tau\left[-z(\tau)\frac{d^2z(\tau)}{d\tau^2} + A^2z^2(\tau)\right]\right\} \qquad (4.32)$$

Substituting this value of I_3 in eq. (4.25) we obtain, for P, noting that $Dx = Dz$ since $y(\tau)$ is fixed by eq (4.29)

$$P=\exp\left\{-\frac{A^2}{2\Sigma^2}\left[x^2(T)-x^2(t)\right]-\frac{i\omega B\sqrt{T-t}}{A\sigma}-\frac{1}{2\Sigma^2}\left[\begin{array}{l} A\left[\left(\dfrac{x(T)e^{AT}-x(t)e^{At}}{e^{2AT}-e^{2AT}}\right)^2-\left(\dfrac{x(T)e^{-AT}-x(t)e^{-At}}{e^{-2AT}-e^{-2At}}\right)^2\right] \\ -\left(\dfrac{\omega^2\Sigma^4}{A^3\sigma^2(T-t)}\right)\left[\left(\dfrac{e^{AT}-e^{At}}{e^{2AT}-e^{2At}}\right)^2-\left(\dfrac{e^{-AT}-e^{-At}}{e^{-2AT}-e^{-2At}}\right)^2-A(T-t)\right] \\ +\left(\dfrac{2i\omega\Sigma^2}{A\sigma\sqrt{T-t}}\right)\left[\begin{array}{l}\dfrac{\left(x(T)e^{AT}-x(t)e^{At}\right)\left(e^{AT}-e^{At}\right)}{\left(e^{2AT}-e^{2At}\right)^2}- \\ \dfrac{\left(x(T)e^{-AT}-x(t)e^{-At}\right)\left(e^{-AT}-e^{-At}\right)}{\left(e^{-2AT}-e^{-2At}\right)^2}\end{array}\right]\end{array}\right]\right\}$$

$$\int_{z(t)=0}^{z(T)=0} Dz\exp\left\{\frac{-1}{2\Sigma^2}\int_t^T d\tau\left[-z(\tau)\frac{d^2z(\tau)}{d\tau^2}+A^2z^2(\tau)\right]\right\} \tag{4.33}$$

On exactly same lines, we obtain

$$Q=\exp\left\{-\frac{A^2}{2\Sigma^2}\left[x^2(T)-x^2(t)\right]-\frac{1}{2\Sigma^2}\left[A\left[\left(\frac{x(T)e^{AT}-x(t)e^{At}}{e^{2AT}-e^{2AT}}\right)^2-\left(\frac{x(T)e^{-AT}-x(t)e^{-At}}{e^{-2AT}-e^{-2At}}\right)^2\right]\right]\right\}$$

$$\int_{z(t)=0}^{z(T)=0} Dz\exp\left\{\frac{-1}{2\Sigma^2}\int_t^T d\tau\left[-z(\tau)\frac{d^2z(\tau)}{d\tau^2}+A^2z^2(\tau)\right]\right\} \tag{4.34}$$

Hence

$$
I_1=\exp\left\{\frac{i\omega B\sqrt{T-t}}{A\sigma}-\frac{1}{2\Sigma^2}\left[\begin{array}{l}-\left(\dfrac{\omega^2\Sigma^4}{A^3\sigma^2(T-t)}\right)\left[\left(\dfrac{e^{AT}-e^{At}}{e^{2AT}-e^{2At}}\right)^2-\left(\dfrac{e^{-AT}-e^{-At}}{e^{-2AT}-e^{-2At}}\right)^2-A(T-t)\right]\\ +\left(\dfrac{2i\omega\Sigma^2}{A\sigma\sqrt{T-t}}\right)\left[\begin{array}{l}\dfrac{\left(x(T)e^{AT}-x(t)e^{At}\right)\left(e^{AT}-e^{At}\right)}{\left(e^{2AT}-e^{2At}\right)^2}-\\ \dfrac{\left(x(T)e^{-AT}-x(t)e^{-At}\right)\left(e^{-AT}-e^{-At}\right)}{\left(e^{-2AT}-e^{-2At}\right)^2}\end{array}\right]\end{array}\right]\right\} \tag{4.35}
$$

which when substituted in eqs. (4.20) & (4.22) shall give the values $N(\bar{d}_1)$ and $N(\bar{d}_2)$ respectively as:-

$$
N(\bar{d}_1)=N\left\{\frac{\dfrac{\log\left(\dfrac{S}{E}\right)+\dfrac{1}{2}\sigma^2}{\sigma\sqrt{T-t}}-\dfrac{B\sqrt{T-t}}{A\sigma}-\dfrac{Y}{A\sigma\sqrt{T-t}}}{\left[1-\dfrac{\Sigma^2X}{A^3\sigma^2(T-t)}\right]^{\frac{1}{2}}}\right\} \tag{4.36}
$$

and

$$
N(\bar{d}_2)=N\left\{\frac{\dfrac{\log\left(\dfrac{S}{E}\right)-\dfrac{1}{2}\sigma^2}{\sigma\sqrt{T-t}}-\dfrac{B\sqrt{T-t}}{A\sigma}-\dfrac{Y}{A\sigma\sqrt{T-t}}}{\left[1-\dfrac{\Sigma^2X}{A^3\sigma^2(T-t)}\right]^{\frac{1}{2}}}\right\} \tag{4.37}
$$

where

$$
X=\left(\frac{e^{AT}-e^{At}}{e^{2AT}-e^{2At}}\right)^2-\left(\frac{e^{-AT}-e^{-At}}{e^{-2AT}-e^{-2At}}\right)^2-A(T-t)\ \text{and} \tag{4.38}
$$

$$Y = \frac{\left(x(T)e^{AT} - x(t)e^{At}\right)\left(e^{AT} - e^{At}\right)}{\left(e^{2AT} - e^{2At}\right)^2} \frac{\left(x(T)e^{-AT} - x(t)e^{-At}\right)\left(e^{-AT} - e^{-At}\right)}{\left(e^{-2AT} - e^{-2At}\right)^2} \tag{4.39}$$

To evaluate I_2, we substitute $\omega = i\sigma\sqrt{T-t}$ in eq. (4.35) to get

$$I_2 = \exp\left\{\frac{B(T-t)}{A} \frac{1}{2\Sigma^2}\left[\left(\frac{\Sigma^4}{A^3}\right)X - \left(\frac{2\Sigma^2}{A}\right)Y\right]\right\} \tag{4.40}$$

The closed form solution for the Black Scholes pricing problem with stochastic return on the "hedge portfolio" can now be obtained by substituting the above averages in eq. (4.8).

4.4 CONCLUSION

In this Chapter, we have obtained closed form expressions for the price of a European call option by modifying the Black Scholes formulation to accommodate a stochastic return process for the "hedge portfolio" returns. We have modelled this return process on the basis of the Vasicek model for the short-term interest rates. The need for this extension of the Black Scholes model is manifold. Firstly, the construction of the "hedge portfolio" in the Black Scholes theory implies that the fluctuations in the price of the derivative and that of the underlying exactly and immediately cancel each other when combined in a certain proportion viz. one unit of the derivative with a short sale of

$\frac{\partial C}{\partial S}$ units of the underlying so that the "hedge portfolio" is devoid of any impact of such fluctuations. This mandates an infinitely fast reaction mechanism of the underlying market dynamics whereby any movement in the price of one asset is instantaneously annulled by reactionary response in the other asset constituting the "hedge portfolio". This is, obviously strongly unrealistic and there may subsist brief periods or aberrations when the no arbitrage condition may cease to hold and hence, returns on the "hedge portfolio" may be different from the risk free rate. One way of attending to this anomaly is to model the returns on the "hedge portfolio" as a stochastic process as has been done in this study. The parameters defining the process can be obtained through an empirical study of the market dynamics. Another important justification for adopting a stochastic framework for the "hedge portfolio" return process is that the "hedge portfolio" by its very construction, envisages the neutralization of the fluctuations of the two assets inter se i.e. it assumes a perfect correlation between the two assets. In other words, the "hedge portfolio" may be construed as an isolated system that is such that insofar as factors that influence one component of the system, the same factors influence the other component to an equivalent extent and, at the same time, other factors do not impact the system at all. This is another anomaly that distorts the Black Scholes model. The fact is that while the "hedge portfolio" of the Black Scholes model is immunized against price fluctuations of the underlying and its derivative through mutual interaction, other market factors that would impact the portfolio as a whole are not accounted for e.g. factors affecting bond yields and interest rates etc. Consequently, to assume that the "hedge portfolio" is completely risk free is another aberration – it is risk free only to the extent of risk that emanates from factors that impact the underlying and the derivative in like manner and is still subject to risk and uncertainties that originate from factors that either do not effect the underlying and the derivative to equivalent extent or impact the portfolio as a unit entity. Hence, again, it becomes necessary to model the return on the "hedge portfolio" as some short-term interest rate model as has been done here.

CHAPTER 5
CONSTRUCTION OF DEFORMED LEVY PROCESSES & OPTION PRICING

Abstract

The Black Scholes- Merton theory expounded in a Noble prize winning work in the 1970s still remains the mainstay of valuation of contingent claims. However, with the gradual realization among finance practitioners that significant deviations exist from Gaussian behaviour in stock market price, efforts have been focused on identifying other types of stochastic processes for modeling market evolution. The Levy processes, of which Brownian motion and Poisson processes are special cases, provide a versatile alternative and are considered in this work. The framework of option pricing is extended to Levy processes generated by deformed pseudodifferential operators.

5. 1 INTRODUCTION

The pioneering work of Fischer Black, Myron Scholes [1] and Robert Merton [137-138] in the pricing of contingent claims continues, even after almost four decades, to hold centrestage in option pricing. The merits of the Merton Black Scholes (MBS) theory are two fold. On the one hand, the robustness of the model lends its successful applicability to a wide spectrum of real life financial products. On the other hand, the intrinsic simplicity of the model makes it practically tractable with modest computing tools. The model also works on minimal inputs viz. the risk free rate, the spot price of the underlying and the historic volatility. The last two attributes emanate from the choice of the simplest class of stochastic processes – the Gaussian processes – to model the fluctuations of the underlying asset. This enables the representation of the evolution dynamics of the call option as a partial differential equation amenable to closed form solutions as a boundary value problem [1].

Deviations from Gaussianity in the behavior of stock prices are, now, well accepted with the existence of "fat tails" and a higher and displaced peak [48, 50]. Efforts of analysts have, therefore, been focused on identifying/developing alternative stochastic processes to model the market price dynamics. The existence of the "volatility smile" further endorses the inadequacy of the MBS theory. In this context, it has been shown by empirical studies that that the central part of the probability distribution is relatively better fitted by the stable Levy process [137-138]. Stock prices lend themselves naturally to modeling by the "jump diffusion models" that comprise of a superposition of independent Brownian motion and Poisson processes [139]. However, providing for a large number of jumps makes the model untractable.

5. 2 LEVY PROCESSES [139-140]

Levy processes constitute a very wide family of stochastic processes. Such processes possess the cardinal characteristic of "stationary independent increments". The literature on physical sciences, engineering and, even, economics is abound with applications of such processes. Some of the reasons for the immense popularity of Levy processes in modeling of systems emanates include, in particular:-

(a) A wide variety of processes like the Brownian motion, Poisson processes, subordinators, stable processes can be encompasses in the general formalism of Levy processes by appropriate choice of the parameters describing the process;

(b) They constitute the simplest class of processes consisting of continuous motion interspersed by jumps of random sizes occurring at random intervals of time;

(c) They can also be adapted to model self similar processes (fractals), semimartingales, Feller processes etc.

(d) They result in stochastic differential equations that are largely amenable to analytic and closed form solutions.

5.3 PROPERTIES OF LEVY PROCESSES [139-140]

We enlist here some of the cardinal properties of Levy processes that make them the immensely interesting objects that they are. These properties will also be used in the sequel and hence, facilitate continuity in this article. We shall work in one dimension since that is relevant to the ultimate purpose of this work although generalizations to higher dimensions are straight forward.

An $\mathbf{R}$-valued stochastic process $\{X_t : t \geq 0\}$ is a family of $\mathbf{R}$- random valued variables $X_t(\omega)$ with parameter $t \in (0,\infty)$, defined on a probability space (Ω, F, P).

An $\mathbf{R}$-valued stochastic process $\{X_t : t \geq 0\}$ is called Levy process on $\mathbf{R}$ or 1-dimensional Levy process, if the following five conditions are satisfied:-

a. It has independent increments, that is, for any choice of $n \geq 1$ and $0 \leq t_0 < t_1 < \ldots < t_n$, the random variable $X_{t_0}, X_{t_1} - X_{t_0}, X_{t_2} - X_{t_1}, \ldots, X_{t_n} - X_{t_{n-1}}$ are independent;

b. It starts at the origin, $X_0 = 0$ a.s.(almost surely);

c. It is time homogeneous, that is, the distribution of $\{X_{s+t} - X_s : t \geq 0\}$ does not depend on s;

d. It is stochastically continuous, that is, for any $\varepsilon > 0, P\left[\left|X_{s+t} - X_s\right| > \varepsilon\right] \to 0$ as $t \to 0$;

e. As a function of $t, X_t(\omega)$ is right-continuous with left limits a.s.

5.4 LEVI KHINTCHINE REPRESENTATION [139-140]

The Levi Khintchine representation of a Levy process explicitly brings out the relationship between Levy processes and pseudodifferential operators. We have, if μ is infinitely divisible distribution on $\mathbf{R}$, then there exists a uniquely defined Levy process $\{X_t : t \geq 0\}$ satisfying (a) - (d) such that the distribution of X_t, designated $\Xi(X_1) = \mu$ with the characteristic function $\hat{\mu}(z)$ defined by

$$\hat{\mu}(z) \equiv \int_{\mathbf{R}} e^{izx} \mu(dx), \; z \in \mathbf{R} \tag{5.1}$$

is given by

$$\hat{\mu}(z) = \exp\left[-\frac{1}{2}\sigma^2 z^2 + i\gamma z + \int_{\mathbf{R}} \left(e^{izx} - 1 - izx 1_{\{|x| \le 1\}}(x)\right) \nu(dx)\right] \tag{5.2}$$

where $\sigma \ge 0$, ν is a measure on $\mathbf{R}$ satisfying $\nu(\{0\}) = 0$ and $\int_{\mathbf{R}} \min\left(1, |x|^2\right) \nu(dx) < \infty$, and $\gamma \in \mathbf{R}$ is constant. The representation (5.2) by σ, ν, and γ is unique. σ^2 is called the Gaussian coefficient and ν is called the Levy measure, (σ^2, γ, ν) is called the generating triplet.

For any set B we use $1_B(x)$ for the indicator function of B. It follows that the Levy process $\{X_t\}$ corresponding to μ has the characteristic function given by

$$\begin{aligned} E\left(e^{izX_t}\right) &= e^{-t\psi(z)} = \left[\hat{\mu}(z)\right]^t \\ &= \exp\left\{t\left[-\frac{1}{2}\sigma^2 z^2 + i\gamma z + \int_{\mathbf{R}} \left(e^{izx} - 1 - izx 1_{\{|x| \le 1\}}(x)\right) \nu(dx)\right]\right\} \end{aligned} \tag{5.3}$$

The expression

$$\psi(z) = \left[\frac{1}{2}\sigma^2 z^2 - i\gamma z - \int_{\mathbf{R}} \left(e^{izx} - 1 - izx 1_{\{|x| \le 1\}}(x)\right) \nu(dx)\right] \tag{5.4}$$

is called the characteristic exponent of the given Levy process. We denote $\mu^t = \Xi(X_t)$. The objective of having the γ term is to make $izx 1_{\{|x| \le 1\}}(x)$ in the integrand in (5.2) ν-integrable. If ν satisfies $\int_{|x| \le 1} |x| \nu(dx) < \infty$, then (5.2) can be written as

$$\hat{\mu}(z) = \exp\left[-\frac{1}{2}\sigma^2 z^2 + +i\gamma_0 z + \int_{\mathbf{R}} \left(e^{izx} - 1\right) \nu(dx)\right] \tag{5.5}$$

with some $\gamma_0 \in \mathbf{R}$. This γ_0 is called the drift.

If ν satisfies $\int_{|x|>1} |x| \nu(dx) < \infty$, then we have an expression

$$\hat{\mu}(z) = \exp\left[-\frac{1}{2}\sigma^2 z^2 + i\gamma_1 z + \int_{\mathbf{R}} \left(e^{izx} - 1 - izx\right)\nu(dx)\right] \tag{5.6}$$

with some $\gamma_1 \in \mathbf{R}$, called the centre. In this case, it can be shown that $\gamma_1 = \int_{\mathbf{R}} x\mu(dx)$.

(a) Brownian motion is a Levy process with $\sigma = 1$, $\nu = 0$ and $\gamma = 0$;

(b) Poisson process with intensity $c > 0$ is a Levy process on $\mathbf{R}$ with $\sigma = 0, \gamma_0 = 0$, and $\nu = c\delta_1$, where we denote by δ_a the distribution concentrated at a;

(c) The compound Poisson process corresponds to a Levy process on $\mathbf{R}$ with $\sigma = 0, \nu(\mathbf{R}) < \infty$ and $\gamma_0 = 0$;

(d) The Γ process is a Levy process on $\mathbf{R}$ with $\sigma = 0, \nu(dx) = 1_{(0,\infty)}(x)x^{-1}e^{-qx}dx$, and $\gamma_0 = 0$ so that $\mu^t = 1_{(0,\infty)}(x)\left(q^t/\Gamma(t)\right)x^{t-1}e^{-qx}dx$ $(\Gamma - distribution)$.

5.5 LEVY PROCESSES & PSEUDODIFFERENTIAL OPERATORS [141-144]

Consider a Levy process $\{X_t\}_{t\geq 0}$ on $\mathbf{R}$. Its probability distribution – hence the process – is completely determined through the characteristic exponent $\psi : \mathbf{R} \to \mathbf{C}$ given by eq. (5.4). Let, now, the space of twice continuously differential functions vanishing at infinity. Then for, each $x \in \mathbf{R}$, there exists a limit

$$(Lf)(x) := \lim_{t \downarrow 0} \frac{E[f(x+X_t)] - f(x)}{t} \tag{5.7}$$

with $Lf \in C_0(\mathbf{R})$, being the space of continuous functions vanishing at infinity. We call the map $f \mapsto Lf$ the infinitesimal generator of the process $\{X_t\}_{t\geq 0}$ its explicit representation is given by

$$Lf(x) = \frac{\sigma^2}{2} f''(x) + \gamma f'(x) + \int_{-\infty}^{+\infty} \left(f(x+y) - f(x) - 1_{\{|y|\leq 1\}}(y) f'(x) \right) \nu(dy) \tag{5.8}$$

From eqs. (5.4) and (5.8), it follows that, for $z \in \mathbf{R}$

$$Le^{ixz} = -\psi(z) e^{ixz} \tag{5.9}$$

Assuming that $f(x)$ meets the desired regularity conditions so that its Fourier transform can be defined, we may write

$$f(x) = (2\pi)^{-1} \int_{-\infty}^{+\infty} e^{ixz} \hat{f}(z)\, dz \tag{5.10}$$

where $\hat{f}$ is the Fourier transform of f i.e.

$$\hat{f}(z) = \int_{-\infty}^{+\infty} e^{-ixz} f(x)\, dx \tag{5.11}$$

From eqs. (5.9) & (5.10), we have

$$Lf(x) = (2\pi)^{-1} \int_{-\infty}^{+\infty} e^{ixz} (-\psi(z)) \hat{f}(z)\, dz \tag{5.12}$$

Now, if we define an operator A by its action on $f(x)$ by

$$Af(x) = (2\pi)^{-1} \int_{-\infty}^{+\infty} e^{ixz} a(x,z) \hat{f}(z)\, dz \tag{5.13}$$

then, we have, on comparing eqs. (5.12) & (5.13)

$$-\psi(z) \equiv a(x,z) \tag{5.14}$$

The operator $Af(x)=(2\pi)^{-1}\int_{-\infty}^{+\infty}e^{ixz}a(x,z)\hat{f}(z)dz$ is referred to in the literature as a pseudodifferential operator with the symbol a. The theory of such operators is well developed [145-146].

5.6 OPTION PRICING WITH PSEUDODIFFERENTIAL OPERATORS AS GENERATORS OF LEVY PROCESSES [141-144]

A serious impediment to the shunning of the Black Scholes model framework in that other models for the evolution of stock prices leads to market incompleteness so that a contingent claim cannot be completely hedged and there is no unique Equivalent Martingale Measure (EMM) under which, the discounted stock price process is a martingale. A mechanism, therefore, needs to be evolved to decide on a particular EMM. In the context, most models envisage that the EMM is selected by the market.

Let us consider a market consisting of a riskless bond that evolves with a rate of return $r>0$, and a risky stock price process that evolves as

$$S(t)=\exp X(t) \tag{5.15}$$

where $X(t)$ is a Levy process under a given probability measure $\mathbf{P}$ (for the moment, we assume that such a measure exists).

It is, now, well established that the existence of an equivalent martingale measure, $\mathbf{Q}$ that is absolutely continuous w.r.t. $\mathbf{P}$, is equivalent to the "no arbitrage condition". Assuming no arbitrage at this point and the existence of a unique EMM, $\mathbf{Q}$, corresponding to $\mathbf{P}$, we obtain the discounted stock price

$$Z(t) = e^{-rt} S(t). \tag{5.16}$$

We then have, since $\mathbf{Q}$ is a martingale measure,

$$Z(s) = E^{\mathbf{Q}}\left[Z(t) \middle| \mathrm{F}_s\right] \tag{5.17}$$

where $E^{\mathbf{Q}}\left[S \middle| \mathrm{F}_s\right]$ is the conditional expectation of a random variables S w.r.t. filtration F_s.

Applying this with $s = 0$ to our market model with the riskless bond evolving as

$$B(t) = B(0)e^{rt} \tag{5.18}$$

and

$$S(t) = S(0)e^{X(t)} \tag{5.19}$$

we obtain

$$Z(0) = S(0) = E^{\mathbf{Q}}\left[Z(t) \middle| \mathrm{F}_0\right] = E^{\mathbf{Q}}\left[Z(t)\right] \tag{5.20}$$

Making use of the definition of characteristic exponent ψ given by eq (5.4) and writing $\psi^{\mathbf{Q}}(t)$ as the characteristic exponent under measure $\mathbf{Q}$, we have, from (5.18), (5.19) & (5.20)

$$B(0) = B(0)e^{-t\psi^{\mathbf{Q}}(0)} \tag{5.21}$$

and $S(0) = S(0)e^{-t\left[r + \psi^{\mathbf{Q}}(-i)\right]}$ (5.22)

whence

$$\psi^{Q}(0)=0 \tag{5.23}$$

and

$$r+\psi^{Q}(-i)=0 \tag{5.24}$$

which constitutes the fundamental condition for the existence of an EMM.

The relationship between the measure **P** and EMM **Q** in terms of the characteristic exponent is obtained through the Esscher transform. We look for an EMM **Q** satisfying the differential equation.

$$\left.\frac{dQ}{dP}\right|F_t=\exp\left[X(t)-d(\theta,t)\right] \tag{5.25}$$

Since we want that the discounted price process of the stock $Z(t)=e^{-rt}S(t)$ must be a martingale under **Q** we must have, using eq. (5.25)

$$S(0)=S(0)E^{P}\left[e^{(X(t)-rt)}e^{(\theta X(t)-d(\theta,t))}\right]$$

$$=E^{Q}\left[Z(t)\right]=S(0)E^{P}\left[e^{(1+\theta)X(t)-d(\theta,t)-rt}\right] \tag{5.26}$$

Using the definition of the characteristic exponent ψ of the Levy process $X(t)$ (eq. (5.4)), we have from eq. (5.26)

$$-t\psi^{P}\left[-i(1+\theta)\right]-d(\theta,t)-rt=0 \tag{5.27}$$

Proceeding similarly for the riskless bond, we have

$$-t\psi^{P}(-i\theta)-d(\theta,t)=0 \tag{5.28}$$

whence

$$d(\theta,t)=-t\psi^{P}(-i\theta) \tag{5.29}$$

From eqs. (5.27) & (5.29), we get

$$-r-\psi^{\mathrm{P}}\left[-i(1+\theta)\right]+\psi^{\mathrm{P}}(-i\theta)=0 \tag{5.30}$$

Further eqs. (5.25) & (5.29) yield

$$\left.\frac{dQ}{dP}\right|\mathrm{F}_t=\exp\left[X(t)-d(\theta,t)\right]=\exp\left[\theta X(t)+t\psi^{P}(-i\theta)\right] \tag{5.31}$$

Similarly, using eqs. (5.24) and (5.30), we obtain

$$\psi^{\mathrm{Q}}(\xi)=\psi^{\mathrm{P}}(\xi-i\theta)-\psi^{\mathrm{P}}(-i\theta) \tag{5.32}$$

Let $g(X(T))$ be the terminal payoff for an option on the expiry date T. Then, from the general theory of option pricing, we can obtain the price of the option at an earlier date under the "no arbitrage" conditions as

$$F(S_t,t)=e^{rt}E^{\mathrm{Q}}\left[e^{-rt}g(X(T))\middle|X_t=x\right] \tag{5.33}$$

where $x=\ln S(t)$.

In terms of the probability density of X under $\mathbf{Q}$, we have

$$F(S_t,t)=e^{-r(T-t)}\int_{-\infty}^{+\infty}p_r(y)g(x+y)dy \tag{5.34}$$

Making the substitution $y\mapsto y-x$, we get

$$F(S_t,t)=e^{-r(T-t)}\int_{-\infty}^{+\infty}p_r(y-x)g(y)dy \tag{5.35}$$

Then probability $p_r(x)$ can be written in terms of the characteristic exponent by taking Fourier transform

$$p_r(x)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}e^{-ix\xi-r\psi^{\mathrm{Q}}(\xi)}d\xi \tag{5.36}$$

so that

$$F(S_t,t)=(2\pi)^{-1}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}e^{i(x-y)\xi-(T-t)(r+\psi^{Q}(\xi))}g(y)\,d\xi\,dy \tag{5.37}$$

Assuming regularity conditions enabling the existence of the Fourier transform of $g(y)$, we have

$$\hat{g}(\xi)=\int_{-\infty}^{+\infty}e^{-iy\xi}g(y)\,dy \tag{5.38}$$

Now for Levy process called RLPE (Regular Levy Processes of Exponential Type), Cauchy theorem enables further simplification of the expression (5.38) to

$$F(S_t,t)=(2\pi)^{-1}\int_{-\infty+i\sigma}^{+\infty+i\sigma}e^{ix\xi-(T-t)(r+\psi^{Q}(\xi))}\hat{g}(\xi)\,d\xi \tag{5.39}$$

Writing $f(x,t)=F(e^x,t)$, we obtain, in terms of the pseudodifferential operator ψ

$$f(x,t)=\exp\left[-(T-t)\left(r+\psi^{Q}(D_x)\right)\right]g(x) \tag{5.40}$$

For a call option, the terminal payoff is given by $g(X(T))=\left(e^{X(T)}-E\right)^+$ so that

$$\hat{g}(\xi)=\int_{-\infty}^{+\infty}e^{-ix\xi}\left(e^x-E\right)^+dx==\int_{lnE}^{+\infty}\left(e^{x(1-i\xi)}-Ee^{-i\xi x}\right)dx$$

$$=\frac{e^{(1-i\xi)lnE}}{i(\xi+i)}-\frac{e^{(1-i\xi)lnE}}{i\xi}=\frac{Ee^{-i\xi lnE}}{\xi(\xi+i)} \tag{5.41}$$

whence the price of a call option with exercise price E is

$$F_{call}(S_t,t)=-\frac{E}{2\pi}\int_{-\infty+i\sigma}^{+\infty+i\sigma}\frac{\exp\left[i\xi\ln(S_t/E)-(T-t)\left(r+\psi^{Q}(\xi)\right)\right]}{\xi(\xi+i)}d\xi \tag{5.42}$$

5. 7 DEFORMATIONS OF PSEUDODIFFERENTIAL OPERATORS

In the preceding section, we have discussed the theory of option pricing where the dynamics of the stock price follow a Levy distribution [141-144] making use of the characteristic exponent of such processes that happen to be pseudodifferential operators.

This section and the next following constitute the essence of this work. Our objective, now, is to take up the deformations of Levy processes through a deformation of the pseudodifferential operators that constitute the generators of such processes. The motivation for doing so is manifold. For one thing, such deformed structures enhance the spectrum of practical applications of the mathematical concepts and constitute a significant progression towards bridging the gap between reality and its modeling. The theory of q deformed mathematics since its evolution in the interface between mathematics & physics is gradually pervading into almost all domains including group theory and other algebraic structures, analysis, probability and so on.

Towards deforming the pseudodifferential operators we shall follow the prescription in [147-149] and make use of the following notation and results that are now standard in the theory of q-deformations and quantum groups [147-149]:-

For q-numbers, we have

$$(n)_q = \frac{q^n - 1}{q - 1} \qquad (5.43)$$

$$\binom{m}{l}_q = \frac{(m)_q (m-1)_q \ldots (m-l+1)}{(1)_q (2)_q \ldots (l)_q} \tag{5.44}$$

The q analog of the derivative is defined as

$$D_q f(x) = \frac{f(qx) - f(x)}{q-1} \tag{5.45}$$

and the shift by

$$\tau f(x) = f(qx) \tag{5.46}$$

$$\tau^\beta f(x) = f(q^\beta x) \tag{5.47}$$

We, then have, since τ commutes with D_q so that

$$D_q(fg) = D_q(f)g + \tau(f)D_q(g) \tag{5.48}$$

We then define the q deformed pseudodifferential operator as

$$A(x, D_q) = \sum_{-\infty}^{n} a_i(x) D_q^i \tag{5.49}$$

For $q = 1$, the above operator reduces to the conventional pseudodifferential operator e.g.

$$A(x, \partial) = \sum_{-\infty}^{n} a_i(x) \partial^i \tag{5.50}$$

which leads to the conventional form used in the foregoing as follows:-

Operating the operator $A(x, \partial)$ on an arbitrary function $f(x)$ which possesses the inverse Fourier transform $f(x) = \int e^{i\xi x} \hat{f}(\xi) d\xi$ we have

$$A(x,\partial) f(x) = \sum a_i(x) \int e^{ix\xi} \xi^\alpha \hat{f}(\xi) d\xi$$

$$= \int e^{ix\xi} \sum a_i(x) \xi^\alpha \hat{f}(\xi) d\xi$$

$$= \int e^{ix\xi} a(x,\xi) \hat{f}(\xi) d\xi \tag{5.51}$$

The multiplication law for the deformed operators is defined by

$$D_q \circ a = (D_q a) + \tau(a) D_q \tag{5.52}$$

$$D_q^{-1} \circ a = \sum_{k \geq 0} (-1)^k \left(\tau^{-k\ -1}\left(D_q^k a\right)\right) D_q^{-k\ -1} \tag{5.53}$$

$$D_q^n \circ a = \sum_{l \geq 0} \binom{n}{l} \left(\tau^{n-l}\left(D_q^l a\right)\right) D_q^{n-l} \tag{5.54}$$

and

$$A(x, D_q) \circ B(x, D_q) = \sum_{k \geq 0} \frac{1}{(k)!} \left(\frac{d^k}{dD_q^k} A\right) * \left(D_q^k B\right) \tag{5.55}$$

Proofs of the above are provided in [149]. Corresponding rules for the symbols corresponding to the deformed operators are

$$f * D_q = fD_q \tag{5.56}$$

$$D_q^{-1} * f = \tau^{-1}(f) D_q^{-1} \tag{5.57}$$

$$D_q * f = \tau(f) D_q \tag{5.58}$$

The algebra of the deformed pseudodifferential operators is, thus, well defined, with the appropriate deformed versions of the compositions and operations.

5.8 GENERATION OF LEVY PROCESS FROM DEFORMED PSEUDODIFFERENTIAL OPERATORS

Having accepted the deformed pseudodifferential operators as well defined mathematical structures, the final step is to construct Levy processes conforming to these operators, where after the machinery elucidated earlier for the pricing of options under regular Levy processes can be invoked.

The process of constructing Levy processes from symbols of conventional (undeformed) pseudodifferential operators has been well studied and elaborated in the literature and the following possible routes are advocated: [150]-

1. Use the Hille-Yosida-Ray theorem to construct a Feller semigroup. Apply Feller – Dynkin construction using Kolmogorov's theorem, thereafter, to get a process;
2. Construct a (symmetric) Dirichlet form from the pseudodifferential operator and then apply Fukushima's theory;
3. Find a fundamental solution for the operator $\frac{\partial}{\partial t} + a(x,D)$ by a parametrix method and then prove that this fundamental solution gives rise to a transition function;
4. Establish the martingale problem for $-a(x,D)$;
5. Solve a stochastic differential equation with jumps corresponding to $-a(x,D)$.

To obtain Levy processes corresponding to the deformed pseudodifferential operator, we need therefore, establish the existence of a Feller semigroup corresponding to the deformed operator. For the purpose, we consider the deformation as a perturbation of the standard case and write $-a(x,D_q)$ as a perturbation of an undeformed operator $-a(x,D)$, i.e. an operator generating a Levy process. Therefore, we decompose $a(x,\xi_q)$ into

$a(x,\xi_q) = a(x,\xi) + (a(x,\xi_q) - a(x,\xi)) = a_1(x,\xi) + a_2(x,\xi)$ and denote by $-a_1(x,D)$ and $-a_2(x,D)$ the corresponding Levy generator and the perturbation, respectively. Assume, moreover, that $a(x,\xi_q)$ is real-valued.

The conditions under which $a(x,D_q)$ constitutes a Feller semigroup are examined in [151]. We, briefly elucidate them here.

Let $b^2 : R^n \to R$ is a fixed continues negative definite function such that

$$b^2(\xi) \geq k_0 \|\xi\|^{r_0} \tag{5.59}$$

holds for some $r_0 \in (0,2]$ and $k_0 > 0$. Further, from the boundedness property of pseudodifferential operators, we also have

$$b^2(\xi) \leq c_{b^2}\left(1 + \|\xi\|^2\right), \quad \xi \in \mathbf{R}. \tag{5.60}$$

The conditions for $a(x,\xi_q)$ to generate a Feller semigroup are, then,

(a) Ellipticity Assumption

$$\left(1 + a_1(x,\xi)\right) \geq \gamma_1\left(1 + b^2(\xi)\right) \quad \xi \in \mathbf{R} \tag{5.61}$$

(b) Boundedness Assumptions

$$\left(1 + a_1(x,\xi)\right) \leq \gamma_0\left(1 + b^2(\xi)\right) \quad \xi \in \mathbf{R} \tag{5.62}$$

$$a_2(x,\xi) \in C^m(\mathbf{R}),$$

$$\left|\partial_x^\alpha a_2(x,\xi)\right| \leq \Phi_\alpha(x)\left(1 + b^2(\xi)\right) \quad x,\xi \in \mathbf{R} \tag{5.63}$$

5. 9 CONCLUSION

In this chapter, we have attempted to develop the theory of option pricing in incomplete markets with stock market pricing being simulated by Levy processes. These processes have an intimate connection with pseudodifferential operators in the sense that their characteristic exponent is a pseudodifferential operator. Hence, we can associate a pseudodifferential operator with every Levy processes. The converse of this also holds and Levy processes can be generated by the knowledge of a pseudodifferential symbol. We take advantage of this, and making use of the q deformed pseudodifferential symbols, which have been the substratum of recent research, we attempt to construct the corresponding Levy processes. Treating the deformation as a perturbation, we identify the conditions under which these symbols generate a Feller semigroup for which Levy processes can be constructed in the usual way.

CHAPTER 6
GROUP PROPERTIES OF THE BLACK SCHOLES EQUATION & ITS SOLUTIONS

Abstract

The Black Scholes equation was originally solved and closed form expressions for the pricing of financial derivatives were obtained by converting the problem to a heat equation and then solving it for specific boundary conditions. In this Chapter, we apply the group theoretic methods to obtain various solutions of the Black Scholes equation. We also examine the infinitesimal symmetries of the said equation and explore group transformation properties. The structure of the Lie algebra of the Black Scholes equation is also studied.

6.1 INTRODUCTION

The Black Scholes model, as initially propounded, envisaged the formulation of a partial differential equation for the pricing of an European call option by creating a portfolio that exactly replicated the payoff of the option and the value of whose constituents was known. The theory behind this valuation methodology is well disseminated and can be found in any text on financial derivatives e.g. [2]. The valuation equation of the Black Scholes model is

$$rS\frac{\partial C(S,t)}{\partial S}+\frac{1}{2}\sigma^2 S^2\frac{\partial^2 C(S,t)}{\partial S^2}+\frac{\partial C(S,t)}{\partial t}=rC(S,t), \quad (6.1)$$

This is the fundamental PDE for asset pricing and is referred to as the Black Scholes equation in the sequel.

6.2 TRANSFORMATION TO THE HEAT EQUATION

The transformation of the Black-Scholes equation to the heat equation has been well researched. We make the following transformations:-

$$y = \frac{2}{\sigma^2}\left(r - \frac{1}{2}\sigma^2\right)\ln S - \frac{2}{\sigma^2}\left(r - \frac{1}{2}\sigma^2\right)^2 t - \frac{2}{\sigma^2}\left(r - \frac{1}{2}\sigma^2\right)\ln S_0 + \frac{2}{\sigma^2}\left(r - \frac{1}{2}\sigma^2\right)^2 t_0, \quad (6.2)$$

$$\tau = -\left[\frac{2}{\sigma}\left(r - \frac{1}{2}\sigma^2\right)\right]^2 t + \frac{2}{\sigma^2}\left(r - \frac{1}{2}\sigma^2\right)^2 t_0 \quad (6.3)$$

$$v = C(S,t)e^{rt_0 + \left[\frac{1}{2\sigma^2}\left(r - \frac{1}{2}\sigma\right)^2 - \frac{1}{2}\left(\frac{r}{\sigma} - \frac{\sigma}{2}\right)^2 - r\right]t} S^{\left[\frac{r}{\sigma^2} - \frac{1}{2} - \frac{1}{\sigma^2}\left(r - \frac{1}{2}\sigma\right)\right]} \quad (6.4)$$

On implementing these transformations the Black-Scholes equation gets transformed to the heat equation $\frac{\partial v}{\partial \tau} = \frac{\partial^2 v}{\partial y^2}$ as can be seen by explicit calculations.

The fundamental solution of the heat equation is given by $v = \frac{1}{2\sqrt{\pi\tau}}\exp\left(-\frac{y^2}{4\tau}\right)$ and that of the Black Scholes eq. (6.1) is obtained by substituting back the transformations (6.2-6.4) and we obtain

$$C = \frac{1}{\sigma S_0\sqrt{2\pi(t_0 - t)}}\exp\left\{-\frac{(\ln S - \ln S_0)^2}{2\sigma^2(t_0 - t)} - \left[\frac{1}{2\sigma^2}\left(r - \frac{1}{2}\sigma^2\right)^2 + r\right](t_0 - t) - \frac{1}{\sigma^2}\left(r - \frac{1}{2}\sigma^2\right)(\ln S - \ln S_0)\right\} \quad (6.5)$$

6.3 CONSTRUCTION OF THE SYMMETRY GROUP [152-156]

The Black Scholes equation (6.1) is a partial differential equation in two independent variables viz. the stock price S and time t and one dependent variable in the price of the derivative C. Let us consider the following invertible transformations of the three variables S, t, and C

$$\bar{t} = f(t,S,C,a),\ \bar{S} = g(t,S,C,a) \text{ and } \bar{C} = h(t,S,C,a) \tag{6.6}$$

where a is a continuous parameter.

The transformations of eq. (6.6) will constitute symmetry transformations if eq. (6.1) retains its structure in the new variables $\bar{t}$, $\bar{S}$ and $\bar{C}$ and the set of all such transformations constitutes the symmetry group G of the Black Scholes equation.

The generator of the symmetry group G is given by the vector field:-

$$X = \xi^0(t,S,C)\frac{\partial}{\partial t} + \xi^1(t,S,C)\frac{\partial}{\partial S} + \eta(t,S,C)\frac{\partial}{\partial C} \tag{6.7}$$

where $\xi^0(t,S,C)$, $\xi^1(t,S,C)$, $\eta(t,S,C)$ are the parameters of the infinitesimal transformations:-

$$\bar{t} \approx t + a\xi^0(t,S,C),\ \bar{S} \approx S + a\xi^1(t,S,C) \text{ and } \bar{C} \approx C + a\eta(t,S,C) \tag{6.8}$$

They are obtained by solving the following equations:-

$$\frac{d\bar{t}}{da}=\xi^0(\bar{t},\bar{S},\bar{C}),\ \frac{d\bar{S}}{da}=\xi^1(\bar{t},\bar{S},\bar{C})\text{ and }\frac{d\bar{C}}{da}=\eta(\bar{t},\bar{S},\bar{C}) \tag{6.9}$$

with the initial conditions $\bar{t}\big|_{a=0}=t$, $\bar{S}\big|_{a=0}=S$ and $\bar{C}\big|_{a=0}=C$.

The transformations represented by eq. (6.6) would form a symmetry group if $\bar{C}=\bar{C}(\bar{S},\bar{t})$ satisfies the eq. $\frac{\partial\bar{C}}{\partial t}=-\frac{1}{2}\sigma^2\bar{S}^2\frac{\partial^2\bar{C}}{\partial\bar{S}^2}-r\bar{S}\frac{\partial\bar{C}}{\partial\bar{S}}+r\bar{C}$ whenever $C=C(S,t)$ satisfies eq. (6.1).

Our objective here is to determine all possible coefficient functions ξ^0,ξ^1,η such that we are able to obtain the symmetry group of eq. (6.1) by the process of exponentiation. For this purpose we need to obtain the second prolongation of the vector field X of eq. (6.7). In terms of the various partial derivatives, this is given by:-

$$pr^{(2)}X=X+\eta^S\frac{\partial}{\partial C_S}+\eta^t\frac{\partial}{\partial C_t}+\eta^{SS}\frac{\partial}{\partial C_{SS}}+\eta^{St}\frac{\partial}{\partial C_{St}}+\eta^{tt}\frac{\partial}{\partial C_{tt}}$$

where

$$\eta^{SS}=D_S^2(\eta-\xi^1C_S-\xi^0C_t)+\xi^1C_{SSS}+\xi^0c_{SSt}=D_S^2\eta-C_SD_S^2\xi^1-C_tD_S^2\xi^0-2C_{SS}D_S\xi^1-2C_{St}D_S\xi^0$$
$$=C_{SS}+(2C_{SC}-\xi^1_{SS})C_S-\xi^0_{SS}C_t+(\eta_{CC}-2\xi^1_{SC})C_S^2-2\xi^0_{SC}C_SC_t-\xi^1_{CC}C_S^3-\xi^0_{CC}C_S^2C_t$$
$$+(\eta_C-2\xi^1_S)C_{SS}-2\xi^0_SC_{St}-3\xi^1_CC_SC_{SS}-\xi^0_CC_tC_{SS}-2\xi^0_CC_SC_{St}$$

and similar expressions hold for η^{St} and η^{tt}.

The differentials of $\overline{C} = \overline{C}(\overline{S}, \bar{t})$ with respect to $\overline{S}, \bar{t}$ can be expressed in terms of those of $C = C(S,t)$ with respect to S, t through the so called prolongation formulae:-

$$\frac{\partial \overline{C}}{\partial \bar{t}} \approx \frac{\partial C}{\partial t} + a\left[D_t(\eta) - \frac{\partial C}{\partial t} D_t(\xi^0) - \frac{\partial C}{\partial S} D_t(\xi^1)\right] \tag{6.10}$$

$$\frac{\partial \overline{C}}{\partial \overline{S}} \approx \frac{\partial C}{\partial t} + a\left[D_S(\eta) - \frac{\partial C}{\partial t} D_S(\xi^0) - \frac{\partial C}{\partial S} D_S(\xi^1)\right] \tag{6.11}$$

$$\frac{\partial^2 \overline{C}}{\partial S^2} \approx \frac{\partial^2 C}{\partial S^2} + a\left\{D_S\left[D_S(\eta) - \frac{\partial C}{\partial t} D_S(\xi^0) - \frac{\partial C}{\partial S} D_S(\xi^1)\right] - \frac{\partial^2 C}{\partial S^2} D_S(\xi^1) - \frac{\partial^2 C}{\partial S \partial t} D_S(\xi^0)\right\} \tag{6.12}$$

where

$$D_t = \frac{\partial}{\partial t} + \frac{\partial C}{\partial t}\frac{\partial}{\partial C} + \frac{\partial C_t}{\partial t}\frac{\partial}{\partial C_t} + \frac{\partial C_S}{\partial t}\frac{\partial}{\partial C_S} + \ldots \tag{6.13}$$

and

$$D_S = \frac{\partial}{\partial S} + \frac{\partial C}{\partial S}\frac{\partial}{\partial C} + \frac{\partial C_t}{\partial S}\frac{\partial}{\partial C_t} + \frac{\partial C_S}{\partial S}\frac{\partial}{\partial C_S}. \tag{6.14}$$

Using eqs.(6.8, 6.10-6.12), we obtain

$$\frac{\partial \overline{C}}{\partial \bar{t}} + \frac{1}{2}\sigma^2 \overline{S}^2 \frac{\partial^2 \overline{C}}{\partial \overline{S}^2} + r\overline{S}\frac{\partial C}{\partial S} - r\overline{C} \approx \frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + rS\frac{\partial C}{\partial S} - rC + a\Gamma \tag{6.15}$$

where

$$\Gamma = \left[D_t(\eta) - \frac{\partial C}{\partial t} D_t(\xi^0) - \frac{\partial C}{\partial S} D_t(\xi^1) \right] +$$

$$\frac{1}{2}\sigma^2 S^2 \left\{ D_S \left[D_S(\eta) - \frac{\partial C}{\partial t} D_S(\xi^0) - \frac{\partial C}{\partial S} D_S(\xi^1) \right] - \frac{\partial^2 C}{\partial S^2} D_S(\xi^1) - \frac{\partial^2 C}{\partial S \partial t} D_S(\xi^0) \right\} + \quad (6.16)$$

$$rS\frac{\partial C}{\partial S}\left[D_S(\eta) - \frac{\partial C}{\partial t} D_S(\xi^0) - \frac{\partial C}{\partial S} D_S(\xi^1) \right] - r\eta + \sigma^2 S \frac{\partial^2 C}{\partial S^2}\xi^1 + r\frac{\partial C}{\partial S}\xi^1$$

Hence, the determining equation for the problem under reference is of the form $\Gamma = 0$ with Γ being given by eq. (6.16).

Using eqs. (6.13-6.14, 6.16) and equating to zero, the coefficients` of the various monomials of the first and second order partial derivatives of C, we obtain the following equations for the symmetry group of the Black Scholes equation.

$$\xi^0_C = 0 \quad (6.17)$$

$$\xi^0_S = 0 \quad (6.18)$$

$$\xi^0_{CC} = 0 \quad (6.19)$$

$$-\xi^1_C + \frac{1}{2}\sigma^2 S^2 \xi^0_{SC} = 0 \quad (6.20)$$

$$-S\xi^1_S + \xi^1 + \frac{1}{2}rSC\xi^0_C + \frac{1}{2}S\xi^0_t + \frac{1}{4}\sigma^2 S^3 \xi^0_{CC} = 0 \quad (6.21)$$

$$\xi^1_{CC} - rS\xi^0_{CC} = 0 \quad (6.22)$$

$$\frac{1}{2}\eta_{CC} - \xi^1_{SC} + rS\xi^0_{SC} - \frac{1}{2}rC\xi^0_{CC} = 0 \quad (6.23)$$

$$-\xi_t^1+\sigma^2S^2\eta_{SC}-\frac{1}{2}\sigma^2S^2\xi_{SS}^1-rS\xi_S^1+r\xi^1-r^2SC\xi_C^0+rS\xi_t^0+$$
$$r^2S^2\xi_S^0-rC\xi_C^1-\sigma^2rS^2C\xi_{SC}^0+\frac{1}{2}r\sigma^2S^3\xi_{SS}^0=0 \tag{6.24}$$

$$\left(\eta_t+\frac{1}{2}\sigma^2S^2\eta_{SS}+rS\eta_S-r\eta\right)-\left(\xi_t^0+\frac{1}{2}\sigma^2S^2\xi_{SS}^0+rS\xi_S^0-r\xi^0\right)rC-r^2C\xi^0-r^2C^2\xi_C^0+rC\eta_C=0 \tag{6.25}$$

Eqs. (6.17-6.18) require that ξ^0 be a function of t only. Hence, eq. (6.20) reduces to $\xi_C^1=0$ which implies that ξ^1 does not depend on C. Further, eq. (6.21) becomes $-S\xi_S^1+\xi^1+\frac{1}{2}S\xi_t^0=0$ which has the solution

$$\xi^1(S,t)=\frac{1}{2}\xi_t^0(t)S\ln S+M(t)S \tag{6.26}$$

Then eq. (6.23) yields $\frac{1}{2}\eta_{CC}=0$ which mandates that $\eta(t,S,C)$ is a linear function of C and hence can be written as

$$\eta(t,S,C)=\alpha(t,S)C+\beta(t,S) \tag{6.27}$$

With the above constraints for ξ^0 we can write eq. (6.25) as

$$-\xi_t^1+\sigma^2S^2\eta_{SC}-\frac{1}{2}\sigma^2S^2\xi_{SS}^1-rS\xi_S^1+r\xi^1+rS\xi_t^0=0 \tag{6.28}$$

Using eqs. (6.26-6.27), eq. (6.28) reduces to

$$In\,S\xi_{tt}^0-\left(r-\frac{1}{2}\sigma^2\right)\xi_t^0+2M_t(t)-2\sigma^2S\alpha_S(S,t)=0 \tag{6.29}$$

with the solution

$$\alpha(S,t)=\frac{1}{2\sigma^2}\left[\frac{1}{2}(In\,S)^2\xi^0_{ttt}-\left(r-\frac{1}{2}\sigma^2\right)In\,S\xi^0_t+2M_t(t)In\,S+N(t)\right] \tag{6.30}$$

Using eqs. (6.25), (6.27) we find that $\beta(S,t)$ must be a solution of the Black Scholes equation while $\alpha(S,t)$ must satisfy

$$\alpha_t+\frac{1}{2}\sigma^2S^2\alpha_{SS}+rS\alpha_S-r\xi^0_t=0 \tag{6.31}$$

Eqs. (6.30-6.31) yield the following:-

$$\xi^0_{ttt}=0 \text{ so that } \xi^0=Pt^2+Qt+R \tag{6.32}$$

and

$$M_{tt}=0 \text{ so that } M=Ut+V \tag{6.33}$$

We finally end up with the following solutions for ξ^0, ξ^1, η:-

$$\xi^0=Pt^2+Qt+R \tag{6.34}$$

$$\xi^1=\frac{1}{2}(2Pt+Q)S\,In\,S+Ut+V$$

$$\eta=\frac{1}{2\sigma^2}\left\{\begin{array}{l}P(InS)^2-\left(r-\frac{1}{2}\sigma^2\right)(2Pt+Q)InS+2UInS+\left[\left(r-\frac{1}{2}\sigma^2\right)^2+2\sigma^2 r\right]Pt^2+\\ 2\sigma^2\left[\frac{1}{2\sigma^2}\left(r-\frac{1}{2}\sigma^2\right)^2 Q-\frac{1}{2}P+rQ-\frac{1}{\sigma^2}\left(r-\frac{1}{2}\sigma^2\right)U\right]t+W\end{array}\right\}C+\beta(S,t) \quad (6.35)$$

where P,Q,R,U,V,W are arbitrary constants. On substituting these expressions for ξ^0, ξ^1, η in eq. (6.7), we obtain the expressions for the six generators from the coefficients of these constants as follows:-

$$X_1=\frac{\partial}{\partial t} \quad (6.36)$$

$$X_2=S\frac{\partial}{\partial S} \quad (6.37)$$

$$X_3=t\frac{\partial}{\partial t}+\frac{1}{2}S\,InS\frac{\partial}{\partial S}-\frac{1}{2\sigma^2}\left(r-\frac{1}{2}\sigma^2\right)(InS)C\frac{\partial}{\partial C}+\frac{1}{2\sigma^2}\left(r-\frac{1}{2}\sigma^2\right)^2 tC\frac{\partial}{\partial C}+rtC\frac{\partial}{\partial C} \quad (6.38)$$

$$X_4=tS\frac{\partial}{\partial S}+\frac{1}{\sigma^2}(InS)C\frac{\partial}{\partial C}-\frac{1}{\sigma^2}\left(r-\frac{1}{2}\sigma^2\right)tC\frac{\partial}{\partial C} \quad (6.39)$$

$$X_5=t^2\frac{\partial}{\partial t}+(InS)tS\frac{\partial}{\partial S}+\left\{\frac{1}{2\sigma^2}(InS)^2-\frac{1}{\sigma^2}\left(r-\frac{1}{2}\sigma^2\right)(InS)t+\left[\frac{1}{2\sigma^2}\left(r-\frac{1}{2}\sigma^2\right)^2+r\right]t^2-\frac{1}{2}t\right\}C\frac{\partial}{\partial C} \quad (6.40)$$

$$X_6=C\frac{\partial}{\partial C} \qquad X_\beta=\beta(S,t)\frac{\partial}{\partial C} \quad (6.41)$$

Using eq. (6.39), we can present eq.(6.38) in a simplified form as:-

$$X_3=t\frac{\partial}{\partial t}+\frac{1}{2}(InS)S\frac{\partial}{\partial S}+\frac{1}{2}\left(r-\frac{1}{2}\sigma^2\right)tS\frac{\partial}{\partial S}+rtC\frac{\partial}{\partial C} \quad (6.42)$$

The one-parameter groups G_i corresponding to each of the above generators are given by the usual process of exponentiation e.g.

$$G_1 : (t+\varepsilon, S, C) \tag{6.43}$$

$$G_2 : (t, \varepsilon S, C), \varepsilon \neq 0 \tag{6.44}$$

$$G_3 : \left(\varepsilon^2 t, e^{\left(r-\frac{1}{2}\sigma^2\right)\left(\varepsilon^2-\varepsilon\right)t} S^{\varepsilon}, e^{r\left(\varepsilon^2-1\right)t} C \right), \varepsilon \neq 0 \tag{6.45}$$

$$G_4 : \left(t, e^{\varepsilon\sigma^2 t} S, e^{\left[\frac{1}{2}\varepsilon^2\sigma^2-\varepsilon\left(r-\frac{1}{2}\sigma^2\right)\right]t} S^{\varepsilon} C \right) \tag{6.46}$$

$$G_5 : \left(\frac{t}{1-2\varepsilon\sigma^2 t}, S^{\left(1-2\varepsilon\sigma^2 t\right)^{-1} t}, \left(1-2\varepsilon\sigma^2 t\right)^{\frac{1}{2}} e^{\left\{ \frac{\varepsilon\left[\left(\ln S-\left(r-\frac{1}{2}\sigma^2\right)t\right)^2+2r\sigma^2 t^2\right]}{1-2\varepsilon\sigma^2 t} \right\}} C \right) \tag{6.47}$$

$$G_6 : (t, S, \varepsilon C), \varepsilon \neq 0 \tag{6.48}$$

$$G_\beta : (t, S, C+\beta(S,t)) \tag{6.49}$$

We obtain the most general one-parameter symmetry group of the Black Scholes equation as a general linear combination $\sum_{i=1}^{6} c_i X_i + X_\beta$ of the generators given by eqs. (6.36-6.42). We can also represent an arbitrary group transformation g as the composition of transformations in the aforesaid one parameter subgroups.

Since each group G_i is a symmetry group, if $C = C(S,t)$ is a solution of the Black Scholes equation, then so are the functions:-

$$C^{(1)}(S,t) = C(t-\in, S) \tag{6.50}$$

$$C^{(2)}(S,t) = C\left(t, \in^{-1} S\right), \in \neq 0 \tag{6.51}$$

$$C^{(3)}(S,t) = e^{\left(1-\in^{-2}\right)rt} C\left[e^{\left(\in^{-2}-\in^{-1}\right)\left(r-\frac{1}{2}\sigma^2\right)t} S^{\in^{-1}}, \in^{-2} t \right] \tag{6.52}$$

$$C^{(4)}(S,t) = e^{-\left[\frac{1}{2}\in^2\sigma^2 + \in\left(r-\frac{1}{2}\sigma^2\right)\right]t} S^{\in} C\left[Se^{-\in\sigma^2 t}, t \right] \tag{6.53}$$

$$C^{(5)}(S,t) = \left[1 + 2\in\sigma^2 t\right]^{-\frac{1}{2}} e^{\varepsilon\left\{\frac{\left[\log S - \left(r-\frac{1}{2}\sigma^2\right)t\right]^2 + 2r\sigma^2 t^2}{1+2\in\sigma^2 t}\right\}} C\left(\frac{t}{1+2\in\sigma^2 t}, S^{\frac{t}{1+2\in\sigma^2 t}} \right) \tag{6.54}$$

$$C^{(6)}(S,t) = \in C(S,t), \in \neq 0 \tag{6.55}$$

$$C^{(\beta)}(S,t) = C(S,t) + \in \beta(S,t) \tag{6.56}$$

Here $\in$ is any real number and $\beta(S,t)$ any other solution to the Black Scholes equation. It is seen from the symmetry group G_6 and G_β that the solutions of the Black Scholes equation are linear and we can add two solutions and multiply them with a constant. The group G_1 shows time invariance of the solutions. The symmetry group G_2 reflects the scaling symmetry with respect to S.

6.4 STRUCTURE OF THE LIE ALGEBRA $\Lambda=\langle X_1,X_2,X_3,X_4,X_5,X_6\rangle$ [157-159]

We now explore the structure of the finite dimensional Lie algebra generated by $\Lambda=\langle X_1,X_2,X_3,X_4,X_5,X_6\rangle$. The commutator table of Λ is given by:-

	X_1	X_2	X_3	X_4	X_5	X_6
X_1	0	0	$X_1+\frac{K}{2}X_2+rX_6$	$X_2-\frac{K}{\sigma^2}X_6$	$2X_3-KX_4-\frac{1}{2}X_6$	0
X_2	0	0	$\frac{1}{2}X_2$	$\frac{1}{\sigma^2}X_6$	X_4	0
X_3	$-\left(X_1+\frac{K}{2}X_2+rX_6\right)$	$-\frac{1}{2}X_2$	0	$\frac{1}{2}X_4$	X_5	0
X_4	$-X_2+\frac{K}{\sigma^2}X_6$	$\frac{1}{\sigma^2}X_6$	$-\frac{1}{2}X_4$	0	0	0
X_5	$-2X_3+KX_4+\frac{1}{2}X_6$	$-X_4$	$-X_5$	0	0	0
X_6	0	0	0	0	0	0

TABLE 6.1

where $K=r-\frac{1}{2}\sigma^2$. Further,

$$\left[X_1,X_\beta\right]=X_{\beta_t},\ \left[X_2,X_\beta\right]=X_{S\beta_S},\ \left[X_3,X_\beta\right]=X_{t\beta_t+\frac{1}{2}S(\ln S)\beta_S+\frac{1}{2\sigma^2}\left(r-\frac{1}{2}\sigma^2\right)\beta\ln S-\frac{1}{2\sigma^2}\left(r-\frac{1}{2}\sigma^2\right)^2\beta t-r\beta t}$$

$$\left[X_4,X_\beta\right]=X_{tS\beta_S-\frac{1}{\sigma^2}\beta\ln S+\frac{1}{\sigma^2}\left(r-\frac{1}{2}\sigma^2\right)\beta t},\left[X_5,X_\beta\right]=X_{t^2\beta_t+tS(\ln S)\beta_S-\frac{1}{2\sigma^2}\beta(\ln S)^2+\frac{1}{\sigma^2}\left(r-\frac{1}{2}\sigma^2\right)\beta t\ln S-\left[\frac{1}{2\sigma^2}\left(r-\frac{1}{2}\sigma^2\right)^2+r\right]\beta t^2+\frac{1}{2}\beta t}$$

$$\left[X_6,X_\beta\right]=X_{-\beta}\left[X_\beta,X_\beta\right]=0,\text{ where } X_\gamma=\gamma\frac{\partial}{\partial C}$$

From table 6.1, the following readily follow:-

(a) the centralizers of the various elements X_i are:-

$$\chi(X_1)=\langle X_1,X_2,X_6\rangle,\chi(X_2)=\langle X_1,X_2,X_6\rangle,\chi(X_3)=\langle X_3,X_6\rangle,\chi(X_4)=\langle X_4,X_5,X_6\rangle,$$
$$\chi(X_5)=\langle X_4,X_5,X_6\rangle,\chi(X_6)=\langle X_1,X_2,X_3,X_4,X_5,X_6\rangle.$$

(b) the centre of Λ is $\chi(\Lambda)=\bigcap_{i=1}^{6}\chi(X_i)=\langle X_6\rangle$.

(c) $[X_1,\Lambda]=\langle X_1,X_2,X_3,X_4,X_6\rangle,[X_2,\Lambda]=\langle X_2,X_4,X_6\rangle,[X_3,\Lambda]=\langle X_1,X_2,X_4,X_5,X_6\rangle,$

$[X_4,\Lambda]=\langle X_2,X_4,X_6\rangle,[X_5,\Lambda]=\langle X_3,X_4,X_5,X_6\rangle,[X_6,\Lambda]=0.$

(d) $U=\langle X_2,X_4,X_6\rangle$ is a two sided ideal of Λ since $\langle[U,\Lambda]\rangle=\langle[\Lambda,U]\rangle=U$. It is also an invariant subalgebra of Λ.

(e) the Lie algebra Λ is not solvable, since $[\Lambda,\Lambda]=\Lambda$ and hence the derived series of Λ is stationary. However, for the subalgebra U, we have, $[U,U]=\langle X_6\rangle,\left[U^{(2)},U^{(2)}\right]=[X_6,X_6]=0$, so that U is solvable. Being the maximal ideal, it is, therefore, the radical of Λ. Also, $V=\langle X_1,X_3,X_5\rangle$ is a semisimple and simple subalgebra.

(f) in view of (e), the Lie algebra Λ admits the Levi decomposition $\Lambda=U\oplus V$

(g) the adjoint representations of the various elements can be trivially written from the commutator table and, in the ordering $\langle X_1,X_3,X_5,X_2,X_4,X_6\rangle$ take the form:-

$$X_1=\left(a_{21}=-1,a_{24}=-\frac{K}{2},a_{26}=-r,a_{32}=-2,a_{35}=K,a_{36}=\frac{1}{2},a_{54}=-1,a_{56}=\frac{K}{\sigma^2}\right);$$

$$X_2=\left(a_{24}=-\frac{1}{2},a_{35}=-1,a_{56}=-\frac{1}{\sigma^2}\right);$$

$$X_3=\left(a_{11}=1,a_{14}=\frac{K}{2},a_{16}=r,a_{33}=-1,a_{44}=\frac{1}{2},a_{55}=-\frac{1}{2}\right);$$

$$X_4=\left(a_{14}=1,a_{16}=-\frac{K}{\sigma^2},a_{25}=\frac{1}{2},a_{46}=\frac{1}{\sigma^2}\right);$$

$$X_5 = \left(a_{12} = 2, a_{15} = -\mathrm{K}, a_{16} = -\frac{1}{2}, a_{23} = 1, a_{45} = 1 \right); X_6 = \mathrm{O}_{6\times 6}$$

The non-specified elements are $0's$ in the above matrices.

(h) the action, defined by $\varphi_{ij} = \left(e^{\varepsilon adj\, X_i}\right) X_j$, of the adjoints of the various generators X_i on the algebra Λ is summarized below (These constitute the inner automorphism group of the Lie algebra Λ):-

$j \rightarrow$ $i \downarrow$	X_1	X_2	X_3	X_4	X_5	X_6
X_1	$X_1 - \varepsilon X_3$ $+\varepsilon^2 X_5$	$X_2 - \frac{\varepsilon \mathrm{K}}{2} X_3 - \varepsilon X_4$	$X_3 - 2\varepsilon X_5$	$X_4 + \varepsilon \mathrm{K} X_5$	X_5	$-\varepsilon r X_3 + \frac{\varepsilon \mathrm{K}}{\sigma^2} X_4 +$ $\left[\frac{\varepsilon}{2} + \frac{\varepsilon^2}{2}\left(2r + \frac{\mathrm{K}^2}{\sigma^2}\right)\right] X$
X_2	X_1	$X_2 - \frac{\varepsilon}{2} X_3$	X_3	$X_4 - \varepsilon X_5$	X_5	$-\frac{\varepsilon}{\sigma^2} X_4 + \frac{\varepsilon^2}{2\sigma^2} X$ $+X_6$
X_3	$2X_2$	$e^{\frac{\varepsilon}{2}}\left(e^{\frac{\varepsilon}{2}} - 1\right)\mathrm{K} X_1 + e^{\frac{\varepsilon}{2}} X_2$	X_3	$e^{-\frac{\varepsilon}{2}} X_4$	$e^{-\varepsilon} X_5$	X_6
X_4	X_1	$\varepsilon X_1 + X_2$	X_3	$\frac{\varepsilon}{2} X_3 + X_4$	X_5	$\left(\frac{\varepsilon^2}{2\sigma^2} - \frac{\varepsilon \mathrm{K}}{\sigma^2}\right) X_1$ - $\frac{\varepsilon}{\sigma^2} X_2 + X_6$
X_5	X_1	X_2	$2\varepsilon X_1 + X_3$	$-\varepsilon \mathrm{K} X_1 +$ $\varepsilon X_2 + X_4$	$\varepsilon^2 X_1 +$ εX_3 $+X_5$	$-\frac{1}{2}\varepsilon X_1 + X_6$
X_6	X_1	X_2	X_3	X_4	X_5	X_6

TABLE 6.2

CHAPTER 7

A PHENOMENOLOGICAL QUANTUM MECHANICAL MODEL OF FINANCIAL MARKETS

Abstract

Several techniques of fundamental physics like quantum mechanics, field theory and related tools of non-commutative probability, gauge theory, path integral etc. are being applied for pricing of contemporary financial products and for explaining various phenomena of financial markets like stock price patterns, critical crashes etc.. In this Chapter, we apply the well entrenched methods of quantum mechanics and quantum field theory to the modeling of the financial markets and the behaviour of stock prices. After defining the various constituents of the model including creation & annihilation operators and buying & selling operators for securities, we examine the time evolution of the financial markets and obtain the Hamiltonian for the trading activities of the market. We obtain the probability distribution of stock prices in terms of the propagators of the evolution equations. Results on pricing of derivative contracts using quantum mechanical procedures are also presented.

7.1 INTRODUCTION

In this Chapter, the objective is to apply the well entrenched methods of quantum mechanics and quantum field theory to the study of the financial markets and the behaviour of stock prices. Section 2, which forms the essence of this Chapter, arrives at various results for financial markets by modeling them as quantum Hamiltonian systems. The probability distribution for stock prices in efficient markets is also obtained. Results on derivative pricing using the above techniques are also presented in Section 3. Section 4 concludes.

7.2 QUANTUM MODEL OF FINANCIAL MARKETS

We consider an "isolated" financial market comprising of n investors and m type of securities. The market is "isolated" in the sense that new types of securities are neither created nor are existing ones destroyed. Further, the number of investors is also constant. The investor i, $i = 1,2,3......n$ is assumed to possess a cash balance of $x_i, i = 1,2,3.......n$ (which may be negative, representing borrowings) and $y_{ij}(z), i = 1,2,3.....n; j = 1,2,3....m$ units of security j at a unit price of z. Obviously, $y_{ij} \geq 0, \forall i, j$.

Towards constructing a basis for our Hilbert space representing the financial market, we define a pure state of the system as

$$|\Psi_i\rangle = \left|\left\{x_i, \left\{y_{ij}(z), j = 1,2,...m\right\}, i = 1,2,...n\right\}\right\rangle \tag{7.1}$$

Thus, a pure state represents a state of the market where the entire holdings of cash and securities of every investor are known with certainty. This represents a complete measurement of the market and hence, is in conformity with the standard definition of "pure state" of a system.

A basis for our Hilbert space may then be constituted by the set of all the pure states of the type (7.1) i.e.

$$\Psi = \left\{ \left| \left\{ x_i, \left\{ y_{ij}(z), j = 1, 2, \ldots m \right\}, i = 1, 2, \ldots n \right\} \right\rangle \right\} \tag{7.2}$$

The elements of this basis set Ψ satisfy the orthogonality condition $\langle \Psi_i | \Psi_j \rangle = \delta_{ij}$ with respect to the scalar product defined in the sequel. The orthogonality condition makes sense in the financial world – it implies that if a market is in a pure state $|\Psi_i\rangle$ then it cannot be in any other pure state.

However, a complete measurement of the market is, obviously, not practicable in real life. At any point in time, we are likely to have certain information only about a fraction of the market constituents. Hence, the instantaneous state of the market $|\psi(t)\rangle$ may be represented by a linear combination of the pure states $|\Psi_l(t)\rangle$ i.e.

$$|\psi(t)\rangle = \sum_l C_l |\Psi_l(t)\rangle \tag{7.3}$$

We endow our Hilbert space H with the scalar product

$$\langle\psi(t)|\xi(t)\rangle=\sum_{l,m}C_l^*D_m\langle\Psi_l(t)|\Psi_m(t)\rangle=\sum_{l,m}C_l^*D_m\delta_{lm}=\sum_l C_l^*D_l \tag{7.4}$$

where we have assumed the orthogonality of the pure states.

The components of the state space vector $|\psi(t)\rangle$ are given by $C_l=\langle\Psi_l(t)|\psi(t)\rangle$ and are related to the probability of finding the market in the pure state $|\Psi_l(t)\rangle$.

Since our basis comprises of all possible measurable pure states, the completeness of the basis is ensured so that

$$I=\sum_l|\Psi_l(t)\rangle\langle\Psi_l(t)| \tag{7.5}$$

In analogy with the no particle state or ground state in quantum mechanics, we can define a ground state of our financial market as

$$|0\rangle=|x_i=0,y_{ij}(z)=0\forall i,j,z\rangle \tag{7.6}$$

i.e. the ground state is the market state in which no investor has any cash balances nor any securities. This state is, obviously, a pure state being fully measurable and would also not evolve in time since no trade can take place in this market.

We define the cash and security coordinate operators $\hat{x}_i$ & $\hat{y}_{ij}(z)$ by their action on the basis state (7.1) to provide respectively the balances of cash and the j^{th} security (at price z) with the i^{th} investor as the eigenvalues i.e.

$$\hat{x}_i\left|\left\{x_i,\left\{y_{ij}(z),j=1,2,...m\right\},i=1,2,...n\right\}\right\rangle = x_i\left|\left\{x_i,\left\{y_{ij}(z),j=1,2,...m\right\},i=1,2,...n\right\}\right\rangle \quad (7.7)$$

$$\hat{y}_{ij}(z)\left|\left\{x_i,\left\{y_{ij}(z),j=1,2,...m\right\},i=1,2,...n\right\}\right\rangle = y_{ij}(z)\left|\left\{x_i,\left\{y_{ij}(z),j=1,2,...m\right\},i=1,2,...n\right\}\right\rangle \quad (7.8)$$

A cash translation operator $\hat{T}_i(z)$ is also defined by the following

$$\hat{T}_i(z')\left|\left\{x_i,\left\{y_{ij}(z),j=1,2,...m\right\},i=1,2,...n\right\}\right\rangle = \left|\left\{x_i+z',\left\{y_{ij}(z),j=1,2,...m\right\},i=1,2,...n\right\}\right\rangle \quad (7.9)$$

i.e. it transfers an amount of cash z to the i^{th} investor.

The operator $\hat{T}_i(z)$ obviously satisfies the following properties

$$\hat{T}_i(z_1)\hat{T}_i(z_2) = \hat{T}_i(z_1+z_2) \quad (7.10)$$

$$\hat{T}_i(0) = \hat{I} \quad (7.11)$$

$$\left[\hat{T}_i(z),\hat{x}_j\right] = \hat{T}_i(z)\hat{x}_j - \hat{x}_j\hat{T}_i(z) = -\delta_{ij}\hat{T}_i(z) \quad (7.12)$$

$$\hat{T}_i^{\dagger}(z) = \hat{T}_i(-z) \quad (7.13)$$

Towards obtaining an explicit representation of the cash translation operator, we assume $\hat{p}_i = \left.\frac{d\hat{T}_i(z)}{dz}\right|_{z=0}$ as the generator of infinitesimal cash translations dz to the investor i. Expanding $\hat{T}_i(z)$ as a Taylor's series and using eqs. (7.10), (7.11) we have

$$\frac{d\hat{T}_i(z)}{dz} = \lim_{dz\to 0}\frac{\hat{T}_i(z+dz)-\hat{T}_i(z)}{dz} = \lim_{dz\to 0}\frac{\left[\hat{T}_i(dz)-1\right]\hat{T}_i(z)}{dz} = \lim_{dz\to 0}\frac{\left[\hat{T}_i(0)+\left.\frac{d\hat{T}_i(z)}{dz}\right|_{z=0} dz...-1\right]\hat{T}_i(z)}{dz} = \hat{p}_i\hat{T}_i(z) \qquad (7.14)$$

with the solution $\hat{T}_i(z) = e^{z\hat{p}_i}$. Furthermore, we have (suppressing the y_{ij} indices for the sake of brevity)

$$\begin{aligned}&\left|\{x_i+dz, i=1,2...n\}\right\rangle = \hat{T}_i(dz)\left|\{x_i, i=1,2...n\}\right\rangle = \left[\hat{T}_i(0)+\left.\frac{d\hat{T}_i(z)}{dz}\right|_{z=0} dz...\right]\left|\{x_i, i=1,2...n\}\right\rangle \\ &= \left[I+\hat{p}_i dz...\right]\left|\{x_i, i=1,2...n\}\right\rangle\end{aligned} \qquad (7.15)$$

Hence,

$$\begin{aligned}&\frac{\partial\left\langle\{x_i, i=1,2,...n\}\middle|\psi\right\rangle}{\partial x_i} = \lim_{dz\to 0}\frac{\left\langle\{x_i+dz, i=1,2,...n\}\middle|\psi\right\rangle - \left\langle\{x_i, i=1,2,...n\}\middle|\psi\right\rangle}{dz} \\ &= \hat{p}_i^{\dagger}\left\langle\{x_i, i=1,2,...n\}\middle|\psi\right\rangle = -\hat{p}_i\left\langle\{x_i, i=1,2,...n\}\middle|\psi\right\rangle \Rightarrow \hat{p}_i = -\frac{\partial}{\partial x_i}\end{aligned} \qquad (7.16)$$

so that $\hat{T}_i(z) = e^{-z\frac{\partial}{\partial x_i}}$. The following commutation rule holds between $\hat{x}_i$ and $\hat{p}_i$

$$[\hat{x}_i, \hat{p}_j] = \delta_{ij} \tag{7.17}$$

The condition of an isolated market ensures that the basis and hence the Hilbert space does not depend on time. This implies that the temporal evolution of the system is unitary.

7.2.1 CREATION & ANNIHILATION OPERATORS FOR SECURITIES

We define $\hat{a}_{ij}(z)$ as the annihilation operator of the security j from the portfolio of investor i for a price z i.e. when operator $\hat{a}_{ij}(z)$ acts on a state, the number of units of security j is reduced by one from the portfolio of investor i for a price z. Similarly, we define creation operators $\hat{a}^\dagger_{ij}(z)$ as the adjoint of the annihilation operators that increase the number of units of security j in the portfolio of investor i for a price z. The precise action of these operators on a state vector is defined by the following

$$\hat{a}_{ij}(z)\left|\{x_i,\{y_{ij}(z),j=1,2,..m\},i=1,2,..n\}\right\rangle=\sqrt{y_{ij}(z)z}\left|\{x_i,\{y_{ij}(z)-1,j=1,2,..m\},i=1,2,..n\}\right\rangle \tag{7.18}$$

and

$$\hat{a}_{ij}^{\dagger}(z)\left|\left\{x_i,\left\{y_{ij}(z),j=1,2,..m\right\},i=1,2,...n\right\}\right\rangle=\sqrt{(y_{ij}(z)+1)z}\left|\left\{x_i,\left\{y_{ij}(z)+1,j=1,2,..m\right\},i=1,2,..n\right\}\right\rangle \quad (7.19)$$

where the factor ' z ' has been introduced in the eigenvalues to ensure "scale invariance" of the theory.

These operators satisfy the following commutation relations:-

$$\left[\hat{a}_{ij}(z),\hat{a}_{ij}^{\dagger}(z')\right]=z\delta_{zz'}\delta_{ik}\delta_{jl} \quad (7.20)$$

and

$$\left[\hat{a}_{ij}(z),\hat{a}_{kl}(z')\right]=\left[\hat{a}_{ij}^{\dagger}(z),\hat{a}_{ij}^{\dagger}(z')\right]=0 \quad (7.21)$$

$$\left[\hat{T}_i(z),\hat{a}_{jk}(z')\right]=\left[\hat{T}_i(z),\hat{a}_{jk}^{\dagger}(z')\right]=0 \quad (7.22)$$

$$\left[\hat{T}_i^{\dagger}(z),\hat{a}_{jk}(z')\right]=\left[\hat{T}_i^{\dagger}(z),\hat{a}_{jk}^{\dagger}(z')\right]=0 \quad (7.23)$$

Further more

$$\hat{a}_{ij}^{\dagger}(z)\hat{a}_{ij}(z)\left|\left\{x_i,\left\{y_{ij}(z),j=1,2,..m\right\},i=1,2,..n\right\}\right\rangle=zy_{ij}(z)\left|\left\{x_i,\left\{y_{ij}(z),j=1,2,..m\right\},i=1,2,..n\right\}\right\rangle \quad (7.24)$$

which implies that the number operator would be

$$\hat{y}_{ij}(z) = \frac{\hat{a}_{ij}^{\dagger}(z)\hat{a}_{ij}(z)}{z} \tag{7.25}$$

Using the aforesaid operators we can construct an arbitrary basis state from the ground state as follows

$$\left|\left\{x_i, \left\{y_{ij}(z), j=1,2,...m\right\}, i=1,2,...n\right\}\right\rangle = \prod_{i=1}^{n} \hat{T}_i(x_i) \prod_{j=1}^{m} \prod_{\left\{z, y_{ij}(z)\in \mathrm{N}\right\}} \left(\hat{a}_{ij}^{\dagger}(z)\right)^{y_{ij}(z)} \left|0\right\rangle \tag{7.26}$$

7.2.2 BUYING AND SELLING OPERATORS

The buying (selling) operation of a security is, in each case, a composite operation consisting of the following:-

i. the creation (annihilation) of a security at the relevant price z; and

ii. the decrease (increase) in the cash balance by z of the investor undertaking the trade.

Hence we can define the buying (selling) operator as composite of the cash translation operator and the creation (annihilation) operators for securities as follows:-

$$\hat{b}_{ij}^{\dagger}(z) = \hat{a}_{ij}^{\dagger}(z)\hat{T}_i^{\dagger}(z) = \hat{a}_{ij}^{\dagger}(z)\hat{T}_i(-z) \tag{7.27}$$

for the “buying” operation and

$$\hat{b}_{ij}(z) = \hat{a}_{ij}(z)\hat{T}_i(z) \tag{7.28}$$

for the “selling” operation. These operators satisfy the following commutation rules

$$\left[\hat{b}_{ij}(z), \hat{b}_{kl}^{\dagger}(z')\right] = z\delta_{zz'}\delta_{ik}\delta_{jl} \tag{7.29}$$

$$\left[\hat{b}_{ij}(z), \hat{b}_{kl}(z')\right] = \left[\hat{b}_{ij}^{\dagger}(z), b_{kl}^{\dagger}(z')\right] = 0 \tag{7.30}$$

$$\left[\hat{b}_{ij}(z), \hat{T}_k(z')\right] = \left[\hat{b}_{ij}^{\dagger}(z), \hat{T}_k(z')\right] = 0 \tag{7.31}$$

$$\left[\hat{b}_{ij}^{\dagger}(z), \hat{x}_k\right] = z\delta_{ik}\hat{b}_{ij}^{\dagger}(z) \tag{7.32}$$

$$\left[\hat{b}_{ij}(z), \hat{x}_k\right] = -z\delta_{ik}\hat{b}_{ij}(z) \tag{7.33}$$

7.3 TEMPORAL EVOLUTION OF FINANCIAL MARKETS

In analogy with quantum mechanics, we mandate that the state of the market at a given instant of time 't', is represented by a vector in the Hilbert space H whose components determine the statistical nature of the market. Hence the temporal evolution of the market in essentially determined by the evolution of this vector with the flow of time. In the

Schrödinger picture, the time evolution of a system can be characterized through a unitary evolution operator $\hat{U}(t,t_0)$ in H, that acts on the initial state $|\psi(t_0)\rangle$ to transform it to $|\psi(t)\rangle$ i.e

$$|\psi(t)\rangle = \hat{U}(t,t_0)|\psi(t_0)\rangle \tag{7.34}$$

The assumption of the market being isolated and hence $\Psi = \{|\{x_i, \{y_{ij}(z), j=1,2,...m\}, i=1,2,...n\}\rangle\}$ being a complete basis at all times, and the conservation of probability i.e. $\sum_l |C_l(t)|^2 = 1, \forall t$ together with the group property of $\hat{U}(t,t_0)$ implies that the temporal evolution is unitary i.e.

$$U(t,t_0)U^{\dagger}(t,t_0) = U^{\dagger}(t,t_0)U(t,t_0) = 1 \tag{7.35}$$

Furthermore $\hat{U}(t_0,t_0)=1$. Defining the Hamiltonian $\hat{H}(t) = i\frac{\partial}{\partial t}\hat{U}(t,t_0)\Big|_{t=t_0}$ as the infinitesimal generator of time translations (evolution) we obtain, through a Taylor's expansion up to first order $\hat{U}(t+\delta t,t_0) = \hat{U}(t,t_0) + \frac{\partial \hat{U}}{\partial t}(t+\delta t,t_0)\Big|_{\delta t=0} \hat{U}(t,t_0)\delta t + ...$ or

$$\frac{\partial \hat{U}(t,t_0)}{\partial t} = \lim_{\delta t \to 0} \frac{\hat{U}(t+\delta t,t_0)-\hat{U}(t,t_0)}{\delta t} = -i\hat{H}(t)\hat{U}(t,t_0) \quad (7.36)$$

with the immediate solution $\hat{U}(t,t_0) = e^{-\int_{t_0}^{t} \hat{H}(t)dt}$ where time ordering of the operators constituting the Hamiltonian is assumed.

Before progressing further with the development of the model, some observations are in order about the theory developed thus far.

a. In standard quantum mechanics, $\hat{H}(t)$ is usually a bounded operator and hence the exponential series in $\hat{U}(t,t_0) = e^{-\int_{t_0}^{t} \hat{H}(t)dt}$ converges so that its approximation to first order is acceptable giving $i\frac{\partial |\psi(t)\rangle}{\partial t} = \hat{H}(t)|\psi(t)\rangle$ which is the Schrödinger equation of wave mechanics. This may not always be the case in financial markets.

b. Since time evolution of financial market, essentially, occurs through trades in securities, it is appropriate to infer that the Hamiltonian represents the trading activities of the market.

c. In order that the evolution operator $\hat{U}(t,t_0)$ is well defined, we mandate that the Hilbert space H is so constructed that the kernel of $\hat{U}(t,t_0)$ is empty.

7.4 MODELING TIME VALUE OF MONEY

Time value of money and interest rate instruments are classically modeled through the first order differential equation $\frac{dB(t)}{dt} = r(t)B(t)$ with the solution $B(t) = B(0)e^{\int r(t)dt}$.

A possible candidate for the Hamiltonian function H (in the classical picture) that would generate this temporal development as the equations of motion is

$$H(x,p;t) = \sum_{i=1}^{n} H_i(x_i,p_i;t) = \sum_{i=1}^{n} r_i(t)x_i(t)p_i(t) \tag{7.37}$$

This Hamiltonian leads to the following equations of motion

$$\frac{dx_i(t)}{dt} = \frac{\partial H_i(x_i,p_i;t)}{\partial p_i} = r_i(t)x_i(t),\ \frac{dp_i(t)}{dt} = -\frac{\partial H_i(x_i,p_i;t)}{\partial x_i} = -r_i(t)p_i(t) \tag{7.38}$$

While the interpretation of first of these equations is straightforward being the growth of cash reserves of the i^{th} investor with the instantaneous rate $r_i(t)$, the implications of second equation are more subtle. To provide a financial logic to this equation, we note that p_i is the infinitesimal generator of cash translations in the classical picture and hence $\frac{dp_i(t)}{dt} = -\frac{\partial H_i(x_i,p_i;t)}{\partial x_i} = -r_i(t)p_i(t)$ represents the rate of change of the cash translations generator which, given a fixed rate of growth of cash, would decrease with the amount of cash translations.

Using the Weyl formalism for transformation from the classical to the quantum picture, we require that the quantum mechanical analog of $H(x, p; t)$ be Hermitian and symmetric in its component operators. Hence, we postulate the ansatz

$$\hat{H}(\hat{x}(t), \hat{p}(t); t) = \sum_{i=1}^{n} \hat{H}(\hat{x}_i(t), \hat{p}_i(t); t) = \sum_{i=1}^{n} \frac{ir_i(t)}{2}(\hat{x}_i\hat{p}_i + \hat{p}_i\hat{x}_i) = \sum_{i=1}^{n} ir_i(t)\hat{x}_i\left(\hat{p}_i + \frac{1}{2}\hat{I}\right) \quad (7.39)$$

for the quantum mechanical Hamiltonian representing the time value of money, so that the time development operator is

$$\hat{U}(t, t_0) = e^{\left[-i\int_{t_0}^{t} \hat{H}(t)dt\right]} = e^{\sum_{i=1}^{n}\int_{t_0}^{t} r_i(t)x_i\left(\hat{p}_i + \frac{1}{2}\hat{I}\right)dt} \quad (7.40)$$

which may be evaluated using standard methods like Green's functions and Feymann propagator theory.

7.5 REPRESENTATION OF TRADING ACTIVITY

Let us consider a deal in which an investor 'i' buys a security 'j' at a price of 'z' units and immediately thereafter sells the same security to another investor 'k' at a price of 'z'' units and credits/debits the difference $z' - z$ to his cash account. The composite transaction will, in our operator formalism, take the form $\hat{b}_{ij}(z')\hat{b}_{ij}^{\dagger}(z)$. In analogy with this argument, we can represent the Hamiltonian for trading activity of the market as

$$H_{Tr}(t)=\sum_{i,j,k,l}\int_0^{\infty}\frac{dz}{z}\int_0^{\infty}\frac{dz'}{z'}h_{ijkl}(\xi,t)\hat{b}_{ij}^{\dagger}(z)\hat{b}_{kl}(z') \qquad (7.41)$$

where $\xi = ln\frac{z'}{z}$ ensures that the amplitudes are scale invariant.

7.6 PROBABILITY DISTRIBUTION OF STOCK PRICES

We now derive the probability distribution of stock prices in market equilibrium and show that the prices follow a lognormal distribution, thereby vindicating the efficacy of this model.

For this purpose, we assume that an investor $i=\alpha$ buys one unit of a security $j=\beta$ at time $t=t_i$ for a price z. We need to ascertain the probability $P_T(z'|z)$ i.e. the probability of the security $j=\beta$ having a price z' at time $t_f=t_i+T$. We assume that during the period t_f-t_i, investor α holds exactly one unit of β and that before t_i and after t_f, α holds no unit of β.

Let $|\psi^z_{\alpha\beta}(t_i)\rangle$ be the state that represents investor α holding exactly one unit of β at a price z at time t_i in the Hilbert space H. Hence, we have $|\psi^z_{\alpha\beta}(t_i)\rangle=\hat{b}^{\dagger}_{\alpha\beta}(z)|\overline{\psi_{\alpha\beta}(t_i)}\rangle$ where $|\overline{\psi_{\alpha\beta}(t_i)}\rangle$ is the state that represents investor α not holding any unit of β. This also implies that $\hat{b}_{\alpha\beta}(z)\left|\overline{\psi_{\alpha\beta}(t_i)}\right\rangle = 0$. Let us assume that the final state corresponding to the initial state $\left|\psi^z_{\alpha\beta}(t_i)\right\rangle$ is represented by $\left|\psi^z_{\alpha\beta}(t_f)\right\rangle$ so that

$$\left|\psi^z_{\alpha\beta}(t_f)\right\rangle = \hat{U}(t_i,t_f)\left|\psi^z_{\alpha\beta}(t_i)\right\rangle = e^{-i\int_{t_i}^{t_f}\hat{H}dt}\hat{b}^{\dagger}_{\alpha\beta}(z)\left|\overline{\psi_{\alpha\beta}(t_i)}\right\rangle \qquad (7.42).$$

The amplitudes of $\left|\psi^{z'}_{\alpha\beta}(t_f)\right\rangle$ are determined in the usual way by taking scalar product $\left\langle\psi^{z'}_{\alpha\beta}(t_f)\middle|\psi^z_{\alpha\beta}(t_f)\right\rangle$ and we have, for the matrix elements of the propagator

$$G(z',t_f;z,t_i) = \left\langle\overline{\psi_{\alpha\beta}(t_f)}\right|\hat{b}_{\alpha\beta}(z')e^{-i\int_{t_i}^{t_f}\hat{H}dt}\hat{b}^{\dagger}_{\alpha\beta}(z)\left|\overline{\psi_{\alpha\beta}(t_i)}\right\rangle \qquad (7.43)$$

In this case, the trading Hamiltonian will contain creation and annihilation operators relating to the investor α and those relating to the security β i.e., it will be of the form

$$\hat{H}_{Tr}(t) = \sum_{k,l}\int_0^{\infty}\frac{dz}{z}\int_0^{\infty}\frac{dz'}{z'}h_{\alpha\beta kl}(\xi,t)\hat{b}^{\dagger}_{\alpha\beta}(z)\hat{b}_{kl}(z') \qquad (7.44)$$

We further make the assumption that the amplitudes can be approximated by their first two moments about $\xi = 0$, being sharply peaked about $z' = z$ since, in the timescales being considered, most trades would occur around z. Hence, we have

$$h_{\alpha\beta kl} \sim \left[\Omega_{\alpha\beta kl}(t) - i\xi^{-1}\Xi_{\alpha\beta kl}(t)\right]\delta(\xi) \qquad (7.45)$$

Noting that $\xi = ln\frac{z'}{z}$, we have $\xi^{-1} = \left(\ln\frac{z'}{z}\right)^{-1} = \left(\frac{z'}{z}-1\right)^{-1} = \frac{z}{z'-z}$ to first order and $\delta(\xi) = \delta\left(\ln\frac{z'}{z}\right) = \delta(z'-z)\left[\frac{d\left(\ln\frac{z'}{z}\right)}{dz'}\right]^{-1} = z'\delta(z'-z)$. Using these results and eqs. (7.44) & (7.45), we obtain

$$\hat{H}_{Tr}(t) \sim \sum_{k,l}\int_0^\infty \frac{dz}{z}\int_0^\infty \frac{dz'}{z'} z\delta(z'-z)\left[\Omega_{\alpha\beta kl}(t) - i\frac{z}{z'-z}\Xi_{\alpha\beta kl}(t)\right]\hat{b}^\dagger_{\alpha\beta}(z)\hat{b}_{kl}(z')$$

$$= \sum_{k,l}\int_0^\infty \frac{dz}{z}\hat{b}^\dagger_{\alpha\beta}(z)\left[\Omega_{\alpha\beta kl}(t) + iz\Xi_{\alpha\beta kl}(t)\frac{\partial}{\partial z}\right]\hat{b}_{kl}(z') \qquad (7.46)$$

We note that this expression for the Hamiltonian is linear in $\frac{\partial}{\partial z}$ and hence it can be diagonalized in the "momentum space" through a Fourier transformation and we have

$$\hat{H}_{Tr}(t) = \frac{1}{2\pi}\sum_{k,l}\int_0^\infty \frac{dz}{z}\int_0^\infty \frac{dz'}{z'}\int_{-\infty}^\infty dp\,\hat{b}^\dagger_{\alpha\beta}(z)\left[\Omega_{\alpha\beta kl}(t) + i\Xi_{\alpha\beta kl}(t)p\right]\hat{b}_{kl}(z')e^{ip\xi} \qquad (7.47)$$

The assumption of market equilibrium implies that the Hamiltonian should be independent of time over the relevant timescales that would be much smaller than those determining aggregate market behaviour so that we may write eq. (7.43) as

$$G(z',t_f;z,t_i)=\left\langle\overline{\psi_{\alpha\beta}(t_i)}\right|\hat{b}_{\alpha\beta}(z')e^{-i\hat{H}(t_i)T}\hat{b}^{\dagger}_{\alpha\beta}(z)\left|\overline{\psi_{\alpha\beta}(t_i)}\right\rangle \tag{7.48}$$

Because of the Hamiltonian being diagonal in momentum space, it is more convenient to work in momentum space for evaluating the propagators and we have, for the equivalent of eq. (7.48) in momentum space as

$$\tilde{G}(p',p;T,t_i)=\left\langle\overline{\psi_{\alpha\beta}(t_i)}\right|\left[\int_0^{\infty}\frac{dz'}{z'}e^{ip'\ln(z'/\kappa)}\hat{b}_{\alpha\beta}(z')\right]e^{-i\hat{H}(t_i)T}\left[\int_0^{\infty}\frac{dz}{z}e^{-ip\ln(z/\kappa)}\hat{b}^{\dagger}_{\alpha\beta}(z)\right]\hat{b}^{\dagger}_{\alpha\beta}(z)\left|\overline{\psi_{\alpha\beta}(t_i)}\right\rangle \tag{7.49}$$

To solve the problem further, we make use of second order perturbation theory. The first step is to split the Hamiltonian into components as follows

$$\hat{H}(t)=\sum_l\int_0^{\infty}\frac{dz}{z}\hat{b}^{\dagger}_{\alpha\beta}(z)\left[\Omega_{\alpha\beta\alpha l}(t)+\Xi_{\alpha\beta\alpha l}(t)p\right]\hat{b}_{\alpha\beta}(z)+\sum_{k,l,k\neq\alpha}\int_0^{\infty}\frac{dz}{z}\hat{b}^{\dagger}_{\alpha\beta}(z)\left[\Omega_{\alpha\beta kl}(t)+i\Xi_{\alpha\beta kl}(t)p\right]\hat{b}_{kl}(z') \tag{7.50}$$

Let E_i be the energy eigenstate of the unperturbed Hamiltonian i.e. of the state of the market before the purchase of security β by the investor α , then the energy eigenstate of the Hamiltonian $\hat{H}(t_i)$ i.e. after the purchase of security β by the investor α will be of the form $E_p=E_i+\sum_l\left[\Omega_{\alpha\beta\alpha l}(t_i)+i\Xi_{\alpha\beta\alpha l}(t_i)p\right]-ip^2\sigma^2$ where the second term represents the impact on the energy eigenstates of the transactions involving investor α or security β and the last term is the second order perturbation term due to the overall fluctuations of the market. Substituting this value of E_p in eq. (7.49) and noting that the Hamiltonian and hence the propagator $\tilde{G}(p',p;T,t_i)$ is also diagonal in "momentum space", we have

$$\tilde{G}(p',p;T,t_i)\sim 2\pi\delta(p'-p)e^{-iTE_p}=2\pi\delta(p'-p)e^{-iT\left[E_i+\Omega(t_i)+i\Xi(t_i)p-ip^2\sigma^2\right]} \tag{7.51}$$

where $\sum_l \Omega_{\alpha\beta\alpha l}(t_i)=\Omega(t_i)$, $\sum_l \Xi_{\alpha\beta\alpha l}(t_i)=\Xi(t_i)$.

Inverting back to "coordinate space", we obtain

$$G(z',t_f;z,t_i)=\frac{1}{2\pi}\int_{-\infty}^{\infty}dpe^{-iT\left[E_i+\Omega(t_i)+i\Xi(t_i)p-ip^2\sigma^2\right]-ip\ln z'/z}\sim\frac{e^{iT(E_i+\Omega)}}{2\sigma\sqrt{\pi T}}e^{\left[\frac{(ln(z'/z)+\Xi T)^2}{4\sigma^2 T}\right]} \tag{7.52}$$

The probability $P_T(z'|z)$ i.e. the probability of the security $j=\beta$ having a price z' at time $t_f=t_i+T$ will then be proportional to the square of the above amplitude and hence, we finally obtain

$$P_T(z'|z)\alpha\left|G(z',t_f;z,t_i)\right|^2=\left(4\pi\sigma^2 T\right)^{-1}e^{\left[\frac{(ln(z'/z)+\Xi T)^2}{2\sigma^2 T}\right]} \tag{7.53}$$

which agrees perfectly with the standard stochastic theory of finance wherein stock returns are modeled extensively through lognormal distributions.

7.7 BASIC FRAMEWORK FOR OPTION PRICING

In analogy with quantum mechanics, we mandate that the state of a security in the market at a given instant of time 't', is represented by a vector $|C(x,t)\rangle$ in the Hilbert space H. The components of the state vector $|C(x,t)\rangle$ determine the statistical development of the security and its temporal evolution is determined by the evolution of this state vector $|C(x,t)\rangle$ with the flow of time.

In the Schrödinger picture, the time evolution of a system can be characterized through a unitary evolution operator $\hat{U}(t,t_0)$ in H, that acts on the initial state $|C(x,t_0)\rangle$ to transform it to $|C(x,t)\rangle$ i.e

$$|C(x,t)\rangle = \hat{U}(t,t_0)|C(x,t_0)\rangle \tag{7.54}$$

We assume that the evolution of the security is in an isolated market. This assumption implies that the initially chosen basis will remain complete at all times. The conservation of probability together with the group property of $\hat{U}(t,t_0)$,then, implies that the temporal evolution of the security is unitary i.e.

$$U(t,t_0)U^{\dagger}(t,t_0) = U^{\dagger}(t,t_0)U(t,t_0) = 1 \tag{7.55}$$

Furthermore $\hat{U}(t_0,t_0)=1$. Defining the Hamiltonian $\hat{H}(t)=i\frac{\partial}{\partial t}\hat{U}(t,t_0)\Big|_{t=t_0}$ as the infinitesimal generator of time translations (evolution) we obtain, through a Taylor's expansion up to first order

$$\hat{U}(t+\delta t,t_0)=\hat{U}(t,t_0)+\frac{\partial \hat{U}}{\partial t}(t+\delta t,t)\Big|_{\delta t=0}\hat{U}(t,t_0)\delta t+\ldots \text{or}$$

$$\frac{\partial \hat{U}(t,t_0)}{\partial t}=\lim_{\delta t\to 0}\frac{\hat{U}(t+\delta t,t_0)-\hat{U}(t,t_0)}{\delta t}=-i\hat{H}(t)\hat{U}(t,t_0) \tag{7.56}$$

with the immediate solution $\hat{U}(t,t_0)=e^{-\int_{t_0}^{t}\hat{H}(t)dt}$ where time ordering of the operators constituting the Hamiltonian is assumed.

A space translation operator $\hat{T}_i(z)$ is also defined by its action on an arbitrary basis

$$\hat{T}_i(z')\left|\{x_i,i=1,2,\ldots n\}\right\rangle=\left|\{x_i+z',i=1,2,\ldots n\}\right\rangle \tag{7.57}$$

i.e. it translates a distance z in the i^{th} direction.

The operator $\hat{T}_i(z)$ obviously satisfies the following properties

$$\hat{T}_i(z_1)\hat{T}_i(z_2)=\hat{T}_i(z_1+z_2) \tag{7.58}$$

$$\hat{T}_i(0) = \hat{I} \tag{7.59}$$

$$\left[\hat{T}_i(z), \hat{x}_j\right] = \hat{T}_i(z)\hat{x}_j - \hat{x}_j\hat{T}_i(z) = -z\delta_{ij}\hat{T}_i(z) \tag{7.60}$$

$$\hat{T}_i^{\dagger}(z) = \hat{T}_i(-z) \tag{7.61}$$

Towards obtaining an explicit representation of the space translation operator, we assume $\hat{p}_i = -i\left.\frac{d\hat{T}_i(z)}{dz}\right|_{z=0}$ as the generator of infinitesimal space translations dz in the i spatial direction. Expanding $\hat{T}_i(z)$ as a Taylor's series and using eqs. (7.58-7.61) we have

$$\frac{d\hat{T}_i(z)}{dz} = \lim_{dz\to 0}\frac{\hat{T}_i(z+dz)-\hat{T}_i(z)}{dz} = \lim_{dz\to 0}\frac{\left[\hat{T}_i(dz)-1\right]\hat{T}_i(z)}{dz} = \lim_{dz\to 0}\frac{\left[\hat{T}_i(0)+\left.\frac{d\hat{T}_i(z)}{dz}\right|_{z=0} dz ... -1\right]\hat{T}_i(z)}{dz} = i\hat{p}_i\hat{T}_i(z) \tag{7.62}$$

with the solution $\hat{T}_i(z) = e^{iz\hat{p}_i}$. Furthermore, we have

$$\begin{aligned}&\left|\{x_i+dz, i=1,2...n\}\right\rangle = \hat{T}_i(dz)\left|\{x_i, i=1,2...n\}\right\rangle = \left[\hat{T}_i(0)+\left.\frac{d\hat{T}_i(z)}{dz}\right|_{z=0} dz...\right]\left|\{x_i, i=1,2...n\}\right\rangle \\ &= \left[I+i\hat{p}_i dz...\right]\left|\{x_i, i=1,2...n\}\right\rangle\end{aligned} \tag{7.63}$$

Hence,

$$\begin{aligned}&\frac{\partial\left\langle\{x_i, i=1,2,...n\}\middle|\psi\right\rangle}{\partial x_i} = \lim_{dz\to 0}\frac{\left\langle\{x_i+dz, i=1,2,...n\}\middle|\psi\right\rangle - \left\langle\{x_i, i=1,2,...n\}\middle|\psi\right\rangle}{dz} \\ &= -i\hat{p}_i^{\dagger}\left\langle\{x_i, i=1,2,...n\}\middle|\psi\right\rangle = -\hat{p}_i\left\langle\{x_i, i=1,2,...n\}\middle|\psi\right\rangle \Rightarrow \hat{p}_i = -i\frac{\partial}{\partial x_i}\end{aligned} \tag{7.64}$$

The operators $\hat{x}_i$ and $\hat{p}_i$ obey the commutation rule $\left[\hat{x}_i, \hat{p}_j\right] = i\delta_{ij}$.

7.8 PROPERTIES OF THE BLACK SCHOLES HAMILTONIAN

The Black Scholes equation for the instantaneous price of a derivative security is given by:-

$$\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + rS\frac{\partial C}{\partial S} - rC = 0 \tag{7.65}$$

where C is the instantaneous price of a contingent claim, S is the price of the underlying asset at that instant, σ is the volatility (standard deviation) of the underlying and r is the relevant risk free rate. Making the substitution $x = In\,S$ in eq. (7.65), we obtain

$$\frac{\partial C}{\partial t} = \left[-\frac{\sigma^2}{2}\frac{\partial^2}{\partial x^2} + \left(\frac{1}{2}\sigma^2 - r\right)\frac{\partial}{\partial x} + r\right]C \tag{7.66}$$

A comparison with the standard form of the non-relativistic quantum mechanical (Schrödinger) equation identifies

$$\hat{H} = -\frac{\sigma^2}{2}\frac{\partial}{\partial x^2} + \left(\frac{1}{2}\sigma^2 - r\right)\frac{\partial}{\partial x} + r \tag{7.67}$$

with the quantum mechanical Hamiltonian $\hat{H}$, volatility σ with the inverse of mass and the derivative price C with the state vector in Hilbert space.

The occurrence of $\frac{\partial}{\partial x}$ operator in the Hamiltonian $\hat{H}$ due to the drift term in the Black Scholes equation implies that the Hamiltonian in not Hermitian, unlike its quantum mechanical analogue e.g. (in Dirac notation)

$$\langle f|\hat{H}^{\dagger}|g\rangle=\langle g|\hat{H}|f\rangle^{*}=\left[\int_{-\infty}^{\infty}dx\langle g|x\rangle\langle x|\hat{H}|f\rangle\right]^{*}$$

$$=-\frac{\sigma^2}{2}\left[\int_{-\infty}^{\infty}dxg^{*}(x)\frac{\partial^2 f(x)}{\partial x^2}\right]^{*}+\left(\frac{1}{2}\sigma^2-r\right)\left[\int_{-\infty}^{\infty}dxg^{*}(x)\frac{\partial f(x)}{\partial x}\right]^{*}+r\left[\int_{-\infty}^{\infty}dxg^{*}(x)f(x)\right]^{*} \quad (7.68)$$

Now, through an integration by parts and taking relevant limits we obtain:-

$$\int_{-\infty}^{\infty}dxg^{*}(x)\frac{\partial^2 f(x)}{\partial x^2}=-\int_{-\infty}^{\infty}dx\frac{\partial g^{*}(x)}{\partial x}\frac{\partial f(x)}{\partial x}=\int_{-\infty}^{\infty}dx\frac{\partial^2 g^{*}(x)}{\partial x^2}f(x) \quad (7.69)$$

Similarly

$$\int_{-\infty}^{\infty}dxg^{*}(x)\frac{\partial f(x)}{\partial x}=-\int dx\frac{\partial g^{*}(x)}{\partial x}f(x) \quad (7.70)$$

Substituting from eqs. (7.69), (7.70) in (7.68) we get

$$\langle f|\hat{H}^{\dagger}|g\rangle=-\frac{\sigma^2}{2}\int_{-\infty}^{\infty}dxf^*(x)\frac{\partial^2 g(x)}{\partial x^2}-\left(\frac{1}{2}\sigma^2-r\right)\int_{-\infty}^{\infty}dxf^*(x)\frac{\partial g(x)}{\partial x}+r\int_{-\infty}^{\infty}dxf^*(x)g(x)\neq\langle f|\hat{H}|g\rangle \quad (7.71)$$

However we can transform the non-hermitian Hamiltonian $\hat{H}$ to a hermitian Hamiltonian $\hat{H}'$ through a similarity transformation i.e. $\hat{H}'=e^{\left(\frac{r}{\sigma^2}-\frac{1}{2}\right)x}\hat{H}e^{-\left(\frac{r}{\sigma^2}-\frac{1}{2}\right)x}$ for

$$\langle f|\hat{H}'^{\dagger}|g\rangle=\langle g|\hat{H}'|f\rangle^*=\left[\int_{-\infty}^{\infty}dx\langle g|x\rangle\langle x|\hat{H}'|f\rangle\right]^*$$

$$=-\frac{\sigma^2}{2}\left\{\int_{-\infty}^{\infty}dxg^*(x)e^{\left(\frac{r}{\sigma^2}-\frac{1}{2}\right)x}\frac{\partial^2}{\partial x^2}\left[f(x)e^{-\left(\frac{r}{\sigma^2}-\frac{1}{2}\right)x}\right]\right\}^*+\left(\frac{1}{2}\sigma^2-r\right)$$
$$\left\{\int_{-\infty}^{\infty}dxg^*(x)e^{\left(\frac{r}{\sigma^2}-\frac{1}{2}\right)x}\frac{\partial}{\partial x}\left[f(x)e^{-\left(\frac{r}{\sigma^2}-\frac{1}{2}\right)x}\right]\right\}^*+r\left[\int_{-\infty}^{\infty}dxg^*(x)f(x)\right]^* \quad (7.72)$$

Noting that

$$\frac{\partial^2}{\partial x^2}\left[e^{-\left(\frac{r}{\sigma^2}-\frac{1}{2}\right)x}f(x)\right]=\left(\frac{r}{\sigma^2}-\frac{1}{2}\right)^2e^{-\left(\frac{r}{\sigma^2}-\frac{1}{2}\right)x}f(x)-2\left(\frac{r}{\sigma^2}-\frac{1}{2}\right)e^{-\left(\frac{r}{\sigma^2}-\frac{1}{2}\right)x}\frac{\partial f(x)}{\partial x}+e^{-\left(\frac{r}{\sigma^2}-\frac{1}{2}\right)x}\frac{\partial^2 f(x)}{\partial x^2} \quad (7.73)$$

and

$$\frac{\partial}{\partial x}\left[e^{-\left(\frac{r}{\sigma^2}-\frac{1}{2}\right)x}f(x)\right]=-\left(\frac{r}{\sigma^2}-\frac{1}{2}\right)e^{-\left(\frac{r}{\sigma^2}-\frac{1}{2}\right)x}f(x)+e^{-\left(\frac{r}{\sigma^2}-\frac{1}{2}\right)x}\frac{\partial f(x)}{\partial x} \quad (7.74)$$

so that eq. (7.72) reduces to

$$=-\frac{\sigma^2}{2}\left\{\int_{-\infty}^{\infty} dx g^*(x)\frac{\partial^2 f(x)}{\partial x^2}\right\}^* + \frac{1}{2\sigma^2}\left(r+\frac{1}{2}\sigma^2\right)^2\left\{\int_{-\infty}^{\infty} dx g^*(x) f(x)\right\}^*$$

$$=-\frac{\sigma^2}{2}\left\{\int_{-\infty}^{\infty} dx \frac{\partial^2 g^*(x)}{\partial x^2} f(x)\right\}^* + \frac{1}{2\sigma^2}\left(r+\frac{1}{2}\sigma^2\right)^2\left\{\int_{-\infty}^{\infty} dx g^*(x) f(x)\right\}^*$$

$$=-\frac{\sigma^2}{2}\int_{-\infty}^{\infty} dx f^*(x)\frac{\partial^2 g(x)}{\partial x^2} + \frac{1}{2\sigma^2}\left(r+\frac{1}{2}\sigma^2\right)^2\int_{-\infty}^{\infty} dx f^*(x) g(x) = \langle f|\hat{H}'|g\rangle \quad (7.75)$$

so that $\hat{H}'$ is hermitian.

We can show that the Black Scholes Hamiltonian $\hat{H}$ of eq. (7.67) is diagonal in the momentum representation. In the position diagonal (coordinate) representation, the momentum operator is $-i\frac{\partial}{\partial x}$ and hence satisfies the equation

$$\hat{p}\psi(p,x) = -i\frac{\partial\psi(p,x)}{\partial x} = p\psi(p,x) \quad (7.76)$$

where $\psi(p,x)$ is the momentum eigenfunction corresponding to the momentum eigenvalue p.

Eq. (7.76) admits the solution $\psi(p,x) = Ae^{ipx}$ where A is a normalization constant to be determined from the orthogonality condition $\int\psi^*(p',x)\psi(p,x)dx = \delta(p-p')$. Using the definition of the Dirac δ function $\delta(p-p') = \int\frac{dx}{2\pi}e^{i(p-p')x}$ we obtain $A = \frac{1}{\sqrt{2\pi}}$ so that $\psi(p,x) = \frac{1}{\sqrt{2\pi}}e^{ipx}$ are the normalized eigenfunctions of the momentum operator $\hat{p}$ labeled by the index 'p'. The momentum operator can be considered to be a continuous matrix with the elements

$$p(x,x')=\langle x|\hat{p}|x'\rangle=-ip\langle x|x'\rangle=-i\frac{\partial}{\partial x}\delta(x-x') \tag{7.77}$$

The transformation matrix from the coordinate representation to the momentum representation is achieved through a unitary matrix U whose columns are the normalized eigenfunctions of $\hat{p}$ i.e $\psi(p,x)=\frac{1}{\sqrt{2\pi}}e^{ipx}$. The Hamiltonian matrix in the momentum representation is therefore given by

$$\hat{H}_p = \int U^{\dagger}(x,p)\left[-\frac{\sigma^2}{2}\frac{\partial^2}{\partial x^2}+\left(\frac{1}{2}\sigma^2-r\right)\frac{\partial}{\partial x}+r\right]\delta(x-x')U(x',p')\,dxdx'$$

$$=\int U^{\dagger}(x,p)\left[-\frac{\sigma^2}{2}\frac{\partial^2}{\partial x^2}+\left(\frac{1}{2}\sigma^2-r\right)\frac{\partial}{\partial x}+r\right]U(x,p')\,dx$$

$$=\frac{1}{2\pi}\int e^{-ipx}\left[-\frac{\sigma^2}{2}\frac{\partial^2}{\partial x^2}+\left(\frac{1}{2}\sigma^2-r\right)\frac{\partial}{\partial x}+r\right]e^{ip'x}dx$$

$$=\frac{1}{2\pi}\int e^{-ipx}\left[\frac{\sigma^2}{2}p'^2+i\left(\frac{1}{2}\sigma^2-r\right)p'+r\right]e^{ip'x}dx$$

$$=\frac{1}{2\pi}\left[\frac{\sigma^2}{2}p^2+i\left(\frac{1}{2}\sigma^2-r\right)p+r\right]\int e^{i(p-p')x}dx$$

$$=\left[\frac{\sigma^2}{2}p^2+i\left(\frac{1}{2}\sigma^2-r\right)p+r\right]\delta(p-p') \tag{7.78}$$

which is clearly diagonal due to the presence of the Dirac δ function.

Furthermore, we have

$$\hat{H}\left(\frac{1}{\sqrt{2\pi}}e^{ipx}\right)=\frac{1}{\sqrt{2\pi}}\left[-\frac{\sigma^2}{2}\frac{\partial^2}{\partial x^2}+\left(\frac{1}{2}\sigma^2-r\right)\frac{\partial}{\partial x}+r\right]e^{ipx}=\frac{1}{\sqrt{2\pi}}\left[\frac{1}{2}\sigma^2p^2+i\left(\frac{1}{2}\sigma^2-r\right)p+r\right]e^{ipx} \quad (7.79)$$

which shows that $\frac{1}{\sqrt{2\pi}}e^{ipx}$ is also an eigenfunction of the Hamiltonian corresponding to the eigenvalues $\left[\frac{1}{2}\sigma^2p^2+i\left(\frac{1}{2}\sigma^2-r\right)p+r\right]$.

The fact that the eigenfunctions are mutually orthogonal follows from

$$\frac{1}{2\pi}\int e^{-ip'x}e^{ipx}dx=\frac{1}{2\pi}\int e^{i(p-p')x}=\delta\left(p-p'\right). \quad (7.80)$$

The momentum space basis vectors are obtained by

$$|p\rangle=\int U^{\dagger}(x,p)|x\rangle dx=\frac{1}{\sqrt{2\pi}}\int e^{-ipx}|x\rangle dx \quad (7.81)$$

which gives the scalar product

$$\langle x'|p\rangle=\frac{1}{\sqrt{2\pi}}\int e^{-ipx}\langle x'|x\rangle dx=\frac{1}{\sqrt{2\pi}}\int e^{-ipx}\delta\left(x-x'\right)dx=\frac{1}{\sqrt{2\pi}}e^{-ipx'} \text{ which gives } \langle p|x\rangle=\frac{1}{\sqrt{2\pi}}e^{ipx}$$

Hence, $\langle x|x'\rangle = \delta(x-x') = \int \frac{dp}{2\pi} e^{-ip(x-x')} = \int dp \langle x|p\rangle\langle p|x'\rangle$ whence $\int dp|p\rangle\langle p| = I$ which establishes the completeness of the momentum basis.

7.9 THE CALL PRICE (FIRST APPROACH)

The Schrodinger form of the Black Scholes equation is $\frac{\partial C}{\partial t} = -\frac{1}{2}\sigma^2 \frac{\partial^2 C}{\partial x^2} + \left(\frac{1}{2}\sigma^2 - r\right)\frac{\partial C}{\partial x} + rC$ or equivalently $\frac{\partial C}{\partial r} = \hat{H}C$ with a solution $C(x,t) = C(x,0)e^{\hat{H}t}$ which gives $C(x,0) = C(x,T)e^{-\hat{H}T}$ so that $C(x,t) = C(x,T)e^{-\hat{H}(T-t)}$.

The components of the state vector $|C(x,t)\rangle$ in coordinate space are given by $\langle x|C(x,t)\rangle = \langle x|e^{-\hat{H}(T-t)}|C(x,T)\rangle$ which, using the completeness of the coordinate basis gives $\langle x|C(x,t)\rangle = \int dx' \langle x|e^{-\hat{H}(T-t)}|x'\rangle\langle x'|C(x,T)\rangle$.

The matrix elements $\langle x|e^{-\hat{H}(T-t)}|x'\rangle$ are computed by transforming the problem to the momentum basis using the completeness relation $\int dp|p\rangle\langle p| = I$. We have

$$\langle x|e^{-H(T-t)}|x'\rangle = \int \langle x|p\rangle\langle p|e^{-\hat{H}(T-t)}|p'\rangle\langle p'|x'\rangle\delta(p-p')dpdp'$$

$$= \frac{1}{2\pi}\int e^{-ipx+ip'x'}\langle p|e^{-\hat{H}(T-t)}|p'\rangle\delta(p-p')dpdp'$$

$$=\frac{1}{2\pi}\int e^{+ip(x'-x)}\left\langle p\right|e^{-\hat{H}(T-t)}\left|p\right\rangle dp$$

$$=\frac{1}{2\pi}\int e^{ip(x-x')-\left[\frac{\sigma^2}{2}p^2+i\left(\frac{1}{2}\sigma^2-r\right)p+r\right](T-t)}dp$$

$$=\frac{e^{-r(T-t)-\frac{1}{2(T-t)\sigma^2}\left[x'-x+\left(r-\frac{1}{2}\sigma^2\right)(T-t)\right]^2}}{\sqrt{2\pi(T-t)\sigma^2}} \tag{7.82}$$

Hence, we have, finally, for the price of the security

$$\left|C(x,t)\right\rangle=\int dx'\left\langle x\right|e^{-\hat{H}(T-t)}\left|x'\right\rangle C(x',T)$$

$$=\frac{e^{-r(T-t)}}{\sqrt{2\pi(T-t)\sigma^2}}\int dx' e^{-\frac{1}{2(T-t)\sigma^2}\left[x'-x+\left(\frac{1}{2}\sigma^2-r\right)(T-t)\right]^2}C(x',T) \tag{7.83}$$

where $C(x'T)$ is the terminal payoff of the derivative contract given by $\max(S_T-E,0)=(S_T-E)^+$ where S_T is the stock price on the exercise date and E is the exercise price. Substituting this terminal payoff in eq. (7.83), and performing the integration we obtain the standard Black Scholes formula for the pricing of an European call option.

We now show that the Black Scholes Hamiltonian satisfies the martingale condition. Solving the equation $\frac{1}{2}\sigma^2p^2+i\left(\frac{1}{2}\sigma^2-r\right)p+r=0$, we obtain the values of the index p

for which the eigenvalues of the Hamiltonian are zero as $p=-i,\frac{2ir}{\sigma^2}$. The corresponding values of the eigenfunctions are $|C_0\rangle=\frac{1}{\sqrt{2\pi}}e^x=\frac{1}{\sqrt{2\pi}}S, |C_0^{'}\rangle=\frac{1}{\sqrt{2\pi}}e^{-\frac{2rx}{\sigma^2}}=\frac{1}{\sqrt{2\pi}}S^{-\frac{2r}{\sigma^2}}$. For these eigenstates, $\frac{\partial|C_0\rangle}{\partial t}=0, \frac{\partial|C_0^{'}\rangle}{\partial t}=0$ so that these states do not evolve in time, hence, satisfying the martingale property.

7.10 THE CALL PRICE (SECOND APPROACH)

We solve the Black Scholes equation (7.66) through separation of variables. Let $C(x,t)=X(x)Y(t)$ so that

$$\frac{1}{Y(t)}\frac{\partial Y(t)}{\partial t}=\left[-\frac{\sigma^2}{2}\frac{\partial^2 X(x)}{\partial x}+\left(\frac{1}{2}\sigma^2-r\right)\frac{\partial X(x)}{\partial x}+rX(x)\right]\frac{1}{X(x)} \tag{7.84}$$

The left hand side of this equation is a pure function of time and the right hand side is a pure function of x only. Hence each side must be equal to a constant so that

$$\frac{\partial Y(t)}{\partial t}=EY(t) , \tag{7.85}$$

$$-\frac{\sigma^2}{2}\frac{\partial^2 X(x)}{\partial x^2}+\left(\frac{\sigma^2}{2}-r\right)\frac{\partial X(x)}{\partial x}+rX(x)=EX(x) \tag{7.86}$$

Eq. (7.85) has the solution $Y(t) = Ke^{Et}$. Taking T as the exercise point of the derivative, we have $Y(T) = Ke^{ET}$ or $K = Y(T)e^{-ET}$ so that $Y(t) = e^{-E(T-t)}$.Similarly eq. (7.86) has the solution

$$X(x) = e^{\alpha x}\left(Ae^{i\beta x} + Be^{-i\beta x}\right) \text{ where } \alpha = \left(\frac{1}{2} - \frac{r}{\sigma^2}\right), \beta = \left[\frac{2E}{\sigma^2} - \left(\frac{1}{2} + \frac{r}{\sigma^2}\right)^2\right]^{\frac{1}{2}} \quad (7.87)$$

which can also be written in the form $e^{\alpha x}(C\cos\beta x + D\sin\beta x)$ or $Fe^{\alpha x}\sin(\beta x + \gamma)$ where A, B, C, D, F, γ are constants of integration to be determined from the boundary conditions.

Hence, the complete solution of the Black Scholes equation is $C(x,t) = e^{-E(T-t)+\alpha x}\left(Ae^{i\beta x} + Be^{-i\beta x}\right)$. In Dirac notation, the components of the state vector representing the call price are given by $\langle x|C(x,t)\rangle = e^{-E(T-t)+\alpha x}\left(Ae^{i\beta x} + Be^{-i\beta x}\right)$.

In this notation, we have $\langle x|C(x,t)\rangle = e^{-E(T-t)+\alpha x}\left(Ae^{i\beta x} + Be^{-i\beta x}\right)$ and its conjugate $\langle C(x,t)|x\rangle = e^{-E(T-t)-\alpha x}\left(Ae^{-i\beta x} + Be^{i\beta x}\right)$ so that

$$\langle C(x,\beta',t)|C(x,\beta,t)\rangle = \int dx \langle C(x,\beta',t)|x\rangle\langle x|C(x,\beta,t)\rangle$$

$$= e^{-E(T-t)}\left[A^2\int dx e^{i(\beta-\beta')x} + B^2\int dx e^{-i(\beta-\beta')x} + AB\int dx\left(e^{i(\beta+\beta')x} + e^{-i(\beta+\beta')x}\right)\right]$$

$$= 2\pi\left(A^2 + B^2\right)\delta(\beta - \beta')e^{-E(T-t)} + 2\pi AB\delta(\beta + \beta')e^{-E(T-t)} = 2\pi\left(A^2 + B^2\right)\delta(\beta - \beta')e^{-E(T-t)} \quad (7.88)$$

since $\beta + \beta' \neq 0$ which establishes the orthogonality of the call price eigenfunctions.

Noting that $\frac{\delta\beta}{\delta E} = \left[\frac{2E}{\sigma^2} - \left(\frac{1}{2} + \frac{r}{\sigma^2}\right)^2\right]^{-\frac{1}{2}}\frac{1}{\sigma^2}$, we also have

$$\delta(\beta-\beta')=\delta\left\{\frac{1}{\sigma^2}\left[\frac{2E}{\sigma^2}-\left(\frac{1}{2}+\frac{r}{\sigma^2}\right)^2\right]^{-\frac{1}{2}}(E-E')\right\}=\left\{\sigma^2\left[\frac{2E}{\sigma^2}-\left(\frac{1}{2}+\frac{r}{\sigma^2}\right)^2\right]^{\frac{1}{2}}\right\}\delta(E-E')$$

So that the right hand side of eq. (7.88) is equivalent to $2\pi\sigma^2\left(A^2+B^2\right)\left[\frac{2E}{\sigma^2}-\left(\frac{1}{2}+\frac{r}{\sigma^2}\right)^2\right]^{\frac{1}{2}}\delta(E-E')e^{-E(T-t)}$.

thereby also establishing the orthogonality of the energy eigenvalues.

We now examine the set of eigenfunctions for completeness properties. We have

$$\int d\beta\langle x|C(x,\beta,t)\rangle\langle C(x',\beta,t)|x'\rangle=e^{-E(T-t)+\alpha(x-x')}\left[A^2\int d\beta e^{i\beta(x-x')}+B^2\int d\beta e^{-i\beta(x-x')}+AB\int d\beta\left(e^{i\beta(x+x')}+e^{-i\beta(x+x')}\right)\right]$$

$$=2\pi\left(A^2+B^2\right)\delta(\beta-\beta')e^{-E(T-t)+\alpha(x-x')}+2\pi AB\delta(\beta+\beta')e^{-E(T-t)+\alpha(x-x')}=2\pi\left(A^2+B^2\right)\delta(\beta-\beta')e^{-E(T-t)+\alpha(x-x')} \quad (7.89)$$

since $\beta+\beta'\neq 0$ which establishes the completeness of the call price eigenfunctions.

Since the set of call price eigenfunctions is orthogonal and complete, it forms a basis of our Hilbert space and we can evaluate the call price in this basis. This is achieved as follows:-

$$\langle x'|e^{-\hat{H}(T-t)}|x\rangle=\int d\beta\langle x'|C(x',\beta,t)\rangle\langle C(x',\beta,t)|e^{-\hat{H}(T-t)}|C(x,\beta,t)\rangle\langle C(x,\beta,t)|x\rangle$$

Now $\langle C(x',\beta,t)|\hat{H}|C(x,\beta,t)\rangle = E\delta(x-x')$ so that $\langle C(x',\beta,t)|e^{-\hat{H}(T-t)}|C(x,\beta,t)\rangle = e^{-E(T-t)}\delta(x-x')$. Also $\langle x|C(x,t)\rangle = e^{-E(T-t)+\alpha x}\left(Ae^{i\beta x}+Be^{-i\beta x}\right)$ and its conjugate $\langle C(x,t)|x\rangle = e^{-E(T-t)-\alpha x}\left(Ae^{-i\beta x}+Be^{i\beta x}\right)$ so that we finally obtain

$$
\begin{aligned}
&\langle x'|e^{-\hat{H}(T-t)}|x\rangle = \int d\beta e^{-E(T-t)}\langle x'|C(x',\beta,t)\rangle\langle C(x,\beta,t)|x\rangle \\
&= \int d\beta e^{-E(T-t)+\alpha(x'-x)}\left[A^2e^{i\beta(x'-x)}+B^2e^{-i\beta(x'-x)}+AB\left(e^{i\beta(x+x')}+e^{-i\beta(x+x')}\right)\right]
\end{aligned}
$$

which on substituting the value of E in terms of β using $\beta = \left[\frac{2E}{\sigma^2}-\left(\frac{1}{2}+\frac{r}{\sigma^2}\right)^2\right]^{\frac{1}{2}}$ becomes a Gaussian integral and can be evaluated to get the call price.

CHAPTER 8
EMPIRICAL RESULTS ON THE DISTRIBUTION OF RETURNS & MEMORY EFFECTS IN INDIAN CAPITAL MARKETS

Abstract

This chapter examines the various features of the logarithmic return spectrum of the Indian stock market [160-167] thereon the various statistical tests for the normality of data. It also investigates the possible existence of dependencies and memory effects in the return processes. In particular, it performs rescaled range analysis and carries on to compute the Hurst's exponent. The results throw up several intriguing issues of relevance to portfolio managers, stock market players and analysts and academicians.

8.1 INTRODUCTION

There exist two traditional approaches to the modelling of a dynamical system. In the first approach, the dynamical deterministic equations of motion are obtained from first principles as differential / difference equations that are integrated forward in time and solved as an initial value problem. This methodology, although strongly preferred due to its exactness, is sometimes impracticable, particularly when we are analysing the dynamics of many particle systems with complicated interactions among the constituents. In such cases, either the number of degrees of freedom becomes as large as to make the first-principles model intractable or the initial conditions pertaining to each degree of freedom become inaccessible. Attempts are, then, made to model the dynamics as a random process with stochastic, though linear, laws of motion. There was believed to be no region of overlap between these two well-defined approaches.

The modus operandi for studies on stock market phenomena was no different and one could go to the extent of saying that the Efficient Market Hypothesis [79, 96] was formulated with one primary objective – to create a scenario that would justify the use of stochastic calculus [81] for the modeling of capital markets.

The cardinal maxim of the Efficient Market Hypothesis is the existence of a market where all assets are fairly priced according to the information available with neither the buyers nor sellers enjoying any advantage. Market prices are supposed to incorporate all publicly accessible information, both fundamental and price history. It is, further, postulated that prices move only as sequel to new information entering the market. The presence of large number of investors ensures that all prices are fair. Memory effects, if any at all, are extremely short ranging and dissipate rapidly. Feedback effects on prices are, thus, assumed to be marginal. The investor community is considered rational as benchmarked by the traditional concepts of risk and return.

An immediate corollary to the Efficient Market Hypothesis is the independence of single period returns, so that they may be modeled as a random walk and the defining probability distribution, in the limit of the number of observations being large, would be the normal distribution.

This Chapter examines the various features of the logarithmic return spectrum of the Indian stock markets, performing thereon the various statistical tests for the normality of data. It also investigates the possible existence of dependencies and memory effects in the return processes. In particular, it performs rescaled range analysis and carries on to compute the Hurst's exponent. The results throw up several intriguing issues of relevance to portfolio managers, stock market players and analysts and academicians.

8.2 TESTING & EVIDENCE ON THE NORMALITY OF STOCK MARKET RETURNS IN INDIA

As mentioned above, the pivotal fallout of the Efficient Market Hypothesis is that the present price of a security encompasses all the presently available information – including past prices – concerning this security and prices tend to move only if and when a fresh information about the security percolates into the market. Even the seminal work of Fischer Black & Myron Scholes in the pricing of contingent financial claims (that constitutes the cornerstone of contemporary valuation theory) presupposes that the stock prices follow a geometric Brownian motion, the two principal attributes whereof are that:-

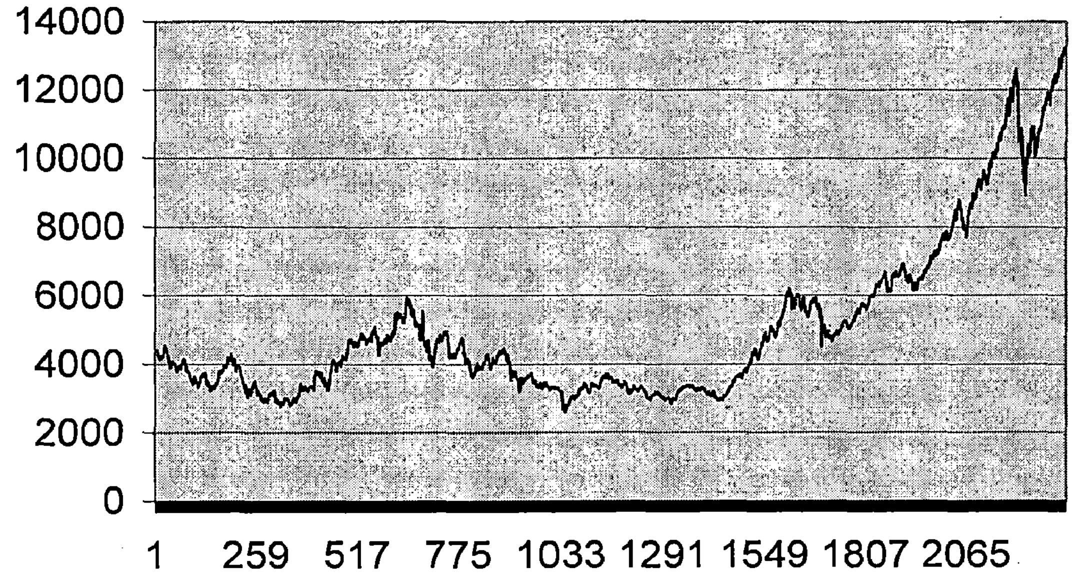
(i) The set of stock prices $S(t), 0 \le t < \infty$ constitute a geometric Brownian motion if, $\forall s,t \ge 0$, the random variable $\frac{S(s+t)}{S(t)}$ is independent of all prices up to time t;

(ii) $\ln\left[\frac{S(s+t)}{S(t)}\right]$ is a normally distributed random variable $\forall s,t \ge 0$ with mean μt and variance $t\sigma^2$ where μ and σ constitute the parameters defining the geometric Brownian motion.

(iii) It follows from (i) & (ii) that the probabilities of the ratio of the price at time s in the future to the present price will not depend on the present price. Additionally, if μ and σ are known, then it is only the present price – and not the history of past prices – that affects the expectations of future prices. Specifically, we have, $E[S(t)] = S(0)e^{\left(\mu+\frac{1}{2}\sigma^2\right)t}$.

In the empirical study that constitutes the substratum of this paper, we test the hypothesis that future price movements are independent of past movements i.e. the stock market logarithmic returns follow a normal distribution in the Indian capital markets. We also examine whether memory effects of any significant duration subsist in these markets. We assume the 30 security BSE SENSEX market index as the proxy for the Indian stock market and conduct the analysis of the Sensex over the period from July 01, 1997 to November 10, 2006. Consisting of 2,317 observations. A chronological plot of the Sensex values for the above period is given in Figure 8.1.

Figure 8.1



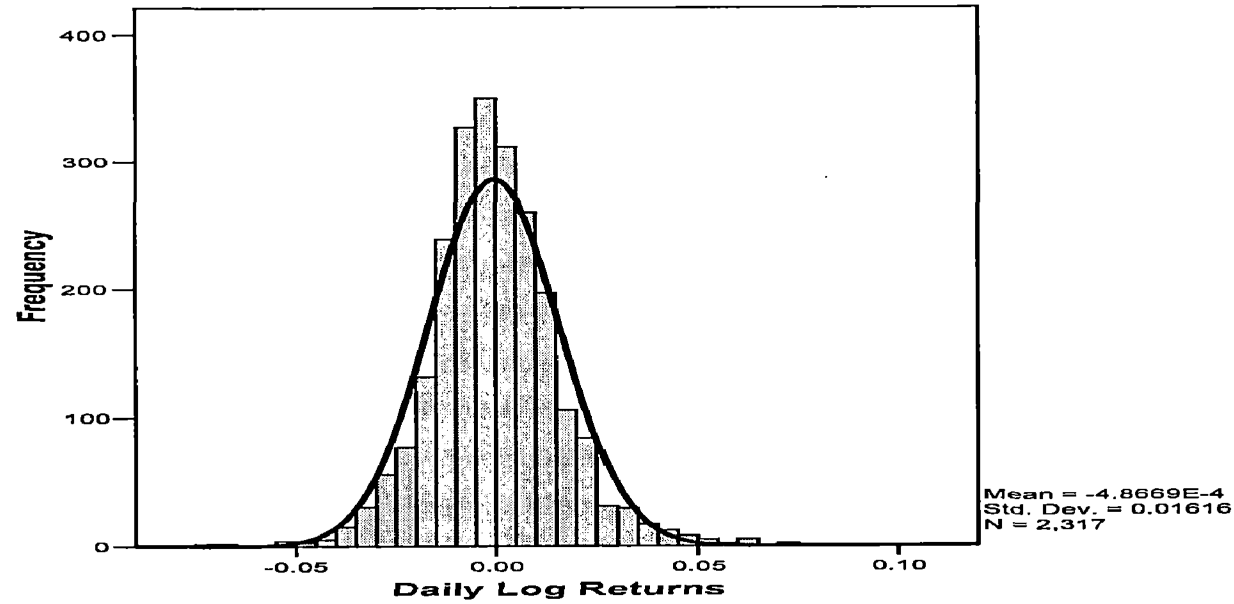
The mean and standard deviation of the daily logarithmic returns of the prices constituting the sample was found to be 0.000486691and 0.016158843 respectively. To test these returns for normality, the number and percentage of observations in the various 0.5σ intervals were calculated and compared with the corresponding values for the standard normal distribution $N(0,1)$. The results are tabulated in Table 8.1 below:-

Interval	No. of Observations	% of Total	Corresponding value for $N(0,1)$
$x \le \bar{x} - 2\sigma$	74	0.031938	0.0228
$\bar{x} - 2\sigma < x \le \bar{x} - 1.5\sigma$	53	0.022874	0.044
$\bar{x} - 1.5\sigma < x \le \bar{x} - \sigma$	157	0.06776	0.0919
$\bar{x} - \sigma < x \le \bar{x} - 0.5\sigma$	331	0.142857	0.1498
$\bar{x} - 0.5\sigma < x \le \bar{x}$	497	0.214502	0.1915
$\bar{x} < x \le \bar{x} + 0.5\sigma$	553	0.238671	0.1915
$\bar{x} + 0.5\sigma < x \le \bar{x} + \sigma$	385	0.166163	0.1498
$\bar{x} + \sigma < x \le \bar{x} + 1.5\sigma$	139	0.059991	0.0919
$\bar{x} + 1.5\sigma < x \le \bar{x} + 2\sigma$	82	0.035391	0.044
$\bar{x} + 2\sigma < x$	46	0.019853	0.0228

Table 8.1

A histogram corresponding to the above data is placed at Figure 8.2. It is clear from the histogram that the assumption of normality of log-returns is, at best, questionable. We pursue the analysis further in the next section.

Figure 8.2



8.3 TESTING FOR MEMORY EFFECTS

As mentioned above, one of the crucial fallouts of the Efficient Market Hypothesis and/or of the assumption of geometric Brownian motion as a model of stock prices is the total absence of any kind of memory effects in the return processes. The above histogram does not provide any level of conclusive evidence for or against the existence of memory effects since it breaks up the range of data values into intervals and then plots the number of data values that fall in each interval. It does not, therefore, provide information about possible dependencies among the data.

To examine the possible existence of dependencies, the daily logarithmic returns (x) are classified into one of six possible class intervals A, B, C, D, E & F viz $x \leq \bar{x} - 2\sigma$ (A),

$\bar{x}-2\sigma < x \le \bar{x}-\sigma$ (B), $\bar{x}-\sigma < x \le \bar{x}$ (C), $\bar{x} < x \le \bar{x}+\sigma$ (D), $\bar{x}+\sigma < x \le \bar{x}+2\sigma$ (E) and $\bar{x}+2\sigma < x$ (F). The observations falling into each of these class intervals are, then, split up on the basis the daily logarithmic return on the next following day. We, thus get a 6×6 square matrix (Table 8.2), the $(i, j)^{th}$ element of which would be the observation that has a logarithmic return falling in the i^{th} class on day n and j^{th} class on day $n+1$. Now, if the price evolution follows a geometric Brownian motion then tomorrow's state should not depend on today's state. In other words, the subset of observations comprising every row should behave as if they are extracted from a normal population. We proceed to test this hypothesis by the well-known χ^2 test, where the expected frequencies are the corresponding values from a normal population. Table 8.3 summarizes the computations (In Table 8.3, we have replaced absolute values of various cells of Table 8.2 by the corresponding percentages. The expected frequencies, also in percentages, are given in parenthesis).

	A	B	C	D	E	F	
A	12	9	15	16	12	10	74
B	13	28	64	74	24	7	210
C	27	89	329	315	58	10	828
D	12	58	339	430	85	14	938
E	6	21	71	86	33	3	220
F	4	5	10	16	8	3	46

Table 8.2

	A (2.28)	B (13.59)	C (34.13)	D (34.13)	E (13.59)	F (2.28)	χ^2
A	16.21622	12.16216	20.27027	21.62162	16.21622	13.51351	151.40
B	6.190476	13.33333	30.47619	35.2381	11.42857	3.333333	7.97
C	3.26087	10.74879	39.7343	38.04348	7.004831	1.207729	6.08
D	1.279318	6.183369	36.14072	45.84222	9.061834	1.492537	10.39
E	2.727273	9.545455	32.27273	39.09091	15	1.363636	2.63
F	8.695652	10.86957	21.73913	34.78261	17.3913	6.521739	32.06

Table 8.3

The overall value of χ^2 is found to be 210.54.

It is clear from Table 8.3 that in both the tails of the distribution representing external cases, there exist very significant memory effects over the one-day period and the distributions are well distorted from the normal distribution – the distortion is particularly massive in the left tail. However, over the intermediate range, the χ^2 values are much closer to the tabulated values, although still well above them even at a 1% significance level establishing that some degree of memory effects do exist even in this range. The histograms of each class interval are placed in Figures 8.3 (A) to 8.3 (F)

Figure 8.3(A)

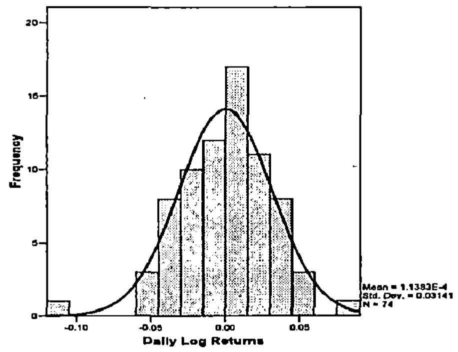


Figure 8.3(B)

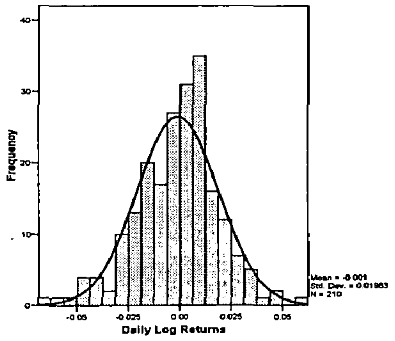


Figure 8.3(C)

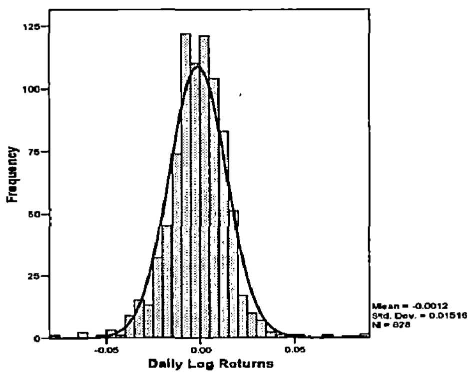


Figure 8.3(D)

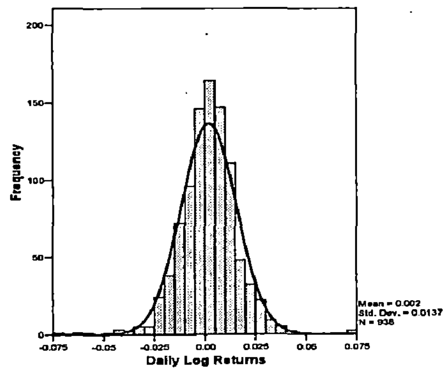


Figure 8.3(E)

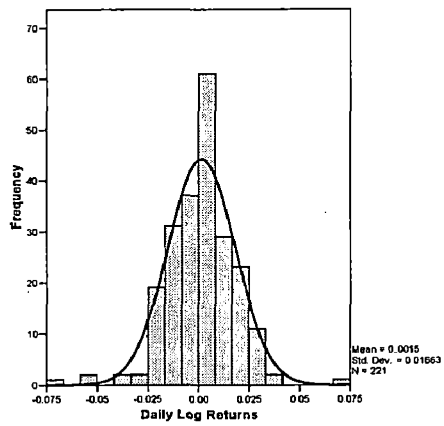
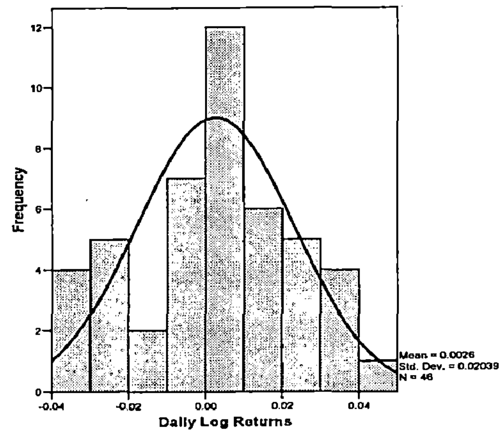


Figure 8.3(F)



To further corroborate the departure of real data from the geometric Brownian motion and the subsistence of significant memory effects we have also performed the one-way ANOVA test to test the hypothesis that all the six data sets describe normal random variables having the same mean and variance. The results of the analysis are tabulated below in Table 8.4:-

	Sum of Squares	df	Mean Square	F	Sig.
Between Groups	.005	5	.001	4.184	.001
Within Groups	.600	2311	.000		
Total	.605	2316			

Table 8.4

The computations provide an F-statistic value of 4.184 that implies that the differences in the means of the six subsets are significant at the 0.05 significant levels. In fact, the F-statistic value of 4.184 implies statistical acceptability of the hypothesis that the subsets A, B, C, D, E & F are normally distributed with the same mean only at a 0.001 level of significance.

8.4 CHAOS [128-132, 169] & INDIAN STOCK MARKETS

Most financial returns, including stock returns have shown deviation from Gaussian behaviour at short time scales with the variance not scaling with the sq. root of timescale, an attribute that is symptomatic of the possible existence of power law distributions. A useful measure of quantifying deviations from the Gaussian distribution is the Hurst's exponent. If a population is Gaussian, a Hurst's exponent of 0.5 is mandated. Empirical evidence, however, shows that the Hurst's exponent for typical stock market data is around 0.6 for small timescales of about a day or less and tends to approach 0.5 asymptotically with the lengthening of the timescales. Empirical evidence also demonstrates the existence of memory effects, particularly in stock price volatilities that show long-term memory effects with lag-s autocorrelations. Further, these effects tend to fall off according to a power law rather than exponentially.

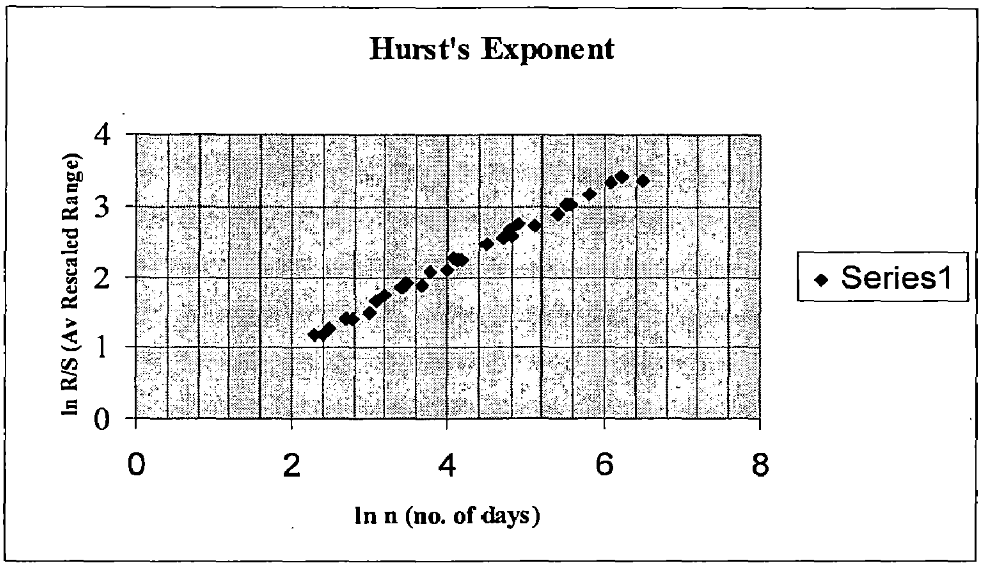
In our study, we have performed the Rescaled Range analysis of the logarithmic returns on the BSE Sensex for the period from July 01, 1997 to November 10, 2006 consisting of 2,317 observations in the manner provided in Ref [126,127]. The results are tabulated in Table 8.5 below:-

Days	Average Rescaled Range	Days	Average Rescaled Range	Days	Average Rescaled Range
10	3.273144	32	6.637159	120	14.08
11	3.230032	33	6.635194	124	13.13
12	3.500606	40	6.585	132	15.54
15	4.048635	44	7.99	165	15.21
16	4.035323	55	8.23	220	17.75
20	4.455719	60	9.61	248	20.76
22	5.146307	62	9.48	264	20.7
24	5.631192	66	9.37	330	23.4
30	6.350008	88	11.6	440	28.09
31	6.306971	110	12.68	496	30.77

Table 8.5

The relationship between the Rescaled Range R/S and the Hurst's Exponent H is given by $R/S = A\times(n)^H$ or equivalently $H = \frac{\ln(R/S)-\ln A}{\ln n}$. Hence, in order to obtain the Hurst's exponent, the logarithm of the Rescaled Range is plotted against the logarithm of the number of days, the slope of this plot (Figure 8.5) being the value of H. The slope and hence, the value of the Hurst's Exponent is found to be 0.58 which is commensurate with similar findings in stock markets of several other countries [126,127].

Figure 8.5



There is a simple mathematical link between the Fractal Dimension or the associated Hurst's Exponent H and the physical dimension. H = 1 corresponds to a perfectly persistent series representable by a unidimensional line whereas H=0.50 corresponds to random or Brownian motion, it is equal to a dimension of 1.50, a fractal or noninteger dimension halfway between a line and a plane. And where H=0, a perfectly antipersistent time series, the corresponding physical dimension is a plane or 2.

The fact that the value of the Hurst's exponent for the time series that is the subject matter of this study has been found to be 0.58 corroborates our earlier findings that stock market returns in the Indian capital markets are not random and hence, do not constitute a population that is normally distributed. Furthermore, geometric Brownian motion cannot accurately model the stock prices. There also exist significant memory effects that result in lumping of observations into some sort of clusters particularly for returns that are located in the tails of the distribution.

Not only do the quantitative tests contradict the validity of the EMH, but the assumption of "Investor Rationality" as envisaged in the EMH may also be questioned on the following grounds:-

- The EMH presupposes that all investors are risk averse. However, investors may not be risk averse in all situations. They may become risk takers in certain situations e.g. when confronted with a situation that involves perceived sure losses. For example, if asked for a trade off between a certain loss of $. 85,000 vs. a loss of $. 1,00,000 with a probability of 0.85 and a zero loss with probability of 0.15 would generally find the investor opting for the latter;

- Investors are usually more confident of their forecasts than is warranted by the available information. They have a tendency to ignore new information if it does not fit in with their current forecasts of the future;

- Investors would not normally react to trends until fully established, a phenomenon that takes some time. They will not begin to accept and extrapolate a set of circumstances until it is firmly established. They then take a decision on the basis of all the information that has accumulated thus far. In other words, reaction to information does not occur in a continuing fashion as and when it is received, but rather in discrete blocks & clumps in a cumulative fashion.

- Only when the level of information reaches a critical level, investors react to all the information received till then. Hence, memory effects subsist

- As a corollary to the above, markets lose their efficiency since all information is not reflected in prices and much of the information is ignored and accumulated till it reaches a threshold level and reaction comes later;

Acceptance of complete randomness in stock prices is beset with questions of consistency as well. One must need appreciate that the vantage point of each investor is different and also that it keeps on changing with the passage of time. That is, they have different points of view. If that is not the case, how does one explain as to why people can be rational investors and still make very different investment decisions? The fact remains that everyone's perspective is different and is varying. Rationality of investors may be construed in that they are internally self-consistent with the information that they possess. However, their decisions may seem illogical from a different informational point of view.

CHAPTER 9
CONCLUSION

9.1 FINDINGS, RESULTS & CONCLUSIONS

In this chapter, the major findings and contributions of the present studies are summarized. The limitations of the study and suggestions for areas of further research are also enumerated.

From the above research the following main conclusions are drawn:-

The objective of this research project was to take the merging of physics and finance program further through a study of the symmetry groups of the dynamical equations relevant to financial processes and, intertwining this two areas through stochastic processes in order to facilitate

(i) the development of a model of financial markets amenable to the quantum mechanical framework and

(ii) the generalizations of extant results to enhance their domain of applicability.

Generalization of the Black Scholes option pricing model by introducing a stochastic process with statistical feedback as a model for stock market returns was achieved. The generalization of financial dynamics of the stock price process by using the deformed Levy process is also studied. A third model using a stochastic return process in lieu of the risk free returns in the Black Scholes partial differential equation is solved. The financial markets model, within the framework of quantum mechanics is constructed. The properties of the Lie algebra being the underlying symmetry of the Black Scholes partial differential equation that represents the dynamics of a financial derivative is studied and interpreted to get new solutions of the above said equation. Empirical study of the Indian capital markets with reference to the normality of return process, existence of significant memory effects and possibility of nonlinear and chaotic behavior is also conducted.

The following are some of the significant contributions emanating from the above research work:-

Generalization of the Black Scholes equation has been attempted in this study by introducing a stochastic process with statistical feedback as a model for stock market returns. This model can embrace in its ambit possible nonlinearities and chaotic behaviour in stock price patterns through the deformation parameter.

The Black Scholes model with a stochastic return process in lieu of the risk free returns in the Black Scholes partial differential equation is studied considering the Black Scholes equation as a partial differential equation in two stochastic processes.

A model of the financial markets has been evolved using the conventional machinery of quantum mechanics and operators pertaining to various trading activities. The symmetry group of the Black Scholes equation has been obtained and the properties of the Lie algebra of the Black Scholes partial differential equation that represents the dynamics of a financial derivative are studied and new solutions of the said equation are obtained.
An empirical study of the Indian capital markets was also conducted with data over the last ten years and it was shown that stock return processes deviate significantly from normality. Performance of R/S analysis also showed that memory effects are prevalent in the price time series with a possibility of nonlinearities and chaos. The result of a generalization of financial dynamics of the stock price process as a deformed Levy process is also discussed.

In the above discussion the following important outcomes of this research work:-

The closed form expression has been obtained for the price of a European call option by modifying the Black Scholes option pricing formula by generalizing the stock return process to a probability dependent deformed Brownian motion that could accommodate "statistical feedback" processes and, thereby, account for the fat tails usually observed in stock market price distributions. It is seen that that in the standard case the exponential is linear in W and the stock price, therefore, is a monotonically increasing function of W. Hence, the condition $S_t - E > 0$ is satisfied for all values of W that exceed a threshold value. However, in this model, consequent to the noise induced drift, the exponential in the stock price process is now a quadratic function of the deformed Brownian motion U. We, therefore, have two roots of U that meet the condition $S_t - E = 0$. Accordingly, there will exist an interval (U_1, U_2) within which the inequality $S_t - E > 0$ will hold. Furthermore, as $q \to 0$, $U_2 \to \infty$ thereby recovering the standard case.
The pricing of financial contingent claims has also been explored when the distribution of the underlying asset is a deformed Levy process. A model of financial markets has been constructed within the quantum mechanical framework, various operators signifying the market processes have been constructed and the market dynamics explored. We derive the probability distribution of stock prices in market equilibrium and show that the prices follow a lognormal distribution, thereby vindicating the efficacy of this model under suitable assumptions as to the quantum mechanical states and amplitudes. Solutions of the Black Scholes equation have been obtained from symmetry considerations and their properties studied with the relevant Lie groups. The various features of the logarithmic return spectrum of the Indian stock markets are examined, performing thereon the various statistical tests for the normality of data like chi-square, ANOVA. The possible existence of dependencies and memory effects in the return processes is also examined. In

particular, the rescaled range analysis is carried out to compute the Hurst's exponent. It is seen that there is unambiguous evidence to the effect that the returns deviate significantly from normal behaviour. There is also evidence of the existence of memory effects and consequential nonlinearity.

Closed form expressions for the price of a European call option by modifying the Black Scholes formulation to accommodate a stochastic return process for the "hedge portfolio" returns. We have modeled this return process on the basis of the Vasicek model for the short-term interest rates. The construction of the "hedge portfolio" in the Black Scholes theory implies that the fluctuations in the price of the derivative and that of the underlying exactly and immediately cancel each other when combined in a certain proportion viz. one unit of the derivative with a short sale of $\frac{\partial C}{\partial S}$ units of the underlying so that the "hedge portfolio" is devoid of any impact of such fluctuations. This mandates an infinitely fast reaction mechanism of the underlying market dynamics whereby any movement in the price of one asset is instantaneously annulled by reactionary response in the other asset constituting the "hedge portfolio". This is, obviously strongly unrealistic and there may subsist brief periods or aberrations when the no arbitrage condition may cease to hold and hence, returns on the "hedge portfolio" may be different from the risk free rate. One way of attending to this anomaly is to model the returns on the "hedge portfolio" as a stochastic process as has been done in this study. The parameters defining the process can be obtained through an empirical study of the market dynamics. Another important justification for adopting a stochastic framework for the "hedge portfolio" return process is that the "hedge portfolio" by its very construction, envisages the neutralization of the fluctuations of the two assets inter se i.e. it assumes a perfect correlation between the two assets. In other words, the "hedge portfolio" may be construed as an isolated system that is such that insofar as factors that influence one component of the system, the same factors influence the other component to an equivalent extent and, at the same time, other factors do not impact the system at all. This is another anomaly that distorts the Black Scholes model. The fact is that while the "hedge portfolio" of the Black Scholes model is immunized against price fluctuations of the underlying and its derivative through mutual interaction, other market factors that would impact the portfolio as a whole are not accounted for e.g. factors affecting bond yields and interest rates etc. Consequently, to assume that the "hedge portfolio" is completely risk free is another aberration – it is risk free only to the extent of risk that emanates from factors that impact the underlying and the derivative in like manner and is still subject to risk and uncertainties that originate from factors that either do not effect the underlying and the derivative to equivalent extent or impact the portfolio as a unit entity. Hence, again, it becomes necessary to model the return on the "hedge portfolio" as some short-term interest rate model as has been done here.

We have attempted to develop the theory of option pricing in incomplete markets with stock market pricing being simulated by Levy processes. These processes have an intimate connection with pseudodifferential operators in the sense that their characteristic exponent is a pseudodifferential operator. Hence, we can associate a pseudodifferential operator with every Levy processes. The converse of this also holds and Levy processes can be generated by the knowledge of a pseudodifferential symbol. We take advantage of this, and making use of the q deformed pseudodifferential symbols, which have been the substratum of recent research, we attempt to construct the corresponding Levy processes. Treating the deformation as a perturbation, we identify the conditions under which these symbols generate a Feller semigroup for which Levy processes can be constructed in the usual way.

Another result in this research work is the use of the quantum mechanical methods for obtaining the instantaneous price of a call option. Traditionally derivatives pricing is done through construction of self financing strategy consisting of Bonds and Shares. However here we have not used this concept and the problem has been modeled has a dynamical system.

9.2 LIMITATIONS OF THE PRESENT STUDY

It needs to be emphasized here that the above models are purely phenomenological models for modeling stock behavior. One could, for instance, postulate that the statistical feedback at the microscopic level represents the actions and interaction of the intra trader interactions among traders constituting the market. The statistical dependency in the noise could, further, be representing the aggregate behavior of these traders. Thus, we could model a market with non homogeneous reactions with consequent biased return structures

It is fair to say that the current stage of research in financial processes is dominated by the postulation of phenomenological models that attempt to explain a limited set of market behavior. There is a strong reason for this. A financial market consists of a huge number of market players. Each of them is endowed with his own set of beliefs about rational behavior and it is this set of beliefs that govern his actions. The market, therefore, invariably generates a heterogeneous response to any stimulus. Furthermore, "rationality" mandates that every market player should have knowledge and understanding about the "rationality" of all other players and should take full cognizance in modeling his response to the market. This logic would extend to each and every market player so that we have a situation where every market player should have knowledge about the beliefs of every other player who should have knowledge of beliefs of every other player and so on. We, thus, end up with an infinitely complicated problem that would defy a solution even with the most sophisticated mathematical procedures. Additionally, unlike as there is in physics, financial economics does not possess a basic set of postulates like General Relativity and Quantum Mechanics that find homogeneous applicability to all systems in their domain of validity.

9.3 SUGGESTIONS FOR FURTHER RESEARCH

It is felt that research carried out in this thesis has been quite extensive that has applications not only in the field of corporate finance, but also in mathematical finance. The present work offers considerable scope and promise for further research. Some possible extensions could be as follows:

Further research projects could target to carry this unification program further. Efforts may be made to apply the contemporary tools of physics to the mechanics of financial markets. Identification of an appropriate action function and the derivation of the mechanics of financial markets, therefore, could be attempted. Properties of financial markets could be examined within the framework of differential geometry with the exploration of the underlying symmetries and structure groups (with a study of their properties) etc.

The theory of pseudo differential operators provides an avenue for obtaining closed form option pricing in markets modeled by Regular Levy processes of exponential type (RLPE), which extend the Black Scholes theory to non Gaussian domains. Various types of deformations of such operators can be constructed and corresponding option pricing models explored.

9.4 CONCLUSION

This chapter synthesized the research work carried out and discussed in the previous chapters. Major contribution of the study, limitation of the study, and the scope of further research work are identified.

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